

Single-class genera of orthogonal groups

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1 Preface

The objective of the present thesis is to give a classification, up to similarity, of single-class, totally definite quadratic forms of rank at least 3 over totally real algebraic number fields.

Motivation. A quadratic form is a homogeneous polynomial f of degree 2 in m variables. The study of quadratic forms over the rational field $\mathbb{Q}$ and over the ring of rational integers $\mathbb{Z}$ is of interest to classical number theory. Basic questions are, among others, the questions of representation (which asks for those numbers that can occur as the value of f), and isometry (which, given another quadratic form f' , asks whether f' results from f via a linear change of base).

In this thesis, we will examine quadratic forms in the more general setting of algebraic number fields K , which are finite extensions of $\mathbb{Q}$. More precisely, we will be interested in quadratic spaces over algebraic number fields K , and in $\mathfrak{o}_K$ -lattices in quadratic spaces over K , where $\mathfrak{o}_K$ is the ring of integers of K . The theorem of HASSE-MINKOWSKI states a “local-global principle” for quadratic spaces: two quadratic spaces over a number field K are isometric if and only if all of their completions are isometric. And an element $a \in K^*$ is represented by a quadratic space over K if and only if it is represented locally at every completion.

However, the local-global principle fails for the integral theory (i.e., lattices in quadratic spaces over $\mathfrak{o}_K$). A classic example given in [Kne02, 21.4] shows that the two $\mathbb{Z}$ -lattices with Gram matrices

$$\begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix} \text{ and } \begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}$$

are isometric over all the completions $\mathbb{Z}_p$ for prime numbers p and over the real field, but are not isometric over $\mathbb{Z}$. Still, the set of lattices in a given quadratic space V which are isometric at every completion is the union of a finite set of isometry classes. This motivates a coarser partitioning of lattices into equivalence classes: lattices which are isometric at every completion are said to be in the same *genus*.

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This thesis examines the special case of “single-class lattices”: whenever the genus of some lattice consists of a single isometry class, the lattice is one of the few that do fulfill a local-global principle.

Historical perspective. The question of classifying single-class quadratic forms has a long history. Dating back to GAUSS is a correspondence between the isometry classes in a genus of binary quadratic forms and certain ideal classes in a quadratic extension of $\mathfrak{o}_K$. This leads to relative ideal class number problems of CM-extensions, which are very hard to solve. Modern approaches to the classification problem usually exclude the binary case, focusing instead on forms in $m \geq 3$ variables.

In 1937, MAGNUS (cf. [Mag37]) showed that single-class, positive definite quadratic forms over $\mathbb{Q}$ can only exist in $m \leq 34$ variables. In a series of articles published between 1962 and 1984, G. L. WATSON went on to improve this bound to $m \leq 10$ (cf. [Wat62]). He then started a classification of single-class, positive definite, primitive quadratic forms of rank $3 \leq m \leq 10$ over $\mathbb{Q}$, using an arithmetic approach. While his classification remains incomplete, the parts that do exist are largely correct.

WATSON’s approach made heavy use of a reduction method he called the “ p -mapping”. Described in the more convenient notation of lattices L in quadratic spaces V , the p -mapping maps L to the intersection $L \cap pL^\#$, where $L^\#$ is the dual lattice of L , and then rescales the underlying quadratic form on V by $\frac{1}{p}$. The p -mapping does not increase the class-number of L , and the resulting lattice may have a simpler local structure at the prime p . After applying the p -mapping repeatedly and for different choices of p to an integral lattice L , one always arrives at a square-free lattice. Being “square-free” is a condition on Jordan decompositions at the completions of L , but can for now be thought of as having “simple enough” local structure. Constraining the classification problem to single-class, positive definite and square-free lattices makes it considerably easier to solve. WATSON’s reduction to square-free lattices can be adapted to lattices over general totally real number fields. This was done in 1972 by GERSTEIN (cf. [Ger72]).

Generalising from $\mathbb{Q}$ to arbitrary totally real number fields, PFEUFFER in 1971 (cf. [Pfe71]) showed that the rank of a single-class, totally positive definite quadratic form over a totally real number K field cannot exceed 16. Moreover, the number of possible base fields K (up to isomorphism) for these forms is finite. His approach makes heavy use of SIEGEL’s mass formula, which relates the numbers of automorphisms of the isometry classes inside a genus to certain local quantities. PFEUFFER also obtained bounds on

the discriminant and the degree of the base field K , but these bounds were unreasonably large for $3 \leq \text{rank}(L) \leq 6$.

Outline. This thesis gives a complete classification of single-class lattices of rank ≥ 3 in totally positive definite quadratic spaces over totally real algebraic number fields. The indefinite case is excluded from the classification (the number of isometry classes in an indefinite genus being essentially a local question). Building on PFEUFFER's results, the mass formula is used to obtain effective bounds on the base fields, and on the local invariants, of single-class, square-free, totally definite lattices over totally real number fields. We then give an algorithm for the enumeration of a genus, which together with the bounds from the mass formula allows for algorithmic enumeration of the subset of single-class genera which are square-free. The general case (i.e., removing the restriction to square-free lattices) is approached with another algorithm, using iterated calculation of pre-images under the mapping GERSTEIN had proposed, and which was inspired by WATSON.

Methods. Chapter 3 states SIEGEL's mass formula in a version that is due to PFEUFFER. The results given there allow to obtain bounds on the discriminant of the number field K ¹, on the invariants of the totally definite quadratic space V over K admitting a single-class lattice L , and on the norm of places $\mathfrak{p}$ where the local structure of L is nontrivial, provided that L is also square-free. This enables an algorithmic approach to our classification problem: one has to construct all candidate square-free lattices in all candidate quadratic spaces over K ("candidate" meaning that they fulfill the various bounds mentioned above) and check whether they are alone in their genus. Sections 2.5 and 2.9 give algorithms to construct quadratic spaces from invariants, and lattices in a given genus.

One obstacle remains before we can complete the classification of single-class, square-free genera: we need to give an algorithm to enumerate all the isometry classes in a genus of totally definite lattices over a general, totally real algebraic number field. This algorithm is developed in Chapter 4. In general, the genus of L partitions into a finite number of *spinor genera*, which in turn consist of a finite number of isometry classes. Thanks to

¹The bounds on the discriminant of K obtained from the mass formula are good enough for $m \geq 4$, in the sense that all number fields (up to isomorphism) fulfilling these bounds can be found in existing tables (e.g., those obtained by VOIGHT in [Voi08]). For $m = 3$, we present an entirely different method to obtain the possible base fields, which is based on a correspondence to quaternion algebras. This correspondence was previously published by M. KIRSCHMER and the author in [KL16].

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strong approximation, KNESER’s neighbour method (cf. [Kne57]), applied at any suitable prime $\mathfrak{p}$, can be used to enumerate all the isometry classes in the proper spinor genus of a totally definite lattice L .

The “Spinor Linkage Theorem” by BENHAM and HSIA ([BH83]) states that given any two proper spinor genera inside the genus of L , there is always some prime $\mathfrak{p}$ such that the set of iterated $\mathfrak{p}$ -neighbours of L is the union of the two proper spinor genera. So for a suitably chosen set of primes P , iterated construction of $\mathfrak{p}$ -neighbours at all primes $\mathfrak{p} \in P$ will cover the entirety of the set of isometry classes in the genus of a totally definite lattice L .

However, existing literature did not provide an effectively computable description of a set P that contains enough primes to link all the different spinor genera in the genus of L . [BH83] merely states that for every two (proper) spinor genera, there are infinitely many primes linking them. To enumerate the entire genus of L , we need to know when to stop generating more neighbours. This problem is solved in the present thesis by identifying the set of spinor genera in the genus of L with the group C_L , which is a quotient of a finite ray class group, according to an idea by WAI-KIU CHAN.

For totally definite lattices L , the isometry classes in the genus of L can now be enumerated by what is essentially an orbit algorithm: choose a set of prime ideal representatives of generators of the group C_L , and perform iterated neighbour generation, using an isometry test to check whether new lattices were found.

For indefinite lattices, the Spinor Linkage Theorem still holds, but an algorithmic isometry test is not available. However, for indefinite lattices of rank ≥ 3 , the proper spinor genus and the proper isometry class coincide. We modify the above approach for indefinite lattices by calculating prime ideal representatives of every element of $C_L/\langle g \rangle$, where $g \in C_L$ is an element calculated in Section 4.5.1 such that $g = 1$ if spinor genera and proper spinor genera coincide in the genus of L , and otherwise g switches between the two proper spinor genera in each spinor genus. Calculating one neighbour at each of these prime ideals gives a complete set of representatives of the isometry classes in the genus of the indefinite lattice L .

This solves the enumeration of a genus of a (definite or indefinite) lattice of rank ≥ 3 over a totally real algebraic number field.

Chapter 5 presents an algorithm based on the ideas of GERSTEIN and WATSON. It allows to construct the remaining single-class lattices (that is, those that are not square-free)

by an iterated calculation of pre-images under a mapping that does not increase the class number.

Chapter 6 gives a summary of the classification of totally definite single-class lattices of rank ≥ 3 over totally real algebraic number fields. Some of these lattices have previously been published – notably, the case $K = \mathbb{Q}$ in [LK13], and the case $\text{rank}(L) = 3$ in [KL16] – so the chapter focuses on the case $K \neq \mathbb{Q}$ and $\text{rank}(L) \geq 4$.

Results. The total number (up to similarity) of single-class, totally definite lattices L with $\text{rank}(L) \geq 3$ in quadratic spaces V over totally real algebraic number fields K is 5903. It turns out that unless $K = \mathbb{Q}$, $\text{rank}(L) \leq 6$. The following list gives the number of these lattices, up to similarity of L , isometry of V and isomorphism of K (cf. Theorem 6.1):

$[K : \mathbb{Q}]$	$\text{rank}(L)$	number of similarity classes	detailed results
1	3, 4, ..., 10	1884	[LK13]
> 1	3	3408 ²	[KL16]
> 1	4	607	Chapter 6
> 1	5	2	Chapter 6
> 1	6	2	Chapter 6

If $K \neq \mathbb{Q}$, the base field in question is always $K = \mathbb{Q}(\sqrt{5})$ for $\text{rank}(L) \in \{5, 6\}$, and is one of 22 different base fields for $\text{rank}(L) = 4$, the largest field degree being 5 (see Section 6.2).

A printed list of representatives of the cases where $K \neq \mathbb{Q}$ and $\text{rank}(L) \in \{4, 5, 6\}$ concludes this thesis.

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²See also the footnote in part 2)b) of Theorem 6.1.

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2.1 Dedekind domains

Definition 2.1. A *Dedekind domain* is an integral domain $\mathfrak{o}$ in which every non-zero ideal is a unique product of prime ideals (up to permutation of the factors).

Let K be the field of fractions of a Dedekind domain $\mathfrak{o}$.

Definition 2.2. A *fractional ideal* of $\mathfrak{o}$ is a non-zero $\mathfrak{o}$ -submodule $\mathfrak{b}$ of K such that there is some $0 \neq \beta \in \mathfrak{o}$ with $\beta\mathfrak{b} \subseteq \mathfrak{o}$. Analogous to ideals, a *principal fractional ideal* is an $\mathfrak{o}$ -submodule of K which is generated by a single element $0 \neq b \in K$.

The set of all fractional ideals of $\mathfrak{o}$ is denoted $\mathbb{I}(\mathfrak{o})$.

If $\mathfrak{a}, \mathfrak{b} \in \mathbb{I}(\mathfrak{o})$, then their *product* is the fractional ideal

$$\mathfrak{a}\mathfrak{b} := \{a_1b_1 + \cdots + a_nb_n \mid n \in \mathbb{N}, a_i \in \mathfrak{a}, b_i \in \mathfrak{b} \text{ for } 1 \leq i \leq n\}.$$

Let $\mathfrak{a} \in \mathbb{I}(\mathfrak{o})$. Then $\mathfrak{a}^{-1} := \{x \in K \mid x\mathfrak{a} \subseteq \mathfrak{o}\} \in \mathbb{I}(\mathfrak{o})$ is called the *inverse* of $\mathfrak{a}$ and satisfies: $\mathfrak{a}\mathfrak{a}^{-1} = \mathfrak{o}$.

Remark 2.3. Dedekind domains are noetherian, integrally closed domains, and every non-zero prime ideal of a Dedekind domain is maximal. Conversely, every such domain is a Dedekind domain (cf. [Neu92, I, §3]).

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Theorem 2.4. The set $\mathbb{I}(\mathfrak{o})$ forms an abelian group under fractional ideal multiplication. It is free on the set of prime ideals of $\mathfrak{o}$. The set of principal ideals $\mathbb{P}(\mathfrak{o}) := \{a\mathfrak{o} \mid a \in K^*\}$ is a subgroup of $\mathbb{I}(\mathfrak{o})$. So we can define the quotient $\text{Cl}_K := \mathbb{I}(\mathfrak{o})/\mathbb{P}(\mathfrak{o})$, the *ideal class group* of K .

The cardinality $h(K)$ of the group Cl_K is called the *class number* of K .

Corollary 2.5. Let $\mathfrak{a}$ be a fractional ideal of $\mathfrak{o}$. Then $\mathfrak{a}$ can be written as a product $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{a})}$, where the product runs over all maximal ideals $\mathfrak{p}$ of $\mathfrak{o}$, with every $\nu_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}$.

For each maximal ideal $\mathfrak{p}$ of $\mathfrak{o}$, this defines a mapping $\nu_{\mathfrak{p}} : \mathbb{I}(\mathfrak{o}) \rightarrow \mathbb{Z}$, $\mathfrak{a} \mapsto \nu_{\mathfrak{p}}(\mathfrak{a})$ and also a mapping

$$\nu_{\mathfrak{p}} : K \rightarrow \mathbb{Z} \cup \{\infty\}, a \mapsto \begin{cases} \infty, & \text{if } a = 0 \\ \nu_{\mathfrak{p}}(a\mathfrak{o}), & \text{otherwise.} \end{cases}$$

Every $\nu_{\mathfrak{p}}$ is a *discrete valuation*, i.e., a map $K \rightarrow \mathbb{Z} \cup \{\infty\}$ satisfying:

1. $\nu_{\mathfrak{p}}(a) = \infty$ if and only if $a = 0$.
2. $\nu_{\mathfrak{p}}(ab) = \nu_{\mathfrak{p}}(a) + \nu_{\mathfrak{p}}(b)$ for all $a, b \in K$.
3. $\nu_{\mathfrak{p}}(a + b) \geq \min(\{\nu_{\mathfrak{p}}(a), \nu_{\mathfrak{p}}(b)\})$ for all $a, b \in K$.

2.2 Algebraic number fields

For proofs of the statements in this section we refer to [Neu92].

Definition 2.6. An *algebraic number field* is a finite extension of $\mathbb{Q}$, the field of rational numbers.

Definition 2.7. Let K be an algebraic number field. An element $a \in K$ is called *integral* if there exists some monic polynomial $\mu \in \mathbb{Z}[x]$ such that $\mu(a) = 0$.

The set $\mathfrak{o}$ of all integral elements of K is called the *ring of integers* of K .

For the remainder of this section, let K be any algebraic number field and $\mathfrak{o}$ its ring of integers.

Example 2.8. 1. Let $K = \mathbb{Q}$. Then $\mathfrak{o} = \mathbb{Z}$.

2. Let $K = \mathbb{Q}(\sqrt{5})$. Then $\mathfrak{o} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$.

Theorem 2.9. 1. The ring of integers $\mathfrak{o}$ of K is a Dedekind domain.

2. The class number of K , $h(K)$, is finite.

Definition 2.10. Any element $a \in K$ defines a $\mathbb{Q}$ -linear mapping $\ell_a : K \rightarrow K, x \mapsto ax$. The *norm* of a is the determinant $\mathfrak{n}(a) := \det(\ell_a)$ of this mapping, and the *trace* of a is its trace $\mathfrak{t}(a) := \text{tr}(\ell_a)$.

The norm $\mathfrak{N}(\mathfrak{a})$ of a fractional $\mathfrak{o}$ -ideal $\mathfrak{a}$ is the fractional $\mathbb{Z}$ -ideal generated by the set

$$\{\mathfrak{n}(a) \mid a \in \mathfrak{a}\}.$$

Remark 2.11. The norm of fractional ideals is multiplicative.

If $\mathfrak{a} \subseteq \mathfrak{o}$ is some ideal, then $\mathfrak{N}(\mathfrak{a})$ is (the $\mathbb{Z}$ -ideal generated by) the cardinality of the *residue class ring* $\mathfrak{o}/\mathfrak{a}$.

Theorem 2.12. Let $p \in \mathbb{Z}$ be a prime number. Write $p\mathfrak{o} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_k^{e_k}$ with $e_i \in \mathbb{N} - \{0\}$ ($1 \leq i \leq k$) and with pairwise distinct prime ideals $\mathfrak{p}_i$ of $\mathfrak{o}$. The $\mathfrak{p}_i$ are called the prime ideals of $\mathfrak{o}$ *over* p , and e_i is called the *ramification index* of $\mathfrak{p}_i$. We say that p is *ramified* if $e_i > 1$ for at least one index $i \in \{1, \dots, k\}$.

For every $1 \leq i \leq k$, the quotient field $\mathfrak{o}/\mathfrak{p}_i$ is a finite extension of the finite field $\mathbb{F}_p$ (of degree f_i , say). The number $f_i \in \mathbb{N}$ is called the *inertia degree* of $\mathfrak{p}_i$.

The Chinese Remainder Theorem yields the fundamental relation

$$[K : \mathbb{Q}] = \sum_{i=1}^k e_i f_i.$$

For a proof, see [Neu92, I, §8, (8.2)].

2.3 Valuations

Definition 2.13. Let K denote any field.

A *valuation* $|\cdot|$ on K is any mapping $K \rightarrow \mathbb{R}^{\geq 0}$ such that the following conditions are met:

1. $|x| = 0 \Leftrightarrow x = 0$.
2. $|xy| = |x| \cdot |y|$ for all $x, y \in K$.
3. $|x + y| \leq |x| + |y|$ for all $x, y \in K$.

If $|\cdot|$ fulfills the *strong triangle inequality* $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$, then $|\cdot|$ is called a *non-archimedean valuation*. This is the case if and only if the set $\{|x| \mid x \in \mathbb{Z}\}$ is bounded on $\mathbb{R}$.

Otherwise, $|\cdot|$ is called an *archimedean valuation*.

Example 2.14. Let K be any field.

1. A valuation $|\cdot|$ on K satisfying $|x| = 1$ for all $x \neq 0$ is called $|\cdot|$ the *trivial valuation* on K .
2. Let $\mathfrak{p}$ be a prime ideal of a Dedekind domain $\mathfrak{o}$ with field of fractions K , and let $\nu_{\mathfrak{p}}$ be defined as in Corollary 2.5. Then

$$|\cdot|_{\mathfrak{p}} : K \rightarrow \mathbb{R}, x \mapsto \mathfrak{N}(\mathfrak{p})^{-\nu_{\mathfrak{p}}(x)}$$

is a non-archimedean valuation of K .

Now let K be an algebraic number field.

3. For every embedding $\iota : K \rightarrow \mathbb{C}$ we obtain an archimedean valuation $|\cdot|_{\iota} : K \rightarrow \mathbb{R}$, $x \mapsto \|\iota(x)\|$ where $\|\cdot\|$ is the usual absolute value on $\mathbb{C}$.
4. Let $\iota : K \rightarrow \mathbb{C}$ be an embedding. It is called *real* if $\iota(K) \subseteq \mathbb{R}$, otherwise it is called *complex*. If ι is a complex embedding of K , then complex conjugation yields a corresponding embedding $\bar{\iota} : K \rightarrow \mathbb{C}, x \mapsto \overline{\iota(x)}$, such that $\iota = \bar{\iota}$. So the complex embeddings of K always come in pairs.

Theorem 2.15. Let $|\cdot|$ be any non-trivial valuation of a field K .

Then there exists a field extension $\widehat{K} \supseteq K$, and an extension of $|\cdot|$ to $\widehat{K}$, such that $\widehat{K}$ is complete (with respect to the extension of $|\cdot|$ to $\widehat{K}$), and K is dense in $\widehat{K}$. Moreover, $\widehat{K}$ is unique up to isomorphism. It is called the *completion* of K (with respect to $|\cdot|$).

Remark 2.16. Two valuations $|\cdot|_1$ and $|\cdot|_2$ on a field K induce the same topology if and only if $|\cdot|_2 = |\cdot|_1^e$ for some $e > 0$. If this is the case, then $|\cdot|_1$ and $|\cdot|_2$ are called *equivalent*. The equivalence classes of non-trivial valuations on K are called (prime) *spots*, or *places* (on K).

$\Omega(K)$ denotes the set of all spots on K . When it is clear which field we are referring to, we will often abbreviate with Ω . Note that the equivalence classes contained in a spot are either all archimedean or all non-archimedean. By $\Omega_{\text{fin}}(K)$ we denote the set of non-trivial spots on K that contain non-archimedean valuations, and by $\Omega_{\infty}(K)$ the set of spots on K that contain archimedean valuations.

Since equivalent valuations induce the same topology on K , the completions of K with respect to equivalent valuations are isomorphic. So it makes sense to speak of “the” completion $K_{\mathfrak{p}}$ of K with respect to a spot $\mathfrak{p} \in \Omega$.

In the following, let K be the field of fractions of a Dedekind domain $\mathfrak{o}$.

Definition 2.17. Let $\mathfrak{p} \in \Omega(K)$ be a spot of $\mathfrak{o}$.

By slight abuse of notation, we will also use the symbol $|\cdot|_{\mathfrak{p}}$ to refer to the extension to $K_{\mathfrak{p}}$ of the valuation defined in Example 2.14.

Then $\mathfrak{o}_{\mathfrak{p}} := \{x \in K_{\mathfrak{p}} \mid |x|_{\mathfrak{p}} \leq 1\}$. Note that $\mathfrak{o}_{\mathfrak{p}}$ is the ring of integers of $K_{\mathfrak{p}}$, and it is the completion of $\mathfrak{o}$ with respect to the spot $\mathfrak{p}$. It is called the *valuation ring* of $K_{\mathfrak{p}}$.

Definition 2.18. A *local ring* is a commutative ring with a unique maximal ideal.

A *discrete valuation ring* is a local principal ideal domain which is not a field.

Remark 2.19. The set $\mathfrak{o}_{\mathfrak{p}}$ is indeed a discrete valuation ring. The set

$$\mathfrak{m}_{\mathfrak{p}} := \{x \in K_{\mathfrak{p}} \mid |x|_{\mathfrak{p}} < 1\}$$

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is its unique maximal ideal, and the set

$$\mathfrak{o}_{\mathfrak{p}}^* := \{x \in K_{\mathfrak{p}} \mid |x|_{\mathfrak{p}} = 1\}$$

is called its group of *units*.

The residue class field $\mathfrak{o}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ is isomorphic to the quotient field $\mathfrak{o}/\mathfrak{p}$ (a finite field). So $K_{\mathfrak{p}}$ is a *local field*, i.e., a field which is complete with respect to a non-archimedean valuation $(|\cdot|_{\mathfrak{p}})$, and which has a finite residue class field.

Moreover, note that $\mathfrak{o} = \bigcap_{\mathfrak{p} \in \Omega_{\text{fin}}(K)} (\mathfrak{o}_{\mathfrak{p}} \cap K)$.

Definition 2.20. Assume $2 \notin \mathfrak{o}_{\mathfrak{p}}^*$. Then $K_{\mathfrak{p}}$ is called *dyadic*. Otherwise, $K_{\mathfrak{p}}$ is called *non-dyadic*.

Theorem 2.21 (Weak Approximation Theorem). Let $|\cdot|_1, \dots, |\cdot|_k$ be pairwise inequivalent non-trivial valuations on a field K , where $2 \leq k \in \mathbb{N}$. Then there exists an element $x \in K$ such that $|x|_1 > 0$ and $|x|_i < 0$ for all $2 \leq i \leq k$.

For the rest of this section, let K be an algebraic number field with ring of integers $\mathfrak{o}$.

Example 2.14 in fact lists all the different spots of K . More precisely, the following important result holds:

Theorem 2.22 (Ostrowski's Theorem (for number fields)). Let $\iota_1, \dots, \iota_r$ denote the real embeddings and $\iota_{r+1}, \overline{\iota_{r+1}}, \dots, \overline{\iota_{r+s}}$ the pairs of complex embeddings of K . Then every archimedean valuation of K is equivalent to ι_i for exactly one $1 \leq i \leq r + s$. Every non-trivial non-archimedean valuation of K is equivalent to $|\cdot|_{\mathfrak{p}}$ for exactly one maximal ideal $\mathfrak{p}$ of $\mathfrak{o}$.

Proposition 2.23. A spot of K is non-archimedean if and only if it is represented by $|\cdot|_{\mathfrak{p}}$ for a maximal ideal $\mathfrak{p}$ of $\mathfrak{o}$. An ideal $\mathfrak{p}$ of $\mathfrak{o}$ is maximal if and only if the localization $\mathfrak{o}_{\mathfrak{p}}$ is a discrete valuation ring.

So there is an identification of Ω_{fin} with the set of all maximal ideals of $\mathfrak{o}$, and from now on we will use finite spots and maximal ideals of $\mathfrak{o}$ interchangeably. The non-archimedean spots are also called *discrete*, or *finite* (seeing that the field $\mathfrak{o}/\mathfrak{p}$ is a finite field). Archimedean spots are also called *infinite*. If an embedding $\iota : K \rightarrow \mathbb{C}$ representing an archimedean spot ν is real, or equivalently, if the completion K_{ν} of K at ν is $\mathbb{R}$, then ν is called *real*. Otherwise ν is called *complex*.

Remark 2.24. An algebraic number field is called *totally real* if it has no complex spots, and *totally complex* if it has no real spots. In general the number of non-archimedean spots of K is $r + s$, where r is the number of real embeddings and $2s$ the number of complex embeddings of K .

Definition 2.25. For a totally real field K , we define $K^{\geq 0}$ to be the set of *totally positive* elements of K , that is, the set of those elements that are positive in every real embedding of K .

The quotient $\text{Cl}_K^+ := \mathbb{I}(\mathfrak{o})/\mathbb{P}(\mathfrak{o})^+$ is the *narrow class group* of K , where $\mathbb{P}(\mathfrak{o})^+$ is the set of principal fractional ideals of K that have a totally positive generator. The cardinality of Cl_K^+ will be denoted by $h^+(K)$.

2.4 Quadratic spaces

Let K be a field of characteristic 0. Let V be a finite-dimensional K -vector space and $\Phi : V \times V \rightarrow K$ a symmetric bilinear form.

Definition 2.26. By the *quadratic space* (V, Φ) we mean the K -vector space V equipped with the quadratic form $q_\Phi : V \rightarrow K, v \mapsto \Phi(v, v)$. The map q_Φ fulfills $q_\Phi(av) = a^2 q_\Phi(v)$ for all $v \in V, a \in K$, and fulfills $2\Phi(v, w) = q_\Phi(v + w) - q_\Phi(v) - q_\Phi(w)$ for all $v, w \in V$.

We always assume that V is *regular* (*non-degenerate*), which means that for any $v \in V$, we have $\Phi(v, V) = \{0\}$ if and only if $v = 0$. A vector $0 \neq v \in V$ is called *isotropic* if $\Phi(v, v) = 0$, and *anisotropic* otherwise. A subspace $\{0\} \neq U \leq V$ is called *totally isotropic* if $\Phi(u, u) = 0$ for all $u \in U$, *isotropic* if U contains an isotropic vector, and *anisotropic* if U does not contain an isotropic vector.

Definition 2.27. A *Gram matrix* of (V, Φ) is any matrix $(\Phi(b_i, b_j))_{1 \leq i, j \leq n} \in K^{n \times n}$ obtained by setting $B = (b_1, \dots, b_n)$ to be some basis of V . The *determinant* $\det((V, \Phi))$ is the square class of the determinant of any Gram matrix of V . So $\det((V, \Phi)) \in K^*/(K^*)^2$.

An element $a \in K$ is *represented* by V , if $a \in q_\Phi(V)$.

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Definition 2.28. Suppose (V', Φ') is another quadratic K -vector space with symmetric bilinear form Φ' . An injective homomorphism $\sigma : V \rightarrow V'$ is called an *isometry* of V into V' if $\Phi'(\sigma(x), \sigma(x)) = \Phi(x, x)$ for all $x \in V$. The set of all isometries from V into V' is written as $O(V, V')$. By $O(V)$ we mean the set $O(V, V)$.

Remark 2.29. It is clear that $O(V)$ is a subgroup of the general linear group $\mathrm{GL}(V)$. We call $O(V)$ the *orthogonal group* of (V, Φ) .

Definition 2.30. If $V = V_1 \oplus V_2$ is an orthogonal direct sum of subspaces V_1, V_2 of V , i.e., $V = V_1 + V_2$, $V_1 \cap V_2 = \{0\}$ and $\Phi(V_1, V_2) = \{0\}$, then we say V has an *orthogonal splitting* into V_1 and V_2 and write $V = V_1 \perp V_2$. We say that the subspaces V_1 and V_2 *split* V .

When $a \in K^*$ is represented by V , then by $\langle a \rangle$ we mean any subspace of V generated by some vector $v \in V$ such that $q_\Phi(v) = a$.

Remark 2.31. Any regular subspace $U \leq V$ splits V . In fact, if U is a regular subspace of V , then $V = U \perp U^*$, where U^* is the *orthogonal complement* of U in V :

$$U^* = \{x \in V \mid \Phi(x, U) = \{0\}\}.$$

In particular, if $a \in K^*$ is some element represented by V , then $V = \langle a \rangle \perp V'$ for some subspace $V' \leq V$. By induction, every quadratic space has an orthogonal base.

Proof. See for example [O'M73, 42:4]. ■

From now on, suppose that K is an algebraic number field. We will often write “the quadratic space V ”, “ $\det(V)$ ” and so on whenever the field K and the symmetric bilinear form Φ are clear from the context.

Remark 2.32. If $\mathfrak{p} \in \Omega(K)$ is any spot of K , then Φ extends to the *completion* of V at $\mathfrak{p}$, which is defined as the tensor product $V_{\mathfrak{p}} := V \otimes_K K_{\mathfrak{p}}$. Hence $(V_{\mathfrak{p}}, \Phi)$ is a quadratic space over $K_{\mathfrak{p}}$.

Theorem 2.33. Questions of isotropy, of representation of elements, and of isometry for regular quadratic spaces can be answered globally by examining their completions:

2.4 Quadratic spaces

1. The quadratic space V is isotropic over the algebraic number field K if and only if it is isotropic at every spot in $\Omega(K)$.
2. An element $a \in K^*$ is represented by V if and only if a is represented by every completion of V at a spot in $\Omega(K)$.
3. Two quadratic K -vector spaces (V, Φ) and (V', Φ') are isometric if and only if each of their completions at spots in $\Omega(K)$ are isometric.

Part 1) (and sometimes part 3), which is a corollary of part 1)) is known as the Theorem of HASSE-MINKOWSKI. As always, we assume that the quadratic spaces in question are regular.

Proof. See for example [O'M73, § 66]. ■

Definition 2.34. Let $\nu \in \Omega$ be any (finite or infinite) place of K .

1. Let $a, b \in K_\nu^*$. The *Hilbert symbol* of $a, b \in K_\nu$ is

$$(a, b)_\nu := \begin{cases} +1, & \text{if the equation } \alpha a^2 + \beta b^2 = 1 \text{ has a solution } \alpha, \beta \in K_\nu \\ -1, & \text{otherwise.} \end{cases}$$

2. Let V be some quadratic K -vector space of dimension n . The *Hasse invariant* $c_\nu((V, \Phi))$, or simply $c_\nu(V)$, is the product $\prod_{1 \leq i < j \leq n} (a_i, a_j)_\nu$ of Hilbert symbols, where the a_i are the diagonal elements of the diagonal matrix $\text{diag}(a_1, \dots, a_n)$, the Gram matrix corresponding to any orthogonal basis of (V_ν, Φ) .

For the well-definedness of $c_\nu(V)$, see [O'M73, § 63B].

Remark 2.35. Some elementary properties of the Hilbert symbol will be used later:

1. If ν is a real place of K and $a, b \in K_\nu^*$, then $(a, b)_\nu = -1$ if both a and b are negative, and $(a, b)_\nu = 1$ otherwise.
2. If ν is a complex place of K , then the Hilbert symbol at ν is trivial.
3. The Hilbert symbol $(\cdot, \cdot)_\nu$ is symmetric, and it is multiplicative in both components.
4. The Hilbert symbol is well-defined on square classes: For any $x, y, z \in K_\nu^*$, $(x, yz^2)_\nu = (x, y)_\nu$.

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5. $(a, -a)_\nu = 1$ for all $a \in K_\nu^*$.
6. Whenever $\mathfrak{p} \in \Omega_{\text{fin}}(K)$ such that $K_\mathfrak{p}$ is a non-dyadic field, the Hilbert symbol is very easy to evaluate. Let $a, b \in \mathfrak{o}_\mathfrak{p}^*$ be units and $c \in \mathfrak{o}_\mathfrak{p}$ be a generator of the maximal ideal $\mathfrak{p}$, then $(a, b)_\mathfrak{p} = 1$ always; and $(a, c)_\mathfrak{p} = 1$ if and only if a is a square.

Proof. For part 6), see [O'M73, 63:12]. ■

Note that our definition of $c_\mathfrak{p}(V)$ differs slightly from the one by O'MEARA in [O'M73, § 63B], where the product runs over $1 \leq i \leq j \leq n$. The additional terms $(a_i, a_i)_\mathfrak{p}$ in his definition incur an additional factor that, by Remark 2.35, equates to $(\det(V), \det(V))_\mathfrak{p}$, or alternatively to $(\det(V), \det(V))_\mathfrak{p} = (\det(V), -1)_\mathfrak{p}$ (see part 5)).

We know from Linear Algebra that for $K = \mathbb{C}$, the rank of V and for $K = \mathbb{R}$, the rank and the usual real signature of V form a complete set of invariants for isometry of quadratic K -vector spaces V . This situation becomes slightly more complicated when K is an algebraic number field. Fortunately, by the Hasse-Minkowski Theorem (cf. Theorem 2.33), isometry of quadratic spaces over number fields is a *local property* – meaning that it can be checked by examining the completions at all spots $\mathfrak{p} \in \Omega$.

Determining a complete set of invariants for the isometry class of a completion $V_\mathfrak{p}$ at any discrete prime $\mathfrak{p}$ is easy, as the following theorem shows:

Theorem 2.36. Let $\mathfrak{p} \in \Omega_{\text{fin}}$ denote a discrete spot of K , and $(V_\mathfrak{p}, \Phi)$ a regular quadratic space over $K_\mathfrak{p}$. Then the isometry class of $V_\mathfrak{p}$ is uniquely determined by the following three invariants:

1. the dimension of $V_\mathfrak{p}$,
2. the determinant of $(V_\mathfrak{p}, \Phi) \in K_\mathfrak{p}^*/(K_\mathfrak{p}^*)^2$, and
3. the Hasse invariant $c_\mathfrak{p}((V_\mathfrak{p}, \Phi))$.

Further, $(V_\mathfrak{p}, \Phi)$ is isotropic if and only if either

1. $m = 2$ and $-d \in (K_\mathfrak{p}^*)^2$,
2. $m = 3$ and $c_\mathfrak{p}((V_\mathfrak{p}, \Phi)) = (-1, -d)_\mathfrak{p}$,
3. $m = 4$ and either $d \notin (K_\mathfrak{p}^*)^2$ or $c_\mathfrak{p}((V_\mathfrak{p}, \Phi)) = (-1, -1)_\mathfrak{p}$, or
4. $m \geq 5$.

Proof. Cf. [O'M73, Ch. 63]. ■

From the Hasse-Minkowski Theorem, we obtain a complete set of invariants to identify a quadratic space over a number field:

Corollary 2.37. A complete set of invariants for the isometry class of a regular quadratic space (V, Φ) over an algebraic number field K is given by:

1. the dimension $\dim(V)$,
2. the determinant $\det((V, \Phi)) \in K^*/(K^*)^2$,
3. the Hasse invariants $c_{\mathfrak{p}}((V, \Phi))$ at all discrete spots $\mathfrak{p} \in \Omega_{\text{fin}}$, and
4. the signature of (V_{ν}, Φ) (that is, the dimension of a maximal positive definite subspace of V_{ν}) at all real spots $\nu \in \Omega_{\infty}$.

We note that for algorithmic determination of the Hasse invariants in part 3) above, it suffices to give the finite set of spots $\mathfrak{p} \in \Omega$ where $c_{\mathfrak{p}}((V, \Phi)) = -1$:

Remark 2.38. Let a, b be non-zero elements of K . Then their Hilbert symbol is 1 at all but finitely many spots $\nu \in \Omega(K)$, and HILBERT's Reciprocity Law holds:

$$\prod_{\nu \in \Omega} (a, b)_{\nu} = 1.$$

In particular, the number of spots with Hilbert symbol -1 is even, and we have the product formula

$$\prod_{\nu \in \Omega} c_{\nu}((V, \Phi)) = 1.$$

Proof. Cf. [O'M73, 71:18]. ■

Definition 2.39. A quadratic space (V, Φ) over a totally real number field K is called (totally) *definite* (positive definite, negative definite) if Φ is definite (positive definite, negative definite) at all real spots of K .

If (V, Φ) is not definite, it is called *indefinite*.

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Corollary 2.40. By Remark 2.38, if (V, Φ) is (totally) positive definite, then the number of spots $\mathfrak{p} \in \Omega_{\text{fin}}$ with $c_{\mathfrak{p}}((V, \Phi)) = -1$ is always even.

We end this section with some further comments on quadratic forms on local fields:

Definition 2.41. 1. A quadratic space V is said to be *universal*, if $q_{\Phi}(V) = K$.

2. A two-dimensional quadratic space V is called a *hyperbolic plane*, if in one of its bases it has the Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Remark 2.42. 1. By definition, all hyperbolic planes are isometric, regular, and have determinant $(-1)(K^*)^2$.

2. In fact any two-dimensional quadratic space that is isotropic and regular, or that has determinant $(-1)(K^*)^2$, is a hyperbolic plane. Every hyperbolic plane is universal.
3. Every isotropic regular quadratic space is split by a hyperbolic plane, so it is universal.
4. If $\mathfrak{p} \in \Omega_{\text{fin}}(K)$ is a discrete spot and $\text{rank}(V_{\mathfrak{p}}) \geq 4$, then $V_{\mathfrak{p}}$ is universal.

Proof. Part 2) is shown in e.g. [O'M73, 42:9]. Part 3) then follows from part 2) and Remark 2.31. Part 4) follows from the conditions for isotropy listed in Theorem 2.36 unless $\dim(V_{\mathfrak{p}}) = 4$, $\det(V_{\mathfrak{p}}) \in (K_{\mathfrak{p}}^*)^2$ and $c_{\mathfrak{p}}((V, \Phi)) \neq (-1, -1)_{\mathfrak{p}}$. In the latter case, $V_{\mathfrak{p}}$ is the unique (up to isometry) anisotropic space of rank 4 over $K_{\mathfrak{p}}$. This space is shown to be universal in [O'M73, 63:17]. ■

2.5 Constructing quadratic spaces with given invariants

We give an algorithm to compute a Gram matrix of a quadratic space provided we know the invariants described in Theorem 2.37. First, a remark about constructing elements of $\mathfrak{o}$ with given Hilbert symbols.

2.5 Constructing quadratic spaces with given invariants

Theorem 2.43. 1. Given a finite set $S \subset \Omega(K)$, there are $a, b \in K^*$ such that

$$(a, b)_{\mathfrak{p}} = \begin{cases} 1, & \mathfrak{p} \in S \\ -1, & \mathfrak{p} \in \Omega - S \end{cases}$$

2. Given a finite subset $S \subset \Omega(K)$ of even cardinality and some $b \in K^*$ such that $b \notin (K_{\nu}^*)^2$ for all $\nu \in S$, there is an explicit algorithm to calculate $a \in \mathfrak{o}$ such that

$$S = \{\nu \in \Omega(K) \mid (a, b)_{\nu} = -1\}.$$

Proof. Part 1) is shown in [O'M73, 71:19]. For part 2), an algorithm was given by KIRSCHMER in [Kir16, Algorithm 3.4.1] (inspired by work of CASSELS on the case $K = \mathbb{Q}$). The idea is to use a randomized approach to find a solution a that is correct at all but possibly one place $\mathfrak{q} \in \Omega_{\text{fin}}(K) - S$, then concluding that $(a, b)_{\mathfrak{q}} = 1$ by the product formula (cf. Remark 2.38). Chebotarev's density theorem guarantees termination of the randomized approach. ■

We can now give an algorithm to construct a quadratic space from its invariants. The algorithm given here is taken from the habilitation thesis of KIRSCHMER ([Kir16, Algorithm 3.4.3]), but a more detailed proof follows.

Algorithm 2.44. Constructing a regular quadratic space with given invariants

Input: A ring of integers $\mathfrak{o}$ of an algebraic number field K with r real embeddings $\iota_1, \dots, \iota_r$, an integer n , an element $0 \neq d \in \mathfrak{o}$, a finite set of places $P \subset \Omega(K)$, and a list $S = (s_i)_{1 \leq i \leq r}$ with $s_i \in \{0, \dots, n\}$ for all i .

Output: A Gram matrix of a quadratic space (V, Φ) of rank n and determinant d (modulo $(K^*)^2$), with $c_{\mathfrak{p}}((V, \Phi)) = -1$ if and only if $\mathfrak{p} \in P$, and with s_i negative elements on the diagonal of any Gram matrix of the completion V_{ι_i} (for $1 \leq i \leq r$).

```

1: function CONSTRUCTSPACE( $\mathfrak{o}, K, n, d, P, S$ )
2:   raise an error if  $\text{sign}(\iota_i(d)) \neq (-1)^{n_i}$  for some  $i$ 
3:   raise an error if  $m = 1$  and  $P = \emptyset$ 
4:   raise an error if  $m = 2$  and  $-d \in (K_{\mathfrak{p}}^*)^2$  for some  $\mathfrak{p}$ 
5:   raise an error if  $\#\{1 \leq i \leq r \mid n_i \equiv 2, 3 \pmod{4}\} + \#P$  is odd.
6:    $D \leftarrow []$ 
7:   while  $n \geq 2$  do
8:     if  $n \geq 4$  then
```

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9:          $a \leftarrow$  some element of  $\mathfrak{o}$  chosen by weak approximation such that  $\iota_i(a) < 0$ 
    if and only if  $s_i < 0$ 
10:     else if  $n = 3$  then
11:          $P' \leftarrow \{\mathfrak{p} \in P \mid (-1, -d)_{\mathfrak{p}} = 1\} \cup \{\mathfrak{p} \in \Omega - P \mid \mathfrak{p} \mid 2d \text{ and } (-1, -d)_{\mathfrak{p}} = -1\}$ 
12:          $a \leftarrow$  some element of  $\mathfrak{o}$  chosen by weak approximation such that  $\iota_i(a) < 0$ 
    if and only if  $s_i < 0$ , and such that  $\nu_{\mathfrak{p}}(ad) \equiv 1 \pmod{2}$  for all  $\mathfrak{p} \in P'$ .
13:     else if  $n = 2$  then
14:          $S \leftarrow P \cup \{\sigma_i \mid 1 \leq i \leq r \text{ and } n_i = 2\}$ 
15:          $a \leftarrow$  some element of  $\mathfrak{o}$  such that  $(a, -d)_{\nu} = -1$  if and only if  $\nu \in S$  (cf.
    Theorem 2.43)
16:     end if
17:      $n \leftarrow n - 1$ 
18:      $P \leftarrow \{\mathfrak{p} \in P \mid (a, -d)_{\mathfrak{p}} = 1\} \cup \{\mathfrak{p} \in \Omega_{\text{fin}}(K) \mid \mathfrak{p} \mid 2ad \text{ and } (a, -d)_{\mathfrak{p}} = -1\}$ 
19:     for  $1 \leq i \leq r$  set  $n_i = \max\{n_i - 1, 0\}$  end for
20:      $D \leftarrow D, a$ 
21:      $d \leftarrow ad$ 
22: end while
23:      $D \leftarrow D, d$ 
24:     return the diagonal matrix  $\text{diag}(D)$ 
25: end function

```

Proof. The conditions checked in lines 2 to 5 have been shown by O'MEARA to be necessary and sufficient for the existence of a quadratic space with these invariants, cf. [O'M73, 63:21, 72:1].

For $m \geq 2$, the algorithm chooses some element a that is represented by (V, Φ) , so a splits (V, Φ) by Remark 2.31. Starting in line 17, an easy calculation then yields the invariants of the form (V', Φ') and one can proceed by induction.

A note on the new value of P in line 18: we have $c_{\mathfrak{p}}((V', \Phi')) = 1$ for all primes not considered in line 18, cf. part 6) of Remark 2.35. Also, for any two quadratic spaces W and W' , $c_{\mathfrak{p}}(W_{\mathfrak{p}} \perp W'_{\mathfrak{p}}) = c_{\mathfrak{p}}(W) \cdot c_{\mathfrak{p}}(W') \cdot (\det W, \det W')_{\mathfrak{p}}$ (see for example the last part of [O'M73, 63:13a]).

It is clear that, at the infinite spots of K , the algorithm produces a quadratic form with the desired signatures.

It remains to show that the elements a obtained in each step are actually represented by (V, Φ) . By the Local-Global Principle (Theorem 2.33), this is equivalent to proving that

$(V, \Phi) \perp \langle -a \rangle$ is isotropic for all primes $\mathfrak{p}$. Recall the necessary and sufficient conditions given in Theorem 2.36 for local isotropy at a prime $\mathfrak{p}$.

1. If $n \geq 4$, then (V, Φ) is universal at $\mathfrak{p}$ by Remark 2.42, so the only thing to ensure is indeed to match the desired signatures at the real spots (which is done by weak approximation in lines 9 and 12).
2. If $n = 3$, then $(V, \Phi) \perp \langle -a \rangle$ is locally isotropic at $\mathfrak{p}$ if either of the following two conditions hold: $c_{\mathfrak{p}}((V, \Phi) \perp \langle -a \rangle) = (-1, -1)_{\mathfrak{p}}$ ($\Leftrightarrow 1 = c_{\mathfrak{p}}((V, \Phi)) \cdot (d, a)_{\mathfrak{p}} \cdot (-d, -1)_{\mathfrak{p}}$), or $-ad \notin (K^*)_{\mathfrak{p}}^2$.

If $\mathfrak{p}$ lies in the finite set P' , then by construction $(V, \Phi) \perp \langle -a \rangle$ is locally isotropic at $\mathfrak{p}$ (the condition $-ad \notin (K^*)_{\mathfrak{p}}^2$ is fulfilled since ad has odd $\mathfrak{p}$ -valuation). Now assume that $\mathfrak{p} \notin P'$ and $-ad \in (K^*)_{\mathfrak{p}}^2$. This implies that $a = -dr^2$ for some $r \in K_{\mathfrak{p}}$, so $(d, a)_{\mathfrak{p}} = (d, -dr^2)_{\mathfrak{p}} = (d, -d)_{\mathfrak{p}} = 1$.

Assume that $c_{\mathfrak{p}}(V, \Phi) \cdot (-d, -1)_{\mathfrak{p}} = -1$. Because $\mathfrak{p} \notin P'$, we then must have $(-1, -d)_{\mathfrak{p}} = -1$. Because $\mathfrak{p} \notin P'$, it follows that $\mathfrak{p} \nmid 2d$. But then $(-1, -d)_{\mathfrak{p}} = 1$ by part 6) of Remark 2.35, a contradiction.

So $c_{\mathfrak{p}}(V, \Phi) \cdot (-d, -1)_{\mathfrak{p}} = 1$. But then we are done because $(d, a)_{\mathfrak{p}} = 1$ as noted above.

3. If $n = 2$, then (V, Φ) is locally isotropic at $\mathfrak{p}$ if and only if $c_{\mathfrak{p}}((V, \Phi) \perp \langle -a \rangle) = (-1, da)_{\mathfrak{p}}$.

This follows from the choice of the element a in line 15, which ensures that $c_{\mathfrak{p}}((V, \Phi)) = (a, -d)_{\mathfrak{p}}$ for all $\mathfrak{p} \in \Omega_{\text{fin}}(K)$, so $c_{\mathfrak{p}}((V, \Phi) \perp \langle -a \rangle) = (a, -d)_{\mathfrak{p}}(d, -a)_{\mathfrak{p}} = (a, d)_{\mathfrak{p}}(a, -1)_{\mathfrak{p}}(d, a)_{\mathfrak{p}}(d, -1)_{\mathfrak{p}} = (-1, da)_{\mathfrak{p}}$.

4. For $n = 1$, there is nothing to show.

■

2.6 Lattices in quadratic spaces

Let $\mathfrak{o}$ be a Dedekind ring with field of fractions K . Let $V = (V, \Phi)$ be a quadratic K -vector space of finite dimension.

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Definition 2.45. An $\mathfrak{o}$ -lattice is a finitely generated, torsion-free $\mathfrak{o}$ -module.

Equivalently, $\mathfrak{o}$ -lattices are the finitely generated $\mathfrak{o}$ -submodules of finite-dimensional vector spaces over K .

If not explicitly stated otherwise, we will always assume that if L is a lattice in a K -vector space V then L is *full*, i.e., L contains a basis of V – or equivalently,

$$KL := \{\alpha x \mid \alpha \in K, x \in L\} = V.$$

Being a Dedekind domain, $\mathfrak{o}$ is noetherian. So if L' is an $\mathfrak{o}$ -submodule of any $\mathfrak{o}$ -lattice L , then L' is finitely generated, hence L' is also a lattice.

Definition 2.46. If $\mathfrak{p}$ is a place of K and L is an $\mathfrak{o}$ -lattice, then the tensor product $L_{\mathfrak{p}} := L \otimes \mathfrak{o}_{\mathfrak{p}}$ is called the *completion* of L at $\mathfrak{p}$.

Lemma 2.47. Let L and L' be $\mathfrak{o}$ -lattices in V . Then there exist non-zero $a, b \in \mathfrak{o}$ such that $aL' \subseteq L$ and $bL \subseteq L'$. Further, the number of spots $\mathfrak{p}$ where $L_{\mathfrak{p}} \neq L'_{\mathfrak{p}}$ is finite.

Proof. Let $B = (b_1, \dots, b_n)$ denote a basis of V that is contained in L . Choose generators $(g_1, \dots, g_k) \in V^k$ for L' . Taking the fractional ideal $\mathfrak{a}$ generated by all the coefficients of the g_i with respect to the basis B , we see that there exists some $a \in \mathfrak{o}$ such that $a\mathfrak{a} \subset \mathfrak{o}$, so $aL' \subseteq L$. Likewise, $bL \subseteq L'$ for some $b \in \mathfrak{o}$.

Hence $L_{\mathfrak{p}} = L'_{\mathfrak{p}}$ whenever both $|a|_{\mathfrak{p}} = 0$ and $|b|_{\mathfrak{p}} = 0$. ■

Example 2.48. If $K = \mathbb{Q}$, then $\mathfrak{o} = \mathbb{Z}$. $\mathbb{Z}$ is a principal ideal domain, so every torsion free $\mathbb{Z}$ -module is free and thus every $\mathbb{Z}$ -lattice has a *basis*, that is, there exist $b_1, \dots, b_m \in V$ such that $L = \oplus_{i=1}^m \mathbb{Z}b_i$. The same holds for any number field K with class-number one (e.g. $K = \mathbb{Q}(\sqrt{5})$).

If K does not have class-number 1, then all we can achieve for general $\mathfrak{o}$ -lattices over K is a *pseudo-basis*:

Theorem 2.49 (Steinitz). Let L denote a (full) $\mathfrak{o}$ -lattice in the K -vector space V . Then there exist a basis $(x_1, \dots, x_m)$ of V and fractional ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ of $\mathfrak{o}$ such that

$$L = \sum_{i=1}^m \mathfrak{a}_i x_i.$$

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The tuple $(x_1, \mathbf{a}_1, \dots, x_m, \mathbf{a}_m)$ is called a *pseudo-basis* of L . Note that the vectors x_i might not belong to L .

Theorem 2.50 (Invariant Factor Theorem). Let $M, N \subseteq V$. Assume M is an $\mathfrak{o}$ -lattice in V and N is an $\mathfrak{o}$ -lattice in $KM = \{km \mid k \in K, m \in M\}$. Then there exist a pseudo basis $B = (x_1, \mathbf{a}_1, \dots, x_n, \mathbf{a}_n)$ of M and fractional ideals $\mathfrak{b}_1 \supseteq \mathfrak{b}_2 \supseteq \dots \supseteq \mathfrak{b}_m$ with $m \leq n$ such that $N = \oplus_{i=1}^m \mathbf{a}_i \mathfrak{b}_i x_i$.

Further, the $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ are independent of the choice of B .

If M and N have the same rank (i.e., if $m = n$), then the *module index* of N in M is defined as

$$\prod_{i=1}^n \mathfrak{b}_i$$

and we will abbreviate it as $[M : N]_{\mathfrak{o}}$.

For a proof, see [O'M73, 81:11].

Remark 2.51. In the situation of Theorem 2.50, and writing $\langle \cdot \rangle_{\mathfrak{o}}$ for the fractional $\mathfrak{o}$ -ideal generated by some set of elements,

$$[M : N]_{\mathfrak{o}} = \prod_{i=1}^m \mathfrak{b}_i = \langle \{\det \varphi : \varphi \in \text{Hom}_{\mathfrak{o}}(M, N)\} \rangle_{\mathfrak{o}}.$$

The full lattices in V satisfy the following local-global principle.

Theorem 2.52. Let M denote a fixed $\mathfrak{o}$ -lattice in V , and let

$$S := \{(L_{(\mathfrak{p})})_{\mathfrak{p} \in \Omega_{\text{fin}}(K)} \mid L_{(\mathfrak{p})} \text{ is a full } \mathfrak{o}\text{-lattice in } V_{\mathfrak{p}}, \text{ and } L_{(\mathfrak{p})} = M_{\mathfrak{p}} \text{ for almost all } \mathfrak{p}\}.$$

There is a one-to-one correspondence

$$\begin{aligned} \{\text{full } \mathfrak{o}\text{-lattices in } V\} &\leftrightarrow S \\ L &\mapsto (L_{\mathfrak{p}})_{\mathfrak{p} \in \Omega_{\text{fin}}(K)} \\ \bigcap_{\mathfrak{p}} (L_{\mathfrak{p}} \cap V) &\leftarrow (L_{(\mathfrak{p})})_{\mathfrak{p} \in \Omega_{\text{fin}}(K)} \end{aligned}$$

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Proof. Cf. [O'M73, 81:14]. ■

Later in this chapter, we will give algorithms to construct lattices that will make extensive use of Theorem 2.52. The situation will usually be the following: suppose an $\mathfrak{o}$ -lattice L is given. Construct a sublattice M of L that is, locally at some prime $\mathfrak{p}$, isometric to a given $\mathfrak{o}_{\mathfrak{p}}$ -lattice L' .

Definition 2.53. Let (V, Φ) be a quadratic K -vector space. Let L be an $\mathfrak{o}$ -lattice in V of rank m , and let $\mathfrak{a}$ be a fractional ideal of $\mathfrak{o}$.

1. Suppose L is free. The *determinant* $\det(L)$ of L is the determinant of any Gram matrix of any $\mathfrak{o}$ -basis of L . It is well-defined up to squares in $\mathfrak{o}$. The *discriminant* $\text{disc}(L)$ of L is defined as $(-1)^{\frac{m(m-1)}{2}} \det(L)$.

If M is any Gram matrix of a lattice in V , then by $L \cong \langle M \rangle$ we mean that L is a free lattice in V with Gram matrix M (after a possible change of base).

2. The *scale* of L is the fractional $\mathfrak{o}$ -ideal $\mathfrak{s}L := \{\Phi(x, y) \mid x, y \in L\}$.
3. The *norm* of L is the fractional $\mathfrak{o}$ -ideal $\mathfrak{n}L := \langle \{\Phi(x, x) \mid x \in L\} \rangle_{\mathfrak{o}}$ (by which we mean the fractional $\mathfrak{o}$ -ideal generated by the $\Phi(x, x)$).
4. The *volume ideal* (*discriminant ideal*) of L is the module index $[L^{\#} : L]_{\mathfrak{o}}$ and is denoted by $\mathfrak{v}L$.
5. The *dual lattice* of L is $L^{\#} := \{x \in V \mid \Phi(x, L) \subseteq \mathfrak{o}\}$.
6. L is $\mathfrak{a}$ -*modular* if $\mathfrak{a}L^{\#} = L$. The $\mathfrak{o}$ -modular lattices are also called *unimodular*. A lattice is called *modular* if it is $\mathfrak{a}$ -modular for some fractional ideal $\mathfrak{a}$.
7. The lattice $L^{\mathfrak{a}}$ is defined as the set $\{x \in L \mid \Phi(x, L) \subseteq \mathfrak{a}\}$.
8. L is called $\mathfrak{a}$ -*maximal*, if $q_{\Phi}(L) \subseteq \mathfrak{a}$ and no proper superlattice of L has this property. The $\mathfrak{o}$ -maximal lattices are simply called *maximal lattices*.
9. L is called *integral*, if $\mathfrak{s}(L) \subseteq \mathfrak{o}$ (i.e., if $L \subseteq L^{\#}$). (Defining integrality in this way, and not via the quadratic form q_{Φ} , means that a maximal lattice is not necessarily integral.)

All of these definitions commute with completions. E.g., $\mathfrak{s}(L)_{\mathfrak{p}} = \mathfrak{s}(L_{\mathfrak{p}})$.

2.6 Lattices in quadratic spaces

Definition 2.54. Assume $\mathfrak{o}$ is the ring of integers of a number field K . Let L be any $\mathfrak{o}$ -lattice in V . Let P be a finite subset of $\Omega_{\text{fin}}(K)$. Then the P -dual (or: *partial dual*) $L^{\#,P}$ of L is the unique $\mathfrak{o}$ -lattice fulfilling the following conditions at all $\mathfrak{p} \in \Omega_{\text{fin}}(K)$:

$$L_{\mathfrak{p}}^{\#,P} := \begin{cases} L_{\mathfrak{p}}, & \text{if } \mathfrak{p} \notin P \\ L_{\mathfrak{p}}^{\#}, & \text{if } \mathfrak{p} \in P \end{cases}$$

If $P = \{\mathfrak{p}\}$ for a single prime $\mathfrak{p}$, we simply write $L^{\#,\mathfrak{p}}$.

The existence and uniqueness of P -duals was shown in Theorem 2.52.

Note that since $(L^{\#,P})^{\#,P} = L$, we must have $h(L^{\#,P}) = h(L)$.

Remark 2.55. If L is given by a pseudo-basis $B = (x_1, \mathfrak{a}_1, \dots, x_n, \mathfrak{a}_n)$, then the above invariants can be computed as follows (cf. [O'M73, § 94]):

1. The scale of L is $\mathfrak{s}(L) = \sum_{1 \leq i, j \leq n} \mathfrak{a}_i \mathfrak{a}_j \Phi(x_i, x_j)$.
Note that if L is $\mathfrak{a}$ -modular, then $\mathfrak{a} = \mathfrak{s}(L)$.
2. The norm of L is $\mathfrak{n}(L) = \sum_{i=1}^n \mathfrak{a}_i^2 \Phi(\mathfrak{a}_i, \mathfrak{a}_i) + 2\mathfrak{s}(L)$.
3. Let $x_1^*, \dots, x_n^*$ be the dual basis of the basis $x_1, \dots, x_n$ of V . Then $L^{\#} = \bigoplus_{i=1}^n \mathfrak{a}_i^{-1} x_i^*$.
4. The volume of L is $\mathfrak{v}(L) = \sum_{1 \leq i, j \leq n} \mathfrak{a}_i \mathfrak{a}_j d$, where d is the determinant of a Gram matrix of $x_1, \dots, x_n$.

Note that $\mathfrak{v}(L)_{\mathfrak{p}} = \det(L_{\mathfrak{p}}) \mathfrak{o}_{\mathfrak{p}}$.

5. To calculate the P -dual of L , choose ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}_{\mathfrak{p}} L \subseteq L_{\mathfrak{p}}^{\#} \subseteq \mathfrak{b}_{\mathfrak{p}} L_{\mathfrak{p}}$ at all $\mathfrak{p} \in P$, and so that the support of $\mathfrak{a}$ and $\mathfrak{b}$ is P .

Then $L^{\#,P} = (L^{\#} + \mathfrak{a}L) \cap \mathfrak{b}L$.

Definition 2.56. Analogously to Definition 2.30, for a lattice L in V , we say $L = L_1 \perp L_2$ when $L = L_1 \oplus L_2$ with $\Phi(L_1, L_2) = \{0\}$, and we write $L \cong X_1 \perp X_2$ to say that L has a splitting $L = L_1 \perp L_2$ with $L_1 \cong X_1$ and $L_2 \cong X_2$.

2.7 The class, the spinor genus, and the genus

Let $\mathfrak{o}$ be a Dedekind ring with field of fractions K . Let (V, Φ) be a quadratic K -vector space, and let L be any $\mathfrak{o}$ -lattice in V . As usual, we assume that V is regular.

Definition 2.57. An element σ of $O(V)$ is called a *rotation* if $\det(\sigma) = 1$. It is called a *reflection* if $\det(\sigma) = -1$. Since V is regular, $O(V)$ is a disjoint union of rotations and reflections. The group $O^+(V) = \{\sigma \in O(V) \mid \det(\sigma) = 1\}$ is called the *group of rotations* of V .

Definition 2.58. Two $\mathfrak{o}$ -lattices L and L' in V are called *isometric*, if there exists some $\sigma \in O(V)$ such that $L' = \sigma(L)$. They are called *properly isometric*, if this holds for some $\sigma \in O^+(V)$. The set of all lattices in V which are isometric to L is denoted by $\text{cls}(L)$ and is called the *class* of L . For the set of all lattices properly isometric to L , the *proper class* of L , we write $\text{cls}^+(L)$.

Definition 2.59. Let L be some $\mathfrak{o}$ -lattice in (V, Φ) .

1. Let $\alpha \in K^*$. By the *rescaling* L^α we mean the module L regarded in the quadratic space $(V, \alpha\Phi)$. So “rescaling” will always mean rescaling the underlying quadratic form by a scalar, and leaving the module L untouched.
2. Two lattices L, L' in quadratic K -vector spaces (V, Φ) and (V', Φ') are said to be *similar*, if there exists some scalar $\alpha \in K^*$ such that L^α and L' are isometric. Equivalence classes under the equivalence relation on lattices in K -vector spaces defined by similarity are called *similarity classes*.

The rescaling L^α should not be confused with the lattice $L^{\mathfrak{a}}$, where $\mathfrak{a}$ is a fractional $\mathfrak{o}$ -ideal (cf. Definition 2.53).

Definition 2.60. The group

$$O(L) := \text{Aut}(L) := \{\sigma \in O(V) \mid \sigma(L) = L\}$$

is called the *group of units* of L , and the group

$$O^+(L) := \{\sigma \in O^+(V) \mid \sigma(L) = L\}$$

is called the *group of rotations* of L .

2.7 The class, the spinor genus, and the genus

As $O^+(L)$ is clearly the kernel of the restriction of $\det$ to $O(L)$, $[O(L) : O^+(L)] \in \{1, 2\}$. We have $\text{cls}(L) = \text{cls}^+(L)$ if and only if $[O(L) : O^+(L)] = 2$. This holds, for example, whenever $\dim(V)$ is odd, because then $-\text{id}_V \in O(L)$.

Definition 2.61. Let $y \in V$ such that $q_\Phi(y) \neq 0$. Then the *symmetry with respect to y* is an element τ_y of $O(V)$ given by the formula

$$\tau_y(x) = x - \frac{2\Phi(x, y)}{q_\Phi(y)}y.$$

It is clear that τ_y reverses all the vectors in the line Ky , while leaving the hyperplane $(Ky)^* := \{v \in V \mid \Phi(v, Ky) = 0\}$ fixed. Any symmetry τ_y fulfills $\det(\tau_y) = -1$, i.e., symmetries are reflections.

Because every symmetry has determinant -1 , it is now clear that $[O(V) : O^+(V)] = 2$. For a lattice L over a number field, we only know that $[O(L) : O^+(L)] \in \{1, 2\}$ (with $O^+(L) \neq O(L)$ whenever $O(L)$ contains a reflection). For lattices over local fields, however, the group of units always contains at least one symmetry:

Lemma 2.62. Let $\mathfrak{p}$ be any place of K and let v be any vector of $L_\mathfrak{p}$ with the property that $\Phi(v, v)$ generates $\mathfrak{n}(L_\mathfrak{p})$. Then the symmetry τ_v fixes $L_\mathfrak{p}$.

Proof. Clearly $\tau_v \in O(V)$. The quotient

$$\frac{2\Phi(v, w)}{\Phi(v, v)}$$

is always integral, because $\nu_\mathfrak{p}(2\Phi(v, w)) \geq \nu_\mathfrak{p}(2\mathfrak{s}(L_\mathfrak{p})) \geq \nu_\mathfrak{p}(\mathfrak{n}(L_\mathfrak{p})) = \nu_\mathfrak{p}(\Phi(v, v))$. So $\tau_v(L) \subseteq L$. ■

Corollary 2.63. The class and the proper class coincide over local fields: $[O(L_\mathfrak{p}) : O^+(L_\mathfrak{p})] = 2$ and $\text{cls}^+(L_\mathfrak{p}) = \text{cls}(L_\mathfrak{p})$.

Proposition 2.64. Any $\sigma \in O(V)$ can be written as a product of symmetries, $\sigma = \tau_{u_1} \cdots \tau_{u_r}$. While there may never be only one way to write σ as such a product, there is a well-defined homomorphism θ called the *spinor norm*:

$$\theta : O(V) \rightarrow K^*/(K^*)^2, \sigma = \tau_{u_1} \cdots \tau_{u_r} \mapsto q_\Phi(u_1) \cdots q_\Phi(u_r)(K^*)^2$$

The kernel of the restriction $\theta : O^+(V) \rightarrow K^*/(K^*)^2$ will be denoted by $\Theta(V)$.

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Proof. For the well-definedness of θ , see [O'M73, § 55]. ■

From now on, let K be an algebraic number field, and let $\mathfrak{o}$ be its ring of integers.

Definition 2.65. Two $\mathfrak{o}$ -lattices L and L' in V lie in the same *genus*, if for each $\mathfrak{p} \in \Omega_{\text{fin}}$ there is some $\Sigma_{\mathfrak{p}} \in O(V_{\mathfrak{p}})$, such that $L'_{\mathfrak{p}} = \Sigma_{\mathfrak{p}} L_{\mathfrak{p}}$. The set of lattices in V with this property will be denoted by $\text{Genus}(L)$.

Note that there is no need to define a “proper genus” of L (by replacing $O(V_{\mathfrak{p}})$ by $O^+(V_{\mathfrak{p}})$ in the definition of $\text{Genus}(L)$), for we know from Corollary 2.63 that it is identical to the ordinary genus.

Theorem 2.66. $\text{Genus}(L)$ is the union of a finite number $h(L)$ of isometry classes.

Proof. Cf. [O'M73, 103:4]. The proof actually shows that the number of integral lattices with given volume in a given regular quadratic space is finite. ■

Definition 2.67. The number $h(L)$ is called the *class number* of L . Obviously, $h(L)$ depends only on the similarity class of L .

Remark 2.68. The lattices with class-number 1 are precisely the lattices which satisfy a local-global principle. If the space (V, Φ) is indefinite, then determining the class-number of L is mostly a local problem. In most of the indefinite cases, $h(L) = 1$.

In the definite case there are (after similarity) only finitely many lattices with class-number 1. The purpose of this thesis is to enumerate all the definite lattices of rank at least 3.

We will now introduce the concept of spinor equivalence, which gives a coarser partitioning of $\text{Genus}(L)$ than isometry under some conditions. We will see in Chapter 4 that due to the strong approximation theorem for rotations, the concepts of spinor equivalence and of neighbouring lattices are closely related, which ultimately allows for an algorithmic enumeration of $\text{Genus}(L)$.

2.8 Isometry classes of lattices over local fields

Definition 2.69. L and L' are called *spinor equivalent*, if there is some $\sigma \in O(V)$, and some $\Sigma_{\mathfrak{p}} \in \Theta(V_{\mathfrak{p}})$ for each prime $\mathfrak{p} \in \Omega_{\text{fin}}$, such that $L'_{\mathfrak{p}} = \sigma_{\mathfrak{p}} \Sigma_{\mathfrak{p}} L_{\mathfrak{p}}$. If $\sigma \in O^+(V)$, the lattices are called *properly spinor equivalent*.

The notions $\text{cls}(L)$ (the *class* of L), $\text{cls}^+(L)$ (the *proper class* of L), $\text{spn}(L)$ (the *spinor genus* of L), and $\text{spn}^+(L)$ (the *proper spinor genus* of L) will be used to denote the equivalence class of L under isometry, proper isometry, spinor equivalence and proper spinor equivalence, respectively.

Remark 2.70. Note that any two lattices L, L' in V are equal at almost all spots (see Lemma 2.47), so $\Sigma_{\mathfrak{p}} = \text{id}_{V_{\mathfrak{p}}}$ can be assumed to hold for almost all $\mathfrak{p}$ in the definition of the spinor genus.

We clearly have $\text{cls}(L) \subseteq \text{spn}(L) \subseteq \text{Genus}(L)$, and also $\text{cls}^+(L) \subseteq \text{spn}^+(L) \subseteq \text{Genus}(L)$. Recall that the number of isometry classes in a genus and thus also the number of spinor genera in a genus is finite.

2.8 Isometry classes of lattices over local fields

This section is devoted to introducing an important result of O'MEARA from [O'M73], namely, the classification of lattices over local fields, up to isometry, in terms of invariants. We are first going to introduce the quadratic defect, which is necessary to understand some of these invariants. After that, we will give a quick overview of Jordan decompositions, and then state O'MEARA's classification theorem.

Let F be a local field with discrete valuation ν , valuation ring $\mathfrak{o}$ and let $\pi \in \mathfrak{o}$ be a generator of its maximal ideal $\mathfrak{p}$.

2.8.1 The quadratic defect

Theorem 2.71 (Local square theorem). Let $\alpha \in \mathfrak{o}^*$. Then there is some $\beta \in \mathfrak{o}^*$ such that

$$1 + 4\pi\alpha = (1 + 2\pi\beta)^2.$$

The proof is given in [O'M73, 63:1].

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Definition 2.72 (The quadratic defect). For any $a \in \mathfrak{o}$, the *quadratic defect* of a is

$$\mathfrak{d}(a) = \bigcap \{ \beta \mathfrak{o} \mid \beta \in F \text{ such that } a = \beta + \eta^2 \text{ for some } \eta \in F \}.$$

If $a \in \mathfrak{o}$, then the quadratic $\mathfrak{d}(a)$ is the largest power of $\pi \mathfrak{o}$ such that a is a square mod $\mathfrak{d}(a)$.

Proposition 2.73. Let $e := \nu(2)$. We summarise the basic properties of the quadratic defect.

1. $\mathfrak{d}(a) = \{0\}$ if and only if $a \in (F^*)^2$.
2. $\mathfrak{d}(a^2u) = a^2\mathfrak{d}(u)$ for all $a, u \in F$.
3. If $\nu(a)$ is odd, then $a\mathfrak{o} = \mathfrak{d}(a)$.
4. Let $u = 1 + \alpha \in F^*$, F dyadic and $0 < d := \nu(\mathfrak{d}(\alpha)) < 2e$ where d is odd. Then $\mathfrak{d}(u) = \alpha\mathfrak{o}$.
5. Let $u \in \mathfrak{o}^*$. If F is non-dyadic, then either $\mathfrak{d}(u) = 0$ or $\mathfrak{d}(u) = \mathfrak{o}$. If F is dyadic, then $\mathfrak{d}(u)$ is one of the ideals

$$0 \subset 4\mathfrak{o} \subset 4\mathfrak{p}^{-1} \subset 4\mathfrak{p}^{-1} \subset \cdots \subset \mathfrak{p}^3 \subset \mathfrak{p}.$$

Conversely, each of the above ideals occurs as the quadratic defect of a unit $u \in \mathfrak{o}^*$.

Proof. 1. If $a \in (F^*)^2$, then clearly $\mathfrak{d}(a) = (0)$. The other inclusion is an immediate consequence of the Local Square Theorem.

2. This is clear.
3. By part 2) we may assume that $\nu(a) = 1$. Then the result follows from the comment after Definition 2.72.
4. Cf. [O'M73, 63:5].
5. This is shown in [O'M73, 63:2]. Note that the non-dyadic case, and the dyadic case where $\mathfrak{d}(u) \subsetneq 4\mathfrak{o}$, is just the Local Square Theorem.

For the last part, see [O'M73, 63:6].

■

2.8 Isometry classes of lattices over local fields

Proposition 2.74. Any fractional ideal $\mathfrak{a}$ defines an equivalence relation on K^* : $\alpha \sim \beta \pmod{\mathfrak{a}}$ if and only if $\alpha/\beta \in \mathfrak{o}^*$, and $\mathfrak{d}(\alpha/\beta) \subseteq \mathfrak{a}/\beta$.

Corollary 2.75. Let F be a local field. Then there exists an element $\Delta \in \mathfrak{o}^*$ with $\mathfrak{d}(\Delta) = 4\mathfrak{o}$, and Δ is unique up to multiplication with elements of $(\mathfrak{o}^*)^2$.

Proof. The existence is shown in Proposition 2.73.

By the previous remark, we may assume that $\Delta = 1 + 4\delta$ with some $\delta \in \mathfrak{o}^*$. Let $\alpha = \frac{\sqrt{\Delta}-1}{2} \in F[\sqrt{\Delta}]$. Then α is a root of the polynomial $X^2 + X - \delta$. In particular, α is integral over $\mathfrak{o}$. One checks that the Gram matrix of $(1, \alpha)$ with respect to the trace bilinear form is a unit. Hence $F[\sqrt{\Delta}]/F$ is quadratic unramified. So $F[\sqrt{\Delta}]$ is unique up to isomorphism. It is well known that there is only one such extension of F , up to isomorphism. ■

In the following, $\rho \in \mathfrak{o}^*$ will denote some fixed element such that $\Delta = 1 - 4\rho \in \mathfrak{o}^*$ has quadratic defect $4\mathfrak{o}$.

Following the ideas presented in Section 63 of [O'M73], the habilitation thesis of KIRSCHMER gives the following algorithm to compute the quadratic defect:

Algorithm 2.76. Calculating the quadratic defect

Input: Some element $a \in K$.

Output: The quadratic defect $\mathfrak{d}(a)$.

```

1: function QUADRATICDEFECT( $a$ )
2:   if  $\nu(a)$  is odd or  $\infty$  then
3:     return  $a\mathfrak{o}$ 
4:   end if
5:   if  $\nu(a) \neq 0$  then
6:     return  $a \cdot \text{QUADRATICDEFECT}(a/\pi^{\nu(a)})$ 
7:   end if
8:   if  $\nu(2) = 0$  then
9:     if  $a$  is a square mod  $\mathfrak{p}$  then
10:      return  $\{0\}$ 
11:     else
12:       return  $\mathfrak{o}$ 
13:     end if

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```

14:   end if
15:   Compute  $s \in \mathfrak{o}^*$  such that  $s^2 a \equiv 1 \pmod{\mathfrak{p}}$ .
16:    $a \leftarrow as^2$ 
17:    $v \leftarrow \nu(a - 1) \geq 1$ 
18:   while  $v < \nu(4)$  and  $v$  is even do
19:     Compute  $s \in \mathfrak{o}^*$  such that  $s^2 \equiv (a - 1)/\pi^v \pmod{\mathfrak{p}}$ 
20:      $a \leftarrow a/(1 + s\pi^{v/2})^2$ 
21:      $v \leftarrow \nu(a - 1)$ 
22:   end while
23:   if  $v < \nu(4)$  is odd then
24:     return  $\pi^v$ 
25:   else if  $v = \nu(4)$  and  $X^2 + X + (a - 1)/4 \in (\mathfrak{o}/\mathfrak{p})[X]$  is irreducible then
26:     return  $4\mathfrak{o}$ 
27:   else
28:     return  $\{0\}$ 
29:   end if
30: end function

```

Proof. Cf. [Kir16, Algorithm 3.1.3] ■

Corollary 2.77. Every unit square class $u \in \mathfrak{o}_{\mathfrak{p}}^*/(\mathfrak{o}_{\mathfrak{p}}^*)^2$ has a representative $1 + w$ such that $w\mathfrak{o} = \mathfrak{d}(u)$.

2.8.2 Jordan decompositions

Let us consider local lattices for a moment. Up to isometry, these can be described in terms of a number of readily computable invariants.

Definition 2.78. Assume L is an $\mathfrak{o}$ -lattice in a quadratic F -vector space (V, Φ) .

Then a *Jordan decomposition* of L is an orthogonal sum

$$L = L_1 \perp \dots \perp L_k$$

such that each L_i is $\mathfrak{p}^{s_i}$ -modular, for some sequence of integers $s_1 < s_2 < \dots < s_k$.

Theorem 2.79. Every $\mathfrak{o}$ -lattice $L \neq 0$ has a Jordan decomposition

$$L = L_1 \perp \dots \perp L_k \text{ with every } L_i \neq 0.$$

Moreover, assume

$$L' = L'_1 \perp \dots \perp L'_{k'} \text{ with every } L'_i \neq 0$$

is a Jordan decomposition of a lattice L' in the same quadratic space.

1. If $L \cong L'$, then
 - a. $k = k'$
 - b. $\mathfrak{s}(L_i) = \mathfrak{s}(L'_i)$ for all $1 \leq i \leq k$
 - c. $\text{rank}(L_i) = \text{rank}(L'_i)$ for all $1 \leq i \leq k$.
2. Assume $2 \in \mathfrak{o}^*$. Then $L \cong L'$ if the conditions [a\)](#) to [c\)](#) are satisfied, and the determinants $\det(L_i)$ and $\det(L'_i)$ represent the same square classes.

Proof. [[O'M73](#), 91:9 and 92:2] ■

Corollary 2.80. Let F be non-dyadic. Then the Jordan decomposition of an $\mathfrak{o}$ -lattice L is essentially unique. In particular, there are (up to isometry) exactly two unimodular $\mathfrak{o}$ -lattices, namely those given by either of the following diagonal matrices:

1. $L_1 \cong \text{diag}(1, \dots, 1)$
2. $L_2 \cong \text{diag}(1, \dots, 1, \varepsilon)$ for some non-square unit $\varepsilon \in \mathfrak{o}^*$.

They are distinguishable by their determinants. All (not necessarily unimodular) $\mathfrak{o}$ -lattices can be obtained from L_1 and L_2 by taking orthogonal sums and rescaling as indicated by the invariants given in Theorem [2.79](#).

In the dyadic case, a complete set of invariants for isometry is more difficult to obtain. The description of the isometry classes in the dyadic case is one of the principal results of O'MEARA in [[O'M73](#)]. We will use the remainder of this section to present his classification (cf. Theorem [2.85](#) below).

2.8.3 The fundamental invariants and O'Meara's classification

From now on until the end of this section, assume $2 \notin \mathfrak{o}^*$.

Definition 2.81. Let L be an $\mathfrak{o}$ -lattice.

The *norm group* $\mathfrak{g}(L)$ is the additive subgroup $q_\Phi(L) + 2\mathfrak{s}(L)$ of $(F, +)$. An element $a \in \mathfrak{g}(L)$ is a *norm generator* if a generates the norm ideal $\mathfrak{n}(L)$.

The ideal $\mathfrak{m}L$ is the largest fractional ideal contained in the norm group $\mathfrak{g}(L)$.

The *weight* $\mathfrak{w}(L)$ is defined as $\mathfrak{w}(L) = 2\mathfrak{s}(L) + \pi\mathfrak{m}(L)$.

Proposition 2.82. Let $x_1, \dots, x_n$ be a basis of L .

1. If $\mathfrak{n}(L) = 2\mathfrak{s}(L)$, then every generator of $2\mathfrak{s}(L)$ is a norm generator of L . Otherwise, any $a = q_\Phi(x_i)$ for $1 \leq i \leq n$ such that the ideal $q_\Phi(x_i)\mathfrak{o}$ is largest is a norm generator of L .
2. If a is a norm generator of L , then the weight is given by

$$\mathfrak{w}(L) = \sum_i a\mathfrak{d}(q_\Phi(x_i)/a) + 2\mathfrak{s}(L).$$

Proof. Cf. [O'M73, § 94]. ■

Definition 2.83. Let

$$L = L_1 \perp \dots \perp L_k$$

be a Jordan decomposition (each L_i is $\mathfrak{p}^{s_i}$ -modular, for some sequence of integers $s_1 < s_2 < \dots < s_k$).

Let $\mathfrak{w}_i := \mathfrak{w}(L^{\mathfrak{p}^{s_i}})$, and let α_i be a norm generator for $L^{\mathfrak{p}^{s_i}}$.

Then the *fundamental invariants* of L is the tuple

$$(k, s_1, \dots, s_k, \alpha_1, \dots, \alpha_k, \mathfrak{w}_1, \dots, \mathfrak{w}_k).$$

Lastly, we define the ideals $\mathfrak{f}_i$, $1 \leq i < t$, by

2.8 Isometry classes of lattices over local fields

$$\mathfrak{f}_i = \begin{cases} \sum \mathfrak{d}(\alpha\beta), & \text{when } u_i + u_{i+1} \text{ is odd} \\ \sum \mathfrak{d}(\alpha\beta) + 2\mathfrak{p}^{(1/2)(u_i+u_{i+1}+s_i)}, & \text{otherwise,} \end{cases} \quad (2.1)$$

where $u_i = \nu(\alpha_i)$, $\alpha \in \mathfrak{g}_i$, $\beta \in \mathfrak{g}_{i+1}$.

Definition 2.84. Let L, L' be two $\mathfrak{o}$ -lattices with fundamental invariants

$$(k, s_1, \dots, s_k, \alpha_1, \dots, \alpha_k, \mathfrak{w}_1, \dots, \mathfrak{w}_k)$$

and

$$(k', s'_1, \dots, s'_{k'}, \alpha'_1, \dots, \alpha'_{k'}, \mathfrak{w}'_1, \dots, \mathfrak{w}'_{k'}).$$

The lattices are said to have the *same fundamental invariants*, if $k = k'$ and for all $1 \leq i \leq k$:

1. $s_i = s'_i$
2. $\mathfrak{w}_i = \mathfrak{w}'_i$
3. $\alpha_i \sim \alpha'_i \pmod{\mathfrak{w}_i}$ (cf. Proposition 2.74).

O'MEARA has shown that these form a complete set of invariants for the isometry type of a lattice. More precisely, the following result for lattices over dyadic fields holds:

Theorem 2.85 (O'MEARA). Let L, L' be two $\mathfrak{o}$ -lattices in the same quadratic space (V, Φ) . Suppose L and L' have the same fundamental invariants

$$(k, s_1, \dots, s_k, \alpha_1, \dots, \alpha_k, \mathfrak{w}_1, \dots, \mathfrak{w}_k).$$

So L and L' have Jordan decompositions of the same length, say, $L = L_1 \perp \dots \perp L_k$ and $L' = L'_1 \perp \dots \perp L'_k$. Let $L_{(i)} = L_1 \perp \dots \perp L_i$, and $L'_{(i)} = L'_1 \perp \dots \perp L'_i$.

Furthermore, let $\mathfrak{f}_1, \dots, \mathfrak{f}_{k-1}$ be defined as in Eq. 2.1.

Then L and L' are isometric if and only if for all $1 \leq i < k$:

1. $\det(L_{(i)}) / \det(L'_{(i)}) \sim 1 \pmod{\mathfrak{f}_i}$
2. If $\mathfrak{f}_i \subsetneq 4\alpha_{i+1}\mathfrak{w}_{i+1}^{-1}$, then the quadratic space $KL_{(i)}$ embeds into $KL'_{(i)} \perp \langle \alpha_{i+1} \rangle$
3. If $\mathfrak{f}_i \subsetneq 4\alpha_i\mathfrak{w}_i^{-1}$, then the quadratic space $KL_{(i)}$ embeds into $KL'_{(i)} \perp \langle \alpha_i \rangle$.

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Note that if L and L' are unimodular, then only the assumption on having the same fundamental invariants has to be checked.

Proof. Cf. [O'M73, 93:28]. ■

The above theorem yields a classification of all unimodular lattices in the dyadic case. It is explicitly worked out as an example in [O'M73, 93:18]. We will only state the results.

Theorem 2.86. Let L denote a unimodular $\mathfrak{o}$ -lattice of rank m . Let a be a norm generator of L , let b be a generator of $\mathfrak{w}(L)$, and choose a representative $d \in \mathfrak{o}^*$ of $(-1)^{\frac{m(m+1)}{2}} \det(L)$. Let $\alpha \in \mathfrak{o}$ be an element such that $d(\mathfrak{o}^*)^2 = (1 + \alpha)(\mathfrak{o}^*)^2$ with quadratic defect $\mathfrak{d}(d) = \alpha\mathfrak{o}$ (see Lemma 2.77).

For α and $\beta \in \mathfrak{o}$, we define the matrix

$$A(\alpha, \beta) = \begin{pmatrix} \alpha & 1 \\ 1 & \beta \end{pmatrix}.$$

Then:

1. If $m = 2$, then $L \cong \langle A(a, -\alpha a^{-1}) \rangle$.
2. If $m = 3$ and $\nu(a) + \nu(b)$ is odd, then either $L \cong \langle A(b, 0) \rangle \perp \langle -d \rangle$ or $L \cong \langle A(b, 4\rho b^{-1}) \rangle \perp \langle -d(1 - 4\rho) \rangle$. Here ρ is defined as in 2.75.
3. If $m = 4$ and $\nu(a) + \nu(b)$ is odd, then either $L \cong \langle A(a, -\alpha a^{-1}) \rangle \perp \langle A(b, 0) \rangle$ or $L \cong \langle A(a, -(\alpha - 4\rho)a^{-1}) \rangle \perp \langle A(b, 4\rho b^{-1}) \rangle$.
4. If either $m \geq 5$, or $m \geq 3$ and $\nu(a) + \nu(b)$ is even, then the hyperbolic plane $\langle A(0, 0) \rangle$ splits L .

A proof is given in [O'M73, § 93D].

To be able write down representatives of all unimodular $\mathfrak{o}$ -lattices of given rank, we require a list of possible norm generators α . In other words, we need to know representatives of $\mathfrak{o}^*/(\mathfrak{o}^*)^2$.

2.9 Finding a global lattice with given completions

Lemma 2.87. By the Local Square Theorem, the canonical epimorphism $\mathfrak{o} \rightarrow R := \mathfrak{o}/4\mathfrak{p}$ induces an isomorphism

$$\mathfrak{o}^*/(\mathfrak{o}^*)^2 \cong R^*/(R^*)^2.$$

Note that R is a finite Euclidean ring (with zero divisors), hence the calculation of $R^*/(R^*)^2$ is easy.

Proof. See [Neu92, I, §6, (6.3)] ■

In particular, these results yield an algorithm UNIMODULARLATTICES for enumerating, up to isometry, all the unimodular lattices over a given local field.

We note that there is a somewhat nicer presentation of the even unimodular lattices:

Remark 2.88. If L is a binary unimodular lattice over a dyadic field with $\mathfrak{n}(L) \subseteq 2\mathfrak{o}$, then either

- $L \cong \perp_{i=1}^{\frac{m}{2}} A(0, 0)$, or
- $L \cong \perp_{i=1}^{\frac{m}{2}-1} A(0, 0) \perp A(2, 2\rho)$. Here ρ is defined as in 2.75.

Proof. The proof for the binary case is given in [O'M73, 93:11]. As $2\mathfrak{s}(L) \subseteq \mathfrak{w}(L) \subseteq \mathfrak{n}(L)$, we have $\mathfrak{w}(L) = \mathfrak{n}(L)$ for even lattices, so by part 4) of Theorem 2.86, the claim follows. ■

2.9 Finding a global lattice with given completions

The situation in this section is as follows: we are given a quadratic space V , a set of primes $P = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ and for each of these an $\mathfrak{o}_{\mathfrak{p}_i}$ -lattice L_i in the completion $V_{\mathfrak{p}_i}$. How can we construct an $\mathfrak{o}$ -lattice L with the property that $L_{\mathfrak{p}_i} \cong L_i$ for all $\mathfrak{p}_i \in P$, and $L_{\mathfrak{q}}$ is maximal for all $\mathfrak{q} \notin P$? — In other words, we are asking to construct a representative lattice L of some genus of lattices.

Our method for constructing such a lattice L is based on the following result:

Theorem 2.89. Let V be a regular quadratic space over K .

2 Lattices and quadratic forms over number fields

1. Let $M_{\mathfrak{p}}$ and $L_{\mathfrak{p}}$ be maximal lattices in $V_{\mathfrak{p}}$ over the local field $K_{\mathfrak{p}}$, where $\mathfrak{p}$ is any place of K . Then $M_{\mathfrak{p}} \cong L_{\mathfrak{p}}$.
2. The maximal $\mathfrak{o}$ -lattices in V form a single genus.

Proof. Part 1) is shown in [O'M73, 91:2]. Part 2) is an easy consequence: if L and M are maximal $\mathfrak{o}$ -lattices in V , then $L_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ are maximal $\mathfrak{o}_{\mathfrak{p}}$ -lattices in $V_{\mathfrak{p}}$ and thus isometric. So $M \in \text{Genus}(L)$. Conversely, every element of $\text{Genus}(L)$ is clearly again a maximal $\mathfrak{o}$ -lattice. ■

We start by constructing a maximal lattice L . This algorithm was stated by KIRSCHMER in [Kir16, Algorithm 3.5.5].

Algorithm 2.90. Constructing a maximal lattice L

Input: A lattice L and a prime $\mathfrak{p}$, such that $L_{\mathfrak{p}}$ is integral

Output: A list of lattices $L_n \supsetneq L_{n-1} \supsetneq \cdots \supsetneq L_0 = L$ for some $n \in \mathbb{N}$ such that $(L_n)_{\mathfrak{p}}$ is maximal, $(L_n)_{\mathfrak{q}} = L_{\mathfrak{q}}$ for every prime $\mathfrak{q} \neq \mathfrak{p}$, and $L_i/L_{i-1} \cong \mathfrak{o}/\mathfrak{p}$.

```

1: function MAXIMALLATTICE( $L, \mathfrak{p}$ )
2:    $L_0 \leftarrow L$ 
3:    $i \leftarrow 0$ 
4:   while there is a subspace  $0 \neq (\mathfrak{o}/\mathfrak{p})x \subset (2L_i^{\# \cdot \mathfrak{p}} \cap \mathfrak{p}^{-1}L_i)/L_i$  with  $q_{\Phi}(x) \in \mathfrak{o}_{\mathfrak{p}}$  do
5:      $L_{i+1} \leftarrow L_i + \mathfrak{o}x$ 
6:      $i \leftarrow i + 1$ 
7:   end while
8:   return  $L_i, \dots, L_0$ 
9: end function

```

Proof. If $(L_i)_{\mathfrak{p}}$ is not maximal, then by Theorem 2.52 there exists $(L_{i+1}) \supset L_i$ such that $L_{i+1}/L_i \cong \mathfrak{o}/\mathfrak{p}$. Every $x \in L_{i+1} - L_i$ fulfills the conditions of line 4. ■

Remark 2.91. Let L be an integral $\mathfrak{o}$ -lattice. By successively applying Algorithm 2.90 to every prime $\mathfrak{p}$ dividing $2\mathfrak{v}(L)$, we end up with a maximal lattice.

So maximal lattices can easily be constructed from any integral lattice.

2.9 Finding a global lattice with given completions

We are now able to solve the problem stated in this section's introduction. Again, we make use of the local-global principle from [O'M73, 81:14] (cf. Theorem 2.52). The construction stated below was given by KIRSCHMER in [Kir16, Algorithm 3.5.7]:

Algorithm 2.92. Constructing a lattice with given completion at $\mathfrak{p}$

Input: lattices L and M such that $L_{\mathfrak{p}}$ is a maximal $\mathfrak{o}_{\mathfrak{p}}$ -lattice in $V_{\mathfrak{p}}$, $M_{\mathfrak{p}}$ is any lattice with integral norm (i.e., $q_{\Phi}(M_{\mathfrak{p}}) \subseteq \mathfrak{o}_{\mathfrak{p}}$), and the enveloping quadratic spaces $(KL)_{\mathfrak{p}}$ and $(KM)_{\mathfrak{p}}$ are isometric

Output: a lattice $X \leq M$ such that $X_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ and $X_{\mathfrak{q}} = L_{\mathfrak{q}}$ for all $\mathfrak{q} \neq \mathfrak{p}$.

```

1: function LATTICEFROMCOMPLETION( $L, M, \mathfrak{p}$ )
2:    $M_n, \dots, M_0 = M \leftarrow \text{MAXIMALLATTICE}(M, \mathfrak{p})$ 
3:   for  $i = n - 1, \dots, 0$  do
4:     repeat
5:       pick a random maximal sublattice  $Y$  of  $L$  containing  $\mathfrak{p}L$ 
6:     until  $Y_{\mathfrak{p}} \cong (M_i)_{\mathfrak{p}}$ 
7:      $L \leftarrow Y$ 
8:   end for
9: end function
10: return  $L$ 
```

Remark 2.93. If L is definite, Algorithm 2.92 can be sped up by exploiting the action of the automorphism group $O(L)$ on the subspaces of $L/\mathfrak{p}L$.

Remark 2.94. In principle, there is a second possibility to solve this section's problem, following O'MEARA more closely: suppose we find a bijection $\varphi : KM \rightarrow KL$, such that the induced mapping $\varphi : (KM)_{\mathfrak{p}} \rightarrow (KL)_{\mathfrak{p}}$ is "close to" being an isometry (in the sense that $\Phi(\varphi(x), \varphi(y)) \equiv \Phi(x, y) \pmod{\mathfrak{p}^{\kappa}}$ for some large enough κ). Then, due to its invariants being bounded, $\varphi(M_{\mathfrak{p}})$ is isometric to $M_{\mathfrak{p}}$. Then by using the constructive algorithm given in [O'M73, 81:14], find X such that $X_{\mathfrak{q}} = L_{\mathfrak{q}}$ for all $\mathfrak{q} \neq \mathfrak{p}$, and $X_{\mathfrak{p}} = \varphi(M)_{\mathfrak{p}}$.

3 The mass formula

3.1 The mass of a lattice

Let L denote any (full) lattice in a regular, totally definite quadratic space V of rank m over a totally real algebraic number field K . We set n to be the absolute degree $[K : \mathbb{Q}]$, $d_K = |\text{disc}(K : \mathbb{Q})|$ to be the absolute value of the discriminant of K , and as usual denote by $\mathfrak{o}$ the ring of integers of K . Let $q : V \rightarrow K, v \mapsto \Phi(v, v)$ denote the quadratic form associated to Φ .

Remark 3.1 (Calculating isometries). Let L_1 and L_2 denote two $\mathfrak{o}$ -lattices in (V, q) . Then $L_1 \cong L_2$ if and only if there exists a $\mathbb{Z}$ -linear bijection $\varphi : L_1 \rightarrow L_2$ such that tr_1 and tr_α are both φ -invariant, where α is a primitive element of the extension $K/\mathbb{Q}$ and for any $c \in K$,

$$\text{tr}_c : V \rightarrow \mathbb{Q}, v \mapsto \text{tr}_{K/\mathbb{Q}}(c \cdot q(v))$$

This reduces the question of isometry to the calculation of $\mathbb{Z}$ -isometries in the definite quadratic $\mathbb{Q}$ -space (V, tr_1) . These calculations can be done effectively, for example with the PLESKEN-SOUVIGNIER algorithm [PS97].

Corollary 3.2. The automorphism group $\text{Aut}(L)$ is finite.

Proof. Since q is totally definite, the quadratic form tr_1 is definite, hence defines a norm $v \mapsto \sqrt{\text{tr}_1(q(v))}$ on the $(m \cdot [K : \mathbb{Q}])$ -dimensional quadratic $\mathbb{Q}$ -vector space (V, tr_1) . Since all norms on this space are equivalent, this norm is equivalent to the maximum norm $v = \sum a_i b_i \mapsto \max |a_i|$ on V , where the b_i are any fixed $\mathbb{Z}$ -basis of L which is also a basis of V . So for every $\gamma \in \mathbb{Q}$, the number of vectors $l \in L$ with $\text{tr}_1(q(l)) \leq \gamma$ is finite.

Now any $g \in \text{Aut}_{\mathbb{Z}}(L)$ is uniquely determined by the images of the b_i . Since $q(g(b_i)) = q(b_i)$ by definition, the number of possible images of the b_i is finite.

Finally, we have $\text{Aut}(L) \subseteq \text{Aut}_{\mathbb{Z}}(L)$ by Remark 3.1. ■

3 The mass formula

Definition 3.3. Let $L_1, \dots, L_{h(L)}$ be representatives of the isometry classes in $\text{Genus}(L)$. Then the *mass* of L is defined as:

$$\text{Mass}(L) := \sum_{i=1}^{h(L)} \frac{1}{\# \text{Aut}(L_i)}$$

Remark 3.4. $\text{Mass}(L)$ is, of course, always a rational number. If $h(L) = 1$, i.e., L is single-class, $\text{Mass}(L)$ will always be the reciprocal of some even number $n \in \mathbb{N}$, since $-\text{id} : v \mapsto -v$ is always an automorphism of L .

We will now state the *mass formula*. The mass formula relates the quantity $\text{Mass}(L)$ to local properties of L , and is a theorem that has a long history (cf. [CS88, §2]) with important contributions by H. MINKOWSKI, C. SIEGEL and H. J. S. SMITH. The version given below is taken from H. PFEUFFER's 1971 dissertation [Pfe71].

Theorem 3.5 (The mass formula).

$$\text{Mass}(L) = 2^n d_K^{\frac{m(m-1)}{4}} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(L).$$

Here $\mathfrak{p}$ runs through all prime ideals $\mathfrak{p} \in \Omega_{\text{fin}}(K)$, Γ denotes the usual complex Gamma function, and we define the *local density* $\lambda_{\mathfrak{p}}$ for an integral lattice L by

$$\lambda_{\mathfrak{p}}(L) := \mathfrak{N}(\text{disc}(L_{\mathfrak{p}}))^{\frac{m+1}{2}} \cdot d_{\mathfrak{p}}(L, L)^{-1}, \text{ where } d_{\mathfrak{p}}(L, L) = \frac{1}{2} \lim_{v \rightarrow \infty} \frac{\# \text{Aut}_{\mathfrak{p}^v}(L, L)}{\mathfrak{N}(\mathfrak{p}^{vm(m-1)/2})}.$$

Here $\# \text{Aut}_{\mathfrak{p}^v}(L, L)$ is the number of isometries of $L/\mathfrak{p}^v L$.

The evaluation of the local factors $d_{\mathfrak{p}}$ is difficult. PFEUFFER ([Pfe71]) managed to evaluate the $d_{\mathfrak{p}}$ whenever $\mathfrak{p} \nmid 2$, and for the case of $\mathfrak{p} \mid 2$ only under the additional assumption that L is even:

Definition 3.6. A unimodular $\mathfrak{o}_{\mathfrak{p}}$ -lattice $L_{\mathfrak{p}}$ is called *even* if it fulfills $\mathfrak{n}(L_{\mathfrak{p}}) = 2\mathfrak{o}_{\mathfrak{p}}$. An $\mathfrak{o}$ -lattice L is called *even at $\mathfrak{p}$* if $L_{\mathfrak{p}}$ is even, and *even*, if it is even at all the dyadic primes of $\mathfrak{o}$.

3.1 The mass of a lattice

Definition 3.7. By $\left(\frac{a}{\mathfrak{p}}\right)$ we mean the usual Legendre symbol (where $a \in \mathfrak{o}$ and $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}$). It takes the value

$$\left(\frac{a}{\mathfrak{p}}\right) = \begin{cases} 0, & \nu_{\mathfrak{p}}(a) \neq 0 \\ 1, & a \in (K_{\mathfrak{p}}^*)^2 \text{ and } \nu_{\mathfrak{p}}(a) = 0 \\ -1, & a \notin (K_{\mathfrak{p}}^*)^2 \text{ and } \nu_{\mathfrak{p}}(a) = 0 \end{cases}$$

The following is a principal result of PFEUFFER's dissertation. It gives the exact value of the local factors of the mass formula, for those primes $\mathfrak{p}$ where $L_{\mathfrak{p}}$ is even:

Theorem 3.8. Assume $\mathfrak{p}$ is a prime ideal in $\Omega_{\text{fin}}(K)$, and L is an $\mathfrak{o}_{\mathfrak{p}}$ -lattice. If $2 \in \mathfrak{p}$, let $e_{\mathfrak{p}}$ denote the ramification index of $\mathfrak{p}$. Otherwise let $e_{\mathfrak{p}} := 0$.

1. If $L_{\mathfrak{p}}$ is an even unimodular $\mathfrak{o}_{\mathfrak{p}}$ -lattice of rank m , then

$$d_{\mathfrak{p}}(L, L) = \begin{cases} \mathfrak{N}(\mathfrak{p})^{e_{\mathfrak{p}}m} \left(1 - \left(\frac{\text{disc}(KL)}{\mathfrak{p}}\right) \mathfrak{N}(\mathfrak{p})^{-m/2}\right) \prod_{k=1}^{m/2-1} (1 - \mathfrak{N}(\mathfrak{p})^{-2k}), & \text{if } 2 \mid m \\ \mathfrak{N}(\mathfrak{p})^{e_{\mathfrak{p}}m} \prod_{k=1}^{(m-1)/2} (1 - \mathfrak{N}(\mathfrak{p})^{-2k}), & \text{if } 2 \nmid m. \end{cases}$$

2. Now let $L_{\mathfrak{p}}$ denote an arbitrary $\mathfrak{o}_{\mathfrak{p}}$ -lattice. If $L_{\mathfrak{p}} = L_1 \perp \dots \perp L_t$ is a Jordan decomposition at any prime $\mathfrak{p}$, where the L_i ($1 \leq i \leq t$) are $\mathfrak{p}^{s_i}$ -modular $\mathfrak{o}_{\mathfrak{p}}$ -lattices of dimension m_i , $s_0 := 0$ and $\overline{m_i} := m_i + \dots + m_t$, then

$$d_{\mathfrak{p}}(L, L) \leq 2^{t-1} \prod_{i=1}^t \mathfrak{N}(\mathfrak{p})^{(s_i - s_{i-1})\overline{m_i}(\overline{m_i}+1)/2} d_{\mathfrak{p}}(L_i^*, L_i^*).$$

Equality holds whenever all the L_i^* are even lattices.

Here L_i^* denotes the unimodular $\mathfrak{o}_{\mathfrak{p}}$ -lattice obtained from rescaling the modular lattice L_i with a suitable element of the local field $K_{\mathfrak{p}}$.

Proof. Cf. Satz 1 and Hilfssatz 8 in [Pfe71]. ■

Corollary 3.9. Let L be unimodular at a non-dyadic prime $\mathfrak{p}$. If L has rank $m \neq 2$, or $\text{disc}(L_{\mathfrak{p}}) \notin (K_{\mathfrak{p}}^*)^2$ then $d_{\mathfrak{p}}(L, L) < 1$. If L has rank 2 and $\text{disc}(L_{\mathfrak{p}}) \in (K_{\mathfrak{p}}^*)^2$, then $d_{\mathfrak{p}}(L, L) = 1 + \mathfrak{N}(\mathfrak{p})^{-1}$.

3 The mass formula

Proof. If m is odd or $\text{disc}(L_{\mathfrak{p}})$ is not a square, the claim is immediate from Theorem 3.8. If $m = 2$ and $\text{disc}(L_{\mathfrak{p}})$ is a square, the claim is just a restatement of Theorem 3.8. And if $m \geq 4$ is even, then

$$\begin{aligned} & \left(1 - \left(\frac{\text{disc}(KL)}{\mathfrak{p}}\right) \mathfrak{N}(\mathfrak{p})^{-m/2}\right) \prod_{k=1}^{m/2-1} (1 - \mathfrak{N}(\mathfrak{p})^{-2k}) \\ & \leq (1 + \mathfrak{N}(\mathfrak{p})^{-m/2})(1 - \mathfrak{N}(\mathfrak{p})^{-2}) \leq (1 + \mathfrak{N}(\mathfrak{p})^{-2})(1 - \mathfrak{N}(\mathfrak{p})^{-2}) < 1. \end{aligned}$$

■

3.2 Evaluation of the mass formula

The mass formula stated in Theorem 3.5 naturally takes values in the rational numbers. It is not obvious, however, how the right side of the formula can be written as a finite product of rational numbers. By some fairly elementary transformations, we will arrive at an alternative presentation of the mass formula whose factors are all rational. This allows exact computation of an even unimodular lattice's mass, and can be used for more general lattices by applying a finite number of correction factors.

3.2.1 The Dedekind zeta function

In this section, let K denote an arbitrary number field with r_1 real and $2r_2$ complex embeddings, $\mathfrak{o}$ its ring of integers and $n := [K : \mathbb{Q}]$. If $s \in \mathbb{C}$, then by $\Re(s)$ we mean the real part of s . The Dedekind zeta function of K is a generalisation of the Riemann zeta function:

Definition 3.10. The *Dedekind zeta function* of K is the Dirichlet sum

$$\zeta_K : \mathbb{C} \rightarrow \mathbb{C}, s \mapsto \sum_{\mathfrak{a}} \mathfrak{N}(\mathfrak{a})^{-s},$$

where $\mathfrak{a}$ runs over all integral ideals in $\mathbb{I}(\mathfrak{o})$.

Proposition 3.11. The Dedekind zeta function of K has the following essential properties:

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1. $\zeta_K(s)$ converges absolutely for any $s \in \mathbb{C}$ with $\Re(s) > 1$.
2. For any $s \in \mathbb{C}$ with $\Re(s) > 1$, $\zeta_K(s)$ has a representation as an Euler product:

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \mathfrak{N}(\mathfrak{p})^{-s}}$$

where $\mathfrak{p}$ ranges over all prime ideals in $\Omega_{\text{fin}}(K)$.

3. The *completed Dedekind zeta function* of K ,

$$Z_K(s) := \zeta_K(s) |d_K|^{\frac{s}{2}} \pi^{\frac{-ns}{2}} 2^{(1-s)r_2} \Gamma(s/2)^{r_1} \Gamma(s)^{r_2},$$

has an analytic continuation to $\mathbb{C} - \{0, 1\}$, which satisfies the functional equation $Z_K(s) = Z_K(1-s)$. As a consequence, the ordinary Dedekind zeta function $\zeta_K(s)$ has an analytic continuation to $\mathbb{C} - \{1\}$.

Proof. For 1) and 2), cf. [Neu92, VII, 5.2]. For 3), cf. [Neu92, VII, (5.10) and (5.11)]. ■

By identifying ζ_K with its analytic continuation, one can now prove:

Theorem 3.12 (Siegel-Klingen). If K is totally real and s is a negative odd integer, then $\zeta_K(s) \in \mathbb{Q}$.

Proof. Cf. [Sie37, Section 23]. ■

The following statements about ζ_K can now be used for transforming the mass formula in such a way that ζ_K only needs to be evaluated at negative integers:

Proposition 3.13. Assume $s \in \mathbb{C} - \{0, 1\}$.

1. If K is totally real, then

$$\zeta_K(1-s) = \zeta_K(s) \cdot 2^n d_K^{s-\frac{1}{2}} (2\pi)^{-sn} \Gamma(s)^n \left(\cos\left(\frac{\pi s}{2}\right) \right)^n.$$

In particular, $\zeta_K(s) = 0$ for any negative even integer s , and $\zeta_K(s) \neq 0$ for any negative odd integer s .

3 The mass formula

2. If K is totally imaginary, then

$$\zeta_K(1-s) = \zeta_K(s) \cdot 2^n |d_K|^{s-\frac{1}{2}} (2\pi)^{-sn} \Gamma(s)^n \left(\cos\left(\frac{\pi s}{2}\right) \right)^{n/2} \left(\sin\left(\frac{\pi s}{2}\right) \right)^{n/2}.$$

In particular, $\zeta_K(s) = 0$ for any negative integer s .

Proof. This is a direct consequence of the functional equation in part 3) of Proposition 3.11, cf. [Neu92, VII, 5.11(iii)]. ■

3.2.2 The even-dimensional case

Let us for a moment assume that K is totally real, and that L has even rank m . We also define the field extension $E = K(\sqrt{\text{disc}(KL)})$ of K .

Remark 3.14. Assume $E \neq K$. Then by the definition of the discriminant in Definition 2.53, and since K was assumed to be totally real, E is a quadratic extension of K which is totally real if $m \equiv 0 \pmod{4}$, and totally imaginary if $m \equiv 2 \pmod{4}$.

Definition 3.15. Let $\chi_L : \mathbb{I}(\mathfrak{o}) \rightarrow \{-1, 0, 1\}$ be the multiplicative map defined by:

1. if $\text{disc}(KL) \in (K^*)^2$, then $\chi_L(\mathfrak{a}) := 1$ for all $\mathfrak{a} \in \mathbb{I}(\mathfrak{o})$
2. if $\text{disc}(KL) \notin (K^*)^2$, then let χ_L be the multiplicative continuation of the function defined on the prime ideals $\mathfrak{p}$ of $\mathfrak{o}$:

$$\chi_L(\mathfrak{p}) := \begin{cases} 0, & \mathfrak{p} \text{ is ramified in } E/K \\ -1, & \mathfrak{p} \text{ is inert in } E/K \\ 1, & \mathfrak{p} \text{ is split in } E/K \end{cases}$$

χ_L gives rise to an L -function:

$$L(\chi_L, s) := \prod_{\mathfrak{p}} \frac{1}{1 - \chi_L(\mathfrak{p}) \mathfrak{N}(\mathfrak{p})^{-s}}, \text{ defined for } \Re(s) > 1,$$

where $\mathfrak{p}$ runs through the prime ideals in $\Omega_{\text{fin}}(K)$.

Note that clearly $\zeta_K(s) = L(1, s)$ on $\Re(s) > 1$.

Lemma 3.16. Let L be an even unimodular $\mathfrak{o}$ -lattice (again, of even rank m). Then:

1. For any prime $\mathfrak{p}$ of $\mathfrak{o}$, either $\text{disc}(L_{\mathfrak{p}})$ is a square or $\text{disc}(L_{\mathfrak{p}})(\mathfrak{o}_{\mathfrak{p}}^*)^2 = \Delta(\mathfrak{o}_{\mathfrak{p}}^*)^2$, where Δ is chosen as in Definition 2.75.
2. The field extension E/K is unramified.

Proof. If $\text{disc}(KL)$ is a global square, i.e. $\text{disc}(KL) \in (K^*)^2$, then $\text{disc}(L_{\mathfrak{p}}) \in (K_{\mathfrak{p}}^*)^2$ for all primes $\mathfrak{p}$ and $E = K$, so there is nothing to show. So assume $\text{disc}(KL)$ is not a global square and fix any prime $\mathfrak{p}$. Then the extension E/K has degree 2 and its completion at $\mathfrak{p}$ is $E_{\mathfrak{p}} = K(\sqrt{\text{disc}(KL)})_{\mathfrak{p}} = K_{\mathfrak{p}}[x]/(x^2 - \text{disc}(L_{\mathfrak{p}}))$.

If $\text{disc}(L_{\mathfrak{p}})$ is a square, then $E_{\mathfrak{p}} = K_{\mathfrak{p}}$ and there is nothing to show for the prime $\mathfrak{p}$. But if this is not the case, then the square class of $\text{disc}(L_{\mathfrak{p}})$ is uniquely determined:

If $\mathfrak{p}$ is non-dyadic, then $\mathfrak{d}(\text{disc}(L_{\mathfrak{p}})) = 4\mathfrak{o}_{\mathfrak{p}}$ follows from Proposition 2.73. Similarly, if $\mathfrak{p}$ is dyadic, then because L is even and unimodular we have $\mathfrak{n}(L_{\mathfrak{p}}) = 2\mathfrak{o}$. So by Theorem 2.86, $\text{disc}(L_{\mathfrak{p}}) \equiv \Delta \pmod{(K_{\mathfrak{p}}^*)^2}$. The quadratic defect of Δ is $\mathfrak{d}(\Delta) = 4\mathfrak{o}_{\mathfrak{p}}$. So by [O'M73, 63:3], regardless of the choice of $\mathfrak{p}$, the extension of local fields $K_{\mathfrak{p}}(\sqrt{\Delta})/K_{\mathfrak{p}}$ is unramified. ■

Lemma 3.17. If $\text{disc}(KL) \notin (K^*)^2$ and L is an even unimodular lattice, then for all prime ideals $\mathfrak{p}$ of $\mathfrak{o}$:

$$\chi_L(\mathfrak{p}) = \left(\frac{\text{disc}(KL)}{\mathfrak{p}} \right)$$

Note that since L is unimodular, the case $\mathfrak{p} \mid \text{disc}(KL)$ can never occur.

Proof. We have $E_{\mathfrak{p}} = K(\sqrt{\text{disc}(KL)})_{\mathfrak{p}} = K_{\mathfrak{p}}[x]/(x^2 - \text{disc}(L_{\mathfrak{p}}))$. And by Lemma 3.16, for any prime $\mathfrak{p}$, either $\text{disc}(L_{\mathfrak{p}})$ is a square in $K_{\mathfrak{p}}$ or its square class is represented by a unit Δ of quadratic defect $\mathfrak{d}(\Delta) = 4\mathfrak{o}_{\mathfrak{p}}$.

Now if $\text{disc}(L_{\mathfrak{p}})$ is a square, then $E_{\mathfrak{p}} = K_{\mathfrak{p}}[x]/(x-1)(x+1) \cong K_{\mathfrak{p}}[x] \oplus K_{\mathfrak{p}}[x]$. So $\mathfrak{p}$ is split in E/K and $\chi_L(\mathfrak{p}) = 1$. If, on the other hand, $\text{disc}(L_{\mathfrak{p}}) = \Delta$ (modulo squares), then $E_{\mathfrak{p}} \supset K_{\mathfrak{p}}$ is a quadratic unramified field extension by Lemma 3.16. In other words, the polynomial $x^2 - \text{disc}(L_{\mathfrak{p}})$ is irreducible in $K_{\mathfrak{p}}[x]$, so $\mathfrak{p}$ is inert in E/K . ■

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Corollary 3.18. We can write the product over all $\mathfrak{p}$ of the factors containing Legendre symbols in PFEUFFER's mass formula (cf. Theorem 3.8) as an L -series:

$$\prod_{\mathfrak{p}} \left(1 - \left(\frac{\text{disc}(KL)}{\mathfrak{p}} \right) \mathfrak{N}(\mathfrak{p})^{-m/2} \right) = L \left(\chi_L, \frac{m}{2} \right)^{-1}$$

This L -series can be evaluated in terms of Dedekind zeta functions. Similarly to the Dedekind zeta function, we will show that it takes rational values at negative integers $s < -1$ and satisfies a functional equation. Thus, for an evaluation of $\text{Mass}(L)$, the L -series will only be evaluated at places where it takes rational values. Existing algorithms allow exact evaluation of both $L(\chi_L, \cdot)$ and the Dedekind zeta function at these places (see for example [Lou04]).

Proposition 3.19. Let $s \in \mathbb{C}$ with $\Re(s) > 1$.

1. If $\text{disc}(KL) \in (K^*)^2$, then $L(\chi_L, s) = \zeta_K(s)$.
2. If $\text{disc}(KL) \notin (K^*)^2$, then $L(\chi_L, s) = \frac{\zeta_E(s)}{\zeta_K(s)}$ whenever $\Re(s) > 1$, and $L(\chi_L, s)$ has a meromorphic continuation to $\mathbb{C} - \{0, 1\}$.

Proof. 1) is clear from the representation of ζ_K as an Euler product, see Proposition 3.11. The first claim in 2) can be seen in a more general setting from [Neu92, (VII, 10.5)] since χ_L is the non-trivial Galois character of E/K . A direct proof is immediate if we analyse the behaviour of a prime $\mathfrak{p} \in \Omega_{\text{fin}}(K)$ in the field extension E/K , and use the Euler product representation of the Dedekind zeta function. Since E/K is a quadratic extension, the following cases can occur:

1. $\mathfrak{p}$ is ramified: let $\mathfrak{P} \in \Omega_{\text{fin}}(E)$ denote the prime dividing $\mathfrak{p}$ in $\mathfrak{o}_E$. Then $\mathfrak{p} = \mathfrak{P}^2$ and $\mathfrak{N}(\mathfrak{P}) = \mathfrak{N}(\mathfrak{p})$, hence $\mathfrak{P}$ and $\mathfrak{p}$ yield the same factor in ζ_E and ζ_K , respectively, and cancel out.
2. $\mathfrak{p}$ is split: then $\mathfrak{p} = \mathfrak{P}_1 \mathfrak{P}_2$ for some $\mathfrak{P}_i \in \Omega_{\text{fin}}(E)$ with $\mathfrak{N}(\mathfrak{P}_i) = \mathfrak{N}(\mathfrak{p})$, $i = 1, 2$. The resulting contribution to $\zeta_E(s)/\zeta_K(s)$ is $(1 - \mathfrak{N}(\mathfrak{p})^{-s})^{-1}$.
3. $\mathfrak{p}$ is inert: then $\mathfrak{N}(\mathfrak{p}) = \mathfrak{N}(\mathfrak{p})^2$, and the contribution to $\zeta_E(s)/\zeta_K(s)$ is $(1 + \mathfrak{N}(\mathfrak{p})^{-s})^{-1}$.

The second claim in 2) follows from part 3) of Proposition 3.11. ■

3.2 Evaluation of the mass formula

Whenever χ_L is the trivial character, we know from Theorem 3.12 and Proposition 3.13 that $L(\chi_L, s)$ satisfies a functional equation and takes rational values at negative integers. We can now prove a similar result for the case where χ_L is non-trivial:

Definition and Remark 3.20. Let $E \supseteq K$ be a finite extension of the number field K . Let $\mathfrak{O}$ and $\mathfrak{o}$ denote the rings of integers of E and K , respectively. The *relative discriminant* $\mathfrak{d}_{E/K}$ is the ideal of $\mathfrak{o}$ generated by all the discriminants $\det(\mathrm{tr} \alpha_i \alpha_j)_{1 \leq i, j \leq t}$, where $(\alpha_1, \dots, \alpha_t) \in \mathfrak{O}^t$ and $t = [E : K]$. Similarly one defines the relative discriminant $\mathfrak{d}_{E_p/K_p}$ of the completions at a place $\mathfrak{p}$.

We have $\mathfrak{d}_{E/K} = \prod_{\mathfrak{p}} \mathfrak{d}_{E_p/K_p}$. A prime ideal $\mathfrak{p}$ of K ramifies in E if and only if $\mathfrak{p} \mid \mathfrak{d}_{E/K}$. In particular, the field extension $E \supseteq K$ is unramified if and only if $\mathfrak{d}_{E/K} = (1)$. For more details we refer to [Neu92, III, §2].

We can use the functional equation of the Dedekind zeta function to rewrite the mass formula in such a way that the zeta function only needs to be evaluated at negative integers $s \leq -1$, where it takes rational values:

Lemma 3.21. Assume $\mathrm{disc}(KL) \notin (K^*)^2$. Set $\Delta := d_E/d_K$.

1. Then $\Delta = d_K \mathfrak{N}(\mathfrak{d}_{E/K})$.
2. If E is totally real and $s \geq 2$ is an even integer, then

$$L(\chi_L, 1-s) = L(\chi_L, s) \cdot \Delta^{s-\frac{1}{2}} \left((-1)^{\frac{s}{2}} 2^{1-s} \pi^{-s} (s-1)! \right)^n.$$

3. If E is totally imaginary and $s \geq 3$ is an odd integer, then

$$L(\chi_L, 1-s) = L(\chi_L, s) \cdot |\Delta|^{s-\frac{1}{2}} \left((-1)^{\frac{(1-s)}{2}} 2^{1-s} \pi^{-s} (s-1)! \right)^n.$$

Proof. 1) The identity $d_E = d_K^{[E:K]} \mathfrak{N}(\mathfrak{d}_{E/K})$ is shown in [Neu92, III, (2.10)]. Since $[E : K] = 2$, $\Delta = d_E/d_K = d_K \mathfrak{N}(\mathfrak{d}_{E/K})$.

Claim 2) follows directly from Proposition 3.13 together with part 2) of Proposition 3.19.

Let us now show 3). By Proposition 3.19, we can evaluate the meromorphic continuation of $L(\chi_L, \cdot)$ at s and at $1-s$ and use the functional equation of the completed zeta function to arrive at a comparison of the two values:

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$$\begin{aligned}
\frac{Z_E(s)}{Z_K(s)} &= L(\chi_L, s) \cdot |\Delta|^{\frac{s}{2}} \pi^{-\frac{ns}{2}} 2^{(1-s)n} \Gamma\left(\frac{s}{2}\right)^{-n} \Gamma(s)^n \\
&= \lim_{z \rightarrow 1-s} \frac{Z_E(1-s)}{Z_K(1-s)} = L(\chi_L, 1-s) \cdot |\Delta|^{\frac{1-s}{2}} \pi^{-\frac{n(1-s)}{2}} 2^{sn} \left(\lim_{z \rightarrow 1-s} \frac{\Gamma(z)}{\Gamma\left(\frac{z}{2}\right)} \right)^n \\
&= L(\chi_L, 1-s) \cdot |\Delta|^{\frac{1-s}{2}} \pi^{-\frac{n(1-s)}{2}} 2^{sn} 2^{-n} \left(\lim_{z \rightarrow 1-s} \frac{\Gamma(z)(z-(1-s))}{\Gamma\left(\frac{z}{2}\right)\left(\frac{z}{2}-\frac{1-s}{2}\right)} \right)^n \\
&= L(\chi_L, 1-s) \cdot |\Delta|^{\frac{1-s}{2}} \pi^{-\frac{n(1-s)}{2}} 2^{n(s-1)} \left(\lim_{z \rightarrow 1-s} \frac{\text{Res}_{1-s}(\Gamma)}{\text{Res}_{\frac{1-s}{2}}(\Gamma)} \right)^n \\
&\stackrel{(\star)}{=} L(\chi_L, 1-s) \cdot |\Delta|^{\frac{1-s}{2}} \pi^{\frac{n(s-1)}{2}} 2^{n(s-1)} \left((-1)^{\frac{s-1}{2}} \frac{\left(\frac{s-1}{2}\right)!}{(s-1)!} \right)^n
\end{aligned}$$

To see the equality marked with $(\star)$, we note that the Gamma function has simple poles at all negative integers k , with residues given by $\text{Res}_{z=-k}(\Gamma(z)) = \frac{(-1)^k}{k!}$.

The above calculations give the identity

$$L(\chi_L, 1-s) = L(\chi_L, s) \cdot |\Delta|^{s-\frac{1}{2}} \pi^{-ns+\frac{n}{2}} \cdot 2^{2n(1-s)} (-1)^{\frac{n(1-s)}{2}} \left(\frac{(s-1)!}{\Gamma\left(\frac{s}{2}\right)} \cdot \frac{(s-1)!}{\left(\frac{s-1}{2}\right)!} \right)^n.$$

Since s is odd, $\Gamma\left(\frac{s}{2}\right) = \frac{(s-1)!}{2^{s-1}\left(\frac{s-1}{2}\right)!} \sqrt{\pi}$, so

$$L(\chi_L, 1-s) = L(\chi_L, s) \cdot |\Delta|^{s-\frac{1}{2}} \cdot \left((-1)^{\frac{(1-s)}{2}} 2^{1-s} \pi^{-s} (s-1)! \right)^n.$$

■

Theorem 3.22 (KLINGEN-SHINTANI). $L(\chi_L, -s) \in \mathbb{Q}$ for any $1 \leq s \in \mathbb{N}$.

This was shown by KLINGEN in [Kli62] for the case where the class group and the narrow class group of K coincide, and generalised by SHINTANI to arbitrary number fields in [Shi76].

3.2.3 The mass formula for even lattices

We can now calculate the mass of even unimodular lattices as a product of rational numbers. We recall that being “even” is an exceptional case – in general, the norm ideal $\mathfrak{n}(L_{\mathfrak{p}})$ at a dyadic prime $\mathfrak{p}$ can be any of the $\nu_{\mathfrak{p}}(2) + 1$ ideals $\mathfrak{s}(L_{\mathfrak{p}}) \supseteq \cdots \supseteq 2\mathfrak{s}(L_{\mathfrak{p}})$. The very existence of even unimodular lattices in quadratic spaces over general number fields is an exception.

Lemma 3.23. We note some identities used in the proof of the next proposition.

1. Let $m \geq 3$ be odd and $s = \frac{m-1}{2}$. Then

$$\prod_{k=1}^m \Gamma\left(\frac{k}{2}\right) \prod_{i=1}^s \Gamma(2i)^{-1} = 2^{-s^2} \pi^{\frac{s+1}{2}}.$$

2. Let $m \geq 4$ be even and $s = \frac{m}{2} - 1$.

$$\prod_{k=1}^m \Gamma\left(\frac{k}{2}\right) \prod_{i=1}^s \Gamma(2i)^{-1} \Gamma(s+1)^{-1} = 2^{-\frac{m^2}{4} + m - 1} \pi^{\frac{m}{4}}.$$

Proof. These are immediate consequences of the duplication formula for the complex Gamma function,

$$\Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \Gamma(2z) \sqrt{\pi}.$$

■

Proposition 3.24. Assume L is an even unimodular lattice, and $m = \text{rank}(L) > 2$. Let $s := \frac{m}{2} - 1$ and $E = K(\sqrt{\text{disc}(KL)})$ if m is even, and $s := \frac{m-1}{2}$ if m is odd.

1. If m is odd, then

$$\text{Mass}(L) = 2^{-nm+n} (-1)^{\frac{ns(s+1)}{2}} \prod_{i=1}^s \zeta_K(1-2i).$$

2. If m is even and $\text{disc}(KL) \in (K^*)^2$, then

$$\text{Mass}(L) = 2^{-\frac{nm}{2}} \zeta_K(-s) \prod_{i=1}^s \zeta_K(1-2i).$$

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3. m is even and $\text{disc}(KL) \notin (K^*)^2$, then

$$\text{Mass}(L) = 2^{-\frac{nm}{2}} L(\chi_L, -s) \prod_{i=1}^s \zeta_K(1-2i).$$

All the factors occuring here are rational numbers.

Proof. We set $\ell_{L,m} := L(\chi_L, \frac{m}{2})$ if m is even, and $\ell_{L,m} := 1$ if m is odd. Then the mass formula from Theorem 3.5 can be rewritten using Theorem 3.8, Proposition 3.13 of Proposition 3.11, the result from Corollary 3.18, and the identity $\prod_{\mathfrak{p}} \mathfrak{N}(\mathfrak{p})^{m e_{\mathfrak{p}}} = 2^{nm}$. We obtain

$$\begin{aligned} \text{Mass}(L) &= 2^n d_K^{\frac{m(m-1)}{4}} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(L) \\ &= 2^{n-nm} d_K^{\frac{m(m-1)}{4}} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \ell_{L,m} \prod_{i=1}^s \zeta_K(2i) \\ &= 2^{n-nm} d_K^{\frac{m(m-1)}{4}} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \ell_{L,m} \prod_{i=1}^s \left((-1)^i 2^{-n} d_K^{\frac{1}{2}-2i} (2\pi)^{2in} \Gamma(2i)^{-n} \zeta_K(1-2i) \right). \end{aligned}$$

First assume m is odd (so we have $s = \frac{m-1}{2}$). Then we employ the identity from part 1) of Lemma 3.23 and the mass formula simplifies in the way we claimed.

Now assume that m is even.

1. Assume $\text{disc}(KL) \in (K^*)^2$. Then $m \equiv 0 \pmod{4}$ because -1 is not a square in the totally real field K (see the definition of the discriminant in Definition 2.53). By Proposition 3.13,

$$\ell_{L,m} = \zeta_K(-s) d_K^{\frac{1-m}{2}} \left((-1)^{\frac{nm}{2}} 2^{\frac{m}{2}-1} \pi^{\frac{m}{2}} \frac{1}{\Gamma(s+1)} \right)^n.$$

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2. Assume $\text{disc}(KL) \notin (K^*)^2$. We set $\Delta := d_E/d_K$. By part 1) of Lemma 3.21, $|\Delta| = d_E/d_K = d_K \mathfrak{N}(\mathfrak{d}_{E/K})$. And $\mathfrak{N}(\mathfrak{d}_{E/K}) = 1$ because the field extension E/K is unramified, cf. Lemma 3.16 and Remark 3.20. Now using parts 2) and 3) of Lemma 3.21, we see that

$$\ell_{L,m} = L(\chi_L, -s) d_K^{\frac{1-m}{2}} \left((-1)^\beta 2^{\frac{m}{2}-1} \pi^{\frac{m}{2}} \frac{1}{\Gamma(s+1)} \right)^n \text{ where } \beta = \begin{cases} \frac{-s}{2}, & m \equiv 2 \pmod{4} \\ \frac{m}{4}, & m \equiv 0 \pmod{4} \end{cases}$$

In all of the above cases for even m , we can use the identity from part 2) of Lemma 3.23 to eliminate the Gamma functions, and the claim follows after simplification of the resulting formula.

Finally, the claim about the factors being rational numbers follows from Theorem 3.12 and Theorem 3.22. ■

3.3 The mass of single-class genera

We will now see two important applications of the mass formula to single-class lattices. First, there is a bound $\text{sdb}(m)$ to the discriminant of totally real number fields K that allow the existence of single-class, square-free, lattices L of rank m in regular, totally definite vector spaces V over K . And second, given K and m , we can give an upper bound $\text{snb}(K, m)$ on the norm of prime ideals dividing the volume ideal of a single-class, square-free $\mathfrak{o}$ -lattice of rank m in a regular, totally definite quadratic space V over the base field K . In Chapter 5, we will see how to remove the restriction to square-free lattices and complete the classification.

Proposition 3.25. Let V denote a regular, totally definite quadratic space over a totally real number field K . And assume L is a square-free $\mathfrak{o}$ -lattice of rank m in V . Then

1. There is a lower bound on the local factors $\lambda_{\mathfrak{p}}$ of the mass formula, and it grows with $\mathfrak{N}(\mathfrak{p})$. When $2 \notin \mathfrak{p}$,

$$\lambda_{\mathfrak{p}}(L) \geq \frac{3}{8} \mathfrak{N}(\mathfrak{p})^{\frac{m-1}{2}}, \text{ if } m = 3 \text{ or } m \geq 5, \text{ and } \lambda_{\mathfrak{p}}(L) \geq \frac{9}{32} \mathfrak{N}(\mathfrak{p})^{\frac{3}{2}}, \text{ if } m = 4.$$

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2. There is a bound on the mass of L :

$$\text{Mass}(L) \geq \begin{cases} d_K^{\frac{m(m-1)}{4}} \cdot 2^{n-nm} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n & \text{for } m \geq 5 \\ d_K^{\frac{m(m-1)}{4}} \cdot 2^{n-nm} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \cdot \left(\frac{8}{9} \right)^{\#\{\mathfrak{p}|2\mathfrak{o}, \mathfrak{N}(\mathfrak{p})=2\}} & \text{for } m = 4 \\ d_K^{\frac{m(m-1)}{4}} \cdot 2^{n-nm} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \cdot \left(\frac{2}{3} \right)^{\#\{\mathfrak{p}|2\mathfrak{o}, \mathfrak{N}(\mathfrak{p})=2\}} & \text{for } m = 3 \end{cases}$$

Proof. Let $\mathfrak{p}$ be any prime ideal of $\mathfrak{o}$. Take a modular decomposition $L_{\mathfrak{p}} = L_1 \perp L_2$ with $\text{rank}(L_i) = m_i$ ($i = 1, 2$). By Theorem 3.8 (and with the notation L_i^* taken from there), for $i = 1, 2$ we have:

$$d_{\mathfrak{p}}(L_i^*, L_i^*) \begin{cases} = \mathfrak{N}(\mathfrak{p})^{2e_{\mathfrak{p}}}(1 + \mathfrak{N}(\mathfrak{p})^{-1}), & \text{if } L_i^* \text{ is anisotropic, even, and } m_i = 2 \\ \leq \mathfrak{N}(\mathfrak{p})^{e_{\mathfrak{p}}m_i}, & \text{otherwise} \end{cases}$$

The local factors $\lambda_{\mathfrak{p}}$ can then be approximated as follows. Note that the case $m = 4$ is special, for then L_1^* and L_2^* might both be even, anisotropic lattices of rank 2. By the second part of Theorem 3.8:

$$\begin{aligned} \lambda_{\mathfrak{p}}(L) &= \frac{\mathfrak{N}(\text{disc}(L_{\mathfrak{p}}))^{\frac{m+1}{2}}}{d_{\mathfrak{p}}(L, L)} \geq \mathfrak{N}(\mathfrak{p})^{m_2(\frac{m+1}{2})} \cdot \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{-\left(\frac{m_2(m_2+1)}{2}\right)} \frac{1}{d_{\mathfrak{p}}(L_1^*, L_1^*)d_{\mathfrak{p}}(L_2^*, L_2^*)} \\ &= \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m_1m_2}{2}} \frac{1}{d_{\mathfrak{p}}(L_1^*, L_1^*)d_{\mathfrak{p}}(L_2^*, L_2^*)} \\ &\geq \begin{cases} \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m_1m_2}{2}} \mathfrak{N}(\mathfrak{p})^{-me_{\mathfrak{p}}}(1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-1}, & m \neq 4 \\ \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m_1m_2}{2}} \mathfrak{N}(\mathfrak{p})^{-me_{\mathfrak{p}}}(1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-2}, & m = 4 \end{cases} \end{aligned}$$

We quickly see that for $m \neq 4$ and $\mathfrak{p} \nmid 2$, the local factors are

$$\lambda_{\mathfrak{p}}(L) \geq \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m-1}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-1} \geq \frac{3}{8} \mathfrak{N}(\mathfrak{p})^{\frac{m-1}{2}} > 1,$$

while for $m = 4$ and $\mathfrak{p} \nmid 2$,

$$\lambda_{\mathfrak{p}}(L) \geq \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m-1}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-2} \geq \frac{9}{32} \mathfrak{N}(\mathfrak{p})^{\frac{3}{2}} > 1.$$

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This proves part 1) of the claim. For part 2), we further have to examine the total contribution of the dyadic primes to $\text{Mass}(L)$. For $m \neq 4$,

$$\begin{aligned} \prod_{\mathfrak{p}|2\mathfrak{o}} \lambda_{\mathfrak{p}}(L) &= \prod_{\mathfrak{p}|2\mathfrak{o}} \frac{\mathfrak{N}(\text{disc}(L_{\mathfrak{p}}))^{\frac{m+1}{2}}}{d_{\mathfrak{p}}(L, L)} \geq \prod_{\mathfrak{p}|2\mathfrak{o}} \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m-1}{2}} \mathfrak{N}(\mathfrak{p})^{-me_{\mathfrak{p}}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-1} \\ &= \mathfrak{N}(2\mathfrak{o})^{-m} \prod_{\mathfrak{p}|2\mathfrak{o}} \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m-1}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-1} \geq \begin{cases} 2^{-3n} \left(\frac{2}{3}\right)^{\#\{\mathfrak{p}|2\mathfrak{o}, \mathfrak{N}(\mathfrak{p})=2\}}, & \text{if } m = 3 \\ 2^{-mn}, & \text{if } m \geq 5 \end{cases} \end{aligned}$$

We need to be slightly more careful to show a similar result for the case $m = 4$. Setting $\beta \in \{0, 1, 2\}$ to be the number of lattices in $\{L_1^*, L_2^*\}$ which are anisotropic, even, and also fulfill $\text{rank}(L_i) = 2$, we first consider the case where either $\beta \neq 2$ or $\mathfrak{N}(\mathfrak{p}) > 2$. Then,

$$\lambda_{\mathfrak{p}}(L) = \frac{\mathfrak{N}(\text{disc}(L_{\mathfrak{p}}))^{\frac{m+1}{2}}}{d_{\mathfrak{p}}(L, L)} \geq \mathfrak{N}(\mathfrak{p})^{-4e_{\mathfrak{p}}} \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m_1 m_2}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-\beta}.$$

Now if $\mathfrak{N}(\mathfrak{p}) = 2^r$ for some $r > 1$, then $\frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m_1 m_2}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-\beta} \geq \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{4-1}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-\beta} > 1$ for every $\beta \in \{0, 1, 2\}$. Next, assume $\mathfrak{p} | 2\mathfrak{o}$ is a prime ideal of norm $\mathfrak{N}(\mathfrak{p}) = 2$.

1. If $\beta = 0$, then $\frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{4-1}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-\beta} = \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{4-1}{2}} = \sqrt{2} > 1$.
2. If $\beta = 1$, then $\frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{2-2}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-\beta} = \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{2-2}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-1} \geq \frac{4}{3} > 1$.

The last remaining case is when $\mathfrak{N}(\mathfrak{p}) = 2$ and also $\beta = 2$. In this case, we can calculate the exact value of $\lambda_{\mathfrak{p}}(L)$ because L_1 and L_2 are even and anisotropic:

$$\lambda_{\mathfrak{p}}(L) = \frac{\mathfrak{N}(\text{disc}(L_{\mathfrak{p}}))^{\frac{m+1}{2}}}{d_{\mathfrak{p}}(L, L)} = \mathfrak{N}(\mathfrak{p})^{-4e_{\mathfrak{p}}} \frac{1}{2} \mathfrak{N}(\mathfrak{p})^{\frac{m_1 m_2}{2}} (1 + \mathfrak{N}(\mathfrak{p})^{-1})^{-2} = \frac{8}{9} \mathfrak{N}(\mathfrak{p})^{-4e_{\mathfrak{p}}}$$

In the worst case there will be n prime ideals over 2, each being of norm 2, so the best general estimate for the contribution of the dyadic primes to $\prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(L)$ is:

$$\prod_{\mathfrak{p}|2\mathfrak{o}} \lambda_{\mathfrak{p}}(L) \geq \prod_{\mathfrak{p}|2\mathfrak{o}} \frac{8}{9} \mathfrak{N}(\mathfrak{p})^{-4e_{\mathfrak{p}}} = 2^{-4n} \cdot \left(\frac{8}{9}\right)^n$$

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All in all, we have shown:

$$\prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(L) \geq \begin{cases} 2^{-nm} \left(\frac{2}{3}\right)^{\#\{\mathfrak{p}|2\mathfrak{o}, \mathfrak{N}(\mathfrak{p})=2\}}, & m = 3 \\ 2^{-nm} \left(\frac{8}{9}\right)^{\#\{\mathfrak{p}|2\mathfrak{o}, \mathfrak{N}(\mathfrak{p})=2\}}, & m = 4 \\ 2^{-nm}, & m \geq 5. \end{cases}$$

Part 2) of the claim now follows directly from Theorem 3.5. ■

Proposition 3.26. Let K denote any totally real number field and set $n := [K : \mathbb{Q}]$.

1. Lattices L of rank $3 \leq m \leq 14$ with class-number $h(L) \in \{1, 2\}$ do not exist in any totally definite quadratic K -vector space, except when $\sqrt[n]{d_K} \leq \text{sdb}(m)$, where $\text{sdb}(m)$ takes the following values:

m	3	4	5	6	7	8	9	10	11	12	13	14
$\text{sdb}(m)$	24.12	12.07	8.37	6.52	5.33	4.50	3.89	3.42	3.06	2.76	2.52	2.31

2. For $m \geq 15$ and $K \neq \mathbb{Q}$, there are no totally definite lattices L with class-number $h(L) \in \{1, 2\}$ and $\text{rank}(L) \geq m$ over K .
3. For $m \geq 11$, there are no single-class genera of definite lattices of rank m over $\mathbb{Q}$.

Proof. Let L be any lattice of class-number $h(L) \in \{1, 2\}$ and of rank $m \geq 3$ in a totally definite, regular quadratic space V over K . Of course, $\text{Mass}(L) \leq \frac{1}{2} + \frac{1}{2} = 1$ (see Remark 3.4). But then, Proposition 3.25 gives a bound on the root discriminant of K :

$$\sqrt[n]{d_K} \leq \left[2^{m-1} \left(\prod_{i=1}^m \frac{\pi^{i/2}}{\Gamma(i/2)} \right) \cdot \gamma \right]^{\frac{4}{m(m-1)}}$$

with $\gamma = 1$ if $m \geq 5$, $\gamma = \frac{9}{8}$ if $m = 4$, and $\gamma = \frac{3}{2}$ if $m = 3$.

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For $m \in \{3, \dots, 14\}$, the above bound, evaluated numerically, takes values less than the numbers given in the table above. This proves 1). Further, it is strictly decreasing as a function in m . To see this, we rewrite it (for $m \geq 5$, where $\gamma = 1$) as

$$\sqrt[m]{16} \left(\pi^{\frac{m+1}{m-1}} \right) \prod_{i=1}^m \Gamma(i/2)^{\frac{-4}{m(m-1)}}$$

and do a very crude approximation: for $m = 141$, $\prod_{i=1}^m \Gamma(i/2) \geq 2^{(m-1)m}$, so then $\sqrt[m]{d_K} < 1$. The same holds by induction for any $m > 141$, for $\Gamma((m+1)/2) \geq 8^{m/2}$ for these m . For $15 \leq m \leq 140$, it is quickly checked that the bound above takes values less than $2.23 < \sqrt{5}$. It was shown in [Odl90] that $\sqrt{5}$ is a lower bound on the root discriminant $\sqrt[m]{d_K}$ of any number field $K \neq \mathbb{Q}$, which proves 2).

3) was shown by WATSON in [Wat62]. ■

Remark 3.27. A list of all totally real number fields K with root discriminant $\sqrt[m]{d_K} \leq 14$, up to isomorphism, can be found in the tables [Voi08] computed by VOIGHT. By Proposition 3.26, all the possible base fields for single-class genera of rank $m \geq 4$ can be found in these tables.

Remark 3.28. For larger root determinants $\sqrt[m]{d_K} > 14$, a similar computation is not presently feasible. The bound of 24.12 we obtained for the case $m = 3$ is indeed hopelessly large. The case $m = 3$ can be solved by a different approach which is unique to this dimension. In [KL16], the author and M. KIRSCHMER used the correspondence of BRZEZINSKI-PETERS-EICHLER-BRANDT to enumerate ternary genera of lattices of class-number 1 and 2. In Section 3.4, we will review some of the results published there, and obtain an effective bound to $\sqrt[m]{d_K}$.

Corollary 3.29. Let $m \geq 3$, and K any totally real number field. Then there is a bound $\text{snb}(K, m)$, such that if L is a single-class, and square-free $\mathfrak{o}$ -lattice of rank m in a regular, totally definite quadratic K -vector space, then $L_{\mathfrak{p}}$ is modular for every prime $\mathfrak{p}$ of $\mathfrak{o}$ with $\mathfrak{N}(\mathfrak{p}) > \text{snb}(K, m)$.

Proof. Let L be any single-class, totally definite lattice of rank $m \geq 3$ over K . Let $n := [K : \mathbb{Q}]$. Again, $\text{Mass}(L) \leq \frac{1}{2}$. By part 1) of Proposition 3.25, $\lambda_{\mathfrak{p}}(L) > 1$ for all the non-dyadic $\mathfrak{p}$. So we have

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$$\begin{aligned} \frac{1}{2} \geq \text{Mass}(L) &\geq 2^n d_K^{\frac{m(m-1)}{4}} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \lambda_{\mathfrak{p}}(L) \prod_{\mathfrak{q}|2\mathfrak{o}} \lambda_{\mathfrak{q}}(L) \\ &\geq 2^{n-nm} \gamma_m \cdot d_K^{\frac{m(m-1)}{4}} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \lambda_{\mathfrak{p}}(L) \end{aligned}$$

with $\gamma_3 = \left(\frac{2}{3}\right)^{\#\{\mathfrak{q}|2\mathfrak{o}, \mathfrak{N}(\mathfrak{q})=2\}}$, $\gamma_4 = \left(\frac{8}{9}\right)^{\#\{\mathfrak{q}|2\mathfrak{o}, \mathfrak{N}(\mathfrak{q})=2\}}$, and $\gamma_m = 1$ if $m \geq 5$.

Now putting in the lower bound for $\lambda_{\mathfrak{p}}(L)$ from part 1) of Proposition 3.25, it follows that

$$\frac{1}{2} \geq 2^{n-nm} \tilde{\gamma}_m \cdot d_K^{\frac{m(m-1)}{4}} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \mathfrak{N}(\mathfrak{p})^{\frac{m-1}{2}}$$

with $\tilde{\gamma}_3 = \frac{3}{8} \left(\frac{2}{3}\right)^{\#\{\mathfrak{q}|2\mathfrak{o}, \mathfrak{N}(\mathfrak{q})=2\}}$, $\tilde{\gamma}_4 = \frac{9}{32} \left(\frac{8}{9}\right)^{\#\{\mathfrak{q}|2\mathfrak{o}, \mathfrak{N}(\mathfrak{q})=2\}}$, and $\tilde{\gamma}_m = \frac{3}{8}$ for $m \geq 5$.

So finally,

$$\mathfrak{N}(\mathfrak{p}) \leq \text{snb}(K, m) := \left[2^{n-nm+1} \tilde{\gamma}_m \cdot d_K^{\frac{m(m-1)}{4}} \left(\prod_{k=1}^m \frac{\Gamma(k/2)}{\pi^{k/2}} \right)^n \right]^{\frac{-2}{m-1}}.$$

■

In following, we give a mass estimate that will be needed for the descent algorithm in Chapter 5:

Lemma 3.30. Let L and M be two lattices in the same quadratic space over K , and let $\mathfrak{p}$ be a prime ideal of K such that $L_{\mathfrak{q}} = M_{\mathfrak{q}}$ for all places $\mathfrak{q} \neq \mathfrak{p}$. Then

$$\frac{\text{Mass}(M)}{\text{Mass}(L)} = \frac{\lambda_{\mathfrak{p}}(L')}{\lambda_{\mathfrak{p}}(L)}.$$

3.4 Base fields in the ternary case

Proof. Since L and M lie in the same quadratic space, the discriminants $\text{disc}(KL)$ and $\text{disc}(KM)$ are equal (modulo squares of K). This implies that the characters χ_L and χ_M are equal. But then, all the $\lambda_{\mathfrak{q}}$ (for $\mathfrak{q} \neq \mathfrak{p}$) are equal for L and M by Theorem 3.8. ■

Lemma 3.31. Let L be a lattice of rank $m \geq 3$ in a regular, totally definite quadratic space over a totally real number field K . Let $\mathfrak{p}$ be a non-dyadic prime ideal of K such that $L_{\mathfrak{p}}$ is unimodular. Assume we have another lattice M in the same quadratic space, with $M_{\mathfrak{q}} = L_{\mathfrak{q}}$ for all primes $\mathfrak{q} \neq \mathfrak{p}$. At $\mathfrak{p}$, let M have a Jordan decomposition $M_{\mathfrak{p}} = M_0 \perp M_2$, where M_0 is unimodular of rank m_0 and M_2 is $\mathfrak{p}^2$ -modular of rank m_2 .

Then

$$\frac{\text{Mass}(M)}{\text{Mass}(L)} \geq \frac{\mathfrak{N}(\mathfrak{p})^{m-1} d_{\mathfrak{p}}(L, L)}{2(1 + \mathfrak{N}(\mathfrak{p})^{-1})^{\beta}}$$

with $\beta = 2$ if $m = 4$ and $\beta = 1$ if $m \geq 3$ but $m \neq 4$. The value for $d_{\mathfrak{p}}(L, L)$ can be read from Theorem 3.8 to obtain a lower bound for the mass change $\text{Mass}(M)/\text{Mass}(L)$.

Proof. By Corollary 3.9, for the modular components M_i of $M_{\mathfrak{p}}$, we have $d_{\mathfrak{p}}(M_i^*) \leq (1 + \mathfrak{N}(\mathfrak{p})^{-1})$. By Lemma 3.30, $\text{Mass}(M)$ and $\text{Mass}(L)$ can only differ in the local factor at $\mathfrak{p}$. Then using Theorem 3.8,

$$\begin{aligned} \frac{\text{Mass}(M)}{\text{Mass}(L)} &= \frac{\lambda_{\mathfrak{p}}(M)}{\lambda_{\mathfrak{p}}(L)} = \mathfrak{N}(\mathfrak{p})^{m_2(m+1)} \frac{d_{\mathfrak{p}}(L, L)}{d_{\mathfrak{p}}(M, M)} \\ &\geq \frac{\mathfrak{N}(\mathfrak{p})^{m_2(m-m_2)} d_{\mathfrak{p}}(L, L)}{2d_{\mathfrak{p}}(M_0^*, M_0^*) d_{\mathfrak{p}}(M_2^*, M_2^*)} \geq \frac{\mathfrak{N}(\mathfrak{p})^{m-1} d_{\mathfrak{p}}(L, L)}{2(1 + \mathfrak{N}(\mathfrak{p})^{-1})^{\beta}} \end{aligned}$$

with $\beta = 2$ if $m = 4$ and $\beta = 1$ if $m \geq 3$ but $m \neq 4$. ■

3.4 Base fields in the ternary case

In light of Remark 3.28, some more work is needed to obtain an effective bound on the root discriminant of totally real number fields admitting single-class lattices of rank 3. The results in this section are part of a joint work by M. KIRSCHMER and the author and have previously been published in [KL16].¹

¹The two authors' respective contributions to [KL16] are disclosed in the publication list associated with this thesis.

3 The mass formula

In this section, K is always a totally real number field, and $\mathfrak{o}$ its ring of integers.

Definition 3.32. Given $a, b \in K^*$, let $\left(\frac{a,b}{K}\right)$ be the K -algebra with basis $(1, i, j, ij)$ satisfying the relations $i^2 = a, j^2 = b, ij = -ji$.

A *totally definite quaternion algebra* over K is a central simple K -algebra A of dimension 4, such that for all infinite places ν of K the completion A_ν is isomorphic to the real division algebra $\left(\frac{-1,-1}{\mathbb{R}}\right)$.

In particular, totally definite quaternion algebras only exist if the underlying number field is totally real. In the remainder of this section, $\mathcal{Q}$ will always be a totally definite quaternion algebra over K .

Definition 3.33. The algebra $\mathcal{Q}$ is said to be *ramified* at some place $\mathfrak{p}$ of K , if $\mathcal{Q} \otimes_K K_{\mathfrak{p}}$ is a skewfield. The *discriminant* $\mathcal{D}(\mathcal{Q})$ is the product of all the prime ideals $\mathfrak{p} \in \Omega_{\text{fin}}$ such that $\mathcal{Q}$ ramifies at $\mathfrak{p}$.

Definition 3.34. An *order* in $\mathcal{Q}$ is a subring of $\mathcal{Q}$ which is also an $\mathfrak{o}$ -lattice. Given an $\mathfrak{o}$ -lattice L , the sets $\mathcal{O}_l(L) = \{x \in \mathcal{Q} \mid xL \subseteq L\}$ and $\mathcal{O}_r(L) = \{x \in \mathcal{Q} \mid Lx \subseteq L\}$ are orders in $\mathcal{Q}$. They are called the *left* and *right* orders of L , respectively.

An $\mathfrak{o}$ -lattice L is called an (invertible) *right $\mathcal{O}$ -ideal* if for each prime ideal $\mathfrak{p}$ there exists some $x \in \mathcal{O}_{\mathfrak{p}}^*$ such that $L_{\mathfrak{p}} = x_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$. Note that if L is a right $\mathcal{O}$ -ideal, then $\mathcal{O}_r(L) = \mathcal{O}$.

Definition and Remark 3.35. $\mathcal{Q}$ has a unique antiautomorphism

$$\bar{} : \mathcal{Q} \rightarrow \mathcal{Q}, r + si + tj + uij \mapsto r - si - tj - uij$$

with the property that the *reduced norm* $\text{nr} : x \mapsto x\bar{x}$ and the *reduced trace* $\text{tr} : x \mapsto x + \bar{x}$ are contained in K for all $x \in \mathcal{Q}$.

The reduced trace defines the subspace $\mathcal{Q}^0 = \{x \in \mathcal{Q} \mid \text{tr}(x) = 0\}$.

Definition and Remark 3.36. Let $\mathcal{O}$ be an order in $\mathcal{Q}$. The $\mathfrak{o}$ -ideal generated by

$$\{\det(\text{tr}(x_i \bar{x}_j))_{i,j} \mid x_1, \dots, x_4 \in \mathcal{O}\}$$

is always a square (cf. [Rei03, Corollary 25.10]). The square root of this ideal is the (reduced) *discriminant* $\mathcal{D}(\mathcal{O})$. Note that if $\mathcal{O} \subseteq \Lambda$, then $\mathcal{D}(\mathcal{O}) = \mathcal{D}(\Lambda) \cdot [\Lambda : \mathcal{O}]$.

Definition and Remark 3.37. Let $\mathcal{O}$ be an order in $\mathcal{Q}$.

1. $\mathcal{O}$ is called *maximal* if it is not properly contained in another order. If $\mathcal{O}$ is maximal, then $\mathcal{D}(\mathcal{O}) = \mathcal{D}(\mathcal{Q})$. So for every order, the ideal $\mathcal{N}(\mathcal{O}) = \mathcal{D}(\mathcal{O})\mathcal{D}(\mathcal{Q})^{-1}$ is integral.
2. $\mathcal{O}$ is called *hereditary* if one of the following equivalent conditions are met:
 - Every $\mathfrak{o}$ -lattice L with $\mathcal{O} = \mathcal{O}_r(L)$ is a right $\mathcal{O}$ -ideal.
 - The discriminant ideal $\mathcal{D}(\mathcal{O})$ is square-free.

The equivalence of these conditions is shown in [Rei03, Theorem 39.14].

3. $\mathcal{O}$ is called *Gorenstein* if the dual lattice $\mathcal{O}^\#$ is a right $\mathcal{O}$ -ideal.

Remark 3.38. Every maximal order in $\mathcal{Q}$ is hereditary (cf. [Rei03, Ch. 21]), and it is well known that every hereditary order is Gorenstein.

Remark 3.39. By [Kne02, (6.20)], the even part of the Clifford algebra of (V, Φ) is a definite quaternion algebra $\mathcal{Q}_V$. And its trace zero subspace $(\mathcal{Q}_V^0, \text{nr})$, equipped with the reduced norm nr , is isometric to a rescaling $(V, c \cdot \Phi)$ with some totally positive element $c \in K^*$. Since we only wish to classify similarity classes of ternary lattices, we can indeed identify the quadratic space (V, Φ) with the space $(\mathcal{Q}_V^0, \text{nr}) \subset (\mathcal{Q}_V, \text{nr})$.

Definition and Remark 3.40. Two right $\mathcal{O}$ -ideals I and J are isomorphic as right $\mathcal{O}$ -modules if and only if $I = xJ$ for some unit $x \in \mathcal{Q}^*$. By the Jordan-Zassenhaus theorem (cf. [Rei03, Theorem 26.4]), the set of all right $\mathcal{O}$ -ideals is a disjoint union of finitely many isomorphism classes. The number of isomorphism classes of right $\mathcal{O}$ -ideals is called the *ideal class number* $h(\mathcal{O})$ of $\mathcal{O}$. Two-sided $\mathcal{O}$ -ideals are said to be *isomorphic* if they are isomorphic as right $\mathcal{O}$ -ideals. Let $H(\mathcal{O})$ denote the number of isomorphism classes of two-sided $\mathcal{O}$ -ideals.

By the Skolem-Noether theorem, two orders $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic (as $\mathfrak{o}$ -algebras) if and only if they are conjugate in $\mathcal{Q}^*$. The *genus* $\text{Genus}(\mathcal{O})$ of $\mathcal{O}$ is the set of all orders $\mathcal{O}'$ such that $\mathcal{O}_{\mathfrak{p}}$ and $\mathcal{O}'_{\mathfrak{p}}$ are conjugate for all prime ideals $\mathfrak{p}$ of $\mathfrak{o}$. The number of conjugacy classes in $\text{Genus}(\mathcal{O})$ is called the *type number* $t(\mathcal{O})$.

The type and ideal class numbers of $\mathcal{O}$ are related by the following result.

3 The mass formula

Lemma 3.41. Let $\mathcal{O}$ and $\mathcal{O}'$ be orders in the same genus and let S be a set of representatives of the isomorphism classes of right $\mathcal{O}$ -ideals. Then

$$1 \leq H(\mathcal{O}) = \#\{I \in S \mid \mathcal{O}' \text{ is conjugate to } \mathcal{O}_l(I)\} \leq h(\mathcal{O}).$$

In particular, $1 \leq t(\mathcal{O}) \leq h(\mathcal{O})$.

Proof. See for example [Deu68, Section VI.8.2] or [KV10, Proposition 2.10]. ■

Theorem 3.42. Let K be a totally real number field.

There is a bijection between the set of equivalence classes of ternary lattices over K and the set of isomorphism classes of Gorenstein orders over K . Specifically, the maps

$$\mathfrak{D} : L \mapsto 1\mathfrak{o} + \sum_{x,y \in L} \text{nr}(L)^{-1} \cdot xy \quad \text{and} \quad \mathfrak{L} : \mathcal{O} \mapsto \mathcal{D}(\mathcal{O}) \cdot (\mathcal{O}^\# \cap \mathcal{Q}^0)$$

send ternary lattices to Gorenstein orders and vice versa:

- If L is a ternary $\mathfrak{o}$ -lattice in V , then $\mathfrak{D}(L)$ is a Gorenstein order in $\mathcal{Q}_V$. Note that $\mathfrak{D}(L)$ is well-defined because, by virtue of Remark 3.39, we view L as a subset of $(\mathcal{Q}_V^0, \text{nr})$.
- If $\mathcal{O}$ is any order in $\mathcal{Q}$, then $\mathfrak{L}(\mathcal{O})$ is a ternary lattice in $(\mathcal{Q}^0, \text{nr})$.

and are “almost” inverse to each other:

- If $\mathcal{O}$ is a Gorenstein order in $\mathcal{Q}$, then $\mathcal{O} = \mathfrak{D}(\mathfrak{L}(\mathcal{O}))$
- If L is a ternary lattice in V , then there exists a fractional $\mathfrak{o}$ -ideal $\mathfrak{a}$ such that $\mathfrak{a}L = \mathfrak{L}(\mathfrak{D}(L))$.

Finally, $\mathfrak{D}(L)$ and $\mathfrak{D}(L')$ are isomorphic if and only if L is isomorphic to $\mathfrak{a}L'$ for some fractional ideal $\mathfrak{a}$ of K .

Proof. It is shown in [Pet69, Satz 7] and [Brz83, Proposition 2.3] that $\mathfrak{D}(L)$ is indeed a Gorenstein order. For the other claims, see [Brz83, Proposition 3.2], which implies that $\mathfrak{D}(L) = \mathfrak{D}(\mathfrak{L}(\mathfrak{D}(L)))$, and then by [Eic52, Satz 14.1] it follows that L and $\mathfrak{L}(\mathfrak{D}(L))$ differ by a fractional ideal.

The last claim is proven in [Brz83, Corollary 3.10]. ■

3.4 Base fields in the ternary case

Remark 3.43. For every Gorenstein order $\mathcal{O}$, it is possible to specify a canonic order $\mathcal{O}' \supseteq \mathcal{O}$ which is hereditary and has type number $t(\mathcal{O}') \leq t(\mathcal{O})$. For details see [KL16, Lemma (5.1) and Lemma (5.4)]. And vice versa, every hereditary order is Gorenstein. So to classify fields that admit a Gorenstein order of type number t , it suffices to classify fields that admit a hereditary order $\leq t$.

Corollary 3.44. Let L be any ternary lattice over a totally real number field K . Then

$$h(L) = t(\mathfrak{D}(L)) .$$

In particular, for any $c \in \mathbb{N}$, the set of (isomorphism classes of) totally real number fields admitting definite Gorenstein quaternion orders of type number c and the set of (isomorphism classes of) totally real number fields admitting totally definite ternary lattices of class-number c is identical.

We have shown that it suffices to classify definite Gorenstein quaternion orders of type number 1 (up to isometry) to reach our goal of obtaining the list of similarity classes of totally definite ternary lattices of class-number 1. In the present work we will be content with determining the possible base fields, which were unobtainable with the previous results of this section (see Remark 3.28). Once the possible base fields are known, the classification of single-class lattices of rank $m = 3$ can be obtained with the algorithms stated in Chapter 5.

Effective bounds on the base fields can be obtained from a mass formula by EICHLER, given below. It requires hereditary orders, which is fine for us in light of Remark 3.43.

Definition 3.45. Let $\mathcal{O}$ be an order in $\mathcal{Q}$ and suppose that $I_1, \dots, I_{h(\mathcal{O})}$ represent the isomorphism classes of right $\mathcal{O}$ -ideals. By Lemma 3.41, we know that $t(\mathcal{O}) \leq h(\mathcal{O})$, so we may assume that

$$\mathcal{O}_l(I_1), \dots, \mathcal{O}_l(I_{t(\mathcal{O})})$$

represent the conjugacy classes of all orders in the genus of $\mathcal{O}$. Then the *mass* of $\mathcal{O}$ is defined as

$$\text{Mass}(\mathcal{O}) := \sum_{i=1}^{h(\mathcal{O})} \frac{1}{[\mathcal{O}_l(I_i)^* : \mathfrak{o}^*]} = \sum_{i=1}^{t(\mathcal{O})} \frac{H(\mathcal{O}_l(I_i))}{[\mathcal{O}_l(I_i)^* : \mathfrak{o}^*]} .$$

Definition 3.46. Let $\mathfrak{a}$ be any ideal of $\mathfrak{o}$. Then by $\omega(\mathfrak{a})$ we denote the number of prime ideals dividing $\mathfrak{a}$, and by $\omega_2(K)$ we denote the number of prime ideals of norm 2 of K .

3 The mass formula

Theorem 3.47 (Eichler's mass formula). Let $\mathcal{O}$ be a hereditary order in $\mathcal{Q}$. Recall that $h(K)$ denotes the class-number of K . Then

1. EICHLER's mass formula relates $\text{Mass}(\mathcal{O})$ to $\mathcal{D}(\mathcal{O})$:

$$\text{Mass}(\mathcal{O}) = 2^{1-n} \cdot |\zeta_K(-1)| \cdot h(K) \cdot \prod_{\mathfrak{p}|\mathcal{D}(\mathcal{Q})} (\mathfrak{N}_{K/\mathbb{Q}}(\mathfrak{p}) - 1) \prod_{\mathfrak{p}|\mathcal{N}(\mathcal{O})} (\mathfrak{N}_{K/\mathbb{Q}}(\mathfrak{p}) + 1).$$

2. The number $H(\mathcal{O})$ of isomorphism classes of two-sided $\mathcal{O}$ -ideals is:

$$H(\mathcal{O}) = 2^{\omega(\mathcal{D}(\mathcal{O}))} \cdot h(K) \cdot \frac{[\mathcal{O}^* : \mathfrak{o}^*]}{[\mathcal{N}_{\mathcal{Q}^*}(\mathcal{O}) : K^*]}.$$

Proof. Cf. [Eic55, Section 4]. ■

Theorem 3.48. If $\mathcal{O}$ is a hereditary order in $\mathcal{Q}$, then

$$d_K^{1/n} < ((t(\mathcal{O})/2)^{1/n} \cdot 4\pi^2 \cdot (3/2)^{\omega_2(K)/n})^{2/3}.$$

Proof. Let $\{\mathcal{O}_1, \dots, \mathcal{O}_{t(\mathcal{O})}\}$ be a set of representatives of the conjugacy classes in the genus of $\mathcal{O}$. By Theorem 3.47, we conclude that

$$\begin{aligned} t(\mathcal{O}) &\geq \sum_{i=1}^{t(\mathcal{O})} \frac{1}{[\mathcal{N}_{\mathcal{Q}^*}(\mathcal{O}_i) : K^*]} = \frac{1}{2^{\omega(\mathcal{D}(\mathcal{O}))} \cdot h(K)} \cdot \sum_{i=1}^{t(\mathcal{O})} \frac{H(\mathcal{O}_i)}{[\mathcal{O}_i^* : \mathfrak{o}^*]} = \frac{\text{Mass}(\mathcal{O})}{2^{\omega(\mathcal{D}(\mathcal{O}))} \cdot h(K)} \\ &= 2^{1-n} \cdot |\zeta_K(-1)| \prod_{\mathfrak{p}|\mathcal{D}(\mathcal{Q})} \frac{\mathfrak{N}_{K/\mathbb{Q}}(\mathfrak{p}) - 1}{2} \cdot \prod_{\mathfrak{p}|\mathcal{N}(\mathcal{O})} \frac{\mathfrak{N}_{K/\mathbb{Q}}(\mathfrak{p}) + 1}{2} \\ &\geq 2^{1-n} \cdot |\zeta_K(-1)| \cdot 2^{-\omega_2(K)}. \end{aligned}$$

The functional equation of the Dedekind zeta function (cf. Proposition 3.13) gives $|\zeta_K(-1)| = d_K^{3/2} \cdot \zeta_K(2)/(2\pi^2)^n$. Therefore,

$$\begin{aligned} t(\mathcal{O}) &\geq \frac{2d_K^{3/2}}{(2\pi)^{2n}} \cdot \zeta_K(2) \cdot 2^{-\omega_2(K)} > \frac{2d_K^{3/2}}{(2\pi)^{2n}} \cdot (4/3)^{\omega_2(K)} \cdot 2^{-\omega_2(K)} \\ &\geq \frac{2d_K^{3/2}}{(2\pi)^{2n}} \cdot (2/3)^{-\omega_2(K)} \end{aligned}$$

as claimed. ■

3.4 Base fields in the ternary case

Corollary 3.49. Let L be a single-class ternary lattice in a regular, totally definite quadratic space V over a totally real number field K with discriminant d_K and degree $n = [K : \mathbb{Q}]$.

Then $n \leq 8$ and $\sqrt[n]{d_K} < 15.20$.

Proof. By Corollary 3.44, it suffices to bound the set of totally real number fields admitting Gorenstein quaternion orders of type-number 1.

A bound $\sqrt[n]{d_K} < 15.20$ then follows directly from the above result:

$$\sqrt[n]{d_K} < \left(4\pi^2 \left(\frac{3}{2} \right)^{\frac{1}{n} \#\{\mathfrak{p} \mid 2\mathfrak{o}, \mathfrak{N}(\mathfrak{p})=2\}} \right)^{\frac{2}{3}} < 15.20.$$

Now it can be seen using the tables in [BD08] that the degree of K is bounded to $n \leq 10$. VOIGHT's tables contain all possible number fields of degree $n \leq 8$ with root discriminant < 15.5 . It remains to show that there are no possible candidates for K when $n \in \{9, 10\}$. If $n = 10$, then the tables in [BD08] show that $\sqrt[n]{d_K} > 15.20$ unless $\#\{\mathfrak{p} \mid 2\mathfrak{o}, \mathfrak{N}(\mathfrak{p}) = 2\} \leq 1$. Similarly for $n = 9$, these tables show that $\sqrt[n]{d_K} > 15.20$ unless $\#\{\mathfrak{p} \mid 2\mathfrak{o}, \mathfrak{N}(\mathfrak{p}) = 2\} = 0$. But plugging this knowledge about the number of primes of norm 2 back into the above formula yields $\sqrt[n]{d_K} < 11.60$ when $n = 9$, and $\sqrt[n]{d_K} < 11.92$ when $n = 10$. Turning back to the tables in [Voi08], there are no totally real number fields of degree $n = 10$ satisfying $\sqrt[n]{d_K} \leq 14$, and the field with the smallest root discriminant in the case $n = 9$ has $\sqrt[n]{d_K} > 12.86$. ■

Corollary 3.50. All of the possible base fields for the ternary case can be found in [Voi08], as these tables contain all the totally real number fields satisfying both $n \leq 8$ and $\sqrt[n]{d_K} \leq 15.5$. The number of these fields is 358.

Now that the possible base fields are known, the case $m = 3$ does not necessarily need to be treated differently from the case $m \geq 4$. All the algorithms outlined in the present thesis are designed to work in any dimension $m \geq 3$, but are less efficient in the ternary case than the algorithms previously published by M. KIRSCHMER and the author (cf. [KL16, Algorithm 4.5 and Remark 4.6]).

4 Enumerating a genus

In this chapter we will describe an algorithm for enumerating all the isometry classes in the genus of some lattice L . The first step is to recapitulate KNESER's $\mathfrak{p}$ -neighbours algorithm and to introduce the neighbour graph, $\mathcal{N}(L, \mathfrak{p})$. By the Strong Approximation Theorem for rotations, the neighbor graph always comprises full proper spinor genera inside $\text{Genus}(L)$. In fact, $\mathcal{N}(L, \mathfrak{p})$ is the union of either exactly one or exactly two proper spinor genera, and for any two lattices $L, L' \in \text{Genus}(L)$, there is always a prime $\mathfrak{p}$ such that $\mathcal{N}(L, \mathfrak{p}) = \text{spn}^+(L) \cup \text{spn}^+(L')$. This result is known as the “Spinor Linkage Theorem” and was originally shown by BENHAM and HSIA in [BH83].

However, the Spinor Linkage Theorem does not provide an explicit way of computing a set of primes covering the entirety of $\text{Genus}(L)$ via neighbour constructions. Based on an idea that was kindly shared with the author by Prof. WAI KIU CHAN at the AIM conference on algorithms for lattices and algebraic automorphic forms in 2013, we will identify the set of proper spinor genera in $\text{Genus}(L)$ with a quotient of some ray class group. We can then compute a set of primes such that iterated neighbour constructions at these primes cover the entire genus of L .

In the following, K is a totally real number field of degree $n = [K : \mathbb{Q}]$. Its ring of integers is $\mathfrak{o} = \mathfrak{o}_K$, and L will always be an integral $\mathfrak{o}$ -lattice in the regular (and not necessarily definite) quadratic K -vector space (V, Φ) of rank $m \geq 3$.

4.1 The neighbour method

We will first recall the essentials of KNESER's concept of $\mathfrak{p}$ -neighbours. The presentation here closely follows that of SCHIEMANN in [Sch98, Section 2].

Definition 4.1. Assume that $\mathfrak{p}$ is some prime ideal of $\mathfrak{o}$ such that the completion $L_{\mathfrak{p}}$ of L is unimodular. If $\mathfrak{p}$ is dyadic, further assume that $L_{\mathfrak{p}}$ is even (cf. Definition 3.6). Then:

4 Enumerating a genus

1. An integral $\mathfrak{o}$ -lattice $L' \subset V$ is called a $\mathfrak{p}$ -neighbour of L , if $L'_{\mathfrak{p}}$ is unimodular and

$$L/(L \cap L') \cong \mathfrak{o}/\mathfrak{p} \cong L'/(L \cap L')$$

as $\mathfrak{o}$ -modules.

2. Any vector $v \in L - \mathfrak{p}L$ with $\Phi(v, v) \in \mathfrak{p}^2$ (if $\mathfrak{p}$ is non-dyadic), or $\Phi(v, v) \in 2\mathfrak{p}^2$ (if $\mathfrak{p}$ is dyadic) is called $\mathfrak{p}$ -admissible for L .

For a $\mathfrak{p}$ -admissible vector v , the lattice

$$L(v) := L_v + \mathfrak{p}^{-1}v, \text{ where } L_v := \{x \in L \mid (x, v) \in \mathfrak{p}\}$$

is called the $\mathfrak{p}$ -neighbour of L with respect to v .

Proposition 4.2 ([Sch98, 2.2]). Let L be unimodular at $\mathfrak{p}$. If $\mathfrak{p}$ is dyadic, assume that $L_{\mathfrak{p}}$ is even. Then the set of $\mathfrak{p}$ -neighbours of L is equal to $\{L(v) \mid v \text{ is } \mathfrak{p}\text{-admissible for } L\}$.

Proof. Let L' be an integral $\mathfrak{o}$ -lattice in V which is a $\mathfrak{p}$ -neighbour of L . Then the minimal $a \in \mathbb{N}_0$ with $\mathfrak{p}^a L' \subset L$, which exists by Lemma 2.47, cannot be zero. The set $\mathfrak{p}^a L'$ must contain some vector $v \notin \mathfrak{p}L$ by the minimality of a . This vector v lies in $(L \cap \mathfrak{p}L') - \mathfrak{p}L$. Because L' is integral, this implies that $\Phi(v, v) \in \mathfrak{p}^2$. If $\mathfrak{p}$ is dyadic, we also see that $\Phi(v, v) \in \mathfrak{n}(L') = 2\mathfrak{s}(L')$. So v is $\mathfrak{p}$ -admissible.

We claim that $L' = L(v)$.

First note that if $y \in L' \cap L$, then $\Phi(v, y) \in \mathfrak{p}$ because $v \in \mathfrak{p}L'$ and L' is integral, so $y \in L_v$ by definition. Further, $L_v \subsetneq L$, for if $L_v = L$ then we would have $\Phi(L, v) \in \mathfrak{p}$ for the element $v \in L - \mathfrak{p}L$, so $\mathfrak{p} \mid \mathfrak{v}(L)$, a contradiction to L being unimodular at $\mathfrak{p}$. So $L \cap L' \subseteq L_v \subsetneq L$. As an $\mathfrak{o}$ -module, $L/(L \cap L') \cong \mathfrak{o}/\mathfrak{p}$ is simple, so it follows that $L_v = L \cap L'$. Consequently, we also have $L \cap L' \subsetneq L_v + \mathfrak{p}^{-1}v \subseteq L'$, so by the same argument, $L(v) = L'$.

We will now show the reverse inclusion. Let v be $\mathfrak{p}$ -admissible for L . We claim that $L(v)$ is a $\mathfrak{p}$ -neighbour of L .

First we note that as an $\mathfrak{o}$ -submodule of L , L_v is again an $\mathfrak{o}$ -lattice in V (see Definition 2.45). So $L(v)$ is also an $\mathfrak{o}$ -lattice in V . It is easy to verify that $L(v)$ is integral, and that $L(v)_{\mathfrak{q}} = L_{\mathfrak{q}}$ for all finite places $\mathfrak{p} \neq \mathfrak{q}$.

4.1 The neighbour method

Now as $\mathfrak{o}$ -modules,

$$\begin{aligned} L(v)/(L(v) \cap L) &= (L_v + \mathfrak{p}^{-1}v)/(L(v) \cap L) \cong (L + (L_v + \mathfrak{p}^{-1}v))/L \\ &= (L + \mathfrak{p}^{-1}v)/L \cong \mathfrak{p}^{-1}v/(L \cap \mathfrak{p}^{-1}v) = \mathfrak{p}^{-1}v/v \cong \mathfrak{p}^{-1}/\mathfrak{o} \cong \mathfrak{o}/\mathfrak{p}. \end{aligned}$$

Consider the homomorphism

$$L \rightarrow \mathfrak{o}/\mathfrak{p}, y \mapsto \Phi(y, v) + \mathfrak{p}.$$

Its kernel is L_v and it is surjective (because otherwise $\mathfrak{p} \mid \mathfrak{v}(L)$). As $L_v = L(v) \cap L$, $L/(L(v) \cap L)$ is also isomorphic to $\mathfrak{o}/\mathfrak{p}$ as an $\mathfrak{o}$ -module. Further, this shows that L_v is a maximal sublattice of L . But then its volume $\mathfrak{v}(L_v)$ differs from $\mathfrak{v}(L)$ by $\mathfrak{p}^2$. And $L(v) = \mathfrak{p}^{-1}v + L_v$ is integral and properly contains L_v , so $\mathfrak{v}(L(v)) = \mathfrak{o}$. So $L(v)$ is indeed unimodular at $\mathfrak{p}$.

If $\mathfrak{p}$ is dyadic, then a direct verification shows that $\Phi(av + w, av + w) \in 2\mathfrak{o}$ for all $a \in \mathfrak{p}^{-1}$ and $v \in L, w \in L_v$, so $\mathfrak{n}(L(v)_{\mathfrak{p}}) = 2\mathfrak{s}(L(v)_{\mathfrak{p}}) = 2\mathfrak{o}_{\mathfrak{p}}$.

All in all, we have shown that the lattice $L(v)$ is in fact a $\mathfrak{p}$ -neighbour of L . ■

Lemma 4.3. All the $\mathfrak{p}$ -neighbours of L lie in $\text{Genus}(L)$.

Proof. We have previously seen that any $\mathfrak{p}$ -neighbour of L can be written as some $L(v)$ with v as in part 2) of Definition 4.1, where $L(v)$ is again unimodular at $\mathfrak{p}$.

First assume $\mathfrak{p}$ is non-dyadic. The invariants for local isometry for the unimodular lattices $L_{\mathfrak{p}}$ and $L(v)_{\mathfrak{p}}$ are then determined purely by the enveloping space V , see [O'M73, § 92]. So $L(v)_{\mathfrak{p}} \cong L_{\mathfrak{p}}$. Now $L(v)$ lies in the same vector space as L by definition, and $L_{\mathfrak{q}} = L(v)_{\mathfrak{q}}$ at all primes $\mathfrak{q} \neq \mathfrak{p}$, so $L(v) \in \text{Genus}(L)$.

Now assume $\mathfrak{p}$ is dyadic, in which case our definition of a neighbour demands that $2\mathfrak{s}(L_{\mathfrak{p}}) = \mathfrak{n}(L_{\mathfrak{p}})$. We have seen (in the last part of the proof of Proposition 4.2) that $L(v)$ is an even unimodular lattice. By [O'M73, § 93 E], the local unimodular lattices $L_{\mathfrak{p}}$ and $L(v)_{\mathfrak{p}}$ are isometric if and only if their norm groups $\mathfrak{g}(L_{\mathfrak{p}})$ and $\mathfrak{g}(L(v)_{\mathfrak{p}})$ (as defined in [O'M73, § 93 A]) coincide. For any $\mathfrak{o}_{\mathfrak{p}}$ -lattice M , the norm group $\mathfrak{g}(M)$ always fulfills $2\mathfrak{s}(M) \subseteq \mathfrak{g}(M) \subseteq \mathfrak{n}(M)$, which in the case of even unimodular lattices implies $\mathfrak{g}(L_{\mathfrak{p}}) = \mathfrak{g}(L(v)_{\mathfrak{p}}) = 2\mathfrak{o}$. ■

Remark 4.4. At dyadic primes $\mathfrak{p}$ where L is not even, the norm group $\mathfrak{n}(L_{\mathfrak{p}})$ can change when passing to a $\mathfrak{p}$ -neighbour of L . In this case, not all $\mathfrak{p}$ -neighbours of L will lie in $\text{Genus}(L)$.

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To generate the set of $\mathfrak{p}$ -neighbours of L , it suffices to run through the elements of an m -dimensional vector space over the finite field $\mathfrak{o}/\mathfrak{p}$:

Definition 4.5. For any vector $v \in L$, let $[v]$ denote the class of v in the $\mathfrak{o}/\mathfrak{p}$ -vector space $L/\mathfrak{p}L$.

Lemma 4.6 ([Sch98, Section 3]). If v, w are both $\mathfrak{p}$ -admissible for L , then

- $L(v) = L(w)$ if and only if $[v] = [w]$, assuming $\mathfrak{p}$ is non-dyadic
- $L(v) = L(w)$ if and only if $[v] = [w]$ and $\Phi(v, w) \in \mathfrak{p}^2$, assuming $\mathfrak{p}$ is dyadic.

Proof. First assume $[v] = [w]$. In other words, there are some $\lambda, \mu \in \mathfrak{o}^*$ and $x, y \in \mathfrak{p}L$ such that $v = \lambda w + x$ and $w = \mu v + y$.

If $\mathfrak{p}$ is non-dyadic, we observe that $\Phi(v, v) = \lambda^2 \Phi(w, w) + 2\lambda \Phi(w, x) + \Phi(x, x) \in \mathfrak{p}^2$. Because $\Phi(w, w) \in \mathfrak{p}^2$ and $x \in \mathfrak{p}L$, this implies that $2\lambda \Phi(w, x) \in \mathfrak{p}^2$, hence $\Phi(w, x) \in \mathfrak{p}^2$. But we can rewrite $\Phi(v, w) = \Phi(\lambda w + x, w) = \lambda \Phi(w, w) + \Phi(x, w)$, so indeed $\Phi(v, w) \in \mathfrak{p}^2$.

Now take any vector $\nu v + z \in L(v)$ where $\nu \in \mathfrak{p}^{-1}$, $z \in L_v$. We have $\nu v + z = \nu \lambda w + \nu x + z$ and note that $\nu \lambda w \in \mathfrak{p}^{-1}w$, so it suffices to show that $\nu x + z \in L_w$. To see this, consider

$$\begin{aligned} \Phi(\nu x + z, w) &= \Phi(\nu x + z, \mu v + y) \equiv \mu \nu \Phi(x, v) \\ &\equiv \mu \nu \Phi(v - \lambda w, v) \pmod{\mathfrak{p}} = \mu \nu (\Phi(v, v) - \lambda \Phi(w, v)) \pmod{\mathfrak{p}}. \end{aligned}$$

But $\mu \nu (\Phi(v, v) - \lambda \Phi(w, v)) \in \mathfrak{p}$, for we know that both $\Phi(v, v) \in \mathfrak{p}^2$ and $\Phi(v, w) \in \mathfrak{p}^2$, $\lambda, \mu \in \mathfrak{o}$ and $\nu \in \mathfrak{p}^{-1}$. So $\nu v + z \in L(w)$ as claimed, and hence $L(v) \subseteq L(w)$. It follows from symmetry that $L(v) = L(w)$.

Conversely, assume that $L(v) = L(w)$. Then certainly $\Phi(v, w) \in \mathfrak{p}^2 \Phi(\mathfrak{p}^{-1}v, \mathfrak{p}^{-1}w) \subseteq \mathfrak{p}^2$ because $\mathfrak{p}^{-1}w \in L(w) = L(v)$ and the lattice $L(v)$ is integral. It remains to show that $[v] = [w]$. This holds because we have $\mathfrak{p}^{-1}v + L_v = \mathfrak{p}^{-1}w + L_w$, which implies that $\mathfrak{p}L_v + \mathfrak{o}v = \mathfrak{p}L_w + \mathfrak{o}w$. Both L_v and L_w are contained in L , so it immediately follows that $\mathfrak{o}v - \mathfrak{o}w \subseteq \mathfrak{p}L$. But then there is some $\lambda \in \mathfrak{o}$ such that $\lambda v \equiv w \pmod{\mathfrak{p}L}$, i.e., $[v] = [w]$. ■

We can now state an explicit algorithm for calculating the $\mathfrak{p}$ -neighbours of L . For the sake of simplicity, we will only state the algorithm for non-dyadic primes. Note that we do allow the calculation of neighbours of modular lattices, which can easily be done by a suitably rescaling the lattice, calculating the neighbours, and then rescaling back.

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Definition 4.7. A prime ideal $\mathfrak{p} \in \Omega_{\text{fin}}$ is called *good for neighbour generation of L* or simply *good*, if it is non-dyadic and $L_{\mathfrak{p}}$ is a modular lattice.

Algorithm 4.8. Calculating the set of $\mathfrak{p}$ -neighbours of L

Input: An $\mathfrak{o}$ -lattice L in the space (V, Φ) , and a good prime ideal $\mathfrak{p}$ of $\mathfrak{o}$.

Output: The set of $\mathfrak{p}$ -neighbours of L .

```

1: function NEIGHBOURS( $L, \mathfrak{p}$ )
2:    $\pi \leftarrow$  a uniformising element of  $\mathfrak{p}$ 
3:    $\tau \leftarrow$  an element of  $K^*$  such that  $(L_{\mathfrak{p}}, \tau\Phi)$  is unimodular
4:    $L \leftarrow (L, \tau\Phi)$ 
5:    $S \leftarrow$  a subset of  $L$  representing the 1-dimensional subspaces of the  $\mathfrak{o}/\mathfrak{p}$ -space  $L/\mathfrak{p}L$ .
6:   result  $\leftarrow []$ 
7:   for  $x \in S$  do
8:     if  $\Phi(x, x) \in \mathfrak{p}$  then
9:        $z_x \leftarrow$  a vector in  $L$  with  $\Phi(z_x, x) \notin \mathfrak{p}$ 
10:       $b_x \leftarrow$  an element of  $\mathfrak{o}$  such that  $b_x \equiv -\frac{\Phi(x, x)}{2\pi\Phi(z_x, x)} \pmod{\mathfrak{p}^2}$ 
11:       $x' \leftarrow x + \pi b_x z_x$ 
12:      result  $\leftarrow$  result  $\cup \{(L(x'), \tau^{-1}\Phi)\}$ 
13:     end if
14:   end for
15:   return result
16: end function

```

Proof. By Proposition 4.2, we have to show that the above algorithm calculates the set $\{L(v) \mid v \text{ is } \mathfrak{p}\text{-admissible for } L\}$. First, if $x \in L - \mathfrak{p}L$, then clearly $\Phi(x, x) \equiv \Phi(x + \lambda w, x + \lambda w) \pmod{\mathfrak{p}}$ for all $w \in \mathfrak{p}L$, $\lambda \in \mathfrak{o}^*$. So if $\Phi(x, x) \notin \mathfrak{p}$ in step 8, then no representative of the class $[x]$ is necessary for generating the set N and x can be skipped.

It is clear from Lemma 4.6 that we just need to pick any single $\mathfrak{p}$ -admissible representative vector x' from every class $x \in S$ and add the lattice $L(x')$ to the output. This will hit every lattice in N exactly once.

The vector z_x from line 9 exists because $L_{\mathfrak{p}}$ is unimodular, and b_x from line 10 exists because of our assumption on $\mathfrak{p}$. In line 11, $[x] = [x']$ is clear from the choice of b_x and

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z_x , and we can check that

$$\begin{aligned}\Phi(x', x') &= \Phi(x, x) + 2\pi b_x \Phi(z_x, x) + \pi^2 b_x^2 \Phi(z_x, z_x) \\ &\equiv \Phi(x, x) - \Phi(x, x) + \pi^2 b_x^2 \Phi(z_x, z_x) \equiv 0 \pmod{\mathfrak{p}^2}.\end{aligned}$$

Note that in the definite case, the set S in step 5 can instead be chosen as a system of representatives of the $O(L)$ -orbits of the 1-dimensional subspaces of the $\mathfrak{o}/\mathfrak{p}$ -space $L/\mathfrak{p}L$, speeding up the calculation. $\blacksquare$

4.2 The strong approximation theorem for rotations

In this section, we will state the principle of strong approximation in the context of orthogonal groups. A direct consequence is that the set of vertices of the neighbour graph (defined below) always comprises full proper spinor genera.

Definition 4.9. Whenever two $\mathfrak{o}$ -lattices $L, L' \subset V$ differ only at a single place $\mathfrak{p} \in \Omega_{\text{fin}}(K)$, i.e., $L_{\mathfrak{q}} = L'_{\mathfrak{q}}$ for all finite places $\mathfrak{p} \neq \mathfrak{q}$, we call them *linked at $\mathfrak{p}$* . Two proper isometry classes $\text{cls}^+(L)$ and $\text{cls}^+(L')$ (or two isometry classes $\text{cls}(L)$ and $\text{cls}(L')$) are *linked at $\mathfrak{p}$* if they have some linked representatives.

Let $\mathcal{N}^*(L, \mathfrak{p})$ denote the set of *iterated neighbours* of L , that is, the set of all lattices M in V such that there exists a chain $L = L_1, L_2, \dots, L_r = M$ for some $r \in \mathbb{N}$, where L_i is a $\mathfrak{p}$ -neighbour of L_{i-1} for all $2 \leq i \leq r$.

Further, let $\mathcal{N}(L, \mathfrak{p}) = \{\text{cls}(L) \mid L \in \mathcal{N}^*(L, \mathfrak{p})\}$ and $\mathcal{N}^+(L, \mathfrak{p}) = \{\text{cls}^+(L) \mid L \in \mathcal{N}^*(L, \mathfrak{p})\}$.

By the *neighbour graph of L* , we mean the graph whose vertex set is $\mathcal{N}(L, \mathfrak{p})$, and where two vertices $\text{cls}^+(L)$ and $\text{cls}^+(L')$ are adjacent whenever there exists some lattice $L'' \in \text{Genus}(L)$ which is a $\mathfrak{p}$ -neighbour of L and isometric to L' .

For a definite lattice L , there is an algorithmic isometry test available (cf. Remark 3.1), so the set $\mathcal{N}^*(L, \mathfrak{p})$ can be easily computed as follows.

Algorithm 4.10. Iterated construction of neighbours

Input: A definite $\mathfrak{o}$ -lattice L , and a good prime $\mathfrak{p}$ of $\mathfrak{o}$.

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Output: A set of representatives of $\mathcal{N}(L, \mathfrak{p})$

```

1: function ITERATEDNEIGHBOURS( $L, \mathfrak{p}$ )
2:   todo  $\leftarrow [L]$ 
3:   result  $\leftarrow []$ 
4:   while there exists some  $K \in \text{todo}$  do
5:     todo  $\leftarrow \text{todo} - \{K\}$ 
6:      $N \leftarrow \text{NEIGHBOURS}(K, \mathfrak{p})$ 
7:     for  $J \in N$  do
8:       if there is no  $J' \in \text{result}$  such that  $\text{ISISOMETRIC}(J, J')$  then
9:         result  $\leftarrow \text{result} \cup \{J\}$ 
10:      end if
11:    end for
12:  end while
13:  return result
14: end function

```

Note that it is not necessary to restrict the definition of $\mathcal{N}(L, \mathfrak{p})$, and line 7 of Algorithm 4.10 to proper isometry, as isometric lattices will have isometric $\mathfrak{p}$ -neighbours.

Linked lattices are in fact iterated neighbours of each other:

Lemma 4.11. Let $\mathfrak{p} \in \Omega_{\text{fin}}$ such that $\mathfrak{p}$ is good for neighbour generation of L . Let $L' \in \text{Genus}(L)$ and assume that L and L' are linked at $\mathfrak{p}$. Then $L' \in \mathcal{N}^*(L, \mathfrak{p})$.

Proof. We construct a chain of $\mathfrak{p}$ -neighbours linking L to L' . Consider the module index e , where $\mathfrak{p}^e = [L : L \cap L']_{\mathfrak{o}}$. If $e = 0$ then $L = L'$ and we are done. Assume $e > 0$. As in the proof of Proposition 4.2, there exists a $\mathfrak{p}$ -admissible vector $v \in (L \cap \mathfrak{p}L') - \mathfrak{p}L$. We set the neighbour $L(v)$ (which lies in $\text{Genus}(L)$ by Lemma 4.3) to be the next element of our chain. By the choice of v , $L(v) \cap L' \supsetneq L \cap L'$. Consequently, $[L : (L(v) \cap L')]_{\mathfrak{o}} = \mathfrak{p}^b$ for some $b < e$, and the claim follows by induction. ■

In the following, we will describe $\mathcal{N}(L, \mathfrak{p})$ in detail in terms of the proper spinor genus $\text{spn}^+(L)$. We will first show that every isometry class of a lattice in $\text{spn}^+(L)$ lies in $\mathcal{N}(L, \mathfrak{p})$:

Theorem 4.12 (Strong approximation for rotations). Assume V is a regular quadratic K -vector space with $\dim(V) \geq 3$, S is the set of all finite places except for a single place

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$\mathfrak{q} \in \Omega$ such that $V_{\mathfrak{q}}$ is isotropic, and $T \subset S$ is a finite subset. A rotation $\varphi_{\mathfrak{p}}$ is given in $\Theta(V_{\mathfrak{p}})$ at each spot $\mathfrak{p} \in T$ (for the definition of Θ , check Proposition 2.64). Then for each $\varepsilon > 0$ there is a rotation $\sigma \in \Theta(V)$ such that

$$\|\sigma - \varphi_{\mathfrak{p}}\|_{\mathfrak{p}} < \varepsilon \text{ for all } \mathfrak{p} \in T$$

and

$$\|\sigma\|_{\mathfrak{p}} = 1 \text{ for all } \mathfrak{p} \in S - T.$$

Proof. Cf. [O'M73, 104:4]. ■

The above theorem is false if we replace $\Theta(V)$ by the larger group $O^+(V)$. For a counterexample, we refer to [Cas78, Chapter 10, Theorem 7.1].

For indefinite lattices of rank 3 and above, strong approximation implies that the concepts of (proper) isometry class and (proper) spinor genus coincide:

Corollary 4.13. If L is a lattice in a regular indefinite quadratic space V of dimension $\dim(V) \geq 3$ over a number field K , then $\text{cls}^+(L) = \text{spn}^+(L)$.

A proof is given in [O'M73, 104:5].

With definite lattices, we are not quite as lucky. However, the fact that $\Theta(V)$ allows strong approximation has the nice consequence that the neighbour graph $\mathcal{N}(L, \mathfrak{p})$ is a union of spinor genera:

Lemma 4.14. Let L be an integral lattice of rank $m \geq 3$ and $\mathfrak{p}$ such that $\mathfrak{p} \nmid 2\mathfrak{v}(L)$. Then $V_{\mathfrak{p}}$ is isotropic.

Proof. Because L is integral and $\mathfrak{p} \nmid \mathfrak{v}(L)$, $L_{\mathfrak{p}}$ is unimodular. And because $\mathfrak{p} \nmid 2$, $L_{\mathfrak{p}}$ has a diagonal Gram matrix $\text{diag}(a_1, \dots, a_m)$ where $a_i \in \mathfrak{o}_{\mathfrak{p}}^*$ for all i , which is also a Gram matrix of the quadratic form of the enveloping space. It follows that the quadratic $\mathfrak{o}/\mathfrak{p}$ -vector space $(L/\mathfrak{p}L, \Phi)$ is isotropic, cf. [O'M73, 62:1b]. By HENSEL's lemma, $V_{\mathfrak{p}}$ is isotropic. ■

Corollary 4.15. Assume L is integral and $\text{rank}(L) \geq 3$. Let $\mathfrak{p}$ denote any prime ideal of $\mathfrak{o}$ which does not divide $\mathfrak{v}(L)$. If $\mathfrak{p} \mid 2$, further assume that the completion $V_{\mathfrak{p}}$ is isotropic. Then every isometry class of a lattice in $\text{spn}^+(L)$ lies in $\mathcal{N}(L, \mathfrak{p})$.

4.3 The spinor linkage theorem

Proof. Let $L' \in \text{spn}^+(L)$. Then there exists $\sigma \in O^+(V)$ and for every $\mathfrak{q} \in \Omega_{\text{fin}}$ there exists a $\Sigma_{\mathfrak{q}} \in \Theta(V_{\mathfrak{q}})$, such that $L_{\mathfrak{q}} = \sigma_{\mathfrak{q}} \Sigma_{\mathfrak{q}} L'_{\mathfrak{q}}$.

The completion $V_{\mathfrak{p}}$ is isotropic (by assumption if $\mathfrak{p} \mid 2$, and by Lemma 4.14 otherwise). But then the sets $S = \{\mathfrak{q} \in \Omega_{\text{fin}} \mid \mathfrak{q} \neq \mathfrak{p}\}$ and $T := \{\mathfrak{q} \in \Omega_{\text{fin}} \mid L_{\mathfrak{q}} \neq L'_{\mathfrak{q}}\}$, which is a finite subset of S , meet the conditions of Theorem 4.12. Hence for any $\varepsilon > 0$ there exists a global rotation $\tau = \tau(\varepsilon) \in \Theta(V)$ such that $\|\tau\|_{\mathfrak{q}} = 1$ (hence $\tau_{\mathfrak{q}}(L'_{\mathfrak{q}}) = L_{\mathfrak{q}}$) for all $\mathfrak{q} \in S - T$, and $\|\tau_{\mathfrak{q}} - \Sigma_{\mathfrak{q}}\|_{\mathfrak{q}} < \varepsilon$ for all $\mathfrak{q} \in T$. By [O'M73, 101:4(iii)], this implies that $\tau_{\mathfrak{q}}$ is indistinguishable from $\sigma_{\mathfrak{q}}$, i.e. $\tau_{\mathfrak{q}}(L'_{\mathfrak{q}}) = \Sigma_{\mathfrak{q}}(L'_{\mathfrak{q}})$, for small enough ε . So we fix an ε_0 small enough such that this condition holds for all $\mathfrak{q} \in T$ simultaneously (note that T is finite). Let $\tau_0 := \tau(\varepsilon_0)$, and $L'' := \sigma(\tau_0(L'))$. Then $L'' \in \text{cls}^+(L')$ because $\tau_0 \in \Theta(V)$ is a global rotation, and $L_{\mathfrak{q}} = \sigma_{\mathfrak{q}} \Sigma_{\mathfrak{q}}(L'_{\mathfrak{q}}) = \sigma_{\mathfrak{q}}(\tau_0)_{\mathfrak{q}}(L'_{\mathfrak{q}}) = L''_{\mathfrak{q}}$ for all finite places $\mathfrak{p} \neq \mathfrak{q}$. So $L'' \in \mathcal{N}^*(L, \mathfrak{p})$ by Lemma 4.11. ■

Remark 4.16. We have seen that to enumerate the classes in $\text{spn}^+(L)$, one can simply choose a suitable prime ideal of smallest norm and enumerate the classes in $\mathcal{N}(L, \mathfrak{p})$. Speed improvements may be gained by using an algorithm that works with suitable dyadic primes ($\mathfrak{p} \mid 2$, $\mathfrak{p} \nmid \mathfrak{v}(L)$, and with $V_{\mathfrak{p}}$ isotropic). After all, the number of elements in the $\mathfrak{o}/\mathfrak{p}$ -vector space $L/\mathfrak{p}L$ is $((\mathfrak{N}\mathfrak{p})^k - 1)/(\mathfrak{N}(\mathfrak{p}) - 1)$. For a neighbours algorithm improved in this way, see the work [SH98] of SCHARLAU and HEMKEMEIER.

4.3 The spinor linkage theorem

In this section, we will give a proof of the Spinor Linkage Theorem (Corollary 4.28), and then lay out an explicit way to find a set of prime ideals such that iterated neighbour construction at these primes covers the entirety of $\text{Genus}(L)$. This allows to state Algorithm 4.36, which calculates representatives of all isometry classes in $\text{Genus}(L)$.

In this section, let K be any algebraic number field, $\mathfrak{o}$ its ring of integers, and let L be an $\mathfrak{o}$ -lattice in the quadratic K -vector space V . L does not need to be definite, but we will always require $\text{rank}(L) \geq 3$.

Definition 4.17. We will need some new notation in this section, most of which is taken from O'MEARA and BENHAM-HSIA ([O'M73], [BH83]):

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- $J = \{j \in \prod_{\mathfrak{p} \in \Omega} K_{\mathfrak{p}}^* \mid \nu_{\mathfrak{p}}(j_{\mathfrak{p}}) = 0 \text{ for almost all } \mathfrak{p} \in \Omega\}$ is the idele group of K . We will sometimes speak of the “support” of an idele j , and mean the set of primes $\mathfrak{p}$ such that $j_{\mathfrak{p}} \neq 1$.
- $J_V = \{\Sigma \in \prod_{\mathfrak{p} \in \Omega} O^+(V_{\mathfrak{p}}) \mid \|\Sigma_{\mathfrak{p}}\|_{\mathfrak{p}} = 1 \text{ for almost all } \mathfrak{p} \in \Omega\}$ is the adèle group of V . Here the $\|\cdot\|_{\mathfrak{p}}$ are fixed norms on the sets of automorphisms $\text{Hom}_{K_{\mathfrak{p}}}(V_{\mathfrak{p}}, V_{\mathfrak{p}})$ with respect to a base of V .
- $P_V = \{\prod_{\mathfrak{p} \in \Omega} \sigma \mid \sigma \in O^+(V)\} \leq J_V$ is the group of *principal split rotations*, and we will also need the subgroups $J'_V = \{\Sigma \in J_V \mid \Sigma_{\mathfrak{p}} \in \Theta(V_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \in \Omega\} \leq J_V$, and $J_L = \{\Sigma \in J_V \mid \Sigma_{\mathfrak{p}} \in O^+(L_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \in \Omega_{\text{fin}}\} \leq J_V$. Note: J'_V contains the commutator subgroup of J_V , so it is a normal subgroup and the quotient J_V/J'_V is abelian.
- By $j(\mathfrak{p}, x)$, for an element $x \in K_{\mathfrak{p}}^*$, we denote an idele satisfying $j(\mathfrak{p}, x)_{\mathfrak{p}} = x$, and $j(\mathfrak{p}, x)_{\mathfrak{q}} = 1$ for all places $\mathfrak{q} \neq \mathfrak{p}$. We write $j(\mathfrak{p})$ as an abbreviation for $j(\mathfrak{p}, \pi)$ where π can be any uniformising element of $\mathfrak{p}$.
- By J^L we denote those ideles which lie in the spinor norms of L at all finite places: $J^L = \{j \in J \mid j_{\mathfrak{p}} \in \theta(O^+(L_{\mathfrak{p}})) \text{ for all } \mathfrak{p} \in \Omega_{\text{fin}}\}$.
- Let S be the set of those real places v of K such that the quadratic space (V_v, Φ) is anisotropic. By K_S^* we mean the set of elements of K^* which are positive at the places in S . Similarly, for any subgroup X of the idele group J , let X_S denote the subgroup $\{x \in X \mid x_v > 0 \text{ for all } v \in S\}$. We say that these ideles are *positive at the places in S* .
- We set $P_{\theta(O^+(V))} \subseteq J$ to be the set of principal ideles coming from the set $\theta(O^+(V))$, that is, for $j \in P_{\theta(O^+(V))}$, there exists some $d \in \theta(O^+(V))$ such that $j_{\mathfrak{p}} = d$ for all $\mathfrak{p} \in \Omega$.

Remark 4.18. Let $\Sigma \in \prod_{\mathfrak{p} \in \Omega} O(V_{\mathfrak{p}})$ such that $\|\Sigma_{\mathfrak{p}}\|_{\mathfrak{p}} = 1$ for almost all $\mathfrak{p}$. Then there is exactly one $\mathfrak{o}$ -lattice L' in V such that $L'_{\mathfrak{p}} = \Sigma_{\mathfrak{p}}(L_{\mathfrak{p}})$ for every prime ideal $\mathfrak{p}$ of K , cf. Theorem 2.52. We set ΣL to be this lattice L' . In particular, the group J_V acts $\text{Genus}(L)$.

It is clear that for any $\varphi \in O(V)$, we have $\Sigma_{\varphi}(L) = \varphi(L)$ under the action described above, where the adèle Σ_{φ} is the diagonal embedding of φ into the adelic orthogonal group $\prod_{\mathfrak{p} \in \Omega} O(V_{\mathfrak{p}})$.

4.3 The spinor linkage theorem

The first part of the following proposition was shown by O'MEARA in [O'M73, § 102]. We outline his proof, then continue with the proof of an isomorphism stated in [EH90], which overall leads to an identification of the set of proper spinor genera in $\text{Genus}(L)$ with the group $J/K^*J_S^L$.

Proposition 4.19. The set $\{\text{spn}^+(L) \mid L \in \text{Genus}(L)\}$ of proper spinor genera in $\text{Genus}(L)$ can be identified with the group $J_V/J'_V P_V J_L$. Further, when $\text{rank}(L) \geq 3$, there are group isomorphisms

$$J_V/J'_V P_V J_L \cong J/K_S^* J^L \cong J/K^* J_S^L.$$

Proof. The group of split rotations J_V acts transitively on the set $\{\text{spn}^+(L) \mid L \in \text{Genus}(L)\}$, and the kernel of this action is $J'_V P_V J_L$, which is discussed in detail in [O'M73, 102:7]. So the abelian group $J_V/J'_V P_V J_L$ acts faithfully and transitively (hence, regularly) on the set of proper spinor genera in $\text{Genus}(L)$. So via this operation, there is a one-to-one correspondence between $J_V/J'_V P_V J_L$ and the set of proper spinor genera in $\text{Genus}(L)$. Note that this correspondence depends on the choice of a basis of V , cf. the definition of J_V .

O'MEARA continues his proof (parts 3) and 4) of [O'M73, 102:7]) by showing that the mapping $J_V \rightarrow J/P_{\theta(O^+(V))} J^L$ is a well-defined homomorphism. Here an adele Σ is mapped to the class of an idele j such that $j_{\mathfrak{p}} \in \theta(\Sigma_{\mathfrak{p}})$ for all $\mathfrak{p} \in \Omega$. Now whenever $\text{rank}(L) \geq 3$, this homomorphism is surjective with kernel $J'_V P_V J_L$. By [O'M73, 101:8], $P_{\theta(O^+(V))}$ is identical to the set K_S^* whenever $\text{rank}(V) \geq 3$, hence $J_V/J'_V P_V J^L \cong J/K_S^* J^L$.

To see the existence of the second isomorphism we claimed, $J/K^* J_S^L \cong J/K_S^* J^L$, consider the mapping $\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$ given by the composition of:

$$J/K^* \xrightarrow[\varphi_1]{\sim} J_S/K_S^* \xrightarrow[\varphi_2]{\hookrightarrow} J/K_S^* \xrightarrow[\varphi_3]{\twoheadrightarrow} J/K_S^* J^L$$

Here φ_2 and φ_3 are just the canonical embedding and projection, respectively. The mapping φ_1 maps a class $jK^* \in J/K^*$ to cjK_S^* , where c is some element in K^* obtained via approximation such that $(cj)_v > 0$ for the infinite places $v \in S$. Then φ_1 is a well-defined homomorphism, for choosing a different approximation c' implies $c(c')^{-1} \in K_S^*$. It is clear that φ_1 is a bijection.

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The composite mapping φ is surjective although φ_2 is not, for all ideles which differ only at infinite places lie in the same residue class modulo J^L .

The claim follows once we show that $\ker(\varphi) = K^* J_S^L / K^*$:

$$\begin{aligned}
& \varphi(xK^*) \text{ has a representative } y \in K_S^* J^L \\
& \Leftrightarrow \varphi_2(\varphi_1(xK^*)) \text{ has a representative } z \in J^L \\
& \Leftrightarrow \varphi_1(xK^*) \text{ has a representative } z \in J_S^L \\
& \Leftrightarrow \text{there is some } c \in K^* \text{ such that } cx = z \in J_S^L \\
& \Leftrightarrow xK^* \in K^* J_S^L / K^*.
\end{aligned}$$

■

In particular, the quotient $J/K^* J_S^L$ is finite. It is, however, still a quotient of two infinite groups, and as such is difficult to handle algorithmically. We will now identify it with a quotient of a (finite) ray class group, and then find nice representatives of its elements.

Definition 4.20. A *modulus* is a formal product $D = \prod_{\mathfrak{p} \in \Omega} \mathfrak{p}^{v_{\mathfrak{p}}}$, where $v_{\mathfrak{p}} \in \mathbb{N}_0$ for finite places, $v_{\mathfrak{p}} \in \{0, 1\}$ for real places, $v_{\mathfrak{p}} = 0$ for complex places, and the set $\{\mathfrak{p} \in \Omega \mid v_{\mathfrak{p}} \neq 0\}$ is finite. For a modulus D as above, we write $\nu_{\mathfrak{p}}(D)$ for the exponent $v_{\mathfrak{p}}$, and say that $\mathfrak{p} \mid D$ whenever $\nu_{\mathfrak{p}}(D) > 0$.

We will use the following notion of congruence for moduli. Let $a, b \in K^*$, then:

- If $\mathfrak{p}$ is a finite place of K , then $a \equiv^* b \pmod{\mathfrak{p}^{v_{\mathfrak{p}}}}$ means $\frac{a}{b} \in 1 + \mathfrak{p}^{v_{\mathfrak{p}}}$.
- If $\mathfrak{p}$ is a real place of K , then $a \equiv^* b \pmod{\mathfrak{p}}$ means $\iota_{\mathfrak{p}}(\frac{a}{b}) > 0$ where $\iota_{\mathfrak{p}}$ is the real embedding associated to $\mathfrak{p}$.
- If $D = \prod_{\mathfrak{p} \in \Omega} \mathfrak{p}^{v_{\mathfrak{p}}}$ is a modulus, then $a \equiv^* b \pmod{D}$ means that $a \equiv^* b \pmod{\mathfrak{p}^{v_{\mathfrak{p}}}}$ for all (finite or infinite) places $\mathfrak{p} \mid D$.
- Similarly, the notion of congruence can be extended to apply to ideles: if $\mathfrak{j} \in J$, then $\mathfrak{j} \equiv^* b \pmod{D}$ means $\mathfrak{j}_{\mathfrak{p}} \equiv^* b \pmod{D}$ for all (finite or infinite) places $\mathfrak{p} \mid D$.

We can now define the ray class group with respect to a modulus:

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- If $D = \prod_{\mathfrak{p} \in \Omega} \mathfrak{p}^{m_{\mathfrak{p}}}$ is a modulus, the *ray class group with respect to D* is

$$\mathrm{Cl}_D := \mathbb{I}_D(\mathfrak{o}) / \mathbb{P}_D(\mathfrak{o}),$$

where $\mathbb{I}_D(\mathfrak{o}) = \{\mathfrak{a} \in \mathbb{I}(\mathfrak{o}) \mid \nu_{\mathfrak{p}}(\mathfrak{a}) \neq 0 \Rightarrow m_{\mathfrak{p}} = 0\}$ are the fractional ideals of $\mathfrak{o}$ whose support is relatively prime to D , and $\mathbb{P}_D(\mathfrak{o}) = \{(\alpha) \in \mathbb{I}_D(\mathfrak{o}) \mid \alpha \equiv^* 1 \pmod{D}\}$. Here (α) denotes the principal fractional ideal generated by α .

The elements of Cl_D are called *rays with respect to D* .

Some final definitions will be needed:

- The *set of D -ideles* is $J_{(D)} := \{\mathfrak{j} \in J \mid \mathfrak{j} \equiv^* 1 \pmod{D}\}$. So by slight abuse of notation, $\mathbb{P}_D(\mathfrak{o}) = K^* \cap J_{(D)}$.
- The *set of restricted D -ideles* is

$$J_D := \{\mathfrak{j} \in J \mid \mathfrak{j} \equiv^* 1 \pmod{D}, \mathfrak{j}_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}^* \text{ for all } \mathfrak{p} \in \Omega_{\mathrm{fin}} \text{ with } \mathfrak{p} \nmid D\}.$$

Remark 4.21. Ray class groups are finite. Every element of a ray class group has infinitely many representatives which are prime ideals.

Proof. For the first part, see [Jan73, IV, (1.6)]. By a density theorem from class field theory (see [Jan73, V, (10.3)]), every group H with $\mathbb{I}_D(\mathfrak{o}) \supseteq H \supseteq \mathbb{P}_D(\mathfrak{o})$ contains infinitely many prime ideals of $\mathfrak{o}$. ■

Lemma 4.22. Let D be any modulus. Then $\Xi: J/K^* J_D \xrightarrow{\sim} \mathrm{Cl}_D$ is an isomorphism.

Proof. We start with the *idele class group* J/K^* and note that it is equal to $J_{(D)}K^*/K^*$ by weak approximation, for there is only a finite set of conditions an idele needs to fulfill to lie in $J_{(D)}$. Now consider the composite homomorphism

$$\begin{aligned} \Xi: J/K^* &= J_{(D)}K^*/K^* \cong J_{(D)}/(K^* \cap J_{(D)}) \twoheadrightarrow \mathrm{Cl}_D, \text{ defined by} \\ \mathfrak{j}K^* &= (\alpha\mathfrak{j})\alpha^{-1}K^* \mapsto \alpha\mathfrak{j}(K^* \cap J_{(D)}) \mapsto \left(\prod_{\mathfrak{p} \in \Omega_{\mathrm{fin}}} \mathfrak{p}^{\nu_{\mathfrak{p}}(\alpha\mathfrak{j})} \right) \mathbb{P}_D(\mathfrak{o}) \end{aligned}$$

Here $\alpha \in K^*$ is chosen by weak approximation such that $\alpha\mathfrak{j} \equiv^* 1 \pmod{D}$.

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Then Ξ is clearly well-defined and surjective. It suffices to show that $\ker(\Xi) = J_D K^* / K^*$.

The set $J_D K^* / K^*$ can easily be seen to be a subset of $\ker(\Xi)$, for if $\mathbf{j} \in J_D$, then $\nu_{\mathfrak{p}}(\mathbf{j}) = 0$ for all places $\mathfrak{p} \in \Omega_{\text{fin}}$ with $p \nmid D$, so $\mathbf{j}$ is mapped to 1. On the other hand, if a class $\mathbf{j}(K^* \cap J_{(D)})$ with $\mathbf{j} \in J_{(D)}$ lies in $\ker(\Xi)$, then there is some $\mathbf{a} \in \mathbb{P}_D(\mathfrak{o})$, i.e. $\mathbf{a} = (a)$ for some $a \in J_{(D)} \cap K^*$, such that $\Xi(\mathbf{j}) = \mathbf{a}$. But then the idele $\mathbf{j}a^{-1}$, which is another representative of the class $\mathbf{j}(K^* \cap J_{(D)})$, lies in J_D . $\blacksquare$

For the remainder of this section, we fix a concrete modulus:

$$D := \left(\prod_{v \in S} v \right) \left(\prod_{\mathfrak{p} \in P} \mathfrak{p}^{1+\nu_{\mathfrak{p}}(4)} \right), \text{ where } P := \{\mathfrak{p} \in \Omega_{\text{fin}} \mid \theta(O^+(L_{\mathfrak{p}})) \neq \mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2\} \quad (4.1)$$

The set P can be computed effectively – this will be discussed in Section 4.4.

The following result, kindly shared with me by Prof. WAI KIU CHAN, allows to identify the set of proper spinor genera in $\text{Genus}(L)$ with a quotient of *finite* groups.

Theorem 4.23 (WAI KIU CHAN). Assume that $\text{rank}(L) \geq 3$, and that D is the modulus defined above. Then:

1. The set P is finite.
2. We have the following chain of inclusions:

$$K^* J_D \subseteq K^* J_D J^2 \subseteq K^* J_S^L \subseteq J$$

where $J^2 = \{\mathbf{j} \in J \mid \mathbf{j}_{\mathfrak{p}} \in (K_{\mathfrak{p}}^*)^2 \text{ for every } \mathfrak{p} \in \Omega\}$.

We can further investigate the quotients $K^* J_D J^2 / K^* J_D$ and $K^* J_S^L / K^* J_D$:

3. Let Ξ denote the isomorphism $J / K^* J_D \rightarrow \text{Cl}_D$ from Lemma 4.22. Then Ξ maps the subgroup $K^* J_D J^2 / K^* J_D$ of $J / K^* J_D$ onto $(\text{Cl}_D)^2$.
4. For every $\mathfrak{p} \in P$, let $X_{\mathfrak{p}}$ be a set of representatives of the square classes in $\theta(O^+(L_{\mathfrak{p}}))$ such that $\nu_{\mathfrak{p}}(x) \in \{0, 1\}$ for all $x \in X_{\mathfrak{p}}$.

Let

$$X := \{\mathbf{j} \in J \mid \mathbf{j}_{\mathfrak{p}} \in X_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in P, \mathbf{j}_{\mathfrak{p}} = 1 \text{ for all } \mathfrak{p} \notin P\}.$$

Then the set X is finite, and $\Xi(K^* J_S^L / K^* J_D) = \langle \text{Cl}_D^2, \Xi(X) \rangle$.

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5. There is an isomorphism $\Psi: J/K^*J_S^L \xrightarrow{\sim} C_L$, where $C_L := \text{Cl}_D / \langle \text{Cl}_D^2, \Xi(X) \rangle$. Consequently, the set of proper spinor genera in $\text{Genus}(L)$ can be identified with C_L , an elementary abelian 2-group.

Proof. 1. If $\mathfrak{p}$ is a place of $\mathfrak{o}$ where the spinor norms are not exactly $\mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2$, then either $\mathfrak{p}$ is dyadic or $L_{\mathfrak{p}}$ is not modular (cf. [OM73, 92:5], which holds for $\text{rank}(L) \geq 2$). So the set P is finite.

2. Note that trivially $J_D \subseteq J_D J^2$, and $J^2 \subseteq J_S^L$ holds since the squares $(K_{\mathfrak{p}}^*)^2$ always lie in the spinor norms $\theta(O^+(L_{\mathfrak{p}}))$. To prove the claimed inclusions, it suffices to show that $J_D \subseteq J_S^L$.

This is due to the way the modulus D was chosen: let $\mathfrak{j} \in J_D$. When $\mathfrak{p} \mid D$, then by the definition of J_D , $\mathfrak{j}_{\mathfrak{p}} \in 1 + \mathfrak{p}^{1+\nu_{\mathfrak{p}}(4)}$. By the Local Square Theorem (cf. Theorem 2.71), it follows that $\mathfrak{j}_{\mathfrak{p}} \in (K_{\mathfrak{p}}^*)^2$, so $\mathfrak{j}_{\mathfrak{p}}$ trivially lies in the spinor norms $\theta(O^+(L_{\mathfrak{p}}))$. For the other finite places ($\mathfrak{p} \nmid D$), there is nothing to show. At all infinite places $v \mid S$, we have $\iota_v(\mathfrak{j}_v) > 0$ by definition of D .

3. This is clear.
4. The sets $X_{\mathfrak{p}}$ are finite (cf. Lemma 2.87), so X is finite. Note that since P is finite, the elements of X all have finite support, so do indeed lie in J .

We have shown that J_S^L contains J_D , and trivially J_S^L also contains J^2 , the squares of elements of J . We have noted in part 3) that $\Xi(K^*J_D J^2 / K^*J_D) = (\text{Cl}_D)^2$, so it now suffices to show that the set X generates $K^*J_S^L / K^*J_D \pmod{K^*J_D J^2 / K^*J_D}$. Take a representative $\mathfrak{j} \in K^*J_S^L$ of any element of the group $K^*J_S^L / K^*J_D J^2$. By virtue of multiplying with a suitable element of K^* , we can assume that $\mathfrak{j} \in J_S^L$. Then by multiplying with a suitable element of J^2 , we can achieve that $\mathfrak{j}_{\mathfrak{p}} \in X_{\mathfrak{p}}$ for all $\mathfrak{p} \in P$. For the places $\mathfrak{p} \in \Omega_{\text{fin}}$ with $\mathfrak{p} \notin P$, $\mathfrak{j}_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2$ (by definition, P contains all the places where the spinor norms are not $\mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2$), so by multiplying with another suitable element of $J^2 J_D$ we can achieve $\mathfrak{j}_{\mathfrak{p}} = 1$ for those primes. Finally, at the infinite places $v \notin S$, we can achieve $\iota_v(\mathfrak{j}) = 1$ after multiplying by another suitable element of J_D .

5. $\text{Cl}_D / \langle \text{Cl}_D^2, \Xi(X) \rangle \cong (J/K^*J_D) / (K^*J_S^L / K^*J_D) \cong J/K^*J_S^L$ by combining Lemma 4.22, part 4) of the current proof and the isomorphism theorems.

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So by Proposition 4.19, the group $C_L := \text{Cl}_D / \langle \text{Cl}_D^2, \Xi(X) \rangle$ (which is clearly an elementary abelian 2-group) can be identified with the set of proper spinor genera in $\text{Genus}(L)$.

■

Part 5) of Theorem 4.23 has laid the ground for explicit enumeration of $\text{Genus}(L)$, where L is a lattice of rank $m \geq 3$ over an algebraic number field. Algorithm 4.36, given later in this chapter, is based on choosing prime ideal representatives of the ray class group quotient C_L , and then iterating the neighbours algorithm with these primes. This yields a complete set of representatives for the proper isometry classes in $\text{Genus}(L)$.

In the remainder of this section, we will make the correspondence between prime ideals and C_L more explicit, and formulate the Spinor Linkage Theorem (Corollary 4.28).

Definition 4.24. Using Remark 4.21, we can fix a prime ideal representative for every element of C_L :

- Let $g \in C_L$. Then by $\mathfrak{p}_g$ we denote a prime ideal of $\mathfrak{o}$ which is a representative of the ideal class $g \in C_L$, such that
 1. $\mathfrak{p}_g \nmid D$,
 2. $\mathfrak{p}_g$ is of smallest possible norm, and:
 3. $\mathfrak{p}_g$ is good for neighbour generation of L .
- Let $\mathfrak{p} \in \Omega_{\text{fin}}$. Then by $g_{\mathfrak{p}}$ we denote the image of $\mathfrak{j}(\mathfrak{p})$ in C_L .

Remark 4.25. Note that the image of the idele $\mathfrak{j}(\mathfrak{p}_g)$ in C_L is indeed g , and that whenever $\mathfrak{p} \in \Omega_{\text{fin}}$ with $\mathfrak{p} \nmid D$, then $g_{\mathfrak{p}}$ is represented by the class $\mathfrak{p}\mathbb{P}_D(\mathfrak{o})$ of the ideal $\mathfrak{p}$.

Lemma 4.26. Assume $\mathfrak{p}$ is a good prime ideal of K and $L_{\mathfrak{p}}$ is unimodular. Let π be any uniformising element of $\mathfrak{p}$ and let L' be any $\mathfrak{p}$ -neighbour of L . Then there is a rotation $\sigma \in O^+(V_{\mathfrak{p}})$ with spinor norm $\theta(\sigma) = \pi(K_{\mathfrak{p}}^*)^2$, such that $\sigma(L_{\mathfrak{p}}) = L'_{\mathfrak{p}}$.

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Proof. By Proposition 4.2, $L' = L(v)$ for some $\mathfrak{p}$ -admissible vector v . Since $v \notin \mathfrak{p}L$ and $L_{\mathfrak{p}}$ is unimodular, we can pick some $w \in L$ such that $\Phi(v, w) = 1$. So for the $\mathfrak{o}_{\mathfrak{p}}$ -lattice $M_{\mathfrak{p}} := \langle v, w \rangle$ we have $\mathfrak{s}(M_{\mathfrak{p}}) = \mathfrak{s}(L_{\mathfrak{p}})$. But then $M_{\mathfrak{p}}$ splits $L_{\mathfrak{p}}$ by [O'M73, 82:15a], i.e. $L_{\mathfrak{p}} = M_{\mathfrak{p}} \perp M'_{\mathfrak{p}}$ for some $\mathfrak{o}_{\mathfrak{p}}$ -lattice $M'_{\mathfrak{p}}$. Now $M'_{\mathfrak{p}} \subseteq (L_v)_{\mathfrak{p}}$ by definition, and thus $L'_{\mathfrak{p}} = L(v)_{\mathfrak{p}} = \langle \pi^{-1}v, \pi w \rangle \perp M'_{\mathfrak{p}}$. So to prove the claim, it suffices to give a mapping $\sigma \in O^+(K_{\mathfrak{p}} M_{\mathfrak{p}})$ with $\sigma(M_{\mathfrak{p}}) = \langle \pi^{-1}v, \pi w \rangle$ and spinor norm $\pi(K_{\mathfrak{p}}^*)^2$, cf. [O'M73, 55:4]. This reduces the problem to the two-dimensional case.

So in the following we can assume that L has rank 2.

We will first find a more suitable base of $L_{\mathfrak{p}}$. We claim that $L_{\mathfrak{p}}$ contains a vector $v' = v + \lambda \pi w$ for some $\lambda \in \mathfrak{o}_{\mathfrak{p}}$ that fulfills $\Phi(v', v') = 0$. For if this is the case, then $[v'] = [v]$, so by Lemma 4.6, v and v' generate the same $\mathfrak{p}$ -neighbours of L . To show the existence of v' , we need to find $\lambda \in \mathfrak{o}_{\mathfrak{p}}$ such that

$$\Phi(v, v) + 2\lambda\pi\Phi(v, w) + \lambda^2\pi^2\Phi(w, w) = 0.$$

Because $\Phi(v, v) \in \mathfrak{p}$, this is still an equation in $\mathfrak{o}_{\mathfrak{p}}$ if we divide by π . We set $a = \frac{\Phi(v, v)}{\pi}$, $b = 2\Phi(v, w)$ and $c = \pi\Phi(w, w)$. Now the equation $f(\lambda) = a + b\lambda + c\lambda^2 \equiv a + b\lambda \pmod{\mathfrak{p}}$, so certainly has a solution $\pmod{\mathfrak{p}}$. And the derivative is $f'(\lambda) = 2c\lambda + b$. Now if $\lambda \in \mathfrak{o}_{\mathfrak{p}}$ such that $f(\lambda) \equiv 0 \pmod{\mathfrak{p}}$, then $f'(\lambda) \in \mathfrak{o}_{\mathfrak{p}}^*$ (note that $2 \notin \mathfrak{p}$ by assumption). So $\nu_{\mathfrak{p}}(f(\lambda)) > 2\nu_{\mathfrak{p}}(f'(\lambda))$ and the claim follows from HENSEL's lemma.

But then $L_{\mathfrak{p}}$ is isotropic, so we can find w' such that (v', w') is a base for $L_{\mathfrak{p}}$ fulfilling $\Phi(v', v') = 0 = \Phi(w', w')$ and $\Phi(v', w') = 1$. Consider the mapping σ given by the matrix $\text{diag}(\pi^{-1}, \pi)$ on the base (v', w') . Then σ sends L to its $\mathfrak{p}$ -neighbour $L(v)$, because we have shown that $[v] = [v']$.

But applying σ to $L_{\mathfrak{p}}$ is the same as applying the product of symmetries (cf. Definition 2.61) $\tau_{v'-\pi w'} \circ \tau_{v'-w'} \in O^+(V_{\mathfrak{p}})$, which has spinor norm $\theta(\sigma) = Q_{\Phi}(v' - \pi w')Q_{\Phi}(v' - w')(K_{\mathfrak{p}}^*)^2 = 4\pi(K_{\mathfrak{p}}^*)^2 = \pi(K_{\mathfrak{p}}^*)^2$. So indeed $\sigma = \tau_{v'-\pi w'} \circ \tau_{v'-w'}$. It follows that σ has the spinor norm we claimed. $\blacksquare$

We can now make the action of C_L on the proper spinor genera in $\text{Genus}(L)$ explicit:

Corollary 4.27. Every element of C_L is equal to some $g_{\mathfrak{p}}$ for a good prime $\mathfrak{p}$. The group C_L acts regularly on the set of proper spinor genera in $\text{Genus}(L)$, $\{\text{spn}^+(K) \mid K \in \text{Genus}(L)\}$ by

$$(g_{\mathfrak{p}}, \text{spn}^+(L)) \mapsto \text{spn}^+(L'), \text{ where } L' \text{ is some } \mathfrak{p}\text{-neighbour of } L.$$

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Proof. The first claim was shown in Remark 4.21, and it became clear in part 5) of Theorem 4.23 that C_L indeed acts regularly on the proper spinor genera (by transfer of structure from the group $J_V/J'_V P_V J^L$). By Lemma 4.26, if L and another $\mathfrak{o}$ -lattice $L' \leq V$ are $\mathfrak{p}$ -neighbours, there is some adele $\Sigma \in J_V$ with $\Sigma L = L'$ and which has spinor norm $\theta(\Sigma) = j(\mathfrak{p})$. So the class of Σ is mapped to the class of $j(\mathfrak{p})$ under the isomorphism $J_V/J'_V P_V J^L \xrightarrow{\sim} J/K_S^* J^L$. And because $\mathfrak{p}$ is a good prime (cf. Remark 4.25), $j(\mathfrak{p})$ in turn is mapped to $g_{\mathfrak{p}}$ under the isomorphism $J/K_S^* J^L \xrightarrow{\sim} C_L$. ■

Corollary 4.28 (Spinor Linkage Theorem, see also [BH83]). Let $\mathfrak{p} \in \Omega_{\text{fin}}$ be good for neighbour generation of L .

1. If $g_{\mathfrak{p}} = 1$, then $\mathcal{N}^+(L, \mathfrak{p})$ comprises exactly one proper spinor genus.
2. If $g_{\mathfrak{p}} \neq 1$, then $\mathcal{N}^+(L, \mathfrak{p})$ comprises exactly two proper spinor genera in $\text{Genus}(L)$, and adjacent vertices lie in different proper spinor genera.
3. For $L' \in \text{Genus}(L)$, there exist infinitely many primes $\mathfrak{q} \in \Omega_{\text{fin}}$ such that a $\mathfrak{q}$ -neighbour of L is isometric to L' .

Proof. By Corollary 4.27, passing from $\text{spn}^+(L)$ to $\text{spn}^+(L')$, where L' is a $\mathfrak{p}$ -neighbour of L , is achieved by applying the element $g_{\mathfrak{p}}$ to $\text{spn}^+(L)$. And $g_{\mathfrak{p}}$ has order 2 if $g_{\mathfrak{p}} \neq 1$, for C_L has exponent 2 by Theorem 4.23.

The existence of a prime $\mathfrak{q}$ in part 3) is just a restatement of Corollary 4.27, and there are infinitely many such primes as was discussed in Remark 4.21. ■

4.4 Calculating the spinor norms

Before we can give an algorithm for enumerating the isometry classes in $\text{Genus}(L)$, we need to elaborate on how to determine the set $P = \{\mathfrak{p} \in \Omega_{\text{fin}} \mid \theta(O^+(L_{\mathfrak{p}})) \neq \mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2\}$ defined in Equation 4.1, and used in Theorem 4.23. We will give no proofs in this section but rather refer to the extensive literature.

The local spinor norms $\theta(O^+(L_{\mathfrak{p}}))$ can be computed fairly easily whenever $\mathfrak{p}$ is odd, thanks to the following theorem:

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Theorem 4.29 (KNESER 1956). Assume $\mathfrak{p}$ is a non-dyadic prime ideal of $\mathfrak{o}$. Let $\text{rank}(L) \geq 3$, and let $L_{\mathfrak{p}} = \perp_{i=1}^t L_i$ denote a Jordan decomposition of $L_{\mathfrak{p}}$. Let

$$\mathfrak{m}(K_{\mathfrak{p}}) := \bigcup_{i=1}^t \{\Phi(x, x) \mid x \in L_i \text{ and } \Phi(x, x) \notin \mathfrak{ps}(L_i)\}$$

Then $\theta(O^+(L_{\mathfrak{p}})) = \{(\prod_{i=1}^{2l} m_i)(K_{\mathfrak{p}}^*)^2 \mid m_i \in \mathfrak{m}(K_{\mathfrak{p}}) \text{ and } l \in \mathbb{N}\}$. In particular, whenever $\text{rank}(L_i) > 1$ for any $1 \leq i \leq t$, then $\mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2 \subseteq \theta(O^+(L_{\mathfrak{p}}))$. If L is a modular lattice, then $\mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2 = \theta(O^+(L_{\mathfrak{p}}))$.

Proof. Cf. [Kne56, Satz 3]. Note that the prerequisite $\text{rank}(L) \geq 3$ is crucial here. ■

In the rest of this section, assume that $\mathfrak{p}$ is a dyadic prime ideal of $\mathfrak{o}$.

Determining the spinor norms is a lot more involved for dyadic primes. BELI solved this problem as recently as 2003, and we now summarise the relevant results from [Bel03] and [Bel06]:

Definition 4.30. 1. An element $x \in L_{\mathfrak{p}}$ is called a *norm generator* if $\mathfrak{n}(L_{\mathfrak{p}}) = \Phi(x, x)\mathfrak{o}_{\mathfrak{p}}$. A *base of norm generators* (BONG) is defined recursively as a sequence $x_1, \dots, x_m$ of vectors in $V_{\mathfrak{p}}$ such that x_1 is a norm generator for $L_{\mathfrak{p}}$, and $x_2, \dots, x_m$ is a base of norm generators of the projection $\text{pr}_{x_1^\perp} L_{\mathfrak{p}}$.

For the determination of the spinor norms, we are primarily interested in the values of the quadratic form at the x_i . We write $L_{\mathfrak{p}} \cong \prec a_1, \dots, a_m \succ$ to say that there exists some BONG $x_1, \dots, x_m$ of $L_{\mathfrak{p}}$ with $\Phi(x_i, x_i) = a_i$.

2. A BONG $x_1, \dots, x_m$ for $L_{\mathfrak{p}}$ with $\nu_{\mathfrak{p}}(\Phi(x_i, x_i)) = R_i$ is called *good* if $R_i \leq R_{i+2}$ for all $1 \leq i \leq m-2$. Good BONGs are crucial for calculating the spinor norms of $L_{\mathfrak{p}}$.
3. The lattice $L_{\mathfrak{p}}$ has *property A* if $L_{\mathfrak{p}}$ has a Jordan splitting $L_{\mathfrak{p}} = \perp_{i=1}^t L_i$ such that $\text{rank}(L_i) \leq 2$ for all i , and for every pair $1 \leq i < j \leq t$ we have

$$0 < \nu_{\mathfrak{p}}(\mathfrak{n}(L_j)) - \nu_{\mathfrak{p}}(\mathfrak{n}(L_i)) < 2(\nu_{\mathfrak{p}}(\mathfrak{s}(L_j)) - \nu_{\mathfrak{p}}(\mathfrak{s}(L_i))).$$

4. A *maximal norm splitting* for $L_{\mathfrak{p}}$ is a splitting $L_{\mathfrak{p}} = \perp_{i=1}^t L_i$ such that each L_i is either one-dimensional or two-dimensional modular with $\mathfrak{s}(L_1) \supseteq \mathfrak{s}(L_2) \supseteq \dots \supseteq \mathfrak{s}(L_t)$ and $\mathfrak{n}(L_i) = \mathfrak{n}(L_{\mathfrak{p}}^{\mathfrak{s}(L_i)}) = \mathfrak{n}(\{x \in L_{\mathfrak{p}} \mid \Phi(x, L_{\mathfrak{p}}) \subseteq \mathfrak{s}(L_i)\})$. Note that a maximal norm splitting is not in general a Jordan splitting.

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The condition that $\mathbf{n}(L_i) = \mathbf{n}(L_{\mathfrak{p}}^{\mathfrak{s}(L_i)})$ for all $1 \leq i < j \leq t$ in the definition of a maximal norm splitting is equivalent to a relaxation of the definition of having property A, namely to

$$0 \leq \nu_{\mathfrak{p}}(\mathbf{n}(L_j)) - \nu_{\mathfrak{p}}(\mathbf{n}(L_i)) \leq 2(\nu_{\mathfrak{p}}(\mathfrak{s}(L_j)) - \nu_{\mathfrak{p}}(\mathfrak{s}(L_i))) \text{ for all } 1 \leq i < j \leq t.$$

It is easy to see that if $L_{\mathfrak{p}}$ has property A, then every Jordan splitting will also be a maximal norm splitting.

We note that a lattice is uniquely determined by a BONG, cf. [Bel03, (2.6)]. While BONGs can be trivially seen to exist for every lattice, in general the orthogonal decompositions in Jordan splittings are not BONGs (let alone good BONGs). Further, every BONG is good if the lattice has property A, cf. [Bel03, (4.2)], but there are obviously lattices that do not have property A – for example, those with a Jordan component of dimension 3 or larger. Fortunately, good BONGs can always be obtained via maximal norm splittings:

- Remark 4.31.** 1. $L_{\mathfrak{p}}$ always admits a maximal norm splitting. There is an effective algorithm for computing such a splitting.
2. Given a maximal norm splitting $L_{\mathfrak{p}} = \perp_{i=1}^t L_i$, the sequence obtained by concatenating BONGs for the individual one- and two-dimensional components L_i ($1 \leq i \leq t$) is a good BONG for $L_{\mathfrak{p}}$. Hence, good BONGs always exist.

Proof. The existence of a maximal norm splitting is shown in [Bel03, (4.6)]. For an explicit algorithm, see [Bel06, section 7]. To see 2), cf. [Bel03, (4.1)(i), (4.2)(ii)]. Note that any BONG of a two-dimensional lattice is automatically good. ■

Theorem 4.32 (BELI 2003). Let $L_{\mathfrak{p}} \cong \langle a_1, \dots, a_n \rangle$ relative to a good BONG of L . Let the function $G: K_{\mathfrak{p}}^*/(K_{\mathfrak{p}}^*)^2 \rightarrow \theta(O^+(L_{\mathfrak{p}}))$ be defined as in [Bel03, Definition 4] (which is omitted here for the sake of brevity).

1. Assume L has property A. Then the spinor norms are

$$\theta(O^+(L_{\mathfrak{p}})) = G(a_2/a_1)G(a_3/a_2) \cdots G(a_m/a_{m-1})(1 + \mathfrak{p}^{\alpha})(K_{\mathfrak{p}}^*)^2$$

where $\alpha = \min\{\frac{\nu_{\mathfrak{p}}(a_{i+2}) - \nu_{\mathfrak{p}}(a_i)}{2} \mid 1 \leq i \leq m-2\}$, and $(1 + \mathfrak{p}^{\alpha}) = \{x \in K_{\mathfrak{p}}^* \mid \nu_{\mathfrak{p}}(x^{-1}\mathfrak{d}(x)) \geq \alpha\}$ (recall the definition of $\mathfrak{d}$ in Definition 2.72).

2. Assume L does not have property A. Then either $\theta(O^+(L_{\mathfrak{p}})) = K_{\mathfrak{p}}^*$ or $\theta(O^+(L_{\mathfrak{p}})) = \mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2$, and $\theta(O^+(L_{\mathfrak{p}})) = \mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2$ if and only if

4.5 An algorithm for enumerating the genus

- a. $G(a_{i+1}/a_i) \subseteq \mathfrak{o}_{\mathfrak{p}}^*(K_{\mathfrak{p}}^*)^2$ for each $1 \leq i \leq m-1$, and:
- b. for each $1 \leq i \leq m-2$ with $\nu_{\mathfrak{p}}(a_i) = \nu_{\mathfrak{p}}(a_{i+2})$, we have $\frac{\nu_{\mathfrak{p}}(a_{i+1}) - \nu_{\mathfrak{p}}(a_i)}{2} \equiv \nu_{\mathfrak{p}}(2) \pmod{2}$.

Proof. 1) is [Bel03, Theorem 1]. The first claim in 2) is shown in [O'M73, 91:8], while the remainder is [Bel03, Theorem 3]. ■

4.5 An algorithm for enumerating the genus

We have identified three isomorphic groups – the first is a quotient of the adelic orthogonal group J_V , the second is a quotient of the idele group J , and the third is a quotient C_L of the ray class group Cl_D – which act regularly on the set of proper spinor genera in $\text{Genus}(L)$, $\{\text{spn}^+(L') \mid L' \in \text{Genus}(L)\}$, where L is any full lattice of rank $m \geq 3$ in a regular quadratic space over a totally real number field K .

The group C_L is a quotient of a finite group and can be computed efficiently. We have seen in Remark 4.21 that nice representations of its elements can be found by mapping prime ideals into C_L . Taking $g \in C_L$ and writing $g = g_{\mathfrak{p}}$ for a sufficiently chosen prime $\mathfrak{p}$, we know from Corollary 4.27 that g acts on $\text{spn}^+(L)$ by sending it to $g.\text{spn}^+(L) = \text{spn}^+(L')$, where L' is any $\mathfrak{p}$ -neighbour of L .

4.5.1 Enumerating the genus of an indefinite lattice

In the following, assume that L is an indefinite lattice in (V, Φ) of $\text{rank}(L) \geq 3$. For indefinite lattices, no algorithmic isometry test is available. The results about the group C_L that have been developed in this chapter, however, still hold. Because $\text{cls}^+(L) = \text{spn}^+(L)$ in the indefinite case (see Corollary 4.13), we could just apply each element of $C_L - \{1\}$ once to L to obtain a representative of every *proper* isometry class in $\text{Genus}(L)$. Unfortunately, whenever $\text{cls}^+(L) \subsetneq \text{cls}(L)$, this will hit every isometry class in $\text{Genus}(L)$ twice. A possible solution is to factor out a suitable element of C_L :

Proposition 4.33. Let $v \in L$ be any anisotropic vector.

1. For every $L' \in \text{Genus}(L)$, $[O(L') : O^+(L')] = [O(L) : O^+(L)]$.

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2. Consider the symmetry τ_v with respect to v (cf. Definition 2.61). Then $\text{spn}(L) = \text{spn}^+(L) \cup \text{spn}^+(\tau_v(L))$.
3. Set $\varphi := \tau_v$. Let Σ_φ denote the diagonal embedding of φ into the adelic group $\prod_{\mathfrak{p} \in \Omega} O(V_{\mathfrak{p}})$ (i.e., $(\Sigma_\varphi)_{\mathfrak{p}} = \varphi$ at every place $\mathfrak{p}$). Note that $\Sigma_\varphi \notin J_V$, because φ has determinant -1 .

Define the set $Q = \{\mathfrak{p} \in \Omega_{\text{fin}} \mid \nu_{\mathfrak{p}}(\Phi(v, v)) \geq \nu_{\mathfrak{p}}(\mathfrak{n}(L_{\mathfrak{p}}))\}$. Of course Q is finite.

For $\mathfrak{p} \in \Omega$, define $x_{\mathfrak{p}} := v$ if $\mathfrak{p} \notin Q$, and when $\mathfrak{p} \in Q$ set $x_{\mathfrak{p}}$ to be any vector such that $\Phi(x_{\mathfrak{p}}, x_{\mathfrak{p}})$ generates the norm ideal $\mathfrak{n}(L_{\mathfrak{p}})$. Let the adele Σ' be defined as $\Sigma'_{\mathfrak{p}} = \tau_{x_{\mathfrak{p}}}$ for all $\mathfrak{p} \in \Omega$.

The product $\Sigma_\varphi \Sigma'$ lies in the group J_V .

4. Let the idele $\mathfrak{j}$ be defined as $\mathfrak{j}_{\mathfrak{p}} = 1$ for all $\mathfrak{p} \notin Q$, and as $\mathfrak{j}_{\mathfrak{p}} = \Phi(v, v)\Phi(x_{\mathfrak{p}}, x_{\mathfrak{p}})$ for all $\mathfrak{p} \in Q$ (with $x_{\mathfrak{p}}$ as defined above). Then $\mathfrak{j}$ is a representative of the image of $\Sigma_\varphi \Sigma'$ under the isomorphism $J_V/J'_V P_V J_L \xrightarrow{\sim} J/K^* J_S^L$.

Set g to be the image of (the class of) $\Sigma_\varphi \Sigma'$ under the isomorphism $J_V/J'_V P_V J_L \xrightarrow{\sim} C_L$ (cf. Theorem 4.23). Then $g = 1$ if and only if $\text{spn}^+(L) = \text{spn}(L)$. If $g \neq 1$, then for every lattice $L' \in \text{Genus}(L)$, $g \cdot \text{spn}^+(L') = \text{spn}^+(L'')$, where $\text{spn}(L') = \text{spn}^+(L') \sqcup \text{spn}^+(L'')$.

So the element g switches between proper spinor genera inside $\text{Genus}(L)$. And g can easily be calculated, because it is the image of the idele $\mathfrak{j}$ which has finite support.

Proof. Part 1) is clear. Part 2) is clear from the definition of the spinor genus (symmetries have determinant -1). Of course $\text{spn}^+(L)$ and $\text{spn}^+(\tau_v(L))$ could still be identical.

Now for part 3) and 4). We have $\Sigma_\varphi(L) = \varphi(L)$ under the group action discussed in Remark 4.18. And $\Sigma'(L) = L$ because $\Sigma'_{\mathfrak{p}}(L_{\mathfrak{p}}) = L_{\mathfrak{p}}$ for every place $\mathfrak{p}$, cf. Lemma 2.62. Locally at every prime $\mathfrak{p}$, $\det((\Sigma_\varphi \Sigma')_{\mathfrak{p}}) = +1$ by construction. So $\Sigma_\varphi \Sigma' \in J_V$. And of course, $\Sigma_\varphi \Sigma'(L) = \Sigma_\varphi(\Sigma'(L)) = \Sigma_\varphi(L) = \tau_v(L)$. So by part 2), $\mathfrak{j}$, which corresponds to $\Sigma_\varphi \Sigma'$, switches between the proper spinor genera in $\text{Genus}(L)$. The rest is clear from Theorem 4.23. ■

Corollary 4.34. There is an element $g \in C_L$ (cf. Theorem 4.23), namely the element g constructed in part 4) of Proposition 4.33, such that either

4.5 An algorithm for enumerating the genus

- a) $g = 1$, and $\text{spn}^+(L') = \text{spn}(L')$ for any $L' \in \text{Genus}(L)$, or
- b) $g \neq 1$, $\text{spn}^+(L') \neq \text{spn}(L')$ for any $L' \in \text{Genus}(L)$, and the action of g on the set $\{\text{spn}^+(L') \mid L' \in \text{Genus}(L)\}$ is switching between the two proper spinor genera in any spinor genus inside $\text{Genus}(L)$.

4.5.2 The algorithms

The previous sections provide us with an explicit way to calculate a set of primes linking all the proper spinor genera in $\text{Genus}(L)$. We can now give an algorithm to enumerate $\text{Genus}(L)$. In case where L is totally definite, this can be done efficiently with an orbit algorithm (because an isometry test is available). In the indefinite case, $\text{Genus}(L)$ can still be enumerated by applying every element of C_L once to L , provided we account for the possible case of two distinct isometry classes lying in $\text{spn}^+(L)$.

Algorithm 4.35. Primes for genus enumeration

Input: Any $\mathfrak{o}$ -lattice L , and a (possibly empty) set of good primes B

Output: A set R of good primes for L , such that the set $\{g_{\mathfrak{r}} \mid \mathfrak{r} \in R\}$ generates the quotient $C_L / \langle g_{\mathfrak{q}} \mid \mathfrak{q} \in B \rangle$.

```

1: function GETPRIMES( $L, B$ )
2:    $P, D \leftarrow$  the set of primes and the modulus defined in Equation 4.1
3:    $R \leftarrow$  the ray class group  $\text{Cl}_D$ .
4:    $H \leftarrow \{x^2 \mid x \in R\}$ .
5:   for all  $\mathfrak{p} \in P$  do
6:     for all representatives  $x$  of the square classes in  $\theta(O^+(L_{\mathfrak{p}}))$  do
7:        $H \leftarrow \langle H, \Xi(j(\mathfrak{p}, x))K_S^* J^L \rangle$ , where  $\Xi$  is the epimorphism  $J/K^* \twoheadrightarrow \text{Cl}_D$  from
       Lemma 4.22.
8:     end for
9:   end for
10:   $C_L \leftarrow R/H$ .
11:   $g \leftarrow$  the element of  $C_L$  from Corollary 4.34.
12:   $G \leftarrow$  a minimal generating set of  $C_L / \langle \{g_{\mathfrak{p}} \mid \mathfrak{p} \in B\} \cup \{g\} \rangle$ .
13:  return the set  $R := \{\mathfrak{p}_g \mid g \in G\}$ .
14: end function
```

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Proof. The ideles $j(\mathfrak{p}, x)$ in step 7 are the elements of the finite set X from part 4) of Theorem 4.23, so the group generated in step 10 is in fact the group C_L from part 5) of Theorem 4.23. ■

Note that the set P in Step 2, and the spinor norms in step 6 can be calculated using Theorem 4.29 if $\mathfrak{p}$ is non-dyadic, and by calculating a maximal norm splitting of L (see Remark 4.31) and then applying Theorem 4.32 if $\mathfrak{p}$ is dyadic. For the elements $\mathfrak{p}_g$, see Definition 4.24. These can be found by simply walking through the prime ideals of $\mathfrak{o}$, starting with those of the smallest norm, and mapping them to C_L . There are infinitely many prime ideal representatives for each group element (see Remark 4.21).

To explicitly enumerate the genus, we need to calculate representatives of every proper spinor genus in $\text{Genus}(L)$. This can now be done using the neighbours algorithm, as follows:

Algorithm 4.36. Genus of a definite lattice

Input: An $\mathfrak{o}$ -lattice L in a regular, totally definite quadratic K -vector space of rank $m \geq 3$

Output: A set of representatives of the isometry classes in $\text{Genus}(L)$

```

1: function GENUSDEFINITE( $L$ )
2:    $R \leftarrow []$ 
3:    $T \leftarrow [L]$ 
4:    $\mathfrak{p} \leftarrow$  a good prime ideal of  $L$  of smallest norm
5:    $P \leftarrow \text{GETPRIMES}(L, \{\mathfrak{p}\})$ 
6:   while  $T$  is not empty do
7:     remove a lattice  $M$  from  $T$ 
8:     append representatives of all the previously unencountered isometry classes in
       the set  $\text{ITERATEDNEIGHBOURS}(M, \mathfrak{p})$  to  $R$ 
9:     for all  $\mathfrak{q} \in P$  do
10:       $N \leftarrow$  a random  $\mathfrak{q}$ -neighbour of  $M$ 
11:      if  $N$  is not isometric to some lattice in  $R$  then append  $N$  to  $T$  end if
12:    end for
13:  end while
14:  return  $R$ 
15: end function

```

Proof. This is basically an orbit algorithm.

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Line 8 ensures that for each proper spinor genus $\text{spn}^+(L') \subseteq \text{Genus}(L)$, representatives of all the isometry classes in $\mathcal{N}^*(L', \mathfrak{p})$ are added to the result. $\mathcal{N}^*(L', \mathfrak{p})$ is either equal to $\text{spn}^+(L')$ (in which case $\mathfrak{p}$ represents the identity element of C_L), or to $\text{spn}^+(L') \sqcup \text{spn}^+(L'')$ for some $L'' \in \text{Genus}(L)$ such that $g_{\mathfrak{p}} \cdot \text{spn}^+(L') = \text{spn}^+(L'')$. See Corollary 4.28.

So $\mathfrak{p}$, chosen for its small norm, can be factored out of the group calculated by GETPRIMES. Indeed, the primes returned by GETPRIMES($L, \mathfrak{p}$) generate the group $C_L / \langle g_{\mathfrak{p}}, g \rangle$, where g is the element from Corollary 4.34 (and which does not yield any new isometry classes). ■

Algorithm 4.37. Genus of an indefinite lattice

Input: An $\mathfrak{o}$ -lattice L in a regular indefinite quadratic K -vector space of rank $m \geq 3$

Output: A set of representatives of the isometry classes in $\text{Genus}(L)$

```

1: function GENUSINDEFINITE( $L$ )
2:    $R \leftarrow []$ 
3:    $P = [\mathfrak{p}_1, \dots, \mathfrak{p}_s] \leftarrow \text{GETPRIMES}(L, \{ \})$ 
4:    $N = [N_1, \dots, N_s] \leftarrow$  a  $\mathfrak{p}_i$ -neighbour of  $L$ , for every  $1 \leq i \leq s$ .
5:   for all subsets  $I \subseteq \{1, \dots, s\}$  do
6:      $M \leftarrow$  the lattice in  $V$  with  $M_{\mathfrak{p}_i} = N_i$  for every  $i \in I$ , and with  $M_{\mathfrak{q}} = L_{\mathfrak{q}}$  for
       all other places  $\mathfrak{q} \in \Omega$ .
7:     append  $M$  to  $R$ .
8:   end for
9:   return  $R$ 
10: end function
```

Proof. This variant of the genus algorithm works without an isometry test, which is not available in the indefinite case. GETPRIMES factors out the element g from Corollary 4.34, which (possibly) switches between two proper spinor genera inside a spinor genus in $\text{Genus}(L)$. The remaining elements of C_L , which are all applied once to L , result in 2^s lattices representing the 2^s distinct proper spinor in $\text{Genus}(L)$, with no isometry class hit twice. ■

5 Completing the classification of single-class lattices

In this chapter we will present an algorithm for classifying, up to isometry, all similarity classes of lattices L with $\text{rank}(L) \geq 3$ in regular, totally definite quadratic spaces (V, Φ) over a given totally real number field K with the property that $\text{cls}(L) = \text{Genus}(L)$.

Remark 5.1. In the following, we will focus entirely on (totally) definite lattices. The indefinite case does not seem very interesting, and can in large parts be answered purely from the local theory. Whenever $\dim(V) \geq 3$ and L is indefinite, $\text{cls}^+(L) = \text{spn}^+(L)$ (see Corollary 4.13). Further, O'MEARA gives sufficient conditions for when $\text{spn}^+(L) = \text{Genus}(L)$ (see [O'M73, 102:9, 102:10]).

For example, from [O'M73, 102:9, 102:10, 104:5] one can conclude that there are infinitely many isometry classes of quadratic $\mathbb{Q}$ -vector spaces admitting lattices L with $\text{cls}^+(L) = \text{spn}^+(L) = \text{Genus}(L)$. Essentially, any integral, indefinite $\mathbb{Z}$ -lattice of dimension 3 whose local structure is simple enough that $\theta(O^+(L_{\mathfrak{p}})) \supseteq \mathfrak{o}_{\mathfrak{p}}^*$ for all $\mathfrak{p}$ is alone in its genus.

5.1 A method of descent

We will quickly recall a method used by WATSON to reduce any lattice L of class-number c to a lattice with simpler local structure of class-number $\leq c$. An adaptation designed by GERSTEIN is fit for use in arbitrary number fields.

We recall Definition 2.53: Given any fractional ideal $\mathfrak{a}$ of $\mathfrak{o}$, we have

$$L^{\mathfrak{a}} = \{x \in L \mid \Phi(x, L) \in \mathfrak{a}\} = \mathfrak{a}L^{\#} \cap L.$$

The effect of passing from L to $L^{\mathfrak{a}}$, locally at a prime $\mathfrak{p}$, can be described in terms of the Jordan decomposition of L at $\mathfrak{p}$:

5 Completing the classification of single-class lattices

Remark 5.2. If $\mathfrak{a}$ is a fractional ideal of K , and L is $\mathfrak{b}$ -modular for some fractional ideal $\mathfrak{b}$, then

$$L^{\mathfrak{a}} = \begin{cases} L, & \text{if } \mathfrak{b} \subseteq \mathfrak{a} \\ \mathfrak{a}\mathfrak{b}^{-1}L, & \text{if } \mathfrak{b} \supseteq \mathfrak{a} \end{cases}.$$

Further, for any prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, $(L^{\mathfrak{a}})_{\mathfrak{p}} = (L_{\mathfrak{p}})^{\mathfrak{a}_{\mathfrak{p}}}$.

For the (easy) proofs of these statements, see [O'M73, § 82I].

Corollary 5.3. Let $L_{\mathfrak{p}} = L_0 \perp L_1 \perp \dots \perp L_k$ be a Jordan decomposition of the completion of some integral lattice L at $\mathfrak{p}$, where each L_i is a $\mathfrak{p}^i$ -modular lattice in $V_{\mathfrak{p}}$, and $k \geq 3$ (we allow that any L_i can be the zero lattice). Then $L_{\mathfrak{q}} = (L^{\mathfrak{p}})_{\mathfrak{q}}$ for all primes $\mathfrak{q} \neq \mathfrak{p}$, and

$$(L^{\mathfrak{p}})_{\mathfrak{p}} = L_1 \perp (\mathfrak{p}L_0 \perp L_2) \perp \dots \perp L_k.$$

Definition 5.4. Let L be a lattice in $(V_{\mathfrak{p}}, \Phi)$ with Jordan decomposition

$$L_{\mathfrak{p}} = L_j \perp L_{j+1} \perp \dots \perp L_k,$$

for some integers $j \leq k \in \mathbb{Z}$, such that L_i is $\mathfrak{p}^i$ -modular for all $j \leq i \leq k$, and where L_j and L_k are not the zero lattice (though we do allow that $j = k$, i.e., $L_{\mathfrak{p}}$ is modular). Then by the $\mathfrak{p}$ -length $\text{len}_{\mathfrak{p}}(L_{\mathfrak{p}})$, we mean the number $k - j + 1$.

Definition 5.5. An $\mathfrak{o}$ -lattice L is called *square-free at $\mathfrak{p}$* , if $L_{\mathfrak{p}}$ is integral and $L \subseteq \mathfrak{p}L^{\#}$. If this property holds for all $\mathfrak{p} \in \Omega_{\text{fin}}$, L is called *square-free*.

Equivalently, being square-free at $\mathfrak{p}$ means that any Jordan decomposition of $L_{\mathfrak{p}}$ can only have a unimodular and a $\mathfrak{p}$ -modular component.

We have seen that passing to the lattice $L^{\mathfrak{p}}$ merely “glues” the $\mathfrak{o}_{\mathfrak{p}}$ -modular and the $\mathfrak{p}^2$ -modular Jordan components together. And we can say with some justification that the resulting lattice is less complex than L , for if $\mathfrak{s}(L_{\mathfrak{p}}) = \mathfrak{o}_{\mathfrak{p}}$ and $\text{len}_{\mathfrak{p}}(L) \geq 2$, then $\text{len}_{\mathfrak{p}}(L^{\mathfrak{p}}) < \text{len}_{\mathfrak{p}}(L)$.

These properties of the lattice $L^{\mathfrak{p}}$ have been used by WATSON (cf. [Wat62]) to reduce any definite $\mathbb{Z}$ -lattice to a square-free lattice, without increasing the class number. Note that if $\mathfrak{s}(L_{\mathfrak{p}}) = \mathfrak{o}_{\mathfrak{p}}$, then $\mathfrak{s}((L^{\mathfrak{p}})_{\mathfrak{p}}) \subseteq \mathfrak{p}$ and $(L^{\mathfrak{p}})^{\mathfrak{p}} = L^{\mathfrak{p}}$. This is why the reduction method Watson used (and which he called the “ $\mathfrak{p}$ -mapping”) maps the module L to $L^{\mathfrak{p}}$ but also

5.1 A method of descent

rescales the form Φ by a totally positive generator of $\mathfrak{p}^{-1}$. This rescaling is not in general possible when the narrow class group of K is nontrivial.

We will now present a slight adaptation by GERSTEIN (cf. [Ger72]) to Watson's method that works over general number fields. The basic idea is that when Watson's $\mathfrak{p}$ -mapping is applied twice, then Φ is rescaled by a square, and so the square of the Watson $\mathfrak{p}$ -mapping can be obtained from L by changing only the module L and not the enveloping quadratic space.

Definition 5.6. Let L be any $\mathfrak{o}$ -lattice in (V, Φ) , and $\mathfrak{a}$ any fractional ideal of K . The *Gerstein $\mathfrak{a}$ -mapping* of L is

$$\text{Ger}_{\mathfrak{a}}(L) = L + \mathfrak{a}^{-1}(L^{\mathfrak{a}^2}).$$

Lemma 5.7. Assume L is integral, with a Jordan decomposition $L_{\mathfrak{p}} = L_0 \perp \dots \perp L_k$ locally at the prime $\mathfrak{p}$, where each L_i is a $\mathfrak{p}^i$ -modular $\mathfrak{o}_{\mathfrak{p}}$ -lattice, for $0 \leq i \leq k$.

Then

1. $\text{Ger}_{\mathfrak{p}}(L)_{\mathfrak{q}} = L_{\mathfrak{q}}$ at all primes $\mathfrak{q} \neq \mathfrak{p}$, and $\text{Ger}_{\mathfrak{p}}(L)_{\mathfrak{p}}$ has a Jordan decomposition

$$\text{Ger}_{\mathfrak{p}}(L)_{\mathfrak{p}} = L_0 \perp L_1 \perp \mathfrak{p}^{-1}(L_2 \perp \dots \perp L_k).$$

It follows that Gerstein maps commute:

$$\text{Ger}_{\mathfrak{p}}(\text{Ger}_{\mathfrak{q}}(L)) = \text{Ger}_{\mathfrak{q}}(\text{Ger}_{\mathfrak{p}}(L)) \text{ for any } \mathfrak{p}, \mathfrak{q} \in \Omega_{\text{fin}}(K).$$

2. $\text{Ger}_{\mathfrak{p}}(L) \neq L$ if and only if L has a non-zero $\mathfrak{p}^j$ -modular component for any $j \geq 2$. In particular, the Gerstein map is not well-defined on the similarity class of L (if L is similar to L' , then $\text{Ger}_{\mathfrak{p}}(L)$ is not necessarily similar to $\text{Ger}_{\mathfrak{p}}(L')$).
3. $h(\text{Ger}_{\mathfrak{p}}(L)) \leq h(L)$. The Gerstein $\mathfrak{p}$ -map is well-defined on isometry classes (if $L \cong L'$, then $\text{Ger}_{\mathfrak{p}}(L) \cong \text{Ger}_{\mathfrak{p}}(L')$), and similarly on genera of lattices.
4. $L \subseteq \text{Ger}_{\mathfrak{p}}(L) \subseteq \mathfrak{p}^{-1}L$.
5. $\text{Ger}_{\mathfrak{p}}(L)$ is integral, and $\mathfrak{s}(\text{Ger}_{\mathfrak{p}}(L)) \supseteq \mathfrak{s}(L)$. If $L_{\mathfrak{p}}$ is not squarefree and $\text{len}_{\mathfrak{p}}(L) = \text{len}_{\mathfrak{p}}(\text{Ger}_{\mathfrak{p}}(L))$, then $\mathfrak{s}(\text{Ger}_{\mathfrak{p}}(L)_{\mathfrak{p}}) \supsetneq \mathfrak{s}(L_{\mathfrak{p}})$.

5 Completing the classification of single-class lattices

Proof. By Remark 5.2, $(L^{\mathfrak{p}^2})_{\mathfrak{p}} = \mathfrak{p}^2 L_0 \perp \mathfrak{p} L_1 \perp (L_2 \perp \dots \perp L_k)$, so indeed $(L + \mathfrak{p}^{-1} L^{\mathfrak{p}^2})_{\mathfrak{p}} = L_0 \perp L_1 \perp \mathfrak{p}^{-1} (L_2 \perp \dots \perp L_k)$. Now claims 2), 4) and 5) follow immediately. Part 3) holds because taking sums, intersections, or duals of lattices commutes with isometries – see [Ger72, Theorem 3.6]. ■

The Gerstein map provides a way of reducing any integral lattice to a squarefree lattice:

Corollary 5.8. Let L be an integral $\mathfrak{o}$ -lattice. Then there is a sequence $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ ($s \geq 0$) of prime ideals such that the $\mathfrak{o}$ -lattice $(\text{Ger}_{\mathfrak{p}_s} \circ \dots \circ \text{Ger}_{\mathfrak{p}_1})(L)$ is squarefree. These prime ideals are unique up to permutation.

Proof. This is just a restatement of part 5) of Lemma 5.7. ■

5.2 Explicit enumeration of single-class lattices in quadratic spaces

Recall the definition of square-free lattices in Definition 5.5. We will need another, slightly stronger definition:

Definition 5.9. We fix a set $P^+(K)$ of representatives of Cl_K^+ , the narrow class group of K , such that $P^+(K)$ always includes $\mathfrak{o}$, and every other element of $P^+(K)$ is a prime ideal.

Let L be an integral $\mathfrak{o}$ -lattice in V . We call L *terminal*, if each of the following conditions hold:

1. L_{ν} is positive definite for every $\nu \in \Omega_{\infty}$.
2. $L_{\mathfrak{p}}$ is square-free for every $\mathfrak{p} \in \Omega_{\text{fin}}$.
3. There is a place $\mathfrak{q} \in P^+(K)$ such that at every place $\mathfrak{q} \neq \mathfrak{p} \in \Omega_{\text{fin}}$, the rank of the $\mathfrak{p}^1$ -Jordan component of L (denoted $L_{\mathfrak{p},1}$) is at most as large as the rank of the $\mathfrak{p}^0$ -Jordan component (denoted $L_{\mathfrak{p},0}$).

Remark 5.10. The notion of a terminal lattice obviously depends on the choice of the set $P^+(K)$. We usually assume that this set is fixed for each number field K that will occur.

5.2 Explicit enumeration of single-class lattices in quadratic spaces

Remark 5.11. Let L be an integral $\mathfrak{o}$ -lattice in V .

1. If L is terminal, then $\mathfrak{s}(L) = \mathfrak{o}$ or $\mathfrak{s}(L) = \mathfrak{q}$, where $\mathfrak{q} \in P^+(K)$ as mentioned in Definition 5.9.
2. The similarity class of L has a representative L' such that there is a sequence $\mathfrak{p}_1, \dots, \mathfrak{p}_t \in \Omega_{\text{fin}}(K)$ such that $\text{Ger}_{\mathfrak{p}_t} \circ \dots \circ \text{Ger}_{\mathfrak{p}_1}(L')$ is a terminal lattice.

Proof. Part 1) is clear. To see part 2), define the finite set

$$S := \left\{ \mathfrak{p} \in \Omega_{\text{fin}}(K) \mid \sum_{i \in 2\mathbb{N}_0} L_{\mathfrak{p},i} < \frac{1}{2} \text{rank}(L) \right\},$$

where $L_{\mathfrak{p},i}$ denotes the rank of the Jordan component of $L_{\mathfrak{p}}$ with scale $\mathfrak{p}^i$. Now choose $\mathfrak{q} \in P^+(K)$ such that $\mathfrak{q} \prod_{\mathfrak{p} \in S} \mathfrak{p}$ has a totally positive generator $\alpha \in K$, and replace L by the rescaling $L' := L^\alpha$. Then L' has the desired property: certainly L' can be reduced to a square-free lattice L'' by repeated applications of Gerstein maps. And for any lattice M , $\sum_{i \in 2\mathbb{N}_0} M_{\mathfrak{p},i} = \sum_{i \in 2\mathbb{N}_0} \text{Ger}_{\mathfrak{p}}(M)_{\mathfrak{p},i}$ (see part 1) of Lemma 5.7), so for every $\mathfrak{q} \neq \mathfrak{p} \in \Omega_{\text{fin}}(K)$, $L''_{\mathfrak{p},1} \leq L''_{\mathfrak{p},0}$. ■

In this section, we will establish that, up to isometry, there is only a finite number of terminal lattices in regular quadratic spaces over totally real number fields which have class-number 1. We will state an explicit algorithm to calculate these. We need the following Lemma for preparation:

Lemma 5.12. Let L be an integral $\mathfrak{o}$ -lattice in (V, Φ) . Let $\mathfrak{p} \in \Omega_{\text{fin}}$.

1. If $\nu_{\mathfrak{p}}(\det(V, \Phi))$ is odd, then $\mathfrak{p} \mid \mathfrak{v}(L)$.
2. If $\nu_{\mathfrak{p}}(\det(V, \Phi))$ is even, $2 \notin \mathfrak{p}$ and $c_{\mathfrak{p}}(V) = -1$ then $\mathfrak{p}^2$ divides $\mathfrak{v}(L)$.

Proof. Part 1): L is integral, so $\nu_{\mathfrak{p}}(\mathfrak{v}(L)) \geq 0$, and we always have $\nu_{\mathfrak{p}}(\det(V, \Phi)) \equiv \nu_{\mathfrak{p}}(\mathfrak{v}(L)) \pmod{2}$. Now for proving part 2), the local lattice $L_{\mathfrak{p}}$ can be diagonalised because $2 \notin \mathfrak{p}$. Assume that $L_{\mathfrak{p}}$ is unimodular. Then a Gram matrix of $L_{\mathfrak{p}}$ has the form $\langle \varepsilon_1, \dots, \varepsilon_m \rangle$ where the ε_i are units of $\mathfrak{o}_{\mathfrak{p}}$, and this diagonal matrix is of course also a Gram matrix of the quadratic space $(V_{\mathfrak{p}}, \Phi)$. But units in a non-dyadic local field always have Hilbert symbol $+1$ (see Remark 2.35), so $c_{\mathfrak{p}}(V) = 1$ in contradiction to our assumption. By the same argument as in part 1), we see that indeed $\nu_{\mathfrak{p}}(\mathfrak{v}(L)) \geq 2$. ■

5 Completing the classification of single-class lattices

The following algorithm constructs candidates of all quadratic spaces possibly admitting single-class lattices:

Algorithm 5.13. Candidate spaces for single-class lattices

Input: K a totally real number field with ring of integers $\mathfrak{o}$, and $m \geq 3$ some integer.

Output: A finite list T of all (totally) positive definite regular quadratic spaces of rank m over K , such that if V is another (totally) positive definite regular quadratic space of rank m over K that admits a single-class lattice, then V is isometric to some space $V' \in T$.

```

1: function SPACESWITHTERMINALLATTICES( $K, m$ )
2:    $P \leftarrow \{\mathfrak{p} \in \Omega_{\text{fin}}(K) \mid 2 \in \mathfrak{p}\} \cup \{\mathfrak{p} \in \Omega_{\text{fin}}(K) \mid \mathfrak{N}(\mathfrak{p}) \leq \text{snb}(K, m)\} \cup (P^+(K) \cap \Omega_{\text{fin}}(K))$ . For the definition of  $\text{snb}(K, m)$ , cf. Corollary 3.29.
3:    $[\mathfrak{a}_1, \dots, \mathfrak{a}_h] \leftarrow$  a set of representatives of  $\text{Cl}(K)$ .
4:    $[u_1, \dots, u_s] \leftarrow$  a set of representatives of  $(K^{\geq 0} \cap \mathfrak{o}^*)/(\mathfrak{o}^*)^2$ .
5:    $T \leftarrow []$ .
6:   for all  $I \subseteq P$  and  $1 \leq i \leq h$  do
7:     if  $\mathfrak{a}_i^2 \prod_{\mathfrak{p} \in I} \mathfrak{p}$  is generated by some totally positive element  $d$  then
8:       for all  $J \subseteq P$  where the cardinality  $\#J$  is an even number do
9:         for all  $j \in \{1, \dots, s\}$  do
10:           $(V, \Phi) \leftarrow \text{CONSTRUCTSPACE}(\mathfrak{o}, m, du_j, J, \{\})$ 
11:          Append  $(V, \Phi)$  to  $T$ .
12:         end for
13:       end for
14:     end if
15:   end for
16:   return  $T$ .
17: end function

```

Proof. Let (V, Φ) be a (totally) positive definite, regular quadratic space of rank m over K , and suppose that L is a single-class, terminal lattice in (V, Φ) . Choose a prime $\mathfrak{p}$ of $\mathfrak{o}$ such that $\mathfrak{p} \nmid 2$, and $\mathfrak{p} \notin P^+(K)$. Then either $L_{\mathfrak{p}}$ is unimodular or $\mathfrak{N}(\mathfrak{p}) \leq \text{snb}(K, m)$.

By Lemma 5.12, we know $\nu_{\mathfrak{p}}(\det(L_{\mathfrak{p}}))$ is even for all $\mathfrak{p} \notin P$, where P is chosen as in line 2.

Hence $\det(V, \Phi)$ is represented by some totally positive generator of $\mathfrak{a}^2 \prod_{\mathfrak{p} \in I} \mathfrak{p}$ for some ideal $\mathfrak{a}$, and some subset $I \subseteq P$. So in one of the iterations $1 \leq j \leq s$, $\det(V, \Phi)$ is represented by du_j .

5.2 Explicit enumeration of single-class lattices in quadratic spaces

If $\mathfrak{p} \nmid 2$, $\mathfrak{p} \notin P^+(K)$ with $c_{\mathfrak{p}}(V, \Phi) = -1$, then by Lemma 5.12 it follows that $L_{\mathfrak{p}}$ is not unimodular (but squarefree). So $\mathfrak{N}(\mathfrak{p})$ must be less than $\text{snb}(K, m)$.

So, line 10 will at some point construct a space isometric to (V, Φ) , since it considers all possible determinants and sets $\{\mathfrak{p} \mid c_{\mathfrak{p}}(V, \Phi) = -1\}$. ■

Algorithm 5.14. Calculate single-class, terminal lattices

Input: K a number field with ring of integers $\mathfrak{o}$, and $m \geq 3$ some integer

Output: A list of all terminal lattices in regular, totally positive definite quadratic spaces of rank m over $\mathfrak{o}$ which have class-number 1 (up to similarity).

```

1: function SINGLECLASSTERMINALLATTICES( $K, m$ )
2:    $S \leftarrow \text{SPACESWITHTERMINALLATTICES}(K, m)$ 
3:    $T \leftarrow \emptyset$ 
4:   for all  $(V, \Phi) \in S$  do
5:      $P \leftarrow \{\mathfrak{p} \in \Omega_{\text{fin}}(K) \mid \mathfrak{p} \mid 2\mathfrak{o} \text{ or } c_{\mathfrak{p}}(V, \Phi) = -1 \text{ or } \nu_{\mathfrak{p}}(\det(V, \Phi)) \text{ is odd}\} \cup$ 
        $(P^+(K) \cap \Omega_{\text{fin}}(K))$ 
6:      $L \leftarrow$  any integral lattice in  $(V, \Phi)$ 
7:      $M \leftarrow \text{MAXIMALLATTICE}(L)$ 
8:     for all  $\mathfrak{p} \in P$  do
9:        $\mathcal{L}_{\mathfrak{p}} \leftarrow$  a system of representatives of the genera represented by lattices in
       the following set:

```

$$\{\text{full } \mathfrak{o}\text{-lattices } L \text{ in } V \mid L_{\mathfrak{q}} = M_{\mathfrak{q}} \text{ for } \mathfrak{q} \neq \mathfrak{p}, L_{\mathfrak{p}} \text{ is squarefree, and } \text{rank}(L_{\mathfrak{p},0}) \geq \frac{m}{2} \text{ if } \mathfrak{p} \notin P^+(K)\}$$

```

10:     $\mathcal{L} \leftarrow \{\bigcap_{\mathfrak{p} \in P} L_{\mathfrak{p}} \mid L_{\mathfrak{p}} \in \mathcal{L}_{\mathfrak{p}}\}$ 
11:     $\mathcal{L} \leftarrow \{L \in \mathcal{L} \mid h(L) = 1 \text{ and } L \text{ is terminal}\}$ 
12:     $T \leftarrow T \cup \mathcal{L}$ .
13:  end for
14: end for
15:  $T \leftarrow$  a system of representatives of the similarity classes represented by  $T$ .
16: return  $T$ 
17: end function

```

Proof. Let L be a terminal lattice in one of the spaces V contained in S , and suppose $h(L) = 1$. Assume $\mathfrak{p} \notin P$. Then $L_{\mathfrak{p}}$ is unimodular, and hence isometric to $M_{\mathfrak{p}}$. In particular, L is in the genus of some lattice $L' \in \mathcal{L}$, where $\mathcal{L}$ is defined as in line 11. ■

5 Completing the classification of single-class lattices

Remark 5.15. 1. If $L \subseteq (V, \Phi)$ is any (terminal) lattice, then $L_{\mathfrak{p}}$ can be described by local invariants, cf. Theorem 2.79 and Theorem 2.85.

Algorithm 2.92 finds a sublattice $X \subseteq M$ such that $X_{\mathfrak{q}} = M_{\mathfrak{q}}$ for all $\mathfrak{q} \neq \mathfrak{p}$ and that $X_{\mathfrak{p}}$ has exactly these invariants. So this is how $\mathcal{L}_{\mathfrak{p}}$ can be constructed in line 9.

2. The set $(K^{\geq 0} \cap \mathfrak{o}^*)/(\mathfrak{o}^*)^2$ acts on the lattices in the set T . This is why, in the last line 15, one has to remove duplicates. This can generally be done as follows:

If L and M are two totally positive definite $\mathfrak{o}$ -lattices of rank m , then by setting $\mathfrak{a} = \mathfrak{s}(M)/\mathfrak{s}(L)$ one has either $1 \neq \mathfrak{a} \in \text{Cl}^+(K)$, in which case L and M are not similar, or $\mathfrak{a} = a \cdot \mathfrak{o}$ for some totally positive $a \in K$. Test the rescalings L^{au} for isometry to M , where u runs through a system of representatives of $(\mathfrak{o}^* \cap K^{\geq 0})/(\mathfrak{o}^*)^2$. A necessary condition for this isometry to hold is the easier test for $\mathfrak{n}(M) = \mathfrak{a}^m \mathfrak{n}(L)$.

Note that, in our setting, $\mathfrak{s}(M)$ and $\mathfrak{s}(L)$ are both elements of $P^+(K)$, so if M and L are similar, then $\mathfrak{s}(M) = \mathfrak{s}(L)$, and one only has to test for isometry arising from rescalings with units.

We are now prepared to solve the classification problem underlying this thesis.

5.2 Explicit enumeration of single-class lattices in quadratic spaces

Algorithm 5.16. Calculate single-class lattices

Input: K a number field with ring of integers $\mathfrak{o}$, and $m \geq 3$ some integer.

Output: A list of all lattices in regular, totally positive definite quadratic spaces of rank m over $\mathfrak{o}$ which have class-number 1 (up to similarity).

```

1: function SINGLECLASSLATTICES( $K, m$ )
2:    $T \leftarrow \text{SINGLECLASSTERMINALLATTICES}(K, m)$ 
3:   for all  $L \in S'$  do
4:      $P \leftarrow \{\mathfrak{p} \in \Omega_{\text{fin}}(K) \mid \mathfrak{p} \mid 2\mathfrak{v}(L)\} \cup \{\mathfrak{p} \in \Omega_{\text{fin}}(K) \mid \frac{\mathfrak{N}(\mathfrak{p})^{m-1}d_{\mathfrak{p}}(L, L)}{2(1+\mathfrak{N}(\mathfrak{p})^{-1})^\beta} \leq \frac{1}{2}\#O(L)\}$ 
       with  $\beta = 2$  when  $m = 4$  and  $\beta = 1$  otherwise, cf. Lemma 3.31, where  $d_{\mathfrak{p}}(L, L)$  is
       given by Theorem 3.8.
5:      $\mathcal{L} \leftarrow [L]$ 
6:     for all  $\mathfrak{p} \in P$  do
7:        $i \leftarrow 1$ 
8:       while  $i \leq \#\mathcal{L}$  do
9:          $U \leftarrow$  a set of representatives of the genera in the full pre-image  $\text{Ger}_{\mathfrak{p}}^{-1}(\mathcal{L}[i])$ .
10:        for all  $X \in U$  do
11:          If  $h(X) = 1$ ,  $\mathfrak{s}(X) \in P^+(K)$  and  $X$  is not similar to any  $Y \in \mathcal{L}$ ,
          then append  $X$  to  $\mathcal{L}$ .
12:        end for
13:         $i \leftarrow i + 1$ 
14:      end while
15:    end for
16:     $T \leftarrow T \cup \{L \in \mathcal{L} \mid L \text{ is not similar to any lattice in } T\}$ 
17:  end for
18:  return  $T$ 
19: end function

```

Proof. Assume the algorithm does not terminate. Then it constructs an infinite sequence of lattices $L_1, L_2, \dots$ and some fixed prime $\mathfrak{p} \in \Omega_{\text{fin}}(K)$ such that $\text{Ger}_{\mathfrak{p}}(L_i) = L_{i-1}$ for all $i \geq 2$. Now $\mathfrak{s}(L_i) \in P^+(K)$, so this means that the set $\{\text{len}_{\mathfrak{p}}(L_i) \mid i \in \mathbb{N}\}$ is not bounded. Then Theorem 3.8 shows that $\text{Mass}(L_i) \rightarrow \infty$ as $i \rightarrow \infty$. This contradicts the assumption that $h(L_i) = 1$, i.e., $\text{Mass}(L_i) \leq \frac{1}{2}$.

Let L be any single-class, (totally) positive definite $\mathfrak{o}$ -lattice of rank $m \geq 3$. After rescaling we may assume that L is integral and that $\mathfrak{s}(L) \in P^+(K)$. By Corollary 5.8 there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$, such that $M := \text{Ger}_{\mathfrak{p}_1} \circ \text{Ger}_{\mathfrak{p}_2} \circ \dots \circ \text{Ger}_{\mathfrak{p}_s}(L)$ is square-free. We have $h(M) = 1$ because the Gerstein mappings do not increase the class-number.

5 Completing the classification of single-class lattices

And $\mathfrak{s}(M) \in P^+(K)$ due to the properties of the Gerstein mapping.

Set $B := \{\mathfrak{p} \in \Omega_{\text{fin}}(K) \mid \mathfrak{p} \notin P^+(K) \text{ and } \text{rank}(M_{\mathfrak{p},0}) < \frac{m}{2}\}$.

Choose some prime ideal $\mathfrak{q} \in P^+(K)$ such that $\mathfrak{a} := \mathfrak{q} \prod_{\mathfrak{p} \in B} \mathfrak{p}$ has a totally positive generator a . Then $M' := (M^{\#, \mathfrak{a}})^a$ is terminal and also has class-number 1 (see the remark at the end of Definition 2.54). So by construction, M' is similar to a lattice in S .

If $\mathfrak{p} \in B$, then $M_{\mathfrak{p}}$ is not unimodular and $\mathfrak{s}(M_{\mathfrak{p}}) \neq \mathfrak{p}$. Hence $M_{\mathfrak{p}}$ is not modular, so neither can $M'_{\mathfrak{p}}$ be modular. This shows that M is similar to a lattice in S' , so without loss of generality we may assume that $M \in S$.

Because $\text{Ger}_{\mathfrak{p}_1} \circ (\text{Ger}_{\mathfrak{p}_2} \circ \dots \circ \text{Ger}_{\mathfrak{p}_s})(L) = M \in S'$, it follows that there exists a pre-image $X \in \text{Ger}_{\mathfrak{p}_1}^{-1}(M)$ such that $h(X) = 1$. But then $\mathfrak{p}_1 \in P$ as defined in line 4. Because all Gerstein mappings commute (cf. Lemma 5.7), it follows that all the $\mathfrak{p}_i$ lie in P . By induction on s it follows that the algorithm will at some point construct a lattice similar to L . This proves completeness of the list. $\blacksquare$

Remark 5.17. The full pre-image $\text{Ger}_{\mathfrak{p}}^{-1}(L)$ in line 9 can be calculated as follows: let $K \in \text{Ger}_{\mathfrak{p}}^{-1}(L)$, and let $L_{\mathfrak{p}} = L_0 \perp \dots \perp L_j$ be a Jordan decomposition of L at $\mathfrak{p}$ (as usual, $\mathfrak{s}(L_i) = \mathfrak{p}^i$ for every $0 \leq i \leq j$). Then by part 1) of Lemma 5.7, $K_{\mathfrak{p}} = K_0 \perp K_1 \perp K_2 \perp K_3 \perp \mathfrak{p}(L_2 \perp L_3 \perp \dots \perp L_j)$ is a Jordan decomposition of K at $\mathfrak{p}$, where:

1. $\mathfrak{s}(K_0) = \mathfrak{o}_{\mathfrak{p}}$, $\mathfrak{s}(K_2) = \mathfrak{p}^2$, and $K_0 \perp \mathfrak{p}^{-1}K_2 = L_0$, and
2. $\mathfrak{s}(K_1) = \mathfrak{p}$, $\mathfrak{s}(K_3) = \mathfrak{p}^3$, and $K_1 \perp \mathfrak{p}^{-1}K_3 = L_1$.

So to calculate the full pre-image, one has to write down all possible Gram matrices for the local lattices $K_0, \dots, K_3$ in $(V_{\mathfrak{p}}, \Phi)$, such that the conditions 1. and 2. are fulfilled. Because $L \subseteq \text{Ger}_{\mathfrak{p}}(L) \subseteq \mathfrak{p}^{-1}L$ (see part 4) of Lemma 5.7), a lattice K in (V, Φ) fulfilling $K_{\mathfrak{p}} = K_0 \perp K_1 \perp K_2 \perp K_3 \perp \mathfrak{p}(L_2 \perp L_3 \perp \dots \perp L_j)$ can then be found by Algorithm 2.92.

6 Results

6.1 Overview of results

Algorithm 5.16 was implemented in the MAGMA computer algebra system by the author. The underlying MAGMA package for computations with quadratic lattices over number fields ([Kir]) is a joint work by M. KIRSCHMER and the author. The result is a classification of single-class lattices in regular, definite quadratic spaces over number fields. We begin this chapter with a summary of the classification:

Theorem 6.1. Suppose L is a lattice in a regular, totally definite quadratic space V over a totally real number field K . Suppose that the class number of L is 1, and suppose $m = \text{rank}(L) \geq 3$.

1. Then $m \leq 10$ if $K = \mathbb{Q}$, and $m \leq 6$ otherwise.
2. Up to isometry of V and isomorphism of K , the similarity class of L is one of 5903 distinct similarity classes:
 - a. 1884 similarity classes of one-class genera of lattices over $K = \mathbb{Q}$. The following is the number of similarity classes for each rank $3 \leq \text{rank}(L) \leq 10$:

$\text{rank}(L)$	3	4	5	6	7	8	9	10
no. of classes	794	481	295	186	86	36	4	2

In a previous publication by the author and M. KIRSCHMER, these lattices were classified using the *genus symbol* developed by CONWAY and SLOANE in [CS99, Chapter 15]. For details, see [LK13].¹ Gram matrices of these rational lattices can be found in the Catalogue of Lattices ([NS]).

¹The two authors' respective contributions to [LK13] are disclosed in the publication list associated with this thesis.

6 Results

- b. 3408 similarity classes of one-class genera of ternary lattices over 29 different base fields with $[K : \mathbb{Q}] > 1$. The largest field degree is 5. These lattices were previously published by M. KIRSCHMER and the author in [KL16].² A principal result of this previous publication is a bound to the root discriminant of possible base fields in the ternary case, where no usable result can be obtained from the general results developed in the present thesis (cf. Proposition 3.26 and Remark 3.28). The proof of the bound from [KL16] was summarised in Section 3.4.
- c. 607 similarity classes of one-class genera of quaternary (i.e., $m = 4$) lattices over 22 different base fields with $[K : \mathbb{Q}] > 1$. The largest field degree is 5.
- d. 2 similarity classes of one-class genera of quinary (i.e., $m = 5$) lattices with $[K : \mathbb{Q}] > 1$, both over $K = \mathbb{Q}(\sqrt{5})$. Setting L to be a representative of one of these similarity classes, its dual $L^\#$ represents the other. The lattice H_4 from R. SCHARLAU's classification of even unimodular lattices over real quadratic fields ([Sch94, Thm. 3.4]) is similar to an orthogonal summand of L .
- e. 2 similarity classes of one-class genera of senary (i.e., $m = 6$) lattices with $[K : \mathbb{Q}] > 1$, both over $K = \mathbb{Q}(\sqrt{5})$. Again, H_4 appears as an orthogonal summand, and the similarity classes have representatives which are each other's dual.

²By Theorem 3.42, the isomorphism class of $\mathfrak{D}(L)$ contains $h(K)$ similarity classes of ternary lattices. Under the broader definition of similarity that was used in [KL16], there are 3400 (broad) similarity classes of (totally) positive definite single-class ternary lattices over totally real number fields. More precisely, [KL16, Thm. 1] gives a total of 4194 which includes 794 lattices over $K = \mathbb{Q}$. Under the notion of similarity used in the present work (cf. Definition 2.59), any (broad) similarity class is the union of $h(K)$ similarity classes. There is only one base field K admitting single-class ternary lattices that fulfills $h(K) > 1$, namely $K = \mathbb{Q}(\sqrt{15})$. Now $h(\mathbb{Q}(\sqrt{15})) = 2$ and there are 8 (broad) similarity classes of single-class lattices over $\mathbb{Q}(\sqrt{15})$. This explains the discrepancy between the number 3408 given here, and the number 3400 given in [KL16].

6.2 List of base fields

In the remainder of this chapter, we give representatives of the similarity classes of all 611 single-class lattices L of rank $m \geq 4$ in regular, totally definite quadratic spaces over totally real number fields $K \neq \mathbb{Q}$ (up to isometry / isomorphism). We have seen in Theorem 6.1 that unless $m \in \{4, 5, 6\}$, no such lattices over $K \neq \mathbb{Q}$ exist. Now when $m \in \{5, 6\}$, the only possible base field $K \neq \mathbb{Q}$ is $K = \mathbb{Q}(\sqrt{5})$. And when $\text{rank}(L) = 4$, the following are the 22 admissible base fields $K \neq \mathbb{Q}$, their discriminants, their narrow class number $h^+(K)$, their class number $h(K)$, and the number of distinct similarity classes of single-class lattices that they admit:

Defining polynomial of K	$\text{disc}(K)$	$h^+(K)$	$h(K)$	single-class lattices		
				$m = 4$	$m = 5$	$m = 6$
$x^2 - x - 1$	5^1	1	1	98	2	2
$x^2 - 2$	2^3	1	1	173	—	—
$x^2 - 3$	$2^2 3^1$	2	1	137	—	—
$x^2 - x - 3$	13	1	1	30	—	—
$x^2 - x - 4$	17	1	1	21	—	—
$x^2 - x - 5$	$3^1 7^1$	2	1	5	—	—
$x^2 - 6$	$2^3 3^1$	2	1	14	—	—
$x^3 - x^2 - 2x + 1$	7^2	1	1	24	—	—
$x^2 - 15$	$2^2 3^1 5^1$	4	2	3	—	—
$x^3 - 3x - 1$	3^4	1	1	29	—	—
$x^3 - x^2 - 3x + 1$	$2^2 3 7^1$	1	1	11	—	—
$x^3 - x^2 - 4x - 1$	13^2	1	1	10	—	—
$x^3 - 4x - 1$	$2 29^1$	2	1	4	—	—
$x^3 - x^2 - 4x + 2$	$2^2 7 9^1$	1	1	4	—	—
$x^3 - x^2 - 4x + 1$	$3^1 10 7^1$	1	1	3	—	—
$x^4 - x^3 - 3x^2 + x + 1$	$5^2 29^1$	1	1	10	—	—
$x^4 - x^3 - 4x^2 + 4x + 1$	$3^2 5^3$	2	1	3	—	—
$x^4 - 4x^2 - x + 1$	$19^1 10 3^1$	1	1	12	—	—
$x^4 - 5x^2 + 5$	$2^4 5^3$	2	1	2	—	—
$x^4 - 4x^2 + 1$	$2^8 3^2$	2	1	2	—	—
$x^4 - x^3 - 4x^2 + x + 2$	$2 7 7 7^1$	1	1	9	—	—
$x^5 - 5x^3 - x^2 + 3x + 1$	$61^1 3 9 7^1$	1	1	3	—	—

6.3 List of single-class lattices

How to read the tables: We order the lattices by rank, and by the discriminant of their base field. For each rank and base field, we give a list $L_1, L_2, \dots, L_k$ of k representative lattices. Each lattice L_i , $1 \leq i \leq k$, will be stated in one of three ways:

1. Whenever L_i ($2 \leq i \leq k$) is similar to the dual of L_j for an index $j < i$, we merely state that $L_i = L_j^\#$.
2. Whenever there is a set of primes $P = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ such that L_i is similar to the partial dual $L_j^{\#,P}$ (cf. Definition 2.54), we state that $L_i = L_j^{\#, \mathfrak{p}_1, \dots, \mathfrak{p}_s}$. The primes needed will be stated once per number field (“Additional primes needed for partial duals:”).
3. For all other lattices L_i of rank m , we give the lattice itself. We follow [Sch98] here: Of course, L_i is not necessarily a free lattice. Still, even if L_i is not free, we can choose a pseudo-base (cf. Theorem 2.49) of L_i , that is, a basis $(x_1, \dots, x_m)$ of V and fractional ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ of $\mathfrak{o}$ such that

$$L = \sum_{\ell=1}^m \mathfrak{a}_\ell x_\ell.$$

Recall the set $P^+(K)$ from Definition 5.9. Possibly passing to a different representative of the similarity class of L_i , it is possible to choose the fractional ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ in such a way that each $\mathfrak{a}_\ell$, $1 \leq \ell \leq m$, is either $\mathfrak{o}$ or is taken from the set $P^+(K)$. To state the similarity class of L_i , it now suffices to give the fractional ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_m$, and the Gram matrix $(a_{r,s})_{1 \leq r,s \leq m}$.

We will do this by stating the fractional ideals in the first column, and next to it the lower triangle of the Gram matrix. The following example is for $m = 4$:

$\mathfrak{a}_1$	$a_{1,1}$				
$\mathfrak{a}_2$	$a_{2,1}$	$a_{2,2}$			
$\mathfrak{a}_3$	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$		
$\mathfrak{a}_4$	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	$a_{4,4}$	

To conserve space, we will usually use the upper triangle of the above matrix to state the Gram matrix, and the rightmost column to state the fractional ideals, for the lattice L_{i+1} , in the same way.

6.3.1 Single-class lattices of rank 4

Base fields with discriminant 5^1

Number field $K = \mathbb{Q}(\alpha)$ with defining polynomial: $x^2 - x - 1$

98 similarity classes of single-class genera.

Narrow class group is the trivial group.

Additional primes needed for partial duals:

Ideal	generators
$\mathfrak{q}_1$	$5, a + 2$
$\mathfrak{q}_2$	2
$\mathfrak{q}_3$	$41, a + 6$
$\mathfrak{q}_4$	3
$\mathfrak{q}_5$	$41, a + 34$

Lattices L_1 and L_2 (of 98):

$\mathfrak{o}$	$-2a + 4$	$2a + 2$	$2a + 1$	0	0	$\mathfrak{o}$
$\mathfrak{o}$	$-a + 2$	2	$2a + 1$	4	0	$\mathfrak{o}$
$\mathfrak{o}$	$a - 2$	$a - 1$	2	0	0	$\mathfrak{o}$
$\mathfrak{o}$	-1	-1	1	$2a + 4$	0	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_1) = 2^4 \cdot 3^1 \cdot 5^1$. $\# \text{Aut}(L_2) = 2^4 \cdot 3^1 \cdot 5^1$.

(Partial) duals of the above: $L_3 = L_1^{\# \cdot \mathfrak{q}_1}$, $L_4 = L_1^{\# \cdot \mathfrak{q}_1}$, $L_5 = L_1^{\#}$, $L_6 = L_2^{\# \cdot \mathfrak{q}_1}$, $L_7 = L_2^{\# \cdot \mathfrak{q}_4}$, $L_8 = L_2^{\#}$.

Lattices L_9 and L_{10} (of 98):

$\mathfrak{o}$	$-16a + 26$	2	0	0	$-a + 1$	$\mathfrak{o}$
$\mathfrak{o}$	$-3a + 4$	$8a + 6$	0	$2a + 2$	a	$\mathfrak{o}$
$\mathfrak{o}$	$-3a + 5$	$-2a$	2	0	a	$\mathfrak{o}$
$\mathfrak{o}$	$-2a + 3$	$2a + 2$	$-a$	4	$-a + 1$	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_9) = 2^4 \cdot 3^1 \cdot 5^1$. $\# \text{Aut}(L_{10}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{11} = L_9^{\#}$, $L_{12} = L_{10}^{\# \cdot \mathfrak{q}_5}$, $L_{13} = L_{10}^{\# \cdot \mathfrak{q}_1}$, $L_{14} = L_{10}^{\#}$.

Lattices L_{15} and L_{16} (of 98):

$\mathfrak{o}$	$-2a + 4$	1	0	0	0	$\mathfrak{o}$
$\mathfrak{o}$	$-a + 1$	$2a + 2$	0	1	0	$\mathfrak{o}$
$\mathfrak{o}$	-1	a	2	0	0	$\mathfrak{o}$
$\mathfrak{o}$	1	$-a$	$-a$	2	0	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_{15}) = 2^6 \cdot 3^2 \cdot 5^2$. $\# \text{Aut}(L_{16}) = 2^7 \cdot 3^1$.

Lattices L_{17} and L_{18} (of 98):

0	$-2a+4$	2	1	1	1	0
0	$-a+1$	$2a+2$	1	2	1	0
0	1	0	2	1	1	0
0	0	0	0	2	1	0

Number of automorphisms: $\# \text{Aut}(L_{17}) = 2^4 \cdot 3^1 \cdot 5^1$. $\# \text{Aut}(L_{18}) = 2^7 \cdot 3^2$.

(Partial) duals of the above: $L_{19} = L_{17}^\#$.

Lattices L_{20} and L_{21} (of 98):

0	2	2	0	$-a+1$	a	0
0	a	2	0	2	$-a+1$	0
0	0	0	$2a+2$	$-a+1$	$-a+1$	0
0	0	0	$-a-1$	2	a	0

Number of automorphisms: $\# \text{Aut}(L_{20}) = 2^5 \cdot 5^2$. $\# \text{Aut}(L_{21}) = 2^6$.

Lattices L_{22} and L_{23} (of 98):

0	$2a+2$	$-2a+4$	$a-1$	1	$2a-3$	0
0	1	$-2a+4$	$a-1$	$2a+2$	0	0
0	0	0	$2a+2$	1	0	0
0	0	0	$2a+1$	4	$2a-3$	0

Number of automorphisms: $\# \text{Aut}(L_{22}) = 2^5 \cdot 3^2$. $\# \text{Aut}(L_{23}) = 2^3 \cdot 5^2$.

Lattices L_{24} and L_{25} (of 98):

0	2	2	a	1	$a-1$	0
0	1	$-4a+8$	a	2	0	0
0	$-a-1$	$-2a$	$4a+4$	1	0	0
0	a	$-a+1$	a	$6a+6$	$a-1$	0

Number of automorphisms: $\# \text{Aut}(L_{24}) = 2^4$. $\# \text{Aut}(L_{25}) = 2^3 \cdot 5^2$.

Lattices L_{26} and L_{27} (of 98):

0	2	2	1	$-a$	$a+1$	0
0	-1	$4a+4$	1	2	$-a$	0
0	$-a+1$	-2	4	$-a$	$-a$	0
0	$a-1$	$-2a+1$	$a-3$	$-6a+12$	$a+1$	0

Number of automorphisms: $\# \text{Aut}(L_{26}) = 2^4$. $\# \text{Aut}(L_{27}) = 2^3 \cdot 3^2$.

Lattices L_{28} and L_{29} (of 98):

0	$-2a+4$	2	0	1	$-a+1$	0
0	$-a+2$	$-2a+4$	0	2	a	0
0	$a-2$	0	4	1	a	0
0	$a-1$	$a-1$	$-2a$	$4a+4$	$-a+1$	0

Number of automorphisms: $\# \text{Aut}(L_{28}) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_{29}) = 2^5$.

Lattices L_{30} and L_{31} (of 98):

$\emptyset$	2	$-a$	-1	$-a$	$\emptyset$
$\emptyset$	0	2	$-2a+6$	$3a-4$	$\emptyset$
$\emptyset$	a	-1	-1	$3a-4$	$\emptyset$
$\emptyset$	1	a	4	$-a$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{30}) = 2^5$, $\# \text{Aut}(L_{31}) = 2^3$.

Lattices L_{32} and L_{33} (of 98):

$\emptyset$	2	1	0	0	$\emptyset$
$\emptyset$	$a-1$	4	0	$2a+2$	$-a-1$
$\emptyset$	$-a-1$	$a+1$	$4a+4$	0	$-a-1$
$\emptyset$	1	$a-1$	$-2a-1$	6	0

Number of automorphisms: $\# \text{Aut}(L_{32}) = 2^3$, $\# \text{Aut}(L_{33}) = 2^4 \cdot 5^1$.

Lattices L_{34} and L_{35} (of 98):

$\emptyset$	$2a+2$	2	0	$-a$	a
$\emptyset$	$-a-1$	2	0	2	$-a$
$\emptyset$	0	0	4	$-a$	$-a$
$\emptyset$	0	0	$2a+2$	$4a+4$	a

Number of automorphisms: $\# \text{Aut}(L_{34}) = 2^4 \cdot 5^2$, $\# \text{Aut}(L_{35}) = 2^6$.

Lattices L_{36} and L_{37} (of 98):

$\emptyset$	2	1	0	0	$\emptyset$
$\emptyset$	0	2	0	2	1
$\emptyset$	$-a+1$	$-a+1$	$-2a+6$	0	1
$\emptyset$	$a-1$	$a-1$	$a-2$	$-2a+6$	0

Number of automorphisms: $\# \text{Aut}(L_{36}) = 2^6$, $\# \text{Aut}(L_{37}) = 2^3 \cdot 5^1$.

(Partial) duals of the above: $L_{38} = L_{37}^\#$.

Lattices L_{39} and L_{40} (of 98):

$\emptyset$	$2a+2$	2	1	0	$a+1$
$\emptyset$	$-2a-1$	$2a+2$	1	2	0
$\emptyset$	a	$-a-1$	$4a+4$	0	0
$\emptyset$	$a+1$	$-a-1$	$-a-1$	4	$a+1$

Number of automorphisms: $\# \text{Aut}(L_{39}) = 2^3 \cdot 5^1$, $\# \text{Aut}(L_{40}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{41} = L_{39}^\#$, $L_{42} = L_{39}^\#$, $L_{43} = L_{39}^\#$, $L_{44} = L_{39}^\#$, $L_{45} = L_{40}^\#$, $L_{46} = L_{40}^\#$.

Lattices L_{47} and L_{48} (of 98):

$\emptyset$	1	2	1	1	$a+1$
$\emptyset$	0	1	1	2	0
$\emptyset$	0	0	1	1	0
$\emptyset$	0	0	0	$a+2$	$a+1$

Number of automorphisms: $\# \text{Aut}(L_{47}) = 2^5 \cdot 3^1$, $\# \text{Aut}(L_{48}) = 2^5 \cdot 3^1$.

(Partial) duals of the above: $L_{49} = L_{47}^\#$, $L_{50} = L_{47}^\#$, $L_{51} = L_{48}^\#$, $L_{52} = L_{48}^\#$.

Lattices L_{53} and L_{54} (of 98):

$\emptyset$	$-2a+4$	4	1	1	$a+1$	$\emptyset$
$\emptyset$	$a-2$	2	1	4	-1	$\emptyset$
$\emptyset$	$a-2$	$-a+1$	2	1	-1	$\emptyset$
$\emptyset$	$a-2$	1	$-a+1$	4	$a+1$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{53}) = 2^4 \cdot 3^1 \cdot 5^1$, $\# \text{Aut}(L_{54}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{55} = L_{53}^\#$, $L_{56} = L_{54}^{\#,\text{q1}}$, $L_{57} = L_{54}^{\#,\text{q4}}$, $L_{58} = L_{54}^\#$.

Lattices L_{59} and L_{60} (of 98):

$\emptyset$	2	$-2a+4$	-1	$-a+1$	$2a-3$	$\emptyset$
$\emptyset$	$-a$	$-2a+8$	-1	$2a+2$	$-a-1$	$\emptyset$
$\emptyset$	$a-1$	2	$2a+6$	$-a+1$	$-a-1$	$\emptyset$
$\emptyset$	a	$-3a+2$	3	$2a+6$	$2a-3$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{59}) = 2^4$, $\# \text{Aut}(L_{60}) = 2^3 \cdot 3^2$.

(Partial) duals of the above: $L_{61} = L_{59}^{\#,\text{q1}}$, $L_{62} = L_{59}^{\#,\text{q4}}$, $L_{63} = L_{59}^\#$.

Lattices L_{64} and L_{65} (of 98):

$\emptyset$	$2a+2$	4	-1	1	1	$\emptyset$
$\emptyset$	$a+1$	2	-1	4	1	$\emptyset$
$\emptyset$	1	0	$-2a+6$	1	1	$\emptyset$
$\emptyset$	$2a+1$	$a+1$	$-2a+3$	6	1	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{64}) = 2^3 \cdot 5^2$, $\# \text{Aut}(L_{65}) = 2^4 \cdot 3^1 \cdot 5^1$.

(Partial) duals of the above: $L_{66} = L_{65}^\#$.

Lattices L_{67} and L_{68} (of 98):

$\emptyset$	2	2	0	$-a+1$	$-a$	$\emptyset$
$\emptyset$	a	$-2a+8$	0	2	a	$\emptyset$
$\emptyset$	$-a$	$2a-3$	$-2a+8$	$-a+1$	a	$\emptyset$
$\emptyset$	$-a$	$-3a+2$	$-2a+3$	$-2a+8$	$-a$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{67}) = 2^4 \cdot 3^1 \cdot 5^1$, $\# \text{Aut}(L_{68}) = 2^5$.

(Partial) duals of the above: $L_{69} = L_{67}^\#$.

Lattices L_{70} and L_{71} (of 98):

$\emptyset$	2	$-8a+14$	$4a-7$	$-2a+2$	2	$\emptyset$
$\emptyset$	-1	2	$4a-7$	$-2a+8$	$a-1$	$\emptyset$
$\emptyset$	$-a+1$	$a-1$	$-4a+8$	$-2a+2$	$a-1$	$\emptyset$
$\emptyset$	$-a$	a	-1	$4a+4$	2	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{70}) = 2^3 \cdot 3^2$, $\# \text{Aut}(L_{71}) = 2^3 \cdot 5^1$.

(Partial) duals of the above: $L_{72} = L_{71}^\#$.

Lattices L_{73} and L_{74} (of 98):

0	$a+2$	$-a+4$	$-a$	$-a$	$-2a-1$	0
0	$-a$	$a+2$	$-a$	$-a+4$	$-a+2$	0
0	1	1	$-a+3$	$-a$	$-a+2$	0
0	0	0	0	4	$-2a-1$	0

Number of automorphisms: $\# \text{Aut}(L_{73}) = 2^4 \cdot 3^1 \cdot 5^1$. $\# \text{Aut}(L_{74}) = 2^4 \cdot 3^1 \cdot 5^1$.

(Partial) duals of the above: $L_{75} = L_{73}^\#$, $L_{76} = L_{74}^\#$.

Lattices L_{77} and L_{78} (of 98):

0	4	4	2	2	-1	0
0	2	6	2	$-4a+8$	1	0
0	$a-1$	$-3a+2$	$2a+8$	2	1	0
0	$2a-3$	$a-4$	$a-1$	$-12a+22$	-1	0

Number of automorphisms: $\# \text{Aut}(L_{77}) = 2^4$. $\# \text{Aut}(L_{78}) = 2^3 \cdot 5^1$.

(Partial) duals of the above: $L_{79} = L_{77}^\#$.

Lattices L_{80} and L_{81} (of 98):

0	4	4	-2	$a-1$	$a+1$	0
0	$-3a-2$	$12a+8$	-2	$4a+4$	$-2a$	0
0	$a-2$	$3a+2$	4	$a-1$	$-2a$	0
0	$-2a+3$	$-a$	-2	$-12a+22$	$a+1$	0

Number of automorphisms: $\# \text{Aut}(L_{80}) = 2^3 \cdot 5^1$. $\# \text{Aut}(L_{81}) = 2^4 \cdot 5^2$.

(Partial) duals of the above: $L_{82} = L_{80}^\#$.

Lattices L_{83} and L_{84} (of 98):

0	2	4	$-2a$	$-2a+2$	2	0
0	$-a+1$	2	$-2a$	4	1	0
0	0	0	$-2a+6$	$-2a+2$	1	0
0	0	0	$-a-2$	$2a+4$	2	0

Number of automorphisms: $\# \text{Aut}(L_{83}) = 2^4 \cdot 5^2$. $\# \text{Aut}(L_{84}) = 2^5$.

(Partial) duals of the above: $L_{85} = L_{84}^\#$.

Lattices L_{86} and L_{87} (of 98):

0	$-58a+94$	$-4a+8$	$-3a+4$	$a-3$	$-a$	0
0	$-11a+18$	$4a+8$	$-3a+4$	$-4a+8$	$a-2$	0
0	$3a-5$	$-4a-4$	$8a+6$	$a-3$	$a-2$	0
0	$4a-7$	$-a-3$	$3a+2$	$2a+4$	$-a$	0

Number of automorphisms: $\# \text{Aut}(L_{86}) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_{87}) = 2^3 \cdot 5^1$.

(Partial) duals of the above: $L_{88} = L_{87}^\#$.

Lattices L_{89} and L_{90} (of 98):

$\mathfrak{o}$	$12a + 8$	$-a + 4$	-2	$-a + 2$	a	$\mathfrak{o}$
$\mathfrak{o}$	$-3a - 2$	$-2a + 6$	-2	$4a + 4$	$2a$	$\mathfrak{o}$
$\mathfrak{o}$	$7a + 4$	$-2a$	$12a + 8$	$-a + 2$	$2a$	$\mathfrak{o}$
$\mathfrak{o}$	$-5a - 3$	3	$-2a - 2$	$-34a + 60$	a	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_{89}) = 2^3 \cdot 5^1$, $\# \text{Aut}(L_{90}) = 2^2 \cdot 5^1$.

(Partial) duals of the above: $L_{91} = L_{89}^\#$, $L_{92} = L_{90}^\#$.

Lattices L_{93} and L_{94} (of 98):

$\mathfrak{o}$	$-2a + 5$	$-2a + 5$	$-a + 1$	$2a - 3$	$2a - 3$	$\mathfrak{o}$
$\mathfrak{o}$	$-a + 1$	$5a + 4$	$-a + 1$	$5a + 4$	$a - 1$	$\mathfrak{o}$
$\mathfrak{o}$	$2a - 3$	$-a - 1$	$-2a + 5$	$2a - 3$	$a - 1$	$\mathfrak{o}$
$\mathfrak{o}$	$a - 1$	a	$-a + 1$	$a + 4$	$2a - 3$	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_{93}) = 2^2 \cdot 5^1$, $\# \text{Aut}(L_{94}) = 2^4 \cdot 3^1$.

(Partial) duals of the above: $L_{95} = L_{93}^\#$, $L_{96} = L_{94}^\#$.

Lattice L_{97} (of 98):

$\mathfrak{o}$	$2a + 3$					
$\mathfrak{o}$	$-a - 1$	$-a + 4$				
$\mathfrak{o}$	2	0	4			
$\mathfrak{o}$	$2a + 1$	$-a - 1$	0	$2a + 3$		

Number of automorphisms: $\# \text{Aut}(L_{97}) = 2^4 \cdot 3^1$.

(Partial) duals of the above: $L_{98} = L_{97}^\#$.

Base fields with discriminant 2^3

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^2 - 2$

173 similarity classes of single-class genera.

Narrow class group is the trivial group.

Additional primes needed for partial duals:

Ideal	generators
$\mathfrak{q}_1$	3
$\mathfrak{q}_2$	$17, a + 11$
$\mathfrak{q}_3$	$2, a$
$\mathfrak{q}_4$	$17, a + 6$

Lattices L_1 and L_2 (of 173):

$\mathfrak{o}$	2	2	1	$-a + 1$	0	$\mathfrak{o}$
$\mathfrak{o}$	1	2	1	2	$-a + 1$	$\mathfrak{o}$
$\mathfrak{o}$	$a + 1$	0	$a + 3$	$-a + 1$	$-a + 1$	$\mathfrak{o}$
$\mathfrak{o}$	$-a + 1$	0	-1	$-2a + 4$	0	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_1) = 2^2 \cdot 3^1$, $\# \text{Aut}(L_2) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_3 = L_1^\#$, $L_4 = L_2^\#$.

Lattices L_5 and L_6 (of 173):

$\emptyset$	2	2	1	$-a-1$	$a-1$	$\emptyset$
$\emptyset$	0	$-a+2$	1	2	$-a$	$\emptyset$
$\emptyset$	$-a$	$-a+1$	$-3a+6$	$-a-1$	$-a$	$\emptyset$
$\emptyset$	$-a$	0	$a-1$	$-a+4$	$a-1$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_5) = 2^3$. $\# \text{Aut}(L_6) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_7 = L_5^\#$, $L_8 = L_6^\#$, $L_9 = L_6^{\#,\text{q3}}$, $L_{10} = L_6^\#$.

Lattices L_{11} and L_{12} (of 173):

$\emptyset$	1	2	a	$a+1$	$\emptyset$
$\emptyset$	0	1	a	2	1
$\emptyset$	0	0	1	a	1
$\emptyset$	0	0	0	1	$a+1$

Number of automorphisms: $\# \text{Aut}(L_{11}) = 2^7 \cdot 3^1$. $\# \text{Aut}(L_{12}) = 2^7 \cdot 3^1$.

Lattices L_{13} and L_{14} (of 173):

$\emptyset$	2	1	0	0	0
$\emptyset$	a	2	0	1	0
$\emptyset$	a	1	2	0	0
$\emptyset$	1	a	a	2	0

Number of automorphisms: $\# \text{Aut}(L_{13}) = 2^8 \cdot 3^2$. $\# \text{Aut}(L_{14}) = 2^7$.

Lattices L_{15} and L_{16} (of 173):

$\emptyset$	2	2	a	$a+1$	0
$\emptyset$	$a+1$	$a+2$	a	2	$a+1$
$\emptyset$	0	0	2	$a+1$	$a+1$
$\emptyset$	0	0	$a+1$	$a+2$	0

Number of automorphisms: $\# \text{Aut}(L_{15}) = 2^9$. $\# \text{Aut}(L_{16}) = 2^7$.

(Partial) duals of the above: $L_{17} = L_{16}^\#$.

Lattices L_{18} and L_{19} (of 173):

$\emptyset$	2	2	$a+1$	0	0
$\emptyset$	$a+1$	$a+2$	$a+1$	$a+2$	0
$\emptyset$	0	0	$a+2$	0	0
$\emptyset$	0	0	0	$-a+2$	0

Number of automorphisms: $\# \text{Aut}(L_{18}) = 2^7$. $\# \text{Aut}(L_{19}) = 2^8$.

(Partial) duals of the above: $L_{20} = L_{18}^\#$.

Lattices L_{21} and L_{22} (of 173):

0	2	2	a	0	a+1	0
0	-a	2	a	2	0	0
0	-a	1	2	0	0	0
0	-a	1	1	2	a+1	0

Number of automorphisms: $\# \text{Aut}(L_{21}) = 2^7 \cdot 3^1$. $\# \text{Aut}(L_{22}) = 2^9$.

(Partial) duals of the above: $L_{23} = L_{21}^\#$.

Lattices L_{24} and L_{25} (of 173):

0	2	4a+6	0	a+1	a+2	0
0	1	2	0	-7a+10	5a-7	0
0	a-1	0	-2a+4	a+1	5a-7	0
0	-1	-1	-a	2a+4	a+2	0

Number of automorphisms: $\# \text{Aut}(L_{24}) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_{25}) = 2^4$.

Lattices L_{26} and L_{27} (of 173):

0	2	2	-1	-a+1	1	0
0	-a-1	14a+20	-1	2	a	0
0	-a+1	a+2	-2a+4	-a+1	a	0
0	a	-a-2	0	4	1	0

Number of automorphisms: $\# \text{Aut}(L_{26}) = 2^4$. $\# \text{Aut}(L_{27}) = 2^3 \cdot 3^2$.

Lattices L_{28} and L_{29} (of 173):

0	-a+2	2	a-1	-a-1	1	0
0	0	2	a-1	-14a+20	a-2	0
0	a-1	a+1	a+4	-a-1	a-2	0
0	1	a	a+1	4	1	0

Number of automorphisms: $\# \text{Aut}(L_{28}) = 2^4$. $\# \text{Aut}(L_{29}) = 2^4$.

Lattices L_{30} and L_{31} (of 173):

0	2	2	a+1	-1	0	0
0	-a+1	-14a+20	a+1	14a+20	-5a-7	0
0	-1	5a-7	6	-1	-5a-7	0
0	0	2a-3	-3a-2	6a+10	0	0

Number of automorphisms: $\# \text{Aut}(L_{30}) = 2^3$. $\# \text{Aut}(L_{31}) = 2^3$.

Lattices L_{32} and L_{33} (of 173):

0	2	2	a	-a-1	-a-1	0
0	a	2	a	2	-a-1	0
0	a-1	-a+1	6	-a-1	-a-1	0
0	-a+1	a-1	-a-3	6	-a-1	0

Number of automorphisms: $\# \text{Aut}(L_{32}) = 2^8$. $\# \text{Aut}(L_{33}) = 2^8$.

Lattices L_{34} and L_{35} (of 173):

0	2	2	1	0	0	0
0	-1	2	1	2	0	0
0	0	0	-a+2	0	0	0
0	0	0	-a+1	-2a+4	0	0

Number of automorphisms: $\# \text{Aut}(L_{34}) = 2^4 \cdot 3^1$, $\# \text{Aut}(L_{35}) = 2^4 \cdot 3^2$.

Lattices L_{36} and L_{37} (of 173):

0	2	2	a	0	a+1	0
0	0	-2a+4	a	2	0	0
0	a+1	-2a+2	6	0	0	0
0	0	a-2	a-1	2a+6	a+1	0

Number of automorphisms: $\# \text{Aut}(L_{36}) = 2^4$, $\# \text{Aut}(L_{37}) = 2^6$.

(Partial) duals of the above: $L_{38} = L_{37}^{\#, q3}$, $L_{39} = L_{37}^{\#, q1}$, $L_{40} = L_{37}^{\#}$.

Lattices L_{41} and L_{42} (of 173):

0	2	2	a+1	0	0	0
0	-3a-4	14a+20	a+1	a+2	0	0
0	-1	a+1	-a+3	0	0	0
0	a-1	-a-2	a-1	-8a+12	0	0

Number of automorphisms: $\# \text{Aut}(L_{41}) = 2^5$, $\# \text{Aut}(L_{42}) = 2^6$.

(Partial) duals of the above: $L_{43} = L_{41}^{\#}$, $L_{44} = L_{42}^{\#}$.

Lattices L_{45} and L_{46} (of 173):

0	2	2	1	a+1	33a+46	0
0	-1	2	1	2	a+1	0
0	a+1	-a	a+3	a+1	a+1	0
0	a+1	0	a+1	2a+4	33a+46	0

Number of automorphisms: $\# \text{Aut}(L_{45}) = 2^2 \cdot 3^1$, $\# \text{Aut}(L_{46}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{47} = L_{45}^{\#}$, $L_{48} = L_{46}^{\#}$.

Lattices L_{49} and L_{50} (of 173):

0	2	2	1	-a-1	2a-3	0
0	0	a+2	1	2	0	0
0	a	a+1	4	-a-1	0	0
0	0	a+1	1	-a+4	2a-3	0

Number of automorphisms: $\# \text{Aut}(L_{49}) = 2^3$, $\# \text{Aut}(L_{50}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{51} = L_{49}^{\#}$, $L_{52} = L_{50}^{\#, q3}$, $L_{53} = L_{50}^{\#, q4}$, $L_{54} = L_{50}^{\#}$.

Lattices L_{55} and L_{56} (of 173):

0	4	2	-1	-a-1	0	0
0	-2a	-a+4	-1	2	a+1	0
0	-2	-11a+17	-135a+192	-a-1	a+1	0
0	2a	-a-1	-13a+17	-2a+6	0	0

Number of automorphisms: $\# \text{Aut}(L_{55}) = 2^3$. $\# \text{Aut}(L_{56}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{57} = L_{55}^{\#q_3}$, $L_{58} = L_{55}^{\#q_2}$, $L_{59} = L_{55}^{\#q_2}$, $L_{60} = L_{56}^{\#q_3}$, $L_{61} = L_{56}^{\#q_2}$, $L_{62} = L_{56}^{\#}$.

Lattices L_{63} and L_{64} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 2 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 2 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} a+1 \\ 2 \\ a+1 \\ -a \end{smallmatrix}$	$\begin{smallmatrix} -a+1 \\ 1 \\ 1 \\ 3 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 1 \\ -a+1 \\ 0 \end{smallmatrix}$
--	--	--	---	---	--

Number of automorphisms: $\# \text{Aut}(L_{63}) = 2^5 \cdot 3^1$. $\# \text{Aut}(L_{64}) = 2^4 \cdot 3^1$.

(Partial) duals of the above: $L_{65} = L_{63}^{\#}$, $L_{66} = L_{64}^{\#}$.

Lattices L_{67} and L_{68} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ -1 \\ a+1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 2 \\ -1 \\ a+1 \end{smallmatrix}$	$\begin{smallmatrix} a-1 \\ a-1 \\ 2 \\ a+1 \end{smallmatrix}$	$\begin{smallmatrix} a-1 \\ -a+3 \\ a-1 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ -a+2 \\ -a+2 \\ 2a+4 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{smallmatrix}$
--	---	---	--	---	--	---

Number of automorphisms: $\# \text{Aut}(L_{67}) = 2^5 \cdot 3^1$. $\# \text{Aut}(L_{68}) = 2^5$.

(Partial) duals of the above: $L_{69} = L_{67}^{\#}$.

Lattices L_{70} and L_{71} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 0 \\ 0 \\ a+1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 2 \\ 0 \\ -1 \end{smallmatrix}$	$\begin{smallmatrix} -1 \\ -1 \\ 2 \\ a+1 \end{smallmatrix}$	$\begin{smallmatrix} -1 \\ 2 \\ -1 \\ 2a+4 \end{smallmatrix}$	$\begin{smallmatrix} -1 \\ 1 \\ 1 \\ -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	---	--	---	--	--

Number of automorphisms: $\# \text{Aut}(L_{70}) = 2^7$. $\# \text{Aut}(L_{71}) = 2^7 \cdot 3^2$.

Lattices L_{72} and L_{73} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} -a+2 \\ -a+2 \\ 0 \\ a-1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ a+2 \\ -a-1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 2 \\ 0 \\ 3 \end{smallmatrix}$	$\begin{smallmatrix} a-1 \\ 0 \\ 0 \\ a-1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	--	---	--	--	--

Number of automorphisms: $\# \text{Aut}(L_{72}) = 2^5$. $\# \text{Aut}(L_{73}) = 2^6$.

(Partial) duals of the above: $L_{74} = L_{72}^{\#}$, $L_{75} = L_{73}^{\#}$.

Lattices L_{76} and L_{77} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -a \\ -a-1 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ a+1 \\ a+1 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} a+1 \\ a+1 \\ a+3 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ a+2 \\ 0 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	--	--	--	--

Number of automorphisms: $\# \text{Aut}(L_{76}) = 2^6$. $\# \text{Aut}(L_{77}) = 2^8$.

(Partial) duals of the above: $L_{78} = L_{76}^{\#}$.

Lattices L_{79} and L_{80} (of 173):

$\emptyset$	$a+2$	$-a+2$	0	$-a+1$	$a-1$	$\emptyset$
$\emptyset$	0	$-a+2$	0	$a+2$	$-a-1$	$\emptyset$
$\emptyset$	$a+1$	$-a+1$	4	$-a+1$	$-a-1$	$\emptyset$
$\emptyset$	$-a-1$	$a-1$	$-a-2$	4	$a-1$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{79}) = 2^7$. $\# \text{Aut}(L_{80}) = 2^6$.

(Partial) duals of the above: $L_{81} = L_{79}^\#$.

Lattices L_{82} and L_{83} (of 173):

$\emptyset$	2	2	0	$-a+1$	$-a$	$\emptyset$
$\emptyset$	$-a$	2	0	2	1	$\emptyset$
$\emptyset$	$a+1$	$-a-1$	$a+4$	$-a+1$	1	$\emptyset$
$\emptyset$	$-a-1$	$a+1$	$-2a-2$	$a+4$	$-a$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{82}) = 2^8$. $\# \text{Aut}(L_{83}) = 2^4$.

(Partial) duals of the above: $L_{84} = L_{83}^\#$.

Lattices L_{85} and L_{86} (of 173):

$\emptyset$	2	2	0	-1	$-a+1$	$\emptyset$
$\emptyset$	0	2	0	2	$a-1$	$\emptyset$
$\emptyset$	1	$a-1$	$-a+4$	-1	$a-1$	$\emptyset$
$\emptyset$	-1	$-a+1$	-2	$-a+4$	$-a+1$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{85}) = 2^5$. $\# \text{Aut}(L_{86}) = 2^5$.

(Partial) duals of the above: $L_{87} = L_{85}^\#$, $L_{88} = L_{86}^\#$.

Lattices L_{89} and L_{90} (of 173):

$\emptyset$	$a+2$	1	0	0	0	$\emptyset$
$\emptyset$	$-a-1$	4	0	2	0	$\emptyset$
$\emptyset$	$-a-1$	$-a+2$	4	0	0	$\emptyset$
$\emptyset$	$-a-1$	$a+1$	1	$-a+4$	0	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{89}) = 2^4$. $\# \text{Aut}(L_{90}) = 2^6$.

(Partial) duals of the above: $L_{91} = L_{89}^\#$.

Lattices L_{92} and L_{93} (of 173):

$\emptyset$	$a+2$	4	$2a+2$	$-a-2$	$-a+2$	$\emptyset$
$\emptyset$	0	$-a+2$	$2a+2$	$2a+4$	$-2a-2$	$\emptyset$
$\emptyset$	0	0	4	$-a-2$	$-2a-2$	$\emptyset$
$\emptyset$	$a+1$	$a-1$	$2a+2$	$2a+6$	$-a+2$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{92}) = 2^7$. $\# \text{Aut}(L_{93}) = 2^7$.

(Partial) duals of the above: $L_{94} = L_{92}^\#$.

Lattices L_{95} and L_{96} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ a \\ 0 \\ -a \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -a+3 \\ 1 \\ -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ a+3 \\ a+1 \end{smallmatrix}$	$\begin{smallmatrix} a+1 \\ 2 \\ a+1 \\ a+3 \end{smallmatrix}$	$\begin{smallmatrix} -a \\ a+1 \\ a+1 \\ -a \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	---	--	--	--	--	--

Number of automorphisms: $\# \text{Aut}(L_{95}) = 2^6$. $\# \text{Aut}(L_{96}) = 2^5$.

(Partial) duals of the above: $L_{97} = L_{96}^\#$.

Lattices L_{98} and L_{99} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ -1 \\ -a \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 0 \\ -1 \\ -a \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 2 \\ 2 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} a-1 \\ 2 \\ a-1 \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} -a+1 \\ -1 \\ -1 \\ -a+1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	--	--	--	--	--

Number of automorphisms: $\# \text{Aut}(L_{98}) = 2^5 \cdot 3^1$. $\# \text{Aut}(L_{99}) = 2^5$.

(Partial) duals of the above: $L_{100} = L_{98}^\#$.

Lattices L_{101} and L_{102} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -1 \\ 1 \\ 3a+4 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 4 \\ -a-1 \\ -15a-21 \end{smallmatrix}$	$\begin{smallmatrix} a-1 \\ a-1 \\ 4 \\ 7a+10 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ -8a+12 \\ 1 \\ 188a+266 \end{smallmatrix}$	$\begin{smallmatrix} -a+1 \\ 2a-3 \\ 2a-3 \\ -a+1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	---	--	--	--	--

Number of automorphisms: $\# \text{Aut}(L_{101}) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_{102}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{103} = L_{101}^\#$, $L_{104} = L_{102}^\#$.

Lattices L_{105} and L_{106} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 0 \\ 0 \\ a-1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -2a+6 \\ 3a-2 \\ -2a-1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ -2a+6 \\ a-3 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 2a+6 \\ 0 \\ 2a+10 \end{smallmatrix}$	$\begin{smallmatrix} -1 \\ 3a+2 \\ 3a+2 \\ -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	---	--	---	--	--

Number of automorphisms: $\# \text{Aut}(L_{105}) = 2^6$. $\# \text{Aut}(L_{106}) = 2^6$.

(Partial) duals of the above: $L_{107} = L_{105}^\#$, $L_{108} = L_{106}^\#$.

Lattices L_{109} and L_{110} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ 2a \\ 2a+2 \\ a \end{smallmatrix}$	$\begin{smallmatrix} -a+3 \\ 2a+2 \\ a+2 \\ a+3 \end{smallmatrix}$	$\begin{smallmatrix} -a \\ -a \\ -a \\ a+4 \end{smallmatrix}$	$\begin{smallmatrix} a-1 \\ a+3 \\ a+1 \\ a-1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	--	---	--	--

Number of automorphisms: $\# \text{Aut}(L_{109}) = 2^4$. $\# \text{Aut}(L_{110}) = 2^3$.

(Partial) duals of the above: $L_{111} = L_{109}^\#$, $L_{112} = L_{109}^\#$, $L_{113} = L_{109}^\#$, $L_{114} = L_{110}^\#$, $L_{115} = L_{110}^\#$, $L_{116} = L_{110}^\#$.

Lattices L_{117} and L_{118} (of 173):

$\emptyset$	2	2	$-a+1$	a	-1	$\emptyset$
$\emptyset$	a	2	$-a+1$	4	a	$\emptyset$
$\emptyset$	0	0	$-14a+20$	a	a	$\emptyset$
$\emptyset$	$a-1$	1	$-7a+10$	$-6a+10$	-1	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{117}) = 2^6$. $\# \text{Aut}(L_{118}) = 2^3$.

(Partial) duals of the above: $L_{119} = L_{117}^{\#q3}$, $L_{120} = L_{117}^{\#q1}$, $L_{121} = L_{117}^{\#}$, $L_{122} = L_{118}^{\#q3}$, $L_{123} = L_{118}^{\#q4}$, $L_{124} = L_{118}^{\#}$.

Lattices L_{125} and L_{126} (of 173):

$\emptyset$	2	2	1	$a-1$	$-a-1$	$\emptyset$
$\emptyset$	1	2	1	2	-1	$\emptyset$
$\emptyset$	a	0	$-10a+16$	$a-1$	-1	$\emptyset$
$\emptyset$	0	0	$-4a+6$	$-2a+4$	$-a-1$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{125}) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_{126}) = 2^2 \cdot 3^1$.

(Partial) duals of the above: $L_{127} = L_{125}^{\#q3}$, $L_{128} = L_{125}^{\#q4}$, $L_{129} = L_{125}^{\#}$, $L_{130} = L_{126}^{\#}$.

Lattices L_{131} and L_{132} (of 173):

$\emptyset$	2	2	-1	$a+1$	$-a+1$	$\emptyset$
$\emptyset$	-1	2	-1	2	$-a-1$	$\emptyset$
$\emptyset$	$a-1$	$-a+1$	$-a+3$	$a+1$	$-a-1$	$\emptyset$
$\emptyset$	$a+1$	$-a$	1	4	$-a+1$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{131}) = 2^2 \cdot 3^1$. $\# \text{Aut}(L_{132}) = 2^2 \cdot 3^1$.

(Partial) duals of the above: $L_{133} = L_{131}^{\#}$, $L_{134} = L_{132}^{\#}$.

Lattices L_{135} and L_{136} (of 173):

$\emptyset$	2	$a+3$	$a+1$	1	1	$\emptyset$
$\emptyset$	$a-1$	$-a+3$	$a+1$	$a+3$	1	$\emptyset$
$\emptyset$	-1	$-a+1$	$-a+3$	1	1	$\emptyset$
$\emptyset$	-1	1	1	$a+3$	1	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{135}) = 2^2$. $\# \text{Aut}(L_{136}) = 2^3$.

(Partial) duals of the above: $L_{137} = L_{135}^{\#}$.

Lattices L_{138} and L_{139} (of 173):

$\emptyset$	$-3a+5$	2	$-a-1$	$a-1$	$-a$	$\emptyset$
$\emptyset$	$a+1$	$9a+13$	$-a-1$	$2a+4$	0	$\emptyset$
$\emptyset$	1	$2a+3$	$a+3$	$a-1$	0	$\emptyset$
$\emptyset$	-1	$-2a-3$	$-a-1$	$a+3$	$-a$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{138}) = 2^4 \cdot 3^1$. $\# \text{Aut}(L_{139}) = 2^3$.

(Partial) duals of the above: $L_{140} = L_{138}^{\#}$, $L_{141} = L_{139}^{\#}$.

Lattices L_{142} and L_{143} (of 173):

0	2	$a+3$	$-a$	0	-1	0
0	-1	2	$-a$	$-a+3$	-1	0
0	$-a$	0	4	0	-1	0
0	$-a-1$	a	$2a+2$	$2a+6$	-1	0

Number of automorphisms: $\# \text{Aut}(L_{142}) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_{143}) = 2^2$.

(Partial) duals of the above: $L_{144} = L_{142}^\#$.

Lattices L_{145} and L_{146} (of 173):

0	2	2	1	$-a$	a	0
0	$a-1$	$-2a+4$	1	2	$-a$	0
0	$-a-1$	0	$2a+4$	$-a$	$-a$	0
0	1	a	$-a$	4	a	0

Number of automorphisms: $\# \text{Aut}(L_{145}) = 2^3$. $\# \text{Aut}(L_{146}) = 2^3 \cdot 3^2$.

Lattices L_{147} and L_{148} (of 173):

0	2	2	0	0	0	0
0	$-a$	$-a+3$	0	$a+3$	1	0
0	0	0	$2a+4$	0	1	0
0	a	-1	$-2a-2$	$a+5$	0	0

Number of automorphisms: $\# \text{Aut}(L_{147}) = 2^4$. $\# \text{Aut}(L_{148}) = 2^5$.

(Partial) duals of the above: $L_{149} = L_{148}^\#$.

Lattices L_{150} and L_{151} (of 173):

0	$a+2$	2	0	$a-1$	-1	0
0	0	4	0	2	-1	0
0	$a+1$	2	$-2a+6$	$a-1$	-1	0
0	$a+1$	$2a+2$	$a+3$	$3a+6$	-1	0

Number of automorphisms: $\# \text{Aut}(L_{150}) = 2^4$. $\# \text{Aut}(L_{151}) = 2^5$.

(Partial) duals of the above: $L_{152} = L_{150}^\#$.

Lattices L_{153} and L_{154} (of 173):

0	4	2	$-a+1$	1	0	0
0	$-2a$	4	$-a+1$	$-a+4$	1	0
0	$-a$	$a+2$	$a+4$	1	1	0
0	$-a$	$-a$	$-2a-1$	$a+4$	0	0

Number of automorphisms: $\# \text{Aut}(L_{153}) = 2^5$. $\# \text{Aut}(L_{154}) = 2^2$.

(Partial) duals of the above: $L_{155} = L_{154}^\#$.

Lattices L_{156} and L_{157} (of 173):

0	2	2	$-a$	0	0	0
0	$-a$	$a+3$	$-a$	$a+3$	1	0
0	0	1	$-3a+5$	0	1	0
0	0	$-a-1$	$-a+1$	$a+3$	0	0

Number of automorphisms: $\# \text{Aut}(L_{156}) = 2^4$, $\# \text{Aut}(L_{157}) = 2^4$.

(Partial) duals of the above: $L_{158} = L_{156}^\#$, $L_{159} = L_{157}^\#$.

Lattices L_{160} and L_{161} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} -3a+5 \\ 1 \\ -1 \\ -a+1 \end{smallmatrix}$	$\begin{smallmatrix} -3a+5 \\ a+3 \\ -a-1 \\ -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ a+3 \\ -1 \end{smallmatrix}$	$\begin{smallmatrix} a-2 \\ 4 \\ a-2 \\ a+5 \end{smallmatrix}$	$\begin{smallmatrix} a \\ 2a-2 \\ 2a-2 \\ a \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_{160}) = 2^2 \cdot 3^1$, $\# \text{Aut}(L_{161}) = 2^2 \cdot 3^1$.

(Partial) duals of the above: $L_{162} = L_{160}^\#$, $L_{163} = L_{161}^\#$.

Lattices L_{164} and L_{165} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 3a+5 \\ 0 \\ a+2 \\ -a \end{smallmatrix}$	$\begin{smallmatrix} -a+3 \\ 4 \\ 2a+2 \\ -2a \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 2a+6 \\ -2a-2 \end{smallmatrix}$	$\begin{smallmatrix} a-1 \\ 3a+5 \\ a-1 \\ -2a+6 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ -1 \\ -1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_{164}) = 2^2 \cdot 3^1$, $\# \text{Aut}(L_{165}) = 2^2 \cdot 3^1$.

(Partial) duals of the above: $L_{166} = L_{164}^\#$, $L_{167} = L_{165}^\#$.

Lattices L_{168} and L_{169} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ -2a \\ 2 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} a+3 \\ 2a+6 \\ -a+2 \\ -2a-2 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ -3a+7 \\ -a-2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ -8a+12 \\ 1 \\ 2a+6 \end{smallmatrix}$	$\begin{smallmatrix} a+1 \\ -2a+4 \\ -2a+4 \\ a+1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	--	---	--	--	--

Number of automorphisms: $\# \text{Aut}(L_{168}) = 2^4$, $\# \text{Aut}(L_{169}) = 2^2$.

(Partial) duals of the above: $L_{170} = L_{168}^\#$, $L_{171} = L_{169}^\#$.

Lattice L_{172} (of 173):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} a+3 \\ -a-1 \\ -a+1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} a+3 \\ -a+1 \\ -7a+11 \\ -2a+3 \end{smallmatrix}$	$\begin{smallmatrix} a+5 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_{172}) = 2^4$.

(Partial) duals of the above: $L_{173} = L_{172}^\#$.

Base fields with discriminant $2^2 \cdot 3^1$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^2 - 3$

137 similarity classes of single-class genera.

Narrow class group prime ideal representatives:

Ideal	generators
$\mathfrak{p}_1$	$-a$

Additional primes needed for partial duals:

Ideal	generators
$\mathfrak{q}_1$	5
$\mathfrak{q}_2$	$11, a + 5$
$\mathfrak{q}_3$	$2, a + 1$
$\mathfrak{q}_4$	$11, a + 6$

Lattices L_1 and L_2 (of 137):

	2	$\frac{2}{3}(a+1)$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(a+1)$	$-\frac{1}{3}$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}(2a+3)$	$\frac{1}{3}(2a+6)$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(a+3)$	$\frac{1}{3}(a+2)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}(a-3)$	$-\frac{1}{3}a$	$\frac{1}{3}(-2a+6)$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(a+2)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{2}{3}(-a+3)$	0	$\frac{2}{3}(2a-3)$	2	$-\frac{1}{3}$	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_1) = 2^2$. $\# \text{Aut}(L_2) = 2^7$.

(Partial) duals of the above: $L_3 = L_1^\#$.

Lattices L_4 and L_5 (of 137):

	2	$\frac{2}{3}(a+3)$	$\frac{1}{3}(a-1)$	$\frac{1}{3}(a-1)$	$\frac{1}{3}(-a+2)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{2}{3}(-a-1)$	$\frac{2}{3}(a+3)$	$\frac{1}{3}(a-1)$	$\frac{1}{3}(-2a+4)$	$\frac{1}{3}(a-1)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$-\frac{1}{3}a$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(a-1)$	$\frac{1}{3}(a-1)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$-\frac{2}{3}a$	$\frac{1}{3}(a-1)$	$\frac{1}{3}(a-1)$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(-a+2)$	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_4) = 2^5$. $\# \text{Aut}(L_5) = 2^6$.

(Partial) duals of the above: $L_6 = L_4^\#$, $L_7 = L_5^\#$.

Lattices L_8 and L_9 (of 137):

	$\frac{2}{3}(a+3)$	$\frac{2}{3}(-a+3)$	$\frac{1}{3}(a-1)$	$\frac{1}{3}a$	$\frac{1}{3}(2a+3)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}a$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(a-1)$	$\frac{1}{3}(-2a+4)$	$\frac{1}{3}(a-1)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}(-a-1)$	$\frac{2}{3}(a+3)$	$\frac{1}{3}(a+3)$	$\frac{1}{3}a$	$\frac{1}{3}(a-1)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}a$	$\frac{2}{3}(a-2)$	$\frac{1}{3}(-a-2)$	$\frac{1}{3}(-2a+3)$	$\frac{1}{3}(2a+3)$	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_8) = 2^5 \cdot 3^1$. $\# \text{Aut}(L_9) = 2^7$.

(Partial) duals of the above: $L_{10} = L_8^\#$, $L_{11} = L_9^\#$.

Lattices L_{12} and L_{13} (of 137):

$\mathbf{p}_1$	$\frac{2}{3}$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(a+1)$	$\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(a+1)$	$\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	0	$\frac{1}{3}a$	$\frac{1}{3}(-a+2)$	$\frac{1}{3}(2a-3)$	$\frac{1}{3}(-4a+8)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}$	$\frac{1}{3}(-a+1)$	$\frac{1}{3}(-a+2)$	$\frac{1}{3}(2a-3)$	$\frac{1}{3}(-4a+8)$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{12}) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_{13}) = 2^4$.

(Partial) duals of the above: $L_{14} = L_{12}^\#$, $L_{15} = L_{13}^\#$.

Lattices L_{16} and L_{17} (of 137):

$\mathbf{p}_1$	$\frac{2}{3}$	1	$-\frac{1}{3}a$	0	$\frac{1}{3}a$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}a$	$\frac{4}{3}$	$-\frac{1}{3}a$	1	$-\frac{1}{3}a$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}a$	$\frac{1}{3}(a+2)$	$\frac{4}{3}$	0	$-\frac{1}{3}a$	$\mathbf{p}_1$
$\mathbf{p}_1$	0	$\frac{1}{3}(a+1)$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}a$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{16}) = 2^4$. $\# \text{Aut}(L_{17}) = 2^5 \cdot 3^1$.

(Partial) duals of the above: $L_{18} = L_{16}^{\# \cdot \mathbf{p}_1}$.

Lattices L_{19} and L_{20} (of 137):

$\mathbf{p}_1$	2	1	$\frac{1}{3}(2a-3)$	0	$\mathbf{p}_1$
$\mathbf{p}_1$	$a-2$	$-2a+4$	$\frac{1}{3}(2a-3)$	$\frac{1}{3}(-5a+9)$	0
$\mathbf{p}_1$	$\frac{2}{3}a$	$\frac{1}{3}(-a+3)$	2	0	0
$\mathbf{p}_1$	$-\frac{2}{3}a$	$\frac{1}{3}(2a-3)$	$\frac{1}{3}a$	$\frac{1}{3}(2a+6)$	0

Number of automorphisms: $\# \text{Aut}(L_{19}) = 2^6 \cdot 3^2$. $\# \text{Aut}(L_{20}) = 2^5$.

Lattices L_{21} and L_{22} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}(a+3)$	$\frac{2}{3}$	$\frac{1}{3}(-a+3)$	$-\frac{1}{3}$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(-2a-2)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}a$	$\frac{1}{3}(-a+3)$	$-\frac{1}{3}$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(-2a-2)$	$\mathbf{p}_1$
$\mathbf{p}_1$	0	0	$\frac{1}{3}a$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(a+3)$	$\frac{1}{3}$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{21}) = 2^7$. $\# \text{Aut}(L_{22}) = 2^3$.

(Partial) duals of the above: $L_{23} = L_{22}^\#$.

Lattices L_{24} and L_{25} (of 137):

$\mathbf{p}_1$	$\frac{2}{3}$	$\frac{1}{3}(-6a+14)$	$\frac{1}{3}(a+2)$	0	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(2a-3)$	$\frac{1}{3}(-6a+14)$	$\frac{1}{3}(a+2)$	$\frac{1}{3}(2a+4)$	0
$\mathbf{p}_1$	0	$\frac{1}{3}(5a+9)$	$\frac{1}{3}(26a+46)$	0	0
$\mathbf{p}_1$	$\frac{1}{3}a$	$\frac{1}{3}a$	$\frac{1}{3}(5a+8)$	$\frac{4}{3}$	0

Number of automorphisms: $\# \text{Aut}(L_{24}) = 2^3$. $\# \text{Aut}(L_{25}) = 2^6 \cdot 3^1$.

(Partial) duals of the above: $L_{26} = L_{24}^\#$, $L_{27} = L_{25}^\#$.

Lattices L_{28} and L_{29} (of 137):

$\mathbf{p}_1$	$\frac{2}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}(a+1)$	$\frac{1}{3}(4a+7)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}a$	0	0	$\frac{1}{3}(2a+4)$	0	$\mathbf{p}_1$
$\mathbf{p}_1$	0	0	$\frac{1}{3}(-2a+6)$	$\frac{1}{3}(a+1)$	0	$\mathbf{p}_1$
$\mathbf{p}_1$	0	0	$-a+1$	$\frac{1}{3}(-2a+6)$	$\frac{1}{3}(4a+7)$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{28}) = 2^6 \cdot 3^2$. $\# \text{Aut}(L_{29}) = 2^4$.

(Partial) duals of the above: $L_{30} = L_{28}^{\#}, \mathbf{p}_1, L_{31} = L_{29}^{\#}$.

Lattices L_{32} and L_{33} (of 137):

$\mathbf{p}_1$	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	$\mathbf{p}_1$
$\mathbf{p}_1$	0	$\frac{1}{3}(-2a+4)$	0	$\frac{1}{3}(a+2)$	0	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(a+1)$	0	2	0	0	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(2a-3)$	$\frac{1}{3}(-7a+12)$	$\frac{1}{3}(-9a+16)$	$-24a+42$	0	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{32}) = 2^4$. $\# \text{Aut}(L_{33}) = 2^5$.

(Partial) duals of the above: $L_{34} = L_{32}^{\#}, L_{35} = L_{33}^{\#}$.

Lattices L_{36} and L_{37} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}(-2a+4)$	$-\frac{1}{3}$	$\frac{1}{3}(a-1)$	0	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}a$	$\frac{1}{3}(a+3)$	$-\frac{1}{3}$	$\frac{1}{3}(-a-2)$	$\mathbf{p}_1$
$\mathbf{p}_1$	0	0	$\frac{1}{3}(a+3)$	$\frac{1}{3}(a-1)$	$\mathbf{p}_1$
$\mathbf{p}_1$	0	0	$-\frac{1}{3}(a+3)$	0	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{36}) = 2^5$. $\# \text{Aut}(L_{37}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{38} = L_{37}^{\#}, L_{39} = L_{37}^{\#}, \mathbf{p}_1, L_{40} = L_{37}^{\#}, \mathbf{q}_2$.

Lattices L_{41} and L_{42} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}(-2a+4)$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}(-a+1)$	$-\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-a+2)$	$\frac{1}{3}(-2a+4)$	$-\frac{1}{3}$	$\frac{1}{3}(2a-2)$	$\mathbf{p}_1$	
$\mathbf{p}_1$	$-\frac{1}{3}$	$\frac{1}{3}(a-2)$	$\frac{4}{3}$	$\frac{1}{3}(-a+1)$	$\frac{1}{3}(2a-2)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(a-2)$	$\frac{1}{3}(4a-7)$	$\frac{1}{3}(-a+2)$	$\frac{1}{3}(-12a+22)$	$-\frac{1}{3}$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{41}) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_{42}) = 2^3$.

(Partial) duals of the above: $L_{43} = L_{41}^{\#}, L_{44} = L_{41}^{\#}, \mathbf{p}_1, L_{45} = L_{41}^{\#}, \mathbf{q}_4, L_{46} = L_{42}^{\#}, \mathbf{q}_3, L_{47} = L_{42}^{\#}, \mathbf{q}_1, L_{48} = L_{42}^{\#}$.

Lattices L_{49} and L_{50} (of 137):

$\mathbf{p}_1$	2	$\frac{1}{3}(a+3)$	0	0	$\frac{1}{3}a$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{2}{3}a$	$\frac{1}{3}(2a+6)$	0	$\frac{1}{3}(-a+3)$	$\frac{1}{3}a$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{2}{3}a$	$\frac{1}{3}a$	2	0	$\frac{1}{3}a$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}a$	$-\frac{1}{3}a$	$-\frac{2}{3}a$	$\frac{1}{3}(-2a+6)$	$\frac{1}{3}a$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{49}) = 2^2 \cdot 5^1$. $\# \text{Aut}(L_{50}) = 2^3$.

(Partial) duals of the above: $L_{51} = L_{49}^{\#}, L_{52} = L_{50}^{\#}$.

Lattices L_{53} and L_{54} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}(-2a+6)$	$\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{2}{3}a$	2	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-2a+3)$	$-\frac{1}{3}(-a+3)$	$\frac{1}{3}(-2a+6)$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-a+3)$	$-\frac{1}{3}a$	$\frac{1}{3}(-2a+3)$	2	$\frac{1}{3}$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{53}) = 2^2$. $\# \text{Aut}(L_{54}) = 2^6$.

(Partial) duals of the above: $L_{55} = L_{53}^\#$.

Lattices L_{56} and L_{57} (of 137):

$\mathbf{p}_1$	$\frac{2}{3}$	$\frac{1}{3}(-a+3)$	$-\frac{1}{3}a$	$\frac{1}{3}(a-1)$	$\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	0	$-\frac{1}{3}a$	$-\frac{1}{3}a$	$-\frac{1}{3}(a+3)$	$-\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}a$	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}(a-1)$	$-\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(2a+3)$	$\frac{1}{3}(-a-2)$	$a+2$	$\frac{1}{3}(10a+18)$	$\frac{1}{3}$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{56}) = 2^5$. $\# \text{Aut}(L_{57}) = 2^4$.

(Partial) duals of the above: $L_{58} = L_{56}^\#$.

Lattices L_{59} and L_{60} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}$	$\frac{1}{3}(-a+3)$	$-\frac{1}{3}a$	0	$-\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}$	$\frac{1}{3}(-a+2)$	$\frac{1}{3}(-2a+4)$	$\frac{4}{3}$	$-\frac{1}{3}(-a-2)$	$\mathbf{p}_1$
$\mathbf{p}_1$	0	$\frac{1}{3}(-a+1)$	0	0	$-\frac{1}{3}(-a-2)$	$\mathbf{p}_1$
$\mathbf{p}_1$	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{59}) = 2^3$. $\# \text{Aut}(L_{60}) = 2^3$.

(Partial) duals of the above: $L_{61} = L_{59}^\#$, $L_{62} = L_{60}^\#$.

Lattices L_{63} and L_{64} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(a+3)$	$\frac{1}{3}(a+1)$	$-\frac{1}{3}$	$\frac{1}{3}(a+1)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-2a+4)$	$\frac{1}{3}(-4a+8)$	$\frac{1}{3}(a+1)$	$\frac{1}{3}(a+3)$	$\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{2}{3}$	$\frac{1}{3}(2a+4)$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-a+1)$	$\frac{1}{3}(-2a+2)$	$\frac{1}{3}(-a-1)$	$\frac{1}{3}(2a+6)$	$\frac{1}{3}(a+1)$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{63}) = 2^5$. $\# \text{Aut}(L_{64}) = 2^5$.

(Partial) duals of the above: $L_{65} = L_{63}^\#$, $L_{66} = L_{64}^\#$.

Lattices L_{67} and L_{68} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}(a+3)$	$\frac{1}{3}(a+3)$	$-\frac{1}{3}a$	$-\frac{1}{3}a$	$\frac{1}{3}(a+1)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}$	$\frac{1}{3}(-a+3)$	$-\frac{1}{3}a$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(-a+2)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}a$	$-\frac{1}{3}$	$\frac{4}{3}$	$-\frac{1}{3}a$	$\frac{1}{3}(-a+2)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{2}{3}$	0	$\frac{1}{3}a$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(a+1)$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{67}) = 2^3$. $\# \text{Aut}(L_{68}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{69} = L_{67}^\#$, $L_{70} = L_{68}^\#$.

Lattices L_{71} and L_{72} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}(-2a+6)$	$\frac{1}{3}(-2a+3)$	-1	$-a+2$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-2a+3)$	$\frac{1}{3}(-2a+3)$	$\frac{1}{3}(-2a+6)$	0	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(2a+1)$	$\frac{1}{3}(a+7)$	-1	0	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}$	$\frac{1}{3}(a+2)$	$\frac{1}{3}(3a+7)$	$-a+2$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{71}) = 2^3$. $\# \text{Aut}(L_{72}) = 2^3 \cdot 3^2$.

(Partial) duals of the above: $L_{73} = L_{71}^{\#}, \mathbf{p}_1, L_{74} = L_{71}^{\#}, \mathbf{p}_1^3, L_{75} = L_{71}^{\#}$.

Lattices L_{76} and L_{77} (of 137):

$\mathbf{p}_1$	$\frac{2}{3}(-a+2)$	$\frac{1}{3}(-4a+8)$	$\frac{1}{3}a$	$\frac{1}{3}(-a+2)$	$\frac{1}{3}(a-1)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}a$	$\frac{2}{3}(-a+3)$	$\frac{1}{3}a$	$\frac{1}{3}(2a+6)$	$\frac{1}{3}(a-3)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{2}{3}(-a+2)$	2	$\frac{2}{3}(-a+3)$	$\frac{1}{3}(-a+2)$	$\frac{2}{3}(a-3)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-a+2)$	$\frac{1}{3}(-a+2)$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(-4a+8)$	$\frac{1}{3}(a-1)$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{76}) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_{77}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{78} = L_{76}^{\#}, L_{79} = L_{77}^{\#}$.

Lattices L_{80} and L_{81} (of 137):

$\mathbf{p}_1$	$\frac{2}{3}(-a+2)$	$-\frac{1}{3}a$	$-\frac{1}{3}a$	0	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-a-2)$	$-\frac{1}{3}a$	$\frac{1}{3}a$	$\frac{1}{3}(a+1)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-a+2)$	$\frac{1}{3}(4a+8)$	$-\frac{1}{3}a$	$\frac{2}{3}(a+1)$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}$	$\frac{1}{3}(-a-2)$	$\frac{1}{3}(4a+8)$	0	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{80}) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_{81}) = 2^3$.

(Partial) duals of the above: $L_{82} = L_{81}^{\#}$.

Lattices L_{83} and L_{84} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}(-a+3)$	$\frac{2}{3}(2a+6)$	0	$\frac{1}{3}(a+1)$	$\frac{1}{3}a$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}a$	$\frac{1}{3}(2a+6)$	0	$\frac{1}{3}(a+3)$	$-\frac{1}{3}a$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}a$	$\frac{1}{3}(5a+9)$	$\frac{1}{3}(11a+21)$	$\frac{1}{3}(a+1)$	$-\frac{1}{3}a$	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}a$	$\frac{1}{3}(2a+3)$	$\frac{1}{3}(2a+6)$	$\frac{1}{3}(-a+9)$	$\frac{1}{3}a$	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{83}) = 2^4$. $\# \text{Aut}(L_{84}) = 2^4$.

Lattices L_{85} and L_{86} (of 137):

$\mathbf{p}_1$	$\frac{1}{3}(2a+6)$	$\frac{1}{2}(2a+6)$	$\frac{1}{3}(a-3)$	-1	-1	$\mathbf{p}_1$
$\mathbf{p}_1$	$-\frac{1}{3}$	2	$\frac{1}{3}(a-3)$	$\frac{1}{3}(-4a+9)$	$-a+1$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(a+3)$	-1	$\frac{1}{3}(2a+6)$	-1	$-a+1$	$\mathbf{p}_1$
$\mathbf{p}_1$	$\frac{1}{3}(-a+3)$	-1	$\frac{1}{3}(a+3)$	$\frac{1}{3}(-4a+12)$	-1	$\mathbf{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{85}) = 2^4$. $\# \text{Aut}(L_{86}) = 2^3$.

(Partial) duals of the above: $L_{87} = L_{85}^{\#}, L_{88} = L_{86}^{\#}$.

Lattices L_{89} and L_{90} (of 137):

p_1	$\frac{1}{3}(-2a+6)$	1	$-\frac{1}{3}a$	0	$\frac{1}{3}a$	p_1
p_1	$\frac{1}{3}(-a-3)$	$\frac{1}{3}(4a+9)$	$-\frac{1}{3}a$	1	$-\frac{1}{3}a$	p_1
p_1	1	$-a-1$	3	0	$-\frac{2}{3}a$	p_1
p_1	$\frac{1}{3}(-2a+3)$	$-\frac{1}{3}a$	$-a+1$	$\frac{1}{3}(-2a+9)$	$\frac{1}{3}a$	p_1

Number of automorphisms: $\# \text{Aut}(L_{89}) = 2^3$. $\# \text{Aut}(L_{90}) = 2^2 \cdot 3^1$.

(Partial) duals of the above: $L_{91} = L_{89}^\#$, $L_{92} = L_{90}^\#$.

Lattices L_{93} and L_{94} (of 137):

p_1	1	1	0	$\frac{1}{3}a$	$\frac{1}{3}a$	p_1
p_1	0	1	0	1	$\frac{1}{3}a$	p_1
p_1	$-\frac{1}{3}a$	$-\frac{1}{3}a$	2	$\frac{1}{3}a$	$\frac{2}{3}a$	p_1
p_1	$-\frac{1}{3}a$	$-\frac{1}{3}a$	$\frac{2}{3}a$	2	$\frac{1}{3}a$	p_1

Number of automorphisms: $\# \text{Aut}(L_{93}) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_{94}) = 2^3$.

(Partial) duals of the above: $L_{95} = L_{93}^\#$.

Lattices L_{96} and L_{97} (of 137):

p_1	$\frac{1}{3}(-2a+6)$	2	$a-2$	$-\frac{2}{3}a$	1	p_1
p_1	$\frac{1}{3}(a-3)$	$\frac{1}{3}(-2a+6)$	$a-2$	$-2a+4$	$\frac{1}{3}(2a-3)$	p_1
p_1	$\frac{2}{3}a$	$-\frac{1}{3}a$	$\frac{1}{3}(-2a+12)$	$-\frac{2}{3}a$	$\frac{2}{3}(2a-3)$	p_1
p_1	$\frac{2}{3}a$	$-\frac{2}{3}a$	$\frac{1}{3}(8a-12)$	$\frac{1}{3}(-14a+30)$	1	p_1

Number of automorphisms: $\# \text{Aut}(L_{96}) = 2^4 \cdot 3^1$. $\# \text{Aut}(L_{97}) = 2^5 \cdot 3^2$.

(Partial) duals of the above: $L_{98} = L_{96}^\#$.

Lattices L_{99} and L_{100} (of 137):

p_1	2	$\frac{1}{3}(2a+6)$	$-\frac{2}{3}a$	$\frac{2}{3}a$	$\frac{2}{3}a$	p_1
p_1	$\frac{1}{3}(2a-3)$	$\frac{1}{3}(-2a+6)$	$-\frac{2}{3}a$	$\frac{1}{3}(-2a+6)$	$\frac{1}{3}(2a-3)$	p_1
p_1	$-\frac{2}{3}a$	-1	2	$\frac{2}{3}a$	$\frac{2}{3}(2a-3)$	p_1
p_1	$-2a-3$	$\frac{1}{3}(-a-3)$	$\frac{1}{3}(4a+9)$	$\frac{1}{3}(20a+36)$	$\frac{2}{3}a$	p_1

Number of automorphisms: $\# \text{Aut}(L_{99}) = 2^5$. $\# \text{Aut}(L_{100}) = 2^4$.

(Partial) duals of the above: $L_{101} = L_{100}^\#$.

Lattices L_{102} and L_{103} (of 137):

p_1	$\frac{1}{3}(-a+3)$	1	$-\frac{1}{3}a$	$\frac{1}{3}a$	$-\frac{1}{3}a$	p_1
p_1	0	$\frac{1}{3}(-a+3)$	$-\frac{1}{3}a$	1	$\frac{1}{3}a$	p_1
p_1	$\frac{1}{3}a$	$\frac{2}{3}a$	2	$\frac{1}{3}a$	$\frac{2}{3}a$	p_1
p_1	$\frac{2}{3}a$	$\frac{1}{3}(a+3)$	2	$-\frac{1}{3}a$	p_1	p_1

Number of automorphisms: $\# \text{Aut}(L_{102}) = 2^6$. $\# \text{Aut}(L_{103}) = 2^4$.

Lattices L_{104} and L_{105} (of 137):

$\mathfrak{p}_1$	$\frac{1}{3}(a+3)$	$\frac{1}{3}(-a+3)$	$\frac{1}{3}a$	0	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}a$	$\frac{1}{3}(-2a+6)$	$\frac{1}{3}a$	$\frac{1}{3}(a+3)$	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	0	0	2	0	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	0	0	$a-2$	$-2a+4$	0	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{104}) = 2^5 \cdot 3^1$. $\# \text{Aut}(L_{105}) = 2^5 \cdot 3^1$.

Lattices L_{106} and L_{107} (of 137):

$\mathfrak{p}_1$	2	$\frac{1}{3}$	0	0	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$a-2$	$-2a+4$	0	$\frac{2}{3}$	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	0	0	$\frac{1}{3}(-2a+6)$	0	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	0	0	$\frac{1}{3}(a-3)$	$\frac{1}{3}(-2a+6)$	0	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{106}) = 2^5 \cdot 3^2$. $\# \text{Aut}(L_{107}) = 2^5$.

(Partial) duals of the above: $L_{108} = L_{106}^\#$.

Lattices L_{109} and L_{110} (of 137):

$\mathfrak{p}_1$	$\frac{2}{3}$	$\frac{1}{3}(-a-1)$	0	0	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}(a+4)$	0	$\frac{1}{3}(-2a+4)$	$\frac{1}{3}(a-1)$	$\mathfrak{p}_1$	
$\mathfrak{p}_1$	$-\frac{1}{3}$	$\frac{1}{3}(-a+4)$	0	$\frac{1}{3}(a-1)$	$\mathfrak{p}_1$	
$\mathfrak{p}_1$	0	$\frac{1}{3}(a-1)$	$\frac{1}{3}(-2a+5)$	0	$\mathfrak{p}_1$	

Number of automorphisms: $\# \text{Aut}(L_{109}) = 2^5$. $\# \text{Aut}(L_{110}) = 2^5$.

(Partial) duals of the above: $L_{111} = L_{109}^\#$, $L_{112} = L_{110}^\#$.

Lattices L_{113} and L_{114} (of 137):

$\mathfrak{p}_1$	1	$\frac{1}{3}(a+3)$	0	0	$\frac{1}{3}a$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$-\frac{1}{3}a$	1	0	$\frac{1}{3}(-a+3)$	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$-\frac{1}{3}a$	$-\frac{1}{3}a$	$\frac{1}{3}(2a+6)$	0	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}(2a-3)$	$\frac{1}{3}(2a-3)$	-1	$-2a+4$	$\frac{1}{3}a$	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{113}) = 2^5$. $\# \text{Aut}(L_{114}) = 2^5$.

(Partial) duals of the above: $L_{115} = L_{113}^\#$, $L_{116} = L_{114}^\#$.

Lattices L_{117} and L_{118} (of 137):

$\mathfrak{p}_1$	$\frac{2}{3}$	$\frac{1}{3}(a+3)$	$\frac{1}{3}(-a-1)$	$\frac{1}{3}(-a-1)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$-\frac{1}{3}$	$\frac{1}{3}(a+3)$	$\frac{1}{3}(a+3)$	$\frac{1}{3}(-a-1)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	0	$\frac{1}{3}(a+1)$	$\frac{1}{3}(-a-1)$	$\frac{1}{3}(-a-1)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}(-a+1)$	$\frac{1}{3}(a-1)$	$\frac{1}{3}(-2a+6)$	$\frac{1}{3}(-a-1)$	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{117}) = 2^3$. $\# \text{Aut}(L_{118}) = 2^3$.

Lattices L_{119} and L_{120} (of 137):

p_1	$\frac{1}{3}(-2a+9)$	1	$-\frac{2}{3}a$	$\frac{1}{3}a$	$-\frac{1}{3}a$	p_1
p_1	$\frac{1}{3}(-4a+3)$	$\frac{1}{3}(-2a+9)$	$-\frac{2}{3}a$	$\frac{1}{3}a$	$\frac{1}{3}(-2a-6)$	p_1
p_1	$\frac{1}{3}a$	$\frac{1}{3}(2a-6)$	4	$\frac{2}{3}a$	$\frac{1}{3}(-2a-6)$	p_1
p_1	0	$\frac{2}{3}a$	$\frac{2}{3}a$	$\frac{1}{3}(-4a+12)$	$-\frac{1}{3}a$	p_1

Number of automorphisms: $\# \text{Aut}(L_{119}) = 2^2 \cdot 3^1$. $\# \text{Aut}(L_{120}) = 2^2 \cdot 3^1$.

Lattices L_{121} and L_{122} (of 137):

p_1	$\frac{1}{3}(4a+12)$	1	$\frac{1}{3}a$	$\frac{1}{3}(-2a+3)$	$-\frac{1}{3}a$	p_1
p_1	$\frac{1}{3}(2a+6)$	$\frac{1}{3}(-2a+12)$	$\frac{1}{3}a$	$\frac{1}{3}(2a+6)$	$\frac{1}{3}(2a-3)$	p_1
p_1	$-a-1$	$2a-5$	$-6a+12$	$\frac{1}{3}(-2a+3)$	$\frac{1}{3}(2a-6)$	p_1
p_1	$-a-1$	$a-2$	$-3a+6$	6	$-\frac{1}{3}a$	p_1

Number of automorphisms: $\# \text{Aut}(L_{121}) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_{122}) = 2^2$.

(Partial) duals of the above: $L_{123} = L_{121}^\#$, $L_{124} = L_{122}^\#$.

Lattices L_{125} and L_{126} (of 137):

p_1	$\frac{1}{3}(2a+6)$	$\frac{1}{3}(2a+9)$	$\frac{1}{3}(-4a-3)$	$\frac{1}{3}(2a-3)$	$\frac{1}{3}(-2a+3)$	p_1
p_1	$\frac{1}{3}(-2a-3)$	2	$\frac{1}{3}(-4a-3)$	$\frac{1}{3}(2a+9)$	-1	p_1
p_1	$\frac{1}{3}(2a+3)$	-1	$\frac{1}{3}(2a+6)$	$\frac{1}{3}(2a-3)$	-1	p_1
p_1	$\frac{1}{3}(-a+3)$	$\frac{1}{3}(a-3)$	0	$-2a+4$	$\frac{1}{3}(-2a+3)$	p_1

Number of automorphisms: $\# \text{Aut}(L_{125}) = 2^3$. $\# \text{Aut}(L_{126}) = 2^4$.

(Partial) duals of the above: $L_{127} = L_{125}^\#$.

Lattices L_{128} and L_{129} (of 137):

p_1	$\frac{1}{3}(2a+6)$	$\frac{1}{3}(-2a+9)$	$\frac{1}{3}(-2a-3)$	$\frac{2}{3}a$	$-a-1$	p_1
p_1	0	$\frac{1}{3}(-2a+6)$	$\frac{1}{3}(-2a-3)$	$\frac{1}{3}(2a+9)$	$-\frac{4}{3}a$	p_1
p_1	0	$\frac{2}{3}(-4a+6)$	$\frac{2}{3}(-7a+15)$	$\frac{2}{3}a$	$-\frac{2}{3}a$	p_1
p_1	$\frac{1}{3}(a+3)$	$-a+1$	$\frac{1}{3}(-4a+3)$	$\frac{1}{3}(a+12)$	$-a-1$	p_1

Number of automorphisms: $\# \text{Aut}(L_{128}) = 2^3$. $\# \text{Aut}(L_{129}) = 2^2$.

(Partial) duals of the above: $L_{130} = L_{128}^\#$, $L_{131} = L_{129}^\#$.

Lattices L_{132} and L_{133} (of 137):

p_1	$\frac{1}{3}(2a+9)$	2	$-\frac{1}{3}a$	$\frac{1}{3}(a+3)$	$-a-1$	p_1
p_1	$\frac{1}{3}(2a-3)$	$\frac{1}{3}(-2a+9)$	$-\frac{1}{3}a$	$\frac{1}{3}(a+3)$	$-a-1$	p_1
p_1	$-a+1$	$\frac{2}{3}(5a-3)$	$\frac{1}{3}(-2a+12)$	$\frac{1}{3}(a+3)$	$-a-1$	p_1
p_1	$\frac{1}{3}(-2a-3)$	$\frac{1}{3}(-2a-3)$	$\frac{1}{3}(-a-3)$	$\frac{1}{3}(2a+9)$	$-a-1$	p_1

Number of automorphisms: $\# \text{Aut}(L_{132}) = 2^2$. $\# \text{Aut}(L_{133}) = 2^1$.

(Partial) duals of the above: $L_{134} = L_{132}^\#$, $L_{135} = L_{133}^\#$.

Lattices L_{136} and L_{137} (of 137):

p_1	$\frac{1}{2}(-2a+9)$	$\frac{1}{3}(2a+9)$	$-\frac{2}{3}a$	a	p_1
p_1	$\frac{2}{3}a$	4	$-\frac{2}{3}a$	4	p_1
p_1	a	$\frac{1}{3}(-4a+6)$	$\frac{1}{3}(-2a+15)$	a	$\frac{1}{3}(-4a-6)$
p_1	$a-1$	$-2a+2$	$-a+3$	$\frac{1}{3}(-10a+24)$	$\frac{2}{3}(-4a-6)$
				a	p_1

Number of automorphisms: $\# \text{Aut}(L_{136}) = 2^1$, $\# \text{Aut}(L_{137}) = 2^1$.

Base fields with discriminant 13^1

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^2 - x - 3$

30 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 30):

0	2	2	1	$-a+1$	0	0
0	-1	2	1	2	0	0
0	$-a+1$	a	4	$-a+1$	0	0
0	1	0	$-a+1$	2	0	0

Number of automorphisms: $\# \text{Aut}(L_1) = 2^6 \cdot 3^2$, $\# \text{Aut}(L_2) = 2^3 \cdot 3^2$.

Lattices L_3 and L_4 (of 30):

0	2	2	0	1	-1	0
0	$3a-6$	$-18a+42$	0	2	1	0
0	a	$-2a+6$	$2a+4$	1	1	0
0	$-a-1$	$-2a+3$	$-2a-2$	$2a+6$	-1	0

Number of automorphisms: $\# \text{Aut}(L_3) = 2^3 \cdot 3^2$, $\# \text{Aut}(L_4) = 2^6$.

Lattices L_5 and L_6 (of 30):

0	2	2	$-a+1$	$a-1$	a	0
0	0	2	$-a+1$	$-2a+6$	$a-3$	0
0	1	1	$2a+4$	$a-1$	$a-3$	0
0	-1	-1	-1	$2a+4$	a	0

Number of automorphisms: $\# \text{Aut}(L_5) = 2^6$, $\# \text{Aut}(L_6) = 2^3$.

Lattices L_7 and L_8 (of 30):

0	4	2	a	$a-1$	1	0
0	$-a+2$	4	a	$2a+6$	$-2a$	0
0	$a-2$	$a-1$	4	$a-1$	$-2a$	0
0	$a-2$	$a-1$	$-a+1$	4	1	0

Number of automorphisms: $\# \text{Aut}(L_7) = 2^4 \cdot 3^1$, $\# \text{Aut}(L_8) = 2^4 \cdot 3^1$.

(Partial) duals of the above: $L_9 = L_7^\#$, $L_{10} = L_8^\#$.

Lattices L_{11} and L_{12} (of 30):

0	2	2	a	a	0	0
0	0	2	a	4	a+2	0
0	-1	a-1	4	a	a+2	0
0	a-1	-1	-a+1	4	0	0

Number of automorphisms: $\# \text{Aut}(L_{11}) = 2^5$, $\# \text{Aut}(L_{12}) = 2^3$.

Lattices L_{13} and L_{14} (of 30):

0	4	2	-a+1	-a	-a+1	0
0	-a-1	4	-a+1	-2a+8	-2a+5	0
0	a+1	-a	4	-a	-2a+5	0
0	a+1	-a	a	4	-a+1	0

Number of automorphisms: $\# \text{Aut}(L_{13}) = 2^4 \cdot 3^1$, $\# \text{Aut}(L_{14}) = 2^4 \cdot 3^1$.

(Partial) duals of the above: $L_{15} = L_{13}^\#$, $L_{16} = L_{14}^\#$.

Lattices L_{17} and L_{18} (of 30):

0	2	2	a-1	-a+1	0	0
0	0	2	a-1	4	-1	0
0	1	-a	4	-a+1	-1	0
0	-a	-1	0	4	0	0

Number of automorphisms: $\# \text{Aut}(L_{17}) = 2^5$, $\# \text{Aut}(L_{18}) = 2^3$.

Lattices L_{19} and L_{20} (of 30):

0	4	4	2a-1	a+1	-a-1	0
0	-a-1	4	2a-1	4	a+1	0
0	a-2	a-2	-2a+6	a+1	a+1	0
0	-a+2	-a+2	a-2	-2a+6	-a-1	0

Number of automorphisms: $\# \text{Aut}(L_{19}) = 2^3 \cdot 3^1$, $\# \text{Aut}(L_{20}) = 2^3 \cdot 3^2$.

(Partial) duals of the above: $L_{21} = L_{19}^\#$, $L_{22} = L_{20}^\#$.

Lattices L_{23} and L_{24} (of 30):

0	4	4	2a-1	a-2	-a+2	0
0	-a+2	4	2a-1	4	-a+2	0
0	a+1	-a-1	2a+4	a-2	-a+2	0
0	-a-1	a+1	-a-1	2a+4	-a+2	0

Number of automorphisms: $\# \text{Aut}(L_{23}) = 2^3 \cdot 3^1$, $\# \text{Aut}(L_{24}) = 2^3 \cdot 3^2$.

(Partial) duals of the above: $L_{25} = L_{23}^\#$, $L_{26} = L_{24}^\#$.

Lattices L_{27} and L_{28} (of 30):

0	4	4	2	-2	2a	0
0	-2	-2a+6	2	2a+4	-1	0
0	2	-1	-2a+6	-2	-1	0
0	-2a+2	a-1	-a+1	-2a+8	2a	0

Number of automorphisms: $\# \text{Aut}(L_{27}) = 2^5$, $\# \text{Aut}(L_{28}) = 2^5$.

(Partial) duals of the above: $L_{29} = L_{27}^\#$, $L_{30} = L_{28}^\#$.

Base fields with discriminant 17^1

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^2 - x - 4$

21 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 21):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -1 \\ -7a-11 \\ a \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 2 \\ 0 \\ -a+1 \end{smallmatrix}$	$\begin{smallmatrix} a-1 \\ a-1 \\ 146a+228 \\ -9a-14 \end{smallmatrix}$	$\begin{smallmatrix} a \\ -a+4 \\ a \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} -a-2 \\ 2 \\ 2 \\ -a-2 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	---	--	---	--	--

Number of automorphisms: $\# \text{Aut}(L_1) = 2^4 \cdot 3^2$. $\# \text{Aut}(L_2) = 2^5$.

Lattices L_3 and L_4 (of 21):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ a \\ -a+1 \\ a-3 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ a+3 \\ -2 \\ -a+2 \end{smallmatrix}$	$\begin{smallmatrix} a \\ a \\ 4 \\ 2a-5 \end{smallmatrix}$	$\begin{smallmatrix} a+2 \\ a \\ a+2 \\ -5a+13 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ a+1 \\ a+1 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_3) = 2^5$. $\# \text{Aut}(L_4) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_5 = L_4^\#$.

Lattices L_6 and L_7 (of 21):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -a \\ a-1 \\ -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 4 \\ -2 \\ 2a-4 \end{smallmatrix}$	$\begin{smallmatrix} -1 \\ -1 \\ 4 \\ -1 \end{smallmatrix}$	$\begin{smallmatrix} -a \\ 2 \\ -a \\ -6a+16 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ a-1 \\ a-1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	--	---	---	--	--

Number of automorphisms: $\# \text{Aut}(L_6) = 2^5$. $\# \text{Aut}(L_7) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_8 = L_7^\#$.

Lattices L_9 and L_{10} (of 21):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -a \\ -a \\ -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 4 \\ 3 \\ a \end{smallmatrix}$	$\begin{smallmatrix} -a+1 \\ -a+1 \\ 4 \\ a-1 \end{smallmatrix}$	$\begin{smallmatrix} -1 \\ 4 \\ -1 \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ a-1 \\ a-1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	---	--	--	--	--	--

Number of automorphisms: $\# \text{Aut}(L_9) = 2^5$. $\# \text{Aut}(L_{10}) = 2^3$.

Lattices L_{11} and L_{12} (of 21):

0	2	-a	1	-1	0
0	1	2	-a	4	0
0	-1	-1	-2a+6	1	0
0	1	0	a-3	-2a+6	-1
0					0

Number of automorphisms: $\# \text{Aut}(L_{11}) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_{12}) = 2^3$.

Lattices L_{13} and L_{14} (of 21):

0	2	a	-a+1	0	0
0	-1	2	a	-a+4	-a-2
0	1	-1	2a+4	-a+1	-a-2
0	-1	1	a+1	2a+4	0
0					0

Number of automorphisms: $\# \text{Aut}(L_{13}) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_{14}) = 2^3$.

Lattices L_{15} and L_{16} (of 21):

0	2	-a+3	-1	-1	-a+2
0	-a+1	-a+5	-1	a+3	a+1
0	a	-a-1	a+5	-1	a+1
0	0	-a+1	2	-a+5	-a+2
0					0

Number of automorphisms: $\# \text{Aut}(L_{15}) = 2^4$. $\# \text{Aut}(L_{16}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{17} = L_{16}^\#$.

Lattices L_{18} and L_{19} (of 21):

0	2	-3a+8	-a+3	-a+3	0
0	-1	2a+4	-a+3	4	a+1
0	1	a+1	2a+4	-a+3	a+1
0	-a	-1	1	6	0
0					0

Number of automorphisms: $\# \text{Aut}(L_{18}) = 2^4$. $\# \text{Aut}(L_{19}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{20} = L_{19}^\#$.

Lattice L_{21} (of 21):

0	2				
0	1	-2a+6			
0	1	a-2	-2a+6		
0	-a+1	-1		6	
0					

Number of automorphisms: $\# \text{Aut}(L_{21}) = 2^4$.

Base fields with discriminant $3^1 \cdot 7^1$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^2 - x - 5$

5 similarity classes of single-class genera.

Narrow class group prime ideal representatives:

Ideal	generators
$\mathfrak{p}_1$	$a + 1$

Lattices L_1 and L_2 (of 5):

$\mathfrak{p}_1$	$2a + 4$	$\frac{1}{3}(4a + 10)$	$\frac{1}{3}(8a + 14)$	$\frac{1}{3}(2a + 2)$	$\frac{1}{3}(a + 1)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}(-5a - 8)$	$2a + 4$	$\frac{1}{3}(8a + 14)$	$\frac{1}{3}(22a + 40)$	$\frac{1}{3}(-5a - 8)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}(2a + 2)$	$-a - 2$	$\frac{1}{3}(4a + 10)$	$\frac{1}{3}(2a + 2)$	$\frac{1}{3}(-5a - 8)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{3}(a + 4)$	$-a - 1$	$\frac{1}{3}(2a + 2)$	2	$\frac{1}{3}(a + 1)$	$\mathfrak{p}_1$

Number of automorphisms: $\# \operatorname{Aut}(L_1) = 2^4 \cdot 3^2$. $\# \operatorname{Aut}(L_2) = 2^2 \cdot 3^1$.

(Partial) duals of the above: $L_3 = L_2^\#$.

Lattice L_4 (of 5):

$\mathfrak{p}_1$	$\frac{1}{3}(14a + 26)$
$\mathfrak{p}_1$	$\frac{1}{3}(-16a - 28)$
$\mathfrak{p}_1$	$\frac{1}{3}(20a + 38)$
$\mathfrak{p}_1$	$\frac{1}{3}(8a + 14)$
$\mathfrak{p}_1$	$\frac{1}{3}(14a + 26)$
$\mathfrak{p}_1$	$\frac{1}{3}(23a + 41)$
$\mathfrak{p}_1$	$\frac{1}{3}(20a + 35)$
$\mathfrak{p}_1$	$\frac{1}{3}(44a + 80)$

Number of automorphisms: $\# \operatorname{Aut}(L_4) = 2^2 \cdot 3^1$.

(Partial) duals of the above: $L_5 = L_4^\#$.

Base fields with discriminant $2^3 \cdot 3^1$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^2 - 6$

14 similarity classes of single-class genera.

Narrow class group prime ideal representatives:

Ideal	generators
$\mathfrak{p}_1$	$a + 1$

Lattices L_1 and L_2 (of 14):

$\mathfrak{p}_1$	$\frac{1}{25}(-4a + 14)$	$\frac{1}{25}(-4a + 14)$	0	0	$\frac{1}{5}(a - 1)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{25}(3a + 2)$	$\frac{1}{25}(8a + 22)$	0	$\frac{1}{25}(-4a + 14)$	$\frac{1}{25}(7a - 12)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{25}(9a - 19)$	$\frac{1}{25}(-2a + 7)$	$\frac{1}{25}(-34a + 94)$	0	$\frac{1}{25}(7a - 12)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{25}(-4a - 11)$	$\frac{1}{25}(-4a - 11)$	$\frac{1}{25}(-7a - 38)$	$\frac{1}{25}(62a + 158)$	$\frac{1}{5}(a - 1)$	$\mathfrak{p}_1$

Number of automorphisms: $\# \operatorname{Aut}(L_1) = 2^6$. $\# \operatorname{Aut}(L_2) = 2^5$.

(Partial) duals of the above: $L_3 = L_1^\#$, $L_4 = L_2^\#$.

Lattices L_5 and L_6 (of 14):

$\mathfrak{p}_1$	$\frac{1}{5}(-6a+16)$	$\frac{1}{5}(-3a+8)$	$\frac{1}{5}(-4a+9)$	0	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$a-2$	$\frac{1}{5}(-2a+12)$	$\frac{1}{5}(-4a+9)$	$\frac{1}{5}(-6a+16)$	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{5}(-4a+9)$	$\frac{1}{5}(3a-8)$	$\frac{1}{5}(-6a+16)$	0	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{5}(a-1)$	$\frac{1}{5}(-a+1)$	$\frac{1}{5}(a-6)$	$\frac{1}{5}(a+9)$	0	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_5) = 2^4$, $\# \text{Aut}(L_6) = 2^5$.

Lattices L_7 and L_8 (of 14):

$\mathfrak{p}_1$	$\frac{1}{25}(8a+22)$	$\frac{1}{25}(-8a+28)$	$\frac{1}{25}(-18a+38)$	$\frac{1}{25}(-7a+12)$	$\frac{1}{25}(3a+2)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{25}(-11a-24)$	$\frac{1}{25}(24a+66)$	$\frac{1}{25}(-18a+38)$	$\frac{1}{25}(-34a+94)$	$\frac{1}{25}(-16a+31)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	0	0	$\frac{1}{25}(-26a+66)$	$\frac{1}{25}(-7a+12)$	$\frac{1}{25}(-16a+31)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{25}(4a+11)$	$\frac{1}{5}(-2a-3)$	$\frac{1}{25}(-6a+21)$	$\frac{1}{25}(-6a+46)$	$\frac{1}{25}(3a+2)$	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_7) = 2^5$, $\# \text{Aut}(L_8) = 2^3$.

(Partial) duals of the above: $L_9 = L_7^\#$, $L_{10} = L_8^\#$.

Lattices L_{11} and L_{12} (of 14):

$\mathfrak{p}_1$	$\frac{1}{5}(-6a+16)$	$\frac{1}{5}(-a+6)$	$\frac{1}{5}(2a-2)$	0	$\frac{1}{5}(3a-3)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{5}(-a+1)$	$-a+3$	$\frac{1}{5}(2a-2)$	$\frac{1}{5}(-a+6)$	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{5}(4a-9)$	$\frac{1}{5}(-11a+26)$	$\frac{1}{5}(-27a+67)$	0	0	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{5}(-2a+2)$	$\frac{1}{5}(a-1)$	$\frac{1}{5}(2a-7)$	$\frac{1}{5}(2a+8)$	$\frac{1}{5}(3a-3)$	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_{11}) = 2^2$, $\# \text{Aut}(L_{12}) = 2^5$.

(Partial) duals of the above: $L_{13} = L_{11}^\#$, $L_{14} = L_{12}^\#$.

Base fields with discriminant 7²

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^3 - x^2 - 2x + 1$

24 similarity classes of single-class genera.

Narrow class group is the trivial group.

Additional primes needed for partial duals:

Ideal	generators
$\mathfrak{q}_1$	3
$\mathfrak{q}_2$	7, $a+2$

Lattices L_1 and L_2 (of 24):

0	2	2	a^2+a-1	0	0	0
0	-1	2	a^2+a-1	$2a^2+2a$	0	0
0	-1	1	2	0	0	0
0	1	0	-1	2	0	0

Number of automorphisms: $\# \text{Aut}(L_1) = 2^7 \cdot 3^2$. $\# \text{Aut}(L_2) = 2^5 \cdot 7^2$.

Lattices L_3 and L_4 (of 24):

$\emptyset$	2	$2a^2$	a^2	$3a^2 + 3a - 2$	$a^2 - a - 2$	$\emptyset$
$\emptyset$	$-a$	2	a^2	2	$2a^2 + a - 1$	$\emptyset$
$\emptyset$	$a^2 + a - 1$	$-a^2 - a + 1$	$4a^2$	$3a^2 + 3a - 2$	$2a^2 + a - 1$	$\emptyset$
$\emptyset$	$-a^2 - a + 1$	a^2	$-3a^2 + 1$	$4a^2$	$a^2 - a - 2$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_3) = 2^3 \cdot 7^2$. $\# \text{Aut}(L_4) = 2^3 \cdot 7^2$.

Lattices L_5 and L_6 (of 24):

$\emptyset$	$2a^2 - 4a + 2$	$-2a + 4$	$-2a^2 + 3a + 1$	$-a^2 + a$	$-a + 1$	$\emptyset$
$\emptyset$	$-a + 1$	$2a^2 + 2a$	$-2a^2 + 3a + 1$	$2a^2 - 8a + 8$	$-a + 2$	$\emptyset$
$\emptyset$	$-a^2 + 2a$	$-2a^2 + a + 3$	$-6a^2 + 20$	$-a^2 + a$	$-a + 2$	$\emptyset$
$\emptyset$	$-a^2 + 2a - 1$	a	$-2a^2 - a + 8$	$-2a^2 + 8$	$-a + 1$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_5) = 2^3 \cdot 7^2$. $\# \text{Aut}(L_6) = 2^3 \cdot 3^2$.

Lattices L_7 and L_8 (of 24):

$\emptyset$	2	$2a^2 + 2a$	$-a^2 + 1$	$a^2 + a - 1$	$a^2 + a$	$\emptyset$
$\emptyset$	$-a^2 + 1$	$8a^2 - 16a + 6$	$-a^2 + 1$	2	0	$\emptyset$
$\emptyset$	$-a^2 - a$	$3a^2 - 2a$	$6a^2 + 4a - 2$	$a^2 + a - 1$	0	$\emptyset$
$\emptyset$	$3a^2 - a - 7$	$4a^2 - 8a$	2	$-14a^2 + 6a + 38$	$a^2 + a$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_7) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_8) = 2^3 \cdot 3^2$.

Lattices L_9 and L_{10} (of 24):

$\emptyset$	2	2	0	$-a + 1$	$-a^2$	$\emptyset$
$\emptyset$	0	2	0	2	$-a^2 + 1$	$\emptyset$
$\emptyset$	$-a^2 + 1$	$a^2 - a - 1$	$-14a^2 + 8a + 34$	$-a + 1$	$-a^2 + 1$	$\emptyset$
$\emptyset$	$a - 1$	$a^2 + a - 1$	0	$6a^2 - 2a$	$-a^2$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_9) = 2^5$. $\# \text{Aut}(L_{10}) = 2^5$.

Lattices L_{11} and L_{12} (of 24):

$\emptyset$	2	$-2a + 4$	$2a^2 - 4a + 1$	$a^2 - 2a$	$-a + 1$	$\emptyset$
$\emptyset$	0	2	$2a^2 - 4a + 1$	$10a^2 - 22a + 8$	$-2a^2 + 4a - 1$	$\emptyset$
$\emptyset$	-1	$-a^2 + a + 1$	$-8a^2 + 4a + 20$	$a^2 - 2a$	$-2a^2 + 4a - 1$	$\emptyset$
$\emptyset$	$-2a^2 - a + 1$	a	a^2	$12a^2 + 4a - 4$	$-a + 1$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{11}) = 2^5$. $\# \text{Aut}(L_{12}) = 2^4 \cdot 3^1$.

(Partial) duals of the above: $L_{13} = L_{12}^{\#q1}$, $L_{14} = L_{12}^{\#q2}$, $L_{15} = L_{12}^{\#}$.

Lattices L_{16} and L_{17} (of 24):

$\emptyset$	$-4a^2 + 2a + 12$	$2a + 4$	$2a^2 - 1$	$-2a^2 + 2a + 5$	$-a^2 - 2a$	$\emptyset$
$\emptyset$	$-4a^2 + 2a + 10$	$-4a^2 + 14$	$2a^2 - 1$	$2a^2 - 2a + 2$	0	$\emptyset$
$\emptyset$	$-a^2 + a$	$a^2 - a - 3$	6	$-2a^2 + 2a + 5$	0	$\emptyset$
$\emptyset$	$2a^2 - a - 4$	-3	$a^2 + a - 1$	$6a^2 + 2a$	$-a^2 - 2a$	$\emptyset$

Number of automorphisms: $\# \text{Aut}(L_{16}) = 2^3$. $\# \text{Aut}(L_{17}) = 2^3 \cdot 7^2$.

(Partial) duals of the above: $L_{18} = L_{16}^\#$.

Lattices L_{19} and L_{20} (of 24):

$\mathfrak{o}$	4	4	$-2a + 1$	$a^2 - 3a$	$-a^2 + a$
$\mathfrak{o}$	$-2a^2 + 3$	4	$-2a + 1$	4	$-a + 2$
$\mathfrak{o}$	$-a^2 - 3a + 1$	$2a^2 - a$	$12a^2 + 4a - 4$	$a^2 - 3a$	$-a + 2$
$\mathfrak{o}$	$-2a^2 + a + 3$	$-a^2 + a + 2$	$-2a^2$	$-2a^2 + 4a + 10$	$-a^2 + a$

Number of automorphisms: $\# \text{Aut}(L_{19}) = 2^3 \cdot 7^1$. $\# \text{Aut}(L_{20}) = 2^3 \cdot 7^1$.

(Partial) duals of the above: $L_{21} = L_{19}^\#$, $L_{22} = L_{20}^\#$.

Lattice L_{23} (of 24):

$\mathfrak{o}$	4	$-4a^2 + a + 9$	$-18a^2 + 10a + 42$	
$\mathfrak{o}$	$-2a^2 - 2a + 1$	$2a + 2$	$6a^2 + 4a - 2$	
$\mathfrak{o}$	1	$-a^2 + 3$	$-a^2$	$4a^2$

Number of automorphisms: $\# \text{Aut}(L_{23}) = 2^3 \cdot 7^1$.

(Partial) duals of the above: $L_{24} = L_{23}^\#$.

Base fields with discriminant $2^2 \cdot 3^1 \cdot 5^1$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^2 - 15$

3 similarity classes of single-class genera.

Narrow class group prime ideal representatives:

Ideal	generators
$\mathfrak{p}_1$	$11, a + 9$
$\mathfrak{p}_2$	$5, a$
$\mathfrak{p}_3$	$3, a$

Lattices L_1 and L_2 (of 3):

$\mathfrak{p}_1$	$\frac{1}{11}(5a + 21)$	$\frac{1}{11}(10a + 42)$	$\frac{1}{11}(-21a - 75)$	0	$\frac{1}{11}(-2a - 4)$
$\mathfrak{p}_1$	$\frac{1}{11}(3a + 6)$	2	$\frac{1}{11}(-21a - 75)$	$\frac{1}{11}(38a + 164)$	$\frac{1}{11}(3a + 6)$
$\mathfrak{p}_1$	0	$\frac{1}{11}(-18a - 69)$	$16a + 62$	0	$\frac{1}{11}(3a + 6)$
$\mathfrak{p}_2$	$\frac{1}{5}(-2a - 5)$	$\frac{1}{5}(-a - 15)$	$\frac{1}{5}(17a + 65)$	$\frac{1}{5}(3a + 27)$	$\frac{1}{11}(-2a - 4)$

Number of automorphisms: $\# \text{Aut}(L_1) = 2^2$. $\# \text{Aut}(L_2) = 2^2$.

(Partial) duals of the above: $L_3 = L_2^\#$.

Base fields with discriminant 3^4

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^3 - 3x - 1$

29 similarity classes of single-class genera.

Narrow class group is the trivial group.

Additional primes needed for partial duals:

Ideal	generators
$\mathfrak{q}_1$	$71, a + 46$
$\mathfrak{q}_2$	$3, a + 2$
$\mathfrak{q}_3$	$71, a + 55$
$\mathfrak{q}_4$	$71, a + 41$

Lattices L_1 and L_2 (of 29):

$\mathfrak{o}$	2	$-2a + 4$	$-a^2 + a + 2$	0	0
$\mathfrak{o}$	0	2	$-a^2 + a + 2$	$2a^2 + 4a + 2$	0
$\mathfrak{o}$	$a^2 - 2$	$a^2 - a - 1$	$2a^2 - 4a + 2$	0	0
$\mathfrak{o}$	$-a^2 + 1$	$-a^2 - a + 1$	$-a - 2$	$4a^2 + 8a + 4$	0

Number of automorphisms: $\# \text{Aut}(L_1) = 2^3$. $\# \text{Aut}(L_2) = 2^5 \cdot 3^4$.

(Partial) duals of the above: $L_3 = L_1^{\# \cdot \mathfrak{q}_1}$, $L_4 = L_1^{\# \cdot \mathfrak{q}_2}$, $L_5 = L_1^{\#}$.

Lattices L_6 and L_7 (of 29):

$\mathfrak{o}$	2	2	0	a	$2a^2 - a - 1$	0
$\mathfrak{o}$	0	2	0	2	$a^2 - 1$	0
$\mathfrak{o}$	$-a^2 + a$	$-a^2 + a$	$4a^2 - 4a$	a	$a^2 - 1$	0
$\mathfrak{o}$	$-a^2 + a$	$-a^2 + a$	$4a^2 - 5a - 2$	$4a^2 - 4a$	$2a^2 - a - 1$	0

Number of automorphisms: $\# \text{Aut}(L_6) = 2^6$. $\# \text{Aut}(L_7) = 2^5$.

Lattices L_8 and L_9 (of 29):

$\mathfrak{o}$	2	2	0	$a + 1$	$a^2 - 2a - 2$	0
$\mathfrak{o}$	0	$-2a^2 + 2a + 8$	0	2	$a^2 - 2$	0
$\mathfrak{o}$	$a - 1$	$-a^2 + 2$	$18a^2 - 28a - 10$	$a + 1$	$a^2 - 2$	0
$\mathfrak{o}$	$a^2 - 6$	$2a^2 - 5$	$-2a + 2$	$-4a^2 - 2a + 22$	$a^2 - 2a - 2$	0

Number of automorphisms: $\# \text{Aut}(L_8) = 2^5$. $\# \text{Aut}(L_9) = 2^5$.

Lattices L_{10} and L_{11} (of 29):

$\mathfrak{o}$	$-8a^2 + 6a + 28$	2	$-a^2$	$-a^2 + a$	-1	0
$\mathfrak{o}$	$-3a^2 - 2a + 4$	$2a^2 + 2a + 2$	$-a^2$	$2a^2 + 2a + 2$	$2a^2 - a - 1$	0
$\mathfrak{o}$	$-3a^2 + 7$	$-a^2 + 2a + 2$	4	$-a^2 + a$	$2a^2 - a - 1$	0
$\mathfrak{o}$	$3a^2 - 7$	$-2a^2 + 2a$	$2a^2 - 4a - 3$	$28a^2 - 42a - 16$	-1	0

Number of automorphisms: $\# \text{Aut}(L_{10}) = 2^3$. $\# \text{Aut}(L_{11}) = 2^3$.

Lattices L_{12} and L_{13} (of 29):

0	2	2	0	$-a^2$	$a^2 - a - 1$	0
0	$a^2 - 1$	4	0	2	-1	0
0	a	$2a + 1$	$4a^2 + 6a + 2$	$-a^2$	-1	0
0	$-a^2 + 1$	$-a^2 - a - 1$	$a^2 + a$	$18a^2 + 28a + 8$	$a^2 - a - 1$	0

Number of automorphisms: $\# \text{Aut}(L_{12}) = 2^3$. $\# \text{Aut}(L_{13}) = 2^3$.

(Partial) duals of the above: $L_{14} = L_{13}^{\#}, q_2$, $L_{15} = L_{13}^{\#}, q_3$, $L_{16} = L_{13}^{\#}$.

Lattices L_{17} and L_{18} (of 29):

0	2	4	-2	$-2a + 2$	$2a - 1$	0
0	0	2	-2	4	$a - 1$	0
0	$a^2 - a - 2$	$-a^2 + 1$	4	$-2a + 2$	$a - 1$	0
0	$a + 1$	$a^2 + a$	$-a^2 - a - 1$	$4a^2 + 6a + 2$	$2a - 1$	0

Number of automorphisms: $\# \text{Aut}(L_{17}) = 2^3$. $\# \text{Aut}(L_{18}) = 2^4 \cdot 3^2$.

(Partial) duals of the above: $L_{19} = L_{17}^{\#}, q_2$, $L_{20} = L_{17}^{\#}, q_4$, $L_{21} = L_{17}^{\#}$, $L_{22} = L_{18}^{\#}$.

Lattices L_{23} and L_{24} (of 29):

0	4	4	$-2a^2 + a + 6$	$a^2 + a - 2$	-2	0
0	$-2a^2 + 2a$	$4a^2 - 4a$	$-2a^2 + a + 6$	$-4a + 8$	$a - 1$	0
0	$-2a + 2$	$-2a^2 + 4a + 1$	$2a^2 - 4a + 2$	$a^2 + a - 2$	$a - 1$	0
0	$2a - 1$	$a^2 - 4a$	$a - 1$	$2a^2 + 2$	-2	0

Number of automorphisms: $\# \text{Aut}(L_{23}) = 2^3 \cdot 3^3$. $\# \text{Aut}(L_{24}) = 2^3 \cdot 3^3$.

(Partial) duals of the above: $L_{25} = L_{24}^{\#}$.

Lattices L_{26} and L_{27} (of 29):

0	4	2	$-a^2 - a$	0	0	0
0	$-2a^2 + 2a$	$4a^2 - 4a$	$-a^2 - a$	$2a^2 + 4a + 2$	0	0
0	$a^2 - a - 3$	$-a^2 + a + 3$	$-2a^2 + 8$	0	0	0
0	$a^2 + a - 2$	$-a + 1$	$-2a^2 + 5$	$2a + 4$	0	0

Number of automorphisms: $\# \text{Aut}(L_{26}) = 2^4 \cdot 3^4$. $\# \text{Aut}(L_{27}) = 2^4 \cdot 3^4$.

Lattice L_{28} (of 29):

0	4	$2a^2 - 2a$	$4a^2 - 4a$			
0	$2a^2 - 2a$	$4a^2 - 5a - 2$	$4a^2 - 4a$			
0	$-2a + 2$	$2a^2 - 4a - 1$	$2a^2 - 4a - 1$	$2a^2 - 4a + 2$		

Number of automorphisms: $\# \text{Aut}(L_{28}) = 2^5$.

(Partial) duals of the above: $L_{29} = L_{28}^{\#}$.

Base fields with discriminant $2^2 \cdot 37^1$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^3 - x^2 - 3x + 1$

11 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 11):

$\mathfrak{o}$	2	2	-1	$a^2 - 1$	$-a^2$	$\mathfrak{o}$
$\mathfrak{o}$	$a^2 - a - 2$	$-a^2 + 5$	-1	2	0	$\mathfrak{o}$
$\mathfrak{o}$	a	$a - 1$	$5a^2 + 2a - 1$	$a^2 - 1$	0	$\mathfrak{o}$
$\mathfrak{o}$	$-a^2 + 2a + 1$	$a - 2$	a	$2a^2 - 6a + 4$	$-a^2$	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_1) = 2^7 \cdot 3^1$. $\# \text{Aut}(L_2) = 2^7 \cdot 3^2$.

Lattices L_3 and L_4 (of 11):

$\mathfrak{o}$	2	2	-1	$-a$	$-a$	$\mathfrak{o}$
$\mathfrak{o}$	$a^2 - 1$	$2a + 4$	-1	2	1	$\mathfrak{o}$
$\mathfrak{o}$	$-a^2 + 2$	$-a - 2$	$2a + 4$	$-a$	1	$\mathfrak{o}$
$\mathfrak{o}$	$-a^2 + 1$	$-a^2 - a$	$2a^2 + 2a - 1$	$4a^2 + 2a$	$-a$	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_3) = 2^6$. $\# \text{Aut}(L_4) = 2^3 \cdot 3^2$.

Lattices L_5 and L_6 (of 11):

$\mathfrak{o}$	2	2	$-3a^2 + 8a - 2$	$-a^2 + 1$	$a^2 - 3$	$\mathfrak{o}$
$\mathfrak{o}$	$a^2 - a - 3$	$-2a^2 + 2a + 8$	$-3a^2 + 8a - 2$	$52a^2 - 130a + 38$	$-7a^2 + 16a - 4$	$\mathfrak{o}$
$\mathfrak{o}$	0	-1	$a^2 - 2a + 1$	$-a^2 + 1$	$-7a^2 + 16a - 4$	$\mathfrak{o}$
$\mathfrak{o}$	$-a$	a	0	$6a^2 - 8a + 2$	$a^2 - 3$	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_5) = 2^4$. $\# \text{Aut}(L_6) = 2^4$.

Lattices L_7 and L_8 (of 11):

$\mathfrak{o}$	2	2	$-a^2 + 1$	$-a^2 + 1$	$-a^2 + a + 1$	$\mathfrak{o}$
$\mathfrak{o}$	$-a^2$	$2a^2 + 2$	$-a^2 + 1$	$2a + 6$	$3a + 3$	$\mathfrak{o}$
$\mathfrak{o}$	$a - 1$	$a^2 - 4a + 1$	$4a^2 - 10a + 4$	$-a^2 + 1$	$3a + 3$	$\mathfrak{o}$
$\mathfrak{o}$	$a^2 - 1$	$-a^2 - 2a$	$2a^2 - 2a - 3$	$-4a^2 + 4a + 18$	$-a^2 + a + 1$	$\mathfrak{o}$

Number of automorphisms: $\# \text{Aut}(L_7) = 2^3$. $\# \text{Aut}(L_8) = 2^5$.

(Partial) duals of the above: $L_9 = L_8^\#$.

Lattice L_{10} (of 11):

$\mathfrak{o}$	$-2a^2 + 2a + 8$	4	$2a^2 - 6a + 4$	$a^2 - 8a + 4$
$\mathfrak{o}$	a	1	$3a^2 - 7a + 2$	
$\mathfrak{o}$	$-a^2 + a + 3$	$a^2 - 2$	$4a^2 - 8a + 4$	
$\mathfrak{o}$	$a + 1$			

Number of automorphisms: $\# \text{Aut}(L_{10}) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{11} = L_{10}^\#$.

Base fields with discriminant 13^2

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^3 - x^2 - 4x - 1$

10 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 10):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2a^2+4a+2 \\ -2a^2-4a-1 \\ 2a^2+3a \\ a^2+2a+1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 4a^2-2a+2 \\ -a^2-5a-1 \\ -2a^2+1 \end{smallmatrix}$	$\begin{smallmatrix} a^2-a-2 \\ a^2-a-2 \\ 2a^2+2a+2 \\ a^2+2a \end{smallmatrix}$	$\begin{smallmatrix} a^2-1 \\ -6a^2+8a+24 \\ a^2-1 \\ 2a^2-2a \end{smallmatrix}$	$\begin{smallmatrix} -a^2-a+1 \\ -5a^2+8a+21 \\ -5a^2+8a+21 \\ -a^2-a+1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_1) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_2) = 2^3 \cdot 3^2$.

Lattices L_3 and L_4 (of 10):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ a^2 \\ 0 \\ a^2-2a-4 \end{smallmatrix}$	$\begin{smallmatrix} 10a^2+10a+2 \\ -2a^2-4a-1 \\ 9a^2+14a+3 \end{smallmatrix}$	$\begin{smallmatrix} -a^2+1 \\ -a^2+1 \\ 2a+4 \\ -4a^2-5a+1 \end{smallmatrix}$	$\begin{smallmatrix} -a^2+a+1 \\ 4a^2-6a+6 \\ -a^2+a+1 \\ 8a^2+34a+32 \end{smallmatrix}$	$\begin{smallmatrix} -a-1 \\ 3a+2 \\ 3a+2 \\ -a-1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_3) = 2^3 \cdot 3^2$. $\# \text{Aut}(L_4) = 2^3$.

Lattices L_5 and L_6 (of 10):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ -a^2+3a+2 \\ a \\ a-2 \end{smallmatrix}$	$\begin{smallmatrix} 2a^2-4a \\ 2a^2-4a-2 \\ a^2-2a \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ 2a^2-3a-1 \\ 2a^2-3a-1 \\ a^2-2a-1 \end{smallmatrix}$	$\begin{smallmatrix} a-1 \\ 2a^2-4a+4 \\ a-1 \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} a^2-2a \\ -2a^2+4a+4 \\ -2a^2+4a+4 \\ a^2-2a \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_5) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_6) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_7 = L_5^\#$, $L_8 = L_6^\#$.

Lattice L_9 (of 10):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ a^2-3a-2 \\ a^2-a-1 \\ -2a^2+3a+7 \end{smallmatrix}$	$\begin{smallmatrix} 4a^2-8a-2 \\ -2a^2+2a+1 \\ -a-2 \end{smallmatrix}$	$\begin{smallmatrix} 4a^2 \\ a^2+2a \\ -8a^2+12a+32 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_9) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_{10} = L_9^\#$.

Base fields with discriminant 229¹

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^3 - 4x - 1$

4 similarity classes of single-class genera.

Narrow class group prime ideal representatives:

Ideal	generators
$\mathfrak{p}_1$	$a^2 - 2$

Lattices L_1 and L_2 (of 4):

$\mathfrak{p}_1$	$\frac{1}{49}(2a^2 - 8a + 24)$	$\frac{1}{49}(2a^2 - 8a + 24)$	$\frac{1}{49}(-17a^2 + 19a + 41)$	$\frac{1}{49}(-18a^2 + 23a + 29)$	$\frac{1}{49}(-52a^2 + 110a + 13)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{49}(-19a^2 + 27a + 17)$	$\frac{1}{49}(40a^2 - 62a - 10)$	$\frac{1}{49}(-17a^2 + 19a + 41)$	$\frac{1}{49}(6a^2 - 24a + 72)$	$\frac{1}{49}(-16a^2 + 15a + 53)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	0	0	$\frac{1}{49}(2a^2 - 8a + 24)$	$\frac{1}{49}(-18a^2 + 23a + 29)$	$\frac{1}{49}(-16a^2 + 15a + 53)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{7}(2a^2 - a - 4)$	$\frac{1}{49}(-16a^2 + 15a + 4)$	$\frac{1}{49}(-12a^2 - a + 52)$	$\frac{1}{7}(-4a^2 + 2a + 22)$	$\frac{1}{49}(-52a^2 + 110a + 13)$	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_1) = 2^5$, $\# \text{Aut}(L_2) = 2^4$.

(Partial) duals of the above: $L_3 = L_1^\#$, $L_4 = L_2^\#$.

Base fields with discriminant 2² · 79¹

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^3 - x^2 - 4x + 2$

4 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 4):

$\mathfrak{o}$	2	2	1	$a^2 - 1$	$-a^2 + 1$	0
$\mathfrak{o}$	1	2	1	2	$a^2 - a - 1$	0
$\mathfrak{o}$	$a^2 - 2a + 1$	$a^2 - 3a + 2$	$7a^2 - 19a + 8$	$a^2 - 1$	$a^2 - a - 1$	0
$\mathfrak{o}$	$-a + 1$	1	$a^2 - 4a + 2$	$a^2 - a + 2$	$-a^2 + 1$	0

Number of automorphisms: $\# \text{Aut}(L_1) = 2^3 \cdot 3^1$, $\# \text{Aut}(L_2) = 2^3 \cdot 3^2$.

Lattices L_3 and L_4 (of 4):

$\mathfrak{o}$	$8a^2 - 8a + 2$	2	-1	1	$-2a^2 + a + 8$	0
$\mathfrak{o}$	$-a^2$	$-a^2 + 8$	-1	$-4a^2 + 2a + 18$	$-2a^2 + a + 8$	0
$\mathfrak{o}$	$2a^2 + 3a - 1$	$6a^2 - 5a - 29$	$-53a^2 + 34a + 236$	1	$-2a^2 + a + 8$	0
$\mathfrak{o}$	a	$4a^2 - 2a - 16$	$-15a^2 + 7a + 62$	$-10a^2 + 6a + 44$	$-2a^2 + a + 8$	0

Number of automorphisms: $\# \text{Aut}(L_3) = 2^4$, $\# \text{Aut}(L_4) = 2^4$.

Base fields with discriminant $3^1 \cdot 107^1$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^3 - x^2 - 4x + 1$

3 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 3):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -a^2 + a \\ -a^2 + a \\ a - 1 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ 2a^2 - 2a + 2 \\ 2a^2 - 2a + 1 \\ -2a + 1 \end{smallmatrix}$	$\begin{smallmatrix} -a^2 + a + 1 \\ -a^2 + a + 1 \\ 2a^2 - 2a + 2 \\ -3a + 1 \end{smallmatrix}$	$\begin{smallmatrix} -2a + 2 \\ 4 \\ -2a + 2 \\ 2a^2 - 2a + 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ a - 1 \\ a - 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
--	--	--	--	--	--	--

Number of automorphisms: $\# \text{Aut}(L_1) = 2^5$, $\# \text{Aut}(L_2) = 2^4$.

Lattice L_3 (of 3):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -a \\ a \\ -a^2 + a + 1 \end{smallmatrix}$	$\begin{smallmatrix} -18a^2 + 14a + 78 \\ a^2 - 3a - 9 \\ a^2 + a - 4 \\ -3a^2 - 6a + 2 \end{smallmatrix}$	$\begin{smallmatrix} 18a^2 + 26a - 6 \\ -3a^2 - 6a + 2 \\ 2a^2 - 2a + 2 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_3) = 2^4$.

Base fields with discriminant $5^2 \cdot 29^1$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^4 - x^3 - 3x^2 + x + 1$

10 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 10):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} -8a^3 + 4a^2 + 22a + 10 \\ -3a^3 + 11a + 5 \\ -2a^3 + 2a^2 + 4a + 1 \\ a^3 - 4a^2 + 3a + 2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 6a^3 - 10a^2 - 12a + 18 \\ -a^2 + a + 2 \\ -2a^3 + 4a^2 + 4a - 3 \end{smallmatrix}$	$\begin{smallmatrix} -a^3 + a^2 + 2a + 1 \\ -a^3 + a^2 + 2a + 1 \\ -2a^3 + 4a^2 + 2a \\ 2a^3 - 2a^2 + a + 1 \end{smallmatrix}$	$\begin{smallmatrix} a^3 - a^2 + 1 \\ -2a^3 + 2a^2 + 4a + 2 \\ a^3 - a^2 + 1 \\ 4a^3 + 14a^2 - 6a - 4 \end{smallmatrix}$	$\begin{smallmatrix} a^3 + 1 \\ -a^2 + 2a + 1 \\ -a^2 + 2a + 1 \\ a^3 + 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_1) = 2^6 \cdot 3^2 \cdot 5^2$, $\# \text{Aut}(L_2) = 2^3 \cdot 5^2$.

Lattices L_3 and L_4 (of 10):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ -a^3 + 2a + 1 \\ -a^3 + a^2 + 3a - 1 \end{smallmatrix}$	$\begin{smallmatrix} -2a^3 + 2a^2 + 4a + 2 \\ -a^3 + a^2 + a \\ -a^3 + 2a^2 + a - 1 \\ -a^3 + 2a^2 + a - 1 \end{smallmatrix}$	$\begin{smallmatrix} -a^3 + a^2 \\ 2a^3 + 4a^2 \\ 2a^2 + a \\ 12a^3 - 16a^2 - 28a + 26 \end{smallmatrix}$	$\begin{smallmatrix} -a^3 + a^2 + a - 1 \\ -2a^3 + 6a^2 + 2a \\ -a^3 + a^2 + a - 1 \\ -a^3 - a^2 - a + 1 \end{smallmatrix}$	$\begin{smallmatrix} a^3 + a^2 - a - 1 \\ a^3 + a^2 - a - 1 \\ -a^3 - a^2 + a + 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_3) = 2^3 \cdot 5^2$, $\# \text{Aut}(L_4) = 2^3 \cdot 3^2$.

Lattices L_5 and L_6 (of 10):

$\mathfrak{o}$	2	2	0	$3a^3 - 3a^2 - 6a - 2$	-1	0
$\mathfrak{o}$	$a^3 - a^2 + 1$	$2a^3 + 2a^2 - 4a + 2$	0	2	$a^3 - a^2 - a$	0
$\mathfrak{o}$	a	$2a^2 - 1$	$2a^3 - 2a^2 - 6a + 6$	$3a^3 - 3a^2 - 6a - 2$	$a^3 - a^2 - a$	0
$\mathfrak{o}$	-1	$-a^3 + a^2 + a - 2$	$-a^3 + 4a - 1$	$-2a^3 + 2a^2 + 4a + 4$	-1	0

Number of automorphisms: $\# \text{Aut}(L_5) = 2^3 \cdot 3^2$, $\# \text{Aut}(L_6) = 2^5$.

Lattices L_7 and L_8 (of 10):

$\mathfrak{o}$	$-116a^3 + 32a^2 + 368a + 156$	2	$-a^3 + a^2 + a$	$-a^2 + 1$	$-a^3 + 2a^2 + 2a - 1$	0
$\mathfrak{o}$	$-42a^3 + 13a^2 + 131a + 55$	$-18a^3 + 8a^2 + 52a + 22$	$-a^3 + a^2 + a$	6	$a^3 - a^2 + 2a + 1$	0
$\mathfrak{o}$	$7a^3 - 8a^2 - 10a - 2$	$6a^3 - 8a^2 - 4a + 1$	$18a^3 + 62a^2 - 28a - 20$	$-a^2 + 1$	$a^3 - a^2 + 2a + 1$	0
$\mathfrak{o}$	$9a^3 - 5a^2 - 25a - 10$	$4a^3 - 4a^2 - 9a - 3$	$-11a^3 + 8a^2 + 5a - 3$	$6a^2 + 2$	$-a^3 + 2a^2 + 2a - 1$	0

Number of automorphisms: $\# \text{Aut}(L_7) = 2^3 \cdot 5^1$, $\# \text{Aut}(L_8) = 2^3 \cdot 5^1$.

(Partial) duals of the above: $L_9 = L_7^\#$, $L_{10} = L_8^\#$.

Base fields with discriminant $3^2 \cdot 5^3$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^4 - x^3 - 4x^2 + 4x + 1$

3 similarity classes of single-class genera.

Narrow class group prime ideal representatives:

Ideal	generators
$\mathfrak{p}_1$	$a^2 - 1$

Lattices L_1 and L_2 (of 3):

$\mathfrak{p}_1$	$2a^2 - 2a$	$\frac{1}{5}(18a^3 + 14a^2 - 46a - 2)$	$\frac{1}{5}(31a^3 + 23a^2 - 72a - 19)$	$\frac{1}{5}(19a^3 + 12a^2 - 43a - 11)$	$8a^3 + 6a^2 - 20a - 3$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$-2a^3 + a^2 + 2a$	$2a^3 + 6a^2 - 10a - 2$	$\frac{1}{5}(31a^3 + 23a^2 - 72a - 19)$	$\frac{1}{5}(52a^3 + 56a^2 - 144a - 28)$	$\frac{1}{5}(23a^3 + 34a^2 - 76a - 17)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{5}(-a^3 - 13a^2 + 12a + 4)$	$\frac{1}{5}(9a^3 - 8a^2 - 3a - 1)$	$\frac{1}{5}(12a^3 + 26a^2 - 44a - 8)$	$\frac{1}{5}(19a^3 + 12a^2 - 43a - 11)$	$\frac{1}{5}(23a^3 + 34a^2 - 76a - 17)$	$\mathfrak{p}_1$
$\mathfrak{p}_1$	$\frac{1}{5}(a^3 - 7a^2 + 3a - 4)$	$2a^3 - 3a$	$\frac{1}{5}(-4a^3 + 13a^2 + 3a - 4)$	$\frac{1}{5}(-6a^3 + 12a^2 + 2a + 14)$	$8a^3 + 6a^2 - 20a - 3$	$\mathfrak{p}_1$

Number of automorphisms: $\# \text{Aut}(L_1) = 2^4 \cdot 3^2 \cdot 5^2$, $\# \text{Aut}(L_2) = 2^2 \cdot 3^1 \cdot 5^1$.

(Partial) duals of the above: $L_3 = L_2^\#$.

Base fields with discriminant $19^1 \cdot 103^1$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^4 - 4x^2 - x + 1$

12 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 12):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ 0 \\ a^2 - a \end{smallmatrix}$	$\begin{smallmatrix} -1 \\ 2 \\ -1 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} -a^2 + a + 1 \\ 2 \\ -a^2 + a + 1 \\ -2a^3 + 4a^2 + 2a \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 1 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_1) = 2^6 \cdot 3^2$. $\# \text{Aut}(L_2) = 2^3 \cdot 3^2$.

Lattices L_3 and L_4 (of 12):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ -a^3 + 2a^2 + 3a - 3 \\ a^3 - 2a^2 - 3a + 3 \\ a^3 - 2a^2 - 3a + 3 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 4 \\ -a^3 + 4a - 1 \\ -a^3 + 4a - 1 \end{smallmatrix}$	$\begin{smallmatrix} a^3 - a^2 - a + 1 \\ a^3 - a^2 - a + 1 \\ -a^2 \\ a^3 - 4a + 1 \end{smallmatrix}$	$\begin{smallmatrix} -a^2 \\ -2a^3 + 2a^2 + 4a + 8 \\ -a^2 \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} -a^3 + 3a + 1 \\ -3a^3 + 5a + 1 \\ -3a^3 + 5a + 1 \\ -a^3 + 3a + 1 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_3) = 2^4 \cdot 3^1$. $\# \text{Aut}(L_4) = 2^4 \cdot 3^1$.

(Partial) duals of the above: $L_5 = L_3^\#$, $L_6 = L_4^\#$.

Lattices L_7 and L_8 (of 12):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 0 \\ a^2 \\ a^3 - a^2 - 3a - 1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 2 \\ -a^3 + a^2 + a - 1 \\ -a^2 \end{smallmatrix}$	$\begin{smallmatrix} a^2 - a - 1 \\ a^2 - a - 1 \\ -2a^3 + 6a^2 + 4a \\ 2a^3 - 3a^2 - 2a + 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ -2a^3 + 2a^2 + 4a + 2 \\ 0 \\ -2a^3 + 4a^2 + 6a + 2 \end{smallmatrix}$	$\begin{smallmatrix} -a \\ -a^3 - a^2 + 5a + 3 \\ -a^3 - a^2 + 5a + 3 \\ -a \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_7) = 2^5$. $\# \text{Aut}(L_8) = 2^3$.

Lattices L_9 and L_{10} (of 12):

$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ a^3 - a^2 - a + 1 \\ -a^3 + a + 1 \\ a^2 - 1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ -2a^3 + 2a^2 + 4a + 8 \\ -a^3 - 4a^2 + 6a + 5 \\ -a^2 + 4a \end{smallmatrix}$	$\begin{smallmatrix} -1 \\ -1 \\ -2a^3 + 4a^2 + 8a + 4 \\ -a^2 - 2a \end{smallmatrix}$	$\begin{smallmatrix} -a^3 + a^2 - 1 \\ 2 \\ -a^3 + a^2 - 1 \\ -2a^3 + 6a^2 + 2a \end{smallmatrix}$	$\begin{smallmatrix} -a^3 + a^2 + a + 1 \\ 0 \\ a^3 - 2a \\ a^3 - 2a \\ -a^3 + a^2 + a + 1 \end{smallmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_9) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_{10}) = 2^3 \cdot 3^2$.

(Partial) duals of the above: $L_{11} = L_9^\#$, $L_{12} = L_{10}^\#$.

Base fields with discriminant $2^4 \cdot 5^3$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^4 - 5x^2 + 5$

2 similarity classes of single-class genera.

Narrow class group prime ideal representatives:

Ideal	generators
$\mathfrak{p}_1$	$a^2 + a - 4$

Lattice L_1 (of 2):

$\mathfrak{p}_1$	$\frac{1}{361}(-36a^3 + 12a^2 + 176a + 182)$	$\frac{1}{361}(-2232a^3 + 2910a^2 + 4414a - 2434)$
$\mathfrak{p}_1$	$\frac{1}{361}(281a^3 - 214a^2 - 732a - 117)$	$\frac{1}{361}(869a^3 - 1132a^2 - 1200a + 1483)$
$\mathfrak{p}_1$	$\frac{1}{361}(-117a^3 + 39a^2 + 211a + 50)$	$\frac{1}{361}(-198a^3 + 788a^2 + 246a - 804)$
$\mathfrak{p}_1$	$\frac{1}{361}(16a^3 + 115a^2 + 2a - 121)$	$\frac{1}{361}(-14a^3 + 125a^2 - 92a - 210)$
		$\frac{1}{361}(50a^3 + 224a^2 - 84a + 28)$

Number of automorphisms: $\# \operatorname{Aut}(L_1) = 2^4$.

(Partial) duals of the above: $L_2 = L_1^\#$.

Base fields with discriminant $2^8 \cdot 3^2$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^4 - 4x^2 + 1$

2 similarity classes of single-class genera.

Narrow class group prime ideal representatives:

Ideal	generators
$\mathfrak{p}_1$	$a^3 - a^2 - 4a + 1$

Lattice L_1 (of 2):

$\mathfrak{p}_1$	$\frac{1}{529}(-172a^3 - 82a^2 + 612a + 464)$	$\frac{1}{23}(-4a^3 + 12a^2 + 26a + 14)$
$\mathfrak{p}_1$	$\frac{1}{529}(-157a^3 - 81a^2 + 411a + 239)$	$\frac{1}{23}(41a^3 + 84a^2 - 2a - 17)$
$\mathfrak{p}_1$	$\frac{1}{529}(-408a^3 - 662a^2 + 283a + 301)$	$\frac{1}{529}(22396a^3 + 43672a^2 - 5136a - 11134)$
$\mathfrak{p}_1$	$\frac{1}{529}(-86a^3 - 41a^2 + 306a + 232)$	$\frac{1}{529}(-134a^3 - 150a^2 + 526a + 423)$
		$\frac{1}{529}(-166a^3 + 130a^2 + 320a + 374)$

Number of automorphisms: $\# \operatorname{Aut}(L_1) = 2^4$.

(Partial) duals of the above: $L_2 = L_1^\#$.

Base fields with discriminant 2777¹

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^4 - x^3 - 4x^2 + x + 2$

9 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 9):

$\begin{bmatrix} 0 & 2 \\ 0 & -a^3 + 2a^2 + a - 1 \\ 0 & -1 \\ 0 & a^2 + a - 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -8a^3 + 20a^2 + 2a - 10 \\ 2a^3 - 4a^2 - 2a + 1 \\ a^2 + a + 1 \end{bmatrix}$	$\begin{bmatrix} a^2 - 1 \\ a^2 - 1 \\ -2a^3 + 8a + 8 \\ a^3 - a^2 - 5a - 1 \end{bmatrix}$	$\begin{bmatrix} -a^3 + a^2 + 4a + 1 \\ 2a^2 \\ -a^3 + a^2 + 4a + 1 \\ 8a^3 + 10a^2 - 6a - 4 \end{bmatrix}$	$\begin{bmatrix} -a^3 + a^2 + a \\ a^2 + a \\ a^2 + a \\ -a^3 + a^2 + a \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_1) = 2^4 \cdot 3^2$. $\# \text{Aut}(L_2) = 2^5$.

Lattices L_3 and L_4 (of 9):

$\begin{bmatrix} 0 & 2 \\ 0 & -1 \\ 0 & -a^2 + 1 \\ 0 & a^2 \end{bmatrix}$	$\begin{bmatrix} 2 & a^2 + a - 1 \\ 2 & a^2 + a - 1 \\ a^2 - a - 1 \\ a^3 - a^2 - a \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2a^3 - 2a + 4 \\ 0 \\ 16a^3 + 34a^2 - 14a - 20 \end{bmatrix}$	$\begin{bmatrix} -a^3 + a^2 + a + 1 \\ -a^3 + a^2 + 3a - 1 \\ -a^3 + a^2 + 3a - 1 \\ -a^3 + a^2 + a + 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_3) = 2^3 \cdot 3^1$. $\# \text{Aut}(L_4) = 2^5$.

(Partial) duals of the above: $L_5 = L_3^\#$.

Lattices L_6 and L_7 (of 9):

$\begin{bmatrix} 0 & 2 \\ 0 & -a^3 + a - 1 \\ 0 & a^3 - a^2 - a \\ 0 & -a^2 + a + 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 4a^3 + 8a^2 - 4a - 4 \\ -3a^3 - a^2 + 2a + 1 \\ 5a^2 - a - 3 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -1 \\ -2a^3 + 8a^2 + 2a - 2 \\ -3a^3 + 6a^2 - 2a - 5 \end{bmatrix}$	$\begin{bmatrix} a^3 - a \\ 2 \\ a^3 - a \\ -4a^3 + 12a^2 + 4a - 4 \end{bmatrix}$	$\begin{bmatrix} a^3 + a^2 \\ -a^2 \\ -a^2 \\ a^3 + a^2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_6) = 2^3$. $\# \text{Aut}(L_7) = 2^3 \cdot 3^2$.

Lattice L_8 (of 9):

$\begin{bmatrix} 0 & a^3 - 3a^2 - 2a + 9 \\ 0 & -4a^3 + 7a^2 + 10a - 11 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 99a^3 - 165a^2 - 286a + 293 \\ a^3 - 2a^2 - 4a + 4 \\ 14a^3 - 23a^2 - 39a + 40 \end{bmatrix}$	$\begin{bmatrix} 2a^2 \\ 2a + 2 \\ 2a^3 + 2a^2 + 6a + 12 \end{bmatrix}$
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Number of automorphisms: $\# \text{Aut}(L_8) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_9 = L_8^\#$.

Base fields with discriminant $61^1 \cdot 397^1$

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^5 - 5x^3 - x^2 + 3x + 1$

3 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattices L_1 and L_2 (of 3):

$\begin{smallmatrix} 0 & 2 \\ 0 & -1 \\ 0 & a^3 - a^2 - a \\ 0 & a^4 - a^3 - 2a^2 + a \end{smallmatrix}$	$\begin{smallmatrix} 4 & & & \\ 2 & & & \\ 4a^4 - 2a^3 - 20a^2 + 4a + 10 & & & \\ -a^4 + a^3 + 2a^2 + 1 & & & \end{smallmatrix}$	$\begin{smallmatrix} -35a^4 + 18a^3 + 166a^2 - 49a - 78 & 2a^4 - 3a^2 - a - 1 \\ -35a^4 + 18a^3 + 166a^2 - 49a - 78 & 152a^4 - 22a^3 - 778a^2 - 18a + 548 \\ 16a^4 - 8a^3 - 66a^2 + 10a + 40 & 2a^4 - 3a^2 - a - 1 \\ 2a^4 + 2a^3 - 14a^2 + 6 & 4a^4 - 10a^3 + 10a + 4 \end{smallmatrix}$	$\begin{smallmatrix} -a^3 - a^2 + 3a \\ 13a^4 - 7a^3 - 56a^2 + 19a + 25 \\ 13a^4 - 7a^3 - 56a^2 + 19a + 25 \\ -a^3 - a^2 + 3a \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}$
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Number of automorphisms: $\# \operatorname{Aut}(L_1) = 2^3 \cdot 3^2$, $\# \operatorname{Aut}(L_2) = 2^3 \cdot 3^1$.

(Partial) duals of the above: $L_3 = L_2^\#$.

6.3.2 Single-class lattices of rank 5

Discriminant 5

Base ring 1 of 1

Base fields with discriminant 5^1

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^2 - x - 1$

2 similarity classes of single-class genera.

Narrow class group is the trivial group.

Lattice L_1 (of 2):

$\begin{smallmatrix} 0 & 2 \\ 0 & -a \\ 0 & a - 1 \\ 0 & -a + 1 \\ 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 2 & & & & \\ & 2 & & & \\ & -1 & 2 & & \\ & 1 & a - 1 & 2 & \\ & 0 & 0 & 0 & 2 \end{smallmatrix}$
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Number of automorphisms: $\# \operatorname{Aut}(L_1) = 2^7 \cdot 3^2 \cdot 5^2$.

(Partial) duals of the above: $L_2 = L_1^\#$.

6.3.3 Single-class lattices of rank 6

Discriminant 5
Base ring 1 of 1

Base fields with discriminant 5¹

Number field $K = \mathbb{Q}(a)$ with defining polynomial: $x^2 - x - 1$
2 similarity classes of single-class genera.
Narrow class group is the trivial group.
Lattice L_1 (of 2):

0	2				
0	$a - 2$	$-2a + 4$			
0	$-a$	0	2		
0	$a + 1$	-1	$-a$	$2a + 2$	
0	0	0	0	0	2
0	0	0	0	$a + 1$	$2a + 2$

Number of automorphisms: $\# \operatorname{Aut}(L_1) = 2^8 \cdot 3^2 \cdot 5^3$.
(Partial) duals of the above: $L_2 = L_1^\#$.

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