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Characterising clique convergence for locally cyclic graphs of minimum degree $\delta \geq 6$

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A B S T R A C T

The clique graph kG of a graph G has as its vertices the cliques (maximal complete subgraphs) of G, two of which are adjacent in kG if they have non-empty intersection in G. We say that G is clique convergent if $k^nG \cong k^mG$ for some $n \neq m$, and that G is clique divergent otherwise. We completely characterise the clique convergent graphs in the class of (not necessarily finite) locally cyclic graphs of minimum degree $\delta \geq 6$, showing that for such graphs clique divergence is a global phenomenon, dependent on the existence of large substructures. More precisely, we establish that such a graph is clique divergent if and only if its universal triangular cover contains arbitrarily large members from the family of so-called "triangular-shaped graphs".

1. Introduction

Given a (not necessarily finite) simple graph G, a **clique** $Q \subseteq G$ is an inclusion maximal complete subgraph. The **clique graph** $\boldsymbol{kG}$ has as its vertices the cliques of G, two of which are adjacent in kG if they have a non-empty intersection in G. The operator $\boldsymbol{k}$ is known as the **clique graph operator** and the behaviour of the sequence $G, kG, k^2G, \dots$ is the **clique dynamics** of G. The graph is **clique convergent** if the clique dynamics cycles eventually and it is **clique divergent** otherwise. It is an ongoing endeavour to understand which graph properties lead to convergence and divergence respectively, however, since clique convergence is known to be undecidable in general [2], this investigation often restricts to certain graph classes, such as graphs of low degree [17], circular arc graphs [13], or locally H graphs (e.g. locally cyclic graphs [5] or shoal graphs [12]).

The focus of the present article is on **locally cyclic** graphs, that is, graphs for which the neighbourhood of each vertex induces a cycle. Such graphs can be interpreted as triangulations of surfaces (always to be understood as "without boundary"), and it was recognized early that the study of their clique dynamics can be informed by topological considerations.

So it is known that each closed surface (i.e., compact and without boundary) has a clique divergent triangulation [8], but that convergent triangulations exist on all closed surfaces of negative Euler characteristic [6]. It has furthermore been conjectured that there are no convergent triangulations on closed surfaces of non-negative Euler characteristic (for a precise statement one requires minimum degree $\delta \geq 4$; see Conjecture 5.1). For example, the 4-regular [3] and 5-regular [14]

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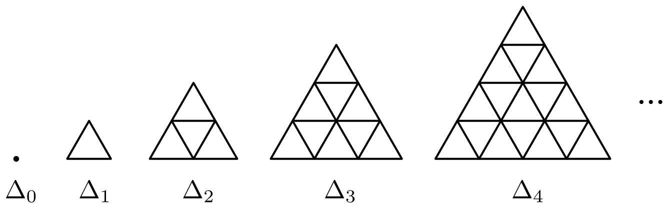

Fig. 1. The triangular-shaped graphs Δ_m for $m \in \{0, \ldots, 4\}$.

triangulations of the sphere (i.e., the octahedral and icosahedral graph) are clique divergent; as is any 6-regular triangulation of the torus or Klein bottle [4,5]. On the other hand, a triangulation of minimum degree $\delta \geq 7$ (necessarily of a closed surface of higher genus) is clique convergent [7]. Triangulations that mix degrees above and below six are still badly understood.

Baumeister & Limbach [1] broadened these investigations to triangulations of non-compact surfaces, that is, to infinite locally cyclic graphs. They gave an explicit description of $k^n G$ in terms of so-called **triangular-shaped subgraphs** of G (see Fig. 1), where G is a triangulation of minimum degree $\delta \geq 6$ of a (not necessarily compact) simply connected surface (see Section 2.3 for details).

The goal of this article is to bring the investigation of [1] to a satisfying conclusion: we apply their explicit construction of $k^n G$ to completely characterise the clique convergent triangulations in the class of (not necessarily finite) locally cyclic graphs of minimum degree $\delta \geq 6$. We thereby answer the open questions from Section 9 of [1].

Our first main result concerns locally cyclic graphs that are **triangularly simply connected**, that is, they correspond to triangulations of simply connected surfaces (see Section 4.2 for a rigorous definition). We identify the clique divergence of these graphs as a consequence of the existence of arbitrarily large triangular-shaped subgraphs.

Theorem A *(Characterisation theorem for triangularly simply connected graphs). A triangularly simply connected locally cyclic graph of minimum degree $\delta \geq 6$ is clique divergent if and only if it contains arbitrarily large triangular-shaped subgraphs.*

The difficulty in proving Theorem A lies in establishing divergence for a sequence of *infinite* graphs. Divergence is usually shown by observing the divergence of some numerical graph parameter, such as the vertex count or graph diameter. As our graphs are potentially infinite, this fails since the straightforward quantities might be infinite to begin with. The quest then lies mainly in identifying an often more contrived graph invariant which is still finite yet unbounded.

As a consequence of Theorem A we find that the 6-regular triangulation of the Euclidean plane (aka. the hexagonal lattice) is clique divergent.

By applying Theorem A to the universal triangular cover (see Section 4.2), we obtain the following more general result.

Theorem B *(General characterisation theorem). A (not necessarily finite) connected locally cyclic graph of minimum degree $\delta \geq 6$ is clique divergent if and only if its universal triangular cover contains arbitrarily large triangular-shaped subgraphs.*

The "only if" direction of Theorem B was supposedly proven in [1], but the proof contains a gap, which we close in Section 4.

As a consequence of Theorem B, a triangulation of minimum degree $\delta \geq 6$ of a closed surface is clique divergent if and only if it is 6-regular (cf. [1, Lemma 8.10]).

We mention two further recent results on clique dynamics that are in a similar spirit. In 2017, Larrión, Pizaña, and Villarroel-Flores [11] showed that the clique operator preserves (finite) triangular graph bundles, which are a generalisation of finite triangular covering maps. Also, just recently in 2022, Villarroel-Flores [17] showed that among the (finite) connected graphs with maximum degree at most four, the octahedral graph is the only one that is clique divergent.

1.1. Structure of the paper

In Section 2, we recall the fundamental concepts and notations used throughout the paper. In particular, in Section 2.3 we recall the geometric clique graph G_n and the relevant statements of [1] that established the explicit description of $k^n G$ in terms of G_n.

In Section 3, we prove Theorem A. To show that a sequence of infinite graphs is divergent, we identify a finite yet unbounded graph invariant $D(H)$ (see (3.1)) based on the distribution of vertices of degree 26.

In Section 4, we prove Theorem B. We extend the divergence results of Section 3 to graphs that are not necessarily triangularly simply connected by exploiting that covering relations interact well with the clique operator and the geometric clique graph.

Section 5 summarizes the results and lists related open questions.

We also include an appendix which recalls helpful background theory for Section 4. Appendix A gives a proof that triangular simple connectivity is preserved under the clique operator while Appendix B focuses on the existence and uniqueness

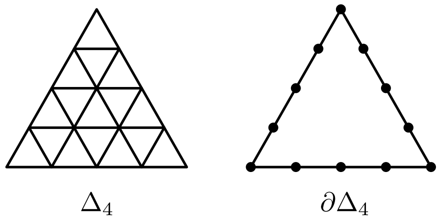

Fig. 2. The triangular-shaped graph Δ_4 and its boundary $\partial\Delta_4$.

of a triangularly simply connected triangular cover for any connected graph. Appendix C recalls the construction of the Γ-isomorphism $\psi_n\colon G_n \to k^n G$ from [1].

2. Notation and background

2.1. Basic notation

All graphs in this article are simple, non-empty and potentially infinite. If not stated otherwise, they are connected and locally finite. For a graph G we write $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. The adjacency relation is denoted by $\sim$. We define the closed and the open neighbourhood of a set $U \subseteq V(G)$ of vertices as

$$N_G[U] := \{v \in V(G) \mid v \in U \text{ or } v \sim w \text{ for some } w \in U\} \text{ and}$$
$$N_G(U) := \{v \in V(G) \mid v \notin U \text{ and } v \sim w \text{ for some } w \in U\},$$

respectively.

For $v \in V(G)$, we write $N_G[v]$ instead of $N_G[\{v\}]$ and $N_G(v)$ instead of $N_G(\{v\})$. We write $\deg_G(v) := |N_G(v)|$ for the degree of v, and $\operatorname{dist}_G(v, w)$ for the graph-theoretic distance between two vertices $v, w \in V(G)$. For $v \in V(G)$ and $U, U' \subseteq V(G)$ we write

$$\operatorname{dist}_G(v, U) := \min_{w \in U} \operatorname{dist}_G(v, w) \quad \text{and} \quad \operatorname{dist}_G(U, U') := \min_{w \in U,\, w' \in U'} \operatorname{dist}_G(w, w').$$

We write G-degree, G-neighbourhood, or G-distance to emphasize the graph with respect to which these quantities are computed. Finally, we use $\cong$ to denote isomorphy between graphs.

We write $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for the sets of natural numbers without and with zero. We write $k\mathbb{N}$ and $k\mathbb{N}_0$ to denote multiples of k.

2.2. Cliques, clique graphs, and clique dynamics

A **clique** in G is an inclusion maximal complete subgraph. The **clique graph** kG has vertex set $V(kG) := \{\text{cliques of } G\}$, and distinct cliques $Q, Q' \in V(kG)$ are adjacent in kG if they have vertices in common. We consider k as an operator, the **clique graph operator**, mapping a graph to its clique graph. By k^n, we denote its n-th iterate.

A sequence $G^0, G^1, G^2, \ldots$ of graphs is said to be **convergent** if it is eventually periodic, that is, if for some $r \in \mathbb{N}$ and all sufficiently large $n \in \mathbb{N}$ we have $G^n \cong G^{n+r}$. The sequence is said to be **divergent** otherwise. A graph G is said to be **clique convergent** if the sequence $k^0G, k^1G, k^2G, \ldots$ is convergent, and is called **clique divergent** otherwise.

2.3. Locally cyclic graphs, triangular-shaped subgraphs, and the geometric clique graph

A graph G is **locally cyclic** if the (open) neighbourhood of each vertex induces a cycle. In particular, a locally cyclic graph is locally finite. Such graphs can also be interpreted as triangulations of surfaces. We shall however use this geometric perspective only informally, and work with the purely graph theoretic definition given above. A fundamental example of a locally cyclic graph is the hexagonal triangulation of the Euclidean plane.

We use the class of **triangular-shaped graphs $\boldsymbol{\Delta_m}$** from [1], which are subgraphs of the hexagonal lattice, and the smallest five of which are depicted in Fig. 1. The parameter m is called the **side length** of Δ_m, and the boundary $\boldsymbol{\partial\Delta_m}$ is the subgraph of Δ_m that consists of the vertices of degree less than six and the edges that lie in only a single triangle (Fig. 2).

In [1], it was shown that the n-th iterated clique graph k^nG of a triangularly simply connected locally cyclic graph G of minimum degree $\delta \geq 6$ (also called “pika” in [1]) can be explicitly constructed based on triangular-shaped subgraphs of G (see Definition 2.1 and Theorem 2.2 below). Hereby “triangularly simply connected” means “triangulation of a simply connected surface”, but a precise definition is postponed until Section 4.2 (or see [1]). For now it suffices to use these terms as a black box, merely to apply Theorem 2.2. Note however that such a graph is in particular connected.

The explicit construction of k^nG is captured by the following definition:

Definition 2.1 *([1, Definition 4.1]).* Given a triangularly simply connected locally cyclic graph G of minimum degree $\delta \geq 6$, its ***n*-th geometric clique graph G_n** ($n \geq 0$) has the following form:

(i) the vertices of G_n are the triangular-shaped subgraphs of G of side length $m \leq n$ with $m \equiv n \pmod 2$.
(ii) two distinct triangular-shaped subgraphs $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m+s}$ with $s \geq 0$ are adjacent in G_n if and only if any of the following applies:
  a. $s = 0$ and $S_1 \subset N_G[S_2]$ (or equivalently, $S_2 \subset N_G[S_1]$).
  b. $s = 2$ and $S_1 \subset S_2$.
  c. $s = 4$ and $S_1 \subset S_2 \setminus \partial S_2$.
  d. $s = 6$ and $S_1 = S_2 \setminus N_G[\partial S_2]$.

Note that $G_0 \cong G$. We then have

Theorem 2.2 *([1, Theorem 6.8 + Corallary 7.8]). If G is locally cyclic, triangularly simply connected and of minimum degree $\delta \geq 6$, then $G_n \cong k^n G$ for all $n \in \mathbb{N}_0$.*

We refer to the four types of adjacencies listed in Definition 2.1 as adjacencies of type $0, \pm 2, \pm 4$ and ± 6 respectively. For a triangular-shaped graph $S \in V(G_n)$ of side length m, we refer to a neighbour $T \in N_{G_n}(S)$ of side length $m+s$ as being of type $s \in \{-6, -4, -2, 0, +2, +4, +6\}$. Some visualisations for the various configurations of triangular-shaped graphs that correspond to adjacency in G_n can be seen in Figs. 3 to 5 in the next section.

The following example demonstrates how Theorem 2.2 can be used to establish clique convergence in non-trivial cases:

Example 2.3. A locally cyclic and triangularly simply connected graph G of minimum degree $\delta \geq 7$ does not contain any triangular-shaped graphs of side length ≥ 3 (because such have vertices of degree six). Hence, $k^n G \cong G_n = G_{n+2} \cong k^{n+2} G$ whenever $n \geq 1$. Such a graph G is therefore clique convergent.

3. Proof of Theorem A

Throughout this section, we assume that G is a locally cyclic graph that is triangularly simply connected and has minimum degree $\delta \geq 6$. We can then apply Theorem 2.2 and investigate the dynamics of the sequence of geometric clique graphs G_n in place of $k^n G$.

One direction of Theorem A follows immediately from the definition of the geometric clique graph (Definition 2.1) together with Theorem 2.2. We make a remark for later reference:

Remark 3.1. If all triangular-shaped subgraphs of G are of side length $\leq m \in 2\mathbb{N}$, then $G_m \cong G_{m+2}$, that is, the sequence cycles, and G is clique convergent by Theorem 2.2. This reasoning can also be found in [1, Theorem 7.9].

The remainder of this section is devoted to proving the other direction of Theorem A: if G contains arbitrarily large triangular-shaped subgraphs, then G is clique divergent. For this, we identify a graph invariant that is both finite and unbounded for the sequence G_n as $n \to \infty$, as long as G contains arbitrarily large triangular-shaped subgraphs. It turns out that a suitable graph invariant can be built from measuring distances between vertices of certain degrees. Curiously, the degree 26 plays a special role, and the following notation comes in handy:

$$\mathbf{DEG_{26}}(\boldsymbol{H}) := \{v \in V(H) \mid \deg_H(v) = 26\}$$

$$\overline{\mathbf{DEG}}_{\mathbf{26}}(\boldsymbol{H}) := \{v \in V(H) \mid \deg_H(v) \neq 26\}$$

The corresponding graph invariant is the following:

$$\boldsymbol{D}(\boldsymbol{H}) := \max_{v \in V(H)} \operatorname{dist}_H\big(v, \overline{\mathrm{DEG}}_{26}(H)\big). \tag{3.1}$$

The significance of the number 26 stems from the observation that most vertices of G_n have G_n-degree ≤ 26; and have G_n-degree *exactly* 26 only in very special circumstances that can be expressed as the existence of certain triangular-shaped subgraphs in G. This is proven in Lemma 3.2 and Lemma 3.3. Finitude and divergence of $D(G_n)$ as $n \to \infty$ are proven afterwards in Lemma 3.4 and Lemma 3.5.

In the following, we generally consider G_n only for even $n \in 2\mathbb{N}$, as this cuts down on the cases we need to investigate, and is still sufficient to show that $D(G_n)$ is unbounded. Note that each $S \in V(G_n)$ is then of even side length $m \in \{0, 2, 4, 6, \ldots\}$.

Lemma 3.2. *Let $S \in V(G_n)$ be a triangular-shaped graph of side length $m \geq 6$. Then $\deg_{G_n}(S) \leq 26$, with equality if and only if S has a neighbour of type $+6$.*

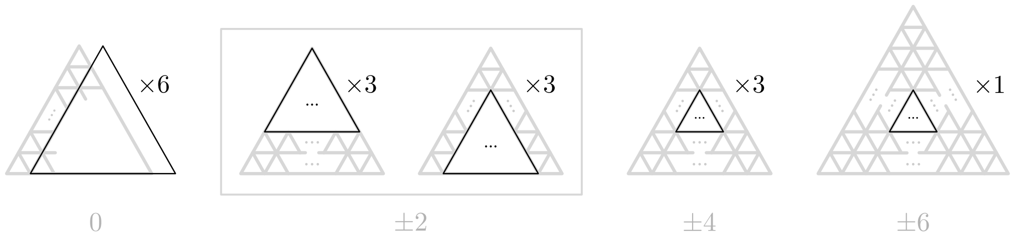

Fig. 3. The 26 possible ways in which a triangular-shaped graph $S \in V(G_n)$ of side length $m \geq 6$ can be G_n-adjacent to another triangular-shaped graph $T \in V(G_n)$ of side length $m+s$, where $s \in \{-6,-4,-2,0,+2,+4,+6\}$. Two configurations may differ merely by a symmetry (one of the six "reflections" and "rotations" of a triangular-shaped graph), and we always show only a single configuration with the multiplication factor next to it indicating the number of equivalent configuration related by symmetry. Note that for the types ± 2, ± 4 and ± 6, the configurations must be accounted for twice in the G_n-degree of S: once with S being the larger graph (in grey), and once with S being the smaller graph (in black). Then $26 = 6 + 2 \cdot (3+3+3+1)$.

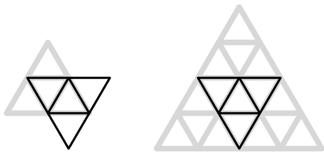

Fig. 4. For $m \in \{2,4\}$, there also exist the following "twisted adjacencies".

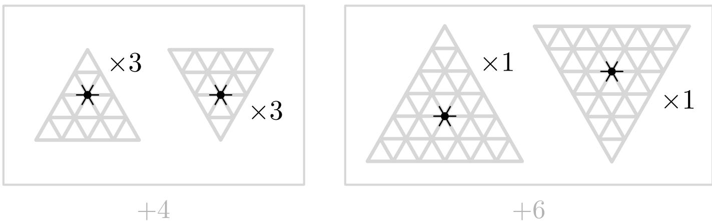

Fig. 5. The eight possible neighbours of a triangular-shaped graph of side length $m=0$ of type $+4$ and $+6$. See the caption of Fig. 3 for an explanation of the multiplicities.

Lemma 3.2 actually holds unchanged for $m \geq 2$. Since we do not need these cases to prove Theorem A, and since verifying them requires a distinct case analysis (because of "twisted adjacencies", cf. Fig. 4), we do not include them here.

Proof of Lemma 3.2. Fig. 3 shows all potential configurations of S and a G_n-neighbour of S according to Definition 2.1 (here we need $m \geq 6$, as there are exceptional "twisted adjacencies" for smaller m, see Fig. 4). In total this amounts to a degree of at most 26. In particular, if just one of the neighbours is missing, say the neighbour of type $+6$, then S must have a G_n-degree of less than 26.

Conversely, one can verify that if S has a neighbour of type $+6$, say $T \in N_{G_n}(S)$, then all other neighbours of types $-6,-4,-2,0,+2$, and $+4$ can be found as subgraphs of T. Therefore, all 26 neighbours are present and the degree is 26. □

For $m=0$ only one direction holds, which is also sufficient for our purpose.

Lemma 3.3. *Let $n \in 2\mathbb{N}$ and $s \in V(G_n)$ be a triangular-shaped graph of side length $m=0$ (that is, s is a vertex of G). If s has no G_n-neighbour of type $+6$, then $\deg_{G_n}(s) \neq 26$.*

Proof. Clearly, s has no neighbours of type $-6,-4$ or -2. The G_n-neighbours of type 0 are exactly the vertices that are also adjacent to s in G, that is, there are *exactly* $\deg_G(s)$ many. The potential neighbours of type $+4$ and $+6$ are shown in Fig. 5, which amount to *at most* eight neighbours of these types. Note that these can exist only if $\deg_G(s) = 6$.

It remains to count the neighbours of type $+2$, which will turn out at *exactly* $2\deg_G(s)$, independent of the specifics of G. Observe first that there can be two types of neighbours $T \in N_{G_n}(s)$ of type $+2$ distinguished by the T-degree of s, which is either two or four (cf. Fig. 6). We shall say that these neighbours are of type $+2_2$ and $+2_4$ respectively.

In the following, an *r-chain* is an inclusion chain $s \subset \Delta \subset T$, where Δ is an s-incident triangle in G, and T is a neighbour of s of type $+2_r$. The following information can be read from Fig. 6: a neighbour of s of type $+2_r$ can be extended to an r-chain in exactly n_r ways (where $n_2 = 1$ and $n_4 = 3$). Likewise, an s-incident triangle can be extended to an r-chain in exactly n_r ways as well. By double counting, we find that $1/n_r$ times the number of r-chains equals both the number of

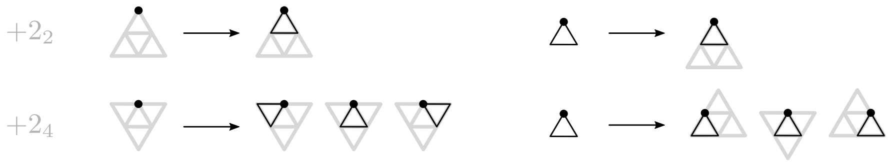

Fig. 6. Row $+2_r$ shows the ways in which an inclusion $s \subset T$ (left; T being a G_n-neighbour of s of type $+2_r$) or an inclusion $s \subset \Delta$ (right; Δ being an triangle in G) extends to an r-chain in $n_r = r-1$ ways.

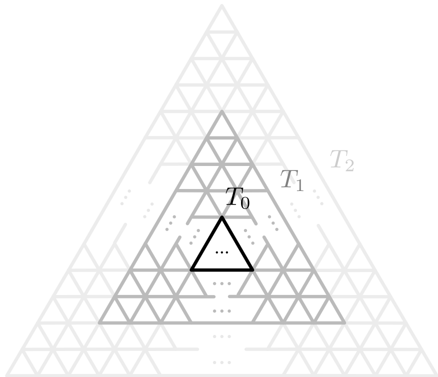

Fig. 7. Initial segment $T_0 T_1 T_2 \dots$ of an increasing path of triangular-shaped subgraphs of G where T_i and T_{i+1} are adjacent of type ± 6.

s-incident triangles (which is exactly $\deg_G(s)$) and the number of neighbours of s of type $+2_r$. In conclusion, the number of neighbours of s of type $+2$ is *exactly* $2\deg_G(s)$.

Taking together all of the above, we count

$$\deg_{G_n}(s) \begin{cases} = \deg_G(s) + 2\deg_G(s) = 3\deg_G(s) & \text{if } \deg_G(s) \neq 6 \\ \leq 6 + 2 \cdot 6 + 8 = 26 & \text{if } \deg_G(s) = 6 \end{cases}.$$

Since $26 \not\equiv 0 \pmod 3$, if $\deg_G(s) \neq 6$ we obtain $\deg_{G_n}(s) \neq 26$ right away. If $\deg_G(s) = 6$ and if there is no G_n-neighbour of type $+6$, then the maximal amount of 26 neighbours cannot have been attained, and $\deg_{G_n}(s) \neq 26$ as well. □

It remains to show that if G contains arbitrarily large triangular-shaped subgraphs, then the graph invariant $D(G_n)$ is both finite and unbounded as $n \to \infty$. We first prove finitude of $D(G_n)$ if $n \in 2\mathbb{N}$ (in particular, $n \geq 2$, as $D(G_0) = D(G)$ might be infinite).

Lemma 3.4. *If $n \in 2\mathbb{N}$, then each $S \in V(G_n)$ has a distance to $\overline{\text{DEG}}_{26}(G_n)$ of at most $n/6 + 1$. That is, $D(G_n) \leq n/6 + 1$.*

Proof. Suppose $S \cong \Delta_m$ with $m \in 2\mathbb{N}$. We distinguish two cases.

Case 1: there is a $T \in V(G_n)$ of side length $\mu \geq 6$ and $\operatorname{dist}_{G_n}(S, T) \leq 2$. We then fix a maximally long path $T_0 T_1 \dots T_\ell$ in G_n with $T_0 := T$ and $T_i \cong \Delta_{\mu + 6i}$ (i.e., T_i and T_{i+1} are adjacent of type ± 6; see Fig. 7). Since the path is maximal, T_ℓ has no G_n-neighbour of type $+6$, and since T_ℓ is of side length $\mu + 6\ell \geq \mu \geq 6$, we have $T_\ell \in \overline{\text{DEG}}_{26}(G_n)$ by Lemma 3.2. As a vertex of G_n, T_ℓ is of side length at most n, and hence $\mu + 6\ell \leq n \Longrightarrow \ell \leq n/6 - \mu/6 \leq n/6 - 1$. We conclude

$$\begin{aligned} \operatorname{dist}_{G_n}(S, \overline{\text{DEG}}_{26}(G_n)) &\leq \operatorname{dist}_{G_n}(S, T) + \operatorname{dist}_{G_n}(T, \overline{\text{DEG}}_{26}(G_n)) \\ &\leq 2 + (n/6 - 1) = n/6 + 1. \end{aligned}$$

Case 2: there is *no* $T \in V(G_n)$ of side length $\mu \geq 6$ and $\operatorname{dist}_{G_n}(S, T) \leq 2$. Then we can conclude two things: first, $m < 6$ (otherwise, choose $T := S$) and so there is an $s \in N_{G_n}(S)$ of side length zero. Second, s has no neighbour of type $+6$ (otherwise, set T to be this neighbour). But then s cannot have degree 26 by Lemma 3.3, and therefore

$$\operatorname{dist}_{G_n}(S, \overline{\text{DEG}}_{26}(G_n)) \leq \operatorname{dist}_{G_n}(S, s) = 1 \leq n/6 + 1. \quad \square$$

Finally, we show that $D(G_n)$ is unbounded as $n \to \infty$, assuming that there are arbitrarily large triangular-shaped subgraphs of G.

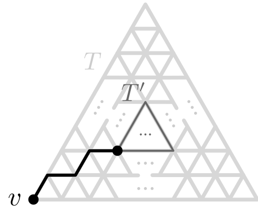

Fig. 8. The "corner vertex" v of $T \in V(G_n)$ (light grey) has G-distance four to the neighbour $T' \in N_{G_n}(T)$ of type -6 (dark grey).

Lemma 3.5. *If G contains a triangular-shaped subgraph of side length $n \in 48\mathbb{N}$, then there exists an $S' \in V(G_n)$ with distance to $\overline{\text{DEG}}_{26}(G_n)$ of more than $n/48$. That is, $D(G_n) > n/48$.*

Proof. Choose a triangular-shaped graph $S \in V(G_n)$ of side length $n \in 48\mathbb{N}$. Roughly, the idea is to define a set $\mathcal{M} \subseteq \text{DEG}_{26}(G_n)$ that contains "deep vertices", i.e., vertices that have no "short" G_n-paths that lead out of $\mathcal{M}$. We claim that the following set has all the necessary properties:

$$\mathcal{M} := \left\{ T \in V(G_n) \;\middle|\; \begin{array}{l} T \subseteq S, \\ T \text{ has side length } m \geq 6 \text{ and} \\ \text{dist}_G(T, \partial S) \geq 4 \end{array} \right\}.$$

The following observation will be used repeatedly and we shall abbreviate it by $(*)$: if $T \in V(G_n)$ is of side length $m \geq 6$ (e.g. if $T \in \mathcal{M}$) and if $T' \in N_{G_n}(T)$ is some G_n-neighbour, then $\text{dist}_G(T, v) \leq 4$ for all $v \in T'$. This can be verified by considering the configurations shown in Fig. 3. The bound ≤ 4 is best possible as seen in Fig. 8.

We first verify $\mathcal{M} \subseteq \text{DEG}_{26}(G_n)$. Fix $T \in \mathcal{M}$ and consider an embedding of S into the hexagonal lattice. In this embedding, $T \subseteq S$ has a neighbour T' of type $+6$ that, for all we know, might partially lie outside of S; though we now show that actually $T' \subseteq S$: in fact, for all $v \in V(T')$ holds

$$\text{dist}_G(v, \partial S) \geq \text{dist}_G(T, \partial S) - \text{dist}_G(T, v) \geq 4 - 4 = 0,$$

where we used both $(*)$ and $T \in \mathcal{M}$ in the second inequality. Thus $T' \subseteq S$ and T' also exists in G. Note that this argument shows that all G_n-neighbours of T are contained in S. We denote the latter fact by $(**)$ as we reuse it below. For now we conclude that since T has a G_n-neighbour of type $+6$, we have $T \in \text{DEG}_{26}(G_n)$ by Lemma 3.2.

Next we identify a "deep vertex" in $\mathcal{M}$, that is, a vertex with distance to $V(G_n) \setminus \mathcal{M}$ of more than $n/48$. We claim that we can choose for this the "central" triangular-shaped subgraph $S' \cong \Delta_{n/2}$. By that we mean the triangular-shaped graph obtained from S by repeatedly deleting the boundary $n/6$ times. The resulting triangular-shaped subgraph has side length $n/2$ and $\text{dist}_G(S', \partial S) = n/6$. Since $n \geq 48$, we have both $m_0 := n/2 \geq 6$ and $\text{dist}_G(S', \partial S) = n/6 \geq 4$, and therefore $S' \in \mathcal{M}$. It remains to show that we have $\ell := \text{dist}_{G_n}(S', V(G_n) \setminus \mathcal{M}) > n/48$. Let $S'_0 \dots S'_\ell$ be a path in G_n from $S'_0 := S'$ to some $S'_\ell \notin \mathcal{M}$. Let $m_i \in \mathbb{N}_0$ be the side length of S'_i. Since $S'_{\ell-1} \in \mathcal{M}$, by $(**)$ we have $S'_\ell \subseteq S$. Thus, for S'_ℓ to be not in $\mathcal{M}$, only two reasons are left, and we verify that either implies $\ell > n/48$:

- Case 1: $m_\ell < 6$. Since S'_{i-1} and S'_i are adjacent in G_n they can differ in side length by at most six (via an adjacency of type ± 6). That is, $m_{i-1} - m_i \leq 6$, and thus

$$6\ell \geq m_0 - m_\ell > n/2 - 6 \implies \ell > n/12 - 1 \geq n/48.$$

- Case 2: $\text{dist}_G(S'_\ell, \partial S) < 4$. Note first that for all $i \in \{1, \dots, \ell\}$ holds

$$\text{dist}_G(S'_{i-1}, \partial S) - \text{dist}_G(S'_i, \partial S) \leq \text{dist}_G(S'_{i-1}, S'_i) \overset{(*)}{\leq} 4.$$

  It then follows

$$4\ell \geq \text{dist}_G(S'_0, \partial S) - \text{dist}_G(S'_\ell, \partial S) > n/6 - 4 \implies \ell > n/24 - 1 \geq n/48.$$

In both cases, the right-most inequality was obtained using $n \geq 48$. □

Since in our setting we have $G_n \cong k^n G$, and since $D(\cdot)$ is a graph invariant, we have $D(k^n G) = D(G_n)$. We can then conclude

Corollary 3.6. *If G contains Δ_n as a subgraph for $n \in 48\mathbb{N}$, then*

$$D(k^n G) \in \left(\tfrac{n}{48}, \tfrac{n}{6} + 1\right],$$

where $D(\cdot)$ is the graph invariant defined in (3.1). In particular, if G contains arbitrarily large triangular-shaped subgraphs, then $D(k^nG)$ is unbounded as $n \to \infty$, and G is therefore clique divergent.

Together with Remark 3.1 we conclude the characterisation of clique convergent triangularly simply connected locally cyclic graphs of minimum degree $\delta \geq 6$.

Theorem A *(Characterisation theorem for triangularly simply connected graphs). A triangularly simply connected locally cyclic graph of minimum degree $\delta \geq 6$ is clique divergent if and only if it contains arbitrarily large triangular-shaped subgraphs.*

4. Proof of Theorem B

In this section we prove Theorem B. We need to recall basic facts about group actions and graph coverings, which we do in Section 4.1 and Section 4.2 below.

4.1. Group actions, Γ-isomorphisms, and quotient graphs

We say that a group Γ **acts** on a graph G if we have a group homomorphism $\sigma : \Gamma \to \mathrm{Aut}(G)$. For every $\gamma \in \Gamma$ and every $v \in V(G)$, we define $\gamma v := \sigma(\gamma)(v)$. The graph G together with this action is called a **Γ-graph**. For every subgroup $\Gamma \leq \mathrm{Aut}(G)$, G is a Γ-graph in a natural way. For two Γ-graphs G and H, we call a graph isomorphism $\phi\colon G \to H$ a **Γ-isomorphism**, if $\phi(\gamma v) = \gamma\phi(v)$ for each $v \in V(G)$ and each $\gamma \in \Gamma$.

Remark 4.1. If G is a Γ-graph, so is kG with respect to the induced action $\gamma Q = \{\gamma v \mid v \in Q\}$. Note that in [5] this action is denoted as the natural action of the group $\Gamma_k \leq \mathrm{Aut}(kG)$, which is isomorphic to Γ. For a second Γ-graph H and a Γ-isomorphism $\phi\colon G \to H$, the map $\boldsymbol{\phi_k}\colon kG \to kH$, $Q \mapsto \{\phi(v) \mid v \in Q\}$ is a Γ-isomorphism.

Remark 4.2. If a Γ-graph G is locally cyclic, triangularly simply connected and of minimum degree $\delta \geq 6$, the action of Γ on G induces an action on the triangular-shaped subgraphs of G which makes the geometric clique graph G_n into a Γ-graph as well. Moreover, the isomorphism $\boldsymbol{\psi_n}\colon G_n \to k^nG$, that exists according to Theorem 2.2, is a Γ-isomorphism. This can be seen easily from its explicit construction in [1, Corollary 6.9], though in order to be self-contained, the argument is summarized in Appendix C.

For any vertex $v \in V(G)$ of a Γ-graph G, we denote the orbit of v under the action of Γ by $\boldsymbol{\Gamma v}$. These orbits form the vertex set of the **quotient graph G/Γ**, two of which are adjacent if they contain adjacent representatives. Note that if two graphs G and H are Γ-isomorphic, the quotient graphs G/Γ and H/Γ are isomorphic.

4.2. Triangular covers

In the following, we transfer the convergence criterion of Theorem A from the triangularly simply connected case to the general case using the triangular covering maps from [5].

We define the topologically inspired term of "triangular simple connectivity" via the concept of walk homotopy. As usual, a **walk of length ℓ** in a graph G is a finite sequence of vertices $\alpha = v_0 \dots v_\ell$ such that each pair $v_{i-1}v_i$ of consecutive vertices is adjacent. The vertex v_0 is called the **start vertex**, the vertex v_ℓ is called the **end vertex**, a walk is called **closed** if start and end vertices coincide, and it is called **trivial** if it has length zero.

In order to define the homotopy relation on walks, we define four types of **elementary moves** (see also Fig. 9). Given a walk that contains three consecutive vertices that form a triangle in G, the **triangle removal** shortens the walk by removing the middle one of them. Conversely, if a walk contains two consecutive vertices that lie in a triangle of G, the **triangle insertion** lengthens the walk by inserting the third vertex of the triangle between the other two. The **dead end removal** shortens a walk that contains a vertex twice with distance two in the walk by removing one of the two occurrences as well as the vertex between them. Conversely, the **dead end insertion** lengthens a walk by inserting behind one vertex an adjacent one and then the vertex itself again.

Note that elementary moves do not change the start and end vertices of walks, not even of closed ones.

Two walks are called **homotopic** if it is possible to transform one into the other by performing a finite number of elementary moves. The graph G is called **triangularly simply connected** if it is connected and if every closed walk is homotopic to a trivial one.

A **triangular covering map** is a homomorphism $p\colon \tilde{G} \to G$ between two connected graphs which is a local isomorphism, i.e., the restriction $p|_{N[\tilde{v}]}\colon N[\tilde{v}] \to N[p(\tilde{v})]$ to the closed neighbourhood of any vertex $\tilde{v}$ of $\tilde{G}$ is an isomorphism and in this case, $\tilde{G}$ is called a **triangular cover** of G. The term "triangular" refers to the **unique triangle lifting property** which can be used as an alternative definition and is defined in Appendix B. For a triangular covering map $p\colon \tilde{G} \to G$, we define the map $\boldsymbol{p_{k^n}}\colon k^n\tilde{G} \to k^nG$ which is constructed from p recursively by $p_{k^0} = p$ and $p_{k^n}(\tilde{Q}) = \{p_{k^{n-1}}(\tilde{v}) \mid \tilde{v} \in \tilde{Q}\}$ for $n \geq 1$. By [5, Proposition 2.2], p_{k^n} is a triangular covering map, as well.

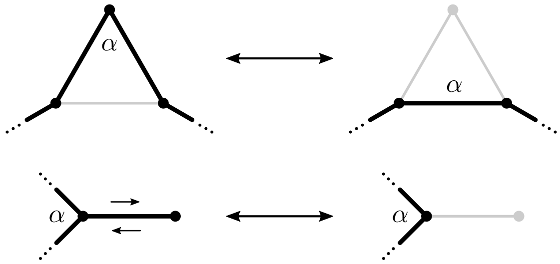

Fig. 9. Visualisations of the elementary moves.

A triangular covering map $p\colon \tilde{G} \to G$ is called **universal** if $\tilde{G}$ is triangularly simply connected, and in this case $\tilde{G}$ is called the **universal (triangular) cover** of G. Note that every connected graph has a universal cover that is unique up to isomorphism. A proof can be found in [16, Theorem 3.6] or in the appendix in Theorem B.5.

For the following lemma, we need to use that triangular simple connectivity is preserved under the clique operator. This is proven in [9], but we also provide an elementary proof in the appendix in Lemma A.2.

Lemma 4.3. *If a connected graph G is clique convergent, so is its universal triangular cover $\tilde{G}$.*

Proof. Let the clique operator be convergent on G, i.e., there are $n, r \in \mathbb{N}$ such that $k^n G \cong k^{n+r} G$, and let $p\colon \tilde{G} \to G$ be a universal triangular covering map. As p_{k^n} and $p_{k^{n+r}}$ are triangular covering maps and $k^n \tilde{G}$ and $k^{n+r}\tilde{G}$ are triangularly simply connected by Lemma A.2, they are universal triangular covering maps. As the universal cover is unique up to isomorphism (Theorem B.5), $k^n \tilde{G} \cong k^{n+r}\tilde{G}$ and $\tilde{G}$ is clique convergent. □

In the following, we show that for locally cyclic graphs with minimum degree $\delta \geq 6$ the converse implication is true as well. This has been stated in [1] as Lemma 8.8, but the proof contains a gap, as it does not show that $k^n \tilde{G}$ and $k^{n+r}\tilde{G}$ are Γ-isomorphic (in fact, this is still unknown if $\tilde{G}$ is a cover of a general graph G; see also Question 5.5). We will close this gap in the remainder of this section.

In order to do this, we need the definition of Galois maps. For a group Γ, we call a triangular covering map $p\colon \tilde{G} \to G$ **Galois with Γ** if $\tilde{G}$ is a Γ-graph such that the vertex preimages of p are exactly the orbits of the action, which implies $\tilde{G}/\Gamma \cong G$. By [5, Proposition 3.2], if p is Galois with Γ, so is p_{k^n}.

The following lemma is proven in [1, Lemma 8.7], but again, an elementary proof is provided in Lemma B.6.

Lemma 4.4 *(from [1, Lemma 8.7]). A universal triangular covering map $p\colon \tilde{G} \to G$ is Galois with $\Gamma := \{\gamma \in \mathrm{Aut}(\tilde{G}) \mid p \circ \gamma = p\}$, which is called the* ***deck transformation group*** *of p. Consequently, $(k^n \tilde{G})/\Gamma \cong k^n G$.*

We are now able to deduce the clique convergence of a graph from the clique convergence of its universal cover.

Lemma 4.5. *Let G be a locally cyclic graph with minimum degree $\delta \geq 6$ and $\tilde{G}$ its universal triangular cover. If $\tilde{G}$ is clique convergent, then so is G.*

Proof. We start with the universal triangular cover $\tilde{G}$ being clique convergent. By Theorem A, there is an $m \in \mathbb{N}$ such that G does not contain Δ_m as a subgraph. Consequently, $\tilde{G}_{m-2}$ and $\tilde{G}_m$ are identical and thus Γ-isomorphic (for every Γ).

Let Γ be the deck transformation group of the universal covering map $p\colon \tilde{G} \to G$. By Lemma 4.4, this implies $k^n G \cong (k^n \tilde{G})/\Gamma$ for each $n \in \mathbb{N}_0$. Using the Γ-isomorphism $\psi_n\colon k^n \tilde{G} \to \tilde{G}_n$ from Remark 4.2, we conclude that G is clique convergent via

$$k^{m-2} G \cong (k^{m-2}\tilde{G})/\Gamma \cong \tilde{G}_{m-2}/\Gamma = \tilde{G}_m/\Gamma \cong (k^m \tilde{G})/\Gamma \cong k^m G. \quad \square$$

By joining Lemma 4.3, Lemma 4.5, and Theorem A, we conclude the characterisation of clique convergent locally cyclic graphs with minimum degree $\delta \geq 6$.

Theorem B *(General characterisation theorem). A (not necessarily finite) connected locally cyclic graph of minimum degree $\delta \geq 6$ is clique divergent if and only if its universal triangular cover contains arbitrarily large triangular-shaped subgraphs.*

5. Conclusion and further questions

In this article, we completed the characterisation of locally cyclic graphs of minimum degree $\delta \geq 6$ with a convergent clique dynamics, first in the triangularly simply connected case (Theorem A) and then in the general case (Theorem B).

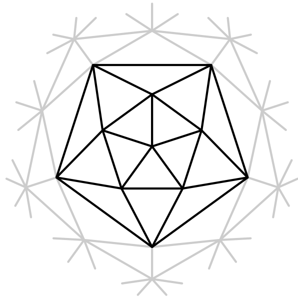

Fig. 10. An "almost 7-regular" triangulation of the Euclidean plane, that is, it is 7-regular outside a small region.

Our findings turned out to be consistent with the geometric intuition from the finite case: the hexagonal lattice is clique divergent, as is any of its quotients. The finite analogues are the 6-regular triangulations of surfaces with Euler characteristic zero, which were known to be clique divergent by [4,5]. We are tempted to say that the hexagonal lattice is clique divergent because it has a "flat geometry".

Theorem A may allow for a similar interpretation: if a triangularly simply connected locally cyclic graph G of minimum degree $\delta \geq 6$ is clique divergent, then it contains arbitrarily large triangular-shaped subgraphs. As a consequence, vertices of degree ≥ 7 cannot be distributed densely everywhere in G. Since degrees ≥ 7 can be interpreted as a discrete analogue of negative curvature (we think of the 7-regular triangulation of the hyperbolic plane), a potential geometric interpretation of Theorem A is that G is clique divergent because it is "close to being flat" on large parts, which then dominate the clique dynamics.

To consolidate this interpretation, it would be helpful to shed more light on the lower degree analogues: locally cyclic graphs of minimum degree $\delta = 5$ or even $\delta = 4$. There however, the clique dynamics might be governed by different effects. In a sense, it was surprising to find that for minimum degree $\delta \geq 6$, the asymptotic behaviour of the clique dynamics is determined only on the global scale, that is, by the presence or absence of subgraphs in G from a relatively simple infinite family (the triangular-shaped graphs). Such a description should not be expected for smaller minimum degree: for $\delta \leq 5$ there exist finite graphs that are clique divergent – even simply connected ones – and such clearly cannot contain "arbitrarily large" forbidden structures in any sense.

It might be worthwhile to first study triangulations of the plane of minimum degree $\delta = 5$ or $\delta = 4$, since those are not subject to the same argument of "finite size". Yet, as far as we are aware, it is already unknown which of the following graphs are clique divergent: consider a triangulated sphere of minimum degree $\delta \in \{4, 5\}$ (e.g. the octahedron or icosahedron). Remove a vertex or edge together with all incident triangles – which leaves us with a triangulated disc – and extend this to a triangulation of the Euclidean plane that is 7-regular outside the interior of the disc (see Fig. 10). For all we know, it is at least conceivable that below minimum degree $\delta = 6$ divergence can appear as a local phenomenon that does not require arbitrarily large "bad regions".

For triangulations of closed surfaces (and further mild assumptions, see below), the most elementary open question is whether non-negative Euler characteristic already implies clique-divergence. This has previously been conjectured by Larrión, Neumann-Lara and Pizaña [7], and we shall repeat it here.

Conjecture 5.1. *If a locally cyclic graph G of minimum degree $\delta \geq 4$ triangulates a closed surface of Euler characteristic $\chi \geq 0$ (i.e., a sphere, projective plane, torus or Klein bottle), then G is clique divergent.*

To shed further light on the perceived connection between topology and clique dynamics, the study of further topologically motivated generalisations appears worthwhile. We briefly mention two of them.

First, one could turn to higher-dimensional analogues, that is, triangulations of higher-dimensional manifolds and their 1-skeletons.

Question 5.2. Can something be said about when the clique dynamics of the triangulation of a manifold converges depending on the topology of the manifold?

The second generalisation is to allow for triangulations of surfaces *with boundary*. Such triangulations can be formalised as graphs for which each open neighbourhood is either a cycle (of length at least four) or a path graph – we shall call them **locally cyclic with boundary**. Triangulations of bordered surfaces have already received some attention: in [10, Theorem 1.4] the authors show that, except for the disc, each compact surface (potentially with boundary) admits a clique divergent triangulation. In contrast, they conjecture that discs do not have divergent triangulations:

Conjecture 5.3. *If a locally cyclic graph G with boundary and of minimum degree $\delta \geq 4$ triangulates a disc, then it is clique convergent (actually,* ***clique null****, that is, it converges to the one-vertex graph).*

This is known to be true if all interior vertices of the triangulation have degree ≥ 6 [6, Theorem 4.5].

Moving on from the topologically motivated investigations, yet another route is to generalise from locally cyclic graphs of a particular minimum degree to graphs of a lower-bounded **local girth** (that is, the girth of each open neighbourhood is bounded from below). In fact, it has already been noted by the authors of [7] that their results apply not only to locally cyclic graphs of minimum degree ≥ 7, but equally to general graphs of local girth ≥ 7.

Question 5.4. Can the results for locally cyclic graphs of minimum degree $\delta \geq 6$ be generalized to graphs of local girth ≥ 6?

Various other open questions emerge from the context of graph coverings. As we have seen in Lemma 4.3, if a graph G is clique convergent, so is its universal triangular cover $\tilde{G}$. Even stronger: if $k^n G \cong G$, then $k^n \tilde{G} \cong \tilde{G}$. If G is locally cyclic of minimum degree $\delta \geq 6$, then conversely, by Lemma 4.5 convergence of $\tilde{G}$ implies convergence of G.

For general triangular covers $p\colon \tilde{G} \to G$ (between connected locally finite graphs) however, such connections are not known. If both $\tilde{G}$ and G are finite, then a straightforward pigeon hole argument shows that clique convergence of G and of $\tilde{G}$ are equivalent. Yet, whether finite or infinite, it is generally unknown whether the statements $k^n G \cong G$ and $k^n \tilde{G} \cong \tilde{G}$ are always equivalent. We summarize all of this in the following question:

Question 5.5. Let $p\colon \tilde{G} \to G$ be a triangular covering map between two connected locally finite graphs. Is $\tilde{G}$ clique convergent if and only if G is clique convergent? To consider the directions separately, we ask:

(i) Is there an analogue of Lemma 4.3 for non-universal covering maps: if G is clique convergent but p is not universal, is $\tilde{G}$ clique convergent as well?
(ii) If $\tilde{G}$ is clique convergent, is G clique convergent as well?

An even stronger version of the question is: is $k^n \tilde{G} \cong \tilde{G}$ equivalent to $k^n G \cong G$ for every $n \in \mathbb{N}$? Is this at least true for finite graphs?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. The clique graph operator and simple connectivity

In this section, we show that triangular simple connectivity is preserved under the clique graph operator. A weaker version was obtained by Prisner [15] in 1992, who proved that the clique graph operator preserves the first $\mathbb{Z}_2$ Betti number. Larrión and Neumann-Lara [5] then extended this in 2000 to the isomorphism type of the triangular fundamental group. An extension to more general graph operators (including the clique graph operator and the line graph operator) was proven by Larrión, Pizaña, and Villarroel-Flores [9] in 2009. The proof presented here is completely elementary, as it explicitly constructs a sequence of elementary moves that transforms a given closed walk to the trivial one.

As triangular simple connectivity requires connectivity, we start with a lemma about connectivity before we show that closed walks are equivalent to trivial ones.

Lemma A.1. *For a connected graph G, the clique graph kG is also connected.*

Proof. Let $Q, Q' \in V(kG)$ be two cliques of G. We choose two vertices $v \in Q$ and $v' \in Q'$. As G is connected, there is a shortest walk $v_0 \dots v_\ell$ in G connecting $v_0 = v$ to $v_\ell = v'$. For each $i \in \{1, \dots, \ell\}$ we choose a clique Q_i that contains the pair of consecutive vertices v_{i-1} and v_i of this walk. Thus, for each $i \in \{1, \dots, \ell - 1\}$, the cliques Q_i and Q_{i+1} intersect in v_i and they are distinct, as otherwise the vertices v_{i-1} and v_{i+1} would be adjacent, in contradiction to the minimality of the walk $v_0 \dots v_\ell$. Thus, $Q_1 \dots Q_\ell$ is a walk in kG. If $Q \neq Q_1$ we add Q to the start of the walk and if $Q_\ell \neq Q'$ we append Q'. The resulting walk connects Q and Q' in kG and, thus, kG is connected. $\square$

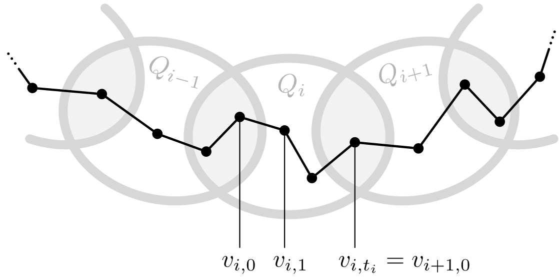

Fig. A.11. The correspondence relation between a walk in G and one in kG.

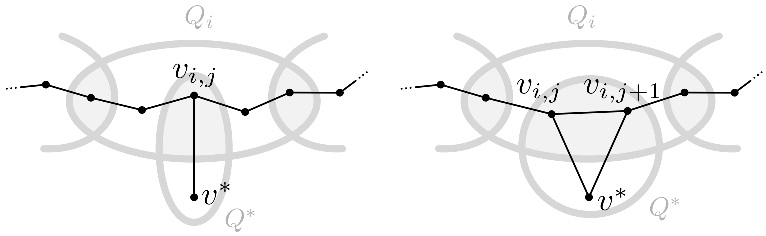

Fig. A.12. The elementary move in kG that corresponds to a dead end insertion (left) or triangle insertion (right) of a vertex which is not in Q_i.

We establish a concept of correspondence between a walk in G and a walk in kG in order to use the elementary moves that morph the former one into a trivial one as a guideline for doing the same with the latter one.

We say that a closed walk α in G and a closed walk $\alpha' = Q_0 \dots Q_\ell$ in kG with $Q_0 = Q_\ell$ **correspond** if for each $i \in \{0, \dots, \ell - 1\}$ there is a walk $v_{i,0} \dots v_{i,t_i}$ of length $t_i \in \mathbb{N}_0$ that lies completely in Q_i and α is the concatenation of those walks, i.e., $v_{i,t_i} = v_{i+1,0}$ for each $i \in \{0, \dots, \ell - 1\}$. As α is closed, we the have $v_{0,0} = v_{\ell-1,t_{\ell-1}} =: v_{\ell,0}$ (see Fig. A.11).

Note that for every closed walk in kG there is a corresponding one in G, which is obtained as follows. Let $\alpha' = Q_0 \dots Q_\ell$ with $Q_0 = Q_\ell$ be a closed walk in kG. For every $i \in \{1, \dots, \ell\}$, we choose $w_i \in Q_{i-1} \cap Q_i$, we define $w_0 := w_\ell$, and we drop repeated consecutive vertices. This way, we obtain a walk $\alpha = w_0 \dots w_\ell$ which clearly corresponds to α'.

Lemma A.2. *If G is a triangularly simply connected graph, so is kG.*

Proof. Let G be a triangularly simply connected graph. Thus, G is connected and, by Lemma A.1, so is kG. Next, we show that every closed walk in kG can be morphed to a single vertex by a sequence of elementary moves. Let $\alpha' = Q_0 \dots Q_\ell$ with $Q_0 = Q_\ell$ be a closed walk in kG. Let α be any corresponding walk in G, thus α consists of subwalks $v_{i,0} \dots v_{i,t_i}$ as described above.

Since G is triangularly simply connected, there is a sequence of elementary moves from α to a trivial walk. We now describe how we use the first of these moves as a guideline for elementary moves on α'; for the other moves in the sequence, it works by induction.

Let β be the walk in G that is reached from α by the first move. We now perform two steps in order to construct a walk β' in kG, which is homotopic to α' and which corresponds to β.

The first step consists of repeated triangle removals and dead end removals on α' that preserve the correspondence to α until α' cannot be shortened any further in that way. As no elementary move can change the start and end vertex of a walk, we do not remove $Q_0 = Q_\ell$ this way. As for every $i \in \{1, \dots, \ell - 1\}$ with $t_i = 0$, the clique Q_i can be removed in a triangle or dead end removal; the only t_i that can be zero is t_0. For the second step, we distinguish two cases.

<u>Case 1:</u> insertion moves. If the elementary move from α to β is a triangle insertion or dead end insertion, let the indices $i \in \{0, \dots, \ell - 1\}$ and $j \in \{0, \dots, t_i - 1\}$ be chosen such that the additional one or two vertices are inserted between $v_{i,j}$ and $v_{i,j+1}$. For the triangle insertion, the subwalk $v_{i,0} \dots v_{i,t_i}$ becomes $v_{i,0} \dots v_{i,j} v^* v_{i,j+1} \dots v_{i,t_i}$ and for the dead end insertion, it becomes $v_{i,0} \dots v_{i,j} v^* v_{i,j} v_{i,j+1} \dots v_{i,t_i}$. If $v^* \in Q_i$, $\beta' := \alpha'$ corresponds to β and we are finished. If $v^* \notin Q_i$, let Q^* be a clique that contains v^*, $v_{i,j}$ and, in the case of a triangle insertion, also $v_{i,j+1}$. Then, the dead end inclusion of Q^* and Q_i behind Q_i yields a walk β'. In the case of a dead end inclusion, it corresponds to β because $v_{i,0} \dots v_{i,j}$ and $v_{i,j} \dots v_{i,t_i}$ lie in Q_i and $v_{i,j} v^* v_{i,j}$ lies in Q^*. In the case of a triangle inclusion, it corresponds to β because $v_{i,0} \dots v_{i,j}$ and $v_{i,j+1} \dots v_{i,t_i}$ lie in Q_i and $v_{i,j} v^* v_{i,j+1}$ lies in Q^* (see Fig. A.12).

<u>Case 2:</u> removal moves. If the elementary move from α to β is a triangle removal or dead end removal, let the indices $i \in \{0, \dots, \ell - 1\}$ and $j \in \{0, \dots, t_i - 1\}$ be chosen such that $v_{i,j}$ (triangle removal) or $v_{i,j}$ and $v_{i,j+1}$ (dead end removal) are removed from Q_i. This choice is possible, as the (first) removed vertex and its successor lie in a common Q_i. If $j \geq 1$, the walk $\beta' = \alpha'$ corresponds to β as $v_{i,0} \dots v_{i,j-1} v_{i,j+1} \dots v_{i,t_i}$ or $v_{i,0} \dots v_{i,j-1} v_{i,j+2} \dots v_{i,t_i}$ respectively, still lie in Q_i. In case of a dead end removal, this works even if $t_i = j + 1$, as then $v_{i,j-1} = v_{i,j+1} = v_{i+1,0}$.

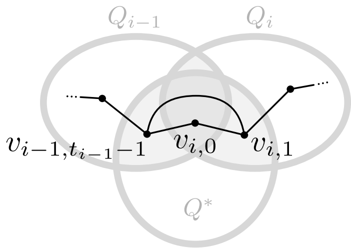

Fig. A.13. The elementary move in kG that corresponds to triangle removal in G.

If $j=0$, we know that $i \neq 0$, as otherwise $v_{i,j}=v_{0,0}$ would be removed. Furthermore, we know that if $i=1$, $t_0 \neq 0$ as this also would imply that $v_{0,0}=v_{1,0}$ is removed. In any case, $v_{i,j}$ lies between $v_{i-1,t_{i-1}-1}$ and $v_{i,1}$. We now distinguish between two subcases.

Case 2.1: $v_{i-1,t_{i-1}-1} \notin Q_i$ and $v_{i,1} \notin Q_{i-1}$. As $v_{i,1} \in Q_i$, it is immediately clear that $v_{i-1,t_{i-1}-1} \neq v_{i,1}$, thus it is a triangle removal step and $v_{i-1,t_{i-1}-1}v_{i,0}v_{i,1}$ is a triangle. Let Q^* be a clique that contains $v_{i-1,t_{i-1}-1}$ and $v_{i,1}$. As Q^* is neither Q_{i-1} nor Q_i, the insertion of Q^* between Q_{i-1} and Q_i is a triangle insertion and thus the resulting walk β' is homotopic to α'. Furthermore, β and β' correspond, because $v_{i-1,0}\dots v_{i-1,t_{i-1}-1}$ lies in Q_{i-1}, $v_{i-1,t_{i-1}-1}v_{i,1}$ lies in Q^* and $v_{i,1}\dots v_{i,t_i}$ lies in Q_i (see Fig. A.13).

Case 2.2: $v_{i-1,t_{i-1}-1} \in Q_i$ or $v_{i,1} \in Q_{i-1}$. We start by assuming that $v_{i,1} \in Q_{i-1}$. We subdivide α differently in pieces that each lie in one clique Q_i. Let $t'_{i-1} := t_{i-1}+1$, $t'_i := t_i - 1$ and $t'_s := t_s$ for every $s \in \{0,\dots,\ell-1\} \setminus \{i-1,i\}$. Furthermore, let $v'_{i-1,t'_{i-1}} := v_{i,1}$, let $v'_{i,u} := v_{i,u+1}$ for every $u \in \{0,\dots,t'_i\}$, and let $v'_{s,u} := v_{s,u}$ for every $s \in \{0,\dots,\ell-1\} \setminus \{i-1,i\}$ and every $u \in \{0,\dots,t'_s\}$. Now, the removed vertex is $v'_{i-1,t'_{i-1}}$ and as $t'_{i-1} \geq 1$ we are in the $(j \geq 1)$-part of Case 2, which we have already dealt with. The step for $v_{i-1,t_{i-1}-1} \in Q_i$ is analogous.

After proceeding inductively for the other moves of the sequence, we reach a closed walk in kG which corresponds to a trivial walk in G. Thus, all vertices of that walk in kG are pairwise connected, as they all contain the single vertex of that trivial walk, and the walk can easily be morphed into a trivial one. □

Appendix B. Some background on (universal) triangular covers

In this section, we provide some background on triangular covering maps. We start with some preliminaries on walk homotopy in the preimage and image of a triangular covering map. After that, we spend the main part of this section showing that the universal cover of a connected graph is unique up to isomorphism and covers every other triangular cover of a connected graph. Afterwards, we show that the universal covering map is Galois, i.e., that it can be interpreted as factoring out a group of symmetries from a graph. Most of the proofs are based on ideas from [16], but they only use basic concepts and they are much more concise as they use stronger prerequisites than the respective theorems in [16] have.

We remark that every triangular covering map $p\colon \tilde{G} \to G$ fulfils the **unique edge lifting property**, i.e., for each pair of adjacent vertices $v, w \in V(G)$ and each $\tilde{v} \in V(\tilde{G})$ such that $p(\tilde{v}) = v$, there is a unique $\tilde{w} \in V(\tilde{G})$ such that $\tilde{v}$ and $\tilde{w}$ are adjacent and $p(\tilde{w}) = w$. This property is equivalent to the **unique walk lifting property**, which says that for each walk α in G and each preimage of its start vertex there is a unique walk $\tilde{\alpha}$ in $\tilde{G}$ which is mapped to α. Furthermore, triangular covering maps fulfil the **triangle lifting property**, i.e., for each triangle (i.e. 3-cycle) $\{u,v,w\}$ in G and each preimage $\tilde{u}$ of u, there exists a unique triangle $\{\tilde{u},\tilde{v},\tilde{w}\}$ in $\tilde{G}$ that is bijectively mapped to $\{u,v,w\}$. Lastly, it follows from the unique walk lifting property that every triangular covering map between two connected graphs is surjective.

Throughout this section, we repeatedly make use of the following lemma connecting triangular covering maps and homotopy of walks. The lemma is equivalent to [16, Lemma 2.2] but instead of proving this equivalence, we reprove it in the language of [5].

Lemma B.1. *Given a triangular covering map $p\colon \tilde{G} \to G$ and two homotopic walks $\alpha = v_0 \dots v_\ell$ and $\beta = v'_0 \dots v'_{\ell'}$ in G, for a fixed vertex $\tilde{v}$ from the preimage of their common start vertex $v_0 = v'_0$ the unique walks $\tilde{\alpha} = \tilde{v}_0 \dots \tilde{v}_\ell$ with $p(\tilde{v}_i) = v_i$ and $\tilde{\beta} = \tilde{v}'_0 \dots \tilde{v}'_{\ell'}$ with $p(\tilde{v}'_i) = v'_i$ are homotopic as well. Especially, they have the same end vertex $\tilde{v}_\ell = \tilde{v}'_{\ell'}$.*

Proof. As homotopy is defined by a finite sequence of elementary moves, it suffices to show that an elementary move in the image implies an elementary move in the preimage. Thus, let $\alpha = v_0 \dots v_\ell$ be a walk in G and let $\tilde{\alpha} = \tilde{v}_0 \dots \tilde{v}_\ell$ be from its preimage with $p(\tilde{v}_i) = v_i$. Let β be reached from α by inserting a vertex v^* and possibly v_i again between v_i and v_{i+1} for some $i \in \{0,\dots,\ell-1\}$. As lifting a walk is done vertex by vertex from start to end, the lift of β begins with the vertices $\tilde{v}_0$ to $\tilde{v}_i$. As the restriction of p to the neighbourhood of v_{i-1} is an isomorphism, the lift of β starting in $\tilde{v}_0$ still has $\tilde{v}_{i+1}$ as the preimage of v_{i+1} and consequently the lift of β agrees with that of α in all following vertices. Thus, the lift of β arises from the lift of α by inserting a vertex $\tilde{v}^*$ and possibly $\tilde{v}_i$ between $\tilde{v}_i$ and $\tilde{v}_{i+1}$, which is an elementary move. For the elementary moves that remove vertices, exchange α and β. □

Next, we show that every connected graph has a universal triangular cover. The proof of the following lemma is influenced by a combination of [16, Theorem 2.5, 2.8, and 3.6].

Lemma B.2. *For every connected graph G, there is a universal triangular covering map $p\colon \tilde{G} \to G$, i.e., a triangular covering map with a triangularly simply connected graph $\tilde{G}$.*

Proof. We give a construction for a graph $\tilde{G}$ and a map p. Then, we show that p is in fact a triangular covering map, that $\tilde{G}$ is connected and that $\tilde{G}$ is triangularly simply connected.

Construction of $\tilde{G}$ and p: We fix a vertex v of G. For each walk α, we denote by $[\alpha]$ its homotopy class, i.e., the set of walks that can be reached from α by a finite sequence of elementary moves. A walk β is called a continuation of a walk α if β arises from α by appending exactly one vertex to its end. Now we can define the graph $\tilde{G}$ by

$$V(\tilde{G}) = \{[\alpha] \mid \alpha \text{ is a walk in } G \text{ starting at vertex } v\} \text{ and}$$
$$E(\tilde{G}) = \{[\alpha][\beta] \mid \beta \text{ is a continuation of } \alpha\}.$$

Note that $[\alpha][\beta] \in E(\tilde{G})$ does not imply that β is a continuation of α, but there is a $\beta' \in [\beta]$ such that β' is a continuation of α. We define

$$p\colon \tilde{G} \to G, [\alpha] \mapsto \text{end}(\alpha),$$

in which end(α) is the end vertex of α. The map p is well defined as homotopic walks have the same start and end vertex.

Triangular covering map: For an edge $[\alpha][\beta] \in E(\tilde{G})$, let without loss of generality β be a continuation of α. Thus, the end vertices of the two walks are adjacent and p is a graph homomorphism. Next we show that the restriction of p to neighbourhoods is bijective. Thus, let $[\alpha_w]$ be a class of walks from v to some vertex w. As noted above, the neighbourhood of $[\alpha_w]$ consists of the classes of continuations of α_w to the neighbours of w. Especially, the restriction of p to the neighbourhoods of $[\alpha_w]$ and w, respectively, is bijective. Let now α_x and α_y be the continuations of α_w by two distinct neighbours x and y of w. As we have already shown that the adjacency of $[\alpha_x]$ and $[\alpha_y]$ implies the adjacency of x and y, it remains to show the reverse. Thus, let x and y be adjacent. Hence, we can construct the walk α'_y as the continuation of α_x by the vertex y, thus, $[\alpha_x]$ and $[\alpha'_y]$ are adjacent. Since α'_y is reached from α_y by the elementary move of inserting x between w and y, they are homotopic and thus $[\alpha_y]$ is also adjacent to $[\alpha_x]$.

Connectivity: We show that every vertex $[\alpha]$ is connected to the trivial walk α_v that consists only of the vertex v. Thus, let α be any walk in G. The vertices $[\alpha_v]$ and $[\alpha]$ are connected by the walk $[\beta_0]\dots[\beta_\ell]$ in $\tilde{G}$, where ℓ is the length of α, and β_i is the initial subwalk of length i of α.

Triangular simple connectivity: For a closed walk $[\alpha_0]\dots[\alpha_\ell]$ with $[\alpha_0] = [\alpha_\ell]$ in $\tilde{G}$, we can assume without loss of generality that α_i is a continuation of α_{i-1} for each $i \in \{1,\dots,\ell\}$. Furthermore, we can assume that α_0 is the trivial walk as all the walks $\alpha_0,\dots,\alpha_\ell$ coincide with α_0 on their initial subwalks, anyway. We prove that the closed walk $[\alpha_0]\dots[\alpha_\ell]$ and the trivial walk $[\alpha_0]$ are homotopic.

As α_0 and α_ℓ are homotopic, there is a finite sequence of elementary moves that morphs α_ℓ into α_0. To each walk α' in G that occurs in this homotopy between α_0 and α_ℓ, we associate the walk $[\alpha'_0]\dots[\alpha'_{\ell'}]$ where ℓ' is the length of α', and α'_i is the initial subwalk of length i of α'. Then, $[\alpha'_0]\dots[\alpha'_{\ell'}]$ is a walk by construction and it fulfils $\alpha'_0 = \alpha_0$ and $\alpha'_{\ell'} = \alpha'$. This way, we associate the final (trivial) walk α_0 to the trivial walk $[\alpha_0]$. If the walks α' and α'' are connected by an elementary move in G, their associated walks in $\tilde{G}$ are connected by the corresponding elementary move in the following way: a triangle insertion move that inserts v^* after v_i corresponds to the insertion of the class of the continuation of α_i by v^* and changing the representative of the following classes to the one, in which v^* is inserted after v_i. The other elementary moves work analogously. □

In the next lemma, we show that universal triangular covers are in fact universal objects. The proof is a combination of special cases from the proofs of [16, Theorem 3.2 and Theorem 3.3].

Lemma B.3. *The universal triangular covering map $p\colon \tilde{G} \to G$ fulfils the following universal property: for each triangular covering map $q\colon \bar{G} \to G$ there exists a triangular covering map $\tilde{q}\colon \tilde{G} \to \bar{G}$ such that $p = q \circ \tilde{q}$ (see the commuting diagram in Fig. B.14). Furthermore, for any pair of fixed vertices $\tilde{v} \in V(\tilde{G})$ and $\bar{v} \in V(\bar{G})$ such that $p(\tilde{v}) = q(\bar{v})$, we get a unique triangular covering map $\tilde{q}_{\tilde{v},\bar{v}}\colon \tilde{G} \to \bar{G}$ with $p = q \circ \tilde{q}_{\tilde{v},\bar{v}}$ and $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{v}) = \bar{v}$.*

Proof. Let $p\colon \tilde{G} \to G$ be a triangular covering map such that $\tilde{G}$ is triangularly simply connected and let $q\colon \bar{G} \to G$ be any triangular covering map. We fix a vertex $v \in V(G)$ as well as vertices $\tilde{v} \in V(\tilde{G})$ and $\bar{v} \in V(\bar{G})$ that are in the preimage of v under p and q, respectively. We construct $\tilde{q}_{\tilde{v},\bar{v}}$ from p and q and show that it is in fact a well-defined triangular covering map.

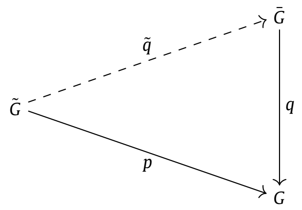

Fig. B.14. The commuting diagram depicting the property from Lemma B.3.

<u>Construction of $\tilde{q}_{\tilde{v},\bar{v}}$:</u> For each $\tilde{u} \in V(\tilde{G})$, we choose a walk $\alpha_{\tilde{v},\tilde{u}}$ from $\tilde{v}$ to $\tilde{u}$. The image of $\alpha_{\tilde{v},\tilde{u}}$ under p is a walk, which we call $\beta_{\tilde{u}}$, from $p(\tilde{v})$ to $p(\tilde{u})$. As $p(\tilde{v}) = v = q(\bar{v})$, by the unique walk lifting property, there is exactly one walk $\alpha_{\bar{v},\bar{u}}$ starting at $\bar{v}$ that is mapped to $\beta_{\tilde{u}}$ by q. We define $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{u})$ to be the end vertex $\bar{u}$ of $\alpha_{\bar{v},\bar{u}}$.

<u>Well-Definedness:</u> We need to show that $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{u})$ is independent of the choice of the walk $\alpha_{\tilde{v},\tilde{u}}$. Thus, let $\alpha'_{\tilde{v},\tilde{u}}$ be a different walk from $\tilde{v}$ to $\tilde{u}$. Its image under p is called $\beta'_{\tilde{u}}$ which has the same start and end vertices as $\beta_{\tilde{u}}$. As $\tilde{G}$ is triangularly simply connected, the walks $\alpha_{\tilde{v},\tilde{u}}$ and $\alpha'_{\tilde{v},\tilde{u}}$ are homotopic and, consequently, so are $\beta_{\tilde{u}}$ and $\beta'_{\tilde{u}}$. By Lemma B.1 also the preimages under q, which are called $\alpha_{\bar{v},\bar{u}}$ and $\alpha'_{\bar{v},\bar{u}}$, are homotopic and, thus, have the same end vertex, implying $\tilde{q}_{\tilde{v},\bar{v}}$ being well defined. Additionally, $p = q \circ \tilde{q}_{\tilde{v},\bar{v}}$ holds by construction.

<u>Homomorphy:</u> Let $\tilde{x}$, $\tilde{y}$ be adjacent vertices in $\tilde{G}$. Let $\alpha_{\tilde{v},\tilde{y}}$ be a walk from $\tilde{v}$ to $\tilde{y}$ such that $\tilde{x}$ is its penultimate vertex. Via the same construction as above, we obtain a walk $\alpha_{\bar{v},\bar{y}}$ such that its penultimate vertex $\bar{x}$ fulfils $p(\tilde{x}) = q(\bar{x})$. Consequently, $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{x}) = \bar{x}$ and $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{y}) = \bar{y}$ are adjacent and thus $\tilde{q}_{\tilde{v},\bar{v}}$ is a graph homomorphism.

<u>Triangular covering map:</u> Let $\tilde{u}$ be a vertex of $\tilde{G}$ and let $u = p(\tilde{u})$ and $\bar{u} = \tilde{q}_{\tilde{v},\bar{v}}(\tilde{u})$ be its images. As $p|_{N[\tilde{u}]} : N[\tilde{u}] \to N[u]$ and $q|_{N[\bar{u}]} : N[\bar{u}] \to N[u]$ are isomorphisms, so is $\tilde{q}_{\tilde{v},\bar{v}}|_{N[\tilde{u}]} = q|^{-1}_{N[\bar{u}]} \circ p|_{N[\tilde{u}]}$.

<u>Uniqueness of $\tilde{q}_{\tilde{v},\bar{v}}$:</u> Let $\tilde{q} : \tilde{G} \to \bar{G}$ be any triangular covering map such that $p = q \circ \tilde{q}$ and $\tilde{q}(\tilde{v}) = \bar{v}$. With the definitions from above, both the image of $\alpha_{\tilde{v},\tilde{u}}$ under $\tilde{q}$ and $\alpha_{\bar{v},\bar{u}}$ are lifts of the walk $\beta_{\tilde{u}}$ and they share the start vertex $\bar{v}$. By the unique walk lifting property, they are equal and so is their end vertex, implying $\tilde{q}(\tilde{u}) = \bar{u} = \tilde{q}_{\tilde{v},\bar{v}}(\tilde{u})$. □

Lemma B.4. *If for a graph G there are two graphs $\tilde{G}$ and $\bar{G}$ and two triangular covering maps $p : \tilde{G} \to G$ and $q : \bar{G} \to G$ such that p and q both fulfil the universal property from Lemma B.3, $\tilde{G}$ and $\bar{G}$ are isomorphic.*

Proof. Let $p : \tilde{G} \to G$ and $q : \bar{G} \to G$ be two triangular covering maps which both fulfil the universal property. Furthermore, let $\tilde{v} \in V(\tilde{G})$ and $\bar{v} \in V(\bar{G})$ be chosen such that $p(\tilde{v}) = q(\bar{v})$. By the universal properties, there are (unique) triangular covering maps $\tilde{p} : \bar{G} \to \tilde{G}$ and $\tilde{q} : \tilde{G} \to \bar{G}$ such that $p = q \circ \tilde{q}$, $\tilde{q}(\tilde{v}) = \bar{v}$, $q = p \circ \tilde{p}$, and $\tilde{p}(\bar{v}) = \tilde{v}$. Consequently, $p = p \circ \tilde{p} \circ \tilde{q}$ and $(\tilde{p} \circ \tilde{q})(\tilde{v}) = \tilde{v}$. As the identity map $id : \tilde{G} \to \tilde{G}$ is a triangular covering map that fulfils $p = p \circ id$ and $id(\tilde{v}) = \tilde{v}$, we know by the uniqueness of the universal property of p that $\tilde{p} \circ \tilde{q} = id$, which implies that $\tilde{q} : \tilde{G} \to \bar{G}$ is an isomorphism. □

Theorem B.5. *Every connected graph has a universal triangular cover, which is unique up to isomorphism.*

Proof. By Lemma B.2, the graph G has a universal triangular cover. Let $p : \tilde{G} \to G$ and $q : \bar{G} \to G$ be two universal triangular covering maps. By applying Lemma B.3, they both have the universal property. By Lemma B.4, the universal triangular covers are isomorphic. □

Now we can look at the universal triangular cover through the lens of quotient graphs by using Galois covering maps. We reprove this lemma from [1] using only basic notation.

Lemma B.6. *A universal triangular covering map $p : \tilde{G} \to G$ is Galois with $\Gamma := \{\gamma \in \mathrm{Aut}(\tilde{G}) \mid p \circ \gamma = p\}$, which is called the **deck transformation group** of p. Moreover, it holds that $(k^n\tilde{G})/\Gamma \cong k^n G$.*

Proof. As each $\gamma \in \Gamma$ fulfils $p \circ \gamma = p$, the group Γ acts on every vertex preimage of p individually. Thus, it suffices to show that for each pair of vertices $\tilde{v}, \tilde{w}$ with $p(\tilde{v}) = p(\tilde{w})$ there is a $\gamma \in \Gamma$ such that $\gamma(\tilde{v}) = \tilde{w}$. If we apply Lemma B.3 with $q = p$, we get a triangular covering map $\tilde{q}_{\tilde{v},\tilde{w}}$ which maps $\tilde{v}$ to $\tilde{w}$ and which is an isomorphism by Theorem B.5, thus $\gamma = \tilde{q}_{\tilde{v},\tilde{w}}$ fulfils the condition. As p is a Galois covering map, by [5, Proposition 3.2] so is p_{k^n}. Consequently, it holds that $(k^n\tilde{G})/\Gamma \cong k^n G$. □

Appendix C. The isomorphism between G_n and $k^n G$

The proof of Theorem 2.2 as presented in [1] provides an explicit construction for the isomorphism ψ_n between the clique graph $k^n G$ and the geometric clique graph G_n (Definition 2.1). More precisely, isomorphisms C_n are constructed between G_n and kG_{n-1}.

In this section we repeat the construction (Appendix C.1) and give a short argument for why this yields Γ-isomorphisms (Appendix C.2) as required in Section 4.

C.1. Notation and isomorphisms

We define the hexagonal grid as well as triangular-shaped graphs in a way that enables precise definition of maps.

Definition C.1. Define the coordinate set

$$\vec{\boldsymbol{D}}_{\mathbf{0}} := \big\{(1,-1,0),(1,0,-1),(-1,1,0),(0,1,-1),(-1,0,1),(0,-1,1)\big\}.$$

For $m \in \mathbb{Z}$, the **hexagonal grid of height m** is the graph $\mathbf{Hex_m} = (V_m, E_m)$ with

$$\boldsymbol{V_m} := \{(x_1,x_2,x_3) \in \mathbb{Z}^3 \mid x_1+x_2+x_3 = m\} \text{ and}$$
$$\boldsymbol{E_m} := \{\{x,y\} \subset V_m \mid x-y \in \vec{D}_0\}.$$

For $m \geq 0$ the **triangular-shaped graph $\boldsymbol{\Delta_m}$ of side length $\boldsymbol{m}$**, is defined as the induced subgraph $\mathrm{Hex}_m[V_m \cap \mathbb{Z}^3_{\geq 0}]$. The boundary $\boldsymbol{\partial\Delta_m}$ is the subgraph of Δ_m that consists of the vertices of degree less than six and the edges that lie in only a single triangle.

For a locally cyclic graph G, a **hexagonal chart** is a graph isomorphism $\mu: H \to F$ (also written $H \overset{\mu}{\Longrightarrow} F$) with induced subgraphs $H \subseteq \mathrm{Hex}_m$ and $F \subseteq G$. For $(t_1,t_2,t_3) \in \mathbb{Z}^3$, we define the **triangle inclusion map**:

$$\boldsymbol{\Delta_m^{t_1,t_2,t_3}}: \Delta_m \to \mathrm{Hex}_{m+t_1+t_2+t_3}, \qquad (a_1,a_2,a_3) \mapsto (a_1+t_1, a_2+t_2, a_3+t_3).$$

Furthermore, we define

$$\vec{\boldsymbol{E}} := V_1 \cap \mathbb{N}_0^3 = \{(1,0,0),(0,1,0),(0,0,1)\},$$
$$\boldsymbol{\nabla_1} := \mathrm{Hex}_2[(1,1,0),(0,1,1),(1,0,1)], \text{ and}$$
$$\boldsymbol{\nabla_2'} := \mathrm{Hex}_1[(1,1,-1),(-1,1,1),(1,-1,1)].$$

For a hexagonal chart $\mu: \Delta_{m+1} \to S$ and $(t_1,t_2,t_3) \in \mathbb{Z}^3$, we denote the image of $\mu \circ \Delta^{t_1,t_2,t_3}_{m+1-t_1-t_2-t_3}$ by $\boldsymbol{\mu^{t_1,t_2,t_3}}$. The following remark is a combination of [1, Corollary 6.9] and [1, Corollary 7.8].

Remark C.2. If G is a triangularly simply-connected locally cyclic graph of minimum degree $\delta \geq 6$, then for each $n \in \mathbb{N}_0$, there is an isomorphism $C_{n+1}: G_{n+1} \to kG_n$, for which a direct construction is given as follows:

(a) Let $\Delta_m \overset{\mu}{\Longrightarrow} S \in V(G_{n+1})$, for $m \geq 1$. If $m = 1$ let $\hat{\mu}: \nabla_2' \to G$ be the hexagonal chart extending μ. It exists and is unique, as each pair of vertices of S has one common neighbour outside S. Then,

$$C_{n+1}(S) = \underbrace{M_{m-1}}_{|\cdot|=3} \cup \underbrace{M_{m+1}}_{\substack{|\cdot|\leq 3\\ |\cdot|=0,\text{ if } n=m,}} \cup \underbrace{M_{m+3}}_{\substack{|\cdot|\leq 1\\ |\cdot|=0,\text{ if } n\leq m+2,}} \cup \begin{cases} \varnothing, & \text{if } m=1 \text{ and } n \leq 1,\\ \{\hat{\mu}(\nabla_2')\}, & \text{if } m=1 \text{ and } n \geq 2,\\ \{\mu(\nabla_1)\}, & \text{if } m=2,\\ \{S \setminus \partial S\}, & \text{if } m \geq 3.\end{cases}$$

- M_{m-1} consists of the elements $\Delta_{m-1} \cong \mu^{\vec{e}}$ for $\vec{e} \in \vec{E}$.
- M_{m+1} consists of the elements $\Delta_{m+1} \overset{\nu}{\Longrightarrow} T$ fulfilling $\mu = \nu \circ \Delta^{\vec{e}}_{m-1}$ for an $\vec{e} \in \vec{E}$.
- M_{m+3} consists of the element $\Delta_{m+3} \cong T$ enclosing S with distance 1, i.e., $S = T \setminus \partial T$.

(b) For $\Delta_0 \cong S \in V(G_{n+1})$, we denote the vertex of S by v. In this case,

$$C_{n+1}(S) = \underbrace{\{T \in V(G_n) \mid T \cong \Delta_1,\ S \subseteq T\}}_{|\cdot|=\deg_G(v)} \cup \underbrace{\{T \in V(G_n) \mid T \cong \Delta_3,\ S \subseteq T \setminus \partial T\}}_{\substack{|\cdot|=0,\text{ if } \deg_G(v)\geq 7 \text{ or } n\leq 2,\\ |\cdot|=2,\text{ if } \deg_G(v)=6 \text{ and } n\geq 3,}}.$$

In conclusion, for each $m \geq 0$ and $S \cong \Delta_m$ the elements of $C_{n+1}(S)$ can only be isomorphic to Δ_{m-3}, Δ_{m-1}, Δ_{m+1}, or Δ_{m+3}.

C.2. Γ-isomorphisms

Let Γ be any group acting on G. Let Γ act on G_n and $k^n G$ as described in Remark 4.1 and Remark 4.2, respectively.

By close inspection of Remark C.2, it can be seen that the isomorphism $C_n\colon G_n \to kG_{n-1}$ is a Γ-isomorphism in the following way: The elements of the clique $C_n(S)$ for some $\Delta_m \overset{\mu}{\Rightarrow} S \in V(G_n)$ are each defined by hexagonal charts or by subgraph inclusions, which behave well towards the automorphisms induced by the elements from Γ. For example, the triangular-shaped graphs from M_{m-1} fulfil the following equivalences and similar calculations can be given for the other types of triangular-shaped graphs in the clique:

$$\begin{aligned} T \in M_{m-1}(S) &\iff T = \mu^{\vec{e}} \text{ for some } \vec{e} \in \vec{E} \\ &\iff \gamma(T) = (\gamma \circ \mu)^{\vec{e}} \text{ for some } \vec{e} \in \vec{E} \\ &\iff \gamma(T) \in M_{m-1}(\gamma(S)). \end{aligned}$$

Thus, a Γ-isomorphism $\psi_n\colon G_n \to k^n G$ is obtained from the following chain of Γ-isomorphisms:

$$\begin{aligned} G_n \xrightarrow{C_n} kG_{n-1} &\xrightarrow{(C_{n-1})_k} k(kG_{n-2}) = k^2 G_{n-2} \\ &\longrightarrow \cdots \longrightarrow k^{n-2}(kG_1) = k^{n-1} G_1 \xrightarrow{(C_1)_{k^{n-1}}} k^{n-1}(kG) = k^n G. \end{aligned}$$

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