

On Projective Delineability

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Abstract—We consider cylindrical algebraic decomposition (CAD) and the key concept of delineability which underpins CAD theory. We introduce the novel concept of projective delineability which is easier to guarantee computationally. We prove results about this which can allow reduced CAD computations.

Index Terms—Cylindrical Algebraic Decomposition, Non-linear Real Arithmetic, (Projective) Delineability

I. INTRODUCTION

Cylindrical algebraic decomposition (CAD) is a well-known tool for computing with polynomial constraints over the reals, developed in the 1970s by Collins [5] as the basis for quantifier elimination (QE) over real-closed fields. CAD has been continuously researched, improved, and applied in the half century since its inception.

Recently, CAD theory has been adapted and specialized for *satisfiability modulo theories* (SMT) solvers which check the satisfiability of Boolean combinations of real polynomial constraints: e.g. the *NuCAD* [4], the *NLSAT* [9], and the *CAIC* [1] algorithms. The corresponding *QF_NRA* category of the SMT-LIB benchmarks contains instances of such problems originating from e.g. verification systems, the life sciences and economics.

Central to CAD is the notion of *delineability* of a polynomial over a connected set (*cell*). Roughly speaking, an n -variate polynomial P is delineable over a cell $S \subseteq \mathbb{R}^{n-1}$ if the set of zeros of P in the cylinder $S \times \mathbb{R}$ is the disjoint union of the graphs of continuous real-valued functions of $n - 1$ variables (we call these *real root functions*). These graphs cut the cylinder $S \times \mathbb{R}$ into cells which the sign of P is invariant within. We order these cells bottom-to-top to form a *stack*. The key observation is that the number of cells, their ordering, and the sign of P within cells, are

all the same no matter which point of $s \in S$ we are over. Thus the presence of delineability allows for construction at a sample point to be concluded valid over the cell.

We are particularly interested by the *single cell construction* [13], which is used as a subroutine by the three SMT algorithms mentioned earlier [4], [9], [1]. This uses CAD projection theory to compute a single CAD cell around a given sample point within which a set of polynomials is sign-invariant. Compared to traditional CAD algorithms, the single cell construction allows us to use the sample point to detect which projection polynomials are not relevant for the cell under construction, and omit them in the computation.

In this paper we introduce the new notion of *projective delineability*: easier to guarantee computationally than delineability but still useful. Our new notion allows us to further omit leading coefficients during single cell construction in cases where projective delineability of a polynomial is sufficient instead of its delineability. Indeed, projective delineability is equivalent to delineability in the case where the leading coefficient of P never vanishes, but may still be guaranteed when the leading coefficient has zeroes (see Proposition 3 for a more precise statement).

II. DELINEABILITY AND LEADING COEFFICIENTS

A CAD is a decomposition of $\mathbb{R}^n$, consisting of cells which maintain properties for the input; usually that a finite family of input polynomials $\mathcal{F}$ in n variables are sign-invariant within each cell. These cells are *cylindrically arranged*, meaning that the projection of a CAD of $\mathbb{R}^n$ is a CAD of $\mathbb{R}^{n-1}$, and over each cell S of the latter, the cylinder $S \times \mathbb{R}$ is decomposed by cells of the former. A CAD may be computed recursively: for obtaining the CAD of $\mathbb{R}^n$, we compute a *projection* $\mathcal{F}'$ in $n - 1$ variables of $\mathcal{F}$ such that $\mathcal{F}'$ defines a suitable CAD of $\mathbb{R}^{n-1}$. The following notion formalizes the property that the CAD of $\mathbb{R}^{n-1}$ needs to fulfil if we are to extend it to a CAD of $\mathbb{R}^n$.

Definition 1 ([5], [10]). We say that $P \in \mathbb{R}[\mathbf{x}; x_n]$ is *delineable* on a subset S of $\mathbb{R}^{n-1}$ if there exist $k \in \mathbb{N}$ and some continuous functions $\theta_1, \dots, \theta_k : S \rightarrow \mathbb{R}$ such that:

- 1) the set of zeros of P in $S \times \mathbb{R}$ is the disjoint union $\text{Graph}(\theta_1) \sqcup \dots \sqcup \text{Graph}(\theta_k)$;
- 2) for all $l \in \{1, \dots, k\}$, there exists $m_l \in \mathbb{N}^*$ such that for all $\mathbf{x} \in S$, the multiplicity of the root $\theta_l(\mathbf{x})$ is m_l .

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We say that $\theta_1, \dots, \theta_k$ are the *real root functions* of P over S . The graph of θ_l is called a *real P -section* over S .

The computation of the projection $\mathcal{F}'$ has been an active area of research. Our work is based on the projection operator of McCallum [11] with the optimization by Brown [3]: $\mathcal{F}'$ contains the discriminant and leading coefficient of each $P \in \mathcal{F}$ and the resultant of each pair $P, Q \in \mathcal{F}$.

The new notion of projective delineability is designed for the *single cell construction* [13] discussed above. Here the cell is already constructed using a partial CAD projection (given the sample point we identify that some projection polynomials for the full CAD are not needed for the region of interest). Prior work reduced the amount of resultants and some discriminants. Projective delineability allows detecting irrelevant leading coefficients also.

Example 1. Consider the polynomials $P = x_1^2 + x_2^2 - 1$ and $Q = x_1x_2 - 1$ and the sample $s = (-\frac{1}{2}, 0)$ as depicted in Figure 1 (left). The darker shaded region around the sample is a CAD cell bounded by the roots of P in the x_2 -axis. The bounds on the x_1 -axis are defined by the projection: using regular delineability means x_1 is bounded by the discriminant of P from below and the leading coefficient of Q from above. However, this upper bound on x_1 is too tight, since the singularity of Q does not affect the sign-invariance property of the cell. We could thus extend the single cell to include also the lighter shaded region.

In this paper, we formalize what “*delineability without leading coefficient*” means. To do so, we view the roots of a polynomial in real projective space and discover that these roots behave nicely, i.e. we can describe the variety of a polynomial using continuous functions (see Figure 1, right). Projective delineability of Q and the order-invariance of the resultant of the two polynomials is sufficient to show that no real root of Q can enter or pop-up within the larger cell. We can thus safely leave out the leading coefficient of Q in this example.

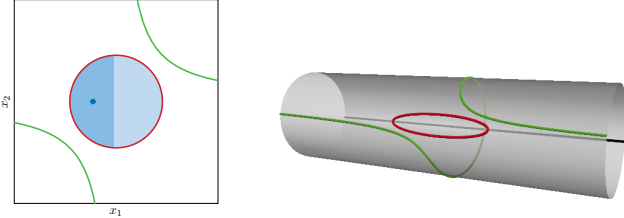


Fig. 1. The sets of roots of P (red) and Q (green) seen in $\mathbb{R}^2$ (left) and in $\mathbb{R} \times \mathbb{RP}^1$ (right)

III. PRELIMINARIES

A. The projective line

The points of the *real projective line* $\mathbb{RP}^1$ are the lines of $\mathbb{R}^2$ passing through $(0, 0)$. Such a line is determined by any of its nonzero vectors $(x, y) \in \mathbb{R}_*^2 = \mathbb{R}^2 \setminus \{(0, 0)\}$, and so $\mathbb{RP}^1$ is equivalently defined as the quotient set of $\mathbb{R}_*^2$ by the equivalence relation given by proportionality. We denote

the quotient map by $\pi : \mathbb{R}_*^2 \rightarrow \mathbb{RP}^1$ and the equivalence class of (x, y) by $\pi(x, y)$ or $(x : y)$.

The topology of $\mathbb{RP}^1$ is the usual quotient topology: $U \subset \mathbb{RP}^1$ is open if $\pi^{-1}(U)$ is open in $\mathbb{R}_*^2$. The map $\psi(x : y) = \frac{y}{x^2 + y^2}(x, y)$ is a homeomorphism from $\mathbb{RP}^1$ to a circle.

The real line $\mathbb{R}$ is usually identified with an open and dense subset of $\mathbb{RP}^1$ by first identifying $\mathbb{R}$ to a (parameterized) line of $\mathbb{R}_*^2$ and then composing with the projection:

$$\iota : \mathbb{R} \rightarrow \mathbb{R}_*^2 : x \mapsto (x, 1), \quad \varphi : \mathbb{R} \rightarrow \mathbb{RP}^1 : x \mapsto (x : 1). \quad (1)$$

The only point in $\mathbb{RP}^1$ that is not in the range of the continuous injective map φ is the point $(1 : 0)$. For this reason it is sometimes denoted by ∞ . The inverse map

$$\varphi^{-1} : \mathbb{RP}^1 \setminus \{\infty\} \rightarrow \mathbb{R} : (x : y) \mapsto \frac{x}{y}$$

is also continuous. The choice to identify $\mathbb{R}$ with a given line in $\mathbb{R}^2$ (not containing $(0, 0)$) is arbitrary: any other choice of parameterized line reads $\iota_A(\mathbb{R})$ where $A = (a_{ij})_{i,j \in \{1,2\}}$ is a nonsingular matrix ($A \in GL(2, \mathbb{R})$) and

$$\iota_A : \mathbb{R} \rightarrow \mathbb{R}_*^2 : x \mapsto A \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Setting $\varphi_A = \pi \circ \iota_A$, the point at infinity (with respect to A) is the point that is not in the range of φ_A , and thus corresponds to the first column of A . The maps $(\varphi_A)^{-1}$, for $A \in GL(2, \mathbb{R})$, endow $\mathbb{RP}^1$ with the structure of an analytic manifold. Note that we have $\varphi_A = \mu_A \circ \varphi$, where for any $A \in GL(2, \mathbb{R})$, the map μ_A is defined by

$$\mu_A : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1 : (x : y) \mapsto \pi \left(A \begin{pmatrix} x \\ y \end{pmatrix} \right).$$

B. Binary forms, homogenizations and pull-backs

Later we deal with roots of polynomials at ∞ . The framework of binary forms allows us to deal with these roots at infinity on an equal footing with the real roots.

Definition 2. A *binary form* of degree d is a function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ expressed as

$$g(x, y) = \sum_{k=0}^d c_k x^k y^{d-k}, \quad c_0, \dots, c_d \in \mathbb{R}. \quad (2)$$

The set of binary forms of degree d is denoted by $\mathbb{R}[x, y]_d$.

In view of Definition 2, the space $\mathbb{R}[x, y]_d$ is clearly isomorphic to the vector space $\mathbb{R}[x]_{\leq d}$ of univariate polynomial functions of degree less than or equal to d . An isomorphism is given by the pull-back and its inverse, the homogenization with respect to degree d .

Definition 3. The *pull back of binary forms* of degree $d \in \mathbb{N}$ is the map $\iota^{*d} : \mathbb{R}[x, y]_d \rightarrow \mathbb{R}[x]_{\leq d} : g \mapsto g \circ \iota$. The *homogenization* with respect to degree d is $H^d = (\iota^{*d})^{-1}$.

Remark 1.

- 1) If g is a binary form of degree d given by (2), then $\iota^{*d}g(x) = \sum_{k=0}^d c_k x^k$. It has degree less than or equal to $p < d$ if and only if $c_d = \dots = c_{p+1} = 0$.

- 2) We have $H^d(P)(x, y) = y^d P(\frac{x}{y})$ for every $P \in \mathbb{R}[x]_{\leq d}$ and $(x, y) \in \mathbb{R}^2$ such that $y \neq 0$.
- 3) By definition, pull-back and homogenization with respect to a same fixed degree are the inverse of each other. This fails to be true if they are not performed with respect to the same fixed degree. For instance, if $g(x, y) = x^2 y + x y^2 + y^3$, then $\iota^{*3} g(x) = x^2 + x + 1$, and we have $H^2(\iota^{*3} g)(x, y) \neq g(x, y)$.

As we continue it will be useful to consider the pull-back and homogenization with respect to other lines in $\mathbb{R}^2$.

Definition 4. For every $A \in GL(2, \mathbb{R})$, the pull-back with respect to A is $\iota_A^{*d} : \mathbb{R}[x, y]_d \rightarrow \mathbb{R}[x]_{\leq d} : g \mapsto g \circ \iota_A$ and the corresponding homogenization is $H_A^d = (\iota_A^{*d})^{-1}$. Finally, we set $A^{*d} : \mathbb{R}[x]_{\leq d} \rightarrow \mathbb{R}[x]_{\leq d} : P \mapsto \iota_A^{*d}(H^d(P))$.

Remark 2. If $P(x) = \sum_{k=0}^d c_k x^k$, we compute

$$A^{*d}P(x) = \sum_{k=0}^d c_k (a_{11}x + a_{12})^k (a_{21}x + a_{22})^{d-k}$$

for $A \in GL(2, \mathbb{R})$. When $a_{21}x + a_{22} \neq 0$, this expression also reads $(a_{21}x + a_{22})^d P\left(\frac{a_{11}x + a_{12}}{a_{21}x + a_{22}}\right)$.

The next lemma gathers useful formulas to deal efficiently with these operators. We denote by R_A the right composition with A (seen as an operator from $\mathbb{R}^2$ to $\mathbb{R}^2$), namely $R_A : \mathbb{R}[x, y]_d \rightarrow \mathbb{R}[x, y]_d : g \mapsto g \circ A$.

Lemma 1. For every $d \in \mathbb{N}$ and $A, B \in GL(2, \mathbb{R})$, we have $\iota_A^{*d} \circ H_B^d = \iota^{*d} \circ R_{B^{-1}A} \circ H^d$. In particular, we have $\iota_A^{*d} = \iota^{*d} \circ R_A$, and $H_A^d = R_{A^{-1}} \circ H^d$.

Proof. It is sufficient to prove the particular cases. The first one follows from the very definition of ι_A^{*d} , while the second is obtained by inversion. $\square$

C. Projective roots, resultants and discriminants

Projective¹ roots are defined for binary forms and then, via homogenization, for univariate polynomials.

Definition 5. A point $(x : y) \in \mathbb{RP}^1$ is a *projective root* of $g \in \mathbb{R}[x, y]_d$ if $g(x, y) = 0$. It is a projective root of $P \in \mathbb{R}[x]_{\leq d}$, (w.r.t. d) if it is a projective root of $H^d(P)$.

Note that this definition is valid since, by homogeneity, when g vanishes at a point $(x, y) \in \mathbb{R}_*^2$, it vanishes on the whole line $(x : y)$. We can define the multiplicity of the roots of a binary form using the following result².

Lemma 2. All nonzero $g \in \mathbb{R}[x, y]_d$ factor uniquely³ as

$$g(x, y) = y^{d-\deg(\iota^{*d}g)} (x - x_1 y)^{m_1} \dots (x - x_t y)^{m_t} Q(x, y), \quad (3)$$

where $x_1, \dots, x_t$ are the distinct roots of $\iota^{*d}g$ with multiplicities $m_1, \dots, m_t$ and Q is a binary form without projective roots.

¹Here and throughout this work, by projective, we mean in $\mathbb{RP}^1$.

²See also Lemma 6, §7, of [6, Chapter 8].

³Up to permutation.

Proof. If $g \in \mathbb{R}[x, y]_d$ is not zero, then $\iota^{*d}g \in \mathbb{R}[x]_{\leq d}$ factors as follows, where $x_1, \dots, x_t \in \mathbb{R}$ are the distinct roots of $\iota^{*d}g$ and R has no real root:

$$(\iota^{*d}g)(x) = (x - x_1)^{m_1} \dots (x - x_t)^{m_t} R(x). \quad (4)$$

We obtain the decomposition (with $Q = H^{\deg(R)}(R)$) and the properties of the multiplicities by computing $g(x, y) = H^d(\iota^{*d}g)(x, y)$ for $y \neq 0$ (using Remark 1) and extending the equality to $\mathbb{R}^2$ by continuity. The uniqueness directly follows from the uniqueness of the factorization of $\iota^{*d}g$. $\square$

To sum up, the projective roots of a binary form g are on the one hand the roots of $\iota^{*d}g$, seen as points in $\mathbb{RP}^1$ through the map φ , with the same multiplicities (these are called real roots); and on the other hand $\infty = (1 : 0)$ (the root at infinity) with multiplicity $d - \deg(\iota^{*d}g)$ (when this number is positive). The projective roots with respect to degree d of a univariate polynomial $P \in \mathbb{R}[x]_{\leq d}$ are its real roots, seen as elements of $\mathbb{RP}^1$ via φ and the root at infinity with multiplicity $d - \deg(P)$. In particular, P always has a projective root at infinity when $d > \deg(P)$.

Lemma 3. If $g \in \mathbb{R}[x, y]_d$ and $A \in GL(2, \mathbb{R})$, then $(x : y)$ is a root of multiplicity m of $R_A g$ if and only if $\mu_A(x : y)$ is a root of multiplicity m of g . The same result holds for the projective roots with respect to degree d of the polynomials $A^{*d}P$ and $P \in \mathbb{R}[x]_{\leq d}$, respectively.

Proof. In view of Lemma 2, the binary form g factors as

$$g(x, y) = \prod_{i=0}^t l_i(x, y)^{m_i} Q(x, y)$$

and the projective roots of g are the roots of the linearly independent linear forms l_i , with multiplicities m_i for $i \in \{0, \dots, t\}$, while Q has no projective root. We then deduce the factorization of $R_A g$:

$$R_A g(x, y) = \prod_{i=0}^t (R_A l_i(x, y))^{m_i} R_A Q(x, y).$$

If $(x_0 : y_0)$ is a projective root of $R_A g$, then $\mu_A(x_0 : y_0)$ is a projective root of one, and only one, linear form l_i . Hence it is a projective root of g with multiplicity m_i . The projective roots of $A^{*d}P$ with respect to d are the projective roots of $H^d(A^{*d}P) = H^d \circ \iota_A^{*d} \circ H^d(P) = R_A(H^d(P))$ by Lemma 1 and the result follows. $\square$

To emphasize that resultants and discriminants are computed with respect to fixed degree we use the notations $\text{Res}^{p,q}(P, Q)$ for the resultant with respect to degrees (p, q) of $(P, Q) \in \mathbb{R}[x]_{\leq p} \times \mathbb{R}[x]_{\leq q}$. We denote by $\text{Disc}^p(P)$ the *discriminant* of P with respect to degree p . We recall the following result (see [8, Chapter 12]).

Proposition 1. For every $p, q \in \mathbb{N}$, $P \in \mathbb{R}[x]_{\leq p}$, $Q \in \mathbb{R}[x]_{\leq q}$ and $A \in GL(2, \mathbb{R})$, we have

$$\begin{aligned} \text{Res}^{p,q}(A^{*p}P, A^{*q}Q) &= \det(A)^{pq} \text{Res}^{p,q}(P, Q), \\ \text{Disc}^p(A^{*p}P) &= \det(A)^{p(p-1)} \text{Disc}^p(P). \end{aligned}$$

D. Extensions to multivariate polynomials

CAD projection is calculated with respect to the last variable x_n . Hence we now extend the previous constructions to n -variate polynomials by acting on the last variable only. We denote the extended operators by the same symbol, and use bold $\mathbf{x}$ to denote a $(n-1)$ -tuple.

Definition 6. If f is a function in more than $n-1$ variables, for every $\mathbf{x}_0 \in \mathbb{R}^{n-1}$, we define the *evaluation* $E_{\mathbf{x}_0}(f)$ to be the function obtained from f by assigning the value $\mathbf{x}_0$ to the $(n-1)$ -tuple of first variables.

Definition 7.

- 1) We denote by $\mathbb{R}[\mathbf{x}; x_n, y]_d$ the set of polynomials of degree d in x_n, y , i.e. those $g \in \mathbb{R}[x_1, \dots, x_n, y]$ such that $E_{\mathbf{x}_0}(g) \in \mathbb{R}[x_n, y]_d$, for all $\mathbf{x}_0 \in \mathbb{R}^{n-1}$.
- 2) We denote by $\mathbb{R}[\mathbf{x}; x_n]_{\leq d}$ the set of polynomials of degree at most d in x_n , i.e. those $P \in \mathbb{R}[x_1, \dots, x_n]$ such that $E_{\mathbf{x}_0}(P) \in \mathbb{R}[x_n]_{\leq d}$, for all $\mathbf{x}_0 \in \mathbb{R}^{n-1}$. The degree in x_n of P is defined by $\deg_{x_n}(P) = \min \{d \in \mathbb{N} \mid P \in \mathbb{R}[\mathbf{x}; x_n]_{\leq d}\}$.

Using again the evaluation operators, we now define the extension of the operators defined so far. These extensions are all defined by first evaluating an n -variate or $(n+1)$ -variate polynomial at a point $\mathbf{x}_0$ of $\mathbb{R}^{n-1}$ to obtain a univariate or bivariate polynomial, and then using the corresponding operator defined in Section III-C.

Definition 8. For $d \in \mathbb{N}$, $A \in GL(2, \mathbb{R})$ and $T \in \{H_A^d, \iota_A^{*d}, R_A, A^{*d}, \text{Disc}^p\}$ from Section III-C, we define the action of the extended operator T on f in $\mathbb{R}[\mathbf{x}; x_n]_{\leq d}$ or $\mathbb{R}[\mathbf{x}; x_n, y]_d$ by defining the evaluations of $T(f)$ at every point $\mathbf{x}_0 \in \mathbb{R}^{n-1}$, setting $E_{\mathbf{x}_0} \circ T = T \circ E_{\mathbf{x}_0}$. For $p, q \in \mathbb{N}$, we extend similarly $\text{Res}^{p,q}$ on $\mathbb{R}[\mathbf{x}; x_n]_{\leq p} \times \mathbb{R}[\mathbf{x}; x_n]_{\leq q}$.

In the above definition, we described formally the usual pointwise extensions of operators. For example, writing $P(\mathbf{x}, x_n) = \sum_{k=0}^{d_n} c_k(\mathbf{x})x_n^k$, we have

$$H^{d_n}(P)(\mathbf{x}, (x_n, y)) = E_{\mathbf{x}}(H^{d_n}(P))(x_n, y) = \sum_{k=0}^{d_n} c_k(\mathbf{x})x_n^k y^{d-k},$$

$$A^{*d_n}P(\mathbf{x}, x_n) = \sum_{k=0}^{d_n} c_k(\mathbf{x})(a_{11}x_n + a_{12})^k(a_{21}x_n + a_{22})^{d_n-k}. \quad (5)$$

Similarly, we extend the definitions of $\varphi_A, \mu_A, \iota_A$, keeping again the same symbol and acting on the last variable only.

Definition 9. For $A \in GL(2, \mathbb{R})$ and $F \in \{\varphi_A, \mu_A, \iota_A\}$ defined on the set X as in Section III-A, we define the extended function F on $\mathbb{R}^{n-1} \times X$ by setting $F(\mathbf{x}, u) = (\mathbf{x}, F(u))$, i.e. the extended function is given by $\text{id} \times F$ where id is the identity map of $\mathbb{R}^{n-1}$.

Pointwise properties established in the previous sections readily generalize to the extended operators and functions defined above. However, it is important to fix a priori the degree with respect to which homogenizations, resultants and discriminants are computed.

Remark 3. One could also adopt another equivalent point of view to define these extensions to multivariate polynomials: first, define the operators of Section III-C in general for polynomials with coefficients in an integral domain and then, study whether or not these operators commute with ring homomorphisms ([2, Section 4.2.1] for the resultant) and in particular with the evaluation $E_{\mathbf{x}}$.

We will analyze the behaviour of the order of polynomials with respect to homogenization and composition with a linear transformation. Note that the second part of Proposition 2 justifies the definition of order of a binary form at a point of the projective line.

Proposition 2.

- 1) Let $f \in \mathbb{R}[x_1, \dots, x_m]$ and $A \in GL(m, \mathbb{R})$. We have $\text{ord}(f \circ A, v) = \text{ord}(f, Av)$ for all $v \in \mathbb{R}^m$.
- 2) If $g \in \mathbb{R}[\mathbf{x}; x_n, y]_d$, then we have $\text{ord}(g, (\mathbf{x}, (x_n, y))) = \text{ord}(g, (\mathbf{x}, k(x_n, y)))$ for every $\mathbf{x} \in \mathbb{R}^{n-1}, (x_n, y) \in \mathbb{R}_*^2$ and $k \neq 0$.
- 3) If $P \in \mathbb{R}[\mathbf{x}; x_n]_{\leq d}$. We have $\text{ord}(P, (\mathbf{x}, x_n)) = \text{ord}(H^d(P), (\mathbf{x}, (x_n, 1)))$ for every $\mathbf{x} \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$.

The first assertion of Proposition 2 is a particular case of a general theorem ([10, Theorem 2.1]). We give an elementary proof of the particular case under consideration.

Proof. Using the chain rule, a simple induction shows that for $\forall k \in \mathbb{N}, i_1, \dots, i_k \leq m$,

$$\partial_{i_1} \dots \partial_{i_k}(f \circ A) = \sum_{j_1, \dots, j_k=1}^m A_{j_1 i_1} \dots A_{j_k i_k} (\partial_{j_1} \dots \partial_{j_k} f) \circ A.$$

It follows from the definition of the order that $\text{ord}(f \circ A, v) \geq \text{ord}(f, Av)$, and the first result follows by symmetry. The second one follows by using $A = \text{id}_{n-1} \times (\text{id}_2)$ and noticing that $g \circ A = k^d g$ has the same order as g . Finally, denote by k the order of P at $(\mathbf{x}_0, u)$. Using the multi-index notation, we have $P(\mathbf{x}, x_n) = \sum_{|\alpha|+l \geq k} D_{\alpha, l}(\mathbf{x} - \mathbf{x}_0)^\alpha (x_n - u)^l$ and there exists (α_0, l_0) such that $|\alpha_0| + l_0 = k$ and $D_{\alpha_0, l_0} \neq 0$. We compute

$$H^d(P)(\mathbf{x}, (x_n, y)) = \sum_{|\alpha|+l \geq k}^d D_{\alpha, l}(\mathbf{x} - \mathbf{x}_0)^\alpha (x_n - uy)^l y^{d-l}.$$

It is easily seen that $(\partial_{\mathbf{x}}^\beta \partial_{x_n}^a \partial_y^b H^d(P))$ vanishes at $(\mathbf{x}_0, (u, 1))$ whenever $|\beta| + a + b < k$ and that it is nonzero if $\beta = \alpha_0, a = l_0$ and $b = 0$. $\square$

IV. PROJECTIVE DELINEABILITY

In this section, we define the new concept of projective delineability of a polynomial over a subset $S \subset \mathbb{R}^{n-1}$, which extends the classical notion of delineability [10] by allowing the root functions to reach infinity.

Definition 10. For $P \in \mathbb{R}[\mathbf{x}; x_n]$ of degree $d_n = \deg_{x_n}(P)$ in x_n and $S \subset \mathbb{R}^{n-1}$, we define

$$Z_{\mathbb{R}}(P, S) = \{(\mathbf{x}, x_n) \in S \times \mathbb{R} \mid P(\mathbf{x}, x_n) = 0\},$$

$$Z_{\mathbb{RP}^1}(P, S) = \{(\mathbf{x}, (x_n : y)) \in S \times \mathbb{RP}^1 \mid H^{d_n}(P)(\mathbf{x}, (x_n, y)) = 0\}.$$

We say that x_n is a *real root* of P above $\mathbf{x}_0 \in S$ if $(\mathbf{x}_0, x_n) \in Z_{\mathbb{R}}(P, S)$. Similarly, we say that $(x_n : y)$ is a *projective root* of P above $\mathbf{x}_0$ if $(\mathbf{x}_0, (x_n : y)) \in Z_{\mathbb{RP}^1}(P, S)$.

As already mentioned, the projective roots of P above $\mathbf{x}_0$ consist of the real roots above $\mathbf{x}_0$ (seen in the projective space) and the root at infinity above $\mathbf{x}_0$. This latter root appears precisely when $\deg(E_{\mathbf{x}_0}P) < d_n$ as explained in Lemma 2 and the subsequent discussion.

Definition 11. We say that $P \in \mathbb{R}[\mathbf{x}; x_n]$ is *projectively delineable* on a subset S of $\mathbb{R}^{n-1}$ if there exist $k \in \mathbb{N}$ and some continuous functions $\theta_1, \dots, \theta_k : S \rightarrow \mathbb{RP}^1$ such that

- 1) $Z_{\mathbb{RP}^1}(P, S) = \text{Graph}(\theta_1) \sqcup \dots \sqcup \text{Graph}(\theta_k)$;
- 2) for all $l \in \{1, \dots, k\}$, there exists $m_l \in \mathbb{N}^*$ such that for all $\mathbf{x} \in S$, the multiplicity of the projective root $\theta_l(\mathbf{x})$ of $E_{\mathbf{x}}P$ (w.r.t. $\deg_{x_n}(P)$) is m_l .

We say that $\theta_1, \dots, \theta_k$ are the *projective root functions* of P over S . The graph of θ_l is a *projective P -section* over S .

Requiring analyticity for S and the root functions leads to the notion of analytic projective delineability, extending the notion of analytic delineability from [10].

Example 2. Polynomial $P(x_1, x_2) = (x_1x_2 - 1)(x_2 + x_1^3)$ is projectively delineable on $\mathbb{R}$. Indeed, $Z_{\mathbb{RP}^1}(P, \mathbb{R})$ is exactly the disjoint union of the graphs of the continuous functions $\theta_1, \theta_2 : \mathbb{R} \rightarrow \mathbb{RP}^1$ defined by $\theta_1(x_1) = (1 : x_1)$ and $\theta_2(x_1) = (-x_1^3 : 1)$ (respectively depicted in green and red in Figure 2, top). Furthermore, $\theta_1(x_1)$ and $\theta_2(x_1)$ are both simple roots of $E_{x_1}(P)$ for all $x_1 \in \mathbb{R}$.

In general, the delineability of P on S does not imply the projective delineability of P on S , nor in the other direction. E.g. consider the polynomials $P(x_1, x_2) = x_1^2x_2^2 + 1$ and $Q(x_1, x_2) = x_1x_2 - 1$ and $S = \mathbb{R}$. The relations between these two concepts are investigated in the following.

Proposition 3. Let $S \subset \mathbb{R}^{n-1}$ and $P \in \mathbb{R}[\mathbf{x}; x_n]$ of degree d_n in x_n be defined by $P(\mathbf{x}, x_n) = \sum_{k=0}^{d_n} c_k(\mathbf{x})x_n^k$. If there exists $k \in \{0, \dots, d_n\}$ such that for all $\mathbf{x} \in S$, we have

$$c_{d_n-k}(\mathbf{x}) \neq 0 \wedge \forall j \in \mathbb{N} : 0 \leq j \leq k-1, c_{d_n-j}(\mathbf{x}) = 0, \quad (6)$$

then P is projectively delineable on S if and only if P is delineable on S .

Remark 4. (6) means that for all $\mathbf{x} \in S$, P admits ∞ as a projective root of multiplicity k (w.r.t. d_n) above $\mathbf{x}$.

Proof. If (6) holds and P is delineable on S , then $Z_{\mathbb{R}}(P, S)$ is the disjoint union of the graphs of the continuous functions $\eta_1, \dots, \eta_p : S \rightarrow \mathbb{R}$, and therefore

$$Z_{\mathbb{RP}^1}(P, S) = \begin{cases} \bigsqcup_{i=1}^p \text{Graph}(\varphi \circ \eta_i) & \text{if } k = 0, \\ \bigsqcup_{i=1}^p \text{Graph}(\varphi \circ \eta_i) \sqcup \text{Graph}(\theta_{p+1}) & \text{if } k \neq 0, \end{cases}$$

where θ_{p+1} is constantly equal to ∞ on S and φ is defined in (1). Hence P is projectively delineable since the condition on multiplicities is satisfied.

If (6) holds and if P is projectively delineable on S , then $Z_{\mathbb{RP}^1}(P, S)$ is the disjoint union of the graphs of the

continuous functions $\theta_1, \dots, \theta_p : S \rightarrow \mathbb{RP}^1$. If $k = 0$, there is no root at infinity. We thus have $Z_{\mathbb{R}}(P, S) = \bigsqcup_{i=1}^p \text{Graph}(\varphi^{-1} \circ \theta_i)$. Hence P is delineable since the condition on multiplicities is satisfied by Lemma 2. If $k > 0$, we introduce new projective root functions $\theta'_1, \dots, \theta'_p$ of P on S such that one of them is constantly equal to ∞ in the following way. Considering for all $i \in \{1, \dots, p\}$ the closed set $C_i = \theta_i^{-1}(\{\infty\})$, we have $S = \bigsqcup_{i=1}^p C_i$. We pick $i_0 \in \{1, \dots, p\}$ such that $C_{i_0} \neq \emptyset$ and define the new functions by $\theta'_j|_{C_i} = \theta_{\sigma_i(j)}|_{C_i}$, for all $i, j \in \{1, \dots, p\}$, where σ_i is the transposition $(i \ i_0)$ if $i \neq i_0$ and the identity if $i = i_0$. It follows from the pasting lemma that these functions are continuous. The definition ensures that θ'_{i_0} is constantly equal to ∞ and $Z_{\mathbb{RP}^1}(P, S) = \bigsqcup_{i=1}^p \text{Graph}(\theta'_i)$. We thus have $Z_{\mathbb{R}}(P, S) = \bigsqcup_{i=1, i \neq i_0}^p \text{Graph}(\varphi^{-1} \circ \theta'_i)$. We show that for $i \in \{1, \dots, p\} \setminus \{i_0\}$, the multiplicity of $\theta'_i(\mathbf{x})$ as a projective root of $E_{\mathbf{x}}P$ (w.r.t. d_n) is equal to that of $\theta_i(\mathbf{x})$ and hence, does not depend on $\mathbf{x} \in S$. If $C_i = \emptyset$, then $\theta'_i = \theta_i$ on S and the conclusion is direct. Otherwise, the multiplicity of $\theta_i(\mathbf{x})$ is k for every $\mathbf{x} \in C_i$, hence for every $\mathbf{x} \in S$. The multiplicity of $\theta'_i(\mathbf{x})$ is that of $\theta_i(\mathbf{x})$ or that of $\theta_{i_0}(\mathbf{x})$, and both are equal to k . By Lemma 2, P is therefore delineable on S . $\square$

If we only assume the sign-invariance of the leading coefficient of P on S , then the delineability of P on S does not imply the projective delineability of P on S . A counter example is given by the polynomial $P(x_1, x_2, x_3) = x_1x_3^3 + (x_1^2 + x_2^2)x_3^2 + 1$ over the line $S \equiv x_1 = 0$. The converse is true if we suppose that S is connected.

Corollary 1. Using the notation of Proposition 3,

- 1) if c_{d_n} never vanishes on S , then P is projectively delineable on S if and only if P is delineable on S ;
- 2) if c_{d_n} vanishes identically on a connected S and P is projectively delineable on S , then P is delineable on S .

Proof. The first item is the particular case $k = 0$ of Proposition 3. For the second item, the projective delineability of P on the connected set S implies that there is a projective root function, with a constant multiplicity k on S , that is identically equal to ∞ on S . By Remark 4, Condition (6) is then satisfied. From Proposition 3 we deduce the delineability of P on S . $\square$

Corollary 1 (2) fails without the connectedness assumption. A counter example is given by the polynomial

$$P(x_1, x_2) = (x_1x_2 - 1)((x_1 - 1)x_2 - 1)^2$$

and the set $S = \{0, 1\}$.

While projective delineability is not a theoretically weaker property than delineability, the corresponding projection (discriminants and resultants only, see below) that ensures projective delineability is a proper subset of the reduced McCallum projection to ensure delineability [3]. Thus projective delineability offers computational savings.

V. LOCAL RESULT

We prove a first projective analogue of the classical results concerning delineability [5], [11], [3]. This result is local since it only holds in a suitable neighbourhood of each point in a connected analytic submanifold of $\mathbb{R}^{n-1}$.

Theorem 1. Consider S a connected analytic submanifold of $\mathbb{R}^{n-1}$ and $P \in \mathbb{R}[\mathbf{x}; x_n]$ a polynomial of degree d_n in x_n that never nullifies on S , such that $\text{Disc}_{x_n}^{d_n}(P)$ is not the zero polynomial and is order-invariant on S . For each $\mathbf{s} \in S$, there exists a connected neighbourhood $N_{\mathbf{s}}$ of $\mathbf{s}$ in S such that P is projectively delineable on $N_{\mathbf{s}}$ and $H^{d_n}(P)$ is order-invariant on each projective P -section on $N_{\mathbf{s}}$.

The idea of the proof is to use Theorem 2 of [11] for delineability. When the degree in x_n of $E_{\mathbf{x}}P$ is not constant on S , this result does not apply, and the multiplicity of ∞ as a projective root of $E_{\mathbf{x}}P$ is not constant on S . Intuitively, we bypass this issue by using a geometric transformation to modify the zero set of the polynomial P in order to guarantee that it does not reach infinity anymore, as shown in Example 3. To do so, we replace P by the polynomial $A^{*d_n}P$ on a neighbourhood of $\mathbf{s} \in S$ for a well chosen matrix A and ensure that this new polynomial has all the required properties (via Lemma 4). We then obtain locally the delineability of $A^{*d_n}P$ and deduce the desired result for P .

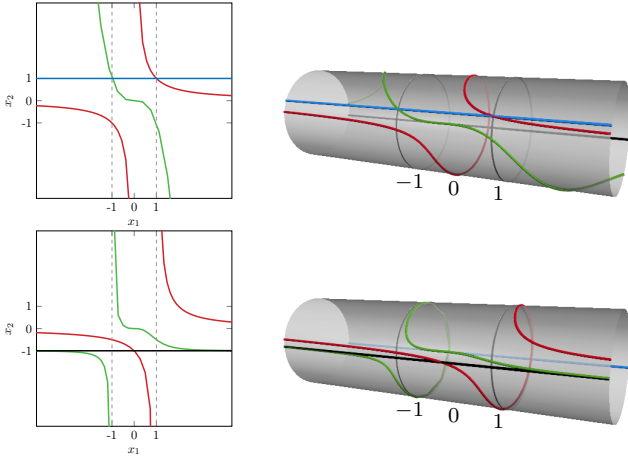


Fig. 2. A visualization of $Z_{\mathbb{RP}^1}(P, \mathbb{R})$ with the blue line $x_2 = 1$ (top) and $Z_{\mathbb{RP}^1}(A^{*2}P, \mathbb{R})$ with the black line $x_2 = -1$ (bottom).

Example 3. Continuing Example 2 and setting $\mathbf{s} = 0$, we observe that the root θ_1 goes through ∞ exactly above $\mathbf{s}$: this is precisely where its leading coefficient vanishes. If we set A to the 2×2 lower triangular matrix of 1s, then

$$A^{*2}P(x_1, x_2) = ((x_1 - 1)x_2 - 1)((1 + x_1^3)x_2 + x_1^3).$$

The leading coefficient of this polynomial vanishes exactly when $x_1 = \pm 1$. In particular, on the open neighbourhood $N_{\mathbf{s}} = (-1, 1)$ of $\mathbf{s}$, $A^{*2}P$ does not admit any root at infinity (see Figure 2, bottom). The delineability of this polynomial is given by Theorem 2 of [11] (or can be checked by hand as in Example 2). Denoting by η_1 and η_2 its real root

functions on $N_{\mathbf{s}}$, we can show that their compositions with φ_A give the two projective root functions of P on $N_{\mathbf{s}}$.

Lemma 4. Let $P \in \mathbb{R}[\mathbf{x}; x_n]$ be a nonzero polynomial of degree d_n in x_n and $A \in GL(2, \mathbb{R})$.

- 1) The polynomial $A^{*d_n}P \in \mathbb{R}[\mathbf{x}; x_n]$ has degree less than or equal to d_n in x_n and its coefficient of degree d_n in x_n is $H^{d_n}(P)(\mathbf{x}, (a_{11}, a_{21}))$.
- 2) The map μ_A is a self-homeomorphism of $\mathbb{R}^{n-1} \times \mathbb{RP}^1$ such that $\mu_A(Z_{\mathbb{RP}^1}(A^{*d_n}P)) = Z_{\mathbb{RP}^1}(P)$, preserving the multiplicities of roots.
- 3) The map μ_A preserves the order. More precisely,

$$\begin{aligned} \text{ord}(H^{d_n}(A^{*d_n}P), (\mathbf{x}, (x_n : y))) \\ = \text{ord}(H^{d_n}(P), \mu_A(\mathbf{x}, (x_n : y))) \end{aligned}$$

for all $(\mathbf{x}, (x_n : y)) \in \mathbb{R}^{n-1} \times \mathbb{RP}^1$.

- 4) If $H^{d_n}(P)(\mathbf{s}, (a_{11}, a_{21})) \neq 0$ for some $\mathbf{s} \in \mathbb{R}^{n-1}$, then the polynomial $A^{*d_n}P$ has degree d_n and the set $V_{\mathbf{s}} = \{\mathbf{x} \in \mathbb{R}^{n-1} \mid H^{d_n}(P)(\mathbf{x}, (a_{11}, a_{21})) \neq 0\}$ is an open neighbourhood of $\mathbf{s}$ such that $A^{*d_n}P$ has no root at infinity above $V_{\mathbf{s}}$.

Proof.

- 1) This follows from Equation (5).
- 2) The correspondence of the roots and multiplicities is discussed in Lemma 3 for univariate polynomials and directly extends to n -variate polynomials.
- 3) By definition of order and Lemma 1, we need to show

$$\begin{aligned} \text{ord}(R_A \circ H^{d_n}(P), (\mathbf{x}, (x_n, y))) \\ = \text{ord}(H^{d_n}(P), (\text{id} \times A)(\mathbf{x}, (x_n, y))) \end{aligned}$$

for all $(\mathbf{x}, x_n, y) \in \mathbb{R}^{n+1}$. Furthermore, we have $R_A \circ H^{d_n}(P) = H^{d_n}(P) \circ (\text{id} \times A)$. The conclusion follows from the first item of Proposition 2.

- 4) The result concerning the degree of $A^{*d_n}P$ follows from the first assertion. The continuity of the map $\mathbf{x} \mapsto H^{d_n}(P)(\mathbf{x}, (a_{11}, a_{21}))$ implies that $V_{\mathbf{s}}$ is an open neighbourhood of $\mathbf{s}$. The leading coefficient (of degree d_n) of $A^{*d_n}P$ is $H^{d_n}(P)(\mathbf{x}, (a_{11}, a_{21}))$ and therefore does not vanish above $V_{\mathbf{s}}$. $\square$

Proof of Theorem 1. Fix $\mathbf{s} \in S$. Since $E_{\mathbf{s}}(P)$ is not the zero polynomial, we can choose $(a_{11}, a_{21}) \in \mathbb{R}_*^2$ such that $H^{d_n}(P)(\mathbf{s}, (a_{11}, a_{21})) \neq 0$ and then $a_{12}, a_{22} \in \mathbb{R}$ such that $\det(A) = 1$. The fourth assertion of Lemma 4 guarantees the existence of a neighbourhood $V_{\mathbf{s}}$ of $\mathbf{s}$ in $\mathbb{R}^{n-1}$ such that $A^{*d_n}P$ has no root at infinity above $V_{\mathbf{s}}$. Then $N_{\mathbf{s}} = V_{\mathbf{s}} \cap S$ is an open subset of S (hence an analytic submanifold), which we may assume to be connected, since S is locally connected. We now apply the delineability result to $A^{*d_n}P$ above $N_{\mathbf{s}}$ (Theorem 2 of [11]). To this aim, we first show that the hypotheses of this result are satisfied:

- 1) The leading coefficient of $A^{*d_n}P$ (i.e. the coefficient of degree d_n) does not vanish on $N_{\mathbf{s}}$.
- 2) The discriminant $\text{Disc}_{x_n}^{d_n}(A^{*d_n}P)$ equals $\text{Disc}_{x_n}^{d_n}(P)$ by Proposition 1. It is therefore not the zero polynomial and is order-invariant on $N_{\mathbf{s}}$ by assumption.

Thus $A^{*d_n}P$ is delineable on N_s and is order-invariant in each $A^{*d_n}P$ -section over N_s , i.e. there exist continuous functions $\eta_1 < \dots < \eta_k : N_s \rightarrow \mathbb{R}$ such that:

- 1) $Z_{\mathbb{R}}(A^{*d_n}P, N_s) = \text{Graph}(\eta_1) \sqcup \dots \sqcup \text{Graph}(\eta_k)$;
- 2) the multiplicity of $\eta_l(\mathbf{x})$ as a root of $E_{\mathbf{x}}(A^{*d_n}P)$ is constant on N_s ; and
- 3) the order of $A^{*d_n}P$ at $(\mathbf{x}, \eta_l(\mathbf{x}))$ is constant on N_s .

Hence, by assertion 2 of Lemma 4, the continuous maps $\theta_l = \varphi_A \circ \eta_l = \mu_A \circ \varphi \circ \eta_l$ with $l \in \{1, \dots, k\}$ are precisely the projective root functions of P . Indeed, we check that these functions satisfy the requirements of Definition 11:

- 1) For all $\mathbf{x} \in N_s$, the points $\theta_1(\mathbf{x}), \dots, \theta_k(\mathbf{x})$ are distinct by injectivity of $\mu_A \circ \varphi$.
- 2) We have $\varphi(Z_{\mathbb{R}}(A^{*d_n}P, N_s)) = Z_{\mathbb{RP}^1}(A^{*d_n}P, N_s)$ since $A^{*d_n}P$ has no root at infinity above N_s . Hence,

$$\begin{aligned} Z_{\mathbb{RP}^1}(P, N_s) &= \mu_A(Z_{\mathbb{R}}(A^{*d_n}P, N_s)) \\ &= \text{Graph}(\theta_1) \cup \dots \cup \text{Graph}(\theta_k). \end{aligned}$$

- 3) For all $\mathbf{x} \in N_s$, $\varphi(\eta_l(\mathbf{x}))$ is a projective root of $A^{*d_n}P$ of multiplicity m_l by Lemma 2 and the subsequent discussion. Furthermore, Lemma 4 guarantees that these multiplicities are preserved by μ_A , that is $\theta_l(\mathbf{x})$ is a projective root of multiplicity m_l of P .

The third assertion of Lemma 4 and Proposition 2 give the conclusion concerning the order-invariance of $H^{d_n}(P)$ on the projective P -sections. $\square$

VI. GLOBAL RESULT AND COVERINGS

In this section, we prove a global counterpart of Theorem 1, finding this requires some additional assumptions which are classical in the theory of covering spaces [7].

Definition 12. Let X be a topological space. A *covering space* of X is a topological space Y together with a continuous map $p : Y \rightarrow X$ such that, for each point $x \in X$, there exist an open neighbourhood U of x in X and a non-empty family of disjoint open sets $(V_i)_{i \in I}$ of Y such that $p^{-1}(U) = \sqcup_{i \in I} V_i$ and $p|_{V_i} : V_i \rightarrow U$ is a homeomorphism. The covering space Y is trivial if these conditions hold for $U = X$.

Remark 5. Note that in this setting, defining for $i \in I$ the *local section* $s_i = (p|_{V_i})^{-1}$ we obtain a family of continuous functions $(s_i : U \rightarrow Y)_{i \in I}$, such that $p \circ s_i = \text{id}_U$, $s_i(U)$ is open in $p^{-1}(U)$, and $p^{-1}(U)$ is the union of these sets.

Recall that a topological space is simply connected if it is path connected (and thus connected), and for any $x \in X$, any loop of X based at x can be continuously deformed (preserving the base point) onto the constant loop.

Theorem 2. Under the hypotheses of Theorem 1, and the further assumption that S is simply connected, we have that P is projectively delineable on S and $H^{d_n}(P)$ is order-invariant on each projective P -section over S .

Proof. Theorem 1 guarantees that the set $Z_{\mathbb{RP}^1}(P, S)$ is a covering space of S for the usual (continuous) projection

$$p : Z_{\mathbb{RP}^1}(P, S) \rightarrow S : (\mathbf{x}, (x_n : y)) \rightarrow \mathbf{x}.$$

Indeed, with the notation of Theorem 1, for all $\mathbf{s} \in S$, projective delineability of P over the open neighbourhood N_s implies that $p^{-1}(N_s) = Z_{\mathbb{RP}^1}(P, N_s)$ is the disjoint union $\sqcup_{i=1}^k \text{Graph}(\theta_i)$ where $\theta_1, \dots, \theta_k$ are the projective root functions from N_s to $\mathbb{RP}^1$. Then for all $i \in \{1, \dots, k\}$, $V_i = \text{Graph}(\theta_i)$ is homeomorphic to N_s through p and $\text{Id} \times \theta_i$. Finally, since θ_i is a continuous function from N_s to the Hausdorff space $\mathbb{RP}^1$, its graph is closed in $N_s \times \mathbb{RP}^1$, hence in $p^{-1}(N_s) \subset N_s \times \mathbb{RP}^1$. Thus $V_i = p^{-1}(N_s) \setminus \cup_{j \neq i} V_j$ is open in $p^{-1}(N_s)$, and therefore in $Z_{\mathbb{RP}^1}(P, S)$.

Since S is a simply connected analytic submanifold, any covering space over S is trivial [7, Corollary 13.8], so there exist global sections $s_1, \dots, s_k$ of $p : Z_{\mathbb{RP}^1}(P, S) \rightarrow S$. These functions read $s_i(\mathbf{x}) = (\mathbf{x}, t_i(\mathbf{x}))$ and t_i are by definition continuous projective root functions globally defined on S , such that $Z_{\mathbb{RP}^1}(P, S)$ is the disjoint union of their graphs. Finally for every $\mathbf{s} \in S$ the function t_i coincides over N_s with one of the functions $\theta_1, \dots, \theta_k$. Therefore, the multiplicity of t_i as a projective root of $E_{\mathbf{s}}(P)$ is constant on N_s as is the order of $H^{d_n}(P)$ at t_i on N_s . We conclude by the connectedness of S . $\square$

Remark 6. Observe that the projective root functions in Theorems 1 and 2 are analytic functions. Indeed, their local expressions in suitable (analytic) charts are exactly the root functions given by Theorem 2 of [11].

Theorem 2 fails without the additional assumption:

Proposition 4. There exist a connected analytic submanifold S of $\mathbb{R}^2$ and $P \in \mathbb{R}[x_1, x_2; x_3]_{\leq 4}$ such that

- 1) P is never nullified on S ;
- 2) the discriminant $\text{Disc}_{x_3}^4(P)$ of P is not the zero polynomial of $\mathbb{R}[x_1, x_2]$ and is order-invariant on S ;
- 3) P is not projectively delineable over S .

Proof. We define $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 - 1 = 0\}$ and $P \in \mathbb{R}[x_1, x_2; x_3]_{\leq 4}$ such that $P(x_1, x_2, x_3)$ is given by

$$(1 - x_1)x_3^4 + 4x_2x_3^3 + (2 + 6x_1)x_3^2 - 4x_2x_3 + (1 - x_1).$$

See Figure 3. Then P satisfies the requested properties: it is never nullified on S and we compute that

$$\text{Disc}_{x_3}^4(P)(x_1, x_2) = 2^{14}(-1 + x_1^2 + x_2^2)^2(x_1^2 + x_2^2).$$

Elementary computations show that it has order 2 on S .

To prove that P is not projectively delineable over S , we first compute the set of projective roots of P above every $\mathbf{x} = (x_1, x_2) \in S$. If we write $\mathbf{x} = (\cos(t), \sin(t))$ for $t \in \mathbb{R}$ then we can compute that $H^4(P)(\mathbf{x}, (x_3, y))$ is

$$8(\cos(t/4)x_3 - \sin(t/4)y)^2(\sin(t/4)x_3 + \cos(t/4)y)^2,$$

implying that P has exactly two projective roots over (x_1, x_2) of multiplicity two given by

$$(\sin(t/4) : \cos(t/4)) \text{ and } (-\cos(t/4) : \sin(t/4)).$$

If we assume that P is projectively delineable over S , then there exists two continuous functions $\theta_1, \theta_2 : S \rightarrow \mathbb{RP}^1$ such that $Z_{\mathbb{RP}^1}(P, S) = \text{Graph}(\theta_1) \sqcup \text{Graph}(\theta_2)$. Hence, $Z_{\mathbb{RP}^1}(P, S)$ is not connected because the graphs $\text{Graph}(\theta_1)$ and $\text{Graph}(\theta_2)$ are non-empty and closed. However $Z_{\mathbb{RP}^1}(P, S)$ happens to be connected as the image of $\mathbb{R}$ by the continuous function $\Upsilon : \mathbb{R} \rightarrow \mathbb{R}^2 \times \mathbb{RP}^1$ given by

$$t \mapsto (\cos(t), \sin(t), (\sin(t/4) : \cos(t/4))). \quad \square$$

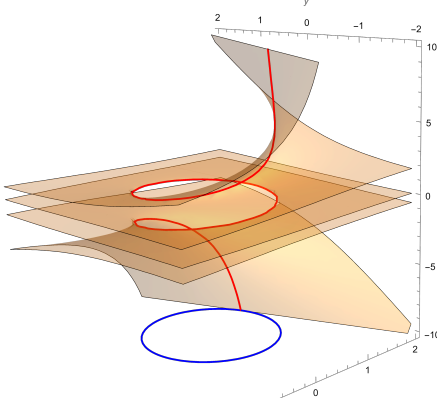


Fig. 3. S (blue), $Z_{\mathbb{R}}(P, \mathbb{R}^2)$ (orange) and $Z_{\mathbb{R}}(P, S)$ (red)

We show that Theorem 2.2 of [12] extends naturally in the projective framework. In particular, the order-invariance of the resultant of two projectively delineable polynomials on S guarantees the projective delineability of their product on S , allowing the notion to generalise from one polynomial to a set, as with classical delineability.

Theorem 3. Let S be a connected analytic submanifold of $\mathbb{R}^{n-1}$ and $P, Q \in \mathbb{R}[\mathbf{x}; x_n]$ of respective positive degree p_n and q_n in x_n . Assume P, Q are never nullified on S and P is projectively delineable over S . If $\text{Res}_{x_n}^{p_n, q_n}(P, Q)$ is not the zero polynomial and is order-invariant over S , then along each projective P -section over S , either $H^{q_n}(Q)$ vanishes completely or it never vanishes at all.

Proof. By assumption, P is projectively delineable over S and we denote by $\theta_1, \dots, \theta_k : S \rightarrow \mathbb{RP}^1$ its projective root functions over S . For every $l \in \{1, \dots, k\}$, we consider

$$\Omega_l = \{\mathbf{x} \in S \mid (\mathbf{x}, \theta_l(\mathbf{x})) \in Z_{\mathbb{RP}^1}(Q, S)\}$$

and we have to prove that this set is either empty or equal to S . We do it by using similar arguments as in Theorem 1 and the connectedness of S . We may suppose that Ω_l is non-empty and take $\mathbf{s}$ in its closure.

Since $E_{\mathbf{s}}(PQ) = E_{\mathbf{s}}(P)E_{\mathbf{s}}(Q)$ is nonzero, we can choose $(a_{11}, a_{21}) \in \mathbb{R}_*^2$ such that $H^{p_n}(P)H^{q_n}(Q)(\mathbf{s}, (a_{11}, a_{21})) \neq 0$ and then $a_{12}, a_{22} \in \mathbb{R}$ such that $\det A = 1$. We then obtain a neighbourhood $N_{\mathbf{s}}$ of $\mathbf{s}$ as in the beginning of the proof of Theorem 1 (but for the polynomial PQ). The projective root functions of P above $N_{\mathbf{s}}$ are given by the restrictions of $\theta_i, i \in \{1, \dots, k\}$ to $N_{\mathbf{s}}$. By Lemma 3 the projective root

functions of $A^{*p_n}P$ above $N_{\mathbf{s}}$ are given by the continuous functions $\mu_{A^{-1}} \circ \theta_i$. Hence $A^{*p_n}P$ is projectively delineable over $N_{\mathbf{s}}$ and has, by the choice of A , no root at infinity. By Proposition 3, we obtain the delineability of $A^{*p_n}P$ and denote by $\eta_i = \varphi^{-1} \circ \mu_{A^{-1}} \circ \theta_i, i \in \{1, \dots, k\}$ its root functions. We now check the hypotheses of Theorem 2.2 of [12], with $A^{*p_n}P$ and $A^{*q_n}Q$:

- 1) They have respective degree p_n and q_n in x_n by the choice of A . Hence their degrees are positive.
- 2) By Proposition 1, $\text{Res}^{p_n, q_n}(A^{*p_n}P, A^{*q_n}Q) = \text{Res}^{p_n, q_n}(P, Q)$, which is therefore not the zero polynomial and order-invariant over the connected analytic submanifold $N_{\mathbf{s}}$.

In particular, we conclude that $A^{*q_n}Q$ is sign invariant on the graph of η_l over $N_{\mathbf{s}}$. Since $\mathbf{s}$ is in the closure of Ω_l , there exists $\mathbf{s}_0 \in N_{\mathbf{s}} \cap \Omega_l$. Then $\theta_l(\mathbf{s}_0)$ is a projective root of Q above $\mathbf{s}_0$, so $\mu_{A^{-1}} \circ \theta_l(\mathbf{s}_0)$ is a projective root of $A^{*q_n}Q$ above $\mathbf{s}_0$ and finally $A^{*q_n}Q(\mathbf{s}_0, \eta_l(\mathbf{s}_0)) = 0$. By the sign invariance $A^{*q_n}Q(\mathbf{x}, \eta_l(\mathbf{x})) = 0$ for every $\mathbf{x} \in N_{\mathbf{s}}$, and it follows that $(\mathbf{x}, \theta_l(\mathbf{x})) \in Z_{\mathbb{RP}^1}(Q, S)$ for every $\mathbf{x} \in N_{\mathbf{s}}$. We just showed that $N_{\mathbf{s}} \subset \Omega_l$ for all $\mathbf{s}$ in the closure of Ω_l , ensuring that the closure of Ω_l is a subset of the interior of Ω_l . Then Ω_l is open and closed and therefore $\Omega_l = S$ by connectedness. $\square$

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