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Invariant sets for a class of nonlinear control systems tractable by symbolic computation

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Abstract: A set S is said to be controlled invariant with respect to a control system if a state feedback law exists such that the closed loop system has S as an invariant set. In the present paper we generalise results on input-affine polynomial control systems and algebraic varieties (i.e. sets described by the zeros of polynomial equations) considered in Zerz and Walcher (2012) to an extended class of vector fields. More precisely, we consider vector fields of the form $f = F \circ h$, where F is a polynomial vector and h is a continuously differentiable function with certain (algebraic) properties, as well as sets V_h as the preimages of varieties under h. We will see that for example polynomial expressions in sine and cosine satisfy the mentioned properties. The main advantage of the considered function class is that it is accessible to symbolic computation. We give computational methods (based on the theory of Gröbner bases) to decide the controlled invariance of V_h.

Keywords: Nonlinear control systems; Polynomial methods; Structural properties; Invariance; State feedback; Algebraic systems theory; Symbolic computation.

1. INTRODUCTION

A milestone for the modern control theory was laid in the books of Basile and Marro (1992) and Wonham (1985), which were focused on linear invariant sets for linear control systems. Based on these results Ilchmann (1989) treated the time-varying case and approaches in Isidori (1995) led to a theory also for more general, nonlinear control systems and nonlinear invariant sets. For autonomous systems, the concept of invariant sets has been widely studied by several authors. First characterisations of invariance via tangential cones are given in Bony (1969) and in Brezis (1970), with extensions in Redheffer (1972). A special kind of these invariant sets, namely algebraic curves, were considered in Christopher et al. (2009) for polynomial ODE systems.

These publications serve as a starting point for several results on polynomial control systems with focus on symbolic computation. While Zerz et al. (2010) deals with controlled invariant algebraic varieties in general, we have more specific results for hypersurfaces presented in Zerz and Walcher (2012). Both take use of the theory of Gröbner bases (see, for instance, Greuel and Pfister (2008)) to compute all polynomial vector fields which has a given variety V as an invariant set, to decide if V is controlled invariant for a given polynomial control system and how the corresponding state feedback laws can be computed explicitly. A similar approach for controlled invariance has been presented in Yuno and Ohtsuka (2014). The works of Harms et al. (2017) and Yuno et al. (2020) generalised these concepts to some classes of semi-algebraic sets and Schilli et al. (2020) to rational feedback systems.

This present work now regards control systems of compositions of polynomial functions and a continuously differentiable function h with certain algebraic properties. We point out that, among other functions, polynomial expressions in sine and cosine have these assumed properties. Constructing sets V_h as the preimage of varieties under h, we derive conditions for the controlled invariance of these sets for the given control systems. Although the systems might be non-polynomial, the presented methods in the area of symbolic computation and Gröbner bases allow us to decide the controlled invariance of the sets V_h with respect to the given non-polynomial system.

2. PRELIMINARIES AND KNOWN RESULTS

Consider the ordinary differential equation (ODE)

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $f : U \to \mathbb{R}^n$ is a $\mathcal{C}^1$-function defined on some open set $U \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}$. For any $x^0 \in U$, the initial value problem (IVP)

$$\dot{x}(t) = f(x(t)), \quad x(0) = x^0$$

has a unique $\mathcal{C}^1$-solution $\varphi(\cdot, x^0) : J(x^0) \to U$ on the maximal existence interval $J(x^0) \subseteq \mathbb{R}$, which is an open neighbourhood of 0.

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Definition 1. A subset $S \subseteq U$ is called *invariant* for the ODE $\dot{x} = f(x)$ if for any vector $x^0 \in S$, the solution $\varphi(\cdot, x^0)$ to the IVP $\dot{x} = f(x)$, $x(0) = x^0$ fulfils $\varphi(t, x^0) \in S$ for all $t \in J(x^0)$.

We are particularly interested in a special class of sets. Let $\mathcal{P} := \mathbb{R}[Z_1, ..., Z_m]$ be the polynomial ring over $\mathbb{R}$ in $m \in \mathbb{N}$ variables. A subset $P \subseteq \mathcal{P}$ defines the variety

$$\mathcal{V}(P) := \{z \in \mathbb{R}^m \mid p(z) = 0 \text{ for all } p \in P\}$$

and a subset $M \subseteq \mathbb{R}^m$, in turn, defines the set

$$\mathcal{J}(M) := \{p \in \mathcal{P} \mid p(z) = 0 \text{ for all } z \in M\},$$

which is an ideal of $\mathcal{P}$. Since $\mathcal{P}$ is Noetherian, any ideal of $\mathcal{P}$ is finitely generated. The following results are folklore and easy to see.

Lemma 2. The operators $\mathcal{V}$ and $\mathcal{J}$ have the following properties:

1.) $\mathcal{V}$ and $\mathcal{J}$ are inclusion-reversing.
2.) $P \subseteq \mathcal{J}(\mathcal{V}(P))$ and $S \subseteq \mathcal{V}(\mathcal{J}(S))$ for $P \subseteq \mathcal{P}$, $S \subseteq \mathbb{R}^m$.
3.) $\mathcal{J} \circ \mathcal{V} \circ \mathcal{J} \equiv \mathcal{J}$ and $\mathcal{V} \circ \mathcal{J} \circ \mathcal{V} \equiv \mathcal{V}$.
4.) $\mathcal{V}(I) \cap \mathcal{V}(J) = \mathcal{V}(I + J)$ for ideals $I, J \subseteq \mathcal{P}$.
5.) $\mathcal{J}(\mathcal{V}(I)) = \sqrt[\mathbb{R}]{I}$ for any ideal $I \subseteq \mathcal{P}$, where $\sqrt[\mathbb{R}]{I}$ denotes the real radical of I (see e.g. Bochnak et al. (1998)).

For arbitrary $l, k \in \mathbb{N}$ and a function $q \in \mathcal{C}^1(\widetilde{U}, \mathbb{R}^l)$ with an open subset $\widetilde{U} \subseteq \mathbb{R}^k$ we define the *Jacobian matrix* of q by

$$Dq := \begin{bmatrix} \frac{\partial q_1}{\partial x_1} & \cdots & \frac{\partial q_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial q_l}{\partial x_1} & \cdots & \frac{\partial q_l}{\partial x_k} \end{bmatrix}.$$

Furthermore, for $q \in \mathcal{C}^1(\widetilde{U}, \mathbb{R})$, let the *Lie derivative* of q along $g \in \mathcal{C}^1(\widetilde{U}, \mathbb{R}^k)$ be defined by

$$L_g(q) := Dq \cdot g = \sum_{i=1}^{k} \frac{\partial q}{\partial x_i} \cdot g_i,$$

which is again a $\mathcal{C}^1$-function on $\widetilde{U}$.
Now if we have two polynomial functions $f \in \mathcal{P}^m$ and $q \in \mathcal{P}$, then also $L_f(q) \in \mathcal{P}$ is polynomial and we get the following folklore result, which provides the key to symbolic computations for polynomial systems (see, for instance, Zerz and Walcher (2012) or Harms et al. (2017)).

Lemma 3. Let $f \in \mathcal{P}^m$ and a variety $V \subseteq \mathbb{R}^m$ be given. Then V is invariant for the ODE $\dot{x} = f(x)$ if and only if

$$L_f(\mathcal{J}(V)) \subseteq \mathcal{J}(V).$$

3. AUTONOMOUS CASE

We wish to extend the class of functions for which this last result stays true. A standard result is that in the more general setting of the beginning of Section 2 at least one direction of Lemma 3 stays true. We state this here with proof for the sake of completeness:

Lemma 4. Let $f \in \mathcal{C}^1(U, \mathbb{R}^n)$, $q \in \mathcal{C}^1(U, \mathbb{R})$ and $S \subseteq U$. If S is invariant for the ODE $\dot{x} = f(x)$ and q vanishes on S, then $L_f(q)$ vanishes on S.

Proof. Let S be invariant and let q vanish on S. Then, for all $x^0 \in S$ and all times $t \in J(x^0)$ we have

$$q(\varphi(t, x^0)) = 0,$$

where $\varphi(\cdot, x^0)$ denotes the solution to the IVP $\dot{x} = f(x)$, $x(0) = x^0$. Differentiation yields

$$0 = \frac{\partial}{\partial t} q(\varphi(t, x^0)) = L_f(q)(\varphi(t, x^0))$$

for all $t \in J(x^0)$, which implies $L_f(q)(x^0) = 0$. Hence, $L_f(q)$ vanishes on S. □

3.1 Extension of the function class

In order to generalise the pure polynomial case as treated in Lemma 3, we first construct a more general version of varieties and of our considered ODE systems and give assumptions on these constructions with which we are allowed to find invariance criteria afterwards.

For arbitrary $h \in \mathcal{C}^1(U, \mathbb{R}^m)$ and $P \subseteq \mathcal{P}$ we define the set

$$\mathcal{V}_h(P) := \{x \in U \mid p(h(x)) = 0 \text{ for all } p \in P\}$$

and for $S \subseteq U$ the set

$$\mathcal{J}_h(S) := \{p \in \mathcal{P} \mid p(h(x)) = 0 \text{ for all } x \in S\}.$$

Clearly, the latter is a finitely generated ideal of $\mathcal{P}$. The connection to the operators $\mathcal{V}$ and $\mathcal{J}$ is as follows. We have

$$\mathcal{V}_h(P) = h^{-1}(\mathcal{V}(P)) \quad \text{and} \quad h(\mathcal{V}_h(P)) = \mathcal{V}(P) \cap \operatorname{im}(h)$$

as well as

$$\mathcal{J}_h(S) = \mathcal{J}(h(S)).$$

Again, $\mathcal{V}_h$ and $\mathcal{J}_h$ are inclusion-reversing, we obtain

$$P \subseteq \mathcal{J}(\mathcal{V}(P)) \subseteq \mathcal{J}_h(\mathcal{V}_h(P)) \quad \text{and} \quad S \subseteq \mathcal{V}_h(\mathcal{J}_h(S))$$

for all $P \subseteq \mathcal{P}$, $S \subseteq U$, as well as $\mathcal{J}_h \circ \mathcal{V}_h \circ \mathcal{J}_h \equiv \mathcal{J}_h$ and $\mathcal{V}_h \circ \mathcal{J}_h \circ \mathcal{V}_h \equiv \mathcal{V}_h$ (cf. Lemma 2).

Example 5. It is $\mathcal{V}_h(\mathcal{P}) = \emptyset$, $\mathcal{V}_h(\{0\}) = U$, and $\mathcal{J}_h(\emptyset) = \mathcal{P}$. The ideal

$$\mathcal{J}_h(U) = \{p \in \mathcal{P} \mid p \circ h = 0\}$$

consists of all algebraic dependencies between the components of h and is contained in every ideal $\mathcal{J}_h(S)$ for $S \subseteq U$. This implies

$$I + \mathcal{J}_h(U) \subseteq \mathcal{J}_h(\mathcal{V}_h(I))$$

and $\mathcal{V}_h(I + \mathcal{J}_h(U)) = \mathcal{V}_h(I)$ for every ideal $I \subseteq \mathcal{P}$.

After these preparations, we consider more general ODE systems as in (1) in which the vector field f takes the form

$$f = F \circ h,$$

where $F \in \mathcal{P}^n$ is polynomial and $h \in \mathcal{C}^1(U, \mathbb{R}^m)$ fulfils the additional condition that the Jacobian matrix of h can be written as

$$Dh = Q \circ h, \quad \text{for some } Q \in \mathcal{P}^{m \times n}. \tag{2}$$

Hence, the pure polynomial case as treated in Zerz and Walcher (2012) corresponds to the choices $m = n$ and $h = \mathrm{id}_{\mathbb{R}^n}$, the identity on $\mathbb{R}^n$. In this case Q is just the identity matrix of size n.

Condition (2) allows to prove a similar statement as in Lemma 3 for this more general setting (see Theorem 8) and is motivated by computing the Lie derivative of $p \circ h$, for $p \in \mathcal{P}$, along $f = F \circ h$:

$$\begin{aligned} L_{F \circ h}(p \circ h) &= D(p \circ h) \cdot (F \circ h) \\ &= ((Dp) \circ h) \cdot Dh \cdot (F \circ h) \\ &= (Dp \cdot Q \cdot F) \circ h. \end{aligned}$$

Clearly, condition (2) is indeed a hard restriction on the set of admissible $\mathcal{C}^1$-functions h. However, there are some interesting examples which satisfy (2).

Example 6. 1.) For $h(x) = \mathrm{e}^x$ we have $Dh(x) = \mathrm{e}^x$. Thus, $Q = Z \in \mathcal{P}$ fulfils condition (2).

2.) For $h_1(x) = \sin(x)$ and $h_2(x) = \cos(x)$ we may write

$$Dh(x) = \begin{pmatrix} \cos(x) \\ -\sin(x) \end{pmatrix} = \begin{pmatrix} h_2(x) \\ -h_1(x) \end{pmatrix} = (Q \circ h)(x),$$

where $Q = (Z_2, -Z_1)^T \in \mathcal{P}^{2\times 1}$.

3.) Also for the trigonometric function $h(x) = \tan(x)$ on $U = (-\frac{\pi}{2}, \frac{\pi}{2})$ we obtain

$$Dh(x) = 1 + \tan(x)^2 = 1 + h(x)^2 = (Q \circ h)(x),$$

where $Q = 1 + Z^2 \in \mathcal{P}$.

4.) For $U = \mathbb{R} \setminus \{0\}$ and $h(x) = \frac{1}{x}$ we have

$$Dh(x) = -\frac{1}{x^2} = -h(x)^2,$$

hence, $Q = -Z^2 \in \mathcal{P}$.

Remark 7. Since the composition of a polynomial and a $\mathcal{C}^1$-function is again a $\mathcal{C}^1$-function one may think that we can just drop F in $f = F \circ h$ and consider just arbitrary $\mathcal{C}^1$-functions h. But then condition (2) gets harder to fulfil. For instance, let $\tilde{h}(x) = (\sin(x)^2, \cos(x)^2)^T$. We have

$$D\tilde{h}(x) = \begin{pmatrix} 2\sin(x)\cos(x) \\ -2\sin(x)\cos(x) \end{pmatrix},$$

which clearly can not be written as a polynomial expression of h_1 and h_2. But by setting $F = (Z_1^2, Z_2^2)$ and $h(x) = (\sin(x), \cos(x))^T$ we get $\tilde{h} = F \circ h$ and h fulfils condition (2) which has been shown in Example 6. Thus, one should choose h cleverly "as simple as possible".

Theorem 8. Let $F \in \mathcal{P}^n$ and let $h \in \mathcal{C}^1(U, \mathbb{R}^m)$ with $Dh = Q \circ h$, for some $Q \in \mathcal{P}^{m\times n}$. Set $V_h = \mathcal{V}_h(I)$ where $I \subseteq \mathcal{P}$ is an ideal. The following are equivalent:

1.) The set $V_h \subseteq U$ is invariant for the differential equation $\dot{x} = f(x) = F(h(x))$.

2.) For all $p \in \mathcal{J}_h(V_h)$ the Lie derivative

$$L_{F\circ h}(p \circ h)$$

vanishes on V_h.

3.) Every $p \in \mathcal{J}_h(V_h)$ fulfils

$$Dp \cdot Q \cdot F \in \mathcal{J}_h(V_h). \tag{3}$$

Proof. By Lemma 4, statement 1.) implies 2.). Since

$$L_{F\circ h}(p \circ h) = (Dp \cdot Q \cdot F) \circ h,$$

the equivalence of 2.) and 3.) is clear by the definition of $\mathcal{J}_h(V_h)$. We show "3.) $\Rightarrow$ 1.)". Let $x^0 \in V_h$ be given and let $\varphi(\cdot, x^0)$ denote the solution to the IVP $\dot{x} = f(x)$, $x(0) = x^0$. We have $\mathcal{J}_h(V_h) = \langle p_1, ..., p_k\rangle$ for some $p_i \in \mathcal{P}$ and $V_h = \mathcal{V}_h(\mathcal{J}_h(V_h))$. Setting

$$y_i(t) := (p_i \circ h)(\varphi(t, x^0)),$$

the invariance of the set V_h is equivalent to $y_i(t) = 0$ for all $i \in \{1, ..., k\}$ and $t \in J(x^0)$. Assuming 3.), we have

$$Dp_i \cdot Q \cdot F = \sum_{j=1}^{k} l_{ij} \cdot p_j$$

for some $l_{ij} \in \mathcal{P}$. Hence, differentiation yields

$$\begin{aligned}\dot{y}_i(t) &= L_{F\circ h}(p_i \circ h)(\varphi(t, x^0)) = (Dp_i \cdot Q \cdot F)(h(\varphi(t, x^0))) \\ &= \sum_{j=1}^{k} l_{ij}(h(\varphi(t, x^0)))y_j(t).\end{aligned}$$

Defining the $k \times k$-matrix A by $A_{ij}(t) = l_{ij}(h(\varphi(t, x^0)))$, we obtain the IVP

$$\dot{y}(t) = A(t)y(t), \quad y(0) = 0.$$

Since A is a matrix of continuous functions on $J(x^0)$, we deduce $y \equiv 0$ on $J(x^0)$, which implies 1.). □

3.2 Computational aspects

To turn (3) into an algorithmic test for invariance, let $\mathcal{J}_h(V_h) = \langle p_1, ..., p_k\rangle$ for some suitable $p_i \in \mathcal{P}$ and set $\underline{p} := (p_1, ..., p_k)^T$. As a consequence of the linearity and the product rule of differentiation, the ideal membership (3) is equivalent to the existence of $\ell \in \mathcal{P}^{k^2}$ such that

$$D\underline{p} \cdot Q \cdot F = [p_1 I_k, \ldots, p_k I_k] \cdot \ell, \tag{4}$$

where I_k denotes the identity matrix of size k. This condition can be tested using Gröbner basis algorithms (see Greuel and Pfister (2008) for theoretical aspects), which are implemented in the computer algebra system SINGULAR (Decker et al. (2022)). Moreover, we can compute the module

$$\mathcal{M} := \pi_n(\ker \left[D\underline{p} \cdot Q,\ p_1 I_k,\ \ldots,\ p_k I_k\right]) \subseteq \mathcal{P}^n,$$

where π_n denotes the projection onto the first n components.

Corollary 9. In the situation of Theorem 8, the set V_h is invariant for $\dot{x} = F(h(x))$ if and only if $F \in \mathcal{M}$.

Hence, for a fixed function class via a suitable $\mathcal{C}^1$-function h and a set $V_h = \mathcal{V}_h(I)$, the module $\mathcal{M}$ describes all polynomial vectors F such that V_h is invariant for the system $\dot{x} = (F \circ h)(x)$.

The question remains how to find generators of $\mathcal{J}_h(V_h)$, where $V_h = \mathcal{V}_h(I)$ for an ideal $I \subseteq \mathcal{P}$. Observe that

$$\mathcal{J}_h(\mathcal{V}_h(I)) = \mathcal{J}(h(\mathcal{V}_h(I))) = \mathcal{J}(\mathcal{V}(I) \cap \mathrm{im}(h)).$$

We always have $\mathrm{im}(h) \subseteq \mathcal{V}(\mathcal{J}_h(U))$ which, by Lemma 2, implies

$$\mathcal{J}(\mathcal{V}(I) \cap \mathrm{im}(h)) \supseteq \mathcal{J}(\mathcal{V}(I + \mathcal{J}_h(U))) = \sqrt[\mathbb{R}]{I + \mathcal{J}_h(U)},$$

but we cannot guarantee equality here. Recall, that $\mathcal{J}_h(U)$ is the ideal of all algebraic dependencies between the components of h.

The following theorem gives us sufficient criteria for which we may derive an equality for $\mathcal{J}_h(\mathcal{V}_h(I))$.

Theorem 10. Let $h \in \mathcal{C}^1(U, \mathbb{R}^m)$ and $I \subseteq \mathcal{P}$ an ideal.

1.) If the image of h is a variety, that is, $\mathrm{im}(h) = \mathcal{V}(J)$ for some ideal $J \subseteq \mathcal{P}$, we obtain

$$\mathcal{J}_h(\mathcal{V}_h(I)) = \sqrt[\mathbb{R}]{I + J}.$$

In particular, $\mathcal{J}_h(U) = \sqrt[\mathbb{R}]{J}$.

2.) If $\mathcal{V}(I) \subseteq \mathrm{im}(h)$ or equivalently $h(\mathcal{V}_h(I)) = \mathcal{V}(I)$, then

$$\mathcal{J}_h(\mathcal{V}_h(I)) = \sqrt[\mathbb{R}]{I}.$$

In particular, $\mathcal{J}_h(U) \subseteq \sqrt[\mathbb{R}]{I}$.

Proof. Part 2.) follows directly from the consideration above this theorem. For part 1.), Lemma 2 yields

$$\begin{aligned}\mathcal{J}_h(\mathcal{V}_h(I)) &= \mathcal{J}(\mathcal{V}(I) \cap \mathrm{im}(h)) = \mathcal{J}(\mathcal{V}(I) \cap \mathcal{V}(J)) \\ &= \mathcal{J}(\mathcal{V}(I + J)) = \sqrt[\mathbb{R}]{I + J}\end{aligned}$$

and $\mathcal{J}_h(U) = \mathcal{J}(\mathrm{im}(h)) = \mathcal{J}(\mathcal{V}(J)) = \sqrt[\mathbb{R}]{J}$. □

Let us consider a few examples which meet the assumptions of part 1.) or part 2.), respectively.

Example 11. 1.) If $h \in \mathcal{C}^1(U, \mathbb{R}^m)$ is surjective, then

$$\mathrm{im}(h) = \mathbb{R}^m = \mathcal{V}(\{0\}).$$

In particular, no algebraic dependencies between the components of h occur, that is, $\mathcal{J}_h(U) = \{0\}$ and $\mathcal{J}_h(\mathcal{V}_h(I)) = \sqrt[\mathbb{R}]{I}$ for any ideal I, using Theorem 10. For example, the tangent function $h(x) = \tan(x)$ is surjective on $U = (-\frac{\pi}{2}, \frac{\pi}{2})$.

2.) It is easy to see that

$$\mathrm{im}((\sin(x), \cos(x))^T) = \mathcal{V}(Z_1^2 + Z_2^2 - 1).$$

So, this example fulfils the assumptions of the first part of Theorem 10 and for an arbitrary ideal I of $\mathbb{R}[Z_1, Z_2]$ we have $\mathcal{J}_h(\mathcal{V}_h(I)) = \sqrt[\mathbb{R}]{I + \langle Z_1^2 + Z_2^2 - 1\rangle}$. Furthermore, the observation

$$\mathrm{im}((x, \tfrac{1}{x})^T) = \mathcal{V}(Z_1 \cdot Z_2 - 1)$$

leads to $\mathcal{J}_h(\mathcal{V}_h(I)) = \sqrt[\mathbb{R}]{I + \langle Z_1 \cdot Z_2 - 1\rangle}$.

3.) On the other side we cannot write the image of $h(x_1, x_2) = (\mathrm{e}^{x_1}, \mathrm{e}^{x_2})$ as a zero set of polynomials. But for every ideal $I \subseteq \mathbb{R}[Z_1, Z_2]$ with $\mathcal{V}(I) \subseteq \mathbb{R}^2_{>0}$ we have $\mathcal{J}_h(\mathcal{V}_h(I)) = \sqrt[\mathbb{R}]{I}$ by part 2.) of Theorem 10.

Remark 12. In the situation of Theorem 8, let $q \in \mathcal{J}_h(U)$ be an algebraic dependence between the components of h. Then we also have

$$Dq \cdot Q \cdot F \in \mathcal{J}_h(U) \subseteq \mathcal{J}_h(V_h)$$

for every $F \in \mathcal{P}^n$. Moreover, the module $\mathcal{M}$ always contains the submodule $\mathcal{J}_h(U) \cdot \mathcal{P}^n$ whose elements F yield the zero vector field $F \circ h = 0$. Thus, we may also do calculations in the factor ring $\mathcal{P}/\mathcal{J}_h(U)$ instead of $\mathcal{P}$ if $\mathcal{J}_h(U)$ is known. Otherwise, we still can simplify condition (4) as follows. If one of the generators $p_1, ..., p_k$ of $\mathcal{J}_h(V_h)$ lies in $\mathcal{J}_h(U)$, say $p_k \in \mathcal{J}_h(U)$, we may omit p_k in the computation of the matrix $D\underline{p}$ in (4).

4. CONTROLLED INVARIANCE

Consider the control system

$$\dot{x}(t) = F(h(x(t))) + G(h(x(t)))u(t), \tag{5}$$

where $F \in \mathcal{P}^n$ and $G \in \mathcal{P}^{n \times l}$ are polynomial and $h \in \mathcal{C}^1(U, \mathbb{R}^m)$ is a $\mathcal{C}^1$-function defined on some open set $U \subseteq \mathbb{R}^n$. Let I be an ideal of $\mathcal{P}$. One says that $V_h = \mathcal{V}_h(I)$ is *controlled invariant* for system (5) if there exists an $\alpha \in \mathcal{P}^l$ such that the feedback law $u(t) = \alpha(h(x(t)))$ will lead to a closed loop system $\dot{x}(t) = \widetilde{F}(h(x(t)))$ with $\widetilde{F} := F + G\alpha$ for which V_h is invariant.

Based on the theory of the previous section, we derive the following recipe to check the controlled invariance of V_h for system (5):

First, we check if h fulfils the condition

$$Dh = Q \circ h, \quad \text{for some } Q \in \mathcal{P}^{m \times n}.$$

If it is possible, determine an ideal $J \subseteq \mathcal{P}$ such that

$$\mathrm{im}(h) = \mathcal{V}(J)$$

or validate that

$$\mathrm{im}(h) \subseteq \mathcal{V}(I).$$

Note, that system (5) might not be expressible by unique F, G, and h and different choices of these functions may also lead to different results in this validation process (see Remark 7).

Afterwards, determine generators for $\mathcal{J}_h(\mathcal{V}_h(I))$ (Theorem 10) and the module $\mathcal{M}$, generated by $m_1, ..., m_s \in \mathcal{P}^n$, using (4) and Remark 12. Then, the set V_h is controlled invariant for system (5) if and only if $\widetilde{F} = F + G\alpha \in \mathcal{M}$ for some $\alpha \in \mathcal{P}^l$ which is equivalent to

$$F \in \mathcal{M} + \mathrm{im}(G\,\cdot). \tag{6}$$

If this is the case, we may choose the feedback law

$$u(t) = \alpha(h(x(t)))$$

with $\alpha = \pi_l(y) \in \mathcal{P}^l$, where $\pi_l(\cdot)$ denotes the projection onto the last l components and $y \in \mathcal{P}^{s+l}$ is a solution of

$$F = [m_1, \ldots, m_s, -G] \cdot y.$$

The set of all such solutions $\alpha = \pi_l(y)$ is either empty or has the structure of an affine module.

We close this section applying the developed theory to a commonly known nonlinear control system:

Example 13. Consider the unicycle model

$$\dot{x}_1 = \cos(x_3) \cdot u_1$$
$$\dot{x}_2 = \sin(x_3) \cdot u_1$$
$$\dot{x}_3 = u_2.$$

The state variables x_1 and x_2 indicate the position of the unicycle in the plane $\mathbb{R}^2$ and x_3 defines the unicycle orientation. The speed of the unicycle is set by the input variable u_1 while u_2 sets the angular velocity of the unicycle orientation.

In order to apply the theory developed in this paper, we set $\mathcal{P} = R[Z_1, Z_2, Z_3, S, C]$, $U = \mathbb{R}^3$, and

$$h(x) = (x_1, x_2, x_3, \sin(x_3), \cos(x_3))^T.$$

Then the system is given by $\dot{x} = F(h(x)) + G(h(x)) \cdot u$, where

$$F = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{P}^3 \quad \text{and} \quad G = \begin{bmatrix} C & 0 \\ S & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{P}^{3 \times 2}.$$

The desired conditions on h are fulfilled, since we obtain $Dh = Q \circ h$ for

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & C \\ 0 & 0 & -S \end{bmatrix} \in \mathcal{P}^{5 \times 3}$$

and

$$\mathrm{im}(h) = \mathbb{R}^3 \times \mathrm{im}((\sin(x_3), \cos(x_3))^T) = \mathcal{V}(q)$$

for $q = S^2 + C^2 - 1 \in \mathcal{P}$.

Now consider the set $V_h = \mathcal{V}_h(p)$ defined by

$$p = Z_1^2 + Z_2^2 - 2Z_1 S + 2Z_2 C \in \mathcal{P}.$$

Note that for fixed $x_3 \in \mathbb{R}$, the set V_h contains the circle with radius 1 around $(\sin(x_3), -\cos(x_3))^T \in \mathbb{R}^2$. From Theorem 10 we derive

$$\mathcal{J}_h(V_h) = \sqrt[\mathbb{R}]{\langle p, q\rangle}$$

and using SINGULAR (Decker et al. (2022)), we compute the module $\mathcal{M}$:

$$\mathcal{M} = \langle \begin{bmatrix} C \\ S \\ 1 \end{bmatrix}, \begin{bmatrix} -Z_2 \\ Z_1 \\ 1 \end{bmatrix} \rangle + p \cdot \mathcal{P}^3 + q \cdot \mathcal{P}^3.$$

Setting the input to zero, i.e. $u = (0, 0)^T$, we obtain the trivial system $\dot{x} = 0$ for which every set is invariant. Apart from the zero solution, we can choose $\alpha = (1, 1)^T \in \mathcal{P}^2$.

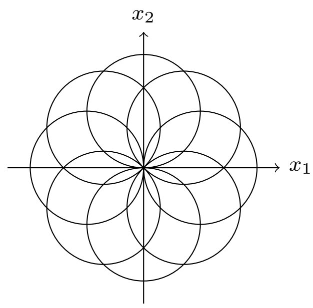

Fig. 1. Unicycle: Trajectories of the unicycle starting in $(0, 0, \frac{k\pi}{4})^T$, $k \in \{0, ..., 7\}$, with constant input $u_1 = 1$ and $u_2 = 1$, projected into the plane $\mathbb{R}^2$

Then the feedback law $u = \alpha(h(x)) = (1, 1)^T$ yields the closed loop system $\dot{x} = \widetilde{F}(h(x))$ with

$$\widetilde{F} = F + G\alpha = (C, S, 1)^T \in \mathcal{M},$$

such that V_h is controlled invariant by (6).

For illustration, we consider some trajectories of this closed loop system. We have $x^0 = (0, 0, \theta)^T \in V_h$ for all $\theta \in \mathbb{R}$ and obtain

$$\varphi(t, x^0) = (\sin(t + \theta) - \sin(\theta), -\cos(t + \theta) + \cos(\theta), t + \theta)^T$$

for all $t \in \mathbb{R}$. Figure (1) shows some of this trajectories projected into the plane $\mathbb{R}^2$. Since we chose constant speed ($u_1 = 1$) and constant angular velocity ($u_2 = 1$), the unicycle moves along different circles, when starting in $(x_1^0, x_2^0)^T = (0, 0)^T$ with different angles $x_3^0 = \theta \in \mathbb{R}$. But we also observe that the first two components of $\varphi(t, x^0)$ lie on the circle with radius 1 around $(\sin(t+\theta), -\cos(t+\theta))^T$ for all $t \in \mathbb{R}$, hence, $\varphi(t, x^0) \in V_h$ for all $t \in \mathbb{R}$, which illustrates the invariance of V_h.

5. CONCLUSION

In this paper, we have presented criteria to decide the controlled invariance of a set $V_h = \mathcal{V}_h(I)$ for a control system of the form (5). A natural question arising in this context is whether it is possible to choose an output feedback instead of a state feedback to render the set V_h invariant. In the case of polynomial systems, this has been studied in Yuno and Ohtsuka (2015) and Schilli et al. (2020) and their results may be generalised to the setting of this present paper. This is a topic for future work.

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