J. Math. Anal. Appl. 548 (2025) 129376

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Metastability of almost minimizers in a 1-d Allen-Cahn equation on a compact domain

Maria G. Westdickenberg

RWTH Aachen University, Germany

A R T I C L E   I N F O

Article history:
Received 11 November 2024
Available online 12 February 2025
Submitted by C.E. Wayne

Keywords:
Energy–energy–dissipation
Nonlinear PDE
Gradient flow
Dynamic metastability

A B S T R A C T

We study metastability of the one-dimensional Allen-Cahn equation on a bounded interval with pinned ± 1 boundary conditions. We are particularly interested in considering the case of an asymmetric, nondegenerate double-well potential, meaning that the second derivative of the potential is nonzero in each of the wells but $G''(-1) \neq G''(1)$ is allowed. Under fairly general conditions on the initial data, we show that the solution is drawn quickly into a small neighborhood of the so-called slow manifold and remains stuck there for an exponentially long time (which we quantify), even though the unique global minimizer of the energy is far away.

1. Introduction

Dynamic metastability is a phenomenon in which generic solutions of a time-dependent system rapidly relax to a configuration *that is not a stationary solution* but then eventually—canonically after an *exponentially long slow motion phase*—change dramatically. The seminal example of dynamical metastability is the one-dimensional Allen-Cahn equation, whose metastability was described and analyzed in unpublished notes of John Neu and the landmark papers of Carr and Pego [11] and Fusco and Hale [17]. The first energy method for its analysis was introduced by Bronsard and Kohn in [9]. The Allen-Cahn equation and variants thereof has since been studied by many authors and we do not attempt a comprehensive review, but point for instance to [12,14,20,24,25] and the references therein.

A related, fourth-order equation for which metastability has been numerically and analytically captured is the one-dimensional Cahn-Hilliard equation; see for instance [1–4,7,8,15,19,22,23]. Unlike the Allen-Cahn equation, the Cahn-Hilliard equation preserves mass.

For both the Allen-Cahn and Cahn-Hilliard equations, there are slowly evolving configurations consisting of multiple layers (see below for more details). The closest pair of layers attracts, however the attraction

E-mail address: maria@math1.rwth-aachen.de.

https://doi.org/10.1016/j.jmaa.2025.129376

force is so weak (exponentially weak in the distance between layers) that the time it takes for the layers to come together and annihilate is exponentially long.

Relaxation to equilibrium—not metastability—for the Cahn-Hilliard equation on the line was analyzed in [21] and a *two-scale phenomenon* was observed: For initial data not too far from a kink (a single transition layer satisfying ± 1 boundary conditions at $\pm\infty$), the system relaxes quickly to *a kink* because of energetic preferences and then more slowly to *the kink* that is singled out by the conserved quantity. The Allen-Cahn equation on the line does not display this behavior: If the same ± 1 boundary conditions are imposed, the solution passes through metastable phases as excess transition layers annihilate (cf. [25]), but once only one transition layer remains, convergence is exponential in time with an order one rate. Heuristically, this is due to the fact that *any* center of the transition layer is equally fine; there is no conserved quantity to single out one kink as being better than any other.

In this paper, we consider the Allen-Cahn equation with a selection mechanism that generates metastable behavior even for single layer solutions. Namely, we consider the equation on a bounded interval $(0, L)$ with $u(0) = -1$, $u(L) = 1$. Whereas on the line or on a bounded interval with periodic boundary conditions, there is a continuum of energy minimizers, our problem has a unique energy minimizer—but a continuum of *near minimizers*, which will be the source of the metastability. For an appropriate class of initial data, the solution will converge in an order one time to a small neighborhood of a near minimizer and then remain close to that near minimizer for an exponentially long period of time before eventually converging to the true minimum. The cartoon that one should have in mind is that of a Mexican hat potential where the ring at the base is not flat but exponentially close to being flat.

Our motivation is three-fold. On the one hand, the paper can be considered didactic in the sense that it demonstrates an application of the metastability framework developed in [20,22]. For simplicity, we restrict our attention to the convergence to a single-layer profile and to uniformly bounded disturbances, so that the presentation simplifies in several ways compared to [25]. On the other hand, we attempt to shine a light on the metastability and weak energetic preferences that can be captured with current tools of analysis. While the Allen-Cahn equation may be considered a toy model, the phenomenon is of broader interest. See [18] for a recent application in the context of a self-attention model (a toy model arising in machine learning). A final motivation is to reveal the effect of *asymmetry* of the potential in the sense that *unequal second derivatives* control the "timescale of metastability" in the sense of Theorem 1.5 and Remark 1.7 below. While this effect can be understood heuristically from a harmonic approximation and has been studied in the physics literature [13], to the best of our knowledge this is a new result in the mathematics literature; see also (1.3) and the accompanying discussion below..

The reason that we do not also study the Cahn-Hilliard equation is that its conserved quantity makes the equation overconstrained for our purpose. The Allen-Cahn equation, with its freedom to consider arbitrarily shifted kinks, is the right setting to observe the effect of the asymmetry.

1.1. Setting and result

We study the one-dimensional Allen–Cahn equation

$$\partial_t u = \partial_{xx} u - G'(u), \qquad x \in (0, L),\ t > 0 \tag{1.1}$$

subject to the boundary conditions

$$u(0) = -1, \quad u(L) = 1. \tag{1.2}$$

Here $G(u)$ is a double-well potential with unique and nondegenerate global minima normalized to be at ± 1 with $G(\pm 1) = 0$. Here "nondenerate" means that the second derivatives at the minima are strictly positive: $0 < G''(\pm 1)$. For recent work on degenerate potentials, we refer to [6,16].

While previously the literature has often focused on potentials with equal second derivatives at the minima or "dismissed" asymmetry by reducing to estimates in terms of $\min\{G''(-1), G''(1)\}$, we are specifically interested in allowing—and probing the influence of—asymmetry in the form

$$G''(-1) \neq G''(1). \tag{1.3}$$

For the rest of the paper, when we refer to an "asymmetric potential," we mean that (1.3) holds. Equality of the second derivatives was assumed in [17, p. 86] and a slow motion timescale based on the *minimum* of the second derivatives was established in [11, p. 547], [1, p. 94], and [19, p. 23]. The effect of asymmetry is captured in the weighted coarsening timescale derived by Chen [12, p. 402, point (3)] and Bethuel and Smets [5, p. 86], however in [12] the case of $n \geq 2$ interfaces on $\mathbb{R}$ is investigated, and in [5] well-prepared data with $n \geq 2$ on $\mathbb{R}$ is the focus ($n = 1$ would not be metastable). In particular, our result is not contained in and cannot be deduced from the results of [5,12].

Equation (1.1) is often studied with a small parameter ε appearing in the equation, in which case typical transition layers connecting ± 1 occur on scale ε. This is equivalent via a rescaling of space to studying (1.1) for L large and order-one transition layers.

The Allen-Cahn equation is the L^2 gradient flow of the scalar Ginzburg–Landau energy

$$E(u) := \int_0^L \frac{1}{2}(\partial_x u)^2 + G(u)\, dx. \tag{1.4}$$

While for Neumann or periodic boundary conditions, the solution for generic initial data converges for $t \to \infty$ to $u \equiv -1$ or $u \equiv 1$, under the "pinned" Dirichlet conditions, the solution approaches the (unique) function

$$v_* := \operatorname{argmin} E \quad \text{subj to } v_*(0) = -1,\ v_*(L) = 1. \tag{1.5}$$

We denote the minimal energy by

$$e_* := E(v_*).$$

The solution v_* is monotone and has a unique zero x_*. For a symmetric potential, $x_* = L/2$. For an asymmetric potential, there holds

$$x_* = \frac{L\sqrt{G''(1)}}{\sqrt{G''(-1)} + \sqrt{G''(1)}} + O(1). \tag{1.6}$$

See Subsection 2.1 below for proofs of these facts. To fix ideas and without loss of generality, we will assume that

$$G''(1) \geq G''(-1)$$

so that the zero x_* is located in the right half-interval $[L/2, L)$.

We will now introduce our "slow manifold" of approximate minimizers. In order to get universal bounds on our constants, we will work with functions whose zero is uniformly bounded of order L away from the endpoints and x_*. Without loss of generality, we will consider centers *to the left of* x_*, but we could just as well consider centers bounded to the right of x_*. Choosing the left-hand side, we restrict to the approximate minimizers:

Definition 1.1. We define the approximate minimizers or energy optimal profiles as

$$\mathcal{N}(L) := \{v_\xi : [0,L] \to \mathbb{R} \colon \text{for some } \xi \in (x_*/2, 3x_*/4),$$
$$v_\xi = \operatorname{argmin} E \text{ subj to } v_\xi(0) = -1,\, v_\xi(\xi) = 0,\, v_\xi(L) = 1\}.$$

More generally, one could consider $\xi \in (\alpha x_*, \beta x_*)$ for any $\alpha < \beta$ in $(0,1)$, but the constants in the theorem would depend on α and β.

Notice that $v \in \mathcal{N}(L)$ is continuous but typically not differentiable, but the jump in the first derivative is exponentially small in L. Also the dependence of $E(v)$ on ξ is exponentially weak. (We will make these statements more precise in Subsection 2.1 below.)

In our theorem we will select initial conditions from $\mathcal{N}(L)$ and then during the evolution consider approximate minimizers whose zeros are allowed to take values in a larger interval:

$$\mathcal{N}_+(L) := \{v_\xi : [0,L] \to \mathbb{R} \colon \text{for some } \xi \in (x_*/4, 7x_*/8),$$
$$v_\xi = \operatorname{argmin} E \text{ subj to } v_\xi(0) = -1,\, v_\xi(\xi) = 0,\, v_\xi(L) = 1\}.$$

Remark 1.2. We emphasize for future reference that

$$x_* \overset{(1.6)}{=} O(L)$$

and for $v_\xi \in \mathcal{N}_+(L)$,

$$\xi = O(L).$$

The expected behavior is the following: An appropriate initial condition with projection in $\mathcal{N}(L)$, and hence with a large region of $u_0 < 0$ and a large region of $u_0 > 0$, will *quickly relax* to some almost minimizer $v_\xi \in \mathcal{N}_+(L)$ and then *slowly evolve* towards the absolute minimizer v_*, with the zero $\mathbf{x}$ of the solution slowly approaching x_*. Our goal is to capture the sharp exponential timescale of this metastable phase. The timescale is controlled by the projection onto the slow manifold: If u_0 is order-one away from a function $v_0 \in \mathcal{N}(L)$ with zero x_0, then it is trapped near v_0 for a time of order $\exp(\sqrt{G''(-1)}x_0)$. If on the other hand u_0 is order-one away from an energy optimal profile with a zero to the right of x_*, then it is trapped nearby for a time of order $\exp(\sqrt{G''(1)}(L - x_0))$. See Remark 1.7 for a more detailed statement. We do not consider initial data such that the projection has a zero close to x_* since this is the stable rather than metastable regime.

The metastable set with which we will work is the set of functions with order one energy and order one L^2-distance to a member of the slow manifold:

$$\mathcal{M}(L, C_1) := \Big\{u \in H^1\left((0,L)\right),\, u(0) = -1,\, u(L) = 1, \text{ such that } E(u) \le C_1 \text{ and}$$
$$\text{there exists } v \in \mathcal{N}(L) \text{ such that } \|u - v\|_{L^2}^2 \le C_1\Big\}.$$

We will often abbreviate

$$\mathcal{N} = \mathcal{N}(L), \quad \mathcal{M} = \mathcal{M}(L, C_1).$$

Throughout the paper we will for $u \in \mathcal{M}(L, C_1)$ let $\mathbf{x}$ and $v_\mathbf{x}$ denote any zero of u and a corresponding constrained minimizer $v = v_\mathbf{x}$ (a "projection onto $\mathcal{N}_+$") such that

$$\|u - v\|_{L^2}^2 \le C_1. \tag{1.7}$$

A solution u may (initially) have additional zeros for which the associated almost minimizer v does not satisfy (1.7); these zeros are not admissible for the projection.

Notice that $\mathbf{x}$ and v depend on time. To get fine results, we will track the v-related energy gap

$$\mathcal{E} = \mathcal{E}(t) := E\big(u\big) - E\big(v\big).$$

By construction of v, there holds $\mathcal{E} \geq 0$.

For initial data $u_0 \in \mathcal{M}$ and the associated (constant in time) function v_0, we will for the solution u of the Allen-Cahn equation (1.1) sometimes track the simpler quantities

$$\bar{\mathcal{E}} = \bar{\mathcal{E}}(t) := E\big(u\big) - E(v_0) \qquad \text{and} \qquad \bar{H} = \bar{H}(t) := \|u - v_0\|_{L^2}^2.$$

We define in addition the dissipation

$$D = D(t) := \int \Big(\partial_{xx} u - G'\big(u\big)\Big)^2 \, dx.$$

Because of the gradient flow structure and since v_0 is fixed in time, there holds

$$-\frac{d}{dt}\bar{\mathcal{E}} = D, \tag{1.8}$$

and hence $\bar{\mathcal{E}}$ is decreasing in time (not true for $\mathcal{E}$).

Remark 1.3 *(Zero(s) of the solution).* Here $u \in \mathcal{M}$ does not rule out—and is not disturbed by—the existence of more than one zero of u. However we will show that after an order-one initial relaxation stage, u has precisely one zero. As a consequence, the zero and the associated $v \in \mathcal{N}$ are from that point on uniquely defined (and continuous in time).

Remark 1.4 *(Initial data).* For simplicity of presentation and in order to have only universal constants in our estimates, we restrict to initial data such that

$$E(u_0) \leq 4e_*, \qquad \|u_0 - v_0\|_{L^2}^2 \leq 4e_*.$$

As was carried out in [25], one can extend to initial data with larger energy/distance at the expense of an initial layer in time in which bounds depend on these values. Notice that according to Remark 1.2 above, the zero x_0 in the statement of the theorem is $O(L)$ and δ is exponentially small.

Theorem 1.5. *Let C_{ed} be the constant from Lemma 2.5 below. There exist L_1, $\varepsilon \in (0,1)$ such that for any*

$$L \geq L_1, \quad u_0 \in \mathcal{M}(L, 4e_*), \tag{1.9}$$

the following holds true. Let x_0 denote the zero of the projection $v_0 \in \mathcal{N}_+(L)$ associated to u_0 and define

$$\delta = \exp(-2\sqrt{G''(-1)}x_0) \qquad \text{and} \qquad T_* := \varepsilon\delta^{-1}. \tag{1.10}$$

For the solution $u(t)$ of the Allen–Cahn equation (1.1) with initial condition u_0 and all $t \in [0, T_*]$, there exists a projection $v = v(t) \in \mathcal{N}_+(L)$ and u, v satisfy:

(i) *There exists an order one time* s_1 *such that on* $[0, s_1]$, *there holds*

$$\|u(t)-v(t)\|_{H^1}^2+\mathcal{E}(t)\lesssim \mathcal{E}_0\exp\left(-t/(2C_{\mathrm{ed}})\right)+\left(\mathcal{E}_0^{\frac{1}{2}}+\mathcal{E}_0^{\frac{1}{4}}\right)\delta, \tag{1.11}$$

$$|\mathbf{x}(t)-\mathbf{x}(0)|\lesssim \mathcal{E}_0+\mathcal{E}_0^{\frac{1}{4}}, \tag{1.12}$$

$$\|u(t)-u_0\|_{L^2}\lesssim \left(\mathcal{E}_0^{\frac{1}{2}}+\mathcal{E}_0^{\frac{1}{4}}\right)(1+\delta t). \tag{1.13}$$

(ii) *On* $[s_1, T_*]$, *the solution has exactly one simple zero and is drawn into an exponentially small neighborhood of the slow manifold and remains trapped in the sense that*

$$\|u(t)-v(t)\|_{H^1}^2+\mathcal{E}(t)\lesssim \mathcal{E}(s_1)\exp\left(-t/(2C_{\mathrm{ed}})\right)+\delta^2, \tag{1.14}$$

$$|\mathbf{x}(t)-\mathbf{x}(s_1)|\lesssim \mathcal{E}(s_1)^{\frac{1}{2}}, \tag{1.15}$$

$$\|u(t)-u(s_1)\|_{L^2}\lesssim \mathcal{E}(s_1)^{\frac{1}{2}}. \tag{1.16}$$

The theorem is proved in Subsection 3.

Remark 1.6 *(Timescale separation).* From (1.14) we read off that at latest by the order L time

$$s_2 := 8C_{\mathrm{ed}}\sqrt{G''(-1)}x_0,$$

the solution is δ^2 close to $\mathcal{N}_+(L)$, whereas the time s_3 to come within an order-one neighborhood of v_* is, according to (1.16), exponentially long of order $s_3 \geq \delta^{-1}$.

Remark 1.7 *(Metastable set and the clock).* Whether $G''(-1)$ or $G''(1)$ appears in the "clock" of the metastable lifetime is controlled by whether the zero of the initial projection is to the right or the left of the zero of the energy minimizer v_*. If instead of $\mathcal{N}$ as defined above we consider initial data not far from a near minimizer v with zero $x_0 \in (x_* + \alpha(x_* + L)/2, \beta L)$ to the right of x_*, the metastable timescale is

$$\delta = \exp(-2\sqrt{G''(1)}(L-x_0)).$$

Remark 1.8 *(Small initial data).* For initial data such that are well-prepared in the sense that $\mathcal{E}_0 \leq \delta^2$, the solution already enters stage (ii) within an order one time (cf. Lemma 2.8), the motion of the zero (or zeros, since multiple zeros are possible within the initial layer) is estimated by $\delta^{1/4}$ (cf. (1.12) and (1.15)), and not much happens for a long time in the sense that:

$$\inf\left\{t>0\colon \|u(t)-u(0)\|_{L^2}=\delta^{1/2}\right\}\gtrsim \delta^{-1}\sim \exp\bigl(2\sqrt{G''(-1)}x_0\bigr).$$

Remark 1.9 *(Going through collisions).* We do not track multi-kink metastable solutions in this paper, but the theorem can be used for going through collisions in the sense that after two transition layers have come close to each other, the configuration can be viewed as an order-one disturbance of a single layer state.

Assumption 1.10 *(Potential).* The potential G satisfies

- G is C^2,
- the only zeros of G' are $\pm 1, 0$ and $s = \pm 1$ are the unique local and global minima of G normalized so that $G(\pm 1) = 0$,
- $G''(\pm 1) > 0$.

Notation 1.11. We use the Landau $O(\cdot)$ and $o(\cdot)$ notation and sometimes write

$$A \lesssim B \qquad \text{if} \qquad \frac{A}{B} = O(1).$$

If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$. We occasionally use C to represent a constant whose definition may change from line to line and for exponentially small terms when we are not interested in a sharp constant.

1.1.1. Organization

We begin in Subsection 2.1 by collecting, with proof, the facts about the energy landscape in the asymmetric setting. Then in Subsections 2.2-2.4 we reprise the facts about the energy–energy–dissipation estimates, the differential inequality for the dissipation, and the metastability tools that will serve as the backbone of our main proof. These results are imported from previous papers. While this unfortunately means that this paper is not self-contained, repeating the proofs here would take up more space than appropriate. We comment on minor adjustments as we present the results and simplify the statement of the metastability tools to the Allen-Cahn setting, which hopefully makes the results very accessible for the reader who wants to quickly understand the implications for the current setting. In Section 3 we assemble the proof of Theorem 1.5.

2. Ingredients

In this section we collect the tools that will be necessary to prove the theorem. In the first subsection, we focus on the energy of the minimizer and almost minimizers, especially in the asymmetric case. In the following subsections we review the algebraic and differential estimates and metastability framework developed by the author and co-authors in [20–22,25].

2.1. The minimizer and energy landscape in the asymmetric case

We collect in this subsection the facts and proofs related to an asymmetric potential. To be self-contained and to highlight the dependence on unequal second derivatives, we include the details of the proofs. However we emphasize that the idea of the expansions is not new; they have deep roots in the applied literature and in the math literature go back at least to [10] but earlier references may exist. We begin with the properties of the minimizer.

Lemma 2.1. *There exist $C, L_1 \in (0, \infty)$ with the following property. For any $L \geq L_1$ there exists a unique v_* satisfying (1.5) and such that*

$$e_* := E(v_*) = \int_{-1}^{1} \sqrt{2G(v)}dv + \exp(-L/C)$$

in the sense that

$$\frac{1}{L}\log(e_* - \int_{-1}^{1} \sqrt{2G(v)}dv) = -\frac{1}{C} + o(1)_{L\to\infty}. \tag{2.1}$$

The unique zero of v_* is given by

$$x_* = \frac{L\sqrt{G''(1)}}{\sqrt{G''(-1)} + \sqrt{G''(1)}} + O(1)_{L\to\infty} \tag{2.2}$$

and v_* satisfies the ordinary differential equation

$$v' = \sqrt{2G(v) + c^2} \qquad \text{on} \ \ (0, L) \tag{2.3}$$

for a unique constant c_* estimated by

$$c_* = v_*'(0) = v_*'(L) = \exp\left(-L \frac{\sqrt{G''(-1)}\sqrt{G''(1)}}{\sqrt{G''(-1)} + \sqrt{G''(1)}} + O(1)_{L\to\infty} \right). \tag{2.4}$$

Proof. Existence of a minimizer v follows from the direct method of the calculus of variations and v satisfies the Euler-Lagrange equation

$$-\partial_{xx} v + G'(v) = 0.$$

Multiplying with $\partial_x v$ and integrating from 0 to x gives

$$\frac{(v')^2}{2} - G(v) = \frac{(v'(0))^2}{2} = \frac{c^2}{2} \quad \text{for} \ \ c := v'(0) = v'(L), \tag{2.5}$$

where for simplicity of notation we have set $v' = \partial_x v$. From (2.5) and the boundary conditions we deduce (2.3) and

$$c > 0.$$

The only question that remains for uniqueness is whether there can be more than one c for which a solution exists. For this, we use the familiar trick of expressing the interval length via

$$L = \int_0^L dx = \int_{-1}^1 \frac{dv}{v'} = \int_{-1}^1 \frac{dv}{\sqrt{2G(v) + c^2}}, \tag{2.6}$$

for any solution v (by substitution). Regarding L as a function of the parameter c, we compute

$$L'(c) = -c \int_{-1}^1 \frac{dv}{(2G(v) + c^2)^{3/2}} < 0$$

and deduce that there exists a unique c_* for which (2.6) holds. This value uniquely determines the minimizer, which we henceforth refer to as v_*.

From the representation formula (2.6) we also deduce that in the regime $L \gg 1$ of interest, there holds

$$c_* \ll 1,$$

so that an expansion in c_* will be justified.

We now turn to the derivation of (2.2). Similar computations to above will yield the formula for the zero x_*, which because of the monotonicity from (2.3) is uniquely determined. We calculate on the one hand

$$\sqrt{G''(-1)} x_* = \sqrt{G''(-1)} \int_{-1}^0 \frac{dv}{\sqrt{2G(v) + c_*^2}} \tag{2.7}$$

and on the other hand

$$\sqrt{G''(1)}(L-x_*) = \sqrt{G''(1)}\int_0^1 \frac{dv}{\sqrt{2G(v)+c_*^2}}. \tag{2.8}$$

We then add and subtract the leading order term in each and expand the first integral in c_* while calculating the second integral explicitly. In (2.7) this yields

$$\begin{aligned}\sqrt{G''(-1)}x_* &= \sqrt{G''(-1)}\int_{-1}^0 \frac{1}{\sqrt{2G(v)+c_*^2}} - \frac{1}{\sqrt{(v+1)^2+c_*^2}}dv + \int_{-1}^0 \frac{1}{\sqrt{(v+1)^2+c_*^2}}dv \\ &= C_- + o(c_*) + \log(c_*^{-1}) + \log(1+\sqrt{(1+c_*^2)}) \\ &= \log(c_*^{-1}) + C_- + \log(2) + o(c_*),\end{aligned} \tag{2.9}$$

for

$$C_- = \int_{-1}^0 \frac{\sqrt{G''(-1)}}{\sqrt{2G(v)}} - \frac{1}{v+1}\,dv.$$

In (2.8), on the other hand, we obtain

$$\begin{aligned}\sqrt{G''(1)}(L-x_*) &= \sqrt{G''(1)}\int_0^1 \frac{1}{\sqrt{2G(v)+c_*^2}} - \frac{1}{\sqrt{(v-1)^2+c_*^2}}dv + \int_0^1 \frac{1}{\sqrt{(v-1)^2+c_*^2}}dv \\ &= C_+ + o(c_*) + \log(c_*) - \log(-1+\sqrt{(1+c_*^2)}) \\ &= \log(c_*^{-1}) + C_+ + \log(2) + o(c_*),\end{aligned} \tag{2.10}$$

for

$$C_+ = \int_0^1 \frac{\sqrt{G''(1)}}{\sqrt{2G(v)}} - \frac{1}{v-1}\,dv.$$

Solving (2.9) and (2.10) for $\log(c_*^{-1})$, equating them, and solving for x_* yields

$$x_* = \frac{L\sqrt{G''(1)}}{\sqrt{G''(-1)}+\sqrt{G''(1)}} + \frac{C_- - C_+}{\sqrt{G''(-1)}+\sqrt{G''(1)}} + o(c_*).$$

Substituting this expression back into (2.9) leads to the quantitative estimate of c_*:

$$\log(c_*^{-1}) = L\frac{\sqrt{G''(-1)}\sqrt{G''(1)}}{\sqrt{G''(-1)}+\sqrt{G''(1)}} + O(1)_{L\to\infty}, \tag{2.11}$$

which gives (2.4).

Finally, we express the energy as

$$E(v_*) \;=\; \int_0^L \frac{(v_*')^2}{2} + G(v_*)\,dx = \int_0^L \frac{1}{2}(v_*' - \sqrt{2G(v_*)})^2\,dx + \int_0^L \sqrt{2G(v_*)}\,v_*'\,dx$$

$$\overset{(2.3)}{=} \int_{-1}^{1} \sqrt{2G(v)}\, dv + O(L\, c_*^2),$$

which together with (2.11) establishes (2.1). □

We now turn to the almost minimizers.

Lemma 2.2. *There exists $L_1 < \infty$ such that for any $L \geq L_1$, the following holds true. Let $v, \tilde{v}$ be two functions that minimize the energy subject to ± 1 boundary conditions at 0 and L and zeros x_v, $x_{\tilde{v}}$, respectively. Then the energy satisfies a Lipschitz condition*

$$|E(v) - E(\tilde{v})| \leq \delta |x_v - x_{\tilde{v}}|. \tag{2.12}$$

If the zeros of $v, \tilde{v}$ satisfy

$$\frac{x_*}{4} \leq x_v < x_{\tilde{v}} < \frac{7}{8} x_*, \tag{2.13}$$

then (2.12) holds with

$$\log(\delta^{-1}) = 2\sqrt{G''(-1)}\, x_v + O(1)_{L\to\infty}. \tag{2.14}$$

Similarly, if the zeros of $v, \tilde{v}$ satisfy

$$\frac{L + x_*}{2} \leq x_{\tilde{v}} < x_v \leq \frac{3}{4} L, \tag{2.15}$$

then (2.12) holds with

$$\log(\delta^{-1}) = 2\sqrt{G''(1)}\, (L - x_v) + O(1)_{L\to\infty}. \tag{2.16}$$

Remark 2.3. The restrictions on $x_v, x_{\tilde{v}}$ can be loosened to belonging to the intervals $[\alpha x_*, \beta x_*]$ or $[x_* + \alpha(x_* + L)/2, \beta L]$ for any α, β in $(0, 1)$ at the expense of increasing L_1.

Proof. For any $\xi \in (0, L)$, there exists a unique, continuous minimizing v that satisfies

$$v(0) = -1, \qquad v(\xi) = 0, \qquad v(L) = 1,$$

and on each of the subintervals $(0, \xi)$ and (ξ, L), this minimizer satisfies

$$-v''(x) + G'(v(x)) = 0.$$

The energy of a minimizer v with zero ξ is naturally a function of ξ. With a slight abuse of notation, we denote the energy as $E = E(\xi)$ and by the mean value theorem, it suffices to estimate $|E'(\xi)|$ appropriately.

Arguing as in the previous proof, we deduce the existence of unique and, for L large, exponentially small in L (cf. (2.11)) values $c_- = c_-(\xi)$, $c_+ = c_+(\xi)$ such that

$$v'(0) = c_- \qquad \text{and} \qquad v'(x) = \sqrt{2G(v) + (c_-)^2} \text{ on } (0, \xi), \tag{2.17}$$

$$v'(L) = c_+ \qquad \text{and} \qquad v'(x) = \sqrt{2G(v) + (c_+)^2} \text{ on } (\xi, L). \tag{2.18}$$

We then represent the energy as

$$E(\xi) = \int_0^L \frac{(v')^2}{2} + G(v)\,dx \overset{(2.17),(2.18)}{=} \int_0^\xi 2G(v) + \frac{c_-^2}{2}\,dx + \int_\xi^L 2G(v) + \frac{c_+^2}{2}\,dx$$

$$\overset{(2.17),(2.18)}{=} \int_{-1}^0 \sqrt{2G(v) + c_-^2}\,dv + \int_0^1 \sqrt{2G(v) + c_+^2}\,dv - \frac{c_-^2}{2}\xi - \frac{c_+^2}{2}(L-\xi).$$

Differentiating with respect to ξ and using the length representation formulas (as for (2.7), (2.8)), we obtain the simplified expression:

$$E'(\xi) = -\frac{c_-^2}{2} + \frac{c_+^2}{2}, \tag{2.19}$$

so that (2.12) holds with

$$\delta := \sup_{\xi\in[x_v,x_{\tilde v}]} \left| \frac{c_+^2}{2} - \frac{c_-^2}{2} \right|.$$

It remains to estimate the right-hand side in case of (2.13) or (2.15). Estimating c_-, c_+ as in (2.9) and (2.10) gives

$$\log(c_-^{-2}(\xi)) = 2\xi\sqrt{G''(-1)} + O(1)_{L\to\infty}, \qquad \log(c_+^{-2}(\xi)) = 2(L-\xi)\sqrt{G''(1)} + O(1)_{L\to\infty}. \tag{2.20}$$

The result then follows from (2.19) and (2.20) using the dominance conditions in (2.13) or (2.15). For instance (2.13) implies that $\xi \in [x_v, x_{\tilde v}]$ fulfills

$$\xi \le \frac{7}{8}x_* \overset{(2.2)}{<} \frac{L\sqrt{G''(1)}}{\sqrt{G''(-1)} + \sqrt{G''(1)}}$$

and hence

$$\sqrt{G''(1)}(L-\xi) > \sqrt{G''(-1)}\xi.$$

We conclude from here and (2.20) that both c_-^2 and c_+^2 are of exponential order $\exp(-2\sqrt{G''(-1)}\xi)$ and that δ satisfies (2.14).

Similarly, in the case (2.15), $\xi \in [x_{\tilde v}, x_v]$ obeys

$$\xi \ge \frac{L + x_*}{2} \overset{(2.2)}{>} \frac{L\sqrt{G''(1)}}{\sqrt{G''(-1)} + \sqrt{G''(1)}}$$

and hence

$$\sqrt{G''(1)}(L-\xi) < \sqrt{G''(-1)}\xi,$$

and (2.16) follows analogously to (2.14). □

2.2. Algebraic properties including energy–energy–dissipation

For the initial phase of the evolution, we will use the squared L^2 distance $\bar H$ to the fixed function v_0 to measure changes in the solution. It will be important to know that the solution u still has a projection onto a near minimizer whose zero is not too far away from that of v_0. This fact is contained in the following lemma.

Lemma 2.4. *For any $C_1 \in (0,\infty)$, there exists $L_1 \in (0,\infty)$ with the following property. For any $L \geq L_1$ and $u \in \mathcal{M}(L, C_1)$, let x_v denote the zero of a function $v_0 \in \mathcal{N}(L)$ and let $\bar{H} := \|u - v_0\|_{L^2}^2$. Then u has a zero $\mathbf{x} \in (x_*/4, 7x_*/8)$ such that*

$$|\mathbf{x} - x_v| \lesssim \bar{H} + \bar{H}^{\frac{1}{3}} \tag{2.21}$$

and such that the energy-optimal profile $v \in \mathcal{N}_+(L)$ associated to $\mathbf{x}$ satisfies

$$\|u - v\|_{L^2}^2 \lesssim \bar{H} + \bar{H}^{\frac{1}{3}}. \tag{2.22}$$

The main workhorse of the metastability proof is the following nonlinear energy–energy–dissipation (EED) estimate.

Lemma 2.5. *For any $C_1 \in (0,\infty)$, there exist $L_1, C_{\mathrm{ed}} \in (0,\infty)$ such that, for any $L \geq L_1$, $u \in \mathcal{M}(L, C_1)$, and the projection $v \in \mathcal{N}_+(L)$ of u as in Lemma 2.4, there holds:*

$$\frac{1}{C_{\mathrm{ed}}}\|u - v\|_{H^1}^2 \leq \mathcal{E} \leq C_{\mathrm{ed}} D. \tag{2.23}$$

The proofs are long but elementary. The main idea is to consider the dichotomy that either the energy gap is small or not, or that the dissipation is small or not, respectively. In the former case, one linearizes, while in the latter case, a rough bound suffices. We refer to [20,22,25] for the proofs. Equality of the second derivatives is not necessary in any of the arguments (only nondegeneracy). The only nontrivial difference in the current setting is that our energy-optimal profiles satisfy the Euler-Lagrange equations (2.17), (2.18) rather than

$$v_x = \sqrt{2G(v) + 2G(v_{max})}$$

as in [20], but $c_\pm$ and $G(v_{max})$ have in common that they are exponentially small in L, which is the relevant fact.

The second ingredient for the metastability proof is a Lipschitz condition on $\mathcal{N}_+(L)$. To pass from shifts in zeros to shifts in L^2, we will make use of the following elementary lemma, a proof of which can be found in [20]. Because our slow manifold consists of single kinks, the statement and proof simplify slightly.

Lemma 2.6 *(Lemma 3.2 of [20]). There exists $L_1 \in (0,\infty)$ such that for any $L \geq L_1$ and for any two $v, \tilde{v} \in \mathcal{N}_+(L)$ with zeros x, $\tilde{x}$, there holds:*

$$|x - \tilde{x}| \lesssim \|v - \tilde{v}\|_{L^2}^2 + \|v - \tilde{v}\|_{L^2} \quad \text{and} \quad \|v - \tilde{v}\|_{L^2} \lesssim |x - \tilde{x}| + 1. \tag{2.24}$$

We will use Lemma 2.6 together with the Lipschitz condition from Lemma 2.2 to deduce

$$|E(v) - E(\tilde{v})| \lesssim \delta\|v - \tilde{v}\|_{L^2}$$

when $|x - \tilde{x}| \lesssim 1$.

2.3. Small dissipation and one simple zero

In order to deduce that the solution has only one simple zero after the energy has relaxed, we employ the following differential inequality for the dissipation.

Lemma 2.7. *[25, Lemma 2.6] There exist ε, $L_1 \in (0, \infty)$ such that, for any $L \geq L_1$ and a solution $u \in \mathcal{M}(L, C_1)$ of the Allen-Cahn equation (1.1), the following holds true. If u has a projection $v \in \mathcal{N}_+(L)$ such that $\|u - v\|_\infty \leq \varepsilon$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} D \lesssim D. \tag{2.25}$$

Smallness of the dissipation then follows as in [22] from the following lemma.

Lemma 2.8. *[22, Lemma 2.12] There exist $L_1, \gamma_1 \in \mathbb{R}^+$ such that if $L \geq L_1$ and $u \in \mathcal{M}(L, C_1)$ satisfies $\mathcal{E} \leq \gamma$ on $[s, t]$ for $\gamma \leq \gamma_1$ and some $t \geq s + 1$, then*

$$\max_{[s+1,t]} D \lesssim \gamma.$$

Notice that the choice of $s + 1$ is arbitrary; it would do to take $s + a$ for any fixed constant $a > 0$.

Although the condition in [22, Lemma 2.12] is stated in terms of $\bar{\mathcal{E}}$ rather than $\mathcal{E}$, we can infer $\bar{\mathcal{E}} \leq 2\gamma$ from $\mathcal{E} \leq \gamma$ and by possibly increasing L_1, since for any two $v, \tilde{v} \in \mathcal{N}_+(L)$ the energy difference is exponentially small in L.

As in [22], smallness of the dissipation then implies simple zeros.

Lemma 2.9. *[22, Lemma 2.13] There exist $L_1, \gamma \in \mathbb{R}^+$ such that if $L \geq L_1$ and $u \in \mathcal{M}(L, C_1)$ satisfies $\mathcal{E} + D \leq \gamma$, then u has exactly one simple zero.*

2.4. Metastability tools

We now state (without proof) two metastability propositions from [20,22], used also in [25]. For simplicity of presentation, we specify them to our setting. For the more general results, we refer to [22,25].

During the initial relaxation phase, we will use Proposition 2.10 to establish exponential in time convergence of $\bar{\mathcal{E}}$ (instead of $\mathcal{E}$).

Proposition 2.10 *(Proposition 2.9 of [22]). Let $C_{\mathrm{ed}} \in (0, \infty)$. Suppose that for a solution u of the Allen-Cahn equation on $[0, T]$, there holds*

(i) *For every u there is an energy-optimal profile $v \in \mathcal{N}_+(L)$ such that*

$$\frac{1}{C_{\mathrm{ed}}} \|u - v\|_{L^2}^2 \leq E(u) - E(v) \leq C_{\mathrm{ed}}\, D, \tag{2.26}$$

(ii) *There is a constant $\delta \in (0, 1)$ such that for all appropriately chosen energy-optimal profiles $v_1, v_2 \in \mathcal{N}_+(L)$ we have*

$$|E(v_1) - E(v_2)| \leq \delta \|v_1 - v_2\|_{L^2}. \tag{2.27}$$

Define the initial energy gap $\mathcal{E}_0 := E\big(u(0)\big) - E\big(v(0)\big)$. For all $t \leq \min\{T, \delta^{-1}\}$ there holds

$$\Big|E\big(u(t)\big) - E\big(v(0)\big)\Big| \lesssim \mathcal{E}_0 \exp\big(-t/(2C_{\mathrm{ed}})\big) + \Big(\mathcal{E}_0^{\frac{1}{2}} + \mathcal{E}_0^{\frac{1}{4}}\Big)\, \delta. \tag{2.28}$$

Furthermore, changes in the solution and along the slow manifold are controlled by

$$\sup_t \Big(\|u(t) - u(0)\|_{L^2} + \|v(t) - v(0)\|_{L^2}\Big) \lesssim \Big(\mathcal{E}_0^{\frac{1}{2}} + \mathcal{E}_0^{\frac{1}{4}}\Big)\Big(1 + \delta t\Big). \tag{2.29}$$

After the order-one time such that the energy and dissipation have become so small that (according to Lemma 2.9) the solution has only one simple zero, we will use Proposition 2.11 to deduce the exponentially long metastable timescale.

Proposition 2.11 *(Proposition 2.10 of [22]). Consider the conditions of Proposition 2.10. Suppose moreover that $t \mapsto E(v(t))$ is integrable on $[0,T]$. Then for every $\epsilon \in (0,1)$ there is a constant C_ϵ such that*

$$\begin{aligned}\|u(t)-v(t)\|_{L^2}^2 + E\big(u(t)\big) - E\big(v(t)\big)\\ \lesssim \exp\big(-(1-\epsilon)t/C_{\mathrm{ed}}\big)\mathcal{E}_0 + C_\epsilon \delta^2.\end{aligned} \tag{2.30}$$

Furthermore, changes in the solution and along the slow manifold are controlled by

$$\|u(t)-u(0)\|_{L^2} + \|v(t)-v(0)\|_{L^2} \lesssim \mathcal{E}_0^{\frac{1}{2}} + \delta(t+1). \tag{2.31}$$

3. Proof of Theorem 1.5

We are now ready to prove our theorem.

Proof of Theorem 1.5. We will control two phases of the evolution, an initial phase $[0,s_1]$ in which we use rougher control on the energy decay from Proposition 2.10 and a second phase $[s_1,T_*]$ in which u has only one simple zero moving continuously in time, so that we can apply Proposition 2.11.

Step 1. The proof will rely on a buckling argument. A priori, we will need to control changes in $\bar{H}(t)$ and define

$$T_1 := \inf\big\{t \geq 0\colon \bar{H}(t) \geq C_2\big\} \tag{3.1}$$

for a universal constant $C_2 \geq 4e_*$ to be specified below. Notice that on $[0,T_1]$ there holds $u(t) \in \mathcal{M}(L,C_2)$ and hence, by Lemma 2.4, $v(t) \in \mathcal{N}_+(L)$ is well-defined. Note also that according to (2.21) the motion of the zeros on $[0,T_1]$ is bounded according to

$$\Delta\mathbf{x}(t) := |\mathbf{x}(t) - x_0| \leq C_3 = C_3(C_2), \tag{3.2}$$

so that

$$x_0 - C_3 \leq \mathbf{x}(t) \leq x_0 + C_3. \tag{3.3}$$

In order to apply Proposition 2.10, we need to verify (2.26) and (2.27). Since $v \in \mathcal{N}_+(L)$, the EED estimate for $\mathcal{E}$ follows directly from Lemma 2.5. According to Lemma 2.2, there is a Lipschitz condition of the form:

$$|E(v) - E(\tilde{v})| \overset{(2.14)}{\lesssim} \tilde{\delta}|\mathbf{x} - \tilde{\mathbf{x}}| \tag{3.4}$$

with

$$\tilde{\delta} \overset{(3.3)}{\leq} C\exp(C\sup_{[0,\hat{T}]}\Delta\mathbf{x}(t))\exp(-2\sqrt{G''(-1)}x_0)$$

and hence

$$
\begin{aligned}
|E(v)-E(\tilde v)| &\overset{(2.24)}{\lesssim} \tilde\delta(1+\Delta\mathbf{x}(T_*))\|v-\tilde v\|_{L^2} \\
&\lesssim \tilde{\tilde\delta}\|v-\tilde v\|_{L^2}, \qquad (3.5)
\end{aligned}
$$

for

$$
\tilde{\tilde\delta} = (1+\sup_{[0,\hat T]}\Delta\mathbf{x}(t))\exp(C\sup_{[0,\hat T]}\Delta\mathbf{x}(t))\exp(-2\sqrt{G''(-1)}x_0),
$$

which by (3.2) is of exponential order $\exp(-2\sqrt{G''(-1)}x_0$.

We set $\hat T := \min\{T_1, \tilde{\tilde\delta}^{-1}\}$. We will show that

$$
\hat T < T_1 \qquad (3.6)
$$

and that $\Delta\mathbf{x}(t)$ satisfies a sharper bound than (3.2).

We now apply Proposition 2.10 and measure changes to the fixed function $v_0 \in \mathcal{N}(L)$ that is associated to u_0. We obtain up to time $\hat T$ the estimates

$$
\bar{\mathcal{E}}(t) \lesssim \mathcal{E}_0\exp(-t/(2C_{\mathrm{ed}})) + \left(\mathcal{E}_0^{\frac12}+\mathcal{E}_0^{\frac14}\right)\tilde{\tilde\delta}, \qquad (3.7)
$$

$$
\sup_t\left(\|u(t)-u(0)\|_{L^2}+\|v(t)-v(0)\|_{L^2}\right) \lesssim \left(\mathcal{E}_0^{\frac12}+\mathcal{E}_0^{\frac14}\right)\left(1+\tilde{\tilde\delta}t\right) \lesssim \mathcal{E}_0^{\frac12}+\mathcal{E}_0^{\frac14} \lesssim 1. \qquad (3.8)
$$

Choosing C_2 sufficiently large with respect to the implicit constant in (3.8) implies (3.6) and uniformly bounds

$$
\tilde{\tilde\delta} \lesssim \exp\left(-2\sqrt{G''(-1)}x_0\right).
$$

This gives

$$
\sup_{t\le\tilde{\tilde\delta}^{-1}}\Delta\mathbf{x}(t) \overset{(2.24),(3.8)}{\lesssim} \mathcal{E}_0+\mathcal{E}_0^{\frac14}, \qquad (3.9)
$$

which is (1.12), and substituting the uniform bound on $\tilde{\tilde\delta}$ into the first estimate in (3.8) implies (1.13).

Also since

$$
|\mathcal{E}-\bar{\mathcal{E}}| \lesssim \exp(-2\sqrt{G''(-1)}x_0), \qquad (3.10)
$$

(cf. Lemma 2.2) and using the EED inequality (cf. Lemma 2.5), we obtain (1.11) for $\mathcal{E}$ and $\|u(t)-v(t)\|_{H^1}^2$.

According to (1.11) together with Lemmas 2.8 and 2.9, and for L_1 sufficiently large (with respect to the order-one constant γ), there exists an order-one time s_1 such that the solution has one simple zero on $[s_1, \tilde{\tilde\delta}^{-1}]$. But then because of the boundary conditions and Lemma 2.1, the solution has one simple zero for all $t \ge s_1$.

Step 2. We now examine the relaxation of the energy gap more carefully with Proposition 2.11, which will give δ^2 (rather than just δ-) distance to the slow manifold. Rather than tracking $\bar{\mathcal{E}}$, we here track $\mathcal{E} := E(u(t)) - E(v(t))$ on $[s_1, T_*]$, taking the solution at time s_1 as initial data. We again define

$$
T_1 := \inf\{t > s_1 \colon \|u(t)-v(s_1)\|_{L^2}^2 \ge C_2\}, \qquad \hat{\hat T} := \min\{T_1, T_*\}
$$

for an ε to be specified below. Because of the unique, simple zero, $t \mapsto E(v(t))$ is measurable and, in particular, integrable, and we may apply Proposition 2.11. Proceeding as in the previous step and setting $\epsilon = 1/2$, we obtain

$$\|u(t) - v(t)\|_{L^2}^2 + \mathcal{E}(t) \lesssim \exp\big(-(t - s_1)/(2C_{\text{ed}})\big)\mathcal{E}_0 + \tilde{\delta}^2, \tag{3.11}$$

$$\|u(t) - u(s_1)\|_{L^2} + \|v(t) - v(s_1)\|_{L^2} \lesssim \mathcal{E}(s_1)^{\frac{1}{2}} + \tilde{\delta}(t - s_1 + 1) \tag{3.12}$$

on $[s_1, \hat{\hat{T}}]$ with a $\tilde{\delta}$ of exponential order $\exp(-\sqrt{G''(-1)}x_0)$ defined via

$$\tilde{\delta} = (1 + \sup_{[0,\hat{\hat{T}}]} \Delta_s \mathbf{x}(t)) \exp(C \sup_{[0,\hat{\hat{T}}]} \Delta_s \mathbf{x}(t)) \exp(-2\sqrt{G''(-1)}x_0),$$

for

$$\Delta_s \mathbf{x}(t) := |\mathbf{x}(t) - \mathbf{x}(s_1)|.$$

Because the zero moves continuously, we can bound $\Delta_s \mathbf{x}(t)$ with a continuity argument as follows: Using

$$\Delta_s \mathbf{x}(t) \overset{(2.24)}{\lesssim} \sup_{t \in [0,\hat{\hat{T}}]} \Big(\|v(t) - v(s_1)\|_{L^2} + \|v(t) - v(s_1)\|_{L^2}^2 \Big) \tag{3.13}$$

and bounding the L^2 norm via (2.31), we obtain

$$\begin{aligned} \Delta_s \mathbf{x}(t) &\lesssim \mathcal{E}(s_1)^{\frac{1}{2}} + \tilde{\delta}(\hat{\hat{T}} - s_1 + 1) \\ &\lesssim \mathcal{E}(s_1)^{\frac{1}{2}} + \varepsilon C \exp(C \Delta_s \mathbf{x}(t))(1 + \Delta_s \mathbf{x}(t)). \end{aligned}$$

Continuity of $\Delta_s \mathbf{x}$, $\Delta_s \mathbf{x} = 0$, and choosing $\varepsilon \in (0, 1)$ small enough in the definition of T_* implies

$$\Delta_s \mathbf{x}(t)) \lesssim \mathcal{E}(s_1)^{\frac{1}{2}}.$$

Hence we have a uniform bound on $\tilde{\delta}$, and Lemma 2.6 implies $\hat{\hat{T}} > T_1$, so that we obtain (1.15) up to T_*. Moreover (3.11) yields (1.14). Finally (1.16) follows from (3.12). □

References

[1] Nicholas Alikakos, Peter W. Bates, Giorgio Fusco, Slow motion for the Cahn-Hilliard equation in one space dimension, J. Differ. Equ. 90 (1) (1991) 81–135.

[2] Mederic Argentina, M.G. Clerc, R. Rojas, E. Tirapegui, Coarsening dynamics of the one-dimensional Cahn-Hilliard model, Phys. Rev. E 71 (4) (2005) 046210.

[3] Peter W. Bates, Jian Ping Xun, Metastable patterns for the Cahn-Hilliard equation. I, J. Differ. Equ. 111 (2) (1994) 421–457.

[4] Peter W. Bates, Jian Ping Xun, Metastable patterns for the Cahn-Hilliard equation. II. Layer dynamics and slow invariant manifold, J. Differ. Equ. 117 (1) (1995) 165–216.

[5] Fabrice Bethuel, Didier Smets, On the motion law of fronts for scalar reaction-diffusion equations with equal depth multiple-well potentials, Chin. Ann. Math., Ser. B 38 (1) (2017) 83–148.

[6] Fabrice Bethuel, Didier Smets, The motion law of fronts for scalar reaction-diffusion equations with multiple wells: the degenerate case, in: Partial Differential Equations Arising from Physics and Geometry, in: London Math. Soc. Lecture Note Ser., vol. 450, Cambridge Univ. Press, Cambridge, 2019, pp. 88–171.

[7] Sarah Biesenbach, Richard Schubert, Maria G. Westdickenberg, Optimal relaxation of bump-like solutions of the one-dimensional Cahn-Hilliard equation, Commun. Partial Differ. Equ. 47 (3) (2022) 489–548.

[8] Lia Bronsard, Danielle Hilhorst, On the slow dynamics for the Cahn-Hilliard equation in one space dimension, Proc. R. Soc. Lond. A 439 (1907) (1992) 669–682.

[9] Lia Bronsard, Robert V. Kohn, On the slowness of phase boundary motion in one space dimension, Commun. Pure Appl. Math. 43 (8) (1990) 983–997.
[10] J. Carr, M.E.Gurtin, M.Slemrod, Structured phase transitions on a finite interval, Arch. Rational Mech. Anal. 86 (4) (1984) 317–351.
[11] J. Carr, R.L. Pego, Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$, Commun. Pure Appl. Math. 42 (5) (1989) 523–576.
[12] Xinfu Chen, Generation, propagation, and annihilation of metastable patterns, J. Differ. Equ. 206 (2) (2004) 399–437.
[13] E.N.M. Cirillo, N. Ianiro, G. Sciarra, Kink localization under asymmetric double-well potentials, Phys. Rev. E 86 (4) (2012).
[14] J.-P. Eckmann, J. Rougemont, Coarsening by Ginzburg-Landau dynamics, Commun. Math. Phys. 199 (2) (1998) 441–470.
[15] Charles M. Elliott, Donald A. French, Numerical studies of the Cahn-Hilliard equation for phase separation, IMA J. Appl. Math. 38 (2) (1987) 97–128.
[16] Raffaele Folino, Ramón G. Plaza, Marta Strani, Long time dynamics of solutions to p-Laplacian diffusion problems with bistable reaction terms, Discrete Contin. Dyn. Syst. 41 (7) (2021) 3211–3240.
[17] G. Fusco, J.K. Hale, Slow-motion manifolds, dormant instability, and singular perturbations, J. Dyn. Differ. Equ. 1 (1) (1989) 75–94.
[18] Borjan Geshkovski, Hugo Koubbi, Yury Polyanskiy, Philippe Rigollet, Dynamic metastability in the self-attention model, 2024, arXiv:2410.06833.
[19] Christopher P. Grant, Slow motion in one-dimensional Cahn-Morral systems, SIAM J. Math. Anal. 26 (1) (1995) 21–34.
[20] Felix Otto, Maria G. Reznikoff, Slow motion of gradient flows, J. Differ. Equ. 237 (2) (2007) 372–420.
[21] Felix Otto, Maria G. Westdickenberg, Relaxation to equilibrium in the one-dimensional Cahn-Hilliard equation, SIAM J. Math. Anal. 46 (1) (2014) 720–756.
[22] Sebastian Scholtes, Maria G. Westdickenberg, Metastability of the Cahn-Hilliard equation in one space dimension, J. Differ. Equ. 265 (4) (2018) 1528–1575.
[23] Xiaodi Sun, Michael J. Ward, Dynamics and coarsening of interfaces for the viscous Cahn-Hilliard equation in one spatial dimension, Stud. Appl. Math. 105 (3) (2000) 203–234.
[24] Michael J. Ward, Metastable patterns, layer collapses, and coarsening for a one-dimensional Ginzburg-Landau equation, Stud. Appl. Math. 91 (1) (1994) 51–93.
[25] Maria G. Westdickenberg, On the metastability of the 1-d Allen-Cahn equation, J. Dyn. Differ. Equ. 33 (4) (2021) 1853–1879.