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Stabilization of a Multi-Dimensional System of Hyperbolic Balance Laws – A Case Study

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In a recent work we studied a stabilization problem for a multi-dimensional system of n hyperbolic partial differential equations. Using a novel Lyapunov function taking into account the multi-dimensional geometry we show stabilization in L^2 for the arising system using a suitable feedback control. The aim of the present work is to show how this approach can be applied to a particular system. Moreover we present numerical results supporting the theoretical results.

1 Introduction

Stabilization of spatially one–dimensional systems of hyperbolic balance laws has recently attracted research interest in the mathematical and engineering community. We refer to the monographs [1–4] for further references and an overview also on related controllability problems. A key tool in the analysis has been the introduction of a Lyapunov function as a weighted L^2 (or H^s) function that allows to estimate deviations from steady states, see e.g. [1]. Exponential decay of a Lyapunov function under general dissipative conditions has been established for a variety of problem formulations [15–18] and a comparison to other stability concepts is presented e.g. in [19].

However, to the best of our knowledge the presented results are limited to the spatially one–dimensional case. In a recent work, see [25], we introduced a new approach which is able to deal with multi–dimensional hyperbolic balance laws. The reminder of the present work is devoted to a particular system highlighting the capability of the recently presented approach. In Section 2 we provide the basic framework and the key result for our purpose. The system subject to study is presented in Section 3 and we also derive all needed quantities. Finally numerical results are presented in Section 4.

2 Stabilization of Multi-Dimensional Linear Hyperbolic Balance Laws

We are interested in the following initial boundary value problem (IBVP) for the given system of PDEs

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{w} + \sum_{k=1}^{n} \mathbf{A}^{(k)}(\mathbf{x})\frac{\partial}{\partial x_k}\mathbf{w}(t,\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{w}(t,\mathbf{x}) & = 0,\ (t,\mathbf{x}) \in [0,T)\times\Omega \\ \mathbf{w}(0,\mathbf{x}) & = \mathbf{w}_0(\mathbf{x}),\ \mathbf{x}\in\Omega, \\ \mathbf{w}(t,\mathbf{x}) & = \mathbf{w}_{BC}(t,\mathbf{x}),\ (t,\mathbf{x})\in[0,T)\times\partial\Omega \end{cases} \tag{1}$$

Here $\mathbf{w} \equiv (w_1(t,\mathbf{x}),\ldots,w_n(t,\mathbf{x}))^T$ is the vector of unknowns and $\Omega \subset \mathbb{R}^n$ a bounded domain with sufficiently smooth boundary $\partial\Omega$. Moreover, $\mathbf{A}^{(k)}$ and $\mathbf{B}$ are sufficiently smooth and bounded $n\times n$ real matrices. Additionally, the matrices $\mathbf{A}^{(k)}(\mathbf{x}) = (a_{ii}^{(k)}(\mathbf{x}))_{i=1,\ldots,n}$ are assumed to be diagonal matrices and hence have a full set of eigenvectors. The system is hyperbolic. The boundary condition will be made more precise later on. According to the references [26–29] there exists a solution $\mathbf{w} \in C^1((0,T),H^s(\Omega))^n$ if the initial and boundary data are sufficiently smooth. In order to study the Lyapunov function it is beneficial to rewrite the equation for each unknown w_i as follows

$$\frac{\partial}{\partial t}w_i(t,\mathbf{x}) + \mathbf{a}_i(\mathbf{x})\cdot\nabla w_i(t,\mathbf{x}) + \mathbf{b}_i(\mathbf{x})\cdot\mathbf{w}(t,\mathbf{x}) = 0,$$
$$\text{with}\quad \mathbf{a}_i := (a_{ii}^{(1)},\ldots,a_{ii}^{(n)}),\ \text{and}\quad \mathbf{b}_i := (b_{i1},\ldots,b_{in}).$$

For later use, let

$$\mathcal{E}(\mu(\mathbf{x})) := \mathrm{diag}(\exp(\mu_1(\mathbf{x}),\ldots,\exp(\mu_n(\mathbf{x}))),$$
$$\mathcal{A}(t,\mathbf{x}) := \left(\mathbf{w}^T\mathbf{A}^{(1)}\mathcal{E}\mathbf{w},\ldots,\mathbf{w}^T\mathbf{A}^{(n)}\mathcal{E}\mathbf{w}\right)^T,$$
$$\mathcal{M}^{(k)}(\mathbf{x}) := \mathrm{diag}\left(\frac{\partial}{\partial x_k}\mu_1(\mathbf{x}),\ldots,\frac{\partial}{\partial x_k}\mu_n(\mathbf{x})\right).$$

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The boundary $\partial\Omega$ will be separated in the controllable and uncontrollable part, i.e. for $i = 1, \ldots, n$

$$\Gamma_i^+ := \{\mathbf{x} \in \partial\Omega \mid \mathbf{a}_i \cdot \mathbf{n} \geq 0\}, \quad \Gamma_i^- := \{\mathbf{x} \in \partial\Omega \mid \mathbf{a}_i \cdot \mathbf{n} < 0\}.$$

The part Γ_i^- is then split disjointly into the part $\mathcal{C}_i$ where we apply a feedback control u_i and a part $\mathcal{Z}_i$ where we prescribe $w_i = 0$.
Now, we cite the Corollary needed for the focus of this work and which is a special case of the main theorem given in [25]. The remarkable fact is that here the Lyapunov function is equal to the exponential function.

Corollary 2.1 *Let $\mathbf{w}(t,\mathbf{x}) \in C^1\left((0,T), H^s(\Omega)\right)^n$, $s \geq 1 + n/2$, be a solution to the IBVP* (1) *and additionally assume that $\mathbf{B}$ is a diagonal matrix. The Lyapunov function is given as*

$$L(t) = \int_\Omega \mathbf{w}(t,\mathbf{x})^T \mathcal{E}(\mu(\mathbf{x}))\mathbf{w}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{2}$$

where we assume that there exists $\mu_i(\mathbf{x}) \in H^s(\Omega)$ satisfy

$$\begin{aligned} &\sum_{k=1}^{n}\left(\mathcal{M}^{(k)}\mathbf{A}^{(k)} + \frac{\partial}{\partial x_k}\mathbf{A}^{(k)}\right) - 2\mathbf{B} = -C_L\mathbf{Id} \\ &\Leftrightarrow \quad \mathbf{a}_i \cdot \nabla\mu_i(\mathbf{x}) + \nabla\cdot\mathbf{a}_i - 2\mathbf{B}_{ii} = -C_L,\ C_L > 0 \end{aligned} \tag{3}$$

for some value $C_L \in \mathbb{R}_{>0}$. The boundary condition for (1) *is given by*

$$\mathbf{w}_{BC,i}(t,\mathbf{x}) = \begin{cases} 0 & , \mathbf{x} \in \mathcal{Z}_i \\ u_i(t,\mathbf{x}) & , \mathbf{x} \in \mathcal{C}_i \end{cases} \quad t \in [0,T),\ i = 1,\ldots,n$$

where the u_i satisfies

$$-\sum_{i=1}^{n}\int_{\mathcal{C}_i} u_i(t,\mathbf{x})^2\,(\mathbf{a}_i\cdot\mathbf{n})\exp(\mu_i(\mathbf{x}))\,\mathrm{d}\mathbf{x} = \sum_{i=1}^{n}\int_{\Gamma_i^+} w_i^2\,(\mathbf{a}_i\cdot\mathbf{n})\exp(\mu_i(\mathbf{x}))\,\mathrm{d}\mathbf{x}. \tag{4}$$

Then the Lyapunov function satisfies

$$L(t) = L(0)\exp(-C_L t).$$

and thus the solution $\mathbf{w}$ of the IBVP (1) *is asymptotically stable in the L^2-sense.*

3 Case Study of a Two Dimensional Linear System

We want to study a system of two linear equations in two dimensions which has the form

$$\frac{\partial}{\partial t}\mathbf{w}(t,\mathbf{x}) + \mathbf{A}^{(1)}\frac{\partial}{\partial x_1}\mathbf{w}(t,\mathbf{x}) + \mathbf{A}^{(2)}\frac{\partial}{\partial x_2}\mathbf{w}(t,\mathbf{x}) = 0 \tag{5}$$

with the matrices

$$\mathbf{A}^{(1)} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{A}^{(2)} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}. \tag{6}$$

Furthermore we consider the spatial domain $\Omega = [0,1]\times[0,1]$. From the matrices (6) we obtain the following vectors

$$\mathbf{a}_1 = (4,2) \quad \text{and} \quad \mathbf{a}_2 = (2,-2). \tag{7}$$

The matrix $\mathbf{B}$ is identically zero in this system and does not need to be considered. For the normal of the domain Ω we obtain

$$\mathbf{n}(\mathbf{x}) = \begin{cases} (1,0), & \mathbf{x} \in \{1\}\times[0,1], \\ (0,1), & \mathbf{x} \in [0,1]\times\{1\}, \\ (-1,0), & \mathbf{x} \in \{0\}\times[0,1], \\ (0,-1), & \mathbf{x} \in [0,1]\times\{0\} \end{cases}$$

and thus we yield for the associated products $\mathbf{a}_i \cdot \mathbf{n}$

$$\mathbf{a}_1 \cdot \mathbf{n}(\mathbf{x}) = \begin{cases} 4, & \mathbf{x} \in \{1\} \times [0,1], \\ 2, & \mathbf{x} \in [0,1] \times \{1\}, \\ -4, & \mathbf{x} \in \{0\} \times [0,1], \\ -2, & \mathbf{x} \in [0,1] \times \{0\} \end{cases} \quad \text{and} \quad \mathbf{a}_2 \cdot \mathbf{n}(\mathbf{x}) = \begin{cases} 2, & \mathbf{x} \in \{1\} \times [0,1], \\ -2, & \mathbf{x} \in [0,1] \times \{1\}, \\ -2, & \mathbf{x} \in \{0\} \times [0,1], \\ 2, & \mathbf{x} \in [0,1] \times \{0\} \end{cases}. \tag{8}$$

Hence we have the following partitioning of the boundary

$$\begin{aligned} \Gamma_1^+ &:= \{1\} \times [0,1] \cup [0,1] \times \{1\}, \quad \Gamma_1^- := \{0\} \times [0,1] \cup [0,1] \times \{0\}, \\ \Gamma_2^+ &:= [0,1] \times \{0\} \cup \{1\} \times [0,1], \quad \Gamma_2^- := \{0\} \times [0,1] \cup [0,1] \times \{1\}. \end{aligned} \tag{9}$$

In what follows we apply the control on the part $\mathcal{C} := \mathcal{C}_1 = \mathcal{C}_2 = \{0\} \times [0,1]$ and prescribe zero boundary conditions on $\mathcal{Z}_1 = [0,1] \times \{0\}$ and $\mathcal{Z}_2 = [0,1] \times \{1\}$. Note that the boundary parts correspond to the respective component of $\mathbf{w}$. Next we determine the weight functions $\mu_1(\mathbf{x})$ and $\mu_2(\mathbf{x})$. Due to the given structure we obtain for the weight functions

$$\begin{aligned} &\mathbf{a}_i \cdot \nabla \mu_i(\mathbf{x}) = -C_L^{(i)} \quad \Leftrightarrow \quad \frac{\partial}{\partial x_1}\mu_i(\mathbf{x}) + \frac{\mathbf{a}_i^{(2)}}{\mathbf{a}_i^{(1)}}\frac{\partial}{\partial x_2}\mu_i(\mathbf{x}) = -\frac{1}{\mathbf{a}_i^{(1)}}C_L^{(i)} \\ \Rightarrow \quad &\mu_i(\mathbf{x}) = g_i\left(x_2 - \frac{\mathbf{a}_i^{(2)}}{\mathbf{a}_i^{(1)}}x_1\right) - \frac{1}{\mathbf{a}_i^{(1)}}C_L^{(i)}x_1. \end{aligned}$$

This holds for arbitrary $g_i \in C^1$ and for the sake of simplicity we assume $g_i(\sigma) = \sigma$ and further $C_L^{(1)} = C_L^{(2)} = C_L > 0$. Thus we yield the following weight functions

$$\mu_1(\mathbf{x}) = x_2 - \left(\frac{1}{2} + \frac{1}{4}C_L\right)x_1 \quad \text{and} \quad \mu_2(\mathbf{x}) = x_2 + \left(1 - \frac{1}{2}C_L\right)x_1. \tag{10}$$

Next we specify the constraint for the boundary control and we assume $u(t) := u_1(t,\mathbf{x}) = u_2(t,\mathbf{x})$ which leads to

$$u(t)^2 \leq -\left(\sum_{i=1}^{2}\int_{\mathcal{C}_i}(\mathbf{a}_i \cdot \mathbf{n})\exp(\mu_i(\mathbf{x}))\,\mathrm{d}\mathbf{x}\right)^{-1}\sum_{i=1}^{2}\int_{\Gamma_i^+} w_i^2\,(\mathbf{a}_i \cdot \mathbf{n})\exp(\mu_i(\mathbf{x}))\,\mathrm{d}\mathbf{x}. \tag{11}$$

Inserting the obtained results we get

$$\begin{aligned} &-\sum_{i=1}^{2}\int_{\mathcal{C}_i}(\mathbf{a}_i \cdot \mathbf{n})\exp(\mu_i(\mathbf{x}))\,\mathrm{d}\mathbf{x} = 6\int_0^1 \exp(x_2)\,\mathrm{d}x_2 = 6(e-1), \\ &\mathcal{I}(t) := \sum_{i=1}^{2}\int_{\Gamma_i^+} w_i^2\,(\mathbf{a}_i \cdot \mathbf{n})\exp(\mu_i(\mathbf{x}))\,\mathrm{d}\mathbf{x} \\ &\qquad = 2\int_0^1 2w_1(t,1,\sigma)^2\exp(\mu_1(1,\sigma)) + w_1(t,\sigma,1)^2\exp(\mu_1(\sigma,1))\ldots \\ &\cdots + w_2(t,\sigma,0)^2\exp(\mu_2(\sigma,0)) + w_2(t,1,\sigma)^2\exp(\mu_2(1,\sigma))\,\mathrm{d}\sigma \\ \Rightarrow \quad &u(t)^2 \leq \frac{1}{6(e-1)}\mathcal{I}(t). \end{aligned} \tag{12}$$

Thus a possible boundary control satisfying Corollary 2.1 is given by

$$u(t) = \sqrt{\frac{1}{6(e-1)}\mathcal{I}(t)}. \tag{13}$$

Now we have specified all the details needed to stabilize the solution of the given PDE (5) in Ω. This will be demonstrated in the following section numerically.

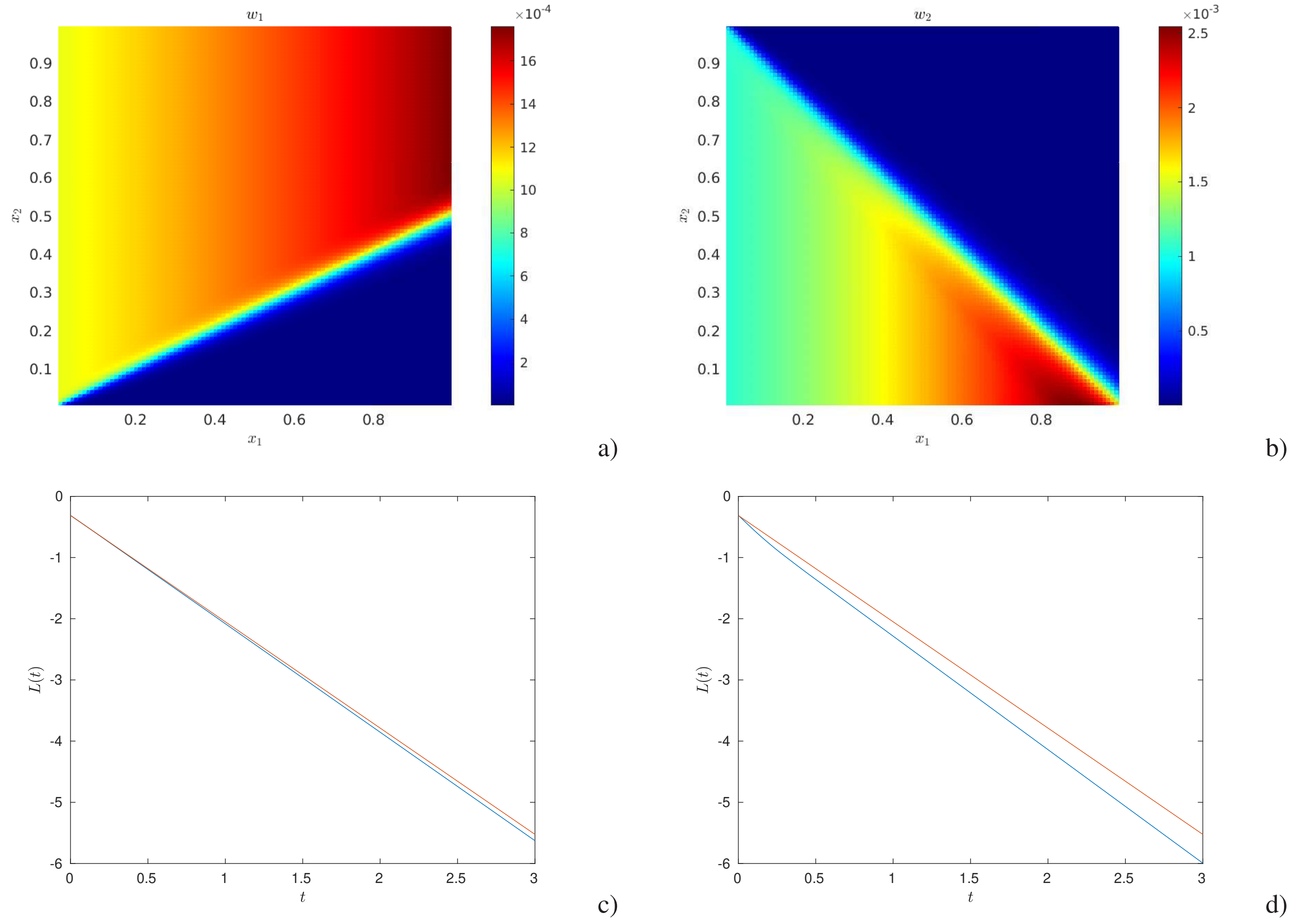

Fig. 1: Numerical results at $t_{end} = 3$: In (a) and (b) the solution for the components w_1 and w_2 are given. In (c) the computed (blue) decay rate for the Lyapunov function obtained with the MUSCL-FV scheme is given and compared to the exact decay rate (red). In (d) the computed (blue) decay rate for the Lyapunov function obtained with the Rusanov scheme is given and compared to the exact decay rate (red).

4 Numerical Results

In this section the theoretical estimate on the decay of the Lyapunov function is confirmed using a numerical discretization of the dynamics. The MUSCL second–order finite–volume scheme, see [30], is used on a regular mesh for Ω with grid size Δx to solve the discretized equation (5). The cell average of $\mathbf{W}$ on $C_{i,j} = [x_{1,i} - \frac{\Delta x}{2}, x_{1,i} + \frac{\Delta x}{2}] \times [x_{2,j} - \frac{\Delta x}{2}, x_{2,j} + \frac{\Delta x}{2}]$, $x_{k,i} = i\Delta x$ for $k = 1, 2$ and time $t_n = n\Delta t$ is given by

$$\mathbf{W}^n_{i,j} = \frac{1}{|C_{i,j}|} \int_{C_{i,j}} \mathbf{W}(t_n, x)\, \mathrm{d}x,$$

for $i, j = 0, \ldots, N_x$ and $n \geq 0$ and where $N_x \Delta x = 1$. The cell averages of the initial data $\mathbf{W}_0$ are obtained analogously and define $\mathbf{W}^0_{i,j}$. As boundary conditions we use transmissive boundary conditions on Γ_1^+ and Γ_2^+, see [30]. Zero boundary conditions are prescribed on $\mathcal{Z}_1, \mathcal{Z}_2$ and the boundary control is imposed on $\mathcal{C}$. Here, the control u^n is obtained using a numerical quadrature formula applied to (13). The Lyapunov function itself (2) is approximated at time t^n by L^n using a numerical quadrature rule with equi-distant grid

$$L^n := \Delta x^2 \sum_{i,j=0}^{N_x} \left[(w^n_{1;i,j})^2 \exp(\mu_1(x_i, x_j)) + (w^n_{2;i,j})^2 \exp(\mu_2(x_i, x_j)) \right]. \tag{14}$$

As initial data a sinusoidal function is chosen, i.e. $w_i(0, \mathbf{x}) = \sin(2\pi x_1) \sin(2\pi x_2)$. We report on computational results for $L^n, n \geq 0$ for the following computational setup $\Delta x = 10^{-2}, t_{end} = 3$ and $C_{CFL} = 0.5$. In Figure 1 the results for the control (13) are presented. It is visible that the numerical decay of the Lyapunov function agrees well with the theoretical decay, as expected from Corollary 2.1. In Figure 1 (d) the results for the same setup and the Rusanov scheme, see [30], is given. There the observed numerical decay is stronger compared with the theoretical estimate possibly due to additional diffusive terms in the numerical approximation. This is confirmed by the observation that coarser grids lead to stronger decay compared with refined meshes, see [31].

5 Summary

A novel Lyapunov function for L^2-control of a class of multi–dimensional systems of hyperbolic equations has been presented in [25]. A stabilizing feedback control has been derived and exponential decay of a weighted L^2-norm has been established. In the present work the obtained results are applied to a particular system of two equations in two dimensions. Further numerical results are presented which support the theoretical results. Future extensions could be given towards numerical discretizations as well as stabilization in H^s-norm.

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