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Error Estimates for First- and Second-Order Lagrange–Galerkin Moving Mesh Schemes for the One-Dimensional Convection–Diffusion Equation

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Abstract

A new moving mesh scheme based on the Lagrange–Galerkin method for the approximation of the one-dimensional convection–diffusion equation is studied. The mesh movement is prescribed by a discretized dynamical system for the nodal points. This system is related to the velocity and diffusion coefficient in the convection–diffusion equation such that the nodal points follow the convective flow of the model. It is shown that under a restriction of the time step size the mesh movement cannot lead to an overlap of the elements and therefore an invalid mesh. Using a piecewise linear approximation, optimal error estimates in the $\ell^\infty(L^2)\cap\ell^2(H_0^1)$ norm are proved in case of both, a first-order backward Euler method and a second-order two-step method in time. These results are based on new estimates of the time dependent interpolation operator derived in this work. Preservation of the total mass is verified for both choices of the time discretization. Numerical experiments are presented that confirm the error estimates and demonstrate that the proposed moving mesh scheme can circumvent limitations that the Lagrange–Galerkin method on a fixed mesh exhibits.

Keywords Finite element method · Lagrange–Galerkin scheme · Moving mesh method · Error estimate · Convection–diffusion system

Mathematics Subject Classification 65M15 · 65M25 · 65M50

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1 Introduction

Apart from their application in fluid flow problems convection–diffusion equations have recently been extensively used in the modeling of chemical and biological processes such as pollutant transport, immune system dynamics and cancer growth, see e.g. [2, 28]. In many of these applications, migration of cells or transport of solutes in fluids are involved, which leads to convection-dominant problems. In these cases the Galerkin finite element scheme can produce oscillating solutions. Hence, a plethora of extended and alternative numerical methods have been developed to perform stable computation, e.g., upwind methods [3, 23, 34] and characteristics methods [17, 18, 30, 35]. Among the latter ones the Lagrange–Galerkin (LG) method has been shown to be an effective and efficient method to deal with convection-dominated problems, see for example [4–6, 19, 20, 31, 32]. Not only the convection-dominant nature but also the rich dynamics of the biological applications, which include traveling waves and aggregation phenomena, pose a challenge for the numerical schemes. Different approaches related to mesh adaptation have been recently proposed to improve the accuracy in these cases, including a mass-transport approach for the one-dimensional problem [12] and adaptive mesh refinement [1, 25].

The application of the LG method to convection–diffusion problems offers the advantage that the time step size is not constrained by traditional CFL-type conditions. However, it is important to note that while a time step restriction is indeed necessary, this restriction is not directly proportional to the spatial discretization, as is typically the case with CFL conditions. This flexibility has motivated the use of the LG method in solving the Navier–Stokes equations and models of viscoelastic fluid flow, to name only a few [26, 27, 29]. Various LG methods have been proposed for convection–diffusion problems: Some have been concerned with higher order approximations in time using single- and multi-step methods, e.g., [4, 5, 7]. Others have focused on maintaining the mass balance on the discrete level, e.g. [16, 32]. In [20, 32] such mass-preserving LG schemes of first- and second-order in time have been proposed and error estimates have been provided. As the LG scheme relies on an upwind-interpolation of the numerical solution that follows the velocity field backwards in time a promising approach is to introduce mesh movement along the velocity field. From a computational point of view this might ease the identification of the upwind points and reduce the interpolation error.

Different kinds of moving mesh methods have been considered to numerically solve convection–diffusion problems. A common approach is the redistribution of mesh cells according to a monitor function that depends on local features of the numerical solution or an a posteriori error estimator, see e.g. [1]. In other approaches separate moving mesh PDEs and transformations obtained from the solution of the Monge–Kantorovich problem are used, see [22, 33]. In the context of hyperbolic balance laws and fluid dynamics a variety of schemes, which also entail mesh movement, has been derived from the Lagrangian formulation of the problem, such as the hydrodynamic *GLACE* scheme [11] and the arbitrary Lagrangian–Eulerian finite element method [13, 36]. In this context high order of accuracy has been achieved by adopting high order essentially non-oscillatory reconstructions, see e.g., [9, 14]. In some applications error estimates have been derived taking the movement of the mesh into account, e.g. [15, 21]. We consider here a Lagrangian grid approach based on a dynamical system for the nodal points that both exhibits significant benefits over static grids in numerical simulations and allows for an error analysis of the extended LG scheme.

In this work we are concerned with a moving mesh approach within LG schemes of first- and second-order in time and corresponding error estimates. We introduce the Lagrange–

Galerkin Moving Mesh (LGMM) schemes, which combine the LG schemes derived in [20, 32] with a moving mesh, in order to improve the performance and efficiency over the original LG schemes with fixed mesh. The mesh movement we consider is inspired by [12]; although we focus on the one-dimensional case in this work our mesh movement is expressed in a form suitable for higher dimensional cases. We derive a condition under which the mesh movement is applicable. We generalize the mass-conservation property and stability results from the static grid versions of the schemes to the new LGMM schemes. Based on the idea of temporal derivatives on deforming grids in [24] we derive bounds for the time derivative of the dynamic interpolation operator, which then allow us to prove optimal error estimates for the LGMM scheme on piecewise linear elements in the $\ell^\infty(L^2) \cap \ell^2(H_0^1)$ norm. Moreover, we present numerical experiments that verify the error estimates. They further show that in case of aggregation the LGMM method eliminates oscillations of the numerical solution that the LG method produces.

The rest of the paper is organized as follows: In Sect. 2 we present the mass-preserving LG schemes of first- and second-order in time for the convection–diffusion problem. This scheme is equipped with a mesh moving technique in Sect. 3, for which we state various properties. In Sects. 4 and 5, we provide the main results concerned with the mass-conservation property, the stability, and the error estimates for the schemes of order one and two, respectively, which are afterwards proven in Sect. 6. To show the advantages of the LGMM schemes, two numerical simulations are given in Sect. 7, followed by the conclusions in Sect. 8.

2 Lagrange–Galerkin Schemes

2.1 Statement of the Problem

Let $\Omega = (a, b)$ be a bounded interval in $\mathbb{R}$. We denote by $\Gamma := \partial\Omega$ the two point boundary of Ω and by T a positive constant. In this paper we use the Lebesgue spaces $L^2(\Omega)$, $L^\infty(\Omega)$ and the Sobolev spaces $W^{m,p}(\Omega)$, $W_0^{1,\infty}(\Omega)$, $H^m(\Omega)$, $H_0^1(\Omega)$, for $m \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$. We use the notation $(\cdot, \cdot)$ to represent the $L^2(\Omega)$ inner products for both scalar and vector-valued functions. The norm in $L^2(\Omega)$ is simply denoted as $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$. For any normed space Y with norm $\|\cdot\|_Y$, we define the function spaces $H^m(0, T; Y)$ and $C^0(0, T; Y)$ consisting of Y-valued functions in $H^m(0, T)$ and $C^0([0, T])$, respectively. For the two real numbers $t_0 < t_1$ we introduce the function space

$$Z^m(t_0, t_1) := \{\psi \in C^j(t_0, t_1; H^{m-j}(\Omega));\ j = 0, \ldots, m,\ \|\psi\|_{Z^m(t_0,t_1)} < \infty\},$$

with the norm

$$\|\cdot\|_{Z^m(t_0,t_1)} := \left(\sum_{j=1}^{m} \|\cdot\|^2_{C^j(t_0,t_1;H^{m-j}(\Omega))} \right)^{1/2},$$

and set $Z^m := Z^m(0, T)$. We often omit Ω and $[0, T]$ if there is no confusion and write, e.g., $C^0(L^\infty)$ in place of $C^0([0, T]; L^\infty(\Omega))$. Although we are concerned with a one-dimensional domain we use the general notations $\nabla := \partial_x$, $\nabla\cdot := \partial_x$, $\Delta := \partial_x^2$, and $\frac{\partial}{\partial n} := n\partial_x$ to refer to spatial derivatives in order to allow for a straightforward application of the multi-dimensional theory for LG schemes. We use c and C (with or without subscript or superscript) to denote generic positive constant independent of discretization parameters and solutions.

We consider a convection–diffusion problem, in which we aim to find $\phi : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (u\phi) - \nu \Delta \phi = f \quad \text{in } \Omega \times (0, T), \tag{1a}$$

$$\nu \frac{\partial \phi}{\partial n} - \phi u \cdot n = g \quad \text{on } \Gamma \times (0, T), \tag{1b}$$

$$\phi = \phi^0 \quad \text{in } \Omega, \text{at } t = 0, \tag{1c}$$

where $u : \Omega \times (0, T) \to \mathbb{R}$, $f : \Omega \times (0, T) \to \mathbb{R}$, $g : \Gamma \times (0, T) \to \mathbb{R}$ and $\phi^0 : \Omega \to \mathbb{R}$ are given functions, $n : \Gamma \to \{\pm 1\}$ is the outward unit normal vector and $\nu > 0$ is the diffusion coefficient.

Let $\Psi := H^1(\Omega)$ and Ψ' be the dual space of Ψ. The weak formulation corresponding to problem (1) is to find $\{\phi(t) = \phi(\cdot, t) \in \Psi; t \in (0, T)\}$ such that for $t \in (0, T)$ the variational equality

$$\left(\frac{\partial \phi}{\partial t}(t), \psi \right) + a_0(\phi(t), \psi) + a_1(\phi(t), \psi; u(t)) = \langle F(t), \psi \rangle, \quad \forall \psi \in \Psi \tag{2}$$

holds in addition to $\phi(0) = \phi^0$. The bilinear forms $a_0(\cdot, \cdot)$ and $a_u(\cdot, \cdot) = a_1(\cdot, \cdot; u)$ and for $t \in (0, T)$ in (2) are defined such that by

$$a_1(\phi, \psi; u) := -(\phi, u \nabla \psi)$$

and $F(t) \in \Psi'$ is a functional given by

$$\langle F(t), \psi \rangle = (f(t), \psi) + [g(t), \psi]_\Gamma,$$
$$[g(t), \psi]_\Gamma := \int_\Gamma g(t)\psi \, ds = g(a, t)\psi(a) + g(b, t)\psi(b),$$

for $f(t) = f(\cdot, t) \in L^2(\Omega)$ and $g(t) = g(\cdot, t) \in L^2(\Gamma)$.

By substituting $1 \in \Psi$ into ψ in (2), and integrating over $(0, t)$, we derive the mass balance identity

$$\int_\Omega \phi(x, t)\, dx = \int_\Omega \phi^0(x)\, dx + \int_0^t \int_\Omega f(x, \tau) dx\, d\tau + \int_0^t \int_\Gamma g(s, \tau) ds\, d\tau \tag{3}$$

that holds for all $t \in (0, T)$. This property is desired to be maintained also at the discrete level, which is indeed achieved by the Lagrange–Galerkin schemes of first- and second-order in time proposed in [32] and [20], respectively.

2.2 The First-Order Lagrange–Galerkin Scheme

Let $\Delta t > 0$ be a time step size, $t^n := n\Delta t$ for $n \in \mathbb{N} \cup \{0\}$ an equidistant discretization of the time domain and $N_T := \lfloor T/\Delta t \rfloor$. For a function ρ defined in $\Omega \times (0, T)$ and $0 \le t^n \le T$ the function $\rho(\cdot, t^n)$ in Ω is denoted by ρ^n. Let $\mathcal{T}_h^n := \{K^n\}$, $n \in \{0, \ldots, N_T\}$ be a time-dependent partition of $\bar{\Omega}$ with K^n representing an element in $\mathcal{T}_h^n$. Let $h := \max_{n=0,\ldots,N_T} \max_{K^n \in \mathcal{T}_h^n} \operatorname{diam}(K^n)$ denote the global mesh size. Let $\Psi_h^n \subset \Psi$ be a time varying finite element space defined by

$$\Psi_h^n := \left\{ \psi_h \in C^0(\bar{\Omega});\ \psi_{h|K^n} \in P_1(K^n), \forall K^n \in \mathcal{T}_h^n \right\},$$

where $P_1(K^n)$ is the space of linear polynomial functions on $K^n \in \mathcal{T}_h^n$. Details about the spaces Ψ_h^n and $\mathcal{T}_h^n$ are discussed in Sect. 3.1, the rest of this section as well as Sect. 2.3 address the first- and second-order schemes assuming Ψ_h^n is known.

For a given velocity $v : \Omega \to \mathbb{R}$, we define the upwind point of x with respect to v and Δt using the mapping $X_1(v, \Delta t) : \Omega \to \mathbb{R}$,

$$X_1(v, \Delta t)(x) := x - v(x)\Delta t.$$

With respect to the velocity u we define the mapping $X_1^n : \Omega \to \mathbb{R}$ and its corresponding Jacobian $\gamma^n : \Omega \to \mathbb{R}$ by

$$X_1^n(x) := X_1(u^n, \Delta t)(x) = x - u^n(x)\Delta t, \qquad \gamma^n(x) := \det\left(\frac{\partial X_1^n}{\partial x}(x)\right).$$

Suppose an approximate function $\phi_h^0 \in \Psi_h^0$ of ϕ^0 is given. In the first-order Lagrangian moving mesh scheme we look for $\{\phi_h^n \in \Psi_h^n;\ n = 1, \ldots, N_T\}$ such that for $n = 1, \ldots, N_T$ it holds

$$\left(\frac{\phi_h^n - \phi_h^{n-1} \circ X_1^n \gamma^n}{\Delta t}, \psi_h\right) + a_0(\phi_h^n, \psi_h) = \langle F^n, \psi_h\rangle, \quad \forall \psi_h \in \Psi_h^n. \tag{4}$$

The functional $F^n \in (\Psi_h^n)'$ on the right hand side of (4) is defined by

$$\langle F^n, \psi_h\rangle := (f^n, \psi_h) + [g^n, \psi_h]_\Gamma.$$

2.3 The Second-Order Lagrange–Galerkin Scheme

To obtain a higher order discretization in time we define the additional mapping $\tilde{X}_1^n : \Omega \to \mathbb{R}$ and its Jacobian $\tilde{\gamma}^n : \Omega \to \mathbb{R}$ by

$$\tilde{X}_1^n(x) := X_1(u^n, 2\Delta t)(x) = x - 2u^n(x)\Delta t, \qquad \tilde{\gamma}^n(x) := \det\left(\frac{\partial \tilde{X}_1^n}{\partial x}(x)\right).$$

Suppose an approximation $\phi_h^0 \in \Psi_h^0$ of ϕ^0 is given. Then the second-order Lagrangian moving mesh scheme aims to find $\{\phi_h^n \in \Psi_h^n;\ n = 1, \ldots, N_T\}$ satisfying

$$\left(\frac{\phi_h^n - \phi_h^{n-1} \circ X_1^n \gamma^n}{\Delta t}, \psi_h\right) + a_0(\phi_h^n, \psi_h) = \langle F^n, \psi_h\rangle, \quad \forall \psi_h \in \Psi_h^n, \quad n = 1, \tag{5a}$$

$$\left(\frac{3\phi_h^n - 4\phi_h^{n-1} \circ X_1^n \gamma^n + \phi_h^{n-2} \circ \tilde{X}_1^n \tilde{\gamma}^n}{2\Delta t}, \psi_h\right) + a_0(\phi_h^n, \psi_h) = \langle F^n, \psi_h\rangle,$$
$$\forall \psi_h \in \Psi_h^n, \quad n \geq 2. \tag{5b}$$

In the following, we rewrite scheme (5) as

$$(\mathcal{A}_{\Delta t}\phi_h^n, \psi_h) + a_0(\phi_h^n, \psi_h) = \langle F^n, \psi_h\rangle, \ \ \forall \psi_h \in \Psi_h^n, \tag{6}$$

where, for a series $\{\rho^n\}_{n=0}^{N_T} \subset \Psi$, the function $\mathcal{A}_{\Delta t}\rho^n : \Omega \to \mathbb{R}$ is given by

$$\mathcal{A}_{\Delta t}\rho^n := \begin{cases} \mathcal{A}_{\Delta t}^{(1)}\rho^n := \dfrac{1}{\Delta t}\left(\rho^n - \rho^{n-1} \circ X_1^n \gamma^n\right), & n = 1, \\ \mathcal{A}_{\Delta t}^{(2)}\rho^n := \dfrac{1}{2\Delta t}\left(3\rho^n - 4\rho^{n-1} \circ X_1^n \gamma^n + \rho^{n-2} \circ \tilde{X}_1^n \tilde{\gamma}^n\right), & n \geq 2, \end{cases}$$

We introduce some discrete norms in the following. Let Y be a normed space, $m \in \{0, \dots, N_T\}$ be an integer, and $\{\rho^n\}_{n=0}^{N_t} \subset Y$. We define the norms $\|\cdot\|_{\ell_m^\infty(Y)}$ and $\|\cdot\|_{\ell_m^2(Y)}$ by

$$\|\rho\|_{\ell_m^\infty(Y)} := \max_{n=m,\dots,N_T} \|\rho^n\|_Y, \quad \|\rho\|_{\ell_m^2(Y)} := \left(\Delta t \sum_{n=m}^{N_T} \|\rho^n\|_Y^2\right)^{1/2}.$$

Additionally, we define the following norm over the time varying finite element spaces

$$\|\rho\|_{\ell_m^2(\Psi_h')} := \left(\Delta t \sum_{n=m}^{N_T} \|\rho^n\|_{(\Psi_h^n)'}^2\right)^{1/2}.$$

If there is no confusion we omit the subscript i.e, $\|\rho\|_{\ell_1^\infty(Y)} =: \|\rho\|_{\ell^\infty(Y)}$ and $\|\rho\|_{\ell_1^2(Y)} =: \|\rho\|_{\ell^2(Y)}$.

3 Moving Mesh Method for the Lagrange–Galerkin Scheme

3.1 Moving Mesh

In this section the construction and evolution of the partitions $\mathcal{T}_h^n$ is considered. To this end a moving mesh is employed that in this work is defined as follows.

Definition 1 For a given partition $\{t^n : n = 0, \dots, N_T\}$ of the time domain $[0, T]$ a *moving mesh* of $\Omega \times [0, T]$ is a set of points $\{P_i^n : i = 1, \dots, N_p, n = 0, \dots, N_T\} \subset \bar{\Omega}$ that satisfy the monotonicity condition

$$a = P_1^n < P_2^n < \cdots < P_{N_p}^n = b, \qquad n \in \{0, \dots, N_T\}. \tag{7}$$

We refer to $N_p \in \mathbb{N}$ as the number of moving mesh points.

A moving mesh allows us to define the partitions introduced in Sect. 2.2 more precisely as

$$\mathcal{T}_h^n = \{K_i^n : i = 1, \dots, N_p - 1\}, \qquad K_i^n := [P_i^n, P_{i+1}^n], \qquad n \in \{0, \dots, N_T\}$$

and therefore determines the nodal points of the finite element spaces Ψ_h^n for $n \in \{0, \dots, N_T\}$. We note that for any fixed $i \in \{1, \dots, N_p\}$ the series $\{P_i^n\}_{n=0}^{N_T}$ can be considered a time discrete trajectory of the moving point P_i. For our analysis we define the velocities of the moving mesh points as

$$w_i^n := \frac{P_i^n - P_i^{n-1}}{\Delta t}, \qquad i \in \{1, \dots, N_p\}, \quad n \in \{1, \dots, N_t\}, \tag{8}$$

which allow us to introduce the time continuous point trajectories

$$P_i(t) := P_i^{n-1} + w_i^n (t - t^{n-1}), \qquad i \in \{1, \dots, N_p\}, \quad t \in [t^{n-1}, t^n]. \tag{9}$$

In addition, for $t \in [0, T]$ we define $w(x, t)$ as an extension of w_i^n, by

$$w(x, t) := \frac{P_{i+1}(t) - x}{P_{i+1}(t) - P_i(t)} w_i^n + \frac{x - P_i(t)}{P_{i+1}(t) - P_i(t)} w_{i+1}^n,$$
$$x \in [P_i(t), P_{i+1}(t)], \quad t \in [t^{n-1}, t^n]. \tag{10}$$

Note that $w \in C^0([0, T]; W_0^{1,\infty}(\Omega))$. Also the basis functions of the finite element spaces Ψ_h^n can be naturally extended using the trajectories (9): for $i \in \{1, \ldots, N_p\}$ let $\psi_i(\cdot, t)$ denote the unique function on Ω that is affine linear restricted to the intervals $[P_j(t), P_{j+1}(t)]$ for $j \in \{1, \ldots, N_p - 1\}$ and satisfies $\psi_i(P_j(t), t) = \delta_{ij}$. Then clearly $\{\psi_i(\cdot, t^n) : i = 1, \ldots, N_p\}$ is a basis of Ψ_h^n.

3.2 Moving Mesh Method

In this section we propose a *moving mesh method* that is used to obtain a moving mesh in the sense of Definition 1 and therefore determines the finite element spaces Ψ_h^n as described in Sect. 3.1. Suppose that the points $P_1^0, \ldots, P_{N_P}^0$ are given and the monotonicity condition (7) is satisfied for $n = 0$. The method we propose determines the position of the points P_i^n iteratively by employing a time discretization of the dynamical system

$$\frac{d\tilde{P}_i}{dt}(t) = u\big(\tilde{P}_i(t), t\big) + \nu_M \sum_{j=1}^{N_p-1} \nabla\psi_i(t)|_{[\tilde{P}_j(t), \tilde{P}_{j+1}(t)]} \tag{11}$$

with initial data $\tilde{P}_i(0) = P_i^0$ for $i \in \{1, \ldots, N_p\}$. Here, ∇ refers to the gradient with respect to the spatial variables, and the parameter $\nu_M \geq 0$ accounts for regularization of the moving mesh. The dynamical system (11) generalizes the mass transport approach from [8, 12]: If $d = 1$, $f = 0$ and $\nu_M = \nu$ hold it provides a semi-discrete scheme for the convection–diffusion equation (1a) in terms of the inverse cumulative distribution function of the state ϕ. Applying it to the nodal points and assuming an exact solution of (1a) yields an equidistribution of the mass of the solution, i.e. $\int_{\tilde{P}_i(t)}^{\tilde{P}_{i+1}(t)} \phi\, dx = Const$ for all $i \in \{1, \ldots, N_p - 1\}$ and $t \geq 0$. By employing (11) in our LG scheme we aim to follow the mass movement due to convection and diffusion with the moving mesh. Note that in this setting the approach can also be used if $f \neq 0$, in which case a parameter $\nu_M \neq \nu$ might yield more accurate results.

Applying a linearly implicit time discretization to the continuous problem (11) gives rise to our moving mesh method: find $\{P_i^n : i = 1, \ldots, N_p,\ n = 0, \ldots, N_T\}$ such that for $n = 1, \ldots, N_T$ it holds

$$\frac{P_i^n - P_i^{n-1}}{\Delta t} = u^{n-1}(P_i^{n-1}) + \nu_M \frac{P_{i+1}^n - 2P_i^n + P_{i-1}^n}{(P_i^{n-1} - P_{i-1}^{n-1})(P_{i+1}^{n-1} - P_i^{n-1})},$$
$$i = 2, \ldots, N_p - 1, \tag{12a}$$

$$P_1^n = a, \qquad P_{N_p}^n = b, \tag{12b}$$

$$\{P_i^0 : i = 1, \ldots, N_p\} \subset \bar{\Omega} \text{ given}; \qquad a = P_1^0 < P_2^0 < \ldots < P_{N_p}^0 = b. \tag{12c}$$

The discretization has been constructed making use of the fact that for $d = 1$ it holds

$$\sum_{j=1}^{N_p-1} \nabla\psi_i|_{[\tilde{P}_j, \tilde{P}_{j+1}]} = \frac{1}{\tilde{P}_i - \tilde{P}_{i-1}} - \frac{1}{\tilde{P}_{i+1} - \tilde{P}_i} = \frac{\tilde{P}_{i+1} - 2\tilde{P}_i + \tilde{P}_{i-1}}{(\tilde{P}_i - \tilde{P}_{i-1})(\tilde{P}_{i+1} - \tilde{P}_i)}.$$

The method is inspired by [12] and can be extended to higher dimensions in a straightforward way. In the case $\nu_M = 0$, the transition from P_i^{n-1} to P_i^n due to (12) and the transition from P_i^{n-1} to $X_1(u^{n-1}, \Delta t)(P_i^{n-1})$ describe movements in opposite directions. In particular, if the velocity field u is smooth we have $P_i^{n-1} \approx X_1^n(P_i^n)$. Hence, a reduction of the computational costs to identify $X_1^n(P_i^n)$ as well as a decrease of the corresponding interpolation error in the scheme are expected. The main idea of the LGMM method is, to combine the LG schemes (4) and (6) with the moving mesh method (12).

Remark 1 While the moving mesh method (12) leads to a well defined set of nodal points $\{P_i^n : i = 1, \ldots, N_p,\ n = 0, \ldots, N_T\}$ it is not clear whether they constitute a moving mesh in the sense of Definition 1 since the condition (7) might not be satisfied.

Remark 2 To obtain the nodal points $P_1^n, \ldots, P_{N_p}^n$ from $P_1^{n-1}, \ldots, P_{N_p}^{n-1}$ according to (12a) and (12b) a sparse linear system is solved. In general, the coefficient matrix of this system is not symmetric.

In fact we show that the method (12) results in a moving mesh for a suitable choice of Δt, see Sect. 1. The other theoretical results we show in this work assume a given moving mesh. While it is important that a positive distance between neighbor points is maintained in the moving mesh, it needs to be also verified that this distance does not become too large. In particular, with respect to the error estimates that we present in the following section we are interested in the situation that the global mesh size h tends to 0. This can be realized by employing an equidistant mesh of size h_0 at the initial time that is iteratively decreased by increasing the number of moving points and ensuring that the distance between neighboring points does not exceed Ch_0 over all time instances for a fixed constant $C > 0$. In practice, positive ν_M in scheme (12) has resulted in a control over the maximal point distance. Next, we state several results concerning the moving mesh method (12). We begin by formulating the following hypotheses.

Hypothesis 1 *The function u satisfies $u \in C^0([0, T]; W_0^{1,\infty}(\Omega))$.*

Hypothesis 2 *The nodal points of the finite element spaces $\Psi_h^0, \ldots \Psi_h^{N_T}$ are given by a moving mesh.*

Hypothesis 3 *The solution ϕ of problem (1) satisfies $\phi \in Z^3 \cap H^2(0, T; H^2(\Omega)) \cap H^1(0, T; H^3(\Omega))$.*

Theorem 1 (Non-overlapping condition for the moving mesh method) *Suppose that Hypothesis 1 holds true. Let $C_0 \in [0, 1)$ be fixed, the set of nodal points $\{P_i^n : i = 1, \ldots, N_p,\ n = 1, \ldots, N_T\}$ be given by method (12), and*

$$\Delta t |u|_{C^0(W^{1,\infty}(\Omega))} \leq C_0, \tag{13}$$

then the set of nodal points describes a moving mesh, i.e., it holds that for any $n \in \{0, \ldots, N_T\}$

$$P_i^n < P_j^n; \quad i < j; \quad i, j \in \{1, \ldots, N_p\}. \tag{14}$$

Proof Refer to Sect. 6.1.1. □

Remark 3 Suppose that the nodal points of the finite element spaces $\Psi_h^0, \ldots \Psi_h^{N_T}$ are governed by the moving mesh method (12) then Hypothesis 2 is implied by condition (13) due to Theorem 1.

Next, we state two results that are necessary in order to derive the error estimates for the LGMM schemes. For $f \in C^0(\bar{\Omega})$, $t \in [0, T]$, and the time dependent P1-basis functions $\psi_i(x, t)$ for $i \in \{1, \ldots, N_p\}$ we define the time dependent Lagrange interpolation of f by

$$[\Pi_h(t) f](x) := \sum_{i=1}^{N_p} f(P_i(t)) \psi_i(x, t). \tag{15}$$

We also denote the difference operator $\bar{D}_{\Delta t} f := \frac{f^n - f^{n-1}}{\Delta t}$.

Theorem 2 *Let $\{\phi(t) = \phi(\cdot, t) \in \Psi; t \in (0, T)\}$ be the solution of problem (1). Suppose that Hypothesis 2 and Hypothesis 3 hold true. Then assuming $w \in C^0(W_0^{1,\infty}(\Omega))$ the following results hold.*

i) There exists a positive constant $C = C(\|w\|_{C^0(L^\infty)})$ independent of Δt and h such that

$$\left\| \bar{D}_{\Delta t}(\Pi_h^n \phi^n) - \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \Pi_h(t) \frac{\partial \phi}{\partial t}(\cdot, t) dt \right\| \leq \frac{Ch}{\sqrt{\Delta t}} \|\phi\|_{L^2(t^{n-1}, t^n; H^2(\Omega))}. \tag{16a}$$

ii) For a positive constant $C' = C'(\|w\|_{C^0(W^{1,\infty})})$ independent of Δt and h it holds

$$\left\| \bar{D}_{\Delta t}(\Pi_h^n \phi^n) - \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \Pi_h(t) \frac{\partial \phi}{\partial t}(\cdot, t) dt \right\|_{\Psi'} \leq Ch^2 \|\phi\|_{H^1(H^3)}. \tag{16b}$$

Proof Refer to Sect. 6.1.2. □

Remark 4 In the case of a fixed mesh (velocity $w = 0$), the bounds on the right hand side of (16b) and (16a) are zero.

Corollary 1 *Let $\{\phi(t) = \phi(\cdot, t) \in \Psi; t \in (0, T)\}$ be the solution of problem (1). Suppose that Hypothesis 2 and Hypothesis 3 hold true. Define $\eta(t) := \phi(t) - \Pi_h(t)\phi(t)$. Then for $w \in C^0(W_0^{1,\infty}(\Omega))$ there exist positive constants $C = C(\|w\|_{C^0(L^\infty)})$ and $C' = C'(\|w\|_{C^0(W^{1,\infty})})$ independent of Δt and h such that the following bounds hold*

$$\left\| \bar{D}_{\Delta t} \eta^n \right\| \leq \frac{Ch}{\sqrt{\Delta t}} \|\phi\|_{H^1(t^{n-1}, t^n; H^2(\Omega))}, \tag{17a}$$

$$\left\| \bar{D}_{\Delta t} \eta^n \right\|_{\Psi'} \leq C' h^2 \left(\frac{1}{\sqrt{\Delta t}} \|\phi\|_{H^1(t^{n-1}, t^n; H^2(\Omega))} + \|\phi\|_{H^1(H^3)} \right). \tag{17b}$$

Proof Refer to Sect. 6.1.3. □

4 Results For The First-Order LGMM Scheme

In this section we state results for the scheme introduced in Sect. 2.2. We start by stating the following hypothesis.

Hypothesis 4 *The time step size Δt satisfies the condition $\Delta t |u|_{C^0(W^{1,\infty})} \leq 1/8$.*

Remark 5 Hypothesis 4 is not a CFL condition since the mesh size h is not included in the inequality. The time step size Δt can be chosen independently of h.

Proposition 1 (Mass preserving property of the first-order LGMM Scheme) *Suppose that Hypotheses 1, 2 and 4 hold true. Let $\{\phi_h^n\}_{n=1}^{N_T}$ be the solution of the numerical scheme (4) for a given initial datum ϕ_h^0. Then it holds for $n = 0, 1, \ldots, N_T$ that*

$$\int_\Omega \phi_h^n dx = \int_\Omega \phi_h^0 dx + \Delta t \sum_{i=1}^{n} \left(\int_\Omega f^i dx + \int_\Gamma g^i ds \right). \tag{18}$$

Proposition 2 (Stability of the first-order LGMM scheme) *Suppose that Hypotheses 1, 2 and 4 hold true. Let $F \in H^1(0, T; \Psi')$ be given. For the given function $\phi_h^0 \in \Psi_h$, let $\{\phi_h^n\}_{n=1}^{N_T} \subset \Psi_h$ be the numerical solutions of scheme (4). Then there exists a constant $C > 0$ independent of h and Δt such that*

$$\|\phi_h\|_{\ell^\infty(L^2)} + \sqrt{\nu}\|\nabla\phi_h\|_{\ell^2(L^2)} \le C\left(\|\phi_h^0\| + \|F\|_{\ell^2(\Psi_h')}\right). \tag{19}$$

Proofs of Proposition 1 and Proposition 2 The proof of the two propositions follows directly from Theorem 1 and Theorem 2 in [32], respectively. For convenience we provide the proofs in Appendix D.1 and Appendix E.1. □

Remark 6 The mass-preserving and stability properties of the first-order Lagrange-Galerkin scheme with fixed mesh (Theorem 1 and Theorem 2 of [32]) are maintained in the first-order LGMM scheme.

Theorem 3 (Error estimates for the first-order LGMM scheme) *Suppose that Hypotheses 1, 2, 4, and 3 hold true. Let $F \in H^1(0, T; \Psi')$ be given. Assuming the initial datum $\phi_h^0 = \Pi_h^0\phi^0 \in \Psi_h$ let $\{\phi(t) = \phi(\cdot, t) \in \Psi; t \in (0, T)\}$ be the solution of problem (1) and $\{\phi_h^n\}_{n=1}^{N_T}$ the numerical solutions of scheme (4). Then there exists a constant $C > 0$ independent of h and Δt such that*

$$\|\phi_h - \phi\|_{\ell^\infty(L^2)} + \sqrt{\nu}\|\nabla(\phi_h - \phi)\|_{\ell^2(L^2)} \le C(\Delta t + h^2)\|\phi\|_{Z^2\cap H^1(H^2)\cap H^1(H^3)}. \tag{20}$$

Proof Refer to Sect. 6.2. □

Remark 7 Using the bound (17a) instead of (17b) in the proof of Theorem 3 a first order bound that requires lower regularity of ϕ is obtained. Namely, under the assumptions of Theorem 3 there exists a constant $C > 0$ independent of h and Δt such that

$$\|\phi_h - \phi\|_{\ell^\infty(L^2)} + \sqrt{\nu}\|\nabla(\phi_h - \phi)\|_{\ell^2(L^2)} \le C(\Delta t + h)\|\phi\|_{Z^2\cap H^1(H^2)\cap H^1(H^2)}.$$

5 Results For The Second-Order LGMM Scheme

The results in this section concern the second-order scheme introduced in Sect. 2.3.

Proposition 3 (Mass preserving property of the second-order LGMM scheme) *Suppose that Hypotheses 1, 2 and 4 hold true. Let $\{\phi_h^n\}_{n=1}^{N_T}$ be the solution of the numerical scheme (6) for a given initial datum ϕ_h^0. Then, we have the following.*

(i) It holds for $n = 1, 2, \ldots, N_T$ that

$$\int_\Omega \left(\frac{3}{2}\phi_h^n - \frac{1}{2}\phi_h^{n-1}\right) dx = \int_\Omega \frac{1}{2}(\phi_h^0 + \phi_h^1) dx + \Delta t \sum_{i=1}^n \left(\int_\Omega f^i dx + \int_\Gamma g^i ds\right). \tag{21}$$

(ii) Assume $f = g = 0$ additionally. Then, it holds for $n = 1, 2, \ldots, N_T$ that

$$\int_\Omega \phi_h^n dx = \int_\Omega \phi_h^0 dx. \tag{22}$$

Proposition 4 (Stability for the Second-Order LGMM Scheme) *Suppose that Hypotheses 1, 2 and 4 hold true. Let $F \in H^1(0, T; \Psi')$ be given. For a given function $\phi_h^0 \in \Psi_h$ let $\{\phi_h^n\}_{n=1}^{N_T} \subset$*

Ψ_h *be the numerical solutions of scheme* (6). *Then, there exists a constant* $C > 0$ *independent of* h *and* Δt *such that*

$$\|\phi_h\|_{\ell^\infty(L^2)} + \sqrt{\nu}\|\nabla\phi_h\|_{\ell^2(L^2)} \leq C\left(\|\phi_h^0\| + \|F\|_{\ell^2(\Psi_h')}\right). \tag{23}$$

Proofs of Proposition 3 and Proposition 4 The proof of both propositions follows from Theorem 1 and Theorem 2 in [20], respectively. For convenience we provide the proofs in Appendices D.2 and E.2. □

Remark 8 Also in case of the second-order Lagrange–Galerkin scheme the mass-preserving and stability properties of the fixed mesh method (Theorem 1 and Theorem 2 of [20]) are maintained in the LGMM scheme.

Theorem 4 (Error Estimates the Second-Order LGMM Scheme) *Suppose that Hypotheses 1, 2, 4, and 3 hold true. Let* $F \in H^1(0, T; \Psi')$ *be given. For a given function* $\phi_h^0 = \Pi_h^0\phi^0 \in \Psi_h$ *let* $\{\phi(t) = \phi(\cdot, t) \in \Psi; t \in (0, T)\}$ *be the solution of problem* (1) *and* $\{\phi_h^n\}_{n=1}^{N_T}$ *be the numerical solutions of scheme* (6). *Then, there exists a constant* $C > 0$ *independent of* h *and* Δt *such that*

$$\|\phi_h - \phi\|_{\ell^\infty(L^2)} + \sqrt{\nu}\|\nabla(\phi_h - \phi)\|_{\ell^2(L^2)} \leq C(\Delta t^2 + h^2)\|\phi\|_{Z^3\cap H^2(H^2)\cap H^1(H^3)}. \tag{24}$$

Proof Refer to Sect. 6.3. □

Remark 9 In analogy to Remark 7 in the proof of Theorem 4 the bound (17a) can be used instead of (17b) to obtain a first order bound that requires lower regularity of ϕ. Namely, under the assumptions of Theorem 4 there exists a constant $C > 0$ independent of h and Δt such that

$$\|\phi_h - \phi\|_{\ell^\infty(L^2)} + \sqrt{\nu}\|\nabla(\phi_h - \phi)\|_{\ell^2(L^2)} \leq C(\Delta t^2 + h)\|\phi\|_{Z^3\cap H^2(H^2)}$$

Remark 10 In case of a static mesh the error estimate (24) is consistent with [20, Theorem 3 (ii)] except for the dependence on $\|\phi\|_{H^1(H^3)}$. Taking into account Remark 4 the exact literature result can easily be recovered. The same is true for the relation between the error estimate (20) and [32, Theorem 3].

6 Proofs

In this section we provide proofs for the results stated in Sects. 3, 4 and 5.

6.1 Proofs of the Results Regarding the Moving Mesh

6.1.1 Proof of Theorem 1

We show property (14) inductively. Hence, suppose $P_i^{n-1} < P_j^{n-1}$; $i < j$; $i, j \in \{1, \ldots, N_p\}$, we show that (14) holds true. Let $h_i^{n-1} := P_{i+1}^{n-1} - P_i^{n-1}$ for $i \in \{1, \ldots, N_p - 1\}$. It is sufficient to show that $h_i^n > 0$ for $i \in \{1, \ldots, N_p - 1\}$. Shifting the index i in scheme (12a), we have

$$\frac{P_{i+1}^n - P_{i+1}^{n-1}}{\Delta t} = u^{n-1}(P_{i+1}^{n-1}) + \nu_M \frac{P_{i+2}^n - 2P_{i+1}^n + P_i^n}{(P_{i+1}^{n-1} - P_i^{n-1})(P_{i+2}^{n-1} - P_{i+1}^{n-1})}, \quad i = 1, \ldots, N_p - 2. \tag{25}$$

By subtracting (12a) from (25) we obtain

$$\frac{h_i^n - h_i^{n-1}}{\Delta t} = u^{n-1}(P_{i+1}^{n-1}) - u^{n-1}(P_i^{n-1}) + \nu_M \left[\frac{h_{i+1}^n - h_i^n}{h_i^{n-1} h_{i+1}^{n-1}} - \frac{h_i^n - h_{i-1}^n}{h_{i-1}^{n-1} h_i^{n-1}} \right],$$
$$i = 2, \dots, N_p - 2.$$

Rearranging the terms it follows for $i = 2, \dots, N_p - 2$

$$\left[\frac{1}{\Delta t} + \nu_M \left(\frac{1}{h_i^{n-1} h_{i+1}^{n-1}} + \frac{1}{h_{i-1}^{n-1} h_i^{n-1}} \right) \right] h_i^n - \nu_M \frac{1}{h_i^{n-1} h_{i+1}^{n-1}} h_{i+1}^n - \nu_M \frac{1}{h_{i-1}^{n-1} h_i^{n-1}} h_{i-1}^n$$
$$= u^{n-1}(P_{i+1}^{n-1}) - u^{n-1}(P_i^{n-1}) + \frac{h_i^{n-1}}{\Delta t}. \tag{26}$$

Using (13) we derive a lower bound of the right hand side in (26) as follows:

$$u^{n-1}(P_{i+1}^{n-1}) - u^{n-1}(P_i^{n-1}) + \frac{h_i^{n-1}}{\Delta t} \geq \frac{h_i^{n-1}}{\Delta t} - |u^{n-1}(P_{i+1}^{n-1}) - u^{n-1}(P_i^{n-1})|$$
$$\geq \frac{h_i^{n-1}}{\Delta t} - h_i^{n-1} |u|_{C^0(W_0^{1,\infty})} = \frac{h_i^{n-1}}{\Delta t} \left(1 - \Delta t |u|_{C^0(W_0^{1,\infty})} \right) > 0. \tag{27}$$

Note that from (12a) for $i = 2$ we have

$$\frac{P_2^n - P_2^{n-1}}{\Delta t} = u^{n-1}(P_2^{n-1}) + \nu_M \frac{h_2^n - h_1^n}{h_1^{n-1} h_2^{n-1}},$$

and from (12b) follows $P_1^{n-1} = a$, which implies $\frac{P_1^n - P_1^{n-1}}{\Delta t} = 0$. Therefore, it holds

$$\left(\frac{1}{\Delta t} + \nu \frac{1}{h_1^{n-1} h_2^{n-1}} \right) h_1^n - \nu_M \frac{1}{h_1^{n-1} h_2^{n-1}} h_2^n = u^{n-1}(P_2^{n-1}) - u^{n-1}(P_1^{n-1}) + \frac{h_1^{n-1}}{\Delta t}. \tag{28}$$

Similarly, from (12a) for $i = N_p - 1$ we have

$$\frac{P_{N_p-1}^n - P_{N_p-1}^{n-1}}{\Delta t} = u^{n-1}(P_{N_p-1}^{n-1}) + \nu_M \frac{h_{N_p-1}^n - h_{N_p-2}^n}{h_{N_p-2}^{n-1} h_{N_p-1}^{n-1}}$$

and from (12b) we obtain $P_{N_p}^{n-1} = b$, which implies $\frac{P_{N_p}^n - P_{N_p}^{n-1}}{\Delta t} = 0$. Therefore, we have

$$\left(\frac{1}{\Delta t} + \nu \frac{1}{h_{N_p-2}^{n-1} h_{N_p-1}^{n-1}} \right) h_{N_p-1}^n - \nu_M \frac{1}{h_{N_p-2}^{n-1} h_{N_p-1}^{n-1}} h_{N_p-2}^n$$
$$= u^{n-1}(P_{N_p}^{n-1}) - u^{n-1}(P_{N_p-1}^{n-1}) + \frac{h_{N_p-1}^{n-1}}{\Delta t}. \tag{29}$$

Proceeding as in (27) positivity of the right hand sides in both (28) and (29) follows. Combining (28), (26) and (29) yields a linear system with unknown variables $h_1^n, \dots, h_{N_p-1}^n$ and a strictly diagonally dominant coefficient matrix, which is thus an M-matrix. Since an M-matrix A has the property that $Ax > 0$ implies $x > 0$, the solution of the linear system is positive, i.e., $h_1^n, \dots, h_{N_p-1}^n > 0$, hence (14) follows. □

6.1.2 Proof of Theorem 2

First, we state the following lemma, which plays an important role in the proof of Theorem 2. The proof of the lemma is given in Appendix A.

Lemma 1 *Let $\{\phi(t) = \phi(\cdot, t) \in \Psi; t \in (0, T)\}$ be the solution of problem (1) and suppose that Hypothesis 2 holds true. For $x \in \bar{\Omega}$ and $t \in [t^{n-1}, t^n]$ we define*

$$I(x,t) := \sum_{i=1}^{N_p} \phi(P_i(t), t) \left[\frac{\partial}{\partial t}\psi_i(x,t)\right],$$

where $P_i(t)$ for $i \in \{1, \dots, N_p\}$ are the nodal point positions defined in (9) and $\psi_i(x,t)$ for $i \in \{1, \dots, N_p\}$ denote the time extended P1 basis functions. We assume $x \in [P_k(t), P_{k+1}(t)]$ for a $k \in \{1, \dots, N_p - 1\}$. Then $I(x,t)$ can be expressed as

$$I(x,t) = -\frac{\phi(P_{k+1}(t), t) - \phi(P_k(t), t)}{P_{k+1}(t) - P_k(t)} \left[w^n(P_{k+1}(t))\psi_{k+1}(x,t) + w^n(P_k(t))\psi_k(x,t)\right]. \tag{30}$$

Proof of Theorem 2 We first define the interpolation operators

$$\left(\Pi_h^\ell \phi^\ell\right)(x) := \sum_{i=1}^{N_p} \phi^\ell(P_i^\ell)\psi_i^\ell(x), \qquad \ell \in \{n-1, n\}.$$

Then we rewrite their difference as

$$\begin{aligned}
&\left(\Pi_h^n\phi^n - \Pi_h^{n-1}\phi^{n-1}\right)(x) \\
&= \int_{t^{n-1}}^{t^n} \frac{d}{dt}(\Pi_h(t)\phi(\cdot,t))(x)dt \\
&= \sum_{i=1}^{N_p} \int_{t^{n-1}}^{t^n} \frac{\partial}{\partial t}[\phi(P_i(t),t)\psi_i(x,t)]dt \\
&= \sum_{i=1}^{N_p} \int_{t^{n-1}}^{t^n} \left(\left[\frac{\partial}{\partial t}\phi(P_i(t),t)\right]\psi_i(x,t) + \phi(P_i(t),t)\left[\frac{\partial}{\partial t}\psi_i(x,t)\right]\right)dt \\
&= \sum_{i=1}^{N_p} \int_{t^{n-1}}^{t^n} \left(\left[\frac{\partial\phi}{\partial t}(P_i(t),t) + \frac{dP_i}{dt}(t)(\nabla\phi)(P_i(t),t)\right]\psi_i(x,t)\right. \\
&\left. \quad + \phi(P_i(t),t)\left[\frac{\partial}{\partial t}\psi_i(x,t)\right]\right)dt \\
&= \sum_{i=1}^{N_p} \int_{t^{n-1}}^{t^n} \left(\left[\frac{\partial\phi}{\partial t}(P_i(t),t) + w^n(P_i(t))(\nabla\phi)(P_i(t),t)\right]\psi_i(x,t)\right. \\
&\left. \quad + \phi(P_i(t),t)\left[\frac{\partial}{\partial t}\psi_i(x,t)\right]\right)dt \\
&= \int_{t^{n-1}}^{t^n} \Pi_h(t)\left[\frac{\partial\phi}{\partial t}(\cdot,t) + w^n(\cdot)\nabla\phi(\cdot,t)\right](x)dt + \int_{t^{n-1}}^{t^n} I(x,t)dt.
\end{aligned} \tag{31}$$

The rest of the proof concerns the last integral in (31). Without loss of generality let $t \in [t^{n-1}, t^n]$ and $x \in K_k(t) := [P_k(t), P_{k+1}(t)]$. For brevity we introduce the notations:

$$w_k^n := w^n(P_k^{n-1}), \qquad w_{k+1}^n := w^n(P_{k+1}^{n-1}), \qquad h_k = P_{k+1}(t) - P_k(t),$$
$$\phi_k := \phi(P_k(t), t), \qquad \phi_{k+1} := \phi(P_{k+1}(t), t).$$

We're now in the position to show i). Due to the Taylor expansions

$$\begin{aligned}
\phi_{k+1} - \phi_k &= \phi(P_k(t) + h_k(t), t) - \phi(P_k(t), t)\\
&= h_k(t)(\nabla\phi)(P_k(t), t) + h_k^2(t) \int_0^1 \int_0^{s_0} (\nabla^2\phi)(P_k(t) + s_1 h_k(t), t) ds_1 ds_0,\\
\phi_{k+1} - \phi_k &= \phi(P_{k+1}(t), t) - \phi(P_{k+1}(t) - h_k(t), t)\\
&= h_k(t)(\nabla\phi)(P_{k+1}(t), t) - h_k^2(t) \int_0^1 \int_0^{s_0} (\nabla^2\phi)(P_{k+1}(t) - s_1 h_k(t), t) ds_1 ds_0.
\end{aligned}$$

we obtain the identity

$$\begin{aligned}
&\frac{\phi_{k+1} - \phi_k}{P_{k+1}(t) - P_k(t)} \left[w_k^n \psi_k(x, t) + w_{k+1}^n \psi_{k+1}(x, t) \right]\\
&= \left[\Pi_h(t) w^n(\cdot) \nabla\phi(\cdot, t) \right](x) + h_k(t) w_k^n \psi_k(x, t) \int_0^1 \int_0^{s_0} (\nabla^2\phi)(P_k(t) + s_1 h_k(t), t) ds_1 ds_0\\
&\quad - h_k(t) w_{k+1}^n \psi_{k+1}(x, t) \int_0^1 \int_0^{s_0} (\nabla^2\phi)(P_{k+1}(t) - s_1 h_k(t), t)\, ds_1 ds_0. \qquad (32)
\end{aligned}$$

By using Lemma 1, we substitute (32) into (31), and through a change of variable, we proceed to compute

$$\begin{aligned}
&\left| \left(\Pi_h^n \phi^n - \Pi_h^{n-1} \phi^{n-1} \right) - \int_{t^{n-1}}^{t^n} \left(\Pi_h(t) \frac{\partial\phi}{\partial t}(\cdot, t) \right) dt \right|\\
&= \left| \int_{t^{n-1}}^{t^n} \Pi_h(t) w^n(\cdot) \nabla\phi(\cdot, t)(x) + \int_{t^{n-1}}^{t^n} I(x, t) dt \right|\\
&\leq c \|w\|_{C^0(L^\infty)} \int_{t^{n-1}}^{t^n} \int_{P_k(t)}^{P_{k+1}(t)} |(\nabla^2\phi)(x, t)| dx dt.
\end{aligned}$$

By taking the L^2-norm over Ω and applying the Cauchy–Schwartz inequality on the right hand side, we obtain

$$\begin{aligned}
&\left\| \left(\Pi_h^n \phi^n - \Pi_h^{n-1} \phi^{n-1} \right) - \int_{t^{n-1}}^{t^n} \left(\Pi_h(t) \frac{\partial\phi}{\partial t}(\cdot, t) \right) dt \right\|^2\\
&\leq c^2 h \Delta t \|w\|_{C^0(L^\infty)}^2 \int_\Omega \int_{t^{n-1}}^{t^n} \int_{K_{\tilde{k}}(t)} (\nabla^2\phi)(y, t)^2 dy dt dx\\
&= c^2 h \Delta t \|w\|_{C^0(L^\infty)}^2 \int_{t^{n-1}}^{t^n} \sum_{k=1}^{N_p - 1} h_{k(t)} \int_{K_k(t)} (\nabla^2\phi)(y, t)^2 dy dt\\
&\leq c^2 h^2 \Delta t \|w\|_{C^0(L^\infty)}^2 ||\nabla^2\phi||_{L^2(t^{n-1}, t^n; L^2(\Omega))}^2, \qquad (33)
\end{aligned}$$

where the dynamic index $\tilde{k} = \tilde{k}(x, t)$ is defined such that $x \in K_{\tilde{k}}(t)$. To complete the proof, we take the square root and divide both sides of (33) by Δt, obtaining (16a).

Next, we proof (ii). Therefore, we first rewrite and then further expand the last double integral in (32) as follows

$$\begin{aligned}&\int_0^1\int_0^{s_0}(\nabla^2\phi)(P_k(t)+(1-s_1)h_k(t),t)ds_1ds_0\\&=\int_0^1\int_0^{s_0}(\nabla^2\phi)(P_k(t)+s_1h_k(t),t)ds_1ds_0\\&\quad+h_k(t)\int_0^1\int_0^{s_0}\int_{s_1}^{1-s_1}(\nabla^3\phi)(P_k(t)+s_2h_k(t),t)ds_2ds_1ds_0.\end{aligned}$$

Additionally we introduce the Taylor expansion

$$w_{k+1}^n=w_k^n+h_k\int_0^1(\nabla w^n)(P_k(t)+sh_k(t))ds.$$

By using Lemma 1 and substituting the above expressions into (31) we compute

$$\begin{aligned}A^n(x):&=\frac{1}{\Delta t}\left[\left(\Pi_h^n\phi^n-\Pi_h^{n-1}\phi^{n-1}\right)(x)-\int_{t^{n-1}}^{t^n}\left(\Pi_h(t)\frac{\partial\phi}{\partial t}(\cdot,t)\right)(x)dt\right]\\&=-\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}h_k(t)w_k^n(\psi_k(x,t)-\psi_{k+1}(x,t))\int_0^1\int_0^{s_0}(\nabla^2\phi)(P_k(t)+s_1h_k(t),t)ds_1ds_0dt\\&\quad+\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}h_k^2(t)\psi_{k+1}(x,t)w_k^n\int_0^1\int_0^{s_0}\int_{s_1}^{1-s_1}(\nabla^3\phi)(P_k(t)+s_2h_k(t),t)ds_2ds_1ds_0dt\\&\quad+\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\left(h_k^2(t)\psi_{k+1}(x,t)\int_0^1(\nabla w^n)(P_k(t)+sh_k(t))ds\right.\\&\quad\left.\int_0^1\int_0^{s_0}(\nabla^2\phi)(P_k(t)+(1-s_1)h_k(t),t)ds_1ds_0\right)dt\\&=:A_1^n(x)+A_2^n(x)+A_3^n(x).\end{aligned}$$

We proceed by estimating the (Ψ')-norm, i.e., $\|A^n\|_{\Psi'}\leq\|A_1^n\|_{\Psi'}+\|A_2^n\|_{\Psi'}+\|A_3^n\|_{\Psi'}$. The following bounds hold

$$\|A_1^n\|_{\Psi'}\leq c_1h^2\|w\|_{C^0(L^\infty)}\|\phi\|_{H^1(H^3)} \tag{34}$$

$$\|A_2^n\|_{\Psi'}\leq c_2h^2\|w\|_{C^0(L^\infty)}\|\phi\|_{H^1(H^3)} \tag{35}$$

$$\|A_3^n\|_{\Psi'}\leq c_3h^2\|w\|_{C^0(W^{1,\infty})}\|\phi\|_{H^1(H^3)} \tag{36}$$

and their detailed proofs are provided in Appendix B. Combining all the bounds, we obtain the estimate

$$\|A\|_{\Psi'}\leq Ch^2\|w\|_{C^0(W^{1,\infty})}\|\phi\|_{H^1(H^3)},$$

which completes the proof of (16b). □

6.1.3 Proof of Corollary 1

We first recall an error estimate for the Lagrange interpolation that follows from [10, Theorem 4.4.20].

Lemma 2 *We suppose that Hypothesis 2 holds true and fix $t \in [0, T]$. Let $\Pi_h = \Pi_h(t) = \sum_{i=1}^{N_p} f(P_i(t))\psi_i(x,t)$ be the Lagrange interpolation operator at time t and $v \in H^2(\Omega)$. Then there exists a constant $C > 0$ independent of h such that*

$$\|\Pi_h v - v\|_{H^s(\Omega)} \leq Ch^{2-s}|v|_{H^2(\Omega)} \quad \textit{for } s \in \{0, 1\}. \tag{37}$$

Proof of Corollary 1 We show (17a) and assume $t \in [t^{n-1}, t^n]$. By applying the bound (16a) from Theorem 2, Cauchy-Schwartz's inequality, as well as Lemma 2, we obtain the estimate,

$$\begin{aligned}
\left\|\eta^n - \eta^{n-1}\right\| &= \left\|(\phi^n - \Pi_h^n\phi^n) - (\phi^{n-1} - \Pi_h^{n-1}\phi^{n-1})\right\| \\
&= \left\|\phi^n - \phi^{n-1} - (\Pi_h^n\phi^n - \Pi_h^{n-1}\phi^{n-1})\right\| \\
&\leq \left\|\int_{t^{n-1}}^{t^n}\left(\frac{\partial\phi}{\partial t} - \Pi_h(t)\frac{\partial\phi}{\partial t}\right)dt\right\| + c_1 h\sqrt{\Delta t}\|\phi\|_{L^2(t^{n-1},t^n);H^2(\Omega))} \\
&\leq \sqrt{\Delta t}\sqrt{\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\phi}{\partial t} - \Pi_h\frac{\partial\phi}{\partial t}\right\|_{L^2(\Omega)}^2 dt} + c_1 h\sqrt{\Delta t}\|\phi\|_{L^2(t^{n-1},t^n);H^2(\Omega))} \\
&\leq c_2\sqrt{\Delta t}\left(h^2\sqrt{\int_{t^{n-1}}^{t^n}\left|\frac{\partial\phi}{\partial t}\right|_{H^2(\Omega)}^2 dt} + h\|\phi\|_{L^2(t^{n-1},t^n);H^2(\Omega))}\right) \\
&\leq c_2 h\sqrt{\Delta t}\left(h\|\phi\|_{H^1(t^{n-1},t^n;H^2(\Omega))} + \|\phi\|_{L^2(t^{n-1},t^n);H^2(\Omega))}\right).
\end{aligned} \tag{38}$$

To complete the proof, we divide both sides of (38) by Δt and obtain (17a). The bound (17b) is obtained repeating the above estimates in the Ψ'-norm, using (16b) instead of (16a) and embedding L^2 in Ψ'. □

6.2 Proof of Theorem 3

To prove Theorem 3, we first state the following lemma.

Lemma 3 (Evaluation of composite functions [20, 32]) *Let a be a function in $W_0^{1,\infty}(\Omega)^d$ satisfying $\Delta t\|a\|_{1,\infty} \leq 1/4$ and consider the mapping $X_1(a, \Delta t)$ defined in (4). Then, the following inequalities hold.*

$$\|\psi \circ X_1(a, \Delta t)\| \leq (1 + c_1\Delta t)\|\psi\|, \qquad \forall\psi \in L^2(\Omega), \tag{39a}$$

$$\|\psi - \psi \circ X_1(a, \Delta t)\| \leq c_2\Delta t\|\psi\|_{H^1(\Omega)}, \qquad \forall\psi \in H^1(\Omega), \tag{39b}$$

$$\|\psi - \psi \circ X_1(a, \Delta t)\|_{H^{-1}(\Omega)} \leq c_3\Delta t\|\psi\|, \qquad \forall\psi \in L^2(\Omega). \tag{39c}$$

Proof of Theorem 3 We define the terms

$$e_h^n := \phi_h^n - \Pi_h^n\phi^n, \qquad \eta(t) := \phi(t) - \Pi_h(t)\phi(t).$$

By substituting the error e_h^n in the numerical scheme (4), we obtain the following expression:

$$\left(\frac{e_h^n - [e_h^{n-1} \circ X_1^n]\gamma^n}{\Delta t}, \psi_h\right) + a_0(e_h^n, \psi_h) = \langle R_h^n, \psi_h\rangle, \quad \forall\psi_h \in \Psi_h^n, \tag{40}$$

where the residual on the right hand side is given by

$$\begin{aligned} R_h^n &:= R_1^n + R_2^n + R_3^n, \\ R_1^n &:= \frac{\partial \phi^n}{\partial t} + \nabla\cdot\ (u^n\phi^n) - \frac{\phi^n - [\phi^{n-1}\circ X_1^n]\gamma^n}{\Delta t}, \\ R_2^n &:= \frac{\eta^n - [\eta^{n-1}\circ X_1^n]\gamma^n}{\Delta t}, \\ \langle R_3^n, \psi_h\rangle &:= a_0(\eta^n, \psi_h). \end{aligned}$$

To obtain an estimate on $\|R_1\|$, we follow the error estimate framework for the convection–diffusion problem on a static mesh (details are given in Appendix C.1), which gives us

$$\|R_1\|_{\ell^2(\Psi_h')} \le c_4 \Delta t \|\phi\|_{Z^2(0,T)}. \tag{41}$$

In case of linear elements in one dimension that are considered here we have $R_3^n = 0$ as is shown in Appendix C.3. To compute a bound for R_2^n we rewrite it as

$$\begin{aligned} R_2^n &= \frac{\eta^n - [\eta^{n-1}\circ X_1^n]\gamma^n}{\Delta t} \\ &= \frac{\eta^n - \eta^{n-1}}{\Delta t} + \frac{\eta^{n-1} - \eta^{n-1}\circ X_1^n}{\Delta t} + \frac{(\eta^{n-1}\circ X_1^n)(1-\gamma^n)}{\Delta t}. \end{aligned}$$

Then, using (39c) and (39a), noting that thanks to Hypothesis 1 it holds $1 - \gamma^n \le c_2\Delta t$, employing (17b), Lemma 2 and embedding $L^2(\Omega)$ in $H^1(\Omega)'$, we obtain the following

$$\begin{aligned} \|R_2^n\|_{(\Psi_h^n)'} &\le \left\| \frac{\eta^n - \eta^{n-1}}{\Delta t} \right\|_{(\Psi_h^n)'} + c_3\|\eta^{n-1}\| + c_6\|\eta^{n-1}\circ X_1^n\| \\ &\le c_7 \left[\frac{h^2}{\sqrt{\Delta t}} \|\phi\|_{H^1(t^{n-1},t^n;H^2(\Omega))} + h^2\|\phi\|_{H^1(H^3)} + \|\eta^{n-1}\| \right] \\ &\le c_8 \left[\frac{h^2}{\sqrt{\Delta t}} \|\phi\|_{H^1(t^{n-1},t^n;H^2(\Omega))} + h^2\|\phi\|_{H^1(H^3)} \right] \end{aligned} \tag{42}$$

Hence, by taking the ℓ^2-norm

$$\|R_2\|_{\ell^2(\Psi_h')} \le c_9 h^2 \left(\|\phi\|_{H^1(0,T;H^2(\Omega))} + \|\phi\|_{H^1(H^3)} \right). \tag{43}$$

By combining the estimates (41), (43), and taking into account the fact $R_3^n = 0$, we get

$$\|R_h\|_{\ell^2(\Psi_h')} \le C \left(\Delta t \|\phi\|_{Z^2(0,T)} + h^2\|\phi\|_{H^1(0,T;H^2(\Omega))} + h^2\|\phi\|_{H^1(H^3)} \right), \tag{44}$$

as estimate for the total residual, where $C > 0$ is independent of h and Δt. Lastly, we apply the stability result from Proposition 2 to problem (40) by substituting ϕ in (19) with $e_h^n = \phi_h^n - \Pi_h^n \phi^n$, initial value $e_h^0 = 0$ and RHS term F^n as R_h^n. We use the bound (44) to obtained the error estimates (20). □

6.3 Proof of Theorem 4

First, we state the following lemma which provides the estimates of the first time step error. The proof is given in Appendix F.

Lemma 4 *Suppose that Hypotheses 1, 2, 4, and 3 hold true. Then, it holds that*

$$||e_h^1|| \le ||e_h^1|| + \sqrt{\nu\Delta t}||\nabla e_h^1|| \le C(\Delta t^2 + h^2)||\phi||_{Z^3\cap H^2(H^2)\cap H^1(H^3)}. \tag{45}$$

Proof of Theorem 4 We substitute e_h^n in the numerical scheme (5) and obtain the following equations for the error:

$$\left(\frac{e_h^n - [e_h^{n-1}\circ X_1^n]\gamma^n}{\Delta t}, \psi_h\right) + a_0(e_h^n, \psi_h) = \langle R_h^n, \psi_h\rangle,\ \forall \psi_h \in \Psi_h^n,\ n=1, \tag{46}$$

$$\left(\frac{3e_h^n - 4e_h^{n-1}\circ X_1^n\gamma^n + e_h^{n-2}\circ \tilde{X}_1^n\tilde{\gamma}^n}{2\Delta t}, \psi_h\right) + a_0(e_h^n, \psi_h) = \langle \tilde{R}_h^n, \psi_h\rangle, \quad \forall \psi_h \in V_h^n,\ n \ge 2, \tag{47}$$

where the residual R_h^n, R_1^n, R_2^n, and R_3^n are given as in the proof of Theorem 3, cf., Sect. 6.2, while the residual on the right hand side of (47) is given by:

$$\begin{aligned}
\tilde{R}_h^n &:= \tilde{R}_1^n + \tilde{R}_2^n + R_3^n,\\
\tilde{R}_1^n &:= \frac{\partial \phi^n}{\partial t} + \nabla\cdot\ (u^n\phi^n) - \frac{3\phi^n - 4\phi^{n-1}\circ X_1^n\gamma^n + \phi^{n-2}\circ\tilde{X}_1^n\tilde{\gamma}^n}{2\Delta t},\\
\tilde{R}_2^n &:= \frac{3\eta^n - 4\eta^{n-1}\circ X_1^n\gamma^n + \eta^{n-2}\circ\tilde{X}_1^n\tilde{\gamma}^n}{2\Delta t}.
\end{aligned}$$

To obtain an estimate for $\|\tilde{R}_1\|$, we follow the error estimate framework for the general convection–diffusion problem on uniform mesh (details are given in Appendices C.2 and C.3), which gives us

$$\|\tilde{R}_1\|_{\ell^2(\Psi_h')} \le C_1\Delta t^2\|\phi\|_{Z^3(0,T)} \tag{48}$$

and as we have shown in Appendix C.3 it holds $R_3^n = 0$. Next, we compute an estimate for $\|\tilde{R}_2\|_{(\Psi_h^n)'}$. For $n \ge 2$ it holds

$$\begin{aligned}
\|\tilde{R}_2^n\|_{(\Psi_h^n)'} &= \frac{1}{2\Delta t}\|3\eta^n - 4\eta^{n-1}\circ X_1^n\gamma^n + \eta^{n-2}\circ\tilde{X}_1^n\tilde{\gamma}^n\|_{(\Psi_h^n)'}\\
&= \left\|\frac{3}{2}\bar{D}_{\Delta t}\eta^n - \frac{1}{2}\bar{D}_{\Delta t}\eta^{n-1} + \frac{2}{\Delta t}(\eta^{n-1} - \eta^{n-1}\circ X_1^n\gamma^n) - \frac{1}{2\Delta t}(\eta^{n-2}\right.\\
&\quad \left. -\eta^{n-2}\circ\tilde{X}_1^n\tilde{\gamma}^n)\right\|_{(\Psi_h^n)'}\\
&\le \frac{3}{2}\|\bar{D}_{\Delta t}\eta^n\| + \frac{1}{2}\|\bar{D}_{\Delta t}\eta^{n-1}\| + \frac{2}{\Delta t}\|(\eta^{n-1} - \eta^{n-1}\circ X_1^n\gamma^n)\|_{(\Psi_h^n)'}\\
&\quad + \frac{1}{2\Delta t}\|(\eta^{n-2} - \eta^{n-2}\circ\tilde{X}_1^n\tilde{\gamma}^n)\|_{(\Psi_h^n)'}\\
&\le C_3(\|\bar{D}_{\Delta t}\eta^n\| + \|\bar{D}_{\Delta t}\eta^{n-1}\| + \|\eta^{n-1}\| + \|\eta^{n-2}\|) \quad (\because (Lem.3))\\
&\le C_4\left(h^2\Delta t^{-1/2}\|\phi\|_{H^1(t^{n-2},t^n;H^2(\Omega))} + h^2\|\phi\|_{H^1(H^3)}\right),
\end{aligned}$$

where the last inequality follows from Lemma 2 and Theorem 2. Taking the ℓ^2-norm in the previous estimate we obtain

$$\|\tilde{R}_2\|_{\ell^2(\Psi_h')} \le C_5 h^2\left(\|\phi\|_{H^1(0,T;H^2(\Omega))} + \|\phi\|_{H^1(H^3)}\right). \tag{49}$$

Combining the bounds (48) and (49) and taking into account the fact $R_3 = 0$, it follows

$$\|\tilde{R}_h\|_{\ell^2(\Psi_h')} \leq C(\Delta t^2 \|\phi\|_{Z^3(0,T)} + h^2 \|\phi\|_{H^1(0,T;H^2(\Omega))} + h^2 \|\phi\|_{H^1(H^3)}), \tag{50}$$

where $C > 0$ is independent of h and Δt. Finally, from the stability result of Proposition 4 and using $e_h^0 = 0$, we get

$$\begin{aligned} ||e_h||_{\ell^\infty(L^2)} + \sqrt{\nu}||\nabla e_h||_{\ell^2(L^2)} &\leq \left(||e_h^1|| + \sqrt{\nu\Delta t}||\nabla e_h^1||\right) + \left(||e_h||_{\ell_2^\infty(L^2)} + \sqrt{\nu}||\nabla e_h^1||_{\ell_2^2(L^2)}\right) \\ &\leq (||e_h^1|| + \sqrt{\nu\Delta t}||\nabla e_h^1||) + C||\tilde{R}_h||_{\ell_2^2(\Psi_h')} \\ &\leq C_1(\Delta t^2 + h^2)||\phi||_{Z^3 \cap H^2(H^2) \cap H^1(H^3)} \end{aligned}$$

which implies the error estimate (24). We employ Lemma 4 for the estimates of the first time step error such that there is no loss of convergence order. □

7 Numerical Experiments

In this section, we present two numerical experiments using the second order LGMM scheme (6) combined with the moving mesh method (12) that show the benefits of the new scheme and verify the error estimate from Theorem 4. As initial data we take $\phi_h^0 = \Pi_h^0 \phi^0$ from the examples below. To compute the integrals that occur in the scheme we employ the Gauss quadrature of order nine. Since linear finite element spaces are used in our proposed scheme we do not consider higher order quadrature formulae as proposed e.g., in [7]. The linear system appearing in (6) and (12) are iteratively solved using the conjugate gradient (CG) method and successive over-relaxation (SOR) method, respectively. In all experiments we start with an equidistant mesh at the initial time, i.e., for a given $h_0 > 0$ the points $P_1^0, \ldots, P_{N_p}^0$ are such that

$$P_{j+1}^0 - P_j^0 = h_0, \qquad \forall i \in \{1, \ldots, N_p\}. \tag{51}$$

The numerical results obtained by the new LGMM scheme are compared to analogous results by the LG scheme with static mesh, which can be interpreted as LGMM scheme with points satisfying $P_j^n = P_j^0$ for all $i \in \{1, \ldots, N_p\}$ and $n \in \{1, \ldots, N_T\}$ in addition to (51).

Example 1 We consider the domain $\Omega = (-1, 1)$, final time $T = 0.5$ and velocity field $u(x, t) = 1 + \sin(t - x)$ in problem (1). No force field is assumed in this example as we set $f = 0$, and for the boundary conditions we set $g = 0$. We take the initial value $\phi^0 = \phi(\cdot, 0)$ according to the exact solution

$$\phi(x, t) = \exp\left(-\frac{1 - \cos(t - x)}{\nu}\right).$$

We solve Example 1 with diffusion coefficient set to $\nu = 0.01$ and $\nu = 0.0001$. In the moving mesh method (12) we set $\nu_M = \nu$. The integer N determines the discretization of the domain as we choose the initial mesh size $h_0 = 2/N$. The time step size is linearly coupled to the initial mesh size through the relation $\Delta t = 4h_0$. In this example, since the velocity u does not satisfy Hypothesis 1, i.e.,$u_{|\Gamma} \neq 0$, the non-overlapping condition (cf. Theorem 1) might not be met at the boundary. In this case, we allow the nodal points $\{P_i^n\}_{i=1}^{N_p}$ to extend beyond the domain.

In Fig. 1 we show the solution of the LGMM scheme for $N = 512$ and $\nu = 0.01$ in terms of the functions ϕ_h^n together with the corresponding local mesh or partition sizes $h_i^n = P_{i+1}^n - P_i^n$

Table 1 Relative errors and EOCs of LG scheme for Example 1 with $\nu = 0.01$

N	Δt	$E_{\ell^\infty(L^2)}$	EOC	$E_{\ell^2(H_0^1)}$	EOC	E_{mass}
128	6.25×10^{-2}	2.795558×10^{-3}	–	4.621785×10^{-3}	–	1.931562×10^{-5}
256	3.12×10^{-2}	8.085728×10^{-4}	1.79	1.296162×10^{-3}	1.83	1.389046×10^{-6}
512	1.56×10^{-2}	2.221100×10^{-4}	1.86	3.445636×10^{-4}	1.91	1.527363×10^{-6}
1024	7.81×10^{-3}	5.927475×10^{-5}	1.91	9.098049×10^{-5}	1.92	8.199302×10^{-8}
2048	3.91×10^{-3}	1.540739×10^{-5}	1.95	2.505214×10^{-5}	1.86	7.994320×10^{-8}
4096	1.95×10^{-3}	3.949271×10^{-6}	1.96	7.976085×10^{-6}	1.64	8.886055×10^{-8}

Table 2 Relative errors and EOCs for of LGMM scheme for Example 1 with $\nu = 0.01$

N	Δt	$E_{\ell^\infty(L^2)}$	EOC	$E_{\ell^2(H_0^1)}$	EOC	E_{mass}
128	6.25×10^{-2}	3.293675×10^{-3}	–	5.441997×10^{-3}	–	1.141478×10^{-6}
256	3.12×10^{-2}	8.756374×10^{-4}	1.91	1.467274×10^{-3}	1.87	5.715086×10^{-6}
512	1.56×10^{-2}	2.265597×10^{-4}	1.95	3.853933×10^{-4}	1.93	9.131147×10^{-7}
1024	7.81×10^{-3}	5.945318×10^{-5}	1.93	8.875689×10^{-5}	2.12	4.549741×10^{-7}
2048	3.91×10^{-3}	1.545287×10^{-5}	1.95	2.439410×10^{-5}	1.87	4.652903×10^{-8}
4096	1.95×10^{-3}	3.948940×10^{-6}	1.96	6.051897×10^{-6}	1.98	1.034399×10^{-7}

Table 3 Relative errors and EOCs of LG scheme for Example 1 with $\nu = 0.0001$

N	Δt	$E_{\ell^\infty(L^2)}$	EOC	$E_{\ell^2(H_0^1)}$	EOC	E_{mass}
128	6.25×10^{-2}	6.127321×10^{-2}	–	1.255443×10^{-1}	–	1.256237×10^{-2}
256	3.12×10^{-2}	1.369196×10^{-2}	2.16	2.916377×10^{-2}	2.10	5.927426×10^{-3}
512	1.56×10^{-2}	3.286310×10^{-3}	2.06	6.026062×10^{-3}	2.27	2.317702×10^{-3}
1024	7.81×10^{-3}	1.045305×10^{-3}	1.66	1.375878×10^{-3}	2.13	1.144369×10^{-3}
2048	3.91×10^{-3}	5.000259×10^{-4}	1.07	5.729551×10^{-4}	1.27	5.870192×10^{-4}
4096	1.95×10^{-3}	2.650173×10^{-4}	0.91	3.289690×10^{-4}	0.78	2.986775×10^{-4}

for $i = 1, \dots, N_P$ with respect to their distribution over the computational domain. Clearly, the LGMM scheme maintains a high resolution, i.e., small mesh sizes, in the region, where ϕ_h is large, whereas regions with small ϕ_h are partly significantly lower resolved.

Tables 1, 2, 3 and 4 show the errors and the corresponding experimental orders of convergence (EOC)[1] of both the LGMM and the LG scheme of second-order after (initial) grid refinement, i.e., iteratively increasing N. In the tables we consider discretization errors with

[1] The EOC is computed by the formula $\text{EOC} = \log_2(E^1/E^2)$ with E^1 and E^2 denoting the corresponding error in two consecutive lines of the table.

Table 4 Relative errors and EOCs of LGMM scheme for Example 1 with $\nu = 0.0001$

N	Δt	$E_{\ell^\infty(L^2)}$	EOC	$E_{\ell^2(H_0^1)}$	EOC	E_{mass}
128	6.25×10^{-2}	1.021001×10^{-1}	–	2.825924×10^{-1}	–	4.833423×10^{-4}
256	3.12×10^{-2}	1.898798×10^{-2}	2.42	4.633479×10^{-2}	2.60	4.112482×10^{-5}
512	1.56×10^{-2}	5.634064×10^{-3}	1.75	1.141006×10^{-2}	2.02	4.714461×10^{-8}
1024	7.81×10^{-3}	8.094441×10^{-4}	2.80	1.244332×10^{-3}	3.20	3.034628×10^{-6}
2048	3.91×10^{-3}	2.574393×10^{-4}	1.66	6.381969×10^{-4}	0.97	5.883933×10^{-7}
4096	1.95×10^{-3}	6.442978×10^{-5}	1.99	1.598421×10^{-4}	1.99	3.556603×10^{-9}

respect to $L^2(\Omega)$, $H^1(\Omega)$ and the loss of total mass, defined as:

$$E_{\ell^\infty(L^2)} := \frac{\|\phi_h - \Pi_h\phi\|_{\ell^\infty(L^2)}}{\|\Pi_h\phi\|_{\ell^\infty(L^2)}}, \qquad E_{\ell^2(H_0^1)} := \frac{\|\phi_h - \Pi_h\phi\|_{\ell^2(H_0^1)}}{\|\Pi_h\phi\|_{\ell^2(H_0^1)}},$$

$$E_{\text{mass}} := \frac{\left|\int_\Omega \phi_h^{N_T}\,dx - \int_\Omega (\Pi_h\phi)^{N_T}\,dx\right|}{\left|\int_\Omega (\Pi_h\phi)^{N_T}\,dx\right|},$$

where $\|\phi\|_{\ell^2(H_0^1)} := \|\nabla\phi\|_{\ell^2(L^2)}$ and Π_h denotes the time dependent Lagrange interpolation operator at time instance t^n given as a mapping $\Pi_h(t^n) : C^0(\bar{\Omega}) \to \Psi_h^n$. Due to Theorem 4 and the coupling between Δt and h we expected experimental convergence order 2 in both the $\ell^\infty(L^2)$ and the $\ell^2(H_0^1)$ (semi-) norm. While the EOCs in the tables mostly support this expectation a decrease in case of higher mesh resolutions for the LG scheme is visible. In the case $\nu = 0.01$ this occurs in $\ell^2(H_0^1)$ and becomes more significant also in $\ell^\infty(L^2)$ in the case $\nu = 0.0001$. The LGMM scheme does not suffer from this decrease in EOC and provides in the affected cases more accurate numerical solution in terms of both norms. The tables further exhibit a low relative loss of mass as E_{mass} is of low magnitude even for coarse grids and further decreases as the mesh is refined. While the mesh movement of the LGMM scheme leads to slightly larger E_{mass} on fine meshes in comparison to the LG scheme if $\nu = 0.01$ the loss of mass for the LGMM scheme is significantly smaller than for the LG scheme if $\nu = 0.0001$.

Remark 11 1. Readers might find some of the EOC result of Example 1 for $N = 2048, 4096$ to be unusual. In fact, we observed that the "strange" errors in Example 1 are due to numerical integration errors. In the current computation, we used a numerical integration formula of degree 9. When we use a numerical integration formula of degree 21, we achieve EOCs of approximately 2. Therefore, we can say that our LGMM scheme reduces numerical integration errors, particularly when using high-degree quadrature formulas. We provide a grid convergence study using numerical integration of degree 21 in Appendix G.1.

2. The theoretical analysis does not require the restriction $g = 0$. We provide an additional example similar to Example 1 with a non-zero boundary condition in Appendix G.2.

Example 2 We consider the domain $\Omega = (-1, 1)$, final time $T = 2$, velocity field $u(x, t) = \sin(2\pi x)$ and diffusion coefficient $\nu = 10^{-5}$ in problem (1). Again we take $f = 0$ and $g = 0$. The initial datum is set to $\phi^0(x) = \exp[-100(1 - \cos(x))]$.

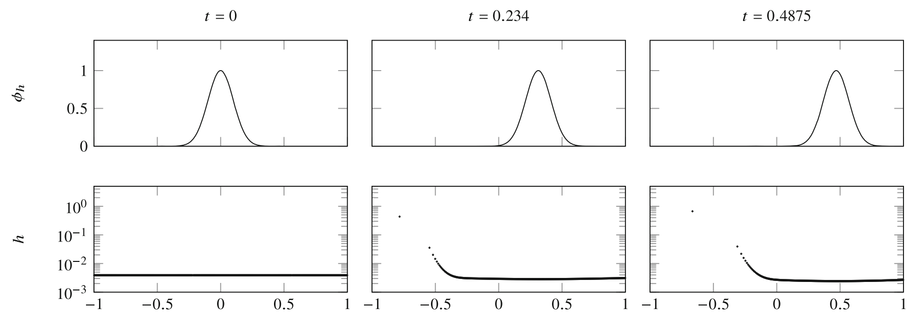

Fig. 1 Numerical solution ϕ_h and corresponding mesh sizes in Example 1 over the computational domain at time instances $t = 0$ (left), $t = 0.2340$ (center) and $t = 0.4875$ (right) obtained by the LGMM scheme for $\nu = 0.01$ and $N = 512$

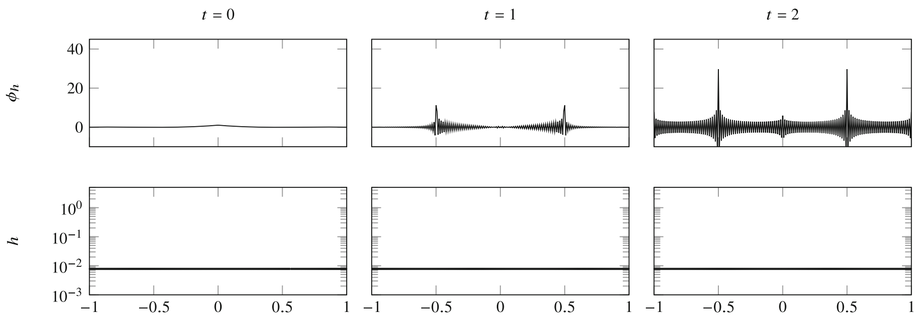

Fig. 2 Numerical solution ϕ_h and corresponding mesh sizes in Example 2 over the computational domain at time instances $t = 0$ (left), $t = 1$ (center) and $t = 2$ (right) obtained by the LG scheme with fixed mesh ($N = 256$). The numerical solution exhibits oscillations

We solve Example 2 using scheme (6) combined with the moving mesh method (12), using the parameter $\nu_M = \nu$, an initial uniform mesh satisfying (51) for $h_0 = 2/1024$ and the fixed time step size $\Delta t = 10^{-4}$, which satisfies condition (13) during the computation. Again the results by the new LGMM scheme are compared to the LG scheme with static mesh. Comparing Figs. 2 and 3, we can observe that while the uniform mesh scheme leads to an oscillating solution, the LGMM scheme is capable to capture the aggregation phenomena. This simulation shows the advantage of the proposed LGMM scheme in capturing sharp spike pattern as observed in bio-medical applications.

8 Conclusion

In this work, we have equipped the mass-preserving Lagrange–Galerkin scheme of second-order in time with a moving mesh method giving rise to the LGMM scheme, which is capable of numerically solving convection–diffusion problems in one space dimension. We also establish the stability and error estimates of the proposed numerical scheme, the latter being with respect to the $\ell^\infty(L^2) \cap \ell^2(H_0^1)$ -norm, of order $O(\Delta t + h^2)$ if the one-step method is used in time and of order $O(\Delta t^2 + h^2)$ if the two-step scheme is used in time. We show numerical results which support the proved error estimates. To this end we have derived a new estimate for the time dependent interpolation operator, which we then embedded in the

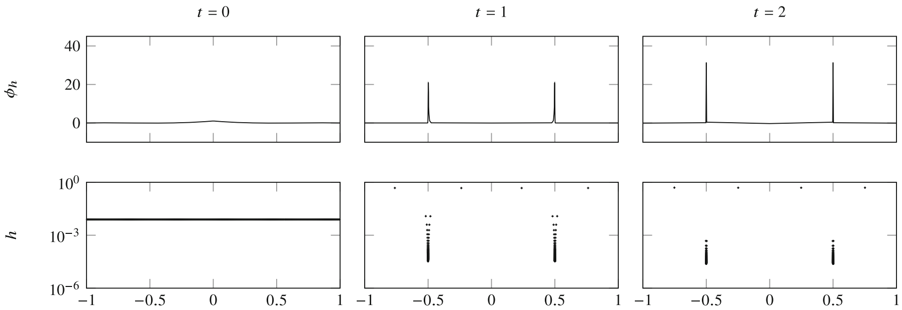

Fig. 3 Numerical solution ϕ_h and corresponding mesh sizes in Example 2 over the computational domain at time instances $t = 0$ (left), $t = 1$ (center) and $t = 2$ (right) obtained by the LGMM scheme with $N = 256$. The nodal points aggregate along with the solution ϕ_h

error estimate framework for the Lagrange–Galerkin method. The numerical simulations also show that the proposed LGMM scheme is capable to capture aggregation phenomena. We believe that the LGMM scheme can be extended to the cases $d = 2, 3$; though this extension may not be straightforward for all element types used in the spatial discretization. While we anticipate that our method will perform well with P1 triangular elements, more complex elements or those employing higher-order interpolation, such as P2, may require additional considerations or modifications to ensure efficiency of the method. In forthcoming research we consider extensions of our scheme to multidimensional problems as well as applications to real-world problems, especially from biology such as immune system dynamics and cancer growth, in which diffusion and aggregation play crucial roles.

Appendix

A Proof of Lemma 1

In the following, we assume $t \in [t^{n-1}, t^n]$ and $x \in [P_k(t), P_{k+1}(t)]$ and use the notations $\phi_k = \phi(P_k(t), t)$ and $\phi_{k+1} = \phi(P_{k+1}(t), t)$. By our choice of t and x the linear basis functions evaluate as

$$\psi_i(x,t) = \begin{cases} \frac{P_{k+1}(t)-x}{P_{k+1}(t)-P_k(t)} & i = k \\ \frac{x-P_k(t)}{P_{k+1}(t)-P_k(t)} & i = k+1 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, we note the identity

$$w_k^n \psi_k + w_{k+1}^n \psi_{k+1} = \frac{x(w_{k+1}^n - w_k^n) + P_{k+1} w_k^n - P_k w_{k+1}^n}{P_{k+1} - P_k}. \tag{52}$$

and obtain the time derivatives

$$\frac{\partial}{\partial t}\psi_k = \frac{x(P_{k+1}' - P_k') - P_{k+1}' P_k + P_{k+1} P_k'}{(P_{k+1} - P_k)^2},$$

$$\frac{\partial}{\partial t}\psi_{k+1} = -\frac{x(P_{k+1}' - P_k') + P_k' P_{k+1} - P_k P_{k+1}'}{(P_{k+1} - P_k)^2}.$$

Employing the fact that due to (9) it holds $P_i'(t) = w_i^n$ we compute

$$\begin{aligned} I(x,t) &= \sum_{i=1}^{N_p} \phi(P_i(t),t)\frac{\partial}{\partial t}\psi_i(x,t) \\ &= \phi_k \frac{\partial}{\partial t}\psi_k + \phi_{k+1}\frac{\partial}{\partial t}\psi_{k+1} \\ &= \phi_k \frac{x(w_{k+1}^n - w_k^n) - w_{k+1}^n P_k + P_{k+1} w_k^n}{(P_{k+1}-P_k)^2} \\ &\quad - \phi_{k+1}\frac{x(w_{k+1}^n - w_k^n) + w_k^n P_{k+1} - P_k w_{k+1}^n}{(P_{k+1}-P_k)^2} \\ &= -\frac{(\phi_{k+1}-\phi_k)}{(P_{k+1}-P_k)^2}\left[x(w_{k+1}^n - w_k^n) + (P_{k+1}w_k^n - P_k w_{k+1}^n)\right] \\ &= -\frac{(\phi_{k+1}-\phi_k)}{P_{k+1}-P_k}\left[w_{k+1}^n\psi_{k+1}(x,t) + w_k^n\psi_k(x,t)\right], \end{aligned}$$

where (52) has been used in the last step. This concludes the proof of Lemma 1. □

B Bounds in the Ψ′-Norm

In this appendix we compute estimates for $\|A_1^n\|_{\Psi'}$, $\|A_2^n\|_{\Psi'}$, and $\|A_3^n\|_{\Psi'}$. To this end we take $v \in H_0^1(\Omega)$ satisfying $\|v\|_{H^1} \leq 1$. We start by showing the estimate (34) on $\|A_1^n\|_{\Psi'}$ and therefore define for $k \in \{1,\dots,N_p-1\}$ the average $v_k := \frac{1}{h_k}\int_{K_k} v(x)\,dx$ and the function

$$Q_k(t) := \int_0^1\int_0^{s_0} (\nabla^2\phi)(P_k(t) + s_1 h_k(t), t)ds_1 ds_0$$

for brevity of notation. Then we estimate

$$\begin{aligned} |(A_1^n, v)| &= \left|\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\sum_{k=1}^{N_p-1} h_k(t)w_k^n Q_k(t)\int_{K_k}\big(\psi_k(x,t) - \psi_{k+1}(x,t)\big)\,v(x)dxdt\right| \\ &\leq c\frac{h}{\Delta t}\|w\|_{C^0(L^\infty)}\int_{t^{n-1}}^{t^n}\|\phi\|_{H^1(H^3)}\sum_{k=1}^{N_p-1}\int_{K_k}\left|\big(\psi_k(x,t)-\psi_{k+1}(x,t)\big)\big(v(x)-v_k\big)\right|dxdt \\ &\leq c\frac{h}{\Delta t}\|w\|_{C^0(L^\infty)}\|\phi\|_{H^1(H^3)}\int_{t^{n-1}}^{t^n}\sum_{k=1}^{N_p-1}\|\psi_k(\cdot,t)-\psi_{k+1}(\cdot,t)\|_{L^2(K_k)}\|v-v_k\|_{L^2(K_k)}dt \\ &\leq c\frac{h}{\Delta t}\|w\|_{C^0(L^\infty)}\|\phi\|_{H^1(H^3)}\int_{t^{n-1}}^{t^n}\left(\sum_{k=1}^{N_p-1}\frac{h_k}{3}\right)^{1/2}\left(\sum_{k=1}^{N_p-1}h_k^2|v|_{H^1(K_k)}^2\right)^{1/2}dt \\ &\leq c'h^2\|w\|_{C^0(L^\infty)}\|\phi\|_{H^1(H^3)}||v||_{H^1}, \end{aligned}$$

where we have used the Cauchy-Schwartz inequality, the Poincaré inequality as well as the following bound and identities

$$Q_k(t) \leq c\|\phi\|_{H^1(H^3)}, \quad \int_{K_k}(\psi_{k+1}-\psi_k)\,dx = 0, \quad \int_{K_k}(\psi_{k+1}-\psi_k)^2\,dx = \frac{h_k}{3}.$$

Hence, we obtain the estimate

$$\|A_1^n\|_{\Psi'} = \sup_{\|v\|_{H^1}\le 1} (A_1^n, v) \le ch^2 \|w\|_{C^0(L^\infty)} \|\phi\|_{H^1(H^3)}.$$

To show the estimate (35) on $\|A_2^n\|_{\Psi'}$ we note that the bound

$$\sum_{k=1}^{N_p-1} \int_{K_k} \left| \int_0^1 \int_0^{s_0} \int_{s_1}^{1-s_1} (\nabla^3\phi)(P_k(t) + s_2 h_k(t), t) ds_2 ds_1 ds_0 \, v(x) \right| dx \\ \le c\|\phi(\cdot, t)\|_{H^3(\Omega)} \|v\|_{L^2(\Omega)}.$$

holds and compute

$$\begin{aligned} |(A_2^n, v)| &= \left| \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \sum_{k=1}^{N_p-1} h_k^2(t) \int_{K_k} \psi_{k+1}(x, t) w_k^n \right. \\ &\quad \left. \times \int_0^1 \int_0^{s_0} \int_{s_1}^{1-s_1} (\nabla^3\phi)(P_k(t) + s_2 h_k(t), t) ds_2 ds_1 ds_0 \, v(x) dx dt \right| \\ &\le c \frac{h^2}{\Delta t} \|w\|_{C^0(L^\infty)} \int_{t^{n-1}}^{t^n} \|\phi(\cdot, t)\|_{H^3} dt \|v\|_{L^2} \\ &= c' \frac{h^2}{\sqrt{\Delta t}} \|w\|_{C^0(L^\infty)} \|\phi\|_{L^2(t^{n-1}, t^n; H^3(\Omega))} \|v\|_{L^2}. \end{aligned}$$

Embedding $L^2(\Omega)$ in Ψ', we thus obtain the estimate

$$\|A_2^n\|_{\Psi'} \le \frac{ch^2}{\sqrt{\Delta t}} \|w\|_{C^0(L^\infty)} \|\phi\|_{L^2(t^{n-1}, t^n; H^3(\Omega))} \le ch^2 \|w\|_{C^0(L^\infty)} \|\phi\|_{H^1(H^3)}.$$

Lastly, to show (36) we define

$$R_k(t) := \int_0^1 \int_0^{s_0} (\nabla^2\phi)(P_k(t) + (1 - s_1) h_k(t), t) ds_1 ds_0$$

and note the bounds

$$|R_k(t)| \le c\|\phi\|_{H^1(H^3)}, \qquad \left| \int_0^1 (\nabla w^n)(P_k(t) + s h_k(t)) ds \right| \le c\|w\|_{C^0(W^{1,\infty})},$$

which allow us to estimate

$$|(A_3^n, v)| = \left| \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \sum_{k=1}^{N_p-1} h_k^2(t) \int_{K_k} \psi_{k+1}(x, t) \int_0^1 (\nabla w^n)(P_k(t) + s h_k(t)) ds R_k(t) dt \, v(x) \, dx \right| \\ \le ch^2 \|w\|_{C^0(W^{1,\infty})} \|\phi\|_{H^1(H^3)} \|v\|_{L^2},$$

where we have used the Cauchy–Schwartz inequality. Hence, we obtain

$$\|A_3^n\|_{\Psi'} \le ch^2 \|w\|_{C^0(W^{1,\infty})} \|\phi\|_{H^1(H^3)},$$

which implies (36). □

C Residual bounds

In this part of the appendix we derive the bounds for the residual terms that are used in the proofs of Theorems 3 and 4 in Sects. 6.2 and 6.3.

C.1 Bound for the Term R_1 in (40)

To derive an estimate for R_1^n we write $R_1^n = I_1^n + I_2^n$, where

$$
\begin{aligned}
I_1^n &:= \frac{\partial \phi^n}{\partial t} + u^n \nabla \phi^n - \frac{\phi^n - \phi^{n-1} \circ X_1^n}{\Delta t}, \\
I_2^n &:= (\nabla \cdot u^n)\phi^n - \frac{\phi^{n-1} \circ X_1^n (1-\gamma^n)}{\Delta t}.
\end{aligned}
$$

We first consider the term I_1^n. The computation

$$
\begin{aligned}
&\phi^n(x) - [\phi^{n-1} \circ X_1^n](x) \\
&\quad = \phi(x, t^n) - \phi(x - u^n(x)\Delta t, t^{n-1}) \\
&\quad = -\int_0^1 \frac{\partial}{\partial s}[\phi(x - su^n(x)\Delta t, t^n - s\Delta t)]\, ds \\
&\quad = \int_0^1 u^n(x)\Delta t\, \nabla \phi(x - su^n(x)\Delta t, t^n - s\Delta t) \\
&\qquad + \Delta t \frac{\partial \phi}{\partial t}(x - su^n(x)\Delta t, t^n - s\Delta t) ds \\
&\quad = \Delta t \int_0^1 \left[\frac{\partial \phi}{\partial t} + u^n(x)\, \nabla \phi\right](x - su^n(x)\Delta t, t^n - s\Delta t) ds.
\end{aligned}
$$

shows that we can write this term in the form

$$
\begin{aligned}
I_1^n(x) &= \int_0^1 \left[\frac{\partial \phi}{\partial t} + u^n(x)\, \nabla \phi\right](x, t^n) ds \\
&\quad - \int_0^1 \left[\frac{\partial \phi}{\partial t} + u^n(x)\, \nabla \phi\right](x - su^n(x)\Delta t, t^n - s\Delta t) ds \\
&= -\int_0^1 \left[\left[\frac{\partial \phi}{\partial t} + u^n(x)\, \nabla \phi\right](x - s_1 u^n(x)\Delta t, t^n - s_1\Delta t)\right]\Bigg|_{s_1=0}^{s} ds \\
&= -\int_0^1 \int_0^s \frac{\partial}{\partial s_1}\left[\frac{\partial \phi}{\partial t} + u^n(x)\, \nabla \phi\right](x - s_1 u^n(x)\Delta t, t^n - s_1\Delta t)\, ds_1\, ds \\
&= \Delta t \int_0^1 \int_0^s \left(\left[\frac{\partial}{\partial t} + u^n(x)\, \nabla\right]^2 \phi\right)(x - s_1 u^n(x)\Delta t, t^n - s_1\Delta t)\, ds_1\, ds.
\end{aligned}
$$

Using the Cauchy-Schwartz inequality, we estimate

$$\begin{aligned}|I_1^n(x)| &\le \Delta t \int_0^1 \left(\int_0^s \left(\left[\frac{\partial}{\partial t} + u^n(x)\,\nabla \right]^2 \phi \right) (x - s_1 u^n(x)\Delta t, t^n - s_1 \Delta t)^2\, ds_1 \right)^{1/2} \left(\int_0^s ds_1 \right)^{1/2} ds \\ &\le \Delta t \int_0^1 \left(\int_0^s \left(\left[\frac{\partial}{\partial t} + u^n(x)\,\nabla \right]^2 \phi \right) (x - s_1 u^n(x)\Delta t, t^n - s_1 \Delta t)^2\, ds_1 \right)^{1/2} ds \\ &\le \Delta t \left(\int_0^1 \left(\left[\frac{\partial}{\partial t} + u^n(x)\,\nabla \right]^2 \phi \right) (x - s_1 u^n(x)\Delta t, t^n - s_1 \Delta t)^2\, ds_1 \right)^{1/2}.\end{aligned}$$

Hence, it holds

$$\begin{aligned}\|I_1^n(x)\|^2 &\le \Delta t^2 \int_\Omega \int_0^1 \left(\left[\frac{\partial}{\partial t} + u^n(x)\,\nabla \right]^2 \phi \right) (x - s_1 u^n(x)\Delta t, t^n - s_1 \Delta t)^2\, ds_1\, dx \\ &\le C_1 \Delta t^2 \int_0^1 \int_\Omega \left(\left[\frac{\partial}{\partial t} + \nabla \right]^2 \phi \right) (x - s_1 u^n(x)\Delta t, t^n - s_1 \Delta t)^2 dx\, ds_1.\end{aligned}$$

Let $y{:=}x - s_1 u^n(x)\Delta t$ and $\tau{:=}t^n - s_1\Delta t$. Then, by a change of variable, we have

$$\begin{aligned}\|I_1^n(x)\|^2 &\le C_2 \Delta t^2 \int_0^1 \int_\Omega \left(\left[\frac{\partial}{\partial t} + \nabla \right]^2 \phi \right) (y, t^n - s_1 \Delta t)^2 dy\, ds_1 \\ &= C_2 \Delta t^2 \int_{t^{n-1}}^{t^n} \frac{1}{\Delta t} \int_\Omega \left(\left[\frac{\partial}{\partial t} + \nabla \right]^2 \phi \right) (y, \tau)^2 dy\, d\tau \\ &= C_2 \Delta t \left\| \left[\frac{\partial}{\partial t} + \nabla \right]^2 \phi \right\|^2_{L^2(t^{n-1}, t^n; L^2(\Omega))} \\ &\le C_3 \Delta t \|\phi\|^2_{Z^2(t^{n-1}, t^n)}.\end{aligned}$$

By taking the square root, we obtain

$$\|I_1^n(x)\|_{L^2(\Omega)} \le C_4 \sqrt{\Delta t} \|\phi\|_{Z^2(t^{n-1}, t^n)}.$$

On the other hand, we note that $\frac{(1-\gamma^n)}{\Delta t} = \nabla \cdot u^n + \mathcal{O}(\Delta t)$, which using (39b) leads to

$$\|I_2^n\| = \|\nabla \cdot u^n (\phi^n - \phi^{n-1} \circ X_1^n) + [\phi^{n-1} \circ X_1^n] \mathcal{O}(\Delta t)\| \le C_5 \sqrt{\Delta t} \|\phi\|_{Z^1(t^{n-1}, t^n)}.$$

By combining the estimates of I_1^n and I_2^n, we have

$$\|R_1\|_{\ell^2(\Psi_h')} \le \left(\Delta t \sum_{n=1}^{N_T} \|R_1^n\|^2 \right)^{1/2} \le C_6 \Delta t \|\phi\|_{Z^2(0,T)},$$

where $C_6 > 0$ is independent of h and Δt. □

C.2 Bound for the Term $\tilde{R}_1$ in (47)

First, we note that γ^n and $\tilde{\gamma}^n$ can be written as

$$\gamma^n(x) = 1 - \Delta t \nabla u^n(x), \qquad \tilde{\gamma}^n(x) = 1 - 2\Delta t \nabla u^n(x).$$

Therefore, the term $\tilde{R}_1^n$ in (47) is recasted as

$$
\begin{aligned}
\tilde{R}_1^n &= \frac{1}{2\Delta t}\left(3\phi^n - 4\phi^{n-1}\circ X_1^n\gamma^n + \phi^{n-2}\circ \tilde{X}_1^n\tilde{\gamma}^n\right) - \left[\frac{\partial \phi}{\partial t} + \nabla\cdot(u^n\phi^n)\right](\cdot, t^n)\\
&= \left[\frac{1}{2\Delta t}\left(3\phi^n - 4\phi^{n-1}\circ X_1^n + \phi^{n-2}\circ \tilde{X}_1^n\right) - \left(\frac{\partial \phi}{\partial t} + u^n\nabla\cdot\phi^n\right)\right]\\
&\quad + \nabla\cdot u^n[2\phi^{n-1}\circ X_1^n - \phi^{n-2}\circ\tilde{X}_1^n - \phi^n]\\
&=: I_1^n + I_2^n.
\end{aligned}
$$

Proceeding in analogy to C.2, let $y(x,s) = y(x,s;n) := x - u^n(x)(1-s)\Delta t$ and $t(s) = t(s;n) := t^{n-1} + s\Delta t$. Then, the terms I_1^n and I_2^n can be expressed in terms of the integrals

$$
I_1^n(x) = -2\Delta t^2 \int_0^1 s\int_{2s-1}^{s}\left[\left(\frac{\partial}{\partial t} + u^n(x)\,\nabla\right)^3\phi\right](y(x,s_1), t(s_1))ds_1\,ds,
$$

$$
I_2^n(x) = -\Delta t^2(\nabla\cdot u^n)(x)\int_0^1\int_{s-1}^{s}\left[\left(\frac{\partial}{\partial t} + u^n(x)\,\nabla\right)^2\phi\right](y(x,s_1), t(s_1))ds_1 ds.
$$

Now, we can estimate

$$
\begin{aligned}
\|I_1^n\| &= 2\Delta t^2\left\|\int_0^1 s\int_{2s-1}^{s}\left[\left(\frac{\partial}{\partial t} + u^n(x)\cdot\nabla\right)^3\phi\right](y(\cdot,s_1), t(s_1))ds_1\,ds\right\|\\
&\le C_1\Delta t^2\int_0^1 s\int_{2s-1}^{s}\left\|\left[\left(\frac{\partial}{\partial t} + \nabla\right)^3\phi\right](y(\cdot,s_1), t(s_1))\right\|ds_1\,ds\\
&\le C_2\Delta t^2\int_0^1 s\int_{2s-1}^{s}\left\|\left[\left(\frac{\partial}{\partial t} + \nabla\right)^3\phi\right](\cdot, t(s_1))\right\|ds_1\,ds\\
&\le C_3\Delta t\int_{t^{n-2}}^{t^n}\left\|\left[\left(\frac{\partial}{\partial t} + \nabla\right)^3\phi\right](\cdot, t)\right\|dt\\
&\le \sqrt{2}C_3\Delta t^{3/2}\left\|\left(\frac{\partial}{\partial t} + \nabla\right)^3\phi\right\|_{L^2(t^{n-2},t^n;L^2)}\\
&\le C_4\Delta t^{3/2}\|\phi\|_{Z^3(t^{n-2},t^n)}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\|I_2^n\| &\le C_5\Delta t^2\int_0^1 s\int_{s-1}^{s}\left\|\left[\left(\frac{\partial}{\partial t} + \nabla\right)^2\phi\right](y(\cdot,s_1), t(s_1))\right\|ds_1\,ds\\
&\le C_6\Delta t\int_{t^{n-2}}^{t^n}\left\|\left[\left(\frac{\partial}{\partial t} + 1\cdot\nabla\right)^2\phi\right](\cdot, t)\right\|dt\\
&\le C_7\Delta t^{3/2}\|\phi\|_{Z^2(t^{n-2},t^n)}.
\end{aligned}
$$

By combining the bounds of I_1^n and I_2^n and taking the ℓ^2-norm over all time instances, we obtain

$$
\|\tilde{R}_1\|_{\ell^2(\Psi_h')} \le \left(\Delta t\sum_{n=1}^{N_T}\|R_1^n\|^2\right)^{1/2} \le C_8\Delta t^2\|\phi\|_{Z^3(0,T)},
$$

where $C_8 > 0$ is independent of h and Δt. □

C.3 Bound for the Term R_3^n in (40) and (47)

We compute an estimate for R_3^n in $(\Psi_h^n)'$. To this end let $v_h \in \Psi_h^n$ be such that $\|v_h\|_{H_1} = 1$ and further let $K_i = [P_i(t^n), P_{i+1}(t^n)]$ for all $i \in \{1, \dots, N_p - 1\}$. Then we have, employing the fact that for $v_h \in \Psi_h^n$ the function $\nabla v_h|_{K_i}$ is constant

$$\begin{aligned} |\langle R_3^n, v_h\rangle| &= |a_0(\eta^n, v_h)| \\ &\le |a_0(\phi, v_h) - a_0(\Pi_h\phi, v_h)| \\ &\le \nu \sum_{i=1}^{N_p-1} \left| \int_{K_i} \left(\nabla\phi\nabla v_h - \frac{\phi^n(P_{i+1}) - \phi^n(P_i)}{h_i} \nabla v_h \right) dx \right| \\ &= \nu \sum_{i=1}^{N_p-1} |\nabla v_h|_{K_i}| \left| \phi^n(P_{i+1}) - \phi^n(P_i) - \frac{\phi^n(P_{i+1}) - \phi^n(P_i)}{h_i} \int_{K_i} dx \right| = 0. \end{aligned}$$

Hence, it follows that $R_3^n = 0 \in (\Psi_h^n)'$. □

D Proofs of the Mass Preserving Properties

To prove Proposition 1 and 3 we first state anotherproposition that will be used in the proofs.

Proposition 5 ([20, 32]) *Suppose that Hypotheses 1, 2 and 4 hold true. Then it holds that* $X_1^n(\Omega) = \tilde{X}_1^n(\Omega) = \Omega$ *and* $1/2 \le \gamma^n, \tilde{\gamma}^n \le 3/2$ *for* $n = 0, \dots, N_T$.

D.1 Proof of Proposition 1

Suppose that Hypotheses 1, 2 and 4 hold true. By Proposition 5 and a change of variable $y = X_1^n(x)$, it holds for all $\rho \in \Psi$, $n = 1, \dots, N_T$ that

$$\int_\Omega \rho \circ X_1^n(x)\gamma^n(x)dx = \int_\Omega \rho dx.$$

We prove the theorem by induction. Let $m \in \{2, \dots, N_T\}$ and assume that (18) holds true for $n = m - 1$. By substituting $1 \in \Psi_h$ into ψ_h in the scheme (4), we obtain

$$\begin{aligned} \int_\Omega \phi_h^m(x)dx &= \int_\Omega \phi_h^{m-1} \circ X_1^m(x)\gamma^m(x)dx + \Delta t \left(\int_\Omega f^m(x)dx + \int_\Gamma g^m(x)ds \right) \\ &= \int_\Omega \phi_h^0(x)dx + \Delta t \sum_{i=1}^m \left(\int_\Omega f^i(x)dx + \int_\Gamma g^i(x)ds \right), \end{aligned}$$

which proves (18). □

D.2 Proof of Proposition 3

Suppose that Hypothesis 1, 2 and 4 holds true. By Proposition 5 and a change of variable $y = \tilde{X}_1^n(x)$, it holds for all $\rho \in \Psi$, $n = 1, \ldots, N_T$ that

$$\int_\Omega \rho \circ \tilde{X}_1^n(x)\tilde{\gamma}^n(x)dx = \int_\Omega \rho dx.$$

We prove (i) of the proposition by induction. Let $m \in \{2, \ldots, N_T\}$ and assume that (21) holds true for $n = m - 1$. By substituting $1 \in \Psi_h$ into ψ_h in scheme (6), we obtain

$$\begin{aligned}
\int_\Omega \left(\frac{3}{2}\phi_h^m - \frac{1}{2}\phi_h^{m-1}\right) dx &= \int_\Omega \left(\frac{3}{2}\phi_h^m - \frac{1}{2}\phi_h^{m-1} \circ X_1^m \gamma^m\right) dx \\
&= \int_\Omega \left(\frac{3}{2}\phi_h^{m-1} \circ X_1^m \gamma^m - \frac{1}{2}\phi_h^{m-2} \circ \tilde{X}_1^m \tilde{\gamma}^m\right) dx + \Delta t \left(\int_\Omega f^m(x)dx + \int_\Gamma g^m(x)ds\right) \\
&= \int_\Omega \left(\frac{3}{2}\phi_h^{m-1} - \frac{1}{2}\phi_h^{m-2}\right) dx + \Delta t \left(\int_\Omega f^m(x)dx + \int_\Gamma g^m(x)ds\right) \\
&= \int_\Omega \left(\frac{3}{2}\phi_h^1 - \frac{1}{2}\phi_h^0\right) dx + \Delta t \sum_{i=2}^m \left(\int_\Omega f^i(x)dx + \int_\Gamma g^i(x)ds\right) \\
&= \int_\Omega \frac{1}{2}(\phi_h^0 + \phi_h^1)dx + \Delta t \sum_{i=1}^m \left(\int_\Omega f^i(x)dx + \int_\Gamma g^i(x)ds\right),
\end{aligned}$$

which implies (21).

We prove (ii). As $f = g = 0$, the identity $\int_\Omega \phi_h^1 dx = \int_\Omega \phi_h^0 dx$ holds from scheme (6) with $n = 1$, which implies similarly, $\int_\Omega \phi_h^2 dx = \int_\Omega \phi_h^0 dx$ from the scheme with $n = 2$. Using the same argument for all n, we obtain (22). □

E Proofs of the Stability Results

E.1 Proof of Proposition 2

We substitute $\phi_h^n \in \Psi_h$ into the numerical scheme (4) and obtain

$$\left(\frac{\phi_h^n - (\phi_h^{n-1} \circ X_1^n)\gamma^n}{\Delta t}, \phi_h^n\right) + \nu\|\nabla\phi_h^n\|^2 = \langle F^n, \phi_h^n\rangle. \tag{53}$$

By Young's inequality, the functional on the right hand side of (53) can be estimated as

$$\langle F^n, \phi_h^n\rangle \le \left(\frac{1}{2\nu}\|F^n\|^2_{(\Psi_h^n)'} + \frac{\nu}{2}\|\phi_h^n\|^2_{(\Psi_h^n)'}\right). \tag{54}$$

Let the first term on the left hand side of (53) be denoted by I_n, then the following lower bound holds

$$\begin{aligned}
I_n &:= \left(\frac{\phi_h^n - (\phi_h^{n-1} \circ X_1^n)\gamma^n}{\Delta t}, \phi_h^n\right) \\
&= \frac{1}{\Delta t}\left[\frac{1}{2}\|\phi_h^n\|^2 - \frac{1}{2}\|(\phi_h^{n-1} \circ X_1^n)\gamma^n\|^2 + \frac{1}{2}\|\phi_h^n - (\phi_h^{n-1} \circ X_1^n)\gamma^n\|^2\right] \\
&\ge \frac{1}{\Delta t}\left[\frac{1}{2}\|\phi_h^n\|^2 - \frac{1}{2}\|(\phi_h^{n-1} \circ X_1^n)\gamma^n\|^2\right].
\end{aligned}$$

Since $\gamma^n - 1 = \mathcal{O}(\Delta t)$ it holds

$$\begin{aligned}\|(\phi_h^{n-1} \circ X_1^n)\gamma^n\| &= \|(\phi_h^{n-1} \circ X_1^n)\gamma^n - \phi_h^{n-1} \circ X_1^n + \phi_h^{n-1} \circ X_1^n\| \\ &\leq \|(\phi_h^{n-1} \circ X_1^n)(\gamma^n - 1)\| + \|(\phi_h^{n-1} \circ X_1^n)\| \\ &\leq C_1 \Delta t \|(\phi_h^{n-1} \circ X_1^n)\| + \|(\phi_h^{n-1} \circ X_1^n)\| \\ &= (1 + C_1 \Delta t)\|(\phi_h^{n-1} \circ X_1^n)\| \\ &\leq (1 + 2C_2 \Delta t)\|\phi_h^{n-1}\|.\end{aligned}$$

Then, I_n can be written as

$$I_n \geq \frac{1}{\Delta t}\left[\frac{1}{2}\|\phi_h^n\|^2 - \frac{1}{2}\|\phi_h^{n-1}\|^2\right] - C_2\|\phi_h^{n-1}\|^2. \tag{55}$$

By substituting (55) and (54) into (53) we obtain

$$\frac{1}{\Delta t}\left[\frac{1}{2}\|\phi_h^n\|^2 - \frac{1}{2}\|\phi_h^{n-1}\|^2\right] + \frac{\nu}{2}\|\nabla\phi_h^n\|^2 \leq \frac{1}{2\nu}\|F^n\|^2_{(\Psi_h^n)'} + c"\|\phi_h^{n-1}\|^2. \tag{56}$$

To complete the proof we apply the Gronwall inequality to (56) and get

$$\|\phi_h\|_{\ell^\infty(L^2)} + \sqrt{\nu}\|\phi_h\|_{\ell^2(H_0^1)} \leq C\left[\|\phi_h^0\| + \|F\|_{\ell^2(\Psi_h')}\right],$$

where the constant $C > 0$ is independent of Δt and h. □

E.2 Proof of Proposition 4

First we state the following lemma that will be used in the last part of the proof.

Lemma 5 (Gronwall's Inequality [20]) *Let a_0, a_1 and a_2 be non-negative numbers such that $a_1 \geq a_2$, let further $\Delta t \in (0, 3/(4a_0)]$ and $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 1}$, $\{z_n\}_{n\geq 2}$, $\{b_n\}_{n\geq 2}$ be non-negative sequences. Suppose that*

$$\frac{1}{\Delta t}\left(\frac{3}{2}x_n - 2x_{n-1} + \frac{1}{2}x_{n-2} + y_n - y_{n-1}\right) + z_n \leq a_o x_n + a_1 x_{n-1} + a_2 x_{n-2} + b_n, \quad \forall n \leq 2 \tag{57}$$

is satisfied. Then it holds that

$$x_n + \frac{2}{3}y_n + \frac{2}{3}\Delta t\sum_{i=2}^{n} z_i \leq (\exp(2(a_0 + a_1 + a_2)n\Delta t) + 1)\left(x_0 + \frac{3}{2}x_1 + y_1 + \Delta t\sum_{i=2}^{n} b_i\right), \quad \forall n \leq 2. \tag{58}$$

Proof of Proposition 4 For $n \geq 2$, note that the scheme (5b) can be written as

$$\left(\frac{3\phi_h^n - 4\phi_h^{n-1} + \phi_h^{n-2}}{2\Delta t}, \psi_h\right) + a_0(\phi_h^n, \psi_h) = \langle F^n, \psi_h\rangle + \langle I_h^n, \psi_h\rangle, \quad \forall \psi_h \in \Psi_h^n, \tag{59}$$

where $I_h^n \in (\Psi_h^n)'$ is given by

$$
\begin{aligned}
I_h^n :=& \frac{1}{2\Delta t}\left[-4(\phi_h^{n-1} - \phi_h^{n-1}\circ X_1^n\gamma^n) + (\phi_h^{n-2} - \phi_h^{n-2}\circ \tilde{X}_1^n\tilde{\gamma}^n)\right],\\
=& \frac{1}{2\Delta t}\left[-4(\phi_h^{n-1} - \phi_h^{n-1}\circ X_1^n) + (\phi_h^{n-2} - \phi_h^{n-2}\circ \tilde{X}_1^n)\right]\\
&+ \frac{1}{2\Delta t}\left[-4(\phi_h^{n-1}\circ X_1^n - \phi_h^{n-1}\circ X_1^n\gamma^n) + (\phi_h^{n-2}\circ \tilde{X}_1^n - \phi_h^{n-2}\circ \tilde{X}_1^n\tilde{\gamma}^n)\right]\\
:=& I_{h1}^n + I_{h2}^n
\end{aligned}
$$

for $n \in \{2, \dots, N_T\}$. By substituting $\phi_h^n \in \Psi_h$ as ψ_h into (6) we have

$$
\left(\frac{3\phi_h^n - 4\phi_h^{n-1} + \phi_h^{n-2}}{2\Delta t}, \phi_h^n\right) + \nu\|\nabla\phi_h^n\|^2 = \langle F^n, \phi_h^n\rangle + \langle I_h^n, \phi_h^n\rangle. \tag{60}
$$

The first term of the left hand side can thereby be estimated as

$$
\begin{aligned}
&\left(\frac{3\phi_h^n - 4\phi_h^{n-1} + \phi_h^{n-2}}{2\Delta t}, \phi_h^n\right)\\
&= \frac{1}{\Delta t}\left[\frac{3}{4}\|\phi_h^n\|^2 - \|\phi_h^{n-1}\|^2 + \frac{1}{4}\|\phi_h^{n-2}\|^2 + \frac{1}{4}\|\phi_h^n - 2\phi_h^{n-1} + \phi_h^{n-2}\|^2\right]\\
&\quad + \frac{1}{\Delta t}\left[\frac{1}{2}\left(\|\phi_h^n - \phi_h^{n-1}\|^2 - \|\phi_h^{n-1} - \phi_h^{n-2}\|^2\right)\right]\\
&\geq \frac{1}{\Delta t}\left[\frac{3}{4}\|\phi_h^n\|^2 - \|\phi_h^{n-1}\|^2 + \frac{1}{4}\|\phi_h^{n-2}\|^2\right]\\
&\quad + \frac{1}{\Delta t}\left[\frac{1}{2}\left(\|\phi_h^n - \phi_h^{n-1}\|^2 - \|\phi_h^{n-1} - \phi_h^{n-2}\|^2\right)\right].
\end{aligned}
$$

Conversely, the terms on the right hand side can be estimated as

$$
\begin{aligned}
\langle F^n, \phi_h^n\rangle &\leq \|F^n\|_{(\Psi_h^n)'}\|\phi_h^n\|_{H^1(\Omega)}\\
&\leq \|F^n\|_{(\Psi_h^n)'}(\|\phi_h^n)\| + \|\nabla\phi_h^n\|)\\
&\leq \frac{1}{8}\|\phi_h^n\|^2 + \frac{\nu}{4}\|\nabla\phi_h^n\|^2 + (2 + 1/\nu)\|F^n\|^2_{(\Psi_h^n)'},\\
\|I_{h1}^n\|_{(\Psi_h^n)'} &\leq C(\|\phi_h^{n-1} + \|\phi_h^{n-2}\|) \quad (\because \text{Lemma 3 (38c)}),\\
\|I_{h2}^n\| &\leq \frac{c}{\Delta t}\left(\|\phi_h^{n-1}\circ X_1^n(1-\gamma^n)\| + \|\phi_h^{n-2}\circ \tilde{X}_1^n(1-\tilde{\gamma}^n)\|\right)\\
&\leq c_1(\|\phi_h^{n-1}\| + \|\phi_h^{n-2}\|) \quad (\because (1-\tilde{\gamma}^n) \leq c_1\Delta t, \text{Lemma 3 (38a)}),\\
\langle I_h^n, \phi_h^n\rangle &\leq \|I_{h1}^n\|_{(\Psi_h^n)'}\|\phi_h^n\|_{\Psi_h} + \|I_{h2}^n\|\|\phi_h^n\|_{\Psi_h}\\
&\leq \|I_{h1}^n\|_{(\Psi_h^n)'}(\|\phi_h^n\| + \|\nabla\phi_h^n\|) + \|I_{h2}^n\|\|\phi_h^n\|_{\Psi_h}\\
&\leq \left(2 + \frac{1}{\nu}\right)\|I_{h1}^n\|^2_{(\Psi_h^n)'} + 2\|I_{h2}^n\|^2 + \frac{1}{4}\|\phi_h^n\|^2 + \frac{\nu}{4}\|\nabla\phi_h^n\|^2\\
&\leq \frac{1}{4}\|\phi_h^n\|^2 + \frac{\nu}{4}\|\nabla\phi_h^n\|^2 + c_2\left(\frac{1}{2}\|\phi_h^{n-1}\|^2 + \frac{1}{2}\|\phi_h^{n-2}\|^2\right).
\end{aligned}
$$

Then, by combining the above estimates (60) can be rewritten as

$$\frac{1}{\Delta t}\left[\frac{3}{4}\|\phi_h^n\|^2 - \|\phi_h^{n-1}\|^2 + \frac{1}{4}\|\phi_h^{n-2}\|^2 + \frac{1}{2}(\|\phi_h^n - \phi_h^{n-1}\|^2 - \|\phi_h^{n-1} - \phi_h^{n-2}\|^2)\right]$$
$$+ \frac{\nu}{2}\|\nabla\phi_h^n\|^2 \leq \frac{3}{8}\|\phi_h^n\|^2 + c_1\left(\frac{1}{2}\|\phi_h^{n-1}\|^2 + \|\phi_h^{n-2}\|^2\right) + c_2\|F^n\|^2_{(\Psi_h^n)'}.$$

To complete the proof of Proposition 4, we apply the Gronwall inequality from Lemma 5 and obtain

$$\|\phi_h\|_{\ell_2^\infty(L^2)} + \sqrt{\nu}\|\nabla\phi_h\|_{\ell_2^\infty(L^2)} \leq C(\|\phi_h^0\| + \|\phi_h^1\| + \|F\|_{\ell^2(\Psi_h')}),$$

where $C > 0$ is independent of Δt and h. By combining this estimate with Proposition 2, the proof of Proposition 4 is completed. □

F Proof of Lemma 4

Proof Recalling the calculation of $||R_1^n||_{(\Psi_h^n)'}$, cf. C.1, the bound of $||R_2^n||_{(\Psi_h^n)'}$ from (42), and taking into account the fact $R_3^n = 0$, it holds that

$$\begin{aligned}||R_h^1|| &\leq c_1\left(\sqrt{\Delta t}||\phi||_{Z^2(t^0,t^1)} + \frac{h^2}{\sqrt{\Delta t}}\|\phi\|_{H^1(t^0,t^1;H^2)} + h^2\|\phi\|_{H^1(H^3)}\right)\\ &\leq c_2\left(\Delta t||\phi||_{Z^3} + h^2||\phi||_{H^2(H^2)\cap H^1(H^3)}\right) \leq c_3(\Delta t + h^2)||\phi||_{Z^3\cap H^2(H^2)\cap H^1(H^3)}.\end{aligned} \tag{61}$$

By substituting e_h^1 into ψ_h in (40), dropping the positive term $a_0(e_h^1, e_h^1)$, taking into account $e_h^0 = 0$, $\langle R_h^1, e_h^1\rangle \leq \|R_h^1\|\|e_h^1\|$, and using (61), we get

$$\begin{aligned}\|e_h^1\| &\leq \Delta t\|R_h^1\| \leq \Delta t c_1(\Delta t + h^2)\|\phi\|_{Z^3\cap H^2(H^2)\cap H^1(H^3)}\\ &\leq c_2(\Delta t^2 + h^2)\|\phi\|_{Z^3\cap H^2(H^2)\cap H^1(H^3)}.\end{aligned}$$

Similarly, substituting e_h^1 into ψ_h in (40), taking into account $e_h^0 = 0$, $\langle R_h^1, e_h^1\rangle \leq \|R_h^1\|\|e_h^1\|$, and using (61), we get

$$\begin{aligned}\|e_h^1\|^2 + \nu\Delta t\|\nabla e_h^1\|^2 &\leq \Delta t\|R_h^1\|\|e_h^1\|\\ &\leq \Delta t c_1(\Delta t + h^2)(\Delta t^2 + h^2)\|\phi\|^2_{Z^3\cap H^2(H^2)\cap H^1(H^3)}\\ &\leq c_2(\Delta t^2 + h^2)^2\|\phi\|^2_{Z^3\cap H^2(H^2)\cap H^1(H^3)},\end{aligned}$$

which implies (45). □

G Additional Numerical Results

G.1 Example 1 with Higher Order Integration Formula

We simulate Example 1 using the Gauss quadrature formula of degree 21. As we can see in Tables 5 and 6, we achieve an EOC of approximately 2. While the LG scheme yields slightly more accurate results in $\ell^2(H_0^1)$ on fine grids we note that a similar accuracy is obtained by the LGMM scheme using the lower order integration formula, which has smaller computational cost.

G.2 Example of Non-zero Boundary Condition Problem

We consider the domain $\Omega = (0, 1)$, final time $T = 0.5$, diffusion coefficient $\nu = 0.0001$, and velocity field $u(x, t) = \sin(\pi x)$ in problem 1. We take the Neumann boundary conditions as

$$g_N(x,t) = \begin{cases} -\nu\pi\cos(\pi(x+t)), & (x,t) \in \{0\} \times [0,T], \\ \nu\pi\cos(\pi(x+t)), & (x,t) \in \{1\} \times [0,T]. \end{cases}$$

We set the external force f and the initial value ϕ^0 appropriately so that the following exact solution is

$$\phi(x,t) = \sin(\pi(x+t)).$$

Table 5 Relative errors and EOCs for the LG scheme with high order quadrature for Example 1 with $\nu = 10^{-4}$

N	Δt	$E_{\ell^\infty(L^2)}$	EOC	$E_{\ell^2(H_0^1)}$	EOC	E_{mass}
128	6.25×10^{-2}	6.838965×10^{-2}	–	1.390206×10^{-1}	–	2.576334×10^{-3}
256	3.12×10^{-2}	1.363359×10^{-2}	2.322	3.159460×10^{-2}	2.140	6.867530×10^{-4}
512	1.56×10^{-2}	3.561592×10^{-3}	1.938	7.527114×10^{-3}	2.072	4.880633×10^{-5}
1024	7.81×10^{-3}	8.752015×10^{-4}	2.022	1.778242×10^{-3}	2.086	2.986503×10^{-5}
2048	3.91×10^{-3}	2.083146×10^{-4}	2.073	4.125544×10^{-4}	2.108	1.518585×10^{-5}
4096	1.95×10^{-3}	4.925582×10^{-5}	2.085	9.721731×10^{-5}	2.089	6.912510×10^{-6}

Table 6 Relative errors and EOCs for the LGMM scheme with high order quadrature for Example 1 with $\nu = 10^{-4}$

N	Δt	$E_{\ell^\infty(L^2)}$	EOC	$E_{\ell^2(H_0^1)}$	EOC	E_{mass}
128	6.25×10^{-2}	1.014336×10^{-1}	–	2.762122×10^{-1}	–	6.870175×10^{-4}
256	3.12×10^{-2}	1.573545×10^{-2}	2.689	3.981974×10^{-2}	2.798	1.210617×10^{-5}
512	1.56×10^{-2}	4.384326×10^{-3}	1.840	1.077595×10^{-2}	1.887	8.170278×10^{-6}
1024	7.81×10^{-3}	1.038217×10^{-3}	2.079	2.580781×10^{-3}	2.066	6.160171×10^{-7}
2048	3.91×10^{-3}	2.823952×10^{-4}	1.881	6.766483×10^{-4}	1.933	2.721251×10^{-8}
4096	1.95×10^{-3}	4.911175×10^{-5}	2.526	1.301657×10^{-4}	2.374	3.394361×10^{-8}

Table 7 Relative errors and EOCs for the LG scheme for the example with non-zero boundary condition

N	Δt	$E_{\ell^\infty(L^2)}$	EOC	$E_{\ell^2(H_0^1)}$	EOC	E'_{mass}
128	3.12×10^{-2}	2.298854×10^{-2}	–	7.654998×10^{-2}	–	6.991448×10^{-3}
256	1.56×10^{-2}	5.857277×10^{-3}	1.973	1.971024×10^{-2}	1.953	1.783230×10^{-3}
512	7.81×10^{-3}	1.479036×10^{-3}	1.983	4.995912×10^{-3}	1.973	4.495670×10^{-4}
1024	3.91×10^{-3}	3.716389×10^{-4}	1.992	1.257303×10^{-3}	1.987	1.127530×10^{-4}
2048	1.95×10^{-3}	9.314610×10^{-5}	1.996	3.153476×10^{-4}	1.996	2.822066×10^{-5}
4096	9.77×10^{-4}	2.331617×10^{-5}	1.998	7.896535×10^{-5}	1.997	7.057941×10^{-6}

Table 8 Relative errors and EOCs for the LGMM scheme for the example with non-zero boundary condition

N	Δt	$E_{\ell^\infty(L^2)}$	EOC	$E_{\ell^2(H_0^1)}$	EOC	E'_{mass}
128	3.12×10^{-2}	2.303668×10^{-2}	–	7.657135×10^{-2}	–	6.997073×10^{-3}
256	1.56×10^{-2}	5.872575×10^{-3}	1.972	1.973625×10^{-2}	1.951	1.783821×10^{-3}
512	7.81×10^{-3}	1.483052×10^{-3}	1.985	5.003768×10^{-3}	1.972	4.496149×10^{-4}
1024	3.91×10^{-3}	3.726406×10^{-4}	1.992	1.259449×10^{-3}	1.987	1.127605×10^{-4}
2048	1.95×10^{-3}	9.341311×10^{-5}	1.996	3.158620×10^{-4}	1.996	2.821955×10^{-5}
4096	9.77×10^{-4}	2.338526×10^{-5}	1.998	7.909588×10^{-5}	1.997	7.057869×10^{-6}

The integer N determines the discretization of the domain as we choose the initial mesh size $h_0 = 1/N$. The time step size is linearly coupled to the initial mesh size through the relation $\Delta t = 4h_0$. Tables 7 and 8 show the numerical convergence of both the LGMM and the LG schemes in this example and again confirm our theoretical results. Since the mass of the exact solution at $t = T$ is 0, we introduce another suitable error for the mass defined by

$$E'_{\text{mass}} := \frac{\max\limits_{n=1,\dots,N_T} \left| \int_\Omega \phi_h^n dx - \int_\Omega (\Pi_h \phi)^n dx \right|}{\max\limits_{n=1,\dots,N_T} \left| \int_\Omega (\Pi_h \phi)^n dx \right|},$$

which is also relatively low.

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