Orthogonal Determinants of Finite Groups of Lie Type

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Abstract

An *orthogonal* representation of a finite group G is a homomorphism $\rho : G \to \mathrm{GL}_n(K)$, for a natural number n and a field $K \subseteq \mathbb{R}$. Analogously, we say a character χ of G is orthogonal if any corresponding representation is orthogonal.

Nebe (2022) showed that for an orthogonal character $\chi \in \mathrm{Irr}(G)$ of even degree ($\chi \in \mathrm{Irr}^+(G)$), there exists a unique element

$$\det(\chi) := d \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2,$$

such that for any representation $\rho : G \to \mathrm{GL}_n(K)$ affording χ over an arbitrary field $K/\mathbb{Q}(\chi)$ and all $\rho(G)$-invariant, non-degenerate bilinear forms β, it holds that

$$\det(\beta) = d \cdot (K^\times)^2.$$

We say that $\det(\chi)$ is the *orthogonal determinant* of χ.

As part of the classification of finite simple groups, the groups of Lie type form the largest class among them. Examples of finite groups of Lie type include $\mathrm{SL}_n(q), \mathrm{GL}_n(q)$ and $\mathrm{SU}_n(q)$ for q a prime power.

The goal of this thesis is to present methods for the calculation of the orthogonal determinants of the finite groups of Lie type. Let $G := G(q)$ be a finite group of Lie type with parameter q and let $\chi \in \mathrm{Irr}^+(G)$. Given that q is odd, we show that there is some sort of "Jordan decomposition" of $\det(\chi) = \det(\chi_U)\det(\chi_T)$, i.e., a decomposition into a unipotent part $\det(\chi_U)$ and a semisimple part $\det(\chi_T)$.

In contrary to the relatively easy determination of $\det(\chi_U)$, the calculation of $\det(\chi_T)$ proves to be a challenge. For that, we apply the theory of Iwahori–Hecke algebras, which are deformations of Coxeter groups, and extensions thereof.

The thesis consists of 6 chapters. After the introduction, the following two chapters establish the theory of orthogonal determinants and finite groups of Lie type. Afterwards we will consider Coxeter groups, where the orthogonal determinants of all Coxeter groups, as well as the alternating groups and some Iwahori–Hecke algebras, are covered. In the fifth chapter, we will describe orthogonal determinants of finite groups of Lie type, where we will also consider some examples like $\mathrm{SL}_3(q)$ and $G_2(q)$. In the final chapter, we handle the groups $\mathrm{GL}_n(q)$, where we accomplish a complete description of the orthogonal determinants.

Zusammenfassung

Eine *orthogonale* Darstellung einer endlichen Gruppe G ist ein Homomorphismus $\rho : G \to \mathrm{GL}_n(K)$, für eine natürliche Zahl n und einen Körper $K \subseteq \mathbb{R}$. Analog nennen wir einen Character χ von G orthogonal, falls eine zugehörige Darstellung orthogonal ist.

Nebe (2022) hat gezeigt, dass wenn $\chi \in \mathrm{Irr}(G)$ ein orthogonaler Character von geradem Grad ist ($\chi \in \mathrm{Irr}^+(G)$), ein eindeutiges Element

$$\det(\chi) := d \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$$

existiert, sodass für alle Darstellungen $\rho : G \to \mathrm{GL}_n(K)$ mit Character χ über einen beliebigen Körper $K/\mathbb{Q}(\chi)$ und alle $\rho(G)$-invarianten, nicht-ausgearteten Bilinearformen β gilt, dass

$$\det(\beta) = d \cdot (K^\times)^2.$$

Wir nennen $\det(\chi)$ die *orthogonale Determinante* von χ.

Im Rahmen der Klassifikation der endlichen einfachen Gruppen bilden die Gruppen vom Lie-Typ die größte Klasse unter diesen. Beispiele von endlichen Gruppen vom Lie-Typ beinhalten $\mathrm{SL}_n(q), \mathrm{GL}_n(q)$ und $\mathrm{SU}_n(q)$ für q eine Primzahlpotenz.

Das Ziel dieser Dissertation ist, Methoden zur Berechnung von den orthogonalen Determinanten für endlichen Gruppe vom Lie-Typ vorzustellen. Sei $G := G(q)$ eine endliche Gruppe vom Lie-Typ mit Parameter q und $\chi \in \mathrm{Irr}^+(G)$. Wir zeigen, dass wenn q ungerade ist, eine Art "Jordan-Zerlegung" von $\det(\chi) = \det(\chi_U)\det(\chi_T)$ existiert, also eine Zerlegung in einen unipotenten Teil $\det(\chi_U)$ und einen halbeinfachen Teil $\det(\chi_T)$.

Im Gegensatz zur relativ einfachen Berechnung von $\det(\chi_U)$ erweist sich die Bestimmung von $\det(\chi_T)$ als eine Herausforderung. Hierfür verwenden wir die Theorie der Iwahori–Hecke Algebren, was Deformationen von Coxeter-Gruppen sind, und Erweiterungen hiervon.

Die Arbeit besteht aus 6 Kapiteln. Nach der Einleitung wird in zwei Kapiteln die Theorie von orthogonalen Determinanten und der Gruppen vom Lie-Typ eingeführt. Anschließend folgt ein Kapitel über Coxeter-Gruppen, wo die orthogonalen Determinanten von allen Coxeter-Gruppen, als auch die der alternierenden Gruppen und einiger Iwahori–Hecke Algebren, bestimmt werden. Im fünften Kapitel beschreiben wir dann die orthogonalen Determinanten für die Gruppen vom Lie-Typ, wo wir auch einige Beispiele wie $\mathrm{SL}_3(q)$ und $G_2(q)$ betrachten. Abschließend behandeln wir im letzten Kapitel die Gruppen $\mathrm{GL}_n(q)$, wo wir eine komplette Beschreibung erreichen.

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Contents

1 Introduction

A common question in mathematics is: Given an object, what is its automorphism group? Representation theory of (finite) groups flips the question around: Given a (finite) group, what are the objects it acts upon? Let us fix a finite group G and a field K. A K-representation of G is a pair (V, ρ) where V is a finite-dimensional K-vector space and ρ is a homomorphism $\rho : G \to \mathrm{GL}(V)$. By abuse of notation, we will often omit the homomorphism ρ and say that V is a representation of G. We say that the representation (V, ρ) is reducible if there is a nontrivial subspace $V' \subsetneq V$ such that (V', ρ) is a subrepresentation, i.e., $\rho(g)v \in V'$ for all $g \in G, v \in V'$, and say that ρ is irreducible else. A K-representation V is absolutely irreducible if for any field extension L of K the L-representation $L \otimes V$ is irreducible. We will restrict ourselves to the case of the characteristic of K being equal to 0, for multiple reasons.

On the one hand, by Maschke's theorem, the group algebra KG is semisimple, i.e., for any representation V and a subrepresentation $V' \subseteq V$, there exists a subrepresentation $V'' \subseteq V$ with $V = V' \bigoplus V''$.

On the other hand, the isomorphism type of (V, ρ) is characterized by its character, which is the map

$$\chi_\rho : G \to K, g \mapsto \mathrm{trace}(\rho(g)).$$

We say that (V, ρ) affords the character χ_ρ. The value of χ_ρ does not depend on the conjugacy class, i.e., $\chi_\rho(C)$ is well-defined for C a conjugacy class of G.

If K is additionally algebraically closed, e.g., $K = \mathbb{C}$, more can be said. Then, there is a 1-to-1 correspondence between the conjugacy classes of G and

$$\mathrm{Irr}(G) := \{\chi_\rho \mid \rho : G \to \mathrm{GL}_n(\mathbb{C}) \text{ is irreducible}, n \geq 1\}.$$

We say that a character is irreducible if it is an element of $\mathrm{Irr}(G)$. If $\{C_1, \ldots, C_n\}$ are the conjugacy classes of G and $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_n\}$ are its irreducible characters, we call the square matrix $(\chi_i(C_j))_{1 \leq i,j \leq n}$ the (ordinary) character table of G. It is clear that any character is a sum of irreducible ones. Moreover, there is an inner product of characters given by

$$\langle \psi_1, \psi_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \psi_1(g)\overline{\psi_2(g)} \in \mathbb{Z}_{\geq 0}$$

for characters ψ_1, ψ_2 such that $\langle \chi_i, \chi_j \rangle_G = \delta_{ij}$. In particular, the decomposition of a character into irreducible ones is fully computable with the character table.

We fix a character $\chi \in \mathrm{Irr}(G)$. What can be said about a representation V affording χ? The simplest available information is the dimension of V. Indeed, $\dim(V) = \chi(1)$, which we will also call the degree of χ. A slightly more involved invariant is the field generated by the values of χ, which we denote by

$$\mathbb{Q}(\chi) := \mathbb{Q}(\{\chi(g) \mid g \in G\})$$

and call the character field of χ. It is not true that an irreducible character can be afforded by a representation over the character field, the smallest counterexample is the character of degree 2 of the quaternion group Q_8, which has the character field $\mathbb{Q}$ but can not be afforded by a representation over the real numbers, although there is a representation over for instance $\mathbb{Q}(i)$ affording it. The minimal index $[L : \mathbb{Q}(\chi)]$ of a field extension $L \supseteq \mathbb{Q}(\chi)$ such that there is a representation over L affording χ is called the Schur index of χ. In general, the Schur index is hard to compute. The related question, of whether a character with real values can be afforded by a representation over the real numbers, has an elegant solution though. The Frobenius–Schur indicator is the value

$$\iota(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^2) \in \{-1, 0, 1\}$$

and we have that

$$\iota(\chi) = \begin{cases} 1, & \text{if } \chi \text{ has real values and can be afforded by a real representation,} \\ -1, & \text{if } \chi \text{ has real values and can not be afforded by a real representation,} \\ 0 & \text{if } \mathbb{Q}(\chi) \text{ is not real.} \end{cases}$$

We say that χ has indicator "+" (resp. "-", resp. "0") if $\iota(\chi) = 1$ (resp. $\iota(\chi) = -1$, resp. $\iota(\chi) = 0$). If $\iota(\chi) = -1$, then 2χ can be afforded by a real representation, and if $\iota(\chi) = 0$, then $\chi + \overline{\chi}$ can.

Let us explore real representations a bit more thorougly. We let ψ be a character of G, not necessarily irreducible. From the previous discussion it follows that ψ can be afforded by a real representation if and only if it is of the form

$$\psi = \sum_{i=1}^{r} a_i \psi_i^{(+)} + 2 \sum_{j=1}^{s} b_i \psi_j^{(-)} + \sum_{k=1}^{t} c_i (\psi_k^{(0)} + \overline{\psi_k}^{(0)}), \tag{1.1}$$

where $\psi_i^{(+)}$ (resp. $\psi_j^{(-)}$, resp. $\psi_k^{(0)}$) are irreducible characters of G with Frobenius–Schur indicator "+" (resp. "-", resp. "0"), and a_i, b_j, c_k are non-negative integers. The character ψ is then also called an orthogonal character, which can be explained by the following construction:

Let (W, ρ) be a K-representation affording the orthogonal character ψ with $K \subseteq \mathbb{R}$, and let $\beta : W \times W \to K$ be a positive-definite symmetric bilinear form. Then,

$$\beta' : W \times W \to K, \beta'(w, w') = \sum_{g \in G} \beta(\rho(g)w, \rho(g)w')$$

is a positive-definite and therefore non-degenerate, symmetric, $\rho(G)$-invariant form, i.e.,

$$\beta'(\rho(g)w, \rho(g)w') = \beta'(w, w')$$

for all $g \in G$, $w, w' \in W$. The other direction also holds, so ψ is afforded by a representation (W, ρ) having a non-degenerate, symmetric, $\rho(G)$-invariant form if and only if ψ can be afforded by a real representation. In particular, the property of a character being orthogonal can be calculated with the character table. A new question arises: Given an orthogonal character, what can we say about these bilinear forms?

Let $w_1, \ldots, w_n$ be a basis of W. The square class

$$\det(\beta') := \det(\beta'(w_i, w_j)_{1\leq i,j\leq n}) \cdot (K^\times)^2$$

is independent of the chosen basis and is called the determinant of β'. If we let $a \in K^\times$, then also $a\beta'$ is a non-degenerate, symmetric, $\rho(G)$-invariant bilinear form and $\det(a\beta') = a^n \det(\beta')$. In particular, if n is odd, no meaningful determinant can be associated to ψ. We call ψ orthogonally stable, if all the $\psi_j^{(+)}$ characters appearing in the sum 1.1 above have even degree. In [NP22] Nebe and Parker showed that if ψ is orthogonally stable, there is a unique element $\det(\psi) := d \in \mathbb{Q}(\psi)^\times/(\mathbb{Q}(\psi)^\times)^2$, called the orthogonal determinant of ψ, such that for any field extension $L \supseteq \mathbb{Q}(\psi)$, any L-representation (U, ρ) affording ψ with any non-degenerate, symmetric, ρ-invariant bilinear form γ on U, it holds that $\det(\gamma) = d \cdot (L^\times)^2$.

We will thus assume that ψ is orthogonally stable. It can be shown that we can reduce the computation of $\det(\psi)$ to the orthogonally simple components, i.e., the individual summands of the form $\psi_i^{(+)}$, $2\psi_j^{(-)}$ and $\psi_k^{(0)} + \overline{\psi_k^{(0)}}$. Further,

$$\det(2\psi_j^{(-)}) = 1 \cdot (\mathbb{Q}(2\psi_j^{(-)})^\times)^2$$

and $\det(\psi_k^{(0)} + \overline{\psi_k^{(0)}})$ can be calculated by the character field. Thus, the case for $\det(\psi_i^{(+)})$ remains. Let

$$\mathrm{Irr}^+(G) := \{\varphi \in \mathrm{Irr}(G) \mid \varphi \text{ is an indicator "+" character of even degree}\}.$$

The knowledge of the orthogonal determinants of the $\mathrm{Irr}^+(G)$-characters, together with the character table, allows to compute the orthogonal determinants of all orthogonally stable characters of G, making them interesting invariants to consider. This gives rise to orthogonal character tables — meaning the ordinary character table, with the data of $\det(\chi)$ for every $\chi \in \mathrm{Irr}^+(G)$. In the sequel, to shorten the language, we will just say that we calculate the orthogonal determinants of G, and mean that we calculate the orthogonal determinants of the $\mathrm{Irr}^+(G)$-characters.

An important observation on how to calculate (and even define!) orthogonal determinants is that for any representation $\rho : G \to \mathrm{GL}_n(K)$ affording a $\chi \in \mathrm{Irr}^+(G)$ character, there is some central simple $\mathbb{Q}(\chi)$-algebra

$$A := \langle \rho(g) \mid g \in G \rangle_{\mathbb{Q}(\chi)}$$

with a natural involution $\iota : A \to A$ generated by $\iota(\rho(g)) := \rho(g^{-1})$. Involutions of central simple algebras and their invariants such as determinants have been studied in more generality by Knus, Merkurjev, Rost and Tignol in the "Book of Involutions" [Knu+98].

By choosing a basis and regarding A is a matrix algebra, we can define determinants of the elements of A — it can now be shown that there is an element $h \in A^\times$ such that $\iota(h) = -h$, i.e., h is skew-symmetric, and such that

$$\det(\chi) = \det(h) \cdot (\mathbb{Q}(\chi)^\times)^2.$$

Since this can be defined in general for involutions on central simple algebras of even dimension, this allows to define "orthogonal determinants" for also more general objects than groups.

So far, we made no assumption on the group G. The building blocks of finite groups are simple groups, and often, one is able to reconstruct a lot of information for G from the normal subgroups and the quotients thereof. Instead of trying to calculate the orthogonal determinants for all finite groups, the goal is to at first gather results for the finite simple groups. In a recent and ongoing long term project, Thomas Breuer, Richard Parker, and the author's PhD advisor, Gabriele Nebe, aim to calculate the orthogonal discriminants (which are orthogonal determinants, up to a sign) of the ordinary and Brauer characters of all finite groups in the ATLAS [Con+85], which includes the "small" finite simple groups, see [BNP24].

After the celebrated classification of finite simple groups, the finite simple groups consist of

- the cyclic groups of prime order,
- the alternating groups $\mathfrak{A}_n$ for $n \geq 5$,
- the finite simple groups of Lie type and
- the 26 sporadic groups.

For G one of the cyclic groups, all irreducible characters have degree 1 so the set $\mathrm{Irr}^+(G)$ is empty. The case of symmetric groups was handled by James and Murphy in their paper [JM79] in 1979. Given that, the alternating groups as index 2 normal subgroups of the symmetric groups also become easy to handle. For more details, regard Subsection 4.1.4. As there is only a finite number of the sporadic groups and thus a finite amount of orthogonally simple characters, a "brute force" approach to the calculation of the orthogonal determinants can thus be attempted. So far, with theoretical and computational means, the orthogonal determinants for the sporadic groups up to the Harada-Norton group HN have been calculated.

This leaves us with the biggest class, the finite simple groups of Lie type, which consists of multiple infinite families of simple groups. Examples include $\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q), \mathrm{G}_2(q)$ for q a power of a prime p. In practice, one often considers the non-simple counterparts like $\mathrm{SL}_n(q)$. Since the different values of q give rise to an infinitude of groups, instead of character tables, one considers generic character tables, which group some characters and conjugacy classes together with some suitable parameters. While in the other classes of finite simple groups all character values are known, the same can not be said for the finite groups of Lie type. As an example, not all character values for the groups $\mathrm{SL}_6(q)$ are known. Nevertheless, there are some groups where we have a full picture. The character table of $\mathrm{SL}_2(q)$ was already known to Jordan and Schur in the early 20th century, regard for instance [Bon11] for a thorough modern treatment. Other examples include $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$ in [SF73] by Simpson and Frame in 1973, the groups $\mathrm{Sp}_4(q)$ in [Sri68] by Srinivasan in 1968 for q odd and in [Eno72] by Enomoto in 1972 for q even, and $\mathrm{G}_2(p^f)$ in [CR74] by Chang and Ree in 1974 for $p \geq 5$, in [Eno76] by Enomoto in 1978 for $p = 3$ and in [EY86] by Enomoto and Yamada in 1986 for $p = 2$. Furthermore, the character tables of all the general linear groups $\mathrm{GL}_n(q)$ are fully known by the work of Green in 1955, see [Gre55].

We now want to extend the generic character tables with the knowledge of the orthogonal determinants of the $\mathrm{Irr}^+(G)$-characters for G a finite group of Lie type, so, giving formulas for the orthogonal determinants in dependence of the parameters of the character table. In conclusion, we want to end up with *generic orthogonal character tables*. This has already been done by Braun and Nebe in their paper [BN17] for the groups $\mathrm{SL}_2(q)$.

The words "generic orthogonal character table" alone imply that there is some pattern to the orthogonal determinants. Indeed, there is one observation easily made by looking at the specific square classes of the orthogonal determinants of finite groups: If the character field of an orthogonally stable character χ is equal to the rational numbers (minor adjustments can be made in the case of the character field being a bigger number field), then there always seems to exist an odd integer d such that

$$\det(\chi) = d \cdot (\mathbb{Q}^\times)^2.$$

So in a certain sense, the orthogonal determinants seem to be "odd". This observation lead to a conjecture of Richard Parker, where he conjectured this oddness pattern to hold for all finite groups. This has been proven to hold for solvable groups, see [Neb22a].

This thesis has two main goals. First, we wish to extend the results in [BN17] to other finite groups of Lie type, at least when the character table is known (and q is odd), and present some general methods and results even when the generic character tables are not (yet) known. This leads to the determination of all orthogonal determinants of for instance the groups $\mathrm{G}_2(q)$ and $\mathrm{GL}_n(q)$, for all n, and q odd. Second, we wish to prove Parker's conjecture about the oddness of orthogonal determinants for some families of groups — most importantly, the symmetric groups. As almost a corollary, we can extend the results to the alternating groups, all finite Coxeter groups, and the groups $\mathrm{GL}_n(q)$ for odd q.

Outline

We will go through the structure of this thesis. In Chapter 2, we will give an introduction to the main focus of this thesis, namely the orthogonal determinants for characters of finite groups (and beyond). For these, we naturally first need some notions of bilinear forms. Next, we will introduce what we call monomial algebras — these are algebras H that behave a lot like group algebras, so for instance, they have some involution

$\dagger : H \to H$ (think of taking the inverse of group elements) and some designated basis that behaves well with that. These are algebras where we can talk about characters and also about orthogonal stability. With the results of the "Book of Involutions" [Knu+98], we can define orthogonal determinants for these algebras in Section 2.2. The main reason we want to introduce orthogonal determinants for more general algebras is specialization, i.e., using maps between monomial algebras to translate information about the orthogonal determinants between each other. We will finish this chapter by regarding the "special case" of the orthogonal determinants for group algebras by discussing some specific strategies and results for their computation, as well as presenting the conjecture of Richard Parker about the oddness of orthogonal determinants. One important result about the orthogonal determinants of finite groups, especially in the context of this thesis, is that they can be easily calculated for the characters of p-groups.

Next, Chapter 3 provides some short overview of the theory of finite groups of Lie type and some very basics of their representation theory. The finite groups of Lie type are special finite subgroups of connected reductive groups defined over an algebraically closed field in positive characteristic p. For every finite group of Lie type G in characteristic p, there is an important class of subgroups, the Borel subgroups. Every Borel subgroup B can be decomposed as a semidirect product $B = U \rtimes T$, where U is a p-Sylow subgroup and T is a maximal torus, an abelian subgroup. One crucial class of groups in the theory are the Coxeter groups; examples include the dihedral groups and the symmetric groups. These are groups having an action on the subgroup T and give important information about the structure of G. For the characters, one important class are the principal series characters, which are characters χ of G appearing in $\mathrm{Ind}_B^G(\mathbf{1}_B)$. Their behavior is determined by the Iwahori–Hecke algebras, these are deformations of Coxeter groups and examples of monomial algebras.

Chapter 4 is about the orthogonal determinants of the finite Coxeter groups (and some Iwahori–Hecke algebras), as well as the alternating groups. The chapter is split into four sections, each highlighting different types of Coxeter groups. For the classical types, as well as the groups $I_2(m)$, there will be a quick reminder on the representation theory of the corresponding groups, before the orthogonal determinants are being handled. We start with type A_n (so, the symmetric groups) in Section 4.1. For these groups, James and Murphy developed a combinatorial formula for the orthogonal determinants in [JM79]. The main ingredients of their formula are Young tableaux and the so called sequences of β-numbers, which are sequences related to partitions. To show Parker's conjecture for the symmetric groups, we will associate a non-negative integer $\mathrm{OddRank}(\lambda)$ to every partition λ which measures how "odd" a partition looks, with a partition having maximal rank if and only if the degree of the character $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ is odd. The proof is then done by induction on $\mathrm{OddRank}(\lambda)$. After this, we are going through some more classes on groups that are closely related to the symmetric groups, namely the alternating groups and the Coxeter groups of type B_n and D_n, where we are able to calculate the orthogonal determinants in all cases. For the groups $I_2(m)$, there is an explicit description of all irreducible representations in terms of matrices, which gives us an easy time calculating the orthogonal determinants. Finally, for the exceptional groups, we used computer help and existing tables. As a side effect, we are able to confirm Parker's conjecture for the alternating groups and for all finite Coxeter groups.

In Chapter 5 we discuss some methods to calculate the orthogonal determinants for finite groups of Lie type, at least if their defining characteristic is odd. Let G be such a finite group of Lie type with Borel subgroup $B = U \rtimes T$. The reason why the group B is important for us is threefold: For once, it is relatively big, meaning that for "most" characters $\chi \in \mathrm{Irr}^+(G)$, the restriction to B remains orthogonally stable. We will say that these characters are *Borel-stable*. Second, it is a semidirect product of a p-group and an abelian group, which are both types of groups where we have a full understanding of the orthogonal determinants. In conclusion, if χ is Borel-stable, then we can calculate $\det(\mathrm{Res}_B^G(\chi))$ and it holds that

$$\det(\chi) = \det(\mathrm{Res}_B^G(\chi)) \cdot (\mathbb{Q}(\chi)^\times)^2.$$

Third, even if our character χ is not Borel-stable, we can reduce the calculation of $\det(\chi)$ to that of some related character $\chi_{\mathcal{H}}$ of the *Hecke algebras* $\mathcal{H} = \mathcal{H}(G, B)$ or $\mathcal{H} = \mathcal{H}(G, U \rtimes T^2)$, which are again monomial algebras. Now, the algebras $\mathcal{H}(G, B)$ are Iwahori–Hecke algebras, for which we have covered some cases in the previous chapter, or are (under some mild conditions) quotients of so called Yokonuma–Hecke algebras, allowing some nice descriptions of the underlying algebras. In the last Section 5.3 of this chapter, we apply the methods developed in this chapter to calculate some generic orthogonal determinants of some finite groups of Lie type of small rank.

In the final Chapter 6 of this thesis, we will describe in more detail how the results of the previous chapter allow for the calculation of the orthogonal determinants for the general linear groups. In particular, we will show that Parker's conjecture holds. We end this thesis by giving tables for the $\mathrm{Irr}^+(\mathrm{GL}_4(q))$-characters for an odd q.

Notation

Throughout this thesis, there are numerous more or less common notations used. We will provide a non-complete list here. Let n be a positive integer.

Important classes of groups:

Symbol	Meaning
$\mathfrak{A}_n$	Alternating group on n letters
$\mathfrak{S}_n$	Symmetric group on n letters
C_n	Cyclic group with n elements
D_{2n}	Dihedral group with $2n$ elements

Products of groups:

Let A, B be finite groups.

Symbol	Meaning
$A \times B$	Direct Product of A and B
$A \rtimes B$	Semidirect product of A and B with A normal
$A \wr C_2$	Wreath product, $(A \times A) \rtimes C_2$

Field theory:

Let A be an integral domain. Let K/L be fields. Let k be an arbitrary integer. Let p be a prime and q be a power of p.

Symbol	Meaning
ζ_n	$\exp\left(\frac{2\pi i}{n}\right)$, primitive complex n-th root of unity
$\vartheta_n^{(k)}$	$\zeta_n^k + \zeta_n^{-k} \in \mathbb{R}$
$[L : K]$	Degree of field extension
$A^\times$	Group of invertible elements in A
$(A^\times)^2$	$\{x^2 \mid x \in A^\times\} \subseteq A^\times$, subgroup of squares
$\mathrm{Quot}(A)$	Quotient field of A
$\mathbb{F}_q$	Finite field with q elements
ε_n	Generator of $\mathbb{F}_{q^n}^\times$

Representation theory of finite groups:

Let G be a finite group with subgroups H and N, where N is a normal subgroup. Let A, B be finite groups. Let χ, χ' be characters of G. Let χ_1, χ_2, ψ and τ be characters of A, B, H and G/N, respectively.

Symbol	Meaning
$\iota(\chi)$	Frobenius–Schur indicator of χ, $\iota(\chi) \in \{-1, 0, 1\}$
$\mathbb{Q}(\chi)$	Character field of χ
$\det(\chi)$	Orthogonal determinant of χ in $(\mathbb{Q}(\chi)^\times)/(\mathbb{Q}(\chi)^\times)^2$, given that χ is orthogonally stable
$\mathrm{Res}^G_H(\chi)$	Restriction of χ from G to H
$\mathrm{Ind}^G_H(\psi)$	Induction of ψ from H to G
$\mathrm{Inf}^G_{G/N}(\tau)$	Inflation of τ from G/N to G
$\mathrm{Irr}(G)$	Irreducible characters of G
$\mathrm{Irr}^+(G)$	Irreducible characters of G of even degree and indicator "+"
$\mathrm{Irr}^+_H(G)$	Irreducible characters in $\mathrm{Irr}^+(G)$ such that the restriction to H is orthogonally stable
$\mathbf{1}_G$	Trivial Character of G, $\mathbf{1}_G(g) = 1$ for all $g \in G$.
$\chi_1 \boxtimes \chi_2$	Outer product of χ_1 and χ_2; character of $A \times B$
$\langle \chi, \chi' \rangle_G$	Inner product of the characters χ and χ'

Finite groups of Lie type:

Let p be a prime, q be a power of p. Let G be a finite group of Lie type and let $L \subseteq G$ be a Levi subgroup. Let χ be a character of G and let θ be a cuspidal character of L.

Symbol	Meaning
$R^G_L(\theta)$	Harish-Chandra induction; character of G
${}^*R^G_L(\chi)$	Harish-Chandra restriction; character of L
χ_L	${}^*R^G_L(\chi)$
$\mathrm{Simp}(G \mid (L, \theta))$	$\{\chi \in \mathrm{Irr}(G) \mid \langle \chi, R^G_L(\theta) \rangle_G \neq 0\}$
$\mathrm{Irr}_{PSU}(G)$	Principal series unipotent characters of G
$\mathbb{G}^n_m(q)$	$\mathbb{F}_q^\times \times \mathbb{F}_q^\times \times \cdots \times \mathbb{F}_q^\times$, where $\mathbb{F}_q^\times$ appears n times
$\mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$, $\ldots$	$\mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{SL}_n(\mathbb{F}_q)$, $\mathrm{SU}_n(\mathbb{F}_q)$, $\ldots$

Combinatorics:

Let λ be a partition of n.

Symbol	Meaning
$\mathscr{P}$	Set of all partitions
$\mathscr{P}_n$	Set of partitions of n
$\mathscr{B}$	Set of all sequences of β-numbers
$\mathscr{B}_n$	Set of sequences of β-numbers of n
$\mathscr{B}(\lambda)$	Set of sequences of β-numbers for λ
$[\lambda]$	Young diagram of λ
λ'	Conjugate partition of λ
χ_λ	Irreducible character of $\mathfrak{S}_n$ corresponding to λ
f_λ	$\chi_\lambda(1)$; degree of χ_λ
$[n]_x$	Gaussian polynomial $1 + x + \cdots + x^{n-1}$
$\Phi_n(x)$	n-th cyclotomic polynomial

Miscellaneous:

Let G be a group with H a subgroup. Let $g \in G$.

Symbol	Meaning
$\mathrm{diag}(A_1, \ldots, A_n)$	Block-diagonal matrix with entries the matrices $A_1, \ldots, A_n$
e_n	Finite or infinite sequence $(0, 0, \ldots, 0, 1, 0, \ldots)$ with the 1 in position n
${}^gH$	gHg^{-1}

2 Orthogonal Determinants

In this chapter, we will set the cornerstone for this thesis by introducing orthogonal determinants of monomial algebras. To be more precise, we will talk about the orthogonal determinants of the $\mathrm{Irr}^+(LH)$-characters for H a semisimple monomial algebra over some integral domain A in characteristic 0 and $L/\mathrm{Quot}(A)$ a splitting field, as well as the orthogonal determinants of the orthogonally stable characters of finite groups.

2.1 Bilinear and Hermitian Forms

In this short first section, we will give a basic introduction to the theory of bilinear and Hermitian forms. Of course, there is some deep theory about these forms and also related notions like quadratic forms, but it will for our purposes suffice to talk only about the very basics. Virtually every book covering these topics could thus be named as a source, we will name just two of them, the book *Quadratische Formen* by Kneser [Kne02], which also handles bilinear forms for modules over general commutative rings, and the book *Quadratic and Hermitian Forms* [Sch85] by Scharlau, which nicely introduces the theory of Hermitian forms. We will also briefly talk about involutions and their relation to bilinear forms. A good source for this is *The Book of Involutions* [Knu+98] by Knus, Merkurjev, Rost and Tignol.

2.1.1 Bilinear Forms

Let A be an integral domain of characteristic unequal 2, n a positive integer and V be a n-dimensional free A-module. Let K be the quotient field of A.

Definition 2.1.1. *We call a function $\beta : V \times V \to A$ a bilinear form, if for all $v_1, v_2, v_3 \in V$, $a \in A$, the following holds:*

(i) $\beta(av_1 + v_2, v_3) = a\beta(v_1, v_3) + \beta(v_2, v_3)$,

(ii) $\beta(v_1, av_2 + v_3) = a\beta(v_1, v_2) + \beta(v_1, v_3)$.

We call the bilinear form β non-degenerate, if for any basis $(e_1, \dots, e_n)$ of V we have that $\det(\beta(e_i, e_j)_{1\leq i,j\leq n}) \neq 0$.

Definition 2.1.2. *Let $\beta : V \times V \to A$ be a bilinear form. We call β symmetric (resp. alternating) if for all $v, w \in V$ we have that $\beta(v, w) = \beta(w, v)$ (resp. $\beta(v, v) = 0$).*

Let from now on β be a symmetric, non-degenerate bilinear form on V. We say that two elements $v, w \in V$ are orthogonal, if $\beta(v, w) = 0$. Further, let $U, U' \subseteq V$ be subspaces. We say that U and U' are orthogonal, if $\beta(u, u') = 0$ for all $u \in U, u' \in U'$. We define

$$U^{\perp} := \{v \in V \mid \beta(v, u) = 0 \text{ for all } u \in U\}.$$

Definition 2.1.3. *Let $(e_1, e_2, \dots, e_n)$ be a basis of V. We call the square class*

$$\det(\beta) := \det(\beta(e_i, e_j)_{1 \leq i,j \leq n}) \cdot (A^{\times})^2 \in A/(A^{\times})^2$$

the determinant of β. It is independent of the choice of basis. We will call

$$\operatorname{disc}(\beta) := (-1)^{n(n-1)/2} \det(\beta) \in A/(A^{\times})^2$$

the discriminant of β.

Let $V_K := K \otimes_A V$ and β_K be the associated extension of β to V_K. We fix a basis $(e_1, e_2, \dots, e_n)$ of V, which we will also consider as a basis of V_K. Given this basis, we will define $B \in A^{n \times n}$ to be the matrix corresponding to β, i.e., $v^{tr} B w = \beta(v, w)$ for all $v, w \in V$. We will associate $\operatorname{End}_A(V)$ (resp. $\operatorname{End}_K(V_K)$) with the matrix rings $A^{n \times n}$ (resp. $K^{n \times n}$).

Definition 2.1.4. *(i) Let M be a set. An involution is a function $\epsilon : M \to M$ such that $\epsilon^2 = \mathrm{id}$.*

(ii) Let H be a finite-dimensional associative A-algebra. An involution on H is a A-linear map $\dagger : H \to H$ such that $(h^{\dagger})^{\dagger} = h$ for all $h \in H$ and $(hh')^{\dagger} = (h')^{\dagger} h^{\dagger}$ for all $h, h' \in H$. Note that an involution is nothing more than an isomorphism $\dagger : H \to H^{op}$, the opposite algebra of H.

Definition 2.1.5. *The adjoint involution $\iota_\beta : \operatorname{End}_K(V_K) \to \operatorname{End}_K(V_K)$ is the involution such that*

$$\beta_K(\alpha(v), w) = \beta_K(v, \iota_\beta(\alpha)(w))$$

for all $v, w \in V_K$.

It is clear that the adjoint involution is well-defined. We have a matrix version, according to the association of the endomorphism algebra with the ring of matrices; we will thus define $\iota_B : K^{n \times n} \to K^{n \times n}$,

$$\iota_B(C) := B^{-1} A^{tr} B$$

for $C \in K^{n \times n}$.

For our ease, we will from now on only work with matrices. We define

$$E_{-}(B) := \{C \in K^{n \times n} \mid C = -\iota_B(C)\}.$$

Lemma 2.1.6. *(cf. [Neb22b, Proposition 2.2]) Assume that $\dim(V_K)$ is even. Then there is a matrix $C \in E_{-}(B)$ with $\det(C) \neq 0$. For any such matrix, we have that $\det(\beta_K) = \det(C) \cdot (K^{\times})^2$.*

2.1.2 Hermitian Forms

We fix a field K of characteristic unequal 2 and a field extension L/K of degree 2. As the characteristic is not 2, there is a $\delta \in K$ such that $L = K[\sqrt{\delta}]$. We have that $\mathrm{Gal}(L/K) = 2$ and we let $\sigma : L \to L$ be the nontrivial element of $\mathrm{Gal}(L/K)$ with

$$\sigma(a + b\sqrt{\delta}) = a - b\sqrt{\delta}$$

for $a, b \in K$.

Let n be a non-negative integer and V be a n-dimensional vector space.

Definition 2.1.7. *We call a function $\gamma : V \times V \to L$ a Hermitian form, if for all $v_1, v_2, v_3 \in V$, $a \in L$, the following holds:*

(i) $\gamma(av_1 + v_2, v_3) = a\gamma(v_1, v_3) + \gamma(v_2, v_3)$,

(ii) $\gamma(v_1, av_2 + v_3) = \sigma(a)\gamma(v_1, v_2) + \gamma(v_1, v_3)$,

(iii) $\gamma(v_1, v_2) = \sigma(\gamma(v_2, v_1))$.

We call the Hermitian form γ non-degenerate, if for any basis $(e_1, \ldots, e_n)$ of V we have that $\det(\gamma(e_i, e_j)_{1 \leq i,j \leq n}) \neq 0$.

Lemma 2.1.8. *(cf. [Sch85, Chapter 10, Remark 1.4]) Let γ be a non-degenerate Hermitian form on V. Let $(e_1, \ldots, e_n)$ be a basis of V. We can regard V as a $2n$-dimensional K-vector space with basis*

$$(e_1, \ldots, e_n, \sqrt{\delta}e_1, \ldots, \sqrt{\delta}e_n)$$

and γ gives rise to a non-degenerate, symmetric bilinear form β over K on that space. Then

$$\mathrm{disc}(\beta) = \delta^n \cdot (K^\times)^2.$$

2.2 Monomial Algebras

In this section, we will define the monomial algebras, which are algebras with "nice enough" properties to *comfortably* define orthogonal determinants. We are here stressing the word comfortably — one can create a more general setup. For instance, we are only working in characteristic 0, so orthogonal discriminants for Brauer characters of finite groups in the context of modular representation theory, as is done in [NP22], are not covered by our approach. Nevertheless, monomial algebras in characteristic 0 still cover a wide enough range of cases to be very useful for us. Monomial algebras are a special case of symmetric algebras, and have already appeared (without being called that) in the paper [Gec14], although we do not assume our algebras to be split. We will need some theory of symmetric algebras which we will recall here, where we have used the book [GP00] by Geck and Pfeiffer as a source.

Definition 2.2.1. *Let A be an integral domain and let H be a finite-dimensional associative A-algebra. An A-linear map $\tau : H \to A$ such that $\tau(hh') = \tau(h'h)$ is called a trace function. We say that the pair (H, τ) is a symmetric algebra if the symmetric bilinear form*

$$H \times H \to A, (h, h') \mapsto \tau(hh')$$

is non-degenerate.

Definition 2.2.2. *Let A be an integral domain and (H, τ) be a symmetric A-algebra. We call H a monomial A-algebra if the following are satisfied:*

(i) There is an involution $\dagger : H \to H$ such that $\tau(h^\dagger) = \tau(h)$ for all $h \in H$.

(ii) There is a finite set W and an A-basis $\{b_w \mid w \in W\}$ of H such that there is an involution $\epsilon : W \to W$ with $b_w^\dagger = b_{\epsilon(w)}$ and

$$\tau(b_w b_{w'}^\dagger) = \begin{cases} a_w & \text{if } \epsilon(w) = w', \\ 0 & \text{if } \epsilon(w) \neq w', \end{cases}$$

where $a_w = a_{\epsilon(w)} \in A$ is invertible.

Example 2.2.3. *(i) Let K be a field with and let V be a n-dimensional vector space over K for some positive integer n. Then $H := \mathrm{End}_K(V)$ is a semisimple monomial algebra: We will fix a basis $e_1, e_2, \ldots, e_n$ of V and regard the entries of H as matrices with respect to the chosen basis. We define the trace function $\tau : H \to K$ be the regular trace of matrices. It is clear that then (H, τ) is a semisimple symmetric algebra with $\tau(\mathrm{id}) = n$.*

We let $\dagger : H \to H$ be the usual matrix transpose. The set $W = \{(i, j) \mid 1 \leq i, j \leq n\}$ gives rise to the K-basis given by the matrices $E_{(i,j)}$ which have a 1 in the (i, j)-position and a 0 everywhere else. Finally, we let

$$\epsilon : W \to W,\ (i, j) \mapsto (j, i)$$

and $a_{(i,j)} = 1$ for all $(i, j) \in W$.

(ii) Let G a finite group and let K be a field. Then KG is a natural monomial algebra with the basis given by the elements of G. We define the trace function by $\tau(1) = 1$ and $\tau(g) = 0$ for all $1 \neq g \in G$. Finally we let $g^\dagger = \epsilon(g) = g^{-1}$. We have $a_g = 1$ for all $g \in G$. The algebra is semisimple if and only if the order of G is invertible in K.

(iii) Let (W, S) be a Coxeter system and let A be a commutative ring with invertible elements $\{a_s, b_s \mid s \in S\} \subseteq A$ such that $a_t = a_t$ and $b_s = b_t$ when s and t are conjugate in W. Let $\mathcal{H} = \mathcal{H}_A(W, S, \{a_s, b_s \mid s \in S\}))$ be an Iwahori–Hecke algebra with basis $\{T_w \mid w \in W\}$ and multiplication

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ a_s T_{sw} + b_s T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for all $s \in S, w \in W$. We now set $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for all $1 \neq w \in W$. Finally we let $\epsilon(w) = w^{-1}$ and $b_w^\dagger = b_{w^{-1}}$. Then $\mathcal{H}$ becomes a monomial algebra and $a_w = a_{s_1} \cdots a_{s_k}$ if $w = s_1 \cdots s_k$ is a reduced expression for $w \in W, s_i \in S$, cf. [GP00, Proposition 8.1.1]. For a criterion of semisimplicity, regard [GP00, Theorem 7.2.6], and also Tits' Deformation Theorem 2.2.15.

We will quickly go through our basic setup and notations used throughout this section. Assume from now on that A is an integral domain in characteristic 0 and let H be a monomial A-algebra. Let K be the quotient field of A. For any ring homomorphism $\phi : A \to M$ for M a field we will write $MH := M \otimes_A H$, where we regard M as an A-module. We assume that KH is semisimple and that there is an extension field L of K such that LH is split. We will fix an extension field M with $K \subseteq M \subseteq L$. Naturally, MH is a semisimple monomial M-algebra by linearly extending τ and $\dagger$.

We will assume that any MH-module V is finitely generated. The representation afforded by V is then the natural homomorphism $\rho_V : MH \to \mathrm{End}_M(V)$. The character of V is defined to be the trace

$$\chi_V : MH \to M, h \mapsto \mathrm{trace}(\rho_V(h)).$$

The degree of a character is defined to be $\deg(\chi_V) := \dim(V)$.

The (finite) set $\mathrm{Irr}(LH)$ will be the set of characters of the isomorphism classes of simple LH-modules.

For every $\chi \in \mathrm{Irr}(LH)$ there is an element $c_\chi \in L$, called the Schur element, such that

$$\tau = \sum_{\chi \in \mathrm{Irr}(LH)} c_\chi^{-1} \chi.$$

Let V be a MH-module. Then $\hat{V} := \mathrm{Hom}_M(V, M)$ also becomes a MH-module by setting

$$(h \cdot \phi)(v) := \phi(h^\dagger \cdot v)$$

for all $h \in MH, \phi \in \hat{V}$ and $v \in V$. We will write $\overline{\chi_V}$ for $\chi_{\hat{V}}$. It is clear that

$$\overline{\chi_V}(h) = \chi_V(h^\dagger)$$

for all $h \in MH$. We say that a non-degenerate bilinear form $\beta : V \times V \to M$ is MH-invariant if

$$\beta(hv, v') = \beta(v, h^\dagger v')$$

for all $v, v' \in V, h \in MH$ and call (V, β) an orthogonal (resp. symplectic) MH-module if β is symmetric (resp. alternating). Further, we will say that a MH-module V' is orthogonal (resp. symplectic) if there exists a bilinear form β' on V' such that (V', β') is orthogonal (resp. symplectic).

Proposition 2.2.4. *(cf. [Gec14, Proposition 2.5]) Let $\chi \in \mathrm{Irr}(LH)$ and let V be a simple LH-module affording χ. Define the Frobenius–Schur indicator*

$$\iota(\chi) := \frac{1}{c_\chi \deg(\chi)} \sum_{w \in W} \frac{1}{a_w} \chi(b_w^2).$$

Then $\iota(\chi) \in \{-1, 0, 1\}$ *and*

$$\iota(\chi) = \begin{cases} 1, & \text{if } \chi = \overline{\chi} \text{ and } V \text{ is orthogonal,} \\ -1, & \text{if } \chi = \overline{\chi} \text{ and } V \text{ is symplectic,} \\ 0 & \text{if } \chi \neq \overline{\chi}. \end{cases}$$

Remark 2.2.5. *The argument in [Gec14] does not require us to work over* L*. More concretely, assume* $\chi \in \mathrm{Irr}(LH)$ *and that there is a field* M *with* $K \subseteq M \subseteq L$ *such that there is a simple* MH*-module* V *affording* χ*. Now, if* $\iota(\chi) = 1$ *(resp.* $\iota(\chi) = -1$*), then* V *is orthogonal (resp. symplectic).*

Note that for any LH-module V, the LH-module $(V \oplus \hat{V}, \beta)$ with

$$\beta((v_1, \phi_1), (v_2, \phi_2)) = \phi_1(v_2) + \phi_2(v_1)$$

is orthogonal. It follows that V is orthogonal if and only if its character is of the form

$$\chi = \sum_{i=1}^{r} a_i \chi_i^{(+)} + 2 \sum_{j=1}^{s} b_i \chi_j^{(-)} + \sum_{k=1}^{t} c_i (\chi_k^{(0)} + \overline{\chi_k}^{(0)}),$$

where $\chi_i^{(+)}$ (resp. $\chi_j^{(-)}$, resp. $\chi_k^{(0)}$) are elements of $\mathrm{Irr}(LH)$ with Frobenius-Schur indicator 1 (resp. -1, resp. 0), and a_i, b_j, c_k are non-negative integers. If additionally, all $\chi_i^{(+)}$ characters have even degree, we say that V is orthogonally stable.

We say that V is orthogonally simple if it has no orthogonal subrepresentation, i.e., if its character is of the form $\chi^{(+)}$, $2\chi^{(-)}$ or $\chi^{(0)} + \overline{\chi}^{(0)}$ for irreducible characters with the evident Frobenius–Schur indicators.

The same terminology will be applied to the characters, so we say that the character χ is orthogonal (resp. orthogonally stable, resp. orthogonally simple) if and only if V is.

We will set

$$\mathrm{Irr}^+(LH) := \{\chi \in \mathrm{Irr}(LH) \mid \iota(\chi) = 1 \text{ and } \deg(\chi) \in 2\mathbb{Z}\}.$$

Definition 2.2.6. *Let* χ *be a character of* LH*. We define the character field of* χ *by*

$$K(\chi) := K(\{\chi(b_w) \mid w \in W\}).$$

2.2.1 Involutions of Central Simple Algebras

After introducing monomial algebras, we will define determinants of involutions of central simple algebra, as was done in [Knu+98].

Let K be a field of characteristic not equal to 2.

Definition 2.2.7. *A finite-dimensional associative* K*-algebra* A *is a central simple algebra if the only two-sided ideals are* $\{0\}$ *and* A*, and the center of* A *is equal to* K*.*

Example 2.2.8. *(i) For n a positive integer, the matrix ring $K^{n\times n}$ is a central simple algebra.*

(ii) Let $a, b \in K^\times$. Define the algebra $Q = (a, b)_K$ with basis $\{1, i, j, k\}$ and relations $i^2 = a, j^2 = b$, $ij = k = -ji$. Then Q is a central simple algebra. It is an example of a quaternion algebra.

Lemma 2.2.9. *(cf. [Knu+98, Theorem 1.1]) A finite-dimensional associative K-algebra A is a central simple algebra if and only if there is a field extension L of K such that $L \otimes_K A \cong L^{n\times n}$. We call such a field a splitting field of A.*

Lemma 2.2.10. *(cf. [Rei75, Theorem 9.3]) Let A be a central simple K-algebra and L be a splitting field of A. Since $L \otimes_K A \cong L^{n\times n}$, we can regard the elements of A as matrices in $L^{n\times n}$. Then for all $a \in A$ it holds that $\det(a) \in K$ and the value is independent of the choice of splitting field. We call this map the reduced norm and denote it by $\mathrm{Nrd} : A \to K$.*

The importance of the next theorem in the context of this thesis can not be understated, as it is the key of defining orthogonal determinants of characters later on.

Theorem 2.2.11. *(cf. [Knu+98, Corollary 2.8, Proposition 7.1]) Let A be an even-dimensional central simple K-algebra with an involution $\iota : A \to A$. Then there is an element $h \in A^\times$ such that $\iota(h) = -h$. Moreover, for any other $h' \in A^\times$ with that property, we have that*

$$\mathrm{Nrd}(h) \equiv \mathrm{Nrd}(h') \mod (K^\times)^2.$$

We denote

$$\det(\iota) := \mathrm{Nrd}(h) \cdot (K^\times)^2 \in K^\times/(K^\times)^2$$

and call it the determinant of ι.

Example 2.2.12. *Regard the dihedral group*

$$D_{12} := \langle s, t \mid s^2 = t^2 = 1, (st)^6 = 1\rangle.$$

Let $\rho : D_{12} \to \mathrm{GL}_2(\mathbb{Q})$ be given by

$$\rho(s) = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \ \rho(t) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

This is an irreducible representation of D_{12}, see for instance [GP00, Theorem 8.3.1]. Then $A := \langle \rho(g) \mid g \in D_{12}\rangle_{\mathbb{Q}} = \mathbb{Q}^{2\times 2}$ is a central simple $\mathbb{Q}$-algebra and we got an involution $\iota : A \to A$ given by $\iota(\rho(g)) := \rho(g^{-1})$. Let $g = st$. We calculate that

$$\rho(g) - \rho(g^{-1}) = \begin{pmatrix} -3 & -6 \\ 6 & 3 \end{pmatrix}$$

with

$$\det(\rho(g) - \rho(g^{-1})) = 27 = 3 \cdot 9,$$

thus

$$\det(\iota) = 3 \cdot (\mathbb{Q}^\times)^2.$$

2.2.2 Orthogonal Determinants of Monomial Algebras over Fields

We will now finally define orthogonal determinants. This was first done for group algebras in [Neb22b], so we will have some slightly more general notion.

Let K be a field in characteristic 0, H be a semisimple monomial K-algebra, and L a splitting field for H.

We will fix a character $\chi \in \mathrm{Irr}^+(LH)$ of degree n, afforded by an orthogonal MH-module (V, β) for a field M sandwiched between K and L. We will choose a basis of V and consider the associated morphism $\rho : H \to M^{n\times n}$ and let the matrix $B \in M^{n\times n}$ represent β with respect to the chosen basis. By definition, we have that

$$\iota_B(\rho(h)) = \rho(h^\dagger).$$

As a direct consequence from the Artin–Wedderburn theorem on semisimple algebras, we have that

$$\langle \rho(h) \mid h \in MH\rangle_M = M^{n\times n}.$$

It follows that

$$E_-(B) = \langle \rho(h) - \rho(h^\dagger) \mid h \in MH\rangle_M.$$

In particular, the space $E_-(B)$ is independent of the choice of β.

We will regard the algebra

$$A := \langle \rho(h) \mid h \in MH\rangle_{K(\chi)}.$$

Lemma 2.2.13. *A is a central simple $K(\chi)$-algebra of dimension n^2.*

Proof. This follows by Lemma 2.2.9, since $M \otimes_{K(\chi)} A = M^{n\times n}$. □

Theorem 2.2.14. *There is a unique $\delta \in K(\chi)^\times/(K(\chi)^\times)^2$ such that for all orthogonal MH-modules (V, β) affording χ over all fields M with $L/M/K(\chi)$, we have that $\det(\beta) = \delta(M^\times)^2$. We define*

$$\det(\chi) := \delta \in K(\chi)^\times/(K(\chi)^\times)^2$$

and call it the orthogonal determinant of χ.

Proof. We have an involution $\iota : A \to A$ given by

$$\iota(\rho(h)) := \rho(h^\dagger)$$

for all $h \in H$. By Theorem 2.2.11, there is thus an element $g \in A^\times$ with $\iota(g) = -g$. In particular, $g \in E_-(B)$. We will denote

$$\delta := \det(\iota) = \det(g) \cdot (K(\chi)^\times)^2,$$

which does not depend on the choice of g. By Lemma 2.1.6, we now have that $\det(\beta) = \delta \cdot (M^\times)^2$, which by the previous discussion does not depend on the particular choice of the MH-invariant bilinear form. □

2.2.3 Specialization

Let H be a monomial algebra over $A := \mathbb{Q}[u, u^{-1}]$ for some indeterminate u. Let K be the quotient field of A. We will follow [GP00, Section 7.4], where a more general picture is presented.

The idea is that the algebra H gives rise to a whole family of $\mathbb{Q}$-algebras H_q for any nonzero integer $q \in \mathbb{Q}$, where we specialize $u \mapsto q$.

Let's make this more formal. We will fix an element $q \in \mathbb{Z}\backslash\{0\}$. In our applications, q will either be equal to 1 or be a power of an odd prime. We define the projection map $\varphi : A \to \mathbb{Q}, u \mapsto q$ and set $H_q := \mathbb{Q}H$ via the map φ. We assume that H_q is semisimple.

Let L be a number field. We denote $A_L := L[u, u^{-1}]$ with quotient field $L(u)$. There is a surjective ring homomorphism $\varphi_L : A_L \to L,\ u \mapsto q$ that extends φ. Accordingly, we arrive at the L-algebra

$$LH_q := L \otimes_{A_L} A_L H = L \otimes_{\mathbb{Q}} H_q.$$

Informally, this algebra just corresponds to replacing every instance of u with q in $A_L H$, and there is a surjective map

$$A_L H \to LH_q, h \mapsto 1 \otimes h.$$

Let now V be a $L(u)H$-module. Let $\mathfrak{p} \subseteq A_L$ be the prime ideal generated by the element $u - q$, and let

$$\mathcal{O} := \{f/g \in L(u) \mid f, g \in A_L,\ g \notin \mathfrak{p}\}$$

be the localization of $\mathfrak{p}$ in A_L. Then $\mathcal{O}$ is a discrete valuation ring, and there is a corresponding extension $\varphi'_L : \mathcal{O} \to L$ of φ_L, again generated by $u \mapsto q$.

By standard results of representation theory, there is a basis of V such that the corresponding homomorphism into the matrix ring is given by $\rho : A_L H \to \mathcal{O}^{n\times n}$. We can now elementwise apply the map φ'_L and arrive at a representation V_q of LH_q.

We will from now on assume that L is a splitting field, i.e., that LH_q and $L(u)H$ are split algebras. For each irreducible character $\chi \in \operatorname{Irr}(L(u)H)$, we get a character χ_q of LH_q coming from the representation V_q , which we do not know a priori to be irreducible. This turns out to be true, which is the content of the famous Tits' deformation theorem:

Theorem 2.2.15 (Tits' Deformation Theorem)**.** *The algebra KH is semisimple. Further, the specialization map $\chi \mapsto \chi_q$ induces a bijection between $\operatorname{Irr}(L(u)H)$ and $\operatorname{Irr}(LH_q)$.*

Next, we will take a closer look at the algebras H_q. We want to talk about orthogonal determinants, so we still want H_q to have the natural structure of a monomial algebra, inherited from H. We let $\tau, \dagger, W, \ldots$ be the data of H as in Definition 2.2.2. Let $\pi : H \to H_q$ be the projection map.

Proposition 2.2.16. *Let $h \in H$. We define the following maps:*

(i) $\tau_q : H_q \to \mathbb{Q}, \pi(h) \mapsto \varphi(\tau(h))$,

(ii) $\dagger_q : H_q \to H_q,\ \pi(h) \mapsto \pi(h^\dagger)$.

Then (H_q, τ_q) *is a symmetric algebra and* $\dagger_q$ *is an involution such that* $\tau_q(h_q^{\dagger_q}) = \tau_q(h_q)$ *for all* $h_q \in H_q$*.* H_q *has a* $\mathbb{Q}$*-basis given by* $\{\pi(b_w) \mid w \in W\}$ *and*

$$\tau_q(\pi(b_w)\pi(b_{w'})^{\dagger_q}) = \begin{cases} \varphi(a_w) & \text{if } \epsilon(w) = w', \\ 0 & \text{if } \epsilon(w) \neq w'. \end{cases}$$

In particular, H_q *is a monomial algebra.*

Proof. Almost everything is a straightforward calculation, except maybe the fact that τ_q is non-degenerate. But this follows since $\{\pi(b_w) \mid w \in W\}$ is an orthogonal basis and since the $a_w \in A$ were defined to be invertible, $\varphi(a_w) \neq 0$. So indeed, the determinant of the Gram matrix is nonzero. □

Corollary 2.2.17. *The specialization map at the level of characters induces a bijection between* $\mathrm{Irr}^+(L(u)H)$ *and* $\mathrm{Irr}^+(LH_q)$*.*

Proof. It holds clearly that $\deg(\chi) = \deg(\chi_q)$ for all $\chi \in \mathrm{Irr}(L(u)H)$. It is thus left to show that $\iota(\chi) = \iota(\chi_q)$, which follows since the formula for the Frobenius–Schur indicator in 2.2.4 clearly behaves well with specialization. □

What is left is now to compare the orthogonal determinants of the $\mathrm{Irr}^+(L(u)H)$-characters and the $\mathrm{Irr}^+(LH_q)$-characters. We immediately run into some issues: Assume for instance that $\chi \in \mathrm{Irr}^+(L(u)H)$ and that $\det(\chi) = u \cdot (K(\chi)^\times)^2$. What we then expect is

$$\det(\chi_q) = \varphi_{\mathbb{Q}(\chi_q)}(u) \cdot (\mathbb{Q}(\chi_q)^\times)^2 = q \cdot (\mathbb{Q}(\chi_q)^\times)^2,$$

and indeed, this will turn out to be the case. The representative we have chosen for the square class is only unique up to a square though, so $u(u-q)^2$ is another valid representative. But $\varphi_{\mathbb{Q}(\chi_q)}(u(u-q)^2) = 0$, so one can not just carelessly choose any representative.

The above problem is easily fixed. Since $A_{\mathbb{Q}(\chi_q)}$ is a unique factorization domain, we can choose a squarefree polynomial $d_\chi(u) \in A_{\mathbb{Q}(\chi_q)}$ in the square class of $\det(\chi)$ for every $\chi \in \mathrm{Irr}^+(G)$.

Theorem 2.2.18. *Let* $\chi \in \mathrm{Irr}^+(L(u)H)$*. Let* $\det(\chi) = d_\chi(u) \cdot (K(\chi)^\times)^2$ *for* $d_\chi(u) \in A_{\mathbb{Q}(\chi_q)}$ *squarefree. Then*

$$\det(\chi_q) = \varphi_{\mathbb{Q}(\chi_q)}(d_\chi(u)) \cdot (\mathbb{Q}(\chi_q)^\times)^2 = d_\chi(q) \cdot (\mathbb{Q}(\chi_q)^\times)^2.$$

Proof. Let $M \subseteq L$ be a field such that there is a representation $\rho : M(u)H \to \mathcal{O}^{n\times n}$ affording the character χ, for a discrete valuation domain $\mathcal{O} \subseteq M$. Let $\rho_q : MH_q \to M^{n\times n}$ be the corresponding specialized representation. We write $\psi : \mathcal{O}^{n\times n} \to M^{n\times n}$ for the evident map that applies $\varphi' : \mathcal{O} \to M$ elementwise. It is clear that for every element $D \in \mathcal{O}^{n\times n}$, it holds that $\varphi'(\det(D)) = \det(\psi(D))$.

Since χ is irreducible,

$$\langle \rho_q(h_q) \mid h_q \in MH_q \rangle_M = M^{n\times n},$$

so we choose an element

$$x_q := \sum_{w \in W} c_w \pi(b_w) \in MH_q$$

with $c_w \in \mathbb{Q}(\chi_q)$ such that $C_q := \rho_q(x_q) - \rho_q(x_q^{\dagger_q})$ is invertible. Recall that by Theorem 2.2.14,

$$\det(\chi_q) = \det(C_q) \cdot (\mathbb{Q}(\chi_q)^\times)^2.$$

The idea now is to lift x_q to $M(u)H$. For that, define

$$x := \sum_{w \in W} c_w b_w \in M(u)H.$$

Further we let $C := \rho(x) - \rho(x^\dagger)$. Clearly, $\psi(C) = C_q$ and by what we have concluded earlier,

$$\varphi'(\det(C)) = \det(\psi(C)) = \det(C_q) \neq 0,$$

so in particular, $\det(C) \neq 0$. So again by Theorem 2.2.14,

$$\det(\chi) = \det(C) \cdot (K(\chi)^\times)^2.$$

Thus by choosing a squarefree representative of the square class of $\det(C) \cdot (K(\chi)^\times)^2$, one arrives at the statement of the theorem. □

2.3 Orthogonal Determinants of Finite Groups

In the last section, we have defined orthogonal determinants for monomial algebras. In this section, we are looking at a special case, namely the orthogonal determinants of characters of group algebras over a field of characteristic 0. In short, we will just say that we are considering orthogonal determinants of finite groups. Of course, much more can be said here than in the more general picture, as we have the full and rich representation theory of finite groups at our disposal. So, subgroups will play a big role, as well as certain important properties a group can enjoy, like being solvable or abelian. We will talk about some techniques that will come in handy for the calculations later on, giving an emphasis on the theory of condensation. At the end of this section, we will discuss a conjecture by Richard Parker about the "oddness" of orthogonal determinants of finite groups. Virtually all of this section is due to the results of Nebe, Parker and Breuer, which can be found in the papers [Neb22b], [Neb22a], [NP22] and [BNP24].

Let G be a finite group. As already mentioned in Example 2.2.3, the algebra $\mathbb{Q}G$ is a semisimple monomial algebra with splitting field $\mathbb{C}$. We denote $\mathrm{Irr}(G) := \mathrm{Irr}(\mathbb{C}G)$ and $\mathrm{Irr}^+(G) := \mathrm{Irr}^+(\mathbb{C}G)$. By abuse of notation, for $L \subseteq \mathbb{C}$ a subfield and (V, β) an orthogonal LG-module, we may also simply say that the bilinear form β is G-invariant.

With Theorem 2.2.14 in mind, we now understand orthogonal determinants of $\mathrm{Irr}^+(G)$-characters. We can generalize the result to all orthogonally stable characters, so let us now fix an orthogonally stable character χ. Let $K := \mathbb{Q}(\chi)$.

Lemma 2.3.1. *(cf. [Tur93, Theorem B]) Assume $\chi = 2\psi$ is orthogonally simple for $\psi \in \mathrm{Irr}(G)$ with $\iota(\psi) = -1$. Let (V, β) be an orthogonal LG-module affording χ for a real-valued field L/K. Then*

$$\det(\beta) = 1 \cdot (L^\times)^2.$$

We will define the orthogonal determinant

$$\det(2\chi) := 1 \cdot (K^\times)^2.$$

The following is a corollary of Lemma 2.1.8:

Lemma 2.3.2. *Assume $\chi = \psi + \overline{\psi}$ is orthogonally simple for $\psi \in \mathrm{Irr}(G)$ with $\iota(\psi) = 0$. Let $L := \mathbb{Q}(\psi)$, we can choose $0 < \delta \in K$ such that $K[\sqrt{-\delta}] = L$. Let (V, β) be an orthogonal MG-module affording χ for a real-valued field M/K. Then*

$$\det(\beta) = \delta^{\psi(1)} \cdot (M^\times)^2.$$

We will define the orthogonal determinant

$$\det(\chi) := \delta^{\psi(1)} \cdot (K^\times)^2.$$

For the case that the field L in the notation of the last lemma is cyclotomic, there is an easy choice for the number δ.

Corollary 2.3.3. *Assume we are in the situation of Lemma 2.3.2. Let m be a positive integer and let*

$$\zeta_m := \exp\left(\frac{2\pi i}{m}\right)$$

be a primitive complex m-th root of unity and let for any integer j

$$\vartheta_m^{(j)} := \zeta_m^j + \zeta_m^{-j} \subseteq \mathbb{R}.$$

Assume that $L = \mathbb{Q}(\zeta_m^j)$. Then $K = \mathbb{Q}(\vartheta_m^{(j)})$. We can thus choose

$$\delta = -(\zeta_m^j - \zeta_m^{-j})^2 = 2 - \vartheta_m^{(2j)}$$

and arrive at

$$\det(\chi) = (2 - \vartheta_m^{(2j)})^{\psi(1)} \cdot (\mathbb{Q}(\vartheta_m^{(j)})^\times)^2.$$

Definition 2.3.4. *(cf. [NP22, Section 5.5]) Let ψ be an orthogonally simple constituent of χ. Let $L := \mathbb{Q}(\psi)$ and let $\Gamma_\psi := \mathrm{Gal}(L/(L \cap K))$. We define*

$$\psi_K := \sum_{\sigma \in \Gamma_\psi} \sigma \cdot \psi,$$

which is clearly again a constituent of χ with $\mathbb{Q}(\psi_K) \subseteq K$. Let $\delta \in L$ be a representative of $\det(\psi) \in L^\times/(L^\times)^2$. We then set

$$N_K(\det(\psi)) := \prod_{\sigma \in \Gamma_\psi} \sigma(\delta) \cdot (K^\times)^2.$$

Lemma 2.3.5. *(cf. [NP22, Proposition 5.17, Remark 5.21]) With the notation of Definition 2.3.4 we write $\chi = \sum_{i=1}^{k} (\psi_i)_K$ for orthogonally stable constituents ψ_i of χ. Let (V, β) be an orthogonal LG-module affording χ for some real valued field L/K. Then*

$$\det(\beta) = \prod_{i=1}^{k} N_K(\det(\psi_i)) \cdot (L^\times)^2.$$

We will define the orthogonal determinant

$$\det(\chi) := \prod_{i=1}^{k} N_K(\det(\psi_i)) \in K^\times/(K^\times)^2.$$

Remark 2.3.6. *By definition, the class function $\mathbf{0} : G \to \mathbb{C}, g \mapsto 0$ is orthogonally stable. It is convenient to set*

$$\det(\mathbf{0}) := 1 \cdot (\mathbb{Q}^\times)^2.$$

We will summarize the results in the following main theorem:

Theorem 2.3.7. *Let (V, β) be an orthogonal LG-module affording the character χ for some real-valued field L/K. Then*

$$\det(\beta) = \det(\chi) \cdot (L^\times)^2.$$

Remark 2.3.8. *Assume we have the following information about our group G:*

(i) The full character table, in particular we have the knowledge about all the Frobenius–Schur indicators and the character fields.

(ii) The orthogonal determinants $\det(\chi)$ for each $\chi \in \mathrm{Irr}^+(G)$.

Then with the statements above, we can calculate the orthogonal determinants of every single orthogonally stable character of G.

Thus we will need to find methods on how to handle the $\mathrm{Irr}^+(G)$-characters. First, we can use representation theoretic methods like restriction and induction of subgroups of G.

Lemma 2.3.9. *Let H be a subgroup of G. Assume that $\mathrm{Res}_H^G(\chi)$ is an orthogonally stable character of H. We will also say that χ is H-stable. Then $\det(\chi) = \det(\mathrm{Res}_H^G(\chi)) \cdot (K^\times)^2$. We denote*

$$\mathrm{Irr}_H^+(G) := \{\psi \in \mathrm{Irr}^+(G) \mid \mathrm{Res}_H^G(\psi) \text{ is orthogonally stable}\}$$

to be the set of H-stable $\mathrm{Irr}^+(G)$-characters.

Lemma 2.3.10. *Let H be a subgroup of G and ψ be an orthogonal character of H such that $\chi = \mathrm{Ind}_H^G(\psi)$ is orthogonally stable with $\mathbb{Q}(\psi) = K$. Then the following holds:*

(i) If the index of H in G is even, then $\det(\chi) = 1 \cdot (K^\times)^2$.

(ii) If the index of H in G is odd, then ψ is orthogonally stable and $\det(\chi) = \det(\psi)$.

Proof. Let (V, β) be an orthogonal LH-module affording ψ for a real valued field L/K. Let $x_1, \ldots, x_N$ be representatives of the cosets G/H and let

$$W = \bigoplus_{i=1}^{n} x_i V$$

be the induced representation. Then we can construct a G-invariant non-degenerate symmetric bilinear form γ on W by letting the individual summands be orthogonal to each other, and choosing the bilinear form β on $x_i V \cong V$. Then $\det(\gamma) = \det(\beta)^N$ from which the statement follows. □

If the character field of an induced character becomes smaller, then the above lemma fails. There is a special case that can be easily handled though:

Lemma 2.3.11. *(cf. [Neb22b, Theorem 4.3]) Assume $G = H \rtimes C_2$ for a normal subgroup H of G. Let $\alpha : H \to H$ be the automorphism of order 2 induced by the action of C_2. Assume that $\chi \in \mathrm{Irr}^+(G)$ such that $\chi = \mathrm{Ind}_H^G(\psi)$, with $\psi \in \mathrm{Irr}(G)$ an odd-degree indicator "+" character such that $\psi \neq \psi \circ \alpha$. Choose a $\delta \in K$ such that $K[\sqrt{\delta}] = \mathbb{Q}(\psi)$. Then*

$$\det(\chi) = \delta \cdot (K^\times)^2.$$

Lemma 2.3.12. *Let N be a normal subgroup of G and let ψ be an orthogonally stable character of G/N. Let $\pi : G \to G/N$ be the projection. The inflation $\mathrm{Inf}_{G/N}^G(\psi)$ is defined to be the character $\psi \circ \pi$ of G. Then*

$$\det(\psi) = \det(\mathrm{Inf}_{G/N}^G(\psi)).$$

Proof. Let $L/\mathbb{Q}(\psi)$ be a field extension and assume that there is an orthogonal $L(G/N)$-module V affording the character ψ with a homomorphism $\rho : L(G/N) \to \mathrm{End}(V)$. Let $h \in L(G/N)$ be an element such that

$$\det(\psi) = \det\left(\rho(h) - \rho(h^\dagger)\right) \cdot (\mathbb{Q}(\psi)^\times)^2.$$

We will denote $\pi' : LG \to L(G/N)$ to be the surjective algebra homomorphism coming from the projection map π. Then we get an according orthogonal LG-module given by the map $\rho' := \rho \circ \pi'$. Let $h' \in LG$ be any lift of h. Then $\rho(h) = \rho'(h')$ and thus

$$\begin{aligned}\det(\mathrm{Inf}_{G/N}^G(\psi)) &= \det\left(\rho'(h') - \rho'((h')^\dagger)\right) \cdot (\mathbb{Q}(\mathrm{Inf}_{G/N}^G(\psi))^\times)^2 \\ &= \det\left(\rho(h) - \rho(h^\dagger)\right) \cdot (\mathbb{Q}(\psi)^\times)^2 = \det(\psi),\end{aligned}$$

which we wanted to show. □

Next, we will regard special classes of groups where we have simple formulas for the orthogonal determinants.

Recall that if G_1, G_2 are finite groups, then

$$\mathrm{Irr}(G_1 \times G_2) = \{\chi_1 \boxtimes \chi_2 \mid \chi_1 \in \mathrm{Irr}(G_1), \chi_2 \in \mathrm{Irr}(G_2)\},$$

where $(\chi_1 \boxtimes \chi_2)(g_1, g_2) := \chi_1(g_1)\chi_2(g_2)$.

Lemma 2.3.13. *Let G_1, G_2 be finite groups, $\chi_1 \in \mathrm{Irr}(G_1), \chi_2 \in \mathrm{Irr}(G_2)$. Let $G = G_1 \times G_2$ and $\chi = \chi_1 \boxtimes \chi_2$. Then $\chi \in \mathrm{Irr}^+(G)$ if and only if one of the following holds:*

(i) We have $\iota(\chi_1) = \iota(\chi_2) = 1$ and at least one of the characters has even degree. Assume without loss of generality that $\chi_1(1)$ is even. Then

$$\det(\chi) = \det(\chi_1)^{\chi_2(1)} \cdot (K^\times)^2.$$

(ii) We have $\iota(\chi_1) = \iota(\chi_2) = -1$ and at least one of the characters has even degree. Then

$$\det(\chi) = 1 \cdot (K^\times)^2.$$

Proof. It is clear by the definition of the Frobenius–Schur indicator given in Proposition 2.2.4 that we have that $\iota(\chi) = \iota(\chi_1)\iota(\chi_2)$. Let us assume that $\chi \in \mathrm{Irr}^+(G)$. In particular, it has even degree and we can assume without loss of generality that χ_1 is a character of even degree. Then $\mathrm{Res}^G_{G_1 \times \{1\}}(\chi) = \chi_2(1)\chi_1$ and the statement follows by Lemma 2.3.9. □

Lemma 2.3.14. *(cf. [Neb22b, Corollary 2.5]) Assume there is an element $g \in G$ such that $\chi(g^2) = -\chi(1)$. Then*

$$\det(\chi) = 1 \cdot (K^\times)^2.$$

Proposition 2.3.15. *(cf. [Neb22a, Theorem 4.7]) Assume that G is a 2-group. Then*

$$\det(\chi) = 1 \cdot (K^\times)^2.$$

Proposition 2.3.16. *(cf. [Neb22a, Theorem 4.3]) Assume that G is a p-group for an odd prime p. Define for $d \geq 1$ the cyclotomic field $Z_d \subseteq \mathbb{C}$ generated by the complex p^d-th roots of unity. Let $Z_d^+ \subseteq \mathbb{R}$ be the biggest real subfield of Z_d. Let $\delta_p > 0$ be such that $Z_1^+(\sqrt{-\delta_p}) = Z_1$. Let f be minimal with the property that $K \subseteq Z_f^+$. Then $[Z_f : K]$ is even and divides $(p-1)$ and $\chi(1)$. Furthermore, the following holds:*

$$\det(\chi) = \left(N_{Z_f^+/K}(\delta_p)\right)^{\chi(1)/[Z_f:K]} \cdot (K^\times)^2.$$

If $p \equiv 3 \mod 4$, then $\det(\chi) = p^{\chi(1)/2} \cdot (K^\times)^2$. If $K = \mathbb{Q}$ and $p \equiv 1 \mod 4$, then $\det(\chi) = p^{\chi(1)/(p-1)} \cdot (K^\times)^2$.

Corollary 2.3.17. *Assume that G is a p-group for an odd prime p. Let $q = p^r$ be a power of p for some positive integer r such that $q - 1 \mid \chi(1)$. Assume that $\mathbb{Q}(\chi) = \mathbb{Q}$. Then*

$$\det(\chi) = q^{\chi(1)/(q-1)} \cdot (\mathbb{Q}^\times)^2.$$

2.3.1 Hecke Algebras and Condensation

Let G be a finite group and let B be a subgroup. Oftentimes, we will be presented with the following problem: Let $\chi \in \mathrm{Irr}^+(G)$ and assume that

$$\mathrm{Res}_B^G(\chi) = \psi + c \cdot \mathbf{1}_B,$$

where c is some (small) non-negative integer and ψ is an orthogonally stable character of B. Assume that we know $\det(\psi)$. If $c = 0$, then of course, by Lemma 2.3.9, we know also $\det(\chi)$. This subsection deals with the case of $c > 0$, which will introduce the Hecke algebras which are certain monomial algebras.

Hecke algebras are well understood and will appear frequently in this thesis. Some nice sources that cover these are [CR81, §11D] and [GP00, Section 8.4]. They also appear in the context of condensation in computational representation theory, regard for instance [MR99]. We also refer to [BNP24] for an application to orthogonal determinants.

Definition 2.3.18. *Let*

$$e_B := \frac{1}{|B|} \sum_{h \in B} h \in \mathbb{Q}G,$$

which is an idempotent in the group algebra. The Hecke algebra $\mathcal{H}(G, B)$ is defined to be the subalgebra $e_B \mathbb{Q}G e_B$ of $\mathbb{Q}G$. If the context is clear, we will just write $\mathcal{H} = \mathcal{H}(G, B)$. For any field extension $K/\mathbb{Q}$, we define the K-algebra $K\mathcal{H} := K \otimes_{\mathbb{Q}} \mathcal{H} = e_B K G e_B$.

Proposition 2.3.19. *(cf. [CR81, Proposition 11.34] Let $D(G, B) = \{x_1, \ldots, x_N\} \subseteq G$ be a set of representatives of the double cosets $B\backslash G/B$, where we assume that $x_1 = 1$. We set $D_i := Bx_iB$ for any $x_i \in D(G, B)$ and define*

$$T_i := \frac{1}{|B|} \sum_{h \in D_i} h \in \mathcal{H}.$$

These form a $\mathbb{Q}$-basis of $\mathcal{H}$, called the Schur basis, as we run over all elements of $D(G, B)$. The element $T_1 = e_B$ is the identity element of $\mathcal{H}$. For any $1 \leq i, j \leq N$, we have that

$$T_i T_j = \sum_{k=1}^{N} \mu_{i,j,k} T_k,$$

where

$$\mu_{i,j,k} := \frac{1}{|B|} |D_i \cap x_k D_j^{-1}|.$$

It holds that $\mathcal{H}$ is a semisimple algebra. Further, it has the structure of a monomial algebra: As a subalgebra of $\mathbb{Q}G$, it inherits many properties of such. Let

$$\tau : \mathbb{Q}G \to \mathbb{Q}, \ \sum_{g \in G} a_g g \mapsto a_1$$

for $a_g \in \mathbb{Q}$, be the trace function on $\mathbb{Q}$, as in Example 2.2.3. Then the restriction of τ on $\mathcal{H}$ makes the pair $(\mathcal{H}, \tau)$ a symmetric algebra, see [CR81, Proposition 11.30(iii)]. Similarly to the trace function on the group algebra, we get that

$$\tau\left(\sum_{i=1}^{N} a_i T_i\right) = a_1,$$

for $a_i \in \mathbb{Q}$. We define an involution on the set $D(G, B)$: For any $x \in D(G, B)$, define $\iota(x) \in D(G, B)$ such that $B\iota(x)B = Bx^{-1}B$. By now setting

$$\dagger : \mathcal{H} \to \mathcal{H},\ T_x^\dagger = T_{\iota(x)},$$

we get an involution on $\mathcal{H}$ which makes $\mathcal{H}$ into a monomial algebra with $a_x = |BxB/B|$ for $x \in D(G, B)$. Note that the involution on $\mathcal{H}$ is just the restriction of the involution of $\mathbb{Q}G$ to $\mathcal{H}$ as a subalgebra.

The character $\operatorname{Ind}_B^G(\mathbf{1}_B)$ is afforded by the left $\mathbb{Q}G$-module $\mathbb{Q}Ge_B$. Note that we have an isomorphism $\mathcal{H} \to \operatorname{End}_{\mathbb{Q}G}(\mathbb{Q}Ge_B)$ given by

$$h \mapsto (v \mapsto vh^\dagger),$$

for $h \in \mathcal{H}, v \in \mathbb{Q}Ge_B$.

We will now investigate the irreducible characters of $\mathbb{C}\mathcal{H}$, see also [CR81, Theorem 11.25]. Let $K/\mathbb{Q}$ be a field extension and V be an absolutely irreducible KG-module such that χ_V appears in $\operatorname{Ind}_B^G(\mathbf{1}_B)$, i.e.,

$$\langle \chi_V, \operatorname{Ind}_B^G(\mathbf{1}_B)\rangle_G \neq 0.$$

Then

$$U := \operatorname{Stab}_{KB}(V) = \{v \in V \mid hv = v \text{ for all } h \in B\}$$

is nontrivial, as $\dim(U) = \langle \chi_V, \operatorname{Ind}_B^G(\mathbf{1}_B)\rangle_G$. In particular, U is a $K\mathcal{H}$-module: For any $x \in KG$, $v \in V$ and $h \in B$, we calculate that

$$h \cdot (e_B x e_B) \cdot v = (e_B x e_B) \cdot v,$$

since $he_B = e_B$. In fact, U is an absolutely irreducible $K\mathcal{H}$-module, and all such modules arise in that way.

Analogously, if $\chi \in \operatorname{Irr}(G)$ is the irreducible character of V, we denote $\chi_{\mathcal{H}} \in \operatorname{Irr}(\mathbb{C}\mathcal{H})$ to be the irreducible character of U and all irreducible characters of $\mathbb{C}\mathcal{H}$ arise that way. It is now also clear that if $L/\mathbb{Q}$ is a field extension such that the LG-module LGe_B splits, then L is a splitting field for $\mathcal{H}$. The degree of $\chi_{\mathcal{H}}$ can be easily calculated from χ: It holds that

$$\deg(\chi_{\mathcal{H}}) = \langle \chi, \operatorname{Ind}_B^G(\mathbf{1}_B)\rangle_G.$$

Theorem 2.3.20. *Let $\chi \in \operatorname{Irr}^+(G)$ such that $\langle \chi, \operatorname{Ind}_B^G(\mathbf{1}_B)\rangle_G \neq 0$ and let $\chi_{\mathcal{H}} \in \operatorname{Irr}(\mathbb{C}\mathcal{H})$ be the corresponding irreducible character. Assume that $\deg(\chi_{\mathcal{H}})$ is even and that*

$$\psi := \operatorname{Res}_B^G(\chi) - \deg(\chi_{\mathcal{H}})\mathbf{1}_B$$

is an orthogonally stable character of B. Then $\chi_{\mathcal{H}} \in \operatorname{Irr}^+(\mathbb{C}\mathcal{H})$ and

$$\det(\chi) = \det(\psi)\det(\chi_{\mathcal{H}}) \cdot (\mathbb{Q}(\chi)^\times)^2.$$

Proof. Let K be a real-valued field such that there is an orthogonal KG-module (V, β) affording χ. Let $U := \mathrm{Stab}_{KB}(V)$ be the associated $K\mathcal{H}$-module; then $(U, \beta_{|U})$ is an orthogonal $K\mathcal{H}$-module. Indeed, $\beta_{|U}$ is non-degenerate because K is real-valued, and it's $K\mathcal{H}$-invariant as $K\mathcal{H}$ is a subalgebra of KG. So $\chi_{\mathcal{H}} \in \mathrm{Irr}^+(\mathbb{C}\mathcal{H})$.

Let $W \subseteq V$ be the KB-submodule affording the character ψ, so $V = U \oplus W$. We choose a basis $(v_1, \ldots, v_n)$ of V with respect to this decomposition, so we get a morphism $\rho : KG \to K^{n\times n}$ with character χ, where n is the K-dimension of V and the vectors $(v_1, \ldots, v_k)$ are a basis of W for $k = \dim(W)$. Accordingly, we also get morphisms $\rho_{KB} : KB \to K^{k\times k}$ and $\rho_{K\mathcal{H}} : K\mathcal{H} \to K^{(n-k)\times(n-k)}$ representing the modules W and U respectively. Let

$$A = \langle \rho(h) \mid h \in KG \rangle_{\mathbb{Q}(\chi)},$$

we similarly define A_{KB} and $A_{K\mathcal{H}}$ with ρ_{KB} and $\rho_{K\mathcal{H}}$ respectively. On each of these, we get an involution ι (resp. ι_{KB}, resp. $\iota_{K\mathcal{H}}$) generated by the involution $\dagger$ on KG (resp. the restriction of $\dagger$ to the subalgebras KB and $K\mathcal{H}$).

So by Theorem 2.2.11, there are invertible elements $g_{KB} \in A_{KB}$, $g_{K\mathcal{H}} \in A_{K\mathcal{H}}$ such that

$$\iota_{KB}(g_{KB}) = -g_{KB}, \ \iota_{K\mathcal{H}}(g_{K\mathcal{H}}) = -g_{K\mathcal{H}}.$$

We now set $g := \mathrm{diag}(g_{KB}, g_{K\mathcal{H}}) \in A$ to be the block diagonal matrix with the blocks g_{KB} and $g_{K\mathcal{H}}$. Then clearly $\iota(g) = -g$ and $\det(g) = \det(g_{KB})\det(g_{K\mathcal{H}})$. The statement now follows by Theorem 2.2.14. □

2.3.2 Parker's Conjecture

Let G be a finite group. If G is small, one can explicitly construct the irreducible representations of G and thus calculate the orthogonal determinants of the $\mathrm{Irr}^+(G)$ by "brute force", as well as the methods discussed in the previous sections (and more). To give a feel on how the orthogonal determinants behave, the reader can for instance take a quick look at Section 4.4, where the orthogonal determinants of the orthogonally stable characters of six finite groups are listed. One quickly observes a pattern, namely, in the listed groups (and for all finite groups investigated so far), for any character $\chi \in \mathrm{Irr}^+(G)$ such that $\mathbb{Q}(\chi) = \mathbb{Q}$, the unique squarefree integer that represents $\det(\chi) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ always seems to be odd.

Richard Parker conjectured that this is always the case, for all finite groups G, also see [Neb22a]. In this subsection, we will present a generalization of this conjecture which we will appropriately call "Parker's Conjecture", for which we will follow the before mentioned paper [Neb22a]. In that paper, Nebe showed Parker's conjecture to hold for all solvable groups. One of the main goal of this thesis is to confirm the conjecture for other families of groups, in particular, it will be shown that Parker's conjecture holds for all finite Coxeter groups in Chapter 4, and for all groups $\mathrm{GL}_n(q)$ for all positive integers n and odd powers of primes $q = p^r$ in Chapter 6.

Definition 2.3.21. *Let K be a number field. A discrete valuation is a non-zero function $\nu : K^\times \to \mathbb{Z}$ such that for all for $a, b \in K^\times$ we have that* *(i)* $\nu(ab) = \nu(a) + \nu(b)$,

(ii) $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$.

We say that ν *is dyadic if* $\nu(2) > 0$.

We will always assume that our discrete valuations are surjective, i.e., $\nu(K^\times) = \mathbb{Z}$. It is a standard fact of algebraic number theory that there is a bijection between discrete valuations of K and non-zero prime ideals of its ring of integers. We will quickly recall the necessary theory. So let K be a number field, $\mathcal{O}$ be its ring of integers and let $\mathfrak{p} \subseteq \mathcal{O}$ be a non-zero prime ideal. Recall that the localization of $\mathcal{O}$ by $\mathfrak{p}$ is defined by

$$\mathcal{O}_\mathfrak{p} := \{\frac{a}{b} \mid a, b \in \mathcal{O},\ b \notin \mathfrak{p}\}.$$

It holds that $\mathcal{O}_\mathfrak{p}$ is a discrete valuation ring (and therefore a principal ideal domain) with unique prime ideal

$$\mathfrak{p}\mathcal{O}_\mathfrak{p} = \pi\mathcal{O}_\mathfrak{p}$$

for some element $\pi \in \mathcal{O}$, called the uniformizer. Now, for any non-zero ideal $I \subseteq \mathcal{O}_\mathfrak{p}$, there is a unique non-negative integer n such that $I = (\pi)^n$. The $\mathfrak{p}$-adic valuation $\nu_\mathfrak{p}(a)$ of an element $a \in \mathcal{O}\backslash\{0\}$ is then defined to be the unique non-negative integer such that

$$a\mathcal{O}_\mathfrak{p} = (\pi)^{\nu_\mathfrak{p}(a)}.$$

We extend this to the $\mathfrak{p}$-adic valuation $\nu_\mathfrak{p} : K^\times \to \mathbb{Z}$ by setting

$$\nu_\mathfrak{p}\left(\frac{a}{b}\right) := \nu_\mathfrak{p}(a) - \nu_\mathfrak{p}(b)$$

for $a, b \in \mathcal{O}\backslash\{0\}$.

It follows that the dyadic valuations correspond exactly to the prime ideals that lie above 2. We will give some examples.

Example 2.3.22. *(i) Let* $K = \mathbb{Q}$. *The* 2*-adic valuation* ν_2 *of a non-zero integer number* a *is then given by* $a = 2^{\nu(2)}a'$, *where* a' *is not divisible by* 2. *For instance,*

$$\nu_2\left(\frac{5}{8}\right) = -3$$

and $\nu_2(2) = 1$.

(ii) Let $K = \mathbb{Q}[i]$. *Then* $\mathcal{O} = \mathbb{Z}[i]$ *and* 2 *ramifies, i.e.,* $2 = -i(1+i)^2$. *Let* $\mathfrak{p} := (1+i)\mathbb{Z}[i]$ *be the prime ideal lying over* 2. *Then* $\nu_\mathfrak{p}$ *is the unique dyadic valuation of* K *and* $\nu_\mathfrak{p}(2) = 2$.

(iii) Let $K = \mathbb{Q}[\sqrt{17}]$. *Then* $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{17}}{2}]$ *and* 2 *splits, i.e.,*

$$2 = \frac{\sqrt{17}+3}{2} \cdot \frac{\sqrt{17}-3}{2}.$$

Let

$$\mathfrak{p}_1 := \frac{\sqrt{17}+3}{2}\mathcal{O},\ \mathfrak{p}_2 := \frac{\sqrt{17}-3}{2}\mathcal{O}$$

be the corresponding prime ideals lying over 2. *We therefore have two dyadic valuations* $\nu_{\mathfrak{p}_1}$ *and* $\nu_{\mathfrak{p}_2}$, *and* $\nu_{\mathfrak{p}_1}(2) = \nu_{\mathfrak{p}_2}(2) = 1$.

Definition 2.3.23. *Let Let* K *be a number field and* $\delta \in K^\times$. *Then we say that* $\delta(K^\times)^2$ *is odd, if and only if* $\nu(\delta) \in 2\mathbb{Z}$ *for all dyadic discrete valuations* ν *on* K.

For instance, let $K = \mathbb{Q}$ and let ν_2 be the unique dyadic valuation on $\mathbb{Q}$. Assume that $\delta \in \mathbb{Q}^\times$ such that $\delta(\mathbb{Q}^\times)^2$ is odd. This is equivalent for there being an $a \in \delta(\mathbb{Q}^\times)^2$ such that $\nu_2(a) = 0$, i.e., a being an odd integer. Thus a square class in the rationals is odd if and only if there is an odd integer in that square class, justifying the name "odd".

Lemma 2.3.24. *Let* L/K *be two number fields,* $\delta \in K^\times$. *If* $\delta(K^\times)^2$ *is odd, then also* $\delta(L^\times)^2$ *is odd.*

Proof. This follows from the well-known fact that discrete valuations can be extended to field extensions, see for instance [Neu92, Chapter II, §8]. □

Note that the converse of this statement is false as can be seen from Example 2.3.22, since the square class of 2 is clearly not odd in $\mathbb{Q}$, but the square class of 2 in $\mathbb{Q}[i]$ becomes odd.

We can now formulate the conjecture by Richard Parker about the orthogonal determinants of orthogonally stable characters of finite groups:

Conjecture 2.3.25 (Parker)**.** *Let* G *be a finite group and* χ *be an orthogonally stable character of* G. *Let* $K = \mathbb{Q}(\chi)$. *Then* $\det(\chi) \in K^\times/(K^\times)^2$ *is odd.*

Let now G be a finite group. We will see that Parker's conjecture can be simplified by only needing to verify it for the $\mathrm{Irr}^+(G)$-characters:

Proposition 2.3.26. *(cf. [Neb22a, Corollary 2.12]) Let* G *be a finite group and assume that for all orthogonally stable constituents* ψ *of* χ *it holds that* $\det(\psi)$ *is odd. Then* $\det(\chi)$ *is odd.*

So we only need to check the conjecture for orthogonally simple orthogonally stable characters χ of G. There are three possibilities of such characters. If $\chi = 2\psi$ for some $\psi \in \mathrm{Irr}(G)$ and $\iota(\psi) = -1$, then $\det(\chi) = 1 \cdot (K^\times)^2$ by Lemma 2.3.1. The next possibility is handled by the following proposition.

Proposition 2.3.27. *(cf. [Neb22a, Lemma 3.2]) Assume* $\chi = \psi + \overline{\psi}$ *for* $\psi \in \mathrm{Irr}(G)$ *and* $\iota(\psi) = 0$. *Then* $\det(\chi)$ *is odd.*

Thus, Parker's conjecture can be reduced to the following:

Conjecture 2.3.28. *Let* G *be a finite group and* $\chi \in \mathrm{Irr}^+(G)$. *Let* $K = \mathbb{Q}(\chi)$. *Then* $\det(\chi) \in K^\times/(K^\times)^2$ *is odd.*

The following is known:

Theorem 2.3.29. *(cf. [Neb22a, Theorem 1.5]) Parker's conjecture holds for solvable groups.*

So, how could one try to prove Parker's conjecture? Certainly, one idea is to somehow show that the oddness of the orthogonal determinants of the $\mathrm{Irr}^+(G)$-characters behaves well with normal subgroups, i.e., to reduce the conjecture to a statement about finite simple groups (or maybe quasisimple groups). Then, by the classification of finite simple groups, one "just" would have to verify the conjecture for each of the (infinitely many!) cases. This approach has shown success for deep conjectures about finite groups, recently Brauer's Height Zero Conjecture was proven with this strategy, see [Mal+24].

While the reduction to finite simple groups has not been proven yet for Parker's conjecture, a proof of the following special case of the conjecture is certainly a big step towards a general solution:

Conjecture 2.3.30. *Let G be a finite simple group and $\chi \in \mathrm{Irr}^+(G)$. Let $K = \mathbb{Q}(\chi)$. Then $\det(\chi) \in K^\times/(K^\times)^2$ is odd.*

In Subsection 4.1.4, we will show that Parker's conjecture holds for the alternating groups, thus confirming it for the first non-trivial infinite class of finite simple groups.

3 Finite Groups of Lie Type

In this chapter, we will introduce the main objects of this thesis, the finite groups of Lie type, as well as some theory about their structure and representation theory that will become essential in the later chapters. Furthermore, we will talk Coxeter groups, which are in a close relationship with the finite groups of Lie type.

3.1 Reductive Algebraic Groups

We assume some basic knowledge in algebraic geometry. There is a vast amount of literature about affine algebraic groups with from time to time conflicting notions, e.g., if a variety is assumed to be irreducible or not. We refer the reader to the standard books [Hum81], [Bor91] and [Spr98], which all have the same title *Linear Algebraic Groups*.

Throughout we fix a prime p and let $K = \overline{\mathbb{F}}_p$ be an algebraically closed field in characteristic p. We let $\mathbb{A}^n_K$ be the affine n-dimensional space. An affine K-variety will always be $X = \mathrm{Spec}(A)$ for A a (commutative) finitely-generated and reduced K-algebra, which we can regard as a closed subvariety of $\mathbb{A}^n_K$ for some n. In particular, we do not assume varieties to be irreducible. We let $X(K)$ be the set of K-points of X.

Definition 3.1.1. *An affine algebraic group* $\mathbf{G}$ *over* K *is an affine* K*-variety, together with morphisms* $\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ *and* $\iota : \mathbf{G} \to \mathbf{G}$ *such that the set* $\mathbf{G}(K)$ *with the multiplication given by* μ *and the inverse given by* ι *is an abstract group.*

Let $\mathbf{G}$ be an affine algebraic group and let $\mathbf{H}$ be a closed subvariety. We say that $\mathbf{H}$ is a closed subgroup of $\mathbf{G}$ if additionally, $\mathbf{H}(K)$ is a subgroup of $\mathbf{G}(K)$ and write $\mathbf{H} \subseteq \mathbf{G}$. We say that $\mathbf{H}$ is a normal or solvable closed subgroup if the same holds for the corresponding groups. If $\mathbf{N}$ is a normal closed subgroup of $\mathbf{G}$, then $\mathbf{G}/\mathbf{N}$ has again the structure of an affine algebraic group. Also the normalizer $N_{\mathbf{G}}(\mathbf{H})$ is a closed subgroup of $\mathbf{G}$. An isomorphism of algebraic groups is an isomorphism of varieties, which is also an isomorphism of the corresponding groups.

Example 3.1.2. *We will give some important examples.*

(i) We define

$$\mathbf{GL}_n := \{(a_{11}, \dots, a_{nn}, b) \in \mathbb{A}^{n^2+1}_K \mid b \det(a_{ij})_{1 \leq i,j \leq n} = 1\}$$

with the obvious multiplication and inversion morphisms to be the general linear algebraic group. We have $\mathbf{GL}_n(K) \cong \mathrm{GL}_n(K)$*. In the sequel, we will thus identify the algebraic group* $\mathbf{GL}_n$ *with the group* $\mathrm{GL}_n(K)$*. By a standard theorem of algebraic groups, see for instance [Hum81, Theorem 8.6], any affine algebraic group is a closed subgroup of* $\mathbf{GL}_n$ *for some* n*.*

(ii) *We set*

$$\mathbf{SL}_n := \{(a_{11}, \dots, a_{nn}, 1) \in \mathbf{GL}_n\}$$

to be the special linear algebraic group.

(iii) *We set* $\mathbb{G}_m := \mathbf{GL}_1$ *to be the multiplicative group. We have that* $\mathbb{G}_m(K) \cong K^\times$. *A torus will be an affine algebraic group isomorphic to* $\mathbb{G}_m^r$ *for some* r.

(iv) *We set* $\mathbb{G}_a := \mathbb{A}_K^1$ *to be the additive group. We have that* $\mathbb{G}_a(K) \cong K^+$.

As any affine algebraic group is a closed subgroup of $\mathbf{GL}_n$ for some n, we can regard which matrices elements of $\mathbf{G}(K)$ correspond to. The closed subgroup $\mathbf{H}$ if $\mathbf{G}$ will be called unipotent, if all elements in $\mathbf{H}(K)$ correspond to unipotent matrices. Equivalently, $\mathbf{H}$ is unipotent if and only if every element of $\mathbf{H}(K)$ has order a power of p.

A class of subgroups of utmost importance are the Borel subgroups - these are maximal closed connected solvable subgroups. Let $\mathbf{U} \subseteq \mathbf{G}$ be a maximal closed connected unipotent subgroup. Then $\mathbf{B} = \mathbf{N_G}(\mathbf{U})$ is a Borel subgroup, and all Borel subgroups arise this way. Any Borel subgroup is a semidirect product of a maximal closed connected unipotent subgroup and a maximal torus. Furthermore, all Borel subgroups are conjugate, and every maximal torus is contained in a Borel subgroup. The non-negative integer r such that $\mathbf{T} \cong \mathbb{G}_m^r$, where $\mathbf{T} \subseteq \mathbf{G}$ is a maximal torus, is called the rank of $\mathbf{G}$. We say that a closed subgroup $\mathbf{P} \subseteq \mathbf{G}$ is a parabolic subgroup, if it contains a Borel subgroup.

Example 3.1.3. *For* $\mathbf{GL}_n$, *a Borel subgroup* $\mathbf{B}_n \subseteq \mathbf{GL}_n$ *is given by the upper triangular matrices, while the maximal closed connected unipotent subgroup* $\mathbf{U}_n$ *contained in* $\mathbf{B}_n$ *is given by the unipotent upper triangular matrices. We denote* $\mathbf{T}_n$ *to be the diagonal, it is clear that it is a maximal torus and* $\mathbf{B}_n = \mathbf{U}_n \rtimes \mathbf{T}_n$. *Further,* $\mathbf{T}_n \cong \mathbb{G}_m^n$ *and so the rank of* $\mathbf{GL}_n$ *is equal to* n. *We can further describe all the parabolic subgroups containing* $\mathbf{B}_n$. *For this, let* $(a_1, \dots, a_m)$ *be positive integers that sum to* n *and consider the closed subgroup*

$$\mathbf{L} := \mathbf{GL}_{a_1} \times \cdots \times \mathbf{GL}_{a_m} \subseteq \mathbf{GL}_n,$$

where we take the diagonal embedding. Then the group generated by $\mathbf{L}$ *and* $\mathbf{B}_n$ *is clearly parabolic, and all parabolic subgroups containing* $\mathbf{B}_n$ *arise that way. In particular, there are* 2^n *many.*

Definition 3.1.4. *The unipotent radical* $\mathbf{R}_u(\mathbf{G})$ *is the maximal closed connected normal unipotent subgroup of* $\mathbf{G}$. *We call* $\mathbf{G}$ *reductive if the unipotent radical is trivial.*

We will from now assume that $\mathbf{G}$ is a connected reductive group. We say that $\mathbf{G}$ is semi-simple, if $Z(\mathbf{G})^0 = \{1\}$, where $Z(\mathbf{G})^0$ is the identity component of the center of $\mathbf{G}$. Note that the derived subgroup $\mathbf{G}' = [\mathbf{G}, \mathbf{G}]$ is semi-simple, and it holds that $\mathbf{G} = Z(\mathbf{G})^0\mathbf{G}'$. We further say that $\mathbf{G}$ is simple, if $\mathbf{G}$ has no nontrivial closed connected normal subgroups. In particular, every simple connected reductive group is semi-simple.

Example 3.1.5. *We have that* $\mathbf{GL}_n$ *is a connected reductive group, while* $\mathbf{B}_n$ *is not reductive, as it has* $\mathbf{U}_n$ *as a maximal closed connected normal unipotent subgroup. The group* $\mathbf{GL}_n$ *is not semi-simple, as* $Z(\mathbf{GL}_n)^0 \cong \mathbb{G}_m$ *is the group of scalar matrices. It is not hard to see that* $\mathbf{GL}_n' = \mathbf{SL}_n$. *Further,* $\mathbf{SL}_n$ *is a simple algebraic group.*

Definition 3.1.6. *Let* $\mathbf{G}$ *be a connected reductive group and let* $\mathbf{T} \subseteq \mathbf{G}$ *be a maximal torus. The group* $W(\mathbf{T}) := \mathbf{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ *is called the Weyl group of* $\mathbf{G}$ *with respect to* $\mathbf{T}$. *It is a finite group and the isomorphism class is independent of the maximal torus chosen. Note that there is a natural action of* $W(\mathbf{T})$ *on* $\mathbf{T}$ *by* $w \cdot t = \dot{w}t\dot{w}^{-1}$ *for* $w \in W(\mathbf{T})$, $t \in \mathbf{T}$ *for any* $\dot{w} \in \mathbf{N}_{\mathbf{G}}(\mathbf{T})$ *representing* w.

Example 3.1.7. *For* $\mathbf{GL}_n$, *we have that* $\mathbf{N}_{\mathbf{G}}(\mathbf{T}_n)$ *is given by the monomial matrices, i.e., the* $n \times n$ *matrices that have exactly one non-zero entry in every row and column. Then* $\mathbf{N}_{\mathbf{G}}(\mathbf{T}_n)/\mathbf{T}_n \cong \mathfrak{S}_n$, *the symmetric group of* n *letters.*

Much of the structure of $\mathbf{G}$ is inherited from its Weyl group. This can mostly be boiled down to the Bruhat decomposition: Let $\mathbf{T} \subseteq \mathbf{B}$ be a Borel subgroup containing the maximal torus $\mathbf{T}$. Then

$$\mathbf{G} = \bigcup_{w \in W(\mathbf{T})} \mathbf{B}w\mathbf{B}$$

is a disjoint union of double cosets.

3.2 Coxeter Groups

The Weyl group of a connected reductive group is a special case of a Coxeter group. Coxeter groups have proven to be an essential part of representation theory and enjoy a very deep and rich theory. Examples include the dihedral groups D_{2n} of order $2n$ and the symmetric groups $\mathfrak{S}_n$. The theory of Coxeter groups is essential for the theory of finite groups of Lie type, and in a very informal way, they can be thought of as "groups of Lie type in characteristic 1". For more details concerning Coxeter groups, regard for instance [GP00] and [BB05].

Definition 3.2.1. *Let* S *be a set and* $m : S \times S \to \mathbb{Z}_{>0} \cup \{\infty\}$ *such that* $m(s, s') = m(s', s)$ *for all* $s, s' \in S$ *and that* $m(s, s') = 1$ *if and only if* $s = s'$. *The group with the presentation*

$$W := \langle S \mid s^2 = 1 \text{ for all } s \in S, (ss')^{m(s,s')} = 1 \text{ for all } s, s' \in S \text{ with } m(s, s') \in \mathbb{Z}_{>0} \rangle$$

is called a Coxeter group and the tuple (W, S) *is called a Coxeter system. The relations*

$$ss's \cdots = s'ss' \ldots,$$

where $m(s, s')$ *many terms appear on each side, are called the braid relations.*

Example 3.2.2. *(i) Take the symmetric group* $\mathfrak{S}_n$ *for* $n > 1$. *We take*

$$S := \{(i, i+1) \mid 1 \leq i \leq n-1\}$$

to be the set of simple transpositions. Then $(\mathfrak{S}_n, S)$ *is a Coxeter system. We denote* $s_i := (i, i+1) \in S$. *Then* $s_i^2 = 1$ *for all* i, $s_i s_{i+1}$ *has order* 3 *for* $i < n-1$, *i.e.,* $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, *and* $s_i s_j = s_j s_i$ *for* $|i - j| > 1$.

(ii) *The dihedral group*

$$D_{2n} = \langle s, t \mid s^2 = t^2 = 1, (st)^n = 1 \rangle$$

of order $2n$ *is a Coxeter group with Coxeter system* $(D_{2n}, \{s, t\})$.

We will now fix a Coxeter system (W, S). Coxeter groups enjoy a variety of nice properties. For once, the order of ss' for $s, s' \in S$ is exactly $m(s, s')$. There is a well-defined length function on W: For any $w \in W$, the length $\ell(w)$ is the minimum number k of generators $s_1, \dots, s_k \in S$ such that $w = s_1 \cdots s_k$. A reduced expression of w means writing $w = s_1 \cdots s_k$ for any $s_i \in S$ such that $\ell(w) = k$. We will now gather some fundamental theorems about Coxeter groups.

Theorem 3.2.3. *Let* $w = s_1 \cdots s_k \in W$ *for* $s_i \in S$ *be a reduced expression.*

(i) *(Deletion property) Assume* $\ell(w) < k$. *Then there are indices* $i < j$ *such that*

$$w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

where the hats mean that we leave these indices out.

(ii) *(Exchange property) Let* $s \in S$. *Then* $\ell(sw) = \ell(w) \pm 1$ *and if* $\ell(sw) < \ell(w)$ *then there is an index* i *such that*

$$sw = s_1 \cdots \hat{s}_i \cdots s_k.$$

(iii) *If* $\ell(w) = k$, *i.e., the expression above is reduced, then the set* $\{s_1, \dots, s_k\} \subseteq S$ *is uniquely determined by* w.

(iv) *If* W *is finite, there is a unique element* $w_0 \in W$ *of maximal length, called the longest element of* W. *It holds that* $w_0^2 = 1$ *and* $\ell(ww_0) = \ell(w_0) - \ell(w)$.

(v) *(Matsumoto's Theorem) Let* $w = s'_1 \cdots s'_k$ *be another reduced expression. Then the two reduced expression can be transformed into each other by braid relations.*

There is a set related to S, which is the set $T := \{wsw^{-1} \mid s \in S, w \in W\}$. For $W = \mathfrak{S}_n$, the set T corresponds to the set of transpositions, i.e., the permutations (i, j) for $1 \leq i < j \leq n$. We will quickly go over some important partial orders on Coxeter groups.

Definition 3.2.4. *Let* $w, w' \in W$.

(i) *(Bruhat order) We say that* $w \leq w'$ *in the Bruhat partial order if there is a sequence* $w = w_1, w_2, \dots, w_k = w'$ *of elements in* W *such that* $\ell(w_i) < \ell(w_{i+1})$ *and* $w_{i+1} = w_i t_i$ *for some* $t_i \in T$.

(ii) *(Weak right order) We say that* $w \leq_R w'$ *in the weak right partial order if* $w' = ws_1 s_2 \cdots s_k$ *for* $s_i \in S$ *such that* $\ell(ws_1 s_2 \cdots s_i) = \ell(w) + i$ *for all* $1 \leq i \leq k$.

(iii) (Weak left order) We say that $w \leq_L w'$ in the weak left partial order if $w' = s_1 s_2 \cdots s_k w$ for $s_i \in S$ such that $\ell(s_1 s_2 \cdots s_i w) = \ell(w) + i$ for all $1 \leq i \leq k$.

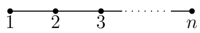
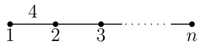
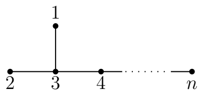
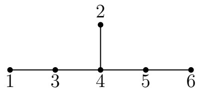
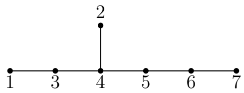
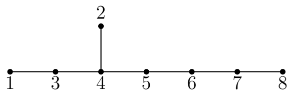
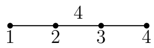
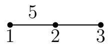
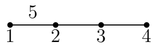
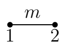
For any subset $I \subseteq S$, there is a group generated by the elements of I, denoted by W_I. These groups are called the parabolic subgroups of W, and (W_I, I) are themselves again Coxeter systems.

We are for our purposes only interested in finite Coxeter groups, so we will from now assume that S is finite and $m(s, s') \in \mathbb{Z}_{>0}$ for all s, s'. There is a complete classification of finite Coxeter groups, given by so called Coxeter diagrams. A Coxeter diagram for the Coxeter system (W, S) is the graph with vertices the set S where we put an edge between two vertices s, s' if $m(s, s') \geq 3$. If $m(s, s') \geq 4$, we label that edge with $m(s, s')$. If the Coxeter graph is disconnected with connected components $S_1, \ldots, S_k$, then

$$W \cong W_{S_1} \times W_{S_2} \times \cdots \times W_{S_k}$$

so we can focus on the case where the Coxeter graph is connected. The groups with connected Coxeter graph are also called irreducible.

We will now give a list of all finite irreducible Coxeter groups, as well as their orders and naming conventions. For a reference, regard for instance [BB05, Appendix A1].

Name	Coxeter Diagram	Group Order
$A_n, n \geq 1$		$(n+1)!$
$B_n, n \geq 2$		$2^n n!$
$D_n, n \geq 4$		$2^{n-1} n!$
E_6		$2^7 \cdot 3^4 \cdot 5$
E_7		$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
E_8		$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
F_4		1152
H_3		120
H_4		14400
$I_2(m), m \geq 3$		$2m$

The index here shows the number of elements in S of the Coxeter system (W, S). The Coxeter groups of type A_n are the symmetric groups $\mathfrak{S}_{n+1}$, and the Coxeter groups of type $I_2(m)$ are the dihedral groups D_{2m}.

There is a representation of finite Coxeter groups which will become important later on, which we will quickly introduce.

Definition 3.2.5. *The sign representation is the homomorphism*

$$\text{sgn} : W \to \{-1, 1\}, \ w \mapsto (-1)^{\ell(w)}.$$

3.3 Root Systems

The Weyl group of a connected reductive group can be realized as the group generated by reflections of a certain finite subset of real vector space, called a root system. Root systems are also the key for the classification of connected reductive groups by so called root data. They appear in a multitude of algebraic situations, for instance they are also crucial for the theory of Lie algebras. For a reference, see for instance [DM20, Chapter 2.2] or [FH91, Lecture 21].

Let V be a finite-dimensional real vector space with $V^* = \mathrm{Hom}(V, \mathbb{R})$ its dual. Let $\alpha \in V\backslash\{0\}$ be a vector and let $\alpha^\vee \in V^*$ such that $\alpha^\vee(\alpha) = 2$. For each such pair, there is a unique element $s_\alpha \in \mathrm{GL}(V)$ of order 2, called a reflection for the pair $(\alpha, \alpha^\vee)$, such that s_α fixes the hyperplane $\ker(\alpha^\vee)$ and $s_\alpha(\alpha) = -\alpha$.

Definition 3.3.1. *A root system is a finite subset $\Phi \subseteq V$, together with a finite subset $\Phi^\vee \subseteq V^*$ and a bijective function $\Phi \to \Phi^\vee, \alpha \mapsto \alpha^\vee$, such that for all $\alpha \in \Phi$, $\alpha^\vee(\alpha) = 2$, the reflection s_α for the pair $(\alpha, \alpha^\vee)$ stabilizes Φ and*

$$\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}.$$

The elements of Φ are called roots and the elements of $\Phi^\vee$ are called coroots. The root system is called crystallographic if $\alpha^\vee(\beta) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Let now $\Phi \subseteq V$ be a root system. We will fix a linear form $\phi \in V^*$ such that Φ does not vanish on ϕ. We get a partition $\Phi = \Phi^+ \cup \Phi^-$, called the positive and negative roots, where $\phi(\alpha) > 0$ (resp. $\phi(\alpha) < 0$) for all $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$). There is a unique minimal subset $\Pi \subseteq \Phi$, called a basis of Φ, such that $\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}\Pi$. The elements of Π are called simple roots and the cardinality of Π is called the rank of the root system. It holds that all $\alpha \in \Phi^+$ that $\alpha \in \mathbb{Z}_{\geq 0}\Pi$.

Let

$$W := \langle \{s_\alpha \mid \alpha \in \Phi\} \rangle, \ S := \{s_\alpha \mid \alpha \in \Pi\}.$$

Then (W, S) is a Coxeter system, called the Weyl group of the corresponding root system. The group W acts transitively on the roots Φ.

It is easy to see that there is a W-invariant inner product $(\cdot, \cdot)$ on V; we will fix such an inner product. This inner product allows for an equivalent approach for root systems by identifying each $\alpha^\vee$ with the element $\frac{2\alpha}{(\alpha,\alpha)} \in V$. It follows that a root system is crystallographic if and only if $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. Any crystallographic root system can thus be realized in a vector space over the rational numbers.

We will from now on assume that Φ is crystallographic. For any two $s, s' \in S$, the order of ss' is in $\{2, 3, 4, 6\}$. This is equivalent with the angle between two elements $\alpha, \alpha' \in \Pi$ being in $\{\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}\}$.

Similarly to the Coxeter diagrams for Coxeter systems, we will now describe Dynkin diagrams for crystallographic root systems. It is the graph with vertices Π, where we put the edges between two vertices as follows:

(i) No edge, if the angle between the roots is $\frac{\pi}{2}$.

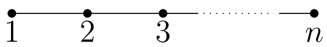
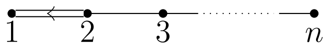
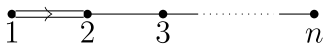
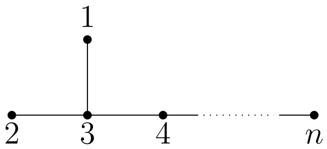
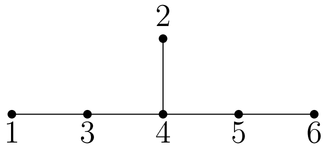
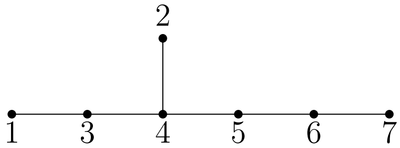
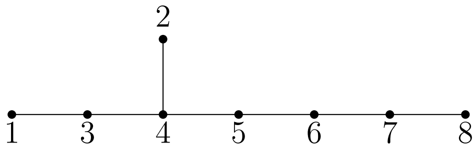
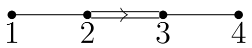
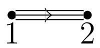
(ii) 1 edge, if the angle between the roots is $\frac{2\pi}{3}$. Note that here, the two roots have the same length.

(iii) 2 edges, if the angle between the roots is $\frac{3\pi}{4}$, with an arrow pointing from the longer to the shorter root.

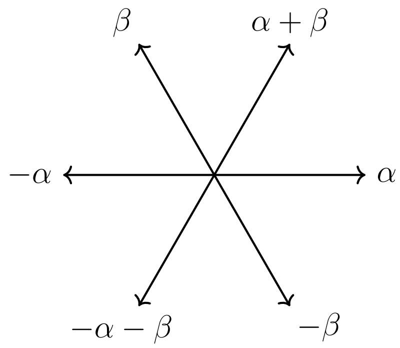
(iv) 3 edges, if the angle between the roots is $\frac{5\pi}{6}$, with an arrow pointing from the longer to the shorter root.

If the Dynkin diagram is disconnected with connected components $\Pi_1, \dots, \Pi_k$, then there is a corresponding partition $\Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_k$ into nonempty disjoint subsets, such that $(\alpha_i, \alpha_j) = 0$ for $\alpha_i \in \Phi_i, \alpha_j \in \Phi_j$, $i \neq j$. If the Dynkin diagram is connected, we also say that the root system is irreducible. Note that the root system is irreducible if and only if its Weyl group is irreducible.

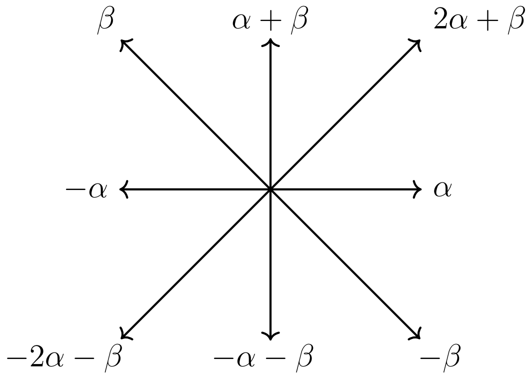
We will now give a full list of all irreducible, crystallographic root systems.

Name	Dynkin Diagram	Type of Weyl Group
$A_n, n \geq 1$		A_n
$B_n, n \geq 2$		B_n
$C_n, n \geq 3$		B_n
$D_n, n \geq 4$		D_n
E_6		E_6
E_7		E_7
E_8		E_8
F_4		F_4
G_2		$I_2(6)$

Example 3.3.2. *(i) The following is the root system A_2, with simple roots α and β.*



(ii) The following is the root system B_2, with simple roots α and β.



3.3.1 Classification of Connected Reductive Groups

We will now describe how root systems can be used to completely classify connected reductive groups, via so called root data. We will follow [DM20, Sections 2.3, 2.4].

We will fix a prime p and let $K = \overline{\mathbb{F}}_p$ be an algebraically closed field in characteristic p. Let $\mathbf{G}$ be a connected reductive group over K and let $\mathbf{T} \cong \mathbb{G}_m^r \subseteq \mathbf{G}$ be a maximal torus, where r is the rank of $\mathbf{G}$.

Definition 3.3.3. *(i) We let $X(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$ be the character group of $\mathbf{T}$. We have that $X(\mathbf{T}) \cong \mathbb{Z}^r$. There is a natural action of the Weyl group $W(\mathbf{T})$ on $X(\mathbf{T})$ given by $(w \cdot \chi)(t) = \chi(w^{-1} \cdot t)$ for $w \in W(\mathbf{T})$, $\chi \in X(\mathbf{T})$ and $t \in \mathbf{T}$.*

(ii) We let $Y(\mathbf{T}) := \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ be the cocharacter group of $\mathbf{T}$. We have that $Y(\mathbf{T}) \cong \mathbb{Z}^r$.

(iii) Note that there is an isomorphism $\tau : \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ given by

$$\tau : (t \mapsto t^n) \mapsto n$$

for any $t \in \mathbb{G}_m$. Let $\chi \in X(\mathbf{T})$, $\phi \in Y(\mathbf{T})$. There is a perfect pairing $X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$ given by

$$(\chi, \phi) \mapsto \tau(\chi \circ \phi).$$

A root subgroup is a non-trivial minimal closed unipotent subgroup $\mathbf{U} \subseteq \mathbf{G}$ that is normalized by $\mathbf{T}$. For any root subgroup, there is an isomorphism $\mathbf{u} : \mathbb{G}_a \to \mathbf{U}$. We define a root $\alpha \in X(\mathbf{T})$ by the property that

$$t\mathbf{u}(x)t^{-1} = \mathbf{u}(\alpha(t)x)$$

for $t \in \mathbf{T}, x \in \mathbb{G}_a$. We write $\mathbf{U} = \mathbf{U}_\alpha$, and let

$$\Phi = \{\alpha \mid \mathbf{U}_\alpha \text{ root subgroup of } \mathbf{G}\}.$$

Note that different root subgroups give rise to different roots.

It now holds that $\Phi = -\Phi$. Let $\alpha \in \Phi$ be a root. Then there is a surjective map $\phi_\alpha : \mathbf{SL}_2 \to \langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} \rangle$ such that

$$\phi_\alpha\left(\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right) = \mathbf{U}_\alpha, \ \phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\right) = \mathbf{U}_{-\alpha}.$$

We then define the coroot $\alpha^\vee \in Y(\mathbf{T})$ by

$$\alpha^\vee(x) = \phi_\alpha\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right)$$

for $x \in \mathbb{G}_m$. The element $\alpha^\vee$ is unique and we let $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ be the set of coroots. Note that the map ϕ_α is either an isomorphism or has kernel $\{I_2, -I_2\}$.

With the perfect pairing we have defined in Definition 3.3.3(iii), it holds that $\alpha^\vee(\alpha) = 2$ for all $\alpha \in \Phi$. Let $V = X(\mathbf{T}) \otimes \mathbb{R}$. Then $(\Phi, \Phi^\vee)$ is a crystallographic root system. The action of $W(\mathbf{T})$ on $X(\mathbf{T})$ translates to a faithful representation $\iota : W(\mathbf{T}) \to \mathrm{GL}(V)$ which gives an isomorphism between $W(\mathbf{T})$ and the Weyl group W of the root system. The reflection $s_\alpha \in W$ then corresponds to the element

$$\phi_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \in \mathbf{N}_{\mathbf{G}}(\mathbf{T}).$$

Let us now fix a Borel subgroup $\mathbf{B} = \mathbf{N}_{\mathbf{G}}(\mathbf{U})$ for some maximal closed connected unipotent subgroup $\mathbf{U} \subseteq \mathbf{G}$. This gives rise to a partition $\Phi = \Phi^+ \cup \Phi^-$ by

$$\mathbf{U} = \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$$

and thus to a set of simple roots $\Pi \subseteq \Phi$ and a corresponding subset $S \subseteq W$ such that (W, S) is a Coxeter system. By the isomorphism ι, also now $W(\mathbf{T})$ is a Coxeter system.

We call the tuple $(X(\mathbf{T}), Y(\mathbf{T}), \Phi, \Phi^\vee)$ the root datum of $\mathbf{G}$. In the other direction, every abstract root datum gives rise to a unique connected reductive group:

Definition 3.3.4. *Let n be a non-negative integer. A lattice X is a free $\mathbb{Z}$-module of rank n. A dual lattice to X is another lattice Y of rank n such that there is a perfect pairing $X \times Y \to \mathbb{Z}$.*

Definition 3.3.5. *A root datum is a tuple $(X, Y, \Phi, \Phi^\vee)$, such that X and Y are dual lattices, $\Phi \subseteq X$, $\Phi^\vee \subseteq Y$ and that $(\Phi, \Phi^\vee)$ is a root system in the vector space $X \otimes \mathbb{R}$. Here, the isomorphism $Y \otimes \mathbb{R} \cong \mathrm{Hom}(X, \mathbb{R})$ is achieved via the perfect pairing of X and Y. We call the root datum crystallographic if the root system $(\Phi, \Phi^\vee)$ is.*

(i) We call the root datum semisimple if Φ spans the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes X$. We call it simple if it further holds that the root system is irreducible.

(ii) If $X = \mathbb{Z}\Phi$ we call the root datum adjoint.

(iii) If $Y = \mathbb{Z}\Phi^\vee$ we call the root datum simply connected.

Theorem 3.3.6. *For every algebraically closed field k and every crystallographic root datum $(X, Y, \Phi, \Phi^\vee)$, there is an up to isomorphism unique connected reductive group over k with the given root datum.*

We call a connected reductive group simple, semisimple, adjoint, or simply connected, if and only if the corresponding root datum has that property.

Remark 3.3.7. *It is well known that a group being simply connected implies that the maps ϕ_α are an isomorphism for any $\alpha \in \Phi$, see for instance [Ste16, p. 31].*

Example 3.3.8. *We will now give examples of some connected reductive groups corresponding to certain irreducible root systems. For simplicity, we will assume that p is odd.*

(i) Type A_n: $\mathbf{GL}_{n+1}$, $\mathbf{SL}_{n+1}$, $\mathbf{PGL}_{n+1}$. The group $\mathbf{SL}_{n+1}$ is simple simply connected and the group $\mathbf{PGL}_{n+1}$ is simple adjoint.

(ii) Type B_n: $\mathbf{SO}_{2n+1}$, the closed subgroup of $\mathbf{GL}_{2n+1}$ that preserves a given non-degenerate symmetric bilinear form.

(iii) Type C_n: $\mathbf{Sp}_{2n}$, the closed subgroup of $\mathbf{GL}_{2n}$ that preserves a given symplectic bilinear form.

(iv) Type D_n: $\mathbf{SO}_{2n}$, the closed subgroup of $\mathbf{GL}_{2n}$ that preserves the non-degenerate symmetric bilinear form given by the matrix $(a_{ij})_{1 \leq i,j \leq 2n}$, where $a_{ij} = 1$ if $i + j = 2n + 1$, and $a_{ij} = 0$ else.

(v) Type G_2: $\mathbf{G}_2$, the closed subgroup of $\mathbf{GL}_8$ of automorphisms of an 8-dimensional non-associative octonion algebra over K. Since $\mathbf{G}_2$ fixes the 7-dimensional subspace of pure octonions, we can also regard $\mathbf{G}_2$ as a closed subgroup of $\mathbf{SO}_7$. Regard for instance [SW15, Section 3] for a more explicit construction.

To conclude this section, we will say a bit about the structure of parabolic subgroups.

Definition 3.3.9. *Let $\mathbf{P} \subseteq \mathbf{G}$ be a parabolic subgroup containing $\mathbf{B}$. Then there is a closed reductive subgroup $\mathbf{L} \subseteq \mathbf{P}$ such that $\mathbf{P} = \mathbf{R}_u(\mathbf{P}) \rtimes \mathbf{L}$. We have that $\mathbf{R}_u(\mathbf{P}) \subseteq \mathbf{U}$ and $\mathbf{T} \subseteq \mathbf{L}$. We call this product a Levi decomposition and $\mathbf{L}$ a Levi subgroup of $\mathbf{G}$.*

Let $I \subset S$. Then $\mathbf{P}_I := \mathbf{B}W_I\mathbf{B}$ is a parabolic subgroup containing $\mathbf{B}$ and all such subgroups arise that way. Here we identified W_I with the corresponding subgroup of $W(\mathbf{T})$. It holds that the Weyl group of $\mathbf{L}$ with respect to $\mathbf{T}$ is isomorphic to W_I.

3.4 Frobenius Maps and Finite Groups of Lie Type

In this section, we will finally introduce the finite groups of Lie type as fixed points of connected reductive groups by some map F. We refer the reader to the sources [Car85], [DM20] and [GM20]. Note that all three books differ slightly in their naming conventions of the maps F involved.

Let q be a power of a prime p and let $K = \overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$. For any power q of p we define the map

$$F_q : \mathbf{GL}_n \to \mathbf{GL}_n, (a_{ij})_{1\leq i,j\leq n} \mapsto (a_{ij}^q)_{1\leq i,j\leq n}.$$

Let $\mathbf{G}$ be an affine K-variety. We will regard $\mathbf{G}$ as a closed subgroup of $\mathbf{GL}_n$ for some n, i.e., there is some inclusion map $i : \mathbf{G} \to \mathbf{GL}_n$. We say that a morphism $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius map, if $i(F(g)) = F_q(i(g))$ for some q and all $g \in \mathbf{G}$. Further, we say that F is a Frobenius root if some power of it is a Frobenius map. Any Frobenius root is bijective and

$$\mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$$

is a finite group.

Definition 3.4.1. *A finite group of Lie type is a group $G = \mathbf{G}^F$ where $\mathbf{G}$ is a connected reductive group over K and $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius root. We will say that the group G is defined in characteristic p.*

We will now fix a connected reductive group $\mathbf{G}$ over K and a Frobenius root $F : \mathbf{G} \to \mathbf{G}$ and let $G := \mathbf{G}^F$ be the finite group of Lie type associated with the data. For any closed subgroup $\mathbf{H} \subseteq \mathbf{G}$ we say that $\mathbf{H}$ is F-stable if $F(\mathbf{H}) = \mathbf{H}$. There are pairs $(\mathbf{T}, \mathbf{B})$ of a maximal torus $\mathbf{T}$ and Borel subgroup $\mathbf{B}$ with $\mathbf{T} \subseteq \mathbf{B}$ that are both F-stable; any such pair is conjugate over G. We fix such a pair and let $T := \mathbf{T}^F$, $B := \mathbf{B}^F$, and call T a quasi-split torus and B a Borel subgroup of G. The order of T is coprime to p. Let $\mathbf{U} \subseteq \mathbf{B}$ be the unipotent radical; it holds that the group $U := \mathbf{U}^F$ is a p-Sylow subgroup of G and we have a semidirect product $B = U \rtimes T$.

We will now discuss the Weyl group of a finite group of Lie type.

Definition 3.4.2. *Let (W, S) be the Coxeter system associated to $\mathbf{G}$ with $W \cong W(\mathbf{T})$. The subgroup $\mathbf{N}_{\mathbf{G}}(\mathbf{T}) \subseteq \mathbf{G}$ is F-stable, we thus get a natural action of F on W which also acts on S. Let S/F denote the set of orbits of the action of F on S. We set $N := \mathbf{N}_{\mathbf{G}}(\mathbf{T})^F$. Then $W^F := N/T = W(\mathbf{T})^F$ is called the Weyl group of G and $(W^F, \{w_I\}_{I\in S/F})$ is a Coxeter system, where we let w_I be the longest element of W_I.*

As in the case of connected reductive groups, there is a Bruhat decomposition of G: It holds that

$$G = \bigcup_{w\in W^F} BwB$$

is a disjoint union of double cosets.

Any subgroup $P \subseteq G$ containing a G-conjugate of B is called a parabolic subgroup; for any such parabolic subgroup, there is a F-stable parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ such that $P = \mathbf{P}^F$. This behaves well with the Levi decomposition: If

$$\mathbf{P} = \mathbf{R}_u(\mathbf{P}) \rtimes \mathbf{L}$$

is the Levi decomposition of $\mathbf{P}$, then also

$$P = \mathbf{R}_u(\mathbf{P})^F \rtimes \mathbf{L}^F,$$

which we will again call the Levi decomposition and say that $\mathbf{L}^F$ is a Levi subgroup of G. Note that the F-stable parabolic subgroups of $\mathbf{G}$ containing $\mathbf{B}$ are exactly the $\mathbf{P}_J$ for $J \subseteq \{w_I\}_{I \in S/F}$.

Definition 3.4.3. *Recall that $X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$. Then F acts on $X(\mathbf{T})$ by*

$$F \cdot \alpha := \alpha \circ F$$

for $\alpha \in X(\mathbf{T})$. Let n be the smallest positive integer such that the action of F^n on $X(\mathbf{T})$ equals $k \cdot \mathrm{id}$ for some positive integer k. It holds that k is a power of p. We define the number $q \in \mathbb{R}_{>0}$ attached to F by $q := \sqrt[n]{k}$ and say that the pair $(\mathbf{G}, F)$ is split if $n = 1$.

Note that if $(\mathbf{G}, F)$ is split, then $W^F = W$. Also, then $T \cong \mathbb{G}_m^r(q)$, where r is the rank of $\mathbf{G}$.

We will now see a nice formula on the order of finite groups of Lie type: It holds that

$$|G| = q^{\ell(w_0)} |T| \sum_{w \in W^F} q^{\ell(w)},$$

where w_0 is the longest element in the Coxeter system (W, S) and ℓ is the length function of W.

Example 3.4.4. *We will give some examples of finite groups of Lie type. Let q be a power of p, and let $F = F_q$ be the (standard) Frobenius map.*

(i) Tori: $\mathbb{G}_m^n(q) := (\mathbb{G}_m^n)^F \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times \times \cdots \times \mathbb{F}_q^\times$.

(ii) Type A_n: $\mathrm{GL}_{n+1}(q) := \mathbf{GL}_{n+1}^F \cong \mathrm{GL}_{n+1}(\mathbb{F}_q)$, $\mathrm{SL}_{n+1}(q) := \mathbf{SL}_{n+1}^F \cong \mathrm{SL}_{n+1}(\mathbb{F}_q)$. *We take the subgroup $\mathbf{B}$ of either $\mathbf{GL}_{n+1}$ or $\mathbf{SL}_{n+1}$ to be the subgroup of upper triangular matrices. Then $\mathbf{B}$ is F-stable and we take $B := \mathbf{B}^F$. The quasi-split torus of either $\mathrm{GL}_{n+1}(q)$ or $\mathrm{SL}_{n+1}(q)$ then just equals the subgroup of diagonal matrices.*

The orders are

$$|\mathrm{GL}_{n+1}(q)| = q^{n(n+1)/2}(q-1)(q^2-1)\cdots(q^{n+1}-1),$$
$$|\mathrm{SL}_{n+1}(q)| = |\mathrm{GL}_{n+1}(q)|/(q-1).$$

(iii) Type B_n: $\mathrm{SO}_{2n+1}(q) := \mathbf{SO}_{2n+1}^F$. *If q is odd, $\mathrm{SO}_{2n+1}(q)$ is isomorphic to the subgroup of $\mathrm{SL}_{2n+1}(q)$ that fixes a given non-degenerate symmetric bilinear form on $\mathbb{F}_q^{2n+1}$.*

(iv) Type C_n: $\mathrm{Sp}_{2n}(q) := \mathbf{Sp}_{2n}^F$*. The group* $\mathrm{Sp}_{2n}(q)$ *is isomorphic to the subgroup of* $\mathrm{SL}_{2n}(q)$ *that fixes a given non-degenerate symplectic bilinear form on* $\mathbb{F}_q^{2n}$*.*

(v) Type D_n: $\mathrm{SO}_{2n}(q) := \mathbf{SO}_{2n}^F$*. If q is odd, the group* $\mathrm{SO}_{2n}(q)$ *is isomorphic to the subgroup of* $\mathrm{SL}_{2n}(q)$ *that fixes a given non-degenerate symmetric bilinear form β on* $\mathbb{F}_q^{2n}$ *such that* $\mathrm{disc}(\beta) = 1 \cdot (\mathbb{F}_q^\times)^2$*.*

(vi) Type G_2: $G_2(q) := \mathbf{G}_2^F$*.*

Example 3.4.5. *Type ${}^2A_n$: We will now regard a non-split pair $(\mathbf{SL}_{n+1}, F)$. Let q be a power of p. We define the matrix*

$$\Omega_{n+1} := \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}$$

with 1s on the antidiagonal. Consider the morphism

$$F : \mathbf{SL}_{n+1} \to \mathbf{SL}_{n+1},\ g \mapsto \Omega_{n+1}(F_q(g)^{tr})^{-1}\Omega_{n+1}.$$

It is clear that $F^2 = F_{q^2}$, so F is a Frobenius root. We set $\mathrm{SU}_{n+1}(q) := \mathbf{SL}_{n+1}^F$*. This group equals the set of matrices in* $\mathrm{SL}_{n+1}(q^2)$ *that fix the Hermitian form on* $\mathbb{F}_{q^2}^{n+1}$ *given by the matrix Ω_{n+1}.*

Let $\mathbf{B} \subseteq \mathbf{SL}_{n+1}$ be the Borel subgroup of upper triangular matrices and let $\mathbf{T} \subseteq \mathbf{B}$ be the subgroup of diagonal matrices, which is a maximal torus. Then both $\mathbf{B}$ and $\mathbf{T}$ are F-stable; we denote $B := \mathbf{B}^F, T := \mathbf{T}^F$ to be a Borel subgroup and quasi-split torus of $\mathrm{SU}_{n+1}(q)$*.*

It is clear that F^2 is the smallest power of F such that the action on $X(\mathbf{T})$ is a multiple of the identity, for which we get that it acts as $q^2 \cdot \mathrm{id}$*. Therefore the number q attached to F is exactly q, so there is no confusion there.*

Let $W \cong \mathfrak{S}_n$ be the Weyl group of $\mathbf{SL}_{n+1}$. Then W^F is isomorphic to a Coxeter group of type $B_{\lceil n/2 \rceil}$. We have that

$$T \cong \begin{cases} \mathbb{G}_m^{(n-1)/2}(q^2) \times \mathbb{G}_m(q), & \text{if } n+1 \text{ is even,} \\ \mathbb{G}_m^{n/2}(q^2), & \text{if } n+1 \text{ is odd.} \end{cases}$$

The order is given by

$$|\mathrm{SU}_{n+1}(q)| = q^{n(n+1)/2}(q^2-1)(q^3+1)\cdots(q^{n+1}-(-1)^{n+1}).$$

3.5 Representation Theory of Finite Groups of Lie Type

One almost cannot talk about the representation theory of finite groups of Lie type without at least slightly diving into Deligne–Lusztig theory, which uses the étale cohomology of certain varieties to construct characters. In the context of this thesis, we are in a privileged position though: It is almost trivially easy to calculate the orthogonal determinants of the characters that are the hardest to construct! In the other direction, the most "basic" characters in the theory, the principal series characters and even more so, the principal series unipotent characters, will be the ones that will give us the most troubles. We will briefly touch upon two important concepts in the overarching theory. First, we will talk about Harish-Chandra theory, which divides the irreducible characters into certain series, the Harish-Chandra series. These arise by inducing certain characters from parabolic subgroups. This is also where the principal series characters appear, these are characters appearing in the induced characters from a Borel subgroup. Second, we want to introduce Iwahori–Hecke algebras. These are deformations of Coxeter groups and appear as the Hecke algebras with respect to Borel subgroups. We will refer to the same sources as in the last section, so [Car85], [DM20] and [GM20].

3.5.1 Harish-Chandra Theory

Let p be a prime, and let $G := \mathbf{G}^F$ be a finite group of Lie type, where $\mathbf{G}$ is a connected reductive group over $\overline{\mathbb{F}}_p$ and F is a Frobenius root on $\mathbf{G}$.

Let $\mathbf{L} \subseteq \mathbf{G}$ be a F-stable Levi subgroup contained in a F-stable parabolic subgroup $\mathbf{P}$. We denote $L = \mathbf{L}^F, P = \mathbf{P}^F$ and assume that we have a Levi-decomposition $P = U \rtimes L$. The idea of Harish-Chandra theory is to use certain "maximal" (called cuspidal) characters of L to partition the characters of G into so called Harish-Chandra series. If L is a quasi-split torus, these characters will be especially important for us; they will be denoted the principal series characters.

Definition 3.5.1. *Let $\theta \in \mathrm{Irr}(L)$, $\chi \in \mathrm{Irr}(G)$ be irreducible characters.*

(i) By the natural projection $P \to L$, the character θ lifts to the character $\mathrm{Inf}_L^P(\theta) \in \mathrm{Irr}(P)$. In the sequel, we will regard $\theta \in \mathrm{Irr}(P)$ as a character of P. The Harish-Chandra induction $R^G_{L\subseteq P}$ of θ is then defined by

$$R^G_{L\subseteq P}(\theta) := \mathrm{Ind}_P^G(\theta)\,.$$

(ii) Regard the character $\mathrm{Res}^G_P(\chi)$. This character is in general not irreducible, we decompose $\mathrm{Res}^G_P(\chi) = \chi_L + \chi_U$, where χ_L is the largest constituent of $\mathrm{Res}^G_P(\chi)$ such that its restriction to U is trivial, i.e.,

$$\mathrm{Res}^P_U(\chi_L) = \langle \mathrm{Res}^G_U(\chi), \mathbf{1}_U\rangle_U \cdot \mathbf{1}_U.$$

We define the Harish-Chandra restriction ${}^*R^G_{L\subseteq P}$ by

$${}^*R^G_{L\subseteq P}(\chi) := \mathrm{Res}^P_L(\chi_L).$$

It is not obvious, but it turns out that the functors defined above do not depend on the choice of parabolic subgroup P:

Proposition 3.5.2. *Let $\mathbf{P}'$ be another F-stable parabolic subgroup containing $\mathbf{L}$, and set $P' = (\mathbf{P}')^F$. Then for any θ in $\mathrm{Irr}(L)$, $\chi \in \mathrm{Irr}(G)$, we have that*

$$R^G_{L\subseteq P}(\theta) = R^G_{L\subseteq P'}(\theta),\ {}^*R^G_{L\subseteq P}(\chi) = {}^*R^G_{L\subseteq P'}(\chi).$$

We will thus from now on write R^G_L and ${}^*R^G_L$ for the Harish-Chandra induction and restriction. In the sequel we will denote

$$\chi_L := {}^*R^G_L(\chi)$$

for $\chi \in \mathrm{Irr}(G)$.

Definition 3.5.3. *Let $\theta \in \mathrm{Irr}(L)$. We say that θ is a cuspidal character of L and that (L, θ) is a cuspidal pair, if for any Levi subgroup $L' \subsetneq L$, it holds that ${}^*R^L_{L'}(\theta) = 0$.*

Definition 3.5.4. *Let θ be a cuspidal character of L. We denote*

$$\mathrm{Simp}(G \mid (L, \theta)) := \{\chi \in \mathrm{Irr}(G) \mid \langle \chi, R^G_L(\theta)\rangle_G \neq 0\}$$

and call it the Harish-Chandra series associated to the cuspidal pair (L, θ).

Note that the group G acts on the set of cuspidal pairs by conjugation, i.e., $g \cdot L := {}^gL$, and $(g \cdot \theta)(glg^{-1}) := \theta(l)$ for any $g \in G, l \in L$.

Theorem 3.5.5. *The following hold:*

(i) The functors R^G_L and ${}^*R^G_L$ are adjoint, i.e., for any $\theta \in \mathrm{Irr}(L)$, $\chi \in \mathrm{Irr}(G)$, it holds that

$$\langle R^G_L(\theta), \chi\rangle_G = \langle \theta, {}^*R^G_L(\chi)\rangle_L.$$

(ii) Let (L, θ), (L', θ') be two cuspidal pairs. Then the sets $\mathrm{Simp}(G \mid (L, \theta))$ and $\mathrm{Simp}(G \mid (L', \theta'))$ have a non-empty intersection if and only if there is a $g \in G$ such that $g \cdot (L, \theta) = (L', \theta')$. If this is the case, then the sets are equal and furthermore, $R^G_L(\theta) = R^G_{L'}(\theta')$.

(iii) All irreducible characters of G are contained in some Harish-Chandra series.

Let now $T \subseteq G$ be a quasi-split torus. Since T is abelian, all its irreducible characters have degree 1 and are especially easy to understand. In particular, for any character $\theta \in \mathrm{Irr}(T)$, the pair (T, θ) is cuspidal. We say that the characters in $\mathrm{Simp}(G \mid (T, \theta))$ for any such θ are in the principal series. If $\theta = \mathbf{1}_T$, we denote $\mathrm{Irr}_{PSU}(G) := \mathrm{Simp}(G \mid (T, \mathbf{1}_T))$ and call the characters appearing in that set the principal series unipotent characters.

3.5.2 Iwahori–Hecke Algebras and Principal Series Unipotent Characters

Let (W, S) be a finite Coxeter system. We denote $\ell : W \to \mathbb{Z}_{>0}$ to be the length function of the Coxeter system (W, S).

Definition 3.5.6. *Let $\{c_s \mid s \in S\}$ be positive integers such that $c_s = c_t$ whenever s and t are conjugate. Let $A = \mathbb{Q}[u, u^{-1}]$ where u is an indeterminate. We define the generic Iwahori–Hecke algebra $\mathcal{H} = \mathcal{H}_A(W, S, \{c_s \mid s \in S\})$ to be the A-algebra with basis $\{T_w \mid w \in W\}$ and generating set $\{T_s \mid s \in S\}$, together with the relations*

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) > \ell(w), \\ (u^{c_s} - 1)T_w + u^{c_s} T_{sw}, & \text{if } \ell(sw) < \ell(w) \end{cases}$$

for $s \in S$ and $w \in W$.

So let now $\mathcal{H} = \mathcal{H}_A(W, S, \{c_s \mid s \in S\})$ be a generic Iwahori–Hecke algebra. Let $K = \mathbb{Q}(u)$ be the quotient field of A. Then $K\mathcal{H}$ is a semisimple algebra, see [DM20, Remark 6.2.13]. We can also explicitly describe a splitting field for $K\mathcal{H}$.

Theorem 3.5.7. *(cf. [DM20, Theorem 6.2.10]) Assume $L \subseteq \mathbb{C}$ is a splitting field for W. Then $L(\sqrt{u})$ is a splitting field for $\mathcal{H}$.*

The above theorem is not optimal, indeed, in all our applications, a splitting field for $\mathcal{H}$ will actually already be equal to $L(u)$ for L a splitting field of $\mathbb{Q}$. Note that then we are in the situation of Subsection 2.2.3.

We want to now apply Iwahori–Hecke algebras to the theory of finite groups of Lie type, so we fix a prime p and let $G = \mathbf{G}^F$ be a finite group of Lie type for a connected reductive group $\mathbf{G}$ over $\overline{\mathbb{F}}_p$ and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius root. We let $B \subseteq G$ be a Borel subgroup and let $T \subseteq B$ be a quasi-split torus. Let (W, S) be the Coxeter system associated with $\mathbf{G}$ and let q be the real number attached to F. Recall that $(W^F, S/F)$ is a Coxeter system, so there are two possible length functions on W^F, the one coming from W and the one coming from W^F. We will always be using the length function coming from W.

As we have seen in the last subsection, we can divide the characters of G into Harish-Chandra series, given by the cuspidal pairs (L, θ) of G. We will now take a closer look at the principal series unipotent characters $\mathrm{Irr}_{\mathrm{PSU}}(G)$, i.e., all the irreducible characters χ of G appearing in $\mathrm{Ind}_B^G(\mathbf{1}_B)$. Let

$$e_B := \frac{1}{|B|} \sum_{h \in B} h \in \mathbb{Q}G$$

be the idempotent corresponding to B. We let $\mathcal{H}_q := e_B \mathbb{Q}G e_B$ be the corresponding Hecke-algebra. This algebra will exactly correspond to the specialization of an Iwahori–Hecke algebra, with the appropriate chosen parameters. As we have already seen, there is an isomorphism $\mathcal{H}_q \cong \mathrm{End}_{\mathbb{Q}G}(\mathrm{Ind}_B^G(\mathbf{1_B}))$ and a 1-to-1 correspondence between

$$\mathrm{Irr}_{\mathrm{PSU}}(G) \longleftrightarrow \mathrm{Irr}(\mathbb{C}\mathcal{H}_q)).$$

Recall the Bruhat decomposition

$$G = \bigcup_{w \in W^F} BwB$$

of G into a disjoint union of double cosets. We thus have a $\mathbb{Q}$-basis of $\mathcal{H}_q$ given by $\{T_w \mid w \in W^F\}$. Further, it can be shown that the set $\{T_{w_I} \mid I \in S/F\}$ is a generating set, together with the relations

$$T_{w_I} T_w = \begin{cases} T_{w_I w}, & \text{if } \ell(w_I w) > \ell(w), \\ (q^{\ell(w_I)} - 1)T_w + q^{\ell(w_I)} T_{w_I w}, & \text{if } \ell(w_I w) < \ell(w) \end{cases}$$

for $I \in S/F$ and $w \in W^F$.

This now allows us to define the Iwahori–Hecke algebra corresponding to a finite group of Lie type, by informally replacing the q in the above relations with an u.

Definition 3.5.8. *Let $A = \mathbb{Q}[u, u^{-1}]$. The Iwahori–Hecke algebra for G is defined to be the algebra*

$$\mathcal{H}_G := \mathcal{H}_A(W^F, S/F, \{l(w_I) \mid I \in S/F\}).$$

The algebra $\mathcal{H}_q$ corresponds to the specialization $u \mapsto q$, and the algebra $\mathbb{Q}W^F$ corresponds to the specialization $q \mapsto 1$. Since now both the irreducible characters of W^F and the principal series characters of G arise from the same Iwahori–Hecke algebra, there now is a 1-to-1 correspondence between the sets $\mathrm{Irr}(W^F)$ and $\mathrm{Irr}_{PSU}(G)$. For any character $\chi \in \mathrm{Irr}_{PSU}(G)$, we let $\tilde{\chi} \in \mathrm{Irr}(W^F)$ be the corresponding character of W^F. This interacts nicely with induction from parabolic subgroups, which is known as the Howlett–Lehrer Comparison Theorem:

Theorem 3.5.9. *(cf. [DM20, Lemma 7.2.11]) Let $J \subseteq S/F$ be a subset of simple roots. Let $W_J^F \subseteq W^F$, resp. $P_J \subseteq G$, be the respective parabolic subgroups. Then for any $\chi \in \mathrm{Irr}_{PSU}(G))$, it holds that*

$$\langle \mathrm{Ind}_{P_J}^G(\mathbf{1}_{P_J}), \chi \rangle_G = \langle \mathrm{Ind}_{W_J^F}^{W^F}(\mathbf{1}_{W_J^F}), \tilde{\chi} \rangle_{W^F}.$$

4 Orthogonal Determinants of Finite Coxeter Groups

In this chapter, we will explore the orthogonal determinants of the finite Coxeter groups. The orthogonal determinants of the symmetric groups have already been fully determined by James and Murphy in [JM79]. Given that result we will be able to give a nice showcase of some of the techniques we learned about in Section 2.3 and obtain the orthogonal determinants of the alternating groups, as well as the Coxeter groups of type B_n and D_n. The remainder of the Coxeter groups will be handled by direct calculations.

For all of the groups considered here, we will also prove Parker's conjecture, where the proof for the symmetric groups will turn out to be the most difficult.

4.1 Type A_n

Let n be a positive integer. To start of, the title of this section is slightly misleading: We will talk about the symmetric groups $\mathfrak{S}_n$, which is a Coxeter group of type A_{n-1} with standard generating set $S = \{s_1, \dots, s_{n-1}\}$ of simple transpositions.

Of course, the symmetric groups can be defined as the permutation group on the set of the first n numbers, so

$$\mathfrak{S}_n := \operatorname{Aut}(\{1, 2, \dots, n\}).$$

They can also be constructed as a subgroup of $\mathrm{GL}_n(K)$ for K any field: It holds that $\mathfrak{S}_n$ is isomorphic to the group of permutation matrices, i.e., the group of matrices with entries in $\{0, 1\}$ such that there is a single 1 in every row and every column. More concretely, the simple transposition s_i corresponds to the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix},$$

where the 2×2-block is in the rows and columns i and $i + 1$.

This approach will be important for the construction of the Coxeter groups of type B_n and D_n in Section 4.2.

As sources, we want to point out the book [JK81] by James and Kerber, which gives a very thorough treatment of the symmetric groups, with a focus on its combinatorics and representation theory. In particular, it contains a nice writeup about the orthogonal determinants of the symmetric groups in [JK81, Section 7.3]. The importance of this result in the context of this thesis can not be stressed enough.

Other sources that should be named are [FH91, Lecture 4] and [BB05].

4.1.1 Representation Theory of the Symmetric Groups

Let n be a positive integer. Let us begin by introducing some crucial notions about the combinatorics of symmetric groups.

Definition 4.1.1. *A partition is a sequence $\lambda = (a_1, \dots, a_m)$ of non-negative integers such that $a_1 \geq a_2 \geq \cdots \geq a_m$. We identify partitions that only differ by a string of zeros at the end. We say that λ is a partition of n if*

$$|\lambda| := \sum_{i=1}^{m} a_i = n.$$

We may combine repeated elements in a partition, e.g., we may also write $(5, 4^{(3)}, 2^{(0)}, 1^{(2)})$ for $(5, 4, 4, 4, 1, 1)$, if convenient. If $\mu = (c_1, \dots, c_l)$ is another partition of n, we write $\lambda \trianglelefteq \mu$ and say that λ proceeds μ in the dominance (partial) order if and only if

$$a_1 + \cdots + a_i \leq c_1 + \cdots + c_i$$

for all $i \geq 1$. We denote by $\mathscr{P}_n$ the set of partitions of n and by

$$\mathscr{P} := \bigcup_{i=0}^{\infty} \mathscr{P}_i$$

the set of all partitions.

For instance,

$$\mathscr{P}_5 = \{(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1^{(3)}), (1^{(5)})\}.$$

We now introduce some fundamental combinatorial objects.

Definition 4.1.2. *Let $\lambda = (a_1, \dots, a_m)$ be a partition.*

(i) We denote

$$[\lambda] = \{(i,j) \mid (i,j) \in \mathbb{Z} \times \mathbb{Z}, 1 \leq i \leq m, 1 \leq j \leq a_m\}$$

to be the Young diagram of λ. The elements of a Young diagram are called cells.

(ii) Put

$$\mathrm{rim}(\lambda) := \{(i,j) \in [\lambda] \mid (i+1, j+1) \notin [\lambda]\}.$$

The rim of a cell $c = (i,j) \in [\lambda]$ *will be the set of cells in the rim that are "between" its head and foot cells, i.e.,*

$$\mathrm{rim}_\lambda(c) = \{(i',j') \in \mathrm{rim}(\lambda) \mid i' \geq i, j' \geq j\}.$$

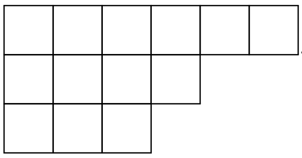
It is clear that $\mathrm{rim}(\lambda) = \mathrm{rim}_\lambda((1,1))$.

(iii) Let $c = (i,j) \in [\lambda]$. *We define* $h_\lambda(c) := |\mathrm{rim}_\lambda(c)|$ *to be the hook length of* c. *Equivalently, it is the number of cells to the right and below* c*, including itself.*

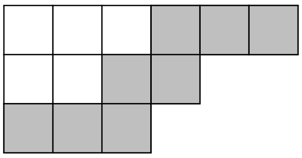
(iv) We say that λ *is* q*-core for some integer* q *if there is no* $c \in [\lambda]$ *such that* q *divides* $h_\lambda(c)$.

There is a convenient way to depict Young diagrams.

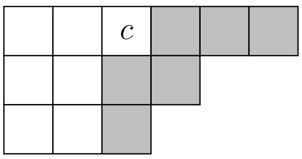
Example 4.1.3. *Let* $\lambda = (6,4,3)$ *be a partition of* 13. *Its Young diagram* $[\lambda]$ *then has the following form:*



Next, the rim $\mathrm{rim}(\lambda)$ *consists of the cells at its border:*



Let $c = (1,3) \in [\lambda]$. *The rim of* c, $\mathrm{rim}_\lambda(c)$*, consists of the following cells:*



Since its rim consists of 6 *cells, we have that* $h_\lambda(c) = 6$. *The hook diagram consists of the Young diagram, where we fill each cell with the corresponding hook length. So the hook diagram of* λ *is*

8	7	6	4	2	1
5	4	3	1		
3	2	1			

Finally λ *is not* 8*-core, since there is a cell* $(1,1)$ *with hook length equal to* 8.

Definition 4.1.4. *Let* λ *be a partition of* n.

(i) *A Young tableau is a bijective mapping of $\{1, \dots, n\}$ to $[\lambda]$. It may also be seen as filling the cells of the Young diagram with the numbers from 1 to n such that every number appears exactly once. There is an obvious regular group action of $\mathfrak{S}_n$ on the set of Young diagrams by permuting the entries.*

(ii) *A Young tableau is called standard if the entries in the cells read from left to right and from top to bottom are always increasing. We write T_λ for the set of standard Young tableaux.*

(iii) *Let t be a Young tableau. We denote by $\{t\}$ the equivalence class of t where two Young tableaux of λ are equivalent if they have the same rows up to reordering. We call these equivalence classes the Young tabloids of λ. There is again an obvious group action of $\mathfrak{S}_n$ on the set of Young tabloids; we denote by M_λ the permutation $\mathbb{Q}$-representation of $\mathfrak{S}_n$ with basis the Young tabloids of λ.*

(iv) *Let t be a Young tableau. We define the group*

$$V_t := \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ fixes every column of } t\}$$

and

$$e_t := \sum_{\sigma \in V_t} \operatorname{sgn}(\sigma)\{\sigma t\} \in M_\lambda.$$

We define the Specht module S_λ to be the subrepresentation of M_λ given by the $\mathbb{Q}$-span of the e_t, where t runs through the Young tableaux of λ.

Example 4.1.5. *There are exactly 5 standard Young tableaux of $(3, 2)$, given by*

$$t_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\,, t_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\,, t_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\,, t_4 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}\,, t_5 = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}\,.$$

A non-trivial element of the Specht module $S_{(3,2)}$ is for instance

$$e_{t_4} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 & 5 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 1 & 5 & 4 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 2 & 5 & 4 \\ \hline 1 & 3 \\ \cline{1-2}\end{array}\,.$$

We are now ready to talk about the irreducible characters of the symmetric groups.

Theorem 4.1.6. *(cf. [JK81, Theorem 7.1.8, Theorem 7.2.7]) The Specht modules S_λ, as λ varies over the set $\mathscr{P}_n$, give a complete list of non-isomorphic absolutely irreducible representations of $\mathfrak{S}_n$. We denote $\chi_\lambda \in \operatorname{Irr}(\mathfrak{S}_n)$ to be the character of S_λ. The standard basis of S_λ is given by $\{e_t \mid t \in T_\lambda\}$.*

There is a convenient way to find the degree of an irreducible character of $\mathfrak{S}_n$.

Proposition 4.1.7. *(cf. [JK81, Theorem 2.3.21]) Let λ be a partition of n. Then*

$$f_\lambda := \chi_\lambda(1) = \frac{n!}{\prod_{c \in [\lambda]} h_\lambda(c)}.$$

It is convenient to also define $f_{(0)} := 1$.

Example 4.1.8. *As we have seen, there are 5 standard Young tableaux of the partition* $(3,2)$*, so* $f_{(3,2)} = 5$*. The hook diagram of* $(3,2)$ *has the form*

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}\,.$$

So with Proposition 4.1.7 we also get that

$$f_{(3,2)} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1^2} = 5.$$

4.1.2 James–Murphy Determinant Formula

In this subsection, we will discuss the formula found by James and Murphy in [JM79] for the orthogonal determinants of the symmetric groups. Their description relies on so called sequences of β-numbers; these are sequences of integers that are in close relation to partitions. We will give a quick introduction on their main properties as well as some explicit examples. While we use our own notation, none of the results in here are original. For more information, we refer the reader to [JK81, Sections 2.7, 7.3]. Let n be a positive integer.

Definition 4.1.9. *Let* λ *be a partition of* n*. Let* γ *be the symmetric bilinear form on the permutation representation* M_λ *where we take the Young tabloids of* λ *as an orthonormal basis. It is clear that* γ *is non-degenerate and* $\mathfrak{S}_n$*-invariant. We define* $\det(\lambda)$ *to be the determinant of the Gram matrix of* $\gamma_{|S_\lambda}$ *with respect to the standard basis of the Specht module* S_λ*.*

Example 4.1.10. *Consider the partition* $\lambda = (3,2)$ *of* 5*. We have determined the standard Young tableaux of* $(3,2)$ *in Example 4.1.5. An easy calculation leads us to*

$$\det(\lambda) = \det \begin{pmatrix} 4 & 2 & 1 & 1 & -1 \\ 2 & 4 & 2 & 2 & 1 \\ 1 & 2 & 4 & 1 & 2 \\ 1 & 2 & 1 & 4 & 2 \\ -1 & 1 & 2 & 2 & 4 \end{pmatrix} = 162 = 2 \cdot 3^4.$$

Remark 4.1.11. *Let* λ *be a partition such that the degree of its character* f_λ *is even. Clearly,* $(S_\lambda, \gamma_{|S_\lambda})$ *is an orthogonal* $\mathbb{Q}\mathfrak{S}_n$*-module affording the character* $\chi_\lambda \in \mathrm{Irr}^+(\mathfrak{S}_n)$*. So Theorem 2.3.7 gives us*

$$\det(\chi_\lambda) = \det(\lambda) \cdot (\mathbb{Q}^\times)^2.$$

James and Murphy found a more combinatorial way to calculate $\det(\lambda)$ involving so called β-numbers. We will give a quick introduction on their main properties.

Definition 4.1.12. *(i) A sequence of integers $\beta = (b_1, \dots, b_m)$ is called a sequence of β-numbers if $b_i \geq 0$ for all i and the entries are pairwise distinct. For each such sequence, there is a unique $\sigma \in \mathfrak{S}_m$ such that $\beta = (b_{\sigma(1)}, \dots, b_{\sigma(m)})$ is a decreasing sequence, i.e., $b_{\sigma(1)} > b_{\sigma(2)} > \cdots > b_{\sigma(m)}$. We set $\operatorname{sgn}(\beta) = \operatorname{sgn}(\sigma)$. We define the partition*

$$\operatorname{Part}(\beta) := (b_{\sigma(1)} - m + 1, b_{\sigma(2)} - m + 2, \dots, b_{\sigma(m)} - m + m)$$

and say that β is a sequence of β-numbers for $\operatorname{Part}(\beta)$. Further, we say that $\ell(\beta) := m$ is the length of β and define $|\beta| := |\operatorname{Part}(\beta)|$.

Let $\mathscr{B}$ be the set of all sequences of β-numbers. We define the subsets

$$\mathscr{B}_n := \{\beta \in \mathscr{B} \mid |\beta| = n\}$$

of the sequences of β-numbers of n.

(ii) Let $\lambda = (a_1, \dots, a_m)$ be a partition where we assume that $a_i > 0$ for all i. We set

$$\operatorname{Beta}(\lambda) := (h_\lambda((1,1)), h_\lambda((2,1)), \dots, h_\lambda((m,1))) \in \mathscr{B}_n$$

to be a sequence of β-numbers for λ. We define

$$\mathscr{B}(\lambda) := \{\beta \in \mathscr{B} \mid \operatorname{Part}(\beta) = \lambda\}.$$

Example 4.1.13. *Let $\lambda = (6, 4, 3)$ as in Example 4.1.3. The first column of its hook diagram consists of the sequence $(8, 5, 3)$, so $(8, 5, 3)$ is a sequence of β-numbers for λ. Further sequences of β-numbers for λ include $(9, 6, 4, 0)$, $(10, 7, 5, 1, 0)$ and $(11, 8, 6, 2, 1, 0)$.*

We are introducing some geometric notion on Young diagrams: Given some Young diagram, we can remove the rim of an arbitrary cell and and up with another Young diagram of a partition of a smaller integer.

Definition 4.1.14. *Let λ be a partition and let $c \in [\lambda]$ be any cell. We let λ_c be the partition such that*

$$[\lambda_c] = [\lambda] \backslash \operatorname{rim}_\lambda(c).$$

We will introduce some notation. Let $e_i := (0, 0, \dots, 0, 1, 0, \dots)$ be the sequence that has a 1 at position i and 0 everywhere else. If $\beta = (b_1, \dots, b_m) \in \mathbb{Z}^m$, $h \in \mathbb{Z}$ and $1 \leq i \leq m$, we define $\beta + he_i \in \mathbb{Z}^m$ to be the sequence that only differs from β in the i-th position, where that entry is $b_i + h$.

The following lemma now hints at the power of the sequences of β-numbers: They allow us to translate the geometric operations of Young diagrams, e.g., removing the rim of a cell, to arithmetic operations on sequences of β-numbers.

Lemma 4.1.15. *(cf. [JK81, Lemma 2.7.13]) Let λ be a partition of n.*

(i) For any cell $c = (i, j) \in [\lambda]$ it holds that

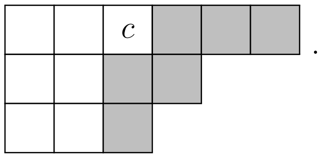
$$\operatorname{Beta}(\lambda) - h_\lambda(c)e_i \in \mathscr{B}(\lambda_c).$$

(ii) In the other direction, if there is a positive integer h and an index i such that

$$\beta := \mathrm{Beta}(\lambda) - he_i \in \mathscr{B},$$

then there is a cell $c = (i, j) \in [\lambda]$ with $h_\lambda(c) = h$ such that $\beta \in \mathscr{B}(\lambda_c)$.

Example 4.1.16. *We are again looking at the example of $\lambda = (6, 4, 3)$ as in Example 4.1.3. Let again $c = (1, 3) \in [\lambda]$; recall that $\mathrm{rim}_\lambda(c)$ has the following form:*



We see easily that if we remove the $\mathrm{rim}_\lambda(c)$ from $[\lambda]$, we end up with the partition $\lambda_c = (3, 2, 2)$ with the following hook diagram:

5	4	1
3	2	
2	1	

In the language of sequences of β-numbers, we have that $\mathrm{Beta}(\lambda) = (8, 5, 3)$. As c is in the first row and $h_\lambda(c) = 6$, by Lemma 4.1.15 we end up with

$$(8, 5, 3) - 6 \cdot e_1 = (2, 5, 3) \in \mathscr{B}((3, 2, 2)).$$

Removing the rim of a cell therefore only changes the sequence of β-numbers in one position, meaning that for a partition λ and a cell $c \in [\lambda]$, the two partitions λ and λ_c for $c \in [\lambda]$ are "close". This motivates the following definition.

Definition 4.1.17. *(i) Let $m > 0$ be an integer and let $\beta = (b_1, \dots, b_m), \beta' = (b'_1, \dots, b'_m) \in \mathbb{Z}^m$. Let $d_m : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}_{\geq 0}$ be defined by*

$$d_m(\beta, \beta') = \#\{i \mid 1 \leq i \leq m,\ b_i \neq b'_j \text{ for all } 1 \leq j \leq m\},$$

i.e., $d_m(\beta, \beta')$ is the number of mismatches of β and β'.

(ii) Let λ, μ be partitions. We choose the integer $m > 0$ large enough such that there are sequences of β-numbers $\beta_\lambda \in \mathscr{B}(\lambda)$, $\beta_\mu \in \mathscr{B}(\mu)$ such that $\ell(\beta_\lambda) = \ell(\beta_\mu) = m$. We define the distance $d : \mathscr{P} \times \mathscr{P} \to \mathbb{Z}_{>0}$ by

$$d(\lambda, \kappa) := d_m(\beta_\lambda, \beta_\kappa).$$

It is clear that the distance of two partitions is well-defined.

Example 4.1.18. *Let $\lambda = (2, 1^{(5)})$, $\mu = (3, 3, 1)$ be two partitions of 7. We choose $\beta_\lambda := \mathrm{Beta}(\lambda) = (7, 5, 4, 3, 2, 1) \in \mathscr{B}(\lambda)$. It is easy to see that $\beta_\mu = (8, 7, 4, 2, 1, 0) \in \mathscr{B}(\mu)$. Comparing the two sequences, we see that there are two mismatches, so $d(\lambda, \mu) = d_6(\beta_\lambda, \beta_\mu) = 2$.*

The following is an easy corollary of Lemma 4.1.15.

Corollary 4.1.19. *Let λ, μ be partitions.*

(i) $(\mathscr{P}, d)$ *is a metric space.*

(ii) If $|\lambda| > |\mu|$, *then* $d(\lambda, \mu) = 1$ *if and only if there is a cell* $c \in [\lambda]$ *such that* $\lambda_c = \mu$.

Thus if $\lambda \neq \mu$ are both partitions of n, the smallest distance between those can be 2. Then it is clear that either $\lambda \trianglelefteq \mu$ or $\mu \trianglelefteq \lambda$. The following gives a characterization of this situation occurring, which also involves some key notions of the James–Murphy determinant formula:

Lemma 4.1.20. *Let* λ, μ *be partitions of* n *such that* $\lambda \trianglelefteq \mu$. *The following are equivalent:*

(i) $d(\lambda, \mu) = 2$.

(ii) There are unique cells $c_1 = (k_1, j), c_2 = (k_2, j) \in [\lambda]$ *with* $k_1 < k_2$ *such that*

$$d(\lambda, \lambda_{c_i}) = 1 = d(\mu, \lambda_{c_i}) \text{ for } i = 1, 2.$$

We define

$$h^i_\lambda(\mu) := h_\lambda(c_i) \text{ for } i = 1, 2$$

and

$$h^{1,2}_\lambda(\mu) := (h^1_\lambda(\mu), h^2_\lambda(\mu)).$$

We set

$$\operatorname{sgn}_\lambda(\mu) := \operatorname{sgn}\left(\operatorname{Beta}(\lambda) + h_\lambda(c_2)e_{k_1} - h_\lambda(c_2)e_{k_2}\right).$$

We will write $\lambda \dot{\trianglelefteq} \mu$ *if any of these apply. Further, we define the set*

$$\mathscr{P}_\lambda := \{\kappa \in \mathscr{P}_n \mid \lambda \dot{\trianglelefteq} \kappa \}.$$

Remark 4.1.21. *Let* $\lambda \dot{\trianglelefteq} \mu$ *be partitions of* n. *There is another way to define* $h^{1,2}_\lambda(\mu)$ *in terms of sequences of* β*-numbers. Since* $d(\lambda, \mu) = 2$, *there are* $\beta_\lambda \in \mathscr{B}(\lambda)$ *(resp.* $\beta_\mu \in \mathscr{B}(\mu)$*) with*

$$\beta_\lambda = (b_1, b_2, b_3, \ldots, b_m),$$
$$\beta_\mu = (c_1, c_2, b_3, \ldots, b_m),$$

such that $b_1 > b_2$, $c_1 > c_2$. *Then*

$$h^1_\lambda(\mu) = b_1 - c_2 = c_1 - b_2,$$
$$h^2_\lambda(\mu) = b_2 - c_2 = c_1 - b_1.$$

Example 4.1.22. *We will revisit Example 4.1.18, so let again* $\lambda = (2, 1^{(5)})$, $\mu = (3, 3, 1)$ *be two partitions of* 7*. Since* $\lambda \trianglelefteq \mu$ *and* $d(\lambda, \mu) = 2$*, it holds that* $\lambda \dot{\trianglelefteq} \mu$*. We choose the sequences of* β*-numbers*

$$\beta_\lambda = (5, 3, 7, 4, 2, 1),$$
$$\beta_\mu = (8, 0, 7, 4, 2, 1).$$

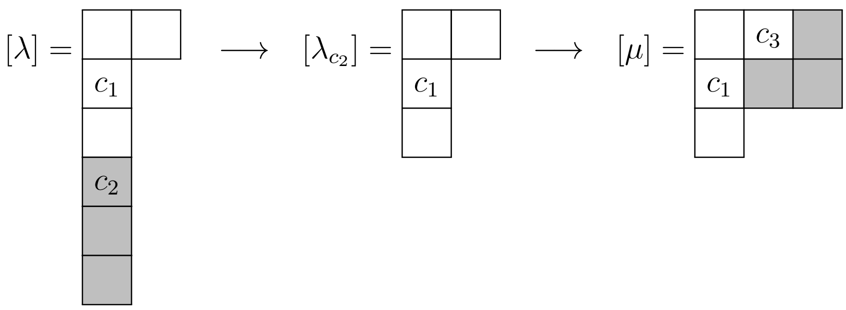
We now calculate

$$h_\lambda^1(\mu) := 5 - 0 = 8 - 3 = 5,$$
$$h_\lambda^2(\mu) := 3 - 0 = 8 - 5 = 3.$$

We know by 4.1.20(iii) that these correspond to the hook lengths of cells $c_1 = (k_1, j), c_2 = (k_2, j) \in [\lambda]$*. We will therefore investigate the hook diagram of* λ*:*

7	1
5	
4	
3	
2	
1	

So $c_1 = (2, 1)$, $c_2 = (4, 1)$*. These two cells now tell us what's "really going on", as these tell us how to go from* $[\lambda]$ *to* $[\mu]$*:*



$$(7, 5, 4, 3, 2, 1) \longrightarrow (7, 5, 4, 0, 2, 1) \longrightarrow (7, 8, 4, 0, 2, 1)$$

This gives us a more geometric way to think about $\lambda \dot{\trianglelefteq} \mu$*: It tells us to take a cell* $c_2 \in [\lambda]$*, to remove its rim, and then to wrap it around the row of the cell* c_1 *to finally end up with* $[\mu]$*. Further, there is a unique cell* $c_3 \in [\mu]$ *with* $h_\mu(c_3) = h_\lambda(c_2)$ *(in our case* $c_3 = (1, 2)$*) such that* $\mu_{c_3} = \lambda_{c_2}$*.*

For the sign it holds that $\mathrm{sgn}_\lambda(\mu) = \mathrm{sgn}((7, 8, 4, 0, 2, 1)) = -1$*.*

We can now formulate the James–Murphy determinant formula.

Theorem 4.1.23. *(cf. [JK81, Theorem 7.3.20]) Let λ be a partition. The following holds:*

$$\det(\lambda) = \prod_{\mu \in \mathscr{P}_\lambda} \left(\frac{h^1_\lambda(\mu)}{h^2_\lambda(\mu)} \right)^{\operatorname{sgn}_\lambda(\mu) f_\mu}.$$

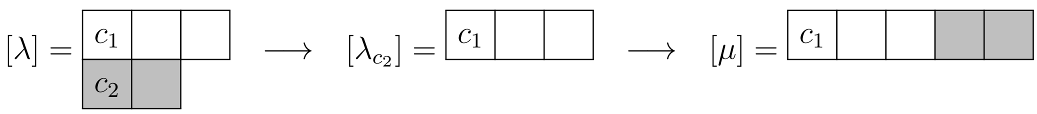
Example 4.1.24. *Let $\lambda = (3, 2)$. We have seen earlier in Example 4.1.10 that $\det(\lambda) = 2 \cdot 3^4$. Let us now use the James–Murphy determinant formula to reconfirm this:*

Recall that the hook diagram of λ has the following form:

$$\begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 & \\ \cline{1-2} \end{array}.$$

It is clear that there are two partitions lying above λ in the dominance order, (5) and $(4, 1)$. Also the distance of these partitions to λ is equal to 2, so $\mathscr{P}_\lambda = \{(5), (4, 1)\}$. There are therefore two factors we have to calculate:

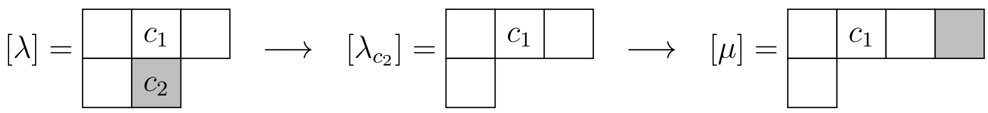
(i) Let $\mu = (5)$. The picture then looks like the following:



$$(4, 2) \longrightarrow (4, 0) \longrightarrow (6, 0)$$

So $h^{1,2}_\lambda(\mu) = (4, 2)$, $f_\mu = 1$ and $\operatorname{sgn}_\lambda(\mu) = 1$.

(ii) Let $\mu = (4, 1)$. The picture then looks like the following:



$$(4, 2) \longrightarrow (4, 1) \longrightarrow (5, 1)$$

So $h^{1,2}_\lambda(\mu) = (3, 1)$, $f_\mu = 4$ and $\operatorname{sgn}_\lambda(\mu) = 1$.

So all in all we end up with

$$\det(\lambda) = \prod_{\mu \in \mathscr{P}_\lambda} \left(\frac{h^1_\lambda(\mu)}{h^2_\lambda(\mu)} \right)^{\operatorname{sgn}_\lambda(\mu) f_\mu} = \left(\frac{4}{2}\right)^{+1} \cdot \left(\frac{3}{1}\right)^{+4} = 2 \cdot 3^4 = 162.$$

Remark 4.1.25. *Let λ be a partition such that f_λ is even. With Remark 4.1.11 and the James–Murphy determinant formula we can simplify*

$$\det(\chi_\lambda) = \prod_{\mu \in \mathscr{P}_\lambda} \left(\frac{h^1_\lambda(\mu)}{h^2_\lambda(\mu)} \right)^{\operatorname{sgn}_\lambda(\mu) f_\mu} \cdot (\mathbb{Q}^\times)^2 = \prod_{\substack{\mu \in \mathscr{P}_\lambda, \\ f_\mu \text{ odd}}} \frac{h^1_\lambda(\mu)}{h^2_\lambda(\mu)} \cdot (\mathbb{Q}^\times)^2.$$

4.1.3 Parker's Conjecture for the Symmetric Groups

We fix a positive integer n. Let r be the integer such that $2^r \leq n < 2^{r+1}$. In this subsection, we want to confirm Parker's conjecture for the symmetric group $\mathfrak{S}_n$. The key is of course the James–Murphy determinant formula in Theorem 4.1.23. The following definition will be useful:

Definition 4.1.26. *Let $M \subseteq \mathscr{P}_\lambda$ be any subset. We say that M is odd if the square class*

$$\prod_{\mu \in M} \frac{h^1_\lambda(\mu)}{h^2_\lambda(\mu)} \cdot (\mathbb{Q}^\times)^2$$

is odd.

Taking a closer look at the formula in Remark 4.1.25, our plan is simply put the following:

(i) Understand when f_λ is odd for a partition λ.

(ii) For any partition λ with f_λ even, find a suitable set partition

$$\{\mu \in \mathscr{P}_\lambda \mid f_\mu \text{ odd}\} = M_1 \cup M_2 \cup \cdots \cup M_k$$

such that each of the M_i is odd.

For the first point, there is a criterion of the parity of f_λ for a partition λ due to MacDonald:

Proposition 4.1.27. *(cf. [Mac71]) Let λ be a partition of n. The following hold:*

(i) If there is a $c \in [\lambda]$ with $h_\lambda(c) = 2^r$, it is unique.

(ii) f_λ is odd if and only if there is a cell $c \in [\lambda]$ with $h_\lambda(c) = 2^r$ and f_{λ_c} is odd.

Corollary 4.1.28. *Let λ be a partition of n and let $\beta \in \mathscr{B}(\lambda)$. There is at most one index i such that $\beta - 2^r e_i \in \mathscr{B}_{n-2^r}$.*

Let us regard some examples.

Example 4.1.29. *(i) Let $n = 7$, so $2^r = 4$. Let $\lambda = (5, 2)$ with the following hook diagram:*

6	5	3	2	1
2	1			

We see that λ is 4-core, i.e., there is no cell $c \in [\lambda]$ such that $h_\lambda(c) = 4$. So by Proposition 4.1.27, f_λ is even.

(ii) *Let* $n = 15$, *so* $2^r = 8$. *Let* $\lambda = (4^{(3)}, 1^{(3)})$ *with the following hook diagram:*

9	5	4	3
8	4	3	2
7	3	2	1
3			
2			
1			

So $c = (2,1) \in [\lambda]$ *is the unique cell such that* $h_\lambda(c) = 8$ *and we see that* $\lambda_c = (4,3)$. *Let us continue the algorithm, so let* $\mu = (4,3)$ *with the following hook diagram:*

5	4	3	1
3	2	1	

So we have a cell $c' = (1,2) \in [\mu]$ *such that* $h_\mu(c') = 4$ *and* $\mu_{c'} = (2,1)$. *We let now* $\kappa = (2,1)$ *with the following hook diagram:*

3	1
1	

So κ *is* 2*-core and thus* f_κ *is even. It follows now by Proposition 4.1.27 that then* f_μ *and* f_λ *are also even.*

(iii) *Let* $\lambda = (3)$ *with the following hook diagram:*

3	2	1

Let $c = (1,2) \in [\lambda]$ *be the cell such that* $h_\lambda(c) = 2$ *and* $\lambda_c = (1)$. *Let now* $\mu = (1)$ *with the following hook diagram:*

1

So there is a cell $c' = (1,1) \in [\mu]$ *such that* $h_\mu(c') = 1$. *Now,* $\mu_{c'} = (0)$. *By definition,* $f_{(0)} = 1$. *So by Proposition 4.1.27, also* f_μ *and* f_λ *are odd.*

This motivates the following definition:

Definition 4.1.30. *Let* λ *be a partition of* n. *We set* $D : \mathscr{P} \to \mathscr{P}$ *to be the following function:*

(i) We set $D((0)) := (0)$.

(ii) If λ *is* 2^r*-core, set* $D(\lambda) := \lambda$.

(iii) Assume λ *is not* 2^r*-core. Let* $c \in [\lambda]$ *be the unique cell such that* $h_\lambda(c) = 2^r$*. We set* $D(\lambda) := \lambda_c$.

Denote $D^0(\lambda) := \lambda$ *and* $D^m(\lambda) := D(D^{m-1}(\lambda))$ *for any integer* $m > 0$*. We define the oddness rank of the partition* λ *by*

$$\mathrm{OddRank}(\lambda) = \min\{m \in \mathbb{Z}_{\geq 0} \mid D^{m+1}(\lambda) = D^m(\lambda)\}.$$

Corollary 4.1.31. *Let* λ *be a partition of* n*. Let*

$$n = \sum_{i=0}^{r} a_i 2^i$$

for $a_i \in \{0,1\}$ *be the decomposition of* n *in binary. Then* f_λ *is odd if and only if*

$$\mathrm{OddRank}(\lambda) = \sum_{i=0}^{r} a_i.$$

Example 4.1.32. *Reconsider the partitions we saw in Example 4.1.29. It is easy to see that* $\mathrm{OddRank}((5,2)) = 0$*,* $\mathrm{OddRank}((4^{(3)}, 1^{(3)})) = 2$ *and* $\mathrm{OddRank}((3)) = 2$.

So we understand pretty well now when f_λ is odd for a partition λ of n. We are now searching for a suitable set partition of

$$\mathscr{P}_\lambda^{\mathrm{odd}} := \{\mu \in \mathscr{P}_\lambda \mid f_\mu \text{ odd}\} = \{\mu \in \mathscr{P}_\lambda \mid \mathrm{OddRank}(\mu) = \sum_{i=0}^{r} a_i\}.$$

As it turns out, this is not the best approach — there is a related set that is better behaved:

Definition 4.1.33. *Let* λ *be a partition of* n*. We define*

$$\mathscr{P}_\lambda^{>0} := \{\mu \in \mathscr{P}_\lambda \mid \mathrm{OddRank}(\mu) > 0\}.$$

We set an equivalence relation on $\mathscr{P}_\lambda^{>0}$ *and say that* $\mu_1 \sim \mu_2$ *if and only if* $D(\mu_1) = D(\mu_2)$ *for* $\mu_1, \mu_2 \in \mathscr{P}_\lambda^{>0}$*. For any* $\mu \in \mathscr{P}_\lambda^{>0}$ *we denote with* $(\mu)_\sim$ *the equivalence class of* μ*. We may also write* $D((\mu)_\sim) := D(\mu)$.

Example 4.1.34. *Let* $\lambda = (6,4,3)$ *be a partition of* 13 *with the hook diagram*

8	7	6	4	2	1
5	4	3	1		
3	2	1			

.

The set $\mathscr{P}_\lambda^{>0}$ *consists of* 8 *partitions that fall into the following* 5 *equivalence classes:*

(i) $M_1 = \{(11, 2), (9, 4)\}$ *with hook diagrams*

12	11	9	8	7	6	5	4	3	2	1
2	1									

,

10	9	8	7	5	4	3	2	1
4	3	2	1					

.

It holds that

$$D(M_1) = (3, 2)$$

and

$$h_\lambda^{1,2}((11, 2)) = (8, 5),\ \ h_\lambda^{1,2}((11, 2)) = (8, 3).$$

(ii) $M_2 = \{(8, 4, 1), (7, 4, 2)\}$ *with hook diagrams*

10	8	7	6	4	3	2	1
5	3	2	1				
1							

,

9	8	6	5	3	2	1
5	4	2	1			
2	1					

.

It holds that

$$D(M_2) = (3, 1, 1)$$

and

$$h_\lambda^{1,2}((8, 4, 1)) = (7, 2),\ \ h_\lambda^{1,2}((7, 4, 2)) = (6, 1).$$

(iii) $M_3 = \{(10, 2, 1), (7, 3, 3)\}$ *with hook diagrams*

12	10	8	7	6	5	4	3	2	1
3	1								
1									

,

9	8	7	4	3	2	1
4	3	2				
3	2	1				

.

It holds that

$$D(M_3) = (2, 2, 1)$$

and

$$h_\lambda^{1,2}((10, 2, 1)) = (7, 4),\ \ h_\lambda^{1,2}((7, 3, 3)) = (4, 1).$$

(iv) $M_4 = \{(6, 5, 2)\}$ *with hook diagram*

8	7	5	4	3	1
6	5	3	2	1	
2	1				

.

It holds that $D(M_4) = (4, 1)$ *and* $h_\lambda^{1,2}((6, 5, 2)) = (3, 1)$.

(v) $M_5 = \{(6,6,1)\}$ *with hook diagram*

8	6	5	4	3	2
7	5	4	3	2	1
1					

It holds that $D(M_5) = (5)$ *and* $h_\lambda^{1,2}((6,6,1)) = (4,2)$.

Remark 4.1.35. *Let* λ *be a partition of* n *and let* $\mu \in \mathscr{P}_\lambda^{>0}$. *Then* $d(\lambda,\mu) = 2$. *Since* d *is a distance function, it follows that* $d(\lambda, D(\mu)) \in \{1,2,3\}$).

The next few lemmas will be quite technical and go over the three possible cases of $d(\lambda, D(\mu))$ in the above remark.

Lemma 4.1.36. *Let* λ *be a partition of* n *with* $\operatorname{OddRank}(\lambda) > 0$. *Let* $\mu \in \mathscr{P}_\lambda^{>0}$. *Then* $d(\lambda, D(\mu)) = 1$, *i.e.,* $D(\lambda) = D(\mu)$, *if and only if* $h_\lambda^i(\mu) = 2^r$ *for either* $i = 1$ *or* $i = 2$.

Proof. Let $\beta_\lambda := (b_1, b_2, \ldots, b_m) \in \mathscr{B}(\lambda)$. Since $\operatorname{OddRank}(\lambda) > 0$, we can after some reordering assume that $(b_1 - 2^r, b_2, \ldots, b_m) \in \mathscr{B}(D(\lambda))$.

Let us now assume that $D(\lambda) = D(\mu)$. So after another reordering

$$(b_1 - 2^r, b_2 + 2^r, \ldots, b_m) \in \mathscr{B}(\mu).$$

The statement follows now by the definition of the $h_\lambda^i(\mu)$. □

Example 4.1.37. *In the situation of Example 4.1.34, the equivalence class* M_1 *has the property described in the above lemma. Indeed,* $D(\lambda) = (3,2)$ *and we see that* $h_\lambda^1(\mu) = 8$ *for both* $\mu \in M_1$.

Lemma 4.1.38. *Let* λ *be a partition of* n. *Let* $\mu \in \mathscr{P}_\lambda^{>0}$. *Assume that* $d(\lambda, D(\mu)) = 3$. *Then* $|(\mu)_\sim| = 1$, $\operatorname{OddRank}(\lambda) > 0$ *and* $D(\mu) \in \mathscr{P}_{D(\lambda)}$. *Further,*

$$h_\lambda^{1,2}(\mu) = h_{D(\lambda)}^{1,2}(D(\mu)).$$

Proof. Let $\beta_\lambda := (b_1, b_2, b_3, \ldots, b_m) \in \mathscr{B}(\lambda)$. After reordering we can assume that there are $h_1, h_2, h_3 \in \mathbb{Z}\backslash\{0\}$ such that

$$\beta_{D(\mu)} := (b_1 - h_1, b_2 - h_2, b_3 - h_3, \ldots, b_m) \in \mathscr{B}(D(\mu)).$$

We know that there is an index i such that $\beta_{D(\mu)} + 2^r e_i \in \mathscr{B}(\mu)$. Since $d(\lambda,\mu) = 2$, it follows that $i \in \{1,2,3\}$, so assume without loss of generality that $i = 3$. It follows that $h_3 = 2^r$. Further, since $d(\lambda, D(\mu)) = 3$, the entry $b_3 - h_3 = b_3 - 2^r$ cannot appear in β_λ. So

$$\beta_\lambda - 2^r e_3 = (b_1, b_2, b_3 - 2^r, \ldots, b_m) \in \mathscr{B}_{n-2^r}.$$

By Corollary 4.1.28 it follows that $\beta_\lambda - 2^r e_3 \in \mathscr{B}(D(\lambda))$. The corollary also tells us that $h_1, h_2 \neq 2^r$ and thus $|(\mu)_\sim| = 1$.

The final statement is easy to see: For the calculation of $h_\lambda^{1,2}(\mu)$, we compare the sequences

$$(b_1, b_2, b_3, \dots, b_m),$$
$$(b_1 - h_1, b_2 - h_2, b_3, \dots, b_m).$$

For the calculation of $h_{D(\lambda)}^{1,2}(D(\mu))$, we compare the sequences

$$(b_1, b_2, b_3 - 2^r, \dots, b_m),$$
$$(b_1 - h_1, b_2 - h_2, b_3 - 2^r, \dots, b_m).$$

So they clearly coincide. This concludes the proof. □

Example 4.1.39. *In Example 4.1.34, the above situation occurs precisely with the equivalence classes M_4 and M_5.*

Let's investigate this a bit more thoroughly: We again set $\lambda = (6, 4, 3)$. Let $\mu_1 := (6, 5, 2)$, $\mu_2 = (6, 6, 1)$. The situation becomes clear by regarding the hook diagrams and by coloring the hook of the cell $(1, 1)$:

$[\lambda] =$

8	7	6	4	2	1
5	4	3	1		
3	2	1			

,

$[\mu_1] =$

8	7	5	4	3	1
6	5	3	2	1	
2	1				

,

$[\mu_2] =$

8	6	5	4	3	2
7	5	4	3	2	1
1					

.

We see that the white cells exactly depict the behavior of the partition $D(\lambda) = (3, 2)$, compare with Example 4.1.24.

Lemma 4.1.40. *Let λ be a partition of n. Let $\mu \in \mathscr{P}_\lambda^{>0}$. Assume that $d(\lambda, D(\mu)) = 2$. The following hold:*

(i) $|(\mu)_\sim| \leq 4$.

(ii) There is a partition $\kappa \in (\mu)_\sim$, which we call a distinguished representative of $(\mu)_\sim$, such that the following holds: Let

$$\beta_\lambda := (b_1, b_2, b_3, \dots, b_m) \in \mathscr{B}(\lambda),$$
$$\beta_\kappa := (b'_1, b'_2, b_3, \dots, b_m) \in \mathscr{B}(\kappa)$$

such that $b_1 > b_2$ and $b'_1 > b'_2$.

Then

$$\beta_\kappa - 2^r e_1 = (b'_1 - 2^r, b'_2, b_3, \dots, b_m) \in \mathscr{B}(D(\mu)).$$

Proof. Let $\beta_\lambda := (b_1, b_2, \ldots, b_m) \in \mathscr{B}(\lambda)$. We will assume that $b_i > 0$ for all i. After reordering we can assume that there are $h_1, h_2 \in \mathbb{Z}\backslash\{0\}$ such that

$$\beta_{D(\mu)} := (b_1 - h_1, b_2 - h_2, \ldots, b_m) \in \mathscr{B}(D(\mu)).$$

Assume without loss of generality that $b_1 > b_2$ and $b_1 - h_1 > b_2 - h_2$.

An upper bound for $|(\mu)_\sim|$ is the number of indices i such that

$$d_m(\beta_\lambda, \beta_{D(\mu)} + 2^r e_i) = 2.$$

There are at most four possibilities for i: Either $i = 1$, or $i = 2$, or if there is an index j (resp. k) such that $b_j = b_1 - 2^r$ (resp. $b_k = b_2 - 2^r$), then i could also be equal to j or k. Thus, $|(\mu)_\sim|$ is at most 4.

For the next claim, first observe that $h_2 > 0$. Indeed, $h_1 + h_2 = 2^r$, so $h_2 < 0$ implies that $h_1 > 2^r$. Then

$$(b_1 - |h_2|, b_2 + |h_2|, \ldots, b_m) \in \mathscr{B}(\mu).$$

But since $b_1 > b_2$, it would follow that $\mu \trianglelefteq \lambda$ which is absurd.

Next, we regard the sequence

$$\alpha := \beta_{D(\mu)} + 2^r e_1 = (b_1 - h_1 + 2^r, b_2 - h_2, \ldots, b_m).$$

If $\alpha \in \mathscr{B}_n$, then by setting $\kappa := \mathrm{Part}(\alpha)$ we have found a distinguished representative $\kappa \in (\mu)_\sim$.

So assume that $\alpha \notin \mathscr{B}_n$. This means that the entry $b_1 - h_1 + 2^r$ appears twice in α, in position 1 and in another position i. Since $b_1 - h_1 > b_2 - h_2$, we know that $i \neq 2$. After a reordering we can assume that $i = 3$, i.e.,

$$\beta_\lambda = (b_1, b_2, b_1 - h_1 + 2^r, \ldots).$$

Consider the sequence

$$\gamma := (b_2 - h_2 + 2^r, b_1 - h_1, b_1 - h_1 + 2^r, \ldots).$$

First, since $b_3 = b_1 - h_1 + 2^r < 2^{r+1}$, it holds that $b_1 - h_1 < 2^r$. Clearly $b_2 - h_2 \geq 0$, it follows that

$$b_2 - h_2 + 2^r > b_1 - h_1.$$

Assume now there is an index $j \geq 4$ such that b_j appears in the first or second entry of γ. Since $\beta_{D(\mu)}$ is a sequence of β-numbers, it follows that $b_j \neq b_1 - h_1$. So then $b_j = b_2 - h_2 + 2^r$. But then both $\beta_\lambda - 2^r e_3$ and $\beta_\lambda - 2^r e_j$ are sequences of β-numbers which contradicts Corollary 4.1.28.

We conclude that $\gamma \in \mathscr{B}_n$. Finally, we can now set $\kappa := \mathrm{Part}(\gamma)$ to be a distinguished representative of $(\mu)_\sim$. □

Example 4.1.41. *We go yet again back to Example 4.1.34. The two equivalence classes M_1 and M_2 fulfill the requirements of the above lemma. In these two classes, all elements $\mu \in M_i$ for $i = 1, 2$ are distinguished representatives. In particular, a distinguished representative is in general not unique.*

The following definition gives us a setup to handle the classification of the distance 2 cases.

Definition 4.1.42. *Let λ be a partition of n. Let $\mu \in \mathscr{P}_\lambda^{>0}$. Assume that $d(\lambda, D(\mu)) = 2$. We further assume that μ is a distinguished representative in its equivalence class. Let $h^{(1)}, h^{(2)}$ be the integers such that $h_\lambda^{1,2}(\mu) = (h^{(1)}, h^{(2)})$. We choose the sequences of β-numbers*

$$\begin{aligned}\beta_\lambda :=& (b_1, b_2, b_3, \ldots, b_m) \in \mathscr{B}(\lambda),\\ \beta_\mu :=& (b_1 + h^{(2)}, b_2 - h^{(2)}, b_3, \ldots, b_m) \in \mathscr{B}(\mu)\end{aligned}$$

with $b_i > 0$ for all i, $b_1 > b_2$ and

$$\beta_\mu - 2^r e_1 = (b_1 + h^{(2)} - 2^r, b_2 - h^{(2)}, b_3, \ldots, b_m) \in \mathscr{B}(D(\mu)).$$

Define the sequence

$$\alpha_\mu := \beta_\mu - 2^r e_1 + 2^r e_2 = (b_1 + h^{(2)} - 2^r, b_2 - h^{(2)} + 2^r, b_3, \ldots, b_m).$$

Lemma 4.1.43. *Assume we are in the situation of Definition 4.1.42. Further assume that $h^{(2)} > 2^r$.*

(i) Assume that $\alpha_\mu \in \mathscr{B}_n$. Then $|(\mu)_\sim| = 4$. Further, $(\mu)_\sim$ is odd.

(ii) Assume that $\alpha_\mu \notin \mathscr{B}_n$. Then $|(\mu)_\sim| = 3$. Further, $\mathrm{OddRank}(\lambda) > 0$, $D(\mu) \in \mathscr{P}_{D(\lambda)}$ and $(\mu)_\sim$ is odd if and only if $\{D(\mu)\} \subseteq \mathscr{P}_{D(\lambda)}$ is odd.

Proof. First note that both $b_1, b_2 > 2^r$. It follows that neither

$$\beta_\lambda - 2^r e_1 = (b_1 - 2^r, b_2, b_3, \ldots, b_m)$$

nor

$$\beta_\lambda - 2^r e_2 = (b_1, b_2 - 2^r, b_3, \ldots, b_m)$$

are sequences of β-numbers. Indeed, if for instance $\beta_\lambda - 2^r e_1$ were a sequence of β-numbers, then there would be a cell $c \in [D(\lambda)]$ with $h_{D(\lambda)}(c) = b_2 > 2^r$ which contradicts the assumption on n.

So there are indices $i, j > 2$ with $b_i = b_1 - 2^r, b_j = b_2 - 2^r$. In other words, after a reordering, we have that

$$\beta_\lambda = (b_1, b_2, b_1 - 2^r, b_2 - 2^r, b_5, \ldots, b_m).$$

We define

$$\gamma_1 := (b_1, b_2 + (h^{(1)} - 2^r), b_1 - 2^r - (h^{(1)} - 2^r), b_2 - 2^r, b_5, \ldots, b_m) \in \mathscr{B}_n$$

and

$$\gamma_2 := (b_1 + (h^{(2)} - 2^r), b_2, b_1 - 2^r, b_2 - 2^r - (h^{(2)} - 2^r), b_5, \ldots, b_m) \in \mathscr{B}_n.$$

Let $\kappa_1 := \text{Part}(\gamma_1)$, $\kappa_2 := \text{Part}(\gamma_1)$. It is clear that both $\gamma_1 - 2^r e_1$ and $\gamma_2 - 2^r e_2$ are sequences of β-numbers for $D(\mu)$. Thus $\kappa_1, \kappa_2 \in (\mu)_\sim$.

If α_μ is a sequence of β-numbers, then also $\alpha_\mu - 2^r e_2$ is a sequence of β-numbers for $D(\mu)$ and thus $\kappa_3 := \text{Part}(\alpha_\mu) \sim \mu$. So

$$(\mu)_\sim = \{\mu, \kappa_1, \kappa_2, \kappa_3\}$$

with

$$\prod_{\nu \in (\mu)_\sim} \frac{h^1_\lambda(\nu)}{h^2_\lambda(\nu)} = \frac{h^{(1)}}{h^{(2)}} \cdot \frac{h^{(2)}}{h^{(1)} - 2^r} \cdot \frac{h^{(1)}}{h^{(2)} - 2^r} \cdot \frac{h^{(1)} - 2^r}{h^{(2)} - 2^r} = \left(\frac{h^{(1)}}{h^{(2)} - 2^r}\right)^2$$

and so $(\mu)_\sim$ is odd.

If on the other hand α_μ is not a sequence of β-numbers, then after reordering, β_λ is of the form

$$(b_1, b_2, b_1 - 2^r, b_2 - 2^r, b_2 - h^{(2)} + 2^r, \dots, b_m).$$

Thus $\text{OddRank}(\lambda) > 0$ and

$$\beta_\lambda - 2^r e_5 = (b_1, b_2, b_1 - 2^r, b_2 - 2^r, b_2 - h^{(2)}, \dots, b_m) \in \mathscr{B}(D(\lambda)).$$

Further,

$$(\mu)_\sim = \{\mu, \kappa_1, \kappa_2\}$$

and it is clear that $D(\mu) \in \mathscr{P}_\lambda$ with

$$h^{1,2}_{D(\lambda)}(D(\mu)) = (h^{(1)} - 2^r, h^{(2)} - 2^r)$$

from which the statement follows. □

Example 4.1.44. *(i) Let $\lambda = (4, 4, 2, 2, 1, 1, 1)$ and let $\mu = (13, 1, 1)$. Then $\mu \in \mathscr{P}^{>0}_\lambda$ with $d(\lambda, D(\mu)) = 2$ and μ is a distinguished representative. Further,*

$$(\mu)_\sim = \{\mu, (5, 5, 2, 2, 1), (5, 4, 2, 2, 1, 1), (5, 3, 2, 2, 1, 1, 1)\}$$

and we are in the situation of Lemma 4.1.43(i).

(ii) Let $\lambda = (2, 1^{(5)})$ and let $\mu = (7)$. Then $\mu \in \mathscr{P}^{>0}_\lambda$ with $d(\lambda, D(\mu)) = 2$ and μ is a distinguished representative. Further,

$$(\mu)_\sim = \{\mu, (3, 3, 1), (3, 1^{(4)})\}$$

and we are in the situation of Lemma 4.1.43(ii).

The proofs of the following two lemmas go similarly to the proof of Lemma 4.1.43 and just involve going through all the cases.

Lemma 4.1.45. *Assume we are in the situation of Definition 4.1.42. Further assume that $h^1_\lambda(\mu) < 2^r$.*

(i) *Assume that* $\alpha_\mu \in \mathscr{B}_n$. *Then* $|(\mu)_\sim| = 2$. *Further,* $(\mu)_\sim$ *is odd.*

(ii) *Assume that* $\alpha_\mu \notin \mathscr{B}_n$. *Then* $|(\mu)_\sim| = 1$. *Further,* $\mathrm{OddRank}(\lambda) > 0$, $D(\mu) \in \mathscr{P}_{D(\lambda)}$ *and* $(\mu)_\sim$ *is odd if and only if* $\{D(\mu)\} \subseteq \mathscr{P}_{D(\lambda)}$ *is odd.*

Proof. One sees easily that the assumption implies that $|(\mu)_\sim| \leq 2$. Thus, if $\alpha_\mu \in \mathscr{B}_n$ then

$$(\mu)_\sim = \{\mu, \mathrm{Part}(\alpha_\mu)\}$$

with

$$\prod_{\nu \in (\mu)_\sim} \frac{h^1_\lambda(\nu)}{h^2_\lambda(\nu)} = \frac{h^{(1)}}{h^{(2)}} \cdot \frac{2^r - h^{(2)}}{2^r - h^{(1)}}$$

and thus $(\mu)_\sim$ is odd.

So assume now that $\alpha_\mu \notin \mathscr{B}_n$. We can now argue as in Lemma 4.1.43 and see that $D(\lambda) > 0$ and $D(\mu) \in \mathscr{P}_{D(\lambda)}$. Further, $(\mu)_\sim = \{\mu\}$ and

$$h^{1,2}_{D(\lambda)}(D(\mu)) = (2^r - h^{(2)}, 2^r - h^{(1)}),$$

which shows the claim. □

Example 4.1.46. (i) *The equivalence classes* M_2 *and* M_3 *from Example 4.1.34 are examples of the situation of Lemma 4.1.45(i).*

(ii) *Let* $\lambda = (10, 8, 6, 3, 2, 1, 1)$ *and let* $\mu = (17, 11, 2, 1)$ *be partitions of* 31. *Then* $\mu \in \mathscr{P}^{>0}_\lambda$ *with* $d(\lambda, D(\mu)) = 2$ *and* μ *is a distinguished representative. Further,*

$$(\mu)_\sim = \{\mu\}$$

and we are in the situation of Lemma 4.1.45(ii).

Lemma 4.1.47. *Assume we are in the situation of Definition 4.1.42. Further assume that* $h^{(1)} > 2^r$, $h^{(2)} < 2^r$.

(i) *If there is an index* $i \geq 3$ *with* $b_1 - b_i = 2^r$, *then* $|(\mu)_\sim| = 2$. *Further,* $(\mu)_\sim$ *is odd.*

(ii) *If there is no index* i *with* $b_1 - b_i = 2^r$, *then* $|(\mu)_\sim| = 1$. *Further,* $\mathrm{OddRank}(\lambda) > 0$, $D(\mu) \in \mathscr{P}_{D(\lambda)}$ *and* $(\mu)_\sim$ *is odd if and only if* $\{D(\mu)\} \subseteq \mathscr{P}_{D(\lambda)}$ *is odd.*

(iii) *If* $b_1 - b_2 = 2^r$, *then* $|(\mu)_\sim| = 1$. *Further,* $(\mu)_\sim$ *is odd.*

Proof. First we argue that $|(\mu)_\sim| \leq 2$. Indeed, unlike in Lemma 4.1.43 and 4.1.45, there is no partition in $(\mu)_\sim$ that can come from α_μ. Assume for a moment that $\alpha_\mu \in \mathscr{B}_n$. Then $b_1 > b_1 + h^{(2)} - 2^r$ since $h^{(1)} > 2^r$ and $b_1 > b_2 - h^{(2)} + 2^r$ since $h^{(2)} < 2^r$. It thus follows that $\mathrm{Part}(\alpha_\mu) \trianglelefteq \lambda$, so indeed, $\mathrm{Part}(\alpha_\mu) \notin \mathscr{P}^{>0}_\lambda$. Next, it is easy to see that $b_2 < 2^r$ or else $b_1 + h^{(2)} > 2^{r+1}$. So $|(\mu)_\sim| \leq 2$ now follows by the discussion in the proof of Lemma 4.1.40(i).

Let us now discuss the case of $|(\mu)_\sim| = 2$. For this to occur, there has to be an index $i \geq 3$ such that $b_1 - b_i = 2^r$. Assume this is the case; after some reordering we can assume that $i = 3$. So

$$\beta_\lambda = (b_1, b_2, b_1 - 2^r, b_4, \ldots, b_m).$$

Let

$$\gamma := (b_1, b_2 - h^{(2)}, b_1 + h^{(2)} - 2^r, b_4, \ldots, b_m) \in \mathscr{B}_n.$$

Let $\kappa := \mathrm{Part}(\gamma)$. It is easy to see that $\kappa \in \mathscr{P}_\lambda^{>0}$ and that $(\mu)_\sim = \{\mu, \kappa\}$. If $b_2 > b_1 - 2^r$, then

$$h_\lambda^{1,2}(\kappa) = (h^{(2)}, h^{(1)} - 2^r).$$

If $b_2 < b_1 - 2^r$, then

$$h_\lambda^{1,2}(\kappa) = (h^{(1)} - 2^r, h^{(2)}).$$

In both cases, $(\mu)_\sim$ is odd.

So in all other cases, $|(\mu)_\sim| = 1$. Assume now that $b_1 - b_2 = 2^r$. Then

$$h_\lambda^{1,2}(\mu) = (h^{(1)}, h^{(2)}) = (2^r + h^{(2)}, h^{(2)})$$

and so $(\mu)_\sim$ is odd.

Finally, assume that there is no index i such that $b_1 - b_i = 2^r$. Then $\mathrm{OddRank}(\lambda) > 0$ and

$$(b_1 - 2^r, b_2, \ldots, b_m) \in \mathscr{B}(D(\lambda)).$$

Further, we see that $D(\mu) \in \mathscr{P}_{D(\lambda)}$. We have the same case distinction as before: If $b_2 > b_1 - 2^r$, then

$$h_{D(\lambda)}^{1,2}(D(\mu)) = (h^{(2)}, h^{(1)} - 2^r).$$

If $b_2 < b_1 - 2^r$, then

$$h_{D(\lambda)}^{1,2}(D(\mu)) = (h^{(1)} - 2^r, h^{(2)}).$$

The statement now follows. □

Example 4.1.48. *(i) Let $\lambda = (4, 2^{(4)}, 1^{(3)})$ and let $\mu = (5, 2^{(4)}, 1, 1)$. Then $\mu \in \mathscr{P}_\lambda^{>0}$ with $d(\lambda, D(\mu)) = 2$ and μ is a distinguished representative. Further,*

$$(\mu)_\sim = \{\mu, (4, 2^{(5)}, 1)\}$$

and we are in the situation of Lemma 4.1.47(i).

(ii) Let $\lambda = (12, 2)$ and let $\mu = (13, 1)$. Then $\mu \in \mathscr{P}_\lambda^{>0}$ with $d(\lambda, D(\mu)) = 2$ and μ is a distinguished representative. Further,

$$(\mu)_\sim = \{\mu\}$$

and we are in the situation of Lemma 4.1.47(ii).

(iii) Let $\lambda = (2, 1^{(5)})$ *and let* $\mu = (5, 1, 1)$*. Then* $\mu \in \mathscr{P}_\lambda^{>0}$ *with* $d(\lambda, D(\mu)) = 2$ *and* μ *is a distinguished representative. Further,*

$$(\mu)_\sim = \{\mu\}$$

and we are in the situation of Lemma 4.1.47(iii).

To summarize the previous lemmas: We have gone through all possible cases and understood pretty well what the parities of the equivalence classes of $\mathscr{P}_\lambda^{>0}$ for a partition λ look like. There have been some cases where we reduced the parities to certain one-element subsets of $\mathscr{P}_{D(\lambda)}$. To finally prove Parker's conjecture for the symmetric groups, we will need to show that the map D allows us to take suitable preimages of $\mathscr{P}_{D(\lambda)}$.

Lemma 4.1.49. *Assume* $\mathrm{OddRank}(\lambda) > 0$ *and let* $\kappa \in \mathscr{P}_{D(\lambda)}$*. Then there is a partition* $\mu \in \mathscr{P}_\lambda^{>0}$ *such that* $D(\mu) = \kappa$ *and* $(\mu)_\sim$ *is odd if and only if* $\{\kappa\} \subseteq \mathscr{P}_{D(\lambda)}$ *is odd.*

Proof. Let

$$\beta_\lambda := (b_1, b_2, b_3, \ldots, b_m) \in \mathscr{B}(\lambda)$$

and assume that, after a suitable reordering,

$$(b_1 - 2^r, b_2, b_3, \ldots, b_m) \in \mathscr{B}(D(\lambda)).$$

There are two indices i, j and a positive integer h such that

$$\beta_\lambda - 2^r e_1 + he_i - he_j \in \mathscr{B}(\kappa)$$

and $b_i > b_j$. There are three possibilities: Either $1 \notin \{i, j\}$, or $i = 1$, or $j = 1$. We will regard

$$\alpha := \beta_\lambda + he_i - he_j.$$

Assume first that $1 \notin \{i, j\}$. After a reordering we can assume that $i = 2, j = 3$. There are three further possibilities: Either $\alpha \in \mathscr{B}_n$, or $b_1 = b_3 - h$, or $b_1 = b_2 + h$.

If $\alpha \in \mathscr{B}_n$, then it is clear that we can set $\mu := \mathrm{Part}(\alpha)$ with $\mu \in \mathscr{P}_\lambda^{>0}$ and that $d(\lambda, \kappa) = 3$. We are thus in the situation of Lemma 4.1.38 and we are done.

Assume now that $b_1 = b_3 - h$. Then

$$\gamma := (b_1 - 2^r, b_2 + h + 2^r, b_3 - h, b_4, \ldots, b_m) \in \mathscr{B}_n.$$

If we now set $\mu := \mathrm{Part}(\gamma)$, then clearly $\mu \in \mathscr{P}_\lambda^{>0}$ and $d(\lambda, \kappa) = 2$. Further, μ is a distinguished representative of its equivalence class and

$$h_\lambda^{1,2}(\mu) = (b_2 - b_3 + h + 2^r, h + 2^r)$$

so we are in the case of Lemma 4.1.43(ii).

Assume now that $b_1 = b_2 + h$. Here, we can set

$$\gamma := (b_1 - 2^r, b_2 + h, b_3 - h + 2^r, b_4, \ldots, b_m) \in \mathscr{B}_n.$$

We again set $\mu := \mathrm{Part}(\gamma)$ and again, $d(\lambda, \kappa) = 2$ and μ is a distinguished representative of its equivalence class. We calculate that

$$h_\lambda^{1,2}(\kappa) = (2^r - h, 2^r - (b_2 - b_3 + h))$$

so we are in the case of Lemma 4.1.45(ii).

Being done with that case, we can now assume that either $i = 1$ or $j = 1$. After reordering, we have

$$(b_1 - 2^r + \delta h, b_2 - \delta h, b_3, \dots, b_m) \in \mathscr{B}(\kappa)$$

for

$$\delta = \begin{cases} 1, & \text{if } i = 1, \\ -1, & \text{if } j = 1. \end{cases}$$

We set

$$\gamma := (b_1 - 2^r + \delta h, b_2 - \delta h + 2^r, b_3, \dots, b_m) \in \mathscr{B}_n$$

and let $\mu := \mathrm{Part}(\gamma)$ be the corresponding partition. It is easy to see that $d(\lambda, \kappa) = 2$ and μ is a distinguished representative of its equivalence class. We quickly confirm that we are in the case of Lemma 4.1.47(iii). This concludes the proof. □

We are now finally able to harvest the fruits of our work:

Theorem 4.1.50. *Parker's conjecture holds for the symmetric groups.*

Proof. Let λ be a partition of n such that f_λ is even. Recall by Remark 4.1.25 that

$$\det(\chi_\lambda) = \prod_{\mu \in \mathscr{P}_\lambda^{\mathrm{odd}}} \frac{h_\lambda^1(\mu)}{h_\lambda^2(\mu)} \cdot (\mathbb{Q}^\times)^2.$$

We will show that $\det(\chi_\lambda)$ is odd by induction on the oddness rank of λ.

So assume first that $\mathrm{OddRank}(\lambda) = 0$. By the lemmas 4.1.36 and 4.1.38 we know that for all $\mu \in \mathscr{P}_\lambda^{\mathrm{odd}}$ it holds that $d(\lambda, D(\mu)) = 2$. So the statement now follows by the lemmas 4.1.43, 4.1.45 and 4.1.47.

Assume now $\mathrm{OddRank}(\lambda) > 0$ and assume that we know the result to hold for $D(\lambda)$. We define the set

$$M = D^{-1}(\mathscr{P}_{D(\lambda)}^{\mathrm{odd}}) \subseteq \mathscr{P}_\lambda^{\mathrm{odd}},$$

where the subset relation holds by Proposition 4.1.27.

We decompose

$$\mathscr{P}_\lambda^{\mathrm{odd}} = M \cup M'$$

for some other set $M' \subseteq \mathscr{P}_\lambda^{\mathrm{odd}}$ into a disjoint union. Lemma 4.1.49 tells us that M is odd if and only if $\mathscr{P}_{D(\lambda)}^{\mathrm{odd}}$ is odd, which we know by the induction hypothesis. Further, the oddness of the set M' follows the same as in the oddness rank 0 case. So the theorem follows. □

4.1.4 Orthogonal Determinants of Alternating Groups

In this subsection, we will describe how to calculate the orthogonal determinants of the alternating groups. Most characters of the alternating groups arise by the restriction of the characters of the symmetric groups; the corresponding orthogonal determinants are therefore known by the James–Murphy determinant formula 4.1.23 and Lemma 2.3.9. For those characters that split, we will give methods to calculate the orthogonal determinants in all cases. This allows us to confirm Parker's conjecture also for the alternating groups.

We recall the basics of the representation theory of the alternating groups. For more information, regard for instance [FH91, Section 5] and [JK81, Section 2.5].

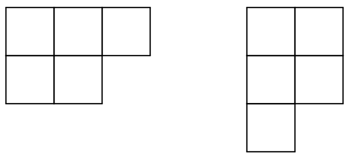
We fix a positive integer $n \geq 2$. Let r be the integer such that $2^r \leq n < 2^{r+1}$.

Definition 4.1.51. *Let $\lambda = (a_1, a_2, \ldots, a_m)$ be a partition of n. We define the conjugate partition λ' by*

$$\lambda' := \mathrm{Part}\left((h_\lambda((1,1)), h_\lambda((1,2)), \ldots, h_\lambda((1,a_1))\right).$$

Equivalently, the Young diagram $[\lambda']$ is obtained by the Young diagram of $[\lambda]$ by flipping along the main diagonal.

For instance, the conjugate partition of $(3, 2)$ is $(2, 2, 1)$:



Theorem 4.1.52. *Let λ be a partition of n.*

(i) If λ is not equal to its conjugate partition λ', then

$$\phi_\lambda := \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(\chi_\lambda) = \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(\chi_{\lambda'})$$

is an irreducible character of $\mathfrak{A}_n$.

(ii) If $\lambda = \lambda'$, then there are two irreducible characters $\phi_\lambda^{(1)}$ and $\phi_\lambda^{(2)}$ of $\mathfrak{A}_n$, each of degree $f_\lambda/2$, such that

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(\chi_\lambda) = \phi_\lambda^{(1)} + \phi_\lambda^{(2)}.$$

Let m be the biggest integer such that $(m, m) \in [\lambda]$. Then

$$\mathbb{Q}(\phi_\lambda^{(1)}) = \mathbb{Q}(\phi_\lambda^{(2)}) = \mathbb{Q}\left(\sqrt{(-1)^{1/2(n-m)} h_\lambda((1,1)) \cdots h_\lambda((m,m))}\right).$$

(iii) The abovementioned characters give a full list of all irreducible characters of $\mathfrak{A}_n$.

The theorem also implies that if μ is any partition with $\mu = \mu'$, then f_μ is odd if and only if $\mu = (1)$.

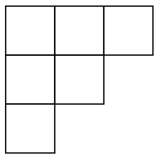
We will need one further result, known as the branching theorem for the symmetric groups.

Proposition 4.1.53. *(cf. [JK81, Theorem 2.4.3]) Let λ be a partition of n. Then*

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(\chi_\lambda) = \sum_{\mu \in \mathscr{P}_{n-1}, d(\mu,\lambda)=1} \chi_\mu,$$

$$\mathrm{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(\chi_\lambda) = \sum_{\nu \in \mathscr{P}_{n+1}, d(\nu,\lambda)=1} \chi_\nu.$$

Example 4.1.54. *Take the partition $(3,2,1)$ with Young tableau*

 .

Then

$$\mathrm{Res}^{\mathfrak{S}_6}_{\mathfrak{S}_5}(\chi_{(3,2,1)}) = \chi_{(2,2,1)} + \chi_{(3,1,1)} + \chi_{(3,2)}$$

and

$$\mathrm{Ind}^{\mathfrak{S}_7}_{\mathfrak{S}_6}(\chi_{(3,2,1)}) = \chi_{(4,2,1)} + \chi_{((3,3,1)} + \chi_{(3,2,2)} + \chi_{(3,2,1,1)}.$$

Since we know how to calculate the orthogonal determinants of the symmetric groups, what is left are the orthogonal determinants of the characters $\phi_\lambda^{(i)} \in \mathrm{Irr}^+(\mathfrak{A}_n)$ for $i = 1, 2$, i.e., for partitions $\lambda = \lambda'$ with $4 \mid f_\lambda$ and $\mathbb{Q}(\phi_\lambda^{(i)}) \subseteq \mathbb{R}$. We will fix such a partition λ of n and let m be the biggest integer such that $(m, m) \in [\lambda]$. The idea is now very simple: We will restrict the character $\phi_\lambda^{(i)}$ to $\mathfrak{A}_{n-1}$ and hope that the restriction turns out to be orthogonally stable. In the few cases that it isn't, we will find some other way to calculate its orthogonal determinant. By induction we then know how to calculate all the orthogonal determinants of the alternating groups.

Lemma 4.1.55. *Assume that $h_\lambda((1,1)) > 2^r + 1$. Then $\mathrm{Res}^{\mathfrak{A}_n}_{\mathfrak{A}_{n-1}}(\phi_\lambda^{(i)})$ is an orthogonally stable character of $\mathfrak{A}_{n-1}$.*

Proof. Note that $h_\lambda((1,2)) = h_\lambda((2,1)) < 2^r$. Indeed, if $m = 1$, then $n = 1 + h_\lambda((1,2)) + h_\lambda((2,1)) < 2^{r+1}$. If $m \geq 2$, then

$$h_\lambda((1,2)) + h_\lambda((2,1)) = h_\lambda((1,1)) + h_\lambda((2,2)) \leq n.$$

Let $\mu \in \mathscr{P}_{n-1}$ with $d(\mu, \lambda) = 1$. Then $2^r \leq |\mu| < 2^{r+1}$ and by the previous discussion there is no cell $c \in [\mu]$ with $h_\mu(c) = 2^r$. By Proposition 4.1.27 we then know that f_μ is even. This already implies that $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(\chi_\lambda)$ is orthogonally stable. So if $\mu \neq \mu'$ for all such μ, we are thus done.

If not, then necessarily $h_\lambda((m,m)) = 1$. But since $\phi_\lambda^{(i)}(1)$ is even, then also $\phi_{\lambda_{(m,m)}}^{(i)}(1)$ has to be even since $\phi_{\lambda_{(m,m)}}^{(i)}$ is the only split character appearing in $\mathrm{Res}^{\mathfrak{A}_n}_{\mathfrak{A}_{n-1}}(\phi_\lambda^{(i)})$. So the statement holds. □

Lemma 4.1.56. *Assume that $h_\lambda((1,1)) \leq 2^r - 1$. Then $\mathrm{Res}^{\mathfrak{A}_n}_{\mathfrak{A}_{n-1}}(\phi_\lambda^{(i)})$ is an orthogonally stable character of $\mathfrak{A}_{n-1}$.*

Proof. If $n > 2^r$, then we can argue just as in the previous lemma. So assume now that $n = 2^r$. The only way that the restriction of $\phi_\lambda^{(i)}$ to $\mathfrak{A}_{n-1}$ could potentially not be orthogonally stable is if there already is a cell $c \in [\lambda]$ with $h_\lambda(c) = 2^{r-1}$. This occurs precisely when

$$(h_\lambda((1,1)), h_\lambda((2,2))) = (2^r - k, k)$$

for some odd $k < 2^{r-1}$. We calculate that $1/2(2^r - 2)$ is an odd number, so by Theorem 4.1.52(ii) $\mathbb{Q}(\phi_\lambda^{(i)})$ is not real and we are done. □

Lemma 4.1.57. *Assume that $h_\lambda((1,1)) = 2^r + 1$. Then $m \geq 2$. If $h_\lambda((2,2)) > 1$, then $\mathrm{Res}^{\mathfrak{A}_n}_{\mathfrak{A}_{n-1}}(\phi_\lambda^{(i)})$ is an orthogonally stable character of $\mathfrak{A}_{n-1}$.*

Proof. Assume that $m = 1$. This implies that $[\lambda]$ consists of a single hook and that $h_\lambda((1,2)) = h_\lambda(2,1)) = 2^{r-1}$. Then

$$f_\lambda = \frac{(2^r+1)!}{(2^r+1)((2^{r-1})!)^2} = \binom{2^r}{2^{r-1}}$$

and it is clear that $4 \nmid f_\lambda$ so $\phi_\lambda^{(i)}(1)$ is odd.

So assume now that $m \geq 2$ and that $h_\lambda((2,2)) > 1$. If

$$\mathrm{Res}^{\mathfrak{A}_n}_{\mathfrak{A}_{n-1}}(\phi_\lambda^{(i)})$$

were not orthogonally stable, then there is a partition μ of $n-1$ such that $d(\mu, \lambda) = 1$ and f_μ is odd. So there is a cell $c \in [\mu]$ with $h_\mu(c) = 2^r$. The only possibility is $c = (1,1)$. Without loss of generality we can take $c' := (1, 2^{r-1}+1) \in [\lambda]$ and let $\mu = \lambda_{c'}$.

Then $D(\mu) = D(\mu)'$ with

$$h_{D(\mu)}((j,j)) = h_\lambda((j+1, j+1))$$

for $1 \leq j \leq m-1$. But then $h_{D(\mu)}((1,1)) > 1$ and thus $f_{D(\mu)}$ is even, since the only self-conjugate partition of odd degree is (1). So by Proposition 4.1.27 f_μ is even. □

Example 4.1.58. *To illustrate the argument in the above lemma, take for instance the partition $\lambda = (5,4,2,2,1)$ of 14. We have $\lambda = \lambda'$ and*

$$(h_\lambda((1,1)), h_\lambda((2,2))) = (9,5).$$

Let $c' = (1,5) \in [\lambda]$ and set $\mu := \lambda_{c'} = (4,4,2,2,1)$. Then $h_\mu((1,1)) = 8$ and $D(\mu) = (3,1,1)$. Below is the hook diagram of λ where $[D(\mu)]$ is colored white.

9	7	4	3	1
7	5	2	1	
4	2			
3	1			
1				

Lemma 4.1.59. *Assume that* $m = 2$, $h_\lambda((1,1)) = 2^r + 1$ *and* $h_\lambda((2,2)) = 1$, *i.e.,*

$$\lambda = (2^{r-1} + 1, 2, 1^{(2^{r-1}-1)}).$$

Assume additionally that $2^r \geq 8$. *Then*

$$\det(\phi_\lambda^{(i)}) = 1 \cdot (\mathbb{Q}(\sqrt{2^r + 1})^\times)^2.$$

Proof. It is easy to see that

$$\mathrm{Ind}_{\mathfrak{A}_n}^{\mathfrak{A}_{n+1}}(\phi_\lambda^{(i)}) = \phi_{\nu_1} + \phi_{\nu_2},$$

where $[\nu_1]$ (resp. $[\nu_2]$) has the extra cell $(2^{r-1} + 2, 1)$ (resp. $(2, 3)$) compared to $[\lambda]$, i.e.,

$$\nu_1 = (2^{r-1} + 1, 2, 1^{(2^{r-1})}), \ \nu_2 = (2^{r-1} + 1, 3, 1^{(2^{r-1}-1)}).$$

Then $h_{\nu_j}((1,1)) > 2^r$ and

$$h_{\nu_j}((2,1)) = h_{\nu_j}((1,2)) + 1 = 2^{r-1} + 2$$

for $j = 1, 2$. Since $2^r \geq 8$ we have that $2^{r-1} + 2 < 2^r$. So $\mathrm{OddRank}(\nu_1) = \mathrm{OddRank}(\nu_2) = 0$ and both ϕ_{ν_1} and ϕ_{ν_2} have even degree and are orthogonally stable. Since n is even, the index of $\mathfrak{A}_n$ in $\mathfrak{A}_{n+1}$ is odd and therefore by Lemma 2.3.10

$$\det(\phi_\lambda^{(i)}) = \det(\phi_{\nu_1}) \cdot \det(\phi_{\nu_2}) \cdot (\mathbb{Q}(\sqrt{2^r + 1})^\times)^2 = \det(\chi_{\nu_1}) \cdot \det(\chi_{\nu_2}) \cdot (\mathbb{Q}(\sqrt{2^r + 1})^\times)^2.$$

We will now calculate the determinants of these characters individually, with the results gathered in Subsection 4.1.3.

(i) Consider the partition ν_1. Then $\mathscr{P}_{\nu_1}^{>0} = M_1 \cup M_2$ consists of two equivalence classes. The first one is

$$M_1 = \{(2^r + 3), (2^{r-1} + 1, 4, 1^{(2^{r-1}-2)})\}, \ D(M_1) = (3).$$

Since $f_{D(M_1)} = 1$ is odd, we regard

$$\prod_{\kappa \in M_1} \frac{h_{\nu_1}^1(\kappa)}{h_{\nu_1}^2(\kappa)} = \frac{2^r + 2}{2^{r-1} + 2} \cdot \frac{2^{r-1} + 2}{2} = 2^{r-1} + 1.$$

The second one is

$$M_2 = \{(2^r, 2, 1), (2^{r-1} + 1, 2^{(3)}, 1^{(2^{r-1}-4)})\}, \ D(M_2) = (1, 1, 1).$$

Since $f_{D(M_2)} = 1$ is odd, we regard

$$\prod_{\kappa \in M_2} \frac{h_{\nu_1}^1(\kappa)}{h_{\nu_1}^2(\kappa)} = \frac{2^r + 2}{2^{r-1} - 1} \cdot \frac{2^{r-1} - 1}{2} = 2^{r-1} + 1.$$

Thus by the James–Murphy determinant formula,

$$\det(\chi_{\nu_1}) = 1 \cdot (\mathbb{Q}^\times)^2.$$

(ii) Consider the partition ν_2. Then $\mathscr{P}_{\nu_2}^{>0} = M_1 \cup M_2$ consists of two equivalence classes. The first one is

$$M_1 = \{(2^r+3), (2^{r-1}+1, 4, 1^{(2^{r-1}-2)})\},\ D(M_1) = (3).$$

Since $f_{D(M_1)} = 1$ is odd, we regard

$$\prod_{\kappa \in M_1} \frac{h^1_{\nu_2}(\kappa)}{h^2_{\nu_2}(\kappa)} = \frac{2^r+1}{2^{r-1}+2} \cdot \frac{2^{r-1}+2}{1} = 2^r+1.$$

The second one is

$$M_2 = \{(2^r, 3), (2^{r-1}+1, 3, 2, 1^{(2^{r-1}-3)})\},\ D(M_2) = (2,1).$$

Since $f_{D(M_2)} = 2$ is even, M_2 does not contribute to the orthogonal determinant. Thus by the James–Murphy determinant formula,

$$\det(\chi_{\nu_2}) = (2^r+1) \cdot (\mathbb{Q}^\times)^2.$$

All in all, $\det(\phi_\lambda^{(i)}) = 1 \cdot (2^r+1) \cdot (\mathbb{Q}(\sqrt{2^r+1})^\times)^2 = 1 \cdot (\mathbb{Q}(\sqrt{2^r+1})^\times)^2$. □

We are left to show a single case:

Lemma 4.1.60. *Let $n = 6$. It holds that $\det(\phi^{(i)}_{(3,2,1)}) = 1 \cdot (\mathbb{Q}(\sqrt{5})^\times)^2$.*

Proof. Note that $\phi^{(i)}_{(3,2,1)}(1) = 8$. It is well known that $\mathfrak{A}_6 \cong \mathrm{PSL}_2(9)$ and we can thus regard the associated characters of $\mathrm{SL}_2(9)$. We will later see in Theorem 5.3.3 that all characters of degree 8 of $\mathrm{SL}_2(9)$ have a square as an orthogonal determinant, so $\det(\phi^{(i)}_{(3,2,1)}) = 1 \cdot (\mathbb{Q}(\sqrt{5})^\times)^2$. □

Remark 4.1.61. *Let us summarize what we have learned. First of, there are no orthogonally stable characters of $\mathfrak{A}_2 \cong \{1\}$. Second, let λ be any partition of n. There are three possibilities for orthogonally simple characters of $\mathfrak{A}_n$:*

(i) If $\lambda \neq \lambda'$ and f_λ is even, then we know that

$$\det(\phi_\lambda) = \det(\chi_\lambda) \cdot (\mathbb{Q}^\times)^2$$

which we can calculate with the James–Murphy determinant formula. In particular, by Theorem 4.1.50, Parker's conjecture holds here.

(ii) If $\lambda = \lambda'$ and $\mathbb{Q}(\phi_\lambda^{(i)})$ is not real for $i = 1, 2$, then

$$\phi_\lambda^{(1)} + \phi_\lambda^{(2)} = \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(\chi_\lambda)$$

is orthogonally stable and

$$\det(\phi_\lambda^{(1)} + \phi_\lambda^{(2)}) = \det(\chi_\lambda) \cdot (\mathbb{Q}(\phi_\lambda^{(1)} + \phi_\lambda^{(2)})^\times)^2.$$

Parker's conjecture holds in that case. Note that we do not need Theorem 4.1.50 here since it already follows by Proposition 2.3.27.

(iii) If $\lambda = \lambda'$, $\mathbb{Q}(\phi_\lambda^{(i)})$ is real and the degree of $\phi_\mu^{(i)}$ is even for $i = 1, 2$, then either the restriction of this character to $\mathfrak{A}_{n-1}$ is orthogonally stable or its orthogonal determinant is a square. By induction, Parker's conjecture also holds here.

We thus arrive at the following main result:

Theorem 4.1.62. *Parker's conjecture holds for the alternating groups.*

4.1.5 Orthogonal Determinants of Iwahori–Hecke Algebras of Type A_n

Let n be a positive integer. We will first recall the definition of Iwahori–Hecke algebras of type A_{n-1}. For this, let $S = \{s_1, \dots, s_{n-1}\}$ be the standard generating set of $\mathfrak{S}_n$. Let u be an indeterminate over $\mathbb{Q}$. We let $A = \mathbb{Q}[u, u^{-1}]$ and let $K = \mathrm{Quot}(A) = \mathbb{Q}(u)$ be the quotient field of A. Let $\mathcal{H} := \mathcal{H}(A_{n-1})$ be the free associative A-algebra with basis $\{T_w \mid w \in \mathfrak{S}_n\}$, together with the relations

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) = \ell(w) + 1, \\ uT_{sw} + (u-1)T_w, & \text{if } \ell(sw) = \ell(w) - 1, \end{cases}$$

for any $s \in S, w \in \mathfrak{S}_n$.

It is well known that K is a splitting field for $\mathcal{H}$ and that the irreducible modules are parameterized by the partitions of n. For each partition λ of n, there is an explicit construction of an irreducible $K\mathcal{H}$-module which we will denote by S'_λ, see [DJ86]. In fact, the modules can be constructed over $\mathcal{H}$, together with a standard basis

$$\{e'_t \in \mathcal{H} \mid t \in T_\lambda\}.$$

So we will from now on regard S'_λ as an $\mathcal{H}$-module. We denote χ'_λ to be the irreducible character afforded by S'_λ. Note that by specializing $A \to \mathbb{Q}, u \mapsto 1$, the character χ'_λ becomes the character $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$, $S_{\lambda'}$ becomes the Specht module S_λ and the standard basis element e'_t gets specialized to e_t for each standard Young tableau $t \in T_\lambda$.

Recall that we have an involution $\dagger$ on $\mathcal{H}$ given by $T_w^\dagger = T_{w^{-1}}$. With that involution in mind, we can again again talk about the orthogonal determinants of the $\mathrm{Irr}^+(K\mathcal{H})$-characters. As in the case of the symmetric groups, S'_λ is a submodule of a "permutation module" (M'_λ, γ'), with a "canonical" symmetric, non-degenerate, $\mathcal{H}$-invariant bilinear form γ'. The restriction of γ' to S_λ now makes $(S'_\lambda, \gamma'_{|S'_\lambda})$ into an orthogonal $\mathcal{H}$-module. We denote $\det(\lambda)'$ to be the Gram matrix of $\gamma'_{|S'_\lambda}$ with respect to the standard basis.

We have discussed in detail the James–Murphy determinant formula for the symmetric groups. There is a generalization of this formula for $\det(\lambda)'$ due to Dipper and James in [DJ87, Theorem 4.11]. This has been further generalized and simplified in [DJM97], where the authors presented explicit formulas for orthogonal determinants of Iwahori–Hecke algebras of type B_n, which contains the orthogonal determinants of Iwahori–Hecke algebras of type A_n as a special case. We will present some of their results now:

So let $\lambda = (c_1, \dots, c_m)$ be a partition of n. Recall that the group $\mathfrak{S}_n$ acts on a standard Young tableau $t \in T_\lambda$ by permuting its entries. We define the standard Young tableau t_λ by putting the numbers $\{1, \dots, c_1\}$ in the first row, the numbers $\{c_1 + 1, \dots, c_1 + c_2\}$ in the second row, and so on. For instance,

$$t_{(4,3,2)} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \cline{1-3} 8 & 9 & & \\ \cline{1-2} \end{array}\ .$$

Definition 4.1.63. *We put a partial order on the set T_λ by saying that*

$$t_1 = w_1 \cdot t_\lambda < t_2 = w_2 \cdot t_\lambda,$$

if and only if $w_1 \leq_L w_2$, for any $t_1, t_2 \in T_\lambda$ and $w_1, w_2 \in \mathfrak{S}_n$.

We shall need the Gaussian polynomials:

Definition 4.1.64. *Let a, b be non-negative integers with $0 \leq b \leq a$. Let x be a variable or an integer. We define the Gaussian polynomials*

(i)

$$[a]_x := \begin{cases} 0, & a = 0, \\ 1 + x + \dots x^{a-1}, & a > 0. \end{cases}$$

(ii)

$$[a]_x! := \begin{cases} 1, & a = 0, \\ (1)_x \cdot (2)_x \cdots (a)_x, & a > 0. \end{cases}$$

(iii)

$$\binom{a}{b}_x := \frac{[a]_x!}{[b]_x! \cdot [a-b]_x!}.$$

As we are not interested in the exact value of $\det(\lambda)'$ but only its square class in $K^\times/(K^\times)^2$, we can slightly simplify the statement given by the authors.

Theorem 4.1.65. *(cf. [DJM97, Section 3]) Assume that $\chi'_\lambda \in \mathrm{Irr}^+(K\mathcal{H})$. We set $a_{t_\lambda} := 1$. Inductively we now define for each $t \in T_\lambda$ the elements $a_t \in A$ by the following condition:*

Let $t_1, t_2 \in T_\lambda$ with $t_1 < t_2$ and assume there is a simple transposition $s = (k, k+1) \in S$ such that $s \cdot t_1 = t_2$. If (i_1, j_1) (resp. (i_2, j_2)) are the positions of k (resp. $k+1$) in t_1, define $r_1 := u^{j_1 - i_1}$ and $r_2 := u^{j_2 - i_2}$. We set

$$a_{t_2} = \frac{(ur_1 - r_2)(r_1 - ur_2)}{(r_1 - r_2)^2} a_{t_1}.$$

Then the a_t are well-defined and

$$\det(\chi'_\lambda) = \prod_{t \in T_\lambda} a_t \cdot (K^\times)^2.$$

We can make further simplifications, for instance, the denominator, as it is a square, can be safely disregarded in actual computations.

Example 4.1.66. *(i) Assume that $\lambda = (2,2)$. Then there are two standard Young tableaux:*

$$t_1 := t_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}\,, t_2 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\,.$$

We have that $s_2 \cdot t_1 = t_2$ with the positions $(1,2)$ and $(2,1)$ getting swapped. So $r_1 = u, r_2 = u^{-1}$, and

$$a_{t_2} = (u^2 - u^{-1})(u-1) \cdot (K^\times)^2 = u(u^2+u+1) \cdot (K^\times)^2,$$

and we arrive at

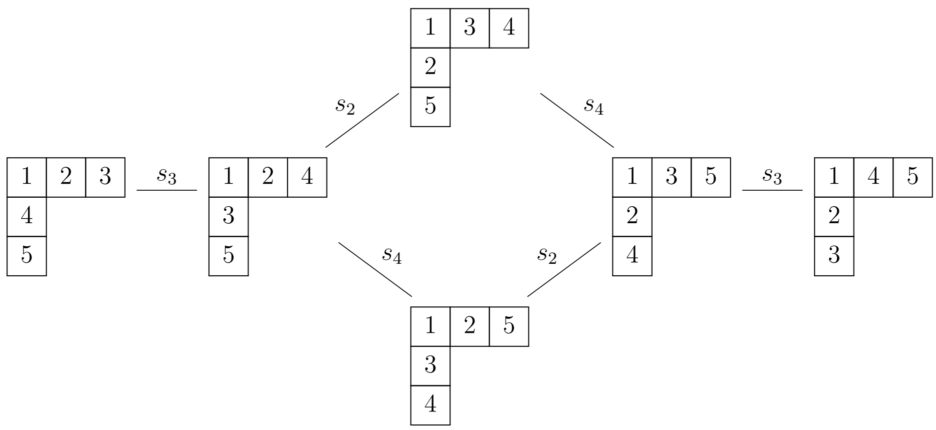
$$\det(\chi'_\lambda) = a_{t_1} a_{t_2} \cdot (K^\times)^2 = u(u^2+u+1) \cdot (K^\times)^2.$$

By specializing (or with the James–Murphy determinant formula) we see that for $\mathfrak{S}_4$,

$$\det(\chi_\lambda) = 3 \cdot (\mathbb{Q}^\times)^2.$$

While we could maybe have guessed a term like u^2+u+1 to appear, the factor of u gets completely lost in the specialization.

(ii) Let $\lambda = (3,1,1)$. There are 6 standard Young tableaux of λ; the following diagram depicts each of these, as well as the simple transpositions that transform them into each other:



An easy calculation now gives us the result

$$\det(\chi'_\lambda) = (u^4+u^3+u^2+u+1) \cdot (K^\times)^2.$$

4.2 Type B_n and Type D_n

We will follow [GP00, Section 1.4] for the description of the Coxeter groups of type B_n and D_n.

Let $n \geq 2$ be an integer and let (W, S) be a Coxeter system of type B_n. We denote $S = \{t, s_1, \dots, s_{n-1}\}$ and recall that W is generated by the set S, with the following relations:

(i) $t^2 = s_i^2 = 1$ for all $1 \leq i \leq n-1$,

(ii) $(ts_1)^4 = 1$ and $ts_i = s_i t$ for $2 \leq i \leq n-1$,

(iii) $(s_i s_{i+1})^3 = 1$ for $1 \leq i \leq n-2$, $s_j s_k = s_k s_j$ for $1 \leq j, k \leq n-1$ and $|k-j| \geq 2$.

Note that we have a parabolic subgroup isomorphic to $\mathfrak{S}_n$, generated by the set $\{s_1, \dots, s_{n-1}\}$, so we will write $\mathfrak{S}_n \subseteq W$ from now on. We have a normal subgroup

$$N := \{wtw^{-1} \mid w \in W\},$$

and $W = N \rtimes \mathfrak{S}_n$ decomposes as a semidirect product.

Similar to the construction of the symmetric groups as the group of permutation matrices, there is an explicit construction of W as a subgroup of $\mathrm{GL}_n(\mathbb{R})$ as the set of monomial matrices, i.e., the matrices that have exactly one non-zero entry in every row and column, where every entry is in $\{-1, 0, 1\}$. Here, $t = \mathrm{diag}(-1, 1, 1, \dots, 1)$ and the s_i are defined just as in Section 4.1. With this embedding in mind, the subgroup $\mathfrak{S}_n$ corresponds to the set of permutation matrices and N corresponds to the intersection of W with the set of diagonal matrices. Accordingly, N is an abelian group with $|N| = 2^n$ and we recover the fact that $|W| = 2^n n!$.

We will now describe a construction of Coxeter groups of type D_n. Let $N' \subseteq N$ be the subgroup of index 2 of diagonal matrices with an even number of -1s, in other words, $N' = N \cap \mathrm{SL}_n(\mathbb{R})$. Define $W' = N' \rtimes \mathfrak{S}_n$. Then $W' \subseteq W$ is a subgroup of index 2 and is in fact a Coxeter system of type D_n. Indeed, let $v := ts_1t = s_1 \cdot \mathrm{diag}(-1, -1, 1, \dots, 1)$. Then $S' = \{v, s_1, s_2, \dots, s_{n-1}\}$ makes (W', S') to a Coxeter system of type D_n, with the additional relations

(i) $v^2 = 1$,

(ii) $vs_i = s_i v$ for $1 \leq i \leq n-1$, $i \neq 2$,

(iii) $vs_2v = s_2vs_2$.

4.2.1 Representation Theory of Coxeter Groups of Types B_n and D_n

The irreducible characters of a semidirect product of the form $G = A \rtimes H$, where A is an abelian group and H is any finite group, can be completely described in term of characters of subgroups of H. In [GP00, Section 5.5], this construction is explicitly applied to the case of Coxeter groups of type B_n, as we have a semidirect product $W = N \rtimes \mathfrak{S}_n$, and N is abelian. This will also be applied to the Coxeter group $W' = N' \rtimes \mathfrak{S}_n$ of type D_n, but here there is also another approach: As a subgroup of index 2 of W, we have that W' is a normal subgroup of W and so we can apply Clifford theory.

We will first describe the general theory of characters of a semidirect product $G = A \rtimes H$ with a normal abelian subgroup A. We have a natural action of H on $\mathrm{Irr}(A)$, given by

$$(h \cdot \theta)(a) := \theta(h^{-1}ah)$$

for $h \in H, \theta \in \mathrm{Irr}(A), a \in A$. Let $\{\theta_1, \dots, \theta_N\} \subseteq \mathrm{Irr}(A)$ be a set of representatives of the action of H on $\mathrm{Irr}(A)$. We define

$$H_i := \mathrm{Stab}_H(\theta_i) = \{h \in H \mid h \cdot \theta_i = \theta_i\}$$

for each $i = 1, \dots, N$. Let $G_i := A \rtimes H_i \subseteq G$. Then θ_i can be considered an irreducible character of G_i, given by

$$\theta_i(ah) = \theta_i(a)$$

for any $a \in A, h \in H_i$. By abuse of notation, we regard any character $\rho \in \mathrm{Irr}(H_i)$ as a character $\rho \in \mathrm{Irr}(G_i)$ by the natural projection.

Theorem 4.2.1. *(cf. [Ser77, Proposition 25]) There is a 1-to-1-correspondence*

$$\bigcup_{1 \leq i \leq N} \mathrm{Irr}(H_i) \leftrightarrow \mathrm{Irr}(G),$$

given by

$$\rho \mapsto \chi_\rho := \mathrm{Ind}_{G_i}^{G}(\rho \cdot \theta_i)$$

for all $\rho \in \mathrm{Irr}(H_i)$ and all i.

We now want to apply the above theorem to the Coxeter groups of type B_n. A system of representatives of $\mathrm{Irr}(N)$ with the action of $\mathfrak{S}_n$ is given by $\{\theta_0, \dots, \theta_n\} \subseteq \mathrm{Irr}(N)$ with

$$\theta_i\left(\mathrm{diag}(a_1, \dots, a_n)\right) = \prod_{j=1}^{i} a_j,$$

for $a_j \in \{-1, 1\}, 0 \leq i \leq n$. Then

$$\mathrm{Stab}_{\mathfrak{S}_n}(\theta_i) = \mathfrak{S}_i \times \mathfrak{S}_{n-i},$$

with the evident generating set $\{s_1, s_2, \dots, s_{n-1}\} \backslash \{s_i\}$. Now,

$$N \rtimes (\mathfrak{S}_i \times \mathfrak{S}_{n-i}) = W_i \times W_{n-i},$$

where W_i and W_{n-i} are again Coxeter groups of type B_i and B_{n-i}, respectively. We can now describe all irreducible characters of W:

Theorem 4.2.2. *[GP00, Theorem 5.5.6] Let $0 \leq i \leq n$, $\lambda \in \mathscr{P}_i$, $\mu \in \mathscr{P}_{n-i}$. We define*

$$\chi_{(\lambda,\mu)} := \mathrm{Ind}_{W_i \times W_{n-i}}^{W}\left((\chi_\lambda \boxtimes \chi_\mu) \cdot \theta_i\right),$$

where χ_λ, χ_μ are the corresponding irreducible characters of $\mathfrak{S}_i$, $\mathfrak{S}_{n-i}$ as described in Section 4.1. As we run over all i and all partitions, we get a complete list of all irreducible characters of W without repeats. It further holds that

$$\chi_{(\lambda,\mu)}(1) = \binom{n}{i} \cdot f_\lambda f_\mu,$$

where $f_\lambda = \chi_\lambda(1)$ and $f_\mu = \chi_\mu(1)$. All characters of W can be realized by representations over the rational numbers.

The proposition above gives us, as a by-product, a characterization of all irreducible characters of W', similar to the case with the alternating groups in Theorem 4.1.52.

Theorem 4.2.3. *[GP00, Section 5.6] Let $0 \leq i \leq n$, $\lambda \in \mathscr{P}_i$, $\mu \in \mathscr{P}_{n-i}$.*

(i) If $\lambda \neq \mu$, then
$$\chi'_{(\lambda,\mu)} := \mathrm{Res}^W_{W'}(\chi_{(\lambda,\mu)}) = \mathrm{Res}^W_{W'}(\chi_{(\mu,\lambda)})$$
is an irreducible character of W'. In particular, $\chi'_{(\lambda,\mu)} = \chi'_{(\mu,\lambda)}$.

(ii) If $\lambda = \mu$, then there are two irreducible characters $\chi'_{(\lambda,+)}$ and $\chi'_{(\lambda,-)}$ of W', each of degree $\chi_{(\lambda,\lambda)}(1)/2$, such that
$$\mathrm{Res}^W_{W'}(\chi_{(\lambda,\lambda)}) = \chi'_{(\lambda,+)} + \chi'_{(\lambda,-)}.$$

(iii) The abovementioned characters give a full list of all irreducible characters of W' and all characters of W' can be realized by representations over the rational numbers.

By Theorem 4.2.1, all characters of W' are induced characters by certain subgroups. We wish to understand the split characters $\chi'_{(\lambda,+)}$ and $\chi'_{(\lambda,+)}$ in this way. So let now $n = 2k$ be even. For the split characters, we need to understand the stabilizer of θ_k in W'. Let $\sigma \in \mathfrak{S}_n$ be the permutation of order 2 defined by

$$\sigma(i) = \begin{cases} i+k, & \text{if } 1 \leq i \leq k, \\ i-k, & \text{if } k \leq i \leq n. \end{cases}$$

Then it is easy to verify that

$$H := \mathrm{Stab}_{\mathfrak{S}_n}(\theta_k) = (\mathfrak{S}_k \times \mathfrak{S}_k) \rtimes \langle \sigma \rangle \cong \mathfrak{S}_k \wr C_2,$$

where $\mathfrak{S}_k \wr C_2$ is the wreath product of $\mathfrak{S}_k$ with C_2. As is now routine, Clifford theory gives us the following:

Lemma 4.2.4. *Let λ be a partition of k. Then there are irreducible characters $\psi_{(\lambda,+)}$, $\psi_{(\lambda,-)}$ of H, each of degree f_λ^2, such that*

$$\mathrm{Res}^H_{\mathfrak{S}_k \times \mathfrak{S}_k}(\psi_{(\lambda,+)}) = \mathrm{Res}^H_{\mathfrak{S}_k \times \mathfrak{S}_k}(\psi_{(\lambda,-)}) = \chi_\lambda \boxtimes \chi_\lambda,$$

and

$$\mathrm{Ind}^{W'}_{N' \rtimes H}(\psi_{(\lambda,+)} \cdot \theta_k) = \chi'_{(\lambda,+)}, \ \mathrm{Ind}^{W'}_{N' \rtimes H}(\psi_{(\lambda,-)} \cdot \theta_k) = \chi'_{(\lambda,-)}.$$

Note that in the notation of the above lemma, the characters $\psi_{(\lambda,+)}$ and $\psi_{(\lambda,-)}$ both take values in the rational numbers and have Schur index 1. If we take $\phi \in \operatorname{Irr}(\langle\sigma\rangle)$ to be the nontrivial character with $\phi(\sigma) = -1$, then

$$\psi_{(\lambda,+)} = \psi_{(\lambda,-)} \cdot \phi.$$

4.2.2 Orthogonal Determinants of Coxeter Groups of Types B_n and D_n

Let again n be an integer. We let W and W' be the Coxeter groups of types B_n and D_n respectively. The statements of the previous subsection now give us a very easy way to get all orthogonal determinants of these two groups:

Theorem 4.2.5. *Let $0 \leq i \leq n$, $\lambda \in \mathscr{P}_i$, $\mu \in \mathscr{P}_{n-i}$ such that $\chi_{(\lambda,\mu)} \in \operatorname{Irr}(W)$ is orthogonally stable. The following hold:*

(i) If $\binom{n}{i}$ is even, then $\det(\chi_{(\lambda,\mu)}) = 1 \cdot (\mathbb{Q}^\times)^2$. If $\binom{n}{i}$ is odd, then

$$\det(\chi_{(\lambda,\mu)}) = \det(\chi_\lambda \boxtimes \chi_\mu) = \begin{cases} 1 \cdot (\mathbb{Q}^\times)^2, & \text{if } f_\lambda, f_\mu \text{ are even,} \\ \det(\chi_\lambda), & \text{if } f_\lambda \text{ is even, } f_\mu \text{ is odd,} \\ \det(\chi_\mu), & \text{if } f_\mu \text{ is even, } f_\lambda \text{ is odd.} \end{cases}$$

(ii) If $\lambda \neq \mu$, then $\det(\chi'_{(\lambda,\mu)}) = \det(\chi_{(\lambda,\mu)})$. Assume now that $\lambda = \mu$ and that $\chi'_{(\lambda,+)}$ is orthogonally stable. Then

$$\det(\chi'_{(\lambda,+)}) = \det(\chi'_{(\lambda,-)}) = 1 \cdot (\mathbb{Q}^\times)^2.$$

Proof. Recall that

$$\chi_{(\lambda,\mu)} = \operatorname{Ind}_{W_i \times W_{n-i}}^{W} \left((\chi_\lambda \boxtimes \chi_\mu) \cdot \theta_i \right).$$

It holds that

$$|W/(W_i \times W_{n-i})| = \binom{n}{i}.$$

So if the index is even, then by Lemma 2.3.10, $\det(\chi_{(\lambda,\mu)}) = 1 \cdot (\mathbb{Q}^\times)^2$. Assume now that the index is odd, so now $(\chi_\lambda \boxtimes \chi_\mu) \cdot \theta_i$ has to be orthogonally stable. We have that

$$\operatorname{Res}_{\mathfrak{S}_i \times \mathfrak{S}_{n-i}}^{W_i \times W_{n-i}} ((\chi_\lambda \boxtimes \chi_\mu) \cdot \theta_i) = \chi_\lambda \boxtimes \chi_\mu.$$

So again by Lemma 2.3.10,

$$\det(\chi_{(\lambda,\mu)}) = \det((\chi_\lambda \boxtimes \chi_\mu) \cdot \theta_i) = \det(\chi_\lambda \boxtimes \chi_\mu).$$

Then (i) now follows by the rules of orthogonal determinants of direct products of groups by Lemma 2.3.13.

For (ii), it suffices to regard the case of $\lambda = \mu$ with $\chi'_{(\lambda,+)}$ orthogonally stable and to just show the result for that character, as it follows analogously for $\chi'_{(\lambda,-)}$. We will again denote $H := \mathrm{Stab}_{\mathfrak{S}_n}(\theta_k)$, in the notation of the previous subsection. The statement is clear if $|W'/(N' \rtimes H)|$ is even, yet again by Lemma 2.3.10. Assume now that the index is odd. It follows by Lemma 4.2.4 that

$$\mathrm{Res}^{N'\rtimes H}_{\mathfrak{S}_k\times\mathfrak{S}_k}(\psi_{(\lambda,+)} \cdot \theta_k) = \chi_\lambda \boxtimes \chi_\lambda,$$

so it has orthogonally stable restriction to $\mathfrak{S}_k \times \mathfrak{S}_k$. Now by the rules of the orthogonal determinants of direct products of groups, $\det(\chi_\lambda \boxtimes \chi_\lambda) = 1 \cdot (\mathbb{Q}^\times)^2$. Bringing everything together, we end up with

$$\det(\chi_{(\lambda,+)}) = \det(\psi_{(\lambda,+)} \cdot \theta_k) = \det(\chi_\lambda \boxtimes \chi_\lambda) = 1 \cdot (\mathbb{Q}^\times)^2,$$

which we wanted to show. □

Corollary 4.2.6. *Parker's conjecture holds for Coxeter groups of types B_n and D_n.*

Proof. This is a direct consequence of the previous theorem and the proof of Parker's conjecture for the symmetric groups in Theorem 4.1.50. □

4.3 Type $I_2(m)$

We recall the presentation of the dihedral groups. Let $m \geq 3$ be an integer and (W, S) be a Coxeter system of type $I_2(m)$. Then $S = \{s, t\}$ and we have the relations

(i) $s^2 = t^2 = 1$,

(ii) $(st)^m = 1$.

We recall the definition of the generic Iwahori–Hecke algebra of type $I_2(m)$:

Definition 4.3.1. *Let u be an indeterminate over $\mathbb{Q}$. We define the Iwahori–Hecke algebra $\mathcal{H} := \mathcal{H}(I_2(m))$ to be the free associative $A := \mathbb{Q}[u, u^{-1}]$-algebra with basis $\{T_w \mid w \in W\}$, together with the relations*

(i) $T_w T_{w'} = T_{ww'}$ for $w, w' \in W$, if $\ell(ww') = \ell(w) + \ell(w')$,

(ii) $T_s^2 = uT_1 + (u-1)T_s$, $T_t^2 = uT_1 + (u-1)T_t$.

Since $\mathcal{H}(I_2(m))$ is an Iwahori–Hecke algebra, it is a semisimple monomial algebra and we can talk about orthogonal determinants. Let $K := \mathrm{Quot}(A)$ be the quotient field of A and let L be an algebraic closure of K.

We set

$$\zeta_m := \exp\left(\frac{2\pi i}{m}\right)$$

to be a primitive complex m-th root of unity. We further set

$$\vartheta_m^{(k)} := \zeta_m^k + \zeta_m^{-k} \in \mathbb{R}$$

for any integer k.

We now describe how to explicitly construct representations for all $\mathrm{Irr}^+(L\mathcal{H})$-characters.

Proposition 4.3.2. *(cf. [GP00, Theorem 8.3.1]) It holds that*

$$|\mathrm{Irr}^+(L\mathcal{H})| = \begin{cases} (m-2)/2, & \text{if } m \text{ is even,} \\ (m-1)/2, & \text{if } m \text{ is odd,} \end{cases}$$

all of which have degree 2. Let j be an integer with $1 \leq j \leq (m-1)/2$. We define the representation ρ_j of $K(\vartheta_m^{(j)})\mathcal{H}$ by

$$\rho_j(T_s) := \begin{pmatrix} -1 & 0 \\ u & u \end{pmatrix}, \; \rho_j(T_t) := \begin{pmatrix} u & 2+\vartheta_m^{(j)} \\ 0 & -1 \end{pmatrix}.$$

Let χ'_j be the character of ρ_j. As we run over all values of j, the χ'_j give a list of all $\mathrm{Irr}^+(L\mathcal{H})$-characters.

Recall that we have an involution $\dagger$ on $\mathcal{H}$ given by $T_w^\dagger = T_{w^{-1}}$ for $w \in W$. So, $T_{st}^\dagger = T_{ts}$. We will use these two elements to determine the orthogonal determinants of the $\mathrm{Irr}^+(L\mathcal{H})$- and the $\mathrm{Irr}^+(W)$-characters:

Theorem 4.3.3. *Let j be an integer with $1 \leq (m-1)/2$. Then*

$$\det\left(\rho_j(T_{ts}) - \rho_j(T_{st})\right) = u(2+\vartheta_m^{(j)})(u^2 - \vartheta_m^{(j)}u + 1).$$

The following now hold:

(i) For the character $\chi'_j \in \mathrm{Irr}^+(L\mathcal{H})$, it holds that

$$\det(\chi'_j) = u(2+\vartheta_m^{(j)})(u^2 - \vartheta_m^{(j)}u + 1) \cdot (K(\vartheta_m^{(j)})^\times)^2.$$

(ii) With the specialization $A \to \mathbb{Q}, u \mapsto 1$, we get irreducible characters $\chi_j \in \mathrm{Irr}^+(W)$. Then

$$\det(\chi_j) = (2 - \vartheta_m^{(2j)}) \cdot (\mathbb{Q}(\vartheta_m^{(j)})^\times)^2.$$

Proof. The proof is a basic calculation. We calculate that

$$\rho_j(T_{ts}) = \begin{pmatrix} (1+\vartheta_m^{(j)})u & (2+\vartheta_m^{(j)})u \\ -u & -u \end{pmatrix}, \; \rho_j(T_{st}) = \begin{pmatrix} -u & -(2+\vartheta_m^{(j)}) \\ u^2 & (1+\vartheta_m^{(j)})u \end{pmatrix},$$

so

$$\rho_j(T_{ts}) - \rho_j(T_{st}) = \begin{pmatrix} (2+\vartheta_m^{(j)})u & -(2+\vartheta_m^{(j)})(u+1) \\ -u(u+1) & -(2+\vartheta_m^{(j)})u \end{pmatrix}.$$

The determinant is thus

$$\det\left(\rho_j(T_{ts}) - \rho_j(T_{st})\right) = u(2+\vartheta_m^{(j)})(u^2 - \vartheta_m^{(j)}u + 1).$$

The rest is a basic consequence of Theorem 2.2.18. □

Again, Parker's conjecture holds here. This also follows by Theorem 2.3.29, since the dihedral groups are solvable. We will capture this result in a corollary:

Corollary 4.3.4. *Parker's conjecture holds for Coxeter groups of types $I_2(m)$.*

4.4 Exceptional Groups

There are in total 6 exceptional Coxeter groups: H_3, H_4, F_4, E_6, E_7 and E_8. Since this is a finite list of finite groups, we will use the aid of a computer to determine the orthogonal determinants of these groups. With the exception of the group H_4, all characters of these groups have rational values and Schur index equal to 1. Now, for the group H_4, theoretical methods can be used to determine the orthogonal determinant of the singular character 48_{rr} of Schur index 2 and of the characters with character field $\mathbb{Q}(\sqrt{5})$. We will now assume that W is a finite Coxeter group of type H_3, H_4, F_4, E_6, E_7 or E_8, and that $\chi \in \mathrm{Irr}^+(W)$ is an orthogonally stable character with Schur index equal to 1. The basic idea of the algorithm to determine $\det(\chi)$ now goes like this:

(i) Construct a $\mathbb{Q}G$-module V that affords the character χ with homomorphism $\rho : \mathbb{Q}G \to \mathrm{End}(V)$.

(ii) Denote $\dagger$ to be the natural involution of $\mathbb{Q}G$. Construct random elements $h \in \mathbb{QG}$ and calculate $\det\left(\rho(h) - \rho(h^\dagger)\right)$.

(iii) If $\det\left(\rho(h) - \rho(h^\dagger)\right) = 0$, construct a new random element h', until

$$\det\left(\rho(h') - \rho((h')^\dagger)\right) \neq 0.$$

(iv) Let $d := \det\left(\rho(h') - \rho((h')^\dagger)\right) \in \mathbb{Q}^\times$. Give out the unique squarefree integer d' that is in the same square class as d.

Recall that by Theorem 2.2.14, such elements always exist and give us the orthogonal determinant.

We thank our colleague Tobias Braun for providing the necessary code, which took care of all exceptional groups except H_4 (here, only the characters with character field $\mathbb{Q}$ were covered) and E_8 (due to the sheer size). For the orthogonal determinants in the case of E_8, we made use of the calculations of Gabriele Nebe, Richard Parker and Thomas Breuer, see [BNP24] for an overview of their methods. In particular, the orthogonal determinants of the $\mathrm{Irr}^+(G)$-characters for $G = \mathrm{GO}_8^+(2)$ (or, in ATLAS notation, $\mathrm{O}_8^+(2).2$) were already determined. For the groups H_4 and E_8, also CHEVIE (cf. [Mic15]) was used, we will indicate on where exactly.

For the notation of the irreducible characters, we will use the one used in [GP00, Appendix C]. The structure of the exceptional Coxeter groups can be found in [Wil09, Section 3.12.4].

Without further ado, we will now give the orthogonal determinants of all $\mathrm{Irr}^+(W)$-characters:

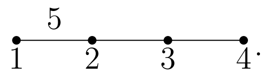
Type H_3

For H_3, there are only two $\mathrm{Irr}^+(W)$-characters, both of degree 4:

χ	$\det(\chi) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$
4_τ, $4'_\tau$	5

Type H_4

Let W be a Coxeter system of type H_4. Recall that H_4 is generated by reflections s_1, s_2, s_3, s_4 with a corresponding Coxeter diagram



Let w_0 be the longest element of W. It holds that the center of W is given by $Z(W) = \{1, w_0\}$. From [Wil09, Section 3.12.4] we gather that

$$W/Z(W) \cong (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes C_2 = \mathfrak{A}_5 \wr C_2.$$

By [GP00, Appendix C, Table C2], all characters of W with a label of r or s correspond to characters of $\mathfrak{A}_5 \wr C_2$. By Lemma 2.3.12, it suffices to calculate the orthogonal determinants of these characters in that group. Since $\mathfrak{A}_5 \wr C_2$ has a normal subgroup $\mathfrak{A}_5 \times \mathfrak{A}_5$ of index 2, almost all of its orthogonal determinants can be gotten by restricting and inducing to and from that subgroup. The only character of $\mathrm{Irr}^+(\mathfrak{A}_5 \wr C_2)$ that can not be calculated in that way is 18_t — it is equal to the induced character

$$\mathrm{Ind}_{\mathfrak{A}_5 \times \mathfrak{A}_5}^{\mathfrak{A}_5 \wr C_2} \left(\phi^{(1)}_{(3,1,1)} \boxtimes \phi^{(2)}_{(3,1,1)}\right).$$

but note that we can not use Lemma 2.3.10 here, since $\mathbb{Q}\left(\phi^{(1)}_{(3,1,1)} \boxtimes \phi^{(2)}_{(3,1,1)}\right) = \mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(18_r) = \mathbb{Q}$. But, the conditions of Lemma 2.3.11 hold here, so

$$\det(18_r) = 5 \cdot (\mathbb{Q}^\times)^2.$$

Let us now regard the characters that do not come from $\mathfrak{A}_5 \wr C_2$. On any of these characters χ, we have that $\chi(w_0) = -1$. With the help of CHEVIE we now construct an element $w \in W$ such that $w^2 = 1$. More explicitly, the element is given by the list

$$[1, 2, 1, 2, 1, 3, 2, 1, 2, 1, 3, 2, 1, 2, 4, 3, 2, 1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 2, 3, 4],$$

meaning that $w = s_1 s_2 s_1 s_2 s_1 s_3 \ldots$. By Lemma 2.3.14, we arrive that

$$\det(\chi) = 1 \cdot (\mathbb{Q}(\chi)^\times)^2$$

for all these characters. In total, the orthogonal determinants of the $\mathrm{Irr}^+(W)$-characters are the following:

χ	$\det(\chi) \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$
18_r	5
All other $\mathrm{Irr}^+(W)$-characters	1

Type F_4

The orthogonal determinants of the $\mathrm{Irr}^+(W)$-characters for W of type F_4 are the following:

χ	$\det(\chi) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$
$2_1,\ 2_2,\ 2_3,\ 2_4$	3
All other $\mathrm{Irr}^+(W)$-characters	1

Type E_6

The orthogonal determinants of the $\mathrm{Irr}^+(W)$-characters for W of type E_6 are the following:

χ	$\det(\chi) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$
$6_p, 6'_p$	3
10_s	3
$20_p,\ 20'_p, 20_s$	1
$24_p, 24'_p$	5
$30_p, 30'_p$	3
$60_p, 60'_p, 60_s$	1
$64_p, 64'_p$	1
80_s	1
90_s	3

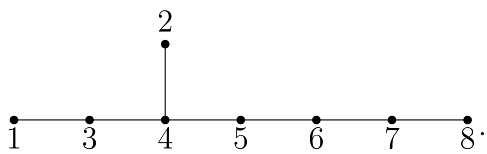
Type E_7

The orthogonal determinants of the $\mathrm{Irr}^+(W)$-characters for W of type E_7 are the following:

χ	$\det(\chi) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$
$56_a, 56'_a$	1
$70_a, 70'_a$	3
$84_a, 84'_a$	5
$120_a, 120'_a$	1
$168_a, 168'_a$	5
$210_a, 210'_a$	1
$210_b, 210'_b$	3
$216_a, 216'_a$	105
$280_a, 280'_a, 280_b, 280'_b$	1
$336_a, 336'_a$	1
$378_a, 378'_a$	15
$420_a, 420'_a$	1
$512_a, 512'_a$	105

Type E_8

We can argue similarly as in the case of H_4. Let W be a Coxeter group of type E_8. Then W is generated by elements $s_1, s_2, \ldots, s_8$ with Coxeter diagram



Let w_0 be the longest element in W. Again, the center is given by $Z(W) = \{1, w_0\}$. Again by [Wil09, Section 3.12.4] we get that

$$W/Z(W) \cong \mathrm{GO}_8^+(2),$$

all of which orthogonal determinants have been calculated by Nebe, Parker and Breuer.

For all of the remaining $\chi \in \mathrm{Irr}^+(W)$, we have that $\chi(w_0) = -\chi(1)$. To use Lemma 2.3.14, we need to find an element $w \in W$ such that $w^2 = w_0$. With the help of CHEVIE, we see that such an element exists and is given by the list

$$[1,2,3,1,4,2,3,1,4,3,5,4,2,3,4,5,6,5,4,2,3,1,4,3,5,4,2,7,6,5,$$
$$4,2,3,1,4,3,5,4,6,5,7,6,8,7,6,5,4,2,3,1,4,3,5,4,6,5,7,6,8,7],$$

meaning that $w = s_1s_2s_3s_1s_4\dots$. So all of these characters have determinant equal to 1 and we arrive at the following table of the orthogonal determinants of the $\mathrm{Irr}^+(W)$-characters:

χ	$\det(\chi) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$
$50_x, 50'_x$	3
$84_x, 84'_x$	5
$210_x, 210'_x$	3
$300_x, 300'_x$	21
$840_x, 840'_x$	5
$972_x, 972'_x$	5
$1050_x, 1050'_x$	3
$2268_x, 2268'_x$	5
$4096_x, 4096'_x$	105
All other $\mathrm{Irr}^+(W)$-characters	1

By looking through the tables, we see that Parker's conjecture holds for all the groups considered in the section:

Corollary 4.4.1. *Parker's conjecture holds for the exceptional Coxeter groups.*

To conclude this chapter, we have checked Parker's conjecture for every irreducible finite Coxeter group. Every finite Coxeter groups is just a direct products of irreducible finite Coxeter groups, and we know that orthogonal determinants behave well with direct products, compare with Lemma 2.3.13. Thus, we have shown that the following theorem holds:

Theorem 4.4.2. *Parker's conjecture holds for every finite Coxeter group.*

5 Orthogonal Determinants of Finite Groups of Lie Type

In this chapter, we will discuss methods for the calculation of the orthogonal determinants of the $\mathrm{Irr}^+(G)$-characters for G a finite group of Lie type defined over a field in odd characteristic, i.e., the corresponding connected reductive group is defined over a field with odd characteristic. Many $\mathrm{Irr}^+(G)$-characters are Borel-stable, i.e., have an orthogonally stable restriction to a Borel subgroup. For those that aren't, we will use condensation techniques to get a hold on the orthogonal determinants. We will finish this chapter with some explicit examples.

5.1 Borel-Stability

Let p be an odd prime and let G be a finite group of Lie type in characteristic p with Borel subgroup $B = U \rtimes T$ for a quasi-split torus T and unipotent radical U. Let $W = N/T$ be the Weyl group of G. In this section, we will consider the orthogonal determinants of the $\mathrm{Irr}^+_B(G)$-characters of G, i.e., the $\mathrm{Irr}^+(G)$-characters that have an orthogonally stable restriction to B. We will in the sequel also say that these characters are *Borel-stable*.

Definition 5.1.1. *Let $\chi \in \mathrm{Irr}(G)$. Recall the Harish-Chandra restriction χ_T from Definition 3.5.1. By inflation we will consider χ_T to also be a character of B. We define*

$$\chi_U := \mathrm{Res}^G_B(\chi) - \chi_T.$$

Definition 5.1.2. *Let $\theta \in \mathrm{Irr}(T)$. Define*

$$\mathrm{GalW}(\theta) = \{\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q}) \mid \sigma \cdot \theta = w \cdot \theta \text{ for some } w \in W\}.$$

Proposition 5.1.3. *Let $\theta \in \mathrm{Irr}(T)$, $\chi \in \mathrm{Simp}(G \mid (T, \theta))$. The following hold:*

(i) Let $\mathrm{Stab}_W(\theta) := \{w \in W \mid w \cdot \theta = \theta\}$. Then

$$\chi_T = \frac{\langle \mathrm{Res}^G_B(\chi), \theta\rangle_B}{|\mathrm{Stab}_W(\theta)|} \sum_{w \in W} w \cdot \theta.$$

(ii) $\mathbb{Q}(\chi_T) = \mathbb{Q}(\theta)^{\mathrm{GalW}(\theta)}$.

(iii) $\mathbb{Q}(\mathrm{Ind}^G_B(\theta)) = \mathbb{Q}(\chi_T)$.

Proof. By Theorem 3.5.5 and Frobenius reciprocity, we have for all $w \in W$ that

$$\langle \mathrm{Res}_B^G(\chi), w \cdot \theta \rangle_B = \langle \mathrm{Res}_B^G(\chi), \theta \rangle_B \neq 0.$$

This already shows that χ_T is a multiple of $\sum_{w \in W} w \cdot \theta$. It is also easy to get the multiplicity right: It clearly holds that

$$\langle \sum_{w \in W} w \cdot \theta, \theta \rangle_T = |\mathrm{Stab}_W(\theta)|,$$

so dividing by that amount makes sure that the irreducible constituents appear exactly once in the sum. Now we just multiply with $\langle \mathrm{Res}_B^G(\chi), \theta \rangle$, i.e., the amount each character "should appear" and the first statement follows.

For the second statement, we first observe by a routine calculation that the actions of W and $\mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$ on $\mathrm{Irr}(T)$ commute. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$. Then

$$\sigma \cdot \sum_{w \in W} w \cdot \theta = \sum_{w \in W} w \cdot (\sigma \cdot \theta).$$

Thus σ leaves the sum invariant if and only if $\sigma \cdot \theta = w' \cdot \theta$ for some $w' \in W$, which shows the statement.

Let us move on to the last statement. Let $x_1, \ldots, x_N$ be representatives of the cosets G/B. Then for $g \in G$, by for instance [FH91, 3.18],

$$\mathrm{Ind}_B^G(\theta)(g) = \sum_{x_i^{-1} g x_i \in B} \theta(x_i^{-1} g x_i).$$

Since for any $w \in W$ it holds that $\mathrm{Ind}_B^G(\theta) = \mathrm{Ind}_B^G(w \cdot \theta)$, we can extend the formula above to

$$\mathrm{Ind}_B^G(\theta)(g) = \frac{1}{|W|} \sum_{x_i^{-1} g x_i \in B} \sum_{w \in W} (w \cdot \theta)(x_i^{-1} g x_i).$$

If no conjugate of g is in B, then $\mathrm{Ind}_B^G(\theta)(g) = 0$. Assume now that some conjugate of g is in B, without loss of generality we can even assume that $g \in B$.

So let $g = tu \in B$ for $t \in T, u \in U$. Let x_i be such that $x_i^{-1} g x_i \in B$. Then $x_i^{-1} g x_i = (x_i^{-1} t x_i)(x_i^{-1} u x_i)$ with $x_i^{-1} t x_i \in T$, $x_i^{-1} u x_i \in U$. Also, there is a $w \in W$ such that $w \cdot t = x_i^{-1} t x_i$. We conclude that there is an integer a such that

$$\mathrm{Ind}_B^G(\theta)(g) = a \cdot \sum_{w \in W} (w \cdot \theta)(t)$$

and the statement follows. □

Remark 5.1.4. *In the above proposition, we have described the field $\mathbb{Q}(\chi_T)$. This leaves open the question about the field $\mathbb{Q}(\chi_U)$. It holds that in many cases $\mathbb{Q}(\chi_U) = \mathbb{Q}$, see also [TZ04] for more information.*

Lemma 5.1.5. *Let $\chi \in \mathrm{Irr}^+(G)$. The following hold:*

(i) Both $\mathbb{Q}(\chi_T)$ *and* $\mathbb{Q}(\chi_U)$ *are subfields of* $\mathbb{Q}(\chi)$*.*

(ii) $\mathrm{Res}_U^B(\chi_U)$ *is orthogonally stable.*

Proof. Assume for a contradiction that $\mathbb{Q}(\chi_T) \not\subseteq \mathbb{Q}(\chi)$. Since $\mathbb{Q}(\chi_T + \chi_U) \subseteq \mathbb{Q}(\chi)$, there have to be irreducible constituents ϕ_T of χ_T and ϕ_U of χ_U such that ϕ_T and ϕ_U are Galois-conjugates over $\mathrm{Gal}\left(\mathbb{Q}(\chi_T, \mathrm{Res}_B^G(\chi))/\mathbb{Q}(\mathrm{Res}_B^G(\chi))\right)$, which is absurd since

$$\phi_U(t) = \phi_U(1) \in \mathbb{Q}$$

for all $t \in T$. This settles for first statement.

For the second statement, recall that for any p-group with p odd, all irreducible characters except the trivial character have Frobenius–Schur indicator "0". Since $\mathbb{Q}(\mathrm{Res}_U^G(\chi_U)) \subseteq \mathbb{R}$ and

$$\langle \mathrm{Res}_U^G(\chi_U)), \mathbf{1}_U \rangle_U = 0$$

by definition, it is orthogonally stable. □

Corollary 5.1.6. *Let* $\chi \in \mathrm{Irr}_B^+(G)$*. Then*

$$\det(\chi) = \det(\chi_T) \cdot \det(\chi_U) \cdot (\mathbb{Q}(\chi)^\times)^2.$$

If further $\mathbb{Q}(\chi_U) = \mathbb{Q}$ *and* $q - 1 \mid \chi_U(1)$ *for* q *a power of* p*, then by Corollary 2.3.17*

$$\det(\chi) = \det(\chi_T) \cdot q^{\chi_U(1)/(q-1)} \cdot (\mathbb{Q}(\chi)^\times)^2.$$

Almost all $\mathrm{Irr}^+(G)$-characters are Borel-stable. We can make precise when a character is not:

Proposition 5.1.7. *Let* $\chi \in \mathrm{Irr}^+(G)$*. Then* $\chi \in \mathrm{Irr}_B^+(G)$ *if and only if* $\chi \notin \mathrm{Simp}(G|(T, \theta))$ *for some* $\theta \in \mathrm{Irr}(T)$ *of order at most* 2*.*

Proof. Assume that $\chi \in \mathrm{Irr}_B^+(G)$. Thus $\chi_T + \chi_U$ is orthogonally stable. By Lemma 5.1.5 also χ_T is orthogonally stable. Since T is an abelian group, all irreducible characters of T have degree 1 and this implies that χ_T does not have any constituents with values in $\mathbb{R}$. The only irreducible characters of any abelian group that have values in $\mathbb{R}$ are characters with values in either $\{1\}$ or $\{-1, 1\}$ and are of order 1 or 2. The other direction is clear. □

Remark 5.1.8. *Note that we can also consider parabolically stable characters, i.e., characters* χ *such that there is a proper parabolic subgroup* $P \subsetneq G$ *with* $\mathrm{Res}_P^G(\chi)$ *orthogonally stable. This is a generalization of being Borel-stable, as there do exist orthogonally stable characters that are parabolically stable but not Borel-stable.*

As an example, regard the group $G = \mathrm{SO}_7(q)$ *for* q *a power of* p*. Let* χ *be the principal series unipotent character of degree* $1/2q(q+1)^2(q^2-q+1)$*. We have that* $\mathbb{Q}(\chi) = \mathbb{Q}$ *and that* $\chi \in \mathrm{Irr}^+(G)$*. It is clear that* χ *is not Borel-stable by Proposition 5.1.7. By the Dynkin diagram of type* B_3*, we see that* G *has a Levi subgroup of type* A_2*; let* $L \subseteq G$ *be the Levi subgroup isomorphic to* $\mathrm{GL}_3(q)$ *and let* P *be a parabolic subgroup containing* L*.*

Accordingly, we have a Levi decomposition $P = U \rtimes L$ *for a* p*-group* U*. We have the Harish-Chandra restriction* χ_L *and as in Definition 5.1.1 the character* χ_U *of* P *such that*

$$\mathrm{Res}_P^G(\chi) = \chi_L + \chi_U,$$

where we regard χ_L *as a character of* P *by inflation.*

With the computer algebra system CHEVIE [Mic15] we see that $\chi_L(1) = q(q+1)$ *and so it corresponds to a unipotent* $\mathrm{Irr}^+(L)$*-character. In Lemma 5.3.6 we will see that* $\det(\chi_L) = (q^2+q+1)\cdot(\mathbb{Q}^\times)^2$.

We have that $\chi_U(1) = 1/2q(q-1)(q+1)(q^2+q+1)$*. This character has orthogonally stable restriction to* U*, so Corollary 2.3.17 now gives us*

$$\det(\chi_U) = \begin{cases} q\cdot(\mathbb{Q}^\times)^2, & \textit{if } q \equiv 1 \mod 4, \\ 1\cdot(\mathbb{Q}^\times)^2, & \textit{if } q \equiv 3 \mod 4. \end{cases}$$

All in all we now arrive at

$$\det(\chi) = \det(\chi_L)\det(\chi_U) = \begin{cases} q(q^2+q+1)\cdot(\mathbb{Q}^\times)^2, & \textit{if } q \equiv 1 \mod 4, \\ (q^2+q+1)\cdot(\mathbb{Q}^\times)^2, & \textit{if } q \equiv 3 \mod 4. \end{cases}$$

There is another way we can arrive at the result: We have that $\mathrm{G}_2(q) \subseteq \mathrm{SO}_7(q)$ *and* $\mathrm{Res}^G_{\mathrm{G}_2(q)}(\chi)$ *remains an orthogonally stable character; in fact it is again a principal series unipotent character which is denoted by* X_{15} *in [His90]. Later in Subsection 5.3.3 we will rediscover the result for* $\mathrm{G}_2(q)$ *with different methods.*

5.2 Orthogonal Determinants of Non-Borel-Stable Characters

We fix an odd prime p. Let $\mathbf{G}$ be a connected reductive group over $\overline{\mathbb{F}}_p$ and let F be Frobenius root. We will, unless otherwise stated, always assume that the pair $(\mathbf{G}, F)$ is split and that $\mathbf{G}$ is simple. Let $\mathbf{T} \subseteq \mathbf{B}$ be a F-stable maximal torus contained in a F-stable Borel subgroup, $\mathbf{N} = \mathbf{N_G}(\mathbf{T})$ and let $(W, S) = W(\mathbf{T}) = \mathbf{N}/\mathbf{T}$ be the Weyl group. Let $G = \mathbf{G}^F$ be the corresponding finite group of Lie type, with Borel subgroup $B^F = U \rtimes T^F$ and the unipotent radical U. Let $N = \mathbf{N}^F$. Then $N/T \cong W$ as $(\mathbf{G}, F)$ is split. Let q be the integer power of p associated to G. We fix some generator ε_1 of $\mathbb{G}_m(q)$.

In the last section we have seen how to calculate the orthogonal determinants of all characters $\chi \in \mathrm{Irr}_B^+(G)$, i.e., all irreducible orthogonally stable characters such that the restriction to B stays orthogonally stable. This leaves the question on how to calculate the orthogonal determinants of the characters $\chi \in \mathrm{Irr}^+(G)\backslash\mathrm{Irr}_B^+(G)$. We will fix such a character χ.

By Proposition 5.1.7, there is a character $\theta \in \mathrm{Irr}(T)$ of order 1 or 2 such that χ appears in $\mathrm{Ind}_B^G(\theta)$. First assume that $\theta = \mathbf{1}_T$. Thus, $\chi \in \mathrm{Irr}_{\mathrm{PSU}}(G)$. We let $A := \mathbb{Q}[u, u^{-1}]$ for u an indeterminate and let $K = \mathbb{Q}(u)$ be the quotient field of A. We arrive at the following important theorem:

Theorem 5.2.1. *Let $\mathcal{H}_G$ be the generic Iwahori–Hecke algebra of G over A. Assume that $\mathbb{Q}(\chi) = \mathbb{Q}$ and that K is a splitting field for $K\mathcal{H}_G$. Denote $\chi' \in \mathrm{Irr}^+(K\mathcal{H}_G)$ to be the character associated to χ. We choose a squarefree $d_{\chi'}(u) \in A$ such that*

$$\det(\chi') = d_{\chi'}(u) \cdot (K^\times)^2.$$

Then

$$\det(\chi) = d_{\chi'}(q) \cdot q^{\chi_U(1)/(q-1)} \cdot (\mathbb{Q}^\times)^2.$$

Proof. Let $\mathcal{H}_q := \mathcal{H}(G, B)$ and let χ_q be the character associated to χ. Then Theorem 2.3.20 tells us that

$$\det(\chi) = \det(\chi_q) \cdot \det(\chi_U) \cdot (\mathbb{Q}^\times)^2.$$

Now, Corollary 2.3.17 gives us $\det(\chi_U)$. The rest now follows since $\mathcal{H}_q$ is the specialization of $\mathcal{H}_G$ via the map $\varphi : A \to \mathbb{Q}, u \mapsto q$, and by Theorem 2.2.18. □

Remark 5.2.2. *Note that an analogous statement holds for the case that the pair $(\mathbf{G}, F)$ is not split or $\mathbf{G}$ not simple. We have assumed further that $\mathbb{Q}(\chi) = \mathbb{Q}$. This turns out to hold for almost all $\mathrm{Irr}_{\mathrm{PSU}}(G)$-characters, except some special cases when $\mathbf{G}$ is of type E_7 or E_8, see also [GM20, Corollary 4.5.6] for a precise statement.*

We will now assume that the order of θ is equal to 2. It is clear that

$$\mathrm{Res}^T_{T^2}(\theta) = \mathbf{1}_{T^2}.$$

By Frobenius reciprocity, χ appears in $\mathrm{Ind}^G_{T^2U}(\mathbf{1}_{T^2U})$. Our goal is to have a similar statement as the above theorem to our new situation. We will have to regard extensions of Coxeter groups by certain abelian groups, which also come up in the context of so called pro-p-Iwahori–Hecke algebras in the representation theory of p-adic groups. In [Vig16], the author describes certain generic algebras that generalize both generic Iwahori–Hecke algebras and pro-p-Iwahohi–Hecke algebras. We will only describe a very special case.

Theorem 5.2.3. *(cf. [Vig16, Theorem 4.7]) Let (W, S) be a finite Coxeter system and let Z be a finite abelian group. Let $W(1)$ be an extension of W by Z, i.e., there is a short exact sequence*

$$1 \longrightarrow Z \longrightarrow W(1) \xrightarrow{\gamma} W \longrightarrow 1.$$

By abuse of notation, we define the length function $\ell : W(1) \to \mathbb{Z}$ to be $\ell(w) := \ell(\gamma(w))$ for all $w \in W(1)$. Let $S(1)$ be the preimage of the set S by the map γ. We write $s \sim s'$ for elements $s, s' \in S(1)$ if $\gamma(s)$ and $\gamma(s')$ are conjugate in W.

Assume there are pairs $(a_s, c_s) \in A \times AZ$ for all $s \in S(1)$ with invertible a_s, such that the following holds:

For all $s, s' \in S(1)$ such that $s \sim s'$, i.e., there is an element $w \in W(1)$ such that $ws'w^{-1}s^{-1} \in Z$, and all $t \in Z$, the following conditions hold:

- $a_s = a_{st} = a_{s'}$,

- $c_{st} = c_s t$,
- $w \cdot c_{s'} = c_{ws'w^{-1}}$, *where we take the action of* $W(1)$ *on* AZ *generated by* $w \cdot t := wtw^{-1}$.

Then there is a unique A*-algebra* $\mathcal{H}_u(a_s, c_s)$ *with basis* $\{T_w \mid w \in W(1)\}$ *with the following relations:*

(i) $T_w T_{w'} = T_{ww'}$ *for all* $w, w' \in W(1)$ *such that* $\ell(ww') = \ell(w) + \ell(w')$.

(ii) $T_s^2 = a_s T_{s^2} + c_s T_s$ *for all* $s \in S(1)$, *where we identify an element* $c_s = \sum_{t \in Z} c_s(t)t$ *with* $\sum_{t \in Z} c_s(t) T_t$, *for* $c_s(t) \in A$.

Proposition 5.2.4. *Let* $\mathcal{H}_u(a_s, c_s)$ *be a generic algebra as in the above theorem.*

(i) Define the function $\tau : \mathcal{H}_u(a_s, c_s) \to A$ *by* $\tau(T_1) := 1, \tau(T_w) = 0$ *for all* $1 \neq w \in W(1)$. *Then* $(\mathcal{H}_u(a_s, c_s), \tau)$ *is a symmetric algebra. Furthermore, define* $a_w := a_{s_1} \cdots a_{s_k}$ *if* $\pi(w) = \pi(s_1) \cdots \pi(s_k)$ *is a reduced expression, for* $w \in W(1), s_i \in S(1)$. *Then*

$$\tau(T_w T_{w'}) = \begin{cases} a_w, & \text{if } w^{-1} = w', \\ 0, & \text{if } w^{-1} \neq w'. \end{cases}$$

(ii) With the involution $\dagger$ *on* $\mathcal{H}_u(a_s, c_s)$ *defined by* $T_w^\dagger := T_{w^{-1}}$ *for* $w \in W(1)$, *the algebra* $\mathcal{H}_u(a_s, c_s)$ *becomes a monomial algebra.*

(iii) The algebra $K\mathcal{H}_u(a_s, c_s)$ *is semisimple.*

Proof. The proof in [GP00, Proposition 8.1.1] for the statement (i) for generic Iwahori–Hecke algebras works here with almost no changes. The semisimplicity follows by Tits' Deformation Theorem, since $(\mathcal{H}_u(a_s, c_s), \tau)$ is a deformation of the group algebra $\mathbb{Q}W(1)$, which is clearly semisimple. □

The above proposition thus allows us to talk about orthogonal determinants of these generic algebras.

We will now see that the above setup naturally comes up in our situation. Let $H \subseteq T$ be a normal subgroup of N. Then $B_H := U \rtimes H$ is a subgroup of B; we define $\mathcal{H}_H := \mathcal{H}(G, B_H)$ to be the Hecke-algebra of the subgroup B_H as in Definition 2.3.18. We define the group $W_H := N/H$. Note that there is a short exact sequence

$$1 \longrightarrow T/H \longrightarrow W_H \longrightarrow W \longrightarrow 1.$$

Thus, the group W_H corresponds to the group $W(1)$ from Theorem 5.2.3; let ℓ be the length function of W_H induced by the one of W.

There are two special cases of H that are named in the literature: Of course, when $H = T$, then $B_H = B$ and the algebra $\mathcal{H}_T$ is an Iwahori–Hecke algebra. There is another special case that we want to point out, the other extreme of $H = \{1\}$. Then $B_H = U$ and the algebra $\mathcal{H}(G, U)$ is called a Yokonuma–Hecke algebra. These were introduced in [Yok67] for the case of $\mathbf{G}$ simple and the pair $(\mathbf{G}, F)$ being split, see also [Juy98] and [JK01].

Lemma 5.2.5. *There is a* 1*-to-*1 *correspondence between the elements of* W_H *and double cosets* $B_H\backslash G/B_H$*, given by sending* $w \in W_H$ *to* $B_H w B_H$*.*

Proof. Recall the Bruhat decomposition

$$G = \bigcup_{w \in W} BwB,$$

which is a disjoint union. We will fix a $w \in W$ and some $\dot{w} \in N$ representing w. Let $w_1, \ldots, w_m \in W_H$ be the elements lying above w with respect to the natural projection $W_H \to W$ and let again $\dot{w}_i \in N$ represent w_i.

The result follows if we can show that

$$BwB = \bigcup_{i=1}^{m} B_H w_i B_H$$

and the union is again disjoint. Let $x \in BwB$. Then there are $b_1, b_2 \in B$ such that $x = b_1 \dot{w} b_2$. There is a unique factoring $b_1 = u_1 t_1, b_2 = t_2 u_2$ with $t_1, t_2 \in T$, $u_1, u_2 \in U$. So

$$x = u_1(t_1 \dot{w} t_2)u_2.$$

Since $t_1 \dot{w} t_2 \in N$, there is a unique index i and an element $t_3 \in H$ such that $t_1 \dot{w} t_2 = \dot{w}_i t_3$. So

$$x = u_1 \dot{w}_i t_3 u_2 \in B_H w_i B_H.$$

In the other direction, it is clear that $B_H w_i B_H \subseteq BwB$ and so the equality of the two sets follows.

Let now i, j with $i \neq j$ be two indices; we want to show that $B_H w_i B_H \neq B_H w_j B_H$. Assume for a contradiction that the two double cosets coincide. This is equivalent to there existing some $b = tu \in B_H$ for some $t \in H, u \in U$ such that $\dot{w}_i b \dot{w}_j^{-1} \in B_H$. We consider the projection to H and arrive at

$$\dot{w}_i t \dot{w}_j^{-1} = \dot{w}_i \dot{w}_j^{-1} t \in H.$$

But this is a contradiction since $\dot{w}_i \dot{w}_j^{-1} \in T\backslash H$. □

Lemma 5.2.6. *Let* $H' \subseteq H$ *be another normal subgroup of* N*. Let* $\pi : N/H' \to N/H$ *be the projection map. Then there is a projection map* $\psi : \mathcal{H}_{H'} \to \mathcal{H}_H$*, given by* $T_w \mapsto T_{\pi(w)}$ *for any* $w \in W_{H'}$*.*

Proof. Let

$$e := \frac{1}{|B_H|} \sum_{h \in B_H} h, \; e' := \frac{1}{|B_{H'}|} \sum_{h' \in B_{H'}} h' \in \mathbb{Q}G$$

be the idempotents corresponding to the algebras $\mathcal{H}_H$ and $\mathcal{H}_{H'}$. We claim that the map

$$\mathcal{H}_{H'} \to \mathcal{H}_H, \; e'ge' \mapsto ee'ge'e$$

for $g \in G$ is equal to ψ. This map is clearly a homomorphism of algebras. First, note that $ee' = e'e = e$; this follows since $B_{H'} \subseteq B_H$ and $be = eb$ for all $b \in B_H$. It remains to show that $eT_we = T_{\pi(w)}$ for $w \in W$. By [CR81, Proposition 11.30],

$$T_w = |B_{H'}/\left({}^wB_{H'} \cap B_{H'}\right)|e'we',$$

so it is left to show that

$$|B_{H'}/\left({}^wB_{H'} \cap B_{H'}\right)| = |B_H/\left({}^wB_H \cap B_H\right)|.$$

Since H is normal in N,

$${}^wB_H = {}^w(HU) = {}^wH \rtimes {}^wU = H{}^wU.$$

In particular, it follows that

$$|B_H/\left({}^wB_H \cap B_H\right)| = |U/\left({}^wU \cap U\right)|.$$

An analogous statement holds for for $B_{H'}$. This proves the statement. □

So if we know the algebra structure for the Yokonuma–Hecke algebra $\mathcal{H}_{\{1\}}$, we can deduce the algebra structure of $\mathcal{H}_{T^2}$. We will follow [Juy98] for the description of the Yokonuma algebra, we denote $\mathcal{Y}(G) := \mathcal{H}_1 = \mathcal{H}(G, U)$.

Let Φ be the root system of $\mathbf{G}$ and let Π be the set of simple roots. Recall from Subsection 3.3.1 that we have surjective maps $\phi_\alpha : \mathbf{SL}_2 \to \langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha}\rangle$ for each $\alpha \in \Pi$ and that we have coroots $\alpha^\vee \in Y(\mathbf{T})$ given by

$$\alpha^\vee(x) = \phi_\alpha\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}\right)$$

for $x \in \mathbb{G}_m$. Since $(\mathbf{G}, F)$ is split, there is by restriction a corresponding map ϕ'_α : $\mathrm{SL}_2(q) \to G$. We set

$$s_\alpha := \phi'_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right) \in N.$$

Note that the elements s_α satisfy the braid relations: Let $\alpha, \beta \in \Pi$ and let $m(s_\alpha, s_\beta)$ be the order of the image of $s_\alpha s_\beta$ in W. Then it holds that

$$s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \ldots$$

where we take $m(s_\alpha, s_\beta)$ products on each side.

Define for any $\alpha \in \Pi$ the element

$$I(\alpha) := \sum_{x \in \mathbb{G}_m(q)} T_{\alpha^\vee(x)} \in \mathcal{Y}(G).$$

Theorem 5.2.7. *(cf. [Juy98, Théorème 2.2]) The basis $\{T_w \mid w \in N\}$ for $\mathcal{Y}(G)$ has the following relations:*

(i) $T_wT_{w'} = T_{ww'}$ *for all* $w, w' \in N$ *such that* $\ell(ww') = \ell(w) + \ell(w')$. *In particular,*

$$T_tT_w = T_{tw} = T_{w(w^{-1}tw)} = T_wT_{w^{-1}tw}$$

for $t \in T$.

(ii) $T_{s_\alpha}^2 = qT_{\alpha^\vee(-1)} + T_{s_\alpha}I(\alpha)$ *for all* $\alpha \in \Pi$.

Lemma 5.2.8. *Let* $S(1)$ *be the preimage of the set* S *in* N. *Let* $(a_s, c_s) \in \mathbb{Q} \times \mathbb{Q}T$ *for* $s \in S(1)$ *be the parameters of* $\mathcal{Y}(G)$, *i.e.,*

$$T_s^2 = a_sT_{s^2} + c_sT_s \text{ for } s \in S(1).$$

Then (a_s, c_s) *fulfill the requirements of Theorem 5.2.3.*

Proof. It is clear that $a_s = q$ for all $s \in S(1)$ by Theorem 5.2.7, so there is nothing to show there.

Let now $s \in S, t \in T$. Then

$$T_{st}^2 = T_s^2T_{s^{-1}ts}T_t = qT_{(st)^2} + c_sT_sT_{s^{-1}ts}T_t = qT_{(st)^2} + c_sT_tT_{st},$$

so indeed, $c_st = c_{st}$.

Assume now that $s, s' \in S(1)$ such that $s \sim s'$. We can without loss of generality assume that $s = s_\alpha$, $s' = s_\beta$ for some elements $\alpha, \beta \in \Pi$. Then clearly there is some $w \in N$ such that

$${}^w\langle \mathbf{U}_\beta^F, \mathbf{U}_{-\beta}^F \rangle = \langle \mathbf{U}_\alpha^F, \mathbf{U}_{-\alpha}^F \rangle.$$

Thus $ws_\beta w^{-1} = s_\alpha t$ for some $t \in \langle \mathbf{U}_\alpha^F, \mathbf{U}_{-\alpha}^F \rangle \cap T$. In $\mathbb{Q}T$, we calculate that

$$w \cdot \sum_{x \in \mathbb{G}_m(q)} \beta^\vee(x) = \sum_{x \in \mathbb{G}_m(q)} \alpha^\vee(x),$$

from which it now follows that $w \cdot c_{s'} = c_{ws'w^{-1}}$. □

Putting the above statements together, we have a system of generators and relations for the algebra $\mathcal{H}_{T^2}$, given by the map $\mathcal{Y}(G) \to \mathcal{H}_{T^2}$ generated by the projection $\pi : N \to N/T^2$. The above lemma then later allows us to regard generic versions of the algebras $\mathcal{H}_{T^2}$. We set $s'_\alpha := \pi(s_\alpha)$, $t'_\alpha := \pi(\alpha^\vee(\varepsilon_1))$ for $\alpha \in \Pi$. Let S_{T^2} be the preimage of S in W_{T^2}.

Example 5.2.9. *(i) Let* $n \geq 2$ *be an integer and let* $G = \mathrm{SL}_n(q)$. *In particular,* $\mathbf{G}$ *is simply connected. There are* $n-1$ *simple roots, so* $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$. *We fix an integer* i *with* $1 \leq i \leq n-1$. *We have maps*

$$\phi'_i : \mathrm{SL}_2(q) \to \mathrm{SL}_n(q),$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a & b & \ldots & 0 \\ 0 & 0 & \ldots & c & d & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix},$$

where the entry a is in position (i,i). We have corresponding elements

$$s_1', \dots, s_{n-1}', t_1', \dots, t_{n-1}' \in N/T^2.$$

Define the projection map $\psi : \mathcal{Y}(G) \to \mathcal{H}(G, T^2U)$.

We see that

$$\psi(I(\alpha_i)) = \sum_{x \in \mathbb{G}_m(q)} T_{\pi(\alpha^\vee(x))} = \frac{q-1}{2}\left(T_1 + T_{t_i'}\right).$$

The following discussion depends on the value of $q \mod 4$. We will first assume that $q \equiv 1 \mod 4$. This implies that -1 is a square in $\mathbb{F}_q$, so $\pi(\alpha^\vee(-1)) = 1$. It thus follows that

$$T_{s_i'}^2 = qT_1 + \frac{q-1}{2} T_{s_i'}\left(T_1 + T_{t_i'}\right).$$

In particular, we see that N/T^2 with the generating set $\{t_1's_1', s_1', s_2', \dots, s_{n-1}'\}$ is a Coxeter system of type D_n. So $\mathcal{H}(G, T^2U)$ is a deformation of a Coxeter group of type D_n, just like the Iwahori–Hecke algebra of type D_n with the parameter q. In fact, via Tits' Deformation Theorem, they are isomorphic. They differ in their natural involutions and thus their orthogonal determinants differ in general.

Assume now that $q \equiv 3 \mod 4$. Then -1 is no longer a square and it holds that

$$-1 \equiv \varepsilon_1 \mod (\mathbb{F}_q^\times)^2.$$

So $\pi(\alpha^\vee(-1)) = t_i$ and we get the square relation

$$T_{s_i'}^2 = qT_{t_i'} + \frac{q-1}{2} T_{s_i'}\left(T_1 + T_{t_i'}\right).$$

So N/T^2 is not a Coxeter system anymore. For instance, for $n = 2$, the group N/T^2 is generated by the element s_1' of order 4 and so is isomorphic to the cyclic group C_4.

(ii) Let $n \geq 2$ be an integer and let $G = \mathrm{PGL}_n(q)$. Again, there are $n-1$ simple roots $\Pi = \{\alpha_1, \dots, \alpha_{n-1}\}$. We will argue that $\alpha^\vee(x) \in T^2$ for all $\alpha \in \Pi$ and $x \in \mathbb{G}_m(q)$. It is enough to show it for $\mathrm{PGL}_2(q)$; it holds that

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x^{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix}^2.$$

Thus the algebra $\mathcal{H}_{T^2}$ has the relations given by the braiding relations and the square relations

$$T_{s_i'}^2 = qT_1 + (q-1)T_{s_i'} \text{ for all } 1 \leq i \leq n-1.$$

(iii) We will now regard the group $G = \mathrm{SU}_4(q)$ with the Borel subgroup B and quasi-split torus T defined as in Example 3.4.5. Recall that the underlying connected reductive group is $\mathbf{SL}_4$ with a Frobenius root F where the pair $(\mathbf{SL}_4, F)$ is not split,

so Theorem 5.2.7 can not be applied. Nevertheless, the algebra $\mathcal{H}_{T^2}$ has a nice set of relations and generators.

It holds that the Coxeter group W^F is of type B_2, so we have two simple roots $\Pi = \{\alpha_1, \alpha_2\}$. We define the two homomorphisms

$$\begin{aligned}\phi'_{\alpha_1} &: \mathrm{SL}_2(q^2) \to G,\\ g &\mapsto \mathrm{diag}(g, F(g))\end{aligned}$$

and

$$\begin{aligned}\phi'_{\alpha_2} &: \mathrm{SU}_2(q) \to G,\\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

Let ε_2 be a generator of $\mathbb{G}_m(q^2)$. We define the elements

$$s_1 = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$s_2 := \phi'_{\alpha_2}\left(\begin{pmatrix} 0 & \varepsilon_2^{(q+1)/2} \\ -\varepsilon_2^{-(q+1)/2} & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2^{(q+1)/2} & 0 \\ 0 & -\varepsilon_2^{-(q+1)/2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Further, we define the elements of the torus

$$\begin{aligned}t_1 &:= \phi'_{\alpha_1}\left(\begin{pmatrix} \varepsilon_2 & 0 \\ 0 & \varepsilon_2^{-1} \end{pmatrix}\right),\\ t_2 &:= \phi'_{\alpha_2}\left(\begin{pmatrix} \varepsilon_2^{q+1} & 0 \\ 0 & \varepsilon_2^{-(q+1)} \end{pmatrix}\right).\end{aligned}$$

Define the projection map $\pi : N \to N/T^2$. We regard the elements s_1, s_2, t_1, t_2 as elements in N/T^2. It is clear that they are generators.

We have the following relations for $\mathcal{H}_{T^2}$:

a) *$T_w T_{w'} = T_{ww'}$ for all $w, w' \in N/T^2$ such that $\ell(ww') = \ell(w) + \ell(w')$.*

b) *The square relations are*

$$\begin{aligned}T_{s_1}^2 &= q^2 T_1 + \frac{q^2-1}{2} T_{s_1}(T_1 + T_{t_1}),\\ T_{s_2}^2 &= q T_{s_2^2} + \frac{q-1}{2} T_{s_2}(T_1 + T_{t_2}).\end{aligned}$$

The parameters $\mu_{s_i,s_i,w}$ for $i = 1, 2$ and $w \in N/T^2$ were calculated by easy explicit calculations. Note that $s_2^2 \in T$ is an element of T^2 if and only if -1 is a square in $\mathbb{F}_q$, meaning that the structure of the algebras differ again on whether $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$.

Remark 5.2.10. *Consider the generic algebras $\mathcal{H}_u(a_s, c_s)$ corresponding to the extension of W by T/T^2, i.e., we have a short exact sequence*

$$1 \longrightarrow T/T^2 \longrightarrow W_{T^2} \longrightarrow W \longrightarrow 1.$$

We are looking for suitable choices of the parameters (a_s, c_s) such that the algebras generalize to $\mathcal{H}_{T_2}$. It is enough to explicitly write out $T^2_{s'_\alpha}$ for all $\alpha \in \Pi$.

We will begin by investigating the group structure of W_{T^2}. Since $(\mathbf{G}, F)$ is split, it holds that $T \cong \mathbb{G}_m^r(q)$, where $r = |\Pi|$ is the rank of $\mathbf{G}$. Thus, $T/T^2 \cong C_2^r$.

The main cases for the generic algebra specializing to $\mathcal{H}_{T^2}$ are the following:

(i) Assume that $\alpha^\vee(-1) \in T^2$ for all $\alpha \in \Pi$; in particular, the above short exact sequence splits. For instance, this is the case if $q \equiv 1 \mod 4$ or when the kernel of ϕ'_α is equal to $\{I_2, -I_2\}$ for all $\alpha \in \Pi$.

- *If $t'_\alpha = 1$ for all $\alpha \in \Pi$, we have the square relations*

$$T_s^2 = u + (u-1)T_s \text{ for all } s \in S_{T^2}$$

in $\mathcal{H}_u(a_s, c_s)$.

- *If $t'_\alpha \neq 1$ for all $\alpha \in \Pi$, we have the square relations*

$$T^2_{s'_\alpha} = u + \frac{u-1}{2} T_{s_\alpha}(T_1 + T_{t'_\alpha}) \text{ for all } \alpha \in \Pi$$

in $\mathcal{H}_u(a_s, c_s)$.

(ii) Assume that $\alpha^\vee(-1) \notin T^2$ for all $\alpha \in \Pi$. By Remark 3.3.7, this for instance happens if $\mathbf{G}$ is simple simply connected and $q \equiv 3 \mod 4$, as then the maps ϕ'_α are isomorphisms and -1 is not a square in $\mathbb{F}_q$. Note that then T/T^2 is generated by $\{(s'_\alpha)^2 \mid \alpha \in \Pi\}$. In particular, $t'_\alpha = (s'_\alpha)^2$.

We have the square relations

$$T^2_{s'_\alpha} = uT_{t'_\alpha} + \frac{u-1}{2} T_{s_\alpha}(T_1 + T_{t'_\alpha}) \text{ for all } \alpha \in \Pi$$

in $\mathcal{H}_u(a_s, c_s)$.

Note that if we can calculate the orthogonal determinants of these generic algebras, we can, with the adjusted version of Theorem 5.2.1, finally calculate the orthogonal determinant $\det(\chi)$. Unfortunately, we do not currently know how to calculate these in general. For instance, there are characters $\chi \in \mathrm{Irr}^+(\mathrm{SL}_6(q))$ for which we do not know how to handle the corresponding characters of the associated generic algebra, so further work is needed.

5.3 Examples

After all this theory building, let us finally regard some actual examples where we calculate the orthogonal determinants of some finite groups of Lie type. We will restrict ourselves to groups of small rank, for multiple reasons.

First, the bigger the rank, the less known the character theory of the group in general becomes. In fact, for all of the groups we will regard, the character tables have been completely determined.

Second, and that is a big point, the orthogonal determinants of the (quotients) of the Yokonuma algebras and the Iwahori–Hecke algebras are in general not known, and one would need either new computational or theoretical tools to handle these in general. So for all groups considered here, all the orthogonally stable irreducible characters that are not Borel-stable are already in the principal series and the corresponding Weyl groups of the groups of Lie type are of type either A_n or $I_2(6)$, where explicit formulas for the orthogonal determinants of the Iwahori–Hecke algebras are known.

We will use some common parameters and notation throughout this section. Let p be an odd prime and let q be a power of p. Let ε_1 be a generator of $\mathbb{G}_m(q)$ and ε_2 be a generator of $\mathbb{G}_m(q^2)$, such that

$$\varepsilon_2^{q+1} = \varepsilon_1.$$

Let m be a positive integer. We define the primitive complex m-th root of unity

$$\zeta_m := \exp\left(\frac{2\pi i}{m}\right)$$

and for any integer j the real numbers

$$\vartheta_m^{(j)} := \zeta_m^j + \zeta_m^{-j}.$$

Further, we define for $k \in \{1, 2\}$ the characters $\alpha_{q^k-1} \in \mathrm{Irr}(\mathbb{G}_m(q^k))$ by

$$\alpha_{q^k-1}(\varepsilon_k) = \zeta_{q^k-1}.$$

It is clear that

$$\mathrm{Irr}(\mathbb{G}_m(q^k)) = \{\alpha_{q^k-1}^j \mid 0 \le j \le q^k - 2\}.$$

5.3.1 $\mathrm{SL}_2(q)$

We first want to point out that the results of this subsection are not new. The character table of $\mathrm{SL}_2(q)$ is very well known and can be found in numerous standard books about the representation theory of finite groups. We want to emphasize one particular source [Bon11], which gives an excellent introduction into the representation theory of finite groups of Lie type. For completeness, we will also list the irreducible characters of $\mathrm{SL}_2(q)$, where we will follow the notation of [Bon11, Section 5.3], slightly adjusted to our notation.

The determination of the orthogonal determinants of the characters of $\mathrm{SL}_2(q)$ can already be found in [BN17], where the authors also calculate another invariant of the associated bilinear forms, namely the Clifford invariant.

Let $G = \mathrm{SL}_2(q)$. We let

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in G \right\}$$

be the Borel subgroup of G. Let $B = U \rtimes T$ be the Levi decomposition with

$$T := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in G \right\}$$

the quasi-split torus and

$$U := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G \right\}$$

the unipotent radical. It is clear that $T \cong \mathbb{G}_m(q)$ and thus

$$\mathrm{Irr}(T) = \{\alpha_{q-1}^j \mid 0 \le j \le q-2\}.$$

We set

$$\delta = \begin{cases} 1, & \text{if } q \equiv 1 \mod 4, \\ -1, & \text{if } q \equiv 3 \mod 4. \end{cases}$$

Theorem 5.3.1. *The following gives the character field, Frobenius–Schur indicator, the degree and a character $\theta \in \mathrm{Irr}(T)$ such that χ appears in $\mathrm{Ind}_B^G(\theta)$ (if such a character exists) for all irreducible characters χ of $\mathrm{SL}_2(q)$.*

χ	$\mathbb{Q}(\chi)$	$\iota(\chi)$	*Principal Series*	*Degree*
$\mathbf{1}_G$	$\mathbb{Q}$	1	$\mathbf{1}_T$	1
$\mathbf{St}_G$	$\mathbb{Q}$	1	$\mathbf{1}_T$	q
$R(\alpha_1^k)$, $1 \le k \le \frac{q-3}{2}$	$\mathbb{Q}(\vartheta_{q-1}^{(k)})$	1, *for k even* -1, *for k odd*	α_1^k	$q+1$
$R_\sigma(\mathbf{sgn}_{q-1})$, $\sigma \in \{1,-1\}$	$\mathbb{Q}(\sqrt{\delta q})$	1, *if $q \equiv 1 \mod 4$* 0, *if $q \equiv 3 \mod 4$*	$\alpha_1^{(q-1)/2}$	$\frac{1}{2}(q+1)$
$R'(\alpha_2^{(q-1)j})$, $1 \le j \le \frac{q-1}{2}$	$\mathbb{Q}(\vartheta_{q+1}^{(j)})$	1, *for j even* -1, *for j odd*	*Not in principal series*	$q-1$
$R'_\sigma(\mathbf{sgn}_{q+1})$, $\sigma \in \{1,-1\}$	$\mathbb{Q}(\sqrt{\delta q})$	-1, *if $q \equiv 1 \mod 4$* 0, *if $q \equiv 3 \mod 4$*	*Not in principal series*	$\frac{1}{2}(q-1)$

Proof. For the Schur indicators and character fields regard [Tur01], where these data are explicitly given for all characters of $\mathrm{SL}_n(q)$ for all n and q. □

Remark 5.3.2. *The first two characters in the table above,* $\mathbf{1}_G$ *and* $\mathbf{St}_G$*, are the principal series unipotent characters of* $\mathrm{SL}_2(q)$*. As we have seen in Subsection 3.5.2 they are in a 1-to-1 correspondence with the characters of the corresponding Weyl group of* G*, which in our case is* $\mathfrak{S}_2$*. The character* $\mathbf{St}_G$ *is called the Steinberg character; these characters exist for any finite group of Lie type and play an important role in the theory, see [DM20, Chapter 7] for more information.*

For the characters $\chi = R(\alpha_1^k)$ *it holds that*

$$R(\alpha_1^k) = \mathrm{Ind}_B^G(\alpha_1^k) = \mathrm{Ind}_B^G(\alpha_1^{-k}).$$

By Frobenius reciprocity and in the notation of Section 5.1, we can decompose

$$\mathrm{Res}_B^G(\chi) = \chi_T + \chi_U, \ \chi_T = \alpha_1^k + \alpha_1^{-k}.$$

This will allow us to easily calculate $\det(\chi)$ *in the case of* k *even.*

Similarly, it holds that

$$\mathrm{Ind}_B^G(\alpha_1^{(q-1)/2}) = R_1(\mathbf{sgn}_{q-1}) + R_{-1}(\mathbf{sgn}_{q-1}).$$

The splitting of the induced characters coming from the (certain) sign characters is a general phenomenon, see [Leh73] for more details. While in the case of $\mathrm{SL}_2(q)$ *these characters are not orthogonally stable, there are orthogonally stable characters arising in that way for* $\mathrm{SL}_n(q)$ *for* $n \geq 6$*. For the calculation of the orthogonal determinants of these characters, the calculation of the orthogonal determinants of the (quotients of the) Yokonuma algebras becomes absolutely necessary.*

The last remaining characters are not in the principal series. As we have seen, the orthogonal determinants of these characters are easy to compute as they have orthogonal stable restriction to the unipotent radical U*, so a* p*-group, of* G*.*

We see that $\mathrm{Irr}^+(G)$ consists of the characters $R(\alpha_1^k)$ and $R'(\alpha_2^{(q-1)j})$ for k, j even. We will now give the orthogonal determinants of these characters.

Theorem 5.3.3. *The following table gives the orthogonal determinants of the* $\mathrm{Irr}^+(G)$*-characters:*

χ	*Description*	$\det(\chi) \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$
$R(\alpha_1^k)$, $1 \leq k \leq \frac{q-3}{2}$, k *even*	B*-stable*	$q(2 - \vartheta_{q-1}^{(2k)})$
$R'(\alpha_2^{(q-1)j})$, $1 \leq j \leq \frac{q-1}{2}$, j *even*	U*-stable*	q

Proof. First we see, for instance by a look at the character tables, that for all $\chi \in \mathrm{Irr}^+(G)$ it holds that $\mathbb{Q}(\chi_U) = \mathbb{Q}$. The degree $\chi_U(1)$ is equal to $q - 1$ in all cases, so we end up with

$$\det(\chi_U) = q \cdot (\mathbb{Q}^\times)^2$$

by Corollary 2.3.17. This already settles the orthogonal determinants of the characters $R'(\alpha_2^{(q-1)j})$ for some even j.

Let now $\chi = R(\alpha_1^k)$. By Remark 5.3.2 it holds that $\chi_T = \alpha_1^k + \alpha_1^{-k}$. Now by Corollary 2.3.3 it holds that

$$\det(\chi_T) = (2 - \vartheta_{q-1}^{(2k)}) \cdot (\mathbb{Q}(\vartheta_{q-1}^{(k)})^\times)^2.$$

Since $\det(\chi)$ is the product of $\det(\chi_U)$ and $\det(\chi_T)$ by Corollary 5.1.6, the result now follows. □

5.3.2 $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$

The complete character tables of the groups $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$ were first determined by Simpson and Frame in [SF73]. We will follow their notation. The reason these two different groups were able to be handled at the same time is because of the "Ennola duality": By formally replacing every q with a $-q$, one can (up to some signs) switch from the character table of $\mathrm{SL}_3(q)$ to the one of $\mathrm{SU}_3(q)$ (so, in a way, $\mathrm{SU}_3(q) = \mathrm{SL}_3(-q)$), see also [Enn63] and [Kaw85]. The statement and proof is there only given for $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$, but nevertheless the statement applies here.

The results of this subsection have already appeared in the paper [HN23] by the author of this thesis and Gabriele Nebe. The author of this thesis contributed fully to all mathematical ideas of the abovementioned paper.

We will begin as in the previous subsection by first defining the relevant subgroups. Recall by Example 3.4.5 that we can define the group $\mathrm{SU}_n(q)$ by choosing a suitable Hermitian form on $\mathbb{F}_{q^2}^3$ and letting $\mathrm{SU}_3(q)$ be the subgroup of $\mathrm{SL}_3(q^2)$ that fixes that form. Let $F : \mathbb{F}_{q^2}^{3\times 3} \to \mathbb{F}_{q^2}^{3\times 3}$ be the Frobenius map that applies the map $x \mapsto x^q$ elementwise. We will make the standard choice for the Hermitian form, i.e., we let

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and define

$$\mathrm{SU}_3(q) = \{A \in \mathrm{SL}_3(q^2) \mid F(A)^{tr} \cdot \Omega \cdot A = \Omega\}.$$

Let G be now equal to either $\mathrm{SL}_3(q)$ or $\mathrm{SU}_3(q)$. We let

$$B := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in G \right\}$$

be the Borel subgroup of G. Let $B = U \rtimes T$ be the Levi decomposition with

$$T := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{pmatrix} \in G \right\}$$

the quasi-split torus and

$$U := \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \in G \right\}$$

the unipotent radical.

Unlike the subgroups B, T and U, the Weyl group W does not allow for such a nice uniform description for the two groups. Since $\mathrm{SL}_3(q)$ is of type A_2, its Weyl group is isomorphic to $\mathfrak{S}_3$. As we have seen in Example 3.4.5, the Weyl group of $\mathrm{SU}_3(q)$ is isomorphic to the cyclic group C_2. So we will denote

$$W := \begin{cases} \mathfrak{S}_3, & \text{if } G = \mathrm{SL}_3(q), \\ C_2, & \text{if } G = \mathrm{SU}_3(q). \end{cases}$$

The isomorphism type of T (and therefore its set of irreducible characters) also depends on the particular group, we have

$$T \cong \begin{cases} \mathbb{G}_m^2(q), & \text{if } G = \mathrm{SL}_3(q), \\ \mathbb{G}_m(q^2), & \text{if } G = \mathrm{SU}_3(q). \end{cases}$$

Accordingly,

$$\mathrm{Irr}(T) = \begin{cases} \{\alpha_1^j \boxtimes \alpha_1^k \mid 0 \leq j, k \leq q-2\}, & \text{if } G = \mathrm{SL}_3(q), \\ \{\alpha_2^j \mid 0 \leq j \leq q^2-2\}, & \text{if } G = \mathrm{SU}_3(q). \end{cases}$$

It will be clear from context if we talk about the subgroups of either $\mathrm{SL}_3(q)$ or $\mathrm{SU}_3(q)$.

Theorem 5.3.4. *The following gives the character field, the degree, and a corresponding character of T if the character is in the principal series for all $\chi \in \mathrm{Irr}^+(\mathrm{SL}_3(q))$ and $\psi \in \mathrm{Irr}^+(\mathrm{SU}_3(q))$.*

(i) $G = \mathrm{SL}_3(q)$*:*

χ	$\mathbb{Q}(\chi)$	*Principal Series*	*Degree*
χ_{qs}	$\mathbb{Q}$	$\mathbf{1}_T$	$q(q+1)$
$\chi_{st}^{(k)}$, $1 \leq k \leq \frac{q-3}{2}$, $k \neq \frac{q-1}{3}$	$\mathbb{Q}(\vartheta_{q-1}^{(k)})$	$\alpha_1^k \boxtimes \alpha_1^{-k}$	$(q+1)(q^2+q+1)$
$\chi_{st'}^{(l)}$, $0 \leq l \leq 2$	$\mathbb{Q}$	$\alpha_1^{(q-1)/3} \boxtimes \alpha_1^{2(q-1)/3}$	$\frac{1}{3}(q+1)(q^2+q+1)$
$\chi_{rt}^{((q-1)j)}$, $1 \leq j \leq \frac{q-1}{2}$	$\mathbb{Q}(\vartheta_{q+1}^{(j)})$	*Not in principal series*	$(q-1)(q^2+q+1)$

(ii) $G = \mathrm{SU}_3(q)$*:*

ψ	$\mathbb{Q}(\psi)$	*Principal Series*	*Degree*
$\psi_{st}^{(k)}$, $1 \leq k \leq \frac{q-1}{2}$, $k \neq \frac{q+1}{3}$	$\mathbb{Q}(\vartheta_{q+1}^{(k)})$	*Not in principal series*	$(q-1)(q^2-q+1)$
$\psi_{st'}^{(l)}$, $0 \leq l \leq 2$	$\mathbb{Q}$	*Not in principal series*	$\frac{1}{3}(q-1)(q^2-q+1)$
$\psi_{rt}^{((q+1)j)}$, $1 \leq j \leq \frac{q-3}{2}$	$\mathbb{Q}(\vartheta_{q-1}^{(j)})$	$\alpha_2^{(q+1)j}$	$(q+1)(q^2-q+1)$

Note that the characters $\chi_{st'}^{(l)}$ *(resp.* $\psi_{st'}^{(l)}$*) only exist if* $3 \mid q-1$ *(resp.* $3 \mid q+1$*).*

Remark 5.3.5. *There are three partitions of the number* 3*, namely* $(3), (2,1)$ *and* $(1,1,1)$*. So there are three principal series unipotent characters of* $\mathrm{SL}_3(q)$*, of degrees* 1*,* $q(q+1)$ *and* q^3 *respectively. We will soon use the results in Subsection 4.1.5 to calculate the orthogonal determinant of the character* χ_{qs}*, so keep in mind that that character corresponds to the partition* $(2,1)$*.*

We note that almost all $\mathrm{Irr}^+(G)$*-characters of* $\mathrm{SL}_3(q)$ *correspond to orthogonally stable characters of* $\mathrm{SU}_3(q)$*, with the exception of* χ_{qs}*. There is a corresponding character* ψ_{qs} *of degree* $q(q-1)$*, but it holds that* $\iota(\psi_{qs}) = -1$*. The characters* ψ_{qs} *is one of the smallest examples of a so called cuspidal unipotent character. These also can be constructed from the trivial character of some maximal torus of* $\mathrm{SU}_3(q)$ *— just not a quasi-split one. The construction of these characters is described by Deligne–Lusztig theory, which we do not cover in this thesis.*

Lemma 5.3.6. *It holds that*

$$\det(\chi_{qs}) = (q^2+q+1) \cdot (\mathbb{Q}^\times)^2.$$

Proof. We will give two separate proofs of this fact. The first one already appeared in [HN23, Theorem 4.7] and uses the fact that χ_{qs} comes from a permutation representation. The second proof uses Theorem 5.2.1 and the explicit formula for the orthogonal determinants of generic Iwahori–Hecke algebras of type A_2.

We are in the situation that $G = \mathrm{SL}_3(q)$. Note that G acts double transitively on the projective space $\mathbb{P}(\mathbb{F}_q^3)$, i.e., the set of 1-dimensional subspaces in $\mathbb{F}_q^3$. It is not hard to see that $|\mathbb{P}(\mathbb{F}_q^3)| = q^2+q+1$. Let

$$V = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle \in \mathbb{P}(\mathbb{F}_q^3).$$

The stabilizer of V is given by the parabolic subgroup

$$P := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \in G \right\}.$$

Let $M_{(2,1)}$ be the $\mathbb{Q}$-representation of G with the basis given by the set $\mathbb{P}(\mathbb{F}_q^3)$ and the natural action of G on the basis elements. Then the induced character $\mathrm{Ind}_P^G(\mathbf{1}_P)$ equals the character of the representation $M_{(2,1)}$. As the action is double transitive, the induced character consists of exactly two irreducible ones and we can decompose

$$\mathrm{Ind}_P^G(\mathbf{1}_P) = \mathbf{1}_G + \chi_{qs}.$$

We have a suitable G-invariant bilinear form β on $M_{(2,1)}$ by letting the set $\mathbb{P}(\mathbb{F}_q^3)$ be an orthonormal basis. We have a one-dimensional subrepresentation with corresponding determinant given by

$$V_1 := \langle \sum_{Q \in \mathbb{P}(\mathbb{F}_q^3)} Q \rangle, \ \det(\beta_{|V_1}) = (q^2+q+1) \cdot (\mathbb{Q}^\times)^2.$$

Thus χ_{qs} is the character of the orthogonal complement $V_1^\perp$ and we calculate that

$$\det(\chi_{qs}) = \det(\beta_{|V_1^\perp}) = \det(\beta)\det(\beta_{|V_1}) = (q^2+q+1) \cdot (\mathbb{Q}^\times)^2.$$

We will now calculate the same result with Theorem 5.2.1. Let $\lambda = (2,1)$ be the partition corresponding to the character χ_{qs}. Let $A := \mathbb{Q}[u, u^{-1}]$ and let $\mathcal{H} := \mathcal{H}_G$ be the generic Iwahori–Hecke algebra of type A_2 over A. Let $K = \mathbb{Q}(u)$ be the quotient field of A and let $\chi' \in \mathrm{Irr}^+(K\mathcal{H})$ be the corresponding irreducible character of $K\mathcal{H}$. There are two standard Young tableaux of λ,

$$t_1 := t_\lambda = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}\, , t_2 := \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}\, .$$

With Theorem 4.1.65 we calculate that

$$\det(\chi') = u(u^2+u+1) \cdot (K^\times)^2.$$

So we now choose $d_{\chi'}(u) := u(u^2+u+1) \in A$ to be the squarefree representative of the orthogonal determinant.

Let us now quickly handle $\det(\chi_U)$. We know by Lemma 5.1.5 that $\mathrm{Res}_U^B(\chi_U)$ is orthogonally stable and has character field equal to $\mathbb{Q}$. Since the degree of χ' is equal to 2, it follows that

$$\det(\chi_U) = q^{((q(q+1)-2)/(q-1)} \cdot (\mathbb{Q}^\times)^2 = q^{q+2} \cdot (\mathbb{Q}^\times)^2 = q \cdot (\mathbb{Q}^\times)^2.$$

We now conclude that

$$\det(\chi_{qs}) = d_{\chi'}(q)\det(\chi_U) \cdot (\mathbb{Q}^\times)^2 = (q^2+q+1) \cdot (\mathbb{Q}^\times)^2.$$

□

Theorem 5.3.7. *The following tables give the orthogonal determinants of all* $\mathrm{Irr}^+(G)$*-characters for* $G = \mathrm{SL}_3(q)$ *and* $\mathrm{SU}_3(q)$*.*

(i) $G = \mathrm{SL}_3(q)$*:*

χ	*Description*	$\det(\chi) \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$
χ_{qs}	*unipotent*	q^2+q+1
$\chi_{st}^{(k)}$	B*-stable*	$q(2-\vartheta_{q-1}^{(2k)})$
$\chi_{st'}^{(l)}$	B*-stable*	$3q$
$\chi_{rt}^{((q-1)j)}$	U*-stable*	q

(ii) $G = \mathrm{SU}_3(q)$*:*

ψ	*Description*	$\det(\psi) \in \mathbb{Q}(\psi)^\times/(\mathbb{Q}(\psi)^\times)^2$
$\psi_{st}^{(k)}$	U*-stable*	q
$\psi_{st'}^{(l)}$	U*-stable*	q
$\psi_{rt}^{((q+1)j)}$	B*-stable*	$q(2-\vartheta_{q-1}^{(2j)})$

Proof. Since we have already handled the character χ_{qs}, the rest of the characters are Borel-stable and we can apply the results of Section 5.1. So let χ be any of the other remaining characters appearing in the tables in Theorem 5.3.4. Either by looking at the character tables or by [TZ04, Theorem 1.9], it holds that $\mathbb{Q}(\chi_U) = \mathbb{Q}$, allowing us to easily calculate $\det(\chi_U)$ with Corollary 2.3.17, once we know its degree. This on the other hand is easily done with Proposition 5.1.3.

Let us now assume that

$$\chi \in \{\chi_{st}^{(k)}, \chi_{st'}^{(l)}, \psi_{rt}^{((q+1)j)}\},$$

i.e., χ_T is nonzero and orthogonally stable.

If $\chi = \chi_{st}^{(k)}$ for some k, then

$$\chi_T = (\alpha_1^k \boxtimes \alpha_1^{-k} + \alpha_1^{-k} \boxtimes \alpha_1^k) + (\alpha_1^{2k} \boxtimes \alpha_1^k + \alpha_1^{-2k} \boxtimes \alpha_1^{-k}) + (\alpha_1^k \boxtimes \alpha_1^{2k} + \alpha_1^{-k} \boxtimes \alpha_1^{-2k}).$$

With Lemma 2.3.5 and Corollary 2.3.3, we compute

$$\det(\chi_T) = (2-\vartheta_{q-1}^{(2j)}) \cdot (\mathbb{Q}(\vartheta_{q-1}^{(k)})^\times)^2.$$

So let now $\chi = \chi_{st'}^{(l)}$ for some l. Then

$$\chi_T = \alpha_1^{(q-1)/3} \boxtimes \alpha_1^{2(q-1)/3} + \alpha_1^{2(q-1)/3} \boxtimes \alpha_1^{(q-1)/3}.$$

So by Corollary 2.3.3,

$$\det(\chi_T) = (2-\vartheta_{q-1}^{(2(q-1)/3)}) \cdot (\mathbb{Q}^\times)^2 = (2-(-1)) \cdot (\mathbb{Q}^\times)^2 = 3 \cdot (\mathbb{Q}^\times)^2.$$

Finally, let $\chi = \psi_{rt}^{((q+1)j)}$ for some j. Then

$$\chi_T = \alpha_2^{(q+1)j} + \alpha_2^{-(q+1)j}$$

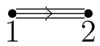
and again by Corollary 2.3.3,

$$\det(\chi_T) = (2 - \vartheta_{q-1}^{(2j)}) \cdot (\mathbb{Q}(\vartheta_{q-1}^{(j)})^\times)^2.$$

This finishes the proof. □

5.3.3 $\mathrm{G}_2(q)$

Let p be an odd prime and let q be a power of p. Recall that $\mathbf{G}_2 \subseteq \mathbf{SO}_7$ over $\overline{\mathbb{F}}_p$ is a simple reductive group with Dynkin diagram



and corresponding Coxeter group

$$W := I_2(6) = \langle w_a, w_b \mid w_a^2 = w_b^2 = 1, (w_a w_b)^6 = 1 \rangle.$$

Let $F_q : \mathbf{G}_2 \to \mathbf{G}_2$ be the standard Frobenius map. Then

$$G := \mathrm{G}_2(q) = \mathbf{G}_2^{F_q}$$

is the corresponding finite group of Lie type. In particular, the pair $(\mathbf{G}_2, F_q)$ is split.

The ordinary character tables of G are known for all possible values of q, by Chang and Ree for $p > 3$ in [CR74] and Enomoto for $p = 3$ in [Eno76]. We will also refer to the habilitation thesis of Hiss [His90] where the character tables for $p > 3$, as well as many other useful pieces of information, are given. In the sequel, we will use the notation used in Hiss' thesis for the irreducible characters of G.

As usual, we let $B = U \rtimes T \subseteq G$ be a Borel subgroup with unipotent radical U and quasi-split torus T. The rank of $\mathbf{G}_2$ is equal to 2, so

$$T \cong \mathbb{G}_m^2(q), \ \mathrm{Irr}(T) = \{\alpha_1^j \boxtimes \alpha_1^k \mid 0 \leq j, k \leq q-2\}.$$

The action of W on $\mathrm{Irr}(T)$ is explicitly given in [His90, Anhang A.3], it holds that

$$\begin{aligned} w_a \cdot (\alpha_1^j \boxtimes \alpha_1^k) &= \alpha_1^j \boxtimes \alpha_1^{j-k}, \\ w_b \cdot (\alpha_1^j \boxtimes \alpha_1^k) &= \alpha_1^k \boxtimes \alpha_1^j. \end{aligned}$$

From the available character tables, we see that there are only two characters that do not have real values, namely $X_{19}(k)$ for $k = 1, 2$. Furthermore, the Schur indices of all characters of G is equal to 1, see [Ohm85]. In particular, the $\mathrm{Irr}^+(G)$ characters are very easily determined, as these are all characters of even degree that are not equal to $X_{19}(k)$.

As with the characters in the previous subsections, the characters of G also come with parameters. For the orthogonal determinants, the values of the parameters mostly don't matter, so we decided to only include these when necessary. For the explicit conditions on the parameters, we refer to [His90, Anhang B.1]. We also decided to omit the character fields.

For a character $\chi \in \mathrm{Irr}^+(G)$, we again decompose

$$\mathrm{Res}_B^G(\chi) = \chi_T + \chi_U,$$

as in Definition 5.1.1.

Theorem 5.3.8. *The following gives information about the principal series and degree of all $\chi \in \mathrm{Irr}^+(G)$-characters.*

χ	q	$\chi_T(1)$	*Principal series*	$\chi(1)$
X_{15}	*all*	2	$\mathbf{1}_T$	$1/2q(q+1)^2(q^2-q+1)$
X_{16}	*all*	2	$\mathbf{1}_T$	$1/6q(q+1)^2(q^2+q+1)$
$X_{1a}(k)$ $X'_{1a}(k)$	*all*	6	$\alpha_1^k \boxtimes \alpha_1^{-k}$	$q(q+1)(q^2+q+1)(q^2-q+1)$ $(q+1)(q^2+q+1)(q^2-q+1)$
$X_{1b}(k)$ $X'_{1b}(k)$	*all*	6	$\alpha_1^k \boxtimes \mathbf{1}_T$	$q(q+1)(q^2+q+1)(q^2-q+1)$ $(q+1)(q^2+q+1)(q^2-q+1)$
$X_1(j,k)$	*all*	12	$\alpha_1^j \boxtimes \alpha_1^k$	$(q+1)(q^2+q+1)(q^3+1)$
X_{31}	$3 \mid q+1$ $3 \mid q-1$	0 2	 $\alpha_1^{(q-1)/3} \boxtimes \alpha_1^{(q-1)/3}$	$q^3(q-1)(q^2+q+1)$ $q^3(q+1)(q^2-q+1)$
X_{32}	$3 \mid q+1$ $3 \mid q-1$	0 2	 $\alpha_1^{(q-1)/3} \boxtimes \alpha_1^{(q-1)/3}$	$(q-1)(q^2+q+1)$ $(q+1)(q^2-q+1)$
X_{33}	$3 \mid q+1$ $3 \mid q-1$	0 4	 $\alpha_1^{(q-1)/3} \boxtimes \alpha_1^{(q-1)/3}$	$q(q-1)^2(q^2+q+1)$ $q(q+1)^2(q^2-q+1)$
X_2	*all*	0		$(q-1)(q^2-q+1)(q^3-1)$
X_{2a}, X_{2b}	*all*	0		$q(q-1)(q^2+q+1)(q^2-q+1)$
X'_{2a}, X'_{2b}	*all*	0		$(q-1)(q^2+q+1)(q^2-q+1)$
X_3	*all*	0		$(q-1)(q^2-1)(q^3+1)$
X_6	*all*	0		$(q+1)(q^2-1)(q^3-1)$
X_a, X_b	*all*	0		q^6-1
X_{17}	*all*	0		$1/2q(q-1)^2(q^2+q+1)$
X_{18}	*all*	0		$1/6q(q-1)^2(q^2-q+1)$

We will now calculate the orthogonal determinants of the characters above.

Lemma 5.3.9. *The following hold:*

$$\det(\chi_{15}) = \begin{cases} q(q^2+q+1)\cdot(\mathbb{Q}^\times)^2, & \text{if } q \equiv 1 \mod 4, \\ (q^2+q+1)\cdot(\mathbb{Q}^\times)^2, & \text{if } q \equiv 3 \mod 4, \end{cases}$$
$$\det(\chi_{16}) = \begin{cases} 3q(q^2-q+1)\cdot(\mathbb{Q}^\times)^2, & \text{if } q \equiv 1 \mod 4, \\ 3(q^2-q+1)\cdot(\mathbb{Q}^\times)^2, & \text{if } q \equiv 3 \mod 4. \end{cases}$$

Proof. It holds that $X_{15}, X_{16} \in \mathrm{Irr}_{\mathrm{PSU}}(G)$ and both characters have rational character fields. So Theorem 5.2.1 allows us to calculate the orthogonal determinant. So let now χ be either χ_{15} or χ_{16}.

We will first calculate $\det(\chi_U)$. The only information we need is the degree of χ_U, which we get by Theorem 5.3.8. A quick calculation now leaves us with

$$\det(\chi_U) = \begin{cases} 1\cdot(\mathbb{Q}^\times)^2, & \text{if } q \equiv 1 \mod 4, \\ q\cdot(\mathbb{Q}^\times)^2, & \text{if } q \equiv 3 \mod 4. \end{cases}$$

Let $A := \mathbb{Q}[u, u^{-1}]$ and let $\mathcal{H} := \mathcal{H}_G$ be the generic Iwahori–Hecke algebra of type $I_2(6)$ over A. Let $\chi' \in \mathrm{Irr}^+(\mathbb{Q}(u)\mathcal{H})$ be the character corresponding to χ. Recall from Proposition 4.3.2 that there are two $\mathrm{Irr}^+(\mathbb{Q}(u)\mathcal{H})$-characters, namely χ_1' and χ_2'. We can for instance use CHEVIE (cf. [Mic15]) to see that χ_2' (resp. χ_1') corresponds to the character χ_{15} (resp. χ_{16}).

By Theorem 4.3.3, we get that

$$\det(\chi_2') = u(u^2+u+1)\cdot(K^\times)^2,$$
$$\det(\chi_1') = 3u(u^2-u+1)\cdot(K^\times)^2.$$

We thus choose the evident representatives

$$d_{\chi_2'}(u) = u(u^2+u+1),$$
$$d_{\chi_1'}(u) = 3u(u^2-u+1)$$

in A. The statement now easily follows with Theorem 5.2.1. □

Theorem 5.3.10. *The following table gives the orthogonal determinants of the* $\mathrm{Irr}^+(G)$*-characters:*

χ	*Description*	q	$\det(\chi) \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$
X_{15}	*unipotent*	$q \equiv 1 \mod 4$ $q \equiv 3 \mod 4$	$q(q^2+q+1)$ q^2+q+1
X_{16}	*unipotent*	$q \equiv 1 \mod 4$ $q \equiv 3 \mod 4$	$3q(q^2-q+1)$ $3(q^2-q+1)$
$X_{1a}(k), X'_{1a}(k),$ $X_{1b}(k), X'_{1b}(k)$	*B-stable*	*all*	$q(2-\vartheta_{q-1}^{(2k)})$
X_1	*B-stable*	*all*	1
X_{31}, X_{32}	*U-stable* *B-stable*	$q \equiv -1 \mod 3$ $q \equiv 1 \mod 3$	q $3q$
X_{33}	*U-stable* *B-stable*	$q \equiv -1 \mod 3$ $q \equiv 1 \mod 3$	1 1
$X_2, X_3, X_6,$ X_a, X_b	*U-stable*	*all*	1
$X_{2a}, X'_{2a},$ X_{2b}, X'_{2b}	*U-stable*	*all*	q
X_{17}, X_{18}	*U-stable*	$q \equiv 1 \mod 4$ $q \equiv 3 \mod 4$	1 q

Proof. Let χ be any of the remaining characters, i.e., $\chi \in \mathrm{Irr}^+(G)$ and χ is not equal to χ_{15} or χ_{16}. Then χ is Borel-stable, it further holds that $\mathbb{Q}(\chi_U) = \mathbb{Q}$ and that $q-1 \mid \chi_U(1)$. By Corollary 5.1.6 it holds that

$$\det(\chi) = \det(\chi_T) q^{\chi_U/(q-1)} \cdot (\mathbb{Q}(\chi)^\times)^2.$$

This already solves the cases of χ being U-stable.

We will now assume that $\chi_T(1) \neq 0$. By Proposition 5.1.3 it follows that

$$\chi_T = \sum_{j=1}^{\chi_T(1)/2} \theta_j + \overline{\theta_j}$$

for certain characters $\theta_j \in \mathrm{Irr}(T)$ and all irreducible characters in that sum are in the same W-orbit. Since $\mathbb{Q}(w \cdot \theta_j) = \mathbb{Q}(\theta_j)$ for all $w \in W$, it follows by Lemma 2.3.5 that

$$\det(\chi_T) = \det\left(\theta_j + \overline{\theta_j}\right)^{\chi_T(1)/2} \cdot (\mathbb{Q}(\chi_T)^\times)^2$$

for any j. In particular, if $4 \mid \chi_T(1)$, then

$$\det(\chi_T) = 1 \cdot (\mathbb{Q}(\chi_T)^\times)^2.$$

The rest of the theorem now follows by Corollary 2.3.3 and Theorem 5.3.8. □

6 Orthogonal Determinants of $\mathrm{GL}_n(q)$

This final chapter is in a way a continuation of Section 5.3, as we are describing the orthogonal determinants of the general linear groups. We will begin by describing the representation theory of these groups; many notions here are just generalizations of the representation theory of the symmetric groups. We will then see how every possible case we have talked about in Chapter 5 does occur here. Since the general linear groups have a particular easy structure, we will be able to give a complete solution. Further, we are able to show that Parker's conjecture holds for the general linear groups — this can yet again be seen as a corollary of the statement for the symmetric groups in Theorem 4.1.50. At the end of this chapter we will regard the example of $\mathrm{GL}_4(q)$.

6.1 Representation Theory of the General Linear Groups

Let p be a prime and q be a power of p. Let n be a positive integer. The general linear groups over a finite field are in a way the easiest groups of Lie type; in contrary to most groups of Lie type, the characters of the groups $\mathrm{GL}_n(q)$ have been fully determined by Green in 1955 in [Gre55]. For a more modern approach, also regard [Mac98]. We will not need the full construction of all irreducible characters, so we will just recall the for our purposes necessary information like the character fields and the Harish-Chandra series the irreducible characters belong to. For this, we will follow [Leh73] and [Tur01].

Let us introduce some notation. We define generators ε_d of $\mathbb{G}_m(q^d)$ for any positive integer d such that for any $d'|d$,

$$\varepsilon_d^{[d]_q/[d']_q} = \varepsilon_{d'}.$$

where $[d]_q$ and $[d']_q$ are the Gaussian polynomials as in Definition 4.1.64. We define the complex numbers

$$\zeta_m := \exp\left(\frac{2\pi i}{m}\right)$$

for any positive integer m and

$$\vartheta_m^{(j)} := \zeta_m^j + \zeta_m^{-j} \subseteq \mathbb{R}$$

for any integer j. We define the element $\alpha_d \in \mathrm{Irr}(\mathbb{G}_m(q^d))$ by

$$\alpha_d(\varepsilon_d) := \zeta_{q^d-1}.$$

It is clear that

$$\mathrm{Irr}(\mathbb{G}_m(q^d)) = \{\mathbf{1}_d, \alpha_d, \alpha_d^2, \dots, \alpha_d^{q^d-2}\},$$

where $\mathbf{1}_d$ is the trivial character of $\mathbb{G}_m(q^d)$. Further, we define

$$\sigma_k : \mathrm{Irr}(\mathbb{G}_m(q^d)) \to \mathrm{Irr}(\mathbb{G}_m(q^d)),\ \sigma_k(\theta) = \theta^k$$

for any integer k. We will now define an equivalence relation on $\mathrm{Irr}(\mathbb{G}_m(q^d))$.

Definition 6.1.1. *We say that two characters $\theta, \theta' \in \mathrm{Irr}(\mathbb{G}_m(q^d))$ are conjugate, if there is an integer k such that $\sigma_{qk}(\theta) = \theta'$. A d-simplex s is a conjugacy class of size d in $\mathrm{Irr}(\mathbb{G}_m(q^d))$. If $\theta \in s$, we also write $s = \langle\theta\rangle$. The degree of a d-simplex s is $d(s) = d$. We let $\mathscr{G}_d$ be the union of all d-simplexes and $\mathscr{G} = \cup_{d=1}^{\infty}\mathscr{G}_d$.*

Example 6.1.2. *Assume that $q = 5$ and $d = 4$. Then*

$$\langle\alpha_4\rangle = \{\alpha_4, \alpha_4^5, \alpha_4^{25}, \alpha_4^{125}\}$$

is a 4*-simplex. On the other hand,*

$$\langle\alpha_4^{26}\rangle = \{\alpha_4^{26}, \alpha_4^{130}\}$$

is not a 4*-simplex.*

It is not hard to see which elements in $\mathrm{Irr}(\mathbb{G}_m(q^d))$ give rise to d-simplexes. Indeed, $\langle\alpha_d^k\rangle$ is a d-simplex if and only if for all $d'|d$ with $d' < d$ we have that

$$\frac{[d]_q}{[d']_q} \nmid k.$$

Definition 6.1.3. *Let $\mathscr{F}$ to be the set of functions $\lambda : \mathscr{G} \to \mathscr{P}$ such that $\lambda(\theta) = (0)$ for almost all $\theta \in \mathscr{G}$ and $\lambda(\sigma_q(\theta')) = \lambda(\theta')$ for all $\theta' \in \mathscr{G}$, i.e., we require that λ is constant on conjugacy classes. The degree of $\lambda \in \mathscr{F}$ is defined to be*

$$\deg(\lambda) = \sum_{\theta\in\mathscr{G}} |\lambda(\theta)|.$$

We define $\mathscr{F}_n$ to be set of $\lambda \in \mathscr{F}$ of degree n.

Theorem 6.1.4. *(cf. [Tur01, Theorem 2.2]) For any positive integer n, there is a bijection $\lambda \mapsto \chi_\lambda$ between $\mathscr{F}_n$ and $\mathrm{Irr}(\mathrm{GL}_n(q))$.*

Remark 6.1.5. *Much more can be said, in the following we will describe how this bijection describes the Harish-Chandra theory of $\mathrm{GL}_n(q)$ for some positive integer n. In particular, we will see that many characters can be constructed by induction from parabolic subgroups, allowing inductive reasoning.*

For $\lambda \in \mathscr{F}$, we write

$$\lambda = (\langle\theta_1\rangle^{\lambda(\theta_1)}, \langle\theta_2\rangle^{\lambda(\theta_2)}, \dots, \langle\theta_k\rangle^{\lambda(\theta_k)})$$

where the $\langle\theta_i\rangle$ are all the pairwise disjoint non-empty simplices with $\lambda(\theta_i) \neq (0)$. Denote

$$l_i := d(\langle\theta_i\rangle)|\lambda(\theta_i)|.$$

By the bijection, each $\langle\theta_i\rangle^{\lambda(\theta_i)}$ *describes an irreducible character of* $\mathrm{GL}_{l_i}(q)$*. We define the Levi subgroup*

$$L_\lambda := \mathrm{GL}_{l_1}(q) \times \mathrm{GL}_{l_2}(q) \times \cdots \times \mathrm{GL}_{l_k}(q) \subseteq \mathrm{GL}_n(q);$$

we get an irreducible character

$$\chi_\lambda^{L_\lambda} := \chi_{(\langle\theta_1\rangle^{\lambda(\theta_1)})} \boxtimes \chi_{(\langle\theta_2\rangle^{\lambda(\theta_2)})} \boxtimes \cdots \boxtimes \chi_{(\langle\theta_k\rangle^{\lambda(\theta_k)})}.$$

Let P_λ *be a parabolic subgroup of* $\mathrm{GL}_n(q)$ *containing* L_λ*. Then*

$$\chi_\lambda = R_{L_\lambda}^{\mathrm{GL}_n(q)}(\chi_\lambda^{L_\lambda}) = \mathrm{Ind}_{P_\lambda}^{\mathrm{GL}_n(q)}(\chi_\lambda^{L_\lambda}).$$

Further, a character χ_λ *is in the principal series if and only if* $d(\langle\theta_i\rangle) = 1$ *for all* i*. It is a unipotent character if and only if* $\lambda = (\langle \mathbf{1}_1\rangle^\mu)$*, where* μ *is a partition of* n*. We will denote this unipotent character by* χ_μ^{GL}*.*

There are explicit formulas for the degrees of the unipotent characters of the finite groups of Lie type, which can be found in [Car85, Section 13]. For the special case of general linear groups, we will use the formula found in [GM20, Proposition 4.3.1], as this formula is close in spirit to the one we had for the symmetric groups in Proposition 4.1.7. Combining this with the explicit formula for the degrees of the irreducible characters of the general linear groups in [His11, 3.3.5], we arrive at the following:

Proposition 6.1.6. *We define the following polynomials:*

(i) Let $\mu = (a_1, \ldots, a_k)$ *be a partition of a positive integer* n *with* $a_k > 0$*. Let* $a(\mu) := \sum_{i=1}^{k}(i-1)a_i$*. Define*

$$f_\mu(x) := q^{a(\mu)} \frac{[n]_x!}{\prod_{c\in[\mu]}[h_\mu(c)]_x}.$$

Then $\chi_\mu^{\mathrm{GL}}(1) = f_\mu(q)$ *and* $f_\mu = \chi_\mu(1) = f_\mu(1)$*, where* $\chi_\mu \in \mathrm{Irr}(\mathfrak{S}_n)$*.*

(ii) Let

$$\lambda = (\langle\theta_1\rangle^{\lambda(\theta_1)}, \langle\theta_2\rangle^{\lambda(\theta_2)}, \ldots, \langle\theta_m\rangle^{\lambda(\theta_m)}) \in \mathscr{F}_n.$$

Let $d_i = d(\theta_i)$ *and* $n_i = |\lambda(\theta_i)|$*. We define*

$$f_\lambda(x) := \frac{(x-1)(x^2-1)\cdots(x^n-1)}{\prod_{i=1}^m[(x^{d_i}-1)(x^{2d_i}-1)\cdots(x^{n_id_i}-1)]} \cdot \prod_{i=1}^m (f_{\lambda(\theta_i)}(x))^{d_i}.$$

Then $\chi_\lambda(1) = f_\lambda(q)$*.*

Definition 6.1.7. *Let* $\lambda = (\langle\theta_1\rangle^{\lambda(\theta_1)}, \langle\theta_2\rangle^{\lambda(\theta_2)}, \ldots, \langle\theta_k\rangle^{\lambda(\theta_k)}) \in \mathscr{F}_n$*. We set* $\mathbb{Q}(\lambda) = \mathbb{Q}(\theta_1, \ldots, \theta_k)$*. Note that* $\mathbb{Q}(\lambda)$ *is generated by a single primitive complex root of unity. For* $\sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$*, define* $\sigma\cdot\lambda \in \mathscr{F}_n$ *by*

$$(\sigma\cdot\lambda)(\theta) = \begin{cases} \lambda(\sigma^{-1}\circ\theta), & \text{if } \mathbb{Q}(\theta) \subseteq \mathbb{Q}(\lambda), \\ (0) & \text{else.} \end{cases}$$

We set

$$\mathrm{Galg}(\lambda) = \{\sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q}) \mid \sigma\cdot\lambda = \lambda\}.$$

Proposition 6.1.8. *(cf. [Tur01, Proposition 2.8]) Let $\lambda \in \mathscr{F}_n$. Then*

$$\mathbb{Q}(\chi_\lambda) = \mathbb{Q}(\lambda)^{\mathrm{Galg}(\lambda)} = \{a \in \mathbb{Q}(\lambda) \mid \sigma(a) = a \text{ for all } \sigma \in \mathrm{Galg}(\lambda)\}.$$

In particular, $\mathbb{Q}(\chi_\lambda) \subseteq \mathbb{R}$ if and only if $\overline{\lambda} = \lambda$, where $\overline{\lambda}$ is the complex conjugation of λ.

Example 6.1.9. *Let $\lambda = (\langle \alpha_2^{q-1} \rangle^{(1)})$. Thus*

$$\mathbb{Q}(\lambda) = \mathbb{Q}(\zeta_{q^2-1}^{q-1}) = \mathbb{Q}(\zeta_{q+1}).$$

The underlying simplex contains two characters:

$$\langle \alpha_2^{q-1} \rangle = \{\alpha_2^{q-1}, \alpha_2^{q(q-1)}\}.$$

Note that $\alpha_2^{q(q-1)} = \alpha_2^{-(q-1)}$. So $\mathrm{Galg}(\lambda) \cong \{\mathrm{id}, \sigma\}$ where σ is complex conjugation. In conclusion, the character field is equal to

$$\mathbb{Q}(\chi_\lambda) = \mathbb{Q}(\zeta_{q+1})^{\mathrm{Galg}(\lambda)} = \mathbb{Q}(\vartheta_{q+1}^{(1)}).$$

The only thing missing to describe all the $\mathrm{Irr}^+(\mathrm{GL}_n(q))$ characters is the Frobenius–Schur indicator. Luckily, for the general linear groups the situation is as nice as one could hope for.

Proposition 6.1.10. *(cf. [Zel81, Proposition 12.6]) The Schur index of all characters of $\mathrm{GL}_n(q)$ is equal to 1.*

If q is odd, we set $\mathbf{sgn} := \alpha_1^{(q-1)/2}$ to be the unique character of $\mathbb{G}_m(q)$ of order 2.

Example 6.1.11. *The irreducible characters of $\mathrm{GL}_2(q)$ are the following:*

λ	*Real Characters*	*Degree*
$(\langle \alpha_1^k \rangle^{(2)})$, $0 \le k \le q-2$	$(\langle \mathbf{1}_1 \rangle^{(2)})$, $(\langle \mathbf{sgn} \rangle^{(2)})$	1
$(\langle \alpha_1^k \rangle^{(1,1)})$, $0 \le k \le q-2$	$(\langle \mathbf{1}_1 \rangle^{(1,1)})$, $(\langle \mathbf{sgn} \rangle^{(1,1)})$	q
$(\langle \alpha_1^k \rangle^{(1)}, \langle \alpha_1^l \rangle^{(1)})$, $0 \le k < l \le q-2$	$(\langle \alpha_1^k \rangle^{(1)}, \langle \alpha_1^{-k} \rangle^{(1)})$, $(\langle \mathbf{1}_1 \rangle^{(1)}, \langle \mathbf{sgn} \rangle^{(1)})$	$q+1$
$(\langle \alpha_2^k \rangle^{(1)})$, $1 \le k \le q^2-2$, $(q+1) \nmid k$	$(\langle \alpha_2^{(q-1)k} \rangle^{(1)})$	$q-1$

Example 6.1.12. *The irreducible characters of $\mathrm{GL}_3(q)$ are the following:*

λ	*Real Characters*	*Degree*
$(\langle\alpha_1^k\rangle^{(3)})$, $0 \le k \le q-2$	$(\langle\mathbf{1}_1\rangle^{(3)})$, $(\langle\mathbf{sgn}\rangle^{(3)})$	1
$(\langle\alpha_1^k\rangle^{(2,1)})$, $0 \le k \le q-2$	$(\langle\mathbf{1}_1\rangle^{(2,1)})$, $(\langle\mathbf{sgn}\rangle^{(2,1)})$	$q(q+1)$
$(\langle\alpha_1^k\rangle^{(1,1,1)})$, $0 \le k \le q-2$	$(\langle\mathbf{1}_1\rangle^{(1,1,1)})$, $(\langle\mathbf{sgn}\rangle^{(1,1,1)})$	q^3
$(\langle\alpha_1^k\rangle^{(2)}, \langle\alpha_1^l\rangle^{(1)})$, $0 \le k,l \le q-2$, $k \ne l$	$(\langle\mathbf{1}_1\rangle^{(2)}, \langle\mathbf{sgn}\rangle^{(1)})$, $(\langle\mathbf{sgn}\rangle^{(2)}, \langle\mathbf{1}_1\rangle^{(1)})$	q^2+q+1
$(\langle\alpha_1^k\rangle^{(1,1)}, \langle\alpha_1^l\rangle^{(1)})$, $0 \le k,l \le q-2$, $k \ne l$	$(\langle\mathbf{1}_1\rangle^{(1,1)}, \langle\mathbf{sgn}\rangle^{(1)})$, $(\langle\mathbf{sgn}\rangle^{(1,1)}, \langle\mathbf{1}_1\rangle^{(1)})$	$q(q^2+q+1)$
$(\langle\alpha_1^k\rangle^{(1)}, \langle\alpha_1^l\rangle^{(1)}, \langle\alpha_1^m\rangle^{(1)})$, $0 \le k < l < m \le q-2$	$(\langle\mathbf{1}_1\rangle^{(1)}, \langle\alpha_1^l\rangle^{(1)}, \langle\alpha_1^{-l}\rangle^{(1)})$, $(\langle\mathbf{sgn}\rangle^{(1)}, \langle\alpha_1^l\rangle^{(1)}, \langle\alpha_1^{-l}\rangle^{(1)})$	$(q+1)(q^2+q+1)$
$(\langle\alpha_2^k\rangle^{(1)}, \langle\alpha_1^l\rangle^{(1)})$, $1 \le k \le q^2-2$, $(q+1) \nmid k$, $0 \le l \le q-2$	$(\langle\alpha_2^{(q-1)k}\rangle^{(1)}, \langle\mathbf{1}_1\rangle^{(1)})$, $(\langle\alpha_2^{(q-1)k}\rangle^{(1)}, \langle\mathbf{sgn}\rangle^{(1)})$	$(q-1)(q^2+q+1)$
$(\langle\alpha_3^k\rangle^{(1)})$, $1 \le k \le q^3-2$, $(q^2+q+1) \nmid k$	*No real characters*	$(q-1)^2(q+1)$

6.2 Orthogonal Determinants of $\mathrm{GL}_n(q)$

In this section, we want to describe how to calculate all orthogonal determinants of the $\mathrm{Irr}^+(G)$-characters. Further, we will see that Parker's conjecture holds for the general linear group. We first do the usual setup, so let p be an odd prime and q be a power of p. Let n be a positiver integer and let $G = \mathrm{GL}_n(q)$. We let $B \subseteq G$ be the Borel subgroup of upper triangular matrices and let $B = U \rtimes T$ be the Levi decomposition, with T be the subgroup of diagonal matrices and U being the subgroup of unipotent upper triangular matrices.

Let us fix a character $\chi \in \mathrm{Irr}^+(G)$. Denote $K := \mathbb{Q}(\chi)$, for which we know an explicit formula by Proposition 6.1.8. Let

$$\lambda = (\langle\theta_1\rangle^{\lambda(\theta_1)}, \langle\theta_2\rangle^{\lambda(\theta_2)}, \ldots, \langle\theta_k\rangle^{\lambda(\theta_k)})$$

be the corresponding element of $\mathscr{F}_n$. Let $\mathrm{Res}_B^G = \chi_T + \chi_U$ be the decomposition as in Section 5.1. Note that $\mathbb{Q}(\chi_U) = \mathbb{Q}$, by either the explicit formula for the character field in Proposition 6.1.8 or by [TZ04, Theorem 1.9].

There is a convenient way to calculate $\chi_T(1)$:

Lemma 6.2.1. *It holds that $\chi_T(1) = f_\lambda(1)$.*

Proof. From the construction of the irreducible characters of the general linear groups, it follows that $\chi_T(1)$ only depends on the "type" of λ, i.e., the information $d(\theta_i)$ and $\lambda(\theta_i)$ for all $1 \leq i \leq k$. In particular, it is independent of q. So we can choose $q > n! + 1$ and show it there.

It follows by the Mackey formula for Harish-Chandra induction and restriction that $\chi_T(1) \leq |\mathfrak{S}_n| = n!$, see for instance [DM20, Theorem 5.2.1]. Since $q-1 \mid \chi_U$, it now follows that

$$\chi_T(1) \equiv \chi(1) \mod q-1.$$

Since $\chi(1) = f_\lambda(q)$, we can also regard the residue of $f_\lambda(x)$ modulo $x-1$, which is equal to $f_\lambda(1)$ by the assumption on q, from which the result follows. □

Remark 6.2.2. *There are the following four possible cases:*

i) χ is U-stable, i.e., χ is not in the principal series. This is equivalent to $d(\theta_i) > 1$ for at least one entry i.

Since $q-1 \mid \chi(1)$, the orthogonal determinant is thus really easy to calculate by Corollary 2.3.17, it holds that

$$\det(\chi) = q^{\chi(1)/(q-1)} \cdot (K^\times)^2.$$

ii) χ is B-stable and not U-stable. Then χ is in the principal series and λ gives us a character $\theta \in \mathrm{Irr}(T)$ such that χ appears in $\mathrm{Ind}_B^G(\theta)$. Recall that

$$\det(\chi) = \det(\chi_T) q^{\chi_U(1)/(q-1)} \cdot (K^\times)^2.$$

Now, here is to calculate χ_T in general. With Lemma 6.2.1 and Proposition 5.1.3, we can easily decompose

$$\chi_T = \sum_{i=1}^{k} \theta_i$$

into its irreducible components, given λ. Let $L := \mathbb{Q}(\theta + \overline{\theta})$. It follows again by Proposition 5.1.3 that $K = \mathbb{Q}(\chi_T)$. Note that it does not always hold that $L = K$. The result now follows by Corollary 2.3.3 and Lemma 2.3.5.

iii) χ is unipotent. Then Theorem 5.2.1 tells us how to calculate $\det(\chi)$. Since we also know explicitly the orthogonal determinants of the generic Iwahori–Hecke algebra of type A_{n-1} by Theorem 4.1.65, $\det(\chi)$ can be calculated.

iv) χ *is neither* B*-stable nor unipotent. Then there is a character* $\theta \in \mathrm{Irr}(T)$ *such that the order of* θ *is equal to* 2 *and that* χ *appears in* $\mathrm{Ind}_B^G(\theta)$*. There is thus an integer* k *with* $0 \leq k \leq n-1$ *and partitions* μ *of* k*,* κ *of* $n-k$ *such that*

$$\lambda = (\langle \mathbf{1}_1 \rangle^{\mu}, \langle \mathbf{sgn} \rangle^{\kappa}).$$

Then $L_\lambda = \mathrm{GL}_k(q) \times \mathrm{GL}_{n-k}(q)$ *and*

$$\chi^{L_\lambda} = \chi_\mu^{\mathrm{GL}} \boxtimes (\mathbf{sgn} \cdot \chi_\kappa^{\mathrm{GL}}).$$

If $\chi_\kappa^{\mathrm{GL}} \in \mathrm{Irr}^+(\mathrm{GL}_{n-k}(q))$*, then clearly*

$$\det(\mathbf{sgn} \cdot \chi_\kappa^{\mathrm{GL}}) = \det(\chi_\kappa^{\mathrm{GL}}).$$

Let $P_\lambda \supseteq L_\lambda$ *be a parabolic subgroup, then we know that*

$$\chi = \mathrm{Ind}_{P_\lambda}^G(\chi^{L_\lambda}).$$

Since we can calculate the orthogonal determinants of unipotent characters, $\det(\chi)$ *can then easily be calculated with Lemma 2.3.10 and Lemma 2.3.13.*

All in all, the mentioned methods allow us to calculate the orthogonal determinant of any orthogonally stable character of G.

We will now tackle Parker's conjecture for the general linear groups. Since we know Parker's conjecture to hold for solvable groups, see Theorem 2.3.29, it holds for all Borel-stable characters. If χ is as in Remark 6.2.2(iv), its orthogonal determinant is either a square (if the index of the corresponding parabolic subgroup is even) or we can reduce the calculation of $\det(\chi)$ to the calculation of the orthogonal determinant of a unipotent character. In either case, we conclude that Parker's conjecture holds for the general linear groups if and only if it holds for all its unipotent characters.

We will need some well-known statements about cyclotomic polynomials. A source is for instance [Nag51, §46, §48].

Definition 6.2.3. *The cyclotomic polynomials* $\Phi_n(x)$ *for positive integers* n *can be inductively defined by the condition*

$$x^n - 1 = \prod_{d \mid n} \Phi_d(x).$$

In particular,

$$[n]_x = \frac{\prod_{d \mid n} \Phi_d(x)}{x-1}.$$

The first few cyclotomic polynomials are

$$\begin{aligned}
\Phi_1(x) &= x - 1,\\
\Phi_2(x) &= x + 1,\\
\Phi_3(x) &= x^2 + x + 1,\\
\Phi_4(x) &= x^2 + 1,\\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1,\\
\Phi_6(x) &= x^2 - x + 1.
\end{aligned}$$

The following gives some very basic properties of cyclotomic polynomials.

Lemma 6.2.4. *Let n be a positive integer.*

i) $\Phi_n(x) \in \mathbb{Z}[x]$.

ii) $\Phi_n(x)$ *is irreducible.*

iii) $\Phi_{2^n}(x) = x^{2^{n-1}} + 1$.

iv) $\Phi_n(1) = \begin{cases} s, & \text{if } n = s^k \text{ for some prime } s, \\ 0, & \text{if } n = 1, \\ 1, & \text{else.} \end{cases}$

So in conclusion, the cyclotomic polynomials are exactly the irreducible factors of the Gaussian polynomials.

Lemma 6.2.5. *Let c be a positive integer. Then the square classes $c(c+2) \cdot (\mathbb{Q}^\times)^2$ and $[c]_q[c+2]_q \cdot (\mathbb{Q}^\times)^2$ have the same parity.*

Proof. Note that since q is odd, for any polynomial $f(u) \in \mathbb{Z}[u]$ it holds that

$$f(1) \equiv f(q) \mod 2.$$

Let $f(u) := [c]_u[c+2]_u$. If c is odd, then clearly both $f(1) = c(c+2)$ and $f(q) = [c]_q[c+2]_q$ are odd integers.

Let us now assume that c is even. Then exactly one of $c/2$ and $(c+2)/2$ is odd. Let 2^k be the biggest power of 2 such that $c(c+2) = 2 \cdot 2^k \cdot m$ for m an odd integer. Clearly $k \geq 2$. By Lemma 6.2.4 we can now write

$$f(u) = [c]_u[c+2]_u = (u+1)^2 \prod_{\substack{d|c \\ d\geq 3}} \Phi_d(u) \prod_{\substack{d'|c+2 \\ d'\geq 3}} \Phi_{d'}(u) = (u+1)^2 \left(\prod_{l=2}^{k} \Phi_{2^l}(u) \right) \cdot f'(u),$$

where $f'(u) \in \mathbb{Z}[u]$ is the product of all cyclotomic polynomials $\Phi_d(u)$ for the divisors d of either c or $c+2$ that are not powers of 2.

We will now go through the three factors of f on the right hand side. Since we regard square classes, we can disregard the term $(u+1)^2$. For the second term, observe that for any $l \geq 2$, by Lemma 6.2.5(iii), it holds that $\Phi_{2^l}(q) = q^{2^{l-1}} + 1 = 2 \cdot r$ for some odd integer r. So, clearly

$$\prod_{l=2}^{k} \Phi_{2^l}(1) \cdot (\mathbb{Q}^\times)^2 = 2^k \cdot (\mathbb{Q}^\times)^2$$

and

$$\prod_{l=2}^{k} \Phi_{2^l}(q) \cdot (\mathbb{Q}^\times)^2$$

have the same parity. Finally, by Lemma 6.2.4(iv), it holds that $f'(1)$ is an odd integer, and therefore $f'(q)$ is also an odd integer. This concludes the proof. $\square$

Theorem 6.2.6. *Let* $\chi \in \mathrm{Irr}^+(G) \cap \mathrm{Irr}_{\mathrm{PSU}}(G)$*. Then* $\det(\chi)$ *is odd.*

Proof. Let $A = \mathbb{Q}[u, u^{-1}]$ and let $\mathcal{H} = \mathcal{H}_G$ the generic Iwahori–Hecke algebra of G over A as in Definition 3.5.8.

Let $\chi' \in \mathrm{Irr}^+(\mathbb{Q}(u)\mathcal{H})$ be the character corresponding to χ. Let $d_{\chi'}(u) \in A$ be a squarefree representative of $\det(\chi')$. Recall by Theorem 5.2.1 that then

$$\det(\chi) = d_{\chi'}(q) \cdot q^a \cdot (\mathbb{Q}^\times)^2$$

for some integer a. So it suffices to show that $d_{\chi'}(q) \cdot (\mathbb{Q}^\times)^2$ is odd.

Let μ be the partition corresponding to χ. Since the generic Iwahori–Hecke algebra specializes to the group algebra $\mathbb{Q}\mathfrak{S}_n$ for the map $A \to \mathbb{Q}, u \mapsto 1$, we know that

$$\det(\lambda) \cdot (\mathbb{Q}^\times)^2 = d_{\chi'}(1) \cdot (\mathbb{Q}^\times)^2$$

is odd by Theorem 4.1.50. The idea now is to show that $d_{\chi'}(1) \cdot (\mathbb{Q}^\times)^2$ and $d_{\chi'}(q) \cdot (\mathbb{Q}^\times)^2$ have the same parity.

By Theorem 4.1.65, it holds that $\det(\chi')$ is a product of terms of the form

$$\frac{(u^{b_1+1} - u^{b_2})(u^{b_1} - u^{b_2+1})}{(u^{b_1} - u^{b_2})^2} = \frac{u[b_1 - b_2 + 1]_u [b_1 - b_2 - 1]_u}{[b_1 - b_2]_u^2}$$

for some integers b_1, b_2 with $b_1 > b_2 + 1$. By now setting $u := b_1 - b_2 - 1$ and by disregarding the square denominator and the term u (since both q and 1 are odd), the statement follows if we can show that $c(c+2) \cdot (\mathbb{Q}^\times)^2$ and $[c]_q[c+2]_q \cdot (\mathbb{Q}^\times)^2$ have the same parity. That is exactly the statement of Lemma 6.2.5 and so we are done. □

Corollary 6.2.7. *Parker's conjecture holds for the groups* $\mathrm{GL}_n(q)$ *for odd* q*.*

6.2.1 Example: Orthogonal Determinants of $\mathrm{GL}_4(q)$

Let p be an odd prime and let q be a power of p. In this final subsection, the orthogonal determinants of the $\mathrm{Irr}^+(G)$-characters of $G = \mathrm{GL}_4(q)$ will be calculated. This will go relatively easily, given that we have already done most of the work.

For completion, we will first give a list of all irreducible characters of G, which easily follows from the discussion in Section 6.1.

Theorem 6.2.8. *Let* k, l, m, r *be suitable integers. The following gives a list of all irreducible characters* χ *of* G *in terms of the elements* $\lambda \in \mathscr{F}_4$*, as well as their degrees and the degree of* χ_T*.*

χ	λ	$(\chi_\lambda)_T(1)$	*Degree*
$\chi_1(k)$	$(\langle\alpha_1^k\rangle^{(4)})$	1	1
$\chi_2(k)$	$(\langle\alpha_1^k\rangle^{(3,1)})$	3	$q(q^2+q+1)$
$\chi_3(k)$	$(\langle\alpha_1^k\rangle^{(2,2)})$	2	$q^2(q^2+1)$
$\chi_4(k)$	$(\langle\alpha_1^k\rangle^{(2,1,1)})$	3	$q^3(q^2+q+1)$
$\chi_5(k)$	$(\langle\alpha_1^k\rangle^{(1,1,1,1)})$	1	q^6
$\psi_1(k,l)$	$(\langle\alpha_1^k\rangle^{(3)}, \langle\alpha_1^l\rangle^{(1)})$	4	$(q+1)(q^2+1)$
$\psi_2(k,l)$	$(\langle\alpha_1^k\rangle^{(2,1)}, \langle\alpha_1^l\rangle^{(1)})$	8	$q(q+1)^2(q^2+1)$
$\psi_3(k,l)$	$(\langle\alpha_1^k\rangle^{(1,1,1)}, \langle\alpha_1^l\rangle^{(1)})$	4	$q^3(q+1)(q^2+1)$
$\psi_4(k,l)$	$(\langle\alpha_1^k\rangle^{(2)}, \langle\alpha_1^l\rangle^{(2)})$	6	$(q^2+1)(q^2+q+1)$
$\psi_5(k,l)$	$(\langle\alpha_1^k\rangle^{(2)}, \langle\alpha_1^l\rangle^{(1,1)})$	6	$q(q^2+1)(q^2+q+1)$
$\psi_6(k,l)$	$(\langle\alpha_1^k\rangle^{(1,1)}, \langle\alpha_1^l\rangle^{(1,1)})$	6	$q^2(q^2+1)(q^2+q+1)$
$\psi_7(k,l,m)$	$(\langle\alpha_1^k\rangle^{(2)}, \langle\alpha_1^l\rangle^{(1)}, \langle\alpha_1^m\rangle^{(1)})$	12	$(q^2+1)[2]_q[3]_q$
$\psi_8(k,l,m)$	$(\langle\alpha_1^k\rangle^{(1,1)}, \langle\alpha_1^l\rangle^{(1)}, \langle\alpha_1^m\rangle^{(1)})$	12	$q(q^2+1)[2]_q[3]_q$
$\psi_9(k,l,m,r)$	$(\langle\alpha_1^k\rangle^{(1)}, \langle\alpha_1^l\rangle^{(1)}, \langle\alpha_1^m\rangle^{(1)}, \langle\alpha_1^r\rangle^{(1)})$	24	$(q^2+1)[2]_q^2[3]_q$
$\pi_1(k,l)$	$(\langle\alpha_2^k\rangle^{(1)}, \langle\alpha_1^l\rangle^{(2)})$	0	$(q-1)(q^2+1)[3]_q$
$\pi_2(k,l)$	$(\langle\alpha_2^k\rangle^{(1)}, \langle\alpha_1^l\rangle^{(1,1)})$	0	$q(q-1)(q^2+1)[3]_q$
$\pi_3(k,l,m)$	$(\langle\alpha_2^k\rangle^{(1)}, \langle\alpha_1^l\rangle^{(1)}, \langle\alpha_1^m\rangle^{(1)})$	0	$(q-1)(q^2+1)[2]_q[3]_q$
$\pi_4(k)$	$(\langle\alpha_2^k\rangle^{(2)})$	0	$(q-1)^2[3]_q$
$\pi_5(k)$	$(\langle\alpha_2^k\rangle^{(1,1)})$	0	$q^2(q-1)^2[3]_q$
$\pi_6(k,l)$	$(\langle\alpha_2^k\rangle^{(1)}, \langle\alpha_2^l\rangle^{(1)})$	0	$(q-1)^2(q^2+1)[3]_q$
$\pi_7(k,l)$	$(\langle\alpha_3^k\rangle^{(1)}, \langle\alpha_1^l\rangle^{(1)})$	0	$(q-1)^2(q^2+1)[2]_q^2$
$\pi_8(k)$	$(\langle\alpha_4^k\rangle^{(1)})$	0	$(q-1)^3[2]_q[3]_q$

Recall that $[2]_q = q+1, [3]_q = q^2+q+1$. Note that the principal series consist of the characters χ_i and ψ_i.

Remark 6.2.9. *We will omit explicitly describing the character fields, as these can involve some case-by-case distinctions which we do not want to get into. For instance, regard the character $\psi_9 = \psi_9(k,l,m,r)$. Assume for a moment that $q = s^2$ is a square. Then $\psi_9(k,-k,sk,-sk) \in \mathrm{Irr}^+(G)$ for any $1 \le k < q-1$ with k not a multiple of $s+1$ or $s-1$. If λ is the corresponding element of $\mathscr{F}_4$, then clearly $\mathbb{Q}(\lambda) = \mathbb{Q}(\zeta_{q-1}^k)$. Then*

$$\mathrm{Galg}(\lambda) = \{\sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})\} = \langle\sigma_{-1}, \sigma_s\rangle \cong C_2^2,$$

where σ_{-1} is complex conjugation and $\sigma_s(\zeta_{q-1}^k) := \zeta_{q-1}^{sk}$. Thus,

$$\mathbb{Q}(\psi_9(k,-k,sk,-sk)) = \mathbb{Q}(\vartheta_{q-1}^{(k)} + \vartheta_{q-1}^{(sk)}, \vartheta_{s-1}^{(k)}, \vartheta_{s+1}^{(k)}).$$

We will now go through all four different classes of the $\mathrm{Irr}^+(G)$-characters we have talked about in the previous section.

Lemma 6.2.10. *The following gives a list of all* $\mathrm{Irr}^+(G)$*-characters that are* U*-stable, as well as their orthogonal determinants.*

χ_λ	*Parameters*	$\det(\chi) \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$
$\pi_1(k(q-1),0)$, $\pi_1(k(q-1),(q-1)/2)$	$1 \leq k \leq \frac{q-1}{2}$	1
$\pi_2(k(q-1),0)$, $\pi_2(k(q-1),(q-1)/2)$	$1 \leq k \leq \frac{q-1}{2}$	1
$\pi_3(k(q-1),0,(q-1)/2)$	$1 \leq k \leq \frac{q-1}{2}$	1
$\pi_3(k(q-1),l,-l)$	$1 \leq k \leq \frac{q-1}{2}, 1 \leq l \leq \frac{q-3}{2}$	1
$\pi_4(k(q-1))$	$1 \leq k \leq \frac{q-1}{2}$	1
$\pi_5(k(q-1))$	$1 \leq k \leq \frac{q-1}{2}$	1
$\pi_6(k,-k)$	$1 \leq k \leq q^2-2$, $(q-1) \nmid k,\ (q+1) \nmid k$	1
$\pi_6(k(q-1),l(q-1))$	$1 \leq k < l \leq \frac{q-1}{2}$	1
$\pi_8(k(q^2-1))$	$1 \leq k \leq \frac{q^2-1}{2}$	1

Proof. As was discussed in the previous section, it holds that

$$\det(\chi) = q^{\chi(1)/q-1)} \cdot (\mathbb{Q}(\chi)^\times)^2$$

for all U-stable characters $\chi \in \mathrm{Irr}^+(G)$. We easily see from the list of the character degrees in Theorem 6.2.8 that even after dividing with $q-1$, the characters degree stay even, so the orthogonal determinant is a square in every case. □

In fact, by the degrees, it is easy to confirm that for any $\chi \in \mathrm{Irr}^+(G)$, it holds that

$$\det(\chi_U) = 1 \cdot (\mathbb{Q}(\chi_U)^\times)^2,$$

so we will tacitly disregard this term in the following calculations.

Lemma 6.2.11. *The following gives a list of all* $\mathrm{Irr}^+(G)$*-characters that are Borel-stable and are in the principal series, as well as their orthogonal determinants.*

χ_λ	*Parameters*	$\det(\chi) \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$
$\psi_4(k,-k)$	$1 \le k \le \frac{q-3}{2}$	$2 - \vartheta_{q-1}^{(2k)}$
$\psi_6(k,-k)$	$1 \le k \le \frac{q-3}{2}$	$2 - \vartheta_{q-1}^{(2k)}$
$\psi_7(0,k,-k)$, $\psi_7((q-1)/2,k,-k)$	$1 \le k \le \frac{q-3}{2}$	1
$\psi_8(0,k,-k)$, $\psi_8((q-1)/2,k,-k)$	$1 \le k \le \frac{q-3}{2}$	1
$\psi_9(0,(q-1)/2,k,-k)$	$1 \le k \le \frac{q-3}{2}$	1
$\psi_9(k,-k,l,-l)$	$1 \le k < l \le \frac{q-3}{2}$	1

Proof. Let χ be any of the above characters. Recall that

$$\det(\chi) = \det(\chi_T)\det(\chi_U) \cdot (\mathbb{Q}(\chi)^\times)^2.$$

Since the determinant of χ_U is a square, we are left to calculate $\det(\chi_T)$. Denote $K = \mathbb{Q}(\chi) = \mathbb{Q}(\chi_T)$. We let θ be one of the irreducible components of χ_T, then $\psi := \theta + \overline{\theta}$ is orthogonally simple. Denote $L = \mathbb{Q}(\psi)$. There are two cases that occur: $L = K$ and $[L/K] = 2$. Recall the character

$$\psi_K := \sum_{\sigma \in \mathrm{Gal}(L/K)} \sigma \cdot \psi,$$

as given in Definition 2.3.4. It follows by Lemma 2.3.5 that

$$\det(\chi_T) = N_K(\det(\psi_K))^{\chi_T(1)/\psi_K(1)} \cdot (K^\times)^2.$$

In particular, if $\chi_T(1)/\psi_K(1)$ is even, then $\det(\chi_T)$ is a square. By definition, it holds that

$$\psi_K(1) = \begin{cases} 2, & \text{if } L = K, \\ 4, & \text{if } [L/K] = 2. \end{cases}$$

For all characters in the table above, except possibly the characters $\psi_9(k,-k,l,-l)$, it holds that $L = K$. There thus are only two cases where $\chi_T(1)/\psi_K(1)$ is not even, namely the characters $\psi_4(k,-k)$ and $\psi_6(k,-k)$, for which we easily calculate with Corollary 2.3.3 that

$$\det(\chi_T) = (2 - \vartheta_{q-1}^{(2k)}) \cdot (K^\times)^2.$$

This proves the statement. □

Lemma 6.2.12. *The orthogonal determinant of the only unipotent* $\mathrm{Irr}^+(G)$ *character* $\chi_3(0) = \chi^{\mathrm{GL}}_{(2,2)}$ *is equal to*

$$\det(\chi^{\mathrm{GL}}_{(2,2)}) = q(q^2+q+1) \cdot (\mathbb{Q}^\times)^2.$$

Proof. Let us abbreviate $\chi = \chi^{\mathrm{GL}}_{(2,2)}$. Similarly to Lemma 5.3.6, we will give two different proofs. First, we will use Theorem 5.2.1, where we actually have already done all the necessary work. In Example 4.1.66 we have calculated the orthogonal determinant of the corresponding character χ' of the generic Iwahori–Hecke algebra for G, which was equal to

$$\det(\chi') = u(u^2 + u + 1) \cdot (\mathbb{Q}(u)^\times)^2,$$

where u is the indeterminate used. So by specializing, we end up with

$$\det(\chi) = q(q^2 + q + 1) \cdot (\mathbb{Q}^\times)^2.$$

Let us now regard a more geometrical proof. Let $\mathcal{F}(1)$ be the set of 1-dimensional subspaces of $\mathbb{F}_q^4$ and $\mathcal{F}(2)$ be the set of 2-dimensional subspaces of $\mathbb{F}_q^4$. Clearly, G acts transitively on these sets and we can regard the associated permutation representation $M_{(1)}$ (resp. $M_{(2)}$) over $\mathbb{Q}$ of $\mathcal{F}(1)$ (resp. of $\mathcal{F}(2)$). We have that $\dim(M_{(1)}) = (q+1)(q^2+1)$ and $\dim(M_{(2)}) = (q^2 + 1)(q^2 + q + 1)$. We denote $\phi_{(1)}$ (resp. $\phi_{(2)}$) to be the characters of the representations $M_{(1)}$ (resp. $M_{(2)}$). We have a decomposition

$$\begin{aligned} \phi_{(1)} &= \mathbf{1}_G + \chi^{\mathrm{GL}}_{(3,1)}, \\ \phi_{(2)} &= \mathbf{1}_G + \chi^{\mathrm{GL}}_{(3,1)} + \chi^{\mathrm{GL}}_{(2,2)}. \end{aligned}$$

The idea now is to use the permutation representations above to construct an explicit G-invariant bilinear form of a representation affording $\chi^{\mathrm{GL}}_{(2,2)}$.

We define

$$f : M_{(1)} \to M_{(2)}, P \mapsto \sum_{L \supseteq P} L$$

where we map any one-dimensional subspace P to the sum of the two-dimensional subspaces L that contain P. This map is clearly a homomorphism of G-modules.

We further argue that f is injective. For that, consider the standard G-invariant bilinear form β on $M_{(2)}$, where the basis given by the set $\mathcal{F}(2)$ is an orthonormal basis. We regard the pullback $\beta' := f^*\beta$ on $M_{(1)}$, defined by

$$\beta'(P, P') := \beta\left(f(P), f(P')\right).$$

This bilinear form is clearly again G-invariant. Now, the map f being injective is equivalent to the bilinear form β' being non-degenerate, so it suffices to calculate the Gram determinant with respect to the standard basis of $M_{(1)}$.

Since every one-dimensional subspace P is contained in exactly q^2+q+1 two-dimensional subspaces and two different one-dimensional subspaces P, P' are only both contained in the two-dimensional subspace $P + P'$, we arrive at

$$\beta'(P, P') = \begin{cases} q^2 + q + 1, & \text{if } P = P', \\ 1, & \text{if } P \neq P'. \end{cases}$$

Thus,

$$\det(\beta') = \det \begin{pmatrix} q^2+q+1 & 1 & \dots & 1 \\ 1 & q^2+q+1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & q^2+q+1 \end{pmatrix} \cdot (\mathbb{Q}^\times)^2 = q(q^2+q+1)\cdot(\mathbb{Q}^\times)^2.$$

So in fact, f is injective and by the decomposition of the characters we see that the orthogonal complement of the image of $M_{(1)}$ in $M_{(2)}$ is isomorphic to a representation that affords the character $\chi^{\mathrm{GL}}_{(2,2)}$. Since the Gram determinant of β is equal to 1 with respect to the standard basis, we finally arrive at

$$\det(\chi) = q(q^2+q+1)\cdot(\mathbb{Q}^\times)^2.$$

□

We now handle the last remaining characters.

Lemma 6.2.13. *The following gives a list of all the* $\mathrm{Irr}^+(G)$*-characters for which* $\chi_T \neq 0$ *and for which any* $\theta \in \mathrm{Irr}(T)$ *appearing in* χ_T *have order* 2*, as well as their orthogonal determinants.*

χ_λ	$\det(\chi) \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$
$\chi_3((q-1)/2)$	$q(q^2+q+1)$
$\psi_1(0,(q-1)/2)$, $\psi_1((q-1)/2,0)$	1
$\psi_2(0,(q-1)/2)$, $\psi_2((q-1)/2,0)$	1
$\psi_3(0,(q-1)/2)$, $\psi_3((q-1)/2,0)$	1
$\psi_4(0,(q-1)/2)$	1
$\psi_5(0,(q-1)/2)$, $\psi_5((q-1)/2,0)$	1
$\psi_6(0,(q-1)/2)$	1

Proof. It holds that

$$\chi_3((q-1)/2) = \mathbf{sgn}\cdot\chi^{\mathrm{GL}}_{(2,2)},$$

where we set $\mathbf{sgn} \in \mathrm{Hom}(G,\mathbb{C}^\times)$ to be the unique character of order 2. Clearly then $\det(\chi_3((q-1)/2)) = \det(\chi^{\mathrm{GL}}_{(2,2)})$.

For the rest of the characters, note that by Remark 6.1.5 these are induced from certain orthogonal characters with rational character fields. As the index of all nontrivial parabolic subgroups of G is even, it follows by Lemma 2.3.10 that the orthogonal determinant of the induced character χ is a square, thus showing the statement. □

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