Research Paper

On Frobenius graphs of diameter 3 for finite groups

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For a subgroup H of a finite group G, the Frobenius graph $\Gamma(G, H)$ records the constituents of the restrictions to H of the irreducible characters of G. We investigate when this graph has diameter 3.

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1. Introduction

Let H be a subgroup of a finite group G. The *Frobenius graph* $\Gamma(G, H)$ is the bipartite graph with vertex set the disjoint union of $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$ and an edge between $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{Irr}(H)$ whenever $[\chi_H, \varphi] \neq 0$. Here $[\alpha, \beta] = [\alpha, \beta]_H$ is the inner

product of (virtual) characters α, β of H, and χ_H denotes the restriction of a (virtual) character χ of G to H. We also define the *Frobenius matrix*

$$F(G, H) := ([\chi_H, \varphi])_{\varphi \in \mathrm{Irr}(H), \chi \in \mathrm{Irr}(G)}.$$

Then the biadjacency matrix of $\Gamma(G, H)$ is obtained from $F(G, H)$ by replacing each nonzero entry by 1.

We will mainly be interested in situations where the *diameter d* of $\Gamma(G, H)$, i.e., the largest distance of two vertices in $\Gamma(G, H)$, is finite. We show that $\Gamma(G, H)$ is connected if and only if the core of H in G is trivial. We note:

- $d = 1$ if and only if $|G| = 1$ holds,
- $d = 2$ if and only if $|G| > 1$ and $|H| = 1$ hold.

Thus $d \geq 3$ holds if and only if H is nontrivial. Observe that for $1_H \neq \varphi \in \mathrm{Irr}(H)$, any path from 1_G to φ in $\Gamma(G, H)$ has odd length > 1.

Frobenius graphs can have arbitrarily large diameters, for example the diameter of $\Gamma(S_{n+1}, S_n)$ is $2n$, where S_n is the symmetric group of degree n. This can be proved in terms of partitions of n, which parametrize the irreducible characters of S_n, as follows: Any partition of n can be transformed into any other partition of n by a sequence of at most $n - 1$ steps, where one step consists of adding an addable node in order to get a partition of $n + 1$, and then removing a removable node. Thus one gets a path of length at most $2n - 2$ between any two characters of S_n in $\Gamma(S_{n+1}, S_n)$. Then it is clear that paths between characters of S_{n+1} have length at most $2n$, and that the path between the trivial and the alternating character of S_{n+1} has length exactly $2n$.

In the following, we will investigate Frobenius graphs $\Gamma(G, H)$ of diameter 3. In this case we call H a *diameter three subgroup* of G.

Section 2 collects properties of pairs (G, H) such that H is a diameter three subgroup of G, Section 3 shows examples, Section 4 studies large subgroups that are diameter three subgroups, Section 5 classifies those quasisimple groups that contain diameter three subgroups, Section 6 deals with the question how minimal groups (w. r. t. inclusion) look like which have a diameter three subgroup. Finally, Section 7 explains some connections between the diameter of a Frobenius graph $\Gamma(G, H)$ and the subgroup depth of H in G.

The direct computations for this paper were done with the help of the computer algebra system Oscar [9,17].

2. Structural properties

The following result is essentially contained in [7, Section 6]. We include a proof for the convenience of the reader.

Proposition 2.1. *Let H be a proper subgroup of a finite group G. Then the connected components of $\Gamma(G, H)$ are in bijection with the G-orbits on $\mathrm{Irr}(K)$ where $K := \mathrm{Core}_G(H) :=$*

$\bigcap_{g \in G} gHg^{-1}$ denotes the core of H in G. In particular, $\Gamma(G, H)$ is connected if and only if $K = 1$.

Proof. Let $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{Irr}(H)$ such that $[\chi_H, \varphi] \neq 0$. By Clifford theory, φ_K is a sum of H-conjugates of a character $\theta \in \mathrm{Irr}(K)$. Thus χ_K is a sum of G-conjugates of θ. If also $\psi \in \mathrm{Irr}(G)$ satisfies $[\psi_H, \varphi] \neq 0$ then ψ_K is again a sum of G-conjugates of θ. Thus, whenever $\chi' \in \mathrm{Irr}(G)$ is contained in the same connected component of $\Gamma(G, H)$ as χ then the irreducible constituents of χ'_K form the G-orbit of θ.

Conversely, let $\chi, \chi' \in \mathrm{Irr}(G)$ such that $0 \neq [\chi_K, \chi'_K] = [1_K, \overline{\chi}_K \chi'_K] = [1_K^G, \overline{\chi}\chi']$. Then $\overline{\chi}\chi'$ has a constituent η whose kernel contains K. On the other hand, the kernel of 1_H^G is K, so that we can view 1_H^G as a faithful character of G/K. By a theorem of Burnside [14, Satz V.10.8], there is a positive integer m such that $0 < [(1_H^G)^m, \eta] \leq [(1_H^G)^m, \overline{\chi}\chi'] = [\chi(1_H^G)^m, \chi']$. Consider the linear map $U : \mathbb{Z}\mathrm{Irr}(G) \longrightarrow \mathbb{Z}\mathrm{Irr}(G)$, $\alpha \longmapsto \alpha_H^G = \alpha(1_H^G)$; here $\mathbb{Z}\mathrm{Irr}(G)$ denotes the group of virtual characters of G. Then $U^n(\alpha) = \alpha(1_H^G)^n$ for all $n > 0$ which implies that $0 \neq [U^m(\chi), \chi']$. Now note that the constituents of $U(\chi) = \chi_H^G$ are contained in the connected component of χ in $\Gamma(G, H)$. Thus also the constituents of $U^m(\chi)$ are contained in the connected component of χ in $\Gamma(G, H)$. In particular, χ' is contained in the connected component of χ in $\Gamma(G, H)$.

This proves the first assertion of our proposition. Now suppose that $\Gamma(G, H)$ is connected. Then $\mathrm{Irr}(K)$ is a single G-orbit, i.e., G acts transitively on $\mathrm{Irr}(K)$. Thus the trivial character 1_K is the only irreducible character of K, so that $K = 1$. The converse is clear. □

By Proposition 2.1 and the remarks from Section 1, a diameter three subgroup H is always nontrivial and *core-free* in G, i.e., $\mathrm{Core}_G(H) = 1$. The following result will be our main tool in order to detect diameter three subgroups.

Proposition 2.2. *Let H be a nontrivial proper subgroup of a finite group G. Then the following assertions are equivalent:*

(a) $\Gamma(G, H)$ *has diameter* 3.
(b)   (i) *For any $\chi \in \mathrm{Irr}(G)$, we have $[\chi_H, 1_H] \neq 0$.*
      (ii) *For any $\varphi, \psi \in \mathrm{Irr}(H)$, we have $[\varphi^G, \psi^G] \neq 0$.*

Proof. Suppose that (a) holds. Then any $\chi \in \mathrm{Irr}(G)$ is connected to the trivial character 1_G of G via a path of length $l \leq 3$ in $\Gamma(G, H)$. Since l must be even, we have in fact $l \leq 2$. Thus χ_H and $(1_G)_H = 1_H$ have a common constituent, which must be 1_H. Hence (i) holds.

Similarly, any two $\varphi, \psi \in \mathrm{Irr}(H)$ are connected via a path of length $l \leq 3$. Again, we must have $l \leq 2$. Thus there is $\chi \in \mathrm{Irr}(G)$ such that $[\chi_H, \varphi] \neq 0 \neq [\chi_H, \psi]$ holds. This implies that $[\varphi^G, \psi^G] \neq 0$, and (ii) holds.

Now suppose that (b) holds. Then (i) implies that $\Gamma(G, H)$ is connected. More precisely, the distance in $\Gamma(G, H)$ between any two characters $\chi, \eta \in \mathrm{Irr}(G)$ is at most 2. Similarly, by (ii) the distance between any two characters $\varphi, \psi \in \mathrm{Irr}(H)$ is at most 2. Thus the distance between any character $\chi \in \mathrm{Irr}(G)$ and any character $\varphi \in \mathrm{Irr}(H)$ is at most 3. Since H is nontrivial, the diameter of $\Gamma(G, H)$ is 3. $\quad\square$

Remark 2.3. We note that the conditions (b) (i) and (b) (ii) in Proposition 2.2 are independent.

In many examples, condition (b) (ii) does not imply condition (b) (i). We can take the symmetric groups $G = S_3$ and $H = S_2$ of degrees 3 and 2, where $\Gamma(G, H)$ is a path of length 4.
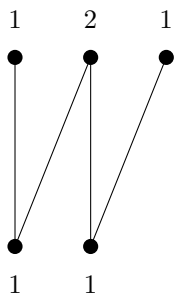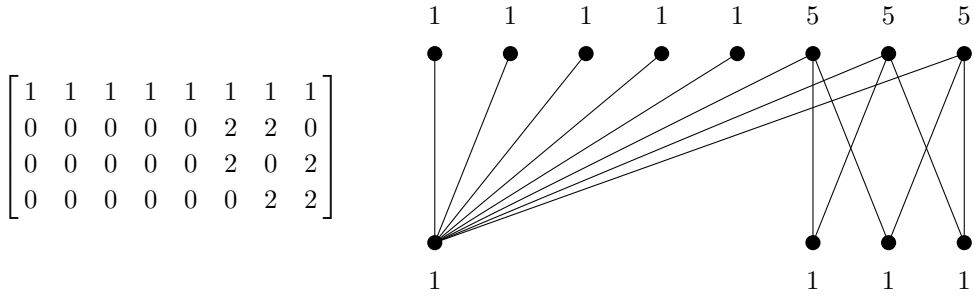

An example where (b) (i) does not imply (b) (ii) is given by the Frobenius group G of order $351 = 3^3 \cdot 13$ with an elementary abelian kernel of order 3^3 and a complement of order 13, and a subgroup H of order 3^2 in G. Then G is a subgroup of index 2 in $\mathrm{AGL}(1, 3^3)$. Moreover, $\mathrm{Irr}(G)$ consists of 13 linear characters and two irreducible characters χ, η of degree 13. Furthermore, condition (b) (i) is satisfied since the permutation character 1_H^G is exactly the sum of all irreducible characters of G. On the other hand, condition (b) (ii) is not satisfied since there are characters $\varphi, \psi \in \mathrm{Irr}(H)$ such that $\varphi^G = 3\chi$ and $\psi^G = 3\eta$, so that $[\varphi^G, \psi^G] = 0$.
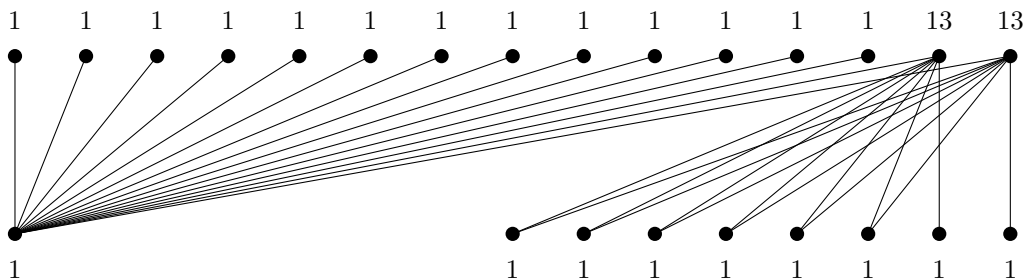
Examples of groups for which (b) (i) does not imply (b) (ii) for some subgroups seem to be rare, see Remark 4.5 and Section 6.3.

Remark 2.4. In the situation of Proposition 2.2, for any two characters $\varphi, \psi \in \mathrm{Irr}(H)$ the induced characters φ^G, ψ^G have a common constituent $\chi \in \mathrm{Irr}(G)$. However, in

general there does not exist a character $\chi \in \mathrm{Irr}(G)$ which is a constituent of φ^G for *every* $\varphi \in \mathrm{Irr}(H)$. As an example, one can take the Frobenius group G of order $2^4 \cdot 5$ and a suitable subgroup H of order 4. (There are seven classes of subgroups of order four, one of them works.) Then the Frobenius matrix $F(G,H)$ and the Frobenius graph $\Gamma(G,H)$ are as follows:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2
\end{bmatrix}
$$

Thus $\Gamma(G,H)$ has diameter 3.

Remark 2.5. Condition (b) (ii) of Proposition 2.2 is equivalent to the following one. For every $\varphi, \psi \in \mathrm{Irr}(H)$, there exists $g \in G$ such that $[\varphi_{H \cap H^g}, \psi^g_{H \cap H^g}] > 0$.

Note that by Frobenius reciprocity and Mackey decomposition,

$$
[\varphi^G, \psi^G] = \sum_{g \in R} [\varphi, (\psi^g_{H^g \cap H})^H] = \sum_{g \in R} [\varphi_{H^g \cap H}, \psi^g_{H^g \cap H}],
$$

where R is a set of representatives of H-H-double cosets in G.

This condition implies the following: For every $\varphi \in \mathrm{Irr}(H)$, there exists $g \in G$ such that $[\varphi_{H \cap H^g}, 1_{H \cap H^g}] > 0$.

The second condition also implies that, for every linear character λ of H, there exists $g \in G$ such that $H \cap H^g$ is contained in the kernel of λ.

The following consequence will be useful in checking examples.

Corollary 2.6. *Let G be a finite group and $H \leq G$. Condition (b) (ii) in Proposition 2.2 is satisfied if one of the following holds.*
   (i) There is $g \in G$ such that $|H^g \cap H| = 1$.
   (ii) H is core-free in G, and all nontrivial elements of H are conjugate in $N_G(H)$.

Proof. (i) Use the reformulation from Remark 2.5, and take g with $|H^g \cap H| = 1$ as one element in R, then $[\varphi_{H^g \cap H}, \psi^g_{H^g \cap H}] \neq 0$.

(ii) We may assume that H is nontrivial. By Brauer's Permutation Lemma, all non-trivial elements in $\mathrm{Irr}(H)$ are conjugate in $N_G(H)$. Since $\Gamma(G,H)$ is connected, there exists a nontrivial character $\varphi \in \mathrm{Irr}(H)$ such that $[1_H^G, \varphi^G] > 0$. Then $[1_H^G, \psi^G] > 0$ for

all $\psi \in \mathrm{Irr}(H)$. Since also $[\varphi^G, \psi^G] = [\varphi^G, \varphi^G] > 0$ for all nontrivial $\varphi, \psi \in \mathrm{Irr}(H)$ the result follows. $\quad \square$

An example where condition (b) (ii) in Proposition 2.2 is satisfied but Corollary 2.6 cannot be applied is the group $G = S_3 \times S_3$ where H is a non-normal S_3 type subgroup. Note that two different G-conjugates of H intersect in a subgroup of order 2 or 3.

Now we record some easy consequences of Proposition 2.2. Several results about diameter three subgroups hold in fact already if condition (b) (i) of this proposition is satisfied. We introduce the following notation.

Definition 2.7. A proper subgroup H of a group G is called *rich in G* if $[\chi_H, 1_H] \neq 0$ holds for all $\chi \in \mathrm{Irr}(G)$.

This terminology is motivated by part (i) of the following corollary.

Corollary 2.8. *Let H be a rich subgroup in a finite group G. Then the following assertions hold:*

(i) *Each character in $\mathrm{Irr}(G)$ is a constituent of the permutation character 1_H^G. In particular, we have $[G : H] \geq \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)$.*
(ii) *H is core-free in G.*
(iii) *The derived subgroup G' of G contains H; moreover, if $G' = H$ then G is abelian.*
(iv) *If χ is an irreducible character of G of degree 2 then the kernel of χ contains H'.*

Proof. (i) follows immediately from the definition of richness.

(ii) follows from the fact that $\Gamma(G, H)$ is connected, and Proposition 2.1.

(iii) follows from the definition of richness since $[\lambda_H, 1_H] \neq 0$ for every linear character λ of G, and that $G' = H$ happens only if H is trivial, by part (ii).

(iv) Suppose that $\chi \in \mathrm{Irr}(G)$ satisfies $[\chi_H, 1_H] \neq 0$ and that $\chi(1) = 2$. Then χ_H must be the sum of two linear characters of H. Thus H' is contained in the kernel of χ. $\quad \square$

Part (i) of Corollary 2.8 implies that rich subgroups must be "small", see Section 4.

Proposition 2.9. *Let G be a finite group with subgroups $1 < L \leq H < K \leq G$. If H is rich in K then L is rich in G, and if $\Gamma(K, H)$ has diameter 3 then $\Gamma(G, L)$ has diameter 3.*

Proof. Let H be rich in K, let $\chi \in \mathrm{Irr}(G)$, and let η be a constituent of χ_K. Then $[\chi_L, 1_L] \geq [\chi_H, 1_H] \geq [\eta_H, 1_H] > 0$, and L is rich in G. Suppose that $\Gamma(K, H)$ has diameter 3, let $\alpha, \beta \in \mathrm{Irr}(L)$, and let $\varphi, \psi \in \mathrm{Irr}(H)$ such that $[\alpha^H, \varphi] \neq 0 \neq [\beta^H, \psi]$. Then $[\alpha^G, \beta^G] \geq [\varphi^G, \psi^G] \geq [\varphi^K, \psi^K] > 0$, and $\Gamma(G, L)$ has diameter 3. $\quad \square$

By Proposition 2.9, a finite group G contains a diameter three subgroup if and only if G contains a diameter three subgroup of prime order. By Corollary 2.6 (i), this is the case if and only if G contains a rich subgroup of prime order, which is the case if and only if G contains a nontrivial rich subgroup. Note also that this property can be decided from the character table of G.

Lemma 2.10. *Let H be a rich subgroup in a finite group G, and let U be a subgroup of G such that $G = HU$. Then $U \cap H$ is rich in U.*

Proof. Let $\chi \in \mathrm{Irr}(U)$. By Frobenius reciprocity and Mackey decomposition, we have

$$[\chi_{U \cap H}, 1_{U \cap H}] = [(\chi_{U \cap H})^H, 1_H] = [(\chi^G)_H, 1_H],$$

and the right hand side is nonzero because any constituent ψ of χ^G satisfies $[\psi_H, 1_H] > 0$, by the assumption that H is rich in G. $\square$

The existence of a diameter three subgroup in the group G does in general not imply the existence of a diameter three subgroup in the factor group of G modulo a normal subgroup, see for example Corollary 2.12 below. However, the following holds.

Lemma 2.11.

1. *Let H be a rich subgroup in G, and let N be a proper normal subgroup of G. Then HN/N is a rich subgroup in G/N.*
2. *Let H be a diameter three subgroup of G, with H of prime order. If N is a normal subgroup of G that does not contain H then HN/N is a diameter three subgroup of G/N.*

Proof. In order to prove part 1, let $\chi \in \mathrm{Irr}(G/N)$, and view χ as a character of G. Since $[\chi_H, 1_H] > 0$ the subgroup H fixes a vector $v \neq 0$ in a module V affording χ. Since N acts trivially on V, the vector v is also fixed by HN, so that $[\chi_{HN}, 1_{HN}] > 0$. Also, choosing χ nontrivial we see that we cannot have $HN = G$. This shows that HN/N is rich in G/N.

Part 2. is just a special case of part 1. where H has prime order, by Corollary 2.6. $\square$

Corollary 2.12. *Let H be a nontrivial subgroup of a supersolvable group G. Then H is not rich in G, and $\Gamma(G, H)$ does not have diameter 3.*

Proof. Let G be a counterexample of minimal order, and let N be a minimal normal subgroup of G. Then N has prime order, so that $N < G$. Since H is core-free in G, N does not contain H. Thus HN/N is a nontrivial rich subgroup of the supersolvable group G/N, by Lemma 2.11. This contradicts the minimality of G. $\square$

Corollary 2.13. *Let H be a rich subgroup in G. If H is maximal in G then G is simple.*

Proof. If H is maximal in G and N is a nontrivial proper normal subgroup of G then either $N \leq H$ or $HN = G$ holds. The former cannot happen because rich subgroups are core-free. The latter cannot happen because of Lemma 2.11. □

See Section 4 for examples of rich subgroups that are maximal in simple groups.

In certain situations, one can go down from a finite group G with a rich subgroup to a smaller group with a rich subgroup.

Lemma 2.14. *Let G be a finite group with subgroups $H < K < G$.*

(i) Suppose that, for $\eta \in \mathrm{Irr}(K)$, there are $m_\eta \in \mathbb{N}$ and $\chi_\eta \in \mathrm{Irr}(G)$ such that $(\chi_\eta)_K = m_\eta \eta$. If the pair (G, H) satisfies Condition (b)(i) in Proposition 2.2 then the pair (K, H) also satisfies this condition.

(ii) Suppose that, for $\chi \in \mathrm{Irr}(G)$, there are $n_\chi \in \mathbb{N}$ and $\eta_\chi \in \mathrm{Irr}(K)$ such that $\chi_K = n_\chi \eta_\chi$. If the pair (G, H) satisfies Condition (b)(ii) in Proposition 2.2 then the pair (K, H) also satisfies this condition.

Proof. (i) Let $\eta \in \mathrm{Irr}(K)$, and let m_η and χ_η be as above. Then Condition (b)(i) for (G, H) implies: $0 < [(\chi_\eta)_H, 1_H] = [m_\eta \eta_H, 1_H]$, so that $[\eta_H, 1_H] > 0$. Thus Condition (b)(i) is satisfied for (K, H).

(ii) Let $\varphi, \psi \in \mathrm{Irr}(H)$. Then, by Condition (b)(ii) for (G, H), φ^G and ψ^G have a common constituent χ. Let n_χ and η_χ be as above. Then we have $0 < [\chi, \varphi^G] = [\chi_H, \varphi] = [n_\chi(\eta_\chi)_H, \varphi] = n_\chi[\eta_\chi, \varphi^K]$, so that $[\eta_\chi, \varphi^K] > 0$ and, similarly, $[\eta_\chi, \psi^K] > 0$. We conclude that $[\varphi^K, \psi^K] > 0$, and Condition (b)(ii) is satisfied for (K, H). □

Remark 2.15. (a) We note that the hypothesis of (i) is satisfied in the special case where every irreducible character of K extends to an (irreducible) character of G. In this case, K is sometimes called a CR-subgroup of G where CR stands for "character restriction"; see for example [16]. In particular, (i) applies whenever K has a normal complement in G.

(b) Similarly, the hypothesis in (ii) is satisfied in the special case where every irreducible character of G restricts to an irreducible character of K.

Lemma 2.16. *Let Z be a central subgroup of a finite group K, and suppose that we have subgroups $1 < H < G \leq K = GZ$. Then H is rich in G if and only if H is rich in K, and the diameter of $\Gamma(G, H)$ is 3 if and only if the diameter of $\Gamma(K, H)$ is 3.*

Proof. The direction from (G, H) to (K, H) follows from Proposition 2.9.

For the other direction, it is easy to see that the irreducible characters of K are extensions of the irreducible characters of G. (Note that K is isomorphic to a quotient of $G \times Z$.) Now apply Lemma 2.14. □

Proposition 2.17. *Let H_1 be a proper subgroup of a finite group G_1, and suppose that H_1 is rich in G_1 (or $\Gamma(G_1, H_1)$ has diameter 3). Moreover, let G_2 be a finite group which is isoclinic to G_1. Then there exists a proper subgroup H_2 of G_2 such that H_2 is rich in G_2 (or $\Gamma(G_2, H_2)$ has diameter 3).*

Proof. Since G_1 and G_2 are isoclinic, there exists a finite group K containing subgroups isomorphic to G_1 and G_2 (which we identify with G_1 and G_2) such that $G_1' = G_2'$ and $K = G_i Z_i$ with a central subgroup Z_i of K, for $i = 1, 2$. (This characterization of isoclinism is often attributed to Conway, see [8, Section 6.7]. A proof that it is equivalent to the usual definition can be found in [13, Theorem 4.2].)

We note that H_1 is contained in $G_1' = G_2'$ by Corollary 2.8 (iii). Thus we can view $H_2 := H_1$ as a subgroup of G_2 as well. Now we apply Lemma 2.16 twice: If $\Gamma(G_1, H_1)$ has diameter 3 then $\Gamma(K, H_1)$ has diameter 3, and therefore $\Gamma(G_2, H_1)$ has diameter 3. □

3. Examples

Example 3.1. Let $G := \mathrm{AGL}(1, p^n)$ be the affine group of degree 1 over a field with p^n elements where p is a prime and $n \geq 2$ is an integer. Then G is a Frobenius group with an elementary abelian kernel E of order p^n and a cyclic complement C of order $p^n - 1$. Moreover, $\mathrm{Irr}(G)$ consists of $p^n - 1$ linear characters and one faithful character χ of degree $p^n - 1$ (cf. [14, Satz V.16.13]). Let H be a subgroup of order p in G. The linear characters of G are trivial on E and thus on H, and χ_E is the sum of all nontrivial irreducible characters of E. Since $n \geq 2$, every irreducible character of H is a constituent of χ_H. Thus H is rich in G. Since also condition (b) (ii) is satisfied by Corollary 2.6 (i), $\Gamma(G, H)$ has diameter 3.

Example 3.2. Let G be the subgroup of order $p^2 d$ in $\mathrm{AGL}(1, p^2)$, where $d > 1$ and d divides $p^2 - 1$. As in Example 3.1, G is a Frobenius group with an elementary abelian kernel E of order p^2 and a cyclic complement C of order d. Then G has a diameter three subgroup (necessarily of order p) if and only if d is divisible by $(p + 1)(p - 1)_2$, where $(p - 1)_2$ is the 2-part of $p - 1$. Equivalently, this happens if $(p^2 - 1)/d$ is an odd divisor of $p - 1$.

First note that d is divisible by $(p + 1)(p - 1)_2$ if and only if all $p + 1$ subgroups of order p in E are conjugate in G. Note that $|N_C(H)| = \gcd(d, p - 1)$, where H is a subgroup of order p. In order to see this, observe that on the one hand, if d is divisible by $(p+1)(p-1)_2$ then $\gcd(d, p-1) = d/(p+1)$, so $|C/N_C(H)| = p+1$ holds, and on the other hand, if there is only one class of order p subgroups in G then $|N_C(H)| = d/(p+1)$, which means that $(p - 1)_2$ divides $d/(p + 1)$.

Let H be a subgroup of order p in E. Then G has $(p^2 - 1)/d$ nonlinear irreducible characters of degree d. The restriction of each such character to E is a sum of d nontrivial irreducibles of E.

If there is only one class of order p subgroups in G then the transitive action of C on these $p + 1$ subgroups means that each C-orbit of nontrivial irreducibles of E contains at least one character with kernel H, thus H is a diameter three subgroup of G.

Conversely, let $H_1, H_2, \ldots, H_k$ be representatives of conjugacy classes of subgroups of order p in G. Each C-orbit of nontrivial irreducibles of E contains characters with kernels only from one class of subgroups of order p. Considering an orbit containing no character with kernel H_i yields that H_i cannot be a diameter three subgroup.

Example 3.3. Let G be the subgroup of order $p^n d$ in $\mathrm{AGL}(1, p^n)$, where d divides $p^n - 1$. Then G acts on the set of $(p^n - 1)/(p - 1)$ subgroups of order p^{n-1} in G by conjugation. They are the kernels of the nontrivial characters of the normal subgroup E of order p^n in G. Let U be a subgroup of order p^{n-1} in G. Its normalizer has order $p^n \gcd(d, p - 1)$. Thus U has precisely $d/\gcd(d, p - 1)$ conjugates in G. Hence there are precisely $(p^n - 1)\gcd(d, p - 1)/((p - 1)d)$ conjugacy classes of subgroups of order p^{n-1} in G. If we can choose one subgroup from each of these conjugacy classes such that their intersection is nontrivial then this intersection contains a diameter three subgroup of order p in G. Since the intersection of k subgroups of order p^{n-1} yields a subgroup of order at least p^{n-k}, such a choice is possible if $n > (p^n - 1)\gcd(d, p - 1)/((p - 1)d)$.

This implies that if $n \geq 3$ and p is odd then the subgroup of index 2 in $\mathrm{AGL}(1, p^n)$ has a diameter three subgroup of order p. Example 3.2 shows that this is not the case for $n = 2$.

(The bound is not sharp. For example, the subgroup of order $7^3 \cdot 19$ in $\mathrm{AGL}(1, 7^3)$ has 3 classes of subgroups of order 7^2, and still has a diameter three subgroup.)

Example 3.4. Let H be a subgroup of order 2 in a nonabelian finite simple group G. Then $\Gamma(G, H)$ has diameter 3. In order to see this, we check that for any $\chi \in \mathrm{Irr}(G)$, $[\chi_H, 1_H] \neq 0$ holds. Write $H = \{1, h\}$. If $\chi \in \mathrm{Irr}(G)$ satisfies $[\chi_H, 1_H] = 0$ then $\chi(h) = -\chi(1)$, and h is contained in $Z(\chi)$, the center of χ. In particular, we have $Z(\chi) \neq 1$. Since G is simple, this implies $Z(\chi) = G$. Since $Z(\chi)/\ker(\chi)$ is cyclic we conclude that $\ker(\chi) = G$. Thus $\chi = 1_G$ which, however, is impossible.

Remark 3.5. Example 3.4 shows that nonabelian finite simple groups always have diameter three subgroups. This fact does not generalize to quasisimple finite groups; in fact, the quasisimple group $\mathrm{SL}(2, 5) = 2.A_5$ does not have a diameter three subgroup. This can be seen by noting that $\mathrm{SL}(2, 5)$ is a Frobenius complement; thus it cannot have nontrivial rich subgroups, by [12, (25.5)].

See Section 5 for more about quasisimple groups.

Proposition 3.6. *Every nonabelian finite simple group has a solvable subgroup which contains a diameter three subgroup of order* 2.

Proof. It is known (cf. the main result of [1]) that every nonabelian finite simple group contains a minimal simple group, i.e., a nonabelian simple group all of whose proper

subgroups are solvable. Thus it suffices to prove that every minimal simple group has a proper subgroup which contains a diameter three subgroup of order 2. The minimal simple groups were classified by Thompson; they are given as follows, cf. [14, Bemerkung II.7.5]:

$$
\begin{array}{ll}
\mathrm{PSL}(2,p), & p > 3 \text{ a prime with } p^2 - 1 \not\equiv 0 \pmod 5, \\
\mathrm{PSL}(2,2^q), & q \text{ a prime}, \\
\mathrm{PSL}(2,3^q), & q \text{ an odd prime}, \\
\mathrm{PSL}(3,3), & \\
\mathrm{Sz}(2^q), & q \text{ an odd prime}.
\end{array}
$$

By part (4) of [14, Satz II.8.27], the groups $\mathrm{PSL}(2,p)$, where $p > 3$ is a prime, and the groups $\mathrm{PSL}(2,3^q)$ contain subgroups isomorphic to the alternating group A_4 which has a diameter three subgroup of order 2. Part (7) of the same theorem yields Frobenius groups with Frobenius kernel of 2-power order, as in Example 3.1, as subgroups of $\mathrm{PSL}(2,2^q)$. The question about $\mathrm{PSL}(3,3)$ can be answered computationally; this group has maximal subgroups of the type S_4, and hence subgroups isomorphic to A_4. Finally, the Suzuki group $G = \mathrm{Sz}(2^q)$ has a Sylow 2-subgroup P of order 2^{2q}, with an elementary abelian center Z of order 2^q (cf. [15, Lemma XI.3.1]). The normalizer of P in G is a semidirect product of P with a cyclic group of order $2^q - 1$, it contains a subgroup S which is the semidirect product of Z with the cyclic group of order $2^q - 1$. This group S is a Frobenius group with Frobenius kernel of 2-power order, as in Example 3.1. □

4. Large rich subgroups

Part (i) of Corollary 2.8 implies that rich subgroups must be "small". Here is a quantitative version of this statement.

Proposition 4.1. Let $T(G) = \sum_\chi \chi(1)$, where χ runs over $\mathrm{Irr}(G)$, $k(G) = |\mathrm{Irr}(G)|$, and $b(G) = \max\{\chi(1); \chi \in \mathrm{Irr}(G)\}$.
  (i) We have

$$
\begin{aligned}
|G| &= \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 \\
&= \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \cdot b(G) - \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \cdot (b(G) - \chi(1)) \\
&= T(G) \cdot b(G) - [G : G'] \cdot (b(G) - 1) - \sum_{1 < \chi(1) < b(G)} \chi(1) \cdot (b(G) - \chi(1)).
\end{aligned}
$$

(ii) We have

$$
|G| \leq T(G) \cdot b(G) \leq T(G) \cdot (T(G) - k(G) + 1).
$$

Moreover, we have $|G| = T(G)b(G)$ if and only if G is abelian.

(iii) If $1 \neq H \leq G$ is rich in G then $|H| \leq [G : H] - k(G) + 1$ and $|H| < b(G)$. In particular, $|H| < \sqrt{|G|}$.

Proof. Part (i) is clear, part (ii) follows easily from part (i), and part (iii) follows from (ii) since $T(G) \leq [G : H]$ by Corollary 2.8. $\quad\square$

Proposition 4.2. *Let H be a rich subgroup in a finite group G, and suppose that $[G : H]$ is a power of a prime p. Then $H = 1$.*

Proof. Let G be a minimal counterexample. Then G is not a p-group, by Corollary 2.12. Let P be a Sylow p-subgroup of G. Then $G = HP$ since $[G : H]$ and $[G : P]$ are coprime. By Lemma 2.10, $H \cap P$ is rich in P. Thus Corollary 2.12 implies that $H \cap P = 1$. Hence $|G| = |H| \cdot |P|$, so that H is a Hall p'-subgroup of G. Thus 1_H^G is the character of the projective cover of the trivial module (in characteristic p). Thus its constituents lie in the principal p-block of G. We conclude that G has only one p-block. Let N be a minimal normal subgroup of G. Then N is isomorphic to S^k where k is a positive integer and S is a simple group. Moreover, N has a unique p-block, and S has a unique p-block; in particular, p divides $|S|$. If S is nonabelian then, as is well-known, we have $p = 2$, and S is isomorphic to M_{22} or M_{24}. On the other hand, N and S both have Hall p'-subgroups which is a contradiction. This shows that S is abelian, i.e., N is an elementary abelian p-group. By Lemma 2.11, HN/N is rich in G/N and of p-power index. Since $|G/N| < |G|$ this implies $HN/N = 1$, i.e., $1 < H < N$. Then G/N and G are p-groups, and we have a contradiction. $\quad\square$

Remark 4.3. If $1 \neq H \leq G$ is rich in G then $1_H^G = \sum_\chi a_\chi \chi$, with $a_\chi > 0$ for all $\chi \in \mathrm{Irr}(G)$, hence the number $[1_H^G, 1_H^G] = \sum_\chi a_\chi^2$ of H-H-double cosets in G, which is equal to the rank of the permutation action of G on the cosets of H, is at least equal to $k(G)$. Note that Proposition 4.1 only yields that this rank is at least 3, because rank 2 would imply a doubly transitive action of G and hence $|G| \geq [G : H]([G : H] - 1)$.

Example 4.4. Let $G := \mathrm{AGL}(1, 2^n)$ be the group from Example 3.1 in the special case $p = 2$, but now choose a subgroup H of order p^{n-1} in G. Then the linear irreducibles of G restrict to 1_H, and because $T(G) = 2(2^n - 1) = [G : H]$ holds, also the unique nonlinear character χ of G occurs with multiplicity 1 in 1_H^G, that is, 1_H^G is exactly the sum of $\mathrm{Irr}(G)$. In order to verify condition (b) (ii) and hence to show that H is indeed a diameter three subgroup of G, we note that χ is a constituent of each φ^G, for $\varphi \in \mathrm{Irr}(H)$.

Now we may take the direct product of G with an abelian group (or more generally, take a group isoclinic with G), and keep the subgroup H, then we get again that H is a diameter three subgroup of G with the property that $[G : H] = T(G)$ holds, that is, H has the largest possible order. Moreover, the rank of the permutation action of G on the cosets of H is exactly $k(G)$.

Table 1
Groups having nontrivial rich subgroups of index at most 45.

n	$\lvert G\rvert$	i	G	diam. 3?	n	$\lvert G\rvert$	i	G	diam. 3?
6	12	3	A_4	+		60	9	$5 \times A_4$	+
12	24	12	S_4	+		240	191	$2^4:15$	+
	24	13	$2 \times A_4$	+	36	72	15	$(2^2:9):2$	+
14	56	11	$2^3:7$	+		72	16	$2 \times 2^2:9$	+
18	36	3	$2^2:9$	+		72	42	$3 \times S_4$	+
	36	11	$3 \times A_4$	+		72	43	$(3 \times A_4):2$	+
20	60	5	A_5	+		72	44	$A_4 \times S_3$	+
	80	49	$2^4:5$	+		72	47	$6 \times A_4$	+
24	48	3	$4^2:3$	+		144	184	$A_4 \times A_4$	+
	48	30	$A_4:4$	+	39	351	12	$3^3:13$	-
	48	31	$4 \times A_4$	+	40	80	49	$2^4:5$	+
	48	48	$2 \times S_4$	+		120	34	S_5	+
	48	49	$2^2 \times A_4$	+		120	35	$2 \times A_5$	+
	48	50	$2^4:3$	+		160	234	$(2^4:5):2$	+
	72	39	$3^2:8$	+		160	235	$2 \times 2^4:5$	+
	72	41	$3^2:Q_8$	+	42	84	10	$7 \times A_4$	+
	96	70	$(2^4:2):3$	+		84	11	$(14 \times 2):3$	+
	96	71	$(4^2:2):3$	+		168	42	$PSL(3,2)$	+
28	56	11	$2^3:7$	+		168	43	$2^3:(7:3)$	+
	112	41	$2 \times 2^3:7$	+		168	44	$3 \times 2^3:7$	+
30	60	5	A_5	+					

Remark 4.5. If $H < G$ is rich in G then the action of G on the cosets of H is faithful. Hence there are, for each prescribed integer n, only finitely many pairs (G, H) such that H is a rich subgroup of index at most n in G. Table 1 lists all groups G with a nontrivial rich subgroup of index at most 45. It was computed using the list of all groups of order at most $2\,000$, up to isomorphism, that contain a nontrivial rich subgroup, see Section 6.3. The columns show $n = [G : H]$, $\lvert G\rvert$, the number i such that G can be obtained as the i-th group of its order, according to the numbering in [2], a structure description of G, and a $+$ sign if the point stabilizer H is a diameter three subgroup of G –the Frobenius group of order $3^3 \cdot 13 = 351$ is the only example where this is not the case, see Remark 2.3.

Proposition 4.6. *Let H be a nontrivial rich subgroup of index $2p$ in a finite group G where p is a prime. Then p is a Mersenne prime, and G is a Frobenius group of order $p(p+1)$.*

Proof. By Proposition 4.2, p is odd. Let P be a Sylow p-subgroup of G. If P is normal in G then Lemma 2.11 implies that either $H \leq P$ or HP/P is a rich subgroup of index 2 in G/P. The former case cannot occur because then H is a p-group, and $\lvert G\rvert < 4p^2$ implies $\lvert H\rvert = p$ and thus $\lvert G\rvert = 2p^2$; however, this means that $\lvert H\rvert < b(G) \leq 2$, hence H is trivial. The latter case cannot occur because then HP/P is normal in G/P which is a contradiction since rich subgroups are core-free. Thus P is not normal in G. By the Itô-Michler theorem, there is $\chi \in \mathrm{Irr}(G)$ such that p divides $\chi(1)$. Since $\chi(1) \leq b(G) < T(G) \leq [G : H] = 2p$ we conclude that $b(G) = \chi(1) = p$. Thus $\lvert H\rvert < b(G) < p + 1 \leq [G : N_G(P)] \leq [G : P] = 2\lvert H\rvert < 2p$. Hence we obtain $P = N_G(P)$ and $\lvert G\rvert = p(p+1)$. Since $N_G(P) = C_G(P)$, a theorem of Burnside implies that G has a normal p-complement N. Since $\lvert N\rvert = p + 1$, P acts transitively on $N \setminus \{1\}$. Since $p + 1$

Table 2
Rich subgroups in simple groups, maximal w. r. t. inclusion.

G	n	g	m	H
A_5	9	2	2	2, 3
$\mathrm{PSL}(3,2)$	15	3	2	3, 4
A_6	22	9	6	5, 4, 2^2, S_3
$\mathrm{PSL}(2,8)$	12	4	3	3, 2^2, 7
$\mathrm{PSL}(2,11)$	16	7	5	2^2, 5, S_3, 6
$\mathrm{PSL}(2,13)$	16	7	5	2^2, 6, S_3, 7
$\mathrm{PSL}(2,17)$	22	10	5	S_3, 8, D_8, 9
A_7	40	16	9	5, S_3, 3^2, A_4, 6×2, $3{:}4$
$\mathrm{PSL}(2,19)$	19	10	6	S_3, 9, D_{10}, 10, A_4
$\mathrm{PSL}(2,16)$	21	11	5	S_3, 2^3, D_{10}, A_4, 15
$\mathrm{PSL}(3,3)$	51	14	6	S_3, Q_8, 8, A_4, $3 \times S_3$
$\mathrm{PSU}(3,3^2)$	36	3	2	3, 4
$\mathrm{PSL}(2,23)$	23	11	5	S_3, D_8, 11, 12
$\mathrm{PSL}(2,25)$	37	19	9	D_8, D_{10}, D_{12}, A_4, 12, 13
M_{11}	39	10	7	2^2, 5, S_3, 6, Q_8, 8
$\mathrm{PSL}(2,27)$	16	10	6	3^2, A_4, 13, 14, D_{14}
$\mathrm{PSL}(2,29)$	22	12	7	S_3, D_{10}, A_4, 14, D_{14}, 15
$\mathrm{PSL}(2,31)$	29	15	8	S_3, D_8, D_{10}, A_4, 15, 16
$\mathrm{PSL}(3,4)$	95	60	15	D_{10}, A_4, $7{:}3$, S_4, $4^2{:}2$, $3^2{:}4$
A_8	137	46	17	7, D_8, D_{10}, A_4, 6×2, $3{:}4$, D_{12}, $2 \times D_8$, $(4 \times 2){:}2$, $3^2{:}2$, $3 \times S_3$, S_4, $2 \times A_4$
$\mathrm{PSL}(2,37)$	23	13	6	D_{12}, A_4, 18, D_{18}, 19
$\mathrm{PSU}(4,2^2)$	116	20	7	2^2, 4, 5, S_3, A_4, $(4 \times 2){:}2$, $3 \times S_3$
$Sz(8)$	22	9	6	5, 7, 4×2, 13
$\mathrm{PSL}(2,32)$	24	16	4	S_3, 11, 2^4, 31
$\mathrm{PSL}(2,41)$	33	21	7	D_{14}, D_{20}, 20, 21, S_4
$\mathrm{PSL}(2,43)$	20	12	7	S_3, A_4, D_{14}, 21, 22, D_{22}
$\mathrm{PSL}(2,47)$	29	19	7	A_4, D_{12}, D_{16}, 23, 24
$\mathrm{PSL}(2,49)$	51	33	10	D_{10}, D_{16}, 24, D_{24}, S_4, 25, $7{:}6$
$\mathrm{PSU}(3,4^2)$	34	4	3	3, 4, 5
$\mathrm{PSL}(2,53)$	20	12	6	A_4, D_{18}, 26, D_{26}, 27
M_{12}	147	64	13	11, A_4, D_{12}, $2 \times D_8$, $3 \times S_3$, $2 \times A_4$, $8{:}2^2$, $4^2{:}2$, $(8{:}2){:}2$, $2 \times 5{:}4$
$\mathrm{PSL}(2,59)$	26	18	7	D_{12}, A_4, D_{20}, 29, 30, D_{30}
$\mathrm{PSL}(2,61)$	32	18	7	A_4, D_{12}, D_{20}, 30, D_{30}, 31
$\mathrm{PSU}(3,5^2)$	80	21	9	5, S_3, 8, $3{:}4$, $7{:}3$, $\mathrm{SL}(2,3)$, $3 \times A_4$
$\mathrm{PSL}(2,67)$	20	12	7	S_3, A_4, D_{22}, 33, 34, D_{34}
J_1	40	30	12	D_{12}, D_{20}, D_{22}, $2 \times A_4$, $3 \times D_{10}$, D_{30}, $5 \times S_3$, $7{:}6$, $11{:}5$, $19{:}3$, A_5

is even, N has to be an elementary abelian 2-group, and G is a Frobenius group with kernel N.   □

Remark 4.7. The statement of Proposition 4.6 generalizes to the situation of nontrivial rich subgroups of index pq where p and q are odd primes. We hope to publish this result in a sequel.

Remark 4.8. For a given group G, we can ask which of its subgroups H are maximal with the property that H is rich in G. Table 2 lists these subgroups H (where duplicate isomorphism types have been removed) for some small simple groups G. The columns labelled by n, g, m list the total number of classes of subgroups of the group G, the number of classes of rich subgroups H, and the number of classes of rich subgroups H of maximal order. In fact, all subgroups listed in the table are diameter three subgroups.

Table 3
Rich maximal subgroups in Lie type simple groups.

G	H
$\mathrm{PSL}(2,27)$	A_4
$\mathrm{PSL}(2,109)$	A_5, A_5
$\mathrm{PSL}(2,113)$	S_4, S_4
$\mathrm{PSL}(2,125)$	A_5
$\mathrm{PSL}(3,7)$	$3^2{:}Q_8$, $19{:}3$
$\mathrm{PSU}(3,8^2)$	$19{:}3$
$\mathrm{PSL}(3,8)$	$7^2{:}S_3$, $\mathrm{PSL}(3,2)$
$\mathrm{PSU}(3,11^2)$	$37{:}3$
$^2G_2(27)$	$(2^2 \times D_{14}){:}3$, $19{:}6$

Table 4
Rich maximal subgroups in sporadic simple groups.

G	H
J_1	$7{:}6$
Suz	A_7
ON	$3^4{:}2^{1+4}D_{10}$, M_{11}, M_{11}, A_7, A_7
$Fi22$	M_{12}
Ly	$67{:}22$, $37{:}18$
Th	$A_5.2$
$J4$	$\mathrm{PGL}(2,23)$, $\mathrm{PSU}(3,3^2)$, $29{:}29$, $43{:}14$, $37{:}12$
F_{3+}	$7{:}6 \times A_7$, $\mathrm{PGL}(2,13)$, $\mathrm{PGL}(2,13)$, $29{:}14$
B	$\mathrm{PSL}(2,49).2_3$, $\mathrm{PSL}(2,31)$, $\mathrm{PSL}(3,3)$, $\mathrm{PGL}(2,17)$, $\mathrm{PGL}(2,11)$, $47{:}23$
M	$(7{:}3 \times He){:}2$, $(5^2{:}[2^4] \times \mathrm{PSU}(3,5^2)).S_3$, $7^{2+1+2}{:}\mathrm{GL}(2,7)$, $(S_5 \times S_5 \times S_5){:}S_3$, $(\mathrm{PSL}(2,11) \times \mathrm{PSL}(2,11)){:}4$, $(7^2{:}(3 \times 2A_4) \times \mathrm{PSL}(2,7)).2$, $(13{:}6 \times \mathrm{PSL}(3,3)).2$, $\mathrm{PSU}(3,4^2).4$, $\mathrm{PSL}(2,71)$, $\mathrm{PSL}(2,59)$, $11^2{:}(5 \times 2.A_5)$, $\mathrm{PSL}(2,41)$, $\mathrm{PGL}(2,29)$, $7^2{:}\mathrm{SL}(2,7)$, $\mathrm{PGL}(2,19)$, $\mathrm{PGL}(2,13)$, $41{:}40$

Remark 4.9. By Corollary 2.13, a maximal subgroup H of G can be rich only if G is simple. Examples where this happens in simple groups G of Lie type are listed in Table 3. Table 4 lists all maximal subgroups H in sporadic simple groups G that are rich. The subgroups shown in these tables are diameter three subgroups.

5. Quasisimple groups

The aim of this section is to classify those quasisimple groups, i.e., perfect central extensions of simple groups, which have a diameter three subgroup.

Theorem 5.1. *Let G be a quasisimple group. Then G has a diameter three subgroup, except if $G \cong \mathrm{SL}(2,5) \cong 2.A_5$ or $G \cong \mathrm{SL}(2,9) \cong 2.A_6$ or $G \cong 6.A_6$.*

The proof uses the classification of finite simple groups.

The idea is to either establish directly the existence of a diameter three subgroup, or to prove the existence of a proper quasisimple subgroup which is already known to have a diameter three subgroup.

We start with computational results in cases where we do not know a conceptual approach.

Table 5
Simple groups with exceptional Schur multiplier.

S			e	S			e
		A_6	3	$B_2(2)$	$\cong$	$\mathrm{PSp}(4,2)$	2
		A_7	3	$^2B_2(2)$	$\cong$	$\mathrm{Sz}(8)$	2^2
$A_1(4)$	$\cong$	$\mathrm{PSL}(2,4)$	2	$B_3(2)$	$\cong$	$\mathrm{PSp}(6,2)$	2
$A_1(9)$	$\cong$	$\mathrm{PSL}(2,9)$	3	$B_3(3)$	$\cong$	$\mathrm{P\Omega}(7,3)$	3
$A_2(2)$	$\cong$	$\mathrm{PSL}(3,2)$	2	$C_3(2)$	$\cong$	$\mathrm{PSp}(6,2)$	2
$A_2(4)$	$\cong$	$\mathrm{PSL}(3,4)$	4×4	$D_4(2)$	$\cong$	$\mathrm{P\Omega}^+(8,2)$	2^2
$A_3(2)$	$\cong$	$\mathrm{PSL}(4,2)$	2	$G_2(3)$			3
$^2A_3(2)$	$\cong$	$\mathrm{PSU}(4,2^2)$	2	$G_2(4)$			2
$^2A_3(3)$	$\cong$	$\mathrm{PSU}(4,3^2)$	3^2	$F_4(2)$			2
$^2A_5(2)$	$\cong$	$\mathrm{PSU}(6,2^2)$	2^2	$^2E_6(2)$			2^2

Lemma 5.2. *Let G be a perfect central extension of a simple group that is either sporadic simple or has an exceptional Schur multiplier. Then G has a diameter three subgroup, except in the exceptional cases of Theorem 5.1.*

Proof. The character tables of all groups G in question are available in the Character Table Library [3], and checking the conditions of Proposition 2.2 requires only the character table of the group.

The simple groups S with exceptional Schur multiplier e are listed in Table 5, cf. Table 6 of [8]. □

Next we deal with alternating groups.

Lemma 5.3. *Let G be a perfect central extension of the alternating group A_n on n points, for $n \geq 7$. Then G has a diameter three subgroup.*

Proof. The claim holds for $n = 7$, by Lemma 5.2, and the alternating group on $n > 7$ points contains A_7. □

It remains to deal with the simple groups of Lie type. The key result is about $\mathrm{PSL}(2,q)$.

Lemma 5.4. *Let q be a prime power with $q > 3$. If $5 \neq q \neq 9$ then $G := \mathrm{SL}(2,q)$ has a diameter three subgroup.*

Proof. If q is even then G is simple. Thus G has a diameter three subgroup of order 2, by Example 3.4. Hence we may assume that q is odd. Let H be a subgroup of order 3 in G. We will show that $\Gamma(G,H)$ has diameter 3, using Proposition 2.2. By Corollary 2.6, it suffices to verify that H is rich in G.

In the following, we use the notation from [11, § 38]. We distinguish several cases.

Case 1: $q \equiv 1 \pmod 3$; in particular, $q \geq 7$.

The subgroup $\langle a \rangle$ of order $q - 1$ in G contains an element $h := a^l$ of order 3, and we may assume that $H = \langle h \rangle$. We need to show that, for $\chi \in \mathrm{Irr}(G)$, we have $0 \neq [\chi_H, 1_H]$, i.e.,

$$\sigma_\chi := \chi(1) + \chi(h) + \overline{\chi(h)} > 0.$$

This is trivial if $\chi = 1_G$ or $\chi(h) = 0$. If $\chi(1) \in \{q, (q+1)/2\}$ then $|\chi(h)| = 1$, and thus $\sigma_\chi \geq 4 - 1 - 1 > 0$. If $\chi(1) = q + 1$ then $|\chi(h)| \leq 2$, and thus $\sigma_\chi \geq 8 - 2 - 2 > 0$, and the result is proved in this case.

Case 2: $q \equiv 2 \pmod 3$; in particular, $q \geq 11$.

In this case we may assume that $H = \langle h \rangle$ is contained in the subgroup $\langle b \rangle$ of order $q + 1$ in G. With notation as above, we need to show that $\sigma_\chi > 0$, and we may again assume that $\chi \neq 1_G$ and $\chi(h) \neq 0$. If $\chi(1) \in \{q, (q-1)/2\}$ then $|\chi(h)| = 1$, and thus $\sigma_\chi \geq 5 - 1 - 1 > 0$. If $\chi(1) = q - 1$ then $|\chi(h)| \leq 2$, and thus $\sigma_\chi \geq 10 - 2 - 2 > 0$. The result follows in this case as well.

Case 3: $q \equiv 0 \pmod 3$; in particular, q is a power of 3, and $q \geq 27$.

In this case we may take $H = \langle h \rangle$ where $h := c$. Again we need to show that $\sigma_\chi > 0$, and we may assume that $\chi \neq 1_G$ and $\chi(h) \neq 0$. If $\chi(1) = q \pm 1$ then $|\chi(h)| = 1$, and thus $\sigma_\chi \geq 26 - 1 - 1 > 0$. If $\chi(1) = (q \pm 1)/2$ then $|\chi(h)| \leq (1 + \sqrt{q})/2$, and thus $\sigma_\chi \geq (q-1)/2 - 1 - \sqrt{q} > 0$. This completes the proof of the lemma. $\quad\square$

Lemma 5.5. *Let G be a quasisimple finite group with $G/Z(G) \cong \mathrm{PSL}(n,q)$, for some positive integer n and some prime power q. Then G has a diameter three subgroup, except when G is isomorphic to one of the groups $\mathrm{SL}(2,5) = 2.A_5$, $\mathrm{SL}(2,9) = 2.A_6$ or $6.A_6$.*

Proof. Since G is quasisimple, we have $n \geq 2$ and $(2,2) \neq (n,q) \neq (2,3)$. Moreover, G is isomorphic to a factor group of the Schur cover X of $S := \mathrm{PSL}(n,q)$, and $Z(X)$ is isomorphic to the Schur multiplier $M(S) := \mathrm{H}^2(S, \mathbb{C}^\times)$. The order of $M(S)$ is $\gcd(n, q-1)$, with the exceptions

$$(n,q) \in \{(2,4), \ (2,9), \ (3,2), \ (3,4), \ (4,2)\},$$

which have been dealt with in Lemma 5.2. Thus from now on we may assume that (n,q) is not one of these exceptional values. Then X is isomorphic to $\mathrm{SL}(n,q)$. By Lemma 5.4, we may assume that $n \geq 3$, and that $Z(G) \neq 1$. Recall that $\mathrm{SL}(n,q)$ contains subgroups isomorphic to $\mathrm{SL}(n-1, q)$ and subgroups isomorphic to $\mathrm{SL}(n, p)$ where p is the prime dividing q. This implies the lemma for $n = 3$. (Note that $M(\mathrm{SL}(3,3))$ and $M(\mathrm{SL}(3,5))$ are trivial.) The result for $n \geq 4$ then follows by induction on n. $\quad\square$

Lemma 5.6. *Let G be a quasisimple finite group with $G/Z(G) \cong \mathrm{PSp}(2n,q)$, for some integer $n \geq 2$ and some prime power q. Then G has a diameter three subgroup.*

Proof. The exceptional case $(n,q) = (3,2)$ has been done in Lemma 5.2, we assume that $(n,q) \neq (3,2)$. Then the Schur multiplier of $\mathrm{PSp}(2n,q)$ is trivial when q is even, and of order 2 when q is odd. Thus we may assume that q is odd. Then $\mathrm{Sp}(2n,q)$ is the Schur cover of $\mathrm{PSp}(2n,q)$. Since our result is known for simple groups, we may assume

Table 6
Subgroups obtained from Dynkin diagrams.

S				T		
$B_n(q)$	$\cong$	$P\Omega(2n+1,q)$	$n \geq 3$	$A_{n-1}(q)$	$\cong$	$\mathrm{PSL}(n,q)$
$D_n(q)$	$\cong$	$P\Omega^+(2n,q)$	$n \geq 4$	$A_{n-1}(q)$	$\cong$	$\mathrm{PSL}(n,q)$
$^2D_n(q)$	$\cong$	$P\Omega^-(2n,q)$	$n \geq 4$	$A_{n-2}(q)$	$\cong$	$\mathrm{PSL}(n-1,q)$
$E_6(q)$				$A_5(q)$	$\cong$	$\mathrm{PSL}(6,q)$
$^2E_6(q)$				$^2A_5(q)$	$\cong$	$\mathrm{PSU}(6,q^2)$
$E_7(q)$				$E_6(q)$		

that $G \cong \mathrm{Sp}(2n,q)$ where $n > 1$ and q is odd. Recall that $\mathrm{Sp}(2n,q)$ contains subgroups isomorphic to $\mathrm{Sp}(2n-2,q)$. Thus it suffices to prove the result for $n = 2$. Since $\mathrm{Sp}(4,q)$ contains subgroups isomorphic to $\mathrm{Sp}(2,q^2) \cong \mathrm{SL}(2,q^2)$, it suffices to consider the case $q = 3$. But then we know that G contains subgroups isomorphic to $\mathrm{PSL}(3,3)$. $\quad\square$

Lemma 5.7. _Let G be a quasisimple finite group with $G/Z(G) \cong \mathrm{PSU}(n,q^2)$ for some integer $n \geq 3$ and some prime power q. Then G has a diameter three subgroup._

Proof. First we consider the case $n = 3$. Then $q > 2$ since $\mathrm{PSU}(3,2^2)$ is solvable. Since the Schur multiplier of $\mathrm{PSU}(3,q^2)$ has order $\gcd(3,q+1)$, we may assume that $q \equiv 2 \pmod 3$ since otherwise G is simple and thus we know already that the result holds. Then G is isomorphic to $\mathrm{SU}(3,q^2)$. Since $\mathrm{SU}(3,q^2)$ contains subgroups isomorphic to $\mathrm{SU}(2,q^2) \cong \mathrm{SL}(2,q)$, Lemma 5.4 implies the lemma unless perhaps $q = 5$. But $\mathrm{SU}(3,5^2)$ contains subgroups isomorphic to A_7.

Next we consider the case $n = 4$. The groups $\mathrm{PSU}(4,2^2) \cong \mathrm{PSp}(4,3)$ and $\mathrm{PSU}(4,3^2)$ have been dealt with in Lemma 5.2. Thus we may assume that $q > 3$. Then $\mathrm{SU}(4,q^2)$ is a Schur cover of $\mathrm{PSU}(4,q^2)$. Since $\mathrm{SU}(4,q^2)$ contains subgroups isomorphic to $\mathrm{SU}(3,q^2)$, the result follows from the previous case.

Finally, we consider the case $n \geq 5$. The group $\mathrm{PSU}(6,2^2)$ has been dealt with in Lemma 5.2. Thus we may assume that $(n,q) \neq (6,2)$. Then $\mathrm{SU}(n,q^2)$ is a Schur cover of $\mathrm{PSU}(n,q^2)$, and $\mathrm{SU}(n,q^2)$ has subgroups isomorphic to $\mathrm{SU}(n-1,q^2)$. Thus our result follows by induction on n. $\quad\square$

Lemma 5.8. _Let G be a quasisimple finite group with $G/Z(G)$ of one of the following types: $P\Omega(2n+1,q)$ $(n \geq 3)$, $P\Omega^+(2n,q)$ $(n \geq 4)$, or $P\Omega^-(2n,q)$ $(n \geq 4)$, where q is some prime power. Then G has a diameter three subgroup._

Proof. In all three cases, Table 6 lists a simple group T such that $S = G/Z(G)$ has a subgroup that is isomorphic to a perfect central extension of T; this subgroup is obtained by removing suitable nodes from the Dynkin diagram of S. Hence the claim follows from Lemma 5.5. $\quad\square$

Proof. (of Theorem 5.1) According to the classification of the nonabelian finite simple groups, each such group is either alternating (see Lemma 5.3), sporadic simple (see Lemma 5.2), or a group of Lie type of the type A (see Lemma 5.4 and Lemma 5.5), C

(see Lemma 5.6), ^2A (see Lemma 5.7), B, D, ^2D (see Lemma 5.8), or of exceptional type.

Thus it remains to show the claim for the latter groups. Table 6 (cf. [8, Table 5]) lists the series of those simple groups S with nontrivial Schur multiplier (omitting the cases that were dealt with in Lemma 5.2) and a simple factor T of a quasisimple subgroup of S, where we know already that any perfect central extension of T has a diameter three subgroup. The groups T can be read off from the Dynkin diagrams of the groups S.

(The Schur multipliers of $^2B_2(q)$, $^3D_4(q)$, $G_2(q)$, $^2G_2(q)$, $F_4(q)$, $^2F_4(q)$, and $E_8(q)$ are trivial, apart from the exceptions listed in Lemma 5.2.)   □

6. Minimal groups with diameter three subgroups

By Proposition 2.9, we can ask for the smallest subgroups of a given group that have a diameter three subgroup. In this section, we study groups that are minimal w. r. t. inclusion in this respect.

Example 3.2 states that $\mathrm{AGL}(1, p^2)$ has a unique subgroup that is minimal in this sense, whose index in $\mathrm{AGL}(1, p^2)$ is the odd part of $p - 1$. Proposition 3.6 states that simple groups are never minimal.

Example 3.2 also implies that the number of prime divisors of the order of minimal groups that contain a diameter three subgroup is not bounded. Namely, for any natural number t we may take the product c of t pairwise different primes, and choose a prime $p = kc - 1$, for some natural number k. (Infinitely many such primes exist by Dirichlet's theorem.) Then Example 3.2 yields a group G of order $p^2 \cdot d$, where d is a multiple of c, such that G contains a diameter three subgroup and is minimal with this property.

Note that the order of minimal non-nilpotent groups is divisible by exactly two different primes. Minimal non-supersolvable groups are solvable, and Proposition 6.2 below shows that minimal groups that have a diameter three subgroup need not be solvable. See [10] for properties of minimal non-nilpotent and minimal non-supersolvable groups.

Remark 6.1. Proposition 2.11 states that factoring out certain normal subgroups of a group with a diameter three subgroup yields again groups with a diameter three subgroup. Thus we could define minimality by going down to subgroups and by factoring out normal subgroups if possible. However, this would yield strange results. For example, we will see in Proposition 6.4 that the group $\mathrm{SL}(2, 7)$ is minimal w.r.t. inclusion; if we allow to take factors then we get $\mathrm{PSL}(2, 7)$, which is *not* minimal because its A_4 type subgroups have diameter three subgroups.

6.1. Series of minimal examples

Proposition 6.2. *Let G be a semidirect product of an elementary abelian p-group E and a cyclic group C of order q, a power of a prime $l \neq p$. Assume that C acts irreducibly on E and that G is minimal with the property that it has a diameter three subgroup.*

Let G_i be a subgroup of index q in the direct product $G \times C_i$, where C_i is a cyclic group of order $q \cdot l^i$, such that the Sylow l-subgroup of G_i is cyclic and acts irreducibly on E. Then G_i has a diameter three subgroup and is minimal with this property.

Proof. The group G_i embeds into $G \times C_i$ by enlarging the center, so G and G_i are isoclinic for all i, see the proof of Proposition 2.17. Hence G_i has a diameter three subgroup by Lemma 2.16.

In order to show the minimality of G_i, we show that no maximal subgroup of G_i has a diameter three subgroup. Let M be a maximal subgroup of G_i. Since G_i is solvable, $[G_i : M]$ is a prime power.

If this prime is l then M contains E, thus M has index l in G_i, and M can be embedded into $U \times C_{i-1}$, where U is the subgroup of index l in G. Thus M is isoclinic with U and hence has no diameter three subgroup, by the minimality of G.

If this prime is p then M is a Sylow l-subgroup of G_i, by the irreducibility of the action on E, thus M has no diameter three subgroup. $\square$

Example 6.3. Applying Proposition 6.2 to the Frobenius group G of order $2^n(2^n - 1)$ from Example 3.1, where $p = 2^n - 1$ is a prime, yields minimal examples $2^2 : 3, 2^2 : 9, 2^2 : 27, \ldots, 2^3 : 7, 2^3 : 49, 2^3 : 343, \ldots, 2^5 : 31, 2^5 : 961, \ldots, 2^7 : 127, 2^7 : 16129$.

Starting from $G = 3^2 : 8$, we get $3^2 : 16, 3^2 : 32, \ldots$, and $G = 2^4 : 5$ yields $2^4 : 25, 2^4 : 25, \ldots$. See Table 7 for more examples.

6.2. The groups $\mathrm{SL}(2, p)$

Proposition 6.4. *Let p be a prime. Then $\mathrm{SL}(2, p)$ is a minimal group (w. r. t. inclusion) that contains a diameter three subgroup if and only if $p > 5$.*

Proof. The group $\mathrm{SL}(2, 2) \cong S_3$ is supersolvable, and $\mathrm{SL}(2, 3)$ and $\mathrm{SL}(2, 5)$ are Frobenius complements, thus we know that these groups do not have diameter three subgroups.

Now assume $p > 5$. Lemma 5.4 shows that $G = \mathrm{SL}(2, p)$ has a diameter three subgroup. We show that the proper subgroups of G do not have such a subgroup. Let $\pi \colon G \to G/Z(G) \cong \mathrm{PSL}(2, p)$ be the natural epimorphism. The subgroups of the simple group $\pi(G)$ are listed in [14, Satz II.8.27], there are eight types of subgroups. The types (1)–(3) and (7) are cyclic or metacyclic, in particular supersolvable, hence also their preimages under π are supersolvable and thus do not have a diameter three subgroup, by Corollary 2.12. The preimages of subgroups of the types (4)–(6) are isomorphic to $\mathrm{SL}(2, 3)$, $\mathrm{SL}(2, 5)$, or the unique group of order 48 with exactly one involution s. t. the factor modulo the center is isomorphic to S_4; the former two groups have been dealt with above, and if the latter had a diameter three subgroup then it would be of order divisible by 3, which would imply a diameter three subgroup of order 3 in S_4, by Proposition 2.11, which contradicts Corollary 2.8. Finally, subgroups of the type (8) do not occur because p is prime. $\square$

Table 7
Small groups that are minimal with a nontrivial rich subgroup.

k	$\lvert G\rvert$	i	G	k	$\lvert G\rvert$	i	G
5	**12**	**3**	$2^2\!:\!3 \cong A_4$	3	**784**	**162**	$7^2\!:\!Q_{16}$
2	**56**	**11**	$2^3\!:\!7$	9	864	676	$3_+^{1+2}\!:\!(8\!:\!4)$
5	**72**	**39**	$3^2\!:\!8$	1	**992**	**194**	$2^5\!:\!31$
9	**72**	**41**	$3^2\!:\!Q_8$	10	1152	4900	$3^2\!:\!(((4 \times 2)\!:\!4)\!:\!4)$
3	**80**	**49**	$2^4\!:\!5$	10	1152	5070	$3^2\!:\!((2 \times (4\!:\!4))\!:\!4)$
2	160	199	$2^{1+4}\!:\!5$	6	1152	5232	$3^2\!:\!((2 \times ((4 \times 2)\!:\!2))\!:\!4)$
4	216	86	$3_+^{1+2}\!:\!8$	4	1152	6492	$3^2\!:\!((8\!:\!4)\!:\!4)$
6	216	88	$3_+^{1+2}\!:\!Q_8$	3	1152	6577	$3^2\!:\!(((2^3)\!:\!4)\!:\!4)$
18	288	393	$3^2\!:\!(8\!:\!4)$	3	1152	6619	$3^2\!:\!(((2^3)\!:\!4)\!:\!4)$
1	336	114	$\mathrm{SL}(2,7)$	3	1152	7054	$3^2\!:\!(32\!:\!4)$
1	**351**	**12**	$3^3\!:\!13$	1	1152	7092	$3^2\!:\!(32\!:\!4)$
6	576	1966	$3^2\!:\!(16\!:\!4)$	1	1320	13	$\mathrm{SL}(2,11)$
9	576	1967	$3^2\!:\!(16\!:\!4)$	1	**1620**	**419**	$3^4\!:\!(5\!:\!4)$
5	576	1973	$3^2\!:\!((8 \times 2)\!:\!4)$	1	**1620**	**420**	$3^4\!:\!20$
9	576	1976	$3^2\!:\!((8\!:\!2)\!:\!4)$	1	1620	421	$3^4\!:\!(5\!:\!4)$
2	**600**	**148**	$5^2\!:\!(3\!:\!8)$	1	1728	2787	$3_+^{1+2}\!:\!(16\!:\!4)$
3	**600**	**149**	$5^2\!:\!24$	3	1728	2788	$3_+^{1+2}\!:\!(16\!:\!4)$
2	**600**	**150**	$5^2\!:\!\mathrm{SL}(2,3)$	2	1728	2794	$3_+^{1+2}\!:\!(4^2\!:\!4)$
2	648	641	$3^3\!:\!\mathrm{SL}(2,3)$	3	1728	2797	$3_+^{1+2}\!:\!((8\!:\!2)\!:\!4)$
2	**784**	**160**	$7^2\!:\!16$	1	1800	270	$5^2\!:\!(9\!:\!8)$

6.3. Small groups

Using the library of small groups [2], we computed the groups G of order up to 2 000, up to isomorphism, that contain nontrivial rich subgroups H. There are exactly 52 239 such isomorphism types, 33 523 of them have order 1 536. A JSON format file containing the data about these groups is publicly available, see [4].

If we consider only those groups G such that no proper subgroup of G has this property, we get exactly 163 such groups, up to isomorphism. These groups lie in 40 isoclinism classes.

(Fortunately, enough information is available such that one need not really run over all isomorphism types of groups. For example, most of the 408 641 062 groups of order 1 536 are supersolvable and hence need not be checked. In the end, no group of this order turned out to be minimal.)

Table 7 shows one representative of each isoclinism class. The first column contains the number k of isomorphism types of those groups in the isoclinism class that are among the 163 minimal examples. The second and third column list the values $\lvert G\rvert$ and i such that the group G is the i-th group of its order, according to the numbering in [2]; the values of Frobenius groups are shown in boldface.

Examples of isoclinic minimal examples are described in Section 6.1.

Diameter three subgroups H in groups of order at most 2 000 have order at most 16.

There are exactly 11 isomorphism types of groups of order at most 2 000 which have nontrivial rich subgroups that are not diameter three subgroups. Among these groups are the Frobenius groups $3^3\!:\!13$ and $2^6\!:\!21$ of the orders 351 and 1 344, respectively, and direct products $A \times 3^3\!:\!13$ where A is an abelian group of order at most 5.

7. Diameter and depth

A notion of depth can be defined for subrings of a ring (cf. [7]). Here we are only interested in complex group algebras of finite groups and their subgroups. In this situation the depth can be computed in terms of the Frobenius matrix. More precisely, let H be a proper subgroup of a finite group G, and set $M := F(G, H)$. Then we have

$$S := MM^\top = ([\varphi^G, \psi^G])_{\varphi, \psi \in \mathrm{Irr}(H)}.$$

For a positive integer m, one has that H is of *depth* $n = 2m + 1$ in G if and only if $S^{m+1} \leq qS^m$ for some $q > 0$, and that H is of *depth* $n = 2m$ in G if and only if $S^m M \leq qS^{m-1}M$ for some $q > 0$. Here the inequality $A \leq B$ between real matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same format is defined by $a_{ij} \leq b_{ij}$ for all i, j.

It is known that depth n always implies depth $n+1$. Thus the *minimal depth* $d(H, G)$ is of particular interest. It is also known that H is of depth 2 in G if and only if H is a normal subgroup of G. Several papers have investigated subgroups of depth 3 (cf. [5,6]). Here we point out connections to the diameter of $\Gamma(G, H)$.

Proposition 7.1. *Let H be a nontrivial core-free subgroup of a finite group G. Then the following assertions hold:*

(i) If $\Gamma(G, H)$ has diameter 3 then $[\varphi^G, \psi^G] > 0$ for all $\varphi, \psi \in \mathrm{Irr}(H)$.

(ii) If $[\varphi^G, \psi^G] > 0$ for all $\varphi, \psi \in \mathrm{Irr}(H)$ then H is of depth 3 in G.

(iii) If H is of depth 3 in G then the diameter of $\Gamma(G, H)$ is 3 or 4.

Proof. (i) is an immediate consequence of condition (b) (ii) in Proposition 2.2.

(ii) follows from the above definition of depth.

(iii) Suppose that H is of depth 3 in G. Since $\Gamma(G, H)$ is connected, [7, Theorem 3.6] implies that the distance between any two distinct $\varphi, \psi \in \mathrm{Irr}(H)$ is 2. Thus the distance between any $\varphi \in \mathrm{Irr}(H)$ and any $\chi \in \mathrm{Irr}(G)$ is at most 3, and the distance between any two distinct $\chi, \eta \in \mathrm{Irr}(G)$ is at most 4. Thus $\Gamma(G, H)$ has diameter 3 or 4. □

Thus diameter 3 implies depth 3, and conversely depth 3 implies diameter 3 or 4. Note that it can happen that the depth is 3 whereas the diameter is 4, as the example $S_2 < S_3$ shows. Note also that the depth is defined also for subgroups with nontrivial core; for example, the depth of the Sylow 2-subgroup H in the dihedral group G of order 12 is 3, but $\Gamma(G, H)$ consists of two connected components, each a path of length 4.

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Data availability

We have used the data from the GAP packages mentioned in the References. We have produced a small file of additional data which is available via its zenodo doi, see the References.

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