



Orthogonal representations of $SL_2(q)$ in defining characteristic

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Abstract

This paper determines the types of the invariant quadratic forms over their respective (finite) fields of definition for all irreducible modules of the groups $SL_2(q)$ in defining characteristic. We prove that for $q > 2$ any absolutely irreducible even dimensional orthogonal $SL_2(q)$ -module W in defining characteristic carries a split invariant quadratic form (i.e. it is of $+$ type) unless $\dim(W) \equiv 4 \pmod{8}$ and the field of definition of W is the subfield of index 2 in \mathbb{F}_q ; in the latter case the type of the invariant quadratic forms is $-$.

Keywords Representations of finite groups of Lie type · Defining characteristic · Invariant quadratic forms · Field of definition

Mathematics Subject Classification 20C20 · 20C33 · 11E08

1 Introduction

Let $\rho : G \rightarrow GL_n(K)$ be an absolutely irreducible representation of a finite group G . Then ρ is called *orthogonal*, if $\rho(G)$ preserves a non-degenerate quadratic form Q ; in this case Q is uniquely determined up to multiplication by non-zero scalars and $\rho(G)$ is a subgroup of the orthogonal group $O(Q)$ of Q .

In a long term project with Richard Parker and Thomas Breuer we aim to determine isomorphism type of $O(Q)$. For a survey see Breuer et al. (2024). If K is a finite field and n is even, there are two isomorphism classes of orthogonal groups, O^+ and O^- . Also for odd dimension n there are two isometry classes of non-degenerate quadratic forms (in odd characteristics), but they are represented by similar forms, having the same orthogonal groups. Therefore for finite fields we only need to handle even degree representations. As field extensions are well controlled (see Sin and Willems 1991,

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Proposition 4.9) it is enough to consider the minimal possible field K , the *field of definition*. Note that we deal with positive characteristic, so there are no Schur indices and the field of definition is generated by the traces of the matrices in $\rho(G)$.

The present paper deals with the smallest infinite series of finite groups of Lie type: the groups $\mathrm{SL}_2(q)$ of 2×2 matrices of determinant 1 over a finite field \mathbb{F}_q with q elements. The paper Braun and Nebe (2017) provides the relevant information for the orthogonal representations of $\mathrm{SL}_2(q)$ over number fields K . This immediately yields the types over finite fields of all characteristics that do not divide the group order. Using the methods of Nebe and Parker (2023) and the decomposition matrices available in Burkhardt (1976) one can also deduce the orthogonal types in non-defining characteristics. The present paper deals with the remaining case, where $\mathrm{char}(K) = p = \mathrm{char}(\mathbb{F}_q)$, the so-called defining characteristic. The main result is given in Theorem 3.7: For $q > 2$ any absolutely irreducible even dimensional orthogonal $\mathrm{SL}_2(q)$ -module W in defining characteristic carries a split invariant quadratic form (i.e. it is of + type) unless $\dim(W) \equiv 4 \pmod{8}$ and the field of definition of W is the subfield of index 2 in \mathbb{F}_q ; in this case the type of the invariant quadratic forms is $-$.

The proof of Theorem 3.7 is based on the observation that the restriction of all relevant representations to a cyclic subgroup $T \leq \mathrm{SL}_2(q)$ of order $|T| = q + 1$ (a non-split torus) is an orthogonal direct sum of irreducible unitary representations.

2 Quadratic forms over finite fields

2.1 Quadratic forms

Let K be a field and V a finite dimensional vector space over K . A quadratic form Q is a map $Q : V \rightarrow K$ such that $Q(ax) = a^2 Q(x)$ for all $a \in K, x \in V$ and such that the *polarisation*

$$B_Q : V \times V \rightarrow K, B_Q(x, y) := Q(x + y) - Q(x) - Q(y)$$

is a bilinear form. The quadratic form Q is called *non-degenerate* if the radical of B_Q is $\{0\}$. Note that the polarisation of a quadratic form is always a symmetric bilinear form. Also $2Q(x) = B_Q(x, x)$, so over a field of characteristic $\neq 2$ quadratic forms and symmetric bilinear forms are equivalent notions. If $\mathrm{char}(K) = 2$ then $B_Q(x, x) = 0$ for all x , so B_Q is alternating, and, in particular, the dimension of a non-degenerate quadratic form is even.

2.2 Quadratic forms over finite fields

Let \mathbb{F}_q denote the field with q elements. Then it is well known that every non-degenerate quadratic form Q of dimension ≥ 3 contains isotropic vectors, i.e. vectors $v \neq 0$ with $Q(v) = 0$. We may conclude that such forms split off a hyperbolic plane

$$\mathbb{H}(\mathbb{F}_q) := ((v, w), Q) \text{ with } Q(av + bw) = ab$$

as an orthogonal summand. There is a unique anisotropic form of dimension 2, $N(\mathbb{F}_q)$. Here the underlying \mathbb{F}_q -vector space is the extension field \mathbb{F}_{q^2} and the quadratic form is the norm form given by $Q(x) := xx^q$ for all $x \in \mathbb{F}_{q^2}$.

Hence on a vector space $V = \mathbb{F}_q^{2m}$ of even dimension there are two non-isometric non-degenerate quadratic forms

$$Q_{2m}^+(q) := \mathbb{H}(\mathbb{F}_q)^m \text{ and } Q_{2m}^-(q) := \mathbb{H}(\mathbb{F}_q)^{m-1} \perp N(\mathbb{F}_q), \quad (1)$$

which we call of + type and of - type respectively.

Remark 2.1 The orthogonal sums of these forms behave as expected:

$$\begin{aligned} Q_{2m}^+(q) \perp Q_{2n}^+(q) &= Q_{2m}^-(q) \perp Q_{2n}^-(q) = Q_{2(m+n)}^+(q), \\ Q_{2m}^-(q) \perp Q_{2n}^+(q) &= Q_{2(m+n)}^-(q). \end{aligned}$$

Fact 2.2 (see for instance Kneser 2002, Kapitel IV)

- The Witt index, i.e. the dimension of a maximal totally isotropic subspace, of $Q_{2m}^+(q)$ is m and $Q_{2m}^-(q)$ has Witt index $m - 1$.
- The number of non-zero isotropic vectors in $Q_{2m}^+(q)$ is $(q^m - 1)(q^{m-1} + 1)$ and in $Q_{2m}^-(q)$ one gets $(q^m + 1)(q^{m-1} - 1)$.

Proposition 2.3 For any non-zero $\alpha \in \mathbb{F}_{q^m}$ the quadratic form

$$Q_\alpha : \mathbb{F}_{q^{2m}} \rightarrow \mathbb{F}_q, Q_\alpha(x) := \text{trace}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha x^{q^m+1})$$

is isometric to $Q_{2m}^-(q)$.

Proof We check that for $x, y \in \mathbb{F}_{q^{2m}}$

$$\begin{aligned} Q_\alpha(x+y) - Q_\alpha(x) - Q_\alpha(y) &= \text{trace}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(x^{q^m}\alpha y + y^{q^m}\alpha x) \\ &= \text{trace}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_q}(x^{q^m}\alpha y) \end{aligned}$$

by the transitivity of the trace. Thus the polarisation of Q_α is given by

$$B_\alpha(x, y) = \text{trace}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_q}(x^{q^m}\alpha y) \text{ for all } x, y \in \mathbb{F}_{q^{2m}}.$$

As the trace form of separable extensions is a non-degenerate bilinear form and the Galois automorphism $x \mapsto x^{q^m}$ of $\mathbb{F}_{q^{2m}}$ is bijective, also B_α is non-degenerate.

One way to see that Q_α is isometric to $Q_{2m}^-(q)$ is to count the number of isotropic vectors: The norm $N : \mathbb{F}_{q^{2m}} \rightarrow \mathbb{F}_{q^m}, x \mapsto x^{q^m}x$ is a surjective anisotropic quadratic form that restricts to a group epimorphism on the multiplicative groups. Hence for any $a \in \mathbb{F}_{q^m} \setminus \{0\}$ the number of $x \in \mathbb{F}_{q^{2m}} \setminus \{0\}$ with $N(x) = a$ is $q^m + 1$. The quadratic form Q_α is the composition of N with multiplication by α followed by the trace. The trace is an \mathbb{F}_q -linear surjective map from \mathbb{F}_{q^m} to \mathbb{F}_q , so the kernel of the trace is an

$(m - 1)$ -dimensional subspace of \mathbb{F}_{q^m} and, in particular, contains $q^{m-1} - 1$ non-zero elements. Consequently the number of isotropic vectors of Q_α is $(q^{m-1} - 1)(q^m + 1)$. \square

Proposition 2.4 *Let $Q : V \rightarrow \mathbb{F}_q$ be a non-degenerate quadratic form and $G \leq O(Q)$ an abelian subgroup of the orthogonal group of Q such that*

- (a) *The \mathbb{F}_q -algebra A spanned by the matrices in G is semi-simple, with $A = \bigoplus_{i=1}^n K_i$ for extension fields K_i of \mathbb{F}_q*
- (b) *All simple summands K_i are invariant under the adjoint involution of B_Q .*
- (c) *The restriction of this involution to K_i is non-trivial for all i .*

Then Q is of $+$ type if and only if the number of composition factors of the A -module V is even.

Proof The set of isomorphism classes of simple A -modules is $\{K_i \mid 1 \leq i \leq n\}$ and the A -module V is hence the direct sum $V \cong \bigoplus_{i=1}^n K_i^{d_i}$ for some $d_i \in \mathbb{N}$. As the adjoint involution fixes each primitive idempotent of A , the summands $K_i^{d_i}$ are pairwise orthogonal. The restriction of the involution to the simple summand K_i of A is the field automorphism F_i of order 2, so the bilinear form B_Q induces Hermitian forms on these orthogonal summands. Hence there are $\alpha_{i1}, \dots, \alpha_{id_i}$ in the fixed field of F_i such that

$$Q = \perp_{i=1}^n \perp_{j=1}^{d_i} Q_{\alpha_{ij}}$$

for quadratic forms $Q_{\alpha_{ij}} : K_i \rightarrow \mathbb{F}_q$ as in Proposition 2.3. As these are of $-$ type, the statement follows by applying the addition formulas from Remark 2.1. \square

We remark that the assumption from Proposition 2.4 is equivalent to the assumption that the representation of G on V is orthogonally stable in the sense of Nebe and Parker (2023, Definition 5.12). In the language of Nebe and Parker (2023) the statement of Proposition 2.4 can also be deduced from Nebe and Parker (2023, Proposition 3.12).

3 The orthogonal representations of $\mathrm{SL}_2(q)$

In this section we fix the following notation:

- p is a prime, $q := p^f$, where f is a positive integer

$$\mathrm{SL}_2(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{F}_q^{2 \times 2} \mid ad - bc = 1 \right\}$$

is the group of determinant 1 matrices over the finite field with q elements.

- $V := \mathbb{F}_q^2$ is the natural $\mathbb{F}_q \mathrm{SL}_2(q)$ -module.

3.1 The irreducible modules and their fields of definition

The irreducible modules of $\mathrm{SL}_2(q)$ in defining characteristic are already described in Brauer and Nesbitt (1941, p. 588–589). We give two facts that can be found in van Ham et al. (1982, Section 1.8).

Fact 3.1 *The irreducible $\mathbb{F}_p \mathrm{SL}_2(p)$ -modules are given by W_0, \dots, W_{p-1} , where*

$$W_k := \mathrm{Sym}_k(V) = \mathbb{F}_p[x, y]_{\deg=k}$$

is the space of homogeneous polynomials on V of degree k . All W_k are absolutely irreducible and the dimension of W_k is $k + 1$.

For arbitrary $f \in \mathbb{N}$ we know that \mathbb{F}_q is a splitting field for $\mathrm{SL}_2(q)$ and the irreducible $\mathbb{F}_q \mathrm{SL}_2(q)$ -modules are given by Steinberg's tensor product theorem: The Galois group $\mathrm{Gal}(\mathbb{F}_{p^f}/\mathbb{F}_p) = \langle F \rangle$ acts on the group $\mathrm{SL}_2(q)$ by applying the Galois automorphism to the entries of the matrices. For $0 \leq i \leq f-1$ let $V^{[i]}$ denote the natural $\mathbb{F}_q \mathrm{SL}_2(q)$ -module V where the action is twisted by F^i and $W_k^{[i]} := \mathrm{Sym}_k(V^{[i]})$.

Fact 3.2 *The irreducible $\mathbb{F}_q \mathrm{SL}_2(q)$ -modules are given by*

$$W(\mathbf{k}) = W(k_0, \dots, k_{f-1}) = W_{k_0}^{[0]} \otimes \dots \otimes W_{k_{f-1}}^{[f-1]}$$

for $\mathbf{k} := (k_0, \dots, k_{f-1}) \in \{0, \dots, p-1\}^f$. The $W(\mathbf{k})$ are pairwise non-isomorphic, absolutely irreducible and of dimension $\dim(W(k_0, \dots, k_{f-1})) = \prod_{i=0}^{f-1} (k_i + 1)$.

The action of the Galois group on these irreducible modules is given by cyclic permutation:

$$W(k_0, \dots, k_{f-1})^F \cong W(k_{f-1}, k_0, \dots, k_{f-2}).$$

As the modules $W(\mathbf{k})$ are pairwise non-isomorphic, the representation on $W(\mathbf{k})$ can be realised over the fixed field of F^ℓ if and only if

$$(k_0, \dots, k_{f-1}) = (k_{f-\ell}, k_{f-\ell+1}, \dots, k_{f-\ell-1}).$$

Remark 3.3 Let $\ell \geq 1$ be minimal such that

$$\mathbf{k} := (k_0, \dots, k_{f-1}) = (k_{f-\ell}, k_{f-\ell+1}, \dots, k_{f-\ell-1}).$$

Then ℓ divides f . Put $\mathbb{F}(\mathbf{k}) := \mathbb{F}_{p^\ell}$ to be the fixed field of F^ℓ in \mathbb{F}_q . Then $\mathbb{F}(\mathbf{k})$ is the field of definition of the module $W(\mathbf{k})$.

By abuse of notation we denote the corresponding $\mathbb{F}(\mathbf{k}) \mathrm{SL}_2(q)$ -module again by $W(\mathbf{k})$.

3.2 Invariant quadratic forms

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{F}_q^{2 \times 2}$ and $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we have $AJA^{tr} = \det(A)J$, so the natural $\mathbb{F}_q \mathrm{SL}_2(q)$ -module $V = \mathbb{F}_q^2$ carries an $\mathrm{SL}_2(q)$ -invariant non-degenerate alternating bilinear form. This yields a non-degenerate $\mathrm{SL}_2(q)$ -invariant bilinear form B_k on the space of homogeneous polynomials

$$B_k : \mathbb{F}_q[x, y]_{deg=k} \times \mathbb{F}_q[x, y]_{deg=k} \rightarrow \mathbb{F}_q : B_k(g, h) := g \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right) (h(x, y)).$$

The form B_k is symmetric if k is even and alternating if k is odd.

Remark 3.4 There is a special case for $q = 2$. Here V carries a non-degenerate quadratic form of $-$ type and $\mathrm{SL}_2(2) \leq O_2^-(2)$.

Remark 3.5 (see Sin and Willems 1991, Proposition 3.4, Garibaldi and Nakano 2016, Proposition 9.1.2) Let G be a group, K a field, and let (V, B) and (W, B') be G -invariant non-degenerate alternating bilinear forms on the KG -modules V and W . Then

$$Q : V \otimes W \rightarrow K, \quad Q \left(\sum_{i=1}^n v_i \otimes w_i \right) := \sum_{i < j} B(v_i, v_j) B'(w_i, w_j)$$

is a G -invariant quadratic form on $V \otimes W$ with polarisation $B \otimes B'$. If $U \leq V$ is an isotropic subspace for the bilinear form B , i.e. $B(U, U) = \{0\}$, then $U \otimes W$ is a totally isotropic subspace for the quadratic form Q , i.e. $Q(U \otimes W) = \{0\}$. In particular, the Witt index of Q is $m := \dim(V \otimes W)/2$. If $K = \mathbb{F}_q$ is a finite field, this shows that Q is isometric to $Q_{2m}^+(q)$.

Proposition 3.6 Assume that $q \neq 2$. Let $\mathbf{k} := (k_0, \dots, k_{f-1})$ and $\mathbb{F}(\mathbf{k})$ be the field of definition of $W(\mathbf{k})$ as in Remark 3.3 and put $e(\mathbf{k}) := |\{i \mid k_i \text{ is odd}\}|$. Then the $\mathbb{F}(\mathbf{k}) \mathrm{SL}_2(q)$ -module $W(\mathbf{k})$ carries a non-degenerate $\mathrm{SL}_2(q)$ -invariant quadratic form $Q_{\mathbf{k}}$ if and only if either

- (i) q is odd and $e(\mathbf{k})$ is even, or
- (ii) q is even and $\dim(W(\mathbf{k})) \geq 4$.

If $[\mathbb{F}_q : \mathbb{F}(\mathbf{k})]$ is odd, then $Q_{\mathbf{k}}$ has maximal Witt index and hence is of $+$ type.

Proof (i) First assume that q is odd and recall that we then may work with symmetric bilinear forms instead of quadratic forms. As $W(\mathbf{k})$ is absolutely irreducible any $\mathrm{SL}_2(q)$ -invariant bilinear form is a scalar multiple of $B_{\mathbf{k}} := B_{k_0} \otimes \dots \otimes B_{k_{f-1}}$. This form is symmetric and hence the polarisation of an invariant quadratic form $Q_{\mathbf{k}}$ if and only if $e(\mathbf{k})$ is even.

- (ii) Now assume that q is even. If $W(\mathbf{k})$ is a proper tensor product, then Remark 3.5 shows that there is a non-degenerate invariant quadratic form $Q_{\mathbf{k}}$ on $W(\mathbf{k})$.

Otherwise $\dim(W(\mathbf{k})) = 2$. Both orthogonal groups $O_2^+(\mathbb{F}_q)$ and $O_2^-(\mathbb{F}_q)$ of dimension 2 are solvable, but $\mathrm{SL}_2(q)$ is not solvable for $q \geq 4$, so there cannot be an invariant quadratic form on $W(\mathbf{k})$ if $\dim(W(\mathbf{k})) = 2$ and $q \geq 4$.

In both cases (q even or odd) Remark 3.5 states that, after extending scalars to the field \mathbb{F}_q , the invariant quadratic form $\mathbb{F}_q \otimes Q_{\mathbf{k}}$ is of maximal Witt index. As odd degree extensions do not change the type of a quadratic form (see for instance Sin and Willems 1991, Proposition 4.9) the type of $\mathbb{F}_q \otimes Q_{\mathbf{k}}$ and $Q_{\mathbf{k}}$ is the same, if $[\mathbb{F}_q : \mathbb{F}(\mathbf{k})]$ is odd.

□

3.3 The type of $Q_{\mathbf{k}}$

In this section we determine the type of the $\mathrm{SL}_2(q)$ -invariant quadratic forms on the simple $\mathrm{SL}_2(q)$ -module $W(\mathbf{k})$ over its field of definition $\mathbb{F}(\mathbf{k})$. We also assume that $W(\mathbf{k})$ is an orthogonal $\mathrm{SL}_2(q)$ -module, i.e. that \mathbf{k} satisfies the conditions of Proposition 3.6 and that the dimension of $W(\mathbf{k})$ is even, i.e. that at least one of the entries of \mathbf{k} is odd.

The case $q = 2$, where $\mathrm{SL}_2(2) \cong S_3$ is a group of order 6, is given in Remark 3.4, so we assume that $q \geq 3$ throughout this section.

Theorem 3.7 *Let $Q_{\mathbf{k}} : W(\mathbf{k}) \rightarrow \mathbb{F}(\mathbf{k})$ be a non-degenerate $\mathrm{SL}_2(q)$ -invariant quadratic form. Then $Q_{\mathbf{k}}$ is of $-$ type if and only if $\dim(W(\mathbf{k})) \equiv 4 \pmod{8}$ and $[\mathbb{F}_q : \mathbb{F}(\mathbf{k})] = 2$.*

Proof Proposition 3.6 proves Theorem 3.7 in the case that $[\mathbb{F}_q : \mathbb{F}(\mathbf{k})]$ is odd so it remains to consider the case where this degree is even, i.e. f is even and

$$\mathbf{k} = (k_0, \dots, k_{f/2-1}, k_0, \dots, k_{f/2-1})$$

where at least one of the k_i is odd. In this case we show that the non-split torus T of $\mathrm{SL}_2(q)$ acts on $W(\mathbf{k})$ such that the image A of $\mathbb{F}(\mathbf{k})T$ in $\mathrm{End}(W(\mathbf{k}))$ is a semi-simple subalgebra that is a direct sum of even degree extension fields of $\mathbb{F}(\mathbf{k})$. Then Proposition 2.4 allows us to conclude that the type of $Q_{\mathbf{k}}$ is $-$ if and only if the number of composition factors of the A -module $W(\mathbf{k})$ is odd.

Let $t \in \mathrm{SL}_2(q)$ denote an element of order $q + 1$. Let $\tau, \tau^q \in \mathbb{F}_{q^2}$ denote the two eigenvalues of t on the natural $\mathrm{SL}_2(q)$ -module $V = \mathbb{F}_q^2$.

Lemma 3.8 *Let $\mathbf{k} = (k_0, \dots, k_{f-1}) \in \{0, \dots, p-1\}^f$ and put $s(\mathbf{k}) := \sum_{i=0}^{f-1} k_i p^i$. Then $s(\mathbf{k}) \leq p^f - 1$. The eigenvalues of t on $W(\mathbf{k})$ are exactly the elements τ^e with*

$$e \in E(\mathbf{k}) := \left\{ s(\mathbf{k}) - 2 \sum_{i=0}^{f-1} x_i p^i \mid x_i \in \{0, \dots, k_i\} \right\} \subseteq \{-s(\mathbf{k}), \dots, s(\mathbf{k})\}.$$

Proof After extending the field to \mathbb{F}_{q^2} we choose a basis of V consisting of eigenvectors of t . Then the monomials in $W_k^{[i]}$ are eigenvectors of t where the eigenvalue of $x^{k-j} y^j$

is τ^e with $e = (k - 2j)p^i$. Therefore the eigenvalues of t on $W(\mathbf{k})$ are the elements τ^e where

$$e \in E(\mathbf{k}) := \left\{ \sum_{i=0}^{f-1} m_i p^i \mid -k_i \leq m_i \leq k_i, k_i - m_i \text{ even} \right\}.$$

Replacing m_i by $k_i - 2x_i$ yields the description in the lemma. \square

Lemma 3.9 *We have $0 \in E(\mathbf{k})$ if and only if all k_i are even.*

If p is odd, f is even, and one of $\pm(p^f + 1)/2 \in E(\mathbf{k})$ then $s(\mathbf{k})$ is odd.

Proof If $0 \in E(\mathbf{k})$ then there are $x_i \in \{0, \dots, k_i\}$ such that $\sum_{i=0}^{f-1} k_i p^i = \sum_{i=0}^{f-1} 2x_i p^i$. Taking the equation mod p , we get that $2x_0 \equiv_p k_0$. As $2x_0 \in \{0, 2, \dots, 2k_0\}$ and $k_0 < p$, we hence have $2x_0 = k_0$ so k_0 is even and $x_0 = k_0/2$. Continuing like this, we obtain that $x_i = k_i/2$ for all i . Now assume that p is odd, f is even, and $\pm(p^f + 1)/2 \in E(\mathbf{k})$. Then $s(\mathbf{k}) = 2 \sum_{i=0}^{f-1} x_i p^i \pm (p^f + 1)/2$. As f is even, $(p^f + 1)/2$ is odd and so is $s(\mathbf{k})$. \square

To finish the proof of Theorem 3.7 note that $s(\mathbf{k}) = (1 + p^{f/2}) \sum_{i=0}^{f/2-1} k_i p^i$ is even if p is odd. The assumption that $\dim(W(\mathbf{k})) = \prod_{i=0}^{f/2-1} (k_i + 1)^2$ is even implies that at least one of the k_i is odd. Hence Lemma 3.9 shows that t has no eigenvalues ± 1 on $W(\mathbf{k})$. Now the order of t is $p^f + 1$. As $\gcd(p^f - 1, p^f + 1) = 2$ (or 1) each eigenvalue of t that does not equal ± 1 generates a quadratic extension of \mathbb{F}_q . Let $A := \mathbb{F}(\mathbf{k})[t] \leq \text{End}(W(\mathbf{k}))$ be the $\mathbb{F}(\mathbf{k})$ -subalgebra generated by the endomorphism t of $W(\mathbf{k})$. Then $A = \bigoplus_{i=1}^n K_i$ is semi-simple and commutative. As the adjoint involution of B_Q inverts the elements of $O(Q)$ and inverting the eigenvalues of t is non-trivial on K_i , this involution is the field automorphism of order 2 on each of the K_i . To apply Proposition 2.4 it is hence enough to determine the parity of the number of composition factors of the A -module $W(\mathbf{k})$. Write $d := [\mathbb{F}_q : \mathbb{F}(\mathbf{k})] = 2^a b$ with $a \geq 1$ and b odd. Then

$$2^{a+1} \text{ is the 2-part of } [K_i : \mathbb{F}(\mathbf{k})] \text{ for all } i \quad (2)$$

because the subfields of 2-power degree in \mathbb{F}_{q^2} are linearly ordered by inclusion.

As \mathbf{k} consists of the d -fold juxtaposition of a sequence of length f/d and one of the k_i is odd, at least d of the k_i are odd and hence

$$\dim(W(\mathbf{k})) = \prod_{i=0}^{f-1} (k_i + 1) \quad (3)$$

is divisible by 2^d .

Accordingly, the number of composition factors of the A -module $W(\mathbf{k})$ is odd, if and only if 2^{a+1} is the maximal 2-power that divides $\dim(W(\mathbf{k}))$. Then

$$a + 1 \geq d = 2^a b,$$

which implies that $a = 1 = b$, i.e. $d = 2$, so $\mathbb{F}(\mathbf{k}) = \mathbb{F}_{p^{f/2}}$. Moreover $\dim(W(\mathbf{k})) \equiv 4 \pmod{8}$. \square

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