Surface Measures on Path Spaces of Riemannian Manifolds

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Summary

This thesis discusses an approach to define surface measures on the path spaces of Riemannian submanifolds given by a Brownian motion. These surface measures have been discussed from various points of view e.g. in [Smo95], [SWW00], [SWW03], [Sid04] and [Wit11]. In this context, it has been helpful to study Brownian motion on tubular neighborhoods of submanifolds and the associated Wiener measure on the infinite-dimensional path spaces. The concept of surface measures is a generalization of the notion of the Lebesgue measure in Euclidean space.
First, the space of continuous paths in $\mathbb{R}^n, n \in \mathbb{N}$, equipped with the Wiener measure $\mathbb{W}_{\mathbb{R}^n}$ and the subspace of continuous paths with values in a compact Lie group $L \subset \mathbb{R}^n$ were studied. Two different ways to define a surface measure were developed and it has been proven that, in this special case, both surface measures coincide with the Wiener measure on the trajectories of the Lie group L. Next, with these approaches on hand, the conditioning of Brownian motion to arbitrary Riemannian submanifolds in Euclidean space $\mathbb{R}^n$ was investigated. Again two different definitions occurred. One result was obtained by the computation of the weak limit of conditional Brownian motion on (small) tubes of decreasing diameter around the submanifold. The other way was provided by the law of reflected Brownian motion in $\mathbb{R}^n$, starting on the submanifold with reflection at the boundary of the tubes (see [Sid03]). Later, these results were also generalized to closed Riemannian submanifolds, i.e. isometrically embedded manifolds, of arbitrary ambient Riemannian manifolds.

In this thesis we deal with the last case. Briefly, let (L, g_L) be a closed Riemannian manifold, isometrically embedded into the complete Riemannian manifold (M, g). We study a Brownian motion on M, starting in $x \in L$. Conditioning Brownian motion to tubular neighborhoods $L(\varepsilon), \varepsilon > 0$, of L is closely related to Brownian motion absorbed at the boundary $\partial L(\varepsilon)$ of the tube. It has been shown (see e.g. [SSWW14]) that the conditional measures correspond to Markov processes and that they converge weakly for $\varepsilon \to 0$. Furthermore, in [Wit11] it has been proved that the solution of the heat equation on tubes of small, decreasing diameters can be described by the solution of the heat equation on L and an additional potential $W_0 \in C^\infty(L)$. It was revealed that W_0 reflects properties of L, M and the embedding $\varphi : L \to M$, i.e. intrinsic and extrinsic geometric quantities.

The aim of this thesis is to investigate and interpret the potential W_0 and its properties. The primary focus is to understand the role W_0 plays in the above described limit process.

Our main result is that for any given closed Riemannian manifold L and $W \in C^\infty(L)$ we are able to construct an embedding of L into an ambient space with W as the associated potential function.

Before we start with the analysis of our main object, we begin with an introduction of the quantities and notation we use throughout this work. Hence, Chapter 1 begins with a detailed description of the initial setting and states the above mentioned results.

In Chapter 2, we explain the behavior of the potential in special cases and elucidate the relevance of W_0 for them. The study of totally geodesic embeddings is remarkably fruitful. In this case, a detailed geometric discussion demonstrates that the potential is related to the first non-zero correction term, which occurs in a well-known volume formula by Gray and Weyl (see [Gra82], p. 203, (1.4)). Hence, we are able to provide an interpretation in this case, since the potential is here closely related to the finite dimensional volume of the tube. These observations will also be used in Chapter 5, where we pick up this connection again. Moreover, we present a coordinate-free version of the volume formula.

Additionally, we develop various formulas for the potential and related relevant objects differing from the ones given in [Wit11] and show that the results depend only on the geometry of the tube (Chapter 3).

In Chapter 4, we prove the main theorem of this work, i.e. Theorem (4.33). As stated above, any given smooth function on a Riemannian manifold occurs as a potential function for a suitable embedding. To prove this, we construct some bilinear form on the fibers of an arbitrary vector bundle $\pi : E \to L$ over L. With the help of this form, we find the so-called perturbed Sasaki metric g_r on a small neighborhood $L(\varepsilon)$ of the zero section in E. We prove that L can be isometrically embedded into $L(\varepsilon)$ equipped with g_r and then we compute the density and the potential in this new setting.

Not only geometric and analytic tools are helpful for our investigations. Another aspect in the study of surface measures is their relation to the probability measures on path spaces. We study this stochastic connection and again the behavior of W_0 in this setting in Chapter 6. Moreover, we investigate the behavior of W_0 for a triple $L_2 \subset L_1 \subset M$ of (isometric) embeddings of Riemannian manifolds: Explicitly, we compare the surface measures obtained by directly conditioning from M to L_2 with the successively conditioning from M to L_1 and afterwards to L_2. We prove that these surface measures differ in general, but we are able to give a cocycle relation between them.

Finally, in Chapter 7, we investigate the connection between Brownian motion and the geometry of the Riemannian manifolds involved, i.e. the Laplace-Beltrami operators according to the Riemannian metrics. In this context, we comment on Riemannian submersions and natural metrics on the ambient space and how these objects simplify the proofs.

Zusammenfassung

In dieser Dissertation werden Oberflächenmaße auf Pfadräumen Riemannscher Mannigfaltigkeiten, die von einer Brownschen Bewegung induziert werden, diskutiert. Hierbei benutzen wir vor allem Hauptresultate aus [Smo95], [SWW00], [SWW03], [Sid04] und [Wit11], in denen diese Thematik bereits vielfältig erörtert wurde. Die Brownsche Bewegung wurde dabei auf Tubenumbegungen von Untermannigfaltigkeiten studiert. Dieses Konzept von Oberflächenmaßen verallgemeinert den Begriff des Lebesguemaßes in einem Euklidischen Raum.
Zunächst wurde der Raum der stetigen Wege in $\mathbb{R}^n$ betrachtet, der mit dem Wiener Maß $\mathbb{W}_{\mathbb{R}^n}$ ausgestattet ist. Als Untermannigfaltigkeit wurde eine kompakte Liegruppe $L \subset \mathbb{R}^n$ untersucht. In dieser Situation konnten Oberflächenmaße auf zwei verschiedene Arten konstruiert werden. Außerdem wurde gezeigt, dass in diesem Spezialfall die Definition der konstruierten Oberflächenmaße mit dem Wiener Maß auf dem Pfadraum der Liegruppe übereinstimmt. Auf Grundlage dieser Resultate wurden anschließend beliebige Riemannsche Untermannigfaltigkeiten des Euklidischen Raumes $\mathbb{R}^n$ studiert. Erneut wurden zwei verschiedene Definitionen angegeben. Hierbei wurde einerseits bedingte Brownsche Bewegung auf immer kleiner werdenden Tuben um die Untermannigfaltigkeit studiert und andererseits wurde die Verteilung von Brownscher Bewegung auf $\mathbb{R}^n$, die am Rand der Tuben reflektiert wird, untersucht (vgl. [Sid03]). Später konnten diese Ergebnisse und Definitionen auch auf beliebige (geschlossene) Riemannsche Mannigfaltigkeiten verallgemeinert werden, die isometrisch in eine umgebende Mannigfaltigkeit eingebettet sind.

In dieser Arbeit behandeln wir den letzten Fall. Es sei (L, g_L) stets eine geschlossene Riemannsche Mannigfaltigkeit, die isometrisch in die vollständige Riemannsche Mannigfaltigkeit (M, g) eingebettet ist. Wir betrachten eine Brownsche Bewegung auf M, die in $x \in L$ startet. Es stellt sich heraus, dass bedingte Brownsche Bewegung auf Tubenumgebungen $L(\varepsilon), \varepsilon > 0$, der Untermannigfaltigkeit, mit der absorbierten Brownschen Bewegung am Rand der Tube zusammenhängt. Für die bedingten Maße wurde in [SSWW14] gezeigt, dass ihnen Markov Prozesse zugrunde liegen, die für $\varepsilon \to 0$ schwach konvergieren. Außerdem wurde in [Wit11] bewiesen, dass die Lösung der Wärmeleitungsgleichung auf Tubenumgebungen von immer kleiner werdendem Radius durch die Lösung der Wärmeleitungsgleichung auf L und einer glatten Potentialfunktion W_0 auf L beschrieben werden kann. Dabei hängt W_0 von den Eigenschaften von L, M und der Einbettung $\varphi : L \to M$, also von intrinsischen und extrinsischen Größen, ab.

Das Hauptziel dieser Arbeit ist die Interpretation und Berechnung des Potentials W_0 und die Analyse seiner Eigenschaften. Hierbei wollen wir vor allem verstehen, welche Rolle W_0 in dem oben genannten Prozess spielt.

Das Hauptresultat ist, dass zu jeder gegebenen glatten Funktion W auf einer geschlossenen Riemannschen Mannigfaltigkeit L eine Einbettung von L in einen umgebenden Raum konstruiert werden kann, sodass die vorgeschriebene Funktion W bereits das Potential dieser Einbettung ist.

Bevor wir mit der Analyse beginnen, werden wir in Kapitel 1 zunächst alle wichtigen Definitionen und Resultate einführen. Außerdem erklären wir die Ausgangssituation und die dieser Dissertation zugrundeliegende Notation.

Im zweiten Kapitel betrachten wir das Potential für verschiedene Spezialfälle von Einbettungen und berechnen die expliziten Formeln. Im Falle total geodätischer Einbettungen ergibt sich ein äußerst bemerkenswertes Resultat. Das Potential ist hier nämlich genau der erste relevante Störungsterm aus der Volumenformel für Tuben von Weyl und Gray (siehe [Gra82], S. 203, (1.4)). Dieses Resultat liefert, durch den Zusammenhang zum endlich dimensionalen Tubenvolumen, eine erste Interpretation für W_0. Die Volumenformel werden wir in Kapitel 5 erneut aufgreifen und in diesem Zusammenhang mit Hilfe unserer bisherigen Ergebnisse eine koordinatenfreie Darstellung für den oben genannten Störungsterm bestimmen.

In Kapitel 3 beschäftigen wir uns mit der Geometrie der Tubenumgebung und zeigen, dass diese die Situation maßgeblich bestimmt. Diese Resultate werden genutzt, um Formeln für W_0 und die damit zusammenhängenden Objekte zu berechnen. Hier weichen die Darstellungen von denen in [Wit11] ab.

Anschließend (Kapitel 4) beweisen wir das Hauptresultat dieser Arbeit, Satz (4.33): Jede glatte Funktion $W \in C^\infty(L)$ kann als Potential einer Einbettung realisiert werden. Wir werden hierzu die Ausgangssituation leicht modifizieren und die Problematik mit Hilfe von Einbettungen in Vektorbündel studieren. Auf einer kleinen Umgebung des Nullschnittes eines beliebigen Vektorbündels über L konstruieren wir durch eine Bilinearform auf den Fasern, eine neue Riemannsche Metrik. Wir zeigen, dass L isometrisch in eine Tube eingebettet werden kann, die mit dieser sogenannten gestörten Sasaki-Metrik ausgestattet wird.

Neben den bisher benutzten analytischen und geometrischen Hilfsmitteln, gibt es weitere Möglichkeiten um W_0 zu studieren. Ein anderer Aspekt der Oberflächenmaße ist der Zusammenhang zu Wahrscheinlichkeitsmaßen auf den Pfadräumen. Diesen stochastischen Zusammenhang und die Rolle, die W_0 darin spielt, werden wir in Kapitel 6 aufgreifen. Außerdem betrachten wir das Tripel von isometrisch eingebetteten Riemannschen Mannigfaltigkeiten $L_2 \subset L_1 \subset M$. Es stellt sich heraus, dass wir im Allgemeinen unterschiedliche Pfadmaße erhalten, wenn wir einmal direkt von M auf L_2 bedingen, oder zunächst auf L_1 und dann auf L_2. Der Unterschied in den Formeln wird durch eine Kozykelbedingung beschrieben.

Abschließend (Kapitel 7) betrachten wir die Thematik wieder von einem eher geometrischen Standpunkt. Wir beschäftigen uns mit den Laplace-Beltrami Operatoren der beteiligten Mannigfaltigkeiten, da diese die Generatoren der Brownschen Bewegung sind. In diesem Kontext werden wir besonders auf Riemannsche Submersionen eingehen und natürliche Metriken einführen, die eng mit diesen verbunden sind. Wir werden zeigen, warum diese Objekte die Situation grundlegend vereinfachen.

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List of Abbreviations

CG	Cheeger-Gromoll
const	Constant
det	Determinant
End	Endomorphism ring
Fix	Group of fixed points
Hom	Homomorphism
id	Identity map
I	Identity matrix
im	Image
Isom	Isometry group
ker	Kernel
rk	Rank
Ric	Ricci map
Sa	Sasaki
Scal	Scalar curvature
spec	Spectrum
Stab	Stabilizer
tr	Trace
T	Transposed
vol	Volume

List of Symbols

$i, j, ..$	Indices on the submanifold (in tangential direction)
$\alpha, \beta, ...$	Indices for normal components
$A, B, ...$	Indices for normal and tangential directions
ϱ^r	Bilinear form induced by algebraic curvature tensor r
Γ_{ij}^k	Christoffel symbols of second type of a Levi-Civita connection
ω	Connection one-form
$C_{i\beta}^{\alpha}$	Connection coefficients of the normal bundle
$K^{\perp}$	Connection map of a normal bundle
K	Connection map of a (tangent) bundle
$\Lambda(E)$	Exterior algebra of E
$\Lambda^k(E)$	k-th exterior power of E
$\star$	Hodge operator
X^h	Horizontal lift of a vector field X
$\mathfrak{H}_W E$	Horizontal subspace of E in W
i	Injectivity radius
G_x	Isotropy group of the element x
$O(\varepsilon), O(\\|W\\|)$	Landau notation (for $\varepsilon, \\|W\\| \to 0$)
$\triangle_g$	Laplace-Beltrami operator with respect to the metric g
∇	Levi-Civita connection
$\mathcal{L}$	Lift
Gx	Orbit of the element x
P	Parallel translation
W_0	Potential on the submanifold
f^*	Pullback of a map f
π_*	Pushforward of a map π
σ_ε	Rescaling map
R	Riemannian curvature tensor
d	Riemannian distance
ρ	Radon-Nikodym density
κ	Sectional curvature
$\mathrm{II}(\varphi)$	Second fundamental form of the embedding φ
$\Omega^n(X)$	Set of all n-forms on X
$\Gamma(E), \Gamma(E, M)$	Set of all sections in the bundle E over M
$\mathcal{R}$	Space of algebraic curvature tensors
$S_0^k(V)$	Space of traceless symmetric k-tensors
$\mathbb{S}^n(r)$	Sphere in $\mathbb{R}^{n+1}$ of radius $r \in \mathbb{R}_+$
μ_0	Surface measure generated by the Wiener measure
$S(V)$	Symmetric algebra of V
$S^k(V)$	k-th symmetric power of V

$\tau, \tau(\phi)$	Tension field of an embedding
$L(\varepsilon)$	Tubular ε-neighborhood of the Riemannian manifold L
V^v	Vertical lift of a vector V
$\mathfrak{V}_W E$	Vertical subspace of E in W
A_W	Weingarten map in direction W

Chapter 1

Setting

This first chapter summarizes the basic facts for our investigations and gives an overview of the relevant quantities addressed in this thesis. Moreover, we briefly sketch the most important results and terminology.

For the preliminaries from Riemannian geometry, see Appendix A. Throughout this thesis, we freely adopt the Einstein summation convention (if not mentioned otherwise), i.e. whenever an index appears twice in the same formula, regardless of its position as an upper or lower index, it is summed up from 1 to the corresponding dimension.

Let $\varphi\colon L \to M$ denote a closed, i.e. compact without boundary, l-dimensional Riemannian submanifold of the m-dimensional Riemannian manifold (M, g). Note that all manifolds, maps and embeddings are assumed to be smooth (unless explicitly indicated otherwise). By $G(\varphi)$, we denote the graph in M. Let d_M denote the Riemannian distance in M and by

$$L(\varepsilon) := \{p \in M | d_M(p, L) < \varepsilon\}$$

define the tubular neighborhood of L in M with radius $\varepsilon > 0$.

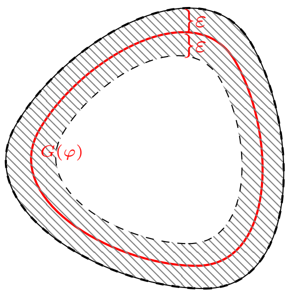

FIGURE 1.1: Illustration of the tubular neighborhood of a closed curve in $\mathbb{R}^2$ that is parametrized by a smooth map φ.

By $\pi\colon NL \to L$, we denote the bundle projection of the normal bundle, which is well-defined according to the embedding of L into M. Without loss of generality, we can assume that the exponential map of M maps some open neighborhood $NL(1)$ of the zero section in the normal bundle NL diffeomorphically onto the tube $L(1)$, see e.g. Chapter III in [Kos07] and Section (1.1) for further details. Hence, we obtain the projection $\pi\colon L(1) \to L$.

Throughout, we assume $L(1)$ to be regularly embedded, i.e. the ε-tubular neighborhood does not intersect itself as the fibers $F_x := \pi^{-1}(x)$ and F_y, for $x \neq y \in L$, have no common points. In the following, we restrict our studies to the cases $0 < \varepsilon \leq 1$.

For each $0 < \varepsilon \leq 1$, we can equip the tubular neighborhood $L(\varepsilon) \subset M$ with the induced metric g of M. We denote this metric by the same symbol.

Let $\Omega_M := C([0,T], M)$, for a fixed $T > 0$, be the path space of the Riemannian manifold (M, g). A study of surface measures on the path spaces of the tubes based on the study of Brown-ian motion in the ambient space showed that it is possible to think of the path spaces as metric measure spaces. See e.g. [Wit11], [SSWW14], [SSWW04b] or Chapter 5 in [BMSW09] for a precise statement of this construction and e.g. [Éme12] or [Ito62] for more information on Brownian motion on manifolds.
The limit behavior of this measure-theoretical process, given by the study of tubular neighborhoods with decreasing radii $\varepsilon \to 0$, constitutes the object of interest for this thesis: the so-called (associated) potential W_0 on L. This potential is given in terms of the intrinsic and extrinsic geometry of L and its embedding into M.

Besides the induced metric on the tubes, we study also another metric, the so-called Sasaki metric g_{Sa}, first introduced by S. Sasaki in [Sas58]. We discuss this metric in detail in Section (1.2). Therefore, we denote by m and m_{Sa} the corresponding volume forms of g and g_{Sa}. Our aim is to investigate their Radon-Nikodym density

$$\rho := \frac{dm}{dm_{Sa}},$$

which is closely related to the potential W_0 as we will see in Chapter 4.

Remark. *Note that we sometimes use μ_M instead of m to denote the volume form, because this notation helps to distinguish between the Riemannian volume forms of different Riemannian manifolds.*

Remark. *Whenever we use the Landau notation in the form $f \in O(g)$, we mean $f \in O(g)$, as $g \to 0$.*

However, before we proceed, it is first necessary to introduce Fermi coordinates to simplify the upcoming computations.

1.1 Fermi coordinates

Local coordinates on the submanifold L are denoted by $x = (x^1, \ldots, x^l) = (x^i)_{i=1,\ldots,l}$, using lower case Latin indices. Let $U \subset L$ be open such that $NL_{|U}$ is trivial (e.g. U starshaped or contractible). There exists a family of sections $e_\alpha : U \to NL_{|U}$, for $\alpha = 1, \ldots, m-l$, which can be chosen orthonormally with respect to the induced metric g. We further denote the fiber coordinates by lower case Greek letters. The local trivialization is given by the map

$$\Psi \colon U \times \mathbb{R}^{m-l} \to NL_{|U};\ (x^i, w^\alpha) \mapsto \varphi(x) + w^\alpha e_\alpha(x).$$

Hence, we obtain the identification $U \times \mathbb{R}^{m-l} \cong NL_{|U}$. For $\varepsilon > 0$ and smaller than the minimum of the injectivity radii of the exponential maps $\exp^M_x : T_xM \to M$, $x \in L$, there exists a diffeomorphism from a subset of $L(\varepsilon)$ to the ε-neighborhood $NL_\varepsilon(0)$ of the zero section in $NL_{|U}$. The existence of such an $\varepsilon > 0$ is justified because L is compact. Explicitly, this map is given by

$$\Phi \colon NL_\varepsilon(0) \to L(\varepsilon);\ (x, w) \mapsto \exp^M_{\varphi(x)}(w^\alpha e_\alpha(x)).$$

These local coordinates $(x, w) := (x^i, w^\alpha)_{i=1,...,l;\ \alpha=1,...,m-l}$ are called (local) **Fermi coordinates** on the tube $L(\varepsilon)$.

Whenever we use upper case Latin indices $A, B, ...$, we either refer to a lower case Latin or a Greek index, i.e. $A, B \in \{1, ..., m\}$.

1.2 The Sasaki metric g_{Sa}

A study of so-called natural metrics on the normal bundle has proved remarkably fruitful. The most commonly known example in the literature is the Sasaki metric.
We restrict our attention to the definition according to Reckziegel (see [Rec79]). For a definition of the Sasaki metric on tangent bundles, see e.g. [Kap01].
It is possible to define the Sasaki metric on every vector bundle $\pi \colon E \to L$ over L by specifying a structure group G, a connection one-form ω and a typical fiber F with fixed G-invariant fiber metric g_F.
Here, we study a Sasaki metric on the normal bundle. By $\pi \colon NL \to L$ we denote the normal bundle with Euclidean fiber F and let g_F be the flat metric on F. Unless noted otherwise, we will henceforth exclusively employ this special Sasaki metric.

Remark. *We call (F, g_F) **prototype** of the fiber or state that F is the **typical fiber**.*

Let $U \subset L$ again be a subset such that $NL_{|U}$ is trivial. Then we are able to identify $NL_{|U}$ and $U \times \mathbb{R}^{m-l}$.
Let ω be the connection one-form on NL given in local coordinates by

$$\omega = C_i dx^i,$$

with coefficients $C_i = C_i(x) \in \mathrm{End}(N_xL) = N_xL \otimes (N_xL)^*$. Using the definitions from Section (1.1), we obtain the characterization

$$\frac{\partial e_\alpha}{\partial x^j} = C^\beta_{j\alpha} e_\beta$$

for every $j = 1, ..., l$ and $\alpha = 1, ..., m - l$.

Note that this definition of a connection differs from the definition of the Levi-Civita connection on the tangent bundle of the ambient manifold M, which is torsion-free and compatible with respect to the metric g. The above concept is more general.
But note, since L is embedded into the ambient space M, it is always possible to equip NL with a metric connection $\nabla^\perp$ induced from the Levi-Civita connection on TM. Hence, this $\nabla^\perp$ is called the induced connection or normal connection.

Let $p \in U$. For an orthonormal basis $(e_{\alpha,p})_{\alpha=1\ldots m-l}$ of sections in N_pL (see Section (1.1)) and for $W \in N_pL$ we want to construct two subspaces of T_WNL to easily describe the action of the Sasaki metric on vector fields. Note that the base point $p \in U$ will be omitted for notational reasons and we just write e_α.

We define the set of vertical vectors by $\mathfrak{V}_W := \{Z \in T_WNL|\ \pi_*Z = 0\}$, which is a $(m-l)$-dimensional subspace. Now define the space of horizontal vectors $\mathfrak{H}_W$ by

$$\mathfrak{H}_W \oplus \mathfrak{V}_W = T_WNL.$$

We define $\mathfrak{H}_W := \ker(K_W^\perp)$, where $K^\perp$ denotes the connection map on the normal bundle, i.e. $K_W^\perp \colon T_WNL \to N_{\pi(W)}L$. Note that $\mathfrak{H}_W = \ker(\mathrm{id}_{\mathfrak{V}} + \omega) \subset T_WNL$.

Further details about the decomposition into the vertical and horizontal subspace and the precise definition of the connection map $K^\perp$ are given in Section 3.1.1.

With the above notations at hand, the construction of the Sasaki metric is now as follows: Define the inner product

$$\langle \cdot, \cdot \rangle_{Sa} := g_{Sa}(\cdot, \cdot) \colon TNL \times TNL \to \mathbb{R}$$

by the following properties:

(1) With respect to g_{Sa} the subspaces $\mathfrak{V}$ and $\mathfrak{H}$ are orthogonal.

(2) The map $\pi \colon (NL, g_{Sa}) \to (L, g_L)$ is a Riemannian submersion. That is, π is a submersion (π has maximal rank, i.e. π_* is surjective) and the tangent map π_* preserves the length of horizontal vectors. For detailed information about Riemannian submersions see [O'N02].

(3) For each $p \in L$ the embedding of the fiber $F_p = \pi^{-1}(p) \subset NL$ is totally geodesic.

(4) For each $p \in L$ the fiber F_p is isometric to (F, g_F).

Remark. *Because the projection map π only projects onto the first component, there are many possibilities to define $\mathfrak{H}_W$. For the definition of the horizontal subspace we need additional information, e.g. a connection to define $K^\perp$.*

The construction leads to the following definition of the Sasaki metric.

Definition 1.1 (Sasaki metric)**.** *For $W \in N_pL$ and $Z, V \in T_WNL$ we define*

$$\langle Z, V \rangle_{W,Sa} := \langle \pi_*Z, \pi_*V \rangle_L + \langle K_W^\perp Z, K_W^\perp V \rangle_{NL}. \tag{1}$$

Next, we present a couple of properties of the Sasaki metric, which follow immediately from the definition.

Corollary 1.2. *1. For $Z, V \in \mathfrak{H}_W$ we have*

$$\langle Z, V \rangle_{W,Sa} = \langle \pi_*Z, \pi_*V \rangle_L.$$

2. For $Z, V \in \mathfrak{V}_W$ the identity

$$\langle Z, V \rangle_{W,Sa} = \langle K_W^\perp Z, K_W^\perp V \rangle_{NL}$$

holds.

3. *Let $Z \in \mathfrak{H}_W$ and $V \in \mathfrak{V}_W$, then*

$$\langle Z, V\rangle_{W,Sa} = 0.$$

Remark. *In most parts of this thesis we merely write $\langle \cdot, \cdot \rangle_{Sa}$.*

We are now able to compute the Sasaki metric in local coordinates. As seen above, we denote by $C_{i\alpha}^{\beta}$ the coefficients of the connection one-form ω, i.e.

$$\omega = C_i dx^i = \left(C_{i\alpha}^{\beta} e_\beta \otimes e_\alpha^* \right) dx^i.$$

Furthermore, we denote by $J_W \colon NL \to T_W N_{\pi(W)} L$ the map given by $J_W(e_\alpha) = \partial_\alpha$, where

$$\partial_\alpha := \frac{\partial}{\partial t} \left(W + te_\alpha \right)_{|t=0} \in T_W N_{\pi(W)} L. \tag{2}$$

In local Fermi coordinates the horizontal lift of a vector field $X = X^i \partial_i$ on L is given by

$$X^h = X^i(\partial_i - w^\mu C_{i\mu}^{\nu} \partial_\nu).$$

A proof is given in Chapter 3, where we also investigate the lifts in a coordinate-free manner.

Remark. 1. *By definition we have $\pi_* X^h = X$.*

2. *The map $K_W^{\perp}$ is the inverse map to J_W on the set of vertical vectors $\mathfrak{V}_W$; i.e.*

$$K_W^{\perp}(w^\alpha C_{i\alpha}^{\beta} \partial_\beta) = w^\alpha C_{i\alpha}^{\beta} e_\beta.$$

3. *Note that $C_{j\nu}^{\mu} g_{Sa\,\mu\alpha} = C_{j\nu}^{\mu} \delta_{\mu\alpha} = C_{j\nu\alpha}$.*

Definition 1.3 (Kronecker delta)**.** *For arbitrary indices A and B we define*

$$\delta_{AB} := \begin{cases} 1, & A = B \\ 0, & A \neq B. \end{cases}$$

Lemma 1.4. *In local Fermi coordinates the Sasaki metric on $L(1)$ is given by*

$$g_{Sa}(x, w) = \left(\begin{array}{c|c} g_{L\,ij}(x) + w^\mu w^\nu C_{i\mu}^{\rho} C_{j\rho\nu} & w^\mu C_{i\mu\beta} \\ \hline w^\nu C_{j\nu\alpha} & \delta_{\alpha\beta} \end{array} \right). \tag{3}$$

Proof. Let $W = w^\alpha e_\alpha \in NL$ and $Z, U \in T_W NL$. With respect to the decomposition $Z = Z^i \partial_i + Z^\alpha \partial_\alpha$ and $U = U^i \partial_i + U^\alpha \partial_\alpha$ we compute

$$\langle Z, U \rangle_{Sa} = \left(Z^i,\ Z^\alpha \right) \left(\begin{array}{c|c} (g_{Sa})_{ij} & (g_{Sa})_{i\beta} \\ \hline (g_{Sa})_{\alpha j} & (g_{Sa})_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} U^i \\ U^\alpha \end{array} \right),$$

where $(g_{Sa})_{AB} = \langle \partial_A, \partial_B \rangle_{W,Sa}$. Note that all ∂_α are vertical vector fields, while ∂_i denotes an arbitrary vector field with horizontal and vertical components. Due to the above representation of the horizontal vectors in local Fermi coordinates, we rewrite

$$\partial_i = \partial_i - w^\alpha C^\mu_{i\alpha}\partial_\mu + w^\alpha C^\mu_{i\alpha}\partial_\mu,$$

where now $\partial_i - w^\alpha C^\mu_{i\alpha}\partial_\mu \in \mathfrak{H}_W$ and $w^\alpha C^\mu_{i\alpha}\partial_\mu \in \mathfrak{V}_W$ hold. Now using the definition of the Sasaki metric, Corollary (1.2), and the fact that

$$K^\perp_W(\partial_\beta) = e_\beta,$$

we first compute

$$\begin{aligned} g_{Sa,\alpha\beta} &= \langle \partial_\alpha, \partial_\beta\rangle_{W,Sa} \\ &= \left\langle K^\perp_W(\partial_\alpha), K^\perp_W(\partial_\beta)\right\rangle_{NL} \\ &= \langle e_\alpha, e_\beta\rangle_{NL} = \delta_{\alpha\beta}. \end{aligned} \tag{4}$$

For the off-diagonal components we obtain

$$\begin{aligned} g_{Sa,i\beta} &= \langle \partial_i, \partial_\beta\rangle_{W,Sa} \\ &= \langle \partial_i - w^\alpha C^\mu_{i\alpha}\partial_\mu + w^\alpha C^\mu_{i\alpha}\partial_\mu, \partial_\beta\rangle_{W,Sa} \\ &= \langle \partial_i - w^\alpha C^\mu_{i\alpha}\partial_\mu, \partial_\beta\rangle_{W,Sa} + \langle w^\alpha C^\mu_{i\alpha}\partial_\mu, \partial_\beta\rangle_{W,Sa} \\ &= w^\alpha C^\mu_{i\alpha}\langle \partial_\mu, \partial_\beta\rangle_{W,Sa} \\ &\overset{(4)}{=} w^\alpha C^\mu_{i\alpha}\delta_{\mu\beta} = w^\alpha C_{i\alpha\beta}. \end{aligned}$$

Due to the remark above, we have $\pi_*(\partial_i - w^\mu C^\nu_{i\alpha}\partial_\nu) = \partial_i$ and thereby conclude

$$\begin{aligned} g_{Sa,ij} &= \langle \partial_i, \partial_j\rangle_{W,Sa} \\ &= \left\langle \partial_i - w^\alpha C^\mu_{i\alpha}\partial_\mu + w^\alpha C^\mu_{i\alpha}\partial_\mu, \partial_j - w^\beta C^\nu_{j\beta}\partial_\nu + w^\beta C^\nu_{j\beta}\partial_\nu\right\rangle_{W,Sa} \\ &= \left\langle \partial_i - w^\alpha C^\mu_{i\alpha}\partial_\mu, \partial_j - w^\beta C^\nu_{j\beta}\partial_\nu\right\rangle_{W,Sa} + \left\langle w^\alpha C^\mu_{i\alpha}\partial_\mu, w^\beta C^\nu_{j\beta}\partial_\nu\right\rangle_{W,Sa} \\ &= \langle \partial_i, \partial_j\rangle_L + w^\alpha C^\mu_{i\alpha} w^\beta C^\nu_{j\beta}\langle \partial_\mu, \partial_\nu\rangle_{W,Sa} \\ &\overset{(4)}{=} g_{L,ij} + w^\alpha w^\beta C^\mu_{i\alpha} C^\nu_{j\beta}\delta_{\mu\nu} \\ &= g_{L,ij} + w^\alpha w^\beta C^\mu_{i\alpha} C_{j\beta\mu}. \end{aligned}$$

□

Corollary 1.5. *The matrix representation directly provides the decomposition*

$$g_{Sa}(x,w) = \left(\begin{array}{c|c} 1 & w^\mu C_{i\mu\beta} \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} g_{L\,ij}(x) & 0 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline w^\nu C_{j\nu\alpha} & 1 \end{array}\right). \tag{5}$$

Proof. Matrix multiplication yields the formula (3). □

From the decomposition and the fact that the determinant of each block is easily computable, the inverse Sasaki metric is given as follows.

Lemma 1.6. *In local Fermi coordinates the dual (resp. inverse) Sasaki metric is given by*

$$g^*_{Sa}(x,w) = \left(\begin{array}{c|c} g_L^{ij}(x) & -w^\mu g_L^{ij}(x) C_{j\mu}^\beta \\ \hline -w^\nu g_L^{ij}(x) C_{i\nu}^\alpha & \delta^{\alpha\beta} + w^\mu w^\nu g_L^{ij}(x) C_{i\mu}^\alpha C_{j\nu}^\beta \end{array}\right). \tag{6}$$

Remark. *From Corollary (1.5) we directly obtain* $\det(g_{Sa}) = \det(g_L)$.

In Chapter 3 we will develop a coordinate invariant expression for the Sasaki metric.

Remark. *NL endowed with the Sasaki metric is a Riemannian manifold; via the exponential map the restriction of g_{Sa} to $U_1(0) \subset NL$ is transported to a Riemannian metric on $L(1)$ and there referred to as reference metric.*
The Dirichlet Laplacian associated to the reference metric can be decomposed into two independent parts; we investigate this behavior later.

1.3 The Riemannian curvature tensor

Throughout this thesis we follow the notation from [Jos98]. Let (M, g, ∇) be a Riemannian manifold equipped with the Levi-Civita connection ∇ to g. From now on we simply write (M, g) or even M. Unless indicated otherwise, we always equip M with its Levi-Civita connection.

For vector fields X and $Y \in \Gamma(TM)$ let

$$g(X,Y) = \langle X, Y\rangle_g,$$

where we omit the subscript on the right-hand side whenever the underlying metric is obvious from the context.

Definition 1.7 ((Riemannian) curvature tensor)**.** *The* ***Riemannian curvature tensor*** *R of a Riemannian Manifold (M, g) is defined by*

$$R\colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM);$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Remark. *We write R^L and R^M to denote the Riemannian curvature tensors of L and M and omit the superscript whenever the context is clear.*

A proof for the following lemma can be found in [Bes87], Lemma (1.85), p. 41.

Lemma 1.8. *Let R be the Riemannian curvature tensor of the Riemannian manifold (M, g) and $X, Y, Z \in \Gamma(TM)$, then*

a) R is a $(3,1)-$tensor,

b) R is skew-symmetric in X and Y, i.e. $R(X,Y)Z = -R(Y,X)Z$,

c) R is skew-symmetric with respect to g, i.e. $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$,

d) the first ***Bianchi identity*** *holds, i.e.*

$$\langle R(X,Y)Z, W\rangle + \langle R(Y,Z)X, W\rangle + \langle R(Z,X)Y, W\rangle = 0,$$

e) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$,

f) the second ***Bianchi identity*** *holds, i.e.*

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

Remark. *Note that from now on we use these symmetry relations without further mentioning.*

By lowering and raising indices we obtain the following object.

Definition 1.9 (Curvature-4-form)**.** *The* ***curvature-4-form*** *of the Riemannian manifold* (M,g) *is defined as the map*

$$(X,Y,Z,W) \mapsto g(R(X,Y)Z, W),$$

where R *is the curvature tensor of* M *and* X, Y, Z, W *are vector fields on* M.

Remark. *Note that we denote the curvature-4-form by the same symbol* R *and that here* $R \in \Gamma(M, \Lambda^2(T^*M) \circ \Lambda^2(T^*M))$, *where* $\circ$ *denotes the symmetric product.*

We set the following conventions for the description of the curvature tensor in local coordinates.
Let $\{\frac{\partial}{\partial x^A}\}_{A=1,\ldots,m}$ denote some orthonormal frame in TM. In local coordinates we use the following expressions for the curvature tensor and curvature-4-form

$$R\left(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B}\right)\frac{\partial}{\partial x^C} = R^D_{CAB}\frac{\partial}{\partial x^D}$$

and

$$R_{DCAB} = g_{DN}R^N_{CAB} = \left\langle R\left(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B}\right)\frac{\partial}{\partial x^C}, \frac{\partial}{\partial x^D}\right\rangle.$$

Remark. *Note that we use this index notation throughout the whole thesis.*

Example 1.10. *Using the expression in local coordinates and the symmetries from Lemma (1.8), we obtain*

$$R_{DCAB} = R_{ABDC} = -R_{ABCD}.$$

1.3.1 Other relevant curvature quantities

The Riemannian curvature tensor is used to define other important objects, which will play significant roles throughout this work.

Definition 1.11 (Sectional curvature)**.** *Let* (M,g) *be a Riemannian manifold with Riemannian curvature tensor* R *and* $x \in M$. *For* $X = X^A\partial_A, Y = Y^B\partial_B \in T_xM$ *we define the* ***sectional curvature*** *of the plane* $X \wedge Y$, *spanned by the vectors* X *and* Y, *as*

$$\kappa(X \wedge Y) := \frac{\langle R(X,Y)Y, X\rangle}{|X \wedge Y|^2}.$$

Definition 1.12. *Let* (M,g) *be a Riemannian manifold with Riemannian curvature tensor* R *and* $x \in M$.

1. *We define the* ***Ricci (curvature) tensor*** *of M in local coordinates as*

$$R_{AB} = g^{KN} R_{AKBN} = R^{K}_{AKB},$$

i.e. the ***Ricci curvature*** *in direction $X \in T_xM$ is given by*

$$\mathrm{Ric}^M(X,X) := g^{KN} \left\langle R\left(X, \frac{\partial}{\partial x^K}\right) \frac{\partial}{\partial x^N}, X \right\rangle.$$

2. *The* ***scalar curvature*** *of M is defined by*

$$\mathrm{Scal}_M := \mathrm{tr}_g \,\mathrm{Ric},$$

i.e. as the trace of the Ricci tensor with respect to the metric g. In local coordinates

$$\mathrm{Scal}_M = g^{AB} R_{AB}.$$

Due to the following fact, we will focus on the sectional curvature more detailed in the following chapters.

Proposition 1.13. *All of the relevant information included in the definition of the Riemannian curvature tensor R^M of a Riemannian manifold (M,g), are already included in the sectional curvature κ. That explicitly means we can reconstruct the scalar curvature and the Ricci tensor from the sectional curvature.*

Proof. Let $(E_A)_{A=1,\dots,m}$ be an orthonormal frame of TM, then

(a) $\mathrm{Scal}_M = \sum_{A,B} \langle R^M(E_A,E_B)E_B, E_A\rangle_g = \sum_{A\neq B} \kappa(E_B \wedge E_A)$,

(b) fix $C \in \{1,\dots,m\}$, then

$$\mathrm{Ric}^M(E_C,E_C) = \sum_A \langle R^M(E_C,E_A)E_A, E_C\rangle = \sum_{A\neq C} \kappa(E_C \wedge E_A).$$

□

Remark. *i) The Ricci tensor can be considered as the trace of the Riemannian curvature tensor.*

ii) The Ricci tensor is symmetric, i.e.

$$R_{AB} = R_{BA}.$$

iii) We think of the Ricci curvature $\mathrm{Ric}(X,X)$ as the average of the sectional curvatures of all planes in T_xM which contain X.

iv) In geodesic normal coordinates the Riemannian volume measure $d\mu_g$ can be expanded such that

$$d\mu_g = \left[1 - \frac{1}{6} R_{AB} x^A x^B + O(\|x\|^3)\right] d\mu_E, \tag{7}$$

where $d\mu_E$ denotes the standard Euclidean measure (see e.g. [Gra04], Corollary 9.9, p. 193).

v) By to the previous observation, in geodesic normal coordinates Scal *is equal to the coefficient of the second order term in the ratio between the volume of a geodesic ball of radius $r > 0$ around $0 \in M$ and the Euclidean volume of the ball of the same radius in $\mathbb{R}^m$. (By differentiating formula (7) twice and then evaluating at zero.)*

1.3.2 Algebraic curvature tensors

In this section we introduce the algebraic curvature tensors and their properties following the explanations in [Bes87], Chapter 1, § G, p. 45-52.

Definition 1.14 (Algebraic curvature tensor)**.** *Let E be a finite dimensional vector space.*

*a) The **Bianchi map** $B\colon S^2(\Lambda^2(E^*)) \to S^2(\Lambda^2(E^*))$ is defined by*

$$B(r)(X,Y,Z,W) = \frac{1}{3}\left(r(X,Y,Z,W) + r(Y,Z,X,W) + r(Z,X,Y,W)\right),$$

where $X, Y, Z, W \in E^*$.

*b) By $\mathcal{R} := \ker(B) \subset S^2(\Lambda^2(E))$ we denote the space of **algebraic curvature tensors**.*

For a Riemannian manifold (M, g) the Bianchi map is analogously defined on the bundle $S^2(\Lambda^2(T^*M))$. We summarize significant properties of the Bianchi map and $\mathcal{R}$ in the following lemma.

Lemma 1.15. *a) $\mathcal{R}$ is a vector space over $\mathbb{R}$.*

b) The Bianchi map B is idempotent and $\mathrm{im}(B) = \Lambda^4(E^*)$.

c) $\mathcal{R} \cong {S^2(\Lambda^2(E^*))}\big/{\Lambda^4(E^*)}$.

Proof. a) Using the vector space structure of E and the linearity of $r \in S^2(\Lambda^2(E))$ yields the statement.

b) Let $r \in S^2(\Lambda^2 E^*)$, i.e. $r = (\alpha \wedge \beta) \circ (\gamma \wedge \delta)$, where $\alpha, \beta, \gamma, \delta \in E^*$. Then

$$\begin{aligned} 3 \cdot B(r) &= (\alpha \wedge \beta) \circ (\gamma \wedge \delta) + (\beta \wedge \gamma) \circ (\alpha \wedge \delta) + (\gamma \wedge \alpha) \circ (\beta \wedge \delta) \\ &= -\left((\alpha \wedge \gamma) \circ (\beta \wedge \delta) + (\gamma \wedge \beta) \circ (\alpha \wedge \delta) + (\beta \wedge \alpha) \circ (\gamma \wedge \delta)\right) \\ &= -3 \cdot B((\alpha \wedge \gamma) \circ (\beta \wedge \delta)). \end{aligned}$$

By $(\beta \wedge \alpha) \circ (\gamma \wedge \delta) = -r = (\alpha \wedge \beta) \circ (\delta \wedge \gamma)$ and

$$\begin{aligned} 3 \cdot B(-r) &= -((\alpha \wedge \beta) \circ (\gamma \wedge \delta) + (\beta \wedge \gamma) \circ (\alpha \wedge \delta) + (\gamma \wedge \alpha) \circ (\beta \wedge \delta)) \\ &= -3 \cdot B(r) \end{aligned}$$

we conclude $B(r) \in \Lambda^4(E^*)$ and $\mathrm{im}(B) \subseteq \Lambda^4(E^*)$. Surjectivity directly follows from the fact that B is idempotent, which we deduce from the following computation.

Let $B(r) = \alpha' \wedge \beta' \wedge \gamma' \wedge \delta' \in \Lambda^4(E^*)$. Then, we have

$$\begin{aligned} B(B(r)) &= B(\alpha' \wedge \beta' \wedge \gamma' \wedge \delta') \\ &= \frac{1}{3}(\alpha' \wedge \beta' \wedge \gamma' \wedge \delta' + \beta' \wedge \gamma' \wedge \alpha' \wedge \delta' + \gamma' \wedge \alpha' \wedge \beta' \wedge \delta') \\ &= 3 \cdot \frac{1}{3}\alpha' \wedge \beta' \wedge \gamma' \wedge \delta' \\ &= \alpha' \wedge \beta' \wedge \gamma' \wedge \delta' = B(r). \end{aligned}$$

c) The splitting follows from the fact that the sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow S^2(\Lambda^2 E^*) \xrightarrow{B} \Lambda^4(E^*) \longrightarrow 0,$$

is exact by definition.

□

One of the most interesting facts about the algebraic curvature tensors is contained in the following theorem. For further information see [DRGR03].

Theorem 1.16. *Every algebraic curvature tensor $r \in \mathcal{R}$ occurs as the Riemannian curvature tensor R of a Riemannian manifold (M, g) at a point $p \in M$.*

Proof. Let $r \in \mathcal{R} \subset S^2\left(\Lambda^2(E)\right)$ be an algebraic curvature tensor and b a metric on E. We need to show that there exists a Riemannian manifold (M, g) such that the associated curvature tensor R to g equals r at $p \in M$, i.e. there exists an isometry $\Phi\colon (E, b) \to (T_pM, g_p)$, where $r = \Phi^* R_p$.
Let $(x^1, ..., x^m)$ be a Riemannian normal coordinate system at the origin of M. We define,

$$g^r_{ij}(x^1, ..., x^m) = \delta_{ij} - \frac{1}{3}\sum_{k,l=1}^{m} r_{ikjl}x^k x^l, \tag{8}$$

which defines a metric on a suitable (with small enough radius) chosen ball B_ε around the origin, for $\varepsilon > 0$. Note that the first order derivatives vanish at the origin of the coordinate system due to the normal coordinates.
In $p = 0$ we conclude that $R_p = r$, because (8) is the exact perturbation of the metric in normal coordinates and r is uniquely defined by (8), because in normal coordinates the curvature is the second derivative of the metric. This is due to the fact that by definition of the Christoffel symbols the Riemannian curvature components can be expressed in the following way:

$$R^M_{ijkl} = \frac{\partial^2}{\partial x^i \partial x^l} g_{kj} - \frac{\partial^2}{\partial x^j \partial x^l} g_{ki}$$

□

For $\dim(E) \geq 4$ and a non-degenerated quadratic form g on E there exists a decomposition of $\mathcal{R}$ as an $O(g)$-module, into three irreducible components; this fact is due to [Bes87], Chapter 1, § G, Theorem 1.114. Before giving the exact statement we first need to define the Kulkarni-Nomizu product.

Definition 1.17 (Kulkarni-Nomizu product)**.** *Let h, g be symmetric $(0,2)$-tensors on E. Then we define their* ***Kulkarni-Nomizu product*** *as the $(0,4)$-tensor given by*

$$
\begin{aligned}
&(h \owedge g)(X, Y, Z, W) \\
:=&h(X,Z)g(Y,W) + h(Y,W)g(X,Z) - h(X,W)g(Y,Z) - h(Y,Z)g(X,W),
\end{aligned}
$$

for all $X, Y, Z, W \in E^*$.

Now we are able to describe the mentioned splitting.

Theorem 1.18. *The space of algebraic curvature tensors $\mathcal{R}$ admits an orthogonal decomposition into three irreducible subspaces*

$$
\mathcal{R} = A \oplus B \oplus C, \tag{9}
$$

where $A = \mathbb{R}g \owedge g$, $B = g \owedge S_0^2 E$ and the remaining part C, which is called the Weyl part. Note that $S_0^2 E$ denotes the space of symmetric 2-tensors, whose trace is zero.

For each $r \in \mathcal{R}$ with $r = a \oplus b \oplus c$ the part in the one-dimensional subspace A is given by

$$
a = \frac{\mathrm{Scal}_r}{m(m-1)} g \owedge g.
$$

The subspace B is called the traceless part of the Ricci tensor and given by

$$
b = \frac{1}{m-2} \left(\mathrm{Ric}_r - \frac{\mathrm{Scal}_r}{m} g \right) \owedge g.
$$

The remaining part, defined to be the **Weyl curvature**, is given by

$$
c = r - a - b = r - \frac{1}{m-2} \left(\mathrm{Ric}_r - \frac{\mathrm{Scal}_r}{2(m-1)} g \right) \owedge g.
$$

Remark. *We will later see that $r \in \mathcal{R}$ corresponds to a symmetric endomorphism $M_r \in$ $\mathrm{End}\left(\Lambda^2(E^*)\right)$. Because this endomorphism splits orthogonally into three blocks, we obtain an explanation for the above decomposition.*

1.4 The second fundamental form and the Weingarten map

In this section, we follow the notation in [Che13]. As a fiberwise direct sum of vector spaces we have $TM_{|_L} = TL \oplus NL$, where NL denotes the orthogonal complement of TL with respect to the metric g of M. Moreover, for each $x \in L$ consider $T_xL \subset T_xM$ as a linear subspace.

Denote by ∇^L and ∇^M the Levi-Civita connections of L and M and their Christoffel symbols by Γ. Then, for X and $Y \in \Gamma(TM)$ the connection ∇^M can be decomposed as

$$
\nabla^M_X Y = p^\perp \left((\nabla^M_X Y)^\perp \right) + p^\perp \left((\nabla^M_X Y)^\top \right),
$$

where $p^\perp$ denotes the orthogonal projection onto TL. The above notations in mind, for $X, Y \in \Gamma(TL)$, we obtain

$$p^\perp \left((\nabla^M_X Y)^\top \right) = \nabla^L_X Y,$$

which implies that the tangential part of the metric connection on M equals the connection on L. This is due to the fact that for vector fields X and Y on L, the expression $\nabla^M_X Y$ is defined for an at least C^2-smooth extension of the vector fields to M. Then a restriction to the submanifold constitutes the above expression. By uniqueness of the Levi-Civita connection the orthogonal projection of $\nabla^M_X Y$ onto TL equals $\nabla^L_X Y$.

Definition 1.19 (Second fundamental form)**.** *Let $\varphi\colon L \to M$ denote the embedding and $X, Y \in \Gamma(TL)$. The **second fundamental form** of φ is defined to be the normal part of the connection, i.e.*

$$\mathrm{II}(\varphi)(X, Y) := \nabla^\perp_X Y := \nabla^M_X Y - \nabla^L_X Y.$$

Remark. *Note that $\mathrm{II} : TL \times TL \to NL$. Let $X, Y \in \Gamma(TL)$. For extensions $\widetilde{X}, \widetilde{Y}$ onto M one has $\nabla^M_{\widetilde{X}} \widetilde{Y} - \nabla^M_{\widetilde{Y}} \widetilde{X} = [\widetilde{X}, \widetilde{Y}]$ and $[\widetilde{X}, \widetilde{Y}]^\perp = 0$ on L; therefore, the second fundamental form is a $(0, 2)$-tensor field on L and symmetric in X and Y.*

Definition 1.20 (Weingarten map)**.** *Let $\varphi\colon L \to M$, $x \in L$ and $W = w^\alpha e_\alpha \in N_x L$. Then there exists a map $A_W : T_x L \to T_x L$, such that*

$$\langle A_W X, Y\rangle_L = \langle \mathrm{II}(\varphi)(X, Y), W\rangle_M \tag{10}$$

*for $X, Y \in T_x L$. A_W is called **Weingarten map** or **shape operator** in direction W.*

Remark. *A_W is symmetric and selfadjoint in the sense that $\langle A_W X, Y\rangle_L = \langle A_W Y, X\rangle_L$, which immediately follows from the properties of the second fundamental form.*

Sometimes (10) is referred to as **Weingarten's equation**.

Proposition 1.21. *For every $Z \in TL$ and $W \in NL$ the action of the Weingarten map in direction of W is given by*

$$A_W X = -\Gamma^k_{j\alpha} w^\alpha X^j \partial_k, \tag{11}$$

where $(\partial_i)_{i=1,\dots,l}$ denotes the orthonormal basis of TL as above.

Proof. For $X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j} \in \Gamma(TL)$ we obtain

$$\begin{aligned} \langle \mathrm{II}(\varphi)(X, Y), W\rangle &= \langle \nabla^\perp_X Y, W\rangle \\ &= -\langle Y, \nabla^\perp_X W\rangle \\ &= \langle A_W X, Y\rangle. \end{aligned}$$

We use the shorthand $\partial_i = \frac{\partial}{\partial x^i}$ for all $i = 1, ..., l$ and for the Greek coordinates in the same manner. In local coordinates we hence find

$$\begin{aligned} \langle Y, \nabla^\perp_X W\rangle &= \langle Y^j \frac{\partial}{\partial x^j}, \nabla^\perp_{X^i \partial_i} w^\alpha \partial_\alpha\rangle \\ &= Y^j X^i \langle \partial_j, \nabla^\perp_{\partial_i} w^\alpha \partial_\alpha\rangle \\ &= Y^j X^i \langle \partial_j, (\partial_i w^\alpha)\partial_\alpha + w^\alpha \nabla^\perp_{\partial_i} \partial_\alpha\rangle \\ &= Y^j X^i w^\alpha \Gamma^k_{i\alpha} \langle \partial_j, \partial_k\rangle. \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}\langle Y, A_W X\rangle &= \langle Y^j\partial_j, A_W X^i\partial_i\rangle\\ &= Y^j\langle\partial_j, W(X^i)\partial_i + X^i A_W{}_i^k\partial_k\rangle\\ &= Y^j X^i A_W{}_i^k\langle\partial_j,\partial_k\rangle.\end{aligned}$$

Comparing the coefficients gives the result. □

The relations between the second fundamental form II and the shape operator A_W are discussed in a similar way in [BCO03] (see p. 7 ff.).

Definition 1.22 (Tension field). *Let $\varphi\colon L \to M$. The trace of the second fundamental form taken over an arbitrary orthonormal basis of the tangent space is called the **tension field** $\tau(\varphi)$ of the embedding, i.e. let $\mathrm{II}(\varphi)\colon T_xL \times T_xL \to N_xL$ be the second fundamental form and $(e_i)_i$ be an arbitrary orthonormal frame in $T_xL, x \in L$ then*

$$\tau(\varphi) := \sum_{i=1}^{l} \mathrm{II}(\varphi)(e_i, e_i).$$

Remark. *Note that the definition is invariant of the choice of the orthonormal basis. One could also define the tension field via*

$$\tau(\varphi) := g^{ij}\mathrm{II}(\varphi)_{ij}.$$

See [Jos98], p. 318 ff., for additional information on the tension field and also the following definition.

Let now $(\widetilde{M}, h)$ be another Riemannian manifold.

Definition 1.23 (Harmonic maps). *A smooth map $\varphi\colon M \to \widetilde{M}$ is called **harmonic map**, if it is a critical point of the Dirichlet energy functional*

$$E(\varphi) = \int_M \|d\varphi\|^2\, \mu_M,$$

i.e. φ is minimal, if the tension field $\tau(\varphi)$ vanishes.

We state another helpful definition, which will be used for the connection to Brownian motion on Riemannian manifolds in later parts of this work.

Definition 1.24 (Harmonic morphism). *Let $\varphi\colon M \to \widetilde{M}$ be smooth and $V \subset \widetilde{M}$ with $\varphi^{-1}(V) \neq \emptyset$. The map φ will be called **harmonic morphism** if for every harmonic map $f\colon V \to \mathbb{R}$ the composition $f \circ \varphi$ is harmonic on $\varphi^{-1}(V)$.*

A definition and useful facts on harmonic morphism can be found in [BW03], e.g. Definition 4.1.1.

Moreover, due to [Woo06], Riemannian subersions with minimal fibers are harmonic morphisms. We will later comment on the properties of Riemannian submersions with totally geodesic, i.e. minimal, fibers in general (Chapter 7).

1.5 The homogenization result

Now we are able to state the homogenization result for Laplace-Beltrami operators on tubular neighborhoods of a closed Riemannian submanifold $L \subset M$, which is crucial for this work. For the precise formulation see [Wit11], Theorem 1.

We already saw that there is a diffeomorphism of an open neighborhood $NL(1)$ of the zero section in the normal bundle NL to $L(1)$. Moreover, we can define the map $\sigma_\varepsilon\colon NL \to NL$ given by $\sigma_\varepsilon(W) = \varepsilon W$, $1 \geq \varepsilon > 0$. With the help of the exponential map and the projection $\pi\colon NL \to L$, we define the map $\exp_\pi^M$ for every $W \in NL$ by $\exp_\pi^M(W) := \exp_{\pi(W)}^M(W)$. Hence, for every $0 < \varepsilon \leq 1$, the diffeomorphisms

$$\sigma_\varepsilon\colon L(1) \to L(\varepsilon)$$

are induced by the commutative diagram

$$\begin{array}{ccc} NL(1) & \xrightarrow{\sigma_\varepsilon} & NL(\varepsilon) \\ \big\downarrow{\scriptstyle \exp_\pi^M} & & \big\downarrow{\scriptstyle \exp_\pi^M} \\ L(1) & \xrightarrow{\sigma_\varepsilon} & L(\varepsilon). \end{array}$$

These maps are denoted by the same symbol.

Remark. *Note that in local Fermi coordinates the map σ_ε is given by $\sigma_\varepsilon(x, w) = (x, \varepsilon w)$.*

Additionally note that $\pi\colon L(1) \to L$ equipped with the Sasaki metric is a Riemannian submersion with totally geodesic fibers. This property is a useful tool in the proof of the forthcoming homogenization result.

Definition 1.25. *We define the Radon-Nikodym density*

$$\rho := \frac{dm}{dm_{Sa}}$$

of the volume forms of g and g_{Sa}.

Remark. *The Radon-Nikodym density ρ is a non-negative function in $C^\infty(\overline{L(1)})$, where the closure is taken with respect to the Riemannian metric g.*

In [Wit11], Lemma 5, it is shown that there exists a unitary isomorphism $\Sigma_\varepsilon\colon L^2(L(\varepsilon), g) \to L^2(L(1), g_{Sa})$. With the help of this map the so-called associated rescaled family is given by

$$\triangle(\varepsilon) := \Sigma_\varepsilon \left(\triangle_\varepsilon - \frac{\lambda_0}{\varepsilon^2} \right) \Sigma_\varepsilon^{-1} \colon H_0^1 \cap H^2(L(1), g_{Sa}) \to L^2(L(1), g_{Sa})$$

according to the family of Laplace-Beltrami operators on the tubes, i.e.

$$\triangle_\varepsilon\colon H_0^1 \cap H^2(L(\varepsilon), g) \to L^2(L(\varepsilon), g). \tag{12}$$

Here λ_0 is the lowest eigenvalue of the Dirichlet Laplacian on a unit ball $B^{m-l}(1) \subset \mathbb{R}^{m-l}$ of dimension $m - l$, note that $\lambda_0 > 0$. Denote by $u_0 \in C(\overline{B^{m-l}(1)}) \cap C^\infty(B^{m-l}(1))$ the associated, normalized and positive eigenfunction.

The following result was proven in [Wit11], Theorem 1.

Theorem 1.26. *1. The operator family $\triangle(\varepsilon)$ is selfadjoint and uniformly bounded from below.*

2. *Let $t_0 > 0$ and $u(\varepsilon)$ be a sequence of functions, which converges to $u \in L^2(L(1), g_{Sa})$. Then, there exists a Schrödinger operator*

$$H_W := \frac{1}{2}(\triangle_L + W_0)$$

on the submanifold L, such that

$$\lim_{\varepsilon\to 0} e^{-\frac{t}{2}\triangle(\varepsilon)}u(\varepsilon) = E_0 e^{-tH_W} E_0 u$$

uniformly in $[t_0, \infty)$. Here E_0 denotes the orthogonal projection onto the eigenspace to the lowest eigenvalue of the flat Dirichlet Laplacian in $\mathbb{R}^{m-l}$.

3. *The smooth potential $W_0 \in C^\infty(L)$ depends on the intrinsic geometry of L and M and properties of the embedding and is given by*

$$W_0 = \left(\frac{1}{2}\mathrm{Scal}_L - \frac{1}{4}\|\tau\|^2 - \frac{1}{6}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L} + \overline{R}_{M/L}\right)\right). \tag{13}$$

By $\overline{\mathrm{Ric}}_{M/L}$ and $\overline{R}_{M/L}$ we further denote specific partial traces of the Ricci and Riemannian curvature tensor of the ambient manifold M. Precisely, for an orthonormal basis $\{e_i\}_{i=1,\dots,l} \subset TL$ we define

$$\overline{\mathrm{Ric}}_{M/L} := \sum_{i=1}^{l} \mathrm{Ric}^M(e_i, e_i),\ \overline{R}_{M/L} := \sum_{i,j=1}^{l} \langle e_i, R^M(e_i, e_j)e_j\rangle.$$

Remark. *Note that in the second statement of the previous theorem, the action of the operator on E_0u is understood in the way that*

$$e^{-tH_W}E_0u(x) := u_0(d_{Sa}(W, \pi(W)))\left(e^{-tH_W}u_a\right)\circ\pi(W),$$

where $W \in L(1)$, $\pi(w) = x \in L$, and $u_a \in L^2(L, g_L)$ is the averaged function to u.

To prove the existence of a limit, it is necessary to consider the rescaled operators. As in [Wit11] and the references given there, a renormalized problem needs to be studied. This helps to deduce epi-convergence and equi-coercivity of the associated quadratic form families. The choice of Fermi coordinates as a suitable microscope on the fibers is a necessary tool, to prove that the limit process does not degenerate.

In subsequent papers (see e.g. [NW19]), it turned out that a different definition of the Schrödinger operator simplifies the proof. We briefly introduce this new operator and state an interesting fact about this approach. Let $\star$ be the Hodge operator associated to g and $\star_{Sa}$ the Hodge operator to g_{Sa}, respectively. Define a modified Hodge operator

$\#_\varepsilon \colon \Lambda^1 T^* L(1) \to \Lambda^{m-1} T^* L(1)$ by

$$\#_\varepsilon := \varepsilon^{l-m} \left(\sigma_\varepsilon^*\right)^{-1} \circ (\rho \circ \star) \circ \sigma_\varepsilon^*,$$

where $\rho = dm/dm_{Sa}$ denotes the Radon-Nikodym density of the volume forms of g and g_{Sa}. Its associated bilinear form is given as

$$b_\varepsilon(h, f) := \int_{L(1)} dh \wedge \#_\varepsilon df.$$

By Friedrichs' construction, which is explained in detail in Appendix C, there exists a selfadjoint and non-negative operator H_ε with domain $\mathcal{D}(H_\varepsilon) = H_0^1 \cap H^2(L(1), \mu_{Sa})$ with

$$H_\varepsilon f = - \star_{Sa} d \#_\varepsilon df.$$

In local coordinates we hence obtain

$$H_\varepsilon f = - \frac{1}{\sqrt{\det g_{Sa}}} \partial_i \left(g^{ij} \sqrt{\det g_{Sa}} \partial_j f \right),$$

for all $f \in \mathcal{D}(H_\varepsilon)$. Therefore, the operator turns out to be a generalized Laplacian, which is still a second order differential operator.

The following result helps us to understand the operator H_ε from a different point of view.

Proposition 1.27. *Let g and h be Riemannian metrics on the smooth manifold M. Then, there exists $a \in \Gamma(\mathrm{End}(TM))$ with*

$$h(aX, Y) = g(X, Y),$$

for all vector fields X and Y on M.

Proof. In local coordinates let $h = (h_{ij})$, $h^{-1} = (h^{ij})$ and $g = (g_{ij})$, respectively. Then, we define an endomorphism by

$$a(\partial_i) = a_i^k(\partial_k) := g_{in} h^{nk}(\partial_k).$$

Let $X = X^l \partial_l$ and $Y = Y^n \partial_n$, then $g(X, Y) = X^l Y^n g_{ln}$. Moreover,

$$\begin{aligned} h(aX, Y) &= h(a(X^l \partial_l), Y^n \partial_n) \\ &= X^l Y^n h(a_l^k \partial_k, \partial_n) \\ &= X^l Y^n a_l^k h_{kn} \\ &= X^l Y^n g_{lm} h^{mk} h_{kn}. \end{aligned}$$

As $h^{nk} h_{kn}$ equals the identity (matrix), by $a^T = g h^{-1}$ we obtain the statement.

□

Using the definitions of the divergence and the gradient (see Appendix A), with the help of the above proposition, we obtain

$$\begin{aligned}
\mathrm{a}(\mathrm{grad}_g f) &= a\left(g^{ik}\frac{\partial f}{\partial x^k}\frac{\partial}{\partial x^i}\right) \\
&= g^{ik}\frac{\partial f}{\partial x^k} a\left(\frac{\partial}{\partial x^i}\right) \\
&= g^{ik}\frac{\partial f}{\partial x^k} g_{in} h^{nl}\left(\frac{\partial}{\partial x^l}\right) \\
&= \delta^k_n h^{nl}\frac{\partial f}{\partial x^k}\frac{\partial}{\partial x^l} = \mathrm{grad}_h f.
\end{aligned}$$

When choosing $h = g_{Sa}$, we hence obtain

$$\begin{aligned}
H_\varepsilon f &= -\mathrm{div}_{g_{Sa}}\left(\mathrm{grad}_g f\right) \\
&= -\mathrm{div}_g\left(a^{-1}\left(\mathrm{grad}_g f\right)\right).
\end{aligned}$$

For an arbitrary endomorphism A on the tangent bundle, we obtain

$$\begin{aligned}
&- \mathrm{div}_g\left(A\,\mathrm{grad}_g f\right) \\
=&\frac{1}{\sqrt{\det g}}\partial_k\left(\sqrt{\det g}A^k_j g^{ij}\partial_i f\right) \\
=&\frac{1}{\sqrt{\det g}}\partial_k\left(\sqrt{\det g}A^k_j g^{ij}\right)\partial_i f + A^k_j g^{ij}\frac{\partial^2 f}{\partial x^k \partial x^i}.
\end{aligned}$$

A simple comparison of the coefficients with the definition of the Laplace-Beltrami operator, in local coordinates, according to an arbitrary metric G shows

$$G^{ki} = A^k_j g^{ij}.$$

Hence, whenever A is symmetric, positive definite, commutes with g as a matrix and by multiplication with g the determinant of the metric remains the same, we obtain a new metric $A \cdot g$ and H_ε is the Laplace-Beltrami operator according to this different metric.

Remark. *Note that there is a one-to-one correspondence between Schrödinger operators $H = \frac{1}{2}\triangle + W$, depending on a smooth potential function, and Laplacians with drift vector fields. By partial integration the associated quadratic forms are related to each other.*

Due to this remark, it is possible to study the problem from other points of view. We return to this observation in Chapter 7, where we comment on the role of the drift and how it appears in this process.

For now, we focus on the approach using the potential function. It is remarkable that we can compute the potential in terms of the Radon-Nikodym density ρ (see Definition (1.25)) on $L(1)$, i.e. define

$$W := \frac{1}{2}\triangle_g \log\rho - \frac{1}{4}\|d\log\rho\|^2_g \in C^\infty\left(\overline{L(1)}\right). \tag{14}$$

Then, in $L(1)$ the potential is given by

$$W(p) = \left(\frac{1}{2}\mathrm{Scal}_L - \frac{1}{4}\|\tau\|^2 - \frac{1}{6}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L} + \overline{R}_{M/L}\right)\right) \circ \pi(p),$$

where $\pi : L(1) \to L$ denotes the projection. A proof of this fact is given in Chapter 4. Note that W restricted to L yields the potential W_0 on L.

Throughout this thesis, we use these and other approaches to compute and understand the potential W_0 by a detailed investigation of the participating quantities.

In Chapter 6 we focus on the Wiener measure associated to Brownian motion and on the associated measures on the path spaces of Riemannian manifolds. Therefore, the following observations will be useful.

Let μ_M denote the Riemannian volume measure of M and $T > 0$ be a finite time horizon. By $\Omega_M := C([0,T], M)$ we denote the space of paths in M, which can be equipped with a metric d, such that (Ω_M, d) is a metric space. Moreover, it is possible to equip this space with a measure μ. Let μ denote the Wiener measure $\mathbb{W}_M^q$ associated to the Brownian motion on M with starting point $q \in M$. We define

$$\Omega_{s,t}^{\varepsilon} := \Omega_{s,t}(\varepsilon) := \{\omega \in \Omega |\; \omega(u) \in L(\varepsilon), \text{ for all } s \leq u \leq t\}$$

as the subset of all paths, whose graphs do not leave the tube $L(\varepsilon)$ between times s and t.

We observe that Ω_L is Minkowski-regular in Ω_M and that there is another characterization of the limit measure. Moreover, note that $\Omega_L(\varepsilon) = \Omega_{L(\varepsilon)}$ holds. In the subsequent chapters we are going to use the homogenization result in the following form. A proof can be found in [SSWW14].

Theorem 1.28. *1. For $x \in L$, and as $\varepsilon > 0$ tends to zero, the conditional Wiener measures*

$$\mu_\varepsilon^x(d\eta) := \mathbb{W}_M^x(d\eta | \eta \in \Omega_L(\varepsilon)) := \mathbb{W}_M(d\eta | \eta \in \Omega_L(\varepsilon), \eta(0) = x)$$

converge weakly to a limit measure μ_0^x on the path space of the submanifold.

2. *The limit measure μ_0^x is equivalent to the Wiener measure $\mathbb{W}_L^x$ on the submanifold and the associated Radon-Nikodym density is given by*

$$\frac{d\mu_0^x}{d\mathbb{W}_L^x}(\eta) = \frac{\exp\left(-\frac{1}{2}\int_0^T W(\eta(s))\, ds\right)}{\int_{\Omega_L} \exp\left(-\frac{1}{2}\int_0^T W(\omega(s))\, ds\right) \mathbb{W}_L^x(d\omega)}. \tag{15}$$

3. *Here W is the potential determined by (14).*

Thereby, the limit of the conditional Brownian motion on $L(\varepsilon)$ is an intrinsic Brownian motion on L related to an effective potential W_0, which reflects the geometry of L and M, and of the embedding into the ambient manifold M.

Definition 1.29. *We define μ_0 as the* ***surface measure*** *on the path space of L generated by the Wiener measure of M.*

Remark. *By formula (15), the conditioned Wiener measure tends to the Wiener measure on L, as ε tends to zero, whenever the potential is constant.*

We now have a closer look at the conditional Brownian motion in the following section.

1.6 Conditional Brownian motion on tubular neighborhoods

Following the construction in Section (1.2) in [Wit11], we want to study a Brownian motion $(B_t)_{t\geq 0}$ on the ambient space, where the starting point is on the submanifold L. For $\varepsilon > 0$ fixed, denote by B_t^ε the process of conditional Brownian motion to the event that the paths remain inside the tubular neighborhood $L(\varepsilon)$ up to time $0 < T < \infty$.

Let $0 \leq s < T$. The distribution of the Brownian motion $B_t^\varepsilon, 0 \leq t < T$, is given by

$$\mathbb{L}_\varepsilon(d\omega) := \mathbb{W}_M(d\omega|\, \omega \in \Omega_{0,T}(\varepsilon)).$$

For $0 \leq s < T$ we want to compute the transition kernel of B_t^ε defined by

$$Q_\varepsilon^{s,x,t}(d\omega) := \mathbb{L}_\varepsilon(\omega(t) \in dy|\, \omega(s) = x).$$

Using the Markov property of the Wiener measure and the definition of conditional probabilities we obtain by $\Omega_{0,T} = \Omega_{s,T} \cap \Omega_{0,s}$ the following result:

$$\begin{aligned}
&Q_\varepsilon^{s,x,t}(d\omega)\\
&= \mathbb{L}_\varepsilon(\omega(t) \in dy|\, \omega(s) = x)\\
&= \mathbb{W}_M(\omega(t) \in dy|\, \omega(s) = x, \omega \in \Omega_{0,T}(\varepsilon))\\
&= \mathbb{W}_M(\omega(t) \in dy|\, \omega(s) = x, \omega \in \Omega_{s,T}(\varepsilon) \cap \Omega_{0,s}(\varepsilon))\\
&= \frac{\mathbb{W}_M(\omega(t) \in dy \cap \omega \in \Omega_{s,T}(\varepsilon)|\, \omega \in \Omega_{0,s}(\varepsilon), \omega(s) = x)}{\mathbb{W}_M(\omega \in \Omega_{s,T}(\varepsilon)|\, \omega(s) = x, \omega \in \Omega_{0,s}(\varepsilon))}\\
&= \frac{\mathbb{W}_M(\omega(t) \in dy \cap \omega \in \Omega_{s,T}(\varepsilon)|\, \omega(s) = x)}{\mathbb{W}_M(\omega \in \Omega_{s,T}(\varepsilon)|\, \omega(s) = x)}\\
&= \frac{\mathbb{W}_M(\omega(t) \in dy \cap \omega \in (\Omega_{s,t}(\varepsilon) \cap \Omega_{t,T}(\varepsilon))|\, \omega(s) = x)}{\mathbb{W}_M(\omega \in \Omega_{s,T})(\varepsilon)|\, \omega(s) = x)}\\
&= \frac{\mathbb{W}_M(\omega(t) \in \Omega_{t,T}|\, \omega(s) = x, \omega(t) = y, \omega \in \Omega_{s,t}^\varepsilon)\mathbb{W}_M(\omega(t) \in dy, \omega \in \Omega_{s,T}|\, \omega(s) = x)}{\mathbb{W}_M(\omega \in \Omega_{s,T}(\varepsilon)|\, \omega(s) = x)}\\
&= \frac{\mathbb{W}_M(\Omega_{t,T}|\, \omega(t) = y) \cdot \mathbb{W}_M(\omega(t) \in dy, \omega \in \Omega_{s,t}(\varepsilon)|\, \omega(s) = x)}{\mathbb{W}_M(\omega \in \Omega_{s,T}(\varepsilon)|\, \omega(s) = x)}.
\end{aligned}$$

Define

$$P_\varepsilon^{s,x,t}(\omega(t) \in dy) := \mathbb{W}_M(\omega(t) \in dy, \omega \in \Omega_{s,t}(\varepsilon)|\, \omega(s) = x).$$

Due to the fact that the right hand side can be expressed as

$$\mathbb{W}_M(\omega(t) \in dy, t < \tau_\varepsilon(\omega)|\, \omega(s) = x),$$

where τ_ε denotes the first exit time of the Brownian motion of the tube $L(\varepsilon)$, P_ε is indeed the transition kernel of the Brownian motion absorbed at the boundary $\partial L(\varepsilon)$ of the tube. Moreover, at the end of Section (1.2) in [Wit11] the following results are stated.

Proposition 1.30. *1. The conditional Brownian motion is a time-homogeneous sub-Markov process, whose transition kernel is uniquely determined by the transition kernel of a Brownian motion absorbed at the boundary of the tube.*

2. The generator of the absorbed Brownian motion on the tube is the Laplace-Beltrami operator $-\triangle_\varepsilon/2$ on M with Dirichlet boundary conditions at $\partial L(\varepsilon)$, i.e. given by (12).

To analyze the potential W_0 in detail, we now proceed our studies with some typical classes of embeddings. We compute W_0 in these cases and give simplified formulas. We present interpretations of the potential in some of these settings. Hence, we get used to the quantities, which determine the potential.

Moreover, we study the geometry of the tubular neighborhoods in detail and use perturbation theory to obtain more insights on W_0. We reformulate the density and the potential more than once before evaluating them.

Chapter 2

Interpretation of the potential W_0 in different settings

In the previous chapter we presented a formula for the potential W_0 on the submanifold, corresponding to the isometric embedding $\varphi\colon L \to M$, under the above introduced limit process. To understand the results given in Section (1.5) and Section (1.6), it is crucial to understand this potential, which reflects intrinsic and extrinsic geometric properties of L, M and the embedding φ.

First of all, note that due to several identities between the curvature tensors of the submanifold L and the ambient manifold M, there is not one specific formula for W_0. To obtain different representations, we use Gauss' equation (see (16)) or different combinations of the involved summations.

To get a better idea of the quantities involved in the formula, we start our analysis with various explicit examples of embeddings. We first simplify formula (13) in some special cases and give interpretations for the occurring terms. For each of the following examples, we hence develop easier formulations for W_0. We begin with the study of embeddings into Euclidean spaces.

Afterwards, we derive a new representation for the potential W_0 of the embedding φ, differing from the one given in [Wit11]. From this new formula, the relevant quantities become clearer. Moreover, we use results of Weyl and Gray (see e.g. [Wey39],[Gra04],[Gra82]) to explain the potential in the specific case of totally geodesic embeddings.

Throughout this thesis, we use the following conventions: Let $\{e_i\}_{i=1,\dots,l}$ in T_pL, $p \in L$, be an orthonormal frame. Following the splitting into Latin and Greek indices from Chapter 1, let $\{e_\alpha\}_{\alpha=l+1,\dots,m}$ be an orthonormal frame in the orthogonal (with respect to g) complement N_pL. Let $\{e_A\}_{A=1,\dots,m}$, where $e_A = e_i$, for $A \leq l$, and $e_A = e_\alpha$, for $A > l$, be a complete orthonormal basis of T_pM. Remember that for summation from 1 to m we always use the Latin indices A and B.

From Theorem (1.26) we know,

$$W_0 = \frac{1}{2}\left(\mathrm{Scal}_L - \frac{1}{2}\|\tau(\varphi)\|^2 - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M\setminus L} + \overline{R}_{M\setminus L}\right)\right).$$

With respect to the given (local) orthonormal frame, the partial traces of R^M are defined as

$$\begin{aligned}\overline{\mathrm{Ric}}_{M/L} &= \sum_{i=1}^{l} \mathrm{Ric}^M(e_i, e_i) = \sum_{i=1}^{l}\sum_{A=1}^{m} \langle e_i, R^M(e_i, e_A)e_A\rangle, \\ \overline{R}_{M/L} &= \sum_{i,j=1}^{l} \langle e_i, R^M(e_i, e_j)e_j\rangle\end{aligned}$$

as usual.

Remark. *Note that we always use the following identification:*

$$\begin{aligned}\mathrm{Scal}_M &= \sum_{A=1}^{m}\sum_{B=1}^{m} \langle e_A, R^M(e_A, e_B)e_B\rangle \\ &= \sum_{i=1}^{l}\sum_{j=1}^{l} \langle e_i, R^M(e_i, e_j)e_j\rangle + 2\sum_{i=1}^{l}\sum_{\alpha=l+1}^{m} \langle e_i, R^M(e_i, e_\alpha)e_\alpha\rangle \\ &\quad + \sum_{\alpha=l+1}^{m}\sum_{\beta=l+1}^{m} \langle e_\alpha, R^M(e_\alpha, e_\beta)e_\beta\rangle\end{aligned}$$

For a suitable interpretation of the formula for the potential W_0 we first introduce a useful tool for our computations and use Gauss' equation in the following form.

Theorem 2.1 (Gauss). *Let $L \subset M$ be a submanifold of a Riemannian manifold and $\varphi\colon L \to M$ the corresponding embedding. Moreover let R^L, R^M be the curvature tensors. For $X, Y, Z, W \in \Gamma(TL)$ the equation*

$$\begin{aligned}\langle R^M(X,Y)Z, W\rangle = \langle R^L(X,Y)Z, W\rangle &+ \langle \mathrm{II}(\varphi)(X,Z), \mathrm{II}(\varphi)(Y,W)\rangle \\ &- \langle \mathrm{II}(\varphi)(Y,Z), \mathrm{II}(\varphi)(X,W)\rangle.\end{aligned} \tag{16}$$

holds.

For a proof and further information about the fundamental equations for the embeddings of submanifolds of Riemannian manifolds see e.g. [Spi75] (p. 51 ff.).

As already announced, this equation ensures that there is more than one possibility to express W_0.

In the following sections we are going to use a generalization of Weyl's famous tube formula (see [Wey39]), as well as comparison theorems for the volumes of tubes in Riemannian manifolds in the version according to A. Gray.

We state the following fact in the form of [Gra04], equation (1.1).

Theorem 2.2. *The volume of a tube of radius $\varepsilon > 0$ around a submanifold L in the Euclidean space $\mathbb{R}^m$ is given by*

$$V_L^{\mathbb{R}^m}(\varepsilon) = \frac{\left(\pi\varepsilon^2\right)^{(m-l)/2}}{((m-l)/2)!} \sum_{c=0}^{\lfloor l/2 \rfloor} \frac{k_{2c}(R^L)\varepsilon^{2c}}{(m-l+2)(m-l+4)\cdots(m-l+2c)} \tag{17}$$
$$= \mathrm{vol}\left(B^{m-l}(\varepsilon)\right) \sum_{c=0}^{\lfloor l/2 \rfloor} \frac{k_{2c}(R^L)\varepsilon^{2c}}{(m-l+2)(m-l+4)\cdots(m-l+2c)},$$

where the coefficients $k_{2c}(R^L)$ are integrals over L of certain polynomials of order $2c$ in the curvature tensor of L and $\mathrm{vol}\left(B^{m-l}(\varepsilon)\right)$ *denotes the volume of an Euclidean ball around zero in $\mathbb{R}^{m-l}$.*

Weyl and Gray refer to these coefficients as a certain type of invariants of the submanifold. For a proof of the formula see [Wey39]. Furthermore, in [Gra82], p. 202, Remark (3), the relation

$$k_0(R^L) = \text{volume of } L$$

is given.

We will prove that there is a correspondence between the potential W_0 and the deviation of measuring the volume of the tubes $L(\varepsilon)$ as subsets of Riemannian manifolds and of Euclidean space.

2.1 Embeddings into Euclidean Spaces

For a submanifold L in an Euclidean space, e.g. $\mathbb{R}^m$, $m \in \mathbb{N}$, the formula for W_0 is easy. Because $\mathbb{R}^m$ is flat, its curvature tensor vanishes identically. Therefore, the partial traces, which occur in the potential, are identically equal to zero and the only remaining terms are the tension field of the embedding $\varphi\colon L \to M$ and the scalar curvature of the underlying submanifold. We then obtain

$$2W_0^{\mathrm{euc}} = \mathrm{Scal}_L - \frac{1}{2}\|\tau(\varphi)\|^2.$$

In particular, if the scalar curvature and the tension field are constant, the surface measure μ_0 from Theorem (1.28) coincides with the Wiener measure on L.

For a deeper discussion of embeddings into Euclidean space, we refer the reader to [Sid03]. A detailed construction of the surface measures using the potential and its associated Radon-Nikodym density is presented there.

Example 2.3. *Study a unit-speed curve $\varphi\colon \mathbb{S}^1 \to \mathbb{R}^2$ without self-intersections, i.e. an isometric embedding into the two-dimensional Euclidean space. Note that we consider only regular embeddings, i.e. $\|\ddot{\varphi}\| < C$ and $C > 0$.*
As the scalar curvature of a one-dimensional object is zero and $\mathbb{R}^2$ is flat, the only remaining term in the potential is the tension field of φ. We obtain

$$W_0^{2d} = -\frac{1}{4}\|\tau(\varphi)\|^2.$$

Here, the tension field equals the acceleration of the curve, i.e. the curvature k of φ and is therefore given by

$$\tau(\varphi) = \frac{d^2}{ds^2}\varphi(s) = k(s).$$

In addition, we obtain that the corresponding density is given by

$$\rho(\omega) = \frac{1}{Z}\exp\left(\frac{1}{8}\int_0^T \|k(s)\|^2(\omega(s))\, ds\right).$$

By Z we denote the normalization.

Example (2.3) is adapted from [BMSW09], where the authors explore the formula for W_0 from the general construction. They compute the metrics g and g_{Sa} in local Fermi coordinates and give the explicit computation of the asymptotic of the induced metric with respect to the reference metric without making use of the general result.

Example 2.4. *Let L denote a two-dimensional submanifold of $\mathbb{R}^3$. By the above considerations we remain only with the scalar curvature of L and the tension field of the embedding, i.e.*

$$W_0 = \frac{1}{2}\mathrm{Scal}_L - \frac{1}{4}\|\tau(\varphi)\|^2.$$

In this setting the scalar curvature is just twice the Gaussian curvature $\kappa_1 \cdot \kappa_2$, where κ_i denote the principal curvatures. The tension field is exactly twice the mean curvature of the hypersurface, i.e.

$$W_0 = \kappa_1\kappa_2 - \frac{1}{4}(\kappa_1 + \kappa_2)^2.$$

Since the scalar curvature of L is an intrinsic invariant, it is a well-known quantity and therefore a well-understood term. Another intrinsic invariant is the Ricci curvature of the submanifold, while the shape operator and the tension field are extrinsic quantities. Therefore, the tension field is probably the most difficult quantity to explain in our context.

Due to a theorem of Bonnet for embeddings into Euclidean space, the second fundamental form is fully determined by fixing sufficient additional data. The statement can be found in a slightly different formulation in Theorem 5.2 of [CL17].

Theorem 2.5 (Theorem of Bonnet). *Let (L, g_L) be a Riemannian manifold of dimension $l \in \mathbb{N}$ and $\pi : E \to L$ a vector bundle of rank $r \in \mathbb{N}$ over L, equipped with a bundle metric b and a metric connection ∇^E. Without loss of generality let L be simply connected and equipped with its Levi-Civita connection.*

If $\omega \in \Omega^1(\mathrm{Hom}(TL, E))$ is symmetric and satisfies the fundamental equations for submanifolds, i.e. the Gauss, Codazzi and Ricci equation, then there exists an isometric immersion

$$\psi\colon L \to \mathbb{R}^{l+r}$$

where $NL \cong E$ and the second fundamental form of the embedding is ω. Moreover ψ is unique up to Euclidean motions.

Remark. *There are not many results on more general settings, i.e. embeddings into arbitrary ambient Riemannian manifolds. Recently, in [LR17], a version of Bonnet's theorem was given for submanifolds of multiproducts of real space forms. The authors introduce a set of so-called compatibility conditions, which do not only involve versions of Gauss', Codazzi's and Ricci's*

equation, but many more restrictions that need to be satisfied.
Due to the high amount of constraints stated in this rather simple case, it is not helpful to ask for a similar result for arbitrary Riemannian manifolds and use this in the following.

2.2 Embeddings of a single point

Let $L = \{p\}$, $p \in M$, be a zero dimensional submanifold of the Riemannian manifold (M, g) and F a fiber of the normal bundle. Then the tangent space is zero dimensional and $\overline{\mathrm{Ric}}_{M/L} = \overline{R}_{M/L} = 0$. Moreover, we obtain

$$\mathrm{Scal}_M = \mathrm{Scal}_F.$$

The scalar curvature of L as well as the tension field τ obviously vanish identically. Therefore, in this case the potential just depends on the scalar curvature of M and is given by

$$W_0^p = -\frac{1}{6}\mathrm{Scal}_M.$$

2.3 Embeddings of hypersurfaces

In this section we give the formula for W_0 in the case that we deal with submanifolds of codimension one. Let therefore $\varphi\colon L \to M$ denote the embedding of the hypersurface L in M and let the dimension of M be $m = l + 1$.

Remark. *Here, following the usual notation, α and β are only equal to m.*

This especially means that the fiber curvature $R^M_{\alpha\beta\alpha\beta} = R^M_{mmmm}$ vanishes.

Remark. *For any hypersurface $L \subset M$ we can simplify the notation of the second fundamental form. Since the normal space N_pL is one-dimensional for all $p \in L$, there exists a unique normal vector n (up to sign), which generates N_pL, $p \in L$. Let then ν be a normal vector field; then, we define*

$$h(\varphi)(e_i, e_j) := \langle \mathrm{II}(\varphi)(e_i, e_j), \nu\rangle$$

for all $i, j = 1, ..., l$ and an orthonormal frame $\{e_i\}_{i=1,...,l} \subseteq TL$.
Hence, $h(e_i, e_j) \in \mathbb{R}$ for all $i, j = 1, ..., l$.

We start with the exploration of an example.

Example 2.6. *Let the embedding φ of the hypersurface L in M have constant second fundamental form, i.e. $h(\varphi)(e_i, e_j) = c$. Note that $c \in \mathbb{R}$, as the remark above demonstrates. Hence, the tension field is given by*

$$\tau(\varphi) = \sum_{i=1}^{l} h(\varphi)(e_i, e_i) = l \cdot c.$$

Using Gauss' equation (16) we obtain (no summation!)

$$R^L_{ijij} = R^M_{ijij} - \langle h(\varphi)(e_i, e_i), h(\varphi)(e_j, e_j)\rangle + \langle h(\varphi)(e_i, e_j), h(\varphi)(e_j, e_i)\rangle,\ i, j = 1, ..., l,$$

where the scalar products are equal to c^2 and therefore the latter terms cancel each other, i.e.

$$R^L_{ijij} = R^M_{ijij}$$

for all $i, j = 1, ..., l$. Therefore, the potential can be computed by

$$\begin{aligned}
&2W_0^{\mathrm{hyp}} \\
&= \mathrm{Scal}_L - \frac{1}{2}\|\tau(\varphi)\|^2 - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M\setminus L} + \overline{R}_{M\setminus L}\right) \\
&= \sum_{i,j} R^L_{ijij} - \frac{1}{2}(c\cdot l)^2 - \frac{1}{3}\left(\sum_{A,B} R^M_{ABAB} + \sum_{i,A} R^M_{iAiA} + \sum_{i,j} R^M_{ijij}\right) \\
&= \sum_{i,j} R^L_{ijij} - \frac{1}{2}(c\cdot l)^2 - \sum_{i,j} R^M_{ijij} - \sum_{i,\alpha} R^M_{i\alpha i\alpha} - \frac{1}{3} R^M_{\alpha\beta\alpha\beta} \\
&= -\frac{1}{2}(c\cdot l)^2 - \sum_i R^M_{imim}.
\end{aligned}$$

For arbitrary embeddings of codimension equal to one (not necessary with constant second fundamental form), using the Gauss' equation (16), the symmetries of the Riemannian curvature tensor, and the definitions of the partial traces, gives the following formula for the corresponding potential:

$$\begin{aligned}
&2W_0^{\mathrm{hyp}} \\
&= \mathrm{Scal}_L - \frac{1}{2}\|\tau(\varphi)\|^2 - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M\setminus L} + \overline{R}_{M\setminus L}\right) \\
&= \sum_{i,j} R^L_{ijij} - \frac{1}{2}\|\sum_i h(\varphi)(e_i,e_i)\|^2 - \frac{1}{3}\left(\sum_{A,B} R^M_{ABAB} + \sum_{i,A} R^M_{iAiA} + \sum_{i,j} R^M_{ijij}\right) \\
&= \sum_{i,j} R^L_{ijij} - \frac{1}{2}\left\langle \sum_i h(\varphi)(e_i,e_i), \sum_j h(\varphi)(e_j,e_j)\right\rangle \\
&\quad - \sum_{i,j} R^M_{ijij} - \sum_{i,\alpha} R^M_{i\alpha i\alpha} - \frac{1}{3}\sum_{\alpha,\beta} R^M_{\alpha\beta\alpha\beta} \\
&\overset{(16)}{=} \sum_{i,j} R^L_{ijij} - \frac{1}{2}\sum_{i,j}\langle h(\varphi)(e_i,e_i), h(\varphi)(e_j,e_j)\rangle \\
&\quad - \sum_{i,j}\left(R^L_{ijij} + \langle h(\varphi)(e_i,e_j), h(\varphi)(e_j,e_i)\rangle - \langle h(\varphi)(e_j,e_j), h(\varphi)(e_i,e_i)\rangle\right) \\
&\quad - \sum_i R^M_{imim} \\
&= \frac{1}{2}\sum_{i,j}\langle h(\varphi)(e_j,e_j), h(\varphi)(e_i,e_i)\rangle - \sum_{i,j}\langle h(\varphi)(e_i,e_j), h(\varphi)(e_j,e_i)\rangle - \sum_i R^M_{imim} \\
&= \frac{1}{2}\|\tau(\varphi)\|^2 - \sum_{i,j}\|h(\varphi)(e_i,e_j)\|^2 - \sum_i R^M_{imim}.
\end{aligned}$$

Therefore, the potential just depends on the second fundamental form and some sectional curvature terms of planes in TM.

Remark. *Note that we drop the summation sign here for the sake of clarity. From now on, we omit these and use the Einstein convention freely, as announced in the introduction.*

Example 2.7. *For a plane spanned by two linear independent vectors in $\mathbb{R}^3$, the potential vanishes, i.e. $W_0 = 0$.*

2.4 Embeddings of one-dimensional submanifolds

2.4.1 General case

Let $\varphi\colon \mathbb{S}^1 \to M$ be the isometric embedding of a closed and at least C^2-smooth curve into the Riemannian manifold M. By c we denote the image of φ in M. Note that the scalar curvature of c vanishes.

In terms of the general formula we obtain

$$\begin{aligned} W_0^c &= \frac{1}{2}\left(\mathrm{Scal}_c - \frac{1}{2}\|\tau(\varphi)\|^2 - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/c} + \overline{R}_{M/c}\right)\right) \\ &= -\frac{1}{4}\|\tau(\varphi)\|^2 - \frac{1}{6}R^M_{ABAB} - \frac{1}{6}R^M_{iAiA} - \frac{1}{6}R^M_{ijij}. \end{aligned}$$

Following the usual notation, we obtain $i = j = 1$; therefore, some more terms cancel. Moreover, the tension field just equals the acceleration vector of the curve and the term of the form R^M_{iAiA} is, in this case, exactly Ric^M_{11}.

Hence, we remain with

$$W_0^c = -\frac{1}{4}\|\tau(\varphi)\|^2 - \frac{1}{2}\mathrm{Ric}^M_{11} - \frac{1}{6}R^M_{\alpha\beta\alpha\beta}.$$

2.4.2 Unit speed geodesic

In this section we study a very special case. Let

$$\gamma\colon \mathbb{S}^1 \to M,$$

be a smooth and closed unit speed geodesic of finite length $\mathcal{L}(\gamma)$, which is topologically embedded into M. We use γ as an abbreviation for its image in M as well and think of the graph as a submanifold of dimension one.

By a result in [ES64] (see Chapter 4, p. 125) the tension field in this situation is given by

$$\tau(\gamma)^A(t) = \frac{d^2\gamma^A}{dt^2} + \Gamma^A_{BC}\frac{d\gamma^B}{dt}\frac{d\gamma^C}{dt},$$

for all components $A = 1, ..., m$ and the Christoffel symbols Γ^A_{BC} of M.

Due to the orthogonality of the first and second derivative of γ with respect to the parameter t, we obtain $\langle \tau(\gamma), \frac{d\gamma}{dt}\rangle = 0$. Therefore, $\tau(\gamma)$ is proportional to the geodesic curvature vector field along γ. We obtain the harmonicity of γ (see next section), i.e. $\tau(\gamma) = 0$, if and only if the graph defines a closed geodesic in M. Hence, we can now assume that the tension field vanishes.

By Theorem 5.3. in [GV82] we obtain in this situation

$$\operatorname{vol}(\gamma(\varepsilon)) = \operatorname{vol}\left(B^{m-1}(\varepsilon)\right) \int_0^1 \left(1 + A\varepsilon^2 + \mathcal{O}\left(\varepsilon^4\right)\right)(\gamma(t))\ dt,$$

where

$$A = -\frac{1}{6(m+1)}\left(\operatorname{Scal}_M + \operatorname{Ric}^M_{11}\right).$$

Remark. *In this case, following the usual notation, i and j are just equal to 1. Therefore, the partial trace R^M_{ijij} vanishes. Moreover, $\operatorname{Scal}_\gamma = 0$ and the tension field vanishes because γ is a geodesic.*

Hence, in local coordinates

$$\begin{aligned} A &= -\frac{1}{6(m+1)}\left(\operatorname{Scal}_M + \operatorname{Ric}^M_{11}\right) \\ &= -\frac{1}{6(m+1)}\left(R^M_{ABAB} + R^M_{1A1A}\right) \\ &= -\frac{1}{6(m+1)}\left(R^M_{ijij} + 2R^M_{i\alpha i\alpha} + R^M_{\alpha\beta\alpha\beta} + R^M_{1A1A}\right) \\ &= -\frac{1}{6(m+1)}\left(3R^M_{1A1A} + R^M_{\alpha\beta\alpha\beta}\right) \\ &= -\frac{1}{2(m+1)}R^M_{1A1A} - \frac{1}{6(m+1)}R^M_{\alpha\beta\alpha\beta}. \end{aligned}$$

Using the above remark for the formula of the potential W_0 leads to

$$\begin{aligned} W_0^\gamma &= \frac{1}{2}\left(\operatorname{Scal}_\gamma - \frac{1}{2}\|\tau\|^2 - \frac{1}{3}\left(\operatorname{Scal}_M + \overline{\operatorname{Ric}}_{M/\gamma} + \overline{R}_{M/\gamma}\right)\right) \\ &= -\frac{1}{2}\left(R^M_{ijij} - R^M_{1\alpha 1\alpha} - \frac{1}{3}R^M_{\alpha\beta\alpha\beta}\right) \\ &= -\left(\frac{1}{2}R^M_{1A1A} + \frac{1}{6}R^M_{\alpha\beta\alpha\beta}\right). \end{aligned}$$

Combining this result with the comparison formula above, we obtain that the deviation of the volumes is basically determined by the potential.

Lemma 2.8. *Let $\gamma\colon \mathbb{S}^1 \to M$ be a smooth, closed unit speed geodesic of finite length in M. Denote by $L^\gamma(\varepsilon)$ the tubular neighborhood of the graph of γ and by $B^{m-1}(\varepsilon)$ the ball of radius $\varepsilon > 0$ in $\mathbb{R}^{m-1}$ around zero. Then, for $\varepsilon \to 0$ we obtain*

$$\frac{\operatorname{vol}\left(L^\gamma(\varepsilon)\right)}{\operatorname{vol}\left(B^{m-1}(\varepsilon)\right)} = \int_0^1 \left\{1 + \frac{\varepsilon^2}{(n+1)}W_0^\gamma + \mathcal{O}\left(\varepsilon^4\right)\right\}(\gamma(t))\, dt.$$

We now generalize the previous result.

2.5 Totally geodesic submanifolds

This previous section was written in order to prepare similar results for embeddings of arbitrary codimension.

Let $L \subset M$ be a totally geodesic submanifold, i.e. the second fundamental form II vanishes identically. Equivalently, let $f\colon L \to M$ denote the embedding of L into M, then f maps geodesics in L into geodesics in M. This especially means that there is no deviation in the length measuring between L and M and hence, the tension field $\tau(\varphi)$ vanishes by definition, i.e. L is also minimal in M.

Remark. *Note that a map $f\colon L \to M$ between Riemannian manifolds, with $\tau(f) = 0$ is often called a harmonic map. For a detailed discussion of harmonic maps see e.g. [ES64].*

In this thesis we use the descriptors harmonic and minimal embedding equally.

Example 2.9. *Let $M = \mathbb{R}^n$, then $\varphi\colon L \to M$ is harmonic if and only if it is constant. This follows from an application of the maximum principle and is discussed in [ES64], p. 118. The characterization gives a hint on the curvature of L.*

For minimal embeddings, we obtain a simplified expression for the potential W_0, i.e.

$$W_0^{\mathrm{tot}}(y) = \frac{1}{2}\left(\mathrm{Scal}_L - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L} + \overline{R}_{M/L}\right)\right)(y). \tag{18}$$

We want to rewrite this formula. It turns out that we can express W_0 just using simple geometric quantities.

Remark. *Note that we will always refer to the scalar curvature of the fiber, when talking about the curvature of the leaf $\pi^{-1}(x) \cap L(\varepsilon)$ at a point $x \in L$.*

Lemma 2.10. *For a totally geodesic submanifold the potential W_0 can be written in the form*

$$W_0^{\mathrm{tot}}(y) = \left(-\frac{1}{4}\mathrm{Scal}_M + \frac{1}{4}\mathrm{Scal}_L + \frac{1}{12}\mathrm{Scal}_{F_y}\right)(y),$$

where Scal_{F_y} denotes the scalar curvature of the (normal) fiber passing through $y \in L$.

Remark. *This result is crucial and very helpful for other computations. The formula from Lemma (2.10) states that whenever $\mathrm{II}(\varphi) = 0$, the formula for the potential is easy and only depends on scalar curvatures. The scalar curvature is a very well-understood intrinsic geometric quantity. Hence, the formula does not only provide a simple condition for W_0 to be constant, but also to prove, if the density is equal to one, i.e. the surface measure coincides with the Wiener measure on the submanifold.*

Proof. Using Gauss' equation and the fact that the second fundamental form vanishes, we obtain

$$\langle R^M(X,Y)Z, W\rangle = \langle R^L(X,Y)Z, W\rangle, \tag{19}$$

for all vector fields X, Y, Z, W on T_yL.

Since L is a totally geodesic submanifold, the embedding of the fibers F_y is totally geodesic as well. Hence, another application of Gauss' equation (16) gives

$$\langle R^M(X,Y)Z, W\rangle = \langle R^F(X,Y)Z, W\rangle, \tag{20}$$

for all $X, Y, Z, W \in \Gamma(N_pL)$.

Because the vectors $\{e_A\}_{A=1,\dots,m}$ form an orthonormal frame, by definition the sectional curvature of an arbitrary tangent plane $e_i \wedge e_\alpha$ is given by

$$\kappa(e_i \wedge e_\alpha) = \langle e_i, R^M(e_i, e_\alpha)e_\alpha \rangle, \tag{21}$$

for every $i \in \{1, ..., l\}$ and $\alpha \in \{l+1, ..., m\}$.

Splitting the sums with respect to the chosen Fermi coordinates, using (21), (20) and the Einstein summation convention, we obtain

$$\begin{aligned} \mathrm{Scal}_M &= \langle e_i, R^M(e_i, e_j)e_j \rangle \\ &= \langle e_i, R^M(e_i, e_j)e_j \rangle + 2\langle e_i, R^M(e_i, e_\alpha)e_\alpha \rangle + \langle e_\alpha, R^M(e_\alpha, e_\beta)e_\beta \rangle \\ &= \mathrm{Scal}_L + \sum_i \sum_\alpha \kappa(e_i \wedge e_\alpha) + \mathrm{Scal}_F. \end{aligned} \tag{22}$$

Therefore, the sum of all sectional curvatures of the planes spanned by the vectors e_i and e_α can be expressed as follows:

$$\sum_i \sum_\alpha \kappa\left(e_i \wedge e_\alpha\right) = \frac{1}{2}\left(\mathrm{Scal}_M - \mathrm{Scal}_L - \mathrm{Scal}_F\right) \tag{23}$$

Finally, we are able to obtain the result by

$$\begin{aligned} & 2W_0^{\mathrm{tot}} \\ =\;& \mathrm{Scal}_L - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L} + \overline{R}_{M/L}\right) \\ \overset{(22)}{=}\;& \mathrm{Scal}_L - \frac{1}{3}\big(\langle e_i, R^M(e_i, e_j)e_j \rangle + \langle e_i, R^M(e_i, e_\alpha)e_\alpha \rangle \\ & \quad + \langle e_\alpha, R^M(e_\alpha, e_\beta)e_\beta \rangle + \langle e_A, R^M(e_A, e_i)e_i \rangle + \langle e_i, R^M(e_i, e_j)e_j \rangle\big) \\ =\;& \mathrm{Scal}_L - \frac{1}{3}\big(2\langle e_i, R^M(e_i, e_j)e_j \rangle + 2\langle e_i, R^M(e_i, e_\alpha)e_\alpha \rangle \\ & \quad + \langle e_\alpha, R^M(e_\alpha, e_\beta)e_\beta \rangle + \langle e_i, R^M(e_i, e_j)e_j \rangle + \langle e_i, R^M(e_i, e_\alpha)e_\alpha \rangle\big) \\ =\;& \mathrm{Scal}_L - \langle e_i, R^M(e_i, e_j)e_j \rangle - \langle e_i, R^M(e_i, e_\alpha)e_\alpha \rangle - \frac{1}{3}\langle e_\alpha, R^M(e_\alpha, e_\beta)e_\beta \rangle \\ \overset{(19)}{=}\;& -\langle e_i, R^M(e_i, e_\alpha)e_\alpha \rangle - \frac{1}{3}\langle e_\alpha, R^M(e_\alpha, e_\beta)e_\beta \rangle \\ \underset{(21)}{\overset{(20)}{=}}\;& -\sum_{i,\alpha} \kappa(e_i \wedge e_\alpha) - \frac{1}{3}\mathrm{Scal}_F \\ \overset{(23)}{=}\;& -\frac{1}{2}\mathrm{Scal}_M + \frac{1}{2}\mathrm{Scal}_L + \frac{1}{6}\mathrm{Scal}_F. \end{aligned}$$

□

Not only the scalar curvatures are helpful in this context. We will later give a sufficient condition for the potential to be constant involving sectional curvature quantities. Moreover, these results support later statements (Chapter 4) and they are the initial point for further investigations on the class of totally geodesic embeddings.

We are now going to use the representation from Lemma (2.10) in a different form.

Remark. *In local coordinates, the potential can be expressed as*

$$\begin{aligned}2W_0^{\mathrm{tot}} &= -\frac{1}{2}\mathrm{Scal}_M + \frac{1}{2}\mathrm{Scal}_L + \frac{1}{6}\mathrm{Scal}_F \\ &= -\frac{1}{2}R^M_{ABAB} + \frac{1}{2}R^M_{ijij} + \frac{1}{6}R^M_{\alpha\beta\alpha\beta} \\ &= -R^M_{i\alpha i\alpha} - \frac{1}{3}R^M_{\alpha\beta\alpha\beta}.\end{aligned}$$

At the beginning of this section we already stated that there is no deviation in the length measuring between L and M. We want to comment on the question, whether the potential is related to a deviation of volume. This relation is more complicated; but at least we obtain a result on the volume of the tubular neighborhood $L(\varepsilon)$ compared to the Euclidean volume.

According to [Gra74], p. 337, Theorem 3.1, there is a formula comparing the volume of a small geodesic ball with radius $r > 0$ in a Riemannian manifold (M, g) around some point $p \in L$ with a ball of the same radius around zero in Euclidean space. We cite

$$\mathrm{vol}^M(B_p(r)) \overset{r\to 0}{\longrightarrow} \mathrm{vol}^{\mathbb{R}^m}(B_0^{\mathbb{R}^m}(r)) \left\{1 - \frac{\mathrm{Scal}_M}{6(m+2)}r^2 + O(r^4)\right\}.$$

Defining V_M as the quotient of the two volumes above, we derive

$$\frac{d^2V_M}{d\varepsilon^2}\Big|_{\varepsilon=0} = -\frac{1}{3}\frac{\mathrm{Scal}_M}{m+2}.$$

As W_0^{tot} depends only on scalar curvatures, this indicates that W_0^{tot} is somehow related to the deviation of the volume measure in the above sense.

The investigation of tubular neighborhoods around Riemannian submanifolds has been presented in different works in the literature. An approach of the volume comparison by Weyl seen in (17) is closely related to our problem. We use the result in the form of Gray (cf. [Gra82], p. 203, (1.4)).

The volume of a tubular neighborhood $L(\varepsilon)$ around a submanifold $L \subseteq M$ with radius $\varepsilon > 0$ is presented as

$$\mathrm{vol}^M(L(\varepsilon)) = \frac{(\pi\varepsilon^2)^{\frac{1}{2}(m-l)}}{\left(\frac{m-l}{2}\right)!}\int_L \left\{1 + A\varepsilon^2 + O(\varepsilon^4)\right\}\mu_L(dy), \tag{24}$$

where A is given by

$$A := \frac{1}{2(m-l+2)}\left(\mathrm{Scal}_L - \sum_{i,j} R^M_{ijij} - \sum_{i,\alpha} R^M_{i\alpha i\alpha} - \frac{1}{3}\sum_{\alpha,\beta} R^M_{\alpha\beta\alpha\beta}\right).$$

For notational reasons, we define $\widetilde{A} := 2(m-l+2)A$.
Again, note that $\frac{(\pi\varepsilon^2)^{\frac{1}{2}(m-l)}}{(\frac{m-l}{2})!}$ is exactly the volume of an Euclidean ball of radius $\varepsilon > 0$ in dimension $m-l$.

In our case, according to the simplified Gauss' equation (19), the terms Scal_L and R^M_{ijij} cancel each other and a comparison with the remark above shows

$$A(y) = \frac{1}{(m-l+2)} W_0^{\mathrm{tot}}(y).$$

Therefore, the volume measure deviation of the tube volume compared to the volume of a Euclidean ball of dimension $m-l$ for $\varepsilon \to 0$ is given by

$$\frac{\mathrm{vol}^M(L(\varepsilon))}{\mathrm{vol}(B^{m-l}(\varepsilon))} = \int_L \left(1 + \frac{\varepsilon^2}{(m-l+2)} W_0^{\mathrm{tot}}(y) + O\left(\varepsilon^4\right)\right) \mu_L(dy).$$

Therefore, we have shown the following result.

Lemma 2.11. *Let L be a totally geodesic submanifold of a Riemannian manifold (M, g), then the first relevant correction term of the volume measure is asymptotically given by the associated potential W_0^{tot}.*

Remark. *By the introductory considerations, this result also holds for minimal submanifolds. Note that this assumption is weaker.*

Remark. *The result of Lemma (2.11) indicates that W_0^{tot} reflects the difference from the finite dimensional volume of the tube.*

We now want to study an example, which fits the above setting. In [Kli95], p. 95, Theorem 1.10.15, a characterization for totally geodesic submanifolds is given. Without proof we state the following result.

Lemma 2.12. *Let $f\colon (M, g) \to (M, g)$ be an isometry of M. Then every connected component X of the set of fixed points*

$$\{y \in M | f(y) = y\}$$

with the induced Riemannian metric is a totally geodesic submanifold.

This characterization helps us to study the following simple example.

Example 2.13. *Let $n > 1$. For the standard euclidean sphere*

$$\mathbb{S}^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} | x_0^2 + ... + x_n^2 = 1\}$$

every k-sphere with $1 \leq k < n$ is a totally geodesic submanifold of the n-sphere $\mathbb{S}^n$. This follows from the observation that $\mathbb{S}^k$ is isometric to the set of fixed points of the map $\mathbb{S}^n \to \mathbb{S}^n$ with

$$f(x_0, ..., x_n) = (x_0, ..., x_k, -x_{k+1}, ..., -x_n)^t.$$

Throughout this thesis, we will repeatedly refer to this example.

Remark. *Note that the fact that L is a totally geodesic submanifold of M, does not give any information about the embedding of the fibers. But, this is another interesting case to study.*

In Section 7.1 of [ELJL10] we find a result, which is important in our context: The authors state that a Riemannian submersion $p\colon M \to L$ maps Brownian motion on the ambient space to Brownian motion on the submanifold, if and only if its fibers are minimal. Since $\pi\colon (L(1), g_{Sa}) \to (L, g)$ is a Riemannian submersion with totally geodesic fibers (see Section (1.2)), we investigate more facts on Riemannian submersions in detail later (see Chapter 7).

2.6 Symmetric spaces and symmetric submanifolds

We first introduce the relevant definition. All facts can be reviewed in [Hel01].

Definition 2.14 ((Riemannian) symmetric space)**.** *A Riemannian symmetric space is a Riemannian manifold (M, g) together with a geodesic reflection $s\colon M \to M$, such that for every $p \in M$ there exists some $s_p \in \mathrm{Isom}(M)$ with the properties*

$$s_p(p) = p \ \textit{and} \ \ (s_p)_{*,p} = -\mathrm{id}_{T_pM}.$$

Remark. *Note that when referring to a symmetric space, we always mean a Riemannian symmetric space.*

In this section, (M, g) always denotes a complete and connected symmetric space and $\varphi\colon L \to M$ the embedding of an isometrically embedded closed submanifold.

We need additional properties of (locally) symmetric spaces to state and prove the result we developed in this setting.

Because every symmetric space is locally symmetric, by [Hel01], Chapter IV, § 1, Theorem 1.3, the sectional curvature of M is invariant under parallel translations. This crucial observation follows from the fact that the Riemannian curvature tensor R^M of M is parallel with respect to the Levi-Civita connection, i.e.

$$\nabla R^M \equiv 0$$

and the fact that g and R^M are invariant under all parallel translations.

2.6.1 Totally geodesic submanifolds of symmetric spaces

Remark. *Note that any totally geodesic submanifold of a symmetric space is a symmetric (sub-) space itself.*

The case of a totally geodesic submanifold L of a locally symmetric space M has been treated in [SSWW04a], p. 20, Corollary 5. It has been shown that in this situation $\mu_0^x = \mathbb{W}_L^x$ always holds.

Hence, the associated potential W_0 from Theorem (1.26) is constant, which we see as follows:
Recall from Lemma (2.10) that

$$W_0^{\mathrm{tot}} = -\frac{1}{2}\sum_{i,\alpha} \kappa(e_i \wedge e_\alpha) - \frac{1}{6}\mathrm{Scal}_F.$$

If L is a totally geodesic submanifold of a symmetric space, then κ is invariant under all parallel translation and is constant. Hence, W_0 is constant because the scalar curvature can be written in terms of the sectional curvature.

We want to use these observations to consider more general cases.

2.6.2 Arbitrary embeddings into symmetric spaces

Using the above facts, we are able to prove the following.

Theorem 2.15. *Let $\varphi\colon L \to M$ denote the isometric embedding of a closed Riemannian submanifold L into a symmetric space M. Then, the associated potential W_0^{sym} from Theorem (1.26) is just determined by the second fundamental form of the embedding.*

Proof. Due to the previous subsection, we know that terms of the form Scal_F, i.e. $R^M_{\alpha\beta\alpha\beta}$ and $\kappa(e_i \wedge e_\alpha) = R^M_{i\alpha i\alpha}$, $i = 1, ..., l$ and $\alpha, \beta = 1, ..., m-l$ are constant.
By using Gauss' equation and the usual splitting notation, we obtain

$$\begin{aligned}
W_0^{\mathrm{sym}} &= \frac{1}{2}\left(\mathrm{Scal}_L - \frac{1}{2}\|\tau(\varphi)\|^2 - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L} + \overline{R}_{M/L}\right)\right) \\
&= \frac{1}{2}\left(R^L_{ijij} - \frac{1}{2}\sum_{i,j}\langle \mathrm{II}(\varphi)(e_i,e_i), \mathrm{II}(\varphi)(e_j,e_j)\rangle\right. \\
&\qquad \left. - R^M_{ijij} - R^M_{i\alpha i\alpha} - \frac{1}{3}R^M_{\alpha\beta\alpha\beta}\right) \\
&\overset{(16)}{=} -\frac{1}{2}\sum_{i,j}\langle \mathrm{II}(\varphi)(e_i,e_i), \mathrm{II}(\varphi)(e_j,e_j)\rangle \\
&\qquad - \sum_{i,j}\left(\langle \mathrm{II}(\varphi)(e_i,e_j), \mathrm{II}(\varphi)(e_j,e_i)\rangle + \langle \mathrm{II}(\varphi)(e_j,e_j), \mathrm{II}(\varphi)(e_i,e_i)\rangle\right) \\
&= \frac{1}{2}\|\tau(\varphi)\|^2 - \sum_{i,j}\|\mathrm{II}(\varphi)(e_i,e_j)\|^2.
\end{aligned}$$

□

Therefore, the potential is completely understood whenever we can easily compute the second fundamental form and the tension field τ.

Remark. *By Theorem 1.3 in [Nai86], we obtain a useful condition: In the case of symmetric submanifolds of symmetric spaces the second fundamental form of the embedding is parallel, i.e. covariantly constant. Hence, the density (15) is equal to one.*

2.7 General setting

Using the usual index notation and the symmetry of the second fundamental form, we are able to state a representation for the potential W_0 differing from the one given in [Wit11], Theorem 1.

As already proven above, the potential depends on curvature quantities of the manifolds, the second fundamental form of the embedding and its tension field.

Remark. *Again note that not only the vanishing terms of W_0 are relevant in our context. For the conditional Brownian motion, we are explicitly interested in all the terms that are non-constant. Every constant quantity (not necessarily zero) cancels in the density (15) due to the normalization, such that the surface measure generated from the conditional Brownian motion equals the Wiener measure whenever W_0 is constant.*

Proposition 2.16. *Let $L \subset M$ be isometrically embedded, then the potential given by (13), can be rewritten as*

$$W_0 = \frac{1}{4}\|\tau(\varphi)\|^2 - \frac{1}{2}\sum_{i,j}\|\mathrm{II}(\varphi)(e_i,e_j)\|^2 - \frac{1}{2}R^M_{i\alpha i\alpha} - \frac{1}{6}\mathrm{Scal}_F. \tag{25}$$

Proof. We take advantage of the computations in the previous sections and derive

$$\begin{aligned}
2W_0 &= \mathrm{Scal}_L - \frac{1}{2}\|\tau(\varphi)\|^2 - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L} + \overline{R}_{M/L}\right)\\
&= R^L_{ijij} - \frac{1}{2}\|\tau(\varphi)\|^2 - R^M_{ijij} - R^M_{i\alpha i\alpha} - \frac{1}{3}R^M_{\alpha\beta\alpha\beta}\\
&\overset{(16)}{=} -\frac{1}{2}\|\tau(\varphi)\|^2 - \sum_{i,j}\langle \mathrm{II}(\varphi)(e_i,e_j), \mathrm{II}(\varphi)(e_j,e_i)\rangle + \|\tau(\varphi)\|^2\\
&\quad - R^M_{i\alpha i\alpha} - \frac{1}{3}R^M_{\alpha\beta\alpha\beta}\\
&= \frac{1}{2}\|\tau(\varphi)\|^2 - \sum_{i,j}\|\mathrm{II}(\varphi)(e_i,e_j)\|^2 - R^M_{i\alpha i\alpha} - \frac{1}{3}R^M_{\alpha\beta\alpha\beta}.
\end{aligned}$$

□

Remark. *According to [Spi75], Ch. 7, p. 61, the summands $R^M_{i\alpha i\alpha}$ are not related to other curvature terms by any fundamental embedding theorem.*

As $\pi\colon L(1) \to L$ is a Riemannian submersion with totally geodesic fibers, soon we will be able to give another simplification of formula (25). We will investigate this fact in Chapter 7, where we take Riemannian submersion into view and study different remarkably properties. Before this detailed discussion, we will first return to the announced connection between the volume and length deviation of the tubes $L(\varepsilon)$ in M and Euclidean balls of the same radius.

First, we use Gauss' equation to compare the second order correction term A in (24) and the expression for W_0 we just found.

Another application of (24) is given by

$$\begin{aligned}
\frac{d^2}{d\varepsilon^2}\frac{\mathrm{vol}^M(L(\varepsilon))}{\mathrm{vol}\left(B^{m-l}_\varepsilon(0)\right)}\Big|_{\varepsilon=0} &= \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0}\int_L \left(1 + \frac{\varepsilon^2}{2(m-l+2)}\widetilde{A}(x) + O\left(\varepsilon^4\right)\right)\,\mu_L(dx)\\
&= \int_L \frac{1}{m-l+2}\widetilde{A}(x)\,\mu_L(dx).
\end{aligned}$$

The principal significance of this expression is that it allows us to write

$$\widetilde{A}(x) = (m-l+2)\frac{d^2}{d\varepsilon^2}\left(\frac{\mathrm{vol}^M(L(\varepsilon)\cap\pi^{-1}(x))}{\mathrm{vol}\left(B_\varepsilon^{m-l}(0)\right)}\right)_{\big|\varepsilon=0}. \tag{26}$$

On the other hand, by definition, we obtain

$$\begin{aligned}
\widetilde{A} &= \mathrm{Scal}_L - R^M_{ijij} - R^M_{i\alpha i\alpha} - \frac{1}{3}R^M_{\alpha\beta\alpha\beta}\\
&= R^L_{ijij} - R^M_{ijij} - R^M_{i\alpha i\alpha} - \frac{1}{3}R^M_{\alpha\beta\alpha\beta}\\
&= R^L_{ijij} - \frac{1}{3}\left(R^M_{ijij} + 2R^M_{i\alpha i\alpha} + R^M_{\alpha\beta\alpha\beta} + R^M_{ijij} + R^M_{i\alpha i\alpha} + R^M_{ijij}\right)\\
&= R^L_{ijij} - \frac{1}{3}\left(R^M_{ABAB} + R^M_{iAiA} + R^M_{ijij}\right).
\end{aligned}$$

Consequently,

$$\widetilde{A} = \mathrm{Scal}_L - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L} + \overline{R}_{M/L}\right). \tag{27}$$

Definition 2.17. *Let $L \subset M$ be isometrically embedded.*

1. *We denote by $k := m - l$ the codimension of the embedding.*
2. *For $x \in L$ we define*

$$q_k(x) := (k+2)\frac{d^2}{d\varepsilon^2}\left(\frac{\mathrm{vol}^M(L(\varepsilon)\cap\pi^{-1}(x))}{\mathrm{vol}\left(B_\varepsilon^{k}(0)\right)}\right)_{\big|\varepsilon=0}.$$

Lemma 2.18. *The potential is in general given by*

$$W_0 = \frac{1}{2}q_k(x) - \frac{1}{4}\|\tau(\varphi)\|^2.$$

Proof. The statement directly follows from formula (27) by

$$\begin{aligned}
W_0 &= \frac{1}{2}\left(\mathrm{Scal}_L - \frac{1}{3}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L} + \overline{R}_{M/L}\right)\right) - \frac{1}{4}\|\tau(\varphi)\|^2\\
&= \frac{1}{2}\widetilde{A}(x) - \frac{1}{4}\|\tau(\varphi)\|^2.
\end{aligned}$$

□

Whenever $\tau(\varphi)$ vanishes, Lemma (2.18) implies that W_0 and q_k coincide up to a constant factor. We summarize the results of this section in the following form.

Theorem 2.19. *Let $L \subseteq M$ be a closed Riemannian submanifold of the Riemannian manifold (M, g).*

1. *If L is minimal in M, i.e. $\tau = 0$, then asymptotically the associated potential W_0 from (13) is the first relevant correction term in the deviation between the volume of the tubular*

neighborhood of L and a ball in a corresponding Euclidean space. For $\varepsilon \to 0$ we have

$$\frac{\mathrm{vol}^M(L(\varepsilon))}{\mathrm{vol}(B^k(\varepsilon))} = \int_L \left\{1 + \frac{\varepsilon^2}{(m-l+2)} W_0^{\mathrm{tot}}(y) + O(\varepsilon^4)\right\} \mu_L(dy). \tag{28}$$

2. *For arbitrary embeddings $\varphi\colon L \to M$ we asymptotically obtain*

$$\frac{\mathrm{vol}^M(L(\varepsilon))}{\mathrm{vol}(B^k(\varepsilon))} = \int_L \left\{1 + \frac{\varepsilon^2}{(m-l+2)} \left(W_0 + \frac{1}{4}\|\tau(\varphi)\|^2\right)(y) + O(\varepsilon^4)\right\} \mu_L(dy) \tag{29}$$

as $\varepsilon \to 0$.

By Theorem (2.19) we are able to interpret the potential in a significant way, whenever we know how the tension field of the associated embedding behaves.

Remark. *In the literature minimal submanifolds are often referred to as Riemannian manifolds of vanishing mean curvature. But here we study isometric immersions $\varphi\colon L \to M$ which means $\mathrm{II}(\varphi)$ equals the second fundamental form of L as an embedded submanifold of M. This also means $\frac{1}{l}\tau(\varphi)$ is exactly the mean curvature of L in M. Therefore, one has $\tau(\varphi) = 0$ if and only if L is a minimal submanifold of M which justifies the formulation in the above statements (see [HW08], [ES64] or [BW03] for further information).*

Proposition 6 in [SWW07], p.20, relates the second fundamental form to a difference in the length measuring between L and M. The statement is the following.

Theorem 2.20. *For an isometric embedding $\varphi\colon L \to M$ and any two points $x, y \in L$, which can be joined by a unique minimizing geodesic γ_{xy} starting at x and parametrized by arc-length one obtains*

$$\lim_{d_L(x,y)\to 0} \frac{d_M(x,y)^2 - d_L(x,y)^2}{d_L(x,y)^4} = -\frac{1}{12}\left\|\mathrm{II}(\varphi)\left(\dot\gamma_{xy}(0), \dot\gamma_{xy}(0)\right)\right\|^2.$$

Let $\varphi\colon L \to M$ be an isometric embedding, $x_0 \in L$ and γ_i a geodesic in L with $\gamma_i(0) = x_0$ and $\dot\gamma_i(0) = e_i$, $i = 1, ..., l$. Choose $y \in \gamma(I)$, where I is the maximal interval on which γ_i realizes the least distance. Then we are able to interpret terms of the form (no summation)

$$\langle \mathrm{II}(\varphi)(e_i, e_i), \mathrm{II}(\varphi)(e_i, e_i)\rangle = \|\mathrm{II}\left(\dot\gamma_i(0), \dot\gamma_i(0)\right)\|^2$$

by Theorem (2.20).

Unfortunately, with the help of this result, we are not able to explain the tension field completely: Due to the fact that

$$\|\tau(\varphi)\|^2 = \sum_{i,j=1}^{l} \langle \mathrm{II}(\varphi)(e_i, e_i), \mathrm{II}(\varphi)(e_j, e_j)\rangle$$

we still remain with the terms of the form $\langle \mathrm{II}(\varphi)(e_i, e_i), \mathrm{II}(\varphi)(e_j, e_j)\rangle$ for $i \neq j$.

But we may conclude that the potential W_0 is related to the perturbation of the tube volume (W_0^{tot}) and the difference in the length measuring (τ).

2.8 Embeddings into space forms

In this section let the ambient Riemannian manifold (M, g) be of constant sectional curvature $K \in \mathbb{R}$, in the literature these spaces are referred to as space forms. The well-known obvious examples are the Euclidean space $\mathbb{R}^n$ with curvature $K = 0$, the hyperbolic space $\mathbb{H}^n$ of constant curvature $K = -1$ and the n-dimensional sphere $\mathbb{S}^n$ of constant curvature $K = 1$.

According to results from [Car92], Chapter 4, § 3, we introduce the definition of a special trilinear map.

Definition 2.21. *Define a* 4*-form* $\widetilde{R} \in \Gamma\left(\Lambda^2(T^*M) \circ \Lambda^2(T^*M)\right)$ *by*

$$\widetilde{R}(X, Y, Z, V) := \langle X, Z\rangle\langle Y, V\rangle - \langle X, Y\rangle\langle Z, V\rangle,$$

for vector fields X, Y, Z, V *on* M *and* $p \in M$.

Remark. *Note that* $g \owedge g = \tilde{R}$.

Lemma 2.22. *Let* $R = R^M$ *be the Riemannian curvature tensor of the Riemannian manifold* (M, g)*. Then* M *has constant sectional curvature* $K \in \mathbb{R}$ *if and only if*

$$R(X, Y, Z, V) = K\widetilde{R}(X, Y, Z, V)$$

for all vector fields X, Y, Z, V *on* M.

Proof. Suppose that the sectional curvature $\kappa(X \wedge Y)$ of the plane spanned by X and Y is constant, i.e. $\kappa(X \wedge Y) =: K$ for each pair $(X, Y) \in T_pM \times T_pM$.
We claim that

$$\langle\widetilde{R}(X, Y)X, Y\rangle = |X \wedge Y|^2.$$

Indeed by definition

$$K = \kappa(X \wedge Y) = \frac{\langle R(X, Y)Y, X\rangle}{|X \wedge Y|^2}$$

and hence

$$\langle R(X, Y)Y, X\rangle = K\langle\widetilde{R}(X, Y)Y, Y\rangle.$$

By [Car92], Lemma 3.3, p. 94, we have that $\kappa(X \wedge Y) = \kappa'(X \wedge Y)$ for every plane $X \wedge Y$, already implies that the corresponding curvature tensors R and R' coincide. The other direction follows immediately. □

With the help of this lemma and the fundamental equations of Riemannian submanifolds we obtain interesting facts about embeddings into space forms. Note that we will use the abbreviation II_{ij} instead of $\mathrm{II}(\varphi)(e_i, e_j)$, for an orthonormal basis $\{e_i\}_{i=1,\dots,l}$ of TL.

Proposition 2.23. *Let* $L \subset M$ *be a closed embedded submanifold and assume* M *has constant sectional curvature* $K \in \mathbb{R}$*. Let* R^M *denote the Riemannian curvature tensor of* M *and* $R^\perp$ *the normal curvature. We denote tangential vector fields on* L *by* X, Y, Z *and normal vector fields by* ζ *and* η*. Then* *(i)* $\langle R^{\perp}(X,Y)\eta,\zeta\rangle = -\langle [A_\eta, A_\zeta]X, Y\rangle$,

(ii) $\langle R^M(X,Y)Z,\eta\rangle = 0$,

(iii) *in local coordinates we obtain*

   (a) $R^M_{ABAB} = -K$, *provided that* $A \neq B$ *and* $R^M_{DCAB} = 0$ *otherwise,*

   (b) $R^L_{ijij} + \langle \mathrm{II}_{ij}, \mathrm{II}_{ji}\rangle - \langle \mathrm{II}_{ii}, \mathrm{II}_{jj}\rangle = -K$ *for* $i \neq j$,

   (c) $R^L_{lkij} + \langle \mathrm{II}_{ik}, \mathrm{II}_{jl}\rangle - \langle \mathrm{II}_{il}, \mathrm{II}_{jk}\rangle = 0$ *in all other cases.*

Remark. *Note that in part* (iii) *above we did not use Einstein summation convention.*

Proof. (i) This is an immediate consequence of Ricci's equation (cf. [Spi75], Chapter 7.C, p. 38) and the fact that by the above lemma

$$\langle R^M(X,Y)\eta,\zeta\rangle = K\left(\langle X,\eta\rangle\langle Y,\zeta\rangle - \langle Y,\eta\rangle\langle X,\zeta\rangle\right) = 0$$

as X and Y are tangential vector fields, while η and ζ are normal vector fields.

(ii) Direct consequence of Lemma (2.22).

(iii) (a) Let $\{e_A\}_{A=1,\ldots,m}$ be an orthonormal basis of T_pM, $p \in M$. Remember, we use the shorthand $R^M_{DCAB} = \langle R^M(e_A, e_B)e_C, e_D\rangle$. From Lemma (2.22) we deduce

$$\begin{aligned} R^M_{DCAB} &= K\left(\langle e_A, e_C\rangle\langle e_B, e_D\rangle - \langle e_B, e_C\rangle\langle e_A, e_D\rangle\right) \\ &= K\left(\delta_{AC}\delta_{BD} - \delta_{BC}\delta_{AD}\right) \end{aligned}$$

and thus

$$R^M_{DCAB} = \begin{cases} K\left(\delta^2_{AB} - 1\right), & D = A, C = B, \\ K\left(1 - \delta^2_{AB}\right), & D = B, C = A. \end{cases}$$

Then, one obtains $R^M_{ABAB} = -K$, provided $A \neq B$ and $R^M_{DCAB} = 0$ otherwise. Note that this is also stated in [Car92], Corollary 3.5.

(b) and (c) directly follow from (a) and Gauss' equation (16).

□

With the help of these observations we are able to obtain a simplification for the formula of the potential W_0.

Corollary 2.24. *Let* $L \subset M$ *be a closed embedded submanifold and assume* M *has constant sectional curvature* $K \in \mathbb{R}$. *Then the coefficients* $R^M_{i\alpha i\alpha}$ *of the Riemannian curvature tensor of* M *are constant.*

Proof. In local coordinates, we obtain

$$\begin{aligned} R_{i\alpha i\alpha} &= \langle R(e_i, e_\alpha)e_\alpha, e_i\rangle \\ &= K\left(\langle e_i, e_\alpha\rangle\langle e_\alpha, e_i\rangle - \langle e_\alpha, e_\alpha\rangle\langle e_i, e_i\rangle\right) \\ &= K\left(-\|e_\alpha\|^2\|e_i\|^2\right) = -K, \end{aligned}$$

due to Lemma (2.22).

□

Therefore, in this special case these coefficients do not occur in the density for the Brownian motion, because every constant summand cancels due to normalization.

Remark. *1. Let L be an $(m-1)$-dimensional compact hypersurface isometrically immersed into the space form M of constant sectional curvature $K \in \mathbb{R}$.
From Proposition (2.23) $(iii)(b)$, by summation over i and j, we obtain*

$$\begin{aligned}\mathrm{Scal}_L &= \sum_{i,j} R^L_{ijij} \\ &= -(m-1)(m-2)K + \tau(\varphi)^2 - \sum_{i,j} \|\mathrm{II}(\varphi)(e_i, e_j)\|^2.\end{aligned}$$

2. In the setting of Example (2.6), with the additional information that M is a space form of sectional curvature $K \in \mathbb{R}$, we hence obtain that the associated potential is constant. Explicitly, it is given by $W_0 = -\frac{1}{4}(c \cdot l)^2 + K$.

Example 2.25. *Let us again have a look at Example (2.13) and define $L := \mathbb{S}^1$ and $M := \mathbb{S}^2$. The Gaussian curvature of the spheres is constant. Moreover, the embedding is totally geodesic and therefore $\tau(\varphi) = 0$. This leads to a constant potential for the embedding $\varphi\colon L \to M$.
Note that we have $\mathrm{Scal}_{\mathbb{S}^2} = 2$ and $\mathrm{Scal}_{\mathbb{S}^1} = 0$. An application of Corollary (2.24) and the fact that the sectional curvature of the spheres is equal to one, together with the definitions of the partial traces gives for the associated potential:*

$$\begin{aligned}W_0 &= \frac{1}{2}\left(-\frac{1}{3}\left(3R^M_{ijij} + 3R^M_{i\alpha i\alpha} + R^M_{\alpha\beta\alpha\beta}\right)\right) \\ &= \frac{1}{2}\left(-R^M_{i\alpha i\alpha}\right) = \frac{1}{2}.\end{aligned}$$

Remark. *Note that using the Differential Geometry package of MAPLE® to compute the right Laplacians and an application of*

$$W_0 = \rho^{-\frac{1}{2}} \triangle_g \rho^{\frac{1}{2}},$$

it is also possible to obtain $W_0 = \frac{1}{2}$. But in this case here, the computation by hand is much faster. For the whole computation in local coordinates with MAPLE® see Appendix E.

Combining all we have computed so far, we obtain that the associated potential for the embedding $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, $n \in \mathbb{N}$, does no longer depend on the path or the position of the particles of the underlying Brownian motion. Therefore, due to the normalization, the corresponding Radon-Nikodym density equals one.

To conclude this chapter, we give some summarizing remarks. In the case that the second fundamental form of the embedding vanishes, the potential just depends on scalar curvatures of the participating manifolds. In this case, we also obtain that W_0 is the first non-zero correction term in the volume formula of Gray (up to some constant). Furthermore, this connection holds true for minimal submanifolds. In the general case, the first relevant correction term includes the tension field as an additional quantity. Hence, we are able to state that the potential W_0 is very well understood, whenever we know how the tension field of the embedding behaves.

Next we analyze the geometry of the tubular neighborhood and develop relevant formulas for the proof of the main theorem.

Chapter 3

Geometry of the tube and the perturbation

In contrast to the representation in local coordinates for the metric g_{Sa} given by formula (3), there is another approach to define the induced metric and the Sasaki metric without using local coordinates. This coordinate-free representation is discussed in the following section and based on geometric arguments using Jacobi field theory.
With the help of these arguments we present another formula for the Radon-Nikodym density ρ from Definition (1.25).

3.1 The induced metric and the Sasaki metric

3.1.1 The geometry of the normal bundle

Let $W \in NL$ and $\pi\colon NL \to L$ denote the normal bundle. Furthermore, we define $\pi W := \pi(W) := x \in L$. As already seen in Chapter 1, we are able to identify the tube $L(1)$ with a neighborhood of the zero section in NL. The normal bundle NL can be equipped naturally with a metric g induced by the metric of the ambient manifold M and an associated metric connection $\nabla = \nabla^M$. Our notation follows [Koz08], Section 1.1.

Let X be vector field in T_xL and ζ a normal vector field. A computation shows that

$$\nabla^{\perp}_X\zeta := \left(\nabla^M_X\zeta\right)^{\perp}$$

defines a linear connection on NL. This connection is related to a bundle morphism $K^{\perp}\colon TNL \to NL$, which is uniquely determined by the following two conditions.

(1) For every $x \in L$ and every $W \in N_xL$ the restriction

$$K^{\perp}_W : T_W N_x L \to N_x L$$

is a canonical isomorphism.

(2) For every section ζ of NL and a vector field X on L, we have

$$K^{\perp}\left(\zeta_*(X)\right) = \nabla^{\perp}_X\zeta.$$

Definition 3.1. *1. By*

$$J_W\colon N_{\pi W}L \to T_W N_{\pi W} L,$$

*we denote the **flat parallel translation** along the fiber.*

2. *The map introduced above is called **connection map** of the normal bundle, i.e. the natural connection $\nabla^\perp$ on NL induces the connection map*

$$K_W^\perp \colon T_W NL \to N_x L,\ K_W^\perp(\zeta_*(X)) = \nabla_X^\perp \zeta.$$

3.1.1.1 Horizontal and vertical lifts

Here we just give a brief summary of the most crucial facts. For more information in a similar case see [Sak96], Chapter 2.4. We use the decomposition of the tangent space $T_W NL$ into horizontal and vertical subbundle, i.e.

$$T_W NL = \mathfrak{H}_W NL \oplus \mathfrak{V}_W NL, \tag{30}$$

where $\mathfrak{H}_W NL := \ker(K_W^\perp)$ and $\mathfrak{V}_W NL := \ker(\pi_{*,W})$.

Due to the decomposition we can now describe the (unique) horizontal and vertical lift of a vector field.

Definition 3.2. *For $Q \in T_{\pi W} NL$ we write $Q = (X, V)$ using (30), with $X \in T_{\pi W} L$ and $V \in N_{\pi W} L$. We now define the **vertical lift** of Q as*

$$V^v := J_W V. \tag{31}$$

*Moreover, we define the unique **horizontal lift** X^h of Q by*

$$\begin{aligned} \pi_{*,W} X^h &= X \\ \text{and}\ \ K_W^\perp X^h &= 0, \end{aligned}$$

such that $V^v + X^h \in T_W NL$.

Remark. 1. *The definition of the horizontal lift depends on the connection, while the vertical lift can be defined independently.*

2. *The covariant derivative of a normal vector field is a (smooth) path in NL. The corresponding velocity vector field evaluated at zero is the horizontal lift of the vector field.*

There are other approaches to define vertical vector fields introducing vertical curves in a vector bundle first. For further details see e.g. Chapter 6 in [DP11].

3.1.1.2 Lifts in local coordinates

To better understand the curvature of the normal bundle and to discuss the connection on NL, we have a closer look at the horizontal lift of a vector field.

Let the standard vector fields be denoted by $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial_\alpha = \frac{\partial}{\partial w^\alpha}$. Moreover, denote by $Z = Z^i \partial_i + Z^\alpha \partial_\alpha \in NL$ a representation of Z. By definition one obtains

$$\pi_* \left(Z_W^h \right) = Z_{\pi(W)} \tag{32}$$

for the horizontal lift of a vector field Z evaluated at $W \in NL$.

Remark. *Equation (32) implies Z^h is π-related to Z. Moreover, by definition of the lifts the zero vector field 0_M in M is π-related to any vertical vector field Y^v.*

Let $U \subset L$ be such that $NL_{|U}$ is trivial.

Lemma 3.3. *The horizontal lift of the standard vector field ∂_i on $\pi^{-1}(U)$ is given by*

$$\partial_i^h = \partial_i - w^\mu C_{i\mu}^\alpha \partial_\alpha.$$

Proof. Assume that $\partial_i^h = Z^j \partial_j + Z^\alpha \partial_\alpha$. Due to equation (32) we directly obtain $Z^j = \delta_i^j$. By definition of the connection map in local coordinates (see e.g. [Yam06], Section 1.1) any horizontal vector field X^h satisfies $K^\perp(X^h) = 0$. Hence, in local coordinates we obtain $Z^\alpha \partial_\alpha + C_{i\mu}^\alpha Z^i w^\mu \partial_\alpha = 0$. Inserting the expression for Z^i yields the formula for the lift. □

A proof for the representation of the horizontal lift in local coordinates on the tangent bundle can be found in [Fàb11], Lemma 1.14.

Next, we prove a connection between the horizontal lift and the curvature. Note that all Lie brackets are taken with respect to the Sasaki metric on NL.

Remark.

1. *By definition of the vertical lift one obtains $[\partial_i^v, \partial_j^v] = 0$.*
2. *The following equations are valid at $(x, w) \in L(1)$: $\partial_i x^j = \delta_i^j$, $\partial_\alpha w^\beta = \delta_\alpha^\beta$, $\partial_i w^\alpha = 0$ and $\partial_\alpha x^i = 0$.*

Lemma 3.4. *Let $R^\perp$ denote the curvature of the normal bundle, i.e. the curvature tensor according to $\nabla^\perp$ on NL. Then*

$$[\partial_i^h, \partial_j^h]_{Sa,W} = {R^\perp}^\nu{}_{\mu ij} w^\mu \partial_\nu.$$

Moreover, it is

$$[X^h, Y^h]_{Sa,W} = [X, Y]^h + \Omega_W(X, Y)^v,$$

for $X, Y \in \Gamma(TL)$, where

$$\Omega_W(X, Y)\Big|_{\pi^{-1}(U)} = w^\alpha R^\perp(X, Y) e_\alpha$$

denotes the curvature two form of $U \subset L$ in local coordinates.

The proof follows the ideas given in [DP11], Proposition 6.6. Moreover, the result is a different formulation of Lemma 5 in [BY89] and a similar result was shown in [YP09] and [GK02] for the tangent bundle instead of the normal bundle.

Proof. First, we outline some results using the second remark above and the fact that the coefficients of the connection $C_{i\beta}^\nu = C_{i\beta}^\nu(x)$ depend only on the coordinates on L, i.e. covariant differentiation into the direction of the fibers is equal to zero. Applying the chain rule we obtain

$$\begin{aligned}\partial_\alpha \left(w^\beta C_{j\beta}^\nu \partial_\nu \right) &= (\partial_\alpha w^\beta) C_{j\beta}^\nu \partial_\nu + w^\beta \left(\partial_\alpha C_{j\beta}^\nu \right) \partial_\nu + w^\beta C_{j\beta}^\nu \partial_\alpha \partial_\beta \\ &= C_{j\alpha}^\nu \partial_\nu + w^\beta C_{j\beta}^\nu \partial_\alpha \partial_\nu\end{aligned}$$

and analogously

$$\partial_i\left(w^\beta C^\nu_{j\beta}\partial_\nu\right) = w^\beta\left(\partial_i C^\nu_{j\beta}\right)\partial_\nu + w^\beta C^\nu_{j\beta}\partial_i\partial_\nu.$$

The computation of the Lie bracket $[\partial_i^h, \partial_j^h]_{Sa,W}$ is now based on the following observations. We know how the horizontal lift of a standard vector field looks like in local coordinates (see Lemma (3.3)). By anti-symmetry, the definition of the Lie bracket and the above remark we obtain

$$\begin{aligned}
&[\partial_i^h, \partial_j^h]_{Sa,W}\\
&= [\partial_i - w^\mu C^\alpha_{i\mu}\partial_\alpha, \partial_j - w^\beta C^\nu_{j\beta}\partial_\nu]\\
&= [w^\mu C^\alpha_{i\mu}\partial_\alpha, w^\beta C^\nu_{j\beta}\partial_\nu] - [\partial_i, w^\beta C^\nu_{j\beta}\partial_\nu] + [\partial_j, w^\mu C^\alpha_{i\mu}\partial_\alpha]\\
&= w^\mu C^\alpha_{i\mu}C^\nu_{j\alpha}\partial_\nu + w^\mu C^\alpha_{i\mu}w^\beta C^\nu_{j\beta}\partial_\alpha\partial_\nu\\
&\quad - w^\beta C^\nu_{j\beta}C^\alpha_{i\nu}\partial_\alpha - w^\beta C^\nu_{j\beta}w^\mu C^\alpha_{j\mu}\partial_\nu\partial_\alpha\\
&\quad - w^\beta\partial_i C^\nu_{j\beta}\partial_\nu - w^\beta C^\nu_{j\beta}\partial_i\partial_\nu + w^\beta C^\nu_{j\beta}\partial_\nu\partial_i\\
&\quad + w^\mu\partial_j C^\alpha_{i\mu}\partial_\alpha + w^\mu C^\alpha_{i\mu}\partial_j\partial_\alpha - w^\mu C^\alpha_{i\mu}\partial_\alpha\partial_j\\
&= w^\mu C^\alpha_{i\mu}C^\nu_{j\alpha}\partial_\nu - w^\beta C^\nu_{j\beta}C^\alpha_{i\nu}\partial_\nu - w^\beta\partial_i C^\nu_{j\beta}\partial_\nu + w^\mu\partial_j C^\alpha_{i\mu}\partial_\alpha.
\end{aligned}$$

By definition of the curvature in local coordinates and renaming the summation indices suitably, this finally equals

$$R^{\perp}{}^{\nu}{}_{\mu ij}w^\mu\partial_\nu.$$

In the general case for $X = X^i\partial_i$ and $Y = Y^j\partial_j$ it directly follows that the horizontal lifts are locally given by

$$X^h = \left(X^i\circ\pi\right)\left(\partial_i - w^\mu C^\alpha_{i\mu}\partial_\alpha\right) \text{ and } Y^h = \left(Y^j\circ\pi\right)\left(\partial_j - w^\beta C^\nu_{j\beta}\partial_\nu\right).$$

To shorten notation we define $d_i := \partial_i - w^\mu C^\alpha_{i\mu}\partial_\alpha$ and $d_j := \partial_j - w^\beta C^\nu_{j\beta}\partial_\nu$, respectively.

By definition of the Lie bracket we finally obtain

$$\begin{aligned}
[X^h, Y^h] &= [\left(X^i\circ\pi\right)d_i, \left(Y^j\circ\pi\right)d_j]\\
&= \left(X^i\circ\pi\right)d_i\left(\left(Y^j\circ\pi\right)d_j\right) - \left(Y^j\circ\pi\right)d_j\left(\left(X^i\circ\pi\right)d_i\right)\\
&= \left(X^i\circ\pi\right)\left(\delta_{ij}d_j + \left(Y^j\circ\pi\right)\left(d_i(d_j)\right)\right) - \left(Y^j\circ\pi\right)\left(\delta_{ji}d_i + \left(X^i\circ\pi\right)\left(d_j(d_i)\right)\right)\\
&= \left([X,Y]^i\circ\pi\right)d_i + \left(X^i\circ\pi\right)\left(Y^j\circ\pi\right)[d_i, d_j].
\end{aligned}$$

Inserting the expression for the Lie bracket of the standard horizontal vector fields yields the statement. □

3.1.1.3 Lift and inverse lift

A special case of (30) is the canonical decomposition

$$T_{\pi W}NL = T_{\pi W}L \oplus N_{\pi W}L.$$

Now we want to study the horizontal and vertical lifts of the respective vectors according to this decomposition.

Definition 3.5. *1. The **lift** $\mathcal{L}_W : T_{\pi W}L \oplus N_{\pi W}L \to T_W NL$ is given by*

$$\mathcal{L}_W(X, V) = \left(X^h, V^v\right) =: Z,$$

where X^h denotes the horizontal lift and V^v the vertical lift as in (3.2).

2. We define $\exp\colon NL \to M$ *by* $\exp(W) := \exp_{\pi W}(W) := \gamma_W(1)$*, where γ_W is the geodesic in M defined by $\gamma_W(0) = x$ and $\dot{\gamma}_W(0) = W$.*

Corollary 3.6. *The **inverse lift** $\mathcal{L}_W^{-1} : T_W NL \to T_{\pi W}L \oplus N_{\pi W}L$ is given by*

$$\mathcal{L}_W^{-1} Z = (\pi_{*,W} Z, K_W Z)\,.$$

Remark. *The map $\mathcal{L}$ only depends on the geometry of the normal bundle, i.e. the metric and the induced connection on NL.*

Lemma 3.7. *The normal bundle can be equipped with a Sasaki metric $\langle \cdot, \cdot \rangle : TNL \times TNL \to \mathbb{R}$ by*

$$\langle Z, Z' \rangle_{Sa,W} := \langle \mathcal{L}_W^{-1} Z, \mathcal{L}_W^{-1} Z' \rangle_{\pi W}. \tag{33}$$

Proof. If NL is equipped with the Sasaki metric g_{Sa}, the projection $\pi\colon NL \to L$ is a Riemannian submersion with totally geodesic fibers. The characterization due to Reckziegel (see Section (1.2)) follows from the decomposition (30) and the definition of the lift. □

3.1.2 The geometry of the embedding

We further study the geometry of the embedding using Jacobi field arguments. First we define the objects involved. We need to fix the initial conditions for the Jacobi field and its first derivative.

Definition 3.8. *Recall that A_W denotes the Weingarten map in direction W. Then we define*

$$S_W : T_W NL \to T_{\pi W} NL,\ S_W Z := -A_W(\pi_* Z) + K_W Z.$$

Remark. *Let $Z \in T_W NL$ be decomposed as $Z = X^h + V^v$. By linearity and the definitions of horizontal and vertical subspaces we obtain*

$$S_W Z = -A_W X + V.$$

Definition 3.9. *Let $P_W : T_{\pi W}L \oplus T_{\pi W}NL \to T_W NL$ denote the **parallel translation** along the geodesic γ_W, defined in Definition* (3.5).

Remark. *If $M = NL$ is equipped with the Sasaki metric g_{Sa}, then $P_W = \mathcal{L}_W$ and the tangent map of the corresponding exponential map is the identity on the tangent space $T_W NL$, i.e.* $\exp_{*,W} = \mathrm{id}_{T_W NL}$.

Definition 3.10. *We define the linear endomorphism $\mathcal{A}_W : T_{\pi W}NL \to T_{\pi W}NL$ by*

$$\mathcal{A}_W := P_W^{-1} \circ \exp_{*,W} \circ \mathcal{L}_W.$$

Remark. *In the situation of the remark above we have* $\mathcal{A}_W = \mathrm{id}_{T_W NL}$.

Proposition 3.11. *a) The endomorphism* $\mathcal{A}_W$ *from Definition (3.10) is given by*

$$\mathcal{A}_W\left(\mathcal{L}_W^{-1} Z\right) = P_W^{-1}\xi(t),$$

for $Z \in T_W NL$ *and a Jacobi field* ξ *along* γ_W.

b) With the notations above we obtain

$$\begin{aligned}\mathcal{A}_W\left(\mathcal{L}_W^{-1} Z\right) = \mathcal{A}_W((X,V)) &= X + V - A_W X - \frac{1}{2}R(W,X)W \\ &- \frac{1}{6}\nabla_W R(W,X)W - \frac{1}{6}R(W, V - A_W X)W + O\left(\|W\|^4\right). \end{aligned} \tag{34}$$

Remark. *Note that, whenever the Landau notation is used, we simply write* $O(\|W\|)$ *instead of* $O(\|W\|)$, *as* $\|W\| \to 0$.

Remark. *For* $(X,V) = \mathcal{L}_W^{-1} Z$ *the Jacobi field* ξ *is given by the initial conditions* $\xi(0) = X$ *and* $\frac{D\xi}{dt}(0) = (-A_W X, V) = S_W \mathcal{L}_W(X,V)$. *Moreover, due to the Jacobi equation we have*

$$\frac{D^2\xi}{dt^2} = -R(\dot\gamma_W, \xi)\dot\gamma_W, \tag{35}$$

where R *denotes the Riemannian curvature tensor corresponding to the induced metric on NL which was chosen above.*

Proof of Proposition (3.11). a) This result directly follows from the standard theory of Jacobi fields, i.e. that the differential of the exponential map defines a Jacobi field (see e.g. [Car92], p. 110 ff.).

b) The basic idea of the proof is to evaluate the Taylor series. With the aforementioned notations, for $Z \in T_W NL$, we obtain

$$\mathcal{A}_W\left(\mathcal{L}_W^{-1} Z\right) = \sum_{k=0}^{n} \frac{1}{k!}\frac{D^{(k)}\xi}{dt^k}(0) + O\left(\|W\|^{n+1}\right) \tag{36}$$

by Taylor's formula.

Substituting the initial conditions of ξ and the Jacobi equation (35) into (36), we are able to evaluate the zero and first order terms explicitly. We now proceed by computing the third derivative of ξ. It remains to prove that the fourth derivative is of order $\|W\|^4$.

For reasons of abbreviation we omit the dependency of the time parameter t in most expressions.

We compute the third derivative by using the Jacobi equation first, then applying the product rule and using the geodesic equation (i.e. $\frac{d\dot{\gamma}_W}{dt} \equiv 0$) for γ_W. We obtain

$$\begin{aligned}\frac{D^3\xi}{dt^3}(0) &= -\frac{D}{dt}\left[R(\dot{\gamma}_W,\xi)\dot{\gamma}_W\right]_{|t=0} \\ &= \left(\frac{-DR}{dt}(\dot{\gamma}_W(0),\xi(0))\dot{\gamma}_W(0)\right)_{|t=0} - R\left(\dot{\gamma}_W(0),\frac{D\xi}{dt}(0)\right)\dot{\gamma}_W(0).\end{aligned}$$

Inserting the initial conditions for γ_W and ξ and using the fact that $\frac{DR}{dt}_{|t=0} = \nabla_W R$, it follows that

$$\frac{D^3\xi}{dt^3}(0) = -\nabla_W R(W,X)W - R(W,-A_W X + V)W.$$

Our next goal is to prove that the fourth derivative is already of order $\|W\|^4$. By an application of our last results, we find

$$\begin{aligned}\frac{D^4\xi}{dt^4}(0) &= -\frac{D}{dt}\left(\frac{DR}{dt}(\dot{\gamma}_W,\xi)\dot{\gamma}_W + R\left(\dot{\gamma}_W,\frac{D\xi}{dt}\right)\dot{\gamma}_W\right)_{|t=0} \\ &= -\left(\frac{D^2R}{dt^2}_{|t=0}(W,X)W + \frac{DR}{dt}_{|t=0}(W,-A_W X+V)W\right. \\ &\qquad \left.+\frac{DR}{dt}_{|t=0}(W,-A_W X+V)W + R\left(W,(-R(W,X)W)\right)W\right) \\ &= O\left(\|W\|^4\right).\end{aligned}$$

Combining these results we conclude

$$\begin{aligned}\mathcal{A}_W(X,V) = X + V - A_W X - \frac{1}{2}R(W,X)W - \frac{1}{6}\nabla_W R(W,X)W \\ - \frac{1}{6}R(W,V - A_W X)W + O\left(\|W\|^4\right).\end{aligned}$$

□

Having this expansion on hand we compute the asymptotic of the induced metric g.

Proposition 3.12. *Let $Z := \mathcal{L}_W(X,V), Z' := \mathcal{L}_W(X',V') \in T_W NL$ and $R_W Z := R(W,Z)W$. Then the induced metric g_W on the tube is given by*

$$\begin{aligned}&\langle Z,Z'\rangle_W \\ &= \begin{pmatrix}X\\V\end{pmatrix}^t \left[\begin{pmatrix}1&0\\0&1\end{pmatrix} - \begin{pmatrix}2A_W&0\\0&0\end{pmatrix} - \begin{pmatrix}R_W^{11} - A_W^2 & \frac{2}{3}R_W^{12}\\ \frac{2}{3}R_W^{21} & \frac{1}{3}R_W^{22}\end{pmatrix}\right]\begin{pmatrix}X\\V\end{pmatrix} + O\left(\|W\|^3\right).\end{aligned}$$

Proof. By the properties of the exponential map, i.e. exp is an isometry on a small neighborhood of the zero section in NL to the tube $L(1)$, and the fact that the inner product is invariant under parallel translation we conclude that

$$\begin{aligned}\langle Z, Z'\rangle_W &= \langle \exp_{*,W} \circ \mathcal{L}_W(X,V), \exp_{*,W} \circ \mathcal{L}_W(X',V')\rangle_M \\ &= \langle P_W^{-1} \circ \exp_{*,W} \circ \mathcal{L}_W(X,V), P_W^{-1} \circ \exp_{*,W} \circ \mathcal{L}_W(X',V')\rangle_{\pi W} \\ &= \langle \mathcal{A}_W(X,V), \mathcal{A}_W(X',V')\rangle_{\pi W}. \end{aligned} \tag{37}$$

We now continue the computation of the metric by using a decomposition of the objects into their horizontal and vertical parts. Therefore, we obtain the representation

$$R_W \circ \mathcal{L}_W \begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} R_W^{11} & R_W^{12} \\ R_W^{21} & R_W^{22} \end{pmatrix} \begin{pmatrix} X \\ V \end{pmatrix}. \tag{38}$$

Due to the fact that $A_W^* = A_W$ and $R_W^* = R_W$, we deduce

$$\langle X, A_W X'\rangle = \langle A_W X, X'\rangle$$

and

$$\langle X', R(W,X)W\rangle = \langle X, R(W,X')W\rangle.$$

Furthermore, note that the Weingarten map A_W acts on the tangent bundle and therefore vertical vectors are orthogonal, with respect to the Sasaki metric, to vectors of the form $A_W X$.

We now combine the upper result (37) with formula (34) and obtain

$$\begin{aligned}
&\langle Z, Z'\rangle_W \\
&= \langle \mathcal{A}_W(X,V), \mathcal{A}_W(X',V')\rangle_{\pi W} \\
&= \Big\langle X + V - A_W X - \frac{1}{2}R(W,X)W - \frac{1}{6}\nabla_W R(W,X)W - \frac{1}{6}R(W, V - A_W X)W, \\
&\quad X' + V' - A_W X' - \frac{1}{2}R(W,X')W - \frac{1}{6}\nabla_W R(W,X')W - \frac{1}{6}R(W, V' - A_W X')W \Big\rangle_{\pi W} \\
&\quad + O\left(\|W\|^4\right) \\
&= \ \left\langle X + V, X' + V'\right\rangle_{\pi W} + \left\langle X, -A_W X'\right\rangle_{\pi W} + \left\langle -A_W X, X'\right\rangle + \left\langle A_W X, A_W X'\right\rangle_{\pi W} \\
&\quad + \left\langle X, -\frac{1}{2}R(W,X')W\right\rangle_{\pi W} + \left\langle X, -\frac{1}{6}R(W, V' - A_W X')W\right\rangle_{\pi W} \\
&\quad + \left\langle V, -\frac{1}{2}R\left(W,X'\right)W\right\rangle_{\pi W} + \left\langle V, -\frac{1}{6}R(W, V' - A_W X')W\right\rangle_{\pi W} \\
&\quad + \left\langle X', -\frac{1}{2}R(W,X)W\right\rangle_{\pi W} + \left\langle X', -\frac{1}{6}R(W, V - A_W X')W\right\rangle_{\pi W} \\
&\quad + \left\langle V', -\frac{1}{2}R\left(W,X\right)W\right\rangle_{\pi W} + \left\langle V', -\frac{1}{6}R(W, V - A_W X')W\right\rangle_{\pi W} + O\left(\|W\|^3\right) \\
&= \ \left\langle \mathcal{L}_W^{-1} Z, \mathcal{L}_W^{-1} Z'\right\rangle_{\pi W} - 2\left\langle A_W X, X'\right\rangle_{\pi W} + \left\langle A_W X, A_W X'\right\rangle_{\pi W} \\
&\quad - \left\langle X, R(W,X')W\right\rangle_{\pi W} + \left\langle X, -\frac{1}{6}R(W,V')W\right\rangle_{\pi W} \\
&\quad + \left\langle V, -\frac{1}{2}R(W,X')W\right\rangle_{\pi W} + \left\langle V, -\frac{1}{6}R(W,V')W\right\rangle_{\pi W}
\end{aligned}$$

$$
\begin{aligned}
&+ \left\langle X', -\frac{1}{6}R(W,V)W \right\rangle_{\pi W} + \left\langle V', -\frac{1}{2}R(W,X)W \right\rangle_{\pi W} \\
&+ \left\langle V', -\frac{1}{6}R(W,V)W \right\rangle_{\pi W} + O\left(\|W\|^3\right)_{\pi W} \\
= &\left\langle \mathcal{L}_W^{-1} Z, \mathcal{L}_W^{-1} Z' \right\rangle_{\pi W} - 2\left\langle A_W X, X' \right\rangle_{\pi W} + \left\langle A_W X, A_W X' \right\rangle_{\pi W} \\
&- \left\langle X, R(W,X')W \right\rangle_{\pi W} - \frac{1}{3}\left\langle V', R(W,V)W \right\rangle_{\pi W} \\
&- \frac{2}{3}\left\langle V, R(W,X')W \right\rangle_{\pi W} - \frac{2}{3}\left\langle X, R(W,V')W \right\rangle_{\pi W} + O\left(\|W\|^3\right).
\end{aligned}
$$

To conclude that the metric can be written in the above block matrix structure, we just have to use the decomposition of the tangent bundle of NL into horizontal and vertical parts. Due to Reckziegel both subspaces are orthogonal with respect to the Sasaki metric which gives

$$
\begin{aligned}
\left\langle \mathcal{L}_W^{-1} Z, \mathcal{L}_W^{-1} Z' \right\rangle_{\pi W} &= \langle X, X' \rangle + \langle V, V' \rangle \\
&= \begin{pmatrix} X \\ V \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X' \\ V' \end{pmatrix}.
\end{aligned}
$$

The representation of the other terms follows by definition of the Weingarten map and formula (38). □

Formula (37) and $\mathcal{L}_W^{-1} = \pi_* \oplus K^\perp$ allow us to reformulate the result in a more compact way.

Corollary 3.13. *Let $Z, Z' \in T_W NL$. The induced metric is given by*

$$
\begin{aligned}
\langle Z, Z' \rangle_W =& \langle \mathcal{U}_W Z, \mathcal{U}_W Z' \rangle_{W,Sa} \\
=& \langle \mathcal{A}_W \mathcal{L}_W^{-1} Z, \mathcal{A}_W \mathcal{L}_W^{-1} Z' \rangle_L + \langle \mathcal{A}_W \mathcal{L}_W^{-1} Z, \mathcal{A}_W \mathcal{L}_W^{-1} Z' \rangle_{N_x L},
\end{aligned}
$$

where $\mathcal{U}_W = \mathcal{L}_W \circ \mathcal{A}_W \circ \mathcal{L}_W^{-1}$.

Remark. *The latter formulation leads to an invariant representation of the Sasaki metric g_{Sa}, i.e. we are able to reformulate the representation in local coordinates given in (5) in a coordinate independent manner. The map $\mathcal{L}_W$ and its inverse $\mathcal{L}_W^{-1}$ can be represented by triangular matrices of the form*

$$
\begin{pmatrix} 1 & -w^\mu C_{i\mu}^\alpha \partial_\alpha \\ 0 & 1 \end{pmatrix}
$$

and its inverse matrix, seen in Chapter 1.

Before rescaling the metric, we first compute the dual metric with the help of the Neumann series (see Appendix $(B.11)$, p. 142). Furthermore, we transfer the concept of the lift to the cotangent bundle $T_W^* NL$.

Lemma 3.14. *The **dual metric** is given by*

$$
g_W^{-1} = 1 + 2\begin{pmatrix} A_W & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} R_W^{11} + 3A_W^2 & \frac{2}{3}R_W^{12} \\ \frac{2}{3}R_W^{21} & \frac{1}{3}R_W^{22} \end{pmatrix} + O\left(\|W\|^3\right). \tag{39}
$$

Proof. We first observe that by Proposition (3.12) the induced metric g_W is given in the form

$$
g_W = 1 - wA - w^2 B + O(w^3),
$$

where A and B represent the parts of the metric of order $\|W\|$ and $\|W\|^2$, respectively. Let us compute the inverse metric as

$$\begin{aligned}&\left(1 - wA - w^2B + O(w^3)\right)^{-1}\\ =&1 + wA + w^2B + (wA + w^2B)^2 + O(w^3)\\ =&1 + wA + w^2(B + A^2) + O(w^3).\end{aligned}$$

Hence, we obtain

$$g_W^{-1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}2A_W & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}R_W^{11} - A_W^2 & \frac{2}{3}R_W^{12}\\ \frac{2}{3}R_W^{21} & \frac{1}{3}R_W^{22}\end{pmatrix} + \begin{pmatrix}4A_W^2 & 0\\ 0 & 0\end{pmatrix} + O(\|W\|^3),$$

which proves the result.

□

Remark. *Note that this formula is coordinate-free.*

More information about the dualization is given by a decomposition of the cotangent bundle. We want to describe the dual lift.

Definition 3.15. *The* ***dual lift*** $\mathcal{L}_W^*\colon T_W^*NL \to T_x^*L \oplus N_x^*L$ *is given by*

$$\mathcal{L}_W^*(\eta) = \left(\kappa_W\eta, J_W^*\eta\right),$$

where κ_W denotes the dualization of the horizontal lift.
The inverse dual lift $\left(\mathcal{L}_W^{-1}\right)^*$ is given by

$$\left(\mathcal{L}_W^{-1}\right)^*(\xi,\zeta) = \pi^*\xi + (K_W^\perp)^*\zeta.$$

Remark. *With the help of this definition every form $\eta \in T_W^*NL$ can be decomposed in the following way. For $\eta^\top \in T_x^*L$ and $\eta^\perp \in N_x^*L$ we have*

$$\eta = \left(\mathcal{L}_W^{-1}\right)^*\left(\eta^\top, \eta^\perp\right).$$

3.1.3 Rescaling the metric

In this subsection we investigate the tubular neighborhoods $L(1)$ and $L(\varepsilon)$ of the Riemannian submanifold L in detail. With the help of a rescaling map we obtain a diffeomorphism between these tubes. According to this relation we are able to translate problems on tubular neighborhoods of variable diameter into perturbed problems on the tube $L(1)$ of fixed diameter.

Definition 3.16. *For $\varepsilon > 0$ the map $\sigma_\varepsilon\colon NL \to NL$ defined by $\sigma_\varepsilon(W) := \varepsilon W$ is called* ***rescaling map****.*

Remark. *Recall that we simply write $\sigma_\varepsilon\colon L(1) \to L(\varepsilon)$.*

Lemma 3.17. *The differential*

$$\sigma_{\varepsilon*,W}\colon T_WNL \to T_{\varepsilon W}NL$$

is given by

$$\mathcal{L}_{\varepsilon W}^{-1} \circ \sigma_{\varepsilon*,W} \circ \mathcal{L}_W(X,V) = (X, \varepsilon V). \tag{40}$$

Proof. This is a direct consequence of the definition of the objects involved. The rescaling map is invariant on the horizontal subspace and rescales vertical vectors with factor ε. Using the decompositions $\mathcal{L}_W(X,V) = Z$ and $\mathcal{L}_W^{-1} Z = (X,V)$, and the definition of

$$\mathcal{L}_{\varepsilon W}^{-1}\colon T_{\varepsilon W} NL \to T_{\pi W} L \oplus N_{\pi W} L$$

we obtain (40). □

Moreover, we study the dual rescaling map which is given as follows.

Lemma 3.18. *Let $\sigma_\varepsilon^*\colon T_{\varepsilon W}^* NL \to T_W^* NL$ and $\eta \in T_{\varepsilon W}^* NL$ be decomposed as*

$$\eta = \pi^* \eta^\top + (K_W^\perp)^* \eta^\perp.$$

Then we obtain

$$\mathcal{L}_{\varepsilon W}^* \sigma_\varepsilon^* \eta = \left(P_T^* + \varepsilon^{-1} R_N^*\right) \mathcal{L}_W^* \eta = \left(\eta^\top, \varepsilon^{-1}\eta^\perp\right),$$

where $P_T\colon T_x NL \to T_x L$ and $P_N\colon T_x NL \to N_x L$ denote the orthogonal projections onto tangent and normal space.

Proof. The projection map π is invariant under composition with the rescaling map, i.e. $\pi \circ \sigma_\varepsilon = \pi$. Now we have

$$\mathcal{L}_{\varepsilon W}^* \sigma_\varepsilon^* \eta = \mathcal{L}_{\varepsilon W}^* \sigma_\varepsilon^* \left(\pi^* \eta^\top + (K_W^\perp)^* \eta^\perp\right).$$

For the tangential part the identity

$$\begin{aligned} \mathcal{L}_{\varepsilon W}^* \sigma_\varepsilon^* \pi^* \eta^\top &= \mathcal{L}_{\varepsilon W}^* (\pi \circ \sigma_\varepsilon)^* \eta^\top = \mathcal{L}_{\varepsilon W}^* \pi^* \eta^\top \\ &= \mathcal{L}_{\varepsilon W}^* \left(\pi^* \eta^\top\right) = \eta^\top, \end{aligned}$$

holds by definition of $\mathcal{L}_W^*$.

Using $K_W^\perp \circ J_W = \mathrm{id}_{NL}$ and the fact that $\sigma_{\varepsilon *} J_{\varepsilon W} = \varepsilon^{-1} J_W$, we prove for the vertical part

$$\begin{aligned} \mathcal{L}_{\varepsilon W}^* \sigma_\varepsilon^* {K_W^\perp}^* \eta^\perp &= J_{\varepsilon W}^* \sigma_\varepsilon^* (K_W^\perp)^* \eta^\perp \\ &= (\sigma_\varepsilon \circ J_{\varepsilon W})^* \, (K_W^\perp)^* \eta^\perp \\ &= \varepsilon^{-1} J_W K_W^\perp \eta^\perp = \varepsilon^{-1} \eta^\perp. \end{aligned}$$

□

Now we are finally able to introduce the rescaled metric.

Definition 3.19. *The rescaled metric $\langle \cdot, \cdot \rangle_{\varepsilon, W}$ in $W \in NL$ is defined by*

$$\langle Z, Z' \rangle_{\varepsilon, W} := \langle \sigma_{\varepsilon *, W} Z, \sigma_{\varepsilon *, W} Z' \rangle_{\varepsilon W}.$$

Remark. *We can express the rescaled metric in the following way:*

$$\begin{aligned} \langle Z, Z' \rangle_{\varepsilon, W} &= \langle \sigma_{\varepsilon *, W} Z, \sigma_{\varepsilon *, W} Z' \rangle_{\varepsilon W} \\ &= \left\langle \sigma_{\varepsilon *, W} \circ \mathcal{L}_W(X,V), \sigma_{\varepsilon *, W} \circ \mathcal{L}_W(X',V') \right\rangle_{\varepsilon W} \\ &= \left\langle \mathcal{L}_{\varepsilon W}^{-1} \circ \sigma_{\varepsilon *, W} \mathcal{L}_W(X,V), \mathcal{L}_{\varepsilon W}^{-1} \circ \sigma_{\varepsilon *, W} \circ \mathcal{L}_W(X',V') \right\rangle_{T_{\pi \varepsilon W} L \oplus N_{\pi \varepsilon W} L} \end{aligned}$$

$$
\begin{aligned}
&= \left\langle \mathcal{A}_{\varepsilon W}\mathcal{L}_{\varepsilon W}^{-1} \circ \sigma_{\varepsilon *, W}\mathcal{L}_W(X,V), \mathcal{A}_{\varepsilon W}\mathcal{L}_{\varepsilon W}^{-1} \circ \sigma_{\varepsilon *, W}\mathcal{L}_W(X',V')\right\rangle_{\pi W} \\
&= \left\langle \mathcal{A}_{\varepsilon W}(X+\varepsilon V), \mathcal{A}_{\varepsilon W}(X'+\varepsilon V')\right\rangle_{\pi W} \\
&= \left\langle \begin{pmatrix} X \\ V \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} g_{\varepsilon W} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} X' \\ V' \end{pmatrix} \right\rangle_{\pi W}.
\end{aligned}
$$

Hence, we obtain a slightly easier formulation, i.e.

$$
g_{\varepsilon,W} := \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} g_{\varepsilon W} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}. \tag{41}
$$

Now we proceed by computing the inverse of the rescaled metric as well.

Lemma 3.20. *The dual rescaled metric $g_{\varepsilon,W}^{-1}$ is given by*

$$
g_{\varepsilon,W}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-2} \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 0 \\ 0 & R_W^{22} \end{pmatrix} + O(\varepsilon). \tag{42}
$$

Proof. By the product representation (41) the inverse metric is given by

$$
g_{\varepsilon,W}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} g_{\varepsilon W}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}.
$$

Moreover, due to (39), we know

$$
g_{\varepsilon W}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} A_{\varepsilon W} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} R_{\varepsilon W}^{11} + 3A_{\varepsilon W}^2 & \frac{2}{3}R_{\varepsilon W}^{12} \\ \frac{2}{3}R_{\varepsilon W}^{21} & \frac{1}{3}R_{\varepsilon W}^{22} \end{pmatrix} + O\left(\varepsilon^3\right).
$$

Matrix multiplication yields

$$
\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} = \begin{pmatrix} A & \varepsilon^{-1}B \\ \varepsilon^{-1}C & \varepsilon^{-2}D \end{pmatrix}
$$

which implies (42).

□

Next we establish a representation of the induced metric as a perturbation of the Sasaki metric.

Definition 3.21. *Let the rescaled Sasaki metric be defined by*

$$
\langle Z, Z'\rangle_{Sa(\varepsilon),W} := \langle \sigma_{\varepsilon *, W} Z, \sigma_{\varepsilon *, W} Z'\rangle_{Sa,\varepsilon W}.
$$

Remark. *According to the definitions of the lifts we obtain*

$$
\begin{aligned}
\langle Z, Z'\rangle_{Sa(\varepsilon),W} &= \langle \sigma_{\varepsilon *, W} Z, \sigma_{\varepsilon *, W} Z'\rangle_{Sa,\varepsilon W} \\
&= \left\langle \mathcal{L}_{\varepsilon W}^{-1} \circ \sigma_{\varepsilon *, W} Z, \mathcal{L}_{\varepsilon W}^{-1}\sigma_{\varepsilon *, W} Z'\right\rangle_{\pi W} \\
&= \left\langle \mathcal{L}_{\varepsilon W}^{-1} \circ \sigma_{\varepsilon *, W} \circ \mathcal{L}_W(X,V), \mathcal{L}_{\varepsilon W}^{-1}\sigma_{\varepsilon *, W} \circ \mathcal{L}_W(X',V')\right\rangle_{\pi W} \\
&= \left\langle \begin{pmatrix} X \\ V \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \begin{pmatrix} X' \\ V' \end{pmatrix} \right\rangle_{\pi W},
\end{aligned}
$$

which demonstrates

$$g_{Sa(\varepsilon),W} := \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}$$

and

$$g^{-1}_{Sa(\varepsilon),W} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-2} \end{pmatrix}.$$

The announced perturbation result is given in the following proposition.

Proposition 3.22. *For the dual induced metric g^{-1} we asymptotically obtain*

$$g^{-1}_{\varepsilon,W} = g^{-1}_{Sa(\varepsilon),W} + \frac{1}{3}\begin{pmatrix} 0 & 0 \\ 0 & R^{22}_W \end{pmatrix} + O(\varepsilon).$$

3.2 The associated Radon-Nikodym density

In this section we want to compute the associated Radon-Nikodym density

$$\rho = \left(\frac{\det g_W}{\det g_{Sa(1),W}}\right)^{\frac{1}{2}}$$

of the volume form of the induced metric with respect to the volume form of the Sasaki metric. We use the expression given in Lemma (3.14) for the metric g_W to compute ρ asymptotically. Set

$$g_W = 1 + X^{(1)} + X^{(2)} + O\left(\|W\|^3\right),$$

where $X^{(1)} \in O(\|W\|)$ and $X^{(2)} \in O(\|W\|^2)$, more precisely

$$X^{(1)} = \begin{pmatrix} -2A_W & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X^{(2)} = -\begin{pmatrix} R^{11}_W - A^2_W & \frac{2}{3}R^{12}_W \\ \frac{2}{3}R^{21}_W & \frac{1}{3}R^{22}_W \end{pmatrix}.$$

Let $x \in L$ be fixed and $(e_i)_{i=1,\dots,l} \subset T_xL$ an orthonormal basis. Recall that the tension field τ is given by

$$\tau = \sum_{i=1}^{l} \mathrm{II}(e_i, e_i).$$

Using the connection between the Weingarten map and the second fundamental form we obtain

$$\begin{aligned}
\operatorname{tr}(A_W) &= \sum_{i=1}^{l} \langle e_i, A_W e_i \rangle \\
&= \sum_{i=1}^{l} \langle \mathrm{II}(e_i, e_i), W \rangle \\
&= \left\langle \sum_{i=1}^{l} \mathrm{II}(e_i, e_i), W \right\rangle = \langle \tau, W \rangle.
\end{aligned}$$

Moreover, the same arguments yield

$$\begin{aligned}
\operatorname{tr}(A_W^2) &= \operatorname{tr}(A_W^* A_W) \\
&= \sum_{i=1}^{l} \langle A_W e_i, A_W e_i \rangle \\
&= \sum_{j=1}^{l} \sum_{i=1}^{l} \langle e_i, A_W e_j \rangle \langle e_i, A_W e_j \rangle \\
&= \sum_{i,j=1}^{l} \langle \mathrm{II}(e_i, e_j), W \rangle^2.
\end{aligned}$$

Let us now compute the determinant of the induced metric with the help of the expansion above. We obtain

$$\begin{aligned}
\det(g_W) &= \det\left(1 + X^{(1)} + X^{(2)} + O\left(\|W\|^3\right)\right) \\
&= \det\left(\exp\left(X^{(1)} + X^{(2)} - \frac{1}{2}\left(X^{(1)}\right)^2 + O\left(\|W\|^3\right)\right)\right) \\
&= \exp\left(\operatorname{tr}\left(X^{(1)} + X^{(2)} - \frac{1}{2}\left(X^{(1)}\right)^2 + O\left(\|W\|^3\right)\right)\right) \\
&= \exp\left(\operatorname{tr}\left(X^{(1)} + X^{(2)} - \frac{1}{2}\left(X^{(1)}\right)^2\right)\right) + O\left(\|W\|^3\right) \\
&= 1 + \operatorname{tr}\left(X^{(1)} + X^{(2)} - \frac{1}{2}\left(X^{(1)}\right)^2\right) + \frac{1}{2}\operatorname{tr}\left(X^{(1)} + X^{(2)} - \frac{1}{2}\left(X^{(1)}\right)^2\right)^2 \\
&\quad + O\left(\|W\|^3\right) \\
&= 1 + \operatorname{tr}\left(X^{(1)} + X^{(2)} - \frac{1}{2}\left(X^{(1)}\right)^2\right) + \frac{1}{2}\operatorname{tr}\left(X^{(1)}\right)^2 + O\left(\|W\|^3\right).
\end{aligned}$$

Because we have the identities

$$\begin{aligned}
\operatorname{tr}\left(X^{(1)}\right) &= -2\operatorname{tr}(A_W), \\
\operatorname{tr}\left(X^{(1)}\right)^2 &= 4\operatorname{tr}(A_W)^2,
\end{aligned}$$

$$\operatorname{tr}\left(\left(X^{(1)}\right)^2\right) = 4\operatorname{tr}\left(A_W^2\right)$$
$$\text{and } \operatorname{tr}\left(X^{(2)}\right) = -\operatorname{tr}\left(R_W^{11}\right) + \operatorname{tr}\left(A_W^2\right) - \frac{1}{3}\operatorname{tr}\left(R_W^{22}\right),$$

we find

$$\det(g_W) = 1 - 2\operatorname{tr}(A_W) - \operatorname{tr}(A_W^2) + 2\operatorname{tr}(A_W)^2 - \operatorname{tr}\left(R_W^{11}\right) - \frac{1}{3}\operatorname{tr}\left(R_W^{22}\right) + O\left(\|W\|^3\right). \tag{43}$$

Moreover, $\det(g_{Sa(1),W}) = 1$, is obvious from the definition.

Now, we have all the information on hand to develop a new formula for the Radon-Nikodym density.

Proposition 3.23. *The density*

$$\rho = \left(\frac{\det g_W}{\det g_{Sa(1),W}}\right)^{\frac{1}{2}},$$

is asymptotically given by

$$\rho = \exp\left(-\operatorname{tr}(A_W) - \frac{1}{2}\operatorname{tr}(A_W^2) - \frac{1}{2}\operatorname{tr}(R_W^{11}) - \frac{1}{6}\operatorname{tr}(R_W^{22})\right) + O\left(\|W\|^3\right). \tag{44}$$

Proof. The basic idea of the proof is to express $\det(g_W)$ given by (43) as an exponential function. We obtain

$$\begin{aligned}\det(g_W) = &\exp\left(-2\operatorname{tr}(A_W) - \operatorname{tr}(A_W^2) + 2\operatorname{tr}(A_W)^2 - \operatorname{tr}(R_W^{11}) - \frac{1}{3}\operatorname{tr}(R_W^{22}) - 2\operatorname{tr}(A_W)^2\right)\\ &+ O\left(\|W\|^3\right).\end{aligned}$$

The asymptotic formula (44) then is an immediate consequence of the definition of ρ. □

This representation enables us to state another density formula.

Proposition 3.24. *The logarithmic density* $\log\rho$ *is asymptotically given by*

$$\log\rho = -\langle\tau, W\rangle - \sum_{i,j}\langle \mathrm{II}(e_i, e_j), W\rangle^2 - \frac{1}{2}\operatorname{tr}R_W^{11} - \frac{1}{6}\operatorname{tr}R_W^{22} + O\left(\|W\|^3\right). \tag{45}$$

Proof. This is a direct consequence of formula (44). □

Proposition (3.24) is a slightly different representation for the logarithmic density than the one given in [Wit11], Lemma 3.

We proceed to discuss the connection between the density ρ and the potential W in more detail in the following chapter.

Chapter 4

The connection between W_0 and ρ in different settings

We briefly review which objects we need to fix to compute the asymptotic behavior of a given metric on the tube in the situation introduced before. At first we focus on the general situation already discussed in Chapters 1 and 3 and use the main results of [Wit11] and [Wit08]. Afterwards, we slightly modify the setting for the proof of our main theorem.

So far, we studied only embeddings $L \subset M$ of closed Riemannian manifolds into some ambient Riemannian manifold. In this chapter, we generalize and transfer the situation to arbitrary vector bundles E over the closed Riemannian manifold (L, g_L), i.e. we study the embedding of L by the zero section into the bundle. And then construct an isometric embedding of L into some neighborhood of the zero section equipped with a new metric.
Moreover, we demonstrate how the involved curvature objects determine the situation. In this context we study small perturbations of the objects and observe the behavior of the potential W_0, as well as the density ρ, in this new setting.

The main theorem of this thesis is given at the end of this chapter where we investigate which smooth functions $f \in C^\infty(L)$ occur as a potential for an embedding $L \subset E$.

4.1 The initial situation

Starting with a closed Riemannian manifold (L, g_L) isometrically embedded into an ambient Riemannian manifold (M, g) we obtain the following inclusions:

$$L \subseteq L(1) \subseteq M.$$

As a direct sum of vector spaces we have

$$N_xL \oplus T_xL = T_xM_{|L},$$

for all $x \in L$. Due to the decomposition of the tangent bundle of M, we can equip NL with a metric using the restriction of g onto NL, i.e.

$$\langle W, V\rangle_{NL} := \langle W, V\rangle_{g,TM},$$

for all $W, V \in NL \subseteq TM$. This defines a bundle metric on the normal bundle NL, which we simply call the induced metric.

Using the tubular neighborhood theorem (cf. [Kos07], Chapter III) and possibly a change of the length scaling, we obtain a diffeomorphism $\exp\colon NL(1) \to L(1)$, i.e. for every $0 < \varepsilon \leq 1$ there exists a diffeomorphism of the ε-neighborhood $L(\varepsilon)$ onto $NL_\varepsilon(0) \subset NL$.

Let further $\pi\colon L(1) \to L$ be the projection of some $q \in L(1)$ onto the point $\tilde{q} \in L$ with the least distance (with respect to the induced distance function of M). Moreover, let $p\colon NL(1) \to L$ be the projection onto the base point. We obtain the following commutative diagram:

$$\begin{tikzcd} L(1) \arrow[r, "\pi"] & L \\ NL(1) \arrow[u, "\exp"] \arrow[ur, "p"'] & \end{tikzcd}$$

Note that π is induced by the projection map of the normal bundle.

We denote the embedding of the tube (inclusion map) and the submanifold by

$$j\colon L(1) \to M \text{ and } \varphi\colon L \to M.$$

We just deal with isometric embeddings φ, i.e. by definition

$$g_L = \varphi^* g.$$

Moreover, the induced metric on $L(1)$ is given by j^*g and therefore denoted by the same symbol g.

On the other hand, we can equip the normal bundle with the Sasaki metric, which is given by (3) in local Fermi coordinates. Remember that π turns out to be a Riemannian submersion, whenever NL is equipped with the Sasaki metric.

Thereby, we are able to compare both metrics on the tubes. By [Wit11], Lemma 1, we know that the induced metric g is a perturbation of the Sasaki metric g_{Sa}. For completeness we give the result obtained there in the following. We first need to express g in local Fermi coordinates as well.

We define $\partial_i := \frac{\partial}{\partial x^i}$ and $\partial_\alpha := \frac{\partial}{\partial w^\alpha}$ for $i = 1, ..., l$ and $\alpha = 1, ..., m-l$. Let $W = w^\mu e_\mu \in NL$ be a section and $A_W\colon TL \to TL$ the Weingarten map in direction W. According to [Wit11], Lemma 1, the induced metric of the tube can be expressed using the Sasaki metric, the Weingarten map and the curvature tensor R of M. We use the abbreviation

$$(\mathrm{Ric}_W)_{AB} := \langle R(\partial_A, W)W, \partial_B \rangle.$$

By the symmetry relations for the curvature tensor R of M and the definition of the Ricci map Ric we obtain the result as

$$
\begin{aligned}
&g(x,w)\\
&= g_{Sa}(x,w) - \left(\begin{array}{c|c} 2A_{W,ij} & 0 \\ \hline 0 & 0 \end{array}\right)\\
&\quad + \left(\begin{array}{c|c} g_{rs}A^r_{W,i}A^s_{W,j} + \langle R(W,\partial_i)W,\partial_j\rangle & \frac{2}{3}\langle \partial_i, R(W,\partial_\beta)W\rangle \\ \hline \frac{2}{3}\langle \partial_\beta, R(W,\partial_i)W\rangle & \frac{1}{3}\langle \partial_\alpha, R(W,\partial_\beta)W\rangle \end{array}\right) + O(\|W\|^3)\\
&= g_{Sa}(x,w) - \left(\begin{array}{c|c} 2A_{W,ij} + (A^2_W)_{ij} & 0 \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} (\mathrm{Ric}_W)_{ij} & \frac{2}{3}(\mathrm{Ric}_W)_{i\beta} \\ \hline \frac{2}{3}(\mathrm{Ric}_W)_{\beta i} & \frac{1}{3}(\mathrm{Ric}_W)_{\alpha,\beta} \end{array}\right)\\
&\qquad + O(\|W\|^3).
\end{aligned}
$$

Additionally, we need a metric connection on NL. On TM we can choose the Levi-Civita connection ∇^g with respect to g. Then, for $W, V \in \Gamma(NL)$ there exist smooth extensions $\widetilde{W}, \widetilde{V}$ on TM. Hence, we are able to evaluate

$$\nabla^g_{\widetilde{W}}\widetilde{V} \in \Gamma(NL).$$

Let $P_{NL}\colon TNL \to NL$ denote the projection onto the normal bundle NL; an application of P_{NL} to a section $Z \in \Gamma(NL)$ again yields a section in NL. Using the properties of the connection and the definition of a projection shows that

$$\widetilde{\nabla}_{\widetilde{W}}\widetilde{V} := P_{NL}(\nabla^g_{\widetilde{W}}\widetilde{V})$$

is linear and the product rule holds. Therefore, $\widetilde{\nabla}$ indeed defines a connection on NL (the induced connection).
Using this information combined with the representations of g and g_{Sa} we are able to give the asymptotic representation of the logarithmic Radon-Nikodym density from [Wit11], Lemma 3, p. 22, i.e.

$$\log(\rho(W)) = -\mathrm{tr}A_W - \frac{1}{2}\mathrm{tr}\left(A^2_W\right) - \frac{1}{2}(\mathrm{Ric}_W)_{ii} - \frac{1}{6}(\mathrm{Ric}_W)_{\alpha\beta} + O\left(\|W\|^3\right),$$

where we used the abbreviations introduced above.

Note that this is basically the same result we computed without the use of local Fermi coordinates in (45).

4.2 Rescaling of the quadratic form family associated to the perturbed metric

Let g_r denote an arbitrary metric on the tube $L(\varepsilon)$. By Friedrichs' construction (see Appendix C) the Dirichlet Laplacian

$$\triangle_\varepsilon\colon H^1_0 \cap H^2\left(L(\varepsilon), g_r\right) \to L^2\left(L(\varepsilon), g_r\right)$$

corresponding to g_r on $L(\varepsilon)$ is associated to the quadratic form

$$q_\varepsilon \colon H_0^1\left(L(\varepsilon), g_r\right) \to \mathbb{R},$$

where

$$q_\varepsilon(f) := \int_{L(\varepsilon)} \|df\|_r^2 \, \mathrm{m}_r(dx).$$

Due to [Wit11], Lemma 5, p. 27, the map $\Sigma_\varepsilon \colon L^2\left(L(\varepsilon), g_r\right) \to L^2\left(L(1), g_{Sa}\right)$ given by

$$\Sigma_\varepsilon f := \left(\sqrt{\varepsilon^{m-l}\rho} f\right) \circ \sigma_\varepsilon = \sigma_\varepsilon^* \left(\sqrt{\varepsilon^{m-l}\rho} f\right)^2 \tag{46}$$

is a unitary isomorphism.

With the help of this map, we investigate our problem on the fixed model space $L(1)$. In this context we first describe the process which explains the connection between the potential W_0 and the density ρ, stated in Section (1.5). The computations in this sequel are based on Chapter 3 in [Wit11].

The potential W_1 on $L(1)$ can be computed from the rescaled form family, which is given as follows.

Definition 4.1. *The **rescaled form family** $q(\varepsilon, \cdot) \colon H_0^1\left(L(1), g_{Sa}\right) \to \mathbb{R}$ is defined by*

$$q(\varepsilon, f) := q_\varepsilon\left(\Sigma_\varepsilon^{-1} f\right). \tag{47}$$

Remark. *The action of the inverse of the rescaling map is given by*

$$\Sigma_\varepsilon^{-1} f = \left(\varepsilon^{m-l}\rho\right)^{-\frac{1}{2}} \sigma_\varepsilon^{-1*} f.$$

In the following discussion we will see that the potential W_0 can be computed from the Radon-Nikodym density ρ in different ways.

Definition 4.2. *Let $\rho = dm_{g_r}/dm_{Sa}$ be the Radon-Nikodym density of the volume forms associated to g_r and g_{Sa}. We define the smooth potential $W_1 \in C^\infty(\overline{L(1)})$ by*

$$W_1 = -\frac{1}{2}\triangle \log\rho + \frac{1}{4}\|d\log\rho\|^2, \tag{48}$$

where the Laplacian and the norm are chosen with respect to g_r.

Lemma 4.3. *The potential can be computed by*

$$W_1 = \rho^{\frac{1}{2}} \triangle \rho^{-\frac{1}{2}}. \tag{49}$$

Proof. We use the definition of the Hodge operator $\star$ associated to g_r and obtain

$$\begin{aligned}
\rho^{\frac{1}{2}} \triangle \rho^{-\frac{1}{2}} &= \rho^{\frac{1}{2}} \left(\star\, d \star\, d\rho^{-\frac{1}{2}}\right) \\
&= \rho^{\frac{1}{2}} \left(\star\, d \star \left(-\frac{1}{2}\rho^{-\frac{3}{2}} d\rho\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\rho^{\frac{1}{2}}\left(\star\, d\, \star \left(\rho^{-\frac{1}{2}}\frac{d\rho}{\rho}\right)\right)\\
&= -\frac{1}{2}\rho^{\frac{1}{2}}\left(\star\, d\left(\rho^{-\frac{1}{2}}\star\; d\log\rho\right)\right)\\
&= -\frac{1}{2}\rho^{\frac{1}{2}}\left(\star\left(-\frac{1}{2}\rho^{-\frac{3}{2}}\; d\rho\wedge\star\, d\log\rho+\rho^{-\frac{1}{2}}\; d\star\; d\log\rho\right)\right)\\
&= \frac{1}{4}\star(d\log\rho\wedge\star\, d\log\rho)-\frac{1}{2}\star d\star d\log\rho.
\end{aligned}
$$

The statement now follows from the fact that $\star\,(d\log\rho\wedge\star\, d\log\rho)=\langle d\log\rho, d\log\rho\rangle$. □

Remark. *1. Note that $W_0 = W_{1|L}$ is the potential introduced in (13).*

2. The sign depends on the choice and sign of the underlying Laplacian and may differ in the literature.

To justify Definition (4.2), we prove the following characterization of the rescaled form family.

Lemma 4.4. *Let $f\in H_0^1\cap H^2\,(L(1), g_{Sa})$ and $\|\cdot\|_\varepsilon$ denote the norm associated to $g_r(\varepsilon)=\sigma_\varepsilon^* g_r$. Then the rescaled form family is given by*

$$
q(\varepsilon,f)=\int_{L(1)}\left(\|df\|_\varepsilon^2-W_\varepsilon f^2\;\right)m_{Sa}(dx),
$$

where $W_\varepsilon := \sigma_\varepsilon^* W_1$.

Proof. By formula (47), first of all we need to compute

$$
\begin{aligned}
d\,\Sigma_\varepsilon^{-1}f &= d\left(\left(\varepsilon^{m-l}\rho\right)^{-\frac{1}{2}}\sigma_\varepsilon^{-1*}(f)\right)\\
&= \left(\varepsilon^{m-l}\rho\right)^{-\frac{1}{2}} d(\sigma_\varepsilon^{-1*}(f))+\sigma_\varepsilon^{-1*}(f)\; d\left(\varepsilon^{m-l}\rho\right)^{-\frac{1}{2}}\\
&= \frac{1}{\sqrt{\varepsilon^{m-l}\rho}}d\sigma_\varepsilon^{-1*}(f)-\frac{1}{2}\sigma_\varepsilon^{-1*}(f)\left(\varepsilon^{m-l}\rho\right)^{-\frac{3}{2}}\varepsilon^{m-l}d\rho\\
&= \frac{1}{\sqrt{\varepsilon^{m-l}\rho}}\left(d\sigma_\varepsilon^{-1*}(f)-\frac{1}{2}\sigma_\varepsilon^{-1*}(f)\; d\log\rho\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|d\,\Sigma_\varepsilon^{-1}f\|_r^2 &= \left\langle d\,\Sigma_\varepsilon^{-1}f, d\,\Sigma_\varepsilon^{-1}f\right\rangle_r\\
&= \frac{1}{\varepsilon^{m-l}\rho}\left\{\left\langle d\sigma_\varepsilon^{-1*}(f), d\sigma_\varepsilon^{-1*}(f)\right\rangle_r-\left\langle d\sigma_\varepsilon^{-1*}(f),\sigma_\varepsilon^{-1*}(f)d\log\rho\right\rangle_r\right.\\
&\quad\left.+\frac{1}{4}\left\langle\sigma_\varepsilon^{-1*}(f)d\log\rho,\sigma_\varepsilon^{-1*}(f)d\log\rho\right\rangle_r\right\}\\
&= \frac{1}{\varepsilon^{m-l}\rho}\left\{\left\langle d\sigma_\varepsilon^{-1*}(f), d\sigma_\varepsilon^{-1*}(f)\right\rangle_r-\left\langle\sigma_\varepsilon^{-1*}(df),\sigma_\varepsilon^{-1*}(f)d\log\rho\right\rangle_r\right.\\
&\quad\left.+\frac{1}{4}\left(\sigma_\varepsilon^{-1*}(f)\right)^2\langle d\log\rho, d\log\rho\rangle_r\right\}\\
&= \frac{1}{\varepsilon^{m-l}\rho}\left\{\langle\sigma_\varepsilon^{-1*}(df),\sigma_\varepsilon^{-1*}(df)\rangle_r-\sigma_\varepsilon^{-1*}(f)\langle\sigma_\varepsilon^{-1*}(df), d\log\rho\rangle_r\right.
\end{aligned}
$$

$$+\frac{1}{4}\left(\sigma_\varepsilon^{-1*}(f)\right)^2 \langle d\log\rho, d\log\rho\rangle_r\Big\}$$

By the definition of the pullback metric we obtain

$$\begin{aligned}\|d\ \Sigma_\varepsilon^{-1} f\|_r^2 = \frac{1}{\varepsilon^{m-l}\rho}\sigma_\varepsilon^{-1*}\Big\{&\langle df, df\rangle_{\sigma_\varepsilon^* g_r} - f\,\langle df, \sigma_\varepsilon^* d\ \log\rho\rangle_{\sigma_\varepsilon^* g_r}\\ &+\frac{1}{4}f^2\langle\sigma_\varepsilon^* d\log\rho, \sigma_\varepsilon^* d\log\rho\rangle_{\sigma_\varepsilon^* g_r}\Big\}.\end{aligned}$$

Note that $\frac{1}{2}d(f^2) = f df$ and ${\sigma_\varepsilon}^{-1}{}_* m_{Sa} = \varepsilon^{m-l} m_{Sa|L(\varepsilon)}$ due to the definition of σ_ε and $m_r(dx) = \rho\, m_{Sa}(dx)$. This gives

$$\begin{aligned}q_\varepsilon\left(\Sigma_\varepsilon^{-1} f\right) &= \int_{L(\varepsilon)} \|d\ \Sigma_\varepsilon^{-1} f\|_r^2\ m_r(dx)\\ &= \int_{L(\varepsilon)} \frac{1}{\varepsilon^{m-l}\rho}\sigma_\varepsilon^{-1*}\left\{\langle df, df\rangle_{\sigma_\varepsilon^* g_r} - \langle f df, \sigma_\varepsilon^* d\log\rho\rangle_{\sigma_\varepsilon^* g_r}\right.\\ &\quad\left.+\frac{1}{4}f^2\langle\sigma_\varepsilon^* d\log\rho, \sigma_\varepsilon^* d\ \log\rho\rangle_{\sigma_\varepsilon^* g_r}\right\}\rho\, m_{Sa}(dx)\\ &= \int_{L(\varepsilon)} \frac{1}{\varepsilon^{m-l}}\sigma_\varepsilon^{-1*} A(x)\ m_{Sa}(dx),\end{aligned}$$

where

$$A(x) = \|df\|^2_{\sigma_\varepsilon^* g_r} - \frac{1}{2}\left\langle df^2, \sigma_\varepsilon^* d\ \log\rho\right\rangle_{\sigma_\varepsilon^* g_r} + \frac{1}{4}f^2\|\sigma_\varepsilon^* d\log\rho\|^2_{\sigma_\varepsilon^* g_r}.$$

Hence, by the transformation formula, we obtain

$$q_\varepsilon\left(\Sigma_\varepsilon^{-1} f\right) = \frac{1}{\varepsilon^{m-l}}\int_{\sigma_\varepsilon^{-1}(L(\varepsilon))} ({\sigma_\varepsilon^{-1}}_* m_{Sa}(dx))A(x) = \int_{L(1)} m_{Sa}(dy)A(y).$$

Now we evaluate A, starting with the second summand above. Note that

$$\nabla\rho^{-1} = -\rho^{-1}\nabla\log\rho$$

and that for differential forms η, ω and on a Riemannian manifold the equation

$$\omega\wedge\star\eta = \langle\omega, \eta\rangle_g \mu_M$$

holds, where μ_M is the volume element of M.
This yields

$$\begin{aligned}&-\frac{1}{2}\int_{L(1)} \langle df^2, \sigma_\varepsilon^* d\ \log\rho\rangle_{\sigma_\varepsilon^* g_r}\ m_{Sa}(dy)\\ &= -\frac{1}{2\varepsilon^{m-l}}\int_{L(\varepsilon)} \rho^{-1}\langle d\sigma_\varepsilon^{-1*} f^2, d\ \log\rho\rangle_r\ m_r(dx)\\ &= \frac{1}{2\varepsilon^{m-l}}\int_{L(\varepsilon)} \langle d\sigma_\varepsilon^{-1*} f^2, d\rho^{-1}\rangle_r\ m_r(dx)\\ &= \frac{1}{2\varepsilon^{m-l}}\int_{L(\varepsilon)} d\sigma_\varepsilon^{-1*} f^2\wedge\star d\rho^{-1}.\end{aligned}$$

Using the fact that $\sigma_\varepsilon^{-1*} f_{|\partial L(\varepsilon)} = 0$ and an integration by parts, we finally have

$$-\frac{1}{2\varepsilon^{m-l}} \int_{L(\varepsilon)} \sigma_\varepsilon^{-1*} f^2 \triangle \rho^{-1} \; m_r(dx) \tag{50}$$

for this part of the formula. We proceed by computing $\triangle \rho^{-1}$. By the product rule we obtain

$$\triangle \left(\rho^{-1}\rho\right) = \rho^{-1}\triangle\rho + \rho\triangle\rho^{-1} + 2\langle d\rho^{-1}, d\rho\rangle_r = 0.$$

This leads to

$$\begin{aligned} \triangle\rho^{-1} &= -\rho^{-1}\left(\rho^{-1}\triangle\rho + 2\langle d\rho^{-1}, d\rho\rangle_r\right) \\ &= -\rho^{-1}\left(\rho^{-1}\triangle\rho - 2\langle d\log\rho, d\log\rho\rangle\right). \end{aligned}$$

Inserting this result into (50) gives

$$\frac{1}{2\varepsilon^{m-l}} \int_{L(\varepsilon)} m_r(dx)\rho^{-1}\sigma_\varepsilon^{-1*} f^2 \left(\rho^{-1}\triangle\rho - 2\langle d\log\rho, d\log\rho\rangle\right). \tag{51}$$

Finally, from a straightforward computation we have

$$\triangle \log\rho = \rho^{-1}\triangle\rho - \langle d\log\rho, d\log\rho\rangle.$$

Therefore (51) equals

$$\frac{1}{2\varepsilon^{m-l}} \int_{L(\varepsilon)} \sigma_\varepsilon^{-1*} f^2 \left(\triangle \log\rho - \langle d\log\rho, d\log\rho\rangle\right) \; m_{Sa}(dx).$$

Combining all these facts indeed gives

$$q(\varepsilon, f) = \int_{L(1)} \|df\|^2_{\sigma_\varepsilon^* g_r} + f^2\sigma_\varepsilon^*\left(\frac{1}{2}\triangle\log\rho - \frac{1}{4}\|d\log\rho\|_r^2\right) m_{Sa}(dx).$$

□

The family $\{q(\varepsilon,\cdot)\}_{\varepsilon>0}$ defines a family of quadratic forms on a common, fixed Sobolev space. For each $\varepsilon > 0$, the form $q(\varepsilon,\cdot)$ defines a self-adjoint operator, which generates a strongly continuous semi-group on $L^2(L(1), g_{Sa})$.

Remark. *This result is Proposition 2 in [Wit11]. Note that due to conventional reasons, the sign of our result differs from the result the author obtained there.*
For a slightly different way of proving the result, we refer to the proof given there.

In the following section we want to find expressions for these objects – potential and density – in a new and more general setting.

4.3 Generalization

Let (L, g_L) denote a closed Riemannian manifold. In analogue to Section (4.1), we first state what data is needed. First, we want to find an isometric embedding of L in some ambient space $(L(\varepsilon), g_r)$.

Let $\pi\colon E \to L$ be a smooth vector bundle of rank $k \in \mathbb{N}$ over L. Denote the embedding of L into E via the zero section by

$$s_0\colon L \to E.$$

We prove that by fixing additional data, we can identify E with the normal bundle of an embedding $L \subset M$ into an ambient Riemannian manifold M of dimension $m \geq l$.
More precisely, we will study the role of the Sasaki metric in this context.

Additionally, we explore perturbations of the metric on the fibers and comment on the behavior of a chosen algebraic curvature tensor on the ambient space, i.e. on the fibers of the bundle. We will prove that the asymptotic behavior in our perturbed setting is completely described by this curvature form.
For this model problem we obtain handy expressions for the density ρ and the potential W_0. We will prove that these quantities depend only on the chosen curvature form.

Let $U \subset L$ be contractible and $e = (e_\alpha)_{\alpha=1,\ldots,k}$ be a local frame of orthonormal sections on E over U, i.e. an ordered basis of (local) sections. Then every local section $\xi \in \Gamma(U, E)$ can be expressed as

$$\xi = \sum_{\alpha=1}^{k} e_\alpha \xi^\alpha(e).$$

Moreover we choose a bundle metric g_ω on E. A metric connection ∇^ω is given according to a connection one-form

$$\omega \in \mathcal{A}^1(L, \mathrm{End}(E)) = \Gamma(L, \Lambda^1 T^*L \otimes \mathrm{End}(E)).$$

This connection gives a canonical decomposition of TE into the direct sum of the horizontal and the vertical subspace (already seen in Section(3.1.1.1)), here denoted by $\mathfrak{H}E$ and $\mathfrak{V}E$. Note that $\mathrm{rk}(\mathfrak{H}E) = \mathrm{rk}(TL)$ and $\mathrm{rk}(\mathfrak{V}E) = \mathrm{rk}(E)$. Recall that the vertical subspace, $\mathfrak{V}E = \ker(\pi_*)$, can be equipped with the fiber metric of E. The horizontal space is defined by the connection, i.e. as kernel of the associated connection map and inherits the induced metric from TL. Therefore, we obtain the usual splitting

$$TE = \mathfrak{H}E \oplus \mathfrak{V}E.$$

To compare our own investigations with the results in [Wit11], we choose the metric in such a way so that

$$\pi\colon E \to L$$

is a Riemannian submersion, i.e. the metric preserves the length of the horizontal vectors. These requirements naturally lead to the characterizations of the Sasaki metric g_{Sa}. Therefore, we want to show that the natural choice is given by

$$g_\omega = g_{Sa}.$$

Before we construct the Sasaki metric for this new setting (on the bundle E) in detail in the following section, we gather more remarkable facts.

Lemma 4.5. *Let $\pi\colon E \to L$ be a vector bundle over L. The embedding $s_0\colon (L, g_L) \subset (E, g_{Sa})$ is isometric.*

Proof. The embedding $s_0 \colon L \to E$ ensures that L is a submanifold of E. Due to the fact that we equipped E with a Sasaki metric, π is a Riemannian submersion and $TL \subseteq \mathfrak{H}E \subseteq TE$. Moreover, the usual identification of $TE_{|L} \cong TL \oplus NL$, where $NL \subseteq \ker(\pi_*) = \mathfrak{V}E$, shows that for any two vectors $V, W \in T_xL$ and $x \in L$ we have

$$\langle V, W\rangle_L \overset{\text{Riem.}}{\underset{\text{sub.}}{=}} \langle \pi_* V, \pi_* W\rangle_L = \langle V, W\rangle_{Sa}$$

because the horizontal and vertical subspaces are orthogonal with respect to the Sasaki metric. □

With the decomposition of TE in mind and the properties of the Sasaki metric at hand, a comparison with a computation for the normal bundle in [BY89] leads to the following observations.

The following lemma naturally identifies E with the complement of TL in $TE_{|L}$ and is therefore as a bundle isomorphic to the normal bundle of an embedding.

Lemma 4.6. *There exists a bundle isomorphism between NL and E.*

Proof. As the projection $\pi \colon E \to L$ is a submersion, it induces the surjective map

$$\pi_* \colon TE \to \pi^* TL.$$

By definition the kernel of the map is exactly the vertical bundle $\mathcal{V}E$. This gives the exact sequence

$$0 \longrightarrow \mathcal{V}E \longrightarrow TE \longrightarrow \pi^* TL \longrightarrow 0.$$

See Chapter 3 in [Spi70] for the exactness. Pulling back this sequence by s_0 yields the exact sequence

$$0 \longrightarrow E \longrightarrow TE_{|L} \longrightarrow TL \longrightarrow 0.$$

Because L is compact, the above sequence splits and we obtain

$$TE_{|L} \cong E \oplus TL.$$

Therefore the bundles E and NL are isomorphic.

□

Remark. *There is no canonical isomorphism. But, any given connection on E induces a canonical splitting, since finding a horizontal subspace at each point is equivalent to the definition of a connection on the ambient space. Note that then $\mathfrak{H}E \cong \pi^*(TL)$.*

We even obtain a more specific result. By the above construction the canonical choice for a metric is the Sasaki metric. We want to study an isomorphism between E and NL.

Theorem 4.7. *For any vector bundle over L with the above structure, equipped with a Sasaki metric, there exists an embedding $L \subset M$ into another Riemannian manifold (M, g), such that E is already isomorphic to the induced vertical subbundle $\mathfrak{V}_{E,W}$, where $x = \pi W$.*

Proof. Let $\pi \colon E \to L$ be the bundle projection and remember that there is a decomposition of TE into the subspaces of vertical and horizontal vectors, i.e. $\mathfrak{V}E := \ker(\pi_*) \subseteq TE$ and $\mathfrak{H}E = \ker(\omega + \mathrm{id}_{\mathfrak{V}}) \subseteq TE$ for the chosen connection one-form $\omega \in \mathcal{A}^1(L, \mathrm{End}(E))$ of L. Recall that these subspaces are orthogonal with respect to the Sasaki metric.

Let now $x \in L$ and $W \in E_x$, i.e. $\pi(W) = x$. Then, due to the bundle properties, the fibers are (linear) vector spaces. Now we study the identification map J_W, already seen in (2), defined as

$$J_W \colon E_x \to T_W E_x, J_W V := \frac{d}{dt}(W + tV)\Big|_{t=0} = V^i \frac{\partial}{\partial W^i}\Big|_W,$$

meaning that for $f_x \in C^\infty(E_x)$ the equality

$$J_W V(f_x) = \frac{d}{dt} f_x(W + tV)\Big|_{t=0}$$

holds.

Next we compute for a smooth f

$$\begin{aligned}
\pi_*(J_W V)(f) &= J_W V(f \circ \pi) \\
&= \frac{d}{dt}\left(f \circ \pi(W + tV)\right)\Big|_{t=0} \\
&= \frac{d}{dt} f(x)\Big|_{t=0} \\
&= 0,
\end{aligned}$$

where we used the fact that the fibers are vector spaces in the penultimate step. As a conclusion we state that

$$\pi_*(J_W V) = 0,$$

for every $V \in E_x$, which means

$$J_W V \subseteq \ker(\pi_*) = \mathfrak{V}_{E,W}.$$

Then $J_W E_x \subseteq \mathfrak{V}_{E,W}$ is a linear subspace. In fact, $J_W E_x = \mathfrak{V}_{E,W}$ due to dimensional reasons. Therefore, J_W is an isomorphism, i.e.

$$J_W \colon E \longrightarrow \mathfrak{V}_{E,W}.$$

Moreover, the embedding of L is given by $s_0 \colon L \to E$ and hence we are able to identify

$$J_0 \colon E_x \xrightarrow{\cong} \mathfrak{V}_{E,0} = s_0^* \mathfrak{V} E\Big|_x.$$

On the other hand, the l-dimensional tangent space TL is isomorphic to $s_0^*(\mathfrak{H}E)$. Let

$$N_x L = \operatorname{span}\{\partial_1, ..., \partial_{m-l}\}.$$

Due to dimensional reasons, the fact that the vertical vectors are tangential to the fibers and $T_x M = T_x M_{|_L} = N_x L \oplus T_x L$ is equipped with

$$g_{Sa,x} = \left(\begin{array}{c|c} g_{L ij} & 0 \\ \hline 0 & \delta_{\alpha\beta} \end{array}\right)$$

for all $x \in L$, we finally obtain $E \cong NL$ with the additional structure information. □

This theorem shows that the natural choice of a metric is the Sasaki metric. Moreover, the previous results show that E is isomorphic to the normal bundle induced by an embedding of L into some ambient space (M, g).

4.3.1 The 2-form and fiber curvature

As announced earlier, we now want to study a new metric on the tubular neighborhoods. First, we explore the vector bundle structure and comment on the construction of the Sasaki metric in this new setting; afterwards we observe how the curvature of the fibers determines the situation basically on its own. We will then define a bilinear form involving curvature quantities to give a representation of the new metric.

Again, note that $s_0 \colon L \to E$ is the embedding of L into the total space via the zero section. We start this section by choosing an arbitrary bundle metric b on E before defining the Sasaki metric with the help of the bundle metric b and a fixed metric connection ∇^b. Finally, we investigate this special metric in detail.

Remark. *A bundle metric on E is a section $b \in \Gamma(E^* \oplus E^*)$, such that for every $x \in L$ the map*

$$b_x(\cdot, \cdot) := \langle \cdot, \cdot \rangle_x \colon E_x \times E_x \to \mathbb{R}$$

is a symmetric and positive definite map on the fibers of E.
It suffices to define a metric on the bundle fiberwise, because

$$T_x E_{|L} = E_x \oplus T_x L$$

and the metric g_L defines an inner product on each tangent space $T_x L, x \in L$.

For more information about the construction of bundle metrics see e.g. [Bau09].

Lemma 4.8. *On every vector bundle $\pi \colon E \to L$ over L there exists a bundle metric b and a compatible connection $\nabla = \nabla^b$.*

Proof. This is a standard result proven by a suitable choice of a partition of unity to define a connection. $\square$

Lemma 4.9. *The set of bundle metrics on E is convex.*

Proof. This follows directly from the fact, that the convex combination of positive definite sections is positive definite. $\square$

The existence of a bundle metric b is not enough. Additionally, let ∇ be a covariant derivative on E, i.e. a connection on E. Moreover, we assume that ∇ is metric with respect to b.

Lemma 4.10. *Let $\pi \colon E \to L$ be a bundle and ∇ a metric connection. The set of metric connections is an affine space, i.e. for each connection $\tilde{\nabla}$ different from ∇, we obtain*

$$\nabla - \tilde{\nabla} =: \eta \in \Omega^1(\mathrm{End}(E)).$$

Proof. There exists a metric connection ∇ on E; therefore, for each $\omega \in \Omega^1(\mathrm{End}(E))$, using linearity, the sum $\nabla + \omega$ is a connection as well. We further need to show that

$$\nabla - \tilde{\nabla} =: \eta \in \Omega^1(\mathrm{End}(E)).$$

Let $\psi \in \Gamma(E)$ be a section and $p \in L$. As the next computation shows, $\eta(\psi)(p)$ does not depend on the choice of ψ but only on its value $\psi(p)$.
For each $f \in C^\infty(L)$, we have

$$\begin{aligned}\eta(f\psi) &= (\nabla - \tilde{\nabla})(f\psi)\\ &= \nabla(f\psi) - \tilde{\nabla}(f\psi)\\ &= \nabla\psi \cdot f + \psi df - \left(\tilde{\nabla}\psi \cdot f + \psi df\right)\\ &= ((\nabla - \tilde{\nabla})\psi)f\\ &= \eta(\psi)f;\end{aligned}$$

i.e. the difference of two connections is a C^∞-module homomorphism in the second argument. Therefore, we obtain

$$\eta \in \Gamma(\mathrm{Hom}(E; \mathrm{Hom}(TL, E))).$$

We then identify η as a section

$$\eta \in \Gamma(\mathrm{Hom}(TL, \mathrm{Hom}(E, E))) = \Gamma(\mathrm{Hom}(TL, \mathrm{End}(E))) = \Omega^1(\mathrm{End}(E)).$$

□

Remark. *In local coordinates, for the difference of two connections ∇ and ∇' on a total space E with connection coefficients C_{ij}^k and ${C'}_{ij}^k$, respectively, we have*

$$(\nabla - \nabla')_X = \left(C_{ij}^k - {C'}_{ij}^k\right) X^i Y^j \partial_k.$$

The coefficients $A_{ij}^k := C_{ij}^k - {C'}_{ij}^k$ act as the components of a $(1,2)$-tensor, which shows that the difference is a one-form with coefficients in $\mathrm{End}(E)$.

We already saw that $E \cong NL(\subseteq TE_{|L})$ as a bundle. Moreover we want to use the above data to construct a Sasaki metric on E explicitly. We follow the construction in [Sak96]. For further computations we always assume that there is no perturbation on the submanifold L, meaning

$$\langle\cdot,\cdot\rangle_{Sa,x|E_x \times E_x} = \langle\cdot,\cdot\rangle_x.$$

For each $W \in E_x$ we need to define the map

$$\langle\cdot,\cdot\rangle_{Sa,W} := g_{Sa,W} \colon T_W E \times T_W E \to \mathbb{R}.$$

Using the connection map $K^\perp$ of NL defined by

$$K^\perp(D\xi(X)) = \nabla_X^\perp \xi,$$

for $X \in \Gamma(E)$, a normal vector field ξ and the induced connection $\nabla^\perp$, we are able to define a Sasaki metric.
Analogously to [Sak96] and Definition (1.1) we define

$$g_{Sa,W}(X,Y) := \langle \pi_* X, \pi_* Y\rangle_L + b(X,Y), \tag{52}$$

where the second part equals $\langle K_W^{\perp} X, K_W^{\perp} Y \rangle_{NL}$.

By construction the following statement is obvious. But for completeness we remark that (52) indeed defines a Sasaki metric in the form of Reckziegel.

Lemma 4.11. *The above defined metric $g_{Sa,W}$ is a Sasaki metric in the form of Reckziegel (see Section (1.2)).*

Proof.
1. By definition the projection $\pi\colon E \to L$ is a Riemannian submersion and by Lemma (4.5) the embedding $(L, g_L) \subset (E, g_{Sa})$ is isometric.
2. Orthogonality of horizontal and vertical vectors is obvious by $\mathfrak{V} = \ker(\pi_*)$ and $\mathfrak{H} = \ker(K^{\perp})$.
3. Because the second fundamental form vanishes at the base points $x \in L$ of the fibers, the fibers are totally geodesically embedded.
4. E defines a vector bundle, hence the fibers are isometric isomorphic to a vector space.

□

As a conclusion we have the following result.

Corollary 4.12. *Whenever a bundle metric b and a metric connection ∇^b on a vector bundle $\pi\colon E \to L$ are given, it is possible to define a Sasaki metric on TE.*

In contrast to the construction in [Wit11] we explore a different perturbation problem and compute the potential for $\varepsilon \to 0$ in this situation. Instead of using the induced metric of the ambient manifold we introduce another metric on the tubes.

Therefore, we first want to describe small perturbations of the Sasaki metric on the normal bundle. The following construction was motivated by [Wit11], Lemma 4, where the author proved that the curvature form determines the asymptotic behavior of the induced metric. Because of this, it seemed to be natural to study small perturbations of the curvature on the fibers.

To find a suitable description, we first introduce a bilinear form which we use for the perturbation afterwards.

Definition 4.13. *For $\pi\colon E \to L$ equipped with a bundle metric b and a metric connection ∇, we define the bilinear form ϱ^r by*

$$\varrho^r_{E_x,W}\colon T_W E_x \times T_W E_x \to \mathbb{R},\ (X,Y) \mapsto b(P_W Y, r(P_W X, W)W),$$

where $W \in E$, $P_W\colon T_W NL \to T_W N_{\pi(W)} L$ the projection map and $r \in \mathcal{R}$.

Recall that $\mathcal{R}$ is the space of algebraic curvature tensors, i.e. r has the same symmetries as the Riemannian curvature tensor.

In Section (4.1) we saw that the induced metric is a perturbation of the Sasaki metric. This is one of the main tools to compute the Radon-Nikodym density ρ (see [Wit11], Lemma 3). In our new setting, we want to define another metric (different from the induced metric) that is a perturbation of the Sasaki metric as well. This new metric depends on the bilinear form ϱ^r from Definition (4.13) and is constructed as follows.

Definition 4.14. *For $W \in E_x$ we define the family of inner products $g_{r,W} : T_W E_x \times T_W E_x \to \mathbb{R}$ in local Fermi coordinates by*

$$g_{r,W}(x,w) := g_{Sa,W}(x,w) + \frac{1}{3}\varrho^r_{E_x,W}(x,w).$$

We call g_r the ***perturbed Sasaki metric*** *on a small neighborhood $E(0)$ of the zero section in E.*

We will identify this small neighborhood $E(0)$ of the zero section again with a tubular neighborhood $L(\varepsilon)$, for $\varepsilon > 0$ small enough.

Remark. *For simplicity, we omit the dependency of the fiber E_x when referring to the perturbed metric g_r or the bilinear form ϱ^r whenever the context is clear. Furthermore, in our computations we use the abbreviation*

$$r_W(X,Y) := \langle X, r(Y,W)W\rangle,$$

where the inner product is taken with respect to the bundle metric. Then in local coordinates the perturbed Sasaki metric is given by

$$g_{r,W}(x,w) = g_{Sa,W}(x,w) + \frac{1}{3}\varrho^r_{W\,\alpha\beta}(x) = g_{Sa,W}(x,w) + \frac{1}{3}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & w^\mu w^\nu r_{W\,\alpha\mu\beta\nu}\end{array}\right). \tag{53}$$

To justify the definition we state the following fact.

Lemma 4.15. *There exists $\varepsilon_0 > 0$, such that the perturbed metric g_r defines a metric on the tubes $L(\varepsilon)$, $0 < \varepsilon \leq \varepsilon_0$.*

Proof. In order to show that g_r defines a metric on a tube we need to assure that g_r is symmetric and positive definite. Due to the fact that L is compact and g_r is continuous on every fiber, it suffices to show both properties fiberwise. The statement can be concluded using a partition of unity subordinate to a local trivialization to glue the fibers together.
Because r is an algebraic curvature tensor and therefore $r(X,Y) = r(Y,X)$, we directly obtain symmetry.
To show that the perturbed metric is positive definite, it is sufficient to proof that the eigenvalues of $g_{Sa} + \varrho^r$ are strictly positive. This follows from the fact that we can diagonalize both of the representing matrices simultaneously. We already know that g_{Sa} is a metric on the normal bundle and therefore induces a metric on the tubes. The eigenvalues can be expressed by the components of r and quadratic terms in entries of W. Therefore, the positivity of the eigenvalues is guaranteed whenever $\|W\|$ and hence the diameter of the tube is small enough. The components of r can be chosen arbitrary in value. □

In the context of natural metrics some authors (see e.g. [Ana99] or [Mun08]) studied different kinds of generalized Sasaki and Cheeger-Gromoll metrics (on the tangent bundle). In [Koz08] a generalized Cheeger-Gromoll metric $h_{p,q}$ is introduced on the normal bundle in a similar manner. The idea behind this family of metrics is that with respect to these metrics the horizontal and vertical subspace are orthogonal and that the projection from the total space equipped with a natural metric is a Riemannian submersion.
As there are not many results on natural Riemannian metrics (unless on the tangent bundle), we elucidate this concept in Chapter 7 in more detail.

A comparison with the so-called general metric $g_{a,b}$ introduced by Munteanu in [Mun08], but originally constructed in [Ana99] leads to the question, whether g_r, $r \in \mathcal{R}$, is a general metric. We define this general metric on the normal bundle.

Definition 4.16 (General metric). *Let $\pi\colon NL \to L$ be the normal bundle over the Riemannian manifold $L \subset (M, g)$. Equip NL with the induced connection to define horizontal lifts for vector fields X, Y on L. Then the general metric g_N in $W \in NL$ is given by*

$$\begin{cases} g_{N,W}(X^H, Y^H) = g_L(X, Y), \\ g_{N,W}(X^H, Y^V) = 0, \\ g_{N,W}(X^V, Y^V) = a(t)g_{NL}(X, Y) + b(t)g_{NL}(X, W)g_{NL}(Y, W), \end{cases} \tag{54}$$

where $a, b\colon [0, \infty) \to [0, \infty)$ and $a > 0$.

Remark. *Note that the choice $a \equiv 1$ and $b \equiv 0$ gives the Sasaki metric.*

Now we define
$$\alpha(X, Y) := g_{NL}(X, W)g_{NL}(Y, W)$$
and note that α is the perturbation that guarantees g_N indeed differs from g_{Sa}, i.e. this is a perturbational problem.
To show which algebraic curvature tensors $r \in \mathcal{R}$ can be realized as perturbations of the form of α, we first consider the following case.

Lemma 4.17. *The 4-form $\tilde{R}$ from Definition (2.21) given by*

$$\tilde{R}(X, Y, Z, V) := \langle X, Z\rangle\langle Y, V\rangle - \langle X, Y\rangle\langle Z, V\rangle,$$

for vector fields X, Y, Z, V on L, is an algebraic curvature tensor.

Proof. We compute

$$\begin{aligned} 3 \cdot B(\tilde{R})(X, Y, Z, V) &= \tilde{R}(X, Y, Z, V) + \tilde{R}(Y, Z, X, V) + \tilde{R}(Z, X, Y, V) \\ &= \langle X, Z\rangle\langle Y, V\rangle - \langle X, Y\rangle\langle Z, V\rangle \\ &\quad + \langle Y, X\rangle\langle Z, V\rangle - \langle Y, Z\rangle\langle X, V\rangle \\ &\quad + \langle Z, Y\rangle\langle X, V\rangle - \langle Z, X\rangle\langle Y, V\rangle = 0. \end{aligned}$$

Thus $\tilde{R} \in \ker(B)$. □

Lemma 4.18. *Let $\tilde{R}$ be as above. Then $g_{\tilde{R}}$ is a general metric in the sense of Definition (4.16).*

Proof. Clearly, the relation for two horizontal vectors is satisfied and orthogonality of the horizontal and vertical subspace holds by construction. For the last equation of (54) we rewrite

$$\begin{aligned} \tilde{R}(X, W, Y, W) &= \langle X, Y\rangle\langle W, W\rangle - \langle X, W\rangle\langle Y, W\rangle \\ &= \|W\|^2 g_{NL}(X, Y) - g_{NL}(X, W)g_{NL}(Y, W). \end{aligned}$$

For any pair X^V, Y^V of vertical vectors, we obtain

$$g_{\tilde{R},W}(X^V, Y^V) = g_{NL}(X, Y) + \frac{1}{3}\tilde{R}(X, W, Y, W)$$

$$= g_{NL}(X,Y) + \frac{1}{3}\left(\|W\|^2 g_{NL}(X,Y) - g_{NL}(X,W)g_{NL}(Y,W)\right)$$
$$= \left(1 + \frac{1}{3}\|W\|^2\right) g_{NL}(X,Y) - \frac{1}{3} g_{NL}(X,W)g_{NL}(Y,W).$$

Choosing $a(t) := 1 + \frac{1}{3}t^2$ and $b(t) \equiv -\frac{1}{3}$ proves the result. □

In the following we want to use the latter construction to decide whether Lemma (4.18) is true for arbitrary $r \in \mathcal{R}$.
With the help of the Riemannian metric on L we are able to define an inner product on $\Lambda^2(TL)$ given as follows. For more information see [Bes87].

Definition 4.19. *On $\Lambda^2(TL)$ we define the inner product $\langle\langle \cdot, \cdot \rangle\rangle \colon \Lambda^2(TL) \times \Lambda^2(TL) \to \mathbb{R}$ by*

$$\langle\langle X \wedge Y, Z \wedge W \rangle\rangle := 2\left(\langle X, Z\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle\right),$$

for vector fields X, Y, Z, W on L.

Remark. *In the same way, an inner product on $\Lambda^2(E)$ for an arbitrary vector bundle E over L can be defined, e.g. the normal bundle, which is our object of interest.*

Remark. *We are able to express the sectional curvature of the plane $X \wedge Y \in \Lambda^2(TM)$ spanned by $X, Y \in \Gamma(TL)$ with the help of the above inner product. Namely,*

$$\kappa(X \wedge Y) = \frac{\langle R(X,Y)Y, X\rangle}{B(X,Y)},$$

with $B(X,Y) := \frac{1}{2}\langle\langle X \wedge Y, X \wedge Y \rangle\rangle$.

We want to explore the connection between this inner product and an element $r \in \mathcal{R} \subset S^2(\Lambda^2(NL))$. Note that any algebraic curvature tensor is a symmetric bilinear form in $S^2(\Lambda^2(NL))$. We obtain the following description.

Lemma 4.20. *Let $r \in S^2(\Lambda^2(NL))$ be a symmetric bilinear form. Then there exists a symmetric endomorphism $M_r \in \mathrm{End}(\Lambda^2(NL))$ associated to the form r via*

$$r(X,Y,Z,W) = \langle\langle X \wedge Y, M_r(Z \wedge W)\rangle\rangle.$$

Vice versa for every symmetric endomorphism $M_r \in \mathrm{End}(\Lambda^2(NL))$ there is an associated 4-form $\tilde{r} \in \mathcal{R}$.

Proof. The first part is clear from the universal property of the exterior algebra and the definition of the inner product. To obtain the next result let $M_r \in \mathrm{End}(\Lambda^2(NL)$, then there exist normal vector fields U and V on L, such that for $Z \wedge W \in \Lambda^2 NL$ the relation $M_r(Z \wedge W) = U \wedge V$ holds. Then we obtain

$$\langle\langle X \wedge Y, M_r(Z \wedge W)\rangle\rangle = \langle\langle X \wedge Y, U \wedge V\rangle\rangle$$
$$= 2\langle X, U\rangle\langle Y, V\rangle - 2\langle Y, U\rangle\langle X, U\rangle =: \frac{1}{2}\tilde{r}(X,Y,U,V).$$

A simple computation shows $\tilde{r} \in \ker(B)$. □

Definition 4.21. *We call M_r the* ***algebraic curvature operator*** *associated to $r \in \mathcal{R}$.*

For every algebraic curvature operator there exists a metric g_{M_r} on a suitable neighborhood $U(0) \subset NL$ in a way that M_r is the curvature operator of g_{M_r} in zero.
Therefore, define

$$g_{M_r,W}(X,Y) := g_L(X,Y) - \frac{1}{3}\langle\langle X \wedge W, M_r(Y \wedge W)\rangle\rangle,$$

which is a symmetric bilinear form for each $W \in NL$. On a sufficiently small tubular neighborhood we obtain that $g_{M_r,W}$ is positive definite, such that it indeed defines a Riemannian metric.

In conclusion we remark that for every algebraic curvature tensor r we can find an $\varepsilon_0 > 0$ and an embedding L into $(L(\varepsilon_0), g_{M_r,W})$ such that $g_{M_r,W}$ is a Riemannian metric on the tube and the asymptotic result is exactly the curvature tensor we started with. We obtain the same result for every $0 < \varepsilon < \varepsilon_0$. Therefore, all of those metrics can be seen as general metrics on the normal bundle.

From this point of view the following observations are interesting as we study the original setting with another admissible choice of metric involved. Instead of choosing another reference metric we perturb the natural choice of metric. First note the following.

Remark. 1. *There are more possibilities than Definition (4.14) to proceed. Another natural way to define the perturbed metric could be*

$$g_{r,W}(x,w) := g_{Sa,W} + \delta\rho_{E_x,W}.$$

By choosing δ sufficiently small, this defines a metric on the whole normal bundle NL.

2. *There is no restriction on the curvature tensor r_W chosen for the perturbation, because whenever the diameter of the tube is small enough, we obtain positivity and hence a metric on $L(\varepsilon)$, $0 < \varepsilon < \varepsilon_0$. This means every element of $\mathcal{R}$ can be used for the perturbation. Note that this fact is closely related to the proof of Theorem (1.16), where the fact that any algebraic curvature tensor occurs as a Riemannian curvature at some point of a Riemannian manifold is given.*

From now on, let $0 < \varepsilon < \varepsilon_0$ be sufficiently small. Next, we derive additional properties of the tube equipped with the perturbed metric.

Lemma 4.22. *The map $s_0 \colon (L, g_L) \to (L(\varepsilon), g_r)$ is an isometric embedding and E is the normal bundle of this embedding.*

Proof. L is embedded into E by the zero section s_0; therefore, the tangent map $s_{0*} \colon T_xL \to T_{s_0(x)}NL$ satisfies $\pi_* \circ s_{0*} = (\pi \circ s_0)_* = \mathrm{id}_*$. By definition, we hence obtain

$$g_L(v,w) = g_r(s_{0*}(v), s_{0*}(w)),$$

for every pair of tangent vectors $v, w \in T_xL$.

□

We want to study whether we are exactly in the same situation as before. The condition that the tube projection is a Riemannian submersion is not always valid, but we will give a sufficient condition for that fact in the following theorem.

Theorem 4.23. *1. Let $L(\varepsilon)$ be equipped with g_r from Definition (4.14). Then the horizontal lift of a vector field $X = X^i\partial_i$ on L is given by*

$$X^H = X^h + X^i Z_i^\alpha \partial_\alpha,$$

where X^h denotes the horizontal lift with respect to the Sasaki metric and the coefficients Z_i are given by

$$Z_i^\alpha := \left(\delta_{\alpha\beta} + \frac{1}{3}w^\sigma w^\tau r_{\alpha\tau\beta\sigma}\right)^{-1}\left(\frac{1}{3}w^\sigma w^\tau r_{\alpha\tau\beta\sigma}w^\mu C_{i\mu}^\alpha\right). \tag{55}$$

2. The projection $\pi\colon (L(\varepsilon), g_r) \to (L, g_L)$ is a Riemannian submersion if and only if the matrix $N = (N_{ij})_{ij}$ given by

$$N_{ij} := Z_i^\alpha C_{j\nu}^\beta w^\nu \delta_{\alpha\beta}$$

vanishes identically.

Remark. *Note that the horizontal lift with respect to the Sasaki metric will always be denoted with the lowercase superscript h, i.e. for any vector field X on L the horizontal Sasaki lift is given by $X^h = X^i(\partial_i - C_{i\mu}^\alpha w^\mu \partial_\alpha)$. The horizontal lift with respect to the metric g_r will further be denoted with the capital letter H as a superscript.*

Proof. 1. Let $X = X^i\frac{\partial}{\partial x^i} = X^i\partial_i$ be a vector field on L. First we compute the horizontal subspace. By definition the perturbed metric is given as $g_r = g_{Sa} + \frac{1}{3}\varrho^r$, we hence obtain the horizontal lift of $X = X^i\partial_i$ as

$$X^H := X^h + X^i Z_i^\alpha\partial_\alpha = X^i(\partial_i - C_{i\mu}^\alpha w^\mu\partial_\alpha) + X^i Z_i^\alpha\partial_\alpha,$$

where by Z_i^α we denote the components in fiber direction occurring due to the perturbation with r. Note that Z_i^α may be a function in $\|W\|$.
By definition and the fact that $\partial_\alpha \in \ker(\pi_*)$ we obtain

$$\pi_* X^H = X.$$

Secondly, we need to verify that X^H is orthogonal with respect to g_R to any $V \in \ker(\pi_*)$, whenever the coefficients Z_i^α are given by (55).
It is

$$\begin{aligned}
\langle X^H, V\rangle_r &= \langle X^H, V\rangle_{Sa} + \frac{1}{3}\varrho^r(X^H, V)\\
&= \langle X^h + X^i Z_i^\alpha\partial_\alpha, V\rangle_{Sa} + \frac{1}{3}\varrho^r\left(X^h + X^i Z_i^\alpha\partial_\alpha, V\right)\\
&= \langle X^i Z_i^\alpha\partial_\alpha, V^\beta\partial_\beta\rangle_{Sa} + \frac{1}{3}\varrho^r\left(X^i\left(Z_i^\alpha - C_{i\mu}^\alpha w^\mu\right)\partial_\alpha, V^\beta\partial_\beta\right)\\
&= X^i Z_i^\alpha V^\beta\langle\partial_\alpha, \partial_\beta\rangle_{Sa} + \frac{1}{3}\langle X^i(Z_i^\alpha - C_{i\mu}^\alpha w^\mu)\partial_\alpha, r(V^\beta\partial_\beta, W)W\rangle_{Sa}\\
&= X^i Z_i^\alpha V^\beta\delta_{\alpha\beta} + \frac{1}{3}X^i\left(Z_i^\alpha - C_{i\mu}^\alpha w^\mu\right)V^\beta w^\sigma w^\tau\langle\partial_\alpha, r(\partial_\beta, \partial_\sigma)\partial_\tau\rangle_{Sa}\\
&= X^i V^\beta\left(Z_i^\alpha\delta_{\alpha\beta} + \frac{1}{3}w^\sigma w^\tau r_{\alpha\tau\beta\sigma}\left(Z_i^\alpha - C_{i\mu}^\alpha w^\mu\right)\right).
\end{aligned}$$

Therefore, we obtain a system of linear equations for the coefficients Z for each pair of i, β given by

$$Z_i^\alpha \delta_{\alpha\beta} + \frac{1}{3} w^\sigma w^\tau r_{\alpha\tau\beta\sigma} \left(Z_i^\alpha - C_{i\mu}^\alpha w^\mu \right) = 0.$$

Equivalently,

$$\left(\delta_{\alpha\beta} + \frac{1}{3} w^\sigma w^\tau r_{\alpha\tau\beta\sigma} \right) Z_i^\alpha - \frac{1}{3} w^\sigma w^\tau r_{\alpha\tau\beta\sigma} w^\mu C_{i\mu}^\alpha = 0.$$

Due to the fact that $\delta_{\alpha\beta} + \frac{1}{3} w^\sigma w^\tau r_{\alpha\tau\beta\sigma}$ is invertible whenever the radius of the tube is small enough, we obtain

$$Z_i^\alpha := \left(\delta_{\alpha\beta} + \frac{1}{3} w^\sigma w^\tau r_{\alpha\tau\beta\sigma} \right)^{-1} \left(\frac{1}{3} w^\sigma w^\tau r_{\alpha\tau\beta\sigma} w^\mu C_{i\mu}^\alpha \right)$$

as desired.
If there exists a choice of coefficients Z_i^α, then this will be the only possible choice.

2. It remains to prove, that g_r is an isometry on the horizontal space, i.e.

$$\langle X^H, Y^H \rangle_r = \langle X, Y \rangle_L.$$

We use the decomposition

$$Y^H = Y^j \partial_j + V_Y,$$

where $V_Y := -Y^j C_{j\nu}^\beta w^\nu \partial_\beta + Y^j Z_j^\beta \partial_\beta \in \ker(\pi_*)$. Moreover, we use that X^H and V_Y are orthogonal with respect to g_r. We derive

$$\begin{aligned}
\langle X^H, Y^H \rangle_r &= \langle X^H, Y^j \partial_j + V_Y \rangle_r \\
&= \langle X^H, Y^j \partial_j \rangle_r + \langle X^H, V_Y \rangle_r \\
&= \langle X^H, Y^j \partial_j \rangle_{Sa} + \frac{1}{3} \varrho^r \left(p_W X^H, p_W Y^j \partial_j \right).
\end{aligned}$$

Because of the properties of the projection map p, the second part vanishes. Using the above decomposition representation again we obtain

$$\begin{aligned}
\langle X^H, Y^H \rangle_r &= \langle X^h, Y^h \rangle_{Sa} + \langle X^i Z_i^\alpha \partial_\alpha, Y^j C_{j\nu}^\beta w^\nu \partial_\beta \rangle_{Sa} \\
&= \langle X, Y \rangle_L + X^i Y^j Z_i^\alpha C_{j\nu}^\beta w^\nu \delta_{\alpha\beta},
\end{aligned}$$

which completes the proof.

□

Remark. 1. *If the normal bundle is flat, the connection coefficients will vanish; therefore, N is identically zero and π is always a Riemannian submersion.*

2. *Whenever all coefficients Z_i^α vanish, we obtain the initial situation, i.e. no perturbation on the fibers.*

3. *From the representation of the coefficients Z_i^α we know that they need to be of infinite order in $\|W\|$, otherwise the orthogonality assumption would never be satisfied.*

In analogy to [Wit11], Chapter 2, we state the following facts.

Definition 4.24. *For an arbitrary metric $\tilde{g}$ on the tubular neighborhood $L(1)$ we define by*

$$\tilde{g}(\varepsilon)\colon\ TL(1)\times TL(1)\to\mathbb{R};\ \tilde{g}(\varepsilon)(X,Y)=\tilde{g}(\sigma_{\varepsilon *}X,\sigma_{\varepsilon *}Y), \tag{56}$$

*i.e. $\tilde{g}(\varepsilon):=\sigma_\varepsilon^*\tilde{g}$, the **rescaled metric**.*

Because the inverse metric corresponds to the principal symbol of the Laplacian, we compute the rescaling of $g_r^*=g_r^{-1}$.
Due to [Wit11], Lemma 4, for an arbitrary chosen metric $\tilde{g}$ and corresponding curvature tensor R we obtain

$$\begin{aligned}\tilde{g}^*(\varepsilon,x,W)&=\left(\begin{array}{c|c}1&0\\\hline 0&\varepsilon^{-1}\end{array}\right)\tilde{g}^*(x,\varepsilon W)\left(\begin{array}{c|c}1&0\\\hline 0&\varepsilon^{-1}\end{array}\right)\\&=g_{Sa}^*(\varepsilon,x,W)-\frac{1}{3}\left(\begin{array}{c|c}0&0\\\hline 0&g_{Sa}^{-1}R_Wg_{Sa}^{-1}\end{array}\right)+O(\varepsilon),\quad\text{as }\varepsilon\to0,\end{aligned}$$

with $R_WX:=R(X,W)W$.

Remark. *Again note that the Landau notation is used for $\varepsilon\to0$ (without further mentioning).*

Therefore, the principal symbol of the Laplacian of $\tilde{g}$ under the rescaling behaves like

$$\tilde{g}^*(\varepsilon)=g_{Sa}^*(\varepsilon)-\frac{1}{3}r_{F,W}+O(\varepsilon).$$

Here $r_{F,W}\colon T^*E\times T^*E\to\mathbb{R}$ is the symmetric bilinear form (on the fibers) given by

$$r_{F,W}(x,W)(X,Y)=\langle P^W(X),R^E(W,P^W(Y))W\rangle,$$

with P^W denoting the dual of the composition of parallel translation in T^*E and the orthogonal projection onto N^*L.
We finally obtain

$$\lim_{\varepsilon\to0}\left[-3\left(\tilde{g}^*(\varepsilon)-g_{Sa}^*(\varepsilon)\right)\right]=r_{F,W} \tag{57}$$

for an arbitrary metric $\tilde{g}$.

We now compute the asymptotic result for the dual perturbed metric g_r^*, showing that this expression depends only on the bilinear form ϱ^r, i.e. the chosen algebraic curvature tensor $r\in\mathcal{R}$ from Definition (4.13).
By Neumann series, we obtain

$$\begin{aligned}g_r^*={g_r}^{-1}&=\left(g_{Sa}+\frac{1}{3}\left(\begin{array}{c|c}0&0\\\hline 0&r_{W\alpha\beta}\end{array}\right)\right)^{-1}\\&=\left(g_{Sa}\left(1+\frac{1}{3}g_{Sa}^{-1}\left(\begin{array}{c|c}0&0\\\hline 0&r_{W\alpha\beta}\end{array}\right)\right)\right)^{-1}\\&=\left(1+\frac{1}{3}g_{Sa}^{-1}\left(\begin{array}{c|c}0&0\\\hline 0&r_{W\alpha\beta}\end{array}\right)\right)^{-1}\cdot g_{Sa}^{-1}\\&=\left(1-\frac{1}{3}g_{Sa}^{-1}\left(\begin{array}{c|c}0&0\\\hline 0&r_{W\alpha\beta}\end{array}\right)+O\left(\|W\|^3\right)\right)g_{Sa}^{-1}\end{aligned}$$

$$
\begin{aligned}
&= g_{Sa}^{-1} - \frac{1}{3} g_{Sa}^{-1} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & r_{W\,\alpha\beta} \end{array} \right) g_{Sa}^{-1} + O\left(\|W\|^3\right) \\
&= \left(\begin{array}{c|c} g_L^{ij} & -w^\mu g_L^{is} C_{s\mu}^\beta \\ \hline -w^\nu g_L^{sj} C_{s\nu}^\alpha & \delta^{\alpha\beta} + w^\mu w^\nu g_L^{ij} C_{i\mu}^\alpha C_{j\nu}^\beta - \frac{1}{3} r_{W\,\alpha\beta} \end{array} \right) + O\left(\|W\|^3\right).
\end{aligned}
$$

For simplicity in this computation and in the following ones, we use the abbreviation

$$
r_{W\,\alpha\beta} := w^\mu w^\nu r_{W\,\alpha\mu\beta\nu}.
$$

To compute the limit (57) we use a result from [Wit11], p. 24 ff. for the renormalized Sasaki metric. We obtain

$$
\begin{aligned}
&-3(g_r^* - g_{Sa}^*)(\varepsilon) \\
&= -3\left(-\frac{1}{3} \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varepsilon^{-1} \end{array} \right) g_{Sa}^{-1}(x, \varepsilon w) \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & r_{\varepsilon W\,\alpha\beta} \end{array} \right) g_{Sa}^{-1}(x, \varepsilon w) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varepsilon^{-1} \end{array} \right) \right) \\
&= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varepsilon^{-1} \end{array} \right) g_{Sa}^{-1}(x, \varepsilon w) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varepsilon^{-1} \end{array} \right) \cdot \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \varepsilon^2 r_{\varepsilon W\,\alpha\beta} \end{array} \right) \cdot \\
&\quad \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varepsilon^{-1} \end{array} \right) g_{Sa}^{-1}(x, \varepsilon w) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varepsilon^{-1} \end{array} \right) \\
&= g_{Sa}^{-1}(\varepsilon, x, w) \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \varepsilon^2 r_{\varepsilon W\,\alpha\beta} \end{array} \right) g_{Sa}^{-1}(\varepsilon, x, w) \\
&= \left(\begin{array}{c|c} O(\varepsilon^4) & O(\varepsilon^2) \\ \hline O(\varepsilon^2) & \varepsilon^{-2} r_{\varepsilon W\,\alpha\beta} + O(\varepsilon^2) \end{array} \right) \\
&= \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \varepsilon^{-2} r_{\varepsilon W\,\alpha\beta} \end{array} \right) + O(\varepsilon^2),
\end{aligned}
$$

where

$$
g_{Sa}^{-1}(\varepsilon, x, w) := \left(\begin{array}{c|c} g_L^{ij} & -w^\mu g_L^{is} C_{s\mu}^\beta \\ \hline -w^\nu g_L^{sj} C_{s\nu}^\alpha & \varepsilon^{-2}\delta^{\alpha\beta} + w^\mu w^\nu g_L^{ij} C_{i\mu}^\alpha C_{j\nu}^\beta \end{array} \right).
$$

Because

$$
\varepsilon^{-2} r_{\varepsilon W\,\alpha\beta} \in O(1),
$$

we obtain that the zero order correction term of the asymptotic corresponds to the algebraic curvature tensor we had chosen in the beginning and hence we have the following statement.

Proposition 4.25. *The perturbation of the metric is defined fiber-wise, depending on $x \in L$. The dependency on the base point is continuous, so that r is a section in the bundle of all algebraic curvature forms over L. Every such r can be realized as the asymptotic result for the above construction.*

Remark. *Due to the fact that the second fundamental form II vanishes at the base points $x \in L$ of the fibers, by Gauss' embedding theorem (see Theorem (2.1)) the curvature of the ambient manifold equals the curvature tensor of the fibers, when inserting normal vectors on N_xL.*

We now want to compute the corresponding potential to the process, where the tube is equipped with the perturbed metric. For the computation we first evaluate the determinant of the perturbed metric and its inverse. We restrict ourselves to the computation of the relevant asymptotic terms to simplify and shorten the computation.

Lemma 4.26. *Let $r \in \mathcal{R}$. Denote by m_r and m_{Sa} the Riemannian volume measures with respect to the perturbed metric g_r and g_{Sa}, respectively. Then their Radon-Nikodym density ρ_r is given by*

$$\rho_r = 1 + \frac{1}{6} w^\mu w^\nu {r_W}^\alpha{}_{\mu\alpha\nu} + O\left(\|W\|^3\right). \tag{58}$$

Remark. 1. *Remember that ρ always denotes a Radon-Nikodym density, while ϱ denotes the bilinear form which induces the perturbation.*

2. *Recall that by the product representation of the Sasaki metric given in (5) and by definition of the density the relations $\det(g_{Sa}) = \det(g_L)$ and $\rho^r \sqrt{\det(g_{Sa})} = \sqrt{\det(g_r)}$ hold.*

Note that we replace the density ρ^r by ρ throughout the following proofs to simplify the notation.

Proof. In local Fermi coordinates using (53) and the remark above we can compute ρ^2 by

$$\begin{aligned}\rho^2 = \frac{\det(g_r)}{\det(g_{Sa})} &= \frac{\det\left(g_{Sa} + \frac{1}{3}\varrho^r_W(x)_{\alpha\beta}\right)}{\det(g_{Sa})} \\ &= \det\left(1 + \frac{1}{3} g_{Sa}^{-1} \varrho^r_W(x)_{\alpha\beta}\right).\end{aligned}$$

Therefore, we need to evaluate

$$\frac{1}{3} g_{Sa}^{-1} \varrho^r(x)_{\alpha\beta}$$

explicitly in local Fermi coordinates. Combining the representation of g_{Sa}^{-1} due to (6) and the representation of the bilinear form ϱ in (53) we conclude

$$g_{Sa}^{-1} \varrho^r_{\alpha\beta} = \left(\begin{array}{c|c} 0 & O\left(\|W\|^3\right) \\ \hline 0 & \delta^{\alpha\beta} r_{W\,\alpha\mu\beta\nu} w^\mu w^\nu + O\left(\|W\|^4\right) \end{array}\right).$$

That gives

$$\begin{aligned}\rho^2 &= \det\left(1 + \frac{1}{3}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & {r_W}^\alpha{}_{\mu\beta\nu} w^\mu w^\nu \end{array}\right) + O\left(\|W\|^3\right)\right) \\ &= \det\left(\exp\left(\frac{1}{3}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & {r_W}^\alpha{}_{\mu\beta\nu} w^\mu w^\nu \end{array}\right)\right) + O\left(\|W\|^3\right)\right).\end{aligned}$$

Now using the definition of the power series of the exponential function and the fact that for any matrix A the identity $\det(\exp(A)) = \exp(\mathrm{tr}(A))$ holds, we finally obtain

$$\begin{aligned}\rho^2 &= \exp\left(\frac{1}{3}\mathrm{tr}\left(\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & {r_W}^\alpha{}_{\mu\beta\nu} w^\mu w^\nu \end{array}\right)\right)\right) + O\left(\|W\|^3\right) \\ &= \exp\left(\frac{1}{3}\sum_\alpha {r_W}^\alpha{}_{\mu\alpha\nu} w^\mu w^\nu\right) + O\left(\|W\|^3\right).\end{aligned}$$

Hence we conclude

$$\rho = \exp\left(\frac{1}{6}\sum_\alpha {r_W}^\alpha{}_{\mu\alpha\nu} w^\mu w^\nu\right) + O\left(\|W\|^3\right),$$

which we denote by

$$\rho = 1 + \frac{1}{6} r_W{}^{22} + O\left(\|W\|^3\right),$$

following the notations above. Using Einstein summation convention yields the statement.

□

Remark. *Lemma (4.26) implies*

$$\log \rho_r = -\frac{1}{6} r_W^{22} + O\left(\|W\|^3\right)$$

in consistency to (45)*, but note that there is a sign shift when comparing the results. This is due to the asymptotic in (57).*

In the original formula for the density the Weingarten map occurred as the term of first order in $\|W\|$. Because there is no first order term in the above representation of ρ_r, it seems as if we had constructed a totally geodesic embedding.

We do not know about the higher order terms and whether they involve relations about the second fundamental form or the tension field of the embedding. But nevertheless we are able to proof the following.

Lemma 4.27. *The embedding* $s_0\colon L \to (L(\varepsilon), g_r)$ *is totally geodesic.*

Proof. We need to compute the Christoffel symbols of both connections. Denote the Christoffel symbols of ∇^{g_r} by ${}^r\Gamma^k_{ij}$.
We use the definition

$$\mathrm{II}(s_0)(X,Y) = \nabla^{g_r}_X Y - \nabla^L_X Y.$$

We first want to show that on the submanifold L, i.e. at every point $p = (x,0) \in NL$, the corresponding connection coefficients coincide. In both cases by construction the connection is the Levi-Civita connection, which means we have to compute the Christoffel symbols ${}^r\Gamma^k_{ij}$ and show they equal Γ^k_{ij} on L.
In local coordinates we have

$${}^r\Gamma^k_{ij}(x,0) = \frac{1}{2} g_r^{kA} \left(\frac{\partial g_{r\,jA}}{\partial x^i} + \frac{\partial g_{r\,iA}}{\partial x^j} - \frac{\partial g_{r\,ij}}{\partial x^A} \right)(x,0).$$

We are going to use that in local coordinates the equation

$$g_{r|L} = \left(\begin{array}{c|c} g_L & 0 \\ \hline 0 & 1 \end{array} \right)$$

holds. Using $g_r = g_{Sa} + O\left(\|W\|^2\right)$, we obtain

$$g_r{}^{kA} \frac{\partial g_{r\,ij}}{\partial x^A} = g_r{}^{ks} \frac{\partial g_{r\,ij}}{\partial x^s} + g_r{}^{k\alpha} \frac{\partial g_{r\,ij}}{\partial w^\alpha} = g_r{}^{ks} \frac{\partial g_{r\,ij}}{\partial x^s}.$$

Moreover, the off-diagonal components of g_r depend on W, which means by covariant differentiation in direction of the submanifold they are still of order $O(\|W\|)$. Inserting $w^\alpha = 0$ leads to

$$g_r{}^{kA} \frac{\partial g_{r\,jA}}{\partial x^i} = g_r{}^{ks} \frac{\partial g_{r\,js}}{\partial x^i} \text{ and } g_r{}^{kA} \frac{\partial g_{r\,iA}}{\partial x^j} = g_r{}^{ks} \frac{\partial g_{r\,is}}{\partial x^j}.$$

Therefore, we finally obtain that

$$ {}^r\Gamma^k_{ij}(x,0) = \Gamma^k_{ij}(x,0). $$

It remains to compute the Christoffel symbols ${}^r\Gamma^\alpha_{ij}$. By definition,

$$ \begin{aligned} {}^r\Gamma^\alpha_{ij}(x,0) =& \frac{1}{2} g_r^{\alpha A} \left(\frac{\partial g_{rjA}}{\partial x^i} + \frac{\partial g_{riA}}{\partial x^j} - \frac{\partial g_{rij}}{\partial x^A} \right)(x,0) \\ =& \frac{1}{2} g_r^{\alpha s} \left(\frac{\partial g_{rjs}}{\partial x^i} + \frac{\partial g_{ris}}{\partial x^j} - \frac{\partial g_{rij}}{\partial x^s} \right)(x,0) + \frac{1}{2} g_r^{\alpha\beta} \left(\frac{\partial g_{rj\beta}}{\partial x^i} + \frac{\partial g_{ri\beta}}{\partial x^j} - \frac{\partial g_{rij}}{\partial w^\beta} \right)(x,0). \end{aligned} $$

The first part equals zero by the same arguments used above. Obviously, this also holds for the first two summands in the second part. For the last remaining term we use the fact that $g_{rij} = g_{Lij} + O\left(\|W\|^2\right)$ and hence obtain it is zero as well. Finally, we have ${}^r\Gamma^\alpha_{ij} = 0$ for all combinations of indices. □

Remark. *If the tube is equipped with the Sasaki metric itself, the embedding is indeed totally geodesic as well.*

Proposition 4.28. *For $\rho := \rho_r = \sqrt{\frac{\det g_r}{\det g_{Sa}}}$ given by (58) the potential is given by*

$$ W_{r\varepsilon} = -\frac{1}{12}\mathrm{Scal}_{F_r} + O(\varepsilon), \tag{59} $$

where Scal_{F_r} denotes the scalar curvature of the fibers equipped with the perturbed metric g_r.

Remark. *Note that $\mathrm{Scal}_{F_r} = \mathrm{Scal}_{F_r}(\pi(W))$ denotes the scalar curvature of the fiber at its base point $x = \pi(W) \in L$.*

Proof. In local Fermi coordinates we already know the expressions for the perturbed metric. Moreover, we already computed the dual perturbed metric in local Fermi coordinates. We use the representation for the density given by (58) to express the logarithmic density in local Fermi coordinates and we obtain

$$ \rho = 1 + \frac{1}{6} w^\mu w^\nu {r_W}^\alpha{}_{\mu\alpha\nu} + O\left(\|W\|^3\right) = \exp\left(\frac{1}{6} {r_W}^\alpha{}_{\mu\alpha\nu} w^\mu w^\nu \right) + O\left(\|W\|^3\right). $$

Therefore,

$$ \log\rho = \frac{1}{6} {r_W}^\alpha{}_{\mu\alpha\nu} w^\mu w^\nu + O\left(\|W\|^3\right). \tag{60} $$

As $g_r^{\alpha\beta} = \delta^{\alpha\beta} + O\left(\|W\|^2\right)$ and the derivatives in direction of the submanifold can be disregarded, we obtain

$$ \begin{aligned} \triangle_r \log\rho &= \frac{1}{\sqrt{\det(g_r)}} \partial_A \left(g^{AB} \sqrt{\det g_r} \partial_B \log\rho \right) \\ &= \frac{1}{\rho\sqrt{\det g_L}} \partial_A \left(g^{AB} \rho \sqrt{\det g_L} \partial_B \log\rho \right) \\ &= \frac{1}{\rho\sqrt{\det g_L}} \partial_A \left(g^{AB} \sqrt{\det g_L} \partial_B \rho \right) \\ &= \frac{1}{\rho\sqrt{\det g_L}} \partial_\sigma \left(g^{\sigma\tau} \sqrt{\det g_L} \partial_\tau \rho \right) + O\left(\|W\|\right) \end{aligned} $$

$$
\begin{aligned}
&= \frac{1}{\rho\sqrt{\det g_L}} \partial_\sigma \left(\delta^{\sigma\tau} \sqrt{\det g_L} \partial_\tau \rho \right) + O\left(\|W\|\right) \\
&= \frac{1}{\rho\sqrt{\det g_L}} \left(\left(\partial_\sigma \delta^{\sigma\tau}\right) \sqrt{\det(g_L)} \partial_\tau \rho + \delta^{\sigma\tau} \left(\partial_\sigma \sqrt{\det g_L} \right) \partial_\tau \rho \right. \\
&\qquad\qquad \left. + \delta^{\sigma\tau} \sqrt{\det g_L} \partial_\sigma \partial_\tau \rho \right) + O\left(\|W\|\right) \\
&= \frac{1}{\rho\sqrt{\det g_L}} \left(\delta^{\sigma\tau} \sqrt{\det g_L} \partial_\sigma \partial_\tau \rho \right) + O\left(\|W\|\right) \\
&= \frac{1}{\rho} \sum_\tau \partial_{\tau\tau} \rho + O\left(\|W\|\right) = \sum_\tau \partial_{\tau\tau} \rho + O\left(\|W\|\right),
\end{aligned}
$$

because $\partial_\alpha \sqrt{\det g_L} = 0$. Moreover, $\log \rho = O\left(\|W\|^2\right)$ holds and hence it is

$$
d \, \log \rho = O\left(\|W\|\right).
$$

Thus, inserting in (48) we obtain

$$
\begin{aligned}
W_r &= -\frac{1}{2} \sum_\tau \partial_{\tau\tau} \rho + O\left(\|W\|\right) \\
&= -\frac{1}{2} \partial_{\tau\tau} \left(1 + \frac{1}{6} w^\mu w^\nu {r_W}^\alpha{}_{\mu\alpha\nu} \right) + O\left(\|W\|\right) \\
&= -\frac{1}{12} \mathrm{Scal}_{F_r} + O\left(\|W\|\right).
\end{aligned}
$$

□

4.3.2 Realization of a smooth function as potential W_0

4.3.2.1 Codimension $k > 1$

The result of Proposition (4.28) is a very simple formula for the potential. This leads to the question, if there exists an embedding of a given closed Riemannian manifold L with an arbitrary smooth function $W_0 \in C^\infty(L)$ into an ambient space, such that the given function is the potential of the above limit process.

Remark. *For every limit process the result is always described by a heat equation and a corresponding Schrödinger operator with potential.*

We prove the above conjecture true, by using a modification of the construction in Section (4.3.1). Then we repeat the computations of the previous sections.

Definition 4.29. *Let $f \in C^\infty(L)$ be smooth (and hence bounded). Define by*

$$
g^f_{r,W} := g_{Sa,W} + \frac{1}{3} f \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & r_{W\,\alpha\mu\beta\nu} w^\mu w^\nu \end{array} \right)
$$

for $W \in E$ another metric on the tubular neighborhood $L(\varepsilon)$.

For the rest of this chapter we study the embedding $(L, g_L) \to (L(\varepsilon), g^f_r)$.

Remark. *Note that the boundedness of f assures that g^f_r is positive definite on a tube $L(\varepsilon)$ of small enough diameter and hence defines a metric.*

Proposition 4.30. *Let $r_W \in \mathcal{R}$ be an algebraic curvature tensor and $f \in C_b^\infty(L)$. Then $f \cdot r_W$ defines an algebraic curvature tensor as well.*

Proof. According to the assumptions on f we obtain that $f \cdot r_W \in S^2(\Lambda^2 E^*)$. Due to the fact that the Bianchi map B is linear we obtain

$$B(f \cdot r_W)(X,Y,Z,W) = f \cdot B(r_W)(X,Y,Z,W) = 0$$

and therefore $f \cdot r_W \in \mathcal{R}$.

□

We now want to compute the associated density and the potential and obtain

$$\begin{aligned}\rho_f^2 &= \frac{\det\left(g_r^f\right)}{\det(g_{Sa})} \\ &= \det\left(1 + \frac{1}{3} f g_{Sa}^{-1} \varrho^r(x)_{\alpha\beta}\right) \\ &= \exp\left(\frac{1}{6} f r_{W\,\alpha\mu\alpha\nu}\right) + O\left(\|W\|^3\right) &(61)\\ &= 1 + \frac{1}{6} \cdot f \cdot r_{W\,\alpha\mu\alpha\nu} + O\left(\|W\|^3\right). &(62)\end{aligned}$$

Hence,

$$\log \rho_f = f \cdot \log \rho^r + O\left(\|W\|^3\right).$$

As $f \in C_b^\infty(L)$ and $\partial_{\alpha\alpha} f = 0$ holds, we obtain

$$W^f = -\frac{1}{12} f \cdot \mathrm{Scal}_r + O\left(\|W\|\right).$$

By a suitably chosen r_W with non-zero, constant scalar curvature Scal_r we are able to realize every potential $W^f \in C^\infty(L)$ by defining

$$f := -12 \frac{W^f}{\mathrm{Scal}_r}.$$

Due to the fact that the scalar curvature is non-zero, this defines a smooth function on L. Our goal is to find such a curvature tensor r_W.

Remark. *Note that constant scalar curvature is not even necessary but can be assumed by the following computation.*

Proposition 4.31. *Let $\pi\colon E \to L$ be a vector bundle and b a bundle metric on E. Then $b \owedge b \in \mathcal{R}$.*

Proof. Clearly, $b \owedge b \in S^2(\Lambda^2 E^*)$. Let now $X, Y, Z, W \in \Gamma(E)$, then

$$\begin{aligned}3B(b \owedge b)(X,Y,Z,W) &= b \owedge b(X,Y,Z,W) + b \owedge b(Y,Z,X,W) + b \owedge b(Z,X,Y,W) \\ &= 2\left(b(X,Z)b(Y,W) - b(X,W)b(Y,Z)\right) \\ &\quad + 2\left(b(Y,X)b(Z,W) - b(Y,W)b(Z,X)\right) \\ &\quad + 2\left(b(Z,Y)b(X,W) - b(Z,W)b(X,Y)\right) = 0.\end{aligned}$$

□

Definition 4.32. *Let $\pi\colon E \to L$ be equipped with a bundle metric b and $f \in C_b^\infty(L)$. In local Fermi coordinates define*

$$g_f(x,w) := g_{Sa}(x,w) + \frac{1}{6} f w^\mu w^\nu \left(b_W \owedge b_W\right)_{\alpha\mu\beta\nu}. \tag{63}$$

We are now able to prove the main result of this section.

Theorem 4.33. *Let (L, g_L) be a closed Riemannian manifold and $W_0 \in C^\infty(L)$ a smooth potential. Then there exists a vector bundle $\pi\colon E \to L$ of rank $k > 1$ over L, an algebraic curvature tensor $r \in \mathcal{R}$ and a bilinear form ϱ^r on the fibers E_x, $x \in L$, such that there is an embedding of L into $(L(\varepsilon), g_r^f)$ with associated potential W_0 according to Theorem (1.26) and g_r^f the metric from Definition (4.29).*

Proof. Let $\pi\colon E \to L$ be an arbitrary vector bundle of rank greater than one over L. Choose a bundle metric b on E. Then by Proposition (4.31) we know that the Kulkarni-Nomizu product $b \owedge b$ is an algebraic curvature tensor. Let $\{e_\alpha\}_{\alpha=1,\dots,k}$, where $k = m - l$, be an orthonormal frame of E and $x = \pi W \in L$, $W \in E$ and $X = v^\alpha \partial_\alpha, Y = u^\beta \partial_\beta \in T_W E_{|L}$. Using the definition of ∂_α from (2) yields

$$\begin{aligned} b_W(X,Y) &= \langle v^\alpha \partial_\alpha, u^\beta \partial_\beta \rangle_{E_x, W} = \langle K_W^\perp(v^\alpha \partial_\alpha), K_W^\perp(u^\beta \partial_\beta) \rangle_{\pi W} \\ &= v^\alpha u^\beta \langle K_W^\perp(\partial_\alpha), K_W^\perp(\partial_\beta) \rangle_{\pi W} = v^\alpha u^\beta \langle e_\alpha, e_\beta \rangle_{\pi W} \\ &= v^\alpha u^\beta \delta_{\alpha\beta}. \end{aligned}$$

Therefore, we obtain in local coordinates

$$\left(b_W \owedge b_W\right)_{\alpha\mu\beta\nu} = 2\left(\delta_{\alpha\beta}\delta_{\mu\nu} - \delta_{\alpha\nu}\delta_{\mu\beta}\right).$$

The corresponding scalar curvature is given by the following trace:

$$\begin{aligned} \mathrm{Scal}_r &= 2 \sum_{\alpha,\mu=1}^{m-l} \left(\delta_{\alpha\alpha}\delta_{\mu\mu} - \delta_{\alpha\mu}\delta_{\mu\alpha}\right) \\ &= 2((m-l)^2 \cdot 1^2 - (m-l) \cdot 1^2) \\ &= 2(k^2 - k) > 0. \end{aligned}$$

Due to the assumption $k > 1$ the constant $k^2 - k$ is greater than zero. We define the smooth function

$$f := -6 \frac{W_0}{k^2 - k} \in C^\infty(L)$$

and equip the tubular neighborhood $L(\varepsilon)$ with the metric g_f from (63). The result then follows from the above computations. $\square$

4.3.2.2 Codimension $k = 1$

Theorem (4.33) does not give any result for codimension one. Moreover, the given proof will not work for submanifolds of codimension one. In case the fibers are one-dimensional a perturbation of the fiber curvature is not helpful. The curvature form is a 2-form, such that in dimension one there is no possibility for a similar construction to the above case.

But with a different construction, using a new metric on the tubes, we are able to prove the result in this case as well.

Proposition 4.34. *Let (L, g_L) be a closed l-dimensional Riemannian manifold and $\widetilde{W} \in C^\infty(L)$. Then there exists a vector bundle $\pi\colon E \to L$ of rank $k = 1$ over L, a metric G on a small neighborhood $L(\varepsilon)$ of the zero section in E and an embedding of L into $(L(\varepsilon), G)$, such that $\widetilde{W}$ is the associated potential according to Theorem (1.26).*

Proof. Let L satisfy the above assumptions and $\pi\colon E \to L$ be a vector bundle of rank $k = 1$, i.e. a line bundle, over L. We obtain

$$TE_{|L} = TL \oplus NL,$$

where $NL = \langle n \rangle$ is a one-dimensional space, spanned by a normal vector field n. Equip E with a bundle metric b and a metric connection ∇^b to define the Sasaki metric in the same way as seen in the previous section. We use the Sasaki metric as reference metric as usual.

Because the connection is metric, i.e. $C^\beta_{i\alpha} = -C^\alpha_{i\beta}$ and $\alpha = \beta = 1$, we obtain $C^\beta_{i\alpha} = 0$, for all $i = 1, ..., l$. Therefore, in our case the Sasaki metric is given by

$$g_{Sa}(x, w) = \left(\begin{array}{c|c} g_L(x)_{ij} & 0 \\ \hline 0 & 1 \end{array}\right).$$

Instead of the perturbed metric from Definition (4.14) we consider another metric on the tubes $L(\varepsilon)$, because in the case of hypersurfaces the curvature of the fibers vanishes identically. Therefore, perturbing the fiber curvature is not helpful at all.

For $f \in C^\infty(L)$ define

$$\widetilde{G}(x, w) := \left(\begin{array}{c|c} g_L(x)_{ij} & 0 \\ \hline 0 & 1 + w^2 f(x) \end{array}\right),$$

which yields a metric on a suitably chosen tubular neighborhood $L(\varepsilon)$, $\varepsilon > 0$, by the same results as seen above.
It is now necessary, to compare this metric with the Sasaki metric and compute the corresponding Radon-Nikodym density and the potential.

Note that $e^{w^2 f(x)} = 1 + w^2 f(x) + O\left(\|W\|^3\right)$ and in local coordinates $W = w^\alpha e_\alpha(x)$. To obtain our result, it shows that using the metric tensor

$$G(x, w) := \left(\begin{array}{c|c} g_L(x)_{ij} & 0 \\ \hline 0 & e^{w^2 f(x)} \end{array}\right)$$

provides easier computations.

We directly dispose the higher order remainder. This is possible due to the fact that we evaluate in $w = 0$ at the end of the computation and we at most differentiate twice with respect to w. Clearly, the determinant of G is given by

$$\det(G) = e^{w^2 f(x)} \det(g_L).$$

We are now able to compute the associated density. It is

$$\rho = \rho^G = \sqrt{\frac{\det G}{\det(g_{Sa})}} = \sqrt{e^{w^2 f(x)}} = e^{\frac{1}{2}w^2 f(x)}.$$

The main step is now to compute

$$W_0 = \left(\rho^{-\frac{1}{2}} \triangle_G \, \rho^{\frac{1}{2}}\right)_{|\,L}.$$

Note that $\rho^{\frac{1}{2}} = e^{\frac{1}{4}w^2 f(x)}$ and the Laplace-Beltrami operator with respect to G is given by

$$\begin{aligned}
\triangle_G &= \frac{1}{\sqrt{\det(G)}}\partial_A \left(G^{AB}\sqrt{\det(G)}\partial_B\right) \\
&= \frac{1}{\sqrt{\det(G)}}\partial_i \left({g_L}^{ij}\sqrt{\det(G)}\partial_j\right) + \frac{1}{\sqrt{\det(G)}}\partial_w \left(e^{-w^2 f(x)}\sqrt{\det(G)}\partial_w\right),
\end{aligned}$$

since $G^{iw} = G^{wj} = 0$.
For the first term, we derive

$$\begin{aligned}
&\frac{1}{\sqrt{\det(G)}}\ \partial_i \left({g_L}^{ij}\sqrt{\det(G)}\partial_j \left(e^{\frac{1}{4}w^2 f(x)}\right)\right) \\
&= \frac{1}{\sqrt{\det(g_L)\ e^{w^2 f(x)}}}\partial_i \left({g_L}^{ij}\sqrt{\det(g_L)e^{w^2 f(x)}} \left(\frac{1}{4}w^2 \frac{\partial f}{\partial x^j} e^{\frac{1}{4}w^2 f(x)}\right)\right) \\
&= \frac{1}{4}\frac{w^2}{\sqrt{\det(g_L)e^{w^2 f(x)}}}\partial_i \left({g_L}^{ij}\sqrt{\det(g_L)e^{w^2 f(x)}}\frac{\partial f}{\partial x^j} e^{\frac{1}{4}w^2 f(x)}\right) \in O\left(\|W\|^2\right).
\end{aligned}$$

Therefore, this part vanishes if restricted to L.

The computation of the other term remains. Using the fact that $\det(g_L)$ is a function in x, we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{\det(G)}}\partial_w \left(e^{-w^2 f(x)}\sqrt{\det(G)}\partial_w \left(e^{\frac{1}{4}w^2 f(x)}\right)\right) \\
&= \frac{1}{\sqrt{\det(g_L)\ e^{w^2 f(x)}}}\partial_w \left(e^{-w^2 f(x)}\sqrt{\det(g_L) + e^{w^2 f(x)}}\frac{1}{2}f(x) w e^{\frac{1}{4}w^2 f(x)}\right) \\
&= \frac{f(x)}{2e^{\frac{1}{2}w^2 f(x)}}\partial_w \left(e^{-\frac{1}{4}w^2 f(x)}\right) \\
&= \frac{f(x)}{2}e^{-\frac{3}{4}w^2 f(x)}\left(-\frac{1}{2}w^2 f(x) + 1\right) \\
&= \frac{f(x)}{2} + O\left(\|W\|^2\right).
\end{aligned}$$

Multiplication of the previous result with $\rho^{-\frac{1}{2}}$ and restriction on L hence gives $\frac{f(x)}{2}$ for this part of the computation.
Finally, the potential is given by

$$W_0(x) = \frac{f(x)}{2}.$$

By the same arguments as in the previous case, we obtain the result when choosing $f = 2\tilde{W}$ and the embedding $L \subset (L(\varepsilon), G)$.

□

The previous two sections combined prove our main result. We summarize that for any given smooth function W on a closed Riemannian manifold (L, g_L), we can find an embedding, such that W is the associated potential. The main point here, is the construction of a suitable metric on a tubular neighborhood of the zero section in an arbitrary vector bundle over the manifold, such that L is isometrically embedded. Moreover, the total space of the bundle is the normal bundle of this embedding.

Chapter 5

Correspondence to the volume formula of Gray

In Chapter 2 we already studied the connection between the potential W_0 and the second order correction of the volume formula (24) of Gray. Note that his comparison result for the volume of the tubular neighborhood around a Riemannian submanifold L of (M, g) is given by

$$\operatorname{vol}^M(L(\varepsilon)) = \frac{(\pi\varepsilon^2)^{\frac{1}{2}(m-l)}}{\left(\frac{m-l}{2}\right)!} \int_L \left\{1 + A(x)\varepsilon^2 + O(\varepsilon^4)\right\} \mu_L(dx),$$

where for $y \in L$

$$A(x) = \frac{1}{2(m-l+2)} \left(\mathrm{Scal}_L - \sum_{i,j} R^M_{ijij} - \sum_{i,\alpha} R^M_{i\alpha i\alpha} - \frac{1}{3} \sum_{\alpha,\beta} R^M_{\alpha\beta\alpha\beta} \right)(x).$$

Remark. *The volume of an Euclidean ball of dimension $m - l$ with radius $\varepsilon > 0$ around zero is given by $\frac{(\pi\varepsilon^2)^{\frac{1}{2}(m-l)}}{(\frac{m-l}{2})!}$.*

5.1 An invariant volume formula for tubes

We want to give a formulation for the volume of a tubular neighborhood using our quantities and express the second order term in a coordinate invariant matter. Remarkably, this coordinate-free representation is basically given by the density ρ in (44), more precisely, a modified version of it.
This proves that the problem depends only on the geometry of the tube. The original idea for the computation was given in [GV81], Theorem 5.3.

In what follows $\pi\colon L(\varepsilon) \to L$ denotes the tube projection which is given in local Fermi coordinates by $\pi(x, w) = x$. Moreover, let $\omega_M = \omega_M(m)$ be the volume form of M and $\omega_L = \omega_L(x)$ the volume form of the submanifold L.

Definition 5.1. *Let $L \subset M$ be isometrically embedded and π as above. For $x \in L$ denote by $F = F_x = \pi^{-1}(x)$ the fiber over x.*

a) Let g_F and ω_F denote the metric on the fibers and the volume form of the fibers, respectively.

b) Define the ***modified density***

$$\widetilde{\rho} := \rho \cdot \det\left(g_F^{-\frac{1}{2}}\right), \tag{64}$$

where $\rho = \sqrt{\frac{\det g_M}{\det g_{Sa}}}$ *as usual.*

Remark. *We call* $\widetilde{\rho}$ *the modified density. It is uniquely determined by*

$$\widetilde{\rho} \cdot \omega_F \wedge \pi^* \omega_L = \omega_M.$$

Note that this is a coordinate-free expression.

Lemma 5.2. *There exists a coordinate-free representation for the function* A *from above, i.e.* A *can be written in the form*

$$A(x) = \frac{1}{2} \frac{d^2}{d\varepsilon^2} \left(\frac{1}{\mathrm{vol}\,(B^{m-l}(\varepsilon))} \int_{F_x(\varepsilon)} \widetilde{\rho}\, \omega_F \right)_{|\varepsilon=0}$$

for $x \in L$.

Proof. Let $\star_M$ denote the Hodge operator on M. By definition $\pi^*\omega_L \in \Lambda^l(T^*L(\varepsilon))$. Using the definition of the exterior product on forms we obtain

$$\begin{aligned} \pi^*\omega_L \wedge \star_M \pi^*\omega_L &= \langle \pi^*\omega_L, \pi^*\omega_L \rangle \omega_M \\ &= \|\pi^*\omega_L\|_M^2 \omega_M. \end{aligned}$$

Thereby,

$$\omega_M = \star_M \frac{\pi^*\omega_L}{\|\pi^*\omega_L\|_M^2} \wedge \pi^*\omega_L. \tag{65}$$

Now we develop (65) in a handy way. By definition, we deduce in local coordinates

(i) $\pi^*\omega_L(x, w) = \sqrt{\det g_L(x)}\, dx^1 \wedge \cdots \wedge dx^l$,

(ii) $\dfrac{\star_M \pi^*\omega_L}{\|\pi^*\omega_L\|_M^2}(x, w) = \alpha\, dw^1 \wedge \cdots \wedge dw^{m-l}$,

where α is some smooth function, which we want to compute now.

As a consequence of (i) and (ii), (65) becomes

$$\begin{aligned} &\sqrt{\det g_M(x,w)}\, dx^1 \wedge \cdots \wedge dx^l \wedge dw^1 \wedge \cdots \wedge dw^{m-l} \\ =&\alpha\, dw^1 \wedge \cdots \wedge dw^{m-l} \wedge \sqrt{\det g_L(x)} dx^1 \wedge \cdots \wedge dx^l, \end{aligned}$$

which means

$$\alpha = \sqrt{\frac{\det g_M(x,w)}{\det g_L(x)}} = \sqrt{\frac{\det g_M(x,w)}{\det g_{Sa}(x,w)}}.$$

The second equality is true due to Corollary (1.5) on a suitably small tubular neighborhood $L(\varepsilon), \varepsilon > 0$.

In order to obtain the statement, it is necessary to write

$$
\begin{aligned}
\frac{\star_M \pi^* \omega_L}{\|\pi^*\omega_L\|_M^2} &= \sqrt{\frac{\det g_M}{\det g_{Sa}}} dw^1 \wedge \cdots \wedge dw^{m-l} \\
&= \sqrt{\frac{\det g_M}{\det g_{Sa} \det g_F}} \sqrt{\det g_F}\, dw^1 \wedge \cdots \wedge dw^{m-l} \\
&= \widetilde{\rho}\, \omega_F .
\end{aligned}
$$

Combining the above equations with the smooth coarea formula (see [Cha06], Chapter III), we deduce that

$$
\begin{aligned}
\mathrm{vol}^M(L(\varepsilon)) &= \int_{L(\varepsilon)} \omega_M \\
&= \int_{L(\varepsilon)} \star_M \left(\frac{\pi^*\omega_L}{\|\pi^*\omega_L\|_M^2} \right) \wedge \pi^*\omega_L \\
&= \int_{L(\varepsilon)} \widetilde{\rho}\, \omega_F \wedge \pi^*\omega_L \\
&= \int_L \omega_L \int_{F_x(\varepsilon)} \widetilde{\rho}\, \omega_F
\end{aligned}
$$

using the fact that

$$
f(x,\varepsilon) := \int_{F_x(\varepsilon)} \star_M \left(\frac{\pi^*\omega_L}{\|\pi^*\omega_L\|_M^2} \right) : L \to \mathbb{R}
$$

is continuous for every $0 < \varepsilon \leq 1$.

It remains to compare this result with Gray's formula (24), which yields

$$
\int_{F_x(\varepsilon)} \widetilde{\rho}\, \omega_F = \mathrm{vol}(B^{m-l}(\varepsilon)) \left(1 + A(x)\varepsilon^2 + O\left(\varepsilon^4\right)\right).
$$

Now it suffices to differentiate both sides with respect to ε twice and evaluate the terms at $\varepsilon = 0$.

□

The coordinate-free representation of A given here is not the only remarkable modification we want to employ. With the help of Proposition (3.24) we obtain a simple representation, i.e. in terms of orders of $\|W\|$.

In local Fermi coordinates formula (44) is equal to

$$
\rho(x,w) = 1 + \rho_\alpha(x) w^\alpha + \frac{1}{2}\rho_{\alpha\beta} w^\alpha w^\beta + O\left(\|W\|^3\right).
$$

To shorten notation we write dW instead of $dw^1 \wedge \cdots \wedge dw^{m-l}$. Combining these expressions with the invariant formulation above yields

$$
\int_{F_x(\varepsilon)} \widetilde{\rho}\, \omega_F = \int_{F_x(\varepsilon)} \rho(x,w)\, dW
$$

$$= \int_{\{\|W\|<\varepsilon\}} \left(1 + \rho_\alpha(x) w^\alpha + \frac{1}{2}\rho_{\alpha\beta} w^\alpha w^\beta\right) \, dW + O\left(\varepsilon^3\right), \varepsilon \to 0.$$

By symmetry arguments the first order term cancels out after integration over the domain, i.e. we remain with the computation of the zero and second order terms. The task now is to evaluate

$$\int_{F_x(\varepsilon)} \widetilde{\rho}\, \omega_F = \mathrm{vol}\left(B^{m-l}(\varepsilon)\right) + \frac{1}{2}\rho_{\alpha\beta} \int_{\{\|W\|<\varepsilon\}} w^\alpha w^\beta \, dW + O\left(\varepsilon^3\right).$$

By rotational symmetry of the coordinate functions we obtain

$$\begin{aligned}
\int_{\{\|W\|<\varepsilon\}} w^\alpha w^\beta \, dW &= \int_0^\varepsilon \left(\int_{\mathbb{S}^{m-l-1}} \theta^\alpha \theta^\beta d\mathbb{S}^{m-l-1} \right) s^{m-l-1} \, ds \\
&= \int_0^\varepsilon \frac{\delta^{\alpha\beta}}{m-l} \mathrm{vol}\left(\mathbb{S}^{m-l-1}(1)\right) s^{m-l+1} \, ds \\
&= \frac{\delta^{\alpha\beta}}{m-l} \mathrm{vol}(\mathbb{S}^{m-l-1}(1)) \frac{\varepsilon^{m-l+2}}{m-l+2}. \qquad (66)
\end{aligned}$$

Parts of this computation can also be found in [VC81], Lemma 6.3. Moreover, we use the following observation for the volume of a Euclidean ball. It is

$$\begin{aligned}
\mathrm{vol}\left(B^{m-l}(\varepsilon)\right) &= \int_0^\varepsilon \mathrm{vol}(\mathbb{S}^{m-l-1}(r)) \, dr \\
&= \int_0^\varepsilon r^{m-l-1} \mathrm{vol}(\mathbb{S}^{m-l-1}(1)) \, dr.
\end{aligned}$$

Hence,

$$\mathrm{vol}\left(B^{m-l}(\varepsilon)\right) = \frac{\varepsilon^{m-l}}{m-l} \mathrm{vol}(\mathbb{S}^{m-l-1}(1)). \qquad (67)$$

Combining all these results gives

$$\begin{aligned}
\int_{F_x(\varepsilon)} \widetilde{\rho}\, \omega_F &= \mathrm{vol}(B^{m-l}(\varepsilon)) + \frac{1}{2}\rho_{\alpha\beta}(x) \frac{\delta^{\alpha\beta}}{m-l} \, \mathrm{vol}\,(\mathbb{S}^{m-l-1}(1)) \frac{\varepsilon^{m-l+2}}{m-l+2} + O\left(\varepsilon^3\right) \\
&= \mathrm{vol}(B^{m-l}(\varepsilon)) + \frac{1}{2} \sum_{\alpha=1}^{m-l} \frac{\rho_{\alpha\alpha}(x) \varepsilon^{m-l} \varepsilon^2}{(m-l)(m-l+2)} \, \mathrm{vol}(\mathbb{S}^{m-l-1}(1)) + O\left(\varepsilon^3\right) \\
&= \mathrm{vol}(B^{m-l}(\varepsilon)) \left(1 + \frac{1}{2} \frac{\varepsilon^2}{m-l+2} \sum_\alpha \rho_{\alpha\alpha}(x) + O\left(\varepsilon^3\right)\right).
\end{aligned}$$

This computation combined with Lemma (5.2) proves the following fact.

Lemma 5.3. *For $x \in L$ the second order correction term can be expressed in terms of the density as*

$$A(x) = \frac{1}{2(m-l+2)} \sum_\alpha \rho_{\alpha\alpha}(x).$$

5.2 Arbitrary correction terms

In [Gra82], Remark (6), p.203, it is stated that the second order term of the volume formula above, is independent of the second fundamental form of the embedding. Intuitively, this statement seems to be wrong due to the Gauss equation, which relates the curvature of the manifold and its submanifold by terms of the second fundamental form. Hence, we use our method from Section (4.3.2) to demonstrate that the second order correction term A does indeed depend on the embedding.

Proposition 5.4. *For a given Riemannian manifold (L, g_L) and a smooth function $\widetilde{A} \in C^\infty(L)$, there exists an embedding $L \to E$, such that the given function $\widetilde{A}$ represents the second order term in the volume formula.*

Proof. Let $L(\varepsilon)$ be equipped with g_r defined by (4.14). Again we study the isometric embedding $(L, g_L) \to (L(\varepsilon), g_r)$. By Lemma (4.26) the associated density is given by

$$\rho_r = \frac{dm_r}{dm_{Sa}} = 1 + \frac{1}{6} w^\mu w^\nu {r_W}^\alpha{}_{\mu\alpha\nu} + O\left(\|W\|^3\right).$$

In the previous section we already computed A to be given by

$$\begin{aligned} A(x) &= \frac{1}{2}\frac{d^2}{d\varepsilon^2}\left(\frac{1}{\mathrm{vol}(B^{m-l}(\varepsilon))}\int_{F_x(\varepsilon)} \widetilde{\rho}\, \omega_F\right)_{\big|\varepsilon=0} \\ &= \frac{1}{2}\frac{d^2}{d\varepsilon^2}\left(\frac{1}{\mathrm{vol}(B^{m-l}(\varepsilon))}\int_{F_x(\varepsilon)} \rho\, dW\right)_{\big|\varepsilon=0}, \end{aligned} \tag{68}$$

for $x \in L$.

Using the coordinate expression for ρ_r yields

$$\begin{aligned} \int_{F_x(\varepsilon)} \rho_r\, dW &= \int_{F_x(\varepsilon)} \left\{1 + \frac{1}{6} w^\mu w^\nu r_{W\,\alpha\mu\alpha\nu} + O\left(\|W\|^3\right)\right\} dW \\ &= \mathrm{vol}\left(B^{m-l}(\varepsilon)\right) + \frac{1}{6} r_{W\,\alpha\mu\alpha\nu}(x) \int_{\{\|W\|<\varepsilon\}} w^\mu w^\nu\, dW + O\left(\varepsilon^3\right). \end{aligned}$$

The integral can be handled in the same way as (66) and (67). We deduce

$$\begin{aligned} &\int_{F_x(\varepsilon)} \rho_r\, dW \\ &= \mathrm{vol}\left(B^{m-l}(\varepsilon)\right) + \frac{1}{6} {r_W}^\alpha{}_{\mu\alpha\nu}(x)\left(\frac{\delta^{\mu\nu}\varepsilon^{m-l+2}}{(m-l)(m-l+2)}\mathrm{vol}\left(\mathbb{S}^{m-l-1}(1)\right)\right) + O\left(\varepsilon^3\right) \\ &= \mathrm{vol}\left(B^{m-l}(\varepsilon)\right)\left(1 + \frac{1}{6} {r_W}^\alpha{}_{\mu\alpha\nu}(x) + \varepsilon^2 \frac{1}{m-l+2}\right) + O\left(\varepsilon^3\right) \\ &= \mathrm{vol}\left(B^{m-l}(\varepsilon)\right)\left(1 + \frac{1}{6}\mathrm{Scal}_{F_r}\frac{\varepsilon^2}{m-l+2}\right) + O\left(\varepsilon^3\right). \end{aligned}$$

Substituting this result into (68) gives

$$A(x) = \frac{1}{6(m-l+2)} \mathrm{Scal}_{F_r}.$$

By the same argument as described in Section (4.3.2) we study the algebraic curvature tensor $f \cdot r_W$, where $f \in C^\infty(L)$ and $r_W \in \mathcal{R}$. Again equip $L(\varepsilon)$ with the metric g_r^f from Definition (4.29).

By formula (62) we obtain for the associated density

$$\rho_r^f = 1 + \frac{1}{6} \cdot f \cdot w^\mu w^\nu {r_W}^\alpha{}_{\mu\alpha\nu} + O\left(\|W\|^3\right).$$

We conclude by the same computation as before that A is now given by

$$A(x) = \frac{1}{6(m-l+2)} f \cdot \mathrm{Scal}_{F_r},\ x \in L.$$

Therefore, an appropriate choice of f and the fact that there exists some algebraic curvature tensor with non-vanishing scalar curvature, we obtain the result that every smooth function A on L can occur as a second order correction term (in the above sense) for an embedding. $\square$

In particular, this proves the announced dependency of the embedding.

Chapter 6

The behavior of W_0 under isometries

Let L be equipped with the induced metric g of the ambient manifold (M, g). This chapter discusses the behavior of the potential W_0 under isometries of the ambient manifold. Here, the primary result is stated in Theorem (6.3), where we prove that the potential is invariant under the subset of isometries of M, which restrict to isometries of L.

For a finite time horizon $T > 0$ we recall the definition of the path space

$$\Omega_{L(\varepsilon)} := C([0, T], L(\varepsilon)) = \{\gamma \colon [0, T] \to L(\varepsilon) | \, \gamma \text{ continuous } \}$$

of the tube.

Remark. *We use the notation of the path space not only for tubes, but for manifolds and their subsets as well, e.g.*

$$\Omega_M := C([0, T], M).$$

Note that $\Omega_{L(\varepsilon)} = \Omega_L(\varepsilon)$.

In the following section we use a different notation of the second fundamental form, which can be found in [Xin96].

Definition 6.1. *Let L, M be Riemannian manifolds. For a smooth embedding $f \colon L \to M$, $p \in L$ and $X, Y \in T_pL$, the* ***second fundamental form*** *of the embedding is defined by*

$$\mathrm{II}(f)(X, Y) := \nabla_X df(Y),$$

where ∇ denotes the induced connection on $T^*L \otimes f^{-1}(TM)$ and

$$df(Y) := f_*Y$$

is a section on $T^*L \otimes f^{-1}(TM)$.

Proposition 6.2. *For $f \colon L \to M$ and $X, Y \in TL$ the equation*

$$\nabla_X df(Y) = \nabla_{f_*X} f_*Y - f_*(\nabla_X Y) \tag{69}$$

holds.

Proof. By using the linearity of the connection and the product rule, we obtain

$$\begin{aligned} \nabla_{f_*X} f_*Y - f_*(\nabla_X Y) &= f_* \nabla_X f_*Y - f_*(\nabla_X Y) \\ &= f_*(\nabla_X f_*Y - \nabla_X Y) \end{aligned}$$

$$= f_*(f_*\nabla_X Y + X f_* Y - \nabla_X Y)$$
$$= f_* X f_* Y = \nabla_X(df(Y))$$

for each pair of tangent vectors X, Y. $\square$

From Theorem (1.26) and Theorem (1.28) in Chapter 1 we obtain the following result. Recall, as ε tends to zero, the conditional Wiener measures

$$\mu_\varepsilon^x = \mathbb{W}_M^x(d\omega |\, \omega \in \Omega_L(\varepsilon))$$

converge weakly to a limit measure μ_0^x, which is equivalent to the Wiener measure on the submanifold with corresponding Radon-Nikodym density δ defined by

$$\delta := \frac{d\mu_0^x}{d\mathbb{W}_L^x}(\omega) = \frac{\exp\left(-\frac{1}{2}\int_0^T \frac{1}{2}\mathrm{Scal}_L - \frac{1}{4}\|\tau(\varphi)\|^2 - \frac{1}{6}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L} + \overline{R}_{M/L}\right) ds\right)}{Z(x)}. \tag{70}$$

Here Z depends on the starting point $x \in L$ of the underlying Brownian motion and normalizes μ_0^x to total mass 1.

Remark. 1. *The normalization is given by*

$$Z(x) := \int_{\Omega(L)} \exp\left(-\frac{1}{2}\int_0^T W(\omega(s))\, ds\right) \mathbb{W}_L^x(d\omega),$$

with W from (14), and ensures that we obtain a probability measure.

2. *The density δ is strictly positive and continuous; therefore, μ_0 and $\mathbb{W}_L$ are equivalent measures on L.*

We now want to investigate the behavior of the potential composed with an isometry of M.

Theorem 6.3. *Let L, M be as above, M geodesically complete, and $j\colon M \to M$ be an isometry, such that the restriction $j_{|L}\colon L \to L$ is an isometry of the submanifold.*
Then the potential W_0 on the path space Ω_L is invariant with respect to j, meaning

$$W_0 \circ j = W_0.$$

Remark. *Equivalently we can use j^{-1} instead of j.*

For the proof of this theorem, we need the following helpful consequence.

Lemma 6.4. *Given the situation of Theorem* (6.3) *and $\varepsilon < i$, where i is the injectivity radius of* $\exp^M$ *on NL, it follows that the tube $L(\varepsilon)$ is invariant under j, which means*

$$j\colon L(\varepsilon) \to L(\varepsilon)$$

is an isometry.

Proof. Due to the tubular neighborhood theorem (cf. [Kos07], Chapter III), using $\varepsilon < i$, there exists a diffeomorphism between $L(\varepsilon)$ and a neighborhood of the zero section in the normal bundle NL of L in M. It suffices to study the properties of the tube and the submanifold. We want to use *Hopf-Rinow's* theorem (see Appendix A). The distance function is invariant under isometries. Therefore, geodesics are invariant under isometries, which means shortest path remain shortest path under application of j. Hence, for each $x \in L(\varepsilon) \setminus L$ the distance to L remains the same. By compactness of L and the Hopf-Rinow theorem, any two points x and y can be connected by a length-minimizing geodesic, hence for every given $x \in L(\varepsilon)$ there exists a unique $y \in L$, such that $d(x, L) = d(x, y) = a$. According to our assumptions $a < \varepsilon$ holds and hence

$$a = d(x, y) = d(j(x), j(y)) < \varepsilon.$$

The isometry j maps L bijectively onto L which means $j(y) \in L$. That implies $j(x) \in L(\varepsilon)$ and shows $j(L(\varepsilon)) \subset L(\varepsilon)$.
The map j is a diffeomorphism, that guarantees the existence of a diffeomorphism j^{-1}, which itself is an isometry as well. The statement follows from the fact that restrictions of homeomorphisms are homeomorphisms.

□

Using the above notations we are now able to prove Theorem (6.3).

Proof of Theorem(6.3). The isometry $j_{|L(\varepsilon)}$ induces a (continuous) map, denoted by J, on the path space $\Omega_{L(\varepsilon)}$. This map is explicitly given by

$$J\colon \Omega_{L(\varepsilon)} \to \Omega_{L(\varepsilon)}, \gamma \mapsto j^{-1} \circ \gamma.$$

According to Lemma (6.4) this map is well-defined and maps into the path space of the tube. Furthermore, we have

$$L = j^{-1}(L) = j(L) \tag{71}$$

and j^{-1} is a diffeomorphism as well. We then derive an equivalent condition for a path ω to be in $\Omega_{L(\varepsilon)}$:

$$\begin{aligned}
\omega \in \Omega_{L(\varepsilon)} &\Leftrightarrow \sup_{s\in[0,T]} d(\omega(s), L) < \varepsilon \\
&\Leftrightarrow \sup_{s\in[0,T]} d(j^{-1}(\omega(s)), j^{-1}(L)) < \varepsilon \qquad (72)\\
&\Leftrightarrow \sup_{s\in[0,T]} d(j^{-1}(\omega(s)), L) < \varepsilon, \qquad (73)
\end{aligned}$$

where we used that j^{-1} is length preserving in (72) and equality (71) for L to obtain the last equivalence. This leads to the following characterization

$$\omega \in \Omega_{L(\varepsilon)} \Leftrightarrow J^{-1}(\omega) \in \Omega_{L(\varepsilon)}. \tag{74}$$

Using the definition of the pushforward (or image) measure we derive

$$\begin{aligned}
J_* \mathbb{W}_L^x(d\omega | \, \omega \in \Omega_{L(\varepsilon)}) &= \mathbb{W}_L^x(J^{-1}(d\omega) | \, \omega \in \Omega_{L(\varepsilon)}) \\
&\overset{(74)}{=} \mathbb{W}_L^x(J^{-1}(d\omega) | \, J^{-1}(\omega) \in \Omega_{L(\varepsilon)}) \\
&= \mathbb{W}_L^{j(x)}(d\omega | \, \omega \in \Omega_{L(\varepsilon)}).
\end{aligned}$$

Here we used again the isometry property of j in the last equation. We obtain a Wiener measure on L with respect to another starting point of the paths.

Using Theorem (1.28), we obtain

$$\mathbb{W}_L^{j(x)}(d\omega|\, \omega \in \Omega_{L(\varepsilon)}) \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{Z(j(x))}\, d\mathbb{W}_L^{j(x)}(\omega) \exp\left(\int_0^T W(\omega(s))\, ds\right),$$

where $Z(j(x))$ denotes the normalization constant depending on the new starting point $j(x)$.

We now want to show that the computation of the pushforward measure commutes with weak limits. Therefore, let $C_b(\Omega_{L(\varepsilon)})$ be the set of continuous and bounded functions and $\mathcal{P}(\Omega_{L(\varepsilon)})$ be the set of finite measures on $\Omega_{L(\varepsilon)}$. Let $(\mu_n)_{n\geq 1}, \mu \in \mathcal{P}(\Omega_{L(\varepsilon)})$ with $\mu_n \to \mu$, $n \to \infty$, in the weak sense, which means

$$\int_{\Omega_{L(\varepsilon)}} f\, d\mu_n \overset{n\to\infty}{\longrightarrow} \int_{\Omega_{L(\varepsilon)}} f\, d\mu,$$

for all $f \in C_b(\Omega_{L(\varepsilon)})$.

We further define $\iota := j_{|L(\varepsilon)}$ and note that for all $f \in C_b(\Omega_{L(\varepsilon)})$, the composition

$$f \circ \iota \in C_b(\Omega_{L(\varepsilon)}),$$

because $\iota(C_b(\Omega_{L(\varepsilon)})) = C_b(\Omega_{L(\varepsilon)})$. Using the transformation formula (see e.g. [Bog07], p. 190 ff.) for the pushforward measure we obtain

$$\int_{\Omega_{L(\varepsilon)}} f \circ \iota\, d\mu = \int_{\Omega_{L(\varepsilon)}} f\, \iota_*(d\mu). \tag{75}$$

Due to our assumptions we have

$$\int_{\Omega_{L(\varepsilon)}} f \circ \iota\, d\mu_n \overset{n\to\infty}{\longrightarrow} \int_{\Omega_{L(\varepsilon)}} f \circ \iota\, d\mu.$$

Because of (75) the left-hand side equals $\int f\, \iota_* d\mu_n$, while the right-hand side is equal to $\int f \iota_* d\mu$, which means

$$\int_{\Omega_{L(\varepsilon)}} f \iota_*\, d\mu_n \overset{n\to\infty}{\longrightarrow} \int_{\Omega_{L(\varepsilon)}} f \iota_*\, d\mu.$$

Therefore, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} J_* \mathbb{W}^x(d\omega|\, \omega \in \Omega_{L(\varepsilon)}) \\
&= J_*\left(\lim_{\varepsilon\to 0} \mathbb{W}^x(d\omega|\, \omega \in \Omega_{L(\varepsilon)})\right) \\
&= J_*\left(\frac{1}{Z(x)} d\mathbb{W}_L^x(\omega') \exp\left(\int_0^T W(\omega'(s))ds\right)\right) \\
&= \frac{1}{Z(x)} \mathbb{W}_L^x(J^{-1}(d\omega')) \exp\left(\int_0^T W \circ \iota^{-1}(\omega'(s))ds\right),
\end{aligned}$$

where the last equality follows from the definition of the pushforward measure. This leads to the equality

$$\begin{aligned}&\frac{1}{Z(j(x))}\, d\mathbb{W}_L^{j(x)}(\omega)\exp\left(\int_0^T W(\omega(s))ds\right)\\ &= \frac{1}{Z(x)}\mathbb{W}_L^x(J^{-1}(d\omega'))\exp\left(\int_0^T W\circ j^{-1}(\omega'(s))ds\right).\end{aligned}$$

To sum up we obtain the equality of the Wiener measure up to the starting point and the densities are equal due to the transformation formula. Now we consider the logarithmic densities and use the properties of the logarithm which leads to

$$\int_0^T \left(W\circ\iota^{-1}(\omega) - W(\omega)\right) ds = \log\left(\frac{Z(x)}{Z(j(x))}\right).$$

The expression on the right-hand side is a constant, as it is the logarithmic quotient of two normalization constants. By the above remark,

$$Z(j(x)) = \int_{\Omega(L)} \exp\left(-\frac{1}{2}\int_0^T W(\omega(s))\, ds\right)\mathbb{W}_L^{j(x)}(d\omega)$$

holds. Using the definition of the pushforward measure, i.e. the identity

$$\mathbb{W}_L^{j(x)}(d\omega) = J_*\mathbb{W}_L^x(d\omega),$$

combined with the fact that j is an isometry restricted to L and the transformation formula we obtain $Z(x) = Z(j(x))$. Note that $W_{|L} = W_0$, therefore we can assume that the right-hand side is always equal to zero and we obtain the equality

$$W_0\circ\iota^{-1}(\omega) = W_0(\omega)$$

almost surely. □

Definition 6.5. *Let $G := \mathrm{Isom}(M)$ denote the isometry group of M. We then define the subgroup*

$$G(L) := \{g\in G \mid g_{|L}\colon L\to L \text{ is an isometry of } L\}\subset \mathrm{Stab}(L).$$

Corollary 6.6. *The potential W_0 is constant on the orbits of the action of $G(L)$ on L.*

Proof. Let $p, q\in L$ be arbitrary points in the same orbit B of the action

$$G(L)\times L\to L, (\phi, p)\to\phi(p).$$

Particularly, there exists $\psi\in G(L)$ with $\psi(q) = p$. Moreover due to Theorem (6.3) we have $W_0\circ\psi = W_0$, which leads to

$$W_0(q) = (W_0\circ\psi)(q) = W_0(p).$$

Thus W_0 is constant on B. □

The same arguments provide the following result.

Corollary 6.7. *Let the action of $G(L)$ be transitive on L. Then W_0 is constant.*

Clearly, the embeddings of lower dimensional spheres $\mathbb{S}^n$ into higher dimensional spheres $\mathbb{S}^m$, $n < m$, yield an example for the latter result.

6.1 Lie groups and symmetric spaces

We investigate in which other contexts Corollary (6.7) can be applied. In this section, we obtain some results with the help of Lie group theory and first want to look at Riemannian symmetric spaces, introduced in Section (2.6), for another application.

Two crucial facts about symmetric spaces used in the following are on the one hand that they are homogeneous spaces, i.e. the isometry group $\mathrm{Isom}(M)$ acts transitively on M. On the other hand their isometry groups are finite dimensional Lie groups.

Example 6.8. *Let $n \geq 1$. The most important examples are*

(i) the Euclidean space $\mathbb{R}^n$ with its standard metric,

(ii) the spheres $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with the induced metric.

The above spaces are of constant sectional curvature. Moreover, the orthogonal groups (as compact Lie groups)

$$O(n) = \{g \in \mathbb{R}^{n\times n} |\; g^T g = \mathrm{I}\},$$

or the Grassmannians

$$G_k(\mathbb{R}^n) = \text{ set of all k-dimensional linear subspaces of } \mathbb{R}^n$$

form symmetric spaces.

By a theorem of E. Cartan we find a remarkable fact, which is helpful for our investigations.

Theorem 6.9 (Closed Subgroup Theorem, Cartan)**.** *Let G be a Lie group and $H \leq G$ a closed subgroup. Then H itself is a Lie group and therefore an embedded submanifold with the induced topology of G.*

Proof. See e.g. [Lee03], Theorem 20.12. □

For the classification of symmetric spaces, a very handy decomposition result is as follows.

Theorem 6.10. *Let G be a connected Lie group with involution σ and a left invariant metric, which is right invariant under the closed subgroup*

$$\hat{K} = \mathrm{Fix}(\sigma) = \{g \in G|\; g^\sigma := \sigma g \sigma^{-1} = g\}.$$

In addition, let $K \leq G$ be a closed subgroup which contains the identity component of $\hat{K}$ and is contained in $\hat{K}$. Then $S := {}^{G}/_{K}$ is a symmetric space.
Moreover, every symmetric space M can be decomposed as $M = {}^{G}/_{K}$ for suitable K and G as above.

For a more precise statement and further details on the classification of symmetric spaces and the connection to Lie groups see e.g. [Hel01], [Lee03] or [Bre72].

Let now M be a Riemannian manifold, such that $G = \mathrm{Isom}(M)$ is a Lie group and let $x \in M$. Denote by

$$G_x := \{g \in G | \; gx = x\}$$

the isotropy group of G and by

$$Gx := \{gx | \; g \in G\}$$

the orbit of x in G. From now on, let G be a compact Lie group and $H \leq G$ a closed subgroup. H is compact. Then let Hp be an orbit of the action of H. Moreover, G acts on Hp and every orbit of this action gives rise to a compact subgroup L which satisfies all assumptions of Corollary (6.7). Hence, L itself needs to be the orbit of a subgroup.

We try to summarize this in the following lemma.

Lemma 6.11. *Let G be a compact Lie group and $H \subset H' \subset G$ be closed subgroups, which are isomorphic to an isotropy group of G. Then, $M := {}^{G}/_{H}$ and $M' := {}^{G}/_{H'}$ yield an embedding of symmetric spaces, with constant associated potential.*

Proof. M and M' are topogical spaces, when equipped with the quotient topology. Moreover, the isometry group of both spaces are equal. Then, using compactness, Corollary (6.7) and the fact that M is conjugated to a subgroup of M' yields the statement. $\square$

Example 6.12. *All of the above listed symmetric spaces fulfill the assumptions, such that e.g. the Grassmannians presented by*

$$G_k(\mathbb{R}^n) \cong {}^{O(n)}/_{O(k)\times O(n-k)},$$

the hyperbolic spaces

$$\mathbb{H}_n = {}^{SO(n,1)^+}/_{SO(n)}$$

or the Stiefel manifolds given as

$$U_k(\mathbb{R}^n) \cong {}^{O(n)}/_{O(n-k)}$$

yield examples.

In [NG97] some of these examples were studied explicitly. The volume of the tubular neighborhoods is computed analogously to the methods used by Gray. For the compact symmetric space $M = {}^{SO(5)}/_{(SO(2)\times SO(3))}$ there are numerical computations for the area and volume functions of tubes around two distinct totally geodesic submanifolds.

Remark.

1. *Whenever the ambient space has a high amount of symmetries, i.e. its isometry group is large, we obtain a couple of examples for Corollary (6.7). For instance, starting with a symmetric space M, one can construct examples of embeddings by studying every closed subgroup of the full isometry group and its orbit under the group action.*

2. *These observations support the facts found in Section (2.6), i.e. that the formula for the potential will be easy, if the ambient space is e.g. of constant sectional curvature or a symmetric space.*

6.2 The cocycle C^{21M}

In [SSWW14] the authors proved that the conditional measures on the tubular neighborhoods $L(\varepsilon)$ described in Chapter 1 correspond to (inhomogeneous) Markov processes and converge weakly to a limit measure for $\varepsilon \to 0$. This limit measure is supported by the path space of the submanifold. This situation was also studied in [JK71], where it is shown that such a limit measure is related to the Schrödinger operator, which describes the dynamic of a quantum particle constrained to the submanifold.

The following section was motivated by this work of H. Jensen and H. Koppe, who especially studied the motion of a particle on a surface by forcing the particle to move between two parallel surfaces with distance $d > 0$ and examine its behavior for $d \to 0$. The authors specify different approaches for the computation of the limit. Moreover, they focus on a motion between equidistant surfaces parallel to a given surface which is parameterized in local coordinates. This construction motivated the so-called *cocyle problem*, which we discuss now.

We introduce the following situation.

Let L_1, L_2, M be Riemannian manifolds, such that $L_2 \subseteq L_1$, $L_1 \subseteq M$ and $L_2 \subseteq M$ are isometrically embedded Riemannian submanifolds. The dimensions are l_1, l_2 and m, respectively. In this case we denote by $W^{L_2L_1}$ or W^{21} the potential according to the embedding of $L_2 \subset L_1$, W^{L_1M} and W^{L_2M}, respectively.

Corollary 6.13. *Let $L_2 \subseteq L_1 \subseteq M$ be as above and $j\colon M \to M$ an isometry with the property that the restrictions of j to the submanifolds are isometries as well.*
Then the expression

$$W_0^{L_2L_1} + W_0{}_{|L_2}^{L_1M} - W_0^{L_2M}$$

is invariant under j.

Proof. This statement is a direct consequence of Theorem (6.3). □

This last corollary initiated the computations in this section.

Let $L_2 \subseteq L_1 \subseteq M$ be Riemannian submanifolds of (M, g_M) as above. Furthermore, let $\varepsilon > 0, A \subset M$ and $\omega \in \Omega(M)$.

Now we want to compare the limit behavior of the conditioned measures we obtain from the conditional limit process according to the embedding $L_2 \subseteq M$, to the surface measure we obtain from conditioning the path measures of M first to L_1 and then afterwards to L_2. We compute the rescaling factors for both processes and investigate, how they are related.
First, compute all of the associated densities in an individual step, each concerning one of the above embeddings. Starting with the latter case we first focus on the embedding $L_1 \subseteq M$. Fix a point $x \in L_1$.

Due to Theorem (1.28) we know the family μ_ε of Wiener measures conditioned to the tubes converges to a probability measure μ_0 on the submanifold L_1. This limit measure μ_0 is equivalent to the Wiener measure on $\mathbb{W}_{L_1}$. Explicitly, we obtain in the given situation

$$
\begin{aligned}
\mu^x(\omega(t) \in A \cap L_1) &= \lim_{\varepsilon \to 0} \frac{\mathbb{W}_M^x(\omega(t) \in A \cap L_1(\varepsilon))}{\mathbb{W}_M^x(\omega([0,1]) \in A \cap L_1(\varepsilon))} \\
&= \lim_{\varepsilon \to 0} \frac{\int_{\Omega_{A \cap L_1(\varepsilon)}} \mathbb{W}_M^x(d\omega)}{\int_{\Omega_{L_1(\varepsilon)}} \mathbb{W}_M^x(d\omega)} \\
&= \frac{\exp\left(-\frac{1}{2}\int_0^1 W^{L_1 M}(\omega(s))ds\right)}{\int_{\Omega_{L_1}} \exp\left(-\frac{1}{2}\int_0^1 W^{L_1 M}(\omega(s))ds\right) \mathbb{W}_{L_1}^x(d\omega)} \mathbb{W}_{L_1}^x(d\omega),
\end{aligned}
$$

where we define the denominator by $Z(x)$ to denote the dependency of the starting point x.

Note that the map

$$
\Omega_{L_1} \to \mathbb{R}, \omega \mapsto \frac{1}{Z(x)} \exp\left(-\frac{1}{2}\int_0^1 W^{L_1 M}(\omega(s))ds\right)
$$

is continuous. We are going to use this result in combination with the second step, where we study L_2 embedded into L_1. Let $x \in L_2$ denote the new starting point. We want to take the result of the conditioned Wiener measure we found above and restrict it to L_2 now.

Analogously, we derive a probability measure ν^x given by

$$
\begin{aligned}
&\nu^x(\omega(t) \in A \cap L_2) \\
&= \lim_{\varepsilon \to 0} \frac{\mu^x(\omega(t) \in A \cap L_2(\varepsilon))}{\mu^x(\omega([0,1]) \in L_2(\varepsilon))} \\
&= \lim_{\varepsilon \to 0} \frac{\int_{\Omega_{A \cap L_2(\varepsilon)}} \frac{1}{Z(x)} \exp\left(-\frac{1}{2}\int_0^1 W^{L_1 M}(\omega(s))ds\right) \mathbb{W}_{L_1}^x(d\omega)}{\int_{\Omega_{L_2(\varepsilon)}} \frac{1}{Z(x)} \exp\left(-\frac{1}{2}\int_0^1 W^{L_1 M}(\omega(s))ds\right) \mathbb{W}_{L_1}^x(d\omega)} \\
&= \lim_{\varepsilon \to 0} \frac{\int_{\Omega_A} \frac{1}{Z(x)} \exp\left(-\frac{1}{2}\int_0^1 W^{L_1 M}(\omega(s))ds\right) \mathbb{W}_{L_1}^x(d\omega)\chi_{L_2(\varepsilon)}}{\int_{\Omega_{L_2(\varepsilon)}} \frac{1}{Z(x)} \exp\left(-\frac{1}{2}\int_0^1 W^{L_1 M}(\omega(s))ds\right) \mathbb{W}_{L_1}^x(d\omega)},
\end{aligned} \tag{76}
$$

where χ_T denotes the characteristic function of T.
Because the limits of denominator and numerator do not exist, we need some help to obtain a suitable result. Therefore, we expand the above expression with

$$
\frac{1}{\int_{\Omega_{L_2(\varepsilon)}} \mathbb{W}_{L_1}^x(d\omega)} = \frac{1}{\int_{\Omega_{L_1}} \chi_{L_2(\varepsilon)} \mathbb{W}_{L_1}^x(d\omega)}.
$$

By Portmanteau's Theorem (cf. [Bil13], Theorem (2.1)) we obtain

$$\nu^x(\omega(t)\in A\cap L_2)=\lim_{\varepsilon\to 0}\frac{\int_{\Omega_A}\frac{\frac{1}{Z(x)}\exp\left(-\frac{1}{2}\int_0^1 W^{L_1M}(\omega(s))ds\right)\chi_{L_2(\varepsilon)}\mathbb{W}^x_{L_1}(d\omega)}{\int_{\Omega_{L_2(\varepsilon)}}\mathbb{W}^x_{L_1}(d\omega)}}{\int_{\Omega_{L_1}}\frac{\frac{1}{Z(x)}\exp\left(-\frac{1}{2}\int_0^1 W^{L_1M}(\omega(s))ds\right)\chi_{L_2(\varepsilon)}\mathbb{W}^x_{L_1}(d\omega)}{\int_{\Omega_{L_2(\varepsilon)}}\mathbb{W}^x_{L_1}(d\omega)}},$$

where we are now able to compute the limits of numerator and denominator independently.

As the potential functions are smooth functions on the submanifolds, and the starting point x is fixed, we can apply a theorem for the weak convergent measure families given by the Wiener measure on L_1 restricted to the tube $L_2(\varepsilon)$. That guarantees the existence of both limits. We use that

$$\lim_{\varepsilon\to 0}\frac{\int_{\Omega_A}\mathbb{W}^x_{L_1}(d\omega)\chi_{L_2(\varepsilon)}}{\int_{\Omega_{L_2(\varepsilon)}}\mathbb{W}^x_{L_1}(d\omega)}=\frac{1}{Z_1(x)}\exp\left(-\frac{1}{2}\int_0^1 W^{L_1L_2}ds\right)$$

and therefore finally obtain a normalization factor, which depends on both restrictions, denoted by Z_f. The final result is

$$\begin{aligned}&\nu^x(\omega(t)\in A\cap L_2)\\ =&\frac{1}{Z_f(x)}\exp\left(-\frac{1}{2}\int_0^1 W^{L_1L_2}(\omega(s))ds\right)\exp\left(-\frac{1}{2}\int_0^1 W^{L_1M}\Big|_{L_2}(\omega(s))ds\right)\mathbb{W}^x_{L_2}(d\omega).\end{aligned}\tag{77}$$

At last for the embedding $L_2\subset M$ and $x\in L_2$ we explore the probability measure λ^x exactly as in the first case and derive

$$\lambda^x(\omega(t)\in A\cap L_2)=\frac{1}{Z_2(x)}\exp\left(-\frac{1}{2}\int_0^1 W^{L_2M}(\omega(s))ds\right)\mathbb{W}_{L_2}(d\omega).\tag{78}$$

We summarize the observations above in the following two lemmata.

Lemma 6.14. *The Wiener measures on M and L_1 are related by*

$$\mathbb{W}_M\Big|_{L_1}=\delta^{1M}\cdot\mathbb{W}_{L_1},$$

where δ^{1M} denotes the corresponding density of the embedding $\psi\colon L_1\to M$.

Proof. By continuity of the density δ^{1M}, a restriction of the Wiener measure $\mathbb{W}_M$ to L_1 means restricting the potential $W_0=\rho^{-\frac{1}{2}}\triangle\rho^{\frac{1}{2}}$ onto the submanifold. By definition of δ (see (70)) we obtain the result. □

Analogously we want to restrict this conditioned measure on the smallest submanifold L_2. We obtain the following connection.

Lemma 6.15. *The Wiener measures on M and L_2 are related by*

$$\left(\mathbb{W}_M\Big|_{L_1}\right)\Big|_{L_2}=\delta^{1M}{}_{|C([0,T],L_2)}\cdot\delta^{21}\cdot\mathbb{W}_{L_2}.$$

Proof. Both densities δ^{1M} and δ^{21} are continuous functions on their domain. Hence, restriction of the density function to a submanifold is equivalent to restricting the corresponding potential to the submanifold. Because the restriction of a probability density to a submanifold provides a probability distribution on the embedded submanifold, this means only paths of this smaller path space are admissible. Applying the previous lemma twice then gives

$$\begin{aligned}\left(\mathbb{W}_{M|L_1}\right)_{|L_2} &= \left(\delta^{1M}\mathbb{W}_{L_1}\right)_{|L_2} \\ &= \delta^{1M}{}_{|C([0,T],L_2)}\mathbb{W}_{L_1|L_2} \\ &= \delta^{1M}{}_{|C([0,T],L_2)}\delta^{21}{}_{|L_2}\cdot\mathbb{W}_{L_2}.\end{aligned}$$

□

We now want to give some conditions that are necessary to obtain the equivalence of ν^x and λ^x.

Since the relation between both conditional measures is somehow described by a cocycle condition, we define the object of interest as the cocycle.

Definition 6.16. *Let $L_2 \subseteq L_1 \subseteq M$ be Riemannian submanifolds of (M, g_M). The* ***cocycle*** *C^{21M}, defined as*

$$\frac{Z_2(x)\exp\left(-\frac{1}{2}\int_0^1 W^{L_1L_2}(\omega(s))ds\right)\exp\left(-\frac{1}{2}\int_0^1 W^{L_1M}\Big|_{L_2}(\omega(s))ds\right)\mathbb{W}^x_{L_2}(d\omega)}{Z_f(x)\exp\left(-\frac{1}{2}\int_0^1 W^{L_2M}(\omega(s))ds\right)\mathbb{W}_{L_2}(d\omega)},$$

denotes the normalization of the limit process that compares the surface measure induced by the embedding of the smallest submanifold L_2 into M with the measure, we obtain by conditioning the surface measure on the path space of M first onto L_1 and then onto L_2. The related potential W_0^{21M} is defined by

$$W_0^{21M} := W^{21M}{}_{|L_2} := W_0^{L_2L_1} + W_0{}^{L_1M}_{|L_2} - W_0^{L_2M},$$

where the subscript 0 at each function always denotes its restriction to the submanifold.

Remark. *Whenever the cocycle equals one, the measures derived from the conditioned processes are not only equivalent, but equal.*

Definition 6.17. *Let (M, g_M) be an n-dimensional Riemannian manifold and R^M the Riemannian curvature tensor according to g_M. Let $L_2 \subseteq L_1 \subseteq M$ be (isometrically) embedded submanifolds of dimension l_2 and l_1, respectively. By $\{e_i\}_{i=1,\ldots,m}$ we denote an orthonormal frame of T_xM, $x \in L_2$, such that the subsets $\{e_j\}_{j=1,\ldots,l_2}$ and $\{e_k\}_{k=1,\ldots,l_1}$ are orthonormal frames of T_xL_2 and T_xL_1. We define the partial trace*

$$R^M_{\frac{L_1}{L_2}} := \sum_{\substack{i\in\{1,\ldots,l_2\}\\ \alpha\in\{l_2+1,\ldots,l_1\}}} R^M_{i\alpha i\alpha}.$$

Remark. 1. *In analogy to the usual notation the index α is used for the fiber components of $NL_2 \subset TL_1$.*

2. *In the same way one defines* $R^M_{\frac{M}{L_1}}$, $R^M_{\frac{M}{L_2}}$ *and* $R^{L_1}_{\frac{L_1}{L_2}}$.
3. *Using the definition of the sectional curvature we obtain*

$$R^M_{\frac{L_1}{L_2}} = \sum_{i,\alpha} \kappa(e_i \wedge e_\alpha).$$

4. *We still sum over repeated indices, such that we will not write down the sums explicitly, whenever there is no reason for confusion.*

Definition 6.18. *For the embeddings $L_2 \subseteq L_1 \subseteq M$ we define another partial trace (over the fiber components) by*

$$R_{M/F_{21}} := \sum_{k,j=l_2+1}^{l_1} R^M_{kjkj}.$$

We can now give a basic property of the cocycle.

Theorem 6.19. *The potential W^{21M} is given as*

$$\begin{aligned} W^{21M} = &-\frac{1}{4}\left(\|\tau(\varphi)\|_1^2 + \|\tau(\psi)\|^2_{M|L_2} - \|\tau(\psi\circ\varphi)\|_M^2\right) \\ &+ \frac{1}{3}\mathrm{Scal}_{F_{21}} - \frac{1}{6}R^{L_1}_{\frac{L1}{L_2}} - \frac{2}{3}R^M_{\frac{L_1}{L_2}} - \frac{1}{3}R_{M/F_{21}} - \frac{1}{6}R^M_{\frac{M}{L_1}} + \frac{1}{6}R^M_{\frac{M}{L_2}}. \end{aligned}$$

Proof. Let $\varphi\colon L_2 \to L_1$ and $\psi\colon L_1 \to M$ denote the embeddings of the submanifolds. Then $\psi\circ\varphi$ is the embedding of L_2 into M. Moreover let $\{e_i\}_{i=1\ldots l_2}$ be an orthonormal frame of TL_2 and with standard theory of linear algebra, we can assume that for an orthonormal frame $\{E_j\}_{j=1\ldots l_1}$ of TL_1 the identification $E_j := e_j$ holds for $1 \le j \le l_2$.

To avoid confusion, we use abbreviations like $\|\cdot\|_1$, whenever the metric of the submanifold L_1 is used.

Note that we just evaluate the potential on points $p \in L_2$ such that some terms directly cancel. We obtain

$$\begin{aligned} &W_0^{L_1L_2} + W_{0|L_2}^{L_1M} - W_0^{L_2M} \\ =& \frac{1}{2}\mathrm{Scal}_{L_2} - \frac{1}{4}\|\tau(\varphi)\|_1^2 - \frac{1}{6}\left(\mathrm{Scal}_{L_1} + \overline{\mathrm{Ric}}_{L_1/L_2} + \overline{R}_{L_1/L_2}\right) \\ &+ \left(\frac{1}{2}\mathrm{Scal}_{L_1} - \frac{1}{4}\|\tau(\psi)\|_M^2 - \frac{1}{6}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L_1} + \overline{R}_{M/L_1}\right)\right)\Big|_{L_2} \\ &- \left(\frac{1}{2}\mathrm{Scal}_{L_2} - \frac{1}{4}\|\tau(\psi\circ\varphi)\|_M^2 - \frac{1}{6}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M/L_2} + \overline{R}_{M/L_2}\right)\right) \\ =& -\frac{1}{4}\left(\|\tau(\varphi)\|_1^2 + \|\tau(\psi)\|^2_{M|L_2} - \|\tau(\psi\circ\varphi)\|_M^2\right) && (79) \\ &+ \frac{1}{3}\mathrm{Scal}_{L_1} - \frac{1}{6}\left(\overline{\mathrm{Ric}}_{L_1/L_2} + \overline{R}_{L1/L2}\right) && (80) \\ &- \frac{1}{6}\left(\overline{\mathrm{Ric}}_{M/L_1} + \overline{\mathbb{R}}_{M/L_1}\right)\Big|_{L_2} + \frac{1}{6}\left(\overline{\mathrm{Ric}}_{M/L_2} + \overline{R}_{M/L_2}\right). && (81) \end{aligned}$$

First consider the summand (80), we explicitly compute the partial traces as

$$\begin{aligned}
\overline{\mathrm{Ric}}_{L_1/L_2} &= \mathrm{tr}_{L_2}(\mathrm{Ric}_{L_1}) \\
&= \sum_{i=1}^{l_2} \mathrm{Ric}_{L_1}(e_i, e_i) \\
&= \sum_{i=1}^{l_2} \sum_{j=1}^{l_1} \langle R^{L_1}(E_j, e_i)e_i, E_j\rangle \\
&= \sum_{i=1}^{l_2} \sum_{j=1}^{l_1} R^{L_1}_{ijij}
\end{aligned}$$

and

$$\overline{R}_{L1/L_2} = \sum_{i,j=1}^{l_2} \langle e_j, R^{L_1}(e_j, e_i)e_i\rangle = \sum_{i,j=1}^{l_2} R^{L_1}_{ijij}.$$

Combining these expressions with the fact that

$$\mathrm{Scal}_{L_1} = \sum_{i,j=1}^{l_1} R^{L_1}_{ijij}$$

we obtain the following simplification

$$\begin{aligned}
(80) &= \frac{1}{3}\mathrm{Scal}_{L_1} - \frac{1}{6}\left(\overline{\mathrm{Ric}}_{L_1/L_2} + \overline{R}_{L1/L2}\right) \\
&= \frac{1}{3}\sum_{i,j=1}^{l_1} R^{L_1}_{ijij} - \frac{1}{6}\sum_{i=1}^{l_2}\sum_{j=1}^{l_1} R^{L_1}_{ijij} - \frac{1}{6}\sum_{i,j=1}^{l_2} R^{L_1}_{ijij} \\
&= \frac{1}{3}\sum_{i,j=1}^{l_1} R^{L_1}_{ijij} - \frac{1}{3}\sum_{i,j=1}^{l_2} R^{L_1}_{ijij} - \frac{1}{6}\sum_{i=1}^{l_2}\sum_{j=l_2+1}^{l_1} R^{L_1}_{ijij} \\
&= \frac{1}{3}\mathrm{Scal}_{F_{21}} - \frac{1}{6}\sum_{i=1}^{l_2}\sum_{j=l_2+1}^{l_1} R^{L_1}_{ijij},
\end{aligned}$$

where F_{21} denotes a fiber of the embedding of L_2 into L_1.

We compute (81) in two steps. On the one hand we have

$$\begin{aligned}
&\frac{1}{6}\left(\overline{\mathrm{Ric}}_{M/L_2} - \overline{\mathrm{Ric}}_{M/L_1|L_2}\right) \\
&= \frac{1}{6}\left(\sum_{k=1}^{m}\sum_{i=1}^{l_2} R^M_{kiki} - \left(\sum_{k=1}^{m}\sum_{j=1}^{l_2} R^M_{kjkj} + \sum_{k=1}^{m}\sum_{j=l_2+1}^{l_1} R^M_{kjkj}\right)_{\big|L_2}\right) \\
&= -\frac{1}{6}\sum_{k=1}^{m}\sum_{j=l_2+1}^{l_1} R^M_{kjkj|L_2}.
\end{aligned}$$

On the other hand, using similar splittings, we obtain

$$\frac{1}{6}\left(\overline{R}_{M/L_2} - \overline{R}_{M/L_{1|L_2}}\right) = -\frac{1}{6}\left(\sum_{i=l_2+1}^{l_1}\sum_{j=1}^{l_2} R^M_{ijij} + \sum_{i=l_2+1}^{l_1}\sum_{j=1}^{l_1} R^M_{ijij|L_2}\right).$$

Combining the latter two results a straight forward computation gives

$$\begin{aligned}(81) &= -\frac{1}{6}\left(\sum_{k=1}^{m}\sum_{j=l_2+1}^{l_1} R^M_{kjkj|L_2} + \sum_{i=l_2+1}^{l_1}\sum_{j=1}^{l_2} R^M_{ijij} + \sum_{i=l_2+1}^{l_1}\sum_{j=1}^{l_1} R^M_{ijij|L_2}\right)\\ &= -\frac{1}{6}\left(3\sum_{k=1}^{l_2}\sum_{j=l_2+1}^{l_1} R^M_{kjkj} + 2\sum_{k=l_2+1}^{l_1}\sum_{j=l_2+1}^{l_1} R^M_{kjkj} + \sum_{k=l_1+1}^{m}\sum_{j=l_2+1}^{l_1} R^M_{kjkj}\right).\end{aligned}$$

The last summand can be rewritten as

$$\sum_{k=l_1+1}^{m}\sum_{j=l_2+1}^{l_1} R^M_{kjkj} = R^M_{\frac{M}{L_1}} - R^M_{\frac{M}{L_2}} + R^M_{\frac{L_1}{L_2}}$$

using the definition above.

Finally, we obtain

$$\begin{aligned}W^{21M} = &-\frac{1}{4}\left(\|\tau(\varphi)\|_1^2 + \|\tau(\psi)\|^2_{M|L_2} - \|\tau(\psi\circ\varphi)\|^2_M\right)\\ &+\frac{1}{3}\mathrm{Scal}_{F_{21}} - \frac{1}{6}R^{L_1}_{\frac{L_1}{L_2}} - \frac{2}{3}R^M_{\frac{L_1}{L_2}} - \frac{1}{3}R_{M/F_{21}} - \frac{1}{6}R^M_{\frac{M}{L_1}} + \frac{1}{6}R^M_{\frac{M}{L_2}}.\end{aligned}$$

□

Remark. *Note that the potential W_{21M} does not depend on the curvature tensor of L_2 any more. Moreover, it does not depend on the embedding of L_2 into L_1, which we will discuss next.*

We want to explain this result in more detail. We can reformulate Theorem (6.19) by the following proposition.

Proposition 6.20. *Given the situation above, let the sectional curvatures of the ambient manifolds L_1 and M be constant. Then the corresponding potential W^{21M} is constant if*

$$\|\tau(\varphi)\|_1^2 + \|\tau(\psi)\|^2_{M|L_2} - \|\tau(\psi\circ\varphi)\|^2_M$$

is constant.

It seems there is another possibility for a simplification due to an exploration of the tension field. By a result of Wood (cf. [Woo13], p. 33, Formula (1.11)) we obtain

$$\tau(\varphi) = \text{tangential part of } \tau(\psi\circ\varphi),$$

where tangential means tangential to L_1.

To understand this result we study the formula of the tension field of the composition of ψ and τ to give a simplification of the latter proposition.

Thus we compute the second fundamental form of the composition of two maps and derive the expression for the tension field by taking the trace over an orthonormal basis $\{e_i\}$.
Let $X, Y \in TL_2$. Using the definitions of the objects and equation (69), according to [Xin96], we have

$$\begin{aligned}
\mathrm{II}(\psi \circ \varphi)(X,Y) &= (\nabla_X d(\psi \circ \varphi))(Y) \\
&= \nabla_{(\psi\circ\varphi)_* X}(\psi \circ \varphi)_* Y - (\psi \circ \varphi)_*(\nabla_X Y) \\
&= \nabla_{(\psi_*\circ\varphi_*) X}(\psi_* \circ \varphi_*) Y - (\psi_* \circ \varphi_*)(\nabla_X Y) \\
&= (\nabla_{\varphi_* X}(d\psi)(\varphi_* Y) + d\psi(\nabla_{\varphi_* X}(\varphi_* Y))) - d\psi(\varphi_* \nabla_X Y) \\
&= \nabla_{\varphi_* X}(d\psi)(\varphi_* Y) + d\psi(\nabla_{\varphi_* X}(\varphi_* Y) - \varphi_* \nabla_X Y) \\
&= \mathrm{II}(\psi)(\varphi_* X, \varphi_* Y) + d\psi(\mathrm{II}(\varphi)(X,Y)),
\end{aligned}$$

where we used the formula for the pushforward of a composition in the third step. Taking the trace using the above orthonormal basis $\{e_i\}$, we derive

$$\tau(\psi \circ \varphi) = \sum_i \mathrm{II}(\psi)(\varphi_* e_i, \varphi_* e_i) + \psi_*(\tau(\varphi)). \tag{82}$$

On the other hand using local Fermi coordinates and the definition of the Christoffel symbols we obtain the following relations. Let

$$(x^1, ..., x^{l_2}, w^1, ..., w^{l_1 - l_2}, z^1, ..., z^{m-l_1})$$

denote the local Fermi coordinates on M. We use $(x^i), (w^\alpha)$ and (z^A) when referring to local coordinates on L_2, L_1 and M. Moreover, the usual notation for the standard vector fields holds, i.e. $\partial_i = \frac{\partial}{\partial x^i}$ et cetera. Let $p \in L_2$. Due to the choice of the coordinates we obtain not only

$$\langle \partial_i, \partial_\alpha \rangle_{L_1,p} = 0,\ \langle \partial_i, \partial_A \rangle_{M,p} = 0 \ \text{ and } \langle \partial_A, \partial_\alpha \rangle_{M,p} = 0,$$

but also

$${\Gamma^{L_1}}^k_{ij} = {\Gamma^{L_2}}^k_{ij}.$$

Moreover, we obtain

$$\begin{aligned}
\mathrm{II}(\varphi)(X,Y) &= \nabla^{L_1}_X Y - \nabla^{L_2}_X Y \\
&= X^i Y^j \left({\Gamma^{L_1}}^k_{ij} - {\Gamma^{L_2}}^k_{ij} \right) \partial_k + X^i Y^j {\Gamma^{L_1}}^\alpha_{ij} \partial_\alpha
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{II}(\psi)(X,Y) &= \nabla^{M}_X Y - \nabla^{L_1}_X Y \\
&= X^r Y^s \left({\Gamma^{M}}^\alpha_{rs} - {\Gamma^{L_1}}^\alpha_{rs} \right) \partial_\alpha + X^r Y^s {\Gamma^{M}}^A_{rs} \partial_A
\end{aligned}$$

for $X, Y \in \Gamma(TL_2)$. Using the composition formula (82) and the fact that we deal with isometric embeddings, we remain with

$$\|\tau(\psi \circ \varphi)\|_M^2 = \|\tau(\psi)_{|L_2}\|_M^2 + \|\psi_* \tau(\varphi)\|_M^2 = \|\tau(\psi)_{|L_2}\|_M^2 + \|\tau(\varphi)\|_M^2.$$

Here $\tau(\psi)_{|L_2}$ denotes the partial trace over the orthonormal basis of L_2 over the tension field $\tau(\psi)$, given in local coordinates by $g_{L_2}^{rs}\left(\Gamma^{M}{}_{rs}^{A}\partial_A - \Gamma^{L_1}{}_{rs}^{\alpha}\partial_\alpha\right)$. Therefore, we have proven the following.

Corollary 6.21. *The potential in the cocycle situation is given as*

$$\begin{aligned} W^{21M} &= -\frac{1}{4}\left(\|\tau(\psi)\|_M^2 - \|\tau(\psi)_{|L_2}\|_M^2\right) && (83)\\ &\quad + \frac{1}{3}\mathrm{Scal}_{F_{21}} - \frac{1}{6}R^{L_1}_{\frac{L_1}{L_2}} - \frac{2}{3}R^{M}_{\frac{L_1}{L_2}} - \frac{1}{3}R_{M/F_{21}} - \frac{1}{6}R^{M}_{\frac{M}{L_1}} + \frac{1}{6}R^{M}_{\frac{M}{L_2}}. && (84)\end{aligned}$$

Remark. *The composition formula (82) directly implies that the first part in the formula for W^{21M} vanishes if ψ is totally geodesic and φ is harmonic, as $\psi \circ \varphi$ is harmonic in this case.*

For more information on harmonic maps also see e.g. [ES64].

We give some consequences of the above result, i.e. study W^{21M} with additional assumptions.

Corollary 6.22. *Assume the embedding ψ is totally geodesic. Then the corresponding potential for the cocycle is given by*

$$W^{21M} = \frac{1}{4}\left(\|\tau(\psi)_{|L_2}\|^2\right) - \frac{5}{6}R^{M}_{\frac{L_1}{L_2}} - \frac{1}{6}R^{M}_{\frac{M}{L_1}} + \frac{1}{6}R^{M}_{\frac{M}{L_2}}.$$

Proof. Using Gauss' equation for the given setting ensures

$$R^M_{ikik} = R^{L_1}_{ikik}$$

for $i, k \leq l_1$. Then we obtain the result by (83).

□

Corollary 6.23. *If $M = \mathbb{R}^n$, $n > 1$, we obtain*

$$W^{21M} = -\frac{1}{4}\left(\|\tau(\varphi)\|_1^2 - \|\tau(\psi)_{|L_2}\|_M^2\right) + \frac{1}{3}\mathrm{Scal}_F - \frac{1}{6}R^{L_1}_{\frac{L_1}{L_2}}.$$

Proposition (1.13) already reviewed the special role of the sectional curvature.

We prove a remarkable fact about embeddings into manifolds of constant sectional curvature.

Corollary 6.24. *Let L_2 be a submanifold of the Riemannian manifold L_1, which itself is a submanifold of the Riemannian manifold (M, g). Assume that the sectional curvatures of the ambient manifolds are constant and all the embeddings are totally geodesic.*
Then the potential W_{21M} is constant and therefore the cocycle C^{21M} equals one, which means the corresponding Wiener measures are equal.

Proof. Because the embeddings are totally geodesic the tension fields $\tau(\varphi)$ and $\tau(\psi)$ vanish. Using the composition formula (82) this yields that the tension field $\tau(\psi \circ \varphi)$ of the composition also vanishes. Moreover because the sectional curvature determines all the other curvature terms the statement follows by Theorem (6.19). $\square$

We study two examples, with the help of MAPLE®. First we discuss an example, where the cocycle differs from one.

Example 6.25. *Let an ellipsoid (spheroid) E with parameters $a = 1$, and $b = c = 2$ be embedded into $\mathbb{R}^3$. Furthermore, let e be an ellipse, embedded into E. In detail: E is parameterized by*

$$\psi\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (0, 2\pi) \to E, (\theta, \phi) \mapsto (2\cos(\theta)\cos(\phi), 2\cos(\theta)\sin(\phi), \sin(\theta))^t,$$

while we have the following parameterization

$$P_e\colon (0, 2\pi) \to E, t \mapsto (0, 2\sin(t), -\cos(t))^t$$

for e. Note e is the intersection of E with the $y - z-$plane (see figure 6.1). The induced metric g on E is given by

$$\begin{pmatrix} -3\cos(\theta)^2 + 4 & 0 \\ 0 & 4\cos(\theta)^2 \end{pmatrix},$$

where we already note that g is independent of the angle ϕ due to symmetry reasons. The explicit computations are presented in the Appendix E. Note that we describe the ε-tube around E by a normalization of the normal vector computed with MAPLE®. A parameterization of the tube is then given by

$$P\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (0, 2\pi) \times (-\varepsilon, \varepsilon) \to E(\varepsilon), (\theta, \phi, u) \mapsto \psi(\theta, \phi) + u \cdot \frac{1}{\|n\|} n(\theta, \phi),$$

where $n(\theta, \phi)$ denotes the normal vector field on E. The Gaussian curvature is given by

$$K_E = \frac{16}{(16\cos(\theta)^2 + 4\sin(\theta)^2)^2}.$$

Considering the $\varepsilon-$tube with respect to E of e, we can parameterize it as a part of the ellipsoid given by

$$\tilde{E}\colon (0, 2\pi) \times (-\rho, \rho) \to e(\varepsilon),\ (\theta, \phi) \mapsto (2\cos(\theta)\cos(\phi), 2\cos(\theta)\sin(\phi), \sin(\theta))^t,$$

for a suitably chosen ρ.

The potential W_0 for the embedding of e into the three-dimensional euclidean space $\mathbb{R}^3$ is given by

$$W_0 = \frac{1}{4}\kappa(t),$$

where κ denotes the curvature of the curve. For this result see e.g. Chapter 2 or [BMSW09]. With the help of MAPLE® and the knowledge about a formula for the curvature of an ellipsoid, we computed the potential for the embeddings of E into $\mathbb{R}^3$ and e into E. The result shows that $W_{eE\mathbb{R}^3}$ is not constant. The complete computation is given in the Appendix.

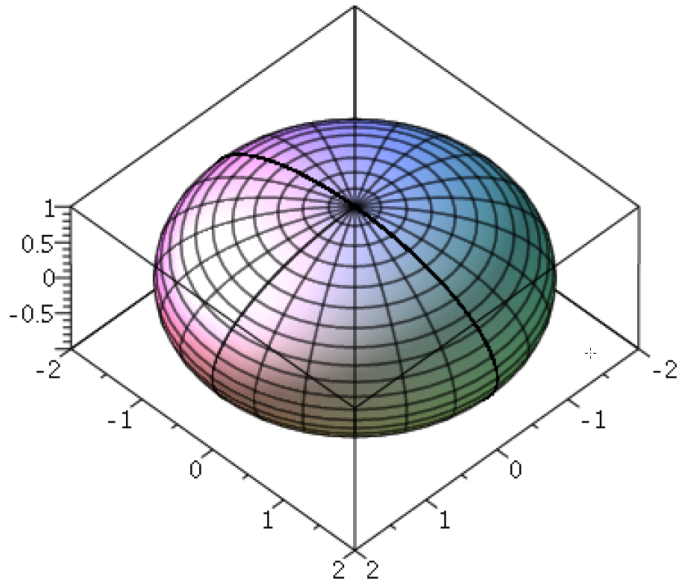

FIGURE 6.1: An ellipsoid with an ellipse.

Remark. *Note that restriction to the two-dimensional problem of the ellipse e embedded into $\mathbb{R}^2$ given by*

$$(0, 2\pi) \to e;\ t \mapsto (2\cos(t), \sin(t))^t\,,$$

gives rise to the formula

$$\|\tau\|^2 = \frac{4}{(4\cos(t)^2 + \sin(t)^2)^3}.$$

According to the formula for the density we can tell that the sojourn probability of the Brownian particles is largest, where the curvature is largest and therefore at the intersection with the x-axis, i.e. the major axis of e.

Example 6.26. *Another example which we state is the embedding of $\mathbb{S}^1$ into $\mathbb{R}^3$ in comparison with the embedding of $\mathbb{S}^1$ into a torus embedded in $\mathbb{R}^3$.*
Here we consider $\mathbb{S}^1$ as a circle lying in the x-z-plane around the torus as given in figure 6.2.
In this setting one obtains for the embedding of $\mathbb{S}^1$ in $\mathbb{R}^3$ the constant potential

$$W_0^{\mathbb{S}^1\mathbb{R}^3} = -\frac{1}{4}.$$

This becomes clear, by restriction of $\mathbb{R}^3$ to the two-dimensional subspace spanned by the x- and z-axis and because the curvature of a circle with radius $r = 1$ is one everywhere.
A concrete computation with the help of MAPLE® showed that this result is not equal to the potential one obtains when first restricting it to the torus and then to $\mathbb{S}^1$.

In Appendix E we present the whole computation.

Remark. 1. *Note that in the case of $\mathbb{S}^1$ given by a circle in the x-y-plane around the torus, the cocycle indeed is equal to one. This is due to the rotational symmetry.*

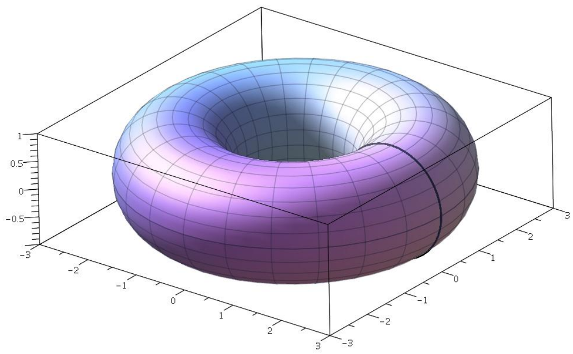

FIGURE 6.2: Torus and $\mathbb{S}^1$ embedded into $\mathbb{R}^3$.

2. *Due to Corollary (6.24) in the case that all Riemannian manifolds are spheres $\mathbb{S}^k, \mathbb{S}^l$ and $\mathbb{S}^n$ with $k < l < n$ and the standard embeddings, we obtain that the cocycle equals one.*

Chapter 7

Riemannian submersions with totally geodesic fibers

We already know that the projection of the tube $\pi\colon (L(1), g_{Sa}) \to L$ is a Riemannian submersion with totally geodesic fibers (see Section (1.2)).
Because there are remarkable results about the connection of the Laplacian according to a metric on the total space of a Riemannian submersion and on the submanifold, we want to study this setting in more detail in this chapter.

Assume the tube $L(1)$ is equipped with the Sasaki metric. Then, the fibers are minimal submanifolds. Therefore, in this special setting, we deal with harmonic morphisms. By a result of Darling (see [Dar82], p. 220) a harmonic morphism $p\colon M \to L$ between two Riemannian manifolds maps a Brownian motion on M with an arbitrary starting point $q \in M$ to a time-changed Brownian motion on L.

Note that due to the remark at the end of Section (2.5) the Riemannian submersions $p\colon M \to L$ with totally geodesic fibers are exactly the ones, which map Brownian motion to Brownian motion.

7.1 The fundamental tensors for a Riemannian submersion

In [O'N02] the author introduces the two so-called fundamental tensors for a Riemannian submersion. We want to introduce both of them and some other remarkable facts. In the following we will use the connection of the first fundamental tensor and the second fundamental form of the fibers. Moreover, we give some relation between the second fundamental tensor an the curvature of the connection.

Let $\pi\colon (M, g) \to (L, g_L)$ be a Riemannian submersion and $\mathfrak{H}, \mathfrak{V}$ denote the horizontal and vertical projection of TM onto the horizontal and vertical subspaces. We denote these spaces by the same symbols.

Definition 7.1 (Basic vector fields). *A horizontal vector field X^H is called basic, if it is $\pi-$related to a vector field X_* on L, i.e. $\pi_* X^H(W) = X_{*,W}$.*

Remark. *Because π_* restricted to $\mathfrak{H}$ is an isometry, the horizontal lift exists and is unique. Therefore, there is a one-to-one correspondence between basic vector fields and the vector fields on L.*

O'Neill (see [O'N02], p. 460 ff.) proved some facts about basic vector fields in connection with the fundamental tensors, which are given as follows.

Definition 7.2 (The fundamental tensors)**.** *Let X, Y be vector fields on M and ∇ the Levi-Civita connection to g. The first fundamental tensor T is defined by*

$$T_X Y = \mathfrak{H}\left(\nabla_{\mathfrak{V}X}\mathfrak{V}Y\right) + \mathfrak{V}\left(\nabla_{\mathfrak{V}X}\mathfrak{H}Y\right).$$

The second fundamental tensor A is given by

$$A_X Y = \mathfrak{V}\left(\nabla_{\mathfrak{H}X}\mathfrak{H}Y\right) + \mathfrak{H}\left(\nabla_{\mathfrak{H}X}\mathfrak{V}Y\right).$$

For more information on the fundamental tensors – besides the facts given in [O'N02] – we e.g. refer to [Bes87], Chapter 9, or [BY17], Section 5.3.

Lemma 7.3. *For U, V vertical vector fields, the first fundamental tensor $T_U V$ equals the second fundamental form of the fibers.*

Proof. See e.g. [BEM16], p. 4. □

Lemma 7.4. *For horizontal vector fields X and Y, the second fundamental tensor A can be computed as*

$$A_X Y = \frac{1}{2}\mathfrak{V}[X, Y].$$

Proof. See e.g. [Bes87], Proposition 9.24. □

Remark. *Note that therefore A is sometimes referred to as integrability tensor.*

7.2 Riemannian submersions and Laplacians

The reason to investigate Riemannian submersions in general is due to the results of Bérard Bergery and Bourguignon in [BB82]. Especially in (3.9) the following is stated: For a function $\overline{f} \in C^2(L)$ and a Riemannian submersion with totally geodesic fibers $\pi \colon M \to L$ one has

$$\left(\triangle^L \overline{f}\right) \circ \pi = \triangle^M \left(\overline{f} \circ \pi\right).$$

This is relevant for the asymptotic behavior, because for a function $f \in C^\infty(M)$ we obtain

$$f_{|L(\varepsilon)} = \left(f_{\big|L}\right) \circ \pi + O(\varepsilon),\ \varepsilon \to 0,$$

and for the corresponding Laplacians

$$\triangle^M f = \left(\triangle^L f_{\big|L}\right) \circ \pi + O(\varepsilon),\ \varepsilon \to 0.$$

Therefore, close to the submanifold every function is basically described by its restriction to L, i.e. the curvature of the fibers is irrelevant. We want to prove this observation.

First, remember every fiber $F_x := \pi^{-1}(x)$ is a Riemannian (sub-)manifold. Furthermore we have an induced connection on the normal bundle and are able to define the horizontal distribution.

By [Her60], Proposition 3.3, and [BB82], (1.11), there is a fiber metric g_{F_x} on F_x given uniquely up to diffeomorphisms. Moreover, for a path $\sigma \colon I \to L$ the diffeomorphism

$h_\sigma \colon \pi^{-1}(\sigma(0)) \to \pi^{-1}(\sigma(1))$, defined by mapping each point $x_0 \in \pi^{-1}(\sigma(0))$ onto the ending point of the horizontal lift of σ starting at x_0, is an isometry.
Without loss of generality (otherwise study a path connected component) we assume L to be path-connected and obtain that the fibers

$$\pi^{-1}(x) \cong \pi^{-1}(y),$$

are isometrically isomorphic for all $x, y \in L$. Further information about the proof can be found in Chapter 9 in [Bes87].

We are now able to explain one more quantity in the potential.

Proposition 7.5. *Let $\pi \colon L(1) \to L$ be a Riemannian submersion with totally geodesic fibers. Then the function*

$$h \colon L \to \mathbb{R};\ y \mapsto \langle R^{\pi^{-1}(y)}(X, W)W, Y\rangle_L$$

is constant for all $W \in NL$ and $X, Y \in \Gamma(L)$.

Proof. As already seen, all fibers F_p are isometrically isomorphic to a typical fiber F and are totally geodesic submanifolds of the tube $L(1)$. Due to Gauss' embedding theorem we obtain the identity

$$R^{\pi^{-1}(y)} = R^M{}_{|TF_y}$$

for the curvature on each fiber, for all $x \in L$. To show that the curvature restricted to the subbundle of the tangent space of the fibers is constant, we use that all F_y, $y \in L$, are isometric to a prototype F with $G-$invariant metric g_F and G is the structure group of NL, i.e.

$$R^M{}_{|TF_y} = R^M{}_{|TF}.$$

□

For the potential W_0 we obtain in fact that all quantities depending on the curvature of the fibers are constant, i.e. $\mathrm{Scal}_F = C$. We obtain for the Radon-Nikodym density defined by (15),

$$\frac{d\mu_0^x}{d\mathbb{W}_L^x}(\eta) = \frac{\exp\left(-\frac{1}{2}\int_0^T(\frac{1}{2}\|\tau(\varphi)\|^2 - \sum_{i,j}\|\mathrm{II}(\varphi)(e_i,e_j)\|^2 - R^M_{i\alpha i\alpha} - \frac{1}{3}\mathrm{C})(\eta(s))\,ds\right)}{\int_{\Omega_L}\exp\left(-\frac{1}{2}\int_0^T(\frac{1}{2}\|\tau(\varphi)\|^2 - \sum_{i,j}\|\mathrm{II}(\varphi)(e_i,e_j)\|^2 - R^M_{i\alpha i\alpha} - \frac{1}{3}\mathrm{C})(\xi(s))\,ds\right)\mathbb{W}_L^x(d\xi)}$$

and the constant terms just cancel each other and do not occur any more.

Another important aspect is given because of the following observation: The connection between Laplacians and Riemannian submersions is worth to study, as a Brownian motion is completely described by the Laplace-Beltrami operator of the underlying manifold. Hence, we can characterize a Brownian motion whenever we are able to compute the Laplacian.

In this context, we state a fact which is given in [Pau90], Theorem 1, p. 426. The author proves that in case that the mean curvature vector – and hence the tension field – is constant along the fibers of a Riemannian submersion $\pi \colon M \to L$, we can effectively describe the projection of Brownian motion. In short, Brownian motion on M projects down to a diffusion process on L, which can be split into Brownian motion on the submanifold L and an additional drift V (remember that due to a remark in Section (1.5) this is basically the same information as a potential given on L), depending on the codimension and the mean curvature. This observation supports our previous investigations and results on the potential and the dependency of the tension field.

7.2.1 Decomposition of the Laplacian

Another important fact is given in [BB82]. In the first section of this paper, the authors introduce the vertical Laplacian as the second-order differential operator acting on a function $f \in C^2(M)$ by

$$(\triangle_v f)(p) := \triangle^{F_p}\left(f_{|F_p}\right)(p),$$

where $\triangle^{F_p}$ denotes the Laplacian on the fiber through $\pi(p)$ with the induced metric from M.
They call the difference operator

$$\triangle_h := \triangle^M - \triangle_v,$$

the horizontal Laplacian.

Remark. *In general $\triangle_h$ and $\triangle_v$ are non-elliptic; therefore, the operators are no Laplacians in the common sense.*

The remarkable observation is now given in Theorem (1.5) in [BB82]. The authors prove that for a Riemannian submersion with totally geodesic fibers the operators $\triangle_h, \triangle_v, \triangle^M$ commute pairwise.
In our setting we therefore obtain for every $f \in C^4(L(1))$ that

$$\left[\triangle^{Sa}, \triangle_v\right] f = \left[\triangle^{Sa}, \triangle_h\right] f = [\triangle_h, \triangle_v] f = 0.$$

For the induced metric on the tube one does not have a similar result. In contrast to the situation there, for a Riemannian submersion with totally geodesic fibers, we obtain a decomposition of the Dirichlet Laplacian into two independent parts. This provides conclusions about the spectrum and the associated quadratic form under the rescaling.

The knowledge about Riemannian submersions seems to be helpful. Hence, we are interested in some kind of classification of those maps. In the next section we want to study and construct Riemannian submersions with totally geodesic fibers which differ from the Sasaki metric and give a sort of classification result.

7.3 A class of natural metrics

Let M and L be complete Riemannian manifolds and $\pi\colon M \to L$ a Riemannian submersion. Whenever the fibers of π are totally geodesic submanifolds, by [Bes87], 9.42, we obtain that π is a smooth fiber bundle with structure group $G = \mathrm{Isom}(F)$ for a typical fiber F.

As the properties of the Sasaki metric turned out to be helpful and the properties of Riemannian submersions with totally geodesic fibers are useful in our situation, we first want to have a look at other metrics on the total space for which π is a Riemannian submersion.

Additionally, we still want the vertical distribution to be orthogonal to the horizontal distribution, which is defined by a fixed connection. This leads to the study of so-called natural metrics on the total space of the bundle.

In the introduction (Chapter 1, p. 1) of [MT88] the authors state the fact that the Sasaki metric g_{Sa} "is perhaps the most natural metric on TM depending only on the Riemannian structure of M, but it is extremely rigid".
It is the only natural metric on the tangent bundle whose induced metric on the fibers is an Euclidean metric.

We first define the objects we are looking at.

Definition 7.6. *Let (L, g_L) be a closed Riemannian manifold and $\pi\colon E \to L$ be a vector bundle over L with bundle metric and metric connection ∇. Denote by $K\colon T_E \to E$ the connection map (according to ∇) of the bundle. Define the horizontal distribution by $\mathfrak{H}_E := \ker K$. We call a metric g on E (or a tubular neighborhood $E(0)$ around the zero section in E) a* ***natural*** *metric, if*

1. *the horizontal and vertical subspace are orthogonal with respect to g,*
2. *$\pi\colon (E, g) \to (L, g_L)$ (or $\pi\colon (E(0), g) \to (L, g_L)$) is a Riemannian submersion.*

Remark.
1. *For each $W \in E$ the connection map is defined fiber-wise by $K_W\colon T_W E \to E_x$, where $x = \pi(W)$.*
2. *Note that by definition the Sasaki metric is indeed a natural metric in the sense of Definition (7.6) on the normal bundle.*
3. *In the literature some authors describe so-called g-natural metrics, which should not lead to any confusion with our setting. They study a special class of metrics on the tangent bundle TM, coming from a first order natural operator D on the bundle of functors of Riemannian metrics to symmetric two-forms. For this definition see Chapter 2 in [AS05]. Note that our definition of a natural metric is a generalization of this concept.*

Moreover, e.g. in [GK02] and [AS05] natural metrics on the tangent bundle of a Riemannian manifold were studied. Abbassi and Sarih studied the g-natural metrics on TM in detail in several papers. In [KS88] the authors give some kind of classification result.
Besides the Sasaki metric there is another very well-studied natural metric on the tangent bundle, the so-called Cheeger-Gromoll metric.

We generalize the constructions from the works mentioned above to arbitrary bundles. Our approach follows the approach of [MT88] for the Sasaki and Cheeger-Gromoll metric on the tangent bundle $\pi\colon TM \to M$ over a Riemannian manifold M.

To our knowledge the following results about natural metrics on arbitrary bundles are new.
We now want to give a partial classification of natural metrics on the total space of a bundle, where the associated projection turns out to be a Riemannian submersion with totally geodesic fibers.

Theorem 7.7. *Let E, L be complete Riemannian manifolds, L as above and $\pi\colon E \to L$ a Riemannian submersion with totally geodesic fibers. Let $W \in E$ and $K_W\colon T_W E \to E_x$ be a connection map.*
Then, a natural metric on $E(0)$ is given by

$$g_R(X,Y) := g_L(\pi_* X, \pi_* Y) + c_{W,Q} Q_{(W)}(K_W X, K_W Y) \tag{85}$$

for vectors $X, Y \in T_W E$.
Here $Q_{(W)}(A,B) := g_{NL}(A,B) + O\left(\|W\|^2\right)$ is a symmetric, positive definite bilinear form on E, where the remainder of higher order is symmetric in W and hence only even powers of $\|W\|$-terms occur. Moreover, $c := c_{W,Q} \in \mathbb{R}$ depends on the choice of the bilinear form and varies continuously in W.

Remark. 1. *$Q_{(W)}$ does not necessarily be of higher order in $\|W\|$-terms, i.e. $Q_{(W)} := g_{NL}$ is an admissible choice for this bilinear form.*

2. *The higher order terms in $Q_{(W)}$ need to be symmetric in W and therefore just even powers of higher order terms occur.*

3. *The second order term of the bilinear form is either of the form*

$$q^1(K_W^\perp X, W, K_W^\perp Y, W),$$

with a symmetric 4-form q^1, or of the form

$$q^a(K_W^\perp X, W) \cdot q^b(K_W^\perp Y, W),$$

for symmetric $(2,0)$-tensors q^a and q^b. Hence, we write the second order term in local coordinates in the form $q_{W\,\alpha\mu\beta\nu} w^\mu w^\nu$.

4. *One obtains the Sasaki metric by choosing $c \equiv 1$ and $Q_{(W)}(A,B) := g_{NL}(A,B)$.*

Proof. We first need to show that g_R is well-defined. By the introduction of this section, we already know that all fibers F_p are isometric to a typical fiber F and π is a fiber bundle.

It is well known (see e.g. [KN96], p. 50 ff.) that there is a (one-to-one) correspondence between fiber bundles and principal bundles. To any given fiber bundle, an associated principal bundle P can be constructed. This principal bundle has the same transition functions $\Phi_{ab}\colon U_a \cap U_b \to G$, where G is the structure group and $\{\pi^{-1}(U_a)\}_a$ is an open cover of trivializations. Vice versa given a principal bundle P and a typical fiber F, a fiber bundle can be constructed, because the structure group G operates on $P \times F$ and by defining the equivalence relation

$$(p,f) \sim (p \cdot g, g^{-1} f),$$

for $g \in G$, $(p,f) \in P \times F$ the associated fiber bundle is given by ${}^{(P\times F)}\!/\!_{\sim}$.

In our case π is associated to a principal bundle $p\colon P \to L$ with structure group $G = \mathrm{Isom}(F)$ and a Riemannian manifold P. Note that G is compact, if F is compact.
We obtain the commutative diagram

$$\begin{array}{ccc} P \times F & \xrightarrow{h} & E \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow{p} & L \end{array}$$

where h is given by the diagonal action of G on $P \times F$. We identify $E \cong P \times_G F$. Note that h is not necessarily a Riemannian submersion if $P \times F$ is endowed with the product metric. But, the product metric yields a metric on NL. A bundle connection on P induces a connection on E and therefore an associated connection map $K_W : T_W E \to E$ is explained.

For vector fields X, Y on E we are now able to define

$$g_R(X, Y) := g_L(\pi_* X, \pi_* Y) + cQ_{(W)}(K_W X, K_W Y),$$

with the above properties. Now it remains to prove, that g_R indeed defines a natural metric on E.

By defining $\mathfrak{H}_W := \ker(K_W)$ as horizontal distribution, we are able to explain horizontal lifts and obtain

$$\langle X^h, Y^v \rangle_R = 0,$$

for each pair $(X^h, Y^v) \in \mathfrak{H} \times \mathfrak{V}$. Moreover, it is clear that g_R is an isometry on the horizontal subspace.

By symmetry and positive definiteness of $Q_{(W)}$, the fact that all higher order terms are symmetric in W and as L and NL are equipped with Riemannian metrics (i.e. positive definite and symmetric tensors), g_R is positive definite and symmetric for a suitable choice of c_Q. $\square$

Remark. *Note that we just need to secure the positive definiteness on a (small) tubular neighborhood $E(0)$.*

Remark. *In [BB90], p. 336, the authors introduce a metric on $P \times F$, such that h is indeed a Riemannian submersion.*

We will now discuss that Theorem (7.7) can be reversed by fixing some additional input data.

Let B, F be Riemannian manifolds and $\pi \colon X \to B$ a (G, F) bundle with structure group G and a G-invariant Riemannian metric g_F on F. According to Theorem (3.5) in [Vil70] it is known that there exists a natural Riemannian metric on X , such that π is a Riemannian submersion with totally geodesic fibers.

We generalize the studies of natural metrics on the tangent bundle given in [AS05], [KS88], [GK02] and [Kap01] to our case and give an explicit choice for the natural metric on the total space.
The most similar description of the natural metrics (on the tangent bundle) is given in [AS05], Proposition 2.2.

As already discussed, natural metrics depend on different quantities. This fact indicates there is are many natural metrics on the total space of a vector bundle or a manifold.

Henceforth, we will later comment on the amount of different natural metrics and their dependency of the fixed data.

Theorem 7.8. *Let (B, g_B) be a closed Riemannian manifold and $\pi\colon E \to B$ a bundle over B. Choose a connection map K on E. Then g_R given by (85) is a natural metric on $E(0)$ such that $\pi\colon (E, g_R) \to (B, g_B)$ is a Riemannian submersion with totally geodesic fibers.*

Proof.

1. We already saw that g_R defines a Riemannian metric.
2. We need to prove that π is a Riemannian submersion. According to the chosen connection map there exists a horizontal distribution $\mathfrak{H} := \ker(K)$ with
$$TE = \mathfrak{H} \oplus \mathfrak{V},$$
where $\mathfrak{V} := \ker(\pi_*)$.
Because we are able to write down the unique horizontal lift of a vector field on L and by definition of g_R we obtain for all $X, Y \in \Gamma(L)$ that
$$\langle X^h, Y^h \rangle_R = \langle X, Y \rangle_L.$$
3. It remains to show that all fibers are totally geodesic submanifolds. Let $\overline{\nabla}$ denote the Levi-Civita connection of g_R. By [Bes87], 9.26, all fibers are totally geodesic submanifolds, if the fundamental tensor T vanishes identically, i.e. in this case we show that
$$T_X Y = \mathfrak{H}\left(\overline{\nabla}_{\mathfrak{V}X} \mathfrak{V} Y\right) + \mathfrak{V}\left(\overline{\nabla}_{\mathfrak{V}X} \mathfrak{H} Y\right)$$
vanishes. By the tensor property of T it suffices to show that T vanishes for every combination of horizontal and vertical vectors, as each vector can uniquely be decomposed into its horizontal and vertical part. Because $\mathfrak{V}\left(X^h\right) = 0$ for every horizontal vector X^h, the definition of T directly leads to
$$T_{X^h} Y^v = T_{X^h} Y^h = 0.$$
Using the fact that the horizontal and vertical subspace are orthogonal with respect to g_R we have the following simplification:
   a) $T_{X^v} Y^h = \mathfrak{H}\overline{\nabla}_{\mathfrak{V}X^v} \mathfrak{V} Y^h + \mathfrak{V}\overline{\nabla}_{\mathfrak{V}X^v} \mathfrak{H} Y^h = \mathfrak{V}\overline{\nabla}_{X^v} Y^h$,
   b) $T_{X^v} Y^v = \mathfrak{H}\overline{\nabla}_{\mathfrak{V}X^v} \mathfrak{V} Y^v + \mathfrak{V}\overline{\nabla}_{\mathfrak{V}X^v} \mathfrak{H} Y^v = \mathfrak{H}\overline{\nabla}_{X^v} Y^v$.

   To prove that these expressions actually vanish in the given situation, we show
   i) $\overline{\nabla}_{X^v} Y^h$ is horizontal,
   ii) $\overline{\nabla}_{X^v} Y^v$ is vertical.

   Let therefore $Z^v \in \mathfrak{V}$ be vertical. In the following all inner products are taken with respect to g_R. Due to the Koszul formula for the Levi-Civita connection we obtain
$$\begin{aligned} 2\langle \overline{\nabla}_{X^v} Y^h, Z^v \rangle =& X^v \langle Y^h, Z^v \rangle + Y^h \langle X^v, Z^v \rangle - Z^v \langle X^v, Y^h \rangle \\ &+ \langle [X^v, Y^h], Z^v \rangle - \langle [Y^h, X^v], Z^v \rangle - \langle X^v, [Y^h, Z^v] \rangle = 0, \end{aligned}$$
using properties of the Lie brackets due to [Dom62], Lemma 2, and [BY89]. This proves i).

For ii) we use that with respect to g_R we know

$$0 = \langle V, H \rangle,$$

for all $V \in \mathfrak{V}$ and $H \in \mathfrak{H}$. Let $U \in \mathfrak{V}$, the product rule yields

$$0 = U\langle V, H \rangle = \langle \overline{\nabla}_U V, H \rangle + \langle V, \overline{\nabla}_U H \rangle.$$

The last term on the right-hand side vanishes due to i), which means

$$0 = \langle \overline{\nabla}_U V, H \rangle,$$

for all $H \in \mathfrak{H}$. Therefore,

$$\overline{\nabla}_U V \in \mathfrak{V}$$

and the proof is complete.

□

Due to the fact that we fixed K as an additional piece of information, we are interested in the behavior of g_R, if K is chosen differently. Moreover, we want to answer how many different Riemannian submersions with totally geodesic fibers can be constructed in this way.

Always have in mind that we already proved $E \cong NL$ for an embedding $L \subset M$. Therefore, the connection map is a quantity depending on the normal bundle and on the amount of metric connections there.
Due to Lemma (4.10) two different connections just differ by a one-form

$$\omega \in \Omega^1(\mathrm{End}(E))$$

such that we see there are as many possible choices as there are one-forms.

Theorem 7.9. *Let (L, g_L) be a Riemannian manifold and $\pi \colon E \to L$ a vector bundle over L. Denote by F the typical fiber and by ∇ a metric connection on the bundle.*
Then by fixing an $O(F)$-invariant Riemannian metric g_F on F, where $O(F)$ denotes the orthogonal group of F, there exists one and only one metric g on $E(0)$ such that $\pi \colon (E(0), g) \to (L, g_L)$ is a Riemannian submersion with totally geodesic fibers and horizontal distribution according to ∇.
Additionally this metric is always given by a metric of the form of (85).

Remark.
1. *The amount of Riemannian submersions with totally geodesic fibers therefore depends on the amount of metric connections on NL and on the amount of $O(F)$-invariant Riemannian metrics on the fiber F.*
2. *Because fiber bundles and principal G-bundles are in one-to-one correspondence we obtain the same result by fixing a principal connection and a typical fiber (F, g_F) with g_F being a G-invariant metric.*

Proof. Due to [Vil70] and [Bes87], 9.59 and 9.60, fixing these data, we obtain one and only one metric such that π is a Riemannian submersion with totally geodesic fibers isometric to (F, g_F) with a suitable horizontal distribution. We already proved that the Riemannian submersions are exactly induced by metrics of g_R-type. Because the additional data of the fiber metric and the connection on NL fixing K can be chosen freely, we obtain the explanation for the amount of the degrees of freedom. □

Due to the results in [YY91], p. 210 ff., this setting is relevant in the theory of Brownian motions. The crucial observation is that a Riemannian submersion $\pi\colon M \to L$ with totally geodesic fibers and the above (G, F)-structure projects a Brownian motion B_t on M down to a Brownian motion on L. Let h_t be the horizontal lift of $\pi(B_t)$ into $G(M)$. Then the inverse of the horizontal lift applied to B_t gives a Brownian motion on the fiber. Note that this Brownian motion may possibly be time-changed.

Thus, by knowing about the Riemannian submersions we know about the existence of Brownian motions on the submanifold. We would like to construct these Brownian motions, i.e. we want to study what happens with the Laplacian and the Brownian motion to the natural metrics on the total space.

Remark. *In case of Riemannian submersions with totally geodesic fibers, a horizontal Brownian motion, i.e. a diffusion process on M with generator $-\frac{1}{2}\triangle_h$, can be constructed as the horizontal lift of a Brownian motion on the base space. More generally, if π is a Riemannian submersion with fibers that have zero mean curvature, the horizontal lift of a Brownian motion on L starting in $x \in L$ is a horizontal Brownian motion on M.*

Now, we give another explicit example of a natural metric, differing from the Sasaki metric and compute the representation in local coordinates.

A natural metric on the normal bundle which has been studied recently (but mainly on the tangent bundle), is the so-called Cheeger-Gromoll metric (see e.g. [GK02]).

Definition 7.10. *Denote by $\pi\colon NL \to L$ the normal bundle over L and by $K^\perp$ its connection map. Let $W \in NL$ and $X, Y \in T_W NL$. Then the Cheeger-Gromoll metric on NL is defined by*

$$\begin{aligned} &g_{CG,W}(X, Y) \\ :=&\langle \pi_* X, \pi_* Y\rangle_L + \frac{1}{1+\|W\|^2}\left(\langle K^\perp_W X, K^\perp_W Y\rangle_{NL} + \langle K^\perp_W X, W\rangle_{NL}\langle K^\perp_W Y, W\rangle_{NL}\right). \end{aligned}$$

Example 7.11. *According to our notation, the Cheeger-Gromoll metric defines a natural metric in the sense of* (85)*, with $c_{W,Q} := \frac{1}{1+\|W\|^2}$ and*

$$Q_{(W)}(A, B) := g_{NL}(A, B) + g_{NL}(A, W) g_{NL}(B, W).$$

In the local coordinate representation from the remark above, we obtain

$$q_{W\,\alpha\mu\beta\nu} w^\mu w^\nu = \delta_{\alpha\mu}\delta_{\beta\mu} w^\mu w^\nu.$$

Let the notation from Definition (7.10) be given. Denote by $\{e_\alpha\}_\alpha$ an orthonormal frame of NL and let $\{\partial_A\}_A$ be a basis of TNL. Then,

$$\pi_*\partial_\alpha = 0,\ K^\perp\partial_\alpha = e_\alpha \ \text{ and } \ K^\perp(\partial_i^h) = 0.$$

Hence,

$$K^\perp_W \partial_i = K^\perp_W\left(\partial_i - w^\mu C^\alpha_{i\mu}\partial_\alpha + w^\mu C^\alpha_{i\mu}\partial_\alpha\right) = w^\mu C^\alpha_{i\mu} K^\perp_W(\partial_\alpha) = w^\mu C^\alpha_{i\mu} e_\alpha.$$

For an arbitrary vector field $X = X^i\partial_i + X^\alpha\partial_\alpha \in T_W NL$ we thereby obtain

$$K_W^\perp(X) = \left(X^i w^\mu C_{i\mu}^\alpha + X^\alpha\right) e_\alpha.$$

Inserting these expressions into the definiteness given above, we obtain that the Cheeger-Gromoll metric is given in local Fermi coordinates by

$$g_{CG}(x,w) = \left(\begin{array}{c|c} g_{CG\,ij} & g_{CG\,i\beta} \\ \hline g_{CG\,\alpha j} & g_{CG\,\alpha\beta} \end{array}\right),$$

where

$$\begin{aligned}
g_{CG\,ij} &= g_{L\,ij}(x) + \frac{1}{1+\|W\|^2}\left(w^\mu w^\nu C_{i\mu}^\alpha C_{j\nu}^\beta \delta_{\alpha\beta} + w^\mu w^\nu w^\sigma w^\tau C_{i\mu}^\alpha C_{j\nu}^\beta \delta_{\alpha\sigma}\delta_{\beta\tau}\right), \\
g_{CG\,i\beta} &= \frac{1}{1+\|W\|^2}\left(w^\mu C_{i\mu\beta} + w^\mu w^\sigma w^\tau C_{i\mu}^\alpha \delta_{\alpha\sigma}\delta_{\beta\tau}\right), \\
g_{CG\,\alpha j} &= \frac{1}{1+\|W\|^2}\left(w^\nu C_{j\nu}^\beta \delta_{\alpha\beta} + w^\sigma w^\mu w^\tau \delta_{\alpha\sigma} C_{j\nu}^\beta \delta_{\beta\tau}\right), \\
g_{CG\,\alpha\beta} &= \frac{1}{1+\|W\|^2}\left(\delta_{\alpha\beta} + \delta_{\alpha\tau}\delta_{\sigma\beta} w^\sigma w^\tau\right).
\end{aligned}$$

For more information on this metric see Chapter 4 of [Kap01], where the author studied the Cheeger-Gromoll metric on the tangent bundle in detail. The representation of the metric in local Fermi coordinates is new, but in this article a formula for the Levi-Civita connection and the curvature tensor R can be found.

With these observations on hand, we explore the metric g_R in more detail.
We develop an expression in local Fermi coordinates for these natural metrics in general. Note that we just compute the metric up to order $O\left(\|W\|^2\right)$.

Corollary 7.12. *Let the setting from Theorem (7.7) be given. Then, in local Fermi coordinates the natural metric g_R, is given by*

$$\begin{aligned}
&g_R(x,w) \\
&= \left(\begin{array}{c|c} g_{L\,ij}(x) + c w^\mu w^\nu C_{i\mu}^\alpha C_{j\nu\alpha} + O\left(\|W\|^4\right) & c\left(w^\alpha C_{i\alpha\beta} + w^\alpha w^\rho w^\nu C_{i\alpha}^\mu q^1_{\mu\rho\beta\nu}\right) + O\left(\|W\|^4\right) \\ \hline c w^\nu C_{j\nu\alpha} + O\left(\|W\|^3\right) & c\left(\delta_{\alpha\beta} + q^1_{\alpha\mu\beta\nu} w^\mu w^\nu + O\left(\|W\|^4\right)\right) \end{array}\right)
\end{aligned}$$

if $Q_{(W)}(A,B) = g_{NL}(A,B) + q^1(A,W,B,W) + O\left(\|W\|^4\right)$ and $c = c_{W,Q} Q_{(W)}$.

Remark. *The results for the different representation of $Q_{(W)}$ follow analogously.*

Proof. Let the metric be given in the form

$$g_R(x,w) = \left(\begin{array}{c|c} g_{R\,ij} & g_{R\,i\beta} \\ \hline g_{R\,\alpha j} & g_{R\,\alpha\beta} \end{array}\right).$$

By definition of the tangent map and the connection map in local coordinates, we compute the entries as follows. First of all we compute the components into fiber direction:

$$\begin{aligned}
g_{R\,\alpha\beta} &= \langle \partial_\alpha, \partial_\beta \rangle_R \\
&= c_{W,Q} Q_{(W)}(e_\alpha, e_\beta)
\end{aligned}$$

$$
\begin{aligned}
&= c_{W,Q}\left(g_{NL}(e_\alpha, e_\beta) + q^1(e_\alpha, W, e_\beta, W) + O\left(\|W\|^4\right)\right) \\
&= c_{W,Q}\left(\delta_{\alpha\beta} + q^1_{\alpha\mu\beta\nu} w^\mu w^\nu + O\left(\|W\|^4\right)\right).
\end{aligned}
$$

With the help of this result, we find

$$
\begin{aligned}
g_{r\,i\beta} &= \langle \partial_i, \partial_\beta \rangle_R \\
&= \langle \partial_i - w^\alpha C^\mu_{i\alpha}\partial_\mu + w^\alpha C^\mu_{i\alpha}\partial_\mu, \partial_\beta \rangle_R \\
&= \langle w^\alpha C^\mu_{i\alpha}\partial_\mu, \partial_\beta \rangle_R \\
&= w^\alpha C^\mu_{i\alpha}\langle \partial_\mu, \partial_\beta \rangle_R \\
&= w^\alpha C^\mu_{i\alpha} c_{W,Q}\left(\delta_{\alpha\beta} + q^1_{\alpha\mu\beta\nu} w^\mu w^\nu + O\left(\|W\|^4\right)\right) \\
&= c_{W,Q} w^\alpha C_{i\alpha\beta} + c_{W,Q} w^\alpha w^\rho w^\nu C^\mu_{i\alpha} q^1_{\mu\rho\beta\nu} + O\left(\|W\|^4\right).
\end{aligned}
$$

Finally, we compute

$$
\begin{aligned}
g_{Rij} &= \langle \partial_i, \partial_j \rangle_R \\
&= \langle \partial^h_i + w^\alpha C^\mu_{i\alpha}\partial_\mu, \partial^h_j + w^\beta C^\nu_{j\beta}\partial_\nu \rangle_R \\
&= g_{Lij} + c_{W,Q} w^\alpha C^\mu_{i\alpha} w^\beta C^\nu_{j\beta}\left(\delta_{\mu\nu} + q^1_{\mu\rho\nu\sigma} w^\rho w^\sigma + O\left(\|W\|^4\right)\right),
\end{aligned}
$$

which gives the result. □

Recently, in [Koz08], the Cheeger-Gromoll metric on the normal bundle has been studied. Moreover, the author gives some kind of generalized Cheeger-Gromoll metric, which he calls (p,q)-metric. This (natural) metric is similar to the metric g_R, whose second and higher order corrections in $Q_{(W)}$ vanish. In Lemma 1.5 the author computes the Levi-Civita connection of the (p,q)-metric in $\theta \in NL$.

One of the main benefits we obtained from the Sasaki metric, is that the computation of the determinant of the metric tensor is simple. This is due to the decomposition of the metric given by (5).

From the representation in local Fermi coordinates, given by Corollary (7.12), we deduce – unless $c \equiv 1$ – that there is no possibility to decompose g_R as a product of simple matrices as this is the case for the Sasaki metric. Even the case $c \equiv 1$ turns out to be very complex.

The asymptotic of the induced metric g is equal to the Sasaki metric, which means that the computations will be similar for any other reference metric on the tube, which asymptotically ($\varepsilon \to 0$) looks like the Sasaki metric. Hence, we demonstrate that there is a way to compute the determinant of a general natural metric in a simple way. We use the block structure of the metric tensor and define

$$
g_R(x,w) := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),
$$

where

$$
\begin{aligned}
A &:= g_{L\,ij}(x) + c w^\mu w^\nu C^\alpha_{i\mu} C_{j\nu\alpha} + O\left(\|W\|^4\right), \\
B &:= c w^\mu C_{i\mu\beta} + O\left(\|W\|^3\right),
\end{aligned}
$$

$$
\begin{aligned}
C &:= cw^{\nu} C_{j\nu\alpha} + O\left(\|W\|^3\right), \\
D &:= c\left(\delta_{\alpha\beta} + q^1_{\alpha\mu\beta\nu} w^{\mu} w^{\nu} + O\left(\|W\|^4\right)\right).
\end{aligned}
$$

Note, that D is invertible, whenever $\|W\|$ is small. The inverse matrix can be computed as a Neumann series. Hence, we can compute the determinant by

$$
\det(g_R) = \det(A - BD^{-1}C)\det(D).
$$

On the one hand, we obtain

$$
\begin{aligned}
(BD^{-1}C)_{ij} &= cw^{\mu} C_{i\mu\beta}\left(c\delta_{\alpha\beta} + cq^1_{\alpha\mu\beta\nu} w^{\mu} w^{\nu}\right)^{-1} cw^{\nu} C_{j\nu\alpha} + O\left(\|W\|^3\right) \\
&= c \cdot w^{\mu} C_{i\mu\beta}\left(\delta^{\alpha\beta} - q^1_{\alpha\mu\beta\nu} w^{\mu} w^{\nu} + O\left(\|W\|^4\right)\right) w^{\nu} C_{j\nu\alpha} + O\left(\|W\|^3\right) \\
&= c \cdot w^{\mu} w^{\nu} C^{\alpha}_{i\mu} C_{j\nu\alpha} + O\left(\|W\|^3\right)
\end{aligned}
$$

and therefore

$$
(A - BD^{-1}C)_{ij} = g_L(x)_{ij} + O\left(\|W\|^3\right),
$$

which implies that the determinant of this matrix equals the determinant of g_L up to higher order terms in W.

On the other hand, we deduce

$$
\begin{aligned}
\det(D) &= \det\left(c(\delta_{\alpha\beta} + q^1_{\alpha\mu\beta\nu} w^{\mu} w^{\nu}) + O\left(\|W\|^4\right)\right) \\
&= c^{m-l} \cdot \det\left(\delta_{\alpha\beta} + q^1_{\alpha\mu\beta\nu} w^{\mu} w^{\nu}\right) + O\left(\|W\|^4\right) \\
&= c^{m-l} \cdot \det\left(\exp\left(q^1_{\alpha\mu\beta\nu} w^{\mu} w^{\nu} + O\left(\|W\|^3\right)\right)\right) \\
&= c^{m-l} \cdot \exp\left(\mathrm{tr}\left(q^1_{\alpha\mu\beta\nu} w^{\mu} w^{\nu} + O\left(\|W\|^3\right)\right)\right) \\
&= c^{m-l} \cdot \exp\left(\delta^{\alpha\beta} q^1_{\alpha\mu\beta\nu} w^{\mu} w^{\nu}\right) + O\left(\|W\|^3\right).
\end{aligned}
$$

Chapter 8

Related questions

Based on our investigations, we introduce further settings and related problems, which could be studied in subsequent works. So far, we just treated the case of closed (smooth) Riemannian submanifolds $L \subset (M, g)$ and obtained surface measures, which are equivalent to the Wiener measure on L.
It is an open question, if the presented approach (see e.g. [Wit11]) to define the surface measure is unique. The main question is if the limit measure does still exist in different situations and if it is still equivalent to the Wiener measure then. We now present possible generalizations and perturbations for this assumption and indicate what kind of new behaviors appear. This should point out that the situation we studied here is indeed peerless. Note that some information of Section (8.2) and (8.3) was already given in the last chapter of [SW09].

8.1 Non-compact submanifolds

In contrast to a submanifold L that is assumed to be compact without boundary, the case of a non-compact Riemannian manifold with or without boundary has not been dealt with so far. In the context of this thesis, one of the significant properties of compact manifolds is used in the proof of the existence of a normal neighborhood. Compactness guarantees that the injectivity radius of the exponential map is bounded from below and hence a suitable neighborhood $NL_\varepsilon(0)$ exists, which is diffeomorphic to the tube $L(\varepsilon)$. In the new setting, it might be sufficient to assume some curvature bounds on the manifold to solve this complication and obtain similar results.

However, compactness is far more important in the proof of the existence of the surface measure μ_0. In Section (7.2), for a Riemannian submersion with totally geodesic fibers, we obtain that the Laplacians on M and L coincide, when applied functions $\bar{f} \in C^2(L)$, up to the projection and some higher order terms in ε as ε tends to zero. The most important fact is that we need significant properties of the spectrum of the (horizontal) Laplacian. Compactness of the manifold secures that the eigenvalues of the Laplace-Beltrami operator are discrete and for any eigenvalue λ (note all eigenvalues are real) the corresponding eigenspace is finite-dimensional. Also all eigenvalues are non-negative.
A possible approach to (hopefully) obtain similar results in the non-compact case is to start with a compact exhaustion of the manifold by manifolds with boundary. These manifolds could possibly be induced by the tubular neighborhoods around the submanifold. Note that in general (in this non-compact case) the tubes will not be regularly embedded, which we need to fix by an approximation. Here we believe that we will have

to impose Neumann boundary conditions to obtain a suitable behavior of the Brownian particles.

8.2 Non-symmetric fibers and tubes of variable diameter

In this thesis all tubes $L(\varepsilon)$, $\varepsilon > 0$, were homogeneous (and isometric) in the sense that all the fibers

$$L_x(\varepsilon) := \pi^{-1}(x) \cap L(\varepsilon)$$

are Euclidean balls $B^{m-l}(\varepsilon)$ centered around the base point $x \in L$ and of constant radius ε. Moreover, the tubes are assumed to be regularly embedded.

It is very challenging to answer how the situation changes if the diameter of the fibers $L_x(\varepsilon)$ depends on the base point, or even if the shape of the fibers is different (or not isometric) at each base point. In the latter case, we again have to distinguish between several cases: For example the fibers could be ellipses, whose principal radii depend on the base point of the fiber. Another possibility is that the shape of the fibers are equal at each base point, but not a Euclidean ball. One can think of arbitrarily difficult examples for this setting.

A priori it is not clear if the corresponding surface measures exist in these cases. But, a suitable result would give a great deal of new information: For example, the connection to the volume formula of Gray (see (24) and Chapter 5). So far, the first order term in his formula canceled due to the symmetry of the fibers (see proof of Lemma (5.2)). If the shape of the fibers differs from a (symmetric) ball, we will in general obtain a non-zero first order term, and hence another dependency of the tension field. Then, the volume deviation is definitely related to the second fundamental form of the embedding.

For the other case, let $f : L \to \mathbb{R}_+$ be a smooth function. We assume the radius of such a ball to be $\varepsilon \cdot f(x)$. Hence, the over-all shape of the (normal) cross-sections of the tubes remains the same as in our initial setting, but varies in its diameter. Note, for $f \equiv c$, $c \in \mathbb{R}$, we basically deal with the same situation as in our usual setting. For arbitrary f, we believe that the limit measure still exists, but differs from the representation we introduced in Section (1.5). Intuitively, the sojourn probability of the Brownian particles should be larger at the maxima of f, this may possibly lead to a limit measure, which is no longer equivalent to the Wiener measure on the submanifold. Furthermore, it is hard to guess, if there is any term of order ε, or if this term vanishes the same way as in our initial setting.

8.3 Non-smooth potentials and manifolds

Another interesting topic is the study of non-smooth potential functions or even of non-smooth (sub-)manifolds. If the Brownian motion is conditioned to tubular neighborhoods of arbitrary manifolds we obtain several difficulties. For example the projection $\pi : L(1) \to L$ is not even well-defined any more. Hence, a priori it is not clear, if a limit measure exists or if the surface measure degenerates.

Moreover, it may be helpful to study the limit behavior close to the singularities of the manifold in more detail, since the tubular neighborhoods behave different at those points. The characterization and shape of the $\varepsilon-$tubes, which turn out to be larger around the singular points, again is connected to the open questions of the previous section. Moreover the curvature of a non-smooth manifold possibly tends to infinity, which makes the computations of the limit measure impossible, since the Radon-Nikodym density degenerates.

One possible approach might be given by our main result in Chapter 4. We know that every smooth function occurs as a potential function for some limit process. This result could probably be used to find some approximation for a singular situation. Given a manifold L and its singularities, we can construct a sequence of smooth functions tending to some singular potential function on the submanifold, with singularities exactly at the singular points of L.

Appendix A

Geometric preliminaries and Brownian motion

In this thesis we follow the conventions from [Jos98]. We just give some basic definitions and facts to introduce the notation and refer the reader to this book for detailed information.

Definition A.1. *A* ***Riemannian manifold*** (M, g) *is a smooth manifold* M *with a* $(2,0)-$ *tensor field* g *(called Riemannian metric), such that for any* $p \in M$ *there exists a map* g_p *which is an inner product on* T_pM.

Remark. *Note* g *induces a natural distance function on* M*, referred to as* d_g*, and therefore* g *defines the Riemannian volume element* $\mu_M := \sqrt{\det(g_{ij})}dp$*. Sometimes we will use* m *for the volume element in order to be consistent to the notation in our main source (see [Wit11]).*

From now on let $(M, g), (M', g')$ always denote Riemannian manifolds.

Definition A.2. *A diffeomorphism* $f : M \to M'$ *is said to be an* ***isometry*** *if*

$$g = f^* g',$$

which means that for all pairs of tangent vectors $v, w \in T_pM$ *for* $p \in M$ *the equality*

$$g(v, w) = g'(f_* v, f_* w)$$

holds.

Remark. *Note that all maps and manifolds occurring in this thesis are smooth (unless explicitly stated otherwise!).*

Definition A.3. *Let* E, M *be smooth manifolds. A differentiable* ***vector bundle*** *of rank* $n \in \mathbb{N}$ *consists of a total space* E*, a base space* M *and a projection* $\pi : E \to M$*, where* π *is differentiable, such that additionally the fibers* $E_x := \pi^{-1}(x)$*,* $x \in M$*, are real* $n-$*dimensional vector spaces. Moreover, the following local triviality condition is satisfied: There exists a covering* $\{U_k | k \in I \subseteq \mathbb{N}\}$ *of* M *and homeomorphisms*

$$\varphi_k : \pi^{-1}(U_k) \to U_k \times \mathbb{R}^n,$$

where for each $y \in U_k$ *the map*

$$\varphi_{y,k} := \varphi_{|E_x} : E_x \to \{y\} \times \mathbb{R}^n$$

is a vector space isomorphism with $\pi_{|\pi^{-1}(U)} = p_1 \circ \varphi$ *and* p_1 *is the projection on the first component.*
Then the maps

$$\psi_{lk} := \varphi_l \circ \varphi_k^{-1} \colon (U_k \cap U_l) \times \mathbb{R}^n \to (U_k \cap U_l) \times \mathbb{R}^n$$

are given by $\psi_{lk}(x, v) = (x, h_{lk}(x) \cdot v)$*, where the so-called transition maps*

$$h_{lk} \colon U_l \cap U_k \to GL(n, \mathbb{R})$$

are differentiable.

We introduce metrics on a bundle.

Definition A.4 (Bundle metric)**.** *A **bundle metric** of a vector bundle* $\pi \colon E \to M$*, over a smooth Riemannian manifold* (M, g) *is a section* $g \in \Gamma(E^* \oplus E^*)$*, which assigns a symmetric, non-degenerated bilinear form, i.e.*

$$g(\cdot, \cdot)_{E_x} := g(\cdot, \cdot)(x) \colon E_x \times E_x \to \mathbb{R}$$

to every $x \in M$*.*

Theorem A.5. *There exists a positive definite bundle metric on every vector bundle.*

A proof can be found e.g. in [Bau09], p. 61 ff.

Definition A.6. *The **Levi-Civita connection*** ∇ *on* M *is the unique connection, which is linear, torsion-free and compatible (or metric) with respect to* g*, i.e. the covariant derivative of* g *vanishes.*

On each Riemannian manifold the unique Levi-Civita connection is given by the following formula.

Theorem A.7 (Koszul formula)**.** *The Levi-Civita connection* ∇ *on the tangent bundle* TM *of a Riemannian manifold* M *is given by*

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle = {} & X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle. \end{aligned}$$

Definition A.8. *The **Christoffel symbols** on* M *are defined by*

$$\Gamma_{ij}^k = \frac{1}{2} g^{mk} \left(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}\right),$$

where (g^{ij}) *denotes the inverse matrix of* (g_{ij})*.*

We study submanifolds of Riemannian manifolds. The fundamental equations and proofs are given in Chapter 7 in [Spi75]. Besides the Gauss equation (16) the following embedding theorems are helpful.

Theorem A.9 (Codazzi-Mainardi equation)**.** *Let* $L \subset M$ *be a submanifold of a Riemannian manifold embedded by* $\varphi \colon L \to M$*. Moreover let* ∇^L, ∇^M *be the corresponding Levi-Civita connections. For* $X, Y, Z \in TL$ *and* ν *a normal vector field we obtain*

$$\langle R^M(X, Y)Z, \nu\rangle = \left(\nabla_X^L \mathrm{II}\right)(Y, Z) - \left(\nabla_Y^L \mathrm{II}\right)(X, Z). \tag{86}$$

Theorem A.10 (Ricci equation)**.** *Let $L \subset M$ be a submanifold of a Riemannian manifold embedded by $\varphi\colon L \to M$. Moreover let $R^\perp, R^M$ be the corresponding Levi-Civita connections on NL and M. For $X, Y \in TL$ and ν, ξ normal vector fields we obtain*

$$\langle R^\perp(X,Y)\xi, \nu\rangle = \langle R^M(X,Y)\xi, \nu\rangle + \langle [A_\xi, A_\nu]X, Y\rangle. \tag{87}$$

For our studies geodesics, the exponential map and Jacobi fields are important. These are defined as follows.

Definition A.11. *A piecewise smooth path, also curve, $\gamma\colon [0,1] \to M$ is called **geodesic** if*

$$\nabla_{\gamma_*}\gamma_* = 0.$$

In local coordinates this means that the second order differential equation

$$\ddot{\gamma}^k + \Gamma^k_{ij}(\gamma(t))(\dot{\gamma}^i)(\dot{\gamma}^j) = 0$$

is satisfied.

Definition A.12. *Let $\gamma\colon [0,1] \to M$ be a geodesic with $\gamma(0) = x \in M$ and $\dot{\gamma}(0) = v \in T_xM$. Then we define*

$$\exp_x(v) := \gamma(1),$$

which defines the exponential map $\exp$ *from a subset of T_xM to M.*

Definition A.13. *A vector field $V\colon [0,1] \to TM$ along a geodesic $\gamma\colon [0,1] \to M$ is called **Jacobi field**, if*

$$\frac{D}{dt}\frac{D}{dt}V(t) + R\left(V(t), \gamma_*\left(\frac{\partial}{\partial t}\right)\right)\gamma_*\left(\frac{\partial}{\partial t}\right) = 0,$$

where R denotes the Riemannian curvature tensor.

Remark. *One can use other ranges than $[0,1]$ to define the above objects.*

A proof of the following theorem can be found in [Jos98], p. 26 ff.

Theorem A.14 (Theorem of Hopf-Rinow)**.** *For a Riemannian manifold (M, g) the following statements are equivalent.*

1. *M is complete as a metric space.*
2. *M is geodesically complete, i.e. for every $p \in M$ the exponential map $\exp_p$ is defined everywhere on T_pM.*
3. *Closed and bounded subsets of M are compact.*

Moreover each statement implies that any two points $x, y \in M$ can be joined by a geodesic of shortest length.

For the calculations we need to define the Laplace operator on a Riemannian manifold.

Definition A.15. *Let (M, g) be a Riemannian manifold. Then $\mathrm{grad}_g f$ denotes the **gradient** of a (smooth) function f on M, which is the vector field $V = V^i\partial_i$ with components $V^i = g^{ik}\frac{\partial f}{\partial x^k}$. For a vector field V, the **divergence** $\mathrm{div}_g V$ is defined by*

$$\mathrm{div}_g V^i = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}V^i\right).$$

*The **Laplace-Beltrami operator** $\triangle_g$ acting on at least twice differentiable functions is in local coordinates given by*

$$\begin{aligned}\triangle_g u &= -\mathrm{div}_g(\mathrm{grad}_g f) \\ &= -\left(\det(g_{ij})\right)^{-\frac{1}{2}} \partial_m \left(\det(g_{ij})^{\frac{1}{2}} g^{mk} \partial_k u\right).\end{aligned}$$

This definition ensures that we are able to define a Brownian motion on a Riemannian manifold analogous to and consistent with the definition on $\mathbb{R}^n$, $n \in \mathbb{N}$, which we give first.

A very classical definition of Brownian motion in Euclidean space is given as follows (see e.g. [SP14], Chapter 1).

Definition A.16. *A (standard) one-dimensional Brownian motion (or Wiener process) is a time-continuous stochastic process $(B_t)_{t\geq 0}$ that satisfies the following properties:*

1. *It is $B_0 = 0$.*
2. *The function $t \mapsto B_t$ is almost surely continuous (in t).*
3. *Let $0 \leq t_0 < t_1 < t_2 < \ldots < t_m$. Then,*

$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}$$

   are (stochastically) independent.

4. *For each $0 \leq s < t$ the random variable $B_t - B_s$ is normally distributed with mean zero and variance $t - s$.*

Let us now study Brownian motion on a closed, i.e. compact without boundary, manifold (M, g).

First note that a time-homogeneous Markov process $X = (X_t)_{t\geq 0}$ on M is uniquely determined by transition kernels q_t as

$$\mathbb{P}_p(X_{t+h} \in dy |\, X_t \in p) = q_t(p, dy),$$

for every $p \in M$ and $t, h \geq 0$.

The following definition of Brownian motion is adapted from [McK69], p. 91.

Definition A.17 (Brownian motion on Riemannian manifolds)**.** *A Brownian motion $B = (B_t)_{t\geq 0}$ on a Riemannian manifold (M, g) is defined to be the diffusion process (Markov process), where the density of its distribution relatively to the volume element $(\det g)^{\frac{1}{2}} dx$ is the minimal (positive) solution q of the heat equation*

$$\frac{\partial}{\partial t} u = -\frac{1}{2} \triangle_g u,$$

with $B_0 = z \in M$. That is, the smallest function $q \geq 0$ on $C^\infty((0,\infty) \times M)$ with $\frac{\partial}{\partial t} q = -\frac{1}{2}\triangle_g q$ and $\lim_{t\searrow 0} \int_U q \cdot (\det g)^{\frac{1}{2}} dx = 1$, for all components $U \subset M$ containing z.

The Laplace-Beltrami operator $L := -\frac{1}{2}\triangle_g$ is called the (infinitesimal) generator of a Brownian motion on a closed Riemannian manifold (M, g).

Remark. *A Brownian motion on a Riemannian manifold is a stochastic process B on M, whose distribution is a Wiener measure on the path space Ω_M.*

A detailed discussion on the definition of Brownian motion in Euclidean space can e.g. be found in [CZ12] and a good overview on Brownian motion on arbitrary Riemannian manifolds is given in [Ito62] or [Éme12].

Appendix B

Functional Analysis

B.1 Hilbert spaces and forms

We recall some definitions and results from [Zei99] and [Wer07] used in this thesis. Most of the quantities occur when using Friedrichs' construction, which we introduce in this chapter. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Note that for this thesis only the case $\mathbb{K} = \mathbb{R}$ is relevant.

Definition B.1. *Let H be a linear space over $\mathbb{K}$. We call $\langle -, - \rangle \colon H \times H \to \mathbb{K}$ an **inner product** on H, whenever the following properties are fulfilled:*

1. $\langle u, u \rangle \geq 0$ *and* $\langle u, u \rangle = 0$*, if and only if* $u = 0$*,*
2. $\langle u, \alpha v_1 + \beta v_2 \rangle = \alpha \langle u, v_1 \rangle + \beta \langle u, v_2 \rangle$*,*
3. $\overline{\langle u, v \rangle} = \langle v, u \rangle$*,*

for $u, v, v_1, v_2 \in \mathbb{K}$ and $\alpha, \beta \in \mathbb{K}$.

That means an inner product is linear in the second component and due to (2) and (3) conjugate linear in the first component.

Definition B.2.
1. *Let H be a linear space over $\mathbb{K}$ with an inner product $\langle -, - \rangle$, then we call $(H, \langle -, - \rangle)$ a **pre-Hilbert space**.*
2. *Let $(H, \langle -, - \rangle)$ be a pre-Hilbert space. If H is complete with respect to the induced norm, which means each Cauchy sequence with respect to this norm is convergent, we say H is a **Hilbert space**.*

Moreover, we state that each pre-Hilbert space H over $\mathbb{K}$ is also a normed space with respect to the **induced norm**

$$||u|| := \langle u, u \rangle^{\frac{1}{2}},$$

for $u \in H$. Therefore, a Hilbert space is a pre-Hilbert space, which is a Banach space with respect to the induced norm.

A sesquilinear form is a generalization of a bilinear form in the following way.

Definition B.3. *Let V be a $\mathbb{C}-$vector space. A map $f \colon V \times V \to \mathbb{C}$ is called **sesquilinear**, if*

$$\begin{aligned} f(x_1 + x_2, y_1 + y_2) &= f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2), \\ f(ax, by) &= \overline{a} b f(x, y), \end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in V$ and $a, b \in \mathbb{C}$.

Lemma B.4. *For a sesquilinear form $\alpha\colon H \times H \to \mathbb{C}$ the following statements are equivalent:*

1. *For all $x, y \in H$ there exists a constant $C > 0$ with*
$$|\alpha(x, y)| \leq C\|x\|\|y\|.$$
2. *The form α is continuous.*

Proof. (a) We first show $(1) \implies (2)$. So assume (1) holds for a $C > 0$. Let (x_n, y_n) be a convergent sequence with limit $(x, y) \in H \times H$. Then using the properties of α, the triangle inequality and (1), we obtain

$$\begin{aligned} &|\alpha(x_n, y_n) - \alpha(x, y)| \\ &= |\alpha(x_n - x, y)| + |\alpha(x, y_n - y)| \\ &= C(\|x_n - x\|\|y_n\| + \|x\|\|y_n - y\|) \overset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

because $(x_n), (y_n)$ are bounded. That shows α is continuous.

(b) Now assume the continuity of α. Assume (1) is wrong, i.e. for all $C \geq 0$ there exist $x, y \in H$ with

$$|\alpha(x, y)| > C\|x\|\|y\|. \tag{88}$$

Without loss of generality assume $x, y \neq 0$ and for $n \in \mathbb{N}$ set $C := n^2$. We define the sequences $x_n := \frac{x}{n\|x\|}$ and $y_n := \frac{y}{n\|y\|}$, which each converge to zero for $n \to \infty$. Using the properties of α and (88) we have

$$\begin{aligned} |\alpha(x_n, y_n)| &= \left|\alpha\left(\frac{x}{n\|x\|}, \frac{y}{n\|y\|}\right)\right| \\ &= \frac{1}{n\|x\|}\frac{1}{n\|y\|}|\alpha(x, y)| \\ &> \frac{1}{n^2\|x\|\|y\|}n^2\|x\|\|y\| = 1, \end{aligned}$$

which is a contradiction to the assumption that α is continuous.

□

Due to [BB03] we recall some notations and definitions in order to explain quadratic forms on Hilbert spaces.

Definition B.5. *Let $(H, \langle -, - \rangle)$ be a Hilbert space. A **quadratic form Q** with a linear domain $\mathcal{D}(Q) \subseteq H$ is a map $Q\colon \mathcal{D}(Q) \times \mathcal{D}(Q) \to \mathbb{C}$, which is conjugate linear in the first and linear in the second argument. We say Q is*

(a) ***symmetric****, if $\overline{Q(f, g)} = Q(g, f)$, for all $f, g \in \mathcal{D}(Q)$,*

(b) ***densely defined****, if $\mathcal{D}(Q) \subset H$ is dense,*

(c) ***semi-bounded****, if there exists a lower bound $\lambda \in \mathbb{R}$, such that for all $f \in \mathcal{D}(Q)$ we have*

$$Q(f, f) \geq -\lambda||f||^2,$$

(d) ***positive****, if Q is semi-bounded with lower bound $\lambda = 0$,*

(e) ***continuous***, *if there is a constant* $C > 0$, *such that*

$$|Q(f,g)| \leq C||f||\,||g||,$$

for all $f, g \in \mathcal{D}(Q)$.

Definition B.6. *(a) A semibounded quadratic form* Q *with lower bound* λ *is called* ***closed***, *if the domain* $\mathcal{D}(Q)$ *is complete with respect to the norm*

$$||f||_Q = \sqrt{Q(f,f) + (\lambda+1)||f||^2},$$

for $f \in \mathcal{D}(Q)$.

(b) *A quadratic form* $\tilde{Q}$ *with domain* $\mathcal{D}(\tilde{Q})$ *is called an* ***extension*** *of* Q *if and only if* $\mathcal{D}(Q) \subset \mathcal{D}(\tilde{Q})$ *and* $\tilde{Q}(f,g) = Q(f,g)$ *for* $f, g \in \mathcal{D}(Q)$.

(c) *A quadratic form is* ***closable*** *if and only if it has a closed extension.*

B.2 Operators.

Let H be a Hilbert space over a field $\mathbb{K}$.

Definition B.7. *Let* $\mathcal{D}(A) \subset H$. *An* ***operator A*** *is a linear map on* $\mathcal{D}(A)$. *We say* A *is* ***closed***, *if for each convergent sequence* $(x_n)_n \subset \mathcal{D}(A)$ *with* $x_n \to x \in H$ *and* $(Ax_n)_n \to y$, *we already obtain* $Ax = y$.
The ***graph G*** *is a linear subspace of* $H \times H$ *with domain*

$$\mathcal{D}(G) := \{f \in H | (f,g) \in G \text{ for some } g \in H\}.$$

By $\overline{G}$ *we denote the* ***closure*** *of the graph* G *in* $H \times H$. *We say* G *is* ***closable***, *if* $\overline{G}$ *is an operator.*

Remark. Equivalently to the definition we call an operator A **closed**, if the graph $G(A)$ is closed in $H \times H$. A graph G defines a linear operator, if for every $(f,g) \in G$ with $f = 0$ already $g = 0$ follows.

Definition B.8. *Let* $\mathcal{L}(H)$ *be the vector space of bounded linear operators from* $H \to H$ *endowed with the* ***operator norm*** $\|A\| = \sup_{\|x\|=1} \|Ax\|$.

Definition B.9. *We call an operator* $T\colon H \to V$ ***continuous***, *if one of the following equivalent conditions is satisfied:*

1. *If* $\lim_{n\to\infty} x_n \to x$, *then* $\lim_{n\to\infty} Tx_n \to Tx$.
2. *For all* $x_0 \in X$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *with*

$$\|x - x_0\| \leq \delta \implies \|Tx - Tx_0\| \leq \varepsilon.$$

3. *For all open sets* $\mathcal{O} \subset V$ *the set* $T^{-1}(\mathcal{O}) \subset H$ *is open.*

Definition B.10. *The set of all continuous and linear functionals on a normed space* X *is called* ***dual space*** *and denoted by* X', *i.e.*

$$X' := \{f : X \to \mathbb{K} | \, f \text{ is continuous and linear}\}.$$

Definition + Theorem B.11. *Let X be a normed vector space and T be a bounded operator on X. If the **Neumann series** $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}(X)$, then the operator $\mathrm{id} - T$ is invertible with*

$$(\mathrm{id} - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

A proof of the following theorem can be found for example in [Alt06].

Theorem B.12 (Lax-Milgram). *Let $\alpha\colon H \times H \to \mathbb{K}$ be a sesquilinear form, with the property*

$$|a(x,y)| \leq C\|x\|\|y\|, \tag{89}$$

for $C > 0$. Then there is a unique operator $A \in \mathcal{L}(H)$ with

$$\alpha(x,y) = \langle x, Ay\rangle_H,$$

for all $x, y \in H$.

Remark. *The condition (89) is equivalent to the continuity of α due to Lemma$(B.4)$.*

From now on let $(H, \langle -.-\rangle_H)$ be a Hilbert space and A a bounded linear operator on H. The **adjoint operator** is a linear operator $A^*\colon H \to H$ with the property

$$\langle Ax, y\rangle = \langle x, A^*y\rangle, \tag{90}$$

for all $x, y \in H$.
The question whether A^* is unique or if it even exists, is answered by the following theorem.

Theorem B.13. *For every $A \in \mathcal{L}(H)$ there is a unique bounded linear operator A^* which satisfies* (90).

Proof. The proof uses the existence of a complete orthonormal system, the *Riesz representation theorem* and the *Schwarz inequality*. □

Some properties of the adjoint operator A^* are listed in the following lemma.

Lemma B.14. *Let $A, B\colon H \to H$ be bounded and $\alpha \in \mathbb{C}$, then*

1. *$(A+B)^* = A^* + B^*$ and $(\alpha A)^* = \overline{\alpha} A^*$,*
2. *$(AB)^* = B^*A^*$,*
3. *$(A^*)^* = A$,*
4. *if A is invertible and A^{-1} is bounded, then $(A^*)^{-1} = (A^{-1})^*$ holds.*

Definition B.15. *The operator $A \in \mathcal{L}(H)$ is called **selfadjoint**, if and only if $A = A^*$.*

Appendix C

Friedrichs extension theorem and boundary value problems

In Chapter 4 of this thesis the computation of the potential W_0 uses some more theory of linear operators. In particular, the Friedrichs extension theorem (see e.g. [Wer07], Theorem VII.2.11), which we refer to as Friedrichs construction, is crucial for our investigations. For completeness we state the most important facts and study an interesting example, which fits the situation.

We follow [Zei99] and [Far75]. Whenever we refer to a field $\mathbb{K}$, then $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Moreover, let V, H be Hilbert spaces over $\mathbb{K}$.

We want to present and explain some facts about the Dirichlet Laplacian (12) on the tube and its associated quadratic form due to the result of Friedrichs.
Therefore, we are going to discuss the Friedrichs construction in detail and obtain that in our case the operator is associated to the quadratic form

$$q_\varepsilon \colon H_0^1 \cap H^2(L(\varepsilon), g) \to \mathbb{R},\ q_\varepsilon(f) := \int_{L(\varepsilon)} \|df\|_g^2 \, m(dx).$$

Moreover, we will study the Laplacian in the context of elliptic boundary value problems.

A quadratic form $q \colon V \to \mathbb{C}$ is associated to a sesquilinear form $s \colon V \times V \to \mathbb{C}$ by assigning f to $s(f, f)$. Vice versa we obtain a sesquilinear form s induced by q on V by the *polarization identity* via

$$4s(f, g) = q(f + g, f + g) - q(f - g, f - g) + iq(f + ig, f + ig) - iq(f - ig, f - ig).$$

C.1 The energetic space and energetic extension

Let us start with a short introduction to the theory of operators for explaining the Friedrichs extension, which was used for the proof of the homogenization result from Section (1.5). We first present the definition of an energetic space and some of its properties.

Definition C.1 (Operators bounded from below). *The operator $B \colon \mathcal{D}(B) \subset H \to H$ is called **bounded from below** or **strongly monotone**, if and only if*

$$\langle Bu, u \rangle \geq c\|u\|^2$$

for all $u \in \mathcal{D}(B)$ *and a constant* $c > 0$.

Let now B be a linear, symmetric and strongly monotone operator on the Hilbert space $(H, \langle -, - \rangle)$. The norm induced by the inner product on H is denoted by $\| \cdot \|$.

Definition C.2 (Energetic space)**.** *Let* B *be given as above.*

1. *We define the* ***energetic inner product*** *as*

$$\langle u, v \rangle_E := \langle Bu, v \rangle,$$

for all $u, v \in \mathcal{D}(B)$.

2. *The corresponding norm is called* ***energetic norm****.*

3. *The* ***energetic space*** H_E *of* B *is the set of all* $u \in H$ *where there exists a sequence* $(u_n) \subset \mathcal{D}(B)$*, such that* $u_n \overset{n\to\infty}{\longrightarrow} u \in H$*, with* (u_n) *a Cauchy sequence with respect to* $\| \cdot \|_E$*.*

One says that every sequence with these properties is an *admissible* sequence for $u \in H_E$.

Proposition C.3. 1. H_E *is a Hilbert space with the energetic inner product.*

2. *The domain* $\mathcal{D}(B) \subset H_E$ *is dense.*

3. H_E *is continuously embedded into* H*.*

4. *The embedding* $H \subset H'_E$ *by*

$$j \colon H \to H'_E, \; j(f)(v) := \langle f, v \rangle,$$

for $v \in H_E$ *and* $f \in H$ *fixed is continuous.*

5. *By identification of* f *and* $j(f)$ *the Hilbert space* H *is a subset of* H'_E*, i.e.*

$$H_E \subset H \subset H'_E.$$

Proof. See [Zei99], Chapter 5, Proposition 3. □

Using these objects we are able to define the *energetic extension*.

Definition C.4 (Energetic extension)**.** *Let* $B \colon \mathcal{D}(B) \to H$ *be as described above, then the* ***duality map***

$$B_E \colon H_E \to H'_E$$

of H_E *is the* ***energetic extension*** *of* B*, where* B_E *is defined by*

$$\langle B_E u, v \rangle_E = \langle u, v \rangle_E.$$

To justify this definition, we show that B_E is indeed an extension of B.
Let therefore $u, v \in \mathcal{D}(B)$, then we obtain

$$\langle B_E u, v \rangle_E = \langle u, v \rangle_E = \langle Bu, v \rangle = \langle Bu, v \rangle_E.$$

Moreover,

$$\langle B_E u, v \rangle_E = \langle Bu, v \rangle_E$$

holds for every $v \in H_E$ because $\mathcal{D}(B)$ is dense. Therefore, we finally obtain $B_E u = Bu$ for all $u \in \mathcal{D}(B)$, which means B_E is an extension of B.

C.2 The Friedrichs construction for quadratic forms

The Friedrichs extension is part of the energetic extension.

Definition C.5 (Friedrichs extension). *Let $B\colon \mathcal{D}(B) \to H$ be linear, symmetric and strongly monotone on H, then the **Friedrichs extension** $A\colon \mathcal{D}(A) \to H$ of B is defined by*

$$Au := B_E u,$$

for $u \in \mathcal{D}(A) := \{u \in H_E | B_E u \in H\}$.

Consider a dense subset $\mathcal{D} \subset V$ and a quadratic form $q\colon \mathcal{D} \to \mathbb{C}$ on $\mathcal{D}$. First, we prove that the densely defined, closed quadratic forms which are bounded from below are in one-to-one correspondence to associated self-adjoint operators bounded from below on V.

Afterwards, we give a densely defined extension A_q for a symmetric, densely defined operator A bounded from below, which is self-adjoint. This remarkable result is due to Friedrichs (see e.g. [Wer07], VII.2.11 or [Zei99], Section 5.5).

Construction. Whenever an operator A is self-adjoint and bounded from below, we are able to define

$$\alpha(f) := \langle f, Af\rangle, \quad \mathcal{D}(\alpha) = \mathcal{D}(A).$$

This indeed defines a quadratic form, which satisfies all of the stated properties except closedness. Even if α is not closed, then it is still closable and the form $\overline{\alpha}$ satisfies all properties stated above.

Let now q be defined on a domain $\mathcal{D}$ satisfying all assumptions. For $f, g \in \mathcal{D}$ we choose the bilinear form $b := b_q$ associated to the given quadratic form q by

$$b\colon \mathcal{D} \times \mathcal{D} \to \mathbb{C};\ b(f,g) := \frac{1}{2}(q(f+g) - q(f) - q(g))$$

with associated inner product

$$\langle f, g\rangle_{|||} := \langle f, g\rangle_V + b(f,g).$$

This inner product induces a metric which will be denoted by $|||\cdot|||$. One shows that these objects are well-defined by using the positivity of q. For the proof of the triangle inequality one applies the polarization identity.

We further set $\tilde{\mathcal{D}} := \overline{\mathcal{D}}^{|||\cdot|||}$, denoting the closure with respect to this norm.

Now an application of *Lax-Milgram's* theorem (see Appendix B) guarantees that q can be represented by

$$q(f) = \langle f, A_q f\rangle,$$

where $A_q\colon V \to V$ is a linear operator.

Then $\tilde{\mathcal{D}}$ can be equipped with the inner product mentioned above and its corresponding norm $|||\cdot|||$. By construction we obtain that $(\tilde{\mathcal{D}}, \langle -, -\rangle_{|||})$ is a Hilbert space.

Using the duality map (see [Zei99], p. 258 ff.) one can show

$$\tilde{\mathcal{D}} \subset V \subset \tilde{\mathcal{D}}^*.$$

Another application of *Lax-Milgram's* theorem results in the existence of an operator

$$\tilde{A}_q \colon \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}^*$$

defined implicitly through the equality

$$\langle f, g \rangle_{|||} = \langle \tilde{A}_q f, g \rangle,$$

for all $g \in \tilde{\mathcal{D}}$.
The solution of our initial problem is now given by

$$\begin{aligned} \mathcal{D}_q &:= \{ f \in \tilde{\mathcal{D}} | \tilde{A}_q f \in V^* \cong V \}, \\ A_q &:= \tilde{A}_{q | \mathcal{D}_q} \end{aligned}$$

and we can finally state the result.

Theorem C.6 (Friedrichs)**.** *The operator $(A_q, \mathcal{D}_q)$ is self-adjoint on V.*

In the following we are going to study characteristics of the Dirichlet Laplacian on different domains. We are interested in the behavior of Brownian motions on tubular neighborhoods of Riemannian submanifolds. Due to Section (1.6) the generator is basically the Dirichlet Laplacian (12), so that we can now study the Friedrichs construction in this special case. We are also going to use this construction later on.

Example C.7. *We roughly follow [Hel13], Section (4.4).*
Let (M, g) be a complete Riemannian manifold, $L \subset M$ an $l-$dimensional compact submanifold and $L(\varepsilon) \subset M$ the tubular neighborhood as usual, for some $0 < \varepsilon \leq 1$. Then $\overline{L(\varepsilon)}$ is compact. We define the operator B by

$$\mathcal{D}(B) = C_0^\infty(L(\varepsilon)), \quad B := -\triangle$$

and the Hilbert space $H := L^2(L(\varepsilon), g)$.
First we want to comment on the properties of the operator. By using partial integration twice we obtain the symmetry of B, because

$$\langle Bu, v \rangle = \int_{L(\varepsilon)} -\triangle u \; v \; d\mu = \int_{L(\varepsilon)} u(-\triangle v) d\mu = \langle u, Bv \rangle.$$

Another partial integration and the Poincare lemma yield

$$\begin{aligned} \langle Bu, u \rangle &= \int_{L(\varepsilon)} -\triangle u \; u \; d\mu \\ &= \int_{L(\varepsilon)} \nabla u \nabla u \; d\mu \geq c \|u\|_{L^2}^2, \end{aligned}$$

with a constant $c > 0$, so that B is non-negative and therefore bounded from below.
The energetic space of B is the space $H_0^1(L(\varepsilon), g)$.
Therefore, we can extend B to $H_0^1(L(\varepsilon), g)$, which is a dense domain of H and the extension A is given by

$$\mathcal{D}(A) := \{ u \in H_0^1(L(\varepsilon), g) | -\triangle u \in L^2(L(\varepsilon), g) \},$$

which can be identified as

$$\mathcal{D}(A) = H_0^1 \cap H^2(L(\varepsilon), g)$$

due to a regularity theorem in [LM68]. To explain this, we see that the domain is exactly the domain of the adjoint B^* intersected with the energetic space (also see [HT08]) and the set of functions, which are at least twice differentiable with respect to the weak topology.

In conclusion, we obtain

$$A := -\triangle, \quad \mathcal{D}(A) := H^2 \cap H_0^1(L(\varepsilon), g),$$

as self-adjoint (Friedrichs) extension of B.

C.2.1 Regular elliptic boundary value problems and example

Once again, let (M, g) be an $m-$dimensional Riemannian manifold and $(x^B)_{B=1,\ldots,m}$ local coordinates on M. Since the tubular neighborhoods of a smooth closed submanifold $L \subset (M, g)$ are manifolds with (smooth) boundary, we give some properties of so-called (regular) elliptic boundary value problems on such domains.

By ∂_B we denote $\partial/\partial x^B$ and let $q \in \mathbb{N}$. Following the conventions in [AS97], we consider the boundary value problem

$$\begin{cases} Au = f, & \text{in } M \setminus \partial M, \\ B_j u = g_j, & j \in \{1, \ldots, q\}, \text{ in } \partial M, \end{cases} \tag{91}$$

where A is a (partial) differential operator of order $m = 2q$ and B_j a boundary partial differential operator of order r_j, for each j, with smooth coefficients. Let α be a multi-index. With the help of the standard notation we write $A = \sum_{|a| \leq m} a_\alpha(x) D^\alpha$ and analogously we obtain a representation for B_j.

Definition C.8 (Principal Symbol)**.** *For the multi-index α with $|\alpha| = \sum \alpha_i$ and a differential operator A the* ***principal symbol*** *a_0, is (in local coordinates) given by*

$$a_0(x, \zeta) = \sum_{|\alpha|=m} a_\alpha(x) \zeta^\alpha.$$

With this definition we are able to define ellipticity for an operator A.

Definition C.9 (Elliptic operators)**.** *An operator A acting on a manifold M is said to be elliptic, if its principal symbol a_0 does not vanish on the cotangent bundle $T^*M \setminus 0$, where 0 denotes the zero section.*

The boundary value problem (91) is called **regular elliptic** if the following conditions are satisfied:

1. The differential operator A is a proper elliptic operator (note that for $m > 2$ proper ellipticity follows from ellipticity and for the general definition see [AS97], p. 6 f.).

2. The system $(A, B_1, ..., B_q)$ satisfies the Shapiro-Lopatinskij conditions (for a precise definition see Section 1.2 and 1.3 in [AS97] or Definition 5 in [Wit11]).

Remark. *If $(A, B_1, ..., B_q)$ defines a regular elliptic boundary value problem, the associated operator $A' := (A, B_1, ..., B_q)$ is called elliptic.*

Lemma C.10 ([Wit11]). *The powers of the Dirichlet-Laplacian*

$$\triangle_{Sa} : H_0^1 \cap H^2(L(1), g_{Sa}) \to L^2(L(1), g_{Sa})$$

satisfy the Shapiro-Lopatinskij conditions.

Proof. See [Wit11], Lemma 12, p. 61. □

This lemma and the theory of regular elliptic boundary value problems is crucial for the convergence result, we use in this thesis.

Next, we give an example. We study the eigenvalue problem of the Dirichlet Laplacian on two domains: a cylinder Z and the Möbius strip M and compare the solutions.
For $I := [0,1]$ we define the square $Q := I \times I \subset \mathbb{R}^2$. Further we denote the Laplacian to the flat metric $g_{ij} = \delta_{ij}$ by $\triangle$, namely $\triangle := \partial_x^2 + \partial_y^2$ on $\mathbb{R}^2$.
Identifying the colored lines in Figure (C.1) by the equivalence relation

$$(0, y) \sim_1 (1, y),\ y \in [0, 1],$$

gives a cylinder $Z := Q/\sim_1$. By the equivalence relation

$$(x, y) \sim_2 (z, w) :\Longleftrightarrow \begin{cases} (x, y) & = (z, w) \\ \{x, z\} & = \{0, 1\} \end{cases}$$

we define the Möbius strip as $M := Q/\sim_2$.

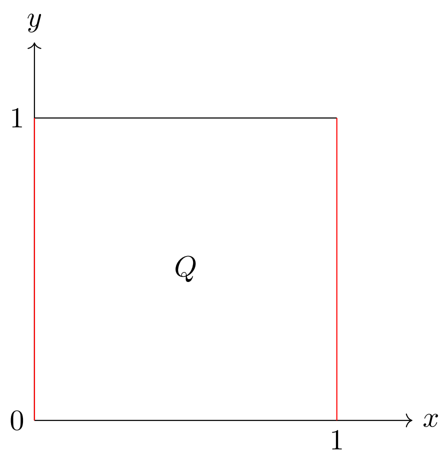

FIGURE C.1: The domain Q with identification of the red lines

Due to the theory of elliptic regularity we are able to state some properties of the Laplacian with Dirichlet boundary conditions, namely

Theorem C.11. *1. $\triangle$ is a selfadjoint operator on $L^2(Z)$ and admits a spectral decomposition.*

2. *The spectrum ρ of the Laplacian is semisimple and* $\mathrm{spec}\triangle \subset (0, \infty)$.

3. *For each* $\lambda \in \mathrm{spec}(\triangle)$ *the associated eigenfunction* ϕ_λ *satisfies* $\phi_\lambda \in C(Z) \cap C^\infty(\mathring{Z})$.

Example C.12. *We consider the boundary value problem*

$$\begin{cases} \triangle u + \lambda u & = 0, \qquad\qquad (x, y) \in \mathring{Q}, \\ u(x, 0) & = u(x, 1) = 0, \quad x \in [0, 1], \\ u(0, y) & = u(1, y), \qquad y \in [0, 1], \\ \frac{\partial u}{\partial x}(0, y) & = \frac{\partial u}{\partial x}(1, y), \qquad y \in [0, 1]. \end{cases} \tag{92}$$

on Z.

It is our aim to solve the differential equation for the Laplacian in direction of x and y separately.
For the sake of completeness, we remark that the operator ∂_y^2 is decomposable with respect to the fibres in x and define an operator on the fibers by

$$\partial_{y,x}^2 : H_0^1 \cap H^2(\{x\} \times (0, 1)) \to L^2(\{x\} \times (0, 1)).$$

Thereby, we obtain a direct integral decomposition of the form

$$\partial_y^2 := \int_{x \in (0,1)}^{\oplus} \partial_{y,x}^2.$$

Let λ be positive. In direction of y we study the one-dimensional problem

$$\begin{cases} \frac{\partial^2 u}{\partial y^2} = u'' = -\lambda u, \quad y \in (0, 1) \\ u(0) = u(1) = 0. \end{cases} \tag{93}$$

Notice that we obtain a second order differential equation with two boundary conditions, so that due to standard results in the theory of differential equations one obtains a unique solution.
We note that by *Gram-Schmidt-Orthonormalization* one has an orthonormal system of $L^2((0, 1))$ consisting of sine and cosine functions.
Naturally, we start with functions of the form

$$u_1(y) := \sin(\sqrt{\lambda} y), \tag{94}$$

$$u_2(y) := \cos(\sqrt{\lambda} y), \tag{95}$$

and check, whether the boundary conditions are satisfied. Note that u_1 and u_2 are not normalized with respect to the L^2-norm.
Inserting the boundary condition into u_1 leads to the equation

$$\sin(\sqrt{\lambda}) = 0,$$

which is satisfied for

$$\lambda = k^2 \pi^2,$$

where $k \in \mathbb{Z}$. Inserting the boundary condition in u_2 yields the contradiction

$$\cos(0) = 0.$$

We conclude that (93) can only be solved by functions of the form of u_1. Due to the point symmetry of the sine function the span of $\sin(k\pi y)$ and the corresponding function $\sin(-k\pi y)$ are equal. So it suffices to consider natural numbers k and thus we obtain the one-dimensional space of eigenfunctions generated by

$$u_k := \sin(k\pi y)$$

with corresponding eigenvalues of the form $\lambda_k := k^2\pi^2$, $k \in \mathbb{N}$.
In direction of x we study the operator $\partial_x^2 \colon H^2((0,1)) \to L^2((0,1))$ and the boundary value problem with periodic boundary conditions given by

$$\begin{cases} \frac{\partial^2 v}{\partial x^2} & = v'' = -\mu v, \quad x \in (0,1) \\ v(1) & = v(0), \\ \frac{\partial v}{\partial x}(1) & = \frac{\partial v}{\partial x}(1). \end{cases} \tag{96}$$

To compute the solution we have a look at

$$v_1(y) := \sin(\sqrt{\mu}x) \tag{97}$$

$$\text{and } \; v_2(y) := \cos(\sqrt{\mu}x). \tag{98}$$

A similar computation shows that we obtain the same result as above with the corresponding eigenvalues already seen. Moreover, inserting the first boundary condition in v_2 leads to eigenvalues of the form

$$\mu_l := 4l^2\pi^2.$$

Due to the the second boundary condition one obtains the two linear independent solutions

$$v_1(x) := \sin(2\pi l x) \text{ and } \; v_2(x) := \cos(2\pi l x).$$

This separation approach allows us to compute possible solutions of our problem (92) by multiplication of the solutions which we obtained in the two separated cases. The corresponding eigenvalues are given by summation of λ_k and μ_l.
We finally combine our results.

Lemma C.13. *The set* $\Lambda := \{\lambda_{kl} := 4\pi^2 l^2 + k^2\pi^2 | \, k, l \in \mathbb{N}\}$ *contains all eigenvalues of* (92) *with corresponding eigenfunctions*

$$f_{1kl} := \sin(k\pi y)\sin(2l\pi x) \text{ and } \; f_{2kl} := \sin(k\pi y)\cos(2l\pi x).$$

Example C.14. *We now consider the same eigenvalue problem on M, given by*

$$\begin{cases} \triangle u + \lambda u & = 0, & (x,y) \in \mathring{Q}, \\ u(x,0) & = u(x,1) = 0, & x \in [0,1], \\ u(0,y) & = u(1,1-y), & y \in [0,1], \\ \frac{\partial u}{\partial x}(0,y) & = \frac{\partial u}{\partial x}(1,1-y), & y \in [0,1]. \end{cases} \tag{99}$$

Once again we try the separation approach to solve the problem. Due to the first and second condition the solutions in direction of y are again given by the functions of the form

$$u_k(y) := \sin(k\pi y), \; k \in \mathbb{Z},$$

which we already obtained for the cylinder. In the next step we need to solve

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} u + \lambda u & = 0, \qquad\qquad (x,y) \in \mathring{Q}, \\ u(0,y) & = u(1,1-y), \quad y \in [0,1], \\ \frac{\partial u}{\partial x}(0,y) & = \frac{\partial u}{\partial x}(1,1-y), \quad y \in [0,1]. \end{cases} \tag{100}$$

With the help of functions of the form

$$v_+(x,y) := \sin(\sqrt{\lambda} x)\sin(k\pi y) \tag{101}$$

and

$$v_-(x,y) := \cos(\sqrt{\lambda} x)\sin(k\pi y) \tag{102}$$

we proceed as above.
Inserting the first boundary condition of (100) into (101) gives $k \in 2 \cdot \mathbb{Z}$. The second equation of (100) leads to

$$\cos(\sqrt{\lambda}) = -1,$$

which holds for $\lambda = \pi^2(2l-1)^2$, $l \in \mathbb{Z}$.
For $l \in \mathbb{Z}$ define

$$\lambda_l := \pi^2(2l-1)^2.$$

Now we check whether (102) fulfills the boundary conditions. By inserting the first one, we obtain the same set of eigenvalues as above, while the second one is only valid for $k \in 2 \cdot \mathbb{Z}$ once more.

Combining our results we finally obtain the solution.

Lemma C.15. *Let* $N := \{n \in \mathbb{N} | n \mod 2 = 1\} = 2\mathbb{N} \pm 1$. *The elements of* $\Lambda_M := \{\lambda_{nm} | \lambda_{nm} = 4\pi^2 m^2 + n^2\pi^2; \text{for } n \in N, m \in \mathbb{N}\}$ *are the eigenvalues of* (99) *with corresponding eigenfunctions*

$$g_{1nm} := \sin(n\pi y)\sin(2m\pi x)$$
$$\textit{and} \;\; g_{2nm} := \sin(n\pi y)\cos(2m\pi x).$$

As a conclusion we state the following.

Remark. *The eigenfunctions of the Dirichlet Laplacian defined on a cylinder and a Möbius strip are different in their periodicity. This is one possibility to tell the different structure of Möbius strip and cylinder apart.*

The above separation approach given by the two different parts of the Laplacian and the decomposition, can be generalized for Laplacians on Riemannian manifolds. Explicitly, for the Laplace Beltrami operator on the total space of a Riemannian submersion.
Here one obtains a decomposition into a vertical and a horizontal part. Whenever the fibers of the Riemannian submersion are totally geodesic submanifolds, those operators commute. We discussed this behavior in Chapter 7. For another precise statement about this content see e.g. [BB82].

Appendix D

Second order elliptic operators

In order to understand the computations in the last two chapters of [Wit11] better, we prove a specific norm equivalence. Just for completeness we give the result in this chapter. Note that these results are not directly necessary for the main part of this thesis but the result is worth to mention. We use the results from Appendix B.

D.1 Basic information

We study operators on Hilbert spaces. To begin we summarize a few important results and list various definitions and properties, mostly proven and stated in [AG13]. Let H again be a Hilbert space and for an operator T on H denote the domain of the operator by $\mathcal{D}_T$.

Definition D.1. *Let $T\colon \mathcal{D}_T \to H$ be an operator. We say the operator T is **symmetric**, if*

(a) its domain $\mathcal{D}_T$ is dense in H and

(b) for all $f, g \in \mathcal{D}_T$ we have

$$\langle Tf, g\rangle = \langle f, Tg\rangle.$$

From the second part in the definition of symmetric operators, one can see that $\langle Tf, f\rangle$ is real. Therefore, at least for symmetric operators, the following definition makes sense as well.

Definition D.2. *If $T\colon \mathcal{D}_T \to H$ is an operator with $\langle Tf, f\rangle \geq 0$ for every $f \in \mathcal{D}_T$, then T is said to be **positive**.*

Later we need the definition of the resolvent for a given operator. But first we introduce some abbreviations to describe the objects more easily. From now on let $T\colon \mathcal{D}_T \to H$ be closed and $\mathcal{D}_T$ be dense in H.

Definition D.3. *We define the operator T_λ by $T_\lambda := T - \lambda I$.*

With the help of these notations we can further define the following objects.

Definition D.4. *If T_λ^{-1} exists and is bounded and defined everywhere in H with $\mathrm{im}(T_\lambda) = H$, then λ is said to be a **regular value** of T. The set of all regular values is called the **resolvent set** and is denoted by $\rho(T)$. All other points in H are the **spectrum** $\sigma(T)$ of T.*

A proof for the following Theorem is given in [AG13], p. 89 f., Theorem 1.

Theorem D.5. *There is a one-to-one correspondence between $\mathcal{D}_T$ and $\mathrm{im}(T_\lambda)$, if and only if λ is not an eigenvalue of T.*

Let again $T\colon \mathcal{D}_T \to H$ be a closed linear operator. We define the resolvent of T as follows.

Definition D.6. *For all $\lambda \in \rho(T)$ we define the **resolvent** of T by*

$$R_\lambda := T_\lambda^{-1}.$$

We directly obtain that for each regular point of T the resolvent R_λ is a bounded operator, which is defined on the whole space and gives rise to a one-to-one correspondence between $\mathcal{D}_T$ and $\mathrm{im}(T_\lambda)$.

D.2 Equivalent norms

We consider a second order operator A on a Hilbert space H, which is symmetric and uniformly elliptic, such that we have the representation

$$Au = -\sum_{ij} \partial_i \left(a_{ij}\partial_j u\right) + \sum_i b_i \partial_i u + cu,$$

where the matrix (a_{ij}) is symmetric and all coefficient functions a_{ij}, b_i and c are smooth.

Definition D.7. *We call the operator A **uniformly elliptic**, if there exists some $\theta > 0$, such that*

$$\sum_{i,j} a_{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2$$

almost everywhere.

Example D.8. *Let $A = -\triangle$, then A is uniformly elliptic with $(a_{ij}) = I$ and $\theta = 1$.*

Let further A always denote a regular elliptic operator and $U \subset H$ open.

We consider the *regular elliptic boundary value problem*, on a relatively compact set $\Omega \subset\subset U$ of the form

$$\begin{cases} Au = f & , \quad \text{in } \Omega \\ u = 0 & , \quad \text{on } \partial\Omega, \end{cases} \tag{103}$$

for a function $f \in C^2(\Omega)$.
The goal of this section will be the proof of the following theorem.

Theorem D.9. *For any solution u of (103) we show $u \in H^2_{\mathrm{loc}}(\Omega)$ and we have an equivalent norm to the regular H^2-norm given by an estimate of the form*

$$c\left(\|u\|_{L^2(\Omega)} + \|Au\|_{L^2(\Omega)}\right) \leq \|u\|_{H^2(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|Au\|_{L^2(\Omega)}\right),$$

for some constants $c, C > 0$.

For every second order elliptic operator, there exists an associated bilinear form.

Definition D.10. *Let $U \subset X$ be open. The bilinear form $B(\cdot,\cdot)$ associated to A is defined by*

$$B(u,v) := \int_U \left[\sum_{ij} a_{ij}\partial_i u \partial_j v + \sum_i b_i \partial_i u v + cuv\right] dx,$$

for all $u, v \in H_0^1(U)$.

For the following statement, we need the *finite difference quotient* and some well-known properties.

Definition D.11. *Let* $u \in L^1_{\text{loc}}(U)$ *and* $\Omega \subset\subset U$. *We define the* $k-th$ ***difference quotient*** *of size* h, *for* $0 < h < dist(\Omega, \partial(U))$, *by*

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h},$$

with $x \in \Omega$ *and* (e_k) *an orthonormal basis.*

We summarize some properties of the difference quotient in the following lemma.

Lemma D.12. *For* $u, v, w \in H_0^1(U)$, *with compact support in* $\Omega \subset\subset U$, *the following statements hold for the difference quotient.*

1. *For sufficiently small* $h > 0$ *the difference quotient* $D_k^h u$ *is in* $H_0^1(U)$.
2. *The identity* $\int_\Omega D_i^h v \cdot w = -\int_\Omega v D_i^{-h} w$ *holds for* h *sufficiently small.*
3. *The product rule, i.e.* $D_i^h(vw) = v^h D_i^h w + (D_i^h v)w$ *with* $v^h = v(x + he_i)$ *holds for all* $0 < h < dist(\Omega, \partial(U))$.
4. *If* $\|D^h u\|_{L^2(\Omega)} \leq C$ *for a constant* $C > 0$ *uniformly in* h, *then* $u \in H^1(\Omega)$ *and* $\|Du\|_{L^2(\Omega)} \leq C$.
5. *The difference quotients* D_k^h *and weak derivatives* ∂_i *commute.*

The part (4) of Lemma (D.12) says that if we are able to bound the difference quotient independently from h, then the weak derivative exists.

Proof of Theorem (*D*.9). We start with the left inequality. Using *Weyl's* theorem, saying that under the assumed conditions the spectrum of the operator A is semisimple, we obtain a representation of the resolvent as

$$R(A, \xi) = \sum_\lambda \frac{1}{\lambda - \xi} E_\lambda.$$

This result is due to the spectral theorem and Neumann's series.

By [AG13] in this case one has the resolvent R given as an operator on $L^2(\Omega)$ which maps to the domain $\mathcal{D}_A$.
Due to the results in Appendix C, we have $\mathcal{D}_A = H_0^1 \cap H^2(\Omega)$, which is compactly embedded into $L^2(\Omega)$ due to Sobolev's embedding theorems. Therefore R is a bounded, compact operator. We then have

$$u = Rf, \tag{104}$$

with

$$\|Rf\|_{\mathcal{D}} \leq c\|u\|_{L^2(\Omega)},$$

where $\|\cdot\|_{\mathcal{D}}$ denotes the *graph-norm* on the domain and $c > 0$ is a constant. By definition of this norm we have

$$\|Rf\|_{L^2(\Omega)} + \|ARf\|_{L^2(\Omega)} = \|Rf\|_{L^2(\Omega)}.$$

Inserting equation (104), we finally obtain the first inequality.

For the second inequality we are going to prove the existence of $u \in H^2(\Omega)$ from some a-priori estimates and the analysis of the ellipticity and regularity theory of the operator. We first sketch a proof of the weaker estimate

$$\|u\|_{H^2(V)} \leq C\left(\|Au\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

for a weak solution u of the elliptic partial differential equation $Au = f$ in Ω and $V \subset\subset \Omega$. The constant C only depends on V, U and the coefficients of A.

We consider a neighborhood $W \subset\subset \Omega$, which contains V and let moreover V be relatively compact in W. Then, we choose a cut-off-function η, such that $0 \leq \eta \leq 1$ and let $\eta \equiv 1$ in V, while its equal to zero in the complement of W in Ω. With the above notation for the difference quotient and the information that $u \in H^1(\Omega)$ we define the compactly supported test function

$$\varphi := -D_k^{-h}(\eta^2 D_k^h u) \in H_0^1(\Omega).$$

Due to Lemma $(D.12(4))$ it suffices to show that $\|D_k^h Du\|_{L^2} \leq \mathrm{const}$ uniformly in h on V to conclude $u \in H^2(V)$.
The weak formulation of the elliptic partial differential equation is

$$-\sum_{ij}\int a_{ij}\partial_i u D_k^{-h}\partial_j(\eta^2 D_k^h u)dx = -\int\left(f - \sum_i b^i\partial_i u - cu\right)\left(D_k^{-h}\left(\eta^2 D_k^h u\right)\right)dx \tag{105}$$

Using $\operatorname{supp}(\eta) \subset\subset W$, integration by parts and the product rule as seen in Lemma $(D.12)$ we derive

$$\begin{aligned}
&-\sum_{ij}\int a_{ij}\partial_i u D_k^{-h}\partial_j(\eta^2 D_k^h u)\,dx\\
&=\sum_{ij}\int D_k^h\left(a_{ij}\partial_i u\right)\partial_j\left(\eta^2 D_k^h u\right)dx\\
&=\sum_{ij}\int D_k^h\left(a_{ij}\partial_i u\right)\left(2\eta\partial_j\eta D_k^h u + \eta^2\partial_j D_k^h u\right)dx\\
&=\sum_{ij}\int D_k^h\left(a_{ij}\partial_i u\right)\left(2\eta\partial_j\eta D_k^h u\right)dx + \sum_{ij}\int D_k^h\left(a_{ij}\partial_i u\right)\left(\eta^2\partial_j D_k^h u\right)dx\\
&=\sum_{ij}\int\left(D_k^h a_{ij}(\partial_i u) + a_{ij}^h D_k^h\partial_i u\right)\left(2\eta\partial_j\eta D_k^h u\right)dx\\
&\quad+\sum_{ij}\left(D_k^h a_{ij}(\partial_i u) + a_{ij}^h D_k^h\partial_i u\right)\left(\eta^2\partial_j D_k^h u\right)dx
\end{aligned}$$

$$= \sum_{ij} \int \eta^2 a_{ij}^h \left(D_k^h \partial_i u\right) \left(D_k^h \partial_j u\right) dx + R,$$

where the error-term R is given by

$$R = \sum_{ij} \int \Big\{ \eta^2 \left(D_k^h a_{ij}\right) (\partial_i u) \left(D_k^h \partial_j u\right)$$
$$+ 2\eta \partial_j \eta \left(a_{ij}^h \left(D_k^h \partial_i u\right) \left(D_k^h u\right) + \left(D_k^h a_{ij}\right) (\partial_i u) \left(D_k^h u\right)\right) \Big\} dx.$$

Using the ellipticity of the operator for the left hand side of (105) the estimate

$$-\sum_{ij} \int a_{ij} \partial_i u D_k^{-h} \partial_j (\eta^2 D_k^h u) dx \geq \theta \int \eta^2 \left| D_k^h Du \right|^2 dx - |R|$$

holds, such that (105) leads to the inequality

$$\theta \int \eta^2 \left| D_k^h Du \right|^2 dx \leq \int \left(f - \sum_i b^i \partial_i u - cu \right) \left(D_k^{-h} \left(\eta^2 D_k^h u \right) \right) dx - R.$$

We now have a closer look at the right hand side. For better understanding we analyze the first term, which can be estimated by the Cauchy-Schwartz inequality in the form

$$\left| \int f D_k^{-h} \left(\eta^2 D_k^h u \right) dx \right| \leq \|f\|_{L^2(\Omega)} \|D_k^{-h} \left(\eta^2 D_k^h u \right) \|_{L^2(\Omega)}.$$

Moreover, with the product rule and the properties of η we can bound the latter norm by

$$\|D_k^{-h} \left(\eta^2 D_k^h u \right) \|_{L^2(\Omega)} \leq \|\eta \partial_k D_k^h u\|_{L^2(\Omega)} + C \|Du\|_{L^2(\Omega)}.$$

It remains to show that the error - term R is bounded, which can be concluded by similar estimates and is given by

$$|R| \leq C_2 \left(\|Du\|_{L^2(\Omega)} \|\eta D_k^h Du\|_+ \|Du\|_{L^2(\Omega)}^2 \right).$$

Combining all estimates we finally obtain

$$\theta \|\eta D_k^h Du\|_{L^2(\Omega)}^2 \leq C_3 \Big(\|f\|_{L^2(\Omega)} \|\eta D_k^h Du\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|Du\|_{L^2(\Omega)}$$
$$+ \|Du\|_{L^2(\Omega)} \|\eta D_k^h Du\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}^2 \Big).$$

Using *Young's inequality* in the $\varepsilon-$version one has

$$\|f\|_{L^2(\Omega)} \|\eta D_k^h Du\|_{L^2(\Omega)} \leq \varepsilon \|\eta D_k^h Du\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon} \|f\|_{L^2(\Omega)}^2,$$
$$\|Du\|_{L^2(\Omega)} \|\eta D_h^k Du\|_{L^2(\Omega)} \leq \varepsilon \|\eta D_k^h Du\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon} \|Du\|_{L^2(\Omega)}^2.$$

By choosing $\varepsilon = \frac{3\theta}{8C_3}$ and absorbing some terms on the right hand side to the left we finally derive

$$\frac{\theta}{4}\|\eta D_k^h Du\|^2_{L^2(\Omega)} \leq C^* \left(\|f\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\right).$$

Due to the choice of the cut-off-function ($\eta \equiv 1$ on V), we then find a constant $\tilde{C}$, such that

$$\|D_k^h Du\|^2_{L^2(V)} \leq \widetilde{C} \left(\|f\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\right). \tag{106}$$

The last estimate is independent from h, which allows to apply Lemma $(D.12)$ and therefore shows that the weak second derivatives of u exist and belong to L^2. Therefore, we finally can express the H^2-norm by

$$\|u\|_{H^2(V)} \leq \widetilde{C} \left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)}\right).$$

To conclude this part of the proof we need to replace the H^1-norm by the L^2-norm. Therefore, we consider another cut-off-function $\eta_1 \in C_0^\infty(\Omega)$ with $\eta_1 \equiv 1$ on W. The ellipticity and the test-function $\nu := \eta_1^2 u$ combined with Young's inequality yield

$$\|Du\|^2_{L^2(W)} \leq \|\eta Du\|^2_{L^2(\Omega)} \leq C \left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right).$$

Inserting this result into (106) we find the stated inequality.
Now we can conclude the forced estimate for the equivalence of the norms.

$\square$

Appendix E

Computations with MAPLE®

Here we give the computations for some of the examples adressed during this thesis. Whenever the MAPLE® output was too long, we omitted the representation in MAPLE®-Inert-2d-View or switched notation to MAPLE®-Inert-3d-View.

E.1 Example 6.25

We give the computations done with the help of MAPLE® to show, that the cocycle differs from one in the setting of Example 6.25.

```
> # Example: Computation for an ellipse e and an
> # ellipsoid E embedded in the Euclidean space R3.
> restart:with(VectorCalculus): with(LinearAlgebra):
> # We give a standard parametrization of the ellipsoid
> # E: psi: (-pi/2, pi/2)x(0,2*Pi)-> E
> psi:= Vector([2*cos(theta)*cos(phi),
> 2*cos(theta)*sin(phi),sin(theta)]);
```

$$\psi := 2\cos(\theta)\cos(\phi)\, e_x + 2\cos(\theta)\sin(\phi)\, e_y + \sin(\theta)\, e_z$$

```
> jac:= Jacobian(psi, [theta,phi]);
```

$$jac := \begin{bmatrix} -2\sin(\theta)\cos(\phi) & -2\cos(\theta)\sin(\phi) \\ -2\sin(\theta)\sin(\phi) & 2\cos(\theta)\cos(\phi) \\ \cos(\theta) & 0 \end{bmatrix}$$

```
> # We obtain the metric tensor by
> F:=simplify(Transpose(jac).jac);
```

$$F := \begin{bmatrix} -3\,(\cos(\theta))^2 + 4 & 0 \\ 0 & 4\,(\cos(\theta))^2 \end{bmatrix}$$

```
> #whose determinant is given by
> de:= simplify(Determinant(F));
```

$$de := -4\left(3\,(\cos(\theta))^2 - 4\right)(\cos(\theta))^2$$

```
> # Proceed with the computation of the tangent vectors
> dthe:=Transpose(Transpose(jac)[1]);
```

$$dthe := \begin{bmatrix} -2\sin(\theta)\cos(\phi) \\ -2\sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}$$

```
> dph:=Transpose(Transpose(jac)[2]);
```

$$dph := \begin{bmatrix} -2\cos(\theta)\sin(\phi) \\ 2\cos(\theta)\cos(\phi) \\ 0 \end{bmatrix}$$

```
> # Test to prove the vectors are perpendicular
> DotProduct(dthe,dph,conjugate=false);
```

$$0$$

```
> #Compute a normal vector
> n:= CrossProduct(dthe,dph);
```

$$n := \begin{bmatrix} -2\,(\cos(\theta))^2\cos(\phi) \\ -2\,(\cos(\theta))^2\sin(\phi) \\ -4\sin(\theta)(\cos(\phi))^2\cos(\theta) - 4\sin(\theta)(\sin(\phi))^2\cos(\theta) \end{bmatrix}$$

```
> # Now we compute a normalized form of n.
> simplify(norm(n,2));
```

$$2\sqrt{\left|(\cos(\theta))^4(\cos(\phi))^2\right| + \left|(\cos(\theta))^4(\sin(\phi))^2\right| + 4\left|(\sin(\theta))^2(\cos(\theta))^2\right|}$$

```
> nor:= simplify(2*cos(theta)*sqrt(cos(theta)^2
> +4*sin(theta)^2));
```

$$nor := 2\cos(\theta)\sqrt{-3\cos(\theta)^2+4}$$

```
> # The normalized vector is then given by
> no:= simplify(1/nor * n );
```

$$no := -\frac{\cos(\theta)\cos(\phi)\,e_x}{\sqrt{-3\,(\cos(\theta))^2+4}} - \frac{\cos(\theta)\sin(\phi)\,e_y}{\sqrt{-3\,(\cos(\theta))^2+4}} - 2\,\frac{\sin(\theta)\,e_z}{\sqrt{-3\,(\cos(\theta))^2+4}}$$

```
> #Description of the parametrization of the tube in R3
> P: (-pi/2, pi/2)x(0,2*Pi)x(-eps,eps)-> E(eps)
> P:= psi + u*no:
> dift:= simplify( Transpose( Transpose(
> Jacobian(P,[ theta,phi,u]))[1]));
```

$$dift := \begin{bmatrix} 2\,\frac{\sin(\theta)\cos(\phi)\left(3\,(\cos(\theta))^2\sqrt{-3\,(\cos(\theta))^2+4}+2\,u-4\sqrt{-3\,(\cos(\theta))^2+4}\right)}{\left(-3\,(\cos(\theta))^2+4\right)^{3/2}} \\ 2\,\frac{\sin(\theta)\sin(\phi)\left(3\,(\cos(\theta))^2\sqrt{-3\,(\cos(\theta))^2+4}+2\,u-4\sqrt{-3\,(\cos(\theta))^2+4}\right)}{\left(-3\,(\cos(\theta))^2+4\right)^{3/2}} \\ -\frac{\cos(\theta)\left(3\,(\cos(\theta))^2\sqrt{-3\,(\cos(\theta))^2+4}+2\,u-4\sqrt{-3\,(\cos(\theta))^2+4}\right)}{\left(-3\,(\cos(\theta))^2+4\right)^{3/2}} \end{bmatrix}$$

```
> difp:= simplify( Transpose( Transpose (
> Jacobian(P, [theta, phi, u]))[2]));
```

$$difp := \begin{bmatrix} -\frac{\cos(\theta)\sin(\phi)\left(2\sqrt{-3\,(\cos(\theta))^2+4}-u\right)}{\sqrt{-3\,(\cos(\theta))^2+4}} \\ \frac{\cos(\theta)\cos(\phi)\left(2\sqrt{-3\,(\cos(\theta))^2+4}-u\right)}{\sqrt{-3\,(\cos(\theta))^2+4}} \\ 0 \end{bmatrix}$$

```
> difu:= simplify( Transpose( Transpose (
> Jacobian(P, [theta, phi, u]))[3]));
```

$$difu := \begin{bmatrix} -\frac{\cos(\theta)\cos(\phi)}{\sqrt{-3(\cos(\theta))^2+4}} \\ -\frac{\cos(\theta)\sin(\phi)}{\sqrt{-3(\cos(\theta))^2+4}} \\ -2\frac{\sin(\theta)}{\sqrt{-3(\cos(\theta))^2+4}} \end{bmatrix}$$

```
> # We know a formula for the curvature of the ellipsoid.
> K:= 16/ (16*cos(theta)^2
> + (4*cos(phi)^2+4*sin(phi)^2)*sin(theta)^2   )^2;
```

$$K := 16\left(16(\cos(\theta))^2 + \left(4(\cos(\phi))^2 + 4(\sin(\phi))^2\right)(\sin(\theta))^2\right)^{-2}$$

```
> simplify(K);
```

$$\left(9(\cos(\theta))^4 + 6(\cos(\theta))^2 + 1\right)^{-1}$$

```
> # With some help MAPLE is able to compute easier
expressions.
> subs([cos(theta)^4=c^2, cos(theta)^2=c],simplify(K));
```

$$\left(9c^2 + 6c + 1\right)^{-1}$$

```
> solve( k= 1/( 9*z^2+6*z+1), z);
```

$$-1/3\frac{\sqrt{k}-1}{\sqrt{k}}, -1/3\frac{\sqrt{k}+1}{\sqrt{k}}$$

```
> # By symmetry of the cosin function we chose an arbitrary
> # solution and insert to obtain the metric in terms of the
> # curvature.
> #The induced metric on the tube is given by F1
> jaco:=simplify(Jacobian(P, [theta, phi, u])):
> F1:=simplify(Transpose(jaco).jaco);
```

$$F1 := \begin{bmatrix} \frac{\left(3(\cos(\theta))^2\sqrt{-3(\cos(\theta))^2+4}+2u-4\sqrt{-3(\cos(\theta))^2+4}\right)^2}{16+9(\cos(\theta))^4-24(\cos(\theta))^2} & 0 & 0 \\ 0 & -\frac{(\cos(\theta))^2\left(2\sqrt{-3(\cos(\theta))^2+4}-u\right)^2}{3(\cos(\theta))^2-4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> # Insert the curvature terms into g
> f2:=subs( [cos(theta)^2= d,cos(theta)^4=d^2], F1);
```

$$f2 := \begin{bmatrix} \frac{\left(3d\sqrt{-3d+4}+2u-4\sqrt{-3d+4}\right)^2}{9d^2-24d+16} & 0 & 0 \\ 0 & -\frac{d\left(2\sqrt{-3d+4}-u\right)^2}{3d-4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> f3:=simplify(subs( d= -(1/3)*(sqrt(k)-1)/sqrt(k), f2));
```

$$f3 := \begin{bmatrix} \frac{\sqrt{k}}{25k^{3/2}-10k+\sqrt{k}}\left(5\sqrt{\frac{5\sqrt{k}-1}{\sqrt{k}}}\sqrt{k} - 2u\sqrt{k} - \sqrt{\frac{5\sqrt{k}-1}{\sqrt{k}}}\right)^2 & 0 & 0 \\ 0 & -1/3\frac{\sqrt{k}-1}{5\sqrt{k}-1}\left(2\sqrt{\frac{5\sqrt{k}-1}{\sqrt{k}}} - u\right)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> # To compute the potential, we need the determinant.
> # Note that we compute it once with the metric given in
terms
> # of the curvature and once just in the local coordinates.
> det3:= simplify( Determinant(f3));
```

$$det3 := -1/3 \frac{\sqrt{k}\left(\sqrt{k}-1\right)}{\left(25\,k^{3/2}-10\,k+\sqrt{k}\right)\left(5\,\sqrt{k}-1\right)}\left(5\sqrt{\frac{5\,\sqrt{k}-1}{\sqrt{k}}}\sqrt{k}-2\,u\sqrt{k}-\sqrt{\frac{5\,\sqrt{k}-1}{\sqrt{k}}}\right)^2\left(2\sqrt{\frac{5\,\sqrt{k}-1}{\sqrt{k}}}-u\right)^2$$

```
> det:= simplify(Determinant(F1));
```

$$\det := -\frac{\left(3\,(\cos(\theta))^2\sqrt{-3\,(\cos(\theta))^2+4}+2\,u-4\sqrt{-3\,(\cos(\theta))^2+4}\right)^2(\cos(\theta))^2\left(2\sqrt{-3\,(\cos(\theta))^2+4}-u\right)^2}{27\,(\cos(\theta))^6-108\,(\cos(\theta))^4+144\,(\cos(\theta))^2-64}$$

```
> # To compute the right Laplacian we use the
> # DifferentialGeometry package.

> with(DifferentialGeometry):  with(Tensor):
```

Now proceed in tensor mode.

```
> DGsetup([theta,phi,u], TE):
> g:= evalDG( (3*cos(theta)^2*sqrt(-3*cos(theta)^2+4)+2*u
> -4*sqrt(-3*cos(theta)^2+4))^2/
> (9*cos(theta)^4-24*cos(theta)^2+16) * dtheta &t dtheta
> - (cos(theta)^2*(2*sqrt(-3*cos(theta)^2+4)-u)^2/
> (3*cos(theta)^2-4))* dphi &t dphi + du &t du );
> # Metric tensor without the curvature expressions

> g3:= evalDG( (5*sqrt((5*sqrt(k)-1)/sqrt(k))*sqrt(k)
> -2*u*sqrt(k)-sqrt((5*sqrt(k)-1)/sqrt(k)))^2/(sqrt(k)
> *(9*cos(theta)^4*sqrt(k)
> +24*sqrt(k)-8))* dtheta&t dtheta
> -(1/3)*(sqrt(k)-1)*(2*sqrt((5*sqrt(k)-1)/sqrt(k))-u)^2/
> (5*sqrt(k)-1)* dphi &t dphi + du &t du  );
> # Metric tensor with the curvature expressions

> # Compute the potential to the right refernce metric
> #on the tube and use the formula
> #W_0= rho^(-1/2)*Laplace(rho^(1/2)).
> # To compute rho, we need the reference metric g_Sa
> # and compute rho= sqrt(det(Fundtens1))/sqrt(det(gSa)).
> rho:= simplify(simplify(sqrt(det))/simplify(sqrt(de))):
> rho2:=simplify(simplify(sqrt(det3))/simplify(sqrt(de))):
> rhot:=subs(cos(theta)^2=-1/3*((sqrt(k)-1)/sqrt(k)),rho2);
> rho3:= simplify(rhot);
```

$$\rho 3 := 1/2\,1\sqrt{-\frac{\sqrt{k}-1}{\left(5\,\sqrt{k}-1\right)^3}\left(5\sqrt{\frac{5\,\sqrt{k}-1}{\sqrt{k}}}\sqrt{k}-2\,u\sqrt{k}-\sqrt{\frac{5\,\sqrt{k}-1}{\sqrt{k}}}\right)^2\left(2\sqrt{\frac{5\,\sqrt{k}-1}{\sqrt{k}}}-u\right)^2}\left(\sqrt{-\frac{\left(5\,\sqrt{k}-1\right)\left(\sqrt{k}-1\right)}{k}}\right)^{-1}$$

```
> lsr:=map ( expand, Laplacian(g, sqrt(rho))):
> lsr3:=map (expand, Laplacian(g3, sqrt(rho31))):
> W:= lsr/ sqrt(rho):
> W3:= lsr3/sqrt(rho31):
> W1:= simplify(W):
> W13:=simplify(W3):
> we:= subs( u= epsilon*u, W1):
> we3:= subs(u=epsilon*u, W13):
> l:=-1/3 * ( sqrt(k)-1)/sqrt(k); # Choose one solution
```

$$l := -\frac{1/3\sqrt{k} - 1/3}{\sqrt{k}}$$

```
> w0:= limit(we, epsilon=0);
> w03:= limit(we, epsilon=0);
> W_0:=simplify(subs( [cos(theta)^2 = l, cos(theta)^4=l^2,
> cos(theta)^6 = l^3], w03));
> # Comuptation of the asymptotic yields finally the
potential
> #(remain exactly the same representation from both
settings).
```

$$w0 := \frac{9\,(\cos(\theta))^4 - 96\,(\cos(\theta))^2 + 128}{432\,(\cos(\theta))^6 - 1728\,(\cos(\theta))^4 + 2304\,(\cos(\theta))^2 - 1024}$$

$$w03 := \frac{9\,(\cos(\theta))^4 - 96\,(\cos(\theta))^2 + 128}{432\,(\cos(\theta))^6 - 1728\,(\cos(\theta))^4 + 2304\,(\cos(\theta))^2 - 1024}$$

$$W\_0 := -1/16\,\frac{161\,k^{3/2} - 34\,k + \sqrt{k}}{125\,k^{3/2} - 75\,k + 15\sqrt{k} - 1}$$

```
> # Note W_0 is just depends on theta.
> restart:
> #Next we study the embedding of the ellipse e into E. e is
> considered as the intersection of E with the y-z-plane.
> #We first give the parametrization of e and then of the
tube
> #to compute the induced metric.
> with(VectorCalculus):  with(LinearAlgebra):
> p:= Vector([0,2*sin(t),-cos(t)]);
```

$$p := 2\,\sin(t)\,e_y - \cos(t)\,e_z$$

```
> #The curvature is given by the following formula
> kappa:= 2/( 4*cos(t)^2 + sin(t)^2)^(3/2);
> solve(k=2/( 4*cos(t)^2 + sin(t)^2)^(3/2), cos(t)^2);
> l:= %[1]:
```

$$\kappa := 2\left(4\,(\cos(t))^2 + (\sin(t))^2\right)^{-3/2}$$

```
> ja:=simplify(Jacobian(p, [t]));
```

$$ja := \begin{bmatrix} 0 \\ 2\,\cos(t) \\ \sin(t) \end{bmatrix}$$

```
> Ft:=simplify(Transpose(ja).ja); #metric on e
```

$$Ft := \left[\; 3\,(\cos(t))^2 + 1 \;\right]$$

```
> # The tube can be described by the parametrization of E
> Pe := Vector([2*cos(theta)*cos(phi), 2*cos(theta)*sin(phi),
> sin(theta)]);
```

$$Pe := 2\,\cos(\theta)\cos(\phi)\,e_x + 2\,\cos(\theta)\sin(\phi)\,e_y + \sin(\theta)\,e_z$$

```
> ab:= Jacobian(Pe,[theta,phi]);
```

$$ab := \begin{bmatrix} -2\,\sin(\theta)\cos(\phi) & -2\,\cos(\theta)\sin(\phi) \\ -2\,\sin(\theta)\sin(\phi) & 2\,\cos(\theta)\cos(\phi) \\ \cos(\theta) & 0 \end{bmatrix}$$

```
> Ft2:= simplify(Transpose(ab).ab); # metric on the tube
> d:= Determinant(Ft2);
```

$$Ft2 := \begin{bmatrix} -3\,(\cos(\theta))^2 + 4 & 0 \\ 0 & 4\,(\cos(\theta))^2 \end{bmatrix}$$

$$d := -4\,\left(3\,(\cos(\theta))^2 - 4\right)(\cos(\theta))^2$$

```
> with(DifferentialGeometry):  with(Tensor):
> DGsetup([theta,phi],E):
> ge:= evalDG( (-3*cos(theta)^2+4 )*(dtheta &t dtheta)
> + (4*cos(theta)^2)*(dphi &t dphi));
> rho:= simplify(simplify(d)/simplify(sqrt(3*cos(t)^2+1) ));
```

$$\rho := -4\,\frac{\left(3\,(\cos(\theta))^2 - 4\right)(\cos(\theta))^2}{\sqrt{3\,(\cos(t))^2 + 1}}$$

```
> map(expand, Laplacian(ge, sqrt(rho)));
> W:= %/ sqrt(rho):
> Wk:=simplify(subs([cos(theta)^2=l , cos(theta)^4=l^2,
> cos(theta)^6 = l^3], W)):
> #Since the result is independent of phi, this is the result
> #for the potential.  (Note the Output is too long.)
> restart:

> # Last consider the embedding of e in R3, the tube is kind
> # of a torus.
> #Since Rn is flat, only Scal_L and the tension field
> #determine the potential.
> # We already know W_0= -1/4 kappa(t)^2, with kappa seen
> # above.
```

E.2 Example 6.26

The following computations were done in the setting of Example 6.26. We used MAPLE® to show that in this setting the cocycle differs from one.
First we studied the embedding of $\mathbb{S}^1$ into $\mathbb{R}^3$.

```
> # We want to compute the potential W_0 for the embedding of
a
> sphere S^1 (in the x-z-plane) into R3.
> # One can compare the result with the embedding into a
torus
> # and then into R3.  We see the results are not equal, such
> # that the corresponding cocyle differs from one.
> with(VectorCalculus):  with(LinearAlgebra):
> with(DifferentialGeometry):  with(Tensor):
> # First we give a parametrization of the tube, which is
kind
> # of a full torus twisted in the 3d space
> # and compute the induced metric.
> P2:= Vector([(1+r*cos(p))*cos(t)+2,-r*sin(p),
(1+r*cos(p))*sin(t)]);
```

$$P2 := ((1 + r\cos(p))\cos(t) + 2)\, e_x - r\sin(p)\, e_y + (1 + r\cos(p))\sin(t)\, e_z$$

```
> ablei:=Jacobian(P2, [r,p,t]);
```

$$ablei := \begin{bmatrix} \cos(p)\cos(t) & -r\sin(p)\cos(t) & -(1 + r\cos(p))\sin(t) \\ -\sin(p) & -r\cos(p) & 0 \\ \cos(p)\sin(t) & -r\sin(p)\sin(t) & (1 + r\cos(p))\cos(t) \end{bmatrix}$$

```
> dr2:= Transpose(Transpose(ablei)[1]);
```

$$dr2 := \begin{bmatrix} \cos(p)\cos(t) \\ -\sin(p) \\ \cos(p)\sin(t) \end{bmatrix}$$

```
> dp2:= Transpose(Transpose(ablei)[2]);
```

$$dp2 := \begin{bmatrix} -r\sin(p)\cos(t) \\ -r\cos(p) \\ -r\sin(p)\sin(t) \end{bmatrix}$$

```
> dt2:= Transpose(Transpose(ablei)[3]);
```

$$dt2 := \begin{bmatrix} -(1 + r\cos(p))\sin(t) \\ 0 \\ (1 + r\cos(p))\cos(t) \end{bmatrix}$$

```
> metri:=Matrix([[simplify(DotProduct(dr2,dr2,
> conjugate=false)),
> simplify(DotProduct(dr2,dp2,conjugate=false)),
> simplify(DotProduct(dr2,dt2,conjugate=false))],
> [simplify(DotProduct(dr2,dp2,conjugate=false)),
> simplify(DotProduct(dp2,dp2,conjugate=false)),
> simplify(DotProduct(dp2,dt2,conjugate=false))],
> [simplify(DotProduct(dt2,dr2,conjugate=false)),
> simplify(DotProduct(dt2,dp2,conjugate=false)),
> simplify(DotProduct(dt2,dt2,conjugate=false))]]);
```

$$metri := \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (1 + r\cos(p))^2 \end{bmatrix}$$

```
> # We find that the reference metric is given by
> gSa:= Matrix(3,3,[[1,0,0],[0,r^2,0],[0,0,1]]);
```

$$gSa := \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> #Therefore we obtain the density
> rho2:= 1+r*cos(p);
```

$$\rho 2 := 1 + r\cos(p)$$

```
> #We start computing with the help of the tensor and the
> DifferentialGeometry package
> DGsetup([r,p,t], Tub):
> metr:= evalDG(dr&t dr+ (r^2)* (dp &t dp)+(1+r*cos(p))^2
> *(dt &t dt ));
```

Now we proceed the computation in the "Tensor mode".

```
> srho:= sqrt(rho2);
```

$$srho := \sqrt{1 + r\cos(p)}$$

```
> # We compute the potential with the help of the correct
> # Laplacian to the metric metr.
> map(expand, Laplacian(metr, srho));
```

$$-1/4\,\frac{1}{(1 + r\cos(p))^{3/2}}\sqrt{\frac{1}{r^2\left(1 + r^2(\cos(p))^2 + 2\,r\cos(p)\right)}}\left(\sqrt{\frac{1}{r^2\,(1 + r\cos(p))^2}}\right)^{-1}$$

```
> simplify(%);
```

$$-1/4\,(1 + r\cos(p))^{-3/2}$$

```
> #Rescaling and the limit process yield
> weps:= subs(r=epsilon*r, %);
```

$$weps := -1/4\,(1 + \epsilon\, r\cos(p))^{-3/2}$$

```
> w0:= limit(%, epsilon=0);
```

$$w0 := -1/4$$

```
> #We finally obtain the same result as already seen,
> #since the curvature of the circle equals one constantly.
```

We proceed with the embedding of the torus in the three-dimensional euclidean space $\mathbb{R}^3$ and derive an expression for the potential.

```
> restart:
> with(LinearAlgebra):  with(VectorCalculus):
> # We first give the embedding of the torus in R3.
> phi:=Vector([(2+cos(p))*cos(t),(2+cos(p))*sin(t),sin(p)]);
```

$$\phi := (2+\cos(p))\cos(t)\,e_x + (2+\cos(p))\sin(t)\,e_y + \sin(p)\,e_z$$

```
> abl:=Jacobian(phi,[p,t]);
```

$$abl := \begin{bmatrix} -\sin(p)\cos(t) & -(2+\cos(p))\sin(t) \\ -\sin(p)\sin(t) & (2+\cos(p))\cos(t) \\ \cos(p) & 0 \end{bmatrix}$$

```
> d1:= Transpose(Transpose(abl)[1]);
```

$$d1 := \begin{bmatrix} -\sin(p)\cos(t) \\ -\sin(p)\sin(t) \\ \cos(p) \end{bmatrix}$$

```
> d2:=Transpose(Transpose(abl)[2]);
```

$$d2 := \begin{bmatrix} -(2+\cos(p))\sin(t) \\ (2+\cos(p))\cos(t) \\ 0 \end{bmatrix}$$

```
> simplify(norm(d1,2));
```

$$\sqrt{\left|(\sin(p))^2(\cos(t))^2\right| + \left|(\sin(p))^2(\sin(t))^2\right| + (|\cos(p)|)^2}$$

```
> # Computation of the normalized normal vector.
> n:= CrossProduct(d1,d2);
> n := -cos(p)*(2+cos(p))*cos(t)*e[x]-cos(p)*(2+cos(p))*sin(t)*e[y]
+(-sin(p)*cos(t)^2*(2+cos(p))-sin(p)*sin(t)^2*(2+cos(p)))*e[z]
> nor:= (2+cos(p))^2;
```

$$nor := (2+\cos(p))^2$$

```
> no:= 1/nor *n;
> no := -cos(p)*cos(t)*e[x]/(2+cos(p))-cos(p)*sin(t)*e[y]/(2+cos(p))
+(-sin(p)*cos(t)^2*(2+cos(p))-sin(p)*sin(t)^2*(2+cos(p)))*e[z]/(2+cos(p))^2
> # Parametrization of the tube in R3.
> P:= phi+ u* no:
> dp1:=simplify(Transpose(
> Transpose(Jacobian(P,[p,t,u]))[1])):
> dt1:=simplify(Transpose(
> Transpose(Jacobian(P,[p,t,u]))[2])):
> dw:=simplify(Transpose(
> Transpose(Jacobian(P,[p,t,u]))[3])):
> # Computation of the induced metric on the tube.
```

```
> Fundtens1:=Matrix([[simplify(DotProduct(dp1,dp1)),
> simplify(DotProduct(dp1,dt1)),
> simplify(BilinearForm(dp1,dw,conjugate=false))],
> [simplify(DotProduct(dp1,dt1)),
> simplify(DotProduct(dt1,dt1)),
> simplify(DotProduct(dt1,dw))],
> [simplify(DotProduct(dw,dp1)),
> simplify(DotProduct(dw,dt1)),
> simplify(DotProduct(dw,dw))]]):
> #We give the nonzero entries.
> Fundtens1[1,1];
> Fundtens1[2,2];
> Fundtens1[3,3];
> Fundtens1[1,3];
> Fundtens1[3,1];
>  (cos(p)^4-2*cos(p)^3*u+8*cos(p)^3-12*cos(p)^2*u+
4*cos(p)*u^2+24*cos(p)^2-24*cos(p)*u+5*u^2+32*cos(p)-16*u+16)
/(cos(p)^4+8*cos(p)^3+24*cos(p)^2+32*cos(p)+16)
```

$$\frac{\left((\cos(p))^2 - \cos(p)\,u + 4\,\cos(p) + 4\right)^2}{(\cos(p))^2 + 4\,\cos(p) + 4}$$

$$\left((\cos(p))^2 + 4\,\cos(p) + 4\right)^{-1}$$

$$\frac{\sin(p)\,u}{(\cos(p))^3 + 6\,(\cos(p))^2 + 12\,\cos(p) + 8}$$

$$\frac{\sin(p)\,u}{(\cos(p))^3 + 6\,(\cos(p))^2 + 12\,\cos(p) + 8}$$

```
> dete:=simplify(Determinant(Fundtens1));
>  dete := (cos(p)^4-2*cos(p)^3*u+cos(p)^2*u^2+8*cos(p)^3
-8*cos(p)^2*u+24*cos(p)^2-8*cos(p)*u
+32*cos(p)+16)*(cos(p)^2-2*cos(p)*u+u^2+4*cos(p)-4*u+4)
/(cos(p)^6+12*cos(p)^5+60*cos(p)^4+160*cos(p)^3+240*cos(p)^2
+192*cos(p)+64)
> # To compute the potential we need the
> # Laplacian to the underlying metric.
> with(DifferentialGeometry):  with(Tensor):
> DGsetup([p,t,u], M):
> met:= evalDG( ((cos(p)^4-2*cos(p)^3*u+8*cos(p)^3
> -12*cos(p)^2*u+4*cos(p)*u^2+24*cos(p)^2-24*cos(p)*u
> +5*u^2+32*cos(p)-16*u+16)
> /(cos(p)^4+8*cos(p)^3
> +24*cos(p)^2+32*cos(p)+16)) * (dp &t dp) +
> (2*simplify(DotProduct(dw,dp1)) )*(dp &t du)
> +( (cos(p)^2-cos(p)*u+4*cos(p)+4)^2/(cos(p)^2+
> 4*cos(p)+4) )*(dt &t dt) + (1/(cos(p)^2+4*cos(p)+4))
> *(du &t du) );
> rho := sqrt( dete):# Potential
> map (expand, Laplacian(met, sqrt(rho))):  #Output too long.
> %/ sqrt(rho):
> W:=simplify(%):
```

```
> subs ( u=epsilon* u , W):
> W_0:=limit(%, epsilon=0);
```

$$W\_0 := \frac{-2\,(\cos(p))^2 - 4\,\cos(p) - 1}{(\cos(p))^2 + 4\,\cos(p) + 4}$$

```
> # We observe that this is neither constant
> # nor the curvature.
```

It remains the embedding of $\mathbb{S}^1$ into the torus.
MAPLE® was not really helpful in this case.
We did not use the DifferentialGeometry package. But computed the potential without it.

```
> #S1 embedded into the torus.
> restart:
> with(VectorCalculus):with(LinearAlgebra):
> # Parametrization of the tube.
> P1:=Vector([(2+cos(p))*cos(t),
> (2+cos(p))*sin(t),sin(p)]);
```

$$P1 := (2+\cos(p))\cos(t)\,e_x + (2+\cos(p))\sin(t)\,e_y + \sin(p)\,e_z$$

```
> abl1:=Jacobian(P1,[p,t]);
```

$$abl1 := \begin{bmatrix} -\sin(p)\cos(t) & -(2+\cos(p))\sin(t) \\ -\sin(p)\sin(t) & (2+\cos(p))\cos(t) \\ \cos(p) & 0 \end{bmatrix}$$

```
> dp1:= Transpose(Transpose(abl1)[1]);
> dt1 := Transpose(Transpose(abl1)[2]);
```

$$dp1 := \begin{bmatrix} -\sin(p)\cos(t) \\ -\sin(p)\sin(t) \\ \cos(p) \end{bmatrix}$$

$$dt1 := \begin{bmatrix} -(2+\cos(p))\sin(t) \\ (2+\cos(p))\cos(t) \\ 0 \end{bmatrix}$$

```
> with(DifferentialGeometry):  with(Tensor):
> Fundtens1:=Matrix([[simplify(BilinearForm(dp1,dp1,
> conjugate=false)),
> simplify(BilinearForm(dp1,dt1,conjugate=false))],
> [simplify(BilinearForm(dp1,dt1,conjugate=false)),
> simplify(BilinearForm(dt1,dt1,conjugate=false))]]);
> mi:= Fundtens1^(-1);
```

$$Fundtens1 := \begin{bmatrix} 1 & 0 \\ 0 & (2+\cos(p))^2 \end{bmatrix}$$

$$mi := \begin{bmatrix} 1 & 0 \\ 0 & (2+\cos(p))^{-2} \end{bmatrix}$$

```
> rho:= 2+ cos(p);
```

$$\rho := 2 + \cos(p)$$

```
> diff(rho, p);
```

$$-\sin(p)$$

```
> diff(rho,t);
```

$$0$$

```
> #We compute the Laplace Beltrami Operator by
> LSR:= (1/(2+cos(p)))* diff( (2+cos(p))*
diff(sqrt(rho),p),p);
```

$$LSR := \frac{1}{2+\cos(p)}\left(1/4\,\frac{(\sin(p))^2}{\sqrt{2+\cos(p)}} - 1/2\,\sqrt{2+\cos(p)}\cos(p)\right)$$

```
> W:= (1/sqrt(rho) )* LSR;
```

$$W := \frac{1}{(2+\cos(p))^{3/2}}\left(1/4\,\frac{(\sin(p))^2}{\sqrt{2+\cos(p)}} - 1/2\,\sqrt{2+\cos(p)}\cos(p)\right)$$

```
> #this equals W_0.
```

E.3 Example 2.13

The following computations are done with MAPLE® and complete the observations made in Example (2.13).
We give the complete code to show how the potential was computed.

```
> #Example: Embedding of S^1 into S^2.
> restart:
> with(LinearAlgebra): with(VectorCalculus):

> #With the help of the standard spherical coordinates, we
give
> # a parametrization of the tube T,
> # given by Psi: (-eps,eps)x(0,2*Pi) -> T as follows.
> psi:=Vector([cos(u)*cos(v), cos(u)*sin(v),sin(u)]);
```

$$\psi := \cos(u)\cos(v)\,e_x + \cos(u)\sin(v)\,e_y + \sin(u)\,e_z$$

```
> #We chose this parametrization since the result turns out
> # to be positive for this case.
> #Now compute the tangent vectors.
> a:=Jacobian(psi,[u,v]);
```

$$a := \begin{bmatrix} -\sin(u)\cos(v) & -\cos(u)\sin(v) \\ -\sin(u)\sin(v) & \cos(u)\cos(v) \\ \cos(u) & 0 \end{bmatrix}$$

```
> d1:=Transpose(Transpose(a)[1]);
> d2:=Transpose(Transpose(a)[2]);
```

$$d1 := \begin{bmatrix} -\sin(u)\cos(v) \\ -\sin(u)\sin(v) \\ \cos(u) \end{bmatrix}$$

$$d2 := \begin{bmatrix} -\cos(u)\sin(v) \\ \cos(u)\cos(v) \\ 0 \end{bmatrix}$$

```
> #We obtain the induced metric on the tube by
> g:=Matrix([[simplify(DotProduct(d1,d1)),
> simplify(DotProduct(d1,d2))],
> [simplify(DotProduct(d2,d1)),
> simplify(DotProduct(d2,d2))]]);
```

$$g := \begin{bmatrix} 1 & 0 \\ 0 & (\cos(u))^2 \end{bmatrix}$$

```
> #The volume element is given by: ds^2= du^2+cos(u)^2 dv^2.
> #The reference metric is as usual the Sasaki.
> #The associated density is given by
> det:=simplify(Determinant(g));
> rho:=sqrt(det) assuming cos(u)>0;
```

$$\det := (\cos(u))^2$$

$$\rho := \cos(u)$$

```
> #This is well defined since the cos function is positive
> # whenever we are close to zero.
> #To determine the potential, we first need to compute
> # the Laplacian with respect to g.
> with(DifferentialGeometry):  with(Tensor):
> DGsetup([u,v], T):
> metr:= evalDG( du &t du+ cos(u)^2 * (dv &t dv)):
```

Now we proceed the computation in the "Tensor mode".

```
> srho:= sqrt(rho);
```

$$srho := \sqrt{\cos(u)}$$

```
> LRR:=map(expand, Laplacian(metr, srho));
```

$$LRR := 1/4\,\frac{\sqrt{(\cos(u))^{-2}}\left(2\,csgn\left(1,(\cos(u))^{-1}\right)\cos(u)\sin(u)+3\,csgn\left((\cos(u))^{-1}\right)(\cos(u))^{2}-csgn\left((\cos(u))^{-1}\right)\right)}{\sqrt{\cos(u)}}$$

```
> #Now we compute W_0 = rho^(-1/2)* LRR, afterwards
> # we derive the expression for the asymptotic.
> # (For overview we do not give the MAPLE output.)
> LRR/srho:
> subs(u=epsilon*u, %):
> limit(%, epsilon=0);
```

$$1/2$$

```
> #Therefore the potential is given constantly by W_0 = 1/2
> # and the associated density is equal to one.
> #Remark:  The Laplacian with respect to g applied to a
> # function f in local coordinates is given by
```

$$\Delta_g f = \frac{\frac{d^2}{dv^2} f}{(\cos(u))^2} - \tan(u)\,\frac{d}{du} f + \frac{d^2}{du^2} f$$

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