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ORIGINAL ARTICLE

2023—A twofold commemoration: the 100th birthday of Walsh functions and the 50th anniversary of Professor Joseph Leonard Walsh's death

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Abstract
In 2023, it is 100 years after the introduction of Walsh functions and 50 years after the death of their introducer Joseph Leonard Walsh. This paper is an homage to J. L. Walsh and presents some insights into the development of Walsh and dyadic analysis with applications of Walsh functions in both mathematics and various areas of computing and engineering. The presentation is based on reminiscences of the authors acting in these areas.

Keywords Walsh functions · Walsh analysis · Dyadic analysis · Dyadic differentiation · Functional approximation · Signal processing · System theory

Mathematics Subject Classification 42C10 · 28A15 · 94A05 · 94A12 · 94A15

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It was Franz Pichler, professor emeritus at the University of Linz, Austria, who suggested this commemorative publication. He sent his draft to PLB via Bernd Lüke, curator at the German Museum of Technology, Berlin. In view of joint academic interests they know each other. Bernd is the son of Professor Hans Dieter Lüke (1935–2005), former head of the Institute for Communications Engineering at RWTH Aachen University, who was indeed a leading communications engineer in Germany, and PLB's best friend.

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1 Introduction

It is a 100 years ago, the year 1923, that *Walsh functions* were introduced into mathematics by the US mathematician Joseph Leonard Walsh with his article "A closed set of orthogonal functions" [1].

Moreover, the year 2023 also recalls the 50th anniversary of Professor Walsh's death: he passed away on 6 December 1973 in College Park, Maryland 5, almost within walking-distance from his home, in the town where he was born, Arlington, VA.

The complete system of orthogonal functions, introduced by Walsh, is characterised by the following properties [1]:

1. Each function of the system is piecewise constant over each of a finite number of sub-intervals of the interval (0, 1), and takes the values +1 and −1, except at a finite number of points of discontinuity, where it takes the value zero.
2. Furthermore, the system forms a group with respect to multiplication, whereby the constant function with value +1 is the unit element and each function is inverse to itself.
3. The n-th Walsh function has $n - 1$ sign-changes, interior the interval (0, 1).
4. Each function is either odd or even with respect to the mid-point of the interval, which leads to the splitting of Walsh functions into subsets of so-called *cal* and *sal* functions [2], emphasizing the similarity with the trigonometric system.

Since the system of Walsh functions is complete, practically every signal in telecommunications (communications engineering) can be represented by means of its corresponding Walsh–Fourier series or Walsh–Fourier integral. For their calculation, the "Fast Walsh–Fourier Transform" provides a high-speed algorithm, like the "Fast Fourier transform" in case of the Fourier representations by trigonometric functions. These properties of the Walsh functions have not only aroused interest in mathematical circles, but various practical applications in communications engineering have been found.

Mention should be made of the contribution of the Austrian scientist from Vienna, Henning F. Harmuth, who recognised the importance of these functions for communications engineering at a very early stage [3]. With his "sequence technology" multi-channel transmission systems in analogue technology were to be realised for telephony. The individual channels were allocated different bandwidths in the Walsh–Fourier spectrum (sequence spectrum) for transmission.

Today, the "Walsh Code" is mainly known in the field of digital signal processing for the realisation of CDMA (Code Division Multiple Access) in the various systems of mobile telephony. In these CDMA systems, the discrete binary Walsh functions are used as carrier functions for the digital modulation of the signals to be transmitted.

The present paper is a recollection of research work in the area of Walsh functions. Applications in mathematics led to the development of a new discipline called dyadic analysis, as an extension of classical Fourier analysis, to a complete different approach to differentiation, derivation of many useful results in approximation theory, and solving partial differential equations in the dyadic sense.

Besides their applications in mathematics, Walsh functions, especially their discrete counterpart, appeared as a very useful tool in solving various tasks in different areas of computing and engineering, starting from initial applications in communications, through signal and image processing, to system theory, digital logic, logic circuits design and verification, and cryptography. Moreover, Walsh functions are used to provide alternative definitions and characterizations of certain important concepts. An example is the definition of bent functions, an important concept in cryptographic applications, in terms of Walsh spectra of Boolean functions.

The reminiscences of the authors of this paper, who are working in different areas of mathematics and engineering, supported and documented by a list of appropriate references, provide a brief introduction into the development of Walsh and dyadic analysis and its worldwide impact in mathematics and engineering. Some initial remarks concern personal data of the life and work of J. L. Walsh highlighting also his attitude in teaching and doing research work with students and colleagues in the fields of his scientific interests.

2 Personal data of J. L. Walsh

Joseph Leonard Walsh (21 September 1895–06 December 1973), was the son of John Leonard Walsh, a Methodist minister, and Sallie Ellicott Jones. After completing the early education at the Baltimore Polytechnic Institute from 1908 to 1912, he joined Columbia University from 1912 to 1913, and then moved to Harvard University where he graduated summa cum laude (Bachelor of Science degree with distinction) in 1916 in mathematics. After receiving a Sheldon Travelling Fellowship for the academic year 1916–1917, Walsh was able to stay partly at Chicago and partly at the University of Wisconsin at Madison.

After being awarded a Master's degree from the University of Wisconsin in 1917, Walsh returned to Harvard and took up a position as an instructor in mathematics. There, Walsh enroled in doctoral studies under the supervision of Maxime Bôcher. However, the political situation was not suitable for scientific study. In a message to Congress on April 2, 1917, the USA president Woodrow Wilson proclaimed that "the world must be made safe for democracy". Walsh was enlisted in the U.S. Navy and served on troopships in the North Atlantic until 1919, when he was able to return to his position at Harvard. Since Bôcher had died in the meantime, he began working with George David Birkhoff, who in turn was inspired by Bôcher. Walsh received his doctorate from Harvard in 1920 with the dissertation "On the location of the roots of a Jacobian of two binary forms, and of the derivative of a rational function".

Another Sheldon Travelling Fellowship allowed Walsh to travel to Paris for the academic year 1920–1921, and work there with Paul Antoine Aristide Montel who was mainly working in the theory of analytic functions of a complex variable, but was also interested in the relation between the coefficients of a polynomial and the location of its zeros in the complex plane, which was probably the point of a common interest between him and Walsh, judging from an earlier publication of the latter at about this time, as it is mentioned below.

In 1921, Walsh again returned to Harvard holding the position of an instructor in mathematics, but being promoted in 1924 to assistant professor.

He spent the academic year 1925–1926 at the University of Munich where he had the opportunity to work with Constantin Carathéodory.

In 1930 he was promoted to associate professor. He spent the academic year 1934 to 1935 at the Institute for Advanced Study in Princeton and was promoted to full professor in 1935. Walsh was chairman of the Department of Mathematics from 1937 to 1942 and again for a short time in the autumn of 1950 [4].

He married Aline Natalie Burgess on 15 July 1931, but the marriage ended in divorce and he married for the second time on 2 April 1946, to Elizabeth Cheney Strayhorn, whom he met when she was an officer in the Women's Army Auxiliary Corps (WAAC). Walsh had two children with his second wife.

During World War II, Walsh served as a Lieutenant Commander from 1942 to 1943. He was promoted to Commander in 1944 and served actively in this rank until 1946; after the war he remained associated with the Navy as a member of the Naval Reserve. He was a Captain for several years in the 1950s before retiring from the service in 1955.

After returning to Harvard in 1946, Walsh was appointed Perkins Professor of Mathematics and held this position until his retirement in 1966, when he moved to the University of Maryland, College Park, where he continued his work with researchers and doctoral students. He chose this university because it was close to his home.

As a teacher, he had a rare privilege of working at the same high institution for most of his career, and promoted the idea that a professor should teach multiple courses. He followed this practice and taught various courses including calculus, algebra, mechanics, differential equations, probability, number theory, potential theory, approximation theory, and complex function theory. Wash used to rotate these courses from year to year [5]. As noticed in [4], his favourite courses were calculus and functions of a complex variable.

Besides this high engagement in teaching, Walsh was renowned as an ultimately devoted researcher, but also as a person who gladly shared his ideas with students and colleagues, motivating them to pursue research and produce good and interesting results. Table 1 shows the list of his 31 doctoral students with the years of their graduation. The last two in the list graduated from the University of Maryland, all the others graduated from Harvard, with the exception of Helen Russel, who graduated from Radcliffe College. The list can be found in [6] or at the Mathematics Genealogy Project [7].

Regarding honours that Walsh received, the most notable are election to the National Academy of Sciences of United States in 1936, and service as vice president of the American Mathematical Society in 1937 and president from 1949 to 1950 (Fig. 1).

3 Early work of J. L. Walsh

As a researcher, Walsh was prolific in terms of publications; see [8]. A full list of his publications was compiled by his first PhD student Morris Marden and includes 7 books and 273 articles, of which 6 are book reviews. One of his books, no. 2 in

Table 1 PhD students of J. L. Walsh

	Name	Graduation year		Name	Graduation year
1.	Marden, Morris	1928	17.	Block, Isaac	1952
2.	Farrell, Orin	1930	18.	Kay, Alan	1952
3.	Holmes, Cecil	1931	19.	Evans, Jacqueline	1953
4.	Doob, Joseph	1932	20.	Rivlin, Theodore	1953
5.	Russell, Helen	1932	21.	Rosenfeld, Lawrence	1953
6.	Curtiss, John	1935	22.	Varga, Richard	1954
7.	Shen, Yu-Cheng	1935	23.	Zedek, Mishael	1956
8.	Sewell, Walter	1936	24.	Landau, Henry	1957
9.	Mosesson, Zehman	1937	25.	Newman, Donald	1958
10.	Ulrich, Floyd	1938	26.	Williams, Vincent	1961
11.	Heins, Maurice	1940	27.	Shaffer, Dorothy	1962
12.	Spitzbart, Abraham	1940	28.	Manjarrez, Victor	1963
13.	Nilson, Edwin	1941	29.	Fields, Jerry	1964
14.	Hershner, Jr., Ivan	1947	30.	Saff, Edward	1968
15.	Elliott, Helen	1948	31.	Ortel, Marvin	1973
16.	Nickerson, Helen	1949			

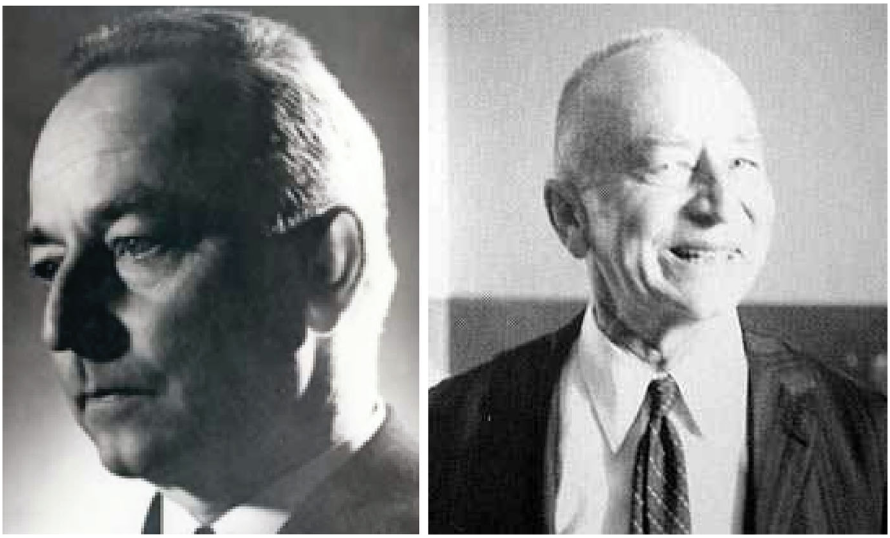

Fig. 1 J.L. Walsh at the time he served as president of the AMS in 1949–1950 (left) (From Pitcher, E.: A history of the second fifty years, American Mathematical Society, 1939–1988, p. 190. American Mathematical Society, Providence, RI (1988). With kind permission of the American Mathematical Society). The photo on the right was taken by Paul Halmos (From MacTutor Archive, under Creative Commons Attribution-ShareAlike 4.0 International License)

Table 2 Co-authors of J. L. Walsh in alphabetical order and the number of joint publications

	Name	# Publ.		Name	# Publ.
1.	Aharonov, Dov	2	13.	Rosenfeld, L.	1
2.	Ahlberg, J. H.	9	14.	Rubinstein, Z.	2
3.	Davis, Philip	2	15.	Russell, Helen G.	4
4.	Elliott, H. M.	4	16.	Saff, E. B.	1
5.	Evans, J. P.	5	17.	Seidel, W.	3
6.	Fekete, M.	3	18.	Sewell, W. E.	6
7.	Gaier, D.[a]	1	19.	Sharma, A.	1
8.	Galbraith, A. S.	1	20.	Shisha, O.	3
9.	Landau, H. J.	1	21.	Sinclair, A.	1
10.	Merriman, G. M.	1	22.	Young, David	3
11.	Motzkin, T. S.	15	23.	Zedek, Mishael	1
12.	Nilson, E. N.	12	24.	Wiener, M.	1

[a]Professor Dieter Gaier (1928–2002) was born in Stuttgart and died in Göppingen, both Germany. He received his doctorate from the U. Rochester in 1951 under Wladimr Seidel and from the U. Stuttgart in 1952 under Werner Meyer-König. He then spent two years at Harvard with J. L. Walsh. In 1955 he habilitated at the U. Stuttgart, and from 1959 he was a professor at the U. Gießen, where he remained until his retirement in 1995. He conducted many conferences in Oberwolfach, especially in complex function theory, and was a visiting professor at Caltech several times. Very popular is his book "Vorlesungen über Approximation im Komplexen", Birkhäuser, 1980, a copy of which he gave to me (PLB) as a gift. He also wrote the book "Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie", Springer, 1986, a reissue of Landau's famous classic with many basic comments and fundamental additions to Landau's original work. Dieter, who was a good friend of mine (PLB), died of a long-term severe disease. He had a very modest and friendly personality. See also Michael von Renteln: "Dieter Gaier (1928–2002) in memoriam". *Jahresber. Deutsch. Math.-Verein*, 107(1), 33–53 (2005), and Wikipedia: Dieter Gaier

Table 4, was even translated into Russian. This full list can be found in [9]. Walsh published mostly as a single author. He had co-authors on 42 of his publications, but never more than two, for a total of 24. Table 2 shows the lists of his co-authors and the number of joint publications with each of them. Table 3 lists his first five publications and Table 4 the titles of the books he wrote.

A brief analysis of Walsh's most important publications is given in [10] on the occasion of his 70th anniversary. A similar overview is also presented in [11]. Selected papers are published in [12].

In the present paper, we focus on a very influential publication by Walsh [1], the results of which made a very significant impact to signal and image processing, and several other areas of computing and engineering.

In this publication, which appeared 100 years ago, Walsh introduced a set of orthogonal functions that are now known as Walsh functions and are used as a basis for the definition of the Walsh transform. At the time of publication, none, including Walsh himself, had anticipated or expected the very high interest in these functions, including their discrete version, and their applications in various fields of computing and electrical and electronic engineering. The impact of the Walsh system on the various fields of mathematics was presented in [13], which appeared in [12]. We mention here

Table 3 First five publications by J. L. Walsh

1.	*Note on Cauchy's integral formula*. Ann. of Math. (2), **18**(2), 79–80 (1916), https://doi.org/10.2307/2007173
2.	On the location of the roots of the Jacobian of two binary forms, and of the derivative of a rational function. Trans. Amer. Math. Soc., **19**(3), 291–298 (1918), https://doi.org/10.2307/1988954
3.	*On the proof of Cauchy's integral formula by means of Green's formula*. Bull. Amer. Math. Soc., **26**(4), 155–157 (1920), https://doi.org/10.1090/S0002-9904-1920-03275-1
4.	*On the solution of linear equations in infinitely many variables by successive approximations*. Amer. J. Math., **42**(2), 91–96 (1920), https://doi.org/10.2307/2370317
5.	*On the location of the roots of the derivative of a polynomial*. Ann. of Math. (2), **22**(2), 128–144 (1920), https://doi.org/10.2307/1967860

Table 4 Books by J. L. Walsh

1.	*Approximation by Polynomials in the Complex Domain*. Mémor. Sci. math., vol. 73. Gauthier-Villars, Paris (1935), ii+70 pp., http://numdam.org/item/MSM_1935__73__1_0/
2.	*Interpolation and Approximation by Rational Functions in the Complex Domain*. *Am. Math. Soc. Colloq. Publ.*, vol. 20. American Mathematical Society, Providence, RI, 1st edn. (1935), ix+382 pp., 2nd edn. (1952), 3rd edn. (1960), 4th edn. (1965), 5th edn. (1969). This book was also translated into Russian by A. A. Gončar and S. Ja. Havinson, with a foreword by S. N. Mergeljan
3.	*A Bibliography on Orthogonal Polynomials* (with J. A. Shohat and E. Hille). *Bulletin of the National Research Council*, vol. 103. National Academy of Sciences-National Research Council, Washington, DC (1940), ix+204 pp
4.	*The Location of Critical Points of Analytic and Harmonic Functions*. Am. Math. Soc. Colloq. Publ., vol. 34. American Mathematical Society, Providence, RI (1950), viii+384
5.	*Approximation by Bounded Analytic Functions*. Mémor. Sci. Math., vol. 144. Gauthier-Villars, Paris (1960), ii+66 pp., http://numdam.org/item/MSM_1960__144__1_0/
6.	*A Rigorous Treatment of Maximum-Minimum Problems in the Calculus*. Heath, Boston, (1962), 22 pp
7.	*The Theory of Splines and Their Applications* (with J. H. Ahlberg, and E. N. Nilson). Academic Press, New York-London (1967), xi+284 pp

just a grasp from the voluminous literature discussing applications of Walsh functions and transforms in different areas, starting from signal and image processing, through digital logic, to system theory and other areas, written by authors from all over the World [3, 14–27].

It should be noted that Paley [28] was one of the first to recognize the importance of the Walsh system by establishing the connection between the Rademacher and Walsh systems and proving the fundamental inequality named after him. The relation with the Rademacher system led to a natural ordering of the Walsh functions, the Walsh–Paley system. This order is the one used in mathematical research. The *sal* and *cal* systems, which follow the sign changes of the trigonometric system and are used in technical applications, and the original Walsh system are both rearrangements of the Walsh–Paley system. It also turned out [23, 29] that even the Hadamard system is related to the Walsh–Paley system via a simple bit transformation.

The great interest in Walsh functions emerged in the early 1970s as a result of the attempt to find a reasonable and computationally efficient replacement for classical Fourier analysis. The limited computing power of the computational devices available at the time was not sufficient for the practical application of many well-developed but computationally intensive signal and image processing algorithms that were based on classical Fourier analysis and required calculation with complex numbers.

Recall that Fourier analysis is based on the exponential functions that can be considered as group characters of the group of real numbers. In this analogy, Walsh functions can be viewed as group characters of the dyadic group, and since they take just two values 1 and -1, they can serve as a computationally very efficient Fourier-like transform, the Walsh transform. This transform was immediately anticipated as a suitable tool for spectral analysis and its applications in computing and engineering [30]. This approach led to the development of a new mathematical discipline, dyadic analysis, including a completely new approach to the concept of differentiation and related differential operators. For more information on this subject, we refer to [23, 31–36].

The monographs [16, 23] summarize the results of the theory of the Walsh system up to 1990. The two fundamental classical works of Fourier analysis by Zygmund [37] and Bary [38] served the authors as a model for the preparation of these monographs.

Another publication by Walsh to be mentioned here is “A property of Haar’s system of orthogonal functions” of 1923 [39]. The reason for mentioning this paper is that the interest of Walsh for the Haar functions is in the same line of thinking as later contributions in dyadic analysis related to the Haar dyadic derivative and its applications [40] as discussed below.

4 Early days of development of Walsh analysis with applications

The interest in Walsh functions sprang up in late 1960s and early 1970s. When speaking about these early days of development of the theory and applications of Walsh functions, it should be recalled that this was the time of the first Cold War and the world was divided into a West and East bloc and separated by the so-called Iron Curtain, which prevented a proper circulation of the literature and a fluent spread of scientific discoveries. This was particularly true in the case of Walsh functions, as the interest in them was motivated by attempts to solve some specific tasks, many of which were related to military purposes. It is not just a random coincidence that the first scientific meetings and then workshops, entitled *Walsh Functions and Applications*, have been organized by the Naval Research Laboratory in Washington, DC, for several consecutive years from 1969 onwards.

Some of the tasks that were in the focus of research interest in these circles were

1. Measurement of thickness of ice over the North Pole so that submarines may sail safely bellow it,
2. Detection of seismic signals caused by underground nuclear experiments,
3. Speech recognition from directed sources, for instance recognition what a particular person is speaking in a crowded room, and echo cancellation.

For more details about these and related tasks, we refer to [14, 41].

At that time, algorithms already existed for solving some of these and related tasks, most of them based on tools from Fourier analysis. It means that these algorithms were computationally demanding as they required extensive calculations over complex numbers. The rather limited computing power of computers at that time did not allow a feasible implementation of these algorithms. Research was consequently focussed on finding alternative approaches. Walsh analysis appeared as a promising approach due to the simplicity of Walsh functions taking just two values 1 and -1. Thus, the computation with Walsh functions is reduced to addition and subtraction without dealing with complex numbers. At the same time, Walsh transform expresses useful properties of the Fourier transform. Actually, the Walsh transform can be viewed as the Fourier transform on the dyadic group.

Research in Walsh analysis was oriented in two main directions,

1. Development of mathematical theory as a foundation for various applications, and
2. Engineering solutions of various concrete tasks, a primarily based on the discrete Walsh functions and transform, with computations based on the Fast Walsh transform (FWT), an analogue of the Fast Fourier transform (FFT) [42].

In such a situation, it may be a good way to trace the early development of Walsh analysis through the reminiscences of the authors of this paper who have worked in these areas, especially as they come from different geographic sides. Germany, at that time West Germany, and Hungary belonged to the Western and Eastern blocs, respectively, Austria was a politically neutral country, while Serbia was a part of the former Yugoslavia, which was considered as an in-between country belonging neither to the West nor the East bloc. At that time, this difference in the geopolitical situation had a strong influence on how the research was performed and how the achieved results were reported.

4.1 Reminiscences of Paul L. Butzer

My personal interest in Walsh functions, or what is now called Walsh analysis, began with H. F. Harmuth's invitation to the symposium and workshop *Applications of Walsh Functions* at the Naval Research Laboratory (NRL), Washington, DC, in 1970. Especially considering the fact that this meeting could possibly bring me in touch with potential applications of our work in approximation theory, harmonic analysis and integral transform theory, I decided to accept the invitation. Moreover, it had always been my desire to meet Joseph L. Walsh, one of the organisers, not least because my first mathematics teacher at Loyola College in Montreal, Father Eric O' Connor S. J. (1907–1980), who claimed he had received his PhD in mathematics from Walsh in the late 1930s, had always spoken very highly of him.

Fortunately, I was one of the 10 or so participants that Walsh invited to his house on the Saturday morning at the conference. Walsh, already 73 at the time, truly impressed me with his broad outlook on mathematics. A lively conversation arose among those present, one of the topics being a possible differentiation-type concept for Walsh functions. The British electrical engineer James Edmund Gibbs of the National Physical Laboratory in Teddington, Middlesex, UK, had expressed important initial ideas in his very first manuscript of January 1967 [30], which led to his concept of a "logical derivative". Also involved in the discussion were the electrical engineer Henning F. Harmuth, stationed at the Catholic University of America, and the system theorist Franz Pichler, who was spending a year at the University of Maryland at the time, both from Austria.

This day also initiated a fundamental research direction at the Lehrstuhl A für Mathematik, Rheinisch-Westfälische Technische Hochschule (RWTH) Aachen, namely one aiming at possible areas of application without, of course, neglecting the usual basic theoretical research. Immediately after my return, I suggested to my student Heinrich Josef Wagner to tackle the problem I had brought with me.

In his report of 1967 [30], Gibbs had defined a derivative, which he called "logical derivative", on the finite dyadic group G_n in such a way that the discrete Walsh functions are solutions of an eigenvalue problem. This report was followed by another basic paper on "Walsh functions as solutions of a logical derivative" written together with Margaret J. Millard in 1969 [43], and a joint paper with Bryan Ireland, University of Bath on "Some generalizations of the logical derivative" [44].

A further ground breaking article to be mentioned in this context is the report [34] by Franz Pichler of 1970, a futuristic sketch of the broad area of dyadic analysis. It concerns the concept of a dyadic derivative of real valued functions of a continuous nonnegative real variable. He credits the definition of the derivative to Gibbs and Millard [43], but comments modestly that "Slight modifications must be made …". It can be said that the term "dyadic differentiation" was coined by Pichler. In addition, Franz applies Kluvánek's sampling theorem of abstract harmonic analysis to the case of the dyadic group in order to obtain the sampling theorem in dyadic harmonic analysis and finally in Walsh analysis, which had never been considered before.

Although in my first paper with H. J. Wagner on this subject [45], written in 1971, we cited the proceedings of the 1970 Washington workshop, containing a shorted version of Pichler's report of 1970, we unfortunately overlooked Pichler's 1970 sketch. Our definition of a dyadic derivative differs from that given by Gibbs in [44] that the factor 2^{-j} occurring in Gibbs' definition of a pointwise dyadic derivative is replaced by 2^j in our definition of a derivative in L^p-norm. In a letter of 1972 to me, Gibbs writes: "I am intrigued by the presence of the factor 2^j in place of the factor 2^{-j} in my definition. I have not yet got to the bottom of this discrepancy, but I associate it with the fact that you set out to define a derivative with properties parallel to those of the classical derivative – and I think you have succeeded". In this respect see also [46].

The mathematicians who put Pichler's sketch on a firm mathematical base were Ferenc Schipp and his school at Eötvös Loránd University, Budapest. Ferenc's paper of 1974 [47], written in German, initiates the many papers of the Budapest-Hungarian school on the subject.

Among the further interesting papers in the two volumes [32, 33] is William R. Wade's "Early history of Walsh analysis" containing, in particular, the "History of the American school of Walsh analysis", in which he showed that Antoni Zygmund, although he never worked on the Walsh system, is essentially the initiator of the American school in view of his 19 descendants who worked in the area. They and in part also their work are listed there. It is of interest that Wade (1943–2016), an academic grandson (via Victor L. Shapiro) of Antoni Zygmund, spent a semester at Moscow State University in 1977 as a Fulbright senior lecturer, where he wrote a joint paper with Valentin Anatol'evich Skvortsov on Walsh series and the dyadic derivative [48].

4.2 Reminiscences of Franz Pichler

I was a PhD student in Innsbruck at the time when a professor in mathematics there, Roman Liedl, was doing research on the topic that he later found to be the Walsh functions. He looked at them from the group theoretical point of view [49–51]. As many other students there, working in generalized Walsh functions, we were studying the work by Lèvy [52], and Vilenkin [53, 54], and Selfridge [55]. At that time, Professor Henning F. Harmuth from Maryland, who was working intensively on engineering applications of Walsh functions, see [3], as an early contribution to the subject, realised that research on Walsh functions was being done in Innsbruck, and generalisations from a mathematical point of view were very topical, and therefore he wanted to establish closed contacts. Professor Liedl knew that I had a telecommunications background and I was working on my PhD thesis on *cal* and *sal* functions in which Harmuth was very interested because of his interest in his so-called meander functions [56] which seemed to be identical with *cal* and *sal* functions that I studied in my PhD thesis [2]. Therefore, I was the right person to be assigned the task of investigating applications for Walsh and related functions in the context of the problems proposed by Harmuth. I also had to discover the concept of the of the dyadic filter, and this is the dyadic convolution, which I did not know before.

In this way, I wrote my first paper on this topic and published it in 1968 in the AEU (Archiv der Elektrischen Übertragung abbreviated as Archiv elektr. Übertragung). The editor or co-editor of AEU at that time was Hansi Piesch, and she was also a reviewer of my paper [57]. Then, another paper came out also in AEU [58], and my research continued [59–61].

I returned to Linz in 1968 and Harmuth contacted me because he needed my help with the mathematical formulation of practical topics. This is why I was invited to the workshops in 1970 and later, and during my stay in Harmuth's lab I wrote two reports. The first, [35] (reprinted in [32]), contains the definition of the dyadic derivative of real valued functions of a continuous nonnegative real variable. The definition is derived as a modification of the corresponding definition of the dyadic derivative on finite discrete structures [43]. The report also contains a discussion of a sampling theorem in abstract harmonic analysis, dyadic harmonic analysis, and Walsh–Fourier analysis.

I had opportunity to visit the National Physical Laboratory at Middlesex a couple of times, where I met and worked with Dr. J. E. Gibbs, which was a very interesting time in my research on Walsh and related topics. For more information on this work, see [62].

4.3 Reminiscences of Ferenc Schipp

The evolution of theory and applications of Walsh functions and dyadic analysis in Eastern Europe can be traced through reminiscences of Ferenc Schipp, born 1939 in Somberek.

In Hungary, Walsh series and related theories have found a very fruitful environment for a further development due to already well-established mathematical schools in Fourier-series, approximation theory, and functional analysis. These foundations were established by Frigyes Riesz, Lipót Fejér, and Alfréd Haar. Research in approximation theory and functional analysis fields has been continued by György Alexits and Pál Turán, in Budapest, and Béla Szőkefalvi-Nagy and Károly Tandori, in Szeged. A founder of the dyadic analysis in Hungary, Ferenc Schipp, recalls [63] that he heard about the Walsh system of functions for the first time in connection with a problem of László Pál, who was working on a problem of Hardy on the convergence of products of multiplication of hyper harmonic series. Their convergence with probability one led to the question of convergence of two-dimensional Rademacher-series, which are Walsh-series of a special type. The paper by Walsh was available in the library of the University of Szeged. This observation is important because at that time there was a strong separation between the countries of Western and Eastern Europe and the circulation of literature was limited.

The paper by Fine [64], where it was pointed out that Walsh functions are characters of the dyadic group, focused the attention to the work of Vilenkin, who considered the topics from a rather broader view [53, 54]. Naum Yakovlevich Vilenkin, was born 1920 in Moscow and also died there in 1991.

The theory presented in [64] was a basis for the work of F. Schipp towards the construction of a continuous function whose Walsh-series diverges at a given point [65, 66].

Another very influential publication for the research in Hungary was written by R. E. A. C. Paley [28], whose ordering of Walsh functions appeared quite convenient for solving certain tasks. In particular, this paper had a great influence on the development of martingale theory.

Research in the Soviet Union, was supported by the Russian translation of the book of Kaczmarz and Steinhaus [67], to which an appendix on Vilenkin systems by P. L. Uljanov was added.

At that time, György Alexits from Hungary and Sergeĭ Mikhaĭlovich Nikolskiĭ from Moscow initiated a series of conferences on approximation theory and Fourier series, and researchers from Bulgaria and Poland joined this programme. These conferences, as well as the meetings and conferences organised at the Banach Centre, a part of the Institute of Mathematics of the Polish Academy of Sciences (IM PAN), provided a suitable environment for further work in these areas, including dyadic analysis. In addition, the journal "Analysis Mathematica" was launched, whose first editors-in-chief were Nikolskiĭ and Szőkefalvi-Nagy. This journal proved to be a good forum for publishing topics related to Walsh functions and dyadic analysis in general. We will mention here only the articles [68–72].

An important impetus for further work in this field was the visit of William R. Wade from the USA to Moscow in 1974, where he had the opportunity to work with Soviet researchers on this topic. Wade also established a connection and collaboration with Ferenc Schipp and his research group, which led to a very influential book [23]. This book was the foundational text for research work in several Hungarian schools in the area, just to name the research groups in Budapest, Nyiregyháza, Debrecen, Pécs, and also for many researchers abroad, such as Serbia and Georgia.

4.4 Reminiscences of Radomir S. Stanković

I heard of Walsh functions for the very first time in February 1976 from my older brother Miomir when I was looking for a suitable subject for my BSc thesis at the Faculty of Electronics in Niš, Serbia. At that time, Miomir was an assistant professor of Mathematics at the Faculty of Occupational Safety, University of Niš, Serbia, and was involved in determining mathematical models for various problems concerning ecological issues and environment protection. This faculty had very good connections with CERN in Switzerland, as there was an efficient exchange of data and statistical results of ecological observations in the Niš region. Thanks to this link, we were able to obtain the "Proceedings of the Workshops on Walsh Functions and Applications", held at the Naval Research Laboratory in Washington, DC, from 1970 to 1974. I then began to learn and continued to study Walsh and Haar functions after graduation.

In the mentioned proceedings, I found the articles and address of Dr. James Edmund Gibbs, and started a long-lasting correspondence with him. I was very fortunate that Dr. Gibbs sent me reports that he wrote for the National Physical Laboratory in Middlesex, Teddington, England, where he worked. In this way I learned about the Gibbs dyadic derivative or logical derivative as he called it, and also on the Butzer–Wagner derivative. Among several NPL reports by J.E. Gibbs, I would especially like to point out [73] since it was also very instructive. I also wrote to Franz Pichler, whose address was in the proceedings mentioned, and I was very happy when he kindly sent me his PhD thesis. At that time it was already known that Walsh and Haar functions were very suitable tools for the analysis of Boolean functions and, furthermore, for the design of Boolean circuits, as explained in the books by Karpovsky and Moskalev [74, 75]. The publications by Lechner were very useful for the work in this area [76–78]. In the preface of [75], it was stated that Fourier analysis can be extended to non-Abelian groups by replacing the group characters, such as the Walsh functions in the case of the dyadic group, by unitary irreducible group representations.

Motivated by this short sentence there, I decided to undertake the task of extending the concept of the Gibbs logical derivative to finite and compact non-Abelian groups, which became the subject of my PhD thesis 10 years after [79]. The applications were in the area of linear system theory which brought me into closer connection with Franz Pichler and Yasushi Endow from Chuo University in Tokyo, Japan, as I was able to attend the *Eighth European Conference on Cybernetics and Systems Research (EMCSR)* in Vienna, Austria, on 1–4 April 1986, and afterwards to study the topic further [80]. My presentation at this conference about linear systems defined in terms of the Gibbs derivative on finite non-Abelian groups [80], was later noticed by Mark Karpovsky, and served as a good introduction to him when I met him for the first time at the *Workshop on Spectral Techniques* in Beijing in 1990, see below. For more information about the Gibbs derivatives on finite non-Abelian groups, we refer to [81].

In 1989, we organized under the auspices of the Mathematical Institute in Belgrade, the *First International Workshop Theory and Applications of Gibbs Derivatives*, in Kupari, Dubrovnik, former Yugoslavia, now in Croatia. We were very happy that the workshop was attended by the most eminent researchers in the area at that time, including P.L. Butzer, F. Pichler, J.E. Gibbs, F. Schipp and his associate J. Pál from Budapest, Naum Aizenberg from Uzhgorod, Ukraine (former USSR), Claudio Moraga from Dortmund, Germany, Yasushi Endow from Japan, William R. Wade from the USA, and a few mathematicians from Belgrade and Niš. The workshop resulted in the proceedings, edited together with P.L. Butzer and published in 1990 [31] by the Mathematical Institute in Belgrade.

5 Remarks on the worldwide interest in applications of Walsh functions outside mathematics

The earliest applications of Walsh functions in engineering can be traced back to the 1960s, as can be seen from a brief overview of such applications in [3]. More can be found in [14, 18, 19, 82].

Thanks to the aforementioned *Workshop on Spectral Techniques* in Beijing, China, in 1990, which was organised under the support of the Volkswagen Foundation by Claudio Moraga, who worked in the area of applications of Walsh and Vilenkin–Chrestenson functions in binary and multiple-valued logic circuit design [22], we were able to established close contacts with researchers in China who are working in theory and applications of Walsh functions. An important centre for such research was the Beijing University of Aeronautic and Astronautics, where certain telemetry systems based on Walsh and Haar functions were built under coordination of Zhang Qi-Shan, who also defined a closely related system of functions, the so-called bridge functions. The name comes from the exploitation of the relationship between Walsh and Haar functions. A couple of books on this topic were published in Chinese by the National Defence Industry Press [27].

At that time, a strong mathematical school was also established at Nanjing University, Nanjing, China, under the guidance of Wei-yi Su with a special interest in dyadic differentiation and generalisations of this concept. This school worked under the influence and friendly guidance of P.L. Butzer, who, among other ways of cooperation and advise, served as the editor of the journal “Approximation Theory and Its Applications” published there. In addition to this journal, good publishing forums were and are the “Journal of Nanjing University”, “Acta Mathematica Sinica”, and others. For more information, see [83] and three translated and reprinted papers of Chinese authors in [33].

In the Soviet Union, in addition to mathematical research [84, 85], another research direction in Walsh functions was focussed on their applications in system and logic design, which led to a series of papers followed by the first book on application of spectral techniques for the design of logic circuits [74]. The books by Harmuth [3, 86], and Ahmed and Rao [87], which were translated with additional chapters on related research by Soviet researchers, certainly contributed to further research in this field, as did articles and books written by Soviet authors [88–91]. The book [92] was also very informative from the mathematical point of view, as it provided a global overview of the field, with an excellent selection of references, making it very influential in more recent earlier work, [16, 26].

These books, published in the former Soviet Union, were printed in modest print quality, but they were distributed in other Eastern Bloc countries and sold extremely cheaply. The prices of the translated books were about 10 times cheaper than the originals published in English. Therefore, we travelled from Serbia to Sofia in Bulgaria to buy these books for pocket money in the popular Russian bookshop named after the famous writer Maksim Gorky.

In the former Soviet Union, besides Moscow and the strong mathematical research there and the aforementioned Uzhgorod with the work of Naum N. Aizenberg, there was also research on Walsh functions and the connection between Walsh and Hadamard matrices in Yerevan, Armenia, under the direction of Sos Agaian [93], Hakub Sarukhanian and others. The applications involved various problems of signal and image processing. Research in these directions continues to this day, as can be seen from the latest publications.

Regarding Japan, in addition to the mathematical work discussed above by Paul Butzer, and the work on applications of Walsh functions in system theory by Yasushi Endow at Chuo University, Tokyo, [94], research on application oriented Walsh analysis has been carried out at other institutions, as for instance an the Research Center of Tsukuba, where Ikuo Fukui worked on various applications of Walsh analysis [95] and published a book on Walsh functions in Japanese.

In Australia, the work on Walsh functions was done by He Zelin [96] and David Mustard as can be seen from their contribution to [31].

Work on applications of Walsh functions in computer science and electrical and electronic engineering continues as we can see from many actual references on the above mentioned topics. Moreover, the range of applications is expanding, and some recent references report the use of Walsh functions in the design of quantum circuits, see for example, [97–99].

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