

A one-dimensional model for aspiration therapy in blood vessels

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Abstract

Aspiration thrombectomy is a treatment option for ischemic stroke due to occlusions in large vessels. During the therapy a device is inserted into the vessel and suction is applied. A new one-dimensional model is introduced that is capable of simulating this procedure while accounting for the fluid-structure interactions in blood flow. To solve the coupling problem at the tip of the device a problem-suited Riemann solver is constructed based on relaxation of the hyperbolic model. Numerical experiments investigating the role of the catheter size and the suction forces are presented.

Keywords: Finite volume method; hyperbolic systems; coupling conditions; Riemann solvers

1 Introduction

Stroke has been found to be the second leading cause of both disability and death worldwide; ischemic stroke, in which the blood supply to parts of the brain is blocked, has accounted for approximately 87% of the cases [15]. The infarction causing blocks of the blood supply in the brain are often due to large vessel occlusions by blood clots. For affected patients endovascular mechanical thrombectomy has recently become the standard treatment option. Aspiration thrombectomy with a distal aspiration catheter is one of the common mechanical thrombectomy treatments.

Mathematical models can be used to obtain detailed insight on the intricate relationship between vascular anatomy, catheter positioning and blood flow physics and to plan clinical interventions by testing various strategies in-silico before their execution. The modeling of blood flow and the circulatory system, has been an active field of research in the last decades attracting engineers, mathematicians and clinicians, see, e.g., [4, 20]. The modeling needs to address various issues, such as the mechanical interaction with the vessel walls and tissue [14]. Applications of computational fluid dynamics (CFD) models based on the Navier-Stokes equations in three dimensions to endovascular mechanical thrombectomy with aspiration catheters have been considered in [16, 12]. These models have so far been limited as they neglect the interactions with the elastic vessel wall and have high computational costs.

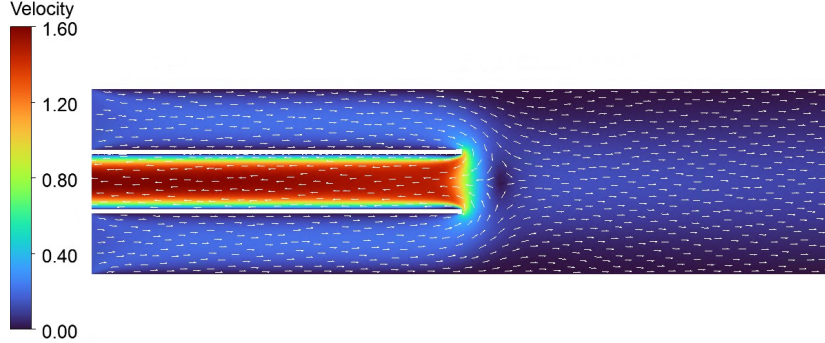


Figure 1: Velocity of the blood flow (in terms of directions and absolute values) in a vessel during aspiration from a steady state CFD simulation in 3D.

Various approaches have been undertaken to reduce the full complexity of 3D vasculature models to 1D network models [7, 2]. Such reduced models are commonly derived from mass conservation, momentum balance and simplified wall mechanics assuming a static equilibrium in the radial direction of a cylindrical vessel, which give rise to a first order PDE system determining the cross-section area and the mass flux depending on time and the axial coordinate. These systems have been studied regarding their numerical solution, well-posedness and smoothness, see e.g., [18, 11]. To account for a full vasculature they have been further extended to networks, see [19], and the resulting framework has been used in the modeling of prosthetics [17]. Occlusion in those network models have moreover been considered in [13] and [1]; the modeling of endovascular therapy in these settings has so far been limited to numerical experiments varying coupling conditions — therapy devices have not been explicitly modeled.

In this work we develop a new one-dimensional model for aspiration thrombectomy in a blood vessel. Thereby we focus on the modeling of the aspiration catheter in the vessel and, in particular, on the coupling problem that arises at the tip of the device. Here, blood flows in possibly opposite directions interact, see Figure 1, which might lead to flow reversal and constriction of the vessel. The rest of this work is organized as follows: we will first recall the one-dimensional modeling and derive a model for a catheterized vessel in Section 2. The coupling problem at the tip of the aspiration catheter is studied in Section 3 and various numerical experiments are presented in Section 4.

2 A Model for blood flow in a catheterized vessel

Blood flow is considered through a cylindrical vessel parameterized by the axial position x . In the vessel an aspiration catheter with cross section area A_c is inserted. For simplicity, we neglect the thickness of the device walls. We further assume that the walls of both the vessel and the device are impermeable. We denote the cross-section of the vessel at axial position x and time t by $S_V(t, x)$. By $S(t, x)$ we refer to the net cross-section deducting the device cross-section from S_V and by $A(t, x)$ to the area of $S(t, x)$. The average flow velocity over the net cross section in axial direction is denoted by $u(t, x)$ and $Q = Au$ denotes the average mass flow. Assuming a flat velocity profile as well as incompressible and inviscid blood flow through the elastic vessel, Reynolds' transport

theorem, conservation of mass and balance of momentum give rise to the model

$$\partial_t A + \partial_x Q = 0, \quad (1a)$$

$$\partial_t Q + \partial_x \left(\frac{Q^2}{A} \right) + \frac{A}{\rho} \partial_x p = 0. \quad (1b)$$

We refer to [11] for a detailed derivation in case of ordinary blood flow through a vessel, which is easily extended to the catheterized case. In (1b) p denotes the pressure, for which we assume the form

$$p = P_{ext} + \Phi(A; A_0, \beta) \quad (2)$$

given the reference vessel cross-section area A_0 and the mechanical parameter

$$\beta = \frac{E h_0 \sqrt{\pi}}{A_0}$$

with Young modulus and vessel thickness denoted by E and h_0 , respectively. The function Φ is strictly increasing in A and satisfies $\Phi(A_0; A_0, \beta) = 0$.

Assuming an instant equilibrium between the vessel wall and forces from the inner vessel an algebraic relation depending on the radius of the vessel, more specifically,

$$\Phi(A; A_0, \beta) = \beta \left(\sqrt{A + A_c} - \sqrt{A_0} \right) \quad (3)$$

is derived that we adopt in this work. Assuming negligible elasticity, we model the catheter as a rigid pipe, in which inviscid and isothermal blood flow is observed. Assuming a uniform density the flow velocity in the device that we denote by w is governed by the inviscid Burgers equation,

$$\partial_t w + w \partial_x w = 0. \quad (4)$$

Basic algebra allows us to rewrite (1) as a conservative system in terms of the section area and the flow velocity. In summary, the blood flow model in the catheterized vessel takes the form

$$\partial_t A + \partial_x (Au) = 0, \quad (5a)$$

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 + \frac{p}{\rho} \right) = 0, \quad (5b)$$

$$\partial_t w + \frac{1}{2} \partial_x w^2 = 0, \quad (5c)$$

where we have assumed that A_0 , P_{ext} and β are uniform in space. Note that (5c) decouples from (5a) and (5b).

Remark 2.1. *The flat velocity profile assumed in (1), which implies a lack of spacial variation of the velocity within the cross-section, makes the model independent of the exact position of the catheter within the catheterized vessel.*

3 The coupling problem at the tip of the device

In this section we consider the transition between the catheterized and un-catheterized segment of the vessel. We assume that a vessel parameterized by its axial position $x \in \mathbb{R}$ has the aspiration device inserted on the negative half axis $(-\infty, 0)$, whereas blood flows freely throughout the full cross-section on the positive half axis $(0, \infty)$. We thus describe flow through the catheterized vessel using model (5). The un-catheterized section is simply

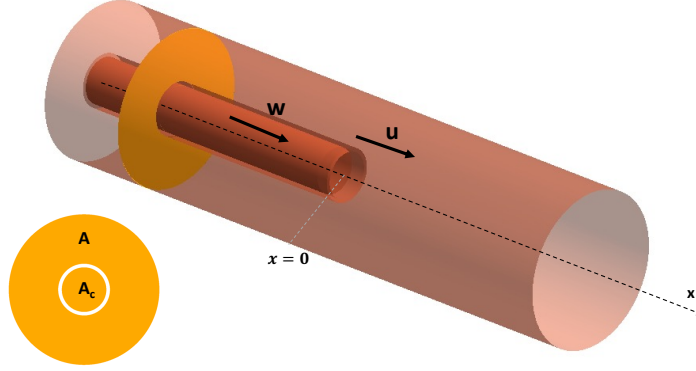


Figure 2: A cylindrical vessel with inserted aspiration catheter ending at position $x = 0$ as considered in model (6).

given by the submodel consisting of (5a), (5b) and an adjusted pressure function. Hence, we have

$$\partial_t A + \partial_x(Au) = 0 \quad (x, t) \in \mathbb{R} \setminus \{0\} \times (0, \infty), \quad (6a)$$

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 + \frac{p}{\rho} \right) = 0 \quad (x, t) \in \mathbb{R} \setminus \{0\} \times (0, \infty), \quad (6b)$$

$$\partial_t w + \frac{1}{2} \partial_x w^2 = 0 \quad (x, t) \in (-\infty, 0) \times (0, \infty) \quad (6c)$$

with pressure function

$$p = \beta \left(\sqrt{A + \mathbb{1}_{(-\infty, 0)}(x) A_c} - \sqrt{A_0} \right). \quad (7)$$

Here $\mathbb{1}$ refers to the indicator function and as the pressure can be considered as a function of the cross section area we will later use the notation $p^1(A)$ and $p^2(A)$ to refer to (7) on the negative and positive half-axis, respectively. We refer to Figure 2 for a visualization of the modeled vessel.

For closure of the system (6) a *coupling condition* at the interface $x = 0$ of the form

$$\Psi(A(0^-, t), u(0^-, t), w(0^-, t), A(0^+, t), u(0^+, t)) = 0 \quad \text{for a.e. } t \in (0, \infty), \quad (8)$$

which is possibly vector valued is required. By 0^- and 0^+ in the second argument in (8) we refer to the limits as x approaches the interface from the left and the right, respectively.

More specifically, we impose continuity of the gross vessel cross-section area at the interface, i.e.,

$$A(0^-, t) + A_c = A(0^+, t) \quad (9a)$$

as the first and continuity of the average flow velocity, i.e.,

$$\frac{A(0^-, t)u(0^-, t) + A_c w(0^-, t)}{A(0^-, t) + A_c} = u(0^+, t) \quad (9b)$$

as the second coupling condition.

3.1 Relaxation system

We follow the recently proposed approach from [6, 5] and proceed by coupling a relaxation version of (6) at the interface. In more details, we introduce the auxiliary variables v^A and v^u as well as a relaxation rate $\varepsilon > 0$ and employ Jin/Xin relaxation for the section area and the average axial velocity, resulting in the system

$$\partial_t A + \partial_x v^A = 0 \quad (x, t) \in \mathbb{R} \setminus \{0\} \times (0, \infty), \quad (10a)$$

$$\partial_t u + \partial_x v^u = 0 \quad (x, t) \in \mathbb{R} \setminus \{0\} \times (0, \infty), \quad (10b)$$

$$\partial_t v^A + \lambda^2 \partial_x A = \frac{1}{\varepsilon} (Au - v^A) \quad (x, t) \in \mathbb{R} \setminus \{0\} \times (0, \infty), \quad (10c)$$

$$\partial_t v^u + \lambda^2 \partial_x u = \frac{1}{\varepsilon} \left(\frac{1}{2} u^2 + \frac{p}{\rho} - v^u \right) \quad (x, t) \in \mathbb{R} \setminus \{0\} \times (0, \infty), \quad (10d)$$

$$\partial_t w + \frac{1}{2} \partial_x w^2 = 0 \quad (x, t) \in (-\infty, 0) \times (0, \infty). \quad (10e)$$

For stability of the relaxation approach the relaxation speed λ is taken as the largest eigenvalue of (5), cf. [9], further details on its choice are discussed below. We note that the state variables governed by system (10) depend on ε , which for brevity is not indicated in the notation, i.e., in the following sections A and u refer to the states of (10). It is well-known, that as $\varepsilon \rightarrow 0$ these quantities recover the states of the original model (5), see [10].

To close this system a coupling condition analogue to (8) is required, that, in addition, imposes rules for the coupling states corresponding to the variables v^A and v^u . For consistency in the relaxation limit $\varepsilon \rightarrow 0$, see [5], we impose the conditions

$$v^A(0^-, t) + A_c w(0^-, t) = v^A(0^+, t), \quad (11a)$$

$$v^u(0^-, t) + \frac{u(0^+, t)^2 - u(0^-, t)^2}{2} = v^u(0^+, t) \quad (11b)$$

in addition to (9). These conditions are obtained assuming that the relaxation variables admit their relaxation limits $v^A = Au$ and $v^u = \frac{1}{2}u^2 + \frac{p}{\rho}$ and using (9).

3.2 Riemann solver at the interface

Given the trace data $A_0^-, u_0^-, v_0^{A-}, v_0^{u-}$ and w_0^- left from the interface as well as $A_0^+, u_0^+, v_0^{A+}, v_0^{u+}$ right from the interface, e.g., from a finite volume scheme, a Riemann solver (RS) for system (10) is a map

$$\mathcal{RS} : (A_0^-, u_0^-, v_0^{A-}, v_0^{u-}, w_0^-, A_0^+, u_0^+, v_0^{A+}, v_0^{u+}) \mapsto (A_R, u_R, v_R^A, v_R^u, w_R, A_L, u_L, v_L^A, v_L^u) \quad (12)$$

that assigns suitable coupling/boundary data. This coupling data both solves the half-Riemann problems at the interface and satisfies the coupling conditions (9) and (11).

As the subsystem given by (10a)–(10d) is linear and the convective terms decouple the former condition implies that the trace states connect to the coupling data by lines in their respective phase space. In more details the connection towards the negative half-axis is given by

$$\begin{aligned} (A_R, v_R^A) &\in L^-(A_0^-, v_0^{A-}), \quad (u_R, v_R^u) \in L^-(u_0^-, v_0^{u-}), \\ L^-(z_1, z_2) &= \{(z_1 - \sigma, z_2 + \lambda\sigma) : \sigma \in \mathbb{R}\} \end{aligned} \quad (13a)$$

and the connection towards the positive half-axis by

$$\begin{aligned} (A_L, v_L^A) &\in L^+(A_0^+, v_0^{A+}), \quad (u_L, v_L^u) \in L^+(u_0^+, v_0^{u+}), \\ L^+(z_1, z_2) &= \{(z_1 + \sigma, z_2 + \lambda\sigma) : \sigma \in \mathbb{R}\}. \end{aligned} \quad (13b)$$

Considering the conditions (9a) and (11a) together with (13) a linear system is obtained that uniquely determines the coupling data for the states A and v^A . Those are given by

$$\begin{aligned} A_R &= \frac{1}{2} (A_0^- + A_0^+ - A_c) + \frac{1}{2\lambda} (v_0^{A-} - v_0^{A+} + A_c w_R), \quad A_L = A_R + A_c, \\ v_R^A &= \frac{1}{2} (v_0^{A-} + v_0^{A+} - A_c w_r) + \frac{\lambda}{2} (A_0^- - A_0^+ + A_c), \quad v_L^A = v_R^A + A_c w_r. \end{aligned} \quad (14)$$

Next, we determine coupling data for u and v^u . To this end we parameterize the tuples (u_R, v_R^u) and (u_L, v_L^u) on their corresponding Lax-curves in (13) by σ_u^- and σ_u^+ , respectively. Given the trace states, A_R and A_L from (14) and w_R we use condition (9b) to eliminate the parameter σ_u^- . Then substituting into (11b) results in a quadratic equation in σ_u^+ , which for $A_c > 0$ and trace states sufficiently close to the coupling data has two real roots, for details we refer to Appendix A. From those roots we select the one with minimal absolute value, which then determines the coupling data u_R , v_R^u , u_L and v_L^u . Since $u_L = u_0^+ + \sigma_0^+$ this choice imposes the minimal change in velocity in the un-catheterized part of the vessel under the given conditions.

For suitable boundary data for the inviscid Burgers equation we require

$$w_R \in (-\infty, \min\{0, -w_0^-\}] \quad (15)$$

and within our RS simply choose $w_R = -|w_0^-|$.

4 Numerical experiments

Numerical scheme To simulate aspiration thrombectomy in a vessel a numerical scheme is required. We consider an adapted variant of a recently introduced central scheme for networks [8]. As the scheme is derived from the relaxation approach that was used to derive (10) the coupling data from the RS naturally enters in the computation of the fluxes at the interface.

Instead of the unbounded setting in Section 3, the catheterized section is parameterized on the interval $\Omega^1 = [-5, 0]$ with tip at interface position $x = 0$ and the un-catheterized section of the vessel is parameterized on the interval $\Omega^2 = [0, 5]$; combined a 10 cm vessel is modeled. We discretize both intervals into the cells $C_j = [x_{j-1/2}, x_{j+1/2}]$ for $j = -N + 1, \dots, N$ with interface points given by $x_{j-1/2} = j\Delta x$ for $\Delta x = 5/N$. Now, let $A_j^{k,n}$ and $u_j^{k,n}$ denote approximate averages of A and u over the cell C_j at time instance $t_n = \sum_{\ell=1}^n \Delta t^\ell$. In this notation k indicates whether C_j is a subset of the negative ($k = 1$) or the positive ($k = 2$) half-axis. Similarly, we denote approximate averages of the velocity w left to the interface by w_j^n . The scheme for model (6) can then be written in conservative form as

$$A_j^{k,n+1} = A_j^{k,n} - \frac{\Delta t}{\Delta x} (F_{j+1/2}^{k,n} - F_{j-1/2}^{k,n}), \quad k \in \{1, 2\}, \quad (16a)$$

$$u_j^{k,n+1} = u_j^{k,n} - \frac{\Delta t}{\Delta x} (G_{j+1/2}^{k,n} - G_{j-1/2}^{k,n}), \quad k \in \{1, 2\}, \quad (16b)$$

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{\Delta x} (H_{j+1/2}^n - H_{j-1/2}^n) \quad (16c)$$

for $-N + 1 \leq j \leq 0$ left and $0 \leq j \leq N$ right from the interface.

The numerical fluxes away from the interface, i.e., for $j \neq 0$, are given by

$$F_{j-1/2}^{k,n} = \frac{1}{2} (A_{j-1}^{k,n} u_{j-1}^{k,n} + A_j^{k,n} u_j^{k,n}) - \frac{\lambda}{2} (A_j^{k,n} - A_{j-1}^{k,n}), \quad (17a)$$

$$G_{j-1/2}^{k,n} = \frac{1}{4} ((u_{j-1}^{k,n})^2 + (u_j^{k,n})^2) + \frac{1}{2\rho} (p^k(A_{j-1}^{k,n}) + p^k(A_j^{k,n})) - \frac{\lambda}{2} (u_j^{k,n} - u_{j-1}^{k,n}), \quad (17b)$$

$$H_{j-1/2}^n = \frac{1}{2} (A_{j-1}^{k,n} u_{j-1}^{k,n} + A_j^{k,n} u_j^{k,n}) - \frac{\lambda}{2} (A_j^{k,n} - A_{j-1}^{k,n}) \quad (17c)$$

and at the interface we have

$$F_{-1/2}^{1,n} = \frac{1}{2} \left(A_{-1}^{1,n} u_{-1}^{1,n} + v_R^{A,n} \right) - \frac{\lambda}{2} (A_R^n - A_{-1}^{1,n}), \quad (18a)$$

$$F_{-1/2}^{2,n} = \frac{1}{2} \left(v_L^{A,n} + A_0^{2,n} u_0^{2,n} \right) - \frac{\lambda}{2} (A_0^{2,n} - A_R^n), \quad (18b)$$

$$G_{-1/2}^{1,n} = \frac{1}{2} \left(\frac{p^1(A_{-1}^{1,n})}{\rho} + \frac{1}{2} (u_{-1}^{1,n})^2 + v_R^{u,n} \right) - \frac{\lambda}{2} (u_R^n - u_{-1}^{1,n}), \quad (18c)$$

$$G_{-1/2}^{2,n} = \frac{1}{2} \left(v_L^{u,n} + \frac{p^2(A_0^{2,n})}{\rho} + \frac{1}{2} (u_0^{2,n})^2 \right) - \frac{\lambda}{2} (u_0^{2,n} - u_L^n). \quad (18d)$$

Here, the coupling data is obtained by applying the RS constructed in Section 3.2 to the numerical trace data, i.e.

$$\begin{aligned} & (A_R^n, u_R^n, v_R^{A,n}, v_R^{u,n}, w_R^n, A_L^n, u_L^n, v_L^{A,n}, v_L^{u,n}) \\ &= \mathcal{RS}(A_{-1}^{1,n}, u_{-1}^{1,n}, A_{-1}^{1,n} u_{-1}^{1,n}, \frac{p^1(A_{-1}^{1,n})}{\rho} + \frac{1}{2} (u_{-1}^{1,n})^2, w_{-1}^n, \\ & \quad A_0^{2,n}, u_0^{2,n}, A_0^{2,n} u_0^{2,n}, \frac{p^2(A_0^{2,n})}{\rho} + \frac{1}{2} (u_0^{2,n})^2). \end{aligned} \quad (19)$$

As we are only interested in the relaxation limit $\varepsilon \rightarrow 0$ of system (10), the fluxes near the interface take the role of the input data $v_0^{A\mp}$ and $v_0^{u\mp}$ in (19).

The time increment is chosen according to the CFL condition

$$\Delta t^n \lambda^n = 0.9 \Delta x$$

and thus adapted in every step to the current relaxation speed λ^n , which is chosen

$$\lambda^n = \max_{j,k} \{|u_j^{k,n} \pm c_j^{k,n}|, |w_j^n|\}, \quad c_j^{1,n} := \sqrt{\frac{\beta}{2\rho}} \frac{\sqrt{A_j^{1,n}}}{\sqrt[4]{A_j^{1,n} + A_c}}, \quad c_j^{2,n} := \sqrt{\frac{\beta}{2\rho}} \sqrt[4]{A_j^{2,n}}. \quad (20)$$

This choice of λ guarantees that the subcharacteristic condition holds in all time steps. The following experiments are conducted with discretization parameter $N = 400$.

Insertion of the device In a first numerical test we investigate the effect of catheter insertion in a vessel. We make the assumption that $u(\cdot, 0) \equiv 254.65$ cm/s and thus blood initially flows uniformly through the vessel. At initial time the vessel is in standard configuration, i.e., $A(\cdot, 0) \equiv A_0 = \pi R_0^2$ and the radius is chosen $R_0 = 0.5$ cm. Moreover, we assume the wall thickness $h_0 = 0.05$ cm, Young modulus $E = 3 \times 10^6$ dyne/cm² and blood density $\rho = 1$ g/cm³. No flow is assumed in the device, i.e. $w \equiv 0$ cm/s. At the boundaries of the vessel we impose homogeneous Neumann conditions.

Figure 3 shows numerical solutions obtained by the scheme (16) upon insertion of the device at time $t = 0$. The pressure decreases in front of the tip of the device in form of a forward propagating shockwave. The flow rate reduces both in front and slightly behind the tip. To equalize pressure the section area decreases and approaches a new equilibrium configuration through a forward propagating shock and a backward propagating rarefaction wave. The extent of these effects is significantly magnified if the size of the device is increased, as can be seen comparing numerical solutions with different values of A_c in Figure 3.

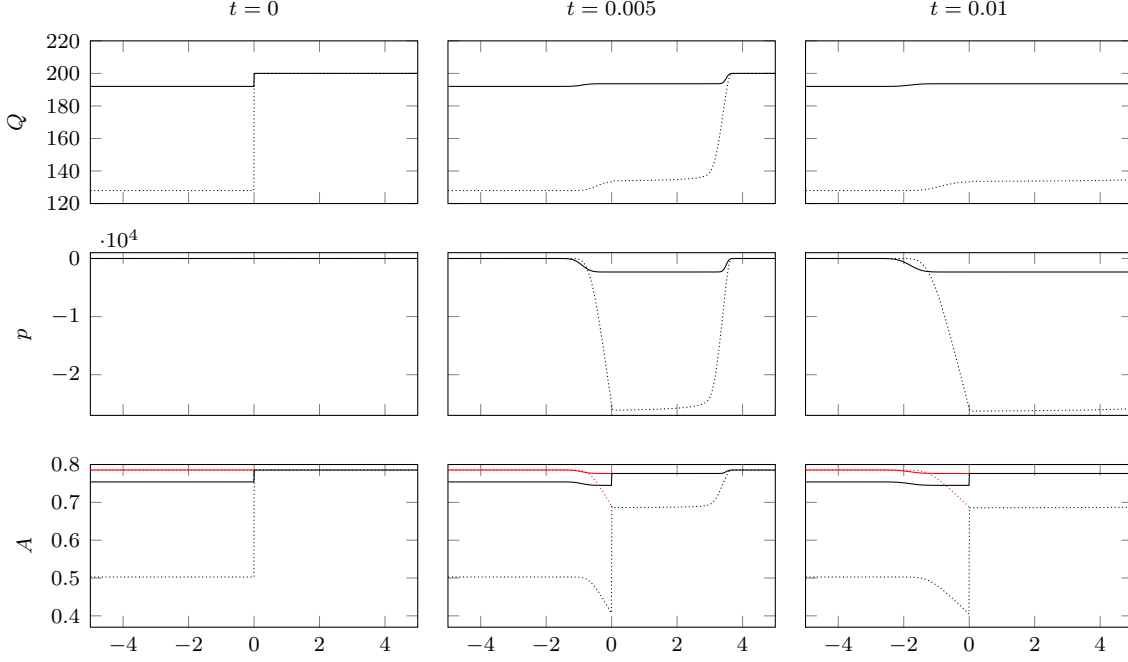


Figure 3: Flow rate, pressure and section area over space in three time instances after insertion of an aspiration device with radius 0.1 cm (solid lines) and 0.25 cm (dashed lines). Red lines in the bottom row indicate the gross section area (including the device).

Varying suction forces In the next numerical experiment, we employ the same configuration as in the previous experiment, fix $A_c = R_c^2 \pi$ for $R_c = 0.1$ cm and activate aspiration. To this end we simulate constant suction with $w \equiv -5000$ cm/s and $w \equiv -10000$ cm/s. Figure 4 shows that strong suction leads to a constriction of the vessel and a high reduction of pressure at the tip of the device. Furthermore, as strong suction leads to a negative flow rate in the vessel the flow rate admits a kink shaped minimum at the tip position.

Occlusion experiment In the last numerical experiment we consider aspiration over a longer time period in the vicinity of an occlusion. Starting from the configuration of the previous experiments we modify the boundary conditions and impose distal condition at the right end of the vessel with a reflection coefficient of $R_T = 0.8$ and proximal condition at the left end of the vessel prescribing the inlet pressure $p_{in}(t) = 8 \sin(2\pi t) \times 10^4$ dyne/cm². For details on the implementations of these conditions, we refer to [3]. Figure 5 shows the dynamics of flow rate and pressure during half a heart beat on the space-time plane in the catheter free setting (left). The occlusion leads to the reflection of pressure waves and a period of constant high pressure in the vessel. For comparison we repeat the simulation with an inserted aspiration catheter of radius $R_c = 0.1$ cm and constant suction with velocity $w \equiv -1000$ cm/s; numerical results are shown in Figure 5 (right). While temporarily the pressure is increased the aspiration in this experiment succeeds to reduce the high pressure period in the vessel.

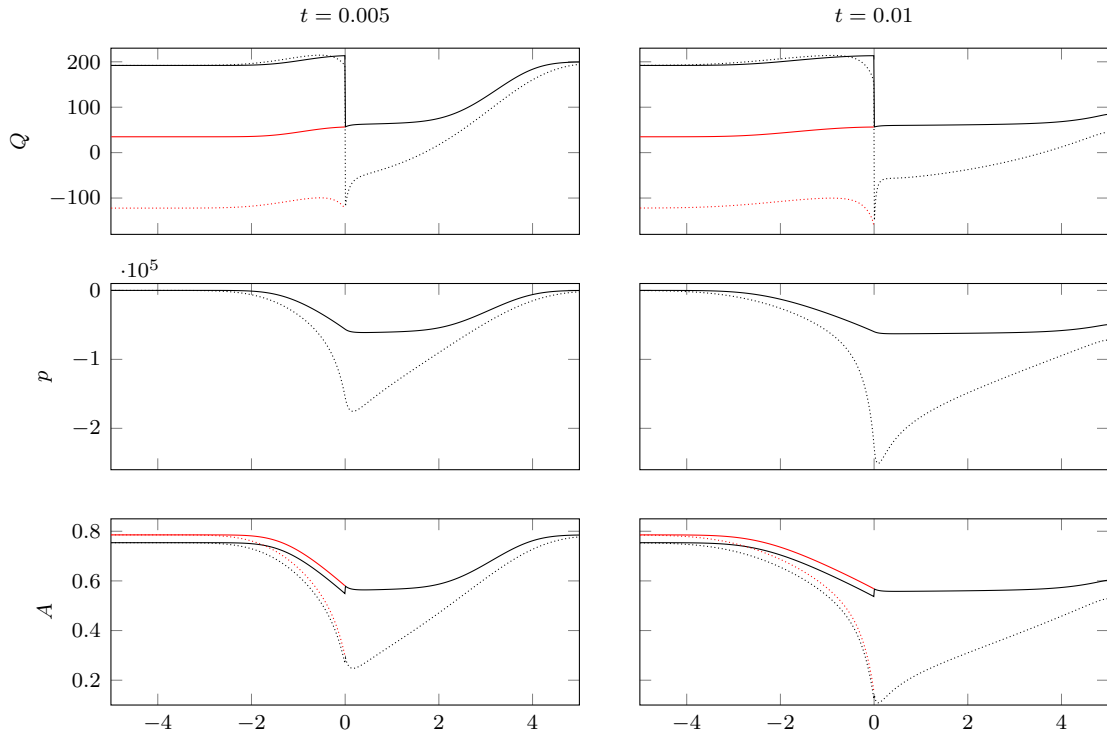


Figure 4: Flow rate, pressure and section area over space in two time instances during aspiration with velocities $w = -5000$ cm/s (solid lines) and $w = -10000$ cm/s (dotted lines) using a catheter with tip at location $x = 0$. Red lines indicate the gross flow rate in the top row and the gross section area in the bottom row (including the device).

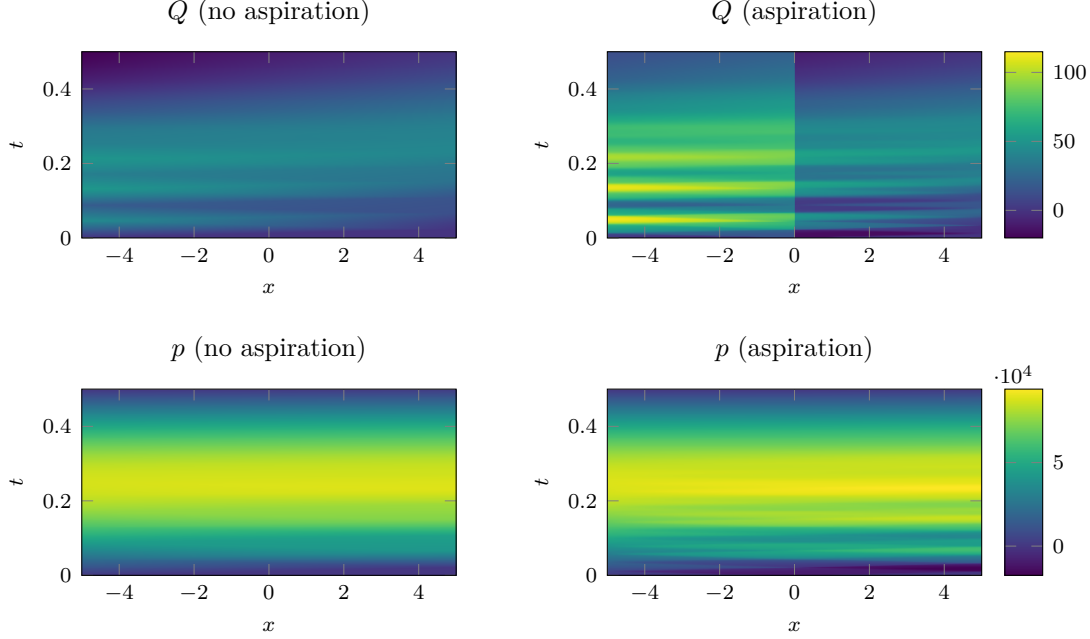


Figure 5: Flow rate and pressure over space and time in case of an occlusion during the systolic phase of a heart beat. An untreated vessel (left) is compared to a vessel during aspiration thrombectomy (right).

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A Coupling of the velocity states

In this appendix we provide details on the computation of u_R , v_R^u , u_L and v_L^u . We parameterize these states on the Lax-curves (13) as follows

$$u_R = u_0^- - \sigma_u^-, \quad v_R^u = v_0^{u-} + \lambda \sigma_u^-, \quad u_L = u_0^+ + \sigma_u^+, \quad v_L^u = v_0^{u+} + \lambda \sigma_u^+.$$

From (9b) and (9a) we get

$$A_R u_R + A_c w_r = A_L u_L$$

and thus after substituting the parametrization we obtain

$$\sigma_u^- = \frac{s_0}{A_R} - \frac{A_L}{A_R} \sigma_u^+, \quad s_0 := A_R u_0^- + A_c w_r - A_L u_0^+. \quad (21)$$

Next, employing the parametrization in (11b) yields

$$s_1 = \lambda(\sigma_u^+ - \sigma_u^-) - u_0^+ \sigma_u^+ - u_0^- \sigma_u^- + \frac{1}{2}(\sigma_u^{-2} - \sigma_u^{+2}), \quad s_1 := v_0^{u-} + \frac{1}{2}(u_0^{+2} - u_0^{-2}) - v_0^{u+}. \quad (22)$$

Substituting now (21) we derive the quadratic equation

$$\begin{aligned} \frac{1}{2} \left[\frac{A_L^2}{A_R^2} - 1 \right] \sigma_u^{+2} + \left[\lambda \left(1 + \frac{A_L}{A_R} \right) + u_0^- \frac{A_L}{A_R} - u_0^+ - \frac{s_0 A_L}{A_R^2} \right] \sigma_u^+ \\ + \left(\frac{s_0}{2A_R} - \lambda - u_0^- \right) \frac{s_0}{A_R} - s_1 = 0. \end{aligned} \quad (23)$$

Due to (9a) it holds that $A_L/A_R > 1$ and therefore the quadratic term does not vanish. Moreover, we note that typically the trace states $u_0^\pm$ and $v_0^{u^\mp}$ are close to the corresponding coupling states and thus due to (9b) and (11b) the terms $|s_0|$ and $|s_1|$ are relatively small. Consequently the constant term in (23) has a relative small absolute value in comparison to the linear term. For these reasons we can expect that (23) has two real roots.

References

- [1] I. Benemerito, A. Mustafa, N. Wang, A. P. Narata, A. Narracott, and A. Marzo. A multiscale computational framework to evaluate flow alterations during mechanical thrombectomy for treatment of ischaemic stroke. *Front. Cardiovasc. Med.*, 10, Mar 2023. URL: <http://dx.doi.org/10.3389/fcvm.2023.1117449>, doi:10.3389/fcvm.2023.1117449.
- [2] L. Formaggia, D. Lamponi, and A. Quarteroni. One-dimensional models for blood flow in arteries. In *Mathematical modelling of the cardiovascular system*, volume 47 of -, pages 251–276. Springer, 2003. doi:10.1023/B:ENGI.0000007980.01347.29.
- [3] L. Formaggia, D. Lamponi, M. Tuveri, and A. Veneziani. Numerical modeling of 1d arterial networks coupled with a lumped parameters description of the heart. *Comput. Methods Biomech. Biomed. Eng.*, 9(5):273–288, Oct 2006. URL: <http://dx.doi.org/10.1080/10255840600857767>, doi:10.1080/10255840600857767.
- [4] L. Formaggia, F. Nobile, A. Quarteroni, and A. Veneziani. Multiscale modelling of the circulatory system: A preliminary analysis. *Comput. Visual Sci.*, 2(2-3):75–83, Dec. 1999. URL: <http://link.springer.com/10.1007/s007910050030>, doi:10.1007/s007910050030.
- [5] M. Herty, N. Kolbe, and S. Müller. A central scheme for two coupled hyperbolic systems. *Communications on Applied Mathematics and Computation*, Nov. 2023. URL: <http://dx.doi.org/10.1007/s42967-023-00306-5>, doi:10.1007/s42967-023-00306-5.
- [6] M. Herty, N. Kolbe, and S. Müller. Central schemes for networked scalar conservation laws. *Netw. Heterog. Media*, 18(1):310–340, 2023. doi:10.3934/nhm.2023012.
- [7] T. J. Hughes and J. Lubliner. On the one-dimensional theory of blood flow in the larger vessels. *Math. Biosci.*, 18(1-2):161–170, Oct 1973. URL: [http://dx.doi.org/10.1016/0025-5564\(73\)90027-8](http://dx.doi.org/10.1016/0025-5564(73)90027-8), doi:10.1016/0025-5564(73)90027-8.
- [8] N. Kolbe. Implementation of central schemes for networks of scalar conservation laws. GitHub repository, <https://github.com/nklb/CentralNetworkScheme>, 2022.
- [9] N. Kolbe, M. Herty, and S. Müller. Numerical schemes for coupled systems of non-conservative hyperbolic equations, 2023.
- [10] T.-P. Liu. Hyperbolic conservation laws with relaxation. *Commun.Math. Phys.*, 108(1):153–175, Mar. 1987. URL: <http://link.springer.com/10.1007/BF01210707>, doi:10.1007/BF01210707.
- [11] J. Peiró and A. Veneziani. Reduced models of the cardiovascular system. In *Cardiovascular mathematics*, volume 1 of *MSE&A. Model. Simul. Appl.*, pages 347–394. Springer Italia, Milan, 2009. URL: https://doi.org/10.1007/978-88-470-1152-6_10, doi:10.1007/978-88-470-1152-6_10.

- [12] G. Pennati, R. Balossino, G. Dubini, and F. Migliavacca. Numerical simulation of thrombus aspiration in two realistic models of catheter tips. *Artificial organs*, 34 4:301–10, 2010. doi:10.1111/j.1525-1594.2009.00770.x.
- [13] T. G. Phan, H. Ma, M. Goyal, J. Hilton, M. Sinnott, V. Srikanth, and R. Beare. Computer modeling of clot retrieval—circle of willis. *Front. Neurol.*, 11, Aug 2020. URL: <http://dx.doi.org/10.3389/fneur.2020.00773>, doi:10.3389/fneur.2020.00773.
- [14] J. A. G. Rhodin. Architecture of the vessel wall. *The Cardiovascular System*, 1980. URL: <https://cir.nii.ac.jp/crid/1573668925426363648>.
- [15] V. Saini, L. Guada, and D. R. Yavagal. Global epidemiology of stroke and access to acute ischemic stroke interventions. *Neurology*, 97(20), Nov. 2021. URL: <http://dx.doi.org/10.1212/wnl.0000000000012781>, doi:10.1212/wnl.0000000000012781.
- [16] Y. Shi, D. Cheshire, F. E. Lally, and C. Roffe. Suction force-suction distance relation during aspiration thrombectomy for ischemic stroke: A computational fluid dynamics study. *Physics in Medicine*, 3:1–8, 2017. doi:10.1016/j.phmed.2016.11.001.
- [17] S. Čanić. Blood flow through compliant vessels after endovascular repair: wall deformations induced by the discontinuous wall properties. *Comput. Vis. Sci.*, 4(3):147–155, 2002. doi:10.1007/s007910100066.
- [18] S. Čanić and E. H. Kim. Mathematical analysis of the quasilinear effects in a hyperbolic model blood flow through compliant axi-symmetric vessels. *Math. Methods Appl. Sci.*, 26(14):1161–1186, 2003. doi:10.1002/mma.407.
- [19] X. Wang. *1D modeling of blood flow in networks: Numerical computing and applications*. PhD thesis, Université Pierre et Marie Curie-Paris VI, 2014.
- [20] J.-M. Zhang, L. Zhong, B. Su, M. Wan, J. S. Yap, J. P. L. Tham, L. P. Chua, D. N. Ghista, and R. S. Tan. Perspective on CFD studies of coronary artery disease lesions and hemodynamics: a review. *Int. J. Numer. Methods Biomed. Eng.*, 30(6):659–680, 2014. doi:10.1002/cnm.2625.