Comparison of approaches for boundary feedback control of hyperbolic systems

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The interest in boundary feedback control of multi-dimensional hyperbolic systems is increasing. In the present work we want to compare some of the recent results available in the literature.

1. Introduction

The stabilization of spatially one–dimensional systems of hyperbolic balance laws is a vivid subject attracting research interest in the mathematical as well as in the engineering community and we refer to the monographs [4, 8, 31, 33] for further references. The mentioned references also provide a comprehensive overview on related controllability problems. A particular focus has been put on problems modeled by the barotropic Euler equations and the shallow water equations which in one space dimension form a 2×2 hyperbolic system to model the temporal and spatial evolution of fluid flows including flows on networks. Analytical results concerning the boundary control of such systems have been studied in several articles, cf. [3, 20, 24, 27] for gas flow and for water flow we refer to [7,13,17,25,26,32]. One key aspect in the analysis is the Lyapunov function which is introduced as a weighted L^2 (or H^s) norm and which allows to estimate deviations from steady states, see e.g. [4]. Under rather general dissipative conditions the exponential decay of the Lyapunov function has been established for various problem formulations and we exemplary refer to [9–11, 14]. For a study on comparisons to other stability concepts we mention [12]. Stability with respect to a higher H^s-norm ($s \geq 2$) gives stability of the nonlinear system [4, 10]. Without aiming at completeness, we mention that recently the results have been extended to also deal with e.g. input-to-state stability [34], numerical methods have been discussed in [2, 22, 23] and for results concerning nonlocal hyperbolic partial differential equations (PDEs) see [15].

However, to the best of our knowledge the presented results are limited to the spatially one–dimensional case and multi–dimensional applications are ubiquitous. Thus there is

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a demand for a multi–dimensional extension of the available theory, yet results in the literature are rare due to the inherent difficulties and mostly focus on particular cases. Based on an application in metal forming processes, see [1, 29], we extended results to multi–dimensional hyperbolic balance laws. In [28] an ansatz for symmetric hyperbolic systems is presented. It relies on the feasibility of an associated linear matrix inequality (LMI). A specific system in two dimensions is discussed in [18] where a control problem for the shallow water equations is studied. There the authors take advantage of the structure of the system and show that the energy is non-increasing upon imposing suited boundary conditions. Just very recently a preprint was published where the boundary stabilization for two dimensional systems is studied using a different Lyapunov function, see [37]. There a stabilizing approach is introduced for a class of multi-dimensional systems with certain structural properties. A further recent result on multi–dimensional hyperbolic scalar conservation laws was presented in [36]. However, the goal of the mentioned paper is different from the one presented here. Here we focus on the relation between the works [28] and [37]. We want to provide a comparative study which hopefully helps to make further progress on this research subject.
The outline of this paper is as follows. In Section 2 we briefly summarize the approach presented in [28], followed by Section 3 where the key points of [37] are given. Then in Section 4 we compare both approaches highlighting commonalities and differences including a discussion of the Lyapunov function. The obtained results are emphasized by discussing the example for the Saint-Venant equations, also presented in [37], in Section 5.

2. The generic LMI approach

In [28] the subsequent system of hyperbolic PDEs is studied

$$\frac{\partial}{\partial t}\mathbf{w}(t,\mathbf{x}) + \sum_{k=1}^{d} \mathbf{A}^{(k)}(\mathbf{x})\frac{\partial}{\partial x_k}\mathbf{w}(t,\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{w}(t,\mathbf{x}) = 0,\ (t,\mathbf{x}) \in [0,T)\times\Omega \tag{2.1}$$

Here $\mathbf{w}(t,\mathbf{x}) \equiv (w_1(t,\mathbf{x}),\dots,w_n(t,\mathbf{x}))^T$ is the vector of unknowns and $\Omega \subset \mathbb{R}^d$ a bounded domain with piecewise C^1 smooth boundary $\partial\Omega$. Moreover, $\mathbf{A}^{(k)}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are sufficiently smooth and bounded $n \times n$ real matrices. The $\mathbf{A}^{(k)}(\mathbf{x})$ are in particular assumed to be symmetric. Usually symmetric hyperbolic systems appear with a general strictly positive definite symmetric matrix $\mathbf{A}^{(0)}$ in front of the time derivative. However, upon applying a suited variable transformation we may transform a symmetric hyperbolic system into the given form with the identity in front of the time derivative, cf. [5]. The assumption of symmetry is no major restriction since it includes all systems equipped with an additional conservation law, cf. [16, 21]. This includes most systems relevant for applications, see [6, 35].

It is further assumed that there exists a *feasible Lyapunov potential* $\overline{\mu}(\mathbf{x})$ such that

$$\overline{\mathbf{m}} := \nabla\overline{\mu}(\mathbf{x}) \quad \text{and} \quad \overline{\mathbf{A}}(\overline{\mathbf{m}}) := -\mathbf{Id} + \sum_{k=1}^{d} \overline{m}_k \mathbf{A}^{(k)} \geq 0. \tag{2.2}$$

Then

$$\mathbf{A}(\mathbf{m}) := C\mathbf{Id} + \sum_{k=1}^{d} m_k \mathbf{A}^{(k)} \leq 0,\ C \in \mathbb{R}_{>0} \tag{2.3}$$

holds for system (2.1) with $\mu(\mathbf{x}) = -C\overline{\mu}(\mathbf{x})$ and $\mathbf{m} = \nabla\mu(\mathbf{x})$. It is remarked in [28] that the LMI (2.3) can be modified if certain reminder terms, such as the coupling matrix $\mathbf{B}$, should be taken into account. Therefore we introduce with $\mathbf{B}^{sym} = \frac{1}{2}\left(\mathbf{B} + \mathbf{B}^T\right)$

$$\mathcal{R}(\mathbf{x}) := \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \mathbf{A}^{(k)}(\mathbf{x}) - 2\mathbf{B}^{sym}(\mathbf{x})$$

and demand

$$\mathbf{A}(\mathbf{m}) := C\mathbf{Id} + \mathcal{R}(\mathbf{x}) + \sum_{k=1}^{d} m_k(\mathbf{x})\mathbf{A}^{(k)}(\mathbf{x}) \leq 0,\ C \in \mathbb{R}_{>0}. \tag{2.4}$$

The LMI (2.4) then replaces (2.3). For example in the case of $\mathcal{R}(\mathbf{x}) < 0$ we could benefit from this additional term in order to find suited coefficients $\mathbf{m}$ as will be demonstrated below. The Lyapunov function is then defined as follows

$$L(t) = \int_{\Omega} \mathbf{w}(t,\mathbf{x})^T \mathbf{w}(t,\mathbf{x}) \exp(\mu(\mathbf{x}))\, \mathrm{d}\mathbf{x}. \tag{2.5}$$

It is then shown, that under suited boundary conditions the Lyapunov function decays exponentially.

3. The approach for SSC systems

In the following we briefly recall the main assumptions given in [37] and we refer to this work for further details. In [37] linear hyperbolic systems with constant coefficients in two dimensions are considered. These fulfill the *structural stability condition* (SSC) and we exemplary refer to [30, 38–41] for further reading. It is shown that such a system can be written in the following form

$$\frac{\partial}{\partial t}\mathbf{U}(t,x,y) + \overline{\mathbf{A}}^{(1)} \frac{\partial}{\partial x}\mathbf{U}(t,x,y) + \overline{\mathbf{A}}^{(2)} \frac{\partial}{\partial y}\mathbf{U}(t,x,y) = -\overline{\mathbf{B}}\mathbf{U}(t,x,y),\ (t,x,y) \in [0,T) \times \Omega \tag{3.1}$$

with $\mathbf{U} = (\overline{\mathbf{u}}, \overline{\mathbf{q}})^T$ where $\overline{\mathbf{u}} \in \mathbb{R}^{n-r}$ and $\overline{\mathbf{q}} \in \mathbb{R}^r$, with $0 < r < n$. Furthermore we have

$$\overline{\mathbf{B}} = \begin{pmatrix} \mathbf{0}_{(n-r)\times(n-r)} & \mathbf{0}_{(n-r)\times r} \\ \mathbf{0}_{r\times(n-r)} & \overline{\mathbf{e}} \end{pmatrix}$$

with $\overline{\mathbf{e}} \in \mathbb{R}^{r\times r}$ being invertible. The Jacobians are given by

$$\overline{\mathbf{A}}^{(i)} = \begin{pmatrix} \overline{\mathbf{a}}_i & \overline{\mathbf{b}}_i \\ \overline{\mathbf{c}}_i & \overline{\mathbf{d}}_i \end{pmatrix}, \, i = 1, 2$$

with $\overline{\mathbf{a}}_i, \overline{\mathbf{b}}_i \in \mathbb{R}^{(n-r)\times(n-r)}$ and $\overline{\mathbf{c}}_i, \overline{\mathbf{d}}_i \in \mathbb{R}^{r\times r}$ According to [37] system (3.1) has the following properties

(i) There exists a symmetric positive definite matrix

$$\mathbf{A}^{(0)} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0}_{(n-r)\times r} \\ \mathbf{0}_{r\times(n-r)} & \mathbf{X}_2 \end{pmatrix}$$

with $\mathbf{X}_1 \in \mathbb{R}^{(n-r)\times(n-r)}$ and $\mathbf{X}_2 \in \mathbb{R}^{r\times r}$, such that $\mathbf{A}^{(0)}\overline{\mathbf{A}}^{(1)}$ and $\mathbf{A}^{(0)}\overline{\mathbf{A}}^{(2)}$ are symmetric.

(ii) $\mathbf{X}_2\overline{\mathbf{e}} + \overline{\mathbf{e}}^T\mathbf{X}_2$ is positive definite.

(iii) There exist real numbers α and β such that $\overline{\mathbf{a}} := \alpha\overline{\mathbf{a}}_1 + \beta\overline{\mathbf{a}}_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ has only negative eigenvalues.

Following [37] the properties (i) and (ii) are implied by the SSC. The Lyapunov function is then defined as follows

$$L(t) = \int_\Omega \lambda(x, y)\mathbf{U}(t, x, y)^T\mathbf{A}^{(0)}\mathbf{U}(t, x, y)\,\mathrm{d}\mathbf{x}. \tag{3.2}$$

with $\lambda(x, y) = K + \alpha x + \beta y$. Note that the positivity of the weight function $\lambda(x, y)$ is an issue needed to be resolved by $K > 0$ with respect to the range of x and y, i.e. the domain Ω. It is then also shown, that under suited boundary conditions the Lyapunov function decays exponentially.

4. Comparison

In the subsequent part we comment on these three properties and their relation to the assumptions in [28].
Property (i) states the existence of a symmetrizer for system (3.1). In particular $\mathbf{X}_1$ and $\mathbf{X}_2$ are positive definite symmetric matrices. Multiplying (3.1) from the left with $\mathbf{A}^{(0)}$ gives

$$\begin{aligned} \mathbf{A}^{(0)}\frac{\partial}{\partial t}\mathbf{U}(t,x,y) + \mathbf{A}^{(0)}\overline{\mathbf{A}}^{(1)}\frac{\partial}{\partial x}\mathbf{U}(t,x,y) + \mathbf{A}^{(0)}\overline{\mathbf{A}}^{(2)}\frac{\partial}{\partial y}\mathbf{U}(t,x,y) &= -\mathbf{A}^{(0)}\overline{\mathbf{B}}\mathbf{U}(t,x,y) \\ \Leftrightarrow: \quad \mathbf{A}^{(0)}\frac{\partial}{\partial t}\mathbf{U}(t,x,y) + \tilde{\mathbf{A}}^{(1)}\frac{\partial}{\partial x}\mathbf{U}(t,x,y) + \tilde{\mathbf{A}}^{(2)}\frac{\partial}{\partial y}\mathbf{U}(t,x,y) &= -\tilde{\mathbf{B}}\mathbf{U}(t,x,y) \end{aligned}$$

with $\tilde{\mathbf{A}}^{(i)}$ being symmetric. Introducing the variables $\mathbf{w} := (\mathbf{A}^{(0)})^{\frac{1}{2}}\mathbf{U}$, $\mathbf{w} \equiv (\mathbf{u}, \mathbf{q})^T$ and multiplying from left by $(\mathbf{A}^{(0)})^{-\frac{1}{2}}$ we can transform the system to

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{w}(t,x,y) + \mathbf{A}^{(1)}\frac{\partial}{\partial x}\mathbf{w}(t,x,y) + \mathbf{A}^{(2)}\frac{\partial}{\partial y}\mathbf{w}(t,x,y) = -\mathbf{B}\mathbf{w}(t,x,y) \\ \text{with} \quad &\mathbf{A}^{(i)} := (\mathbf{A}^{(0)})^{-\frac{1}{2}}\tilde{\mathbf{A}}^{(i)}(\mathbf{A}^{(0)})^{-\frac{1}{2}} \\ \text{and} \quad &\mathbf{B} := (\mathbf{A}^{(0)})^{-\frac{1}{2}}\tilde{\mathbf{B}}(\mathbf{A}^{(0)})^{-\frac{1}{2}} \end{aligned}$$

with $\mathbf{A}^{(i)}$ being symmetric. The transforms are directly applied to the respective block matrices, i.e.

$$\begin{aligned} &\mathbf{a}_i := \mathbf{X}_1^{\frac{1}{2}}\overline{\mathbf{a}}_i\mathbf{X}_1^{-\frac{1}{2}}, \quad \mathbf{b}_i := \mathbf{X}_1^{\frac{1}{2}}\overline{\mathbf{b}}_i\mathbf{X}_1^{-\frac{1}{2}}, \quad \mathbf{c}_i := \mathbf{X}_2^{\frac{1}{2}}\overline{\mathbf{c}}_i\mathbf{X}_2^{-\frac{1}{2}}, \quad \mathbf{d}_i := \mathbf{X}_2^{\frac{1}{2}}\overline{\mathbf{d}}_i\mathbf{X}_2^{-\frac{1}{2}} \\ \text{and} \quad &\mathbf{e} := \mathbf{X}_2^{\frac{1}{2}}\overline{\mathbf{e}}\mathbf{X}_2^{-\frac{1}{2}}. \end{aligned}$$

The stated properties (ii) and (iii) are also transferred to the transformed system, which can be seen as follows. Property (ii) states that $\mathbf{X}_2\overline{\mathbf{e}} + \overline{\mathbf{e}}^T\mathbf{X}_2 > 0$. From this we yield for $\mathbf{v} \in \mathbb{R}^r \setminus \{\mathbf{0}\}$

$$\begin{aligned} 0 &< \mathbf{v}^T\left(\mathbf{X}_2\overline{\mathbf{e}} + \overline{\mathbf{e}}^T\mathbf{X}_2\right)\mathbf{v} \\ &= \mathbf{v}^T\left(\mathbf{X}_2\overline{\mathbf{e}}\mathbf{X}_2^{-\frac{1}{2}}\mathbf{X}_2^{\frac{1}{2}} + \mathbf{X}_2^{\frac{1}{2}}\mathbf{X}_2^{-\frac{1}{2}}\overline{\mathbf{e}}^T\mathbf{X}_2\right)\mathbf{v} \\ &= \mathbf{v}^T\mathbf{X}_2^{\frac{1}{2}}\left(\mathbf{X}_2^{\frac{1}{2}}\overline{\mathbf{e}}\mathbf{X}_2^{-\frac{1}{2}} + \mathbf{X}_2^{-\frac{1}{2}}\overline{\mathbf{e}}^T\mathbf{X}_2^{\frac{1}{2}}\right)\mathbf{X}_2^{\frac{1}{2}}\mathbf{v} \\ &= \mathbf{y}^T\left(\mathbf{e} + \mathbf{e}^T\right)\mathbf{y} \quad \text{with} \quad \mathbf{y} = \mathbf{X}_2^{\frac{1}{2}}\mathbf{v}. \end{aligned}$$

So for the tranformed system property (ii) states $\mathbf{e} + \mathbf{e}^T > 0$.
According to property (iii) there exist real numbers α and β such that $\overline{\mathbf{a}} := \alpha\overline{\mathbf{a}}_1 + \beta\overline{\mathbf{a}}_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ has only negative eigenvalues. We now study the similarity transform

$$\begin{aligned} \mathbf{X}_1^{\frac{1}{2}}\overline{\mathbf{a}}\mathbf{X}_1^{-\frac{1}{2}} &= \mathbf{X}_1^{\frac{1}{2}}\left(\alpha\overline{\mathbf{a}}_1 + \beta\overline{\mathbf{a}}_2\right)\mathbf{X}_1^{-\frac{1}{2}} \\ &= \alpha\mathbf{X}_1^{\frac{1}{2}}\overline{\mathbf{a}}_1\mathbf{X}_1^{-\frac{1}{2}} + \beta\mathbf{X}_1^{\frac{1}{2}}\overline{\mathbf{a}}_2\mathbf{X}_1^{-\frac{1}{2}} \\ &= \alpha\mathbf{a}_1 + \beta\mathbf{a}_2 =: \mathbf{a}. \end{aligned}$$

Since a similarity transform leaves the eigenvalues unchanged, $\mathbf{a}$ has only negative real eigenvalues. In summary we can state, that property (i) stated in [37] allows us to write the studied system in the form

$$\frac{\partial}{\partial t}\mathbf{w}(t,x,y) + \mathbf{A}^{(1)}\frac{\partial}{\partial x}\mathbf{w}(t,x,y) + \mathbf{A}^{(2)}\frac{\partial}{\partial y}\mathbf{w}(t,x,y) + \mathbf{B}\mathbf{w}(t,x,y) = 0. \tag{4.1}$$

This is in particular a symmetric hyperbolic system (3.1) studied in [28].
We study property (ii) and (iii) in terms of the transformed symmetric system (4.1), i.e.,

$\mathbf{e}+\mathbf{e}^T > 0$ and $\mathbf{a}$ has only negative real eigenvalues for suited $\alpha, \beta \in \mathbb{R}$. To understand the implication of property (ii) and (iii) with respect to [28] we study the proof of the main theorem given therein. The first key step is to differentiate the Lyapunov function with respect to time and rearrange the terms, such that the derivative can be written as the sum of a boundary integral $\mathcal{B}(t)$ and a volume integral $\mathcal{I}(t)$. Since [37] deals with constant coefficient matrices we have

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) = -\underbrace{\int_{\partial\Omega} \mathcal{A}(t,x,y)\cdot \mathbf{n}(x,y)\,\mathrm{d}\mathbf{x}}_{=:\mathcal{B}(t)}$$
$$+\underbrace{\int_{\Omega} \mathbf{w}^T(t,x,y)\left[\sum_{k=1}^{2} m_k \mathbf{A}^{(k)} - 2\mathbf{B}\right]\mathbf{w}(t,x,y)\exp(\mu(x,y))\,\mathrm{d}\mathbf{x}}_{=:\mathcal{I}(t)}.$$

Without loss of generality we replace $\mathbf{B}$ by $\mathbf{B}^{sym} = \frac{1}{2}\left(\mathbf{B}+\mathbf{B}^T\right)$. The case of an arbitrary matrix $\mathbf{B}^{sym}$ is excluded in [37] due to the SSC. We hence follow [28] and estimate the quadratic form as follows

$$-2\mathbf{w}^T(t,x,y)\mathbf{B}\mathbf{w}(t,x,y) = -2\mathbf{w}^T(t,x,y)\mathbf{B}^{sym}\mathbf{w}(t,x,y) \le C_B\|\mathbf{w}(t,\mathbf{x})\|_2^2,\ C_B \in \mathbb{R}_{>0}.$$

Since (2.3) is assumed to hold in [28] we can establish the exponential decay. Considering the case $\mathbf{B}^{sym} > 0$ we can simply estimate

$$-2\mathbf{w}^T(t,x,y)\mathbf{B}\mathbf{w}(t,x,y) = -2\mathbf{w}^T(t,x,y)\mathbf{B}^{sym}\mathbf{w}(t,x,y) \le 0.$$

if (2.3) holds. However as noted before, it is also possible to benefit from the coupling term and relax (2.3) to (2.4). This would then correspond to [37] where the SSC is used and the subsequent idea is applied. Using $\mathbf{w} = (\mathbf{u},\mathbf{q})^T$ we yield for the following quadratic form

$$\mathbf{w}^T\mathbf{A}^{(i)}\mathbf{w} = \mathbf{u}^T\mathbf{a}_i\mathbf{u} + \mathbf{u}^T\mathbf{b}_i\mathbf{q} + \mathbf{q}^T\mathbf{c}_i\mathbf{u} + \mathbf{q}^T\mathbf{d}_i\mathbf{q}.$$

During the estimate of the Lyapunov function, the first term of this sum is estimated according to property (iii). Since the term

$$-2\mathbf{w}^T\mathbf{B}\mathbf{w} = -2\mathbf{w}^T\mathbf{B}^{sym}\mathbf{w} = -2\mathbf{q}^T\left(\mathbf{e}+\mathbf{e}^T\right)\mathbf{q}$$

is scaled with a constant $K > 0$ due to the Lyapunov function, all remaining terms are consumed by the coupling term. This can also be in adopted to the approach in [28] as follows. We start from

$$\mathbf{A}(\mathbf{m}) := C\mathbf{Id} - 2\mathbf{B}^{sym} + \sum_{k=1}^{d} m_k\mathbf{A}^{(k)}k = C\mathbf{Id} + \frac{1}{K}\left(-2K\mathbf{B}^{sym} + \sum_{k=1}^{d}\tilde{m}_k\mathbf{A}^{(k)}\right).$$

with $\tilde{m}_k = Km_k$. Now assume the (ii) and (iii) hold. With $\tilde{m}_1 = \alpha$ and $\tilde{m}_2 = \beta$ we then yield for a system with the studied particular structure

$$\begin{aligned}
\mathbf{w}^T\mathbf{A}(\mathbf{m})\mathbf{w} &= C\|\mathbf{w}\|_2^2 + \frac{1}{K}\left(-K\mathbf{q}^T\left(\mathbf{e}+\mathbf{e}^T\right)\mathbf{q}^T + \alpha\mathbf{w}^T\mathbf{A}^{(1)}\mathbf{w}^T + \beta\mathbf{w}^T\mathbf{A}^{(2)}\mathbf{w}^T\right) \\
&= C\|\mathbf{w}\|_2^2 + \frac{1}{K}\Big(- K\mathbf{q}^T\left(\mathbf{e}+\mathbf{e}^T\right)\mathbf{q}^T + \underbrace{\mathbf{u}^T\left(\alpha\mathbf{a}_1+\beta\mathbf{a}_2\right)\mathbf{u}}_{\overset{(ii)}{<}0} \\
&\quad + \mathbf{u}^T\left(\alpha\mathbf{b}_1+\beta\mathbf{b}_2\right)\mathbf{q} + \mathbf{q}^T\left(\alpha\mathbf{c}_1+\beta\mathbf{c}_2\right)\mathbf{u} + \mathbf{q}^T\left(\alpha\mathbf{d}_1+\beta\mathbf{d}_2\right)\mathbf{q}\Big)
\end{aligned}$$

Now $K > 0$ is chosen such that

$$\mathbf{u}^T\left(\alpha\mathbf{b}_1+\beta\mathbf{b}_2\right)\mathbf{q} + \mathbf{q}^T\left(\alpha\mathbf{c}_1+\beta\mathbf{c}_2\right)\mathbf{u} + \mathbf{q}^T\left(\alpha\mathbf{d}_1+\beta\mathbf{d}_2\right)\mathbf{q} - K\mathbf{q}^T\left(\mathbf{e}+\mathbf{e}^T\right)\mathbf{q} < 0$$

Thus there exists a constant $C > 0$ such that (2.4) holds with $m_1 = \alpha/K$ and $m_2 = \beta/K$. With this we can now discuss the Lyapunov function in more detail and in particular the different weight functions. According to [37] we have $\lambda(x,y) = K + \alpha x + \beta y$ and according to [28] we have for the current situation

$$f(x,y) := \exp(\mu(x,y)) \quad \text{with} \quad \mu(x,y) = m_1 x + m_2 y = \frac{\alpha}{K}x + \frac{\beta}{K}y + \ln(K).$$

The additional term $\ln(K)$ is just added for proper scaling and does not affect the results of [28]. For the exponential weight function we thus obtain by Taylor expansion near $(x_0, y_0) = (0,0)$

$$\begin{aligned}
f(x,y) &= f(x_0,y_0) + \nabla f(x_0,y_0)^T\begin{pmatrix} x-x_0 \\ y-y_0\end{pmatrix} + \frac{1}{2}(x-x_0, y-y_0)\mathbf{D}^2 f(\xi_x,\xi_y)\begin{pmatrix} x-x_0 \\ y-y_0\end{pmatrix} \\
&= \exp(\mu(0,0)) + (m_1,m_2)\exp(\mu(0,0))\begin{pmatrix} x \\ y\end{pmatrix} \\
&\quad + \underbrace{\frac{1}{2}(x,y)\begin{pmatrix} m_1^2 & m_1 m_2 \\ m_1 m_2 & m_2^2\end{pmatrix}\exp(\mu(\xi_x,\xi_y))\begin{pmatrix} x \\ y\end{pmatrix}}_{\geq 0} \\
&\geq K + K(m_1 x + m_2 y) = \lambda(x,y).
\end{aligned}$$

Moreover, by choosing the weight functions as stated above the Lyapunov functions are related by

$$\int_\Omega \mathbf{w}(t,x,y)^T\mathbf{w}(t,x,y)\exp(\mu(x,y))\,\mathrm{d}\mathbf{x} \geq \int_\Omega \mathbf{w}(t,x,y)^T\mathbf{w}(t,x,y)\lambda(x,y)\,\mathrm{d}\mathbf{x}. \tag{4.2}$$

This implies in the case of exponential decay in [28]

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}L(t) &\leq -C\int_\Omega \mathbf{w}(t,x,y)^T\mathbf{w}(t,x,y)\exp(\mu(x,y))\,\mathrm{d}\mathbf{x} \\
&\leq -C\int_\Omega \mathbf{w}(t,x,y)^T\mathbf{w}(t,x,y)\lambda(x,y)\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Concerning the boundary condition there is no significant structural difference between the approaches presented in [37] and [28]. Due to the symmetry of $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ the pencil matrix of the system

$$\mathbf{A}^*(\nu) := \sum_{k=1}^{d} \nu_k \mathbf{A}^{(k)}$$

is symmetric and therefore diagonizable on the boundary. In particular we have

$$\mathbf{w}^T \mathbf{A}^*(\mathbf{n})\mathbf{w} = \mathbf{w}^T \left(n_1 \mathbf{A}^{(1)} + n_2 \mathbf{A}^{(2)}\right) \mathbf{w} = \mathbf{w}^T \mathbf{T}\mathbf{T}^T \left(n_1 \mathbf{A}^{(1)} + n_2 \mathbf{A}^{(2)}\right) \mathbf{T}\mathbf{T}^T \mathbf{w} = \mathbf{v}^T \mathbf{\Lambda} \mathbf{v}$$

where $\mathbf{n}$ denotes the outward pointing normal of $\partial\Omega$, $\mathbf{T}$ is the orthogonal transformation matrix, $\mathbf{v} = \mathbf{T}^T\mathbf{w}$ and $\mathbf{\Lambda}$ is the diagonal matrix with the eigenvalues the system.

5. Application to the Saint-Venant Equations

In [37] an example for the Saint-Venant equations is presented and we want to give a short review here. The linearized system in terms of the unknowns $\mathbf{w} = (\tilde{h}, w, v)^T$ is given by

$$\frac{\partial}{\partial t}\mathbf{w}(t,x,y) + \mathbf{A}^{(1)}\frac{\partial}{\partial x}\mathbf{w}(t,x,y) + \mathbf{A}^{(2)}\frac{\partial}{\partial y}\mathbf{w}(t,x,y) = -\mathbf{B}\mathbf{w}(t,x,y) \tag{5.1}$$

with $\mathbf{A}^{(1)} = \begin{pmatrix} w^* & \sqrt{gH^*} & 0 \\ \sqrt{gH^*} & w^* & 0 \\ 0 & 0 & w^* \end{pmatrix}$, $\mathbf{A}^{(2)} = \begin{pmatrix} v^* & 0 & \sqrt{gH^*} \\ 0 & v^* & 0 \\ \sqrt{gH^*} & 0 & v^* \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & -l \\ 0 & l & k \end{pmatrix}$.

where $\tilde{h} = \sqrt{\frac{g}{H^*}}h$ is a scaled perturbed height, w the perturbed velocity in x-direction and v the perturbed velocity in y-direction. The quantity g is the gravity constant, $k > 0$ is the viscous drag coefficient and $l > 0$ is the Coriolis coefficient. Further $(H^*, w^*, v^*)^T$ is a steady state in terms of the height, the velocity in x-direction and the velocity in y-direction, respectively. We want to apply the approach given in [28] and thus want to show that there exist real numbers $\overline{m}_1$ and $\overline{m}_2$ such that

$$\mathbf{A}(\overline{\mathbf{m}}) := -\mathbf{Id} + \overline{m}_1\mathbf{A}^{(1)} + \overline{m}_2\mathbf{A}^{(2)} \geq 0$$

holds. We have

$$\mathbf{A}(\overline{\mathbf{m}}) = \begin{pmatrix} -1 + \overline{m}_1 w^* + \overline{m}_2 v^* & \overline{m}_1\sqrt{gH^*} & \overline{m}_2\sqrt{gH^*} \\ \overline{m}_1\sqrt{gH^*} & -1 + \overline{m}_1 w^* + \overline{m}_2 v^* & 0 \\ \overline{m}_2\sqrt{gH^*} & 0 & -1 + \overline{m}_1 w^* + \overline{m}_2 v^* \end{pmatrix}.$$

A sufficient criterion would be to choose $\overline{m}_1$ and $\overline{m}_2$ such that $\mathbf{A}(\overline{\mathbf{m}})$ becomes strongly diagonal dominant, i.e., the diagonal entries are strict positive and larger than the sum of the absolute row entries. Due to the special structure of the matrix we hence look for coefficients with the property that

$$-1 + \overline{m}_1 w^* + \overline{m}_2 v^* = \sigma\left(|\overline{m}_1| + |\overline{m}_2|\right)\sqrt{gH^*},\ \sigma \in (1, \infty).$$

For $\mathbf{B}$ we have that $\mathbf{B}^{sym} > 0$ which in the approach given in [28] leads to an estimate by zero for the correpsonding term. However, as remarked before we can also make use of $\mathbf{B}^{sym}$ for the estimate as in (2.4) with $\tilde{C} > 0$. We reformulate the inequality to introduce a further scaling for $\mathbf{B}^{sym}$ with $\chi > 0$

$$
\begin{aligned}
& 0 \geq \tilde{\mathbf{A}}(\mathbf{m}) := \tilde{C}\mathbf{Id} - 2\mathbf{B}^{sym} + \sum_{k=1}^{d} \tilde{m}_k \mathbf{A}^{(k)} \\
& \qquad = \frac{1}{\chi}\left(\chi\tilde{C}\mathbf{Id} - 2\chi\mathbf{B}^{sym} + \sum_{k=1}^{d} \chi\tilde{m}_k \mathbf{A}^{(k)}\right) \\
\Leftrightarrow \quad & 0 \geq \mathbf{A}(\mathbf{m}) := C\mathbf{Id} - 2\chi\mathbf{B}^{sym} + \sum_{k=1}^{d} m_k \mathbf{A}^{(k)}
\end{aligned}
$$

We have

$$
\mathbf{A}(\mathbf{m}) = \begin{pmatrix} C + m_1 w^* + m_2 v^* & m_1\sqrt{gH^*} & m_2\sqrt{gH^*} \\ m_1\sqrt{gH^*} & C - 4\chi k + m_1 w^* + m_2 v^* & 0 \\ m_2\sqrt{gH^*} & 0 & C - 4\chi k + m_1 w^* + m_2 v^* \end{pmatrix}.
$$

In [37] the situation is considered with the vertical velocity in the steady state $v^* = 0$ and with the horizontal velocity in the steady state $0 < w^* < \sqrt{gH^*}$, respectively. Thus we have with $m := m_1$

$$
\mathbf{A}(m) = \begin{pmatrix} C + mw^* & m\sqrt{gH^*} & 0 \\ m\sqrt{gH^*} & C - 4\chi k + mw^* & 0 \\ 0 & 0 & C - 4\chi k + mw^* \end{pmatrix}.
$$

We obtain the following conditions on m and χ to have $\mathbf{A}(m) \leq 0$

(i) $C + mw^* \leq 0 \quad \Leftrightarrow \quad m \leq -\frac{C}{w^*}$

(ii) Due to $4\chi k > 0$ (i) also implies $0 \geq C + mw^* > C - 4\chi k + mw^*$

(iii) For the second principle minor we yield

$$
(C + mw^*)(C - 4\chi k + w^*) - m^2 gH^* = (C + mw^*)^2 - 4\chi k \underbrace{(C + mw^*)}_{\leq 0} - m^2 gH^*
$$

and thus there exists a $\chi > 0$ such that this expression becomes positive. Hence for $m < -\frac{C}{w^*}$ and $\chi > 0$ large enough $\mathbf{A}(m) \leq 0$ holds. In the case of $m = -1$ and $\chi = 2L$ these conditions impose a restriction on the decay rate, i.e.

$$
w^* \geq C \geq w^* - \frac{gH^*}{8Lk}.
$$

It remains to study the boundary term and prescribe boundary conditions, such that $\mathcal{BC} \geq 0$. To this end we will make use of the eigenstructure of the system given in the appendix for the readers convenience. Following the example of [37] the boundary of the domain Ω is given by $\partial\Omega = [0, L] \times [0, 1]$.For the boundary integral we have

$$\begin{aligned}\mathcal{BC} &= \int_{\partial\Omega} \mathbf{w}^T \mathbf{A}^*(\mathbf{n})\mathbf{w} \exp(\mu(x, y))\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathbf{v}^T \mathbf{\Lambda}^*(\mathbf{n})\mathbf{v} \exp(\mu(x, y))\,\mathrm{d}\mathbf{x} \\ &= \sum_{i=1}^{3} \int_{\partial\Omega} v_i^2 \lambda_i(\mathbf{n}) \exp(\mu(x, y))\,\mathrm{d}\mathbf{x}\end{aligned}$$

where $\mathbf{v} = \mathbf{T}^T(\mathbf{n})\mathbf{w}$ is calculated according to (A.4) and $\mathbf{n}$ denotes the outward pointing normal of the boundary $\partial\Omega$. Now we identify the controllable and uncontrollable parts of the boundary, i.e. for $i = 1, 2, 3$

$$\begin{aligned}\Gamma_i^+ &:= \{\mathbf{x} \in \partial\Omega \,|\, \lambda_i(\mathbf{x}, \mathbf{n}(\mathbf{x})) \geq 0\}, \\ \Gamma_i^- &:= \{\mathbf{x} \in \partial\Omega \,|\, \lambda_i(\mathbf{x}, \mathbf{n}(\mathbf{x})) < 0\}.\end{aligned}$$

These are given by

$$\begin{aligned}\Gamma_1^- &= \partial\Omega, & \Gamma_1^+ &= \emptyset, \\ \Gamma_2^- &= \{0\} \times (0, 1), & \Gamma_2^+ &= \partial\Omega \setminus \Gamma_2^-, \\ \Gamma_3^- &= \emptyset, & \Gamma_3^+ &= \partial\Omega.\end{aligned} \tag{5.2}$$

Let us denote the general controls by $\varphi_1(t, x, y)$ for the first component and $\psi_2(t, y)$ for the second component, respectively. These have to be chosen such that

$$\begin{aligned}\mathcal{BC} = &\underbrace{\int_{\partial\Omega} \varphi_1(t, x, y)^2 (\mathbf{n}(x, y)w^* - \sqrt{gH^*}) \exp(\mu(x, y))\,\mathrm{d}\mathbf{x} - w^* \int_0^1 \psi_2(t, y)^2 \exp(\mu(0, y))\,\mathrm{d}y}_{\leq 0} \\ &\underbrace{+ w^* \int_{\partial\Omega\setminus\Gamma_2^-} v_2^2 \mathbf{n}(x, y) \exp(\mu(x, y))\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} v_3^2 (\mathbf{n}(x, y)w^* + \sqrt{gH^*}) \exp(\mu(x, y))\,\mathrm{d}\mathbf{x}}_{\geq 0} \geq 0.\end{aligned}$$

Thus one sufficient ansatz would be to choose the controls according to

$$\begin{aligned}&\int_{\partial\Omega} \varphi_1(t, x, y)^2 (\sqrt{gH^*} - \mathbf{n}(x, y)w^*) \exp(\mu(x, y))\,\mathrm{d}\mathbf{x} \leq \int_{\partial\Omega} v_3^2 (\mathbf{n}(x, y)w^* + \sqrt{gH^*}) \exp(\mu(x, y))\,\mathrm{d}\mathbf{x} \\ &\text{and} \quad \int_0^1 \psi_2(t, y)^2 \exp(\mu(0, y))\,\mathrm{d}y \leq \int_{\partial\Omega\setminus\Gamma_2^-} v_2^2 \mathbf{n}(x, y) \exp(\mu(x, y))\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Other choices are of course possible, as it is detailed out in [37] and we refer to this reference for the precise calculation.

Note that the homogeneous part of system (5.1) is exactly the system studied in [18]. In [18] a Lyapunov function of the form

$$L(t) = \int_{\Omega} \mathbf{w}^T \mathcal{L} \mathbf{w} \, \mathrm{d}\mathbf{x} \tag{5.3}$$

is used with $\mathcal{L} := \mathrm{diag}(1, H^*/g, H^*/g)$. It is shown under suited assumptions that the system can be stabilized near $\mathbf{0}$ in terms of $L(t) \leq CL(0)$ with $C > 0$. So while in [18] the boundedness of the Lyapunov function is shown, the approaches presented in [28] and [37] are able to establish exponential decay. It should be remarked that in [19] the same authors established exponential decay in a particular situation and therefore used a reduced one dimensional system.

6. Summary

In the present work we compare novel approaches for stabilizing multi–dimensional hyperbolic systems using a boundary feedback control. Both provide exponential decay under suited conditions. The assumptions are briefly reviewed, compared and discussed. It is shown that the approach given in [28] is always applicable when the prerequisite of [37] hold. Moreover, we highlight the relation between the Lyapunov functions used in these works by showing that a linearization of the weight function used in [28] yields the one used in [37]. Finally we discuss the very relevant example of the Saint-Venant equations to highlight our results. With this work we hope to foster the study of control problems for multi–dimensional hyperbolic systems. The present work thus highlights the applicability of the results obtained in [28] which is now known to be suitable for the Euler equations, the Saint-Venant equations and systems satisfying the SSC condition. It will be interesting to study other examples in complex domains.

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A. Eigenstructure of the Saint-Venant Equations

Subsequently we provide the detailed calculation for the eigenstructure of the considered system (5.1). The system matrix is given by

$$\mathbf{A}^*(\nu) = \nu_1 \mathbf{A}^{(1)} + \nu_2 \mathbf{A}^{(2)}, \ \nu \in \mathbb{S}^2 \tag{A.1}$$

and thus we have with $v^* = 0$

$$\mathbf{A}^*(\nu) = \begin{pmatrix} \nu_1 w^* & \nu_1\sqrt{gH^*} & \nu_2\sqrt{gH^*} \\ \nu_1\sqrt{gH^*} & \nu_1 w^* & 0 \\ \nu_2\sqrt{gH^*} & 0 & \nu_1 w^* \end{pmatrix}.$$

The characteristic polynomial is given by

$$\begin{aligned} \chi(\lambda) &= (\lambda - \nu_1 w^*)^3 - (\lambda - \nu_1 w^*)\nu_1^2 gH^* - (\lambda - \nu_1 w^*)\nu_2^2 gH^* \\ &= (\lambda - \nu_1 w^*)\left((\lambda - \nu_1 w^*)^2 - \left(\nu_1^2 + \nu_2^2\right) gH^*\right) = (\lambda - \nu_1 w^*)\left((\lambda - \nu_1 w^*)^2 - gH^*\right) \end{aligned}$$

and hence we yield the following eigenvalues

$$\begin{aligned} &\lambda_1(\nu) = \nu_1 w^* - \sqrt{gH^*},\ \lambda_2(\nu) = \nu_1 w^*,\ \lambda_3(\nu) = \nu_1 w^* + \sqrt{gH^*} \\ \text{with} \quad &\lambda_1(\nu) < \lambda_2(\nu) < \lambda_3(\nu). \end{aligned} \tag{A.2}$$

The corresponding right eigenvectors are thus obtained to be

$$\mathbf{R}_1(\nu) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\nu_1 \\ -\nu_2 \end{pmatrix}, \quad \mathbf{R}_2(\nu) = \begin{pmatrix} 0 \\ -\nu_2 \\ \nu_1 \end{pmatrix} \quad \text{and} \quad \mathbf{R}_3(\nu) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \nu_1 \\ \nu_2 \end{pmatrix}. \tag{A.3}$$

We hence have the following transformation matrix

$$\mathbf{T}(\nu) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ -\nu_1 & -\sqrt{2}\nu_2 & \nu_1 \\ -\nu_2 & \sqrt{2}\nu_1 & \nu_2 \end{pmatrix}$$

with $\mathbf{\Lambda}^*(\nu) = \mathbf{T}^T(\nu)\mathbf{A}^*(\nu)\mathbf{T}(\nu)$. The transformation of the state vector $\mathbf{w} = (\tilde{h}, w, v)^T$ is given by

$$\mathbf{v} = \mathbf{T}^T(\nu)\mathbf{w} = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{h} - (\nu_1 w + \nu_2 v) \\ -\sqrt{2}(\nu_2 w - \nu_1 v) \\ \tilde{h} + \nu_1 w + \nu_2 v \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \mathbf{T}(\nu)\mathbf{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} v_1 + v_3 \\ -\nu_1(v_1 - v_3) - \sqrt{2}\nu_2 v_2 \\ -\nu_2(v_1 - v_3) + \sqrt{2}\nu_1 v_2 \end{pmatrix} \tag{A.4}$$

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