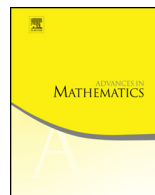




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Genus zero transverse foliations for weakly convex Reeb flows on the tight 3-sphere



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ABSTRACT

A contact form on the tight 3-sphere (S^3, ξ_0) is called weakly convex if the Conley-Zehnder index of every Reeb orbit is at least 2. In this article, we study Reeb flows of weakly convex contact forms on (S^3, ξ_0) admitting a prescribed finite set of index-2 Reeb orbits, which are all hyperbolic and mutually unlinked. We present conditions so that these index-2 orbits are binding orbits of a genus zero transverse foliation whose additional binding orbits have index 3. In addition, we show in the real-analytic case that the topological entropy of the Reeb flow is positive if the branches of the stable/unstable manifolds of the index-2 orbits are mutually non-coincident.

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1. Introduction and main results

The goal of this paper is to construct transverse foliations for Reeb flows on the tight 3-sphere with prescribed binding orbits. Our existence theorem uses specific assumptions that hold for energy levels of Hamiltonians with two degrees of freedom that appear in concrete problems. Hence, we are able to study certain classes of degenerate Reeb flows on the tight 3-sphere and attack questions in celestial mechanics.

The main motivation comes from the quest for homoclinics to the Lyapunov orbits in energy levels of the planar circular restricted 3-body problem. We are particularly interested in the case where the energy is slightly above the first critical value; in this case, a Lyapunov orbit appears between the massive bodies. The notion of chaos in the planar circular restricted 3-body problem is historically connected to such homoclinics. However, the existence of homoclinics to the Lyapunov orbit in this specific regime has only been proved for small mass ratios by Llibre, Martínez, and Simó in [24] and studied numerically in the same work. The existence of such homoclinic orbits for arbitrary mass ratio remains open. This problem is a point of entrance for symplectic methods since the desired homoclinic connection would follow if one can find a transverse foliation with the Lyapunov orbit as part of the binding.

1.1. Transverse foliations and weakly convex Reeb flows

Let ψ_t , $t \in \mathbb{R}$, be a smooth flow on a smooth closed oriented 3-manifold M . A **transverse foliation** adapted to (M, ψ_t) is a singular foliation \mathcal{F} of M satisfying the following:

- (i) The singular set \mathcal{P} of \mathcal{F} consists of finitely many simple periodic orbits P_1, \dots, P_m , called binding orbits. The set \mathcal{P} is called the binding of \mathcal{F} .
- (ii) The complement $M \setminus \cup_{P \in \mathcal{P}} P$ is smoothly foliated by surfaces, called the regular leaves of \mathcal{F} . Every regular leaf of \mathcal{F} is a properly embedded surface $\dot{\Sigma} \hookrightarrow M \setminus \cup_{P \in \mathcal{P}} P$. The closure $\Sigma = \text{cl}(\dot{\Sigma})$ is an immersed compact surface whose boundary is formed by binding orbits in \mathcal{P} . The components of $\partial\Sigma$ are called the asymptotic limits of $\dot{\Sigma}$, and the ends of $\dot{\Sigma}$ are called the punctures of $\dot{\Sigma}$. Each $\dot{\Sigma}$ is transverse to the flow, and each puncture has an associated asymptotic limit.
- (iii) The orientation of M and the flow provide each regular leaf $\dot{\Sigma}$ with an orientation in such a way that trajectories intersect leaves positively. A puncture of $\dot{\Sigma}$ is called positive if the orientation of its asymptotic limit as the boundary of Σ coincides with the orientation of the flow. Otherwise, the puncture is called negative.

A transverse foliation, all of whose regular leaves have genus zero, is called a genus zero transverse foliation. This definition was introduced in [20] and is based on the finite energy foliations introduced by Hofer, Wysocki, and Zehnder in [19], where they prove the following remarkable result for Reeb flows on the tight three-sphere.

Theorem 1.1 (Hofer-Wysocky-Zehnder [19]). *Let $\lambda = f\lambda_0$ be a nondegenerate contact form on the tight three-sphere (S^3, ξ_0) . Then the Reeb flow of λ admits a genus zero transverse foliation whose binding orbits have Conley-Zehnder index equal to 1, 2, or 3.*

We want to construct genus zero transverse foliations adapted to Reeb flows on the tight 3-sphere (S^3, ξ_0) without assuming nondegeneracy on λ . We are particularly interested in transverse foliations whose binding contains a prescribed set of index-2 Reeb orbits.

Here, $S^3 = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1\}$, where (x_1, x_2, y_1, y_2) are canonical coordinates on \mathbb{R}^4 . The (tight) contact structure ξ_0 on S^3 is the one induced by the Liouville form on \mathbb{R}^4 , that is $\xi_0 = \ker \lambda_0$, where λ_0 is the restriction of $\frac{1}{2} \sum_{i=1}^2 (x_i dy_i - y_i dx_i)$ to S^3 .

Let λ be a contact form on (S^3, ξ_0) , that is $\lambda = f\lambda_0$ for some smooth function $f: S^3 \rightarrow (0, +\infty)$. Its Reeb vector field $R = R_\lambda$ is uniquely determined by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$, and its flow ϕ_t is called the Reeb flow of λ .

A periodic orbit of λ , also called a Reeb orbit, is a pair $P = (x, T)$, where $x: \mathbb{R} \rightarrow M$ is a periodic trajectory of the Reeb flow of λ and $T > 0$ is a period of x . If T is the least positive period of x , P is called simple. The Reeb orbits (x, T) and (y, T') satisfying $x(\mathbb{R}) = y(\mathbb{R})$ and $T = T'$ are identified and we denote by $\mathcal{P}(\lambda)$ the set of equivalence

classes of periodic trajectories of λ . We may also denote by P the set $x(\mathbb{R}) \subset S^3$. The action $\int_{[0,T]} x^* \lambda$ of $P = (x, T)$ coincides with its period T since $\lambda(R) = 1$. A periodic orbit $P = (x, T) \in \mathcal{P}(\lambda)$ is said to be nondegenerate if 1 is not an eigenvalue of the linearized map $d\phi_T : \xi_{x(0)} \rightarrow \xi_{x(0)}$. The contact form λ is called nondegenerate if every periodic orbit of λ is nondegenerate.

The symplectic vector bundle $(\xi_0, d\lambda|_{\xi_0}) \rightarrow S^3$ admits a unique symplectic trivialization up to homotopy. Hence, the Conley-Zehnder index $\text{CZ}(P)$ of every periodic orbit $P \in \mathcal{P}(\lambda)$ is uniquely determined by any such trivialization; see section 2 for a definition of the Conley-Zehnder index. We will sometimes refer to the Conley-Zehnder index of a periodic orbit P simply as the index of P . Given $j \in \mathbb{Z}$, denote by $\mathcal{P}_j(\lambda) \subset \mathcal{P}(\lambda)$ the set of Reeb orbits with Conley-Zehnder index j , and denote by $\mathcal{P}_j^{u,-1}(\lambda) \subset \mathcal{P}_j(\lambda)$ the subset of index- j Reeb orbits that are unknotted and have self-linking number -1 ; see section 2 for the definition of self-linking number.

Definition 1.2. A contact form λ on (S^3, ξ_0) is called weakly convex if $\text{CZ}(P) \geq 2$ for every $P \in \mathcal{P}(\lambda)$, that is $\mathcal{P}_j(\lambda) = \emptyset \forall j \leq 1$.

The following theorem is a particular case of Theorem 1.1.

Theorem 1.3 (Hofer-Wysocki-Zehnder [19], Siefring [32]). *If $\lambda = f\lambda_0$ is a nondegenerate weakly convex contact form on the tight three-sphere (S^3, ξ_0) , then its Reeb flow admits a genus zero transverse foliation satisfying the following conditions:*

- (i) *Every binding orbit has index 2 or 3.*
- (ii) *Every regular leaf is either a plane asymptotic to a binding orbit or a cylinder connecting an index-3 binding orbit at a positive end to an index-2 binding orbit at a negative end.*
- (iii) *If there exists only one binding orbit, then its index is 3, and the transverse foliation determines an open book decomposition of (S^3, ξ_0) whose pages are disk-like global surfaces of section.*
- (iv) *Every index-2 binding orbit bounds a pair of rigid planes whose closures form a C^1 two-sphere. The complement of the union of these two-spheres in S^3 contains an index-3 binding orbit in each component. Each index-3 binding orbit admits rigid cylinders connecting it to the index-2 orbits at the boundary of the corresponding component. The complement of the rigid cylinders in that component is foliated by finitely many families of planes asymptotic to the index-3 binding orbit.*

Remark 1.4. Theorem 1.3 is discussed in [19, section 7.2] without further details. The existence of precisely two distinct rigid planes asymptotic to a given index-2 binding orbit follows from standard regularity, compactness, and intersection arguments. See the discussion in [23, Proposition 4.1]. In fact, the orbits that obstruct the existence of these rigid planes are the index-1 orbits, which do not exist by assumption. Siefring's

intersection theory [32], see also [14], implies that these two planes approach the binding orbit through opposite directions forming a C^1 -embedded two-sphere. This is explained in [6, Appendix C].

The following definition was first introduced in [8], motivated both by Theorem 1.3 and the $3 - 2 - 3$ foliations studied in [6,7], see Fig. 1.1 below.

Definition 1.5. A (genus zero) transverse foliation as described in Theorem 1.3 is called a **weakly convex foliation** if it has at least one index-2 binding orbit. In particular, all regular leaves are planes and cylinders.

It follows from Theorem 1.3 that every nondegenerate weakly convex Reeb flow on the tight three-sphere either admits an open book decomposition whose pages are disk-like global surfaces of section, or a weakly convex foliation whose regular leaves are planes and cylinders transverse to the flow. It is our goal to construct weakly convex foliations for a possibly degenerate Reeb flow on the tight three-sphere with a prescribed set of index-2 binding orbits.

1.2. Main results

In the following we assume that the contact form λ on the tight 3-sphere is weakly convex, that $\mathcal{P}_2(\lambda)$ is non-empty and finite, and that every orbit in $\mathcal{P}_2(\lambda)$ is hyperbolic, unknotted, and has self-linking number -1 . Moreover, we assume that the orbits in $\mathcal{P}_2(\lambda)$ are mutually unlinked and that their actions are small when compared to the actions of the simple orbits with higher indices. We also assume that each orbit in $\mathcal{P}_2(\lambda)$ does not link with any orbit in $\mathcal{P}_3^{u,-1}(\lambda)$. Recall that a Reeb orbit $P' = (x', T')$, which is geometrically distinct from an unknotted periodic orbit P , is said to be linked with P if $0 \neq [x'] \in H_1(S^3 \setminus P; \mathbb{Z}) \cong \mathbb{Z}$. Otherwise, we say that P' is not linked with P . The linking number between P and P' is denoted $\text{link}(P, P')$.

Our main result states that under the above-mentioned hypotheses, the Reeb flow of λ admits a weakly convex foliation so that every orbit in $\mathcal{P}_2(\lambda)$ is a binding orbit. Moreover, all other binding orbits have index 3. See also Theorem 3.19.

Theorem 1.6. Let $\lambda = f\lambda_0$ be a weakly convex contact form on (S^3, ξ_0) . Assume that the following conditions are satisfied:

- I. $\mathcal{P}_2(\lambda) = \mathcal{P}_2^{u,-1}(\lambda)$ is a non-empty finite set $\{P_{2,1}, \dots, P_{2,l}\}$ of mutually unlinked hyperbolic periodic orbits.
- II. If $P \in \mathcal{P}(\lambda)$ is geometrically distinct from any orbit in $\mathcal{P}_2^{u,-1}(\lambda)$, then its action is greater than the action of every orbit in $\mathcal{P}_2^{u,-1}(\lambda)$.
- III. If $P \in \mathcal{P}_3^{u,-1}(\lambda)$, then $\text{link}(P, P_{2,i}) = 0, \forall i = 1, \dots, l$.

Then the Reeb flow of λ admits a weakly convex foliation \mathcal{F} so that:

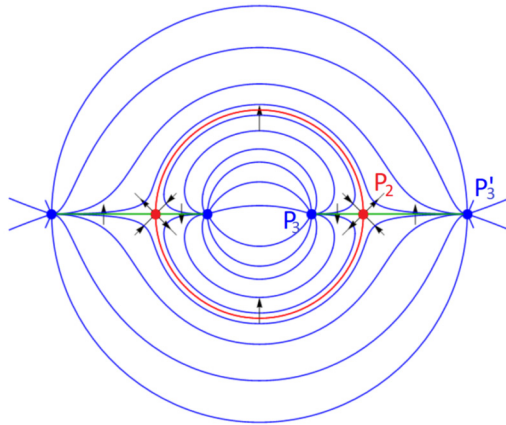


Fig. 1.1. A weakly convex transverse foliation on (S^3, ξ_0) , called 3 – 2 – 3 foliation. The binding is formed by precisely one index-2 orbit P_2 and two index-3 orbits P_3, P'_3 .

- (i) Every orbit in $\mathcal{P}_2(\lambda)$ is a binding orbit.
- (ii) There are $l + 1$ remaining binding orbits $P_{3,1}, \dots, P_{3,l+1} \in \mathcal{P}_3^{u,-1}(\lambda)$.
- (iii) For each $i \in \{1, \dots, l\}$, there exists a pair of rigid planes $U_{i,1}, U_{i,2} \in \mathcal{F}$, so that each one of them is asymptotic to $P_{2,i}$ at its positive puncture. Moreover, $\mathcal{S}_i = U_{i,1} \cup P_{2,i} \cup U_{i,2} \hookrightarrow S^3$ is a C^1 -embedded 2-sphere.
- (iv) The open set $S^3 \setminus \bigcup_{i=1}^l \mathcal{S}_i$ has $l+1$ components $\mathcal{U}_1, \dots, \mathcal{U}_{l+1}$. For each $j \in \{1, \dots, l+1\}$ it holds $P_{3,j} \subset \mathcal{U}_j$.
- (v) If $\mathcal{S}_i \subset \partial \mathcal{U}_j$, then \mathcal{F} contains a rigid cylinder $V_{j,i} \subset \mathcal{U}_j$ asymptotic to $P_{3,j}$ at its positive puncture and to $P_{2,i}$ at its negative puncture.
- (vi) For each $j \in \{1, \dots, l+1\}$, there exist $\tilde{k}_j \geq 1$ families of planes parametrized by the interval $(0, 1)$, so that each such plane is asymptotic to $P_{3,j}$ at its positive puncture. Here, \tilde{k}_j is the number of components of $\partial \mathcal{U}_j$. At each end, every such family of planes breaks onto a rigid cylinder $V_{j,i}$ connecting $P_{3,j}$ to some $P_{2,i}$ and a rigid plane asymptotic to $P_{2,i}$.

Remark 1.7. A similar result holds for general contact forms on (S^3, ξ_0) , not necessarily weakly convex. Indeed, let $\lambda = f\lambda_0$ be a contact form on (S^3, ξ_0) . Let $C = C(\lambda) > 0$ be the constant given in Proposition 2.6. Suppose that there exist l orbits $P_{2,1}, \dots, P_{2,l} \in \mathcal{P}_2^{u,-1}(\lambda)$ as in I, and that every other Reeb orbit with index $-1, 0, 1$ or 2 has action $> C$. Under the additional conditions II and III, the conclusions of Theorem 1.6 still hold.

Theorem 1.6 is inspired by Theorems 1.1 and 1.3. Our results only make non-degeneracy hypotheses on the periodic orbits in $\mathcal{P}_2(\lambda)$ and are intended to apply to classical problems emerging in Celestial Mechanics. (See Fig. 1.1.)

Our next result uses Theorem 1.6 to study certain Reeb flows of real-analytic contact forms that appear in concrete problems. To prepare for the statement we need to introduce some notation.

Consider a contact form λ on (S^3, ξ_0) . Let \mathcal{F} be a weakly convex foliation adapted to the Reeb flow of λ . Let \mathcal{U} be a connected component of the complement of the union of the rigid spheres, and P be a binding orbit in $\partial\mathcal{U}$. The stable manifold $W^s(P)$ is an immersed cylinder transverse to the rigid sphere containing P . Hence, there are two well-defined local branches of $W^s(P)$ through P , and only one of them is contained in \mathcal{U} . We denote by $W_{\mathcal{U}}^s(P)$ the branch of $W^s(P)$ that contains the local branch in \mathcal{U} . The branch $W_{\mathcal{U}}^u(P)$ of the unstable manifold $W^u(P)$ is defined analogously.

Theorem 1.8. *Let $\lambda = f\lambda_0$ be a real-analytic contact form on (S^3, ξ_0) satisfying the conclusions of Theorem 1.6, i.e., admitting a weakly convex foliation as described in Theorem 1.6. Suppose that the actions of the orbits in $\mathcal{P}_2(\lambda)$ coincide. Let \mathcal{U} be a connected component of the complement of the rigid spheres. Then the following statements hold:*

- (i) *Suppose that for every binding orbit P in $\partial\mathcal{U}$ there exist binding orbits P', P'' in $\partial\mathcal{U}$ such that $W_{\mathcal{U}}^s(P) = W_{\mathcal{U}}^u(P')$ and $W_{\mathcal{U}}^u(P) = W_{\mathcal{U}}^s(P'')$. Then there exists an invariant set $A \subset \mathcal{U}$ which admits a cross-section. This cross-section is a punctured disk bounded by the index-3 binding orbit in \mathcal{U} , and the first return map has an infinite twist near each puncture. In particular, A contains infinitely many periodic orbits.*
- (ii) *Suppose that $W_{\mathcal{U}}^s(P) \neq W_{\mathcal{U}}^u(P')$ and that $W_{\mathcal{U}}^u(P) \neq W_{\mathcal{U}}^s(P')$ holds for all pairs of binding orbits P, P' in $\partial\mathcal{U}$. Then there exists an invariant set $\Lambda \subset \mathcal{U}$ such that the Reeb flow restricted to Λ has positive topological entropy.*

The assumption that the actions of the Reeb orbits in $\mathcal{P}_2(\lambda)$ coincide and are small when compared to the actions of the other Reeb orbits are usually verified for Hamiltonian dynamics near certain critical energy surfaces as we later explain.

The foliations obtained and used in this paper, as well as the techniques involved in the proof of Theorem 1.6, were introduced by Hofer, Wysocki and Zehnder in [19]. In the degenerate case, such techniques were further developed in [16]. The argument to get intersections of stable and unstable manifolds of binding orbits with Conley-Zehnder index equal to 2 is originally found in [19]. In the real-analytic case, we use Conley's ideas from [4]. As for the construction of a Bernoulli shift subsystem, we follow Moser's book [27]. Genus zero transverse foliations with a single index-2 binding orbit were treated in [6,7,30], see also [23]. The reader finds more on transverse foliations in the surveys [8] and [20] and references therein.

1.3. Sketch of proof of Theorem 1.6

Consider sequences $\lambda_n \rightarrow \lambda$ of non-degenerate contact forms and $J_n \rightarrow J$ of generic compatible complex structures on $\ker \lambda_n = \ker \lambda$ so that for every n the \mathbb{R} -invariant almost complex structure induced by (λ_n, J_n) admits a finite energy foliation \mathcal{F}_n as in Theorem 2.5 below. We show in Proposition 2.6 that the actions of the binding orbits

of \mathcal{F}_n are uniformly bounded and thus assumptions I and III imply that they consist of continuations of the l orbits in $\mathcal{P}_2(\lambda)$, and of $l + 1$ index-3 orbits $P_{3,1}^n, \dots, P_{3,l+1}^n$. Moreover, \mathcal{F}_n projects to a genus zero transverse foliation whose regular leaves are planes and cylinders asymptotic to the binding orbits, see Proposition 3.1. Now we want to push \mathcal{F}_n to a limiting finite energy foliation \mathcal{F} adapted to (λ, J) . At this point, some difficulties show up in the compactness argument. It is crucial that we start with an almost complex structure J which is generic enough so that some particular low energy pseudo-holomorphic curves asymptotic to the orbits in $\mathcal{P}_2(\lambda)$ do not exist. Here we use assumption II, see Lemma 3.3. For such generic J 's we are able to control the rigid planes asymptotic to the index-2 orbits proving that they converge to corresponding rigid planes associated with (λ, J) , see Proposition 3.5. The action boundedness of the binding orbits $P_{3,1}^n, \dots, P_{3,l+1}^n$ allows us to find distinct limiting Reeb orbits $P_{3,1}^0, \dots, P_{3,l+1}^0 \in \mathcal{P}_3^{u,-1}(\lambda)$ in the complement of the rigid planes. Now some undesired low action Reeb orbits of λ that are unlinked with any of those index-3 orbits may obstruct the existence of a transverse foliation with binding orbits $P_{3,1}^0, \dots, P_{3,l+1}^0$ and the orbits in $\mathcal{P}_2(\lambda)$. To overcome this difficulty we may need to re-start the procedure of taking sequences λ_n, J_n as above so that the previously found index-3 orbits and an undesired unlinked orbit are also Reeb orbits of λ_n for every n . Then we obtain new limiting index-3 Reeb orbits $P_{3,1}^1, \dots, P_{3,l+1}^1$ that necessarily link with the previous unlinked orbit. Repeating the aforementioned procedure of taking new sequences λ_n, J_n that freeze not only an eventual new unlinked Reeb orbit of λ , but also the previously frozen Reeb orbits, we show that the process must terminate after finitely many steps. Hence we eventually find special index-3 orbits $P_{3,1}, \dots, P_{3,l+1} \in \mathcal{P}_3^{u,-1}(\lambda)$ that are necessarily linked with all Reeb orbits that are not covers of the orbits in $\mathcal{P}_2(\lambda)$, see Proposition 3.10. These special orbits and the orbits in $\mathcal{P}_2(\lambda)$ are candidates for binding orbits of the desired foliation. We then take limits of the planes asymptotic to the index-3 orbits and of the rigid cylinders connecting the index-3 to the index-2 orbits, to obtain a genus zero transverse foliation. This foliation, however, may include leaves with more than one negative puncture asymptotic to distinct orbits in $\mathcal{P}_2(\lambda)$. Using uniqueness of such pseudo-holomorphic curves, we show in Proposition 3.15 the existence of at least one plane asymptotic to each index-3 orbit. After slightly changing the almost complex structure near the rigid planes to rule out such non-generic curves with multiple negative ends at index-2 orbits, we obtain the desired foliation from the compactness properties of the planes asymptotic to the index-3 orbits and an application of the gluing theorem.

1.4. Potential applications

Let $H: \mathbb{R}^4 \rightarrow \mathbb{R}$ be a smooth Hamiltonian. Assume that $0 \in \mathbb{R}$ is a critical value of H and that the critical set $H^{-1}(0)$ contains finitely many saddle-center equilibrium points p_1, \dots, p_m . Assume that as the energy changes from negative to positive, a sphere-like component $C_E \subset H^{-1}(E), E < 0$, gets connected to other components of the energy surface, precisely at p_1, \dots, p_m . In particular, C_E becomes a singular sphere-like subset

$C_0 \subset H^{-1}(0)$ with singularities at p_1, \dots, p_m . For every $E > 0$ small, $H^{-1}(E)$ contains an index-2 hyperbolic orbit γ_i in the neck-region about $p_i, i = 1, \dots, m$. The orbit γ_i , called the Lyapunoff orbit, bounds a pair of planes in the energy surface, which are transverse to the flow and form with γ_i a 2-sphere $\mathcal{S}_i \subset H^{-1}(E), \forall i$. We may wish to study the dynamics on the subset $S_E \subset H^{-1}(E)$ near C_0 , which is bounded by the union of the 2-spheres \mathcal{S}_i . The subset S_E is diffeomorphic to a 3-sphere with m disjoint 3-balls removed. In many situations, S_E is the region in $H^{-1}(E)$ where the interesting dynamics takes place. We thus may assume, possibly after changing H away from S_E , that the components which get connected to C_E are formed by m suitable sphere-like hypersurfaces. In particular, $S_E, E > 0$ small, is a subset of a non-convex sphere-like subset $\hat{S}_E \subset H^{-1}(E)$. The Hamiltonian flow on \hat{S}_E is equivalent to a Reeb flow on the tight 3-sphere. If H satisfies some mild convexity conditions on C_0 , then, for energies $E > 0$ sufficiently small, \hat{S}_E satisfies the hypotheses of Theorem 1.6. Indeed, \hat{S}_E admits no periodic orbits with index ≤ 1 , and index-3 orbits do not intersect the 2-spheres \mathcal{S}_i . The transverse foliation given in Theorem 1.6 restricts to a transverse foliation on S_E that contains the Lyapunoff orbits and an index-3 orbit as binding orbits. Dynamical properties such as multiplicity of periodic orbits and the existence of homoclinics/heteroclinics to the Lyapunoff orbits follow from the transverse foliation. If H satisfies a particular \mathbb{Z}_m -symmetry, then Theorem 1.8 applies, deriving more information about the dynamics. As an example, the Hénon-Heiles system is \mathbb{Z}_3 -symmetric and thus the three index-2 Lyapunoff orbits exist and have the same arbitrarily small action for energies slightly above its critical value $\frac{1}{6}$. We will explore this example further in [5]. The circular planar restricted three-body problem is \mathbb{Z}_2 -symmetric and fits a similar setting after regularizing collisions with one of the primaries. We expect to find applications of Theorems 1.6 and 1.8 for certain mass ratios and energies slightly above the first Lagrange value.

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2. Basic definitions

Let M be a closed three-manifold equipped with a contact form λ . Let $P = (x, T) \in \mathcal{P}(\lambda)$, $x_T := x(T \cdot): \mathbb{R}/\mathbb{Z} \rightarrow M$ and $J \in \mathcal{J}(\lambda)$, where $\mathcal{J}(\lambda)$ denotes the set of $d\lambda$ -

compatible almost complex structures on the contact structure $\xi = \ker \lambda$. The asymptotic operator $A_P = A_{P,J}$ is the unbounded self-adjoint operator acting on sections of ξ along P

$$A_P(\eta) := -J|_{x_T} \cdot \mathcal{L}_{\dot{x}_T} \eta \in L^2(x_T^* \xi),$$

for every $\eta \in W^{1,2}(x_T^* \xi)$, where $\mathcal{L}_{\dot{x}_T} \eta$ is the Lie derivative of η in the direction of \dot{x}_T

$$(\mathcal{L}_{\dot{x}_T} \eta)(t) := \left. \frac{d}{ds} \right|_{s=0} \{ D\varphi_s^{-1}(x_T(t+s)) \cdot \eta(t+s) \}.$$

The eigenvalues of A_P are real and accumulate only at $\pm\infty$. A non-trivial eigenvector never vanishes and thus, for a fixed frame Ψ of the contact structure along P , determines a winding number $\text{wind}_\Psi(\mu) \in \mathbb{Z}$ that depends only on the eigenvalue μ . The function $\mu \mapsto \text{wind}_\Psi(\mu)$ is monotonically increasing and surjective and satisfies $\#\text{wind}_\Psi^{-1}(k) = 2, \forall k \in \mathbb{Z}$, where the multiplicities are counted. See [14, Section 3].

The periodic orbit P is degenerate if and only if 0 is an eigenvalue of A_P . Denoting by $\text{wind}_\Psi^{<0}(A_P)$ the winding number of the largest negative eigenvalue and by $\text{wind}_\Psi^{\geq 0}(A_P)$ the winding number of the smallest non-negative eigenvalue, the (generalized) Conley-Zehnder index of P with respect to the frame Ψ is defined as $\text{CZ}_\Psi(P) := \text{wind}_\Psi^{<0}(A_P) + \text{wind}_\Psi^{\geq 0}(A_P)$. It depends on the frame Ψ but not on J . If $(M, \xi) = (S^3, \xi_0)$, we fix a frame induced by a global trivialization of ξ and omit Ψ in the notation.

Now let $x: \mathbb{R}/\mathbb{Z} \rightarrow (S^3, \xi_0)$ be an unknot transverse to the standard contact structure ξ_0 , and let $u: \mathbb{D} \rightarrow S^3$ be an embedded disk, where $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Assume that u is a spanning disk for x , that is $x(t) = u(e^{2\pi i t}), \forall t \in \mathbb{R}/\mathbb{Z}$. Choose a non-vanishing section X of the pullback bundle $u^* \xi_0 \rightarrow \mathbb{D}$. Fix a Riemannian metric g on S^3 and denote by \exp the associated exponential map. If X is sufficiently small, then $x_X(\mathbb{R}/\mathbb{Z}) \cap x(\mathbb{R}/\mathbb{Z}) = \emptyset$, where $x_X(t) := \exp_{x(t)} X(t), \forall t \in \mathbb{R}/\mathbb{Z}$. We may assume that x_X is transverse to u . The *self-linking number* of x , denoted by $\text{sl}(x)$, is defined as the algebraic intersection number $x_X \cdot u$. Here, S^3 is oriented in such a way that $\lambda \wedge d\lambda > 0$. The orientation of x_X is the one induced by x , and x is oriented as the boundary of u . The self-linking number is independent of the involved choices.

2.1. Pseudo-holomorphic curves

In this section, λ is a contact form on a smooth closed three-manifold M and $\xi = \ker \lambda$ is the induced contact structure. The Reeb vector field $R = R_\lambda$ is defined as before. The symplectization of M is the symplectic manifold $(\mathbb{R} \times M, d(e^r \lambda))$, where r is the \mathbb{R} -coordinate. For each $J \in \mathcal{J}(\lambda)$, the pair (λ, J) determines an \mathbb{R} -invariant almost complex structure \tilde{J} on $\mathbb{R} \times M$ so that $\tilde{J} \cdot \partial_r = R$ and $\tilde{J}|_\xi = J$. Let (S, j) be a closed connected Riemann surface, and let $\Gamma \subset S$ be a finite set of punctures. We consider finite energy \tilde{J} -holomorphic curves in $\mathbb{R} \times M$, i.e. smooth maps $\tilde{u} = (a, u): S \setminus \Gamma \rightarrow \mathbb{R} \times M$ satisfying $d\tilde{u} \circ j = \tilde{J}(\tilde{u}) \circ d\tilde{u}$, and having finite Hofer's energy $0 < E(\tilde{u}) = \sup_{\psi \in \Lambda} \int_{S \setminus \Gamma} \tilde{u}^* d(\psi(a)\lambda) < \infty$.

$+\infty$, where Λ is the set of non-decreasing smooth functions $\psi: \mathbb{R} \rightarrow [0, 1]$. If $S = S^2$ and $\#\Gamma = 1$, then \tilde{u} is called a finite energy plane.

Definition 2.1. A \tilde{J} -holomorphic map $\tilde{u}: S \setminus \Gamma \rightarrow \mathbb{R} \times M$ is called somewhere injective if there exists $z_0 \in S \setminus \Gamma$ such that $d\tilde{u}(z_0) \neq 0$ and $\tilde{u}^{-1}(\tilde{u}(z_0)) = \{z_0\}$.

Given a puncture $z_0 \in \Gamma$, choose a holomorphic chart $\phi: (\mathbb{D} \setminus \partial\mathbb{D}, 0) \rightarrow (\phi(\mathbb{D} \setminus \partial\mathbb{D}), z_0)$ centered at z_0 . The map $\tilde{u} \circ \phi(e^{-2\pi(s+it)}), (s, t) \in [0, +\infty) \times \mathbb{R}/\mathbb{Z}$, will be still denoted by $\tilde{u} = (a, u)$. The finite energy condition implies that \tilde{u} is non-constant and the limit $m = m_{z_0} := \lim_{s \rightarrow +\infty} \int_{\{s\} \times S^1} u^* \lambda$ exists. The puncture z_0 is said to be removable if $m = 0$. In this case, \tilde{u} can be smoothly extended over z_0 by Gromov's removable singularity theorem. The puncture z_0 is called positive or negative if $m > 0$ and $m < 0$, respectively. In the following, we tacitly assume that all punctures are non-removable. Stokes' theorem tells us that the set Γ is non-empty. We assign the sign $\varepsilon = \pm 1$ to each puncture, depending on whether it is positive or negative, respectively. This induces a decomposition $\Gamma = \Gamma_+ \cup \Gamma_-$.

Theorem 2.2 (Hofer [13, Theorem 31]). *Let $z_0 \in \Gamma$ be a non-removable puncture of a finite energy pseudo-holomorphic curve $\tilde{u} = (a, u): S \setminus \Gamma \rightarrow \mathbb{R} \times M$. Let $(s, t) \in [0, +\infty) \times \mathbb{R}/\mathbb{Z}$ be holomorphic polar coordinates centered at z_0 as above, and set $\tilde{u}(s, t) = (a(s, t), u(s, t))$. Let $\varepsilon \in \{-1, 1\}$ be the sign of z_0 . Then, for every sequence $s_n \rightarrow +\infty$, there exist a subsequence s_{n_k} of s_n and a periodic orbit $P = (x, T) \in \mathcal{P}(\lambda)$ so that $u(s_{n_k}, \cdot) \rightarrow x(\varepsilon T \cdot)$ in $C^\infty(\mathbb{R}/\mathbb{Z}, M)$ as $k \rightarrow +\infty$.*

The periodic orbit in the previous statement is referred to as an asymptotic limit of \tilde{u} at $z_0 \in \Gamma$. Denote the set of asymptotic limits of \tilde{u} at z_0 by $\Omega = \Omega(z_0) \subset \mathcal{P}(\lambda)$. This set is non-empty, compact and connected. See for instance [12, Lemma 13.3.1]. Explicit examples of finite energy curves with the image of Ω being diffeomorphic to the two-torus are provided by Siefring in [32].

The following theorem due to Hofer, Wysocki and Zehnder tells us that if an asymptotic limit of \tilde{u} at $z_0 \in \Gamma$ is non-degenerate, then Ω consists of a single Reeb orbit. Moreover, \tilde{u} has exponential convergence to the asymptotic limit.

Theorem 2.3 (Hofer-Wysocki-Zehnder [15]). *Let $z_0 \in \Gamma$ be a non-removable puncture of a finite energy pseudo-holomorphic curve $\tilde{u} = (a, u): S \setminus \Gamma \rightarrow \mathbb{R} \times M$. Choose holomorphic polar coordinates $(s, t) \in [0, +\infty) \times \mathbb{R}/\mathbb{Z}$ near z_0 , and set $\tilde{u}(s, t) = (a(s, t), u(s, t))$ as before. Assume that $P = (x, T) \in \mathcal{P}(\lambda)$ is a non-degenerate asymptotic limit of \tilde{u} at z_0 . Then there exist $c, d \in \mathbb{R}$ such that*

- (i) $\sup_{t \in S^1} |a(s, t) - \varepsilon T s - d| \rightarrow 0$ as $s \rightarrow +\infty$.
- (ii) $u(s, \cdot) \rightarrow x(\varepsilon T \cdot + c)$ in $C^\infty(\mathbb{R}/\mathbb{Z}, M)$ as $s \rightarrow +\infty$.
- (iii) let $\pi: TM \rightarrow \xi$ be the projection along the Reeb direction. If $\pi \circ du$ does not vanish identically near z_0 , then $\pi \circ du(s, t) \neq 0$ for every $s \gg 0$.

- (iv) define $\zeta(t, s)$ by $u(s, t) = \exp_{x(\varepsilon Tt+c)} \zeta(\varepsilon Tt + c, s), \forall t \in \mathbb{R}/\mathbb{Z}$. Then there exist an eigenvalue μ of A_P , with $\varepsilon\mu < 0$, and a μ -eigenfunction $e(t) \in \xi|_{x(Tt)}, t \in \mathbb{R}/\mathbb{Z}$, so that $\zeta(t, s) = e^{\varepsilon\mu s} (e(t) + R_1(s, t)), s \gg 0$, where the remainder R_1 and its derivatives converge to 0, uniformly in t , as $s \rightarrow +\infty$.

If \tilde{u} admits a single asymptotic limit P at $z_0 \in \Gamma$ and the asymptotic behavior of \tilde{u} near z_0 is as in Theorem 2.3 for a non-vanishing eigenvalue μ of A_P , then we say that z_0 is a non-degenerate puncture of \tilde{u} and that \tilde{u} exponentially converges to P at z_0 . In the case every asymptotic limit of a finite energy pseudo-holomorphic curve $\tilde{u} = (a, u)$ is non-degenerate, its Conley-Zehnder index and Fredholm index are defined as $\text{CZ}(\tilde{u}) := \sum_{z \in \Gamma_+} \text{CZ}(P_z) - \sum_{z \in \Gamma_-} \text{CZ}(P_z)$, where P_z is the asymptotic limit of \tilde{u} at $z \in \Gamma$, and $\text{Ind}(\tilde{u}) := \text{CZ}(\tilde{u}) - \chi(S) + \#\Gamma$, respectively. Here, CZ is computed in a frame along the asymptotic limits induced by a trivialization of $u^*\xi$. The integer $\text{CZ}(\tilde{u})$ does not depend on this trivialization. Moreover, one can define the following algebraic invariants: suppose that $\pi \circ du$ does not vanish identically. Then $\int_{S \setminus \Gamma} u^*d\lambda > 0$, and Carleman's similarity principle tells us that the zeros of $\pi \circ du$ are isolated. The asymptotic behavior of \tilde{u} described in Theorem 2.3 implies that $\pi \circ du$ does not vanish near the punctures. It follows that the zeros of $\pi \circ du$ are finite, and each zero of $\pi \circ du$ has a positive local degree. We define $\text{wind}_\pi(\tilde{u}) := \#\{\text{zeros of } \pi \circ du\} \geq 0$, where the zeros are counted with multiplicity.

Let $z \in \Gamma$ be a puncture, and let $P = (x, T) \in \mathcal{P}(\lambda)$ be the asymptotic limit of \tilde{u} at z . Let $e \in \Gamma(x_T^*\xi)$ be the associated eigenfunction as in Theorem 2.3-(iv). Define the winding number $\text{wind}_\infty(\tilde{u}; z)$ of \tilde{u} at z to be the winding number of $t \mapsto e(t)$ in a frame of $x_T^*\xi$ induced by a trivialization of $u^*\xi$. The winding number of \tilde{u} is then defined as $\text{wind}_\infty(\tilde{u}) = \sum_{z \in \Gamma_+} \text{wind}_\infty(\tilde{u}; z) - \sum_{z \in \Gamma_-} \text{wind}_\infty(\tilde{u}; z)$. This integer does not depend on the choice of trivialization. The two winding numbers are related by $\text{wind}_\pi(\tilde{u}) = \text{wind}_\infty(\tilde{u}) - \chi(S) + \#\Gamma$, see [14, Proposition 5.6].

Definition 2.4 (Siefring [31]). Let \tilde{u}, \tilde{v} be finite energy planes which are exponentially asymptotic to the same periodic orbit $P \in \mathcal{P}(\lambda)$. Let e_+, e_- be the respective eigensections of A_P that describe \tilde{u}, \tilde{v} near ∞ . We say that \tilde{u} and \tilde{v} are asymptotic to P through opposite directions (resp. through the same direction) if $e_+ = ce_-$ for some $c < 0$ (resp. for some $c > 0$).

2.2. Finite energy foliations

Suppose that the contact form λ on (S^3, ξ_0) is non-degenerate, and choose $J \in \mathcal{J}(\lambda)$. A stable finite energy foliation for (S^3, λ, J) is a two-dimensional smooth foliation $\tilde{\mathcal{F}}$ of $\mathbb{R} \times S^3$ having the following properties:

- (1) Every leaf $\tilde{F} \in \tilde{\mathcal{F}}$ is the image of an embedded finite energy \tilde{J} -holomorphic sphere $\tilde{u}_{\tilde{F}} = (a_{\tilde{F}}, u_{\tilde{F}})$ that has precisely one positive puncture but an arbitrary number

of negative punctures. The energies of such finite energy spheres are uniformly bounded.

- (2) The asymptotic limits of every $\tilde{F} \in \tilde{\mathcal{F}}$, defined as the asymptotic limits of $\tilde{u}_{\tilde{F}}$, have Conley-Zehnder indices belonging to the set $\{1, 2, 3\}$ and self-linking number -1 .
- (3) There exists an \mathbb{R} -action $T: \mathbb{R} \times \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$, defined by $T_c(\tilde{F}) := T(c, \tilde{F}) = \{(c+r, z) \mid (r, z) \in \tilde{F}\} \in \tilde{\mathcal{F}}, \forall c$, so that if $\tilde{F} \in \tilde{\mathcal{F}}$ is not a fixed point of T , then $\text{Ind}(\tilde{u}_{\tilde{F}}) \in \{1, 2\}$, and $u_{\tilde{F}}$ is an embedding, transverse to the Reeb vector field. If \tilde{F} is a fixed point of T , i.e. $T_c(\tilde{F}) = \tilde{F}, \forall c \in \mathbb{R}$, then $\text{Ind}(\tilde{u}_{\tilde{F}}) = 0$, and $\tilde{u}_{\tilde{F}}$ is a cylinder over a periodic orbit.

The following statement on the existence of a stable finite energy foliation is due to Hofer, Wysocki and Zehnder.

Theorem 2.5 (Hofer-Wysocki-Zehnder [19]). *Let λ be a non-degenerate contact form on the tight three-sphere (S^3, ξ_0) . There exists a residual subset $\mathcal{J}_{\text{reg}}(\lambda) \subset \mathcal{J}(\lambda)$, in the C^∞ -topology, such that every $J \in \mathcal{J}_{\text{reg}}(\lambda)$ admits a stable finite energy foliation $\tilde{\mathcal{F}}$. Moreover, the projected foliation $\mathcal{F} := p(\tilde{\mathcal{F}})$, where $p: \mathbb{R} \times S^3 \rightarrow S^3$ denotes the projection onto the second factor, is a genus zero transverse foliation adapted to the flow of λ . If the binding \mathcal{P} consists of a single periodic orbit P , then $\text{CZ}(P) = 3$ and the foliation \mathcal{F} provides an open book decomposition of S^3 whose pages are disk-like global surfaces of section. If $\#\mathcal{P} \geq 2$ and the binding has no periodic orbit with $\text{CZ} = 1$, then the foliation \mathcal{F} is a weakly convex foliation satisfying the following properties:*

- (i) *The binding \mathcal{P} consists of ℓ periodic orbits $P_{2,1}, \dots, P_{2,\ell}$ with Conley-Zehnder index 2 and $\ell+1$ periodic orbits $P_{3,1}, \dots, P_{3,\ell+1}$ with Conley-Zehnder index 3. The binding orbits are unknotted and mutually unlinked and have self-linking number -1 . The orbits $P_{2,1}, \dots, P_{2,\ell}$ are hyperbolic.*
- (ii) *For every $P_{2,i}$, there exists a pair of rigid planes $U_{i,1}, U_{i,2} \in \mathcal{F}$ both asymptotic to $P_{2,i}$ through opposite directions. The set $\mathcal{S}_i := U_{i,1} \cup P_{2,i} \cup U_{i,2}$ is a C^1 -embedded 2-sphere which separates S^3 into two components. In particular, $S^3 \setminus \cup_i \mathcal{S}_i$ has $\ell+1$ components, denoted by $\mathcal{U}_j, j = 1, \dots, \ell+1$.*
- (iii) *Each $P_{3,j}$ is contained in \mathcal{U}_j and spans \tilde{k}_j one-parameter families of planes parametrized by the interval $(0, 1)$. The integer \tilde{k}_j coincides with the number of components of $\partial\mathcal{U}_j$. At their ends, the families break onto a cylinder connecting $P_{3,j}$ to some $P_{2,i} \subset \partial\mathcal{U}_j$ and a plane asymptotic to $P_{2,i}$.*

Let $\lambda = f\lambda_0$ be a possibly degenerate contact form on (S^3, ξ_0) , and let $J \in \mathcal{J}(\lambda)$. As proved in [16, Proposition 6.1], there exists a sequence of non-degenerate contact forms $\lambda_n = f_n\lambda_0$ on (S^3, ξ_0) such that $\lambda_n \rightarrow \lambda$ in C^∞ as $n \rightarrow +\infty$. Since $\mathcal{J}_{\text{reg}}(\lambda_n)$, given in Theorem 2.5, is dense in $\mathcal{J}(\lambda_n) = \mathcal{J}(\lambda)$, for each large n one can find an almost complex structure $J_n \in \mathcal{J}_{\text{reg}}(\lambda_n)$ such that $J_n \rightarrow J$ in C^∞ as $n \rightarrow +\infty$ and, moreover, each

(λ_n, J_n) admits a genus zero transverse foliation \mathcal{F}_n which is the projection to S^3 of a stable finite energy foliation on $\mathbb{R} \times S^3$. The following proposition will be useful in our argument later on.

Proposition 2.6. *Let $\lambda_n \rightarrow \lambda$ and $J_n \rightarrow J$ as $n \rightarrow \infty$, and let \mathcal{F}_n be a genus zero transverse foliation associated with (λ_n, J_n) as in Theorem 2.5. Then there exists a universal constant $C > 0$, depending only on λ , such that the binding orbits of \mathcal{F}_n have action less than C for every large n .*

Proof. The assertion follows from the construction of the foliations \mathcal{F}_n due to Hofer, Wysocki and Zehnder [19]. In fact, the actions of the binding orbits depend only on the C^0 -norm of f . The uniform upper bound for every large n then follows. \square

2.3. Topological entropy

Let X be a nowhere vanishing vector field on a closed manifold M . Abbreviate by ψ_t the flow of X . We fix a metric d that generates the topology of M . For every $T > 0$, we define $d_T(x, y) := \max_{t \in [0, T]} d(\psi_t(x), \psi_t(y))$, $\forall x, y \in M$. Fix $\varepsilon > 0$. A subset U is said to be (T, ε) -separated if $d_T(x, y) \geq \varepsilon$ for every $x \neq y \in U$. Let $N(T, \varepsilon)$ denote the maximal cardinality of a (T, ε) -separated set. The topological entropy $h_{\text{top}}(\psi_t)$ of the flow ψ_t is defined to be the growth rate of $N(T, \varepsilon)$:

$$h_{\text{top}}(\psi^t) := \lim_{\varepsilon \rightarrow 0^+} \limsup_{T \rightarrow +\infty} \frac{1}{T} \log N(T, \varepsilon).$$

It is well-known that the topological entropy of a smooth flow is finite. For more details on topological entropy, we refer to [22, 29].

Remark 2.7. The topological entropy $h_{\text{top}}(f)$ of a continuous map f on a compact Hausdorff metric space (M, d) is defined in the same way as above, with

$$d_n(x, y) := \max\{d(f^k(x), f^k(y)) \mid k = 0, 1, \dots, n-1\}, \quad \forall x, y \in M, \quad n \in \mathbb{N}.$$

If M is a smooth manifold whose topology is determined by the metric d and if the flow ψ_t of a nowhere vanishing vector field X is smooth, then $h_{\text{top}}(\psi_t) = h_{\text{top}}(\psi_1)$. See [22, Proposition 3.1.8].

The following theorem due to Katok [21] relates topological entropy to periodic orbits.

Theorem 2.8 (Katok). *Let ψ_t be the flow of a nowhere vanishing vector field X on a closed three-manifold and let $P_T(\psi_t)$ denote the number of periodic orbits of ψ_t with period smaller than $T > 0$. If $h_{\text{top}}(\psi_t) > 0$, then $\limsup_{T \rightarrow +\infty} \frac{\log P_T(\psi_t)}{T} > 0$.*

A standard way to detect chaotic behavior of a flow is to build a Bernoulli shift. In order to recall its definition, let $A = \{1, \dots, N\}$ be a finite alphabet. The set Σ_N consisting of all doubly infinite sequences $a = (a_j)_{j \in \mathbb{Z}}, a_j \in A$, is equipped with the metric

$$d(a, b) = \sum_{j \in \mathbb{Z}} \frac{1}{2^{|j|}} \frac{|a_j - b_j|}{1 + |a_j - b_j|}, \quad \forall a, b \in \Sigma_N,$$

which makes (Σ_N, d) a compact Hausdorff metric space. The Bernoulli shift on Σ_N is the homeomorphism $\sigma: \Sigma_N \rightarrow \Sigma_N$, defined as $\sigma(a) = (\sigma(a)_j)_{j \in \mathbb{Z}}$, where $\sigma(a)_j := a_{j+1}$.

A homeomorphism ϕ on a compact Hausdorff metric space Λ is said to be semi-conjugate to a Bernoulli shift if there exists a continuous surjective map $\tau: \Lambda \rightarrow \Sigma_N$ for some $N \geq 2$ such that $\tau \circ \phi = \sigma \circ \tau$.

Let $Q \subset \mathbb{R}^2$ be the unit square $[0, 1] \times [0, 1]$. We denote its right, left, upper and lower edges by V_0, V_∞, H_0 and H_∞ , respectively. The compact region bounded by two disjoint and vertically monotone curves connecting H_0 to H_∞ is called a vertical strip in Q . Similarly, one defines a horizontal strip in Q .

We refer the reader to [27, Chapter III] for the proof of the following statement.

Proposition 2.9. *Let $\phi: Q \rightarrow \mathbb{R}^2$ be a mapping that satisfies the following:*

- (N1) *In the square Q , there exist disjoint vertical strips V_1, \dots, V_N and disjoint horizontal strips H_1, \dots, H_N such that $\phi(H_i) = V_i$ for every $i = 1, \dots, N$. The vertical strips and horizontal strips are ordered from right to left and from top to bottom, respectively.*
- (N2) *If $V \subset Q$ is a vertical strip, then for each i , the set $\phi(V) \cap V_i$ contains a vertical strip. Similarly, if $H \subset Q$ is a horizontal strip, then $\phi^{-1}(H) \cap H_i$ contains a horizontal strip for every i .*

Then there exists a compact invariant set $\Lambda \subset Q$ such that $\phi|_\Lambda$ is semi-conjugates to a Bernoulli shift with N symbols. Consequently, $h_{\text{top}}(\phi) > 0$.

Proposition 2.9 generalizes to the case of countably many disjoint vertical strips V_1, V_2, \dots , considering an alphabet with countably many symbols.

3. Proof of Theorem 1.6

Let λ be a weakly convex contact form on (S^3, ξ_0) and let $\mathcal{P}_2(\lambda) = \{P_{2,1}, \dots, P_{2,l}\}$ be a finite set of index-2 periodic orbits satisfying the hypotheses of Theorem 1.6.

Given $C > 0$, we denote by $\mathcal{P}^{\leq C}(\lambda) \subset \mathcal{P}(\lambda)$ the set of periodic orbits with action $\leq C$. For every $j \in \mathbb{Z}$ we define $\mathcal{P}_j^{\leq C}(\lambda) := \mathcal{P}_j(\lambda) \cap \mathcal{P}^{\leq C}(\lambda)$ and $\mathcal{P}_j^{u,-1,\leq C}(\lambda) := \mathcal{P}_j^{u,-1}(\lambda) \cap \mathcal{P}^{\leq C}(\lambda)$, where $\mathcal{P}_j(\lambda)$ and $\mathcal{P}_j^{u,-1,\leq C}(\lambda)$ were established in the introduction.

Take any sequence λ_n of contact forms converging to λ as $n \rightarrow \infty$, and satisfying

- (a) λ_n is non-degenerate, $\forall n \in \mathbb{N}$.
- (b) $P_{2,i} \in \mathcal{P}_2(\lambda_n)$, and $P_{2,i}$ is hyperbolic, $\forall i \in \{1, \dots, l\}, \forall n \in \mathbb{N}$.

As mentioned before, the non-degeneracy in condition (a) is achieved as in [16, Proposition 6.1]. To achieve condition (b), we restrict to the space of contact forms $\lambda_n = f_n \lambda$ satisfying $f_n|_{P_{2,i}} \equiv 1$ and $df_n|_{P_{2,i}} \equiv 0, \forall i, n$, see [16, Lemma 6.8].

Since $P_{2,1}, \dots, P_{2,l}$ are hyperbolic, we can assume, moreover, that for any fixed $C > 0$ sufficiently large, the following assertion holds:

- (c) $\mathcal{P}_2^{\leq C}(\lambda_n) = \{P_{2,1}, \dots, P_{2,l}\}, \forall n$.

Indeed, let $C > 0$ be large enough so that $P_{2,i} \in \mathcal{P}_2^{\leq C}(\lambda_n), \forall i, \forall n$. If for each n we can find an index-2 Reeb orbit Q_n of λ_n , which is geometrically distinct from $P_{2,i}, \forall i = 1, \dots, l$, and whose action is $\leq C$, then the Arzelà-Ascoli Theorem provides us with $Q \in \mathcal{P}^{\leq C}(\lambda)$ so that $Q_n \rightarrow Q$ in C^∞ as $n \rightarrow +\infty$, up to the extraction of a subsequence. The lower semi-continuity of the Conley-Zehnder index and the weak convexity of λ imply that $\text{CZ}(Q) = 2$. Since $\mathcal{P}_2(\lambda) = \{P_{2,1}, \dots, P_{2,l}\}$, Q must coincide with $P_{2,i}$ for some $i = 1, \dots, l$. However, this contradicts the hyperbolicity of the orbits in $\mathcal{P}_2(\lambda)$ and condition (b).

The present goal is to show that λ_n admits a weakly convex foliation \mathcal{F}_n for every large n , so that every $P_{2,i}, i = 1, \dots, l$, is a binding orbit.

Proposition 3.1. *Let $J_n \in \mathcal{J}_{\text{reg}}(\lambda_n)$ be a sequence of almost complex structures satisfying $J_n \rightarrow J \in \mathcal{J}(\lambda)$ in C^∞ as $n \rightarrow +\infty$, where $\mathcal{J}_{\text{reg}}(\lambda_n)$ is given in Theorem 2.5. Let \tilde{J}_n be the almost complex structure on $\mathbb{R} \times S^3$ induced by λ_n and J_n . Then, for every n sufficiently large, the following holds.*

- (i) *The Reeb flow of λ_n admits a weakly convex foliation \mathcal{F}_n , whose leaves are projections to S^3 of embedded finite energy \tilde{J}_n -holomorphic planes and cylinders.*
- (ii) *The binding of \mathcal{F}_n consists of the orbits $P_{2,1}, \dots, P_{2,l} \in \mathcal{P}_2(\lambda_n)$ and $l + 1$ orbits $P_{3,1}^n, \dots, P_{3,l+1}^n \in \mathcal{P}_3^{u,-1,\leq C}(\lambda_n)$, where $C > 0$ is a fixed large number that does not depend on n .*

Fix any $C > 0$ sufficiently large so that property (c) above holds. Before proving Proposition 3.1, we show that for all large n the orbits in $\mathcal{P}_3^{u,-1,\leq C}(\lambda_n)$ do not link with any orbit in $\mathcal{P}_2^{\leq C}(\lambda_n)$ and, moreover, λ_n does not admit orbits with $\text{CZ} = 1$ up to the action C .

Lemma 3.2. *Let $Q_1^n, Q_2^n \in \mathcal{P}_3^{u,-1,\leq C}(\lambda_n)$ such that $Q_1^n \neq Q_2^n, \forall n$. Then*

- (i) *there exist $Q_1^\infty, Q_2^\infty \in \mathcal{P}_3^{u,-1,\leq C}(\lambda)$ so that, up to a subsequence, $Q_1^n \rightarrow Q_1^\infty$ and $Q_2^n \rightarrow Q_2^\infty$ as $n \rightarrow +\infty$. In particular, $\text{link}(Q_j^n, P_{2,i}) = 0, \forall j, i$, and large n .*

- (ii) if $\text{link}(Q_1^n, Q_2^n) = 0, \forall n$, then $Q_1^\infty \neq Q_2^\infty$. In particular, $\text{link}(Q_1^\infty, Q_2^\infty) = 0$.
 (iii) $\mathcal{P}_1^{\leq C}(\lambda_n) = \emptyset, \forall n$ large.

Proof. Because of the uniform upper bound on the actions of $Q_j^n, j = 1, 2$, we can apply the Arzelà-Ascoli Theorem to extract a subsequence, still denoted by Q_j^n , so that $Q_j^n \rightarrow Q_j^\infty \in \mathcal{P}^{\leq C}(\lambda)$ in C^∞ as $n \rightarrow +\infty$. The lower semi-continuity of the Conley-Zehnder index implies that $\text{CZ}(Q_j^\infty) \leq 3$. Since λ is weakly convex, we conclude that Q_j^∞ is simple. Indeed, if $Q_j^\infty = Q^d$ for some $Q \in \mathcal{P}(\lambda)$ and an integer $d > 1$, then $\text{CZ}(Q^d) \geq 4$, a contradiction. It turns out that, as a C^∞ -limit of $Q_j^n \in \mathcal{P}_3^{u, -1, \leq C}(\lambda_n)$, Q_j^∞ is unknotted, has self-linking number -1 and $\text{CZ}(Q_j^\infty) \in \{2, 3\}, j = 1, 2$. For large n , the orbits $P_{2,1}, \dots, P_{2,l}$ are the only orbits of λ_n with $\text{CZ} < 3$ and action $\leq C$. Since these orbits are hyperbolic, we conclude that the limit Q_j^∞ is not an orbit in $\mathcal{P}_2(\lambda)$. Hence $\text{CZ}(Q_j^\infty) = 3$ which implies $Q_j^\infty \in \mathcal{P}_3^{u, -1, \leq C}(\lambda), j = 1, 2$. In view of hypothesis III in Theorem 1.6, Q_j^∞ is not linked with the orbits in $\mathcal{P}_2(\lambda)$. Hence, for every large n , $\text{link}(Q_j^n, P_{2,i}) = 0, \forall j, i$. This proves (i).

Assume now that $\text{link}(Q_1^n, Q_2^n) = 0, \forall n$. Arguing indirectly, suppose that $Q_1^\infty = Q_2^\infty$. Then Q_1^n and Q_2^n are arbitrarily close to each other as $n \rightarrow \infty$. Lemma 5.2 in [16] provides a lower bound on the winding of non-vanishing solutions of the transverse linearized flow along orbits with index 3. In our case, since $\text{CZ}(Q_1^n) = \text{CZ}(Q_2^n) = 3, \forall n$ and both sequences converge to the same limit, which has also index 3, we may apply [16, Lemma 5.2] for every large n to conclude that $\text{link}(Q_1^n, Q_2^n)$ is necessarily positive, which is absurd. Thus $Q_1^\infty \neq Q_2^\infty$ and as C^∞ -limits of $Q_j^n, j = 1, 2$, we conclude that $\text{link}(Q_1^\infty, Q_2^\infty) = 0$. This proves (ii).

Suppose, by contradiction, that $\mathcal{P}_1^{\leq C}(\lambda_n) \neq \emptyset$ for n arbitrarily large. Then, after taking a subsequence, we may assume from the Arzelà-Ascoli Theorem that $P_1^n \rightarrow P_1^\infty$ in C^∞ as $n \rightarrow +\infty$, where $P_1^n \in \mathcal{P}_1^{\leq C}(\lambda_n), \forall n$, and $P_1^\infty \in \mathcal{P}^{\leq C}(\lambda)$. The lower semi-continuity of the Conley-Zehnder index implies that $\text{CZ}(P_1^\infty) \leq 1$, which contradicts the weak convexity of λ . Item (iii) is proved. \square

Proof of Proposition 3.1. Since λ_n is non-degenerate and $J_n \in \mathcal{J}_{\text{reg}}(\lambda_n)$, it follows from Theorem 2.5 that the Reeb flow of λ_n admits a genus zero transverse foliation \mathcal{F}_n whose regular leaves are projections of embedded finite energy \tilde{J}_n -holomorphic curves, where \tilde{J}_n is the \mathbb{R} -invariant almost complex structure in $\mathbb{R} \times S^3$ induced by λ_n and J_n . By Proposition 2.6, there exists $C > 0$ so that the actions of the binding orbits of \mathcal{F}_n are bounded by C for every n . By Lemma 3.2-(iii), $\mathcal{P}_1^{\leq C}(\lambda_n) = \emptyset$, for every large n , and hence we conclude that the binding orbits of \mathcal{F}_n have index 2 or 3 and that regular leaves of \mathcal{F}_n are embedded planes and cylinders. Since, for large n , each $P_{2,i}, i = 1, \dots, l$, is not linked with any orbit in $\mathcal{P}_3^{u, -1, \leq C}(\lambda_n)$, see Lemma 3.2-(i), it follows that each $P_{2,i}, i = 1, \dots, l$, is necessarily a binding orbit of \mathcal{F}_n . Each $P_{2,i}$ bounds a pair of planes, which are regular leaves of \mathcal{F}_n . Together with $P_{2,i}$, these planes form a 2-sphere which separates S^3 into two components. In particular, the complement of the union of these 2-spheres has $l + 1$ components. Each such a component \mathcal{U}_j has a unique index 3 binding

orbit $P_{3,j}$. We conclude that the binding of \mathcal{F}_n is formed by the orbits in $\mathcal{P}_2(\lambda)$ and $l+1$ binding orbits $P_{3,1}, \dots, P_{3,l+1} \in \mathcal{P}_3^{u,-1,\leq C}(\lambda_n)$. \square

In order to construct the desired genus zero transverse foliation \mathcal{F} adapted to the Reeb flow of λ as in Theorem 1.6, we shall study the compactness properties of the finite energy curves in the foliations $\tilde{\mathcal{F}}_n$, which are adapted to (S^3, λ_n, J_n) and project to genus zero transverse foliations \mathcal{F}_n as in Proposition 3.1. Recall that the almost complex structures \tilde{J}_n were taken in the residual set $\mathcal{J}_{\text{reg}}(\lambda_n) \subset \mathcal{J}(\lambda_n) = \mathcal{J}(\lambda)$ in such a way that $J_n \rightarrow J$ in C^∞ as $n \rightarrow +\infty$ for a fixed $J \in \mathcal{J}(\lambda)$. From now on we choose J in a generic set in order to prevent some unsuitable curves that may arise as limits in the compactness argument. More precisely, we need to rule out certain somewhere injective holomorphic curves whose asymptotic limits are contained in $\mathcal{P}_2(\lambda)$. The space of J 's for which such curves do not exist is residual in the C^∞ -topology.

Lemma 3.3. *There exists a residual subset $\mathcal{J}_{\text{reg}}^*(\lambda) \subset \mathcal{J}(\lambda)$ in the C^∞ -topology so that for every $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$, the following assertion holds: let \tilde{J} be the almost complex structure on $\mathbb{R} \times S^3$ induced by λ and J . Let $\tilde{u}: \mathbb{C} \setminus \Gamma \rightarrow \mathbb{R} \times S^3$, $\#\Gamma < +\infty$, be a somewhere injective finite energy \tilde{J} -holomorphic curve having positive $d\lambda$ -area and a unique positive puncture whose asymptotic limit is an orbit in $\mathcal{P}_2(\lambda)$. Then $\Gamma = \emptyset$.*

Proof. An application of [9, Corollary 1.10] provides us with a residual subset $\mathcal{J}_{\text{reg}}^*(\lambda) \subset \mathcal{J}(\lambda)$ such that for every $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$ the following holds: if $\tilde{u}: \mathbb{C} \setminus \Gamma \rightarrow \mathbb{R} \times S^3$ is a somewhere injective finite energy \tilde{J} -holomorphic curve as in the statement, then $\text{Ind}(\tilde{u}) = \text{CZ}(\tilde{u}) - 2 + 1 + \#\Gamma \geq 1$, provided $\pi \circ du \neq 0$. Our standing assumptions on the actions of the orbits in $\mathcal{P}_2(\lambda)$ imply that the asymptotic limits of \tilde{u} at the negative punctures in Γ are covers of orbits in $\mathcal{P}_2(\lambda)$.

Set $\Gamma = \{z_1, \dots, z_{\#\Gamma}\}$ and denote by $N_i \geq 1$ the covering number of the asymptotic limit corresponding to z_i , $i = 1, \dots, \#\Gamma$. Then its Fredholm index satisfies $1 \leq \text{Ind}(\tilde{u}) = 2 - \sum_{i=1}^{\#\Gamma} 2N_i - 1 + \#\Gamma \leq 1 - \#\Gamma$, from which we obtain $\Gamma = \emptyset$. \square

3.1. Rigid planes asymptotic to the index-2 orbits

In this section we prove that for a generic choice of $J \in \mathcal{J}(\lambda)$, each $P_{2,i} \in \mathcal{P}_2(\lambda)$ is the asymptotic limit of a pair of \tilde{J} -holomorphic planes so that the closures of their projections to S^3 form a C^1 -embedded 2-sphere.

Proposition 3.4. *Fix $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$ as in Lemma 3.3. Then for each $i = 1, \dots, l$, there exist embedded finite energy \tilde{J} -holomorphic planes $\tilde{u}_{i,1} = (a_{i,1}, u_{i,1})$, $\tilde{u}_{i,2} = (a_{i,2}, u_{i,2}): \mathbb{C} \rightarrow \mathbb{R} \times S^3$ which are asymptotic to $P_{2,i}$ through opposite directions (see Definition 2.4). In addition, the union $\mathcal{S}_i = u_{i,1}(\mathbb{C}) \cup P_{2,i} \cup u_{i,2}(\mathbb{C})$ is a C^1 -embedded 2-sphere and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset, \forall i \neq j$.*

To prove Proposition 3.4, we choose a sequence of non-degenerate contact forms λ_n converging to λ and satisfying conditions (a), (b) and (c) at the beginning of section 3, and a sequence of almost complex structures $J_n \in \mathcal{J}_{\text{reg}}(\lambda_n)$ converging to $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$ in C^∞ so that the almost complex structure \tilde{J}_n induced by (λ_n, J_n) admits a finite energy foliation $\tilde{\mathcal{F}}_n$ of $\mathbb{R} \times S^3$ whose projection to S^3 is a genus zero transverse foliation \mathcal{F}_n adapted to the flow. Moreover, the binding of \mathcal{F}_n consists of the orbits $P_{2,1}, \dots, P_{2,l} \in \mathcal{P}_2(\lambda_n)$ and $P_{3,1}, \dots, P_{3,l+1} \in \mathcal{P}_3^{-1,u,\leq C}(\lambda_n)$, see Proposition 3.1.

For every large n , $P_{2,i} \in \mathcal{P}_2^{\leq C}(\lambda_n)$ is the boundary of a pair of rigid planes $U_{i,1}^n, U_{i,2}^n \in \mathcal{F}_n$, both transverse to the Reeb vector field R_{λ_n} , so that the 2-spheres $\mathcal{S}_i^n = U_{i,1}^n \cup P_{2,i} \cup U_{i,2}^n$, $i = 1, \dots, l$, are mutually disjoint and do not intersect any $P_{3,j}^n$, $j = 1, \dots, l+1$. The open set $S^3 \setminus \bigcup_{i=1}^l \mathcal{S}_i^n$ contains $l+1$ components \mathcal{U}_j^n such that $P_{3,j}^n \subset \mathcal{U}_j^n$, $\forall j, n$. For each $i = 1, \dots, l$, there exists a pair of embedded \tilde{J}_n -holomorphic planes $\tilde{u}_{i,k}^n = (a_{i,k}^n, u_{i,k}^n): \mathbb{C} \rightarrow \mathbb{R} \times S^3$, $k = 1, 2$, asymptotic to $P_{2,i}$ through opposite directions so that $U_{i,k}^n = u_{i,k}^n(\mathbb{C})$ for every large n .

The following proposition implies Proposition 3.4.

Proposition 3.5. *For each $i = 1, \dots, l$, the embedded \tilde{J}_n -holomorphic rigid planes $\tilde{u}_{i,1}^n, \tilde{u}_{i,2}^n: \mathbb{C} \rightarrow \mathbb{R} \times S^3$ converge in C_{loc}^∞ as $n \rightarrow \infty$, up to reparametrizations and \mathbb{R} -translations, to embedded \tilde{J} -holomorphic rigid planes $\tilde{u}_{i,1} = (a_{i,1}, u_{i,1}), \tilde{u}_{i,2} = (a_{i,2}, u_{i,2}): \mathbb{C} \rightarrow \mathbb{R} \times S^3$ asymptotic to $P_{2,i}$ through opposite directions. The 2-sphere $\mathcal{S}_i = u_{i,1}(\mathbb{C}) \cup P_{2,i} \cup u_{i,2}(\mathbb{C})$ is C^1 -embedded. Moreover, $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$, $\forall i \neq j$, and given neighborhoods $\mathcal{V}_i \subset S^3$ of $\mathcal{S}_i = u_{i,1}(\mathbb{C}) \cup P_{2,i} \cup u_{i,2}(\mathbb{C})$, $i = 1, \dots, l$, we have $\mathcal{S}_i^n \subset \mathcal{V}_i$, $\forall i = 1, \dots, l$, \forall large n .*

Proof. Fix $i \in \{1, \dots, l\}$. For simplicity, denote $\tilde{u}_{i,1}^n$ by $\tilde{u}_n = (a_n, u_n), \forall n$. The case of $\tilde{u}_{i,2}^n$ is treated similarly. Let $\mathcal{U} \subset S^3$ be a small compact tubular neighborhoods $\mathcal{U} \equiv \mathbb{R}/T_{2,i}\mathbb{Z} \times B_\delta(0)$ of $P_{2,i}$, where $\delta > 0$ is small and $T_{2,i} > 0$ denotes the action of $P_{2,i}$ and $B_\delta(0) \subset \mathbb{R}^2$ is the closed ball of radius δ centered at the origin. Since $P_{2,i}$ is hyperbolic, we can take \mathcal{U} sufficiently small so that

- \mathcal{U} contains no periodic orbits that are contractible in \mathcal{U} .
- if $P \subset \mathcal{U}$ is a periodic orbit that is homotopic to $P_{2,i}$ in \mathcal{U} , then $P = P_{2,i}$.

Choose a parametrization of \tilde{u}_n so that

$$\begin{cases} u_n(\mathbb{C} \setminus \mathbb{D}) \subset \mathcal{U}, \\ u_n(1) \in \partial\mathcal{U}, \\ u_n(z_n^*) \in \partial\mathcal{U} \text{ for some } z_n^* \in \partial\mathbb{D} \text{ satisfying } \text{Re}(z_n^*) \leq 0, \\ a_n(2) = 0. \end{cases} \quad (3.1)$$

The existence of such a parametrization is guaranteed as follows. For a fixed parametrization of \tilde{u}_n , the closure K of the set $u_n^{-1}(S^3 \setminus \mathcal{U})$ is compact with non-empty interior. Take

the closed disk $D \subset \mathbb{C}$ containing K which has the smallest radius among all closed disks containing K . Then there exist $w_1 \neq w_2 \in K \cap \partial D$ so that $u_n(w_1), u_n(w_2) \in \partial \mathcal{U}$. Reparametrizing \tilde{u}_n under a map of the form $z \mapsto az + b$, $a, b \in \mathbb{C}$ we may assume that $D = \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and that $w_1 = 1 \in K \subset \mathbb{D}$. If $K \cap \partial \mathbb{D}$ does not contain a point $w_2 \in \partial \mathbb{D}$ with non-positive real part, then shifting K slightly to the left, it is possible to find a disk containing K of radius smaller than 1. This contradicts the minimizing property of D , thus the existence of z_n^* as in (3.1) follows. The last condition in (3.1) may be achieved by considering a suitable \mathbb{R} -translation of \tilde{u}_n for each n .

Lemma 3.6. *Let $\tilde{u}_n = (a_n, u_n) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ satisfy the normalizations in (3.1). Assume that there exist a subsequence of \tilde{u}_n , still denoted by \tilde{u}_n , and a finite energy \tilde{J} -holomorphic map $\tilde{v} = (a, v) : \mathbb{C} \setminus \mathbb{D} \rightarrow \mathbb{R} \times S^3$ so that $\tilde{u}_n|_{\mathbb{C} \setminus \mathbb{D}}$ converges in $C_{\text{loc}}^\infty(\mathbb{C} \setminus \mathbb{D})$ to \tilde{v} as $n \rightarrow +\infty$. Then the following assertions hold:*

- (i) \tilde{v} is non-constant;
- (ii) \tilde{v} is asymptotic to $P_{2,i}$ at ∞ .

Proof. For every $R > 1$ the image of the loop $t \mapsto \gamma_R(t) := v(Re^{it}), \forall t \in \mathbb{R}/2\pi\mathbb{Z}$, is contained in \mathcal{U} since it is the C^∞ -limit of the loops $t \mapsto \gamma_R^n(t) := u_n(Re^{it}), \forall t \in \mathbb{R}/2\pi\mathbb{Z}, \forall n$, which are contained in \mathcal{U} . Hence γ_R is homotopic to γ_R^n in \mathcal{U} for n sufficiently large. Since $u_n(\mathbb{C} \setminus \mathbb{D}) \subset \mathcal{U}$ for every n , and γ_R^n converges to $P_{2,i}$ in \mathcal{U} as $R \rightarrow +\infty$, we conclude that γ_R is homotopic to $P_{2,i}$ in \mathcal{U} for every $R > 1$. This implies, in particular, that γ_R is non-contractible in \mathcal{U} and thus non-constant. As a result, \tilde{v} is non-constant. Moreover, any asymptotic limit $P \subset \mathcal{U}$ of \tilde{v} at ∞ must be homotopic to $P_{2,i}$ in \mathcal{U} since each γ_R has this property for every $R > 1$. Thus our choice of \mathcal{U} implies that the unique asymptotic limit of \tilde{v} at ∞ is $P_{2,i}$. \square

We aim at showing that under the normalizations in (3.1) a bubbling-off phenomenon cannot occur for the sequence \tilde{u}_n , i.e. there is no subsequence of \tilde{u}_n , still denoted by \tilde{u}_n , satisfying $|\nabla \tilde{u}_n(z_n)| \rightarrow +\infty$ as $n \rightarrow \infty$ for a sequence $z_n \in \mathbb{C}$. Here, $|\nabla \tilde{u}_n(z_n)|$ is induced by the inner product on $\mathbb{R} \times S^3$ associated with the pair (λ_n, J_n) . In the absence of bubbling-off, the sequence \tilde{u}_n has gradient bounds which, in this setup and under the normalizations in (3.1), imply C_{loc}^∞ -bounds for \tilde{u}_n from an elliptic bootstrapping argument, see [13]. As a result we will be able to conclude that, up to extraction of a subsequence, \tilde{u}_n converges in C_{loc}^∞ to a \tilde{J} -holomorphic plane $\tilde{u}_{i,1} : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ asymptotic to $P_{2,i}$ as $n \rightarrow \infty$.

An important tool in the bubbling-off analysis is the topological result known as Hofer's Lemma, see [13, Lemma 26]. More specifically, assume \tilde{u}_n admits a subsequence, still denoted by \tilde{u}_n , such that $|\nabla \tilde{u}_n(z_n)| \rightarrow +\infty$ as $n \rightarrow \infty$ for a sequence $z_n \in \mathbb{C}$. Hofer's Lemma allows us to perturb z_n (the new points are still denoted by z_n) and find a sequence of positive numbers $\delta_n \rightarrow 0$, satisfying $r_n := \delta_n |\nabla \tilde{u}_n(z_n)| \rightarrow +\infty$, and so that

an appropriate rescale $\tilde{v}_n: B_{r_n}(0) \rightarrow \mathbb{R} \times S^3$ of $\tilde{u}_n|_{B_{\delta_n}(z_n)}$ has C_{loc}^0 - and C_{loc}^1 -bounds and satisfies $|\nabla \tilde{v}_n(0)| = 1$. To be precise, \tilde{v}_n is defined by

$$\tilde{v}_n(z) = \left(a_n \left(z_n + \frac{\delta_n}{r_n} z \right) - a_n(z_n), u_n \left(z_n + \frac{\delta_n}{r_n} z \right) \right), \quad \forall z \in B_{r_n}(0).$$

From an elliptic bootstrapping argument, we obtain C_{loc}^∞ -bounds and then, up to extraction of a subsequence, \tilde{v}_n converges in C_{loc}^∞ to $\tilde{v}: \mathbb{C} \rightarrow \mathbb{R} \times S^3$, where \tilde{v} is non-constant and has bounded energy by Fatou's Lemma.

If $|z_n| \rightarrow +\infty$ or z_n converges to a point in $\mathbb{C} \setminus \mathbb{D}$, then in view of the normalizations in (3.1), the image $v(\mathbb{C})$ is contained in \mathcal{U} and thus any of its non-trivial asymptotic limits is a contractible periodic orbit in \mathcal{U} , a contradiction to the choice of \mathcal{U} . With this contradiction we conclude that the sequence z_n must be bounded and, up to a subsequence, converges to some point $z_* \in \mathbb{D}$. Such a point is called a bubbling-off point for \tilde{u}_n .

Each bubbling-off point in \mathbb{D} takes away at least $\gamma_0 > 0$ of the $d\lambda_n$ -area of \tilde{u}_n . Here, $\gamma_0 > 0$ is any positive number smaller than the period of the shortest periodic orbit of λ , which exists because of the assumptions on λ . Hence, after passing to a subsequence, we may assume that the set of bubbling-off points $\Gamma \subset \mathbb{D}$ is finite. In particular, $|\nabla \tilde{u}_n|$ is locally bounded on $\mathbb{C} \setminus \Gamma$.

The normalizations in (3.1) provide C_{loc}^0 -bounds for \tilde{u}_n in $\mathbb{C} \setminus \Gamma$. Hence, up to extraction of a subsequence, \tilde{u}_n converges in $C_{\text{loc}}^\infty(\mathbb{C} \setminus \Gamma)$ to a \tilde{J} -holomorphic curve $\tilde{v} = (b, v): \mathbb{C} \setminus \Gamma \rightarrow \mathbb{R} \times S^3$. By Lemma 3.6, \tilde{v} is asymptotic to $P_{2,i}$ at ∞ . Since $P_{2,i}$ is simple, it is somewhere injective.

Let $z^* \in \Gamma$. We claim that

$$\int_{\partial B_\varepsilon(z^*)} v^* \lambda > \gamma_0, \quad \forall \varepsilon > 0 \text{ small.} \quad (3.2)$$

Here, $B_\varepsilon(z^*)$ is the ball centered at z^* of radius $\varepsilon > 0$ and $\partial B_\varepsilon(z^*)$ has the counterclockwise orientation. To prove (3.2), recall that \tilde{u}_n can be appropriately reparametrized in a small neighborhood of $z_n \rightarrow z^*$ so that it converges in C_{loc}^∞ to a non-constant finite energy plane with $d\lambda$ -area $> \gamma_0$. These neighborhoods of z_n are strictly contained in $B_\varepsilon(z^*)$ for every n sufficiently large. Stokes' theorem then gives the desired estimate (3.2). The positivity of the integral in (3.2) implies that every puncture in Γ is negative and therefore ∞ is the only positive puncture of \tilde{v} .

Lemma 3.7. $\Gamma = \emptyset$.

Proof. The first step is to show that the asymptotic limit of \tilde{v} at each $z^* \in \Gamma$ is a cover of an orbit in $\mathcal{P}_2(\lambda)$. Indeed, the hypothesis II in Theorem 1.6 implies that if there exists an asymptotic limit $P = (x, T)$ at z^* which is not a cover of an orbit in $\mathcal{P}_2(\lambda)$, then its period T is greater than $T_{2,i}$. In particular, $\int_{\mathbb{C} \setminus \Gamma} v^* d\lambda < T_{2,i} - T < 0$, a contradiction.

We conclude that \tilde{v} is asymptotic to covers of orbits in $\mathcal{P}_2(\lambda)$ at its negative punctures. Suppose that the $d\lambda$ -area of \tilde{v} vanishes. Then \tilde{v} is a trivial cylinder over $P_{2,i}$. In particular, $\#\Gamma = 1$. If $\Gamma \neq \{1\}$, then $v(1) \in \partial\mathcal{U}$ since $u_n(1) \in \partial\mathcal{U}$ for every n . This is a contradiction. If $\Gamma = \{1\}$, we know from our normalizations in (3.1) that $u_n(z_n^*) \in \partial\mathcal{U}$, and we can assume that $z_n^* \rightarrow z_\infty^* \in \partial\mathbb{D}$, where $\operatorname{Re}(z_\infty^*) \leq 0$, and hence $z_\infty^* \neq 1$ and $v(z_\infty^*) \in \partial\mathcal{U}$, again a contradiction. It follows that the $d\lambda$ -area of \tilde{v} is positive. Since $J \in \mathcal{J}_{\operatorname{reg}}^*(\lambda)$, we conclude that $\Gamma = \emptyset$, see Lemma 3.3. \square

We have proved that, under the normalizations (3.1), we can extract a subsequence of \tilde{u}_n , still denoted by \tilde{u}_n , so that it converges in $C_{\operatorname{loc}}^\infty$ to a finite energy \tilde{J} -holomorphic plane $\tilde{v}: \mathbb{C} \rightarrow \mathbb{R} \times S^3$ asymptotic to $P_{2,i}$ at ∞ . We denote this plane by $\tilde{u}_{i,1} = (a_{i,1}, u_{i,1}): \mathbb{C} \rightarrow \mathbb{R} \times S^3$. Note that it is embedded. Indeed, since $u_{i,1}$ does not intersect its asymptotic limit, an application of Siefring's result [31, Theorem 5.20], see also [12, Theorem 14.5.5], shows that $\tilde{u}_{i,1}$ (and also $u_{i,1}$) is embedded.

Now the analysis of holomorphic cylinders with small area (see [19, Lemma 4.9]) shows that given any S^1 -invariant neighborhood \mathcal{W} of $P_{2,i}$, there exist $R_0 > 0$ and $n_0 \in \mathbb{N}$ such that the loop $t \mapsto \tilde{u}_n(Re^{2\pi it/T_{2,i}})$ belongs to \mathcal{W} for every $R > R_0$ and $n > n_0$. This implies that $u_n(\mathbb{C})$ is arbitrarily C^0 -close to $u_{i,1}(\mathbb{C}) \cup P_{2,i}$. For a proof of these statements in the same setting, see [6, Lemma 7.5].

Considering the sequence of \tilde{J}_n -holomorphic planes $\tilde{u}_{i,2}^n$, we proceed as before to obtain an embedded finite energy \tilde{J} -holomorphic plane $\tilde{u}_{i,2} = (a_{i,2}, u_{i,2}): \mathbb{C} \rightarrow \mathbb{R} \times S^3$, asymptotic to $P_{2,i}$ as the $C_{\operatorname{loc}}^\infty$ -limit of $\tilde{u}_{i,2}^n$. Moreover, $u_{i,2}^n(\mathbb{C})$ is arbitrarily C^0 -close to $u_{i,2}(\mathbb{C}) \cup P_{2,i}$ for n large.

From the uniqueness of \tilde{J} -holomorphic planes asymptotic to $P_{2,i}$ through each direction, see [6, Proposition C.3], we know that there exist at most two \tilde{J} -holomorphic planes asymptotic to $P_{2,i}$, up to reparametrizations and \mathbb{R} -translations. In this case, they are asymptotic to $P_{2,i}$ through opposite directions, as shown below.

Lemma 3.8. $u_{i,m}(\mathbb{C}) \cap u_{j,n}(\mathbb{C}) = \emptyset, \forall (i, m) \neq (j, n)$.

Proof. The transverse foliation \mathcal{F}_n admits $l+1$ binding orbits $P_{3,j}^n, j = 1, \dots, l+1$. By Lemma 3.2 these orbits converge, up to a subsequence, to mutually unlinked periodic orbits $P_{3,1}^\infty, \dots, P_{3,l+1}^\infty \in \mathcal{P}_3^{u,-1,\leq C}(\lambda)$, which are not linked with any index-2 Reeb orbits.

We first show that the planes $u_{i,1}(\mathbb{C})$ and $u_{i,2}(\mathbb{C})$ cannot be the same. Assume by contradiction that this is the case. Since $u_{i,1}^n(\mathbb{C})$ and $u_{i,2}^n(\mathbb{C})$ are C^0 -close to $u_{i,1}(\mathbb{C}) \cup P_{2,i} = u_{i,2}(\mathbb{C}) \cup P_{2,i}$, we conclude that $u_{i,1}^n$ is homotopic to $u_{i,2}^n$, relative to $P_{2,i}$, in a small neighborhood $\mathcal{V} \subset S^3$ of $u_{i,1}(\mathbb{C}) \cup P_{2,i}$. Since the 2-sphere $\mathcal{S}_i^n = u_{i,1}^n(\mathbb{C}) \cup P_{2,1} \cup u_{i,2}^n(\mathbb{C})$ separates S^3 into two components and in each component there exists some index-3 binding orbit of \mathcal{F}_n , any homotopy from $u_{i,1}^n$ to $u_{i,2}^n$ necessarily intersects some $P_{3,j}^n$. Since the limit $P_{3,j}^\infty$ is not linked with $P_{2,i}, \forall i$, we know that for n sufficiently large $P_{3,j}^n \cap \mathcal{V} = \emptyset$.

We now assume by contradiction that $u_{i,1}(\mathbb{C}) \cap u_{i,2}(\mathbb{C}) \neq \emptyset$, and hence $\tilde{u}_{i,1}(\mathbb{C}) \cap \tilde{u}_{i,2}(\mathbb{C}) \neq \emptyset$. Then the positivity and stability of intersections of pseudo-holomorphic curves (see [25, Appendix E]) tell us that $\tilde{u}_{i,1}^n(\mathbb{C}) \cap \tilde{u}_{i,2}^n(\mathbb{C}) \neq \emptyset$ for all n large enough, a contradiction. This finishes the proof. \square

We conclude from Lemma 3.8 and the uniqueness of planes asymptotic to $P_{2,i}$, through each direction, that $\tilde{u}_{i,1}$ and $\tilde{u}_{i,2}$ are asymptotic to $P_{2,i}$ through opposite directions.

Lemma 3.9. *The 2-sphere $\mathcal{S}_i = u_{i,1}(\mathbb{C}) \cup P_{2,i} \cup u_{i,2}(\mathbb{C})$ is C^1 -embedded.*

Proof. Since $\text{CZ}(P_{2,i}) = 2$, the leading eigenvalues of $A_{P_{2,i},J}$, which describe the behavior of $\tilde{u}_{i,1}$ and $\tilde{u}_{i,2}$ near ∞ , coincide with the unique eigenvalue $\mu < 0$ with winding number 1. This means that we can write

$$u_{i,j}(e^{2\pi(s+it)}) = \exp_{x_{2,i}(T_{2,i}t)}\{e^{\mu s}(e_j(t) + R_j(s,t))\}, \quad s \gg 0, \quad j = 1, 2,$$

where $P_{2,i} = (x_{2,i}, T_{2,i})$ and $e_j: \mathbb{R}/\mathbb{Z} \rightarrow x_{2,i}^*\xi$, $j = 1, 2$, is a μ -eigensection with winding number 1. The remainder term R_j and its derivatives converge to 0, uniformly in t as $s \rightarrow +\infty$. See Theorem 2.3.

Since the μ -eigenspace is one-dimensional, we have $e_2 = ce_1$ for some $c \neq 0$. If $c > 0$, then $u_{i,1}(\mathbb{C}) \cap u_{i,2}(\mathbb{C}) \neq \emptyset$, see [6, Proposition C.0]. This contradicts Lemma 3.8, and we conclude that $c < 0$.

Defining $r = e^{\mu s} \Leftrightarrow s = \frac{1}{\mu} \ln r$, we see that the maps

$$v_j(r, t) := u_{i,j}(e^{2\pi(s+it)}) = \exp_{x_{2,i}(T_{2,i}t)}\{r(e_j(t) + \bar{R}_j(r, t))\}, \quad \forall(r, t), \quad j = 1, 2,$$

extend continuously to $[0, \varepsilon] \times \mathbb{R}/\mathbb{Z}$ with $\varepsilon > 0$ small. Since $\lim_{r \rightarrow 0} \bar{R}_j(r, t) = 0$ uniformly in t , we conclude that v_j is at least C^1 . Moreover, the tangent space of v_j along $P_{2,i}$ coincides with $\mathbb{R}e_j \oplus TP_{2,i}$. Now, since $\text{wind}_\pi(\tilde{u}) = \text{wind}_\infty(\tilde{u}; \infty) - 1 = 0$, we conclude that $u_{i,j}$ is an immersion transverse to R_λ , $j = 1, 2$. Hence $u_{i,1}(\mathbb{C}) \cup P_{2,i} \cup u_{i,2}(\mathbb{C})$ is a C^1 -embedded 2-sphere. \square

Given a neighborhood $\mathcal{V} \subset S^3$ of \mathcal{S}_i , we have $\mathcal{S}_i^n \subset \mathcal{V}$ for every large n . Since the same argument holds for every $i = 1, \dots, l$, the proof of Proposition 3.5 is finished. \square

3.2. Special index-3 orbits

Although Lemma 3.2 provides candidates $P_{3,1}^\infty, \dots, P_{3,l+1}^\infty$ in $\mathcal{P}_3^{u,-1,\leq C}(\lambda)$ for the index 3 binding orbits of the desired foliation, we cannot guarantee that every periodic orbit U , which is not a cover of an orbit in $\mathcal{P}_2(\lambda)$ nor a cover of any $P_{3,j}^\infty$, is linked with some $P_{3,i}^\infty$. Such an orbit U might a priori exist. We shall rule out this unpleasant scenario by appropriately choosing new sequences $\lambda_n \rightarrow \lambda$ so that the corresponding limiting orbits $P_{3,1}^\infty, \dots, P_{3,l+1}^\infty$ do not admit such an unlinked orbit U .

Proposition 3.10. *Let $C > 0$ be as in Proposition 2.6, and let $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$. Then there exist a sequence of non-degenerate contact forms $\lambda_n = f_n \lambda$ converging in C^∞ to λ and a sequence of almost complex structures $J_n \in \mathcal{J}_{\text{reg}}(\lambda_n)$ converging in C^∞ to J so that the following holds.*

- (i) $P_{2,1}, \dots, P_{2,l} \in \mathcal{P}_2(\lambda_n), \forall n$, and the almost complex structure \tilde{J}_n on $\mathbb{R} \times S^3$ induced by (λ_n, J_n) admits a stable finite energy foliation $\tilde{\mathcal{F}}_n$ that projects to a genus zero transverse foliation \mathcal{F}_n , whose binding orbits are $P_{2,1}, \dots, P_{2,l}$ and $P_{3,1}^n, \dots, P_{3,l+1}^n \in \mathcal{P}_3^{u,-1,\leq C}(\lambda_n)$.
- (ii) there exist $l+1$ periodic orbits $\mathcal{P}_3^s(\lambda) := \{P_{3,1}, \dots, P_{3,l+1}\} \subset \mathcal{P}_3^{u,-1,\leq C}(\lambda)$, so that for every j , $P_{3,j}^n \rightarrow P_{3,j}$ as $n \rightarrow +\infty$. Moreover,
 - $\text{link}(P_{3,i}, P_{3,j}) = 0, \forall i \neq j$.
 - $\text{link}(P_{3,i}, P_{2,j}) = 0, \forall i, j$.
 - every $P \in \mathcal{P}^{\leq C}(\lambda)$, which is not a cover of any orbit in $\mathcal{P}_2(\lambda) \cup \mathcal{P}_3^s(\lambda)$, is linked with some orbit in $\mathcal{P}_3^s(\lambda)$.
- (iii) let $\tilde{u}_{i,1} = (a_{i,1}, u_{i,1}), \tilde{u}_{i,2} = (a_{i,2}, u_{i,2}): \mathbb{C} \rightarrow \mathbb{R} \times S^3, i = 1, \dots, l$, be the unique \tilde{J} -holomorphic planes asymptotic to $P_{2,i}$ which are C_{loc}^∞ -limits of planes $\tilde{u}_{i,1}^n, \tilde{u}_{i,2}^n$ in $\tilde{\mathcal{F}}_n$, asymptotic to $P_{2,i}$ through opposite directions, and whose existence is assured by Proposition 3.5. Denote by $\mathcal{U}_j \subset S^3, j = 1, \dots, l+1$, the components of $S^3 \setminus \bigcup_{i=1}^l \mathcal{S}_i$, where $\mathcal{S}_i = u_{i,1}(\mathbb{C}) \cup P_{2,i} \cup u_{i,2}(\mathbb{C})$. Then $P_{3,j} \subset \mathcal{U}_j, \forall j = 1, \dots, l+1$.

Proof. Take a sequence of non-degenerate contact forms $\lambda_n = f_n \lambda, n \in \mathbb{N}$, converging in C^∞ to λ and a sequence $J_n \in \mathcal{J}_{\text{reg}}(\lambda_n)$ converging in C^∞ to $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$ as in the previous section. We assume that every λ_n satisfies conditions (a), (b), and (c) from the beginning of section 3. By Lemma 3.2 every orbit in $\mathcal{P}_3^{u,-1,\leq C}(\lambda_n)$ is not linked with any orbit in $\mathcal{P}_2^{\leq C}(\lambda_n)$ for every large n . We conclude in view of Proposition 3.1 that the almost complex structure \tilde{J}_n induced by (λ_n, J_n) admits a stable finite energy foliation which projects to a weakly convex foliation \mathcal{F}_n so that the orbits $P_{2,1}, \dots, P_{2,l} \in \mathcal{P}_2^{\leq C}(\lambda_n)$ are binding orbits of \mathcal{F}_n . Moreover, the remaining binding orbits of \mathcal{F}_n are periodic orbits in the set

$$\mathcal{P}_3^{n,0}(\lambda_n) := \{P_{3,1}^{n,0}, \dots, P_{3,l+1}^{n,0}\} \subset \mathcal{P}_3^{u,-1,\leq C}(\lambda_n).$$

They satisfy the additional properties:

- $\text{link}(P_{3,i}^{n,0}, P_{3,j}^{n,0}) = 0, \forall i \neq j$.
- $\text{link}(P_{3,i}^{n,0}, P_{2,j}) = 0, \forall i, j$.
- given $P \in \mathcal{P}(\lambda_n)$, which is not a cover of any orbit in $\mathcal{P}_2(\lambda_n) \cup \mathcal{P}_3^{n,0}(\lambda_n)$, there exists $j \in \{1, \dots, l+1\}$ so that $\text{link}(P, P_{3,j}^{n,0}) \neq 0$.

We claim that as $n \rightarrow +\infty$ the orbits $P_{3,1}^{n,0}, \dots, P_{3,l+1}^{n,0}$ converge, up to a subsequence, to mutually distinct and mutually unlinked periodic orbits $P_{3,1}^{\infty,0}, \dots, P_{3,l+1}^{\infty,0}$ in $\mathcal{P}_3^{u,-1,\leq C}(\lambda)$.

First of all, the upper bound C on the actions of $P_{3,1}^{n,0}, \dots, P_{3,l+1}^{n,0}$ and the Arzelà-Ascoli Theorem imply that the orbits $P_{3,j}^{n,0}, j = 1, \dots, l+1$, converge, up to a subsequence, to elements $P_{3,j}^{\infty,0} \in \mathcal{P}^{\leq C}(\lambda), j = 1, \dots, l+1$. Their Conley-Zehnder indices are ≤ 3 by the lower semi-continuity. Since λ is weakly convex and the orbits in $\mathcal{P}_2^{u,-1,\leq C}(\lambda_n)$ are the only orbits converging to the corresponding orbits in $\mathcal{P}_2(\lambda)$, we have $\text{CZ}(P_{3,j}^{\infty,0}) = 3, \forall j$. In particular, $P_{3,j}^{\infty,0}$ is simply covered for every j . As limits of the Reeb orbits $P_{3,j}^{n,0}$ as $n \rightarrow +\infty$, we conclude that $P_{3,j}^{\infty,0}$ is unknotted and has self-linking number -1 . If $P_{3,j}^{\infty,0} = P_{3,k}^{\infty,0}$ for some $j \neq k$, then, since $\text{CZ}(P_{3,j}^{\infty,0}) = 3$, we conclude from [16, Lemma 5.2] that, for every large n , $\text{link}(P_{3,j}^{n,0}, P_{3,k}^{n,0}) > 0$, a contradiction. Hence the orbits $P_{3,1}^{\infty,0}, \dots, P_{3,l+1}^{\infty,0}$ are mutually distinct and mutually unlinked. The claim is then proved.

Next we show that the orbits $P_{3,j}^{\infty,0}, j = 1, \dots, l+1$, lie in distinct components \mathcal{U}_j of $S^3 \setminus \bigcup_{i=1}^l \mathcal{S}_i$, where $\mathcal{S}_i = u_{i,1}(\mathbb{C}) \cup P_{2,i} \cup u_{i,2}(\mathbb{C})$ is the C^1 -embedded 2-sphere such that for every i , the maps $u_{i,1}, u_{i,2}$ are the projections to S^3 of the $\tilde{\mathcal{J}}$ -holomorphic planes $\tilde{u}_{i,1} = (a_{i,1}, u_{i,1}), \tilde{u}_{i,2} = (a_{i,2}, u_{i,2}): \mathbb{C} \rightarrow \mathbb{R} \times S^3$, asymptotic to $P_{2,i}$ through opposite directions, and obtained as C_{loc}^∞ -limits of the corresponding $\tilde{\mathcal{J}}_n$ -holomorphic planes of $\tilde{\mathcal{F}}_n$, see Proposition 3.5.

Lemma 3.11. *After reordering \mathcal{U}_j , if necessary, we have $P_{3,j}^{\infty,0} \subset \mathcal{U}_j, \forall j = 1, \dots, l+1$.*

Proof. For every large n and for every i , there exist $\tilde{\mathcal{J}}_n$ -holomorphic planes $\tilde{u}_{i,1}^n = (a_{i,1}^n, u_{i,1}^n), \tilde{u}_{i,2}^n = (a_{i,2}^n, u_{i,2}^n): \mathbb{C} \rightarrow \mathbb{R} \times S^3$ which are asymptotic to $P_{2,i}$ through opposite directions, and which converge in C_{loc}^∞ , up to a subsequence, to $\tilde{\mathcal{J}}$ -holomorphic planes $\tilde{u}_{i,1} = (a_{i,1}, u_{i,1}), \tilde{u}_{i,2} = (a_{i,2}, u_{i,2}): \mathbb{C} \rightarrow \mathbb{R} \times S^3$, also asymptotic to $P_{2,i}$ through opposite directions.

Let $\mathcal{S}_i^n = u_{i,1}^n(\mathbb{C}) \cup P_{2,i} \cup u_{i,2}^n(\mathbb{C}), \forall i = 1, \dots, l$. For every $j = 1, \dots, l+1$, denote by $\mathcal{U}_j^n \subset S^3$ the component of $S^3 \setminus \bigcup_{i=1}^l \mathcal{S}_i^n$, which contains $P_{3,j}^{n,0}$. By Proposition 3.5, given small neighborhoods \mathcal{V}_i of $\mathcal{S}_i, i = 1, \dots, l$, there exists n_0 so that $\mathcal{S}_i^n \subset \mathcal{V}_i$ for every $n > n_0$. Since $P_{3,j}^{\infty,0}$ is not linked with any $P_{2,i}$, the orbits $P_{3,1}^{n,0}, \dots, P_{3,l+1}^{n,0}$ are contained in distinct components $\mathcal{U}_1^n, \dots, \mathcal{U}_{l+1}^n$ for every large n . In particular, after relabelling the components $\mathcal{U}_1, \dots, \mathcal{U}_{l+1}$, if necessary, these orbits are contained in distinct components $\mathcal{U}_1, \dots, \mathcal{U}_{l+1}$ for every large n . Hence $P_{3,j}^{\infty,0} \subset \mathcal{U}_j, \forall j$. The lemma follows. \square

Abbreviate $\mathcal{P}_3^{\infty,0}(\lambda) = \{P_{3,1}^{\infty,0}, \dots, P_{3,l+1}^{\infty,0}\} \subset \mathcal{P}_3^{u,-1,\leq C}(\lambda)$. From Lemma 3.11 we know that $P_{3,j}^{\infty,0} \subset \mathcal{U}_j, \forall j = 1, \dots, l+1$. If every orbit in $\mathcal{P}^{\leq C}(\lambda)$, which is not a cover of an orbit in $\mathcal{P}_2(\lambda) \cup \mathcal{P}_3^{\infty,0}(\lambda)$, is linked with some orbit in $\mathcal{P}_3^{\infty,0}(\lambda)$, then $\mathcal{P}_3^{\infty,0}(\lambda)$ is the desired set of periodic orbits and there is nothing else to be proved. Otherwise, if there exists a simple periodic orbit $U_0 \in \mathcal{P}^{\leq C}(\lambda)$ which is not a cover of an orbit in $\mathcal{P}_2(\lambda) \cup \mathcal{P}_3^{\infty,0}(\lambda)$ and is not linked with any orbit in $\mathcal{P}_3^{\infty,0}(\lambda)$, then we proceed as follows. Consider a new sequence of non-degenerate contact forms λ_n converging in C^∞ to λ as $n \rightarrow +\infty$, so that every orbit in $\mathcal{P}_2(\lambda)$ and every orbit in

$$\mathcal{L}_0 := \mathcal{P}_3^{\infty,0}(\lambda) \cup \{U_0\},$$

is a periodic orbit of $\lambda_n, \forall n$. As before, for a suitable sequence $J_n \in \mathcal{J}_{\text{reg}}(\lambda_n)$ converging to J , we obtain a sequence of weakly convex foliations \mathcal{F}_n whose binding orbits are the orbits in $\mathcal{P}_2(\lambda)$ together with other mutually unlinked orbits $P_{3,1}^{n,1}, \dots, P_{3,l+1}^{n,1} \in \mathcal{P}_3^{u,-1,\leq C}(\lambda_n)$.

Taking the limit $n \rightarrow +\infty$, we obtain a new set of periodic orbits

$$\mathcal{P}_3^{\infty,1}(\lambda) = \{P_{3,1}^{\infty,1}, \dots, P_{3,l+1}^{\infty,1}\} \subset \mathcal{P}_3^{u,-1,\leq C}(\lambda),$$

so that each $P_{3,j}^{\infty,1}$ is the C^∞ -limit of $P_{3,j}^{n,1}$ as $n \rightarrow +\infty$. Arguing as before, we conclude that these orbits are mutually distinct and mutually unlinked and satisfy $P_{3,j}^{\infty,1} \subset \mathcal{U}_j, \forall j$. Some of them may coincide with the corresponding orbits in $\mathcal{P}_3^{\infty,0}(\lambda)$.

We claim that no orbit in $\mathcal{P}_3^{\infty,1}(\lambda)$ coincides with U_0 . Indeed, observe that each $P_{3,j}^{n,1}$ is contained in \mathcal{U}_j for all large n and either coincides with $P_{3,j}^{\infty,0}$ or is linked with $P_{3,j}^{\infty,0}$. Hence $P_{3,j}^{\infty,1}$ either coincides with $P_{3,j}^{\infty,0}$ or is linked with $P_{3,j}^{\infty,0}$. In particular, since U_0 is linked with some $P_{3,j}^{n,1}$ for every large n , we conclude that U_0 is linked with some $P_{3,j}^{\infty,1}$. The claim is proved.

Now if every orbit in $\mathcal{P}^{\leq C}(\lambda)$, which is not a cover of an orbit in $\mathcal{P}_2(\lambda) \cup \mathcal{P}_3^{\infty,1}(\lambda)$, is linked with some orbit in $\mathcal{P}_3^{\infty,1}(\lambda)$, then $\mathcal{P}_3^{\infty,1}(\lambda)$ is the desired set of periodic orbits and there is nothing else to be proved. Otherwise, if there exists a simple periodic orbit $U_1 \in \mathcal{P}^{\leq C}(\lambda)$, which is not a cover of an orbit in $\mathcal{P}_2(\lambda) \cup \mathcal{P}_3^{\infty,1}(\lambda)$ and is not linked with any orbit in $\mathcal{P}_3^{\infty,1}(\lambda)$, then we construct another new sequence of non-degenerate contact forms λ_n converging in C^∞ to λ as $n \rightarrow +\infty$ as before, so that every orbit in $\mathcal{P}_2(\lambda)$ and every orbit in

$$\mathcal{L}_1 := \mathcal{L}_0 \cup \mathcal{P}_3^{\infty,1} \cup \{U_1\},$$

is a periodic orbit of λ_n for every n . Choosing a suitable sequence $J_n \rightarrow J$, we find a new set $\mathcal{P}_3^{\infty,2}(\lambda) = \{P_{3,1}^{\infty,2}, \dots, P_{3,l+1}^{\infty,2}\} \subset \mathcal{P}_3^{u,-1,\leq C}(\lambda)$, with $P_{3,j}^{\infty,2} \subset \mathcal{U}_j, \forall j$, as the limit of the new binding orbits $P_{3,j}^{n,2}, j = 1, \dots, l+1$.

As before, we claim that no orbit in $\mathcal{P}_3^{\infty,2}(\lambda)$ coincides with U_0 or U_1 . To see this, observe that each $P_{3,j}^{n,2}$ is contained in \mathcal{U}_j for all large n and either it coincides with one of the orbits $P_{3,j}^{\infty,0}$ or $P_{3,j}^{\infty,1}$ or is linked with both of them. Hence $P_{3,j}^{\infty,2}$ either coincides with one of the orbits $P_{3,j}^{\infty,0}, P_{3,j}^{\infty,1}$, or is linked with both of them. In particular, since U_0 is linked with some $P_{3,j}^{n,2}$ for every large n , we conclude that U_0 is linked with some $P_{3,j}^{\infty,2}$. The same holds with U_1 . The claim follows.

Again, if we find a simple periodic orbit $U_2 \in \mathcal{P}^{\leq C}(\lambda)$ which is not a cover of an orbit in $\mathcal{P}_2(\lambda) \cup \mathcal{P}_3^{\infty,2}(\lambda)$ and is not linked with any orbit in $\mathcal{P}_3^{\infty,2}(\lambda)$, we define a new set

$$\mathcal{L}_2 := \mathcal{L}_1 \cup \mathcal{P}_3^{\infty,2}(\lambda) \cup \{U_2\},$$

and consider again a new sequence $\lambda_n \rightarrow \lambda$ (and $J_n \rightarrow J$) as before to obtain $\mathcal{P}_3^{\infty,3}(\lambda)$ with similar properties and so on.

Repeating this process indefinitely, if necessary, we end up with sequences

$$\mathcal{P}_3^{\infty,k}(\lambda) \subset \mathcal{P}_3^{u,-1,\leq C}(\lambda) \quad \text{and} \quad U_k \in \mathcal{P}^{\leq C}(\lambda), \quad k \in \mathbb{N},$$

so that

- $\mathcal{P}_3^{\infty,k}(\lambda) \subset \mathcal{P}_3^{u,-1,\leq C}(\lambda)$ is formed by $l+1$ mutually distinct and mutually unlinked orbits $P_{3,j}^{\infty,k} \subset \mathcal{U}_j, \forall j = 1, \dots, l+1$.
- U_k is not a cover of any orbit in $\mathcal{P}_2(\lambda) \cup \mathcal{P}_3^{\infty,k}(\lambda)$.
- U_k is not linked with any orbit in $\mathcal{P}_3^{\infty,k}(\lambda)$.
- for every $p > k$ there exists an orbit in $\mathcal{P}_3^{\infty,p}(\lambda)$ which is linked with U_k .

We may extract a subsequence so that the orbits $P_{3,1}^{\infty,k}, \dots, P_{3,l+1}^{\infty,k} \in \mathcal{P}_3^{\infty,k}(\lambda)$ are converging to $Q_1^\infty, \dots, Q_{l+1}^\infty \in \mathcal{P}_3^{u,-1,\leq C}(\lambda)$ as $k \rightarrow +\infty$. The periodic orbits $Q_j^\infty, j = 1, \dots, l+1$, are mutually distinct, mutually unlinked and $Q_j^\infty \subset \mathcal{U}_j, \forall j$. Moreover, $\text{CZ}(Q_j^\infty) = 3, \forall j$.

Using the Arzelà-Ascoli theorem, we may assume that $U_k \rightarrow U_\infty$ as $k \rightarrow +\infty$, where the action of U_∞ is $\leq C$. Since $P_{2,i}$ is hyperbolic and each U_k is geometrically distinct from the orbits in $\mathcal{P}_2(\lambda)$, we conclude that U_∞ is not a cover of any orbit in $\mathcal{P}_2(\lambda)$. Observe that:

- If U_∞ is a cover of Q_j^∞ for some j , then since $\text{CZ}(Q_j^\infty) = 3$ we conclude that U_k is linked with $P_{3,j}^{\infty,k}$ for every k sufficiently large, a contradiction.
- If U_∞ is not a cover of Q_j^∞ for any j and is linked with some Q_j^∞ , then U_k is linked with $P_{3,j}^{\infty,k}$ for k sufficiently large, a contradiction.
- If U_∞ is not a cover of Q_j^∞ for any j and is not linked with any Q_j^∞ , then for k sufficiently large U_k is not linked with any $P_{3,j}^{\infty,p}$ for every $p > k$. This is also a contradiction.

We have proved that the process of constructing such new sequences of non-degenerate contact forms converging to λ must terminate after finitely many steps. Hence we find a sequence $\lambda_n \rightarrow \lambda$ with the desired properties whose limiting periodic orbits $P_{3,1}, \dots, P_{3,l+1}$ satisfy all properties in the statement of Proposition 3.10. The proof is complete. \square

3.3. Compactness properties of holomorphic cylinders

Let λ_n and $J_n \in \mathcal{J}_{\text{reg}}(\lambda_n)$ be sequences of non-degenerate contact forms and almost complex structures converging to λ and $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$, respectively, as obtained in Proposition 3.10. For every n , the almost complex structure \tilde{J}_n induced by (λ_n, J_n) admits a stable finite energy foliation $\tilde{\mathcal{F}}_n$ that projects to a genus zero transverse foliation \mathcal{F}_n

whose binding orbits are $P_{2,1}, \dots, P_{2,l}, P_{3,1}^n, \dots, P_{3,l+1}^n$, where, for every j , the orbit $P_{3,j}^n$ converges to $P_{3,j} \in \mathcal{P}_3^s(\lambda)$.

Fix $j \in \{1, \dots, l+1\}$, so that $P_{3,j}^n \subset \mathcal{U}_j$ for every n . Choose an arbitrary boundary component, say \mathcal{S}_i , of the closure of \mathcal{U}_j . Then every \mathcal{F}_n contains a unique rigid cylinder connecting $P_{3,j}^n$ to $P_{2,i}$. This cylinder is the projection of an embedded finite energy \tilde{J}_n -holomorphic cylinder $\tilde{v}_n = (b_n, v_n): \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times S^3$. It is asymptotic to $P_{3,j}^n$ at its positive puncture $+\infty$ and to $P_{2,i}$ at its negative puncture $-\infty$.

We shall study the compactness properties of the sequence \tilde{v}_n . Consider $\mathcal{U} \subset \mathcal{U}_j$ a small compact tubular neighborhood of $P_{3,j}$. Since $\text{CZ}(P_{3,j}) = 3$, we can choose \mathcal{U} sufficiently small so that

- \mathcal{U} contains no periodic orbits that are contractible in \mathcal{U} .
- there exists no Reeb orbit $P \subset \mathcal{U}$ of λ which is geometrically distinct from $P_{3,j}$, is homotopic to $P_{3,j}$ in \mathcal{U} and satisfies $\text{link}(P, P_{3,j}) = 0$.

The first property follows from the fact that for \mathcal{U} sufficiently small, every periodic orbit in \mathcal{U} must be homotopic in \mathcal{U} to a positive cover of $P_{3,j}$ and hence is non-contractible in \mathcal{U} . The second property can be achieved since $P_{3,j} \in \mathcal{P}_3^{u,-1}(\lambda)$. Indeed, the flow near $P_{3,j}$ twists fast enough so that any periodic orbit sufficiently close to $P_{3,j}$, if it exists, must be linked with $P_{3,j}$. See Lemma 5.2 in [16].

Using that $P_{3,j}^n \rightarrow P_{3,j}$ as $n \rightarrow +\infty$, we observe that $P_{3,j}^n \subset \text{int}(\mathcal{U})$ for every large n and, moreover, due to the asymptotic properties of \tilde{v}_n , we can normalize \tilde{v}_n to satisfy the following conditions

$$\begin{cases} v_n(s, t) \in \mathcal{U} \text{ for all } s > 0, t \in \mathbb{R}/\mathbb{Z}. \\ v_n(0, 0) \in \partial\mathcal{U}. \\ b_n(1, 0) = 0. \end{cases} \quad (3.3)$$

Lemma 3.12. *Let \tilde{v}_n satisfy the normalizations (3.3). Assume that there exist a subsequence of \tilde{v}_n , still denoted by \tilde{v}_n , and a finite energy \tilde{J} -holomorphic map $\tilde{v} = (b, v): (0, \infty) \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times S^3$ so that $\tilde{v}_n|_{(0, \infty) \times S^1}$ converges in $C_{\text{loc}}^\infty((0, \infty) \times \mathbb{R}/\mathbb{Z})$ to \tilde{v} as $n \rightarrow +\infty$. Then \tilde{v} is asymptotic to $P_{3,j}$ at ∞ .*

Proof. We argue as in Lemma 3.6. Let $R > 0$. Since the loop $t \mapsto \gamma_R^n(t) := v_n(R, t), t \in \mathbb{R}/\mathbb{Z}$, lies in \mathcal{U} for every n and since γ_R^n converges in $C^\infty(\mathbb{R}/\mathbb{Z})$ to the loop $t \mapsto \gamma_R(t) := v(R, t), t \in \mathbb{R}/\mathbb{Z}$, as $n \rightarrow +\infty$, we have $\gamma_R(t) \in \mathcal{U}, \forall t$. In particular, γ_R is homotopic to γ_R^n in \mathcal{U} for every large n .

The fact that $v_n((0, \infty) \times \mathbb{R}/\mathbb{Z}) \subset \mathcal{U}$ for every n and that γ_R^n converges to $P_{3,j}^n$ in \mathcal{U} as $R \rightarrow +\infty$ implies that γ_R is homotopic to $P_{3,j}^n$ in \mathcal{U} for every $R > 0$. Since $P_{3,j}^n \rightarrow P_{3,j}$ as $n \rightarrow +\infty$, γ_R is homotopic to $P_{3,j}$ in \mathcal{U} for every $R > 0$. In particular, γ_R is non-contractible in \mathcal{U} and thus non-constant. Hence \tilde{v} is non-constant as well.

Any asymptotic limit $P \subset \mathcal{U}$ of \tilde{v} at ∞ is homotopic to $P_{3,j}$ in \mathcal{U} since each γ_R is homotopic to $P_{3,j}$ in \mathcal{U} for every $R > 0$. Assume $P \neq P_{3,j}$. Then P must be linked with $P_{3,j}$. This follows from the choice of \mathcal{U} . But since γ_R is the C^∞ -limit of the loops γ_R^n as $n \rightarrow +\infty$, which are all unlinked with $P_{3,j}$, we conclude that γ_R is not linked with $P_{3,j}$, a contradiction. Hence $P_{3,j}$ is the unique asymptotic limit of \tilde{v} at ∞ . This finishes the proof. \square

We next show that, up to a subsequence, the sequence $\tilde{v}_n: \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times S^3$ converges to a \tilde{J} -holomorphic map $\tilde{v} = (b, v): (\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma \rightarrow \mathbb{R} \times S^3$ which is asymptotic to $P_{3,j}$ at its unique positive puncture $+\infty$ and to orbits in $\mathcal{P}_2(\lambda)$ at the negative punctures in $\Gamma \cup \{-\infty\}$.

Proposition 3.13. *Let $\tilde{v}_n = (b_n, v_n): \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times S^3$ be the sequence of embedded finite energy \tilde{J}_n -holomorphic cylinders as above. Under the particular choice of a small compact tubular neighborhood $\mathcal{U} \subset \mathcal{U}_j$ of $P_{3,j}$ and the normalizations (3.3), there exists an embedded finite energy \tilde{J} -holomorphic curve $\tilde{v} = (b, v): (\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma \rightarrow \mathbb{R} \times S^3$, asymptotic to $P_{3,j}$ at its unique positive puncture $+\infty$, to $P_{2,i}$ at $-\infty$ and to other distinct orbits in $\mathcal{P}_2(\lambda)$ at the punctures in Γ , so that, up to a subsequence, $\tilde{v}_n \rightarrow \tilde{v}$ in C_{loc}^∞ as $n \rightarrow +\infty$. Moreover, $v((\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma) \subset \mathcal{U}_j$, and the convergence of \tilde{v} to $P_{3,j}$ at $+\infty$ is exponential.*

Proof. Arguing as in the proof of Proposition 3.5, we take a subsequence of \tilde{v}_n , still denoted by \tilde{v}_n , which admits a sequence $(s_n, t_n) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ so that $|\nabla \tilde{v}_n(s_n, t_n)| \rightarrow +\infty$ as $n \rightarrow +\infty$. Then we have $\limsup_{n \rightarrow \infty} s_n \leq 0$ and, up to a subsequence, we assume that (s_n, t_n) converges to a point in $(-\infty, 0] \times \mathbb{R}/\mathbb{Z}$. Moreover, extracting a subsequence, we can assume that the set of bubbling-off points $\Gamma \subset (-\infty, 0] \times \mathbb{R}/\mathbb{Z}$ is finite. Because of the normalizations (3.3), we can find a \tilde{J} -holomorphic map $\tilde{v} = (b, v): (\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma \rightarrow \mathbb{R} \times S^3$ so that \tilde{v}_n converges to \tilde{v} in $C_{\text{loc}}^\infty((\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma)$ as $n \rightarrow +\infty$. Moreover, \tilde{v} is asymptotic to $P_{3,j}$ at $+\infty$, see Lemma 3.12, and $v((\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma) \subset \mathcal{U}_j$ by positivity and stability of intersections. Every puncture in Γ is negative. For a proof, see the argument just after (3.2). The puncture at $-\infty$ is also negative since $\int_{\{s\} \times \mathbb{R}/\mathbb{Z}} v_n^* \lambda_n > T_{2,i}$ for every n and every fixed $s \ll 0$. Since $+\infty$ is the only positive puncture of \tilde{v} and its asymptotic limit is simple, we also conclude that \tilde{v} is somewhere injective.

Next we claim that the asymptotic limit of \tilde{v} at each $(s^*, t^*) \in \Gamma$ is a cover of an orbit in $\mathcal{P}_2(\lambda)$. Assume by contradiction that there exists an asymptotic limit $Q \subset \mathcal{U}_j$ at $(s^*, t^*) \in \Gamma$, which is not a cover of an orbit in $\mathcal{P}_2(\lambda)$. If Q is not a cover of $P_{3,j}$, then Q is linked with $P_{3,j}$, see Proposition 3.10. Let $\varepsilon > 0$ be sufficiently small so that the loop $v(|(s, t) - (s^*, t^*)| = \varepsilon) \subset S^3$ is arbitrarily close to Q . Then the loop $v_n(|(s, t) - (s^*, t^*)| = \varepsilon)$ is linked with $P_{3,j}$, if n is large enough. In particular, $v_n(|(s, t) - (s^*, t^*)| \leq \varepsilon)$ intersects $P_{3,j}^n$ for every large n , a contradiction. It follows that Q is a cover of $P_{3,j}$. But this implies that $\int_{(\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma} v^* d\lambda \leq T_{3,j} - T_{2,i} - T_{3,j} < 0$, a contradiction. A similar argument shows that the asymptotic limit of \tilde{v} at $-\infty$ is a cover of an orbit in $\mathcal{P}_2(\lambda)$. We

conclude that the asymptotic limits of \tilde{v} at its negative punctures are covers of orbits in $\mathcal{P}_2(\lambda)$.

Now we show that the $d\lambda$ -area of \tilde{v} is positive. Otherwise, \tilde{v} is a cylinder over some periodic orbit P , see [14, Theorem 6.11]. In particular, $\Gamma = \emptyset$ and \tilde{v} is a trivial cylinder over $P_{3,j}$. But this contradicts our normalization (3.3) since it implies $v(0, 0) \in \partial\mathcal{U}$.

We have showed that \tilde{v} is asymptotic to $P_{3,j}$ at $+\infty$, and to covers of orbits in $\mathcal{P}_2(\lambda)$ at its negative punctures in $\Gamma \cup \{-\infty\}$. The usual analysis near $P_{3,j}$ (see [16, Theorem 7.2] and also [6, Proposition 9.3]) implies that the convergence of \tilde{v} to $P_{3,j}$ is exponential with a negative leading eigenvalue of $A_{P_{3,j},J}$, whose eigenvector has winding number 1 with respect to any global trivialization of the contact structure. The asymptotic behavior of \tilde{v} at $+\infty$ is as in Theorem 2.3.

We still need to prove that \tilde{v} is asymptotic to $P_{2,i}$ at $-\infty$ and to other distinct orbits of $\mathcal{P}_2(\lambda)$ at the remaining negative punctures in Γ . Assume, by contradiction, that \tilde{v} is asymptotic to a p_0 -cover, $p_0 > 1$, of some $P_{2,i_0} \in \mathcal{P}_2(\lambda)$ at a negative puncture $(s_0, t_0) \in \Gamma$. In particular, $P_{2,i_0} \subset \partial\mathcal{U}_j$. Since P_{2,i_0} is hyperbolic and satisfies $\mu_{CZ}(P_{2,i_0}) = 2$, the asymptotic operator $A_{P_{2,i_0},J}$ associated with P_{2,i_0} and J admits a unique positive eigenvalue μ with winding number 1 (the least positive eigenvalue) and associated μ -eigenfunctions e, e' which point inside and outside \mathcal{U}_j , respectively. Moreover, the asymptotic operator $A_{P_{2,i_0}^{p_0},J}$ associated with the p_0 -cover $P_{2,i_0}^{p_0}$ of P_{2,i_0} and J admits an eigenfunction e^{p_0} , which equals to e covered p_0 times, whose associated eigenvalue is $p_0\mu$. Its winding number is p_0 with respect to a global trivialization of ξ . Since $P_{2,i_0}^{p_0}$ is also hyperbolic and satisfies $\mu_{CZ}(P_{2,i_0}^{p_0}) = 2p_0$, the eigenvalue $p_0\mu$ is the least positive eigenvalue of $A_{P_{2,i_0}^{p_0},J}$, and the other positive eigenvalues admit winding numbers larger than p_0 . Since the image of v lies in \mathcal{U}_j , this implies that the eigenfunction e^{p_0} describes the asymptotic behavior of \tilde{v} near the puncture (s_0, t_0) . Since $p_0 > 1$, the map v admits self-intersection, see [6, Proposition C.0-i)], which is impossible since the somewhere injective curve \tilde{v} is the C^∞ -limit of embedded curves whose projections to S^3 are embedded surfaces. We conclude that the asymptotic limit of \tilde{v} at a puncture in Γ is an orbit in $\mathcal{P}_2(\lambda)$. In a similar way, if \tilde{v} is asymptotic to the same orbit in $\mathcal{P}_2(\lambda)$ at distinct punctures in $\Gamma \cup \{-\infty\}$, then v also self-intersects, see [6, Proposition C.0-ii)]. Again, this is a contradiction since the somewhere injective curve \tilde{v} is the C_{loc}^∞ -limit of embedded curves whose projections to S^3 are embedded surfaces.

We conclude that, up to extraction of a subsequence, the sequence \tilde{v}_n of embedded finite energy \tilde{J}_n -holomorphic planes, normalized as in (3.3), converges in $C_{loc}^\infty((\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma)$ to a somewhere injective curve \tilde{v} as $n \rightarrow +\infty$. Moreover, \tilde{v} is asymptotic to $P_{3,j}$ at $+\infty$, and to distinct orbits in $\mathcal{P}_2(\lambda)$ at its negative puncture in $\Gamma \cup \{-\infty\}$. Moreover, \tilde{v} and v are embeddings, and $v((\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma) \subset \mathcal{U}_j$.

It remains to show that \tilde{v} is asymptotic to $P_{2,i}$ at $-\infty$. To see this, let $P_{2,l_0} \in \mathcal{P}_2(\lambda)$ be the asymptotic limit of \tilde{v} at $-\infty$. Assume $l_0 \neq i$. For suitable $s_n, c_n \in \mathbb{R}$, with $s_n \rightarrow -\infty$, the shifted maps $\tilde{w}_n(s, t) = (b(s - s_n) + c_n, v(s - s_n)), \forall (s, t) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$,

converges in C_{loc}^∞ to a finite energy \tilde{J} -holomorphic curve $\tilde{w}: (\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma' \rightarrow \mathbb{R} \times S^3$ with Γ' finite, which is asymptotic to P_{2,l_0} at its positive puncture at $+\infty$. Moreover, s_n, c_n can be appropriately chosen so that \tilde{w} is not a trivial cylinder over P_{2,l_0} . Indeed, since all orbits with action $\leq T_{2,l_0}$ are hyperbolic, one can apply the SFT-compactness theorem (see [2]) to obtain a genus zero building of \tilde{J} -holomorphic curves so that the building has a positive puncture at P_{2,l_0} and a negative puncture at $P_{2,i}$. The first level of this building is the non-trivial curve \tilde{w} . Moreover, arguing as above, the punctures of \tilde{w} in $\Gamma' \cup \{-\infty\}$ are negative and \tilde{w} is asymptotic to distinct orbits in $\mathcal{P}_2(\lambda)$ at $\Gamma' \cup \{-\infty\}$. Hence \tilde{w} is somewhere injective and has positive $d\lambda$ -area. Since $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$, we can apply Lemma 3.3 to conclude that the set of negative punctures of \tilde{w} is empty, a contradiction. This proves that $l_0 = i$ and finishes the proof of this proposition. \square

3.4. Compactness properties of holomorphic planes

In this section we study the compactness properties of the families of \tilde{J}_n -holomorphic planes asymptotic to $P_{3,j}^n \in \mathcal{P}_3^{u,-1,\leq C}(\lambda_n)$, $j = 1, \dots, l+1$. Recall that $P_{3,j}^n \rightarrow P_{3,j}$ as $n \rightarrow +\infty$, $\forall j = 1, \dots, l+1$, where $P_{3,j} \in \mathcal{P}_3^{u,-1,\leq C}(\lambda)$. The orbit $P_{3,j}$ lies in the component $\mathcal{U}_j \subset S^3 \setminus \bigcup_{i=1}^l \mathcal{S}_i$, $j = 1, \dots, l+1$, where $\mathcal{S}_i = U_{i,1} \cup P_{2,i} \cup U_{i,2} \subset S^3$ is a C^1 -embedded 2-sphere. We may assume that the sequences λ_n and J_n are given as in Proposition 3.10 so that the orbits $P_{3,j}$ satisfy the properties stated in that proposition.

Fix j and denote by \tilde{k}_j the number of boundary components of \mathcal{U}_j . Let $i_1, \dots, i_{\tilde{k}_j} \in \{1, \dots, l\}$ be such that $\partial \mathcal{U}_j = \bigcup_{k=1}^{\tilde{k}_j} \mathcal{S}_{i_k}$. For each $n \in \mathbb{N}$, $j \in \{1, \dots, l+1\}$ and $k \in \{1, \dots, \tilde{k}_j\}$, there exists a one-parameter family of planes $\mathcal{F}_{k,\tau}^{j,n}$, $\tau \in (0, 1)$, asymptotic to $P_{3,j}^n$. For simplicity, we omit j, k in the notation, i.e. $\mathcal{F}_\tau^{j,n} = \mathcal{F}_{k,\tau}^{j,n}$, $\forall \tau$. Let $\tilde{u}_\tau^n = (a_\tau^n, u_\tau^n): \mathbb{C} \rightarrow \mathbb{R} \times S^3$, $\tau \in (0, 1)$, be the family of \tilde{J}_n -holomorphic planes so that $u_\tau^n(\mathbb{C}) = \mathcal{F}_\tau^{j,n}$, $\forall n, \tau$.

Fix $p \in \mathcal{U}_j \setminus P_{3,j}$ and consider a sequence $p_n \rightarrow p$, where $p_n \in u_{\tau_n}^n(\mathbb{C}) \subset \mathcal{U}_j$ for some $\tau_n \in (0, 1)$. Denote by $\tilde{v}_n = (b_n, v_n): \mathbb{C} \rightarrow \mathbb{R} \times S^3$ the \tilde{J} -holomorphic plane $\tilde{u}_{\tau_n}^n$. In particular, $p_n \in v_n(\mathbb{C})$, $\forall n$.

We consider a small compact tubular neighborhoods $\mathcal{U} \subset \mathcal{U}_j$ of $P_{3,j}$. As in Section 3.3, since $\text{CZ}(P_{3,j}) = 3$, we can choose \mathcal{U} sufficiently small so that

- $p \notin \mathcal{U}$.
- \mathcal{U} contains no periodic orbits that are contractible in \mathcal{U} .
- there exists no periodic orbit $P \subset \mathcal{U}$ which is geometrically distinct from $P_{3,j}$, is homotopic to $P_{3,j}$ in \mathcal{U} and satisfies $\text{link}(P, P_{3,j}) = 0$.

Using that $P_{3,j}^n \rightarrow P_{3,j}$ as $n \rightarrow +\infty$, we observe that $P_{3,j}^n \subset \text{int}(\mathcal{U})$ for every large n and, moreover, we can normalize \tilde{v}_n to satisfy the following conditions

$$\begin{cases} v_n(\mathbb{C} \setminus \mathbb{D}) \subset \mathcal{U}, \\ v_n(1) \in \partial\mathcal{U}, \\ v_n(z_n^*) \in \partial\mathcal{U} \text{ for some } z_n^* \in \partial\mathbb{D} \text{ satisfying } \operatorname{Re}(z_n^*) \leq 0, \\ b_n(2) = 0. \end{cases} \quad (3.4)$$

This normalization is constructed exactly as in (3.1).

The proof of the following lemma is similar to the proof of Lemma 3.12.

Lemma 3.14. *Let \tilde{v}_n satisfy the normalizations in (3.4). Assume that there exist a subsequence of \tilde{v}_n , still denoted by \tilde{v}_n , and a \tilde{J} -holomorphic map $\tilde{v} = (b, v): \mathbb{C} \setminus \mathbb{D} \rightarrow \mathbb{R} \times S^3$ so that $\tilde{v}_n|_{\mathbb{C} \setminus \mathbb{D}}$ converges in $C_{\text{loc}}^\infty(\mathbb{C} \setminus \mathbb{D})$ to \tilde{v} as $n \rightarrow +\infty$. Then \tilde{v} is asymptotic to $P_{3,j}$ at ∞ .*

Recall that $\tilde{v}_n = (b_n, v_n)$ is such that $p_n \in v_n(\mathbb{C}), \forall n$, where $p_n \rightarrow p \in \mathcal{U}_j \setminus P_{3,j}$ is fixed. Next we show that there exists an open subset of $\mathcal{U}_j \setminus P_{3,j}$ of full measure so that if the limit point p is fixed in this subset, then the sequence \tilde{v}_n , normalized as in (3.4) (for particular choices of the small tubular neighborhood $\mathcal{U} \subset \mathcal{U}_j$ of $P_{3,j}$), does not admit bubbling-off points and converges in C_{loc}^∞ as $n \rightarrow +\infty$ to a \tilde{J} -holomorphic plane $\tilde{v}: \mathbb{C} \rightarrow \mathbb{R} \times S^3$ asymptotic to $P_{3,j}$. The cases for which this compactness property fails occur when the sequence \tilde{v}_n admits bubbling-off points in \mathbb{D} , and the limiting curve is asymptotic to $P_{3,j}$ at ∞ and to distinct orbits in $\mathcal{P}_2(\lambda)$ at its negative punctures. As we shall prove below, there are only finitely many such curves. Therefore, if p is taken in the complement of the image of these curves, the compactness property holds, and the limiting curve is a \tilde{J} -holomorphic plane asymptotic to $P_{3,j}$.

Proposition 3.15. *There exists an open subset $\mathcal{U}'_j \subset \mathcal{U}_j \setminus P_{3,j}$ of full measure in $\mathcal{U}_j \setminus P_{3,j}$ so that if $p_n \rightarrow p \in \mathcal{U}'_j$ and $\tilde{v}_n = (b_n, v_n): \mathbb{C} \rightarrow \mathbb{R} \times S^3$ is such that $p_n \in v_n(\mathbb{C}), \forall n$, then, under the particular choice of a small compact tubular neighborhood $\mathcal{U} \subset \mathcal{U}_j$ of $P_{3,j}$ depending on p as above and the normalizations in (3.4), there exists a \tilde{J} -holomorphic plane $\tilde{v} = (b, v): \mathbb{C} \rightarrow \mathbb{R} \times S^3$, exponentially asymptotic to $P_{3,j}$, so that $\tilde{v}_n \rightarrow \tilde{v}$ in C_{loc}^∞ as $n \rightarrow +\infty$ and $p \in v(\mathbb{C})$.*

Proof. Arguing as in the proof of Proposition 3.5, we take a subsequence of \tilde{v}_n , still denoted by \tilde{v}_n , which admits a sequence $z_n \in \mathbb{C}$ so that $|\nabla \tilde{v}_n(z_n)| \rightarrow +\infty$ as $n \rightarrow \infty$. Then z_n is bounded and, up to extraction of a subsequence, converges to a point in \mathbb{D} . Moreover, extracting a subsequence, we can assume that the set of bubbling-off points $\Gamma \subset \mathbb{D}$ is finite. Because of the normalizations (3.4), we find a \tilde{J} -holomorphic map $\tilde{v} = (b, v): \mathbb{C} \setminus \Gamma \rightarrow \mathbb{R} \times S^3$ so that \tilde{v}_n converges to \tilde{v} in $C_{\text{loc}}^\infty(\mathbb{C} \setminus \Gamma)$ as $n \rightarrow +\infty$. Moreover, \tilde{v} is asymptotic to $P_{3,j}$ at ∞ (see Lemma 3.14) and to distinct orbits in $\mathcal{P}_2(\lambda)$ at the negative punctures in Γ . The convergence of \tilde{v} to $P_{3,j}$ is exponential with a negative leading eigenvalue whose eigenfunction has winding number 1 with respect to any global trivialization of the contact structure. We also have $v(\mathbb{C} \setminus \Gamma) \subset \mathcal{U}_j$.

Performing a soft-rescaling of \tilde{v}_n near each puncture in Γ where \tilde{v} is asymptotic to some $P_{2,i} \in \mathcal{P}_2(\lambda)$, we find a new \tilde{J} -holomorphic curve $\tilde{w} = (d, w): \mathbb{C} \setminus \Gamma' \rightarrow \mathbb{R} \times S^3$, with Γ' finite, which is asymptotic to $P_{2,i}$ at ∞ and, at its punctures in $\Gamma' \subset \mathbb{D}$, the curve is asymptotic to covers of orbits in $\mathcal{P}_2(\lambda)$ (see hypothesis II in Theorem 1.6). See also [19] and [6] for more details on the soft-rescaling. Since $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$, we can apply Lemma 3.3 to conclude that \tilde{w} is a plane asymptotic to $P_{2,i}$. In particular, by uniqueness of such planes, $w(\mathbb{C}) \subset \partial\mathcal{U}_j$.

Define the points $q_n \in \mathbb{C}$ by $v_n(q_n) = p_n, \forall n$. We observe that the sequence q_n is bounded and stays away from any puncture of \tilde{w} . Indeed, the usual analysis using cylinders of small area implies that every point sufficiently close to the punctures in Γ are mapped under v_n to a point arbitrarily close to $P_{2,i} \cup w(\mathbb{C}) \subset \partial\mathcal{U}_j$, see [19, Theorem 6.6]. Thus, after taking a subsequence, we may assume that $q_n \rightarrow q_\infty \in \mathbb{D}$, where $q_\infty \notin \Gamma$ and $w(q_\infty) = p$. Denote $\tilde{w}_1 = \tilde{v}$ and $\Gamma_1 = \Gamma$.

Now take $p'_n \rightarrow p' \neq p \in \mathcal{U}_j \setminus (w_1(\mathbb{C} \setminus \Gamma_1) \cup P_{3,j})$, and, as before, consider the \tilde{J}_n -holomorphic planes, suitably normalized as in (3.4) and again denoted by $\tilde{v}_n = (b_n, v_n)$, so that $p'_n \in v_n(\mathbb{C})$. After taking a subsequence, we can assume that there exists $\Gamma_2 \subset \mathbb{D}$, and a \tilde{J} -holomorphic curve $\tilde{w}_2 = (c_2, w_2): \mathbb{C} \setminus \Gamma_2 \rightarrow \mathbb{R} \times S^3$ so that $\tilde{v}_n \rightarrow \tilde{w}_2$ in $C_{\text{loc}}^\infty(\mathbb{C} \setminus \Gamma_2)$ as $n \rightarrow +\infty$. The asymptotic limits of \tilde{w}_2 at the punctures in Γ_2 are distinct orbits in $\mathcal{P}_2(\lambda)$. A soft-rescaling near each $z' \in \Gamma_2$ produces a \tilde{J} -holomorphic plane whose asymptotic limit coincides with the asymptotic limit of \tilde{w}_2 at z' . Moreover, if $q'_n \in \mathbb{D}$ is such that $v_n(q'_n) = p'_n \rightarrow p'$ then, up to a subsequence, $q'_n \rightarrow q'_\infty \notin \Gamma_2$ and $w_2(q'_\infty) = p'$. In particular, the image of w_2 differs from the image of w_1 .

Assume that \tilde{w}_1 and \tilde{w}_2 are asymptotic to the same orbit $P_{2,i} \in \mathcal{P}_2(\lambda)$ at punctures $z_1 \in \Gamma_1$ and $z_2 \in \Gamma_2$, respectively. Then Proposition C.0 in [6] implies that w_1 and w_2 must intersect each other (the intersections of \tilde{w}_1 and \tilde{w}_2 are isolated) and, as C_{loc}^∞ -limits of curves that do not intersect each other, the positivity and stability of intersections of pseudo-holomorphic curves (see [25, Appendix E]) give a contradiction. Thus w_1 and w_2 have mutually distinct asymptotic limits at their negative punctures.

We can repeat the process with a point $p'' \in \mathcal{U}_j \setminus (w_1(\mathbb{C} \setminus \Gamma_1) \cup w_2(\mathbb{C} \setminus \Gamma_2) \cup P_{3,j})$ and find the limiting curve associated with the sequence $\tilde{v}_n = (b_n, v_n)$, suitably normalized and satisfying $p''_n \in v_n(\mathbb{C}) \rightarrow p''$. As before, if the limiting curve $\tilde{w}_3 = (c_3, w_3)$ has negative punctures, then it is asymptotic to an orbit in $\mathcal{P}_2(\lambda)$ at these punctures. These asymptotic limits are distinct from the asymptotic limits of \tilde{w}_1 and \tilde{w}_2 at any of their negative punctures. Hence, we conclude that when we vary the point p in $\mathcal{U}_j \setminus P_{3,j}$ there can be at most \tilde{k}_j of such limiting curves admitting negative punctures. A choice of $p \in \mathcal{U}_j \setminus P_{3,j}$ in the complement of the image of these curves implies that the limiting curve does not admit bubbling-off points and thus is a \tilde{J} -holomorphic plane exponentially asymptotic to $P_{3,j}$. The proof is complete. \square

3.5. Another generic set of almost complex structures

Let $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$ be chosen as in Lemma 3.3. In Section 3 we constructed a transverse foliation \mathcal{F}_J adapted to the Reeb flow of λ , so that the binding is formed by the orbits $P_{2,i} \in \mathcal{P}_2(\lambda)$, $i = 1, \dots, l$, and finitely many orbits $P_{3,j} \in \mathcal{P}_3^{u, -1, \leq C}(\lambda)$, $j = 1, \dots, l+1$, where $C > 0$ is as in Proposition 2.6. Each $P_{2,i}$ is the boundary of a pair of rigid planes $U_{i,1}, U_{i,2} \in \mathcal{F}_J$ so that $\mathcal{S}_i = U_{i,1} \cup P_{2,i} \cup U_{i,2} \subset S^3$ is a C^1 -embedded 2-sphere that separates S^3 into two distinct components. The union of these 2-spheres is denoted by \mathcal{S} . Each $P_{3,j}$, $j = 1, \dots, l+1$, lies in the component \mathcal{U}_j of $S^3 \setminus \bigcup_{i=1}^l \mathcal{S}_i$. The closure of \mathcal{U}_j has \tilde{k}_j boundary components, denoted by $\mathcal{S}_{n_k^j}$, $k = 1, \dots, \tilde{k}_j$, where $n_k^j \in \{1, \dots, l\}$. The leaves of the transverse foliation \mathcal{F}_J are projections to S^3 of embedded finite energy \tilde{J} -holomorphic curves, where \tilde{J} is the almost complex structure in $\mathbb{R} \times S^3$ associated with the pair (λ, J) . For every $i = 1, \dots, l$, there exist two embedded \tilde{J} -holomorphic planes $\tilde{u}_{i,j} = (a_{i,j}, u_{i,j}): \mathbb{C} \rightarrow \mathbb{R} \times S^3$, $j = 1, 2$, asymptotic to $P_{2,i} \in \mathcal{P}_2(\lambda)$ at ∞ , and so that $U_{i,j} = u_{i,j}(\mathbb{C}) \subset \mathcal{S}_i$, $\forall i, j$.

Let $\varepsilon > 0$ be small. For each $j = 1, \dots, l+1$ and for every $k = 1, \dots, \tilde{k}_j$, take a compact ε -neighborhood $\mathcal{U}_{j,n_k^j}^\varepsilon \subset \text{closure}(\mathcal{U}_j)$ of $\mathcal{S}_{n_k^j}$. Abbreviate

$$\mathcal{U}_j^\varepsilon = \bigcup_{k=1}^{\tilde{k}_j} \mathcal{U}_{j,n_k^j}^\varepsilon \subset \text{closure}(\mathcal{U}_j) \quad \text{and} \quad \mathcal{U}^\varepsilon = \bigcup_{j=1}^{l+1} \mathcal{U}_j^\varepsilon \subset S^3.$$

Denote by $\mathcal{J}_j^\varepsilon(\lambda) \subset \mathcal{J}(\lambda)$ the space of $d\lambda$ -compatible almost complex structures J' satisfying $J' = J$ in $(S^3 \setminus \mathcal{U}^\varepsilon) \cup \mathcal{S}$. The set $\mathcal{J}_j^\varepsilon(\lambda)$ inherits the C^∞ -topology from $\mathcal{J}(\lambda)$. Denote by \tilde{J}' the almost complex structure on $\mathbb{R} \times S^3$ determined by λ and $J' \in \mathcal{J}_j^\varepsilon(\lambda)$. In particular, $\tilde{u}_{i,j}$ is \tilde{J}' -holomorphic for every $J' \in \mathcal{J}_j^\varepsilon(\lambda)$.

Taking $\varepsilon > 0$ sufficiently small, we can assure that for every $j = 1, \dots, l+1$, there exists an embedded \tilde{J} -holomorphic plane $\tilde{w}_j = (c_j, w_j): \mathbb{C} \rightarrow \mathbb{R} \times S^3$, which is exponentially asymptotic to $P_{3,j}$ at ∞ and satisfies $w_j(\mathbb{C}) \subset \mathcal{U}_j \setminus \mathcal{U}^\varepsilon$. In particular, \tilde{w} is also \tilde{J}' -holomorphic for every almost complex structure \tilde{J}' associated with λ and $J' \in \mathcal{J}_j^\varepsilon(\lambda)$.

The foliation \mathcal{F}_J constructed in the previous section may contain regular leaves which are projections to S^3 of a \tilde{J} -holomorphic curve $\tilde{w} = (b, w): \mathbb{C} \setminus \Gamma \rightarrow \mathbb{R} \times S^3$, satisfying

- $\Gamma \neq \emptyset$.
- $\int_{\mathbb{C} \setminus \Gamma} w^* d\lambda > 0$.
- ∞ is a positive puncture of \tilde{w} and every puncture in Γ is negative.
- $\exists j \in \{1, \dots, l+1\}$ so that \tilde{w} is exponentially asymptotic to $P_{3,j}$ at ∞ and to distinct orbits in $\mathcal{P}_2(\lambda)$ at the punctures in Γ and $w(\mathbb{C} \setminus \Gamma) \subset \mathcal{U}_j$.

The Fredholm index of \tilde{w} is $\text{Ind}(\tilde{w}) = \text{CZ}(P_{3,j}) - \sum_{z \in \Gamma} \text{CZ}(P_{2,z}) - 1 + \#\Gamma = 2 - \#\Gamma$. Here, $P_{2,z} \in \mathcal{P}_2(\lambda)$ is the asymptotic limit of \tilde{w} at $z \in \Gamma$. Therefore, $\#\Gamma > 1$ implies that $\text{Ind}(\tilde{w}) \leq 0$.

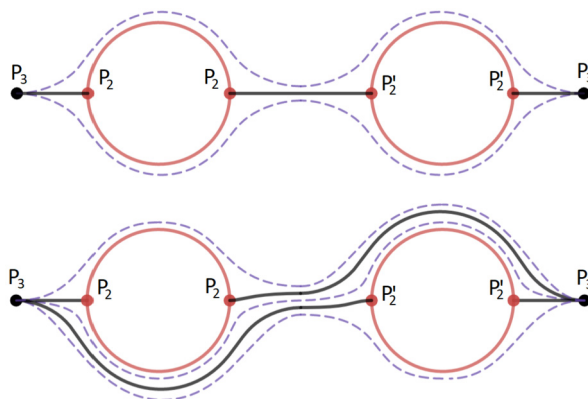


Fig. 3.1. (top) A non-generic \tilde{J} -holomorphic curve $\tilde{v}: \mathbb{C} \setminus \{z_2, z'_2\} \rightarrow \mathbb{R} \times S^3$, asymptotic to P_3 at its positive puncture ∞ and to P_2 and P'_2 at the negative punctures z_2 and z'_2 , respectively. (bottom) For a small generic perturbation of J , this curve unfolds into generic \tilde{J} -holomorphic cylinders, one connecting P_3 to P_2 and the other connecting P_3 to P'_2 .

The following theorem, based on the weighted Fredholm theory developed in [17], states that it is always possible to find $J' \in \mathcal{J}_J^\varepsilon(\lambda)$, which is C^∞ -close to J , so that the Fredholm index of \tilde{w} as above is at least 1. In particular, such curves are rigid cylinders asymptotic to some $P_{3,j}$ at the positive puncture and to an orbit in $\mathcal{P}_2(\lambda)$ at the negative puncture. See Fig. 3.1.

Theorem 3.16 (Hofer-Wysocki-Zehnder [17], Dragnev [9]). *Given $J \in \mathcal{J}_{\text{reg}}^*(\lambda)$ and $\varepsilon > 0$ sufficiently small, there exists a residual set $\mathcal{J}_{J,\text{reg}}^\varepsilon(\lambda) \subset \mathcal{J}_J^\varepsilon(\lambda)$ in the C^∞ -topology so that the following holds: let $J' \in \mathcal{J}_{J,\text{reg}}^\varepsilon(\lambda)$ and let $\tilde{v} = (b, v): \mathbb{C} \setminus \Gamma \rightarrow \mathbb{R} \times S^3$, $\Gamma \neq \emptyset$, be a somewhere injective finite energy \tilde{J}' -holomorphic curve, where \tilde{J}' is the almost complex structure in $\mathbb{R} \times S^3$ induced by the pair (λ, J') . Assume that all punctures in Γ are negative and that \tilde{v} is exponentially asymptotic to some $P_{3,j}$ at the positive puncture $+\infty$ and to distinct orbits in $\mathcal{P}_2(\lambda)$ at the punctures in Γ . Then $\#\Gamma = 1$. In particular, \tilde{v} is a \tilde{J}' -holomorphic cylinder asymptotic to $P_{3,j}$ at ∞ and to an orbit in $\mathcal{P}_2(\lambda)$ at its negative puncture.*

3.6. Finding the desired transverse foliation

Take $\varepsilon > 0$ sufficiently small and let $J' \in \mathcal{J}_{J,\text{reg}}^\varepsilon(\lambda)$ be as in Theorem 3.16. Then for every $i = 1, \dots, l$, the rigid planes $U_{i,1}, U_{i,2}$ are projections of embedded \tilde{J}' -holomorphic planes $\tilde{u}_{i,1}, \tilde{u}_{i,2}: \mathbb{C} \rightarrow \mathbb{R} \times S^3$, which are asymptotic to $P_{2,i}$ at ∞ . For every $j = 1, \dots, l+1$, there exists an embedded \tilde{J}' -holomorphic plane $\tilde{w}_j: \mathbb{C} \rightarrow \mathbb{R} \times S^3$ asymptotic to $P_{3,j}$ at ∞ . Our goal is to construct a transverse foliation which contains only planes and cylinders asymptotic to the orbits $P_{2,i}, i = 1, \dots, l$, and $P_{3,j}, j = 1, \dots, l+1$, and so that the given planes are part of the regular leaves.

Denote by $\mathcal{M}_{\tilde{J}'}(P_{3,j})$ the space of \tilde{J}' -holomorphic planes exponentially asymptotic to $P_{3,j}$ at ∞ . We identify those planes which have the same image in S^3 , that is, $\tilde{w}_1 \sim \tilde{w}_2$ if there exists $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$ so that $\tilde{w}_1(z) = c + \tilde{w}_2(az + b), \forall z \in \mathbb{C}$, where $c + \tilde{w}_2$ is the c -translation of \tilde{w}_2 in the \mathbb{R} -direction. The intersection theory developed in [14,17,31] implies the following proposition.

Proposition 3.17. *Let $j \in \{1, \dots, l+1\}$. Then the following assertions hold:*

- (i) $\mathcal{M}_{\tilde{J}'}(P_{3,j})$ has the structure of a 1-dimensional smooth manifold.
- (ii) if $[\tilde{w} = (c, w)] \in \mathcal{M}_{\tilde{J}'}(P_{3,j})$, then $w(\mathbb{C}) \subset \mathcal{U}_j$.
- (iii) if $[\tilde{w}_1 = (c_1, w_1)] \neq [\tilde{w}_2 = (c_2, w_2)] \in \mathcal{M}_{\tilde{J}'}(P_{3,j})$, then $w_1(\mathbb{C}) \cap w_2(\mathbb{C}) = \emptyset$.

The last step toward the proof of Theorem 1.6 is the following assertion.

Proposition 3.18. *Fix $j \in \{1, \dots, l+1\}$ and let \tilde{k}_j be the number of components of $\partial\mathcal{U}_j$. Then there exist \tilde{k}_j embedded \tilde{J}' -holomorphic cylinders*

$$\tilde{v}_{j,m} = (b_{j,m}, v_{j,m}): \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times S^3, \quad \forall m = 1, \dots, \tilde{k}_j,$$

and \tilde{k}_j families of embedded \tilde{J}' -holomorphic planes $\tilde{w}_{j,m,\tau} \in \mathcal{M}_{\tilde{J}'}(P_{3,j})$,

$$\tilde{w}_{j,m,\tau} = (c_{j,m,\tau}, w_{j,m,\tau}): \mathbb{C} \rightarrow \mathbb{R} \times S^3, \quad \tau \in (0, 1), \quad \forall m = 1, \dots, \tilde{k}_j,$$

so that the following properties hold:

- (i) ∞ is a positive puncture of $\tilde{v}_{j,m}$, where it is exponentially asymptotic to $P_{3,j}$, and $0 \in \mathbb{C}$ is a negative puncture of $\tilde{v}_{j,m}$, where it is asymptotic to $P_{2,n_m^j}, \forall m = 1, \dots, \tilde{k}_j$. Moreover, $v_{j,m}(\mathbb{C} \setminus \{0\}) \subset \mathcal{U}_j, \forall m$, and

$$v_{j,m}(\mathbb{C} \setminus \{0\}) \cap v_{j,n}(\mathbb{C} \setminus \{0\}) = \emptyset, \quad \forall m \neq n.$$
- (ii) ∞ is a positive puncture of $\tilde{w}_{j,m,\tau}$, where it is exponentially asymptotic to $P_{3,j}$, and $w_{j,m,\tau}(\mathbb{C}) \subset \mathcal{U}_j, \forall m$.
- (iii) $v_{j,m}(\mathbb{C} \setminus \{0\})$ and $w_{j,m,\tau}(\mathbb{C}), m = 1, \dots, \tilde{k}_j, \tau \in (0, 1)$, are regular leaves of a transverse foliation of \mathcal{U}_j .
- (iv) for every $m \in \{1, \dots, \tilde{k}_j\}$, $\tilde{w}_{j,m,\tau}$ converges in the SFT-sense (see [2]) to $\tilde{v}_{j,m} \oplus \tilde{u}_{n_m^j,1}$ as $\tau \rightarrow 0^+$ and to $\tilde{v}_{j,m+1} \oplus \tilde{u}_{n_{m+1}^j,2}$ as $\tau \rightarrow 1^-$. Here, $\tilde{u}_{n_m^j,1} = (a_{n_m^j,1}, u_{n_m^j,1})$ and $\tilde{u}_{n_{m+1}^j,2} = (a_{n_{m+1}^j,2}, u_{n_{m+1}^j,2})$ are rigid planes asymptotic to orbits in $\mathcal{P}_2(\lambda)$. Moreover, given neighborhoods $\mathcal{V}_{j,m,1} \subset \text{closure}(\mathcal{U}_j)$ of $u_{n_m^j,1}(\mathbb{C}) \cup P_{2,n_m^j} \cup v_{j,m}(\mathbb{C} \setminus \{0\})$ and $\mathcal{V}_{j,m+1,2} \subset \text{closure}(\mathcal{U}_j)$ of $u_{n_{m+1}^j,2}(\mathbb{C}) \cup P_{2,n_{m+1}^j} \cup v_{j,m+1}(\mathbb{C} \setminus \{0\})$, there exists $\delta > 0$ so that $w_{j,m,\tau}(\mathbb{C}) \subset \mathcal{V}_{j,m,1}, \forall 0 < \tau < \delta$, and $w_{j,m,\tau}(\mathbb{C}) \subset \mathcal{V}_{j,m+1,2}, \forall 1 - \delta < \tau < 1$. By convention, $\tilde{k}_j + 1 \equiv 1$.

Proof. The choice of $\varepsilon > 0$ sufficiently small and the definition of \tilde{J}' guarantee the existence of an embedded \tilde{J}' -holomorphic plane $\tilde{w} = (a, w): \mathbb{C} \rightarrow \mathbb{R} \times S^3$, which is exponentially asymptotic to $P_{3,j}$ and satisfies $w(\mathbb{C}) \subset \mathcal{U}_j \setminus P_{3,j}$. By Proposition 3.17, $\tilde{w} \in \mathcal{M}_{\tilde{J}'}(P_{3,j})$ lies in a 1-parameter family $\tilde{w}_\tau = (c_\tau, w_\tau): \mathbb{C} \rightarrow \mathbb{R} \times S^3, \tau \in (-\delta, \delta)$, of embedded \tilde{J}' -holomorphic planes which are exponentially asymptotic to $P_{3,j}$ for some δ small enough. For each $\tau \in (-\delta, \delta)$, $w_\tau: \mathbb{C} \rightarrow S^3$ is an embedding transverse to the Reeb flow, and $w_{\tau_1}(\mathbb{C}) \cap w_{\tau_2}(\mathbb{C}) = \emptyset, \forall \tau_1 \neq \tau_2$. Consider the maximal one-parameter family of planes containing the family $\tilde{w}_\tau, \tau \in (-\delta, \delta)$, i.e. the smooth family of embedded \tilde{J}' -holomorphic planes $\tilde{w}_\tau \in \mathcal{M}_{\tilde{J}'}(P_{3,j})$ so that the family $w_\tau(\mathbb{C})$ fills the maximal volume in S^3 . Parametrize this maximal family by

$$\tilde{w}_\tau = (c_\tau, w_\tau): \mathbb{C} \rightarrow \mathbb{R} \times S^3, \quad \tau \in (0, 1). \quad (3.5)$$

Such a family is not compact since otherwise the S^1 -family of such planes in the complement of $P_{3,j}$ determines an open book decomposition of S^3 whose binding is $P_{3,j}$ and, as a consequence of the transversality of the pages with respect to the flow, the orbits in $\mathcal{P}_2(\lambda) \neq \emptyset$ are linked with $P_{3,j}$, a contradiction.

We fix the convention that τ increases in the direction of the Reeb flow and that the Reeb vector field points inside \mathcal{U}_j along $U_{n_m^j,1} = u_{n_m^j,1}(\mathbb{C})$ and outside \mathcal{U}_j along $U_{n_m^j,2} = u_{n_m^j,2}(\mathbb{C}), \forall m = 1, \dots, \tilde{k}_j$.

For each $\tau \in (0, 1)$, we choose the following normalization of \tilde{w}_τ . Consider a small compact tubular neighborhood $\mathcal{U} \subset \mathcal{U}_j$ of $P_{3,j}$, so that

- \mathcal{U} contains no periodic orbits that are contractible in \mathcal{U} .
- there exists no Reeb orbit $P \subset \mathcal{U}$ of λ which is geometrically distinct from $P_{3,j}$, is homotopic to $P_{3,j}$ in \mathcal{U} and satisfies $\text{link}(P, P_{3,j}) = 0$.
- $w_\tau(\mathbb{C} \setminus \mathbb{D}) \subset \mathcal{U}$.
- $w_\tau(1) \in \partial \mathcal{U}$.
- $w_\tau(z_\tau^*) \in \partial \mathcal{U}$ for some $z_\tau^* \in \partial \mathbb{D}$ satisfying $\text{Re}(z_\tau^*) \leq 0$.
- $c_\tau(2) = 0$

Let us study the compactness properties of the family (3.5) under the normalizations above. Take a strictly increasing sequence $\tau_n \rightarrow 1^-$ and denote $\tilde{w}_n = \tilde{w}_{\tau_n}$. Arguing as in the proof of Proposition 3.5, we take a subsequence of \tilde{w}_n , still denoted by \tilde{w}_n , which admits a sequence $z_n \in \mathbb{C}$ so that $|\nabla \tilde{w}_n(z_n)| \rightarrow +\infty$ as $n \rightarrow +\infty$. Then z_n is bounded and, up to a subsequence, z_n converges to a point in \mathbb{D} . Moreover, extracting a subsequence, we can assume that the set of bubbling-off points $\Gamma \subset \mathbb{D}$ is finite. The normalizations of \tilde{w}_n imply the existence of a \tilde{J} -holomorphic curve $\tilde{v} = (b, v): \mathbb{C} \setminus \Gamma \rightarrow \mathbb{R} \times S^3$, so that $\tilde{w}_n \rightarrow \tilde{v}$ in $C_{\text{loc}}^\infty(\mathbb{C} \setminus \Gamma)$ as $n \rightarrow +\infty$. By Lemma 3.14, \tilde{v} is non-constant and exponentially converges to $P_{3,j}$ at the positive puncture ∞ . Every puncture in Γ is negative.

Observe that $\Gamma \neq \emptyset$. Indeed, if $\Gamma = \emptyset$, then \tilde{v} is an embedded finite energy \tilde{J}' -holomorphic plane exponentially asymptotic to $P_{3,j}$. In particular, $[\tilde{v}] \in \mathcal{M}_{\tilde{J}'}(P_{3,j})$. By Proposition 3.17, the family (3.5) can be continued, contradicting its maximality and the fact that $\tau_n \rightarrow 1^-$.

The image of v is contained in \mathcal{U}_j since, otherwise, by stability and positivity of intersections of pseudo-holomorphic curves (see [25, Appendix E]) in $\mathbb{R} \times S^3$, for every n sufficiently large, w_n intersects one of the rigid planes asymptotic to some orbit in $\mathcal{P}_2(\lambda)$ which implies that \tilde{w}_n intersects the corresponding \tilde{J}' -holomorphic plane, absurd. Hence, we obtain $v(\mathbb{C} \setminus \Gamma) \subset \mathcal{U}_j$.

Arguing again as in the proof of Proposition 3.5, we obtain that \tilde{v} is asymptotic to $P_{3,j}$ at $+\infty$ and to an orbit in $\mathcal{P}_2(\lambda)$ at each negative puncture in $\Gamma \neq \emptyset$. By Theorem 3.16, $\Gamma = 1$ and thus \tilde{v} is a \tilde{J}' -holomorphic cylinder, exponentially asymptotic to $P_{3,j}$ at $+\infty$, and asymptotic to $P_{2,n_m^j} \in \mathcal{P}_2(\lambda)$ for some $m \in \{1, \dots, \tilde{k}_j\}$ at its unique negative puncture. We can assume that $\Gamma = \{0\}$, so that \tilde{v} is asymptotic to P_{2,n_m^j} at 0.

Performing a soft-rescaling of \tilde{w}_n near the negative puncture 0, we find a new \tilde{J}' -holomorphic curve $\tilde{u} = (a, u): \mathbb{C} \setminus \Gamma_u \rightarrow \mathbb{R} \times S^3$, which is asymptotic to P_{2,n_m^j} at $+\infty$ and to covers of orbits in $\mathcal{P}_2(\lambda)$ at its punctures in Γ_u . As before, the generic choice of J and the soft-rescaling process imply that $\Gamma_u = \emptyset$, and hence \tilde{u} is a plane asymptotic to P_{2,n_m^j} .

We conclude that \tilde{w}_n converges to a 2-level building \mathcal{B} of embedded \tilde{J}' -holomorphic curves. The top level contains the cylinder $\tilde{v}: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times S^3$, exponentially asymptotic to $P_{3,j}$ at ∞ , and asymptotic to P_{2,n_m^j} at its negative puncture $0 \in \mathbb{C}$. The bottom level consists of a plane $\tilde{u}: \mathbb{C} \rightarrow \mathbb{R} \times S^3$, asymptotic to P_{2,n_m^j} at ∞ . The usual analysis of cylinders with small area, see [18] and also [6, Proposition 9.5], implies that given a neighborhood $\mathcal{V} \subset S^3$ of $v(\mathbb{C} \setminus \{0\}) \cup P_{2,n_m^j} \cup u(\mathbb{C})$ we have $w_n(\mathbb{C}) \subset \mathcal{V}$ for n sufficiently large.

The uniqueness of \tilde{J}' -holomorphic planes asymptotic to orbits in $\mathcal{P}_2(\lambda)$, see [6, Proposition C.-3], and the fact that $\tau_n \rightarrow 1^-$ implies that $\tilde{u} = \tilde{u}_{n_m^j, 2}$.

For every large n_0 , we can patch $w_{n_0}(\mathbb{C}) \cup P_{3,j} \cup v(\mathbb{C} \setminus \{0\}) \cup P_{2,n_m^j} \cup u(\mathbb{C})$ to form a topological embedded 2-sphere $S_{n_0} \subset \text{closure}(\mathcal{U}_j)$. The 2-sphere S_{n_0} separates S^3 into two disjoint subsets, one of them, denoted by \mathcal{A}_{n_0} , contains $w_n(\mathbb{C})$ for $n > n_0$. The volume of \mathcal{A}_{n_0} tends to 0 as $n_0 \rightarrow +\infty$. It then follows that for every sequence $\tau_n \rightarrow 1^-$, the image of $w_{\tau_n}(\mathbb{C})$ is contained in \mathcal{A}_{n_0} for every n sufficiently large. This implies that the limiting building \mathcal{B} is the unique SFT-limit of \tilde{w}_τ as $\tau \rightarrow 1^-$.

According to C. Wendl [33, Theorem 1], the curves \tilde{v} and \tilde{u} are automatically transverse. In particular, we can glue $\tilde{v} =: \tilde{v}_{j,m}$ with $\tilde{u}_{n_m^j, 1}$ along P_{2,n_m^j} to form a new family of embedded \tilde{J}' -holomorphic planes, all of them exponentially asymptotic to $P_{3,j}$. See [28, Section 7] or [34, Section 10]. Such planes lie in a maximal family of planes in $\mathcal{M}_{\tilde{J}'}(P_{3,j})$ and will be denoted by $\tilde{w}'_\tau = (c'_\tau, w'_\tau): \mathbb{C} \rightarrow \mathbb{R} \times S^3, \tau \in (0, 1)$, so that $w'_\tau(\mathbb{C}) \subset \mathcal{U}_j, \forall \tau$. Under our parametrizations, \tilde{w}'_τ converges to the holomorphic building formed by \tilde{v} and $\tilde{u}_{n_m^j, 1}$ as $\tau \rightarrow 0^+$. In our notation the family \tilde{w}'_τ now corresponds to $\tilde{w}_{j,m,\tau}, \tau \in (0, 1)$.

If $\tilde{k}_j = 1$, then the family $\tilde{w}'_\tau, \tau \in (0, 1)$, coincides with the family $\tilde{w}_\tau, \tau \in (0, 1)$, and the compactness properties above show that the $\cup_{\tau \in (0,1)} w_\tau(\mathbb{C})$, is open and closed in $\mathcal{U}_j \setminus (P_{3,j} \cup v(\mathbb{C} \setminus \{0\}))$ and thus coincides with $\mathcal{U}_j \setminus (P_{3,j} \cup v(\mathbb{C} \setminus \{0\}))$. If $\tilde{k}_j > 1$, then the families $\tilde{w}_\tau, \tau \in (0, 1)$, and $\tilde{w}'_\tau, \tau \in (0, 1)$, do not coincide, and we consider the compactness properties of the family \tilde{w}'_τ as $\tau \rightarrow 1^-$. As before, this family converges to a building whose top level consists of an embedded \tilde{J}' -cylinder $\tilde{v}': \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times S^3$, which is exponentially asymptotic to $P_{3,j}$ at ∞ and asymptotic to some other $P_{2,i} \in \mathcal{P}_2(\lambda)$ at 0, and whose lower level consists of an embedded \tilde{J}' -holomorphic plane \tilde{u}' , which is asymptotic to $P_{2,i}$ at $+\infty$.

We necessarily have $P_{2,n_m^j} \neq P_{2,i} =: P_{2,n_{m+1}^j}$ since the families \tilde{w}_τ and \tilde{w}'_τ are distinct and hence, by the uniqueness and intersection properties of the \tilde{J}' -holomorphic planes exponentially asymptotic to $P_{3,j}$, points of $w'_\tau(\mathbb{C})$ cannot accumulate at P_{2,n_m^j} as $\tau \rightarrow 1^-$. It follows from the normalizations of $\tilde{w}'_\tau, \tau \in (0, 1)$, that $\tilde{u}' =: \tilde{u}_{n_{m+1}^j, 2}$ and, as before, we glue $\tilde{v}' =: \tilde{v}_{j,m+1}$ with $\tilde{u}_{n_{m+1}^j, 1}$ to obtain a new maximal family of embedded \tilde{J}' -holomorphic planes $\tilde{w}''_\tau, \tau \in (0, 1)$, which are exponentially asymptotic to $P_{3,j}$.

If $\tilde{k}_j = 2$, then the new family \tilde{w}''_τ coincides with the family \tilde{w}_τ and $v''(\mathbb{C} \setminus \{0\}) = v(\mathbb{C} \setminus \{0\})$, where \tilde{v}'' is the embedded \tilde{J}' -holomorphic cylinder consisting of the top level of a holomorphic building associated with the family \tilde{w}''_τ , as $\tau \rightarrow 1^-$, and

$$\bigcup_{\tau \in (0,1)} w_\tau(\mathbb{C}) \cup \bigcup_{\tau \in (0,1)} w'_\tau(\mathbb{C})$$

fills $\mathcal{U}_j \setminus (P_{3,j} \cup v(\mathbb{C} \setminus \{0\}) \cup v'(\mathbb{C} \setminus \{0\}))$. Otherwise, we glue $v'' =: \tilde{v}_{j,m+2}$ with the rigid plane $\tilde{u}_{n_{m+2}^j, 1}$ and continue in a similar manner. It has to stop after a finite number of steps. Indeed, the number of such families of \tilde{J}' -holomorphic planes asymptotic to $P_{3,j}$ is precisely \tilde{k}_j , the number of components in $\partial\mathcal{U}_j$.

We conclude that there exist \tilde{k}_j embedded \tilde{J}' -holomorphic cylinders $\tilde{v}_{j,m}: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times S^3, m = 1, \dots, \tilde{k}_j$, which are exponentially asymptotic to $P_{3,j}$ at the positive puncture ∞ and to $P_{2,n_m^j} \subset \mathcal{S}_{n_m^j}$ at their negative puncture 0. In the complement of such cylinders, there exist \tilde{k}_j families of embedded \tilde{J}' -holomorphic planes $\tilde{w}_{j,m,\tau}, \tau \in (0, 1)$, which are exponentially asymptotic to $P_{3,j}$ at ∞ . Moreover, each family converges to the holomorphic building formed by $\tilde{v}_{j,m} \oplus \tilde{u}_{n_m^j, 1}$ as $\tau \rightarrow 0^+$ and to the building $\tilde{v}_{j,m+1} \oplus \tilde{u}_{n_{m+1}^j, 2}$ as $\tau \rightarrow 1^-$ for every $m = 1, \dots, \tilde{k}_j$. Here, $\tilde{k}_j + 1 \equiv 1$. This finishes the proof. \square

Summarizing the results from Propositions 3.4, 3.10 and 3.18, we obtain the following statement, which implies Theorem 1.6.

Theorem 3.19. *Given a weakly convex contact form $\lambda = f\lambda_0$ on (S^3, ξ_0) satisfying hypotheses I-III of Theorem 1.6, there exists a dense subset $\tilde{\mathcal{J}}_{\text{reg}}(\lambda) \subset \mathcal{J}(\lambda)$ in the C^∞ -topology, so that for every $J \in \tilde{\mathcal{J}}_{\text{reg}}(\lambda)$, the pair (λ, J) admits a stable finite energy foliation $\tilde{\mathcal{F}}$ satisfying the following properties:*

- (i) For each $i = 1, \dots, l$, there exists a pair of embedded finite energy \tilde{J} -holomorphic planes $\tilde{u}_{i,1} = (a_{i,1}, u_{i,1})$, $\tilde{u}_{i,2} = (a_{i,2}, u_{i,2})$: $\mathbb{C} \rightarrow \mathbb{R} \times S^3$ which are asymptotic to $P_{2,i}$. The union $\mathcal{S}_i = u_{i,1}(\mathbb{C}) \cup P_{2,i} \cup u_{i,2}(\mathbb{C})$ is a C^1 -embedded 2-sphere separating S^3 into two components and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$, $\forall i \neq j$. Every component \mathcal{U}_j , $j = 1, \dots, l+1$, of $S^3 \setminus \bigcup_{i=1}^l \mathcal{S}_i$ contains an index-3 orbit $P_{3,j}$ satisfying the linking properties given in Proposition 3.10.
- (ii) Given $j \in \{1, \dots, l+1\}$, denote by \tilde{k}_j the number of boundary components of \mathcal{U}_j , denoted by $\mathcal{S}_{n_k^j}$, $k = 1, \dots, \tilde{k}_j$, where $n_k^j \in \{1, \dots, l\}$.
- (a) Then there exist \tilde{k}_j embedded finite energy \tilde{J} -holomorphic cylinders

$$\tilde{v}_{j,m} = (b_{j,m}, v_{j,m}): \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times S^3, \quad \forall m = 1, \dots, \tilde{k}_j,$$

which is exponentially asymptotic to $P_{3,j}$ at its positive puncture $+\infty$ and to P_{2,n_m^j} at its negative puncture 0, $\forall m = 1, \dots, \tilde{k}_j$. Moreover, they satisfy $v_{j,m}(\mathbb{C} \setminus \{0\}) \subset \mathcal{U}_j$, $\forall m$, and

$$v_{j,m}(\mathbb{C} \setminus \{0\}) \cap v_{j,n}(\mathbb{C} \setminus \{0\}) = \emptyset, \quad \forall m \neq n.$$

- (b) The complement $(\mathbb{R} \times \mathcal{U}_j) \setminus \bigcup_{m=1}^{\tilde{k}_j} \tilde{v}_{j,m}(\mathbb{C} \setminus \{0\})$ is foliated by \tilde{k}_j families of embedded finite energy \tilde{J} -holomorphic planes

$$\tilde{w}_{j,m,\tau} = (c_{j,m,\tau}, w_{j,m,\tau}): \mathbb{C} \rightarrow \mathbb{R} \times S^3, \quad \tau \in (0, 1), \quad \forall m = 1, \dots, \tilde{k}_j,$$

exponentially asymptotic to $P_{3,j}$ at its positive puncture $+\infty$. Moreover, each plane $\tilde{w}_{j,m,\tau}$ satisfies the compactness property described in Proposition 3.18-(iv).

- (iii) Every finite energy \tilde{J} -holomorphic curve described above satisfies $\text{wind}_\pi = 0$, so that its projection to S^3 is transverse to the Reeb vector field of λ .

Consequently, the projection \mathcal{F} of the finite energy foliation $\tilde{\mathcal{F}}$ to S^3 provides the weakly convex foliation as in Theorem 1.6.

4. Transition maps

Throughout this section, we assume that the Reeb flow of λ is real-analytic. Let \mathcal{F} be a genus zero transverse foliation adapted to the Reeb flow of $\lambda = f\lambda_0$ as in Theorem 1.6. We shall set some notations to represent the elements associated with \mathcal{F} , see Fig. 4.1. First recall that all the orbits in $\mathcal{P}_2(\lambda)$ are binding orbits of \mathcal{F} and, for each $i = 1, \dots, l$, the orbit $P_{2,i} \in \mathcal{P}_2(\lambda)$ bounds two rigid planes $U_{i,1}, U_{i,2} \in \mathcal{F}$, so that the embedded 2-sphere $\mathcal{S}_i = U_{i,1} \cup P_{2,i} \cup U_{i,2} \subset S^3$ is C^1 . The 2-spheres \mathcal{S}_i , $i = 1, \dots, l$, are mutually disjoint and each one of them separates S^3 into two components. In this way, the complement of their union is formed by $l+1$ components, denoted by $\mathcal{U}_j \subset S^3 \setminus \bigcup_{i=1}^l \mathcal{S}_i$, $j = 1, \dots, l+1$.

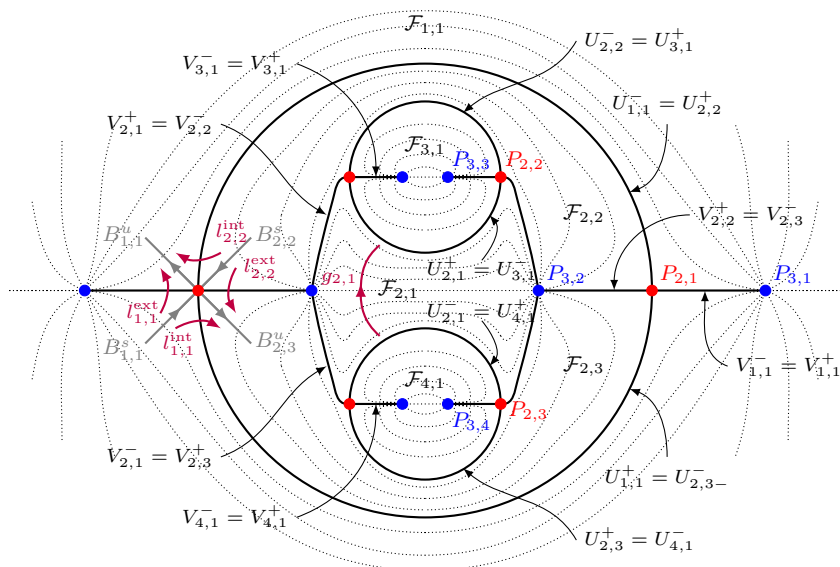


Fig. 4.1. This example illustrates a section of a weakly convex foliation on S^3 with $\ell = 3$. The blue and red dots represent the binding orbits with Conley-Zehnder indices 3 and 2, respectively. The rigid planes and rigid cylinders are represented by bold curves, and the families of planes are represented by dotted curves. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Inside \mathcal{U}_j , there exist a binding orbit $P_{3,j} \in \mathcal{P}_3^{u,-1}(\lambda)$ and \tilde{k}_j one-parameter families of planes asymptotic to $P_{3,j}$, denoted $\mathcal{F}_{j,1}, \dots, \mathcal{F}_{j,\tilde{k}_j} \subset \mathcal{F}, j = 1, \dots, l+1$, where $\tilde{k}_j \in \mathbb{N}^*$ coincides with the number of components in $\partial\mathcal{U}_j$. The family $\mathcal{F}_{j,k} = (\mathcal{F}_{j,k,\tau})_\tau$ is parametrized by $\tau \in (0,1)$. It breaks, as $\tau \rightarrow 0^+$, onto a rigid plane $U_{j,k}^- \in \{U_{1,1}, U_{1,2}, \dots, U_{l,1}, U_{l,2}\} \subset \mathcal{F}$, asymptotic to a binding orbit $\mathcal{P}_{j,k}^- \in \mathcal{P}_2(\lambda)$, and a rigid cylinder $V_{j,k}^- \in \mathcal{F}$, asymptotic to $P_{3,j}$ at its positive puncture and to $\mathcal{P}_{j,k}^-$ at its negative puncture. In a similar manner, as $\tau \rightarrow 1^-$, it breaks onto a rigid plane $U_{j,k}^+ \in \{U_{1,1}, U_{1,2}, \dots, U_{l,1}, U_{l,2}\} \subset \mathcal{F}$, asymptotic to a binding orbit $\mathcal{P}_{j,k}^+ \in \mathcal{P}_2(\lambda)$, and a rigid cylinder $V_{j,k}^+ \in \mathcal{F}$, asymptotic to $P_{3,j}$ at its positive puncture and to $\mathcal{P}_{j,k}^+$ at its negative puncture.

Let $\mathcal{C} := \{(j,k) \in \mathbb{N}^* \times \mathbb{N}^* \mid j = 1, \dots, l+1, k = 1, \dots, \tilde{k}_j\}$. It parametrizes the space of families of planes asymptotic to the index 3 binding orbits: each $(j,k) \in \mathcal{C}$ corresponds to a family $\mathcal{F}_{j,k}$ of planes in \mathcal{U}_j asymptotic to $P_{3,j}$. After relabelling the families of planes, we can assume that $\mathcal{P}_{j,k}^+ = \mathcal{P}_{j,k+1}^-, \forall k \bmod \tilde{k}_j$.

For every $(j,k) \in \mathcal{C}$, there exist a unique branch of the local unstable manifold of $\mathcal{P}_{j,k}^-$ and a unique branch of the local stable manifold of $\mathcal{P}_{j,k}^+$, which intersect $\mathcal{F}_{j,k,\tau}$ for τ sufficiently close to 0 and τ sufficiently close to 1, respectively. Denote these local branches by $\mathcal{B}_{j,k}^u \subset W_{\text{loc}}^u(\mathcal{P}_{j,k}^-)$ and $\mathcal{B}_{j,k}^s \subset W_{\text{loc}}^s(\mathcal{P}_{j,k}^+)$. We shall fix planes $\mathcal{F}_{j,k}^- := \mathcal{F}_{j,k,\tau_-}$ and $\mathcal{F}_{j,k}^+ := \mathcal{F}_{j,k,\tau_+}$, where $\tau_- > 0$ is sufficiently close to 0 and $\tau_+ < 1$ is sufficiently close to 1. In particular, the intersections of $\mathcal{F}_{j,k}^-$ and $\mathcal{F}_{j,k}^+$ with $\mathcal{B}_{j,k}^u$ and $\mathcal{B}_{j,k}^s$, respectively, are simple closed curves, denoted by

$$C_{j,k}^u := \mathcal{F}_{j,k}^- \cap \mathcal{B}_{j,k}^u \quad \text{and} \quad C_{j,k}^s := \mathcal{F}_{j,k}^+ \cap \mathcal{B}_{j,k}^s. \quad (4.1)$$

The closed disks bounded by $C_{j,k}^u$ and $C_{j,k}^s$ will be denoted by

$$\mathcal{D}_{j,k}^u \subset \mathcal{F}_{j,k}^- \quad \text{and} \quad \mathcal{D}_{j,k}^s \subset \mathcal{F}_{j,k}^+, \quad (4.2)$$

respectively. These disks have $d\lambda$ -area equal to the actions of $\mathcal{P}_{j,k}^-$ and $\mathcal{P}_{j,k}^+$, respectively.

For each $(j, k) \in \mathcal{C}$, we have the following transition maps that preserve the area form induced by $d\lambda$.

- The **global transition map** $g_{j,k} : \mathcal{F}_{j,k}^- \rightarrow \mathcal{F}_{j,k}^+$ is defined as the first intersection point of the forward trajectory with the plane $\mathcal{F}_{j,k}^+$. This map is well-defined since $P_{3,j}$ has index 3.
- The **local exterior transition map** $l_{j,k}^{\text{ext}} : \mathcal{F}_{j,k}^+ \setminus \mathcal{D}_{j,k}^s \rightarrow \mathcal{F}_{j,k+1}^- \setminus \mathcal{D}_{j,k+1}^u$ is defined as the first intersection point of the forward trajectory with $\mathcal{F}_{j,k+1}^-$. Any such a trajectory crosses $V_{j,k}^+$ before hitting $\mathcal{F}_{j,k+1}^-$.
- The **local interior transition map** $l_{j,k}^{\text{int}} : \mathcal{D}_{j,k}^s \setminus C_{j,k}^s \rightarrow \mathcal{D}_{j',k'}^u \setminus C_{j',k'}^u$ is defined as the first intersection point of the forward trajectory with $\mathcal{F}_{j',k'}^-$, where (j', k') is such that $U_{j,k}^+ = U_{j',k'}^-$. Any such a trajectory crosses $U_{j,k}^+$ before hitting $\mathcal{F}_{j',k'}^-$.

4.1. Real-analytic models

Let $P_{2,i} = (x_{2,i}, T_{2,i}) \in \mathcal{P}_2(\lambda)$. In this section, we show the existence of suitable real-analytic coordinates near $P_{2,i}$ that will be used to model neighborhoods of $C_{j,k}^+ \subset \mathcal{F}_{j,k}^+$ and $C_{j,k}^u \subset \mathcal{F}_{j,k}^-$.

Proposition 4.1. *There exist real-analytic coordinates $(t, x, y) \in (\mathbb{R}/T_{2,i}\mathbb{Z}) \times B_{\delta'}(0)$, $\delta' > 0$ small, on a small tubular neighborhood $\mathcal{U}_{\delta'} \subset S^3$ of $P_{2,i}$, so that $P_{2,i} \equiv (\mathbb{R}/T_{2,i}\mathbb{Z}) \times \{0\}$, and, up to time reparametrization, the trajectories of the Reeb flow of λ in \mathcal{U}_{δ} coincide with the trajectories of*

$$\dot{t} = 1, \quad \dot{x} = -u(xy)x, \quad \dot{y} = u(xy)y, \quad (4.3)$$

where $u(w) = \ln \eta + \eta_1 w + \eta_2 w^2 + \dots$ ($\eta > 1$) is a convergent power series near $w = 0$. In particular, the quantity xy is preserved by the flow.

Proof. The proof is a direct application of a result due to Moser in [26] which asserts that a real-analytic mapping ϕ defined near a hyperbolic fixed point at $0 \in \mathbb{R}^2$ admits coordinates (x, y) so that it has the form $\phi(x, y) = (xe^{-u(xy)}, ye^{u(xy)})$, where $u(w) = \ln \eta + \eta_1 w + \eta_2 w^2 + \dots$ is a convergent power series near $w = 0$. Such coordinates on a cross section of $P_{2,i}$ induce the desired coordinates on a tubular neighborhood of $P_{2,i}$. \square

In the coordinates given in Proposition 4.1, the local stable and unstable manifolds of $P_{2,i}$ are $W_{\text{loc}}^s(P_{2,i}) \subset (\mathbb{R}/T_{2,i}\mathbb{Z}) \times \mathbb{R} \times \{0\}$ and $W_{\text{loc}}^u(P_{2,i}) \subset (\mathbb{R}/T_{2,i}\mathbb{Z}) \times \{0\} \times \mathbb{R}$, respectively.

4.1.1. The local exterior transition maps

Fix $(j, k) \in \mathcal{C}$ and set $k' = k + 1$. Fix also τ_- and τ_+ close enough to 0 and 1, respectively, so that the circles $C_{j,k}^s$ and $C_{j,k'}^u$, see (4.1), are contained in the tubular neighborhood $\mathcal{U}_{\delta'}$ of $\mathcal{P}_{j,k}^+ = \mathcal{P}_{j,k'}^-$ as in Proposition 4.1. Choose sufficiently small annular neighborhoods $R_{j,k}^s \subset \mathcal{F}_{j,k}^+$ of $C_{j,k}^s$ and $R_{j,k'}^u \subset \mathcal{F}_{j,k'}^-$ of $C_{j,k'}^u$ that are modelled in the real-analytic coordinates (t, x, y) near $P_{2,i}$ by

$$\begin{aligned} R^s &:= \{(t, x, y) \mid x = \frac{\delta}{2}, y \in (-\frac{\delta}{4}, \frac{\delta}{4})\}, \\ R^u &:= \{(t, x, y) \mid x \in (-\frac{\delta}{4}, \frac{\delta}{4}), y = \frac{\delta}{2}\}, \end{aligned}$$

respectively, where $0 < \delta \ll \delta'$. Set $A_{j,k}^{\text{ext},s} := R_{j,k}^s \setminus \mathcal{D}_{j,k}^s$ and $A_{j,k'}^{\text{ext},u} := R_{j,k'}^u \setminus \mathcal{D}_{j,k'}^u$. These annuli are modelled in our real-analytic coordinates by

$$\begin{aligned} A^{\text{ext},s} &:= \{(t, x, y) \mid x = \frac{\delta}{2}, y \in (0, \frac{\delta}{4})\}, \\ A^{\text{ext},u} &:= \{(t, x, y) \mid x \in (0, \frac{\delta}{4}), y = \frac{\delta}{2}\}, \end{aligned} \tag{4.4}$$

respectively. Note that $A_{j,k}^{\text{ext},s}$ is mapped under $l_{j,k}^{\text{ext}}$ onto $A_{j,k'}^{\text{ext},u}$.

Consider the real-analytic maps

$$\begin{aligned} F_{j,k}^{\text{ext},s} &: (\mathbb{R}/T_{2,i}\mathbb{Z}) \times (0, \frac{\delta}{4}) \rightarrow A_{j,k}^{\text{ext},s}, \\ F_{j,k'}^{\text{ext},u} &: (\mathbb{R}/T_{2,i}\mathbb{Z}) \times (0, \frac{\delta}{4}) \rightarrow A_{j,k'}^{\text{ext},u}, \end{aligned} \tag{4.5}$$

given in our coordinates by $(t, y) \mapsto (t, \frac{\delta}{2}, y)$ and $(t, x) \mapsto (t, x, \frac{\delta}{2})$, respectively. In these coordinates, the local exterior transition map $l_{j,k}^{\text{ext}}: A_{j,k}^{\text{ext},s} \rightarrow A_{j,k'}^{\text{ext},u}$ admits a lift

$$\tilde{l}: \mathbb{R} \times (0, \frac{\delta}{4}) \rightarrow \mathbb{R} \times (0, \frac{\delta}{4}), \quad (t, r) \mapsto (t + \Delta t(r), r),$$

where

$$\Delta t(r) = T_{2,i} \frac{1}{u(\delta r/2)} \ln \frac{\delta}{2r} = g(r) - h(r) \ln r, \tag{4.6}$$

for real-analytical functions $g(r), h(r)$ defined near $r = 0$, with $h(0) > 0$. Notice that $\Delta t(r) \rightarrow +\infty$ as $r \rightarrow 0^+$.

Lemma 4.2. *Let $\gamma: [0, 1) \rightarrow \mathbb{R} \times [0, \frac{\delta}{4})$ be a real-analytic curve such that $\gamma(s) \in \mathbb{R} \times \{0\}$ only at $s = 0$. Then $\tilde{l} \circ \gamma$ is a monotone curve in the \mathbb{R} -direction for $s > 0$ sufficiently small. Moreover, writing $\tilde{l} \circ \gamma$ as a graph $r = \eta(t), t \gg 0$, we have $\frac{d\eta}{dt} \rightarrow 0$ as $t \rightarrow +\infty$*

Proof. Write $\gamma(s) = (t(s), r(s))$ such that $t(0) = t_0$ and $r(0) = 0$. Since $r(s)$ is real-analytic, we find $a > 0$ and $n \in \mathbb{N}$ such that

$$r(s) = as^n + O(s^{n+1}) \quad (4.7)$$

and thus $r(s)$ is strictly increasing for $s \geq 0$ sufficiently small. Moreover, there exists $b, B > 0$ such that

$$\frac{B}{s} > \frac{r'(s)}{r(s)} > \frac{b}{s}, \quad \forall s > 0 \text{ small.} \quad (4.8)$$

This implies that there exists $c_1 > 0$ so that

$$\begin{aligned} \frac{d}{ds}(t(s) + \Delta t(r(s))) &= t'(s) + \left(g'(r(s)) - h'(r(s)) \ln r(s) - \frac{h(r(s))}{r(s)} \right) r'(s) \\ &< -\frac{c_1}{s}, \end{aligned} \quad (4.9)$$

for every $s > 0$ sufficiently small. Hence $t(s) + \Delta t(r(s))$ increases monotonically to $+\infty$ as $s \rightarrow 0^+$. From the conclusion above, we can write the curve $\tilde{l} \circ \gamma$ as a graph $r = \eta(t)$, $t \gg 0$, where η is real-analytic. It satisfies $\eta(t(s) + \Delta t(r(s))) = r(s)$, $\forall s > 0$ small. From (4.8) and (4.9), we see that

$$0 > \eta'(t(s) + \Delta t(r(s))) = \frac{r'(s)}{\frac{d}{ds}(t(s) + \Delta t(r(s)))} > -\frac{B}{c_1} r(s) \rightarrow 0$$

as $s \rightarrow 0^+$. Since $t(s) + \Delta t(r(s)) \rightarrow +\infty$ as $s \rightarrow 0^+$, this completes the proof. \square

Now assume that every orbit in $\mathcal{P}_2(\lambda)$ has the same action $T_2 > 0$, so that the disks $\mathcal{D}_{j,k}^s$ and $\mathcal{D}_{j,k}^u$, see (4.2), have the same $d\lambda$ -area equal to T_2 for all $(j, k) \in \mathcal{C}$.

Using an area-preserving argument, we check below that for any fixed $P_{2,i} \in \mathcal{P}_2(\lambda)$, the branch of its unstable manifold in \mathcal{U}_j must intersect the branch in \mathcal{U}_j of the stable manifold of an orbit $P_{2,i'} \in \partial \mathcal{U}_j$. Note that such an intersection corresponds to a heteroclinic/homoclinic trajectory in \mathcal{U}_j connecting $P_{2,i}$ to $P_{2,i'}$.

Let $G : \mathcal{C} \rightarrow \mathcal{C}$ be defined so that $(J, K) = G(j, k)$ is the family of planes so that $\mathcal{D}_{J,K}^s \subset \mathcal{F}_{j,k}^+$ is the first disk intersected by the forward flow of $\mathcal{D}_{j,k}^u$. To be more precise in the definition of G , fix $(j, k) \in \mathcal{C}$ and set $D^0 = \mathcal{D}_{j,k}^u$. If $g_{j,k}(D^0) \cap \mathcal{C}_{j,k}^s \neq \emptyset$, then we define $G(j, k) = (j, k)$. Otherwise, since all index 2 orbits have the same action, we have $g_{j,k}(D^0) \subset \mathcal{F}_{j,k}^+ \setminus \mathcal{D}_{j,k}^s$, and then define $D^1 := l_{j,k}^{\text{ext}}(g_{j,k}(D^0)) \subset \mathcal{F}_{j,k+1}^-$. Now we repeat the procedure with D^1 : if $g_{j,k+1}(D^1) \cap \mathcal{C}_{j,k+1}^s \neq \emptyset$, then we define $G(j, k) = (j, k+1)$. Otherwise, we have $g_{j,k+1}(D^1) \subset \mathcal{F}_{j,k+1}^+ \setminus \mathcal{D}_{j,k+1}^s$ and then define $D^2 := l_{j,k+1}^{\text{ext}}(g_{j,k+1}(D^1)) \subset \mathcal{F}_{j,k+2}^-$. Repeating this process we obtain a sequence $(j, k^{(n)}) \in \mathcal{C}$, where $k^{(n)} \equiv k + n \pmod{\tilde{k}_j}$, and disks $D^n \subset \mathcal{F}_{j,k^{(n)}}^-$, $n \in \mathbb{N}$. By construction and uniqueness of solutions, the disks D^n are mutually disjoint. Since the $d\lambda$ -area of D^n is

independent of n (indeed, it is equal to T_2 for all n) and the available area in all $\mathcal{F}_{j,k}^\pm$ is finite (it coincides with the action of $P_{3,j}$), the procedure above has to terminate after finitely many steps and we find $(j, k^{(N)})$ for some $N \geq 0$ so that $g_{j,k^{(N)}}(D^N) \cap C_{j,k^{(N)}}^s \neq \emptyset$. In this case we set $(J, K) = (j, k^{(N)})$ and define $G(j, k) = (J, K)$, where $J = j$ since along the process we remain inside the same component \mathcal{U}_j via the global and the local exterior transition maps.

We also consider a map

$$\Psi_{j,k}: \mathcal{N}(\mathcal{D}_{j,k}^u) \rightarrow \mathcal{F}_{J,K}^+, (j, k) \in \mathcal{C}, \quad (4.10)$$

where $(J, K) = G(j, k)$ and $\mathcal{N}(\mathcal{D}_{j,k}^u)$ denotes a small neighborhood of $\mathcal{D}_{j,k}^u$ in the plane $\mathcal{F}_{j,k}^-$. The first hit of $\mathcal{N}(\mathcal{D}_{j,k}^u)$ into $\mathcal{F}_{J,K}^+$ under the forward flow is precisely $\Psi_{j,k}$.

Definition 4.3. We say that $(j, k) \in \mathcal{C}$ is **coincident** if $\Psi_{j,k}(\mathcal{D}_{j,k}^u) = \mathcal{D}_{J,K}^s$, where $(J, K) = G(j, k)$. Otherwise, we say it is **non-coincident**.

Pick $(j, k) \in \mathcal{C}$ and abbreviate $(J, K) = G(j, k)$. By definition of G , we have $\Psi_{j,k}(\mathcal{D}_{j,k}^u) \cap C_{J,K}^s \neq \emptyset$. Since all index-2 orbits have the same action and the flow is real-analytic, we find one of the following scenarios associated with the pair (j, k) :

- (a) $\Psi_{j,k}(\mathcal{D}_{j,k}^u) = \mathcal{D}_{J,K}^s$, i.e. (j, k) is coincident.
- (b) $\Psi_{j,k}(\mathcal{D}_{j,k}^u) \cap (\mathcal{D}_{J,K}^s \setminus C_{J,K}^s) = \emptyset$, and $\Psi_{j,k}(\mathcal{D}_{j,k}^u)$ intersects the circle $C_{J,K}^s$ at finitely many points.
- (c) $\Psi_{j,k}(\mathcal{D}_{j,k}^u)$ intersects both $\mathcal{F}_{J,K}^+ \setminus \mathcal{D}_{J,K}^s$ and $\mathcal{D}_{J,K}^s \setminus C_{J,K}^s$. In this case the disk $\Psi_{j,k}(\mathcal{D}_{j,k}^u)$ also intersects the circle $C_{J,K}^s$ at finitely many points.

Notice that every intersection in (b) is tangent and an intersection in (c) is not necessarily transverse.

5. Proof of Theorem 1.8

5.1. Proof of Theorem 1.8-(i)

Fix $j \in \{1, \dots, l\}$ and assume that for every $k \in \{1, \dots, \tilde{k}_j\}$ the branch in \mathcal{U}_j of the unstable manifold of P_{2,n_k^j} coincides with the branch in \mathcal{U}_j of the stable manifold of P_{2,n_l^j} , for some $l \in \{1, \dots, \tilde{k}_j\}$. This means that all pairs $(j, k) \in \mathcal{C}$ are coincident as in scenario (a).

Fix $(j, k_1) \in \mathcal{C}$ as above, let $(j, K_1) = G(j, k_1)$, and let $(j, k_2) := (j, K_1 + 1)$. Our assumptions imply that we can construct an N -periodic sequence of distinct k_1, \dots, k_N, \dots , so that $G(j, k_i) = (j, K_i)$, and $k_{i+1} = K_i + 1, \forall i$. In particular, the mapping $\Psi_{j,k_i}: \mathcal{N}(\mathcal{D}_{j,k_i}^u) \rightarrow \mathcal{F}_{j,K_i}^+$ satisfies $\Psi_{j,k_i}(\mathcal{D}_{j,k_i}^u) = \mathcal{D}_{j,K_i}^s, \forall i = 1, \dots, N$. Moreover, due to the coincidences, the mapping

$$\Psi := \Psi_{j,k_1} \circ l_{j,K_N}^{\text{ext}} \circ \dots \circ \Psi_{j,k_2} \circ l_{j,K_1}^{\text{ext}}, \quad (5.1)$$

is well-defined in $\mathcal{V}_1 \setminus \mathcal{D}_{j,K_1}^s$, where $\mathcal{V}_1 \subset \mathcal{F}_{j,K_1}^+$ is a sufficiently small neighborhood of \mathcal{D}_{j,K_1}^s . For simplicity, assume that $T_{2,i} = 1, \forall i$. Let $\tilde{l}_i: \mathbb{R} \times (0, \delta) \rightarrow \mathbb{R} \times (0, \delta)$ be a lift of l_{j,K_i}^{ext} in coordinates (t, r) , see Section 4.1.1.

Proposition 5.1. *There exist real-analytical functions $g_i, h_i: (-\epsilon, \epsilon) \rightarrow \mathbb{R}$, $h_i(0) > 0$, so that $\tilde{l}_i(t, r) = (t + g_i(r) - h_i(r) \ln r, r)$, $\forall (t, r)$.*

Proof. Recall that $\tilde{l}_i(t, r) = (t + \Delta t_i(r), r)$, where $\Delta t_i(r) = \frac{1}{u_i(\delta r/2)} \ln \frac{\tilde{\delta}_i}{2r}$, $u_i > 0$ is real-analytic and $\tilde{\delta}_i > 0$. This implies the existence of g_i, h_i as in the statement. \square

Now we describe the global mappings $\Psi_{j,k_i}, i = 1, \dots, N$, in coordinates (t, r) . Denote by $\tilde{\psi}_i: \mathbb{R} \times (-\delta, \delta) \rightarrow \mathbb{R} \times (-\delta', \delta')$, $0 < \delta \ll \delta'$, a lift of Ψ_{j,k_i} , defined in a small neighborhood of $C_{j,k_i}^u \subset \mathcal{F}_{j,k_i}^-$.

Proposition 5.2. [7, Lemma 7.1] *The global mapping $\tilde{\psi}_i = (X_i, Y_i)$ has the form $X_i(t, r) = H_i(t) + r\tilde{X}_i(t, r)$ and $Y_i(t, r) = r\tilde{Y}_i(t, r)$, where $H_i - \text{Id}: \mathbb{R} \rightarrow \mathbb{R}$ and $\tilde{X}_i, \tilde{Y}_i: \mathbb{R} \times (-\delta, \delta) \rightarrow \mathbb{R}$ are real-analytic functions, 1-periodic in t , and satisfy $H'_i, \tilde{Y}_i > 0$.*

Consider the sequence $(T_i, R_i) := \tilde{\psi}_{i+1} \circ \tilde{l}_i \circ \dots \circ \tilde{\psi}_2 \circ \tilde{l}_1$, so that (T_N, R_N) represents the mapping Ψ in (5.1). Then $(T_{i+1}, R_{i+1}) = \tilde{\psi}_{i+2} \circ \tilde{l}_{i+1}(T_i, R_i), \forall i$, implying

$$\begin{aligned} T_{i+1} &= H_{i+2}(T_i + g_{i+1}(R_i) - h_{i+1}(R_i) \ln R_i) \\ &\quad + R_i \tilde{X}_{i+2}(T_i + g_{i+1}(R_i) - h_{i+1}(R_i) \ln R_i, R_i), \\ R_{i+1} &= R_i \tilde{Y}_{i+2}(T_i + g_{i+1}(R_i) - h_{i+1}(R_i) \ln R_i, R_i), \end{aligned} \quad (5.2)$$

for every $i \geq 1$, see Propositions 5.1 and 5.2.

Proposition 5.3. *For every $i \geq 1$, there exist $A_i, B_i, C_i > 0$ so that*

$$T_i(t, r) - t > -C_i \ln r \quad \text{and} \quad A_i r < R_i(t, r) < B_i r, \quad (5.3)$$

uniformly in t , for every $r > 0$ sufficiently small.

Proof. We prove by induction. Notice that for $i = 1$, we have

$$\begin{aligned} T_1(t, r) &= H_2(t + g_1(r) - h_1(r) \ln r) + r\tilde{X}_2(t + g_1(r) - h_1(r) \ln r, r), \\ &= t + g_1(r) - h_1(r) \ln r + (H_2 - \text{Id})(t + g_1(r) - h_1(r) \ln r) \\ &\quad + r\tilde{X}_2(t + g_1(r) - h_1(r) \ln r, r), \\ R_1(t, r) &= r\tilde{Y}_2(t + g_1(r) - h_1(r) \ln r, r). \end{aligned} \quad (5.4)$$

The claim follows for $i = 1$ since $H_2 - \text{Id}$, \tilde{X}_2 and \tilde{Y}_2 are 1-periodic in t , and $\tilde{Y}_2(\cdot, 0)$ and $h_1(0)$ are positive.

Next assume (5.3) holds for i . We shall prove that (5.3) holds for $i + 1$. Using that $H_{i+2} = \text{Id} + \tilde{H}_{i+2}$, for some 1-periodic function \tilde{H}_{i+2} , we obtain from (5.2)

$$\begin{aligned} T_{i+1} &= T_i + g_{i+1}(R_i) - h_{i+1}(R_i) \ln R_i + \tilde{H}_{i+2}(T_i + g_{i+1}(R_i) - h_{i+1}(R_i) \ln R_i) \\ &\quad + R_i \tilde{X}_{i+2}(T_i + g_{i+1}(R_i) - h_{i+1}(R_i) \ln R_i, R_i), \\ R_{i+1} &= R_i \tilde{Y}_{i+2}(T_i + g_{i+1} - h_{i+1}) \ln R_i, R_i), \end{aligned}$$

where g_{i+1} and h_{i+1} are real-analytic near 0. Since $\tilde{Y}_{i+2}, \tilde{X}_{i+2}$ are 1-periodic in t , and $\tilde{Y}_{i+2}(\cdot, 0), h_{i+1}(0) > 0$, it follows from the induction hypothesis that (5.3) holds for $i + 1$. \square

Fix any component \mathcal{U}_j of $S^3 \setminus \bigcup_{i=1}^\ell \mathcal{S}_i$ so that $(j, k) \in \mathcal{C}$ is coincident for every $1 \leq k \leq \tilde{k}_j$. Recall that \mathcal{U}_j is homeomorphic to a 3-sphere with \tilde{k}_j disjoint closed 3-balls removed. Fix a plane \mathcal{F}_{j,k_1}^+ of the genus zero transverse foliation in \mathcal{U}_j , bounded by $P_{3,j}$. The branches in \mathcal{U}_j of the stable/unstable manifolds of the orbits in $\mathcal{P}_2(\lambda)$ transversely intersect \mathcal{F}_{j,k_1}^+ at mutually disjoint circles. Indeed, these circles bound closed disks $B_\alpha, \alpha = 1, \dots, \nu$. Since their symplectic areas coincide, B_α have mutually disjoint interiors. The trajectories through the interior of each B_α eventually leave \mathcal{U}_j to an adjacent component $\mathcal{U}_i, i \neq j$. The points in ∂B_α converge to some $P_{2,i} \subset \partial \mathcal{U}_j$.

Consider the connected subset of \mathcal{F}_{j,k_1}^+ given by $\mathcal{A} := \mathcal{F}_{j,k_1}^+ \setminus \bigcup_{\alpha=1}^\nu B_\alpha$. Since the Reeb flow is transverse to \mathcal{F}_{j,k_1}^+ , the successive local and global maps determine a diffeomorphism $\Psi: \mathcal{A} \rightarrow \mathcal{A}$ that preserves the finite area form induced by $d\lambda$. Abbreviating by $\mathcal{U}_\mathcal{A} \subset \mathcal{U}_j$ an invariant open subset, defined by the Reeb trajectories through \mathcal{A} , we conclude that \mathcal{A} is a global surface of section for the Reeb flow restricted to $\mathcal{U}_\mathcal{A}$. The mapping Ψ is the first return map to \mathcal{A} , and thus periodic orbits of Ψ correspond to periodic orbits of the Reeb flow in $\mathcal{U}_\mathcal{A}$.

Notice that the outer boundary component of \mathcal{A} , i.e. the binding orbit $P_{3,j}$, is preserved by Φ , and hence points near $P_{3,j}$ are mapped under Ψ to points near $P_{3,j}$. However, near the inner boundary components, Ψ behaves as a permutation. More precisely, given $\alpha \in \{1, \dots, \nu\}$ there exists α' (possibly $\alpha' = \alpha$) such that Ψ maps points near ∂B_α to points near $\partial B_{\alpha'}$.

Now we proceed as in [7, Section 7]. Define the equivalence relation \sim on \mathcal{F}_{j,k_1}^+ such that $x \sim y \Leftrightarrow x, y \in B_\alpha$ for some α . Abbreviate by $\Pi: \mathcal{F}_{j,k_1}^+ \rightarrow \tilde{\mathcal{F}}_{j,k_1}^+ := \mathcal{F}_{j,k_1}^+ / \sim$ the natural projection. It induces the quotient topology on $\tilde{\mathcal{F}}_{j,k_1}^+$, which becomes an open disk. For each α , put $p_\alpha = \Pi(B_\alpha) \in \tilde{\mathcal{F}}_{j,k_1}^+$. The restriction $\Pi|_\mathcal{A}: \mathcal{A} \rightarrow \hat{\mathcal{F}}_{j,k_1}^+ := \tilde{\mathcal{F}}_{j,k_1}^+ \setminus \{p_\alpha \mid \alpha = 1, \dots, \nu\}$ is a bijection. We endow $\hat{\mathcal{F}}_{j,k_1}^+$ with a natural smooth structure by declaring $\Pi|_\mathcal{A}$ to be a smooth diffeomorphism onto $\hat{\mathcal{F}}_{j,k_1}^+$. In this way, we obtain the finite area form ω on $\hat{\mathcal{F}}_{j,k_1}^+$ which is defined as the push-forward of the area form on \mathcal{A} induced by $d\lambda$. The area form ω naturally extends to a finite area form on $\tilde{\mathcal{F}}_{j,k_1}^+$, still denoted

by ω . This enables us to define a homeomorphism $\Phi: \tilde{\mathcal{F}}_{j,k_1}^+ \rightarrow \tilde{\mathcal{F}}_{j,k_1}^+$, which preserves ω , permutes the points p_1, \dots, p_ν , and satisfies $\Phi(z) = \Pi \circ \Psi \circ \Pi^{-1}(z), \forall z \in \hat{\mathcal{F}}_{k_1+}^j$. In particular, the points p_1, \dots, p_ν are periodic.

Case 1. $\nu \geq 2$.

In this case, Brouwer's fixed point theorem [3] gives a fixed point of Φ , say $p \in \tilde{\mathcal{F}}_{j,k_1}^+$. The restriction $\Phi|_{\tilde{\mathcal{F}}_{j,k_1}^+ \setminus \{p\}}$ is an area preserving homeomorphism of the open annulus $\tilde{\mathcal{F}}_{j,k_1}^+ \setminus \{p\}$, containing at least one periodic point $p_j \neq p$. An application of Franks' Theorem [11] implies that Φ has infinitely many periodic points. It follows that Ψ admits infinitely many periodic points, and hence the Reeb flow admits infinitely many periodic orbits in \mathcal{U}_A .

Case 2. $\nu = 1$.

In this case, there exists only one such disk $B_\alpha \subset \mathcal{F}_{j,k_1}^+$ which coincides with \mathcal{D}_{j,k_1}^s . Its boundary is necessarily the intersection of the unstable manifold of \mathcal{P}_{j,k_1}^- with \mathcal{F}_{j,k_1}^+ , which coincides with the intersection of the stable manifold of \mathcal{P}_{j,k_1}^+ with \mathcal{F}_{j,k_1}^+ . Moreover, $\mathcal{F}_{j,k_1}^+ \setminus B_\alpha$ is mapped under l_{j,k_1}^{ext} to $\mathcal{F}_{j,k_1+1}^- \setminus \mathcal{D}_{j,k_1+1}^u$. In particular, since $\nu = 1$, the branch of the unstable manifold of \mathcal{P}_{j,k_1+1}^- inside \mathcal{U}_j coincides with the branch of the stable manifold of \mathcal{P}_{j,k_1+1}^+ . Now we can consider the global map g_{j,k_1+1} , which necessarily maps $C_{j,k_1+1}^u \subset \mathcal{F}_{j,k_1+1}^-$ to $C_{j,k_1+1}^s \subset \mathcal{F}_{j,k_1+1}^+$. Continuing this process, we conclude that the first return map $\Psi: \mathcal{F}_{j,k_1}^+ \setminus B_\alpha \rightarrow \mathcal{F}_{j,k_1}^+ \setminus B_\alpha$, is given by successive compositions of local and global maps

$$\Psi = g_{j,k_1} \circ l_{j,k_1+\tilde{k}_j-1}^{\text{ext}} \circ \dots \circ g_{j,k_1+2} \circ l_{j,k_1+1}^{\text{ext}} \circ g_{j,k_1+1} \circ l_{j,k_1}^{\text{ext}}, \quad (5.5)$$

where \tilde{k}_j is the number of boundary components of \mathcal{U}_j .

Recall that, in the special coordinates $(t, r) \in \mathbb{R} \times (0, \epsilon_i)$, $\epsilon_i > 0$ small, defined near $C_{j,k_i}^s \subset \mathcal{F}_{j,k_i}^+$, the local mapping l_{j,k_i}^{ext} has a lift of the form $\tilde{l}_i(t, r) = (t + g_i(r) - h_i(r) \ln r, r)$, for real-analytic functions $g_i(r), h_i(r)$ defined near $r = 0$ with $h_i(0) > 0$, see Proposition 5.1.

The global mapping g_{j,k_i} has a lift of the form $\tilde{\psi}(t, r) = (t + H_i(t) + r\tilde{X}_i(t, r), r\tilde{Y}_i(t, r))$, where $\tilde{H}_i, \tilde{X}_i, \tilde{Y}_i$ are real-analytic and 1-periodic in t and satisfy $\tilde{H}_i' > 1$ and $\tilde{Y}_i > 0$, see Proposition 5.2. A lift $\tilde{\Psi}$ of Ψ in coordinates $(t, r) \in \mathbb{R} \times (0, 1)$ is then given by the composition $\tilde{\Psi} = \tilde{\psi}_1 \circ \tilde{l}_{\tilde{k}_j} \circ \dots \circ \tilde{\psi}_2 \circ \tilde{l}_1$, for every $r > 0$ sufficiently small. By Proposition 5.3, $\tilde{\Psi}$ has the form $\tilde{\Psi}(t, r) = (T_{\tilde{k}_j}(t, r), R_{\tilde{k}_j}(t, r))$, where $T_{\tilde{k}_j}(t, r) - t > -C \ln r$ and $Ar < R_{\tilde{k}_j}(t, r) < Br$ for every $(t, r) \in \mathbb{R} \times (0, 1)$ with $r > 0$ sufficiently small. Here, $0 < A < B$ and $0 < C$ are positive constants. In particular, Ψ has infinite twist near $\mathbb{R}/\mathbb{Z} \times \{0\}$ which implies that Ψ has infinitely many fixed points. Let $\tilde{\Psi}_k(x, y) := \tilde{\Psi}(x, y) - (k, 0), \forall k \in \mathbb{N}^*$. Proposition 7.2 from [7] says that $\tilde{\Psi}_k$ has a fixed point for every k large. The proof is based on Franks' generalization of the Poincaré-Birkhoff theorem [10]. Moreover, fixed points of $\tilde{\Psi}_{k_1}$ and $\tilde{\Psi}_{k_2}$, with $k_1 \neq k_2$, correspond to distinct fixed points of Ψ . Therefore, the existence of infinitely many periodic orbits in \mathcal{U}_A follows, and this finishes the proof of Theorem 1.8-(i).

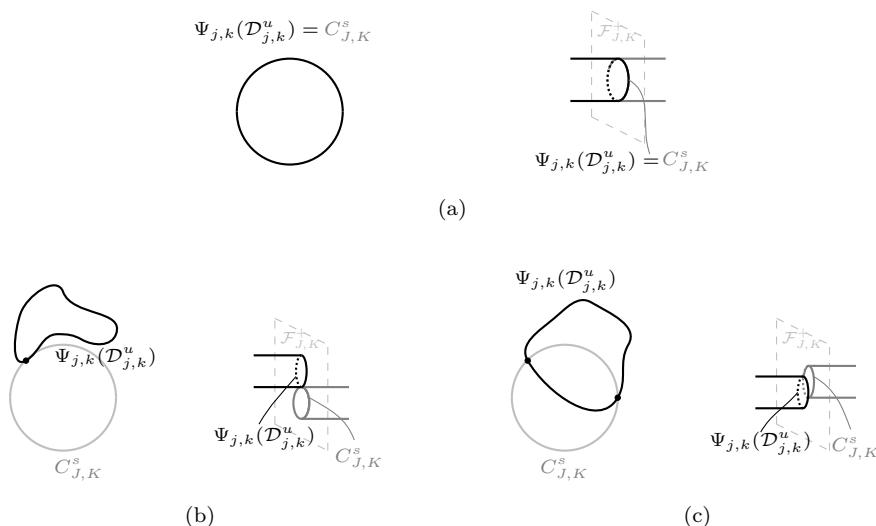


Fig. 5.1. Possible scenarios for $\Psi_{j,k}(\mathcal{D}_{j,k}^u) \cap C_{j,K}^s$.

5.2. Proof of Theorem 1.8-(ii)

Fix $j \in \{1, \dots, l\}$, and assume that $\Psi_{j,k}(C_{j,k}^u) \neq C_{j,K}^s, \forall k \in \{1, \dots, \tilde{k}_j\}$, where the map $\Psi_{j,k}$ is as in (4.10) and $(j, K) = G(j, k)$. This means that for every $k = 1, \dots, \tilde{k}_j$, the branch in \mathcal{U}_j of the unstable manifold of P_{2,n_k^j} does not coincide with the branch in \mathcal{U}_j of the stable manifold of any P_{2,n_l^j} . Hence, for every (j, k) , with $k = 1, \dots, \tilde{k}_j$, we are in scenario (b) or (c), see Fig. 5.1.

Proposition 5.4. *There exists $P_{2,n_k^j} \subset \partial \mathcal{U}_j$ with a transverse homoclinic in \mathcal{U}_j .*

To prove Proposition 5.4, we find $k \in \{1, \dots, \tilde{k}_j\}$ so that $(j, k) \in \mathcal{C}$ is periodic under G . To find such k , choose any $(j, k_1) \in \mathcal{C}$, let $(j, K_1) = G(j, k_1)$, and let $(j, k_2) := (j, K_1 + 1)$. Define the sequences k_1, k_2, k_3, \dots , and K_1, K_2, K_3, \dots , accordingly, as $(j, K_i) = G(j, k_i)$ and $k_{i+1} = K_i + 1$ for every i . The sequence k_1, k_2, \dots , is eventually periodic, so we may ignore the first elements and assume that $k_1, k_2, \dots, k_N, k_1, \dots$ is periodic with least period $N > 0$.

Next we show that for every i the branch in \mathcal{U}_j of the unstable manifold of \mathcal{P}_{j,k_i}^- intersects transversely the branch in \mathcal{U}_j of the stable manifold of $\mathcal{P}_{j,k_{i+m}}^-$, $\forall m \geq 2$. In particular, the orbits \mathcal{P}_{j,k_i}^- admit transverse homoclinics for every i . Take a real-analytic curve $\gamma: [0, \varepsilon) \rightarrow \Psi_{j,k_i}(C_{j,k_i}^u), \varepsilon > 0$ small, such that $\gamma(0) \in \Psi_{j,k_i}(C_{j,k_i}^u) \cap C_{j,K_i}^s$ and $\gamma(t) \notin \mathcal{D}_{j,K_i}^s, \forall t$. Let $\dot{\gamma} = \gamma \setminus \{\gamma(0)\}$.

Recall that the mapping $\Psi_{j,k_{i+1}}$ is defined on a small neighborhood $\mathcal{N}(\mathcal{D}_{j,k_{i+1}}^u)$ of $\mathcal{D}_{j,k_{i+1}}^u$. Let $\beta: [0, \varepsilon) \rightarrow \mathcal{N}(\mathcal{D}_{j,k_{i+1}}^u)$ be a real-analytic arc intersecting $C_{j,k_{i+1}}^u$ only at $t = 0$ and satisfying $\beta(s) \in \mathcal{N}(\mathcal{D}_{j,k_{i+1}}^u) \setminus \mathcal{D}_{j,k_{i+1}}^u, \forall s \in (0, \varepsilon)$, and $\Psi_{j,k_{i+1}}(\beta) \subset C_{j,K_{i+1}}^s$. The following lemma is based on Conley's ideas [4].

Proposition 5.5. *The real-analytic arc $l_{j,K_i}^{\text{ext}}(\dot{\gamma})$ is a spiral around $C_{j,k_{i+1}}^u \subset \mathcal{F}_{j,k_{i+1}}^-$ intersecting β transversely infinitely many times near $C_{j,k_{i+1}}^u$. In particular, the branch in \mathcal{U}_j of the unstable manifold of \mathcal{P}_{j,k_i}^- transversely intersects the branch in \mathcal{U}_j of the stable manifold of $\mathcal{P}_{j,k_{i+2}}^-$.*

Proof. As before, let $\tilde{l}_i: \mathbb{R} \times (0, \delta) \rightarrow \mathbb{R} \times (0, \delta)$ denote a lift of l_{j,K_i}^{ext} . In view of Lemma 4.2, $l_{j,K_i}^{\text{ext}}(\dot{\gamma})$ intersects β infinitely many times. The curve γ is written as $\gamma = \{(t_\gamma(r), r) \mid r \in (0, \varepsilon)\}$ for some real analytic function $t_\gamma: (0, \delta') \rightarrow \mathbb{R} \times (0, \varepsilon)$, which continuously extends over $[0, \delta')$. We then obtain $\tilde{l}(\dot{\gamma}) = \{(\tilde{t}_\gamma(r) = t_\gamma(r) + \Delta t(r), r) \mid r \in (0, \varepsilon)\}$, where $\Delta t(r) = g(r) - h(r) \ln r$, and $g(r), h(r)$ are real-analytic functions defined near $r = 0$, with $h(0) > 0$. See (4.6). Hence $\tilde{t}'_\gamma(r) = t'_\gamma(r) + g'(r) - h'(r) \ln r - \frac{h(r)}{r}$ for every $r \in (0, \varepsilon)$. Due to the estimates in the proof of Lemma 4.2, we find a constant $c_1 > 0$ and $\varepsilon_0 \in (0, \varepsilon)$ such that

$$\tilde{t}'_\gamma(r) < -\frac{c_1}{r}, \quad \forall r \in (0, \varepsilon_0). \quad (5.6)$$

Now we proceed similarly with β . We may assume that $\beta(0) = (0, 0)$. Hence there exists a continuous function $t_\beta: [0, \varepsilon) \rightarrow [0, \varepsilon_0)$, which is real-analytic in $(0, \varepsilon)$ such that $t_\beta(0) = 0$ and $\beta = \{(t_\beta(r), r) \mid r \in [0, \varepsilon)\}$. Because of real analyticity of β , the intersection at $\beta(0) = (0, 0)$ is either transverse to $\{r = 0\}$ or has finite order tangency. Hence, we have either $t_\beta \equiv 0$ or $t_\beta(r) = r^m g(r)$, where g is real-analytic on $r > 0$ and satisfies $g(0) \neq 0$, and $m \in \mathbb{N}^*$ or $\frac{1}{m} \in \mathbb{N}^*$, depending on the way β intersects $C_{j,k_{i+1}}^u$ at $\beta(0) = (0, 0)$. In all cases, we can choose $\lambda \in (0, 1)$ and $c_2 > 0$ such that

$$t'_\beta(r) = m r^{m-1} g + r^m g' > -\frac{c_2}{r^{1-\lambda}} \quad (5.7)$$

for every $r > 0$ small enough. We conclude from (5.6) and (5.7) that $\tilde{t}'_\gamma(r) < t'_\beta(r)$, for every $r > 0$ sufficiently small. \square

Instead of $\dot{\gamma}$, we can now repeat the above construction using a small arc $\dot{\gamma}_1 \subset \Psi_{j,k_{i+1}} \circ l_{j,K_i}^{\text{ext}}(\dot{\gamma})$ in $\mathcal{F}_{j,K_{i+1}}^+ \setminus \mathcal{D}_{j,K_{i+1}}^s$, which corresponds to a transverse intersection of the branch in \mathcal{U}_j of the unstable manifold of \mathcal{P}_{j,k_i}^- with the branch in \mathcal{U}_j of the stable manifold of $\mathcal{P}_{j,k_{i+2}}^-$. Proposition 5.5 then provides a transverse intersection between the branch in \mathcal{U}_j of the unstable manifold of \mathcal{P}_{j,k_i}^- and the branch in \mathcal{U}_j of the stable manifold of $\mathcal{P}_{j,k_{i+3}}^-$. Using the periodicity of the sequence $k_1, k_2, \dots, k_N, \dots$, which satisfies $(j, k_{i+1} - 1) = G(j, k_i), \forall i$, we repeat this construction to find, for every i , a transverse homoclinic to \mathcal{P}_{j,k_i}^- .

The proof of Proposition 5.4 is complete. We are ready to prove Theorem 1.8-(ii).

It is well-known that a transverse homoclinic forces positivity of topological entropy. For completeness, we include the construction of an invariant subset $\Lambda_j \subset \mathcal{U}_j$ whose dynamics contains the Bernoulli shift as a sub-system. We follow Moser's book [27], see also [1].

Let $(j, k) \in \mathcal{C}$ be such that $\mathcal{P}_{j,k}^-$ admits a transverse homoclinic orbit, as obtained in Proposition 5.4. Consider a point $q_0 \in C_{j,k-1}^s \subset \mathcal{F}_{j,k-1}^+$ which corresponds to a transverse homoclinic to $\mathcal{P}_{j,k-1}^+ = \mathcal{P}_{j,k}^-$. This point lies in a small arc $V_\infty \subset \mathcal{F}_{j,k-1}^+$ which is transverse to $C_{j,k-1}^s$ and is the image of an arc $V'_\infty \subset C_{j,k}^u$ under the forward flow. Denote by $\mathcal{G}: \mathcal{V}' \subset \mathcal{F}_{j,k}^- \rightarrow \mathcal{V} \subset \mathcal{F}_{j,k-1}^+$, the corresponding diffeomorphism, given by the first forward hitting point, so that $\mathcal{G}(V'_\infty) = V_\infty$. Notice that $V'_\infty \subset \{r = 0\}$, where (t, r) are the real-analytic canonical coordinates near $C_{j,k}^u \subset \mathcal{F}_{j,k}^-$ as in Section 4.

There exists a small arc $H_\infty \subset C_{j,k-1}^s$, starting from q_0 which, except for q_0 , does not intersect $\mathcal{G}(\mathcal{V}' \cap \mathcal{D}_{j,k}^u)$. In this way, we find a small topological square $Q_0 \subset \mathcal{V}$ near q_0 , whose boundary is formed by the following arcs.

- (i) the two arcs H_∞, V_∞ above.
- (ii) an arc H_0 starting from $\mathcal{G}(\mathcal{V}' \cap C_{j,k}^u)$ which, except for this extreme point, is contained in the exterior of $\mathcal{D}_{j,k-1}^s$ and does not intersect $\mathcal{G}(\mathcal{V}' \cap \mathcal{D}_{j,k}^u)$.
- (iii) an arc V_0 starting from $C_{j,k-1}^s$ which, except for this extreme point, is contained in the exterior of $\mathcal{D}_{j,k-1}^s$ and does not intersect $\mathcal{G}(\mathcal{V}' \cap \mathcal{D}_{j,k}^u)$.

In local coordinates (t, r) defined on a neighborhood of $C_{j,k-1}^s$ we may assume that H_0 is a horizontal segment $r = r_0 > 0$ and V_0 is a horizontal shift of V_∞ .

We can always find coordinates (u, v) near Q_0 so that $Q_0 \equiv [0, 1] \times [0, 1]$, $V_\infty \equiv \{0\} \times [0, 1]$, $H_\infty \equiv [0, 1] \times \{0\}$, $V_0 \equiv \{1\} \times [0, 1]$, $H_0 \equiv [0, 1] \times \{1\}$. We may assume that $q_0 = (0, 0)$ and that the mapping $(t, r) \mapsto (u, v)$ has the form

$$(u, v) = M(t, r) + O(t^2 + r^2), \quad (5.8)$$

for some invertible linear mapping M .

A vertical strip V in Q_0 is a topological closed disk whose boundary consists of horizontal arcs $h_0 \subset H_0, h_\infty \subset H_\infty$ and two regular arcs $v_1, v_2 \subset Q_0$ that connect H_0 and H_∞ . We assume that the arcs v_1, v_2 are transverse to the horizontals $[0, 1] \times \text{const}$. A horizontal strip in Q_0 is similarly defined.

Let $\mathcal{P} := \mathcal{G} \circ l_{j,k-1}^{\text{ext}}: \mathcal{H}_{-1} \subset Q_0 \rightarrow Q_0$, where $\mathcal{H}_{-1} \subset Q_0$ is its domain of definition. Abusing the notation, we denote by $\mathcal{P}^{-1} := (l_{j,k-1}^{\text{ext}})^{-1} \circ \mathcal{G}^{-1}: \mathcal{P}(\mathcal{H}_{-1}) =: \mathcal{V}_1 \rightarrow Q_0$, the first return map under the backward flow.

Lemma 5.6. *If Q_0 is (suitably chosen) sufficiently small, then \mathcal{H}_{-1} is formed by countably many horizontal strips $H_n, n \in \mathbb{N}$, in Q_0 , monotonically accumulating on H_∞ as $n \rightarrow \infty$. Moreover, \mathcal{V}_1 is formed by countably many vertical strips $V_n, n \in \mathbb{N}$, in Q_0 , monotonically accumulating on V_∞ as $n \rightarrow \infty$. For every n , $\mathcal{G} \circ l_{j,k-1}^{\text{ext}}(H_n) = V_n$, and the vertical (horizontal) boundary components of H_n are mapped to the respective vertical (horizontal) boundary components of V_n .*

Proof. Let $\tilde{Q}_0 := \mathcal{G}^{-1}(Q_0) \subset \mathcal{F}_{j,k}^-$. In coordinates (t, r) on $\mathcal{F}_{j,k}^-$, \tilde{Q}_0 is a square whose sides are $\tilde{H}_\infty := \mathcal{G}^{-1}(V_\infty), \tilde{H}_0 := \mathcal{G}^{-1}(V_0), \tilde{V}_\infty := \mathcal{G}^{-1}(H_\infty), \tilde{V}_0 := \mathcal{G}^{-1}(H_0)$. Notice

that $\tilde{H}_\infty = V'_\infty$. Recall that in coordinates (t, r) on $\mathcal{F}_{j,k-1}^+$, we have $H_0 \subset \{r = r_0\}$, for some $r_0 > 0$ small, and $V_0 = V_\infty + (t_0, 0)$ for some $|t_0| > 0$ small. Taking r_0 and $|t_0|$ sufficiently small, we can assume that \tilde{H}_0 is arbitrarily C^1 -close to $\tilde{H}_\infty \subset \{r = 0\}$ and \tilde{V}_0 is arbitrarily C^1 close to \tilde{V}_∞ . Moreover, the images under \mathcal{G}^{-1} of the curves $v_t = V_\infty + (t, 0)$, $|t| \leq |t_0|$, are also C^1 -close to \tilde{H}_0 and \tilde{H}_∞ .

Now observe that $l_{j,k-1}^{\text{ext}}(V_\infty \setminus \{q_0\})$ transversely intersects \tilde{V}_∞ and is arbitrarily C^1 -close to the horizontal line $\{r = 0\}$. This follows from Propositions 5.3 and 5.5. Hence, if $r_0, |t_0| > 0$ are sufficiently small and suitably chosen, then $l_{j,k-1}^{\text{ext}}(Q_0 \setminus H_\infty) \cap \tilde{Q}_0$ consists of countably many horizontal strips \tilde{H}_n , $n \in \mathbb{N}$, in \tilde{Q}_0 . In particular, the images of such strips under \mathcal{G} are vertical strips V_n , $n \in \mathbb{N}$, in Q_0 , ordered from right to left.

Finally, notice that a lift \tilde{l}^{-1} of $(l_{j,k-1}^{\text{ext}})^{-1}$ in coordinates (t, r) has the form $\tilde{l}^{-1}(t, r) = (t - g(r) + h(r) \ln r, r)$, $\forall (t, r)$, for suitable real-analytic functions $g(r), h(r)$, defined near $r = 0$, with $h(0) > 0$. Therefore, \tilde{l}^{-1} shares the same properties of a lift \tilde{l} representing $l_{j,k-1}^{\text{ext}}$ and we can apply Propositions 5.3 and 5.5 to the curve \tilde{V}_∞ (or \tilde{V}_0) to conclude that $H_n := (l_{j,k-1}^{\text{ext}})^{-1}(\tilde{H}_n)$ are horizontal strips in Q_0 that accumulate on H_∞ , and are ordered from top to bottom. Moreover, the horizontal (vertical) boundary components of \tilde{H}_n are mapped under \tilde{l}^{-1} to vertical (horizontal) boundary components of H_n . The interchanging of vertical (horizontal) strips between \tilde{Q}_0 and Q_0 under the mapping \mathcal{G} finishes the proof of this proposition. \square

Each vertical strip $V_n \subset \mathcal{V}_1$ is regarded as a new square and its image under \mathcal{P} consists of infinitely many vertical strips, precisely one sub-strip of each strip in \mathcal{V}_1 . Hence $\mathcal{V}_2 := \mathcal{P}(\mathcal{V}_1) \subset \mathcal{V}_1$ consists of countably many vertical strips, with countably many sub-strips of each strip in \mathcal{V}_1 . Similarly, $\mathcal{V}_2 = \mathcal{P}(\mathcal{H}_{-2})$, where $\mathcal{H}_{-2} \subset \mathcal{H}_{-1}$ consists of countably many horizontal strips, with countably many sub-strips of each strip in \mathcal{H}_{-1} .

Repeating indefinitely this construction, we obtain sequences $\mathcal{V}_1 \supset \mathcal{V}_2 \supset \mathcal{V}_3 \supset \dots$ and $\mathcal{H}_{-1} \supset \mathcal{H}_{-2} \supset \mathcal{H}_{-3} \supset \dots$, so that \mathcal{V}_{n+1} consists of countably many vertical strips, with countably many sub-strips of each strip in \mathcal{V}_n . In the same way, \mathcal{H}_{-n-1} consists of countably many horizontal strips, with countably many sub-strips of each strip in \mathcal{H}_{-n} . The image of \mathcal{H}_{-n} under \mathcal{P} coincides with \mathcal{V}_n .

The non-empty compact subsets of Q_0 , defined as $\bar{\Lambda}_{\mathcal{H}} := \cap_{i=1}^{+\infty} \text{closure}(\mathcal{H}_{-i})$ and $\bar{\Lambda}_{\mathcal{V}} := \cap_{i=1}^{+\infty} \text{closure}(\mathcal{V}_i)$, contain points whose forward and backward trajectories remain in the fixed component \mathcal{U}_j of $S^3 \setminus \cup_{i=1}^l \mathcal{S}_i$, respectively.

The non-empty compact subset $\bar{\Lambda} := \bar{\Lambda}_{\mathcal{H}} \cap \bar{\Lambda}_{\mathcal{V}} \subset Q_0$, contains points whose entire trajectories remain in \mathcal{U}_j . It admits a symbolic dynamics as we outline below. Notice that some points in $\bar{\Lambda}$ are eventually mapped to H_∞ where \mathcal{P} is not well-defined. Similarly, \mathcal{P}^{-1} is not well-defined on V_∞ .

Let $\bar{\Sigma}$ be the set of doubly infinite sequences $a = (a_n)_{n \in \mathbb{Z}}$ of the form

$$\dots, \infty, \infty, a_{l_0}, \dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots, a_{r_0}, \infty, \infty, \dots,$$

where a_n is a positive integer for every $l_0 \leq n \leq r_0$, and $a_n = \infty$ for $n < l_0$ and for $n > r_0$. Here, $-\infty \leq l_0 \leq 0 \leq r_0 \leq +\infty$. The usual shift in $\bar{\Sigma}$ is given by $\sigma(a)_n := a_{n+1}, \forall n$, and is defined only if $a_0 \neq \infty$. Similarly, one has the inverse σ^{-1} .

The conjugation $h: (\bar{\Lambda}, \mathcal{P}) \rightarrow (\bar{\Sigma}, \sigma)$ maps each $x \in \bar{\Lambda}$ to $(a_n)_{n \in \mathbb{Z}}$ satisfying $\mathcal{P}^n(x) \in V_{a_n}, -\infty \leq l_0 \leq n \leq r_0 \leq +\infty$. If $\mathcal{P}^{r_0}(x) \in H_\infty$ for some $0 < r_0 < +\infty$, then $\mathcal{P}^{r_0+1}(x)$ is not well-defined. In this case, we define $h(x)_n := \infty$ for every $n > r_0$. In the same way, if $\mathcal{P}^{l_0}(x) \in V_\infty$ for some $-\infty < l_0 < 0$, then $\mathcal{P}^{l_0-1}(x)$ is not well-defined, and we define $h(x)_n := \infty$ for every $n < l_0$. We also define $h(q_0) = (\dots, \infty, \infty, \infty, \dots)$. Hence $h \circ \mathcal{P} = \sigma \circ h$ wherever defined. By Lemma 5.6 and the reasoning just after it, we see that $h(\bar{\Lambda}) = \bar{\Sigma}$.

The subset $\Sigma \subset \bar{\Sigma}$ of sequences whose entries are positive integers, that is $l_0 = -\infty$ and $r_0 = +\infty$, corresponds to an invariant subset $\Lambda \subset \bar{\Lambda}$, all whose iterates of \mathcal{P} (positive and negative) are defined. Our estimates below show that Λ is in one-to-one correspondence with Σ .

Now we study the hyperbolic structure of the invariant set Λ . To do that, we first compute the derivative of $\mathcal{P} = \mathcal{G} \circ l_{j,k-1}^{\text{ext}}$ in coordinates (t, r) . Since the lift \tilde{l} representing $l_{j,k-1}^{\text{ext}}$ has the form $\tilde{l}(t, r) = (t + g(r) - h(r) \ln r, r), \forall (t, r)$, for suitable real-analytic functions $g(r)$ and $h(r)$ defined near $r = 0$, with $h(0) > 0$, we find

$$d\tilde{l}(t, r) \equiv \begin{pmatrix} 1 & g'(r) - h'(r) - \frac{h(r)}{r} \\ 0 & 1 \end{pmatrix}, \quad \forall (t, r).$$

Observe that the dominating term in $d\tilde{l}(t, r)$ is $L(r) := g'(r) - h'(r) - h(r)/r \rightarrow -\infty$ as $r \rightarrow 0$. A lift of \mathcal{G} is represented by $\tilde{\psi} = \tilde{\psi}(t, r)$. We may assume that $q_0 \equiv (0, 0)$ and $\tilde{q}_0 = \mathcal{G}^{-1}(q_0) \equiv (0, 0)$. Due to the transversality of the homoclinic trajectory, we may also assume that

$$d\tilde{\psi}(t, r) \equiv \begin{pmatrix} A(t, r) & B(t, r) \\ C(t, r) & D(t, r) \end{pmatrix} \rightarrow \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} = d\tilde{\psi}(0, 0) \quad \text{as } (t, r) \rightarrow (0, 0),$$

where $A_0 D_0 - B_0 C_0 = 1$ and $C < 0$. Notice that $(A_0, C_0)^T = d\tilde{\psi}(0, 0) \cdot (1, 0)^T$ is tangent to V_∞ at $(0, 0)$. Hence, $d\mathcal{P} = d\mathcal{G} \cdot dl_{j,k-1}^{\text{ext}}$ is represented by

$$d\tilde{\psi} \cdot d\tilde{l} \equiv \begin{pmatrix} A(t, r) & A(t, r)L(r) + B(t, r) \\ C(t, r) & C(t, r)L(r) + D(t, r) \end{pmatrix},$$

whose eigenvalues are $\lambda_\pm = \frac{\text{tr}}{2} \pm \frac{\sqrt{\text{tr}^2 - 4}}{2}$, where $\text{tr} := A(t, r) + D(t, r) + C(t, r)L(r) \rightarrow +\infty$ as $r \rightarrow 0$. Hence $\lambda_+ \rightarrow +\infty$ and $\lambda_- \rightarrow 0^+$ as $r \rightarrow 0$. The respective eigenspaces $E_+(t, r)$ and $E_-(t, r)$ converge to $E_+ := \mathbb{R}(A_0, C_0)^T$ and $E_- := \mathbb{R}(1, 0)^T$ as $r \rightarrow 0$. Notice that E_+ is tangent to V_∞ and E_- is tangent to H_∞ at $q_0 \equiv (0, 0)$. In coordinates (u, v) , see (5.8), E_+ and E_- converge to $\mathbb{R}(0, 1)^T$ and $\mathbb{R}(1, 0)^T$, respectively, as $(u, v) \rightarrow (0, 0)$.

Fixing $0 < \mu < \frac{1}{2}$ and taking Q_0 sufficiently small, we conclude that the cones $\zeta_{(u,v)} := \{|\delta_v| < \mu|\delta_u|\}$ and $\eta_{(u,v)} := \{|\delta_u| < \mu|\delta_v|\}$, with $\delta_u \frac{\partial}{\partial u} + \delta_v \frac{\partial}{\partial v} \in T_{(u,v)}Q_0 \equiv \mathbb{R}^2$, satisfy $d\mathcal{P} \cdot \eta_{(u,v)} \subset \eta_{\mathcal{P}(u,v)}$, $d\mathcal{P}^{-1} \cdot \zeta_{(u,v)} \subset \zeta_{\mathcal{P}^{-1}(u,v)}$,

$$|d\mathcal{P} \cdot \eta| > \mu^{-1}|\eta|, \quad \forall \eta \in \eta_{(u,v)} \quad \text{and} \quad |d\mathcal{P}^{-1} \cdot \zeta| > \mu^{-1}|\zeta|, \quad \forall \zeta \in \zeta_{(u,v)}.$$

As proved in [27, chapter III], the mapping $h: \bar{\Lambda} \rightarrow \bar{\Sigma}$ is a conjugation between \mathcal{P} and σ . In particular, the topological entropy of \mathcal{P} is positive. The trajectories through Λ form an invariant subset $\Lambda_j \subset \mathcal{U}_j$, where the Reeb flow has positive topological entropy. This completes the proof of Theorem 1.8-(ii).

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