Clifford Orders

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Tobias Braun, M.Sc.

aus Aachen

Berichter: Universitätsprofessorin Dr. Gabriele Nebe

Universitätsprofessor Dr. Daniel Robertz

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Abstract

The aim of this thesis is to investigate the properties of the Clifford algebra of a quadratic lattice over a Dedekind domain and its completions and to compare it with the properties of the Clifford algebra of its ambient quadratic space.

The new object that arises this way - the Clifford order - has not yet been studied extensively as an independent object. The present thesis addresses this, using both the theory of orders and Clifford algebras to extend well-known results that hold for Clifford algebras over fields to this new, more general setting.

It was long known to theory that the centraliser of the even Clifford algebra, the so-called centroid is a cornerstone for describing the Clifford algebra of an orthogonal direct sum of quadratic spaces. This thesis develops the theory of quadratic orders, to describe the centroids of Clifford orders on an abstract level. In this context, a new invariant of a quadratic lattice, the quadratic discriminant, is introduced, allowing for a simplified computation of the centroids in certain situations. As applications, the centroids of the maximal lattices over a Dedekind domain and of an arbitrary root lattice are computed, and an effective way to determine the Clifford order of the orthogonal direct sum of two quadratic lattices is presented. Additionally, an algorithm to compute the centroid of a given Clifford orders over an arbitrary Dedekind domain is described.

Finally, this thesis classifies the Clifford orders and the centroids of all maximal lattices over a complete discrete valuation ring and describes them as a subalgebra of their ambient Clifford algebra.

Zusammenfassung

Das Ziel dieser Dissertation ist es, die Eigenschaften der Clifford-Algebra eines quadratischen Gitters über einem Dedekindbereich und dessen Komplettierungen zu untersuchen und diese mit den Eigenschaften der Clifford-Algebra des umgebenden quadratischen Raums zu vergleichen. Das auf diese Weise entstehende neue Objekt – die Clifford-Ordnung – wurde bisher noch nicht umfassend als eigenständiges Objekt untersucht. Die vorliegende Arbeit widmet sich diesem Thema und nutzt sowohl die Theorie der Ordnungen als auch die der Clifford-Algebren, um bekannte Ergebnisse, welche für Clifford-Algebren über Körpern gelten, auf diesen neuen, allgemeineren Rahmen zu übertragen.

Es war lange Zeit bekannt, dass der Zentralisator der geraden Clifford-Algebra, das sogenannte Zentroid, ein Grundpfeiler zur Beschreibung der Clifford-Algebra einer orthogonalen direkten Summe quadratischer Räume ist. Diese Dissertation entwickelt die Theorie der quadratischen Ordnungen mit dem Ziel, das Zentroid einer Clifford-Ordnung auf einer abstrakten Ebene zu beschreiben. In diesem Zusammenhang wird eine neue Invariante eines quadratischen Gitters, die quadratische Diskriminante, eingeführt, welche eine vereinfachte Berechnung des Zentroids in bestimmten Situationen ermöglicht. Als Anwendung werden die Zentroide der maximalen Gitter über einem Dedekind-Ring und eines beliebigen Wurzelgitters berechnet, und eine effektive Methode zur Bestimmung der Clifford-Ordnung der orthogonalen direkten Summe zweier quadratischer Gitter wird vorgestellt. Darüber hinaus wird ein Algorithmus beschrieben, welcher es ermöglicht, das Zentroid einer Clifford-Ordnung über einem beliebigen Dedekindbereich zu berechnen.

Schließlich werden in dieser Dissertation die Clifford-Ordnungen und die Zentroide aller maximalen Gitter über einem vollständigen diskret bewerteten Ring klassifiziert und als Teilalgebra ihrer umgebenden Clifford-Algebra beschrieben.

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Contents

List of Symbols

Symbol	Description
$\mathbb{N}$	The set of natural numbers without zero, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$.
$\mathbb{N}_0$	The set of natural numbers including zero, i.e. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
$\underline{n}$	The set of natural numbers up to $n \in \mathbb{N}$, i.e. $\underline{n} = \{1, \ldots, n\}$.
$E_1 \perp E_2$	The *orthogonal direct sum* of the quadratic modules E_1, E_2.
$A^\times$	The unit group of the monoid (usually a ring) A.
$A^{n\times n}$	The set of $n \times n$-matrices over the ring A.
$\mathrm{GL}_n(A)$	The group of invertible $n \times n$-matrices over the ring A.
$\mathbb{H}(A)$	The *hyperbolic plane* on the commutative ring A.
$N(R)$	The *norm form* on the local ring R.
$\mathcal{C}(E)$	The *Clifford algebra* of the quadratic module (E, q).
$\mathcal{C}_0(E)$	The *even Clifford algebra* of the quadratic module (E, q).
$\mathcal{C}_1(E)$	The *odd Clifford algebra* of the quadratic module (E, q).
$\mathcal{A} \mathbin{\widetilde{\otimes}} \mathcal{B}$	The *graded tensor product* of the $\mathbb{Z}/2\mathbb{Z}$-graded algebras $\mathcal{A}, \mathcal{B}$.
$\mathcal{Z}(E, q)$	The *centroid* of the quadratic module (E, q).
$\mathrm{disc}(E)$	The *discriminant* of the quadratic module or quadratic lattice (E, q).
$\mathrm{disc}'(V)$	The *half discriminant* of the quadratic space (V, q).
$\mathfrak{n}(L)$	The *norm* of the quadratic lattice (L, q).
$\mathfrak{s}(L)$	The *scale* of the quadratic lattice (L, q).
$\mathfrak{d}(L)$	The *discriminant ideal* of the quadratic lattice (L, q).
$L^\#$	The *dual lattice* of the quadratic lattice (L, q).
$\Lambda(\mathfrak{a}, t, n)$	The *quadratic order* defined by the given parameters over a Dedekind domain.
$\Lambda_i(d)$	The *quadratic order* defined by the given parameters over a local ring.
Λ^o	The *maximal orthogonal suborder* of the quadratic order Λ.
$\mathrm{disq}(L)$	The *quadratic discriminant* of the quadratic lattice or quadratic order L.

The list above presents the most important symbols used throughout this thesis. For explanations of the technical terms that appear in *italics* within the descriptions, please refer to the Index at the end of this thesis, which lists the pages where these terms are defined.

0 Introduction

'In simple words, can you tell me what your PhD thesis is about?' - This is a question that I have encountered frequently in the past few years, and while it has been a bit of a challenge, I find that I am becoming more comfortable with it. The challenge arises, of course, from explaining Clifford orders in simple terms while keeping the person asking engaged, especially if they do not have a strong mathematical background.

The main reason for this is that talking about Clifford orders requires a lot of theory, a large part of which is not found in your standard algebra textbooks. Also note that the title-giving term 'Clifford order' is really a coined phrase. It unifies the concepts of a Clifford algebra and the concept of an order in a separable algebra into a new, more elaborate one. Consequently, at least scratching the surface of both of these is required whenever one wants to initially understand the notion of a Clifford order. Of course, both the theory of Clifford algebras and the theory of orders do come with a rich history of their own.

The theory of Clifford algebras has its roots in the work of William Kingdon Clifford, who introduced these algebras in 1878. In his original paper [Cli78], Clifford extended Hermann Grassmann's ideas by generalising complex numbers and quaternions to higher dimensions, thereby providing a unified framework for these entities. Notably, Clifford did not state his ideas in terms of quadratic forms as we do today; instead, he focused on the algebraic properties and relations within this new system. The modern theory on the geometrical side of Clifford algebras has its roots in the works of Élie Cartan from 1938, see [Car81]. His spinor theory connected the representation theory of reflections in orthogonal groups in higher dimensions to Clifford algebras. At this point in time, quadratic forms were long known to be tightly connected to orthogonal groups; they have been studied extensively in the 18th and 19th century. Thus, it is no surprise that mathematicians started to consider Clifford algebras in the context of quadratic forms. Already in 1954, Chevalley devotes an entire chapter to the Clifford algebra of a quadratic form and studies it as a singular object; see [Che54]. Over subsequent decades, the modern language for Clifford algebras was developed. Today, Clifford algebras play a major role in various fields, including differential geometry, computer graphics, or theoretical physics, where they provide a unified framework for describing orthogonal transformations and have applications in quantum computing (see [NC10]).

The theory of orders in separable algebras connects the fields of algebraic number theory and ring theory and has benefited from advancements in both areas. Notable contributions were made by Richard Dedekind with his ideal theory and by Emmy Noether, who developed the theory of rings and modules from a structural point of view. Noether established the modern concepts of Noetherian rings and modules, which are essential to define *maximal orders*. Additionally, Richard Brauer classified division algebras over number fields in [Bra29], leading to the modern notion of the *Brauer group* and motivating the study of orders within these algebras. Consequently, orders within central simple algebras were considered as these can be regarded as full matrix rings over these division algebras, due to Wedderburn's Theorem. In the subsequent decades many advancements have been made; the book [Rei03] provides a great overview of these developments and is a standard reference for the theory of orders.

Following up on these historical developments, this thesis considers certain orders inside Clifford algebras. To my knowledge, Clifford orders, as defined in this thesis, have not yet been studied as a singular object. However, in his Ph.D. thesis [Bra16], Braun devoted an entire chapter to Clifford orders, albeit primarily from a computational, not a structural point of view.

0.1 The mathematical framework

We briefly describe the setup and the objects that are important for this thesis. Instead of providing extensive definitions and explanations here, we focus on the interplay of these objects and put them inside a broader context. We start with a simpler motivating setting and follow it up with the more general one, that we are interested in.

Consider the ring of rational integers $\mathbb{Z}$ and its field of fraction, the rationals $\mathbb{Q}$. Let V be a finite dimensional $\mathbb{Q}$-space of dimension n that is equipped with a $\mathbb{Z}$-valued quadratic form q. After fixing a basis of V, one can think of the latter as a homogeneous polynomial of degree two in n variables with all its coefficients lying in $\mathbb{Z}$. The pair (V, q) is then called a quadratic $\mathbb{Q}$-space and one can consider its Clifford algebra $\mathcal{C}(V, q)$. This is a finite dimensional associative algebra of dimension 2^n that carries essentially all relevant information on the quadratic space (V, q). For example, from it a complete set of invariants, that is sufficient to determine the isometry type of (V, q), can be computed.

Taking any full $\mathbb{Z}$-lattice L inside V, i.e. a free $\mathbb{Z}$-submodule of V of rank n, the quadratic form q restricts to L, giving rise to the quadratic $\mathbb{Z}$-lattice (L, q). Again, we can consider its Clifford algebra $\mathcal{C}(L, q)$ which is a subring of $\mathcal{C}(V, q)$. Additionally, due to the assumption that q is $\mathbb{Z}$-valued, $\mathcal{C}(L, q)$ is also a full $\mathbb{Z}$-lattice in $\mathcal{C}(V, q)$, regarding the latter as a 2^n-dimensional $\mathbb{Q}$-space. Thus, $\mathcal{C}(L, q)$ is both a full $\mathbb{Z}$-lattice inside $\mathcal{C}(V, q)$ and a subring of $\mathcal{C}(V, q)$; for short, we call it a $\mathbb{Z}$-order in $\mathcal{C}(V, q)$. Of course, orders are not exclusive to the theory of Clifford algebras and in fact any finite dimensional algebra over any field contains orders (see Remark 2.2.2). We call $\mathcal{C}(L, q)$ a *Clifford order*; this is done to emphasise that the Clifford algebra of (L, q) carries additional structure provided by its ambient algebra $\mathcal{C}(V, q)$.

In the more general setting, we consider a Dedekind domain R and its field of fractions K (e.g. $R = \mathbb{Z}$ and $K = \mathbb{Q}$). Let V be a finite dimensional K-space that carries an R-valued quadratic form such that (V, q) is a quadratic K-space. Now take a full R-lattice L in V, which is a torsion-free R-submodule of V that contains a K-basis of V. Then q restricts to L and we obtain the quadratic R-lattice (L, q). Similarly to the example setting above, the Clifford algebra $\mathcal{C}(L, q)$ is both a full R-lattice inside and a subring of $\mathcal{C}(V, q)$, i.e. an R-order in $\mathcal{C}(V, q)$. Again, we call $\mathcal{C}(L, q)$ a Clifford order to emphasise this additional structure of the Clifford algebra of (L, q).

The key takeaway from the above description is that studying Clifford orders involves both the theory of Clifford algebras and orders, necessitating a solid understanding of both these areas of mathematics. The preliminary concepts required are thoroughly covered in the first two chapters of this thesis; however, basic knowledge in algebraic number theory is still needed.

As a consequence of the above, one can pose questions about Clifford orders not only at the level of orders but also at the level of Clifford algebras. Consider again a quadratic K-space (V, q) with an R-valued quadratic form q. On the order theoretic side, one may ask which R-orders within $\mathcal{C}(V, q)$ arise as the Clifford algebra of a quadratic R-lattice, i.e. which of these are Clifford orders. Additionally, it is well known that $\mathcal{C}(V, q)$ is always a separable K-algebra (see Definition 2.2.10), ensuring that any Clifford order is contained in a maximal R-order with respect to set inclusion. Therefore, given a Clifford order $\mathcal{C} = \mathcal{C}(L, q)$ one might ask how to transition from it to a fixed maximal order Γ such that $\mathcal{C} \subseteq \Gamma$. Alternatively one could examine the elementary divisors associated with $\mathcal{C}$ and Γ (see Proposition 2.1.8).

On the side of Clifford algebras one may consider known results that hold for Clifford algebras over fields and try to generalise these to Clifford orders. Interestingly, many authors work with Clifford algebras over fields exclusively, as is illustrated by the various examples [Sch85], [OMe00], [Knu+98], [Shi10] etc. This is due to the fact, that they have no need to consider more general base rings for their respective applications. Notable exceptions to this phenomenon are [Voi21] where Clifford algebras of quadratic lattices over Dedekind domains are considered and, most importantly to this thesis, the treatment of Kneser in [KS02]. Here, Clifford algebras are defined for quadratic modules over arbitrary commutative rings and many results on Clifford algebras are stated in this general framework, so they hold immediately for Clifford orders. However, there are also exceptions to this which this thesis addresses, at least in the case of Dedekind domains.

Finally, we briefly address a point that we have neglected thus far: the choice of the full lattice L within the quadratic K-space (V, q). Selecting an arbitrary lattice without additional structure would make this research unnecessarily complicated and less useful. Therefore, in this thesis we focus on specific classes of lattices that are well-established in the theory. The most notable examples are maximal lattices over complete discrete valuation rings, which are classified in [Eic74], and the root lattices over $\mathbb{Z}$ (see, e.g. [Ebe02]).

0.2 Summary of the main results

Now that the necessary mathematical framework has been established in the previous section, we provide a concise overview of the key findings and most important results within this thesis. Let R be a Dedekind domain with field of fractions K of characteristic zero.

- In Theorem 3.1.9 we show that the R-orders in a two-dimensional étale algebra over K, we call these *quadratic R-orders*, are determined up to isomorphism by their discriminant. Moreover, we show that any such order contains a unique suborder that admits an orthogonal basis and describe its index as a sublattice in Theorem 4.3.9.

- In Definition 3.2.1, we define a new invariant of an arbitrary quadratic R-lattice. The *quadratic discriminant* is a flexible tool for studying quadratic R-lattices, and mostly replaces the a discriminant algebra (see [KS02] (10.5)).

- In Theorem 3.2.11 we give a general description of the centroid of the orthogonal direct sum of two quadratic R-lattices over a Dedekind domain. We also provide a local version of this result in Theorem 4.3.8.

- In Corollary 3.2.17, we show that the centroid (and the quadratic discriminant) of a maximal quadratic R-lattice depend only on its anisotropic orthogonal direct summand. The centroids and quadratic discriminants of the maximal anisotropic quadratic R-lattices over p-adic valuation rings are then classified in Theorem 4.3.11, giving rise to a complete overview of the centroids and quadratic discriminants for arbitrary maximal quadratic lattices over these rings.

- In Theorem 3.2.20 we prove that any two lattices in the same spinor genus have the same centroid.

- In Theorem 3.3.4, we show that the set of isomorphism classes of quadratic R-orders forms a monoid, generalising a construction of [Hah94] over Dedekind domains.

- We generalise two results of Kneser (see [KS02] (7.12), (7.13)) in Theorem 3.4.5 and Theorem 3.4.17 which allow for expressing the (even) Clifford order of the orthogonal direct sum of two quadratic lattices as a suborder of the usual tensor product of their respective (even) Clifford orders over a Dedekind domain. We also provide local versions of these in Theorem 3.4.12 and Theorem 3.4.19.

- In Section 3.5 we give a complete classification of the centroids of a root lattice. We also provide a way to construct a basis of these centroids for the irreducible root lattices. This is sufficient to construct a basis of the centroid of any root lattice.

- In Theorem 4.4.6 and Theorem 4.4.7 we give a complete classification of the Clifford orders and the even Clifford orders of maximal anisotropic quadratic lattices over a p-adic valuation ring. This is sufficient to describe the structure of an arbitrary maximal quadratic lattice over these rings.

0.3 Outline of the thesis

This thesis consists of four chapters. In Chapter 1, we introduce many notations and definitions that are important throughout the thesis. Within the general framework of arbitrary commutative rings, we summarise the known theory on Clifford algebras relevant to our work. Thus, this chapter also prepares the introduction of Clifford orders later on. In particular, we present several standard results on Clifford algebras, which will be generalised to Clifford orders in subsequent chapters.

In Chapter 2, we introduce the central object of this thesis, the Clifford order. We begin by delving into both the general theory of quadratic lattices over Dedekind domains and the general theory of orders in separable algebras. Following this, we present some results on the structure and basic properties of Clifford orders. At the end of this chapter, we discuss a suitable data structure that allows us to work with Clifford orders computationally. We also briefly comment on the implementation of Clifford orders in the OSCAR project [Osc24], which was developed as part of this thesis.

Chapter 3 focuses on the centroid of a quadratic lattice (or more precisely, the centroid of its Clifford order) over an arbitrary Dedekind domain R with a field of fractions K of characteristic zero. We first develop the general theory of so-called *quadratic R-orders*, i.e. orders in two-dimensional étale algebras over K. With this new framework in place, we proceed to study the centroid of a quadratic lattice as a singular object. Here, we obtain many results already mentioned in Section 0.2. Towards the end of this chapter, we explicitly construct the centroids of all irreducible root lattices, which, combined with previous results in this chapter, provides a complete overview of the centroids of all root lattices. Finally, we present an algorithm for efficiently computing the centroid of a quadratic lattice over an arbitrary Dedekind domain; see Algorithm 2.

In Chapter 4, we study Clifford orders over the valuation rings of p-adic number fields, i.e. the valuation rings of finite extensions of the fields $\mathbb{Q}_p$. Note that these rings are still Dedekind domains but with a much simpler structure, simplifying the theory of quadratic lattices and Clifford orders established in Chapter 2. Chapter 4 begins by summarising the general theory of quadratic lattices and quaternion algebras over p-adic number fields. This summary is necessary to achieve the two aims of this chapter: restating many results from Chapter 3 in this simpler setting and providing a full classification of the centroids and Clifford orders of maximal quadratic lattices over the valuation rings of p-adic number fields.

1 Clifford algebras

Throughout this chapter, if not stated otherwise, let A denote an arbitrary commutative ring. This first chapter is devoted to introduce the Clifford algebra of a quadratic A-module (E, q). It is an associative algebra that, as the notation $\mathcal{C}(E, q)$ implies, is unique up to isomorphism. It can be understood as some sort of universal algebra for (E, q) because many of the classical isometry invariants of such a quadratic A-module can be recovered from it. In fact, due to a famous result from Hasse (see [Has24]), if the base ring A is any algebraic number field then one can retrieve a full set of isometry invariants. Still, we consider Clifford algebras in this very general setting, so that we have a lot of useful notation and basic results readily available, once Clifford orders are introduced in Chapter 2.

Additionally, this first chapter contains some results that do not hold in this generality and instead require either more conditions on the base ring A or some restriction on the quadratic A-module (E, q). For this, see especially Section 1.2 concerning the center and the centroid of Clifford algebras. These results are recorded here, so that they can be generalised to Clifford orders later on.

1.1 First definitions and basics

Above, we briefly described the Clifford algebra of a quadratic A-module without giving any definitions at all. Now we provide the relevant theory, following the treatment in [KS02]. We start with a brief summary of the notations and definitions that are used throughout this thesis in the context of quadratic forms.

1.1.1 Quadratic forms

In the following, let E denote an arbitrary A-module.

Definition 1.1.1. A **quadratic form** on E is a map $q : E \to A$ satisfying

(i) $q(ax) = a^2 q(x)$ for all $a \in A$, $x \in E$.

(ii) The **polarisation** $b_q(x, y) \coloneqq q(x+y) - q(x) - q(y)$ with $x, y \in E$ is a symmetric bilinear form on E.

Then we call the pair (E, q) a **quadratic A-module**. Isomorphisms of quadratic A-modules, i.e. isomorphisms of A-modules that preserve the respective quadratic forms are called **isometries**. The set of all isometries of a quadratic A-module (E, q) onto itself is a group, the **orthogonal group** $O(E, q)$ of (E, q).

Remark 1.1.2. In the situation of Definition 1.1.1, if A is a field, we call the pair (E, q) a *quadratic A-space* instead.

Below, in Definition 1.1.3, we summarise essentially all the properties that a quadratic A-module (E, q) may have and are important to this thesis. To prepare it, note that the polarisation b_q always induces the A-module homomorphism

$$\widetilde{b_q} : E \to E^*,\ x \mapsto b_q(x, \cdot)$$

from E into its dual space E^*.

Definition 1.1.3. Let (E, q) be a quadratic A-module. Then (E, q) is called

(i) **non-degenerate**, if $\widetilde{b_q}$ is a monomorphism.

(ii) **regular**, if $\widetilde{b_q}$ is an isomorphism and E is finitely generated and projective as an A-module.

(iii) **anisotropic**, if $q(x) = 0$ implies $x = 0$, i.e. $0 \in E$ is the only *singular* element of E. Otherwise, (E, q) is called **isotropic**.

(iv) **singular**, if $q(E) = \{0\}$, i.e. q is the zero map.

(v) **universal**, if $q(E) = A$, i.e. any element $a \in A$ is represented by (E, q).

Remark 1.1.4. Given a quadratic A-module (E, q) we make it a frequent custom to refer to a property of E or of q (whichever seems better suited) instead of referring to a property of the pair (E, q), provided no confusion arises from this. For example, we would say 'E is non-degenerate' instead of '(E, q) is non-degenerate'.

We continue with two notations that are used throughout this thesis.

Notation 1.1.5. If q is a quadratic form on E then so is aq for each $a \in A$ with polarisation $b_{aq} = ab_q$. We denote the quadratic A-module (E, aq) by ${}^aE$. If we want to emphasise that we consider some $x \in E$ as an element of this *rescaled* quadratic A-module, we write it as ${}^ax$.

Notation 1.1.6 (cf. [KS02] I.(2.5)f.)**.** Let (E, q) be a free quadratic A-module of finite rank n with basis $(e_1, \ldots, e_n)$. Then the form q is uniquely determined by the values $a_{ii} := q(e_i)$ and $a_{ij} := b_q(e_i, e_j)$, for $1 \leq i, j \leq n$. This follows from

$$q\left(\sum_i x_i e_i\right) = \sum_i x_i^2 q(e_i) + \sum_{i<j} x_i x_j b_q(e_i, e_j) = (x_1, \ldots, x_n) Q (x_1, \ldots, x_n)^{\mathrm{tr}},$$

where the defining matrix $Q \in A^{n\times n}$ is given by

$$Q = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix}.$$

We denote the free quadratic A-module (E, q) above by

$$(E, q) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{bmatrix}.$$

Further, if $(e_1, \dots, e_n)$ is an *orthogonal basis* of E, i.e. $a_{ij} = 0$ for $i \neq j$, we write

$$(E, q) = [a_{11}, \dots, a_{nn}] = \mathop{\perp}_{i=1}^{n} [a_{ii}].$$

Here, the symbol $\perp$ denotes the *orthogonal direct sum* of quadratic A-modules (see [KS02] (1.1)).

Remark 1.1.7. In the situation of Notation 1.1.6, the matrix $B := Q + Q^{\mathrm{tr}} \in A^{n\times n}$ is the Gram matrix of the polarisation b_q with respect to the basis $(e_1, \dots, e_n)$. It is clear from Definition 1.1.3 that q is regular, if and only if $\det(B) \in A^\times$ is a unit; and that q is non-degenerate, if and only if $\det(B)$ is not a zero divisor in A.

Example 1.1.8. The free quadratic A-module of rank two

$$\mathbb{H}(A) := \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$$

is called the **hyperbolic plane** on A. By Remark 1.1.7, $\mathbb{H}(A)$ is always regular and, if $2 \in A^\times$, we have $\mathbb{H}(A) = [1, -1]$. Further, an elementary computation shows that

$$O(\mathbb{H}) = \left\langle \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ \beta^{-1} & 0 \end{pmatrix} \;\middle|\; \alpha, \beta \in A^\times \right\rangle,$$

if A is an integral domain.

The quadratic module $\mathbb{H}(A)$ just introduced plays a major role in the theory of quadratic forms and occurs in many canonical decompositions of quadratic modules. We close our summary of quadratic forms with an instance of this phenomenon.

Proposition 1.1.9 (Witt decomposition)**.** *Let A be a field and (E, q) be a finite-dimensional regular quadratic A-space. There is a uniquely determined $k \in \mathbb{N}_0$ such that*

$$(E, q) \cong (F, q_{|F}) \perp \mathop{\perp}_{i=1}^{k} \mathbb{H}(A).$$

The (regular) subspace F is anisotropic and uniquely determined up to isometry. The non-negative integer k is called the ***Witt index*** *of (E, q) and denoted by $\operatorname{ind}(E, q)$ (or just $\operatorname{ind}(E)$). The quadratic A-space $(F, q_{|F})$ is called the* ***anisotropic kernel*** *of E.*

Hence, over a field A, classifying all regular quadratic A-spaces reduces to classifying all the regular anisotropic ones. Over a finite field, the latter have at most dimension two and are well-known in the literatue.

Proposition 1.1.10. *If (E, q) is a regular anisotropic quadratic space of dimension n over the finite field A, then $n = 1$ or $n = 2$. Moreover, if $n = 1$ then $2 \in A^\times$ and $E \cong [1]$ or $E \cong [\varepsilon]$ where $\varepsilon \in A^\times$ is any non-square. If $n = 2$ then E is isometric to the unique degree-two field extension of A equipped with the relative field norm. This quadratic A-space is called the norm form on A and denoted by $N(A)$.*

Proposition 1.1.9 is revisited upon considering quadratic lattices over Dedekind domains in Chapter 2. Proposition 1.1.10 comes into play when dealing with quadratic lattices over complete discrete valuation rings that have a finite residue field, because isometries can be lifted.

1.1.2 The Clifford algebra of a quadratic module

Now that the basics of quadratic A-modules are covered, we may define its Clifford algebra.

Definition 1.1.11. A Clifford algebra for the quadratic A-module (E, q) is an A-algebra $\mathcal{C} = \mathcal{C}(E) = \mathcal{C}(E, q)$ together with an A-module homomorphism $g : E \to \mathcal{C}$ with $g(x)^2 = 1_{\mathcal{C}} \cdot q(x)$, for all $x \in E$ and that satisfies the following universal property: For any A-algebra $\mathcal{B}$ and any homomorphism $f : E \to \mathcal{B}$ with $f(x)^2 = 1_{\mathcal{B}} \cdot q(x)$ there is a unique homomorphism $h : \mathcal{C} \to \mathcal{B}$ with $f = h \circ g$.

The Clifford algebra is defined via a universal property, so the following result is not surprising.

Theorem 1.1.12 ([KS02] II Satz (5.4))**.** *For any quadratic A-module (E, q) there exists a Clifford algebra $\mathcal{C}(E, q)$. This A-algebra is unique up to isomorphism.*

Note that within the Clifford algebra, the relations $g(x)^2 = q(x) \cdot 1_{\mathcal{C}}$ and

$$\begin{aligned} g(x)g(y) + g(y)g(x) &= (g(x) + g(y))^2 - g(x)^2 - g(y)^2 \\ &= (q(x + y) - q(x) - q(y)) \cdot 1_{\mathcal{C}} = b_q(x, y) \cdot 1_{\mathcal{C}} \end{aligned}$$

hold and that these are all one has to infer. Due to this, the Clifford algebra $\mathcal{C} = \mathcal{C}(E, q)$ can be constructed as the quotient of the free A-algebra on any generating set $\{e_i\}_{i \in S}$ of the module E by the two-sided ideal $\mathcal{I}$ that is generated by the relations

(i) $\sum_{i \in S} a_i e_i$ with $a_i \in A$, if $\sum_{i \in S} a_i e_i = 0$ in E.

(ii) $e_i^2 - q(e_i) \cdot 1_{\mathcal{C}}$, for all $i \in S$.

(iii) $e_i e_j + e_j e_i - b_q(e_i, e_j) \cdot 1_{\mathcal{C}}$, for all $i, j \in \underline{n}$, $i \neq j$.

In this construction the map g sends $e_i \in E$ to $e_i + \mathcal{I}$. For details, we refer to [KS02]. Note that due to the relations (ii) and (iii), if S is an ordered set (for example if S finite), a generating system for the Clifford algebra as an A-module is given by

$$\langle g(e_{i_1}) \dots g(e_{i_r}) \mid r \in \mathbb{N}_0, \, i_1 < \dots < i_r, \, i_j \in S \rangle.$$

At first glance, the Clifford algebra appears to be an object that is of purely theoretical interest in the context of quadratic forms. However, many algebras can be interpreted as a Clifford algebra, if one chooses the quadratic module (E, q) suitably. Among the prominent examples are the quaternion algebras, as well as the $\mathbb{R}$-algebra generated by the famous *Pauli matrices* (see Example 1.2.7). Another example is the exterior algebra $\Lambda(E)$ where E is any free A-module; we cover it right below. These examples highlight the relevance of the theory of Clifford algebras, even in research areas that are not purely mathematical, such as physics and computer science.

Example 1.1.13. Let E be any free A-module of finite rank n and choose any basis of E. The free algebra on this basis is the tensor algebra $\mathcal{T}(E)$. If we equip E with the zero form then $g(x)^2 = 0$, for all $x \in E$, so using the construction above, we find that $\mathcal{C}(E)$ is isomorphic to $\mathcal{T}(E)/\mathcal{I}$ where $\mathcal{I}$ is the two-sided ideal generated by the elements $x^2 \in \mathcal{T}(E)$. The A-algebra $\Lambda(E) \coloneqq \mathcal{T}(E)/\mathcal{I}$ is called the *exterior algebra* or *Graßmann algebra* of E.

1.2 The structure of the Clifford algebra

With the definition at hand, we now proceed by recording results on the structure of Clifford algebras. Throughout this section, (E, q) denotes an arbitrary quadratic A-module.

1.2.1 Basis, grading and extending isometries

Theorem 1.2.1 ([KS02] II Satz (5.12))**.** *If E is a free quadratic A-module of rank n with basis $(e_1, \dots, e_n)$, then the Clifford algebra $\mathcal{C}(E, q)$ is a free A-module of rank 2^n with basis*

$$(g(e_{i_1}) \dots g(e_{i_r}) \mid r \in \underline{n}, \, i_1 < \dots < i_r).$$

Proposition 1.2.2. *Let $\{e_i\}_{i\in\underline{n}}$ be a basis of E. Then, as an A-module, the Clifford algebra $\mathcal{C}(E,q)$ can be decomposed into the direct sum $\mathcal{C}(E,q)=\mathcal{C}_0\oplus\mathcal{C}_1$ with*

$$\mathcal{C}_i := \mathcal{C}_i(E,q) := \langle g(e_{i_1})\dots g(e_{i_r}) \mid r\in\mathbb{N}_0,\, r\equiv_2 i,\, i_1<\dots<i_r,\, i_j\in\underline{n}\rangle.$$

Clearly, $\mathcal{C}_i\mathcal{C}_j\subseteq\mathcal{C}_{i+j \bmod 2}$, so $\mathcal{C}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra and $\mathcal{C}_0$ is an A-algebra itself, called the ***even Clifford algebra*** *of (E,q). Moreover, $\mathcal{C}_1$ is a $\mathcal{C}_0$-module and, while not an algebra itself, is called the* ***odd Clifford algebra****.*

Corollary 1.2.3. *If E is a free quadratic A-module of rank n with basis $(e_1,\dots,e_n)$, then $\mathcal{C}_i(E,q)$ is a free A-module of rank 2^{n-1} respectively. An A-basis of $\mathcal{C}_i(E,q)$ is*

$$(g(e_{i_1})\dots g(e_{i_r}) \mid r\in\underline{n}, r\equiv_2 i,\, i_1<\dots<i_r).$$

Another immediate consequence of Theorem 1.2.1 is that if E is free, then the map g is injective. More generally, if F is a direct summand of the free quadratic A-module E, then also the restriction $g_{|F}$ is injective. Thus, we record the following, more general result.

Corollary 1.2.4. *If (E,q) is a finitely generated projective quadratic A-module then the map $g: E\hookrightarrow\mathcal{C}(E)$ is injective, so the submodule $g(E)\leq\mathcal{C}(E)$ can be identified with E.*

From now on, if these conditions are met, we omit the map g and regard elements of E as elements of the Clifford algebra $\mathcal{C}(E)$.

Remark 1.2.5. In Chapter 2 we consider lattices over Dedekind domains that carry a quadratic form. These are always finitely generated and projective, so Corollary 1.2.4 applies in this context, and we may omit the map g.

We continue with some basic facts about the structure of the even Clifford algebra.

Proposition 1.2.6 ([KS02] (5.13)f.)**.** *Let (E,q) be a free quadratic A-module of finite rank and $a\in A^\times$. Then we obtain the following isomorphisms of graded A-algebras.*

(i) $\mathcal{C}_0(E)\cong\mathcal{C}_0({}^aE)$.

(ii) $\mathcal{C}_0(E\perp[-a])\cong\mathcal{C}({}^aE)$.

We do not prove this result here. However, it is instructive for later (see Proposition 2.3.7) to write down the respective isomorphisms explicitly. Let e be a fixed generator of $[-a]$. Then isomorphism from assertion (ii) given by ${}^ax\mapsto xe$ and the isomorphism in (i) is its restriction to the even Clifford algebra $\mathcal{C}_0({}^aE)$.

Example 1.2.7. Consider the so-called *Pauli matrices* which are given by

$$\sigma_1 \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 \coloneqq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{C}^{2\times 2}.$$

and some authors also put $\sigma_0 \coloneqq I_2$. These matrices are fundamental to the theory of quantum computation and quantum information and for further information regarding this, we refer to the book [NC10]. It is easy to see that $\sigma_j^2 = I_2$ and $\sigma_j\sigma_l = -\sigma_l\sigma_j$, for $j, l \in \underline{3}$, with $j \neq l$. Thus, the $\mathbb{R}$-algebra generated by the σ_j is isomorphic to the Clifford algebra of the quadratic $\mathbb{R}$-space $(V, q) = [1, 1, 1]$, which is an eight-dimensional $\mathbb{R}$-algebra. We want to compute $\mathcal{C}_0(V)$, by using Proposition 1.2.6. Writing $[1, 1, 1] = [1, 1] \perp [1]$, we obtain

$$\mathcal{C}_0(V) \cong \mathcal{C}(^{-1}[1, 1]) = \mathcal{C}([-1, -1]) = \langle 1, x, y, xy \mid x^2 = y^2 = -1, xy = -yx\rangle_{\mathbb{R}}.$$

This $\mathbb{R}$-algebra are the famous Hamilton quaternions; we denote them by $\mathcal{H}_{\mathbb{R}}$ for now.
If we instead directly use $\mathcal{C}_0(V) = \langle I_2, \sigma_1\sigma_2, \sigma_1\sigma_3, \sigma_2\sigma_3\rangle$ as $\mathbb{R}$-vector space, together with the easily verified identities $\sigma_1\sigma_2 = \sigma_3 i, \sigma_1\sigma_3 = \sigma_2 i, \sigma_2\sigma_3 = \sigma_1 i$ we find that $\mathcal{H}_{\mathbb{R}} = \langle I_2, \sigma_1 i, \sigma_2 i, \sigma_3 i\rangle$, explicitly realised as a subalgebra of $\mathcal{C}(V)$. In particular, the isomorphism from Proposition 1.2.6 (ii) is just right multiplication with iI_2 in this example.

We close this subsection by recording an important consequence of the universal property of the Clifford algebra.

Proposition 1.2.8 ([KS02] (5.7))**.** *Given an isometry $u \in O(E, q)$, there exists a unique automorphism $\mathcal{C}(u)$ of $\mathcal{C}(E, q)$ that extends u, i.e. $\mathcal{C}(u) \circ g = g \circ u$.*

Example 1.2.9. Suppose that $D = D_0 \oplus D_1$ is an arbitrary $\mathbb{Z}/2\mathbb{Z}$-graded algebra. Then there is a unique involution $\gamma = \gamma_D \in \mathrm{Aut}(D)$, given by

$$\gamma(x) = \begin{cases} x, & x \in D_0 \\ -x, & x \in D_1 \end{cases}.$$

Coming back to the quadratic A-module (E, q) we have $-\mathrm{id} \in O(E)$. If $D = \mathcal{C}(E) = \mathcal{C}$ is a Clifford algebra, its unique extension $\mathcal{C}(-\mathrm{id})$ clearly satisfies the above condition, i.e. $\mathcal{C}(-\mathrm{id})$ is the identity on the even Clifford algebra and its negative on the odd Clifford algebra. We will denote this involution of $\mathcal{C}$ by $\gamma_{\mathcal{C}} = \mathcal{C}(-\mathrm{id})$.

1.2.2 Graded tensor product and regular representation

Suppose that we have a decomposition $E = E_1 \perp E_2$. Then the Clifford algebra $\mathcal{C}(E, q)$ can be expressed in terms of $\mathcal{C}(E_1, q_{|E_1})$ and $\mathcal{C}(E_2, q_{|E_2})$ by using the *graded tensor product*. The latter is a concept that applies to arbitrary $\mathbb{Z}/2\mathbb{Z}$-algebras.

Definition 1.2.10. Let $C = C_0 \oplus C_1$, $D = D_0 \oplus D_1$ be two $\mathbb{Z}/2\mathbb{Z}$-graded A-algebras, i.e. $C_i C_j \subseteq C_{i+j \bmod 2}$, $D_i D_j \subseteq D_{i+j \bmod 2}$. Then the graded tensor product $C \mathbin{\widetilde{\otimes}} D$ is defined as the A-algebra $C \otimes D$ together with the multiplication

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) \coloneqq (-1)^{ij} (x_1 y_1) \otimes (x_2 y_2), \text{ with } x_2 \in D_j,\ y_1 \in C_i.$$

The A-algebra $C \mathbin{\widetilde{\otimes}} D$ is also $\mathbb{Z}/2\mathbb{Z}$-graded via

$$(C \mathbin{\widetilde{\otimes}} D)_0 = (C_0 \otimes D_0) \oplus (C_1 \otimes D_1),\ (C \mathbin{\widetilde{\otimes}} D)_1 = (C_0 \otimes D_1) \oplus (C_1 \otimes D_0),$$

so $\gamma_{C \mathbin{\widetilde{\otimes}} D} = \gamma_C \otimes \gamma_D$, using the notation of Example 1.2.9.

Proposition 1.2.11. *The Clifford algebra of $E = E_1 \perp E_2$ is given by*

$$\mathcal{C}(E, q) = \mathcal{C}(E_1, q_{|E_1}) \mathbin{\widetilde{\otimes}} \mathcal{C}(E_2, q_{|E_2}),$$

with the Clifford algebras $\mathcal{C}(E_i, q_{|E_i})$ being $\mathbb{Z}/2\mathbb{Z}$-graded by Proposition 1.2.2.

This result simplifies the analysis of the structure of the Clifford algebra of E to that of the Clifford algebras of its orthogonal direct summands E_1 and E_2. Consequently, Proposition 1.2.11 is an important result of the theory. However, the graded tensor product is not always that useful.

Suppose that $E = E_1 \perp E_2$ and that we already have matrix representations of both $\mathcal{C}(E_1)$ and $\mathcal{C}(E_2)$ available. Our goal is to obtain a matrix representation of the Clifford algebra of $\mathcal{C}(E) \cong \mathcal{C}(E_1) \mathbin{\widetilde{\otimes}} \mathcal{C}(E_2)$ from these. The problem that arises is that there is no canonical way to do so, even though this is an important practical concern, especially when one wants to implement functionality for Clifford algebras in a computer algebra system. In view of this, one would like to work with the usual tensor product of algebras, because there is no graded tensor product of matrices, but there is the usual *Kronecker product* of matrices. In Proposition 1.2.15 below we present a simple method, to compute the regular representation of $\mathcal{C}(E)$ in terms of the regular representations of $\mathcal{C}(E_1)$ and $\mathcal{C}(E_2)$.

Remark 1.2.12. Let (E, q) be a free quadratic A-module and choose an A-basis $(e_1, \ldots, e_n)$ of E. Then, by Theorem 1.2.1,

$$(1, e_1, e_2, e_1e_2, e_3, e_1e_3, e_2e_3, e_1e_2e_3, \ldots, e_1 \ldots e_n)$$

is an A-basis of the Clifford algebra $\mathcal{C}(E, q)$. We call it the *binary basis* of $\mathcal{C}(E, q)$ associated to the basis $(e_1, \ldots, e_n)$. This name stems from the fact that the basis elements in this ordering correspond to the binary representation of the numbers $0, \dots, 2^n - 1$ in increasing order. From a programmers point of view, this makes the binary basis the canonical choice for implementing Clifford algebras in a computer algebra system and in fact is widely used. It is also used for the implementation of Clifford orders in the OSCAR project [Osc24] that was part of this thesis; see Section 2.4.

The binary basis comes with nice inductive properties regarding the regular representation of a Clifford algebra.

Proposition 1.2.13. *Suppose that the quadratic A-module (E, q) admits an orthogonal basis $(E, q) = \perp_{i=1}^{n} Ae_i$ and let $\mathcal{D}_n$ denote the associated binary basis. Then*

$$\mathcal{C}(E, q) = \widetilde{\bigotimes_{i=1}^{n}} \mathcal{C}(Ae_i, [q(e_i)])$$

by Proposition 1.2.11. In the context of matrices, let $\otimes$ denote the Kronecker product. Now, if $I = I_2$ denotes the 2×2 identity matrix and $D = \operatorname{diag}(1, -1)$ then the map $\rho = \rho_n$ defined by

$$\rho : \mathcal{C}(E, q) \to A^{2^n \times 2^n},\ e_i \mapsto \bigotimes_{j=1}^{n-i} D \otimes \begin{pmatrix} 0 & 1 \\ q(e_i) & 0 \end{pmatrix} \otimes \bigotimes_{j=1}^{i-1} I, \tag{1.1}$$

is a monomorphism of A-algebras. It is the right regular representation of $\mathcal{C}(E, q)$ with respect to the basis $\mathcal{D}_n$.

Proof. For each $n \in \mathbb{N}$, the map ρ_n is an A-algebra homomorphism because the defining relations are satisfied. We proceed by induction to show that ρ_n is indeed the right regular representation, the claim being obvious for $n = 1$. Suppose that this is the case for some fixed $n \in \mathbb{N}$. Notice that we have $\mathcal{D}_{n+1} = (\mathcal{D}_n, \mathcal{D}_n e_{n+1})$ and $\rho_{n+1}(e_i) = D \otimes \rho_n(e_i)$ for $i = 1, \dots, n$, so the latter is the matrix of right multiplication with e_i on $\mathcal{C}(E)$ due to $e_{n+1}e_i = -e_i e_{n+1}$.

By abuse of notation, we have $(\mathcal{D}_n, \mathcal{D}_n e_{n+1})e_{n+1} = (\mathcal{D}_n e_{n+1}, q(e_{n+1})\mathcal{D}_n)$, so the matrix of e_{n+1} under right multiplication is

$$\begin{pmatrix} 0 & 1 \\ q(e_{n+1}) & 0 \end{pmatrix} \otimes \bigotimes_{j=1}^{n} I = \begin{pmatrix} 0 & 1 \\ q(e_{n+1}) & 0 \end{pmatrix} \otimes I_{2^n}.$$

This coincides with the image of e_{n+1} under ρ_{n+1}. □

Remark 1.2.14. Similarly one can show that the left regular representation of

$\mathcal{C}(E,q)$ with respect to the binary basis is given by the map

$$\rho_l : \mathcal{C}(E,q) \to A^{2^n \times 2^n}, e_i \mapsto \bigotimes_{j=1}^{n-i} I \otimes \begin{pmatrix} 0 & q(e_i) \\ 1 & 0 \end{pmatrix} \otimes \bigotimes_{j=1}^{i-1} D.$$

In practice, Proposition 1.2.13 yields an efficient way for storing the regular representations of a Clifford algebra. Instead of explicitly calculating the Kronecker products for the images of the generators e_i, one can maintain a vector of length n that contains the respective 2×2 matrices. This approach allows for easy recovery of the right regular representations of the Clifford algebras $\mathcal{C}(\perp_{i \in I}(Ae_i, [q(e_i)]))$, where $I \subseteq \underline{n}$ is an arbitrary subset. To achieve this, it suffices to take only the components of the tensor product in Equation 1.1 at positions $\{n+1-i \mid i \in I\}$.
In general, we can also use the strategy of Proposition 1.2.13 to compute the right regular representation of the graded tensor product of two Clifford algebras.

Proposition 1.2.15. *Suppose that we are given free quadratic A-modules (E_1, q_1), (E_2, q_2) of ranks m and n. Let $(e_1, \dots, e_m)$ and $(e_{m+1}, \dots, e_{m+n})$ be A-bases of these and write $(E,q) \coloneqq E_1 \perp E_2$. Moreover, let ρ_1, ρ_2 denote the right regular representations of the Clifford algebras $\mathcal{C}(E_1)$ and $\mathcal{C}(E_2)$ with respect to their associated binary bases. Then the right regular representation of $\mathcal{C}(E)$ with respect to the binary basis of $\mathcal{C}(E)$ associated to $(e_1, \dots, e_{m+n})$ is given by*

$$\rho : \mathcal{C}(E,q) \to A^{2^{m+n} \times 2^{m+n}},\ e_i \mapsto \begin{cases} \bigotimes_{j=1}^{n} D \otimes \rho_1(e_i), & i \leq m \\ \rho_2(e_i) \otimes \bigotimes_{j=1}^{m} I, & i > m \end{cases}.$$

with the matrices D, I as in Proposition 1.2.13.

Proof. This follows from the fact that if $\mathcal{D}_1$ denotes the binary basis of $\mathcal{C}(W_1, q_1)$ then the binary basis of $\mathcal{C}(E,q)$ is given by

$$(\mathcal{D}_1, \mathcal{D}_1 e_{m+1}, \mathcal{D}_1 e_{m+2}, \mathcal{D}_1 e_{m+1} e_{m+2}, \dots, \mathcal{D}_1 e_{m+1} \dots e_{m+n}).$$

The rest is a straightforward calculation. □

1.2.3 Center and centroid

Notation 1.2.16. Let (E,q) be a quadratic A-module. By $Z(\mathcal{C})$ and $Z(\mathcal{C}_0)$ we denote the respective centers of $\mathcal{C}(E)$ and $\mathcal{C}_0(E)$.

Definition 1.2.17. The **centroid** of the quadratic A-module (E,q) is

$$\mathcal{Z}(E,q) \coloneqq \{x \in \mathcal{C}(E) \mid xy = yx \text{ for all } y \in \mathcal{C}_0(E)\},$$

i.e. the centraliser of $\mathcal{C}_0(E)$ in $\mathcal{C}(E)$.

Example 1.2.18. If the free quadratic A-module $(E, q) = (E, 0)$ is singular then

$$\mathcal{Z}(E, 0) = \mathcal{C}(E) = \Lambda(E),$$

where the right-hand side is the exterior algebra on E from Example 1.1.13. Further, if $(e_1, \dots, e_n)$ is an A-basis of E and $e_{\underline{n}} \coloneqq e_1 \dots e_n \in \mathcal{C}(E)$, then one computes

$$Z(\mathcal{C}) = \begin{cases} \mathcal{C}_0(E), & n \text{ even} \\ \mathcal{C}_0(E) \oplus Ae_{\underline{n}}, & n \text{ odd} \end{cases}.$$

Remark 1.2.19. For a free quadratic A-module (E, q), the centroid is easy to compute in small dimensions n.

(i) If $n = 1$ with $E = Ae_1$ then $\mathcal{Z}(E, q) = \mathcal{C}(E) = A[e_1] \cong A[X]/(X^2 - q(e_1))$.

(ii) If $n = 2$ with $E = \langle e_1, e_2 \rangle_A$, put $a \coloneqq q(e_1), c \coloneqq q(e_2), b \coloneqq b_q(e_1, e_2)$ and $z \coloneqq e_1 e_2 \in \mathcal{C}_0(E)$. Then $\mathcal{Z}(E, q) = \mathcal{C}_0(E) = A[z]$ and $z^2 - bz + ac = 0$.

There is a useful characterisation of the centroid, connecting it to the center of the Clifford algebra.

Lemma 1.2.20 ([KS02] II (7.5))**.** *Suppose that the quadratic A-module (E, q) satisfies $Aq(E) = A$. There is a unique A-algebra automorphism α of $\mathcal{Z}(E, q)$ such that $xz = \alpha(z)x$, for all $x \in E$ and $\alpha^2 = \mathrm{id}$. Moreover, the center $Z(\mathcal{C}) = \{z \in \mathcal{Z}(E, q) \mid \alpha(z) = z\}$ is the fixed field of α in $\mathcal{Z}(E, q)$.*

Building on this, we present two theorems from [KS02]; the first expresses the center of a Clifford algebra in terms of the centroid, and the second describes the structure of the centroid as an A-algebra. For both theorems, we require the following additional assumptions on (E, q):

(i) E is free of finite rank n.

(ii) If n is even then E is regular; and if n is odd then E is *semi-regular* (see [KS02] (2.13)).

Remark 1.2.21. The first condition is always satisfied if A is a local ring. For the second condition, note that if $2 \in A^\times$ then the notions 'regular' and 'semi-regular' coincide. If $2 \notin A^\times$ then there exist no regular quadratic modules in odd dimension.

Theorem 1.2.22 ([KS02] II (7.10))**.** *Under the two assumptions above, the center of the (even) Clifford algebra is given by*

$$Z(\mathcal{C}) = \begin{cases} A, & n \text{ even} \\ \mathcal{Z}(E,q), & n \text{ odd} \end{cases},$$

$$Z(\mathcal{C}_0) = \begin{cases} \mathcal{Z}(E,q), & n \text{ even} \\ A, & n \text{ odd} \end{cases}.$$

There is also a precise description of the centroid as a free A-algebra of rank two, in [KS02] (7.9) Satz. We restate it for our purposes, but we omit some technical details.

Theorem 1.2.23. *Under the assumptions above, if n is even then*

$$\mathcal{Z}(E,q) \cong A[X]/(X^2 - X + c)$$

as A-algebra with some $c \in A$ such that $1 - 4c \in A^\times$ is a unit. If n is odd then

$$\mathcal{Z}(E,q) \cong A[X]/(X^2 - b)$$

as an A-algebra with a unit $b \in A^\times$. Conversely, if $2 \in A^\times$ then the square classes $(1-4c)\big(A^\times\big)^2$ and $b\big(A^\times\big)^2$ determine the centroid up to isometry in the respective case.

Remark 1.2.24 (cf. [KS02] (10.4)f.)**.** One can define a group structure on the set of isomorphism classes of quadratic A-algebras. In this context, the centroid of the Clifford algebra is also called the *discriminant algebra*, because it generalises the classical discriminant of a quadratic form. Compared to the latter, it is a useful isometry invariant over arbitrary local rings and not just fields of characteristic distinct from two.

Despite this, we now define the discriminant of a quadratic form over a field A of characteristic distinct from two. The reason for this is that, in this thesis, A will usually be the field of fractions of a Dedekind domain of characteristic zero, e.g. an algebraic or a p-adic number field. For the quadratic lattices over these Dedekind domains, which we consider in the next chapter, we provide another definition of the discriminant (see Definition 2.1.13).

Definition 1.2.25. Let A be a field of characteristic distinct from two and let (E,q) be a regular quadratic A-space of finite dimension n. In the notation of

Theorem 1.2.23, the **discriminant** of (E, q) is the A-square class $(1-4c)\big(A^\times\big)^2$ if n is even; and it is the A-square class $2b\big(A^\times\big)^2$ if n is odd. The A-square class $b\big(A^\times\big)^2$ is called the **half discriminant** of (E, q). We denote the discriminant of (E, q) by $\operatorname{disc}(E, q)$ and its half discriminant by $\operatorname{disc}'(E, q)$.

Remark 1.2.26. Putting $\operatorname{disc}(E, q) \coloneqq (-1)^{\binom{n}{2}}\det(G)(A^\times)^2$ yields an equivalent definition of the discriminant. Here, G is any Gram matrix of the polarisation b_q.

With the discriminant available, we close this section with two important results that, under certain conditions, express the (even) Clifford algebra of the orthogonal direct sum $E = E_1 \perp E_2$ in terms of the usual tensor product of the Clifford algebras of its orthogonal summands. As highlighted in the previous subsection, these results are quite important to the representation theory of Clifford algebras.

Theorem 1.2.27 ([KS02] (7.12))**.** *Suppose that $E = E_1 \perp E_2$ with E_1 regular of even rank and put $d \coloneqq \operatorname{disc}(E_1)$. Then*

$$\mathcal{C}(E) \cong \mathcal{C}(E_1) \otimes \mathcal{C}(E)^{\mathcal{C}(E_1)} \cong \mathcal{C}(E_1) \otimes \mathcal{C}({}^{d}E_2),$$

as A-algebras.

Under similar assumptions, there is also a version for the even Clifford algebra.

Theorem 1.2.28 ([KS02] (7.13))**.** *Suppose $E = E_1 \perp E_2$ with E_1 semi-regular of odd rank and put $\delta \coloneqq \operatorname{disc}'(E_1)$. Then*

$$\mathcal{C}_0(E) \cong \mathcal{C}_0(E_1) \otimes \mathcal{C}({}^{-\delta}E_2),$$

as A-algebras.

In Chapter 3 we devote multiple sections to generalise these two theorems to arbitrary Dedekind domains, where the quadratic module E_1 is only required to be non-degenerate.

2 Clifford orders

In this chapter, we introduce the central object of this thesis, the *Clifford order*. Throughout this chapter, if not stated otherwise, we agree on the following setting: Let R be a *Dedekind domain*, i.e. a noetherian integrally closed domain in which each non-zero prime ideal is maximal. Denote its field of fractions by K and assume that $\mathrm{char}(K) = 0$, so that K is either an algebraic or a p-adic number field.

We start with the definition of a Clifford order, even though at this point we have not properly defined all the components that are necessary to do so. Instead, we use the definition below to motivate the next sections in which these missing parts, namely *quadratic R-lattices* and *R-orders*, are addressed individually.

Definition 2.0.1 (Clifford order)**.** Assume that (L, q) is a full even R-lattice in the finite-dimensional quadratic K-space (V, q). Then the Clifford algebra $\mathcal{C}(L, q)$ is a full R-lattice in the Clifford algebra $\mathcal{C}(V, q)$ and $\mathcal{C}(L, q) \subset \mathcal{C}(V, q)$ is an R-order. In this context, we call $\mathcal{C}(L, q)$ the **Clifford order** of (L, q).

Remark 2.0.2. The condition that the quadratic R-lattice (L, q) be even is present to ensure that (L, q) is a quadratic R-module. Thus, the Clifford order $\mathcal{C}(L, q)$ comes with all the general properties of Clifford algebras discussed in the previous chapter.

In the following, we will consider both quadratic R-lattices and R-orders, starting with the former.

2.1 Quadratic lattices

We will begin this chapter with the general theory of finitely generated modules over Dedekind domains, following both [Rei03] and [Coh00]. For the theory of lattices in quadratic spaces we additionally refer to §81 f. in the book of O'Meara [OMe00].

2.1.1 Structure of lattices over Dedekind domains

Definition 2.1.1. An **R-lattice** L in the K-space V is a finitely generated (torsion-free) R-submodule of V. The **rank** of L is the dimension of the subspace $KL := K \otimes_R L \leq V$. If $\mathrm{rank}(L) = n$ then L is called *full*.

Remark 2.1.2. We will frequently use the terminology 'R-lattice' for a finitely generated R-module L that is torsion-free without mentioning the K-space it is contained in. This is because, according to the above definition, we can always consider L as an R-lattice in the so-called *ambient space* $KL = K \otimes_R L$. Note that L is always a full lattice in KL.

Our first goal is to review the structure of lattices over Dedekind domains. Following the above definition, this is just the special case of the general structure theorem for finitely generated modules over Dedekind domains that are torsion-free. To state the latter, we briefly summarise the most important properties of a Dedekind domain.

Notation 2.1.3. Recall that a *fractional ideal* in R is a finitely generated non-zero R-submodule of K or equivalently, a full R-lattice in K. For simplicity, by an 'ideal of R' we refer to a fractional ideal and we call the usual ring-theoretical ideals '*integral ideals* of R', instead. Note that the integral ideals are precisely the fractional ideals that are subsets of R.

Proposition 2.1.4. *Let R be a Dedekind domain with field of fractions K.*

(i) Each ideal J is invertible with inverse $J^{-1} = \{x \in K \mid xJ \subseteq R\}$. Furthermore, each ideal can be generated by at most two elements, the first of which can be chosen arbitrarily.

(ii) Each ideal J can be expressed uniquely as a product of powers of prime ideals in R. Moreover, $J \trianglelefteq R$ is an integral ideal, if and only if all these powers are non-negative.

(iii) The ***class group*** *$\mathfrak{Cl}(R)$ is the set of ideals of R modulo the set of principal ideals of R. It is a finite abelian group and its group order is called the* ***class number*** *of R (or K). The elements of $\mathfrak{Cl}(R)$ are called* ***ideal classes*** *and denoted by $[\mathfrak{a}]$, for an ideal $\mathfrak{a}$ of R. The class group is trivial, if and only if R is a unique factorisation domain, if and only if R is a principal ideal domain.*

Theorem 2.1.5 ([Coh00], Theorem 1.2.12.)**.** *Let M be a finitely generated module over the Dedekind domain R.*

(i) The R-module M is torsion-free if and only if M is projective.

(ii) There exists a torsion-free submodule $N \leq M$, such that

$$M = M_{\text{tors}} \oplus N \text{ and } N \cong M/M_{\text{tors}},$$

where M_{tors} is the torsion submodule of M.

(iii) *If M is torsion-free and $V = KM$ has dimension n, there exist ideals $\mathfrak{a}_i$ of R and elements $\omega_i \in V$ such that*

$$M = \mathfrak{a}_1\omega_1 \oplus \ldots \oplus \mathfrak{a}_n\omega_n.$$

The ideal class of $\mathfrak{a} := \mathfrak{a}_1\mathfrak{a}_2 \ldots \mathfrak{a}_n$ depends only on M; $[\mathfrak{a}] \in \mathfrak{Cl}(R)$ is called the ***Steinitz class*** *of M.*

(iv) *The module M is a free R-module, if and only if its Steinitz class is trivial, i.e. the product $\mathfrak{a}$ is principal.*

(v) *If M is a torsion module, there exist unique non-zero integral ideals δ_i of R and non-unique elements $\omega_i \in M$ such that*

$$M = (R/\delta_1)\omega_1 \oplus \ldots \oplus (R/\delta_n)\omega_n$$

and $\delta_{i-1} \subseteq \delta_i$ for $2 \le i \le n$.

The third statement motivates the following definition.

Definition 2.1.6 ([Coh00] 1.4.1)**.** Let L be an R-lattice and put $V = KL$.

(i) A *pseudo-element* of V is a rank one R-submodule of V of the form $\mathfrak{a}\omega$ or equivalently, it is an equivalence class of pairs $(\omega, \mathfrak{a})$ with $0 \neq \omega \in V$ and $\mathfrak{a}$ an ideal in R. In the latter version, two pairs $(\omega, \mathfrak{a})$ and $(\omega', \mathfrak{a}')$ are equivalent, if and only if $\mathfrak{a}\omega = \mathfrak{a}'\omega'$.

(ii) The pseudo-element $\mathfrak{a}\omega$ is called *integral*, if $\mathfrak{a}\omega \subseteq L$.

(iii) If $\mathfrak{a}_i$ are ideals of R and ω_i are elements of V, we say that $(\omega_i, \mathfrak{a}_i)_{1 \le i \le k}$ is a *pseudo-generating set* for L, if

$$L = \mathfrak{a}_1\omega_1 + \ldots + \mathfrak{a}_k\omega_k.$$

(iv) We say that $(\omega_i, \mathfrak{a}_i)_{1 \le i \le k}$ is a **pseudo-basis** of L, if

$$L = \mathfrak{a}_1\omega_1 \oplus \ldots \oplus \mathfrak{a}_k\omega_k.$$

According to the structure theorem, any R-lattice L has a pseudo-basis. Also note that the number of pseudo-elements in any pseudo-basis of L is equal to the rank of L, hence well-defined. Moreover, given a pseudo-element $\mathfrak{a}\omega$, we can enforce that either $\mathfrak{a}$ is an integral ideal of R or that ω is integral in R, but not necessarily both.

We close this subsection with some technical, but useful results on pseudo-bases.

Proposition 2.1.7 ([Coh00] 1.4.2). *Let $(\omega_i, \mathfrak{a}_i)_{1\leq i\leq n}$ be a pseudo-basis of the R-module M.*

(i) If $(\eta_j, \mathfrak{b}_j)_{1\leq j\leq n}$ is another pseudo-basis of M, put $U = (u_{ij})$ the $n \times n$ matrix satisfying $(\eta_1, \ldots, \eta_n) = (\omega_1, \ldots, \omega_n)U$ and $\mathfrak{a} = \mathfrak{a}_1 \ldots \mathfrak{a}_n$, $\mathfrak{b} = \mathfrak{b}_1 \ldots \mathfrak{b}_n$. Then $u_{ij} \in \mathfrak{a}_i\mathfrak{b}_j^{-1}$ and $\mathfrak{a} = \det(U)\mathfrak{b}$.

(ii) If there exist ideals $\mathfrak{b}_j$ and an $n \times n$ matrix U, such that $u_{ij} \in \mathfrak{a}_i\mathfrak{b}_j^{-1}$ and $\mathfrak{a} = \det(U)\mathfrak{b}_1 \ldots \mathfrak{b}_n$ then $(\eta_j, \mathfrak{b}_j)_{1\leq j\leq n}$ is a pseudo-basis of M, where $(\eta_1, \ldots, \eta_n) = (\omega_1, \ldots, \omega_n)U$.

Proposition 2.1.8 ([Coh00] 1.2.35). *Let M and N be two R-lattices of ranks m and n such that $N \leq M$. There exist compatible pseudo-bases of M and N, that is, there exists a pseudo-basis $(\omega_i, \mathfrak{a}_i)_{1\leq i\leq m}$ of M and integral ideals $\delta_1 \subseteq \ldots \subseteq \delta_n$ such that $(\omega_j, \delta_j\mathfrak{a}_j)_{1\leq j\leq n}$ is a pseudo-basis of N. The ideals $\delta_1, \ldots, \delta_n$ depend only on M and N and are called the* ***elementary divisors*** *associated with M and N.*

Definition 2.1.9. In the situation above, if additionally N has the same rank as M, the integral ideal $\delta := \delta_1 \ldots \delta_m$ is called the **index** of N in M or just the *index*, for short. Sometimes, we write $\delta = [M : N]$.

Remark 2.1.10. If both M and N are free R-modules, e.g. if R is a principal ideal domain, we identify the index of N in M with the determinant of any base change matrix from M to N up to units in R. Then, if specifically $R = \mathbb{Z}$, we may take the absolute value of this determinant, due to $\mathbb{Z}^\times = \{1, -1\}$. Doing so, the index of N in M is just the usual index $[M : N]$ of (abelian) groups, justifying the terminology.

2.1.2 Quadratic forms on lattices

In the previous subsection, the structure of R-lattices over the Dedekind domain R has been clarified. As a next step, we equip the ambient space $V = KL$ with a quadratic form q, giving rise to the quadratic K-space (V, q). In this situation, we can consider the restriction of q to L.

Definition 2.1.11. If (V, q) is a quadratic K-space containing the full R-lattice L then by means of restriction we obtain the **quadratic R-lattice** (L, q). We call (L, q) **non-degenerate**, if (V, q) is non-degenerate in the sense of Definition 1.1.3 (i).

Remark 2.1.12. The quadratic form q need not be R-valued, so (L, q) is not a quadratic R-module in general. Thus, Definition 1.1.3 (i) does not apply directly to L, but if it does, the two definitions of 'non-degenerate' coincide.

Definition 2.1.13. Let (L, q) be a quadratic R-lattice with polarisation $b = b_q$.

(i) The **scale** of L is the ideal $\mathfrak{s}(L) = \langle b(x,y) \mid x,y \in L\rangle_R$; and the **norm** of L is the ideal $\mathfrak{n}(L) = \langle \frac{1}{2} b(x,x) \mid x \in L\rangle_R = \langle q(x) \mid x \in L\rangle_R$.

(ii) Let $(\omega_i, \mathfrak{a}_i)_{1\le i\le n}$ be a pseudo-basis of L. The **discriminant ideal** of L is $\mathfrak{d}(L) = \mathfrak{a}_1^2 \dots \mathfrak{a}_n^2 \det(b(\omega_i,\omega_j)_{i,j})$.

(iii) The **discriminant** of L is the pair

$$\mathrm{disc}(L) = \left(\mathfrak{d}(L), (-1)^{\binom{n}{2}} \det(b(\omega_i,\omega_j)_{i,j})(K^\times)^2\right).$$

Remark 2.1.14. In [OMe00] the norm ideal is defined as $\langle b(x,x) \mid x \in L\rangle_R = 2\langle q(x) \mid x \in L\rangle_R$, so our definition differs by a factor two. We do this so that (L,q) is a quadratic R-module in the sense of Subsection 1.1.1, if and only if $\mathfrak{n}(L)$ is an integral ideal.

We continue with a couple of propositions that summarise the relevant properties of scale, norm and discriminant of a quadratic lattice, as well as state their connection.

Proposition 2.1.15. *Let (L,q) be a quadratic R-lattice with polarisation $b = b_q$.*

(i) $\mathfrak{s}(L) \subseteq \mathfrak{n}(L) \subseteq \frac{1}{2}\mathfrak{s}(L)$.

(ii) Let $(\omega_i, \mathfrak{a}_i)_{1\le i\le n}$ be a pseudo-basis of L. Then

$$\mathfrak{s}(L) = \sum_{i,j\in \underline{n}} \mathfrak{a}_i\mathfrak{a}_j b(\omega_i,\omega_j) \quad \textit{and} \quad \mathfrak{n}(L) = \sum_{i\in\underline{n}} \mathfrak{a}_i^2 q(\omega_i) + \mathfrak{s}(L).$$

(iii) If L is of rank n then $\mathfrak{d}(L) \subseteq \mathfrak{s}(L)^n$.

Proof. Starting with (i), the first inclusion follows from

$$b(x,y) = \frac{1}{2}\left(b(x+y,x+y) - b(x,x) - b(y,y)\right), \quad x,y \in L,$$

while the second one is obvious. Assertion (iii) is the content of 82:10 in [OMe00]. The proof of assertion (ii) is essentially the one of 82:8 in [OMe00], but with our definition of the norm. First, if $n = 1$, we find

$$\mathfrak{n}(L) = \frac{1}{2}\mathfrak{s}(L) = \frac{1}{2}\mathfrak{a}_1^2 b(\omega_1,\omega_1) = \mathfrak{a}_1^2 q(\omega_1).$$

using said reference. Moreover, the general proof for the scale equation is identical. For the general proof of the norm equation one only needs to divide the relevant equations in [OMe00] by two. □

Proposition 2.1.16. *Let (L, q) be a quadratic R-lattice.*

(i) The lattice L is non-degenerate, if and only if $\operatorname{disc}(L) \neq (0, 0(K^\times)^2)$.

(ii) Suppose that $L = L_1 \perp L_2$ and put $s \coloneqq \operatorname{rank}(L_1)\operatorname{rank}(L_2)$. Then

$$\operatorname{disc}(L) = (-1)^s \operatorname{disc}(L_1)\operatorname{disc}(L_2),$$

where the multiplication is defined entry-wise. In particular, the discriminant is multiplicative, if and only if at least one of the summands L_i has even rank.

Remark 2.1.17. If (L, q) is a free quadratic R-lattice of finite rank n then the R-square class $(-1)^{\binom{n}{2}}\det(G)(R^\times)^2$, with G the Gram matrix of b_q with respect to any fixed basis of L, carries the same information as $\operatorname{disc}(L, q)$ from Definition 2.1.13. In view of this and Remark 1.2.26, we make the identification

$$\operatorname{disc}(L, q) = (-1)^{\binom{n}{2}}\det(G)(R^\times)^2,$$

if L is free. The results from Proposition 2.1.16 carry over in the expected way.

Finally, from the elementary divisor theorem over Dedekind domains, Proposition 2.1.8, we obtain the discriminant of a sublattice.

Corollary 2.1.18. *Let M and N be two full quadratic R-lattices in the same m-dimensional quadratic K-space (V, q). Suppose that $N \leq M$ with index $[M : N] = \delta \trianglelefteq R$. Then $\mathfrak{d}(N) = \delta^2\mathfrak{d}(M)$ and if $x \in K^\times/(K^\times)^2$ satisfies* $\operatorname{disc}(M) = (\mathfrak{d}(L), x)$ *then* $\operatorname{disc}(N) = (\delta^2\mathfrak{d}(M), x)$.

Radical splittings and pure sublattices

Recall that we introduced quadratic lattices and non-degeneracy at the same time in Definition 2.1.1. The reason being that we almost exclusively consider non-degenerate lattices and their Clifford orders in this thesis. However, if one encounters a degenerate lattice it can be decomposed uniquely into the orthogonal direct sum of a non-degenerate and a singular sublattice. This decomposition is called the *radical splitting*.

Proposition 2.1.19 (cf. [OMe00] p.226)**.** *Let (L, q) be a quadratic R-lattice with polarisation $b = b_q$. The radical of L is $L^\perp = \{x \in L \mid b(x, L) = 0\}$.*

(i) q induces the quadratic form $\widetilde{q} : L/L^\perp \to R$, $\overline{x} \mapsto q(x)$ on the quotient $L/L^\perp$ and $(L/L^\perp, \widetilde{q})$ is a non-degenerate quadratic R-lattice.

(ii) There exists an R-sublattice $N \leq L$ such that $(N, q) \cong (L/L^{\perp}, \widetilde{q})$ and $(L, q) \cong (N, q) \perp L^{\perp}$ as quadratic R-lattices. This decomposition is called a radical splitting of L.

(iii) If $(L, q) \cong (N, q) \perp L^{\perp}$ and $(L', q') \cong N' \perp (L')^{\perp}$ are radical splittings then $(L, q) \cong (L', q')$ as quadratic R-lattices, if and only if $(N, q) \cong (N', q)$ and $L^{\perp} \cong (L')^{\perp}$ as quadratic R-lattices.

Proof. Assertion (i) is obvious. To see (ii), we first note that $L/L^{\perp}$ is torsion-free, i.e. projective by Theorem 2.1.5 (i). Indeed, if $\overline{x} \in L/L^{\perp}$ is torsion then there is $0 \neq a \in R$ such that $ax \in L^{\perp}$. Thus, $0 = b(ax, L) = a \cdot b(x, L)$, so $x \in L^{\perp}$. Now, as $L/L^{\perp}$ is projective, the short exact sequence

$$0 \longrightarrow L^{\perp} \longrightarrow L \longrightarrow L/L^{\perp} \longrightarrow 0$$

splits, proving (ii). For the non-trivial implication of (iii) let $\sigma : L \to L'$ be an isometry. Clearly, $\sigma(L^{\perp}) \subseteq (L')^{\perp}$ and using the same argument with σ^{-1} we have equality, so $L^{\perp} \cong (L')^{\perp}$. Consequently, σ induces a well-defined isometry $\widetilde{\sigma} : \left(L/L^{\perp}, \widetilde{b}\right) \to \left(L'/(L')^{\perp}, \widetilde{b'}\right)$ on the quotients. Now the assertion follows from (ii). □

Note that in (ii) we only need that the quotient $L/L^{\perp}$ is torsion-free to show the existence of a complement for $L^{\perp}$. There is a general notion for sublattices with this property.

Definition 2.1.20. Let L be an R-lattice. A sublattice $F \leq L$ is called R**-pure**, if and only if the quotient L/F is torsion-free.

In view of the elementary divisor theorem for lattices over Dedekind domains, Proposition 2.1.8, for an R-pure sublattice all the integral ideals δ_i in a compatible pseudo-basis satisfy $\delta_i = R$. As a consequence, we get the following characterisation of R-pure sublattice.

Proposition 2.1.21. *Let L be an R-lattice and $F \leq L$ a sublattice. The following are equivalent:*

(i) F is an R-pure sublattice.

(ii) F is a direct summand of L.

(iii) There is a subspace $W \leq KL$ with $F = W \cap L$. In this case, one can choose $W = KF$.

The dual of a lattice

Definition 2.1.22. Let (L,q) be a non-degenerate quadratic R-lattice. The set

$$L^{\#} := \{x \in KL \mid b_q(x,l) \in R \text{ for all } l \in L\}$$

is also an R-lattice with ambient space KL, called the **dual lattice** of L.

We record the most important properties of the dual lattice.

Proposition 2.1.23. *Let (L,q) be a non-degenerate quadratic R-lattice of rank n with pseudo-basis $(\omega_i, \mathfrak{a}_i)_{1\le i\le n}$.*

(i) $L^{\#}$ has rank n. If $(\omega_1^{\#}, \dots \omega_n^{\#})$ is the dual basis of the K-basis $(\omega_1, \dots, \omega_n)$ of the ambient space KL, then $(\omega_i^{\#}, \mathfrak{a}_i^{-1})_{1\le i\le n}$ is a pseudo-basis of $L^{\#}$.

(ii) $(L^{\#})^{\#} = L$, $KL^{\#} = KL$, $({}^{a}L)^{\#} = a^{-1} \cdot {}^{a}(L^{\#})$ and $(\mathfrak{a}L)^{\#} = \mathfrak{a}^{-1}L^{\#}$, for all $a \in K^{\times}$ and ideals $\mathfrak{a}$ of R. Furthermore, if $L = L_1 \perp L_2$ then $L^{\#} = L_1^{\#} \perp L_2^{\#}$.

(iii) If $\operatorname{disc}(L) = (\mathfrak{d}, x)$ then $\operatorname{disc}(L^{\#}) = (\mathfrak{d}^{-1}, x^{-1}) = (\mathfrak{d}^{-1}, x)$.

Proof. Assertions (i) and (ii) are proved in [OMe00] §82F. Then (iii) follows from (i) and the definition of the discriminant. □

Definition 2.1.24. Let (L,q) be a quadratic R-lattice.

(i) We call L **integral** if $\mathfrak{s}(L) \subseteq R$, i.e. the scale is an integral ideal of R.

(ii) We call L **even** if $\mathfrak{n}(L) \subseteq R$, i.e. the norm is an integral ideal of R.

Remark 2.1.25. The non-degenerate quadratic R-lattice is integral, if and only if $L \subseteq L^{\#}$. In particular, the discriminant ideal $\mathfrak{d} = \mathfrak{d}(L) = [L^{\#} : L]$ is the index of L in $L^{\#}$ for integral lattices L. Moreover, by Proposition 2.1.15 (i), $\mathfrak{s}(L) \subseteq \mathfrak{n}(L) \subseteq \frac{1}{2}\mathfrak{s}(L)$, so an even R-lattice is integral and if $2 \in R^{\times}$ the converse also hold. If $2 \notin R^{\times}$ then L is even, if and only if q is R-valued.

2.1.3 Modular and maximal lattices

Now that the preliminaries of the theory of quadratic R-lattices are established, we use this subsection to introduce two examples of classes of quadratic R-lattices that are well understood and for which a study of their Clifford order seems suitable.

Modular lattices

Note that for a quadratic R-lattice (L, q) of rank n with scale $\mathfrak{s}(L) = \mathfrak{a}$ we have $\mathfrak{d}(L) \subseteq \mathfrak{a}^n$ by Proposition 2.1.15 (iii).

Definition 2.1.26. Let (L, q) be a quadratic R-lattice of rank n and $\mathfrak{a}$ be an ideal of R. We call the lattice L **$\mathfrak{a}$-modular**, if $\mathfrak{s}(L) = \mathfrak{a}$ and $\mathfrak{d}(L) = \mathfrak{a}^n$. We call L **modular** if it is $\mathfrak{a}$-modular for an ideal $\mathfrak{a}$ of R and specifically, if $\mathfrak{a} = R$, the lattice L is called **unimodular**. If the ideal $\mathfrak{a}$ is principal with generator $\alpha \in K$, we say that L is α-modular instead of αR-modular.

Clearly, a modular lattice is non-degenerate. In the propositions to come, we collect the most important properties of modular lattices, starting with two useful descriptions.

Proposition 2.1.27 ([OMe00] 82:14a)**.** *If L is $\mathfrak{a}$-modular then $L = \{x \in KL \mid b_q(x, L) \subseteq \mathfrak{a}\}$.*

Proposition 2.1.28 ([OMe00] 82:14)**.** *Let $\mathfrak{a}$ be an ideal of R and (L, q) be a non-degenerate quadratic R-lattice. Then L is $\mathfrak{a}$-modular if and only if $L = \mathfrak{a}L^{\#}$.*

Corollary 2.1.29. *Let (L, q) be a quadratic R-lattice. Then the following statements are equivalent:*

(i) L is a regular quadratic R-module.

(ii) L is an even unimodular R-lattice.

(iii) $L = L^{\#}$.

Maximal lattices

In the following, we mainly follow Section 14 in [KS02] which deals with maximal lattices over principal ideal domains. We generalise some of the results there to Dedekind domains using [OMe00] §82H. The main reason for considering the Clifford orders of maximal lattices is Corollary 2.1.37. It states that we essentially have a Witt decomposition for maximal lattices; see Proposition 1.1.9. In the following, let $\mathfrak{a}$ denote an ideal of R.

Definition 2.1.30. An R-lattice E in the regular quadratic K-space (V, q) is called **$\mathfrak{a}$-maximal**, if $\mathfrak{n}(E) \subseteq \mathfrak{a}$ and for any other R-lattice L in V with $\mathfrak{n}(L) \subseteq \mathfrak{a}$, we have $L = E$. In the special case $\mathfrak{a} = R$ the lattice E is called **maximal**.

Clearly, an $\mathfrak{a}$-maximal R-lattice is even, if and only if $\mathfrak{a}$ is an integral ideal of R.

Proposition 2.1.31 ([OMe00] 82:18)**.** *Let (L, q) be a non-degenerate quadratic R-lattice with $\mathfrak{n}(L) \subseteq \mathfrak{a}$. Then (L, q) is contained in some $\mathfrak{a}$-maximal R-lattice.*

In order to state Corollary 2.1.37, we first classify maximal lattices in the hyperbolic plane $\mathbb{H}(K)$.

Proposition 2.1.32. *Let $\mathbb{H} = \mathbb{H}(K)$ be the hyperbolic plane on K and let E be a lattice in $\mathbb{H}$.*

(i) E is $\mathfrak{a}$-maximal, if and only if E is $\mathfrak{a}$-modular with $\mathfrak{n}(E) = \mathfrak{a}$.

(ii) Let (x, y) be a K-basis of $\mathbb{H}$ with $q(x) = q(y) = 0$ and $b_q(x, y) = 1$. If (E, q) is $\mathfrak{a}$-maximal then there is an ideal $\mathfrak{b}$ such that $E = \mathfrak{b}x \oplus \mathfrak{a}\mathfrak{b}^{-1}y =: \mathbb{H}_{\mathfrak{b}}(\mathfrak{a})$.

(iii) If $\mathfrak{b}, \mathfrak{c}$ are ideals of R then $\mathbb{H}_{\mathfrak{b}}(\mathfrak{a})$ and $\mathbb{H}_{\mathfrak{c}}(\mathfrak{a})$ are isometric, if and only if $[\mathfrak{c}] = [\mathfrak{a}\mathfrak{b}^{-1}]$ or $[\mathfrak{c}] = [\mathfrak{b}]$ inside the class group $\mathfrak{Cl}(R)$. In particular, if $G := \mathfrak{Cl}(R)$, then the number of isometry classes of $\mathfrak{a}$-maximal R-lattices in $\mathbb{H}$ is $\frac{1}{2}|G|$, if $\mathfrak{a}$ is not a square in G; and it is $\frac{1}{2}(|G| + |G/G^2|)$, otherwise.

(iv) If R a is principal ideal domain and $\alpha R = \mathfrak{a}$ then the rescaled lattice ${}^{\alpha}\mathbb{H}(R)$ is the unique α-maximal R-lattice in $\mathbb{H}$.

Proof. Assertion (i) is 82:21 in [OMe00] and (ii) is 82:21a of said reference. For assertion (iii), note that for $\lambda \in K^{\times}$, we have $\mathbb{H}_{\mathfrak{b}}(\mathfrak{a}) = \mathbb{H}_{\lambda\mathfrak{b}}(\mathfrak{a})$. Assume that $[\mathfrak{b}\mathfrak{c}] = [\mathfrak{a}]$ in $\mathfrak{Cl}(R)$ and choose $\lambda \in K^{\times}$ with $\lambda\mathfrak{b} = \mathfrak{a}\mathfrak{c}^{-1}$. Hence,

$$\mathbb{H}_{\mathfrak{b}}(\mathfrak{a}) = (\mathfrak{a}(\lambda\mathfrak{c})^{-1})x \oplus \lambda\mathfrak{c}y \cong \lambda\mathfrak{c}x \oplus (\mathfrak{a}(\lambda\mathfrak{c})^{-1})y = \mathbb{H}_{\lambda\mathfrak{c}}(\mathfrak{a}) \cong \mathbb{H}_{\mathfrak{c}}(\mathfrak{a}),$$

where the first isometry is given by swapping x and y.

For the remaining direction let E be an $\mathfrak{a}$-maximal R-lattice inside $\mathbb{H}$, isometric to $\mathbb{H}_{\mathfrak{b}}(\mathfrak{a})$ via the isometry φ. Then, by K-linearity, φ extends uniquely to an element in the orthogonal group $O(\mathbb{H})$ and, by Example 1.1.8, there is either $\alpha \in K^{\times}$ with

$$E = \varphi\left(\mathbb{H}_{\mathfrak{b}}(\mathfrak{a})\right) = \alpha\mathfrak{b}x \oplus \mathfrak{a}(\alpha\mathfrak{b}^{-1})y,$$

i.e. $E = \mathbb{H}_{\alpha\mathfrak{b}}(\mathfrak{a})$, or there is $\beta \in K^{\times}$ such that

$$E = \varphi\left(\mathbb{H}_{\mathfrak{b}}(\mathfrak{a})\right) = \beta\mathfrak{b}y \oplus \mathfrak{a}(\beta\mathfrak{b}^{-1})x.$$

In the former case, put $\mathfrak{c} := \alpha\mathfrak{b}$. In the latter case, put $\mathfrak{c} := \mathfrak{a}(\beta\mathfrak{b})^{-1}$ to ensure $[\mathfrak{b}\mathfrak{c}] = [\mathfrak{a}]$ and $E = \mathfrak{c}x \oplus \mathfrak{a}\mathfrak{c}^{-1}y = \mathbb{H}_{\mathfrak{c}}(\mathfrak{a})$. For the final part of (iii) just note that $[\mathfrak{b}] = [\mathfrak{a}\mathfrak{b}^{-1}]$, if and only if $[\mathfrak{b}^2] = [\mathfrak{a}]$. Finally, (iv) is immediate from (iii). $\square$

Remark 2.1.33. The number of isometry classes of $\mathfrak{a}$-maximal lattices in $\mathbb{H}$ is bounded by the class number $|G|$. This maximum is achieved, if and only if G is trivial, or G is an elementary abelian 2-group and $\mathfrak{a}$ is a square in G. If $|G|$ is odd, there are precisely $\frac{1}{2}(|G|+1)$ isometry classes.

Remark 2.1.34. Combining the notions from Proposition 2.1.32 and Example 1.1.8, we have $\mathbb{H}(R) = \mathbb{H}_R(R)$.

Lemma 2.1.35. *Let (V, q) be a regular quadratic K-space with $\operatorname{ind}(V, q) > 0$ and $E \leq V$ be an $\mathfrak{a}$-maximal R-lattice. Put $b := b_q$. Then there is some singular rank 1 sublattice $F \leq E$ such that $b_F(E) = \mathfrak{a}F^*$.*

Proof. This proof is essentially the one of (14.12) in [KS02] but adapted to Dedekind domains. As $\operatorname{ind}(V) > 0$, there is some singular vector $v \in V$ such that $b(V, v) \neq 0$. Put $F := E \cap Kv$ which is a rank 1 sublattice of E. Thus, $F = \mathfrak{b}x$ with some integral pseudo-element $\mathfrak{b}x$ of E, so there is a unique $\lambda \in \mathfrak{b}$ with $v = \lambda x$ whence F is singular. Moreover, $b(E, F) \subseteq b(E, E) \subseteq \mathfrak{n}(E) \subseteq \mathfrak{a}$ as E is $\mathfrak{a}$-maximal. Hence, by the unique ideal factorisation, $J := b(E, F)\mathfrak{a}^{-1}$ is an integral ideal of R. This implies that $E' := E + J^{-1}F$ is an R-lattice containing E that satisfies

$$q(E') \subseteq q(E) + J^{-1}b(E, F) + q(J^{-1}F) \subseteq \mathfrak{a} + J^{-1}(\mathfrak{a}J) + 0 = \mathfrak{a}.$$

By $\mathfrak{a}$-maximality, we conclude $E = E'$, so $J^{-1}F \subseteq E$, also implying $J^{-1}F \subseteq E \cap Kv = F$. As J is integral, we must have $J = R$ and hence, $b(E, F) = \mathfrak{a}$. Because F has rank 1, this implies $b_F(E) = \mathfrak{a}F^*$. $\square$

Theorem 2.1.36. *Under the hypothesis of Lemma 2.1.35 write $V = \mathbb{H}(K) \perp V'$ and the singular R-lattice F of rank 1 as $F = \mathfrak{b}x$. Then there is an $\mathfrak{a}$-maximal R-lattice E' in V' such that*

$$E = \mathbb{H}_{\mathfrak{b}}(\mathfrak{a}) \perp E'.$$

Proof. By Lemma 2.1.35 the sequence of R-modules

$$0 \longrightarrow F^{\perp} \longrightarrow E \longrightarrow \mathfrak{a}F^* \longrightarrow 0$$

is exact and splits because with F also F^* is projective. Thus, there is a direct summand $G \leq E$, isomorphic to $\mathfrak{a}F^*$. More precisely, $b_F(G) = \mathfrak{a}F^*$, implying $G = \mathfrak{a}\mathfrak{b}^{-1}x'$ for some $x' \in V$ with $b(x, x') \in R^\times$, so without loss of generality $b(x, x') = 1$. Putting $H = F \oplus G$ and $y = x' - q(x')x$, we have $b(x, y) = 1$ and $q(y) = 0$. One easily computes

$$\mathfrak{n}(H) = \mathfrak{a}(R + \mathfrak{a}\mathfrak{b}^{-2}q(x'))$$

and this ideal needs to be contained in $\mathfrak{n}(E) \subseteq \mathfrak{a}$. This implies $q(x') \in \mathfrak{a}^{-1}\mathfrak{b}^2$, so $\big((\mathfrak{b}, x), (\mathfrak{a}\mathfrak{b}^{-1}, y)\big)$ is another pseudo-basis of H, by Proposition 2.1.7. Hence,

$$H = \mathfrak{b}x \oplus \mathfrak{a}\mathfrak{b}^{-1}y = \mathbb{H}_{\mathfrak{b}}(\mathfrak{a}),$$

which is an $\mathfrak{a}$-modular sublattice of E. Finally, as E is assumed $\mathfrak{a}$-maximal,

$$b_q(E, H) \subseteq b_q(E, E) = \mathfrak{s}(E) \subseteq \mathfrak{n}(E) \subseteq \mathfrak{a},$$

so we obtain $E = H \perp H^{\perp}$ from [OMe00] 82:15. Both H and $E' := H^{\perp}$ are $\mathfrak{a}$-maximal in their respective ambient spaces, because E is $\mathfrak{a}$-maximal. ☐

Corollary 2.1.37. *Suppose the regular quadratic K-space V decomposes as*

$$V = V' \perp \mathop{\perp}_{i=1}^{k} \mathbb{H}(K)$$

with $V' \leq V$ the anisotropic kernel and $k = \operatorname{ind}(V)$. If E is an $\mathfrak{a}$-maximal R-lattice in V then there is an $\mathfrak{a}$-maximal R-lattice E' in V' and ideals $\mathfrak{b}_i$ of R such that

$$E = E' \perp \mathop{\perp}_{i=1}^{k} \mathbb{H}_{\mathfrak{b}_i}(\mathfrak{a}).$$

Moreover, if R is a principal ideal domain and $\mathfrak{a} = \alpha R$ then this decomposition simplifies to

$$E = E' \perp \mathop{\perp}_{i=1}^{k} {}^{\alpha}\mathbb{H}(R).$$

2.2 Orders

In this section we address the second component of Definition 2.0.1. The standard reference that we use for the theory of orders is the book of Reiner [Rei03]. Note that in this reference, R is a noetherian integral domain that need not be integrally closed. However, in view of the aims of this thesis, we continue to let R denote a Dedekind domain with field of fractions K of characteristic zero.

2.2.1 Basic definitions and properties

Definition 2.2.1. Let A be a finite dimensional K-algebra. An R**-order** in A is a subring $\Lambda \subset A$ with the same unit as A such that Λ is a full R-lattice in A.

Remark 2.2.2 ([Rei03] (8.1))**.** Any K-algebra A contains R-orders.

Proposition 2.2.3 ([Rei03] (8.6) Theorem)**.** *Every element of an R-order Λ is integral over R. Furthermore if R is integrally closed, then for $a \in \Lambda$ we have*

$$\operatorname{min.pol.}_{A/K}(a) \in R[X], \ \operatorname{char.pol.}_{A/K}(a) \in R[X].$$

Corollary 2.2.4. *Let A be a K-algebra with unit $1 = 1_A$. Then, since R is integrally closed, for any R-order Λ in A, we have $K \cdot 1 \cap \Lambda = R \cdot 1$, so $R \cdot 1$ is an R-pure sublattice of Λ. Thus, by Proposition 2.1.21, we always assume that $R \cdot 1$ is the first pseudo-element in any pseudo-basis of Λ.*

Note that the condition that R be integrally closed cannot be dropped. For example, if $A = \mathbb{Q}(\sqrt{5})$, $R = \mathbb{Z}[\sqrt{5}]$ and $\Lambda = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ then $K \cdot 1 \cap \Lambda = \Lambda \supsetneq R$.

Definition 2.2.5. An R-order Λ in the K-algebra A is called **maximal** if it is not properly contained in any other R-order of A.

Example 2.2.6. If Λ is a maximal R-order in the K-algebra A, then for $n \in \mathbb{N}$, we have that $\Lambda^{n \times n}$ is a maximal R-order in $A^{n \times n}$. Specifically, if $R = O_K$ is the ring of integers in the algebraic number field K, or if R is the valuation ring of the p-adic number field K, then $R^{n \times n}$ is a maximal order in $K^{n \times n}$.

Being a maximal order is a local property

Definition 2.2.7. A **place** $\mathfrak{p}$ of the Dedekind domain R is either a *finite place*, given by a non-zero prime of R, or an *infinite place* given by an embedding $K \hookrightarrow \mathbb{C}$. An infinite place $\mathfrak{p}$ is called *real*, if $\mathfrak{p}(K) \subseteq \mathbb{R}$ and *complex* otherwise.

At the moment, we require only finite places.

Notation 2.2.8. For a finite place $\mathfrak{p}$ of R we denote the localisation at $\mathfrak{p}$ by $R_\mathfrak{p}$ and its completion at $\mathfrak{p}$ by $\hat{R}_\mathfrak{p}$. The respective fields of fractions are denoted by $K_\mathfrak{p}$ and $\hat{K}_\mathfrak{p}$. For an R-lattice L, we put $L_\mathfrak{p} := R_\mathfrak{p} \otimes_R L$ and $\hat{L}_\mathfrak{p} := \hat{R}_\mathfrak{p} \otimes_R L$, the localisation and completion of L at $\mathfrak{p}$.

Proposition 2.2.9 ([Rei03] (11.2), (11.6))**.** *Let A be a K-algebra. For an R-order Λ in A, the following are equivalent.*

(i) *Λ is a maximal R-order.*

(ii) *For each non-zero prime $\mathfrak{p}$ of R, $\Lambda_\mathfrak{p}$ is a maximal $R_\mathfrak{p}$-order in A.*

(iii) *For each non-zero prime $\mathfrak{p}$ of R, $\hat{\Lambda}_\mathfrak{p}$ is a maximal $\hat{R}_\mathfrak{p}$-order in $\hat{K}_\mathfrak{p} \otimes A$.*

Existence of maximal orders and well-definedness of the index

As is evident from the definition of Clifford orders, see Definition 2.0.1, we consider R-orders in Clifford algebras of regular quadratic K-spaces. In this brief interlude we ensure that maximal R-orders do always exists inside these. Moreover, we establish that given an R-order Λ in such a Clifford algebra, the index of Λ in all maximal R-orders that contain it coincide and that this index can be read off from $\operatorname{disc}(\Lambda)$.

Definition 2.2.10. Let A be a finite dimensional semisimple K-algebra, so

$$A \cong \bigoplus_{i=1}^{s} D_i^{n_i \times n_i}$$

is a direct sum of full matrix rings over K-division algebras D_i, by the Artin-Wedderburn theorem (see, e.g. 7a in [Rei03]). Then A is called a **separable K-algebra**, if the center $Z(D_i^{n_i \times n_i})$ is a separable field extension of K, for all $i \in \underline{s}$.

Proposition 2.2.11 ([Rei03] (10.4))**.** *Let A be a separable K-algebra. Every R-order in A is contained in a maximal R-order in A. There exists at least one maximal order in A.*

Remark 2.2.12. Let (V, q) be a regular quadratic K-space of finite dimension n and $\mathcal{C}(V, q)$ its Clifford algebra. Then, by [KS02] (7.12)f., [Rei03] (7.8) and [Voi21] (7.6.1), both $\mathcal{C}(V, q)$ and $\mathcal{C}_0(V, q)$ are K-separable. Thus, in all Clifford algebras that do occur in this thesis, there exist maximal orders. Then, given an even R-lattice (L, q) in (V, q), one can ask if $\mathcal{C}(L, q)$ is a maximal order in $\mathcal{C}(V, q)$ and, if not, to which extend it fails to be one. The latter is measured in terms of the index $[\mathfrak{m} : \mathcal{C}(L, q)]$ from Definition 2.1.9, where $\mathfrak{m}$ is a maximal R-order of $\mathcal{C}(V, q)$ that contains $\mathcal{C}(L, q)$. Of course, the same can be done for the even Clifford order $\mathcal{C}_0(L, q)$.

Theorem 2.2.13. *Let Λ be an R-order in the separable K-algebra A. Then the index of Λ in any maximal order in A that contains it is the same. It can be deduced from* $\operatorname{disc}(\Lambda)$*.*

Proof. Being maximal is a local property by Proposition 2.2.9, so we may assume that R is a complete discrete valuation ring. Moreover, using [Rei03] (10.5), we may assume that $A = D^{n \times n}$ is central simple. The maximal R-orders in such an algebra are well understood: By [Rei03] (12.8), the integral closure of the division algebra D, call it Δ, is a ring, i.e. the unique maximal R-order of D. Then, using Example 2.2.6, $\Delta^{n \times n}$ is a maximal R-order in A and, using [Rei03] (17.3) (ii), any other maximal order in A is conjugate to this one. This implies that all maximal R-orders have the same discriminant. The claim now follows from Corollary 2.1.18. □

2.3 Revisiting Clifford orders

Now that we have properly introduced all the necessary components for the definition of Clifford orders to make sense, we repeat it once again. We follow this up with some basic facts about the structure of Clifford orders.

Definition 2.3.1. Let (L, q) be an even R-lattice with ambient space V.

(i) The Clifford algebra $\mathcal{C}(L, q)$ is an R-order in $\mathcal{C}(V, q)$, called the **Clifford order** of (L, q).

(ii) The **even Clifford order** is $\mathcal{C}_0(L, q) := \mathcal{C}_0(V, q) \cap \mathcal{C}(L, q)$; and while not an order itself, the **odd Clifford order** is $\mathcal{C}_1(L, q) := \mathcal{C}_1(V, q) \cap \mathcal{C}(L, q)$.

Remark 2.3.2. We always regard L as a subset of $\mathcal{C}(L)$, as described in Remark 1.2.5.

By definition, the Clifford order $\mathcal{C}(L)$ is an R-lattice, so it has a pseudo-basis. Clearly, the same must hold for the R-pure sublattices $\mathcal{C}_0(L)$ and $\mathcal{C}_1(L)$. We provide the respective pseudo-bases below but first, we introduce some much needed notation.

Notation 2.3.3. Let $n \in \mathbb{N}_0$ and $x_1, \ldots, x_n$ some elements in a monoid $(X, \cdot)$. If $I \subseteq \underline{n}$ is an arbitrary subset with $r := |I|$, then $x_I \in X$ denotes the ordered product $x_I := x_{i_1} \ldots x_{i_r}$, where $I = \{i_1, \ldots, i_r\}$ and $i_1 < \ldots < i_r$.

We will mainly use this notation in the situation where the x_i are ideals of the Dedekind domain R, or elements in some Clifford order or Clifford algebra.

Example 2.3.4. Let $(e_1, \ldots, e_n)$ be a K-basis of the quadratic K-space V. Then $(e_I \mid I \subseteq \underline{n})$ is a K-basis of the Clifford algebra $\mathcal{C}(V)$. The order in which I runs through the subsets $\underline{n}$ is of course arbitrary, but usually one should use the ordering induced by the binary basis associated to $(e_1, \ldots, e_n)$ (see Remark 1.2.12).

Theorem 2.3.5. *Let L be an even R-lattice of rank n with pseudo-basis $(\omega_i, \mathfrak{a}_i)_{i \in \underline{n}}$. Then, using Notation 2.3.3, the following hold.*

(i) A pseudo-basis of $\mathcal{C}(L)$ is $(\omega_I, \mathfrak{a}_I)_{I \subseteq \underline{n}}$.

(ii) A pseudo-basis of $\mathcal{C}_0(L)$ is $(\omega_I, \mathfrak{a}_I)$ where I runs through the subsets of $\underline{n}$ with an even number of elements.

(iii) A pseudo-basis of $\mathcal{C}_1(L)$ is $(\omega_I, \mathfrak{a}_I)$ where I runs through the subsets of $\underline{n}$ with an odd number of elements.

Proof. Only assertion (i) needs a proof, so let Λ be the R-lattice with pseudo-basis $(\omega_I, \mathfrak{a}_I)_{I\subseteq \underline{n}}$. Clearly, Λ is an R-order containing L, so $\mathcal{C}(L) \subseteq \Lambda$, because $\mathcal{C}(L)$ is generated by L as R-algebra. Conversely, the Clifford order $\mathcal{C}(L)$ contains L, so it must contain all the pseudo-elements $(\omega_I, \mathfrak{a}_I)$ with $I \subseteq \underline{n}$. This implies $\Lambda \subseteq \mathcal{C}(L)$. □

Corollary 2.3.6. *For any even R-lattice L in V, $\mathcal{C}(L) = \mathcal{C}_0(L) \oplus \mathcal{C}_1(L)$. In particular, $\mathcal{C}(L)$ inherits the $\mathbb{Z}/2\mathbb{Z}$-grading from $\mathcal{C}(V,q)$ in a canonical way.*

The results above are really just simple generalisations of some of the results from Section 1.2. As a more involved example of a generalisation of a statement, let us again consider Proposition 1.2.6. Applied to a quadratic K-space V, it yields $\mathcal{C}_0(V \perp [-a]) \cong \mathcal{C}({}^aV)$, for any $a \in K^\times$.

Proposition 2.3.7. *Let (L,q) be an even R-lattice of finite rank $n \geq 1$ and $0 \neq a \in R$.*

(i) The even Clifford order $\mathcal{C}_0({}^aL)$ is isomorphic to an R-suborder Λ of $\mathcal{C}_0(L)$. Its index is $[\mathcal{C}_0(L) : \Lambda] = a^sR$, where $s = n2^{n-3}$, if $n \geq 2$; and $s = 0$, if $n = 1$.

In particular, $\mathcal{C}_0({}^aL) \cong \mathcal{C}_0(L)$, if and only if $n = 1$ or $a \in R^\times$.

(ii) The Clifford order $\mathcal{C}({}^aL)$ is isomorphic to an R-suborder Λ of $\mathcal{C}_0(L \perp [-a])$. Its index is $[\mathcal{C}_0(L) : \Lambda] = a^sR$, where $s = (n-1)2^{n-2}$.

In particular, $\mathcal{C}({}^aL) \cong \mathcal{C}_0(L \perp [-a])$, if and only if $n = 1$ or $a \in R^\times$.

Proof. We start with (ii). Let e be a fixed generator of $[-a]$ and $(e_i, \mathfrak{b}_i)_{i\in\underline{n}}$ be a pseudo-basis of (L,q). The isomorphism from Proposition 1.2.6 is given by sending ${}^ax \in KL$ to $xe \in \mathcal{C}_0(KL \perp [-a])$. Thus, it maps the pseudo-basis $({}^ae_I, \mathfrak{b}_I)_{I\subseteq\underline{n}}$ of $\mathcal{C}({}^aL)$ isomorphically to

$$\left((a^{|I|/2}e_I, \mathfrak{b}_I) \mid |I| \equiv_2 0\right) \cup \left((a^{(|I|-1)/2}e_Ie, \mathfrak{b}_I) \mid |I| \equiv_2 1\right).$$

Clearly, $\left((e_I, \mathfrak{b}_I) \mid |I| \equiv_2 0\right) \cup \left((e_Ie, \mathfrak{b}_I) \mid |I| \equiv_2 1\right)$ is a pseudo-basis of $\mathcal{C}_0(L \perp [-a])$, so the index in question is a^sR, where the exponent s is given by

$$\begin{aligned} s &= \frac{1}{2}\left(\sum_{|I|\equiv_2 0} |I| + \sum_{|I|\equiv_2 1} (|I|-1)\right) = \frac{1}{2}\left(\sum_{I\subseteq\underline{n}} |I| - \sum_{|I|\equiv_2 1} 1\right) \\ &= \frac{1}{2}\left(\sum_{k=0}^{n} \binom{n}{k} k - 2^{n-1}\right) = \frac{1}{2}(n2^{n-1} - 2^{n-1}) = (n-1)2^{n-2}. \end{aligned}$$

To show (i), we restrict the map from above to $\mathcal{C}_0(L)$, so this time we only have to consider the pseudo-elements corresponding to subsets $I \subseteq \underline{n}$ with even cardinality. Analogously to part (ii), one sees that the desired index is $\mathfrak{a}^s R$, but this time the exponent is given by

$$s = \frac{1}{2} \sum_{|I| \equiv_2 0} |I| = \frac{1}{2} \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} k = \frac{1}{2} \cdot \begin{cases} n2^{n-2}, & n > 1 \\ 0, & n = 1 \end{cases}. \qquad \square$$

2.4 Data structure

In the final section of this chapter, we describe our implementation of Clifford orders for the OSCAR project [Osc24], which is part of this thesis. For this, we first describe the general framework and data structure, we use. We do not go into too many technical details or analyse the basic methods, because this is not very insightful, and there is also the documentation of the OSCAR project for this. Instead, we focus on the key concepts that make computations with Clifford orders feasible. Throughout this section, let (L, q) denote an even R-lattice with ambient space $V = KL$.

In order to create a suitable data structure for computations in Clifford orders, we revisit the definition of Clifford orders, Definition 2.3.1. In view of this, it is a reasonable choice to implement $\mathcal{C}(L)$ as a subset of the ambient algebra $\mathcal{C}(V)$, which is a Clifford algebra over a field or, more generally, a Clifford algebra of a free module.

2.4.1 The Clifford algebra of a free module

Only for the moment, let A be an arbitrary commutative ring and let (E, q) be a free quadratic A-module of rank n with basis $(e_1, \ldots, e_n)$. Of course, in our applications $A = K$ will always be the field of fractions of the Dedekind domain R and $\operatorname{char}(K) = 0$.

Then, using Theorem 1.2.1, the Clifford algebra $\mathcal{C}(E)$ is also a free A-module of rank 2^n, so $\mathcal{C}(E) \cong A^{2^n}$ via the usual coordinate map of free modules with respect to some fixed basis of $\mathcal{C}(E)$. For the latter, we choose the binary basis associated to $(e_1, \ldots, e_n)$ from Remark 1.2.12, due to its useful properties. We denote the binary basis by $(e_I \mid I \subseteq \underline{n})$, implicitly ordering the subsets of $\underline{n}$. Thus, we represent the elements of $\mathcal{C}(E)$ by vectors of length 2^n with entries in A, identifying the vector $(\lambda_I \mid I \subseteq \underline{n}) \in A^{2^n}$ with the element $\sum \lambda_I e_I \in \mathcal{C}(E)$. This already covers the module-theoretic part of $\mathcal{C}(E)$, by using entrywise addition and the usual scalar multiplication of these vectors. Since $\mathcal{C}(E)$ is a ring, we also need to know its structure constants with respect to the basis $(e_I \mid I \subseteq \underline{n})$. Recall that $\mathcal{C}(E) \cong \mathcal{T}(E)/\mathcal{I}$ as A-algebras, with $\mathcal{T}(E)$ the tensor algebra of E and $\mathcal{I}$ the ideal generated by the relations $e_i^2 = q(e_i)$ and $e_ie_j + e_je_i = b_q(e_i, e_j)$, $i, j \in \underline{n}$, $i \neq j$. Hence, the structure constants of $\mathcal{C}(E)$ can be computed by successively applying these relations. Of course, in practise we would never compute the structure constants explicitly this way; after all there are $2^{3n} = 8^n$ of these and there is no need for this exponential overhead. Instead, we implement the multiplication recursively, the base cases essentially being given by the basic relations $e_i^2 = q(e_i) = \frac{1}{2}b_q(e_i, e_i)$ and $e_ie_j + e_je_i$. This way, it is sufficient to only store the Gram matrix of b_q and look up the necessary parameters on demand. For comparison, this requires only $\mathcal{O}(n^2)$ amount of storage instead of $\mathcal{O}(8^n)$ and speeds up multiplications in $\mathcal{C}(E)$ significantly.

2.4.2 Clifford orders in OSCAR

Having seen that the Gram matrix of a free quadratic module carries sufficient information to construct its Clifford algebra, we return to our usual setting. Since our implementation is aimed to be a part of the OSCAR project, we build it upon already existing data structures for even lattices over number fields. Here, the quadratic R-lattice (L, q) of rank n is stored via a fixed pseudo-basis $(e_i, \mathfrak{a}_i)_{i\in\underline{n}}$ and its ambient quadratic space (V, q). Thus, our data structure for Clifford orders carries the following information:

- The Clifford algebra $\mathcal{C}(V)$. Since (V, q) is a free K-module, we store it as described above.
- The Gram matrix of (V, q) with respect to the K-basis $(e_1, \ldots, e_n)$.
- The *coefficient ideals* $(\mathfrak{a}_I \mid I \subseteq \underline{n})$.

The elements of the Clifford order $\mathcal{C}(L)$ are represented like those of $\mathcal{C}(V)$; they are vectors of length 2^n with entries in K. Of course, given such a vector $(\lambda_I \mid I \subseteq \underline{n}) \in K^{2^n}$ we need to check if $\sum \lambda_I e_I \in \mathcal{C}(L)$. This can easily be done by checking $\lambda_I \in \mathfrak{a}_I$ for each $I \subseteq \underline{n}$. Note that this check is only required upon construction of an element but not during computations, so we can use the same implementations for addition and multiplication in $\mathcal{C}(L)$ as for the ambient algebra $\mathcal{C}(V)$. This is also the reason why we store the Gram matrix separately; we need fast access to its entries for the multiplication. With that, we have a basic framework available in which we can work with Clifford orders over arbitrary Dedekind domains algorithmically.

Note that if R is a principal ideal domain then every R-lattice and hence the Clifford order $\mathcal{C}(L)$ is a free R-lattice. Thus, for Clifford orders over the integers we do not store the coefficient ideals, because they can all be chosen to equal $\mathbb{Z}$. Of course, we could do this for every Dedekind domain with class number one and not just the integers, but in general it is too expensive to compute the class number, if not impossible.

3 Centroids of Clifford orders

Let R be a Dedekind domain with field of fractions K of characteristic zero. In this chapter we study the centroid of a Clifford order. For this, note that Definition 1.2.17 also applies to Clifford orders. Alternatively, the centroid of a Clifford order can be expressed in terms of the centroid of its ambient Clifford algebra.

Remark 3.0.1. Let (L, q) be an even non-degenerate R-lattice with ambient space (V, q) of dimension n. Then, if $n \geq 1$, the centroid $\mathcal{Z}(L, q) = \mathcal{Z}(V, q) \cap \mathcal{C}(L)$ is an R-order in the étale algebra $\mathcal{Z}(V, q) \cong K[X]/(X^2 - \delta)$, where $\delta = \operatorname{disc}(V, q)$, if n is even; and $\delta = \operatorname{disc}'(V, q)$, if n is odd.

Note that Remark 3.0.1 also states that $\mathcal{Z}(L, q)$ is an R-pure sublattice of $\mathcal{C}(L)$, i.e. a direct summand, by Proposition 2.1.21.

At first glance, giving the centroids of Clifford orders an entire chapter of attention seems curious; it is just a certain rank two sublattice. However, a major motivation for studying the centroids of Clifford orders comes from its analogue in the field case. There, in Theorem 1.2.27 and Theorem 1.2.28, it is used implicitly to circumvent the graded tensor product, for describing the Clifford algebra of orthogonal direct sums of quadratic K-spaces. This is important for practical considerations, as we discussed below Proposition 1.2.11. In view of this, the most important results of this chapter are certainly Theorem 3.4.5 and Theorem 3.4.17. They generalise the two above mentioned theorems from Chapter 1 to Dedekind domains. We also provide local versions of these theorems respectively, even though a systematic treatment of Clifford orders over complete discrete valuation rings will happen in Chapter 4. In view of this, it is useful to have the centroids of interesting classes of non-degenerate even R-lattice available. We address this by providing results on the centroids of the maximal lattices. Additionally, we give a complete overview of the centroids of the root lattices in Section 3.5. Here, we not only present the isomorphism type of the respective centroid, but also provide an explicit way to compute a basis of it.

3.1 Quadratic orders over Dedekind domains

In this section we consider the following situation. Let A be a *quadratic étale algebra* over K, i.e. $A = K[X]/(X^2 - d)$ for some $d \in K^\times$. Clearly, the isomorphism type of A depends only on the square class $d(K^\times)^2$. Our aim, inspired by Remark 3.0.1, is to describe the R-orders Λ in A. For obvious reasons, the R-orders in the quadratic étale algebra A are also called **quadratic orders**.

We first note that two quadratic orders in A are isomorphic, if and only if they are equal. Moreover, A is a separable K-algebra, so there do exist maximal R-orders in A, by Proposition 2.2.11.

Remark 3.1.1. The quadratic étale algebra A contains a unique maximal order.

Proof. As A is commutative, its integral closure is the unique maximal order. □

Remark 3.1.2. There is a unique non-trivial K-automorphism β on A, that is given by $\beta(X) = -X$. From this, for any $a \in A$, we obtain the *trace* and the *norm* as the maps

$$t(a) := a + \beta(a), \;\; n(a) := a\beta(a)$$

and the image of both is contained in K. The norm map is a quadratic form $q := n$ on A with polarisation

$$b_q(a_1, a_2) = t(a_1\beta(a_2)) = a_1\beta(a_2) + \beta(a_1)a_2,$$

for $a_1, a_2 \in A$, making the pair (A, q) a two-dimensional quadratic K-space.

By means of restriction, any R-order Λ in A is a quadratic R-lattice (Λ, q) in A. It is even, by Proposition 2.2.3. The following notation is fundamental for the remainder of this thesis.

Notation 3.1.3. For an ideal $\mathfrak{a}$ of R, $t \in \mathfrak{a}^{-1}$ and $n \in \mathfrak{a}^{-2}$ define the R-sublattice

$$\Lambda(\mathfrak{a}, t, n) := R1 \oplus \mathfrak{a}x \subset A,$$

where $x \in A$ satisfies $x^2 - tx + n = 0$.

Remark 3.1.4. For any ideal $\mathfrak{a}$ of R and $t \in \mathfrak{a}^{-1}$, $n \in \mathfrak{a}^{-2}$, the R-lattice $\Lambda(\mathfrak{a}, t, n)$ is an R-order in the quadratic étale algebra $A = K[X]/(X^2 - (t^2 - 4n))$. Further, if $x \in A$ with $\Lambda(\mathfrak{a}, t, n) = R1 \oplus \mathfrak{a}x$, then $t = t(x)$ and $n = n(x)$ are the trace and the norm of x.

Proof. We only need to show that $\Lambda = \Lambda(\mathfrak{a}, t, n)$ is multiplicatively closed, the rest is obvious. For arbitrary $r_1, r_2 \in R$ and $a_1, a_2 \in \mathfrak{a}$, we have

$$(r_1 1 + a_1 x)(r_2 1 + a_2 x) = (r_1 r_2 - a_1 a_2 n)1 + (r_1 a_2 + r_2 a_1 + a_1 a_2 t)x \overset{!}{\in} R1 \oplus \mathfrak{a}x = \Lambda.$$

The coefficients of 1 and x are contained in the ideals $R^2 + \mathfrak{a}^2 n \subseteq R$ and $\mathfrak{a} + \mathfrak{a}^2 t \subseteq \mathfrak{a}$, where the respective inclusions hold, due to the assumptions $t \in \mathfrak{a}^{-1}$ and $n \in \mathfrak{a}^{-2}$. □

Proposition 3.1.5. *Let Λ be a quadratic R-order.*

(i) There exists an ideal $\mathfrak{a}$ and $t \in \mathfrak{a}^{-1}$, $n \in \mathfrak{a}^{-2}$ such that $\Lambda = \Lambda(\mathfrak{a}, t, n)$ as R-algebras.

(ii) For $\lambda \in K^\times$, we have $\Lambda(\mathfrak{a}, t, n) = \Lambda(\lambda^{-1}\mathfrak{a}, \lambda t, \lambda^2 n)$. Moreover, if $\Lambda(\mathfrak{a}, t, n) \cong \Lambda(\mathfrak{a}', t', n')$ then $[\mathfrak{a}] = [\mathfrak{a}']$ in $\mathfrak{Cl}(R)$.

(iii) We have $\operatorname{disc}\big(\Lambda(\mathfrak{a}, t, n), q\big) = \big(\mathfrak{a}^2(t^2 - 4n), (t^2 - 4n)(K^\times)^2\big)$.

Proof. (i) This follows from Theorem 2.1.5, Corollary 2.2.4 and Remark 3.1.4.

(ii) To see the equality $\Lambda(\mathfrak{a}, t, n) = \Lambda(\lambda^{-1}\mathfrak{a}, \lambda^1 t, \lambda^2 n)$, note that for each $\lambda \in K^\times$, we have $\mathfrak{a}x = (\lambda^{-1}\mathfrak{a})(\lambda x)$ and use the fact that $\mu_x = X^2 - tX + n$ implies $\mu_{\lambda x} = X^2 - \lambda X + \lambda^2 n$. The remaining assertion follows from Theorem 2.1.5 (iii), because an isomorphism of R-orders is also an isomorphism of R-lattices and the Steinitz class of $\Lambda(\mathfrak{a}, t, n)$ is the ideal class $[\mathfrak{a}]$.

(iii) This follows from the fact that the free R-order $\Lambda' = \Lambda(R, t, n)$ with basis $(1, x)$ satisfies

$$(\Lambda', q) = \begin{bmatrix} 1 & t \\ & n \end{bmatrix}$$

as a quadratic R-lattice. □

Remark 3.1.6. Let $\Lambda(\mathfrak{a}, t, n)$ be an arbitrary quadratic R-order, with $t \neq 0$. Then $\Lambda(\mathfrak{a}, t, n) = \Lambda(t\mathfrak{a}, 1, t^{-2}n)$ from (ii) and, using (i), we have $1 \in t^{-1}\mathfrak{a}^{-1}$. This implies $t\mathfrak{a} \subseteq R$, so $t\mathfrak{a}$ is an integral ideal of R. To summarise, any quadratic R-order Λ can be written as $\Lambda = \Lambda(\mathfrak{a}, t, n)$, with an integral ideal $\mathfrak{a}$, $n \in \mathfrak{a}^{-2}$ and $t \in \{0, 1\}$. However, this explicit form is usually too inflexible for our purposes, so we do not generally assume it.

3.1.1 Equality of quadratic orders

In this subsection we prove that the quadratic R-order $\Lambda = \Lambda(\mathfrak{a}, t, n)$ is uniquely determined by its Steinitz class $[\mathfrak{a}]$ and the R-square class $(t^2 - 4n)(R^\times)$. Equivalently, knowing $\operatorname{disc}(\Lambda)$ is sufficient. The exact result is found in Theorem 3.1.9. This subsection is motivated by the treatment of free quadratic orders in [Kit73].

Using Proposition 3.1.5 (ii), we only need to determine under which conditions on the parameters t, t', n, n', the two quadratic R-orders $\Lambda(\mathfrak{a}, t, n)$ and $\Lambda(\mathfrak{a}, t', n')$ are equal.

Lemma 3.1.7. *Let $t, t' \in \mathfrak{a}^{-1}$ and $n, n' \in \mathfrak{a}^{-2}$. The following statements are equivalent.*

(i) $\Lambda(\mathfrak{a}, t, n) = \Lambda(\mathfrak{a}, t', n')$.

(ii) There is a unit $u \in R^\times$ and some $w \in \mathfrak{a}^{-1}$, such that $t' = u(t - 2w)$ and $n' = u^2(w^2 - wt + n)$.

(iii) There is a unit $u \in R^\times$, such that $(t')^2 - 4n' = u^2(t^2 - 4n)$ and $t' - ut \in 2\mathfrak{a}^{-1}$.

Proof. (i) ⇒ (ii): Write $\Lambda(\mathfrak{a}, t, n) = R1 \oplus \mathfrak{a}x$ and $\Lambda(\mathfrak{a}, t', n') = R1 \oplus \mathfrak{a}y$ with suitable $x, y \in A$. By Proposition 2.1.7, as $R1 \oplus \mathfrak{a}x = R1 \oplus \mathfrak{a}y$, there are $u \in R$ and $w' \in \mathfrak{a}^{-1}$ such that

$$(1, y) = (1, x) \begin{pmatrix} 1 & w' \\ 0 & u \end{pmatrix}.$$

The determinant of this matrix, this is u, must satisfy $\mathfrak{a} = u\mathfrak{a}$ whence $u \in R^\times$. Thus, $w := -w'u^{-1} \in \mathfrak{a}^{-1}$ and $y = u(x - w)$. An easy computation verifies $t' = t(y) = u(t - 2w)$ and $n' = n(y) = u^2(w^2 - wt + n)$.

(ii) ⇒ (i): Note that $y := u^{-1}x + w$ defines an isomorphism of R-orders $\Lambda(\mathfrak{a}, t, n) \cong \Lambda(\mathfrak{a}, t', n')$, so we have the desired equality. As a side note, this isomorphism is induced by the inverse of the matrix used in the first part of this proof.

(ii) ⇒ (iii): This just a straightforward computation.

(iii) ⇒ (ii): Put $w := \frac{1}{2}(t - u^{-1}t')$. Then $w \in u^{-1}\mathfrak{a}^{-1} = \mathfrak{a}^{-1}$ since $u \in R^\times$ is a unit and

$$u(t - 2w) = u(t - (t - u^{-1}t')) = t'.$$

Moreover, using this identity and the assumption, we compute

$$\begin{aligned} 4n' &= (t')^2 - u^2(t^2 - 4n) = u^2(t - 2w) - u^2(t^2 - 4n) \\ &= u^2(t^2 - 4wt + 4w^2 + 4n - t^2) = 4u^2(w^2 - wt + n) \end{aligned}$$

whence (ii) holds. □

Lemma 3.1.8. *Keep the notation from Lemma 3.1.7. Then $\Lambda(\mathfrak{a}, t, n) = \Lambda(\mathfrak{a}, t', n')$, if and only if there is a unit $u \in R^\times$ such that $(t')^2 - 4n' = u^2(t^2 - 4n)$, i.e. the second condition in Lemma 3.1.7 (iii) can be dropped.*

Proof. Let $u \in R^\times$ with $(t')^2 - 4n' = u^2(t^2 - 4n)$. We are done if we show $t' - ut \in 2\mathfrak{a}^{-1}$. By Proposition 3.1.5 (ii), we have $\Lambda(\mathfrak{a}, t, n) = \Lambda(u\mathfrak{a}, u^{-1}t, u^{-2}n)$, so we may assume that $u = 1$. Then the assumed equality simplifies to

$$(t + t')(t - t') = 4(n - n') \in 4\mathfrak{a}^{-2}.$$

Write $2\mathfrak{a}^{-1} = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_n^{e_n}$ uniquely as a product of powers of prime ideals $\mathfrak{p}_i^{e_i}$. For $i \in \underline{n}$ and $s \in R$, let $\nu_i(s)$ denote the valuation of s at the prime $\mathfrak{p}_i$, i.e. the exponent of the prime factor $\mathfrak{p}_i$ in the unique factorisation of sR. Put $f_i := \nu_i(t + t')$. Then $(t + t')(t - t') \in 4\mathfrak{a}^{-2}$ implies $2e_i \leq f_i + \nu_i(t - t')$. We are finished with the proof if we show that $\nu_i(t-t') \geq e_i$. If $f_i \leq e_i$ then $\nu_i(t-t') \geq 2e_i - f_i \geq e_i$. If $f_i > e_i$ then, by the usual properties of valuations and due to $2t' \in 2\mathfrak{a}^{-1}$, we find

$$\nu_i(t - t') = \nu_i(t + t' - 2t') \geq \min\{f_i, \nu_i(2t')\} \geq e_i,$$

so we are done. □

We state the following summarising result.

Theorem 3.1.9. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of R, $t \in \mathfrak{a}^{-1}, t' \in \mathfrak{b}^{-1}, n \in \mathfrak{a}^{-2}, n' \in \mathfrak{b}^{-2}$. The following are equivalent.*

(i) $\Lambda(\mathfrak{a}, t, n) = \Lambda(\mathfrak{b}, t', n')$.

(ii) $[\mathfrak{a}] = [\mathfrak{b}]$ *in* $\mathfrak{Cl}(R)$*, and for any* $\lambda \in K^\times$ *with* $\mathfrak{b} = \lambda\mathfrak{a}$*, we have*

$$\lambda^2((t')^2 - 4n')(R^\times)^2 = (t^2 - 4n)(R^\times)^2.$$

(iii) $\operatorname{disc}\big(\Lambda(\mathfrak{a}, t, n), q\big) = \operatorname{disc}\big(\Lambda(\mathfrak{b}, t', n'), q\big)$.

Proof. (i) ⇒ (iii): This is obvious.

(iii) ⇒ (ii): By assumption, we have the equality

$$\left(\mathfrak{a}^2(t^2 - 4n), (t^2 - 4n)\left(K^\times\right)^2\right) = \left(\mathfrak{b}^2((t')^2 - 4n'), ((t')^2 - 4n')\left(K^\times\right)^2\right).$$

Comparing the respective second components, there is some $\lambda \in K^\times$, unique up to multiplication with elements in $R^\times$, such that $\lambda^2((t')^2 - 4n') = t^2 - 4n$. Applying this to the respective first components yields

$$\lambda^2\mathfrak{a}^2((t')^2 - 4n') = \mathfrak{a}^2(t^2 - 4n) = \mathfrak{b}^2((t')^2 - 4n')$$

as ideals. Thus, by the unique ideal factorisation in Proposition 2.1.4 (ii), the equality $\mathfrak{b} = \lambda\mathfrak{a}$ holds whence (ii).

(ii) ⇒ (i): Let $\lambda \in K^\times$ with $\mathfrak{b} = \lambda\mathfrak{a}$ and

$$\lambda^2((t')^2 - 4n')(R^\times)^2 = (t^2 - 4n)(R^\times)^2.$$

Using Proposition 3.1.5 (ii), we have $\Lambda(\mathfrak{b}, t', n') = \Lambda(\lambda\mathfrak{a}, t', n') = \Lambda(\mathfrak{a}, \lambda t', \lambda^2 n')$. Thus, $t, \lambda t' \in \mathfrak{a}^{-1}$, $n, \lambda^2 n' \in \mathfrak{a}^{-2}$ satisfy the first condition of Lemma 3.1.7 (iii). The second condition can be dropped due to Lemma 3.1.8. Hence

$$\Lambda(\mathfrak{a}, t, n) = \Lambda(\mathfrak{a}, \lambda t', \lambda^2 n') = \Lambda(\mathfrak{b}, t', n')$$

by Lemma 3.1.7. □

Example 3.1.10. The condition that R is a Dedekind domain cannot be weakened in an obvious way. Let $R = \mathbb{Z}[\sqrt{5}]$. Then, by abuse of notation, $\Lambda(R, 1, 0) = \mathbb{H}(R) \not\cong \Lambda(R, \sqrt{5}, 1)$. However, if instead $R = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ then these orders are isomorphic, as can be witnessed by using the isomorphism induced by the matrix

$$T = \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(R).$$

Note that both orders have discriminant $\big(R, 1(K^\times)^2\big)$, where $K = \mathbb{Q}(\sqrt{5})$.

3.1.2 Orthogonal suborder and quadratic discriminant

In certain situations, see e.g. Theorem 3.4.5 and Theorem 3.4.17, it is sufficient to determine the so-called maximal orthogonal sublattice of the centroid instead of the centroid itself. It is also easier to compute and can be described in terms of the quadratic discriminant, which we introduce in Definition 3.1.15. With this in mind, we continue in the abstract framework of quadratic orders.

Definition 3.1.11. Call a quadratic order Λ **orthogonal**, if it can be written as $\Lambda = R1 \oplus \mathfrak{a}y$ with some ideal $\mathfrak{a}$ of R and $y \in A$ such that $t(y) = 0$. This is equivalent to $\Lambda = \Lambda(\mathfrak{a}, 0, n)$, for some suitable $n \in \mathfrak{a}^{-2}$.

The following result states precisely when a quadratic R-order Λ is orthogonal and if it is not, it gives us the index of the *maximal orthogonal suborder* $\Lambda^o \subseteq \Lambda$.

Theorem 3.1.12. *In our usual notation, let $\Lambda = \Lambda(\mathfrak{a}, t, n) = R1 \oplus \mathfrak{a}x$ be a quadratic R-order in A. Define the rank one sublattices*

$$\Lambda_+ = \{l \in \Lambda \mid \beta(l) = l\} \text{ and } \Lambda_- = \{l \in \Lambda \mid \beta(l) = -l\}.$$

(i) $\Lambda_+ = K1 \cap \Lambda = R1$ and $\Lambda_- = \Lambda \cap K(t - 2x)$ are R-pure sublattices of Λ.

(ii) $\Lambda^o := \Lambda_+ \oplus \Lambda_-$ is the unique maximal orthogonal suborder of Λ.

(iii) $\Lambda^o = \Lambda(\mathfrak{a}_t, 0, -(t^2-4n)) = \Lambda(2\mathfrak{a}_t, t, n) = R1 \oplus 2\mathfrak{a}_t x$ *with* $\mathfrak{a}_t = t^{-1}R \cap \frac{1}{2}\mathfrak{a}$, *if* $t \neq 0$; *and* $\mathfrak{a}_t = \frac{1}{2}\mathfrak{a}$, *if* $t = 0$. *Thus,* $2\mathfrak{a}_t \subseteq \mathfrak{a} \subseteq \mathfrak{a}_t$ *and*

$$[\Lambda : \Lambda^o] = 2\mathfrak{a}_t\mathfrak{a}^{-1} = \begin{cases} 2(t\mathfrak{a})^{-1} \cap R, & t \neq 0 \\ R, & t = 0 \end{cases}.$$

In particular, $2R \subseteq [\Lambda : \Lambda^o] \subseteq R$ *with* $\Lambda = \Lambda^o$, *if and only if* $t \in 2\mathfrak{a}^{-1}$, *so the orthogonality of a quadratic order depends only on the primes dividing* $2R$.

Proof. (i) The elements $1, t-2x \in A$ span the one-dimensional eigenspaces associated to the eigenvalues $1, -1$ of the involution β.

(ii) Putting $y \coloneqq t - 2x$, we find $t(y) = 2t - 2t(x) = 0$, so the suborder Λ^o is orthogonal. Now, if $\Lambda' = R1 \oplus \mathfrak{b}z$ is another orthogonal suborder of Λ then $\mathfrak{b}z \subseteq Kz = Ky$, so $\mathfrak{b}z \subseteq \Lambda \cap Ky = \Lambda_-$ and hence, $\Lambda' \subseteq R1 \oplus \Lambda_- = \Lambda^o$.

(iii) Let $\alpha \in K$ and note that $\alpha y = \alpha t - 2\alpha x$ is contained in Λ, if and only if $\alpha t \in R$ and $2\alpha \in \mathfrak{a}$, which is equivalent to $\alpha \in \mathfrak{a}_t$ whence $\Lambda_- = \mathfrak{a}_t y$. Using this and $n(y) = 4n - t^2$, we compute the discriminant of $\Lambda^o = R1 \oplus \mathfrak{a}_t y = \Lambda(\mathfrak{a}_t, 0, 4n-t^2)$ to be

$$\operatorname{disc}(\Lambda^o) = \left(-4\mathfrak{a}_t^2 n(y), -4n(y)(K^\times)^2\right) = \left(-(2\mathfrak{a}_t)^2 n(y), -n(y)(K^\times)^2\right).$$

The latter coincides with the discriminant of the R-order $\Lambda(2\mathfrak{a}_t, t, n) = R1 \oplus 2\mathfrak{a}_t x$, so the claimed equalities of quadratic orders follow from Theorem 3.1.9. Finally, $\Lambda/\Lambda^o = \mathfrak{a}/2\mathfrak{a}_t$, so the remaining claims follow from a straightforward computation.

□

Being orthogonal is a local property

We briefly consider the completions of the Dedekind domain R. The following results use Notation 2.2.8.

Lemma 3.1.13. *Let* $\mathfrak{p}$ *be a finite place of* R, *denote its associated valuation by* $\nu_\mathfrak{p}$ *and let* $\pi_\mathfrak{p} \in \hat{R}_\mathfrak{p}$ *be a fixed uniformiser of* $\hat{R}_\mathfrak{p}$. *Then* $\Lambda(\mathfrak{a}, t, n)_\mathfrak{p} = \Lambda(\hat{R}_\mathfrak{p}, \pi_\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})} t, \pi_\mathfrak{p}^{2\nu_\mathfrak{p}(\mathfrak{a})} n)$.

Proof. Write $\Lambda \coloneqq \Lambda(\mathfrak{a}, t, n)$. We have the chain of equations of quadratic $\hat{R}_\mathfrak{p}$-orders

$$\Lambda_\mathfrak{p} = \hat{R}_\mathfrak{p} \otimes_R \Lambda(\mathfrak{a}, t, n) = \Lambda(\mathfrak{a}\hat{R}_\mathfrak{p}, t, n) = \Lambda(\pi_\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}\hat{R}_\mathfrak{p}, t, n) = \Lambda(\hat{R}_\mathfrak{p}, \pi_\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})} t, \pi_\mathfrak{p}^{2\nu_\mathfrak{p}(\mathfrak{a})} n).$$

The first equality follows from the distributive properties of the tensor product: If $\Lambda = R1 \oplus \mathfrak{a}x$ with some $x \in K[X]/(X^2-d)$ that satisfies $x^2 - tx + n$, then $\hat{R}_\mathfrak{p} \otimes_R \Lambda = \hat{R}_\mathfrak{p}1 \oplus \mathfrak{a}\hat{R}_\mathfrak{p}x$. The second equality follows from $\mathfrak{a}\hat{R}_\mathfrak{p} = \pi_\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}\hat{R}_\mathfrak{p}$ and the last one from Proposition 3.1.5 (ii). □

Proposition 3.1.14. *The quadratic R-order Λ is orthogonal, if and only if $\Lambda_\mathfrak{p}$ is orthogonal at all primes $\mathfrak{p}$ of R that divide $2R$, if and only if $\Lambda_\mathfrak{p}$ is orthogonal at all places $\mathfrak{p}$ of R.*

Proof. The second equivalence is obvious. For the first one, write $\Lambda = \Lambda(\mathfrak{a}, t, n)$. By Theorem 3.1.12 (iii), $\Lambda^o = \Lambda$, if and only if $t \in 2\mathfrak{a}^{-1}$, if and only if $\nu_\mathfrak{p}(t) \geq \nu_\mathfrak{p}(2) - \nu_\mathfrak{p}(\mathfrak{a})$, for all dyadic primes $\mathfrak{p}$. Similarly, for these dyadic primes, we have

$$(\Lambda_\mathfrak{p})^o = \Lambda_\mathfrak{p} = \Lambda\left(\hat{R}_\mathfrak{p}, \pi_\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}t, \pi_\mathfrak{p}^{2\nu_\mathfrak{p}(\mathfrak{a})}n\right),$$

if and only if $\pi_\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}t \in 2\hat{R}_\mathfrak{p}^{-1} = 2\hat{R}_\mathfrak{p}$, again using Theorem 3.1.12 (iii). This is also equivalent to $\nu_\mathfrak{p}(t) \geq \nu_\mathfrak{p}(2) - \nu_\mathfrak{p}(\mathfrak{a})$, which is equivalent to Λ being orthogonal. □

The quadratic discriminant

In Theorem 3.1.12, we established that any quadratic R-order contains a unique maximal orthogonal suborder. Thus, we can consider its discriminant as an isometry invariant.

Definition 3.1.15. In the notation of Theorem 3.1.12, the **quadratic discriminant** of a quadratic R-order $\Lambda = \Lambda(\mathfrak{a}, t, n)$ is defined as

$$\operatorname{disq}(\Lambda) \coloneqq \frac{1}{4}\operatorname{disc}(\Lambda^o) = \left(\mathfrak{a}_t^2(t^2-4n), (t^2-4n)(K^\times)^2\right).$$

Later, in Definition 3.2.1, we provide a more general definition of the quadratic discriminant that is applicable to arbitrary even R-lattices. In Theorem 3.2.11, we prove that the quadratic discriminant has the same multiplicative properties as the usual discriminant (see Proposition 2.1.16 (ii)). This is also the reason for the scalar $\frac{1}{4}$ in Definition 3.2.1; it ensures this multiplicativity. For the moment, we just state the following lemma.

Lemma 3.1.16. *Let $\mathfrak{a}_1, \mathfrak{a}_2$ be ideals of R and $t_1, t_2 \in K$, possibly zero. Further, put $\mathfrak{b} \coloneqq t_1^{-1}\mathfrak{a}_2 \cap t_2^{-1}\mathfrak{a}_1 \cap \frac{1}{2}\mathfrak{a}_1\mathfrak{a}_2$, omitting the sets $t_1^{-1}\mathfrak{a}_2$ and $t_2^{-1}\mathfrak{a}_1$ in this intersection, if the associated t_i is zero. Then, in the notation of Theorem 3.1.12, $\mathfrak{b}_{t_1t_2} = (\mathfrak{a}_1)_{t_1}(\mathfrak{a}_2)_{t_2}$.*

Proof. Put $t := t_1t_2$. If $t_1, t_2 \neq 0$ are both non-zero then

$$\begin{aligned}\mathfrak{b}_t &= t^{-1}R \cap \frac{1}{2}\mathfrak{b} = t_1^{-1}t_2^{-1}R \cap \frac{1}{2}t_1^{-1}\mathfrak{a}_2 \cap \frac{1}{2}t_2^{-1}\mathfrak{a}_1 \cap \frac{1}{4}\mathfrak{a}_1\mathfrak{a}_2 \\ &= t_1^{-1}\left(t_2^{-1}R \cap \frac{1}{2}\mathfrak{a}_2\right) \cap \frac{1}{2}\mathfrak{a}_1\left(t_2^{-1}R \cap \frac{1}{2}\mathfrak{a}_2\right) = t_1^{-1}(\mathfrak{a}_2)_{t_2} \cap \frac{1}{2}\mathfrak{a}_1(\mathfrak{a}_2)_{t_2} \\ &= \left(t_1^{-1}R \cap \frac{1}{2}\mathfrak{a}_1\right)(\mathfrak{a}_2)_{t_2} = (\mathfrak{a}_1)_{t_1}(\mathfrak{a}_2)_{t_2},\end{aligned}$$

proving the claim in this special case. If $t_i = 0$ then one uses the same computation, but has to omit each ideal in which t_i occurs. □

Remark 3.1.17. For any quadratic R-order Λ, the first coordinate of $\operatorname{disq}(\Lambda)$ is an integral ideal.

Proof. Since $\operatorname{disq}(\Lambda) = \operatorname{disq}(\Lambda^o)$ we may without loss of generality assume that $\Lambda = \Lambda(\mathfrak{a}, 0, n)$ is orthogonal. Now $\operatorname{disq}(\Lambda) = (\mathfrak{a}^2 n, -n(K^\times)^2)$ and $\mathfrak{a}^2 n$ is integral due to $n \in \mathfrak{a}^{-2}$. □

3.2 Properties of the centroid

The previous section provides us with sufficient tools to give a systematic treatment of the centroid of Clifford orders of even R-lattices over a Dedekind domain. In this section, we first define the quadratic discriminant for such a lattice and also record some basic results on their centroids. We follow this up by studying the centroids of orthogonal direct sums of even R-lattices. Afterwards, we describe the centroid of an arbitrary maximal R-lattice. Finally, we prove that any two R-lattices in the same *spinor genus* have the same centroid. The main results of this section are Theorem 3.2.11, Corollary 3.2.17 and Theorem 3.2.20.

Throughout this section, (L, q) denotes an even R-lattice of rank n with ambient space (V, q).

3.2.1 The quadratic discriminant of a lattice

In this subsection, additionally assume that L is non-degenerate.

Definition 3.2.1. By Remark 3.0.1, the centroid $\mathcal{Z}(L, q)$ is an R-order in the étale algebra $\mathcal{Z}(V, q) \cong K[X]/(X^2 - d)$, for some $d \in K^\times$. Define the **quadratic discriminant** of L as $\operatorname{disq}(L) := \operatorname{disq}(\mathcal{Z}(L, q))$.

Remark 3.2.2. One has $\mathcal{Z}(L, q)^o = \mathcal{Z}(L', q')^o$, if and only if $\operatorname{disq}(L) = \operatorname{disq}(L')$. Thus, knowing the quadratic discriminant is equivalent to knowing the maximal orthogonal sublattice of the centroid.

Proof. This immediately follows from Theorem 3.1.9 (iii) and the definition of the quadratic discriminant. □

Remark 3.2.3. Let R be a principal ideal domain. Then $\mathcal{Z}(L,q)^o$ is free, so we can identify $\operatorname{disq}(L)$ with an R-square class, by Remark 2.1.17. More precisely, if d is any representative of the K-square class $\operatorname{disc}(V)$, for n even; or of $\operatorname{disc}'(V)$, for n odd, then $\mathcal{Z}(L,q)^o \cong R[X]/(X^2-\lambda^2 d)$, for some $\lambda \in K^\times$. In this situation, $\operatorname{disq}(L) = \lambda^2 d(R^\times)^2$ under the aforementioned identification.

We provide the version of Remark 1.2.19 over Dedekind domains, that describes $\mathcal{Z}(L,q)$ in small dimensions.

Remark 3.2.4. (i) If $n=1$ with $L=\mathfrak{a}x$, put $b \coloneqq q(x)$. Then $\mathcal{Z}(L,q) = \mathcal{C}(L) = R1 \oplus \mathfrak{a}x$ with $x^2-b=0$. Thus, $\mathcal{Z}(L,q) = \Lambda(\mathfrak{a},0,-b)$ and $\operatorname{disq}(L) = (\mathfrak{a}^2 b, b(K^\times)^2)$.

(ii) If $n=2$ with $L=\mathfrak{a}x \oplus \mathfrak{b}y$, put $a \coloneqq q(x), c \coloneqq q(y), b \coloneqq b_q(x,y)$ and $z \coloneqq xy \in \mathcal{C}_0(L)$. Then $\mathcal{Z}(L,q) = \mathcal{C}_0(L) = R1 \oplus \mathfrak{a}\mathfrak{b}z$ with $z^2-bz+ac=0$. Thus, $\mathcal{Z}(L,q) = \Lambda(\mathfrak{a}\mathfrak{b},b,ac)$ and $\operatorname{disq}(L) = \big(((\mathfrak{a}\mathfrak{b})_b)^2(b^2-4ac),(b^2-4ac)(K^\times)^2\big)$.

Example 3.2.5. Let $\mathfrak{a}$ be an integral ideal of R. Then the $\mathfrak{a}$-maximal R-lattices $\mathbb{H}_\mathfrak{b}(\mathfrak{a})$ from Proposition 2.1.32 are even and, using Remark 3.2.4 (ii), we find that $\mathcal{Z} \coloneqq \mathcal{Z}(\mathbb{H}_\mathfrak{b}(\mathfrak{a})) = \Lambda(\mathfrak{a},1,0)$ is an R-order in $K[X]/(X^2-1)$. Putting $\mathfrak{c} \coloneqq R \cap \frac{1}{2}\mathfrak{a}$, we have $\mathcal{Z}^o = \Lambda(\mathfrak{c},0,-1)$ and $\operatorname{disq}(\mathbb{H}_\mathfrak{b}(\mathfrak{a})) = \big(\mathfrak{c}^2, 1(K^\times)^2\big)$. In particular, if $2R \subseteq \mathfrak{a} \subseteq R$, then $\mathfrak{c}=R$ and $\mathcal{Z}^o = \Lambda(R,0,-1) \cong R[X]/(X^2-1)$.

3.2.2 The centroid of an orthogonal direct sum

In this subsection, we investigate the centroid of an orthogonal direct sum of R-lattices. This has two components to it. On one hand, by Proposition 2.1.19, every even R-lattice L admits a unique radical splitting $L = N \perp L^\perp$ with N non-degenerate. In this case, we can ask for $\mathcal{Z}(L,q)$ in terms of N. On the other hand we can ask for the centroid of the orthogonal direct sum of two non-degenerate even R-lattices $L = L_1 \perp L_2$. Here, we can additionally ask for the quadratic discriminant $\operatorname{disq}(L)$ in terms of $\operatorname{disq}(L_1)$ and $\operatorname{disq}(L_2)$. We answer these questions in Lemma 3.2.7 and the aforementioned Theorem 3.2.11. Beyond this, in Theorem 3.2.14, we give a precise result under which conditions the centroid of the orthogonal direct sum of two lattices is orthogonal in the sense of Definition 3.1.11.

Lemma 3.2.6. *The unique K-algebra automorphism of $\mathcal{Z}(V,q)$ from Lemma 1.2.20 restricts to an R-automorphism α of $\mathcal{Z}(L,q)$, such that*

$$Z(\mathcal{C}(L)) = \{z \in \mathcal{Z}(L,q) \mid \alpha(z) = \alpha\}.$$

Proof. The claimed equality holds for $Z(\mathcal{C}(V))$ instead of $Z(\mathcal{C}(L))$, by Lemma 1.2.20. Intersecting both sides with $\mathcal{C}(L)$ gives the result. □

We give a technical description of the centroid of an orthogonal direct sum. In its core it is just the version of [KS02, II (7.8)] over Dedekind domains. The proof is similar.

Lemma 3.2.7. *Let (L,q) be an even R-lattice in the quadratic K-space (V,q) with a decomposition $(L,q)=(L_1,q_1)\perp(L_2,q_2)$.*

(i) $\mathcal{Z}:=\mathcal{Z}(L,q)\subseteq\mathcal{Z}(L_1,q_1)\,\widetilde{\otimes}\,\mathcal{Z}(L_2,q_2)=:\mathcal{Z}_1\,\widetilde{\otimes}\,\mathcal{Z}_2$ *under the canonical isomorphism of R-algebras $\mathcal{C}(L)\cong\mathcal{C}(L_1)\,\widetilde{\otimes}\,\mathcal{C}(L_2)$ from Proposition 1.2.11.*

(ii) If L_1,L_2 are both non-degenerate then

$$\mathcal{Z}=\{z\in\mathcal{Z}_1\,\widetilde{\otimes}\,\mathcal{Z}_2\mid(\alpha_1\otimes\gamma_2)(z)=(\gamma_1\otimes\alpha_2)(z)\},$$

where α_i denotes the automorphism of $\mathcal{Z}(L_i,q_i)$ from Lemma 3.2.6 and $\gamma_i=\mathcal{C}(L_i)(-1)$ (see Example 1.2.9). Moreover, the automorphism α of $\mathcal{Z}$ is the common restriction of $\alpha_1\otimes\gamma_2$ and $\gamma_1\otimes\alpha_2$.

(iii) If L_1 is non-degenerate and L_2 is singular, e.g. $L_2=L^\perp$, then

$$\mathcal{Z}=\{z\in\mathcal{Z}_1\,\widetilde{\otimes}\,\mathcal{Z}_2\mid(\alpha_1\otimes\mathrm{id})(z)=(\gamma_1\otimes\mathrm{id})(z)\}=1\,\widetilde{\otimes}\,\Lambda(L_2),$$

with α_1,γ_1 as in (ii).

Proof. For assertion (i), one uses the same argument as in the first part of the proof of [KS02, II (7.8)], but has to replace 'basis' by 'pseudo-basis' to see that

$$\mathcal{Z}=\{z\in\mathcal{Z}_1\,\widetilde{\otimes}\,\mathcal{Z}_2\mid z(x_1\,\widetilde{\otimes}\,x_2)=(x_1\,\widetilde{\otimes}\,x_2)z,\ \text{for all } x_i\in L_i\}.$$

This proves (i). Assuming that L_1,L_2 are both non-degenerate, we have the automorphisms α_1,α_2 available. Writing $z=z_1\,\widetilde{\otimes}\,z_2$, we compute

$$\begin{aligned}(x_1\,\widetilde{\otimes}\,x_2)(z_1\,\widetilde{\otimes}\,z_2)&=(x_1\gamma_1(z_1)\,\widetilde{\otimes}\,\alpha_2(z_2)x_2)\\&=((\alpha_1\circ\gamma_1)(z_1)\,\widetilde{\otimes}\,(\gamma_2\circ\alpha_2)(z_2))(x_1\,\widetilde{\otimes}\,x_2),\end{aligned}$$

so $z\in\mathcal{Z}$, if and only if $(\alpha_1\otimes\gamma_2)(z)=(\gamma_1\otimes\alpha_2)(z)$, since α_i and γ_i are involutions respectively. For assertion (iii), we can do the same computation if L_2 is singular but have to replace α_2 by γ_2. This yields $z\in\mathcal{Z}$, if and only if $\alpha_1(z_1)=\gamma_1(z_1)$. Now, by [KS02] (7.9) b) and Lemma 3.2.6, α_1 is the identity, if $n_1:=\mathrm{rank}(L_1)$ is odd and the subset of $\mathcal{Z}_1$ with even grading is precisely R. If, on the other hand, n_1 is even then $\mathcal{Z}_1 \subseteq \mathcal{C}_0(L_1)$, so γ_1 is the identity on $\mathcal{Z}_1$ and $\alpha_1(z_1) = z_1$. By Lemma 3.2.6, this yields $\mathcal{Z}_1 = Z(\mathcal{C}(L))$ and this is again equal to R, by [KS02] (7.9) a). Finally, as in Example 1.2.18, we have $\mathcal{Z}_2 = \mathcal{C}(L_2) = \Lambda(L_2)$, the latter denoting the exterior algebra on L_2. □

Note that the third assertion answers the first question that we posed in this subsection. Thus, in the following, we focus on non-degenerate lattices. Then, to make effective use of Lemma 3.2.7(ii), we need the following fact.

Proposition 3.2.8. *Let (L, q) be non-degenerate and let α be the automorphism of $\mathcal{Z} := \mathcal{Z}(L, q)$ from Lemma 3.2.6. Additionally, put $\gamma := \mathcal{C}(L)(-\mathrm{id})$ and let β denote the unique non-trivial automorphism of $\mathcal{Z}(KL, q)$.*

(i) If the rank n is even, then $\mathcal{Z} \subseteq \mathcal{C}_0(L)$, so the restriction $\gamma_{|\mathcal{Z}}$ is the identity and $\beta = \alpha$.

(ii) If the rank n is odd, then $\mathcal{Z} = \mathcal{Z}^o$ is orthogonal. Thus, α is the identity, so $\mathcal{Z} = Z(\mathcal{C}(L))$ and $\beta = \gamma$. More precisely, $\mathcal{Z} = R1 \oplus \mathfrak{a}z$ with an ideal $\mathfrak{a}$ of R and some $z \in \mathcal{C}_1(L)$, satisfying $t(z) = 0$.

Proof. Assertion (i) follows from [KS02] (7.9) a), intersecting with $\mathcal{C}(L)$ and restricting the automorphisms to $\mathcal{Z}$. To see (ii), suppose that n is odd. Then $\mathcal{Z}(KL, q) = K1 \oplus Kz$ with some $z \in \mathcal{Z}(KL, q)$, that satisfies the desired properties, by [KS02] (7.9) b). If $(\omega_i, \mathfrak{a}_i)_{i \in \underline{n}}$ is a pseudo-basis of L then, as $z \in \mathcal{C}_1(KL)$, there exist unique $\mu_I \in K$, for each $I \subseteq \underline{n}$ with $|I|$ odd such that

$$z = \sum_{\substack{I \subseteq \underline{n}, \\ |I| \text{ odd}}} \mu_I \omega_I.$$

Thus, an arbitrary element in $x \in \mathcal{Z}(KL, q)$ can be uniquely written as

$$x = \lambda_1 + \lambda_2 z = \lambda_1 + \sum_{\substack{I \subseteq \underline{n}, \\ |I| \text{ odd}}} \lambda_2 \mu_I \omega_I$$

with $\lambda_1, \lambda_2 \in K$ and since $(\omega_I, \mathfrak{a}_I)_{I \subseteq \underline{n}}$ is a pseudo-basis of $\mathcal{C}(L)$, we have $x \in \mathcal{Z} = \mathcal{C}(L) \cap \mathcal{Z}(KL, q)$, if and only if $\lambda_1 \in R$ and $\lambda_2 \in \mathfrak{a} := \bigcap_{\mu_I \neq 0} \mu_I^{-1} \mathfrak{a}_I$. Hence, $\mathcal{Z} = R1 \oplus \mathfrak{a}z$ is orthogonal and z has the desired properties. The claim about the automorphisms follows as in (i). □

The proof of part (ii) actually only requires that the coefficient of 1 in z is non-zero, i.e. $\mu_\emptyset \neq 0$. This fact is later used in Algorithm 1.

The next two lemmas give a complete description of the centroid and the quadratic discriminant of the orthogonal direct sum of two non-degenerate lattices. They deal with the different possibilities of the parities that the ranks of the summands may have. Theorem 3.2.11 then summarises these results.

Lemma 3.2.9. *Suppose that the non-degenerate R-lattice (L, q) decomposes as the orthogonal direct sum $(L, q) = (L_1, q_1) \perp (L_2, q_2)$ of non-zero lattices, where L_1 is of even rank. Write $\mathcal{Z}(L_i, q_i) = \Lambda(\mathfrak{a}_i, t_i, n_i)$ and $\mathfrak{b} := t_1^{-1}\mathfrak{a}_2 \cap t_2^{-1}\mathfrak{a}_1 \cap \frac{1}{2}\mathfrak{a}_1\mathfrak{a}_2$, omitting $t_1^{-1}\mathfrak{a}_2$, $t_2^{-1}\mathfrak{a}_1$ in this intersection, if the respective t_i equals zero. Then*

$$\mathcal{Z}(L, q) = \Lambda(\mathfrak{b}, t_1t_2, t_1^2n_2 + t_2^2n_1 - 4n_1n_2),$$

so in particular, $\operatorname{disq}(L) = \operatorname{disq}(L_1)\operatorname{disq}(L_2)$.

Proof. As $\operatorname{rank}(L_1)$ is even, we have $\mathcal{Z}(L_1, q_1) \subseteq \mathcal{C}_0(L_i)$ and thus $\gamma_1 = \mathrm{id}$. Hence, the two orders $\mathcal{Z}(L_i, q_i)$ commute inside the graded tensor product $\mathcal{C}(L) = \mathcal{C}(L_1) \mathbin{\widetilde{\otimes}} \mathcal{C}(L_2)$ and if we write $\mathcal{Z}(L_i, q_i) = \Lambda(\mathfrak{a}_i, t_i, n_i) = R1 \oplus \mathfrak{a}_i z_i$, then

$$\widetilde{\mathcal{Z}} := \mathcal{Z}(L_1, q_1) \mathbin{\widetilde{\otimes}} \mathcal{Z}(L_2, q_2) = R1 \oplus \mathfrak{a}_2(1 \mathbin{\widetilde{\otimes}} z_2) \oplus \mathfrak{a}_1(z_1 \mathbin{\widetilde{\otimes}} 1) \oplus \mathfrak{a}_1\mathfrak{a}_2(z_1 \mathbin{\widetilde{\otimes}} z_2).$$

Using Proposition 3.2.8, we have

$$\alpha_1 \otimes \gamma_2 - \gamma_1 \otimes \alpha_2 = \begin{cases} \beta_1 \otimes \mathrm{id} - \mathrm{id} \otimes \beta_2, & \operatorname{rank}(L_2) \text{ even} \\ \beta_1 \otimes \beta_2 - \mathrm{id} \otimes \mathrm{id}, & \operatorname{rank}(L_2) \text{ odd} \end{cases},$$

so with respect to the above basis of $\widetilde{\mathcal{Z}}$, the matrix of this linear map is

$$\begin{pmatrix} 0 & -t_2 & t_1 & 0 \\ 0 & 2 & 0 & t_1 \\ 0 & 0 & -2 & -t_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & t_1 & 0 \\ 0 & -2 & 0 & -t_1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the first matrix applies, if $\operatorname{rank}(L_2)$ is even. Due to Lemma 3.2.7, a K-basis of the kernel of the respective matrix, say $(1, z)$, is a K-basis of $\mathcal{Z}(KL, q)$. In both cases we may choose $z := t_1(1 \mathbin{\widetilde{\otimes}} z_2) + t_2(z_1 \mathbin{\widetilde{\otimes}} 1) - 2(z_1 \mathbin{\widetilde{\otimes}} z_2)$, because $t_2 = 0$, if L_2 has odd rank. Doing so, we have $\mathcal{Z}(L, q) = R1 \oplus (\widetilde{\mathcal{Z}} \cap Kz) = R1 \oplus \mathfrak{b}z$, with some ideal $\mathfrak{b}$ of R. Comparing the coefficient of z to the pseudo-basis of $\widetilde{\mathcal{Z}}$ above, we must have $t_1 \in \mathfrak{a}_2, t_2 \in \mathfrak{a}_1, 2 \in \mathfrak{a}_1\mathfrak{a}_2$, so $\mathfrak{b} = t_1^{-1}\mathfrak{a}_2 \cap t_2^{-1}\mathfrak{a}_1 \cap \frac{1}{2}\mathfrak{a}_1\mathfrak{a}_2$, possibly omitting

$t_1^{-1}\mathfrak{a}_2$, $t_2^{-1}\mathfrak{a}_1$, as claimed. A straightforward computation shows

$$z^2 - t_1 t_2 z + (t_1^2 n_2 + t_2^2 n_1 - 4n_1 n_2) = 0.$$

Regarding the quadratic discriminants, we have

$$\mathcal{Z}(L_i, q_i)^o = \Lambda((\mathfrak{a}_i)_{t_i}, 0, -(t_i^2 - 4n_i)),\ \mathcal{Z}(L, q)^o = \Lambda(\mathfrak{b}_t, 0, -(t_1^2 - 4n_1)(t_2^2 - 4n_2))$$

with $t = t_1 t_2$. According to Lemma 3.1.16, $\mathfrak{b}_t = (\mathfrak{a}_1)_{t_1}(\mathfrak{a}_2)_{t_2}$, so the claim follows from Proposition 3.1.5 (iii). □

Lemma 3.2.10. *Suppose that the non-degenerate R-lattice (L, q) decomposes as the orthogonal direct sum $(L, q) = (L_1, q_1) \perp (L_2, q_2)$ of two lattices of odd rank. Write $\mathcal{Z}(L_i, q_i) = \Lambda(\mathfrak{a}_i, 0, n_i)$. Then*

$$\mathcal{Z}(L, q) = \Lambda(\mathfrak{a}_1\mathfrak{a}_2, 0, n_1 n_2) = \Lambda\left(\tfrac{1}{4}\mathfrak{a}_1\mathfrak{a}_2, 0, 16 n_1 n_2\right),$$

so in particular, $\mathrm{disq}(L) = -\mathrm{disq}(L_1)\mathrm{disq}(L_2)$.

Proof. The proof of this lemma is identical to the one of Lemma 3.2.9, just with slightly different values. Using the same notation, we get

$$\alpha_1 \otimes \gamma_2 - \gamma_1 \otimes \alpha_2 = \mathrm{id} \otimes \gamma_2 - \gamma_1 \otimes \mathrm{id}$$

and the matrix of this linear map with respect to the basis of $\widetilde{\mathcal{Z}}$ that we used in the proof of Lemma 3.2.9 is $\mathrm{diag}(0, -2, 2, 0)$. The kernel of this matrix is $(1, z)$ with $z = z_1 \mathbin{\widetilde{\otimes}} z_2$, which immediately yields $z^2 = -n_1 n_2$ and thus,

$$\mathcal{Z}(L, q) = \Lambda(\mathfrak{a}_1\mathfrak{a}_2, 0, n_1 n_2) = \Lambda\left(\tfrac{1}{4}\mathfrak{a}_1\mathfrak{a}_2, 0, 16 n_1 n_2\right).$$

The assertion about the quadratic discriminants follows from this equality, using a straightforward computation. □

Theorem 3.2.11. *Suppose that the non-degenerate R-lattice (L, q) decomposes as the orthogonal direct sum $(L, q) = (L_1, q_1) \perp (L_2, q_2)$ of two non-zero lattices and put $s := \mathrm{rank}(L_1)\mathrm{rank}(L_2)$. Write $\mathcal{Z}(L_i, q_i) = \Lambda(\mathfrak{a}_i, t_i, n_i)$ and $\mathfrak{b} := t_1^{-1}\mathfrak{a}_2 \cap t_2^{-1}\mathfrak{a}_1 \cap \frac{1}{2}\mathfrak{a}_1\mathfrak{a}_2$, omitting $t_1^{-1}\mathfrak{a}_2$, $t_2^{-1}\mathfrak{a}_1$ in this intersection, if the respective t_i is equal to zero. Put $t := t_1 t_2$. Then*

$$\mathcal{Z}(L, q) = \Lambda(\mathfrak{b}, t, t_1^2 n_2 + t_2^2 n_1 - (-1)^s 4 n_1 n_2)$$

and t is non-zero, only if both L_1 and L_2 have even rank. Moreover,

$$\mathcal{Z}(L,q)^o = \Lambda\left((\mathfrak{a}_1)_{t_1}(\mathfrak{a}_2)_{t_2}, 0, -(-1)^s(t_1^2-4n_1)(t_2^2-4n_2)\right),$$

so in particular, $\mathrm{disq}(L) = (-1)^s\mathrm{disq}(L_1)\mathrm{disq}(L_2)$.

Remark 3.2.12. This theorem illustrates what we remarked earlier: The maximal orthogonal sublattice is much easier to work with than the centroid itself, as it is essentially multiplicative under orthogonal direct sums.

It follows from Theorem 3.2.11 that $\mathcal{Z}(L_1 \perp L_2, q)$ is orthogonal, if at least one of $\mathcal{Z}(L_i, q_i)$ is orthogonal. In the following, we specify this result.

Lemma 3.2.13. *Let $(L,q) = (L_1,q_1) \perp (L_2,q_2)$ be a non-degenerate R-lattice. Then the centroid $\mathcal{Z}(L,q)$ is orthogonal, if and only if for each dyadic prime $\mathfrak{p}$ of R, at least one of the centroids $\mathcal{Z}(L_i,q_i)_\mathfrak{p}$, $i=1,2$, is orthogonal.*

Proof. For $i=1,2$, write $\mathcal{Z}(L_i,q_i) = \Lambda(\mathfrak{a}_i,t_i,n_i)$ with the standardised form from Remark 3.1.6, so that $\mathfrak{a}_i$ is integral and $t_i \in \{0,1\}$. Then, by Theorem 3.2.11, $\mathcal{Z} := \mathcal{Z}(L,q) = \Lambda(\mathfrak{b}, t, t_1^2 + t_2^2 n_1 - 4n_1n_2)$. Clearly, t equals zero, if and only if at least one of the t_i equals zero, so the claim holds in this case. Otherwise, $t = t_1 = t_2 = 1$ and, by Proposition 3.1.14, $\mathcal{Z}$ is orthogonal, if and only if $\mathcal{Z}_\mathfrak{p}$ is orthogonal for each dyadic prime $\mathfrak{p}$. Fix such a prime $\mathfrak{p}$. By Theorem 3.1.12 (iii), $\mathcal{Z}_\mathfrak{p}$ is orthogonal, if and only if $1 \in 2(\mathfrak{b}\hat{R}_\mathfrak{p})^{-1}$ and this is equivalent to $\nu_\mathfrak{p}(\mathfrak{b}) - \nu_\mathfrak{p}(2) \geq 0$. Now $\mathfrak{b} = \mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \frac{1}{2}\mathfrak{a}_1\mathfrak{a}_2$, so

$$\nu_\mathfrak{p}(\mathfrak{b}) - \nu_\mathfrak{p}(2) = \max\{\nu_\mathfrak{p}(\mathfrak{a}_1), \nu_\mathfrak{p}(\mathfrak{a}_2), \nu_\mathfrak{p}(\mathfrak{a}_1) + \nu_\mathfrak{p}(\mathfrak{a}_2) - \nu_\mathfrak{p}(2)\} - \nu_\mathfrak{p}(2) = \max\{\nu_1, \nu_2, \nu_1 + \nu_2\},$$

with $\nu_i := \nu_\mathfrak{p}(\mathfrak{a}_i) - \nu_\mathfrak{p}(2)$. Clearly, the right-hand side of this equations is non-negative, if and only if at least one of the ν_i is non-negative. By Theorem 3.1.12 (iii), this is equivalent to $\mathcal{Z}(L_i,q_i)_\mathfrak{p}$ being orthogonal. □

Theorem 3.2.14. *Let (L,q) be a non-degenerate R-lattice with a decomposition*

$$(L,q) = (L_1,q_1) \perp \ldots \perp (L_s,q_s).$$

Then the centroid $\mathcal{Z}(L,q)$ is orthogonal, if and only if for each dyadic prime $\mathfrak{p}$ of R, at least one of the centroids $\mathcal{Z}(L_i,q_i)$, $i = 1, \ldots s$, is orthogonal.

Corollary 3.2.15. *Under the assumptions of Theorem 3.2.14, additionally assume that R has only one dyadic prime, e.g. R is a principal ideal domain. Then the centroid $\mathcal{Z}(L,q)$ is orthogonal, if and only if at least one of the centroids $\mathcal{Z}(L_i,q_i)$, $i = 1, \ldots s$ is orthogonal.*

Proof. This follows from Proposition 3.1.14 and Theorem 3.2.14. □

3.2.3 Centroids of maximal lattices

We can use Theorem 3.2.11 to describe the centroid of a maximal lattice (see Definition 2.1.30) over the Dedekind domain R. For this, we first identify the lattices whose centroids behave particularly well under orthogonal direct sums.

Proposition 3.2.16. *Let (L, q) be a non-degenerate R-lattice with centroid $\mathcal{Z}$.*

(i) We have $\mathcal{Z}(L \perp L_1, q \perp q_1) = \mathcal{Z}(L_1, q_1)$, for all non-degenerate even R-lattices (L_1, q_1), if and only if $\mathcal{Z} = \Lambda(R, 1, 0) = \mathbb{H}(R)$ and $\operatorname{rank}(L)$ is even.

(ii) We have $\operatorname{disq}(L \perp L_1) = \operatorname{disq}(L_1)$, for all non-degenerate even R-lattices (L_1, q_1), if and only if $\mathcal{Z} = \Lambda(\mathfrak{a}, 1, 0)$ with an ideal $2R \subseteq \mathfrak{a} \subseteq R$ and $\operatorname{rank}(L)$ is even. In this case, $\mathcal{Z}^o = \mathbb{H}(R)^o = \Lambda(R, 0, -1)$.

Proof. Starting with (i), we use the standardised form of $\mathcal{Z} := \mathcal{Z}(L, q)$ that we briefly discussed in Remark 3.1.6. Thus, write $\mathcal{Z} = \Lambda(\mathfrak{a}, t, n)$ with $\mathfrak{a}$ integral and $t \in \{0, 1\}$. Similarly, let (L_1, q_1) be another non-degenerate even R-lattice with $\mathcal{Z}_1 = \Lambda(\mathfrak{a}_1, t_1, n_1)$ and $t_1 \in \{0, 1\}$ and $\mathfrak{a}_1$ integral. Now, if $\mathcal{Z} = \mathcal{Z}(L \perp L_1, q \perp q_1)$, for all such lattices L_1, then $t = 1$, $n = 0$ and $\operatorname{rank}(L)$ is even, by Theorem 3.2.11. Using this, we find that

$$t_1^{-1}\mathfrak{a} \cap \mathfrak{a}_1 \cap \frac{1}{2}\mathfrak{a}\mathfrak{a}_1 = \mathfrak{a}_1$$

holds (omitting $t_1^{-1}\mathfrak{a}$ for $t_1 = 0$), if and only if $\mathfrak{a}(R \cap \frac{1}{2}\mathfrak{a}_1) \supseteq \mathfrak{a}_1$ (or $2R \subseteq \mathfrak{a}$). Now $\mathfrak{a}$ is integral and $\mathfrak{a}_1$ can be any integral ideal. Choosing $\mathfrak{a}_1 = R$ then forces $\mathfrak{a} = R$. In summary, $\mathcal{Z} = \Lambda(R, 1, 0) = \mathbb{H}(R)$ is the only possibility for the centroid.

For (ii), write $\mathcal{Z} = \Lambda(\mathfrak{a}, t, n)$ and assume that $\operatorname{disq}(L \perp L_1) = \operatorname{disq}(L_1)$, i.e. $(\mathfrak{a}_t)^2(t^2 - 4n) = R$ and $t^2 - 4n \in (K^\times)^2$, by Theorem 3.2.11. This implies that there is some $\lambda \in K^\times$ with $\lambda^2 = t^2 - 4n$ and consequently $(\lambda\mathfrak{a}_t)^2 = R$, forcing $\lambda\mathfrak{a}_t = R$ by the unique ideal factorisation. Applying Proposition 3.1.5 (ii) and Theorem 3.1.12 (iii), we obtain

$$\mathcal{Z}^o = \Lambda(\mathfrak{a}_t, 0, -(t^2 - 4n)) = \Lambda(\lambda\mathfrak{a}_t, 0, -\lambda^{-2}(t^2 - 4n)) = \Lambda(R, 0, -1) = \mathbb{H}(R)^o$$

as the only possibility for the maximal orthogonal sublattice $\mathcal{Z}^o$. In order to determine $\mathcal{Z}$, note that we have the inclusion chain of R-orders $\mathbb{H}(R)^o \subseteq \mathcal{Z} \subseteq \mathbb{H}(R)$, and that $\mathbb{H}(R)$ is the unique maximal order in $K[X]/(X^2 - 1)$. Now, we have $\mathbb{H}(R)/\mathbb{H}(R)^o \cong R/2R$, so $\mathcal{Z} = \Lambda(\mathfrak{a}, 1, 0)$, for some ideal $2R \subseteq \mathfrak{a} \subseteq R$. □

Corollary 3.2.17. *Suppose that the regular quadratic K-space V decomposes as*

$$V = V' \perp \mathop{\perp}_{i=1}^{k} \mathbb{H}(K)$$

with $V' \leq V$ the anisotropic kernel and $k = \operatorname{ind}(V)$.

(i) If $L \leq V$ is maximal then, by Corollary 2.1.37, there is a maximal R-lattice $L' \leq V'$ and ideals $\mathfrak{b}_i$ of R such that

$$L = L' \perp \mathop{\perp}_{i=1}^{k} \mathbb{H}_{\mathfrak{b}_i}(R).$$

In this situation, $\mathcal{Z}(L, q) = \mathcal{Z}(L', q')$.

(ii) If $L \leq V$ is $\mathfrak{a}$-maximal and $2R \subseteq \mathfrak{a} \subseteq R$ then, by Corollary 2.1.37, there is an $\mathfrak{a}$-maximal R-lattice $L' \leq V'$ and ideals $\mathfrak{b}_i$ of R such that

$$L = L' \perp \mathop{\perp}_{i=1}^{k} \mathbb{H}_{\mathfrak{b}_i}(\mathfrak{a}).$$

In this situation, $\operatorname{disq}(L) = \operatorname{disq}(L')$.

Proof. This follows from Example 3.2.5 and Proposition 3.2.16. □

3.2.4 Centroid and spinor genus

Let (V, q) be regular and non-zero. From Proposition 3.2.16 we can derive that the centroids of any two R-lattices in the same spinor genus are isometric. In the following we give an incomplete definition of the spinor genus. It is incomplete, because we do not define the subgroup $O'(V, q)$ of $O(V, q)$; doing so would require too much theory and Theorem 3.2.20 can be understood without it. Instead, we use the definition of $O'(V)$ as presented in §55 in [OMe00].

Definition 3.2.18. Two quadratic R-lattices L, L' in (V, q) are in the same **spinor genus**, if and only if there is some isometry $\sigma \in O(V, q)$ and for each place $\mathfrak{p}$ of R, there is $\tau_{\mathfrak{p}} \in O'(\hat{K}_{\mathfrak{p}} \otimes_K V, q)$, such that $L'_{\mathfrak{p}} = \sigma(\tau_{\mathfrak{p}}(L_{\mathfrak{p}}))$. We denote the spinor genus of L by $\operatorname{spn}(L)$.

Remark 3.2.19. We give a bit of context for Definition 3.2.18 and the general theory connected to it. Spinor genera are studied, because determining R-lattices up to isometry is usually very difficult. The reason is that isometry of R-lattices is not a local property, so an arbitrary quadratic K-space (V, q) may contain R-lattices L, L' that are not isometric, but $L_{\mathfrak{p}} \cong L'_{\mathfrak{p}}$ at all places of $L_{\mathfrak{p}}$. In this case L and L' are said to be in the same *genus*, $\operatorname{gen}(L)$. Clearly, if $\operatorname{cls}(L) = \{L' \subseteq (V, q) \mid L \cong L'\}$, then

$\mathrm{cls}(L) \subseteq \mathrm{spn}(L) \subseteq \mathrm{gen}(L)$ and in many situations (see [OMe00, 104:5]), actually $\mathrm{cls}(L) = \mathrm{spn}(L)$.

Theorem 3.2.20. *If* $\mathrm{spn}(L) = \mathrm{spn}(L')$, *then* $\mathcal{Z}(L,q) \cong \mathcal{Z}(L',q)$.

Proof. In view of Proposition 3.2.16 (i), it is enough to show that $\mathrm{spn}(L) = \mathrm{spn}(L')$ implies $\mathbb{H}(R) \perp L \cong \mathbb{H}(R) \perp L'$. Let σ and $\tau_\mathfrak{p}$ as in Definition 3.2.18. After extending these orthogonal maps to $\hat{\sigma} \in O(\mathbb{H}(K) \perp V)$ and $\hat{\tau}_\mathfrak{p} \in O'(\mathbb{H}(K)_\mathfrak{p} \perp V_\mathfrak{p})$ via the identity map on the respective hyperbolic planes, we have

$$\hat{\sigma}(\hat{\tau}_\mathfrak{p}(\mathbb{H}(R)_\mathfrak{p} \perp L_\mathfrak{p})) = \mathbb{H}(R)_\mathfrak{p} \perp L'_\mathfrak{p}.$$

Thus, $\mathrm{spn}(\mathbb{H}(R) \perp L) = \mathrm{spn}(\mathbb{H}(R) \perp L')$, so $\mathbb{H}(R) \perp L$ and $\mathbb{H}(R) \perp L'$ are in the same spinor genus inside the indefinite, at least three-dimensional regular quadratic K-space $\mathbb{H}(K) \perp V$. This implies $\mathbb{H}(R) \perp L \cong \mathbb{H}(R) \perp L'$, by [OMe00, 104:5]. □

3.3 Quadratic orders revisited

In the previous section, we used the rudimentary notion of a quadratic R-order to derive some basic properties concerning the centroid of a lattice. In this section, we want to consider quadratic R-orders on an abstract level, by defining a monoid structure on the set of isomorphism classes of quadratic R-orders. We call it the **quadratic monoid** of R and denote it by $\mathrm{QM}(R)$. Its definition is strongly motivated by the theory developed in [Hah94]. Here, Hahn studies finitely generated projective separable R-algebras over an arbitrary commutative ring R; note that he uses a more general definition of separability than the one in Definition 2.2.10. He defines a group structure on these algebras that have rank two and shows that they arise precisely as the centroids, Hahn calls them Arf algebras, of regular quadratic R-lattices.

3.3.1 The quadratic monoid

For a quadratic R-order Λ over the Dedekind domain R in characteristic zero, denote its isomorphism class as R-algebra by $[\Lambda]$ and denote the set of all such isomorphism classes by $\mathrm{QM}(R)$. The aim of this section is to define a monoid structure on $\mathrm{QM}(R)$. For this, we define the multiplication of two quadratic R-orders Λ_1, Λ_2 by

$$\Lambda_1 \cdot \Lambda_2 := \{x \in \Lambda_1 \otimes_R \Lambda_2 \mid (\beta_1 \otimes \beta_2)(x) = x\},$$

where β_i denotes the unique non-trivial algebra automorphism of Λ_i, for $i = 1, 2$; see Remark 3.1.2. Writing $\Lambda_i = \Lambda(\mathfrak{a}_i, t_i, n_i) = R1 \oplus \mathfrak{a}_i x_i$, the matrix of β_i with respect to the basis $(1, x_i)$ of the ambient algebra $K\Lambda_i$ is

$$M_i \coloneqq \begin{pmatrix} 1 & t_i \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(R).$$

The 1-eigenspace of $M_1 \otimes M_2$ is the kernel of the matrix

$$M_1 \otimes M_2 - I_4 = \begin{pmatrix} 0 & t_2 & t_1 & t_1t_2 \\ 0 & -2 & 0 & -t_1 \\ 0 & 0 & -2 & -t_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R^{4\times 4}.$$

Intersecting it with $\Lambda_1 \otimes_R \Lambda_2$ yields $(\Lambda_1 \otimes_R \Lambda_2) \cap K\langle 1, z\rangle = R(1\otimes 1) \oplus \mathfrak{a}z$, where

$$z \coloneqq t_1(1\otimes x_2) + t_2(x_1 \otimes 1) - 2(x_1 \otimes x_2) \text{ and } \mathfrak{a} \coloneqq t_1^{-1}\mathfrak{a}_2 \cap t_2^{-1}\mathfrak{a}_1 \cap \tfrac{1}{2}\mathfrak{a}_1\mathfrak{a}_2,$$

omitting the intersections with $t_i^{-1}\mathfrak{a}_j$, if $t_i = 0$. Thus, we are precisely in the situation of the proof of Lemma 3.2.9. From this we obtain the following result.

Lemma 3.3.1. *Let $\Lambda_i \coloneqq \Lambda_i(\mathfrak{a}_i, t_i, n_i)$ be two quadratic R-orders. Then*

$$[\Lambda_1 \cdot \Lambda_2] = [\Lambda(t_1^{-1}\mathfrak{a}_2 \cap t_2^{-1}\mathfrak{a}_1 \cap \tfrac{1}{2}\mathfrak{a}_1\mathfrak{a}_2, t_1t_2, t_1^2n_2 + t_2^2n_1 - 4n_1n_2)] = [\Lambda_2 \cdot \Lambda_1].$$

Moreover, for any quadratic R-order Λ, we have $[\Lambda \cdot \Lambda(R, 1, 0)] = [\Lambda]$.

Proof. The first equality is an immediate consequence of the proof of Lemma 3.2.9 and the expression in the middle is clearly symmetric in Λ_1 and Λ_2. For the last assertion, one can use the proof of Proposition 3.2.16 (i). □

Remark 3.3.2. Let Λ be a quadratic R-order. The invariants $\mathfrak{d}(\Lambda), \mathrm{disc}(\Lambda)$ and $\mathrm{disq}(\Lambda)$ depend only on the isomorphism class of Λ as an R-algebra. In particular, $\mathfrak{d}([\Lambda]), \mathrm{disc}([\Lambda])$ and $\mathrm{disq}([\Lambda])$ are well-defined invariants of the elements of $\mathrm{QM}(R)$.

Lemma 3.3.3. *Let $\Lambda = \Lambda(\mathfrak{a}, t, n)$ be a quadratic R-order with discriminant ideal $\mathfrak{d}(\Lambda) = \mathfrak{a}^2(t^2 - 4n) = R$, i.e. $\mathrm{disc}(\Lambda) \in \{R\} \times K^\times/(K^\times)^2$. Then $\mathfrak{a}_t = \mathfrak{a}$. If in addition $t = 0$, then $\mathfrak{a}_t = \mathfrak{a}$ is equivalent to $2 \in R^\times$.*

Proof. Recall that by definition,

$$\mathfrak{a}_t = \begin{cases} t^{-1}R \cap \tfrac{1}{2}\mathfrak{a}, & t \neq 0 \\ \tfrac{1}{2}\mathfrak{a}, & t = 0 \end{cases}$$

and first consider the case $t = 0$. Clearly, if $2 \in R^\times$, then $\mathfrak{a}_t = \mathfrak{a}$, independently of $\mathfrak{d}(\Lambda)$. Conversely, if $\mathfrak{d}(\Lambda) = R$, then $R = \mathfrak{d}(\Lambda) = 4\mathfrak{a}^2 n \subseteq 4R$, because $n \in \mathfrak{a}^{-2}$. This forces $2 \in R^\times$ and also shows that $\mathfrak{d}(\Lambda) = R$ implies $\mathfrak{a}_t = \mathfrak{a}$, by the first part.

In the remaining case $t \neq 0$, we have $\mathfrak{a}_t = t^{-1}R \cap \frac{1}{2}\mathfrak{a} = \mathfrak{a}$, if and only if $\nu_\mathfrak{p}(\mathfrak{a}t) = 0$, for all primes $\mathfrak{p}$ dividing $2R$. Now, $\mathfrak{a}^2(t^2 - 4n) = R$ implies

$$0 = \nu_\mathfrak{p}(\mathfrak{a}^2(t^2 - 4n)) = \nu_\mathfrak{p}((\mathfrak{a}t)^2 - 4\mathfrak{a}^2 n)) \geq \min\{2\nu_\mathfrak{p}(\mathfrak{a}t), \nu_\mathfrak{p}(4\mathfrak{a}^2 n)\} \geq 0,$$

for all dyadic primes $\mathfrak{p}$. This forces $\nu_\mathfrak{p}(\mathfrak{a}t) = 0$, because either $2 \in R^\times$ and there are no such primes, or $\nu_\mathfrak{p}(4\mathfrak{a}^2 n) \geq \nu_\mathfrak{p}(4) > 0$, so the proof is finished. □

Theorem 3.3.4. $\mathrm{QM}(R)$ *is an abelian monoid via* $[\Lambda_1]\cdot[\Lambda_2] \coloneqq [\Lambda_1 \cdot \Lambda_2]$. *The neutral element is* $1_{\mathrm{QM}(R)} = [\Lambda(R, 1, 0)]$ *and its unit group is the elementary abelian* 2*-group*

$$\mathrm{Qu}(R) \coloneqq \mathrm{QM}(R)^\times = \{[\Lambda] \in \mathrm{QM}(R) \mid \mathfrak{d}([\Lambda]) = R\}, \tag{3.1}$$

the so-called quadratic group of R.

Proof. The product is well-defined due to Proposition 3.1.5 (i). The assertions about the neutral element and the commutativity of the product follow from Lemma 3.3.1. Let $\mathcal{I}(R)$ denote the ideal group of R and let $\mathcal{X} \coloneqq \{[\Lambda] \in \mathrm{QM}(R) \mid \mathfrak{d}([\Lambda]) = R\}$. We first establish $\mathrm{QM}(R)^\times \subseteq \mathcal{X}$. It follows from Theorem 3.2.11 that

$$\mathrm{disq}\colon \mathrm{QM}(R) \to \mathcal{I}(R) \times K^\times/(K^\times)^2,\ [\Lambda] \mapsto \mathrm{disq}\left([\Lambda]\right)$$

is a homomorphism of abelian monoids, which restricts to a group homomorphism $\mathrm{QM}(R)^\times \to \mathcal{I}(R) \times K^\times/(K^\times)^2$. By Remark 3.1.17, the image of this map only contains pairs where the first component is an integral ideal. Thus,

$$\mathrm{disq}(\mathrm{QM}(R)^\times) \subseteq \{R\} \times K^\times/(K^\times)^2.$$

Now let $[\Lambda(\mathfrak{a}, t, n)] \in \mathrm{QM}(R)^\times$ with inverse $[\Lambda(\mathfrak{a}', t', n')]$ and write $d \coloneqq t^2 - 4n$, $d' \coloneqq (t')^2 - 4n'$. Then the above properties of the quadratic discriminant yield

$$(\mathfrak{a}_t)^2 d = R = (\mathfrak{a}'_{t'})^2 d'.$$

Now, using Theorem 3.2.11 we obtain

$$R = \mathfrak{d}(1_{\mathrm{QM}(R)}) = \mathfrak{d}([\Lambda(\mathfrak{a}, t, n) \cdot \Lambda(\mathfrak{a}', t', n')]) = (t\mathfrak{a}' \cap t'\mathfrak{a} \cap \tfrac{1}{2}\mathfrak{a}\mathfrak{a}')^2 dd'.$$

We expand the right-hand side:

$$(t^{-1}\mathfrak{a}' \cap (t')^{-1}\mathfrak{a} \cap \tfrac{1}{2}\mathfrak{a}\mathfrak{a}')^2 dd' = (t^{-1}\mathfrak{a}' \cap \mathfrak{a}(\mathfrak{a}'_{t'}))^2 dd' = \left(t^{-2}(\mathfrak{a}')^2 \cap t^{-1}\mathfrak{a}\mathfrak{a}'\mathfrak{a}'_{t'} \cap \mathfrak{a}^2(\mathfrak{a}'_{t'})^2\right) dd'$$

Because this is an intersection of ideals, $R \subseteq \mathfrak{a}^2(\mathfrak{a}'_{t'})^2 dd' = (\mathfrak{a}^2 d)((\mathfrak{a}'_{t'})^2 d') = \mathfrak{a}^2 d$ must hold. On the other hand, $\mathfrak{a}^2 d = \mathfrak{d}([\Lambda(\mathfrak{a}, t, n)]) \subseteq R$, so in total $\mathfrak{d}([\Lambda(\mathfrak{a}, t, n)]) = R$. Thus, $\mathrm{QM}(R)^\times \subseteq \mathcal{X}$.

For the remaining inclusion, it suffices to show that each $\Lambda(\mathfrak{a}, t, n)$, with $\mathfrak{a}^2(t^2 - 4n) = R$, satisfies $[\Lambda(\mathfrak{a}, t, n)]^2 = 1_{\mathrm{QM}(R)}$. This also establishes that $\mathrm{QM}(R)^\times$ is a 2-group. A direct computation yields $[\Lambda(\mathfrak{a}, t, n)]^2 = [\Lambda(\mathfrak{a}\mathfrak{a}_t, t^2, 2t^2 - 4n^2)]$ and the discriminant of the right-hand side is equal to

$$\left(\mathfrak{a}^2(\mathfrak{a}_t)^2(t^4 - 4(2t^2 n - 4n^2)), t^4 - 4(2t^2 n - 4n^2)(K^\times)^2\right) = \left((\mathfrak{a}\mathfrak{a}_t(t^2 - 4n))^2, 1(K^\times)^2\right).$$

By Lemma 3.3.3, we have $\mathfrak{a}_t = \mathfrak{a}$ and hence $(\mathfrak{a}\mathfrak{a}_t(t^2 - 4n))^2 = (\mathfrak{a}(t^2 - 4n))^2 = R$. To summarise, we have $\mathrm{disc}([\Lambda(\mathfrak{a}, t, n)]^2) = (R, 1(K^\times)^2) = \mathrm{disc}(1_{\mathrm{QM}(R)})$, so $[\Lambda(\mathfrak{a}, t, n)]^2 = 1_{\mathrm{QM}(R)}$ by Theorem 3.1.9. □

Remark 3.3.5. In the case where R is a Dedekind domain of characteristic zero, the group $\mathrm{Qu}(R)$ coincides with that of [Hah94] §12 A. Hahn gives a more general construction of this group, working over an arbitrary commutative ring. He also proves that it is an abelian 2-group.

Proposition 3.3.6. $\mathrm{QM}(R)$ *has the cancellation property, which means that* $ab = ac$ *implies* $b = c$*, for all* $a, b, c \in \mathrm{QM}(R)$*, if and only if* $2 \in R^\times$*. In particular,* $\mathrm{QM}(R)$ *can be embedded into a group, if and only if* $2 \in R^\times$*.*

Proof. A straightforward computation shows $[\Lambda(R, 0, -1)] \cdot [\Lambda] = [\Lambda^o]$, for all $[\Lambda] \in \mathrm{QM}(R)$. In particular, we have

$$[\Lambda(R, 0, -1)] \cdot [\Lambda] = [\Lambda(R, 0, -1)] \cdot [\Lambda^o]. \tag{3.2}$$

Thus, if $2 \notin R$, then choosing $\Lambda = \Lambda(R, 1, 0)$ yields $[\Lambda] \neq [\Lambda^o]$, so $\mathrm{QM}(R)$ does not have the cancellation property. Conversely, if $2 \in R^\times$, $\mathrm{disq}([\Lambda]) = \mathrm{disc}([\Lambda])$, so $\mathrm{disc} : \mathrm{QM}(R) \to \mathcal{I}(R) \times K^\times/(K^\times)^2$ is an injective homomorphism of abelian monoids by Theorem 3.1.9. Because the codomain of this map is a group, it has the cancellation property and so does $\mathrm{QM}(R)$. □

As a consequence of the proofs of Theorem 3.3.4 and Proposition 3.3.6 we record the following result.

Corollary 3.3.7. *The quadratic discriminant induces a homomorphism of abelian monoids*

$$\operatorname{disq}\colon \mathrm{QM}(R) \to \mathcal{I}(R) \times K^\times/(K^\times)^2,\ [\Lambda] \mapsto \operatorname{disq}\left([\Lambda]\right).$$

It is injective, if and only if $2 \in R^\times$.

3.3.2 The orthogonal quadratic monoid

Revisiting Equation 3.2, the element $1^o := [\Lambda(R, 0, -1)] \in \mathrm{QM}(R)$ acts as a neutral element on the subset

$$\mathrm{OQM}(R) := \{[\Lambda] \in \mathrm{QM}(R) \mid \Lambda = \Lambda^o \text{ is orthogonal}\} \subseteq \mathrm{QM}(R).$$

Thus, by restricting the product, $\mathrm{OQM}(R)$ becomes an abelian monoid with neutral element 1^o. We call it the **orthogonal quadratic monoid** of R. Note that $\mathrm{QM}(R) = \mathrm{OQM}(R)$, if and only if $2 \in R^\times$ and otherwise, $\mathrm{OQM}(R)$ is not even a submonoid of $\mathrm{QM}(R)$; the respective neutral elements are distinct. The unit group of $\mathrm{OQM}(R)$ is

$$\mathrm{OQM}(R)^\times = \{[\Lambda] \in \mathrm{OQM}(R) \mid \mathfrak{d}([\Lambda]) = 4R\},$$

because $\mathfrak{d}([\Lambda]) = 4R$ is equivalent to $\operatorname{disq}([\Lambda]) = (R, \lambda(K^\times)^2)$, for some $\lambda \in K^\times$, and because of the following fact.

Remark 3.3.8. The quadratic discriminant induces an injective homomorphism of abelian monoids

$$\operatorname{disq}\colon \mathrm{OQM}(R) \hookrightarrow \mathcal{I}(R) \times K^\times/(K^\times)^2,\ [\Lambda] \mapsto \operatorname{disq}\left([\Lambda]\right).$$

In particular, $\mathrm{OQM}(R)$ has the cancellation property.

Proof. This map being injective is an immediate consequence of Theorem 3.1.12. The assertion about the cancellation property follows because the codomain of this map is a group. □

Proposition 3.3.9. *If M is an abelian monoid, there exists an abelian group $\mathcal{K}$ and a monoid homomorphism $\gamma : M \to \mathcal{K}(M)$, such that for any other abelian group A with a monoid homomorphism $h : M \to A$, there exists a unique group homomorphism $\varphi : \mathcal{K}(M) \to A$ with $h = \varphi \circ \gamma$. The group $\mathcal{K}(M)$ is unique up to isomorphism and called the* ***Grothendieck group*** *of M. The map γ is injective, if and only if M has the cancellation property.*

Proof. This is a well-known construction for abelian monoids. See, for example, [Lan02] p. 39f. □

Performing this construction with the two abelian monoids $\mathrm{QM}(R)$ and $\mathrm{OQM}(R)$ yields the following result.

Theorem 3.3.10. $\mathcal{K} \coloneqq \mathcal{K}(\mathrm{QM}(R)) = \mathcal{K}(\mathrm{OQM}(R))$. *Moreover, the quadratic discriminant yields a well-defined group monomorphism*

$$\mathcal{K} \hookrightarrow \mathcal{I}(R) \times K^\times/(K^\times)^2$$

Proof. For a quadratic R-order $\Lambda = \Lambda(\mathfrak{a}, t, n)$, let $\langle \Lambda \rangle$ denote the element in $\mathcal{K}$ with $\gamma([\Lambda]) = \gamma([\Lambda^o]) = \langle \Lambda \rangle$, where $[\Lambda^o]$ can be regarded as an element of $\mathrm{QM}(R)$ or $\mathrm{OQM}(R)$. Note that in view of Equation 3.2, $\langle \Lambda \rangle = \langle \Lambda^o \rangle$ for each quadratic R-order Λ. Thus, the elements of $\mathcal{K}$, which are equivalence classes of quadratic R-orders, are represented by orthogonal quadratic R-orders. By Remark 3.3.8, any two equivalence classes $\langle \Lambda \rangle$, $\langle \Lambda' \rangle$ coincide, if and only if $\mathrm{disq}(\Lambda) = \mathrm{disq}(\Lambda')$. This implies the assertion about the quadratic discriminant. □

3.4 The orthogonal decomposition

This section is devoted to generalising Theorem 1.2.27 and Theorem 1.2.28 to non-degenerate lattices over the Dedekind domain R, and hence quite technical in nature. We address these theorems in separate subsections because they differ in certain details that require special attention. Before going over to the respective case study, we first describe the universal setting that applies to the generalisations of both theorems. It remains valid throughout this section.

Let (E, q) be an even R-lattice with ambient space V such that $(E, q) = (E_1, q_1) \perp (E_2, q_2)$ with E_1 non-degenerate and $n \coloneqq \mathrm{rank}(E_2) > 0$. Denote the ambient space of E_i by V_i, $i = 1, 2$, so that $V = V_1 \perp V_2$ and write $\mathcal{Z} \coloneqq \mathcal{Z}(E_1, q_1)$. Finally, in this section, we frequently use (graded) tensor products, so we abbreviate $\otimes_R$ by $\otimes$ and $\widetilde{\otimes}_R$ by $\widetilde{\otimes}$, for improved readability.

3.4.1 The orthogonal decomposition for even rank

We begin with the generalisation of Theorem 1.2.27. For this, let $2m \coloneqq \mathrm{rank}(E_1)$ be even, so $\dim(V) = 2m + n$. Using Proposition 1.2.11 and Theorem 1.2.27, we obtain

$$\mathcal{C}(E) \cong \mathcal{C}(E_1) \,\widetilde{\otimes}\, \mathcal{C}(E_2) \subset \mathcal{C}(V_1) \,\widetilde{\otimes}_K\, \mathcal{C}(V_2) \cong \mathcal{C}(V_1) \otimes_K \mathcal{C}(V)^{\mathcal{C}(V_1)} \cong \mathcal{C}(V_1) \otimes_K \mathcal{C}({}^{d'}V_2),$$

where d' is any representative of the K-square class $\mathrm{disc}(V_1)$. This implies that $\mathcal{C}(E)$ and $\mathcal{C}(E_1) \otimes \mathcal{C}(E)^{\mathcal{C}(E_1)}$ are isomorphic to certain R-orders in $\mathcal{C}(V_1) \otimes \mathcal{C}({}^{d'}V_2)$. Thus, we have two objectives: Determine these R-orders up to isomorphism, and find a reasonable substitute for the parameter d'.

By Proposition 3.2.8 (i), an arbitrary element

$$w \in \mathcal{C}(E)^{\mathcal{C}_0(E_1)} \cong \mathcal{Z} \mathbin{\widetilde{\otimes}} \mathcal{C}(E_2) = \mathcal{Z} \otimes \mathcal{C}(E_2)$$

is contained in $\mathcal{C}(E)^{\mathcal{C}(E_1)}$, if and only if it commutes with $x \otimes 1$ for all $x \in E_1$. This is equivalent to $(\alpha_1 \otimes \gamma_2)(w) = w$ with α_1 the automorphism from Lemma 3.2.6 and $\gamma_2 := (\mathcal{C}(E_2))(-\mathrm{id})$. From the field case we know that

$$w \in 1 \otimes \mathcal{C}_0(E_2) \oplus \mathcal{Z}_- \otimes \mathcal{C}_1(E_2)$$

must hold, with $\mathcal{Z}_- \leq \mathcal{Z}$, the rank-one sublattice of elements in $\mathcal{Z}$ that have vanishing trace, as defined in Theorem 3.1.12. Conversely, an easy computation verifies that all elements w in this set satisfy $(\alpha_1 \otimes \gamma_2)(w) = w$, so we get the isomorphism of R-orders

$$\mathcal{C}(E_1) \otimes \mathcal{C}(E)^{\mathcal{C}(E_1)} \cong \mathcal{C}(E_1) \otimes \mathcal{C}_0(E_2) \oplus \mathcal{C}(E_1)\mathcal{Z}_- \otimes \mathcal{C}_1(E_2) \leq \mathcal{C}(E_1) \mathbin{\widetilde{\otimes}} \mathcal{C}(E_2).$$

Writing $\mathcal{Z}_- = \mathfrak{c}z$ with an ideal $\mathfrak{c}$ of R and an element $z \in \mathcal{Z}$, the pseudo-element $\mathfrak{c}z$ is uniquely determined, because $\mathcal{Z}^o = R1 \oplus \mathcal{Z}_-$, by Theorem 3.1.12 (ii). Thus, the set $\mathfrak{d} := \{\lambda^2 z^2 \mid \lambda \in \mathfrak{c}\}$ is also uniquely determined, because putting $d := z^2 \in R$, we have $\operatorname{disq}(E_1) = \big(\mathfrak{c}^2 d, d(K^\times)^2\big)$ and $\mathfrak{d} = \mathfrak{c}^2 d \cap d(K^\times)^2$.

Remark 3.4.1. In the notation above, $d(K^\times)^2 = d'(K^\times)^2$, so $\mathcal{C}({}^{d}V_2) = \mathcal{C}({}^{d'}V_2)$.

We collect some more facts about the set $\mathfrak{d}$, which will serve as our substitute for the parameter d'.

Lemma 3.4.2. *(i) We have $\mathfrak{d} \cap R^\times \neq \emptyset$, if and only if we can choose $\mathfrak{c} = R$ and $z^2 = d \in R^\times$.*

(ii) For each prime $\mathfrak{p}$, of R we have $\nu_\mathfrak{p}(\mathfrak{d}) := 2\nu_\mathfrak{p}(\mathfrak{c}) + \nu_\mathfrak{p}(d) \geq 0$.

(iii) If E_1 is free and we regard $\operatorname{disq}(E_1)$ *as an R-square class, then $\mathfrak{d} = \operatorname{disq}(E_1)R^2$, where $R^2 = \{x^2 \mid x \in R\}$.*

Proof. For the non-trivial direction of assertion (i), let $u \in \mathfrak{d} \cap R^\times$. Then there is some non-zero $\lambda \in \mathfrak{c}$, such that $u = \lambda^2 z^2$, so putting $z' := \lambda z$ and $\mathfrak{c}' := \lambda^{-1} z$, we have $\mathfrak{c}z = \lambda^{-1}\mathfrak{c}\lambda z = \mathfrak{c}'z'$ with $(z')^2 = u \in R^\times$ a unit. Thus, it remains to show that $\mathfrak{c}' = R$. First we show that $\mathfrak{c}' \subseteq R$ holds. Assuming the opposite, there exists some $s \in \mathfrak{c}' - R$ with $sz' \in \mathcal{C}(E_1)$, so sz' must have an integral minimal polynomial over R, by Proposition 2.2.3. However, $(sz')^2 = s^2 u \notin R$, a contradiction. Thus, $\mathfrak{c}' \subseteq R$ holds. If $\mathfrak{c}' \subsetneq R$, then there is a prime $\mathfrak{p}$ of R that divides $\mathfrak{c}'$. For any $\mu \in \mathfrak{p}$, we have $\nu_\mathfrak{p}(\mu^2 u) > 0$ and this implies

$$\mathfrak{d} \cap R^\times = \{\mu^2 u \in R^\times \mid \mu \in \mathfrak{c}'\} \subseteq \{\mu^2 u \in R^\times \mid \mu \in \mathfrak{p}\} = \emptyset.$$

Hence, $\mathfrak{d} \cap R^\times = \emptyset$, which contradicts our assumption. This proves (i). Assertion (ii) is clear, due to $\mathfrak{d} \subseteq R$. Finally, assertion (iii) is immediate from Remark 2.1.17, the definition of the quadratic discriminant and the equality $\mathfrak{d} = \mathfrak{c}^2 d \cap d(K^\times)^2$. □

Notation 3.4.3. Fix pseudo-bases $(e_i, \mathfrak{a}_i)_{i\in\underline{2m}}$ of E_1 and $(f_j, \mathfrak{b}_j)_{j\in\underline{n}}$ of E_2. Using Notation 2.3.3, write e_I and f_J, for $I \subseteq \underline{2n}$ and $J \subseteq \underline{n}$. Furthermore, write ${}^{\mathfrak{d}}f_J$ when considering f_J as an element of $\mathcal{C}({}^{\mathfrak{d}}E_2)$. Here, ${}^{\mathfrak{d}}E_2 = ({}^{\mathfrak{d}}E_2, dq_2)$ denotes the even R-lattice with pseudo-basis $(f_j, \mathfrak{c}\mathfrak{b}_j)_{j\in\underline{n}}$ and the by d rescaled quadratic form q_2.

For example, $({}^{\mathfrak{d}}f_J, \mathfrak{c}^{|J|}\mathfrak{b}_J)_{J\subseteq\underline{n}}$ is a pseudo-basis of the Clifford order $\mathcal{C}({}^{\mathfrak{d}}E_2)$. Also, note that $\mathcal{C}({}^{\mathfrak{d}}E_2)$ is an R-order in $\mathcal{C}({}^{d}V_2)$, so $\mathfrak{d}$ is a reasonable substitute for d'.

Lemma 3.4.4. *Using Notation 3.4.3, the following hold.*

(i) The R-algebra $\mathcal{C}(E)$ has the pseudo-basis $\big((e_I f_J, \mathfrak{a}_I \mathfrak{b}_J)\big)$.

(ii) The R-algebra $\mathcal{C}(E_1) \otimes \mathcal{C}(E)^{\mathcal{C}(E_1)} \leq \mathcal{C}(E)$ has the pseudo-basis

$$\big((e_I f_J, \mathfrak{a}_I \mathfrak{b}_J) \mid |J| \equiv_2 0\big) \cup \big((z e_I f_J, \mathfrak{c}\mathfrak{a}_I \mathfrak{b}_J) \mid |J| \equiv_2 1\big).$$

(iii) The R-algebra $\mathcal{C}(E_1) \otimes \mathcal{C}({}^{\mathfrak{d}}E_2)$ has the pseudo-basis

$$\Big((e_I \otimes {}^{\mathfrak{d}}f_J, \mathfrak{c}^{|J|}\mathfrak{a}_I \mathfrak{b}_J)\Big).$$

Proof. Parts (i) and (iii) are obvious. Part (ii) follows from the isomorphism

$$\mathcal{C}(E_1) \otimes \mathcal{C}(E)^{\mathcal{C}(E_1)} \cong \mathcal{C}(E_1) \otimes \mathcal{C}_0(E_2) \oplus \mathcal{C}(E_1)\mathcal{Z}_- \otimes \mathcal{C}_1(E_2)$$

and $e_I z f_J = (-1)^{|I|} z e_I f_J$. □

Theorem 3.4.5. *Use Notation 3.4.3 and let $\mathcal{X}$ denote the R-suborder of $\mathcal{C}(V_1) \otimes_K \mathcal{C}({}^{d}V_2)$ with pseudo-basis*

$$\left(\left(e_I \otimes {}^{\mathfrak{d}}f_J, d^{-\frac{|J|}{2}}\mathfrak{a}_I \mathfrak{b}_J\right) \,\middle|\, |J| \equiv_2 0\right) \cup \left(\left(z e_I \otimes {}^{\mathfrak{d}}f_J, d^{-\frac{|J|+1}{2}}\mathfrak{a}_I \mathfrak{b}_J\right) \,\middle|\, |J| \equiv_2 1\right).$$

Then $\mathcal{C}(E) \cong \mathcal{X}$ as graded R-algebras, so $\mathcal{Y} := \mathcal{C}(E_1) \otimes \mathcal{C}({}^{\mathfrak{d}}E_2)$ is an R-suborder of $\mathcal{X}$. Moreover, $\mathcal{X} = \mathcal{Y}$, that is, $\mathcal{C}(E) \cong \mathcal{C}(E_1) \otimes \mathcal{C}({}^{\mathfrak{d}}E_2)$, if and only if for all primes $\mathfrak{p}$ of R, we have $\nu_\mathfrak{p}(\mathfrak{d}) = 2\nu_\mathfrak{p}(\mathfrak{c}) + \nu_\mathfrak{p}(d) = 0$, i.e. $\nu_\mathfrak{p}(d) \in 2\mathbb{Z}$, for all $\mathfrak{p}$ and $\mathfrak{c} = \prod_\mathfrak{p} \mathfrak{p}^{-\frac{\nu_\mathfrak{p}(d)}{2}}$.

Proof. By the proof of Theorem 1.2.27, there is an isomorphism of K-algebras $\psi : \mathcal{C}(V_1)\otimes\mathcal{C}(V)^{\mathcal{C}(V_1)} \to \mathcal{C}(V_1)\otimes\mathcal{C}({}^dV_2)$ that respects the gradings and is defined by

$$\begin{aligned}\psi(e_If_J) &:= d^{-\frac{|J|}{2}}e_I\otimes {}^df_J, && \text{for } |J|\equiv_2 0\\ \psi(ze_If_J) &:= (-1)^{|I|}d^{-\frac{|J|-1}{2}}e_I\otimes {}^df_J, && \text{for } |J|\equiv_2 1.\end{aligned}$$

By Proposition 3.2.8 (ii), $z\in\mathcal{C}_0(E_1)$, so $\psi(z) = z\otimes 1$. Moreover, V_1 is regular, because E_1 is non-degenerate. Thus, we have $z^2 = d \neq 0$, i.e. $z^{-1} = d^{-1}z$ exists and $\psi(z^{-1}) = d^{-1}z\otimes 1$. Hence, ψ maps the pseudo-basis $\big((e_If_J, \mathfrak{a}_I\mathfrak{b}_J)\big)$ of $\mathcal{C}(E)$ isomorphically to

$$\left(\left(e_I\otimes {}^{\mathfrak{d}}f_J, d^{-\frac{|J|}{2}}\mathfrak{a}_I\mathfrak{b}_J\right)\,\middle|\,|J|\equiv_2 0\right)\cup\left(\left(ze_I\otimes {}^{\mathfrak{d}}f_J, d^{-\frac{|J|+1}{2}}\mathfrak{a}_I\mathfrak{b}_J\right)\,\middle|\,|J|\equiv_2 1\right),$$

establishing $\mathcal{C}(E)\cong\mathcal{X}$ and also respecting the gradings. As a next step, we compare the R-algebras $\mathcal{X}$ and $\mathcal{Y}$, both of which are R-orders in $\mathcal{C}(V_1)\otimes\mathcal{C}({}^dV_2)$. We denote the pseudo-element associated to the subsets $I\subseteq \underline{2m}$, $J\subseteq\underline{n}$ in the pseudo-basis of $\mathcal{X}$ above by x_{IJ}. From Lemma 3.4.4, we know that $\mathcal{Y}$ has the pseudo-basis $\left((e_I\otimes {}^{\mathfrak{d}}f_J, \mathfrak{c}^{|J|}\mathfrak{a}_I\mathfrak{b}_J)\right)$. The map

$$(\mathfrak{c}^{|J|}z^{|J|}, x_{IJ}) \longrightarrow \mathfrak{c}^{|J|}\mathfrak{a}_I\mathfrak{b}_Je_I\otimes {}^{\mathfrak{d}}f_J,\ \ (\mu z^{|J|}, x)\longmapsto \mu z^{|J|}x$$

is onto for all I, J, because $z^{|J|} = d^{\frac{|J|}{2}}$, if $|J|$ is even; and $z^{|J|} = d^{\frac{|J|+1}{2}}z^{-1}$, if $|J|$ is odd. Moreover, the set $\mathfrak{c}^{|J|}z^{|J|}$ is contained in $\mathcal{C}(E)$, for each $|J|$, so $\psi(\mathfrak{c}^{|J|}z^{|J|}) = \mathfrak{c}^{|J|}z^{|J|}\otimes 1\subseteq\mathcal{X}$. Thus, we have $\mathcal{Y}\subseteq\mathcal{X}$ and therefore $\mathcal{C}(E)\cong\mathcal{Y}$, if and only if $\mathcal{Y}=\mathcal{X}$. This is an equality of R-lattices, which is well known to be a local property (see, e.g. (4.21) in [Rei03]). Thus, we postpone the remainder of this proof after stating and proving the local version of the present theorem. □

Remark 3.4.6. While we are able to precisely describe the index $[\mathcal{X}:\mathcal{Y}]$ above, it is sufficient to do that only in the local version of this theorem, see Theorem 3.4.12.

Remark 3.4.7. Theorem 3.4.5 is very powerful over complete discrete valuation rings, because there we have the Witt decomposition, see Proposition 1.1.9, available. In certain situations, it is also useful over arbitrary Dedekind domains. If, for example, $E_1 = \mathop{\perp}\limits_{i=1}^{k}\mathbb{H}(R)$ and E_2 is arbitrary, we have the isomorphism of R-algebras

$$\mathcal{C}\left(\mathop{\perp}\limits_{i=1}^{k}\mathbb{H}(R)\perp E_2\right)\cong\mathcal{C}(E_2)^{2^k\times 2^k}.$$

Proof. We have $\operatorname{disq}(E_1) = \prod_{i=1}^{k} \operatorname{disq}(\mathbb{H}(R)) = \big(R, 1(K^\times)^2\big)$, by Theorem 3.2.11, Proposition 3.2.16 (ii) and Example 3.2.5. Thus, we obtain the isomorphism

$$\mathcal{C}\left(\mathop{\perp}_{i=1}^{k} \mathbb{H}(R) \perp E_2\right) \cong \mathcal{C}\left(\mathop{\perp}_{i=1}^{k} \mathbb{H}(R)\right) \otimes \mathcal{C}(E_2)$$

from Theorem 3.4.5. Next, in view of Example 1.1.8, we choose an R-basis (x, y) of $\mathbb{H}(R)$, such that $x^2 = y^2 = 0$ and $xy + yx = 0$ in $\mathcal{C}(\mathbb{H}(R))$. This implies that

$$\varphi : \mathcal{C}(\mathbb{H}(R)) \to R^{2\times 2},\ \varphi(x) \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \varphi(x) \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

defines an isomorphism of R-algebras. Thus, we obtain the isomorphisms

$$\mathcal{C}\left(\mathop{\perp}_{i=1}^{k} \mathbb{H}(R)\right) \otimes \mathcal{C}(E_2) \cong R^{2^k\times 2^k} \otimes \mathcal{C}(E_2) \cong \mathcal{C}(E_2)^{2^k\times 2^k}.$$

The first isomorphism holds by Theorem 3.4.5 and then applying the usual tensor product successively. The second one is standard; see, e.g. [HO89] 1.1.1. □

Definition 3.4.8. Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded K-algebra. Then, under the identification $A = A^{1\times 1}$, the K-algebra $A^{2^{k+1}\times 2^{k+1}}$ is also $\mathbb{Z}/2\mathbb{Z}$-graded via

$$(A^{2^{k+1}\times 2^{k+1}})_0 = \begin{pmatrix} (A^{2^k\times 2^k})_0 & (A^{2^k\times 2^k})_1 \\ (A^{2^k\times 2^k})_1 & (A^{2^k\times 2^k})_0 \end{pmatrix},\ (A^{2^{k+1}\times 2^{k+1}})_1 = \begin{pmatrix} (A^{2^k\times 2^k})_1 & (A^{2^k\times 2^k})_0 \\ (A^{2^k\times 2^k})_0 & (A^{2^k\times 2^k})_1 \end{pmatrix},$$

for each $k \in \mathbb{N}_0$. Moreover, given any R-order Λ in $A^{2^k\times 2^k}$, Λ is a $\mathbb{Z}/2\mathbb{Z}$-graded R-algebra by means of restriction.

Corollary 3.4.9. *If $E_1 = \mathop{\perp}_{i=1}^{k} \mathbb{H}(R)$ and E_2 is arbitrary, we have the isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded R-algebras*

$$\mathcal{C}\left(\mathop{\perp}_{i=1}^{k} \mathbb{H}(R) \perp E_2\right) \cong \mathcal{C}(E_2)^{2^k\times 2^k},$$

where the right-hand side is graded as in Definition 3.4.8.

Proof. The isomorphism φ from the proof of Remark 3.4.7 respects the gradings, and so does the tensor product by Definition 3.4.8. □

The orthogonal decomposition for even rank - local version

In this brief interlude we present the local version of Theorem 3.4.12. Thus, only now we assume that R is a complete discrete valuation ring with uniformiser π and field of fractions K of characteristic zero. Since R is a Dedekind domain, all the previous result of Subsection 3.4.1 still apply to (E, q). Thus, we obtain the isomorphism of R-algebras

$$\mathcal{C}(E_1) \otimes \mathcal{C}(E)^{\mathcal{C}(E_1)} \cong \mathcal{C}(E_1) \otimes \mathcal{C}_0(E_2) \oplus \mathcal{C}(E_1)\mathcal{Z}_- \otimes \mathcal{C}_1(E_2) \leq \mathcal{C}(E_1) \,\widetilde{\otimes}\, \mathcal{C}(E_2),$$

but this time we may write $\mathcal{Z}_- = Rz$ with $z \in \mathcal{Z}$, satisfying $z^2 = d$. Here, d denotes a fixed representative of $\operatorname{disq}(E_1)$, the latter being regarded as an R-square class, as described in Remark 3.2.3.

Notation 3.4.10. Fix R-bases $(e_i \mid i \in \underline{2m})$ of E_1 and $(f_j \mid j \in \underline{n})$ of E_2, so that $(e_I \mid I \subseteq \underline{2m})$ and $({}^d f_J \mid J \subseteq \underline{n})$ are R-bases of $\mathcal{C}(E_1)$ and $\mathcal{C}({}^d E_2)$ respectively.

Note that $\mathcal{C}({}^d E_2)$ is an R-order inside $\mathcal{C}({}^d V_2)$ and, due to Lemma 3.4.2 (iii), it is actually the local equivalent of the R-order $\mathcal{C}({}^{\eth} E_2)$ from the global case.

Lemma 3.4.11. *Using Notation 3.4.10, the following hold.*

(i) The R-algebra $\mathcal{C}(E)$ has the basis $(e_I f_J)$.

(ii) The R-algebra $\mathcal{C}(E_1) \otimes \mathcal{C}(E)^{\mathcal{C}(E_1)} \leq \mathcal{C}(E)$ has the basis

$$\big(e_I f_J \mid |J| \equiv_2 0\big) \cup \big(z e_I f_J \mid |J| \equiv_2 1\big).$$

(iii) The R-algebra $\mathcal{C}(E_1) \otimes \mathcal{C}({}^d E_2)$ has the basis $\Big(e_I \otimes {}^d f_J\Big)$.

Theorem 3.4.12. *Use Notation 3.4.10 and let $\mathcal{X}$ denote the R-order in $\mathcal{C}(V_1) \otimes_K \mathcal{C}({}^d V_2)$ with basis*

$$\Big((\lambda_{IJ} e_I) \otimes {}^d f_J \mid I \subseteq \underline{2m}, J \subseteq \underline{n}\Big), \quad \lambda_{IJ} = \begin{cases} d^{-\frac{|J|}{2}}, & |J| \equiv_2 0 \\ (-1)^{|I|} d^{-\frac{|J|+1}{2}} z, & |J| \equiv_2 1 \end{cases}.$$

Then $\mathcal{C}(E) \cong \mathcal{X}$ as graded R-algebras, so $\mathcal{Y} := \mathcal{C}(E_1) \otimes \mathcal{C}({}^d E_2)$ is an R-suborder of $\mathcal{X}$. Moreover, the index of $\mathcal{Y}$ in $\mathcal{X}$ is $[X : Y] = \pi^s$, with $s = \nu(d) 2^{2(m-1)+n} n$. In particular, $\mathcal{X} = \mathcal{Y}$, i.e. $\mathcal{C}(E) \cong \mathcal{C}(E_1) \otimes \mathcal{C}({}^d E_2)$, if and only if $d \in R^\times$ is a unit.

Proof. The isomorphism $\mathcal{C}(E) \cong \mathcal{X}$ is follows immediately what we already proved in Theorem 3.4.5. We also showed $\mathcal{Y} \subseteq \mathcal{X}$, so it remains to consider the quotient of R-modules $\mathcal{X}/\mathcal{Y}$. This is an R-torsion module, uniquely determined by its elementary divisors and we need to determine their product. To compute it, we consider two

R-module endomorphisms of $\mathcal{X}$, that induce automorphisms of the ambient space $K \otimes \mathcal{X}$. The first one, call it φ, is defined as the identity on the basis elements $d^{\frac{|J|}{2}} e_I \otimes {}^d f_J$, if $|J|$ is even; and as left multiplication with z on the basis elements $d^{-\frac{|J|+1}{2}} z e_I \otimes {}^d f_J$, if $|J|$ is odd. Clearly, $\det(\varphi) = d^{2^{2(m-1)+n}}$, due to $z^2 = d$ and $\operatorname{rank}(\mathcal{C}(E_1)) = 2^{2m}$. The second endomorphism, call it τ, is given by

$$\tau(e_I \otimes {}^d f_J) := d^{\frac{|J|}{2}} e_I \otimes {}^d f_J,\ |J| \equiv_2 0 \ \text{ and } \ \tau(e_I \otimes {}^d f_J) := d^{\frac{|J|-1}{2}} e_I \otimes {}^d f_J,\ |J| \equiv_2 1.$$

In the following, I, J run through the subsets of $\underline{2m}$ and $\underline{n}$ and equality only holds up to a sign. We compute

$$\begin{aligned}\det(\tau) &= \prod_{\substack{I,J,\\ |J| \text{ even}}} d^{\frac{|J|}{2}} \prod_{\substack{I,J,\\ |J| \text{ odd}}} (-1)^{|I|} d^{\frac{|J|-1}{2}} = \left(\prod_{|J| \text{ even}} d^{\frac{|J|}{2}} \prod_{|J| \text{ odd}} d^{\frac{|J|-1}{2}} \right)^{2^{2m}} \\ &= d^{\frac{1}{2}\left(\sum_{|J| \text{ even}} |J| + \sum_{|J| \text{ odd}} (|J|-1) \right) 2^{2m}} =: d^x.\end{aligned}$$

Simplifying the exponent yields

$$\begin{aligned}x &= 2^{2m-1} \left(\sum_{|J| \text{ even}} |J| + \sum_{|J| \text{ odd}} (|J|-1) \right) = 2^{2m-1} \left(\sum_{J} |J| - \sum_{|J| \text{ odd}} 1 \right) \\ &= 2^{2m-1} \left(\sum_{k=0}^{n} \binom{n}{k} k - 2^{n-1} \right) = 2^{2m-1} \left(n 2^{n-1} - 2^{n-1} \right) = (n-1) 2^{2(m-1)+n}.\end{aligned}$$

By construction, $\tau \circ \varphi$ induces a K-automorphism of $K \otimes \mathcal{X}$ which maps the basis of $\mathcal{X}$ to a basis of $\mathcal{Y}$. Taking the valuation of $\det(\varphi)\det(\tau)$ gives the result. □

Remaining proof of Theorem 3.4.5. Returning to the situation that R is an arbitrary Dedekind domain with field of fractions K of characteristic zero, we denote the completion of R at the fixed prime $\mathfrak{p}$ by $\hat{R}_\mathfrak{p}$. Put $a_\mathfrak{p} := \nu_\mathfrak{p}(\mathfrak{c})$, so $\nu_\mathfrak{p}(\mathfrak{d}) = 2a_\mathfrak{p} + \nu_\mathfrak{p}(d) \geq 0$ and fix a uniformiser $\pi_\mathfrak{p} \in \hat{R}_\mathfrak{p}$. Then $\hat{R}_\mathfrak{p} \otimes_R \mathfrak{c} z = \hat{R}_\mathfrak{p}(\pi_\mathfrak{p}^{a_\mathfrak{p}} z)$ and $(\pi_\mathfrak{p}^{a_\mathfrak{p}} z)^2 = \pi_\mathfrak{p}^{2a_\mathfrak{p}} d \in \hat{R}_\mathfrak{p}$. Applying Theorem 3.4.12 to $\hat{R}_\mathfrak{p}$, we have

$$\hat{R}_\mathfrak{p} \otimes_R \mathcal{X} = \hat{R}_\mathfrak{p} \otimes_R \mathcal{Y},$$

if and only if $\pi_\mathfrak{p}^{2a_\mathfrak{p}} d \in \hat{R}_\mathfrak{p}^\times$ is a unit. This is equivalent to $\nu_\mathfrak{p}(\mathfrak{d}) = 2a_\mathfrak{p} + \nu_\mathfrak{p}(d) = 0$ and because equality of lattices is a local property, the proof is finished. □

3.4.2 The orthogonal decomposition for odd rank

It remains to generalise Theorem 1.2.28. For this, let $2m+1 := \operatorname{rank}(E_1)$ be odd, so $\dim(V) = 2m+n+1$. By Proposition 3.2.8 (ii), the centroid $\mathcal{Z} := \mathcal{Z}(E_1, q_1)$ is orthogonal and there is $z \in \mathcal{C}_1(E_1)$, with $\mathcal{Z} = R1 \oplus \mathfrak{c}z$ and $\delta := z^2 \in K$. More precisely one has $\delta(K^\times)^2 = \operatorname{disc}'(V_1)$. Put $\mathfrak{d} := \{-\lambda^2 z^2 \mid \lambda \in \mathfrak{c}\}$, so that we have $\operatorname{disq}(E_1) = \big(\mathfrak{c}^2\delta, \delta(K^\times)^2\big)$ and $\mathfrak{d} = -\mathfrak{c}^2\delta \cap -\delta(K^\times)^2$. Then, with exactly the same proof as for Lemma 3.4.4, we obtain the following facts about the set $\mathfrak{d}$.

Lemma 3.4.13. *(i) We have $\mathfrak{d} \cap R^\times \neq \emptyset$, if and only if we can choose $\mathfrak{c} = R$ and $z^2 = -\delta \in R^\times$.*

(ii) For each prime $\mathfrak{p}$ of R, we have $\nu_\mathfrak{p}(\mathfrak{d}) := 2\nu_\mathfrak{p}(\mathfrak{c}) + \nu_\mathfrak{p}(\delta) \geq 0$.

(iii) If E_1 is free and we regard $\operatorname{disq}(E_1)$ as an R-square class, then $\mathfrak{d} = \operatorname{disq}(E_1)R^2$, where $R^2 = \{x^2 \mid x \in R\}$.

Notation 3.4.14. Fix pseudo-bases $(e_i, \mathfrak{a}_i)_{i \in \underline{2m+1}}$ of E_1 and $(f_j, \mathfrak{b}_j)_{j \in \underline{n}}$ of E_2. Moreover, let ${}^\mathfrak{d}E_2 = ({}^\mathfrak{d}E_2, -\delta q_2)$ denote the even R-lattice with pseudo-basis $(f_j, \mathfrak{c}\mathfrak{b}_j)_{j \in \underline{n}}$ and the by $-\delta$ rescaled quadratic form q_2.

For example, $({}^\mathfrak{d}f_J, \mathfrak{c}^{|J|}\mathfrak{b}_J)_{J \subseteq \underline{n}}$ is a pseudo-basis of the Clifford order $\mathcal{C}({}^\mathfrak{d}E_2)$. Also, note that $\mathcal{C}({}^\mathfrak{d}E_2)$ is an R-order in $\mathcal{C}({}^{-\delta}V_2)$.

Lemma 3.4.15. *Using Notation 3.4.14, the following hold.*

(i) The R-algebra $\mathcal{C}_0(E)$ has the pseudo-basis $\Big((e_I f_J, \mathfrak{a}_I\mathfrak{b}_J) \,\Big|\, |I|+|J| \equiv_2 0\Big)$.

(ii) The R-algebra $\mathcal{C}_0(E_1) \otimes \mathcal{C}({}^\mathfrak{d}E_2)$ has the pseudo-basis

$$\Big((e_I \otimes {}^\mathfrak{d}f_J, \mathfrak{c}^{|J|}\mathfrak{a}_I\mathfrak{b}_J) \mid |I| \equiv_2 0\Big).$$

Our aim is to describe the isomorphic image of $\mathcal{C}_0(E)$ under the K-algebra isomorphism $\mathcal{C}_0(V) \cong \mathcal{C}_0(V_1) \otimes_K \mathcal{C}({}^{-\delta}V_2)$ from Theorem 1.2.28. Thus, we need this isomorphism explicitly.

Lemma 3.4.16. *Use Notation 3.4.14, so that $(e_I f_J \mid |I|+|J| \equiv_2 0)$ is a K-basis of $\mathcal{C}(V_1 \perp V_2)$. Then the following hold.*

(i) For any two $J_1, J_2 \subseteq \underline{n}$, let $t := |J_1| + |J_2|$. If

$$f_{J_1} f_{J_2} = \sum_{\substack{J \subseteq \underline{n} \\ |J| \equiv_2 t}} \mu_J f_J \in \mathcal{C}(V_2)$$

with unique $\mu_J \in K$ (equal to zero whenever $|J| > t$), we have

$$^{-\delta}f_{J_1}\,^{-\delta}f_{J_2} = \sum_{\substack{J\subseteq \underline{n}\\ |J|\equiv_2 t}} (-\delta)^{(t-|J|)/2}\mu_J f_J \in \mathcal{C}(^{-\delta}V_2).$$

(ii) The K-algebras $\mathcal{C}_0(V_1 \perp V_2)$ and $\mathcal{C}_0(V_1) \otimes \mathcal{C}(^{-\delta}V_2)$ are isomorphic via

$$\varphi(e_I f_J) := \begin{cases} (-\delta)^{-|J|/2} e_I \otimes {}^{-\delta}f_J, & |I| \equiv_2 |J| \equiv_2 0 \\ (-\delta)^{-(|J|+1)/2} z e_I \otimes {}^{-\delta}f_J, & |I| \equiv_2 |J| \equiv_2 1 \end{cases}.$$

Proof. We start with assertion (i). The left-hand side $f_{J_1}f_{J_2}$ can be transformed into the right-hand side $\sum_{|J|\equiv_2 t} \mu_J f_J \in \mathcal{C}(V_2)$ by successively applying the relations $f_i f_j = b(f_i, f_j) - f_j f_i$ and $f_i^2 = q(f_i)$. By applying the exact same steps to $^{-\delta}f_{J_1}\,^{-\delta}f_{J_2}$, one obtains the claimed right-hand side. This is because the relations in $\mathcal{C}(^{-\delta}V_2)$ can only shorten the length of the initial index sequence (J_1, J_2) by two, while introducing a factor $-\delta$, whenever this happens (and may generate a of copy of the current index sequence). For part (ii), we only need to show that φ is multiplicative, since with 1 and z also the images of the basis vectors of $\mathcal{C}_0(V_1 \perp V_2)$ under φ are K-linearly independent. Consider subsets $I_1, I_2 \subseteq \underline{2m+1}$, $J_1, J_2 \subseteq \underline{n}$, such that $e_{I_1}f_{J_1}, e_{I_2}f_{J_2} \in \mathcal{C}_0(V_1 \perp V_2)$ and put $t := |J_1| + |J_2|$. Then, for all $I \subseteq \underline{2m+1}$, $J \subseteq \underline{n}$ with $|I| \equiv_2 t$ and $|J| \equiv_2 t$, there are unique $\lambda_I, \mu_J \in K$ such that the equations

$$e_{I_1}e_{I_2} = \sum_{|I|\equiv_2 t} \lambda_I e_I \quad \text{and} \quad f_{J_1}f_{J_2} = \sum_{|J|\equiv_2 t} \mu_I f_J$$

hold. Moreover, using $e_i f_j = -f_j e_i$, for all $i \in \underline{2m+1}, j \in \underline{n}$, we have $e_{I_1}f_{J_1}e_{I_2}f_{J_2} = (-1)^{|I_2||J_1|} e_{I_1}e_{I_2}f_{J_1}f_{J_2}$. Using these two facts, we obtain

$$\begin{aligned} \varphi(e_{I_1}e_{I_2}f_{J_1}f_{J_2}) &= \sum_{|I|\equiv_2 |J|\equiv_2 t} \lambda_I \mu_J \varphi(e_I f_J) \\ &= \begin{cases} \sum_{|I|\equiv_2|J|\equiv_2 t} \lambda_I\mu_J(-\delta)^{-|J|/2} e_I \otimes {}^{-\delta}f_J, & t \text{ even} \\ \sum_{|I|\equiv_2|J|\equiv_2 t} \lambda_I\mu_J(-\delta)^{-(|J|+1)/2} z e_I \otimes {}^{-\delta}f_J, & t \text{ odd} \end{cases} \end{aligned}$$

on one hand. On the other hand, we compute

$$(-1)^{|J_1||I_2|}\varphi(e_{I_1}f_{J_1})\varphi(e_{I_2}f_{J_2}) = \begin{cases} (-\delta)^{-t/2}e_{I_1}e_{I_2}\otimes {}^{-\delta}f_{J_1}{}^{-\delta}f_{J_2}, & t \text{ even} \\ (-\delta)^{-(t+1)/2}ze_{I_1}e_{I_2}\otimes {}^{-\delta}f_{J_1}{}^{-\delta}f_{J_2}, & t \text{ odd} \end{cases}$$

$$= \begin{cases} \sum\limits_{|I|\equiv_2 t} \lambda_I(-\delta)^{-t/2}e_I\otimes {}^{-\delta}f_{J_1}{}^{-\delta}f_{J_2}, & t \text{ even} \\ \sum\limits_{|I|\equiv_2 t} \lambda_I(-\delta)^{-(t+1)/2}ze_I\otimes {}^{-\delta}f_{J_1}{}^{-\delta}f_{J_2}, & t \text{ odd} \end{cases},$$

using $t \equiv_2 |J_1|+|I_2|$ and the definition of φ. By part (i) of the present lemma, these two expressions are equal, so we are done. $\square$

Theorem 3.4.17. *Use Notation 3.4.14 and let $\mathcal{X}$ denote the R-order in $\mathcal{C}_0(V_1)\otimes_K \mathcal{C}({}^{-\delta}V_2)$ with pseudo-basis*

$$\left(\left(e_I\otimes {}^{\mathfrak{d}}f_J, (-\delta)^{-\frac{|J|}{2}}\mathfrak{a}_I\mathfrak{b}_J\right)\middle|\, |I|\equiv_2 0\right)\cup\left(\left(ze_I\otimes {}^{\mathfrak{d}}f_J, (-\delta)^{-\frac{|J|+1}{2}}\mathfrak{a}_I\mathfrak{b}_J\right)\middle|\, |I|\equiv_2 1\right),$$

where (I,J) runs through the pairs of subsets of $\underline{2m+1}$ and $\underline{n}$, that satisfy $|I|\equiv_2 |J|$. Then $\mathcal{C}_0(E)\cong\mathcal{X}$ as graded R-algebras, so $\mathcal{Y} := \mathcal{C}_0(E_1)\otimes\mathcal{C}({}^{\mathfrak{d}}E_2)$ is an R-suborder of $\mathcal{X}$. Moreover, $\mathcal{X}=\mathcal{Y}$, that is, $\mathcal{C}_0(E)\cong\mathcal{C}_0(E_1)\otimes\mathcal{C}({}^{\mathfrak{d}}E_2)$, if and only if $\operatorname{rank}(E_1)=\operatorname{rank}(E_2)=1$, *i.e. $m=0$ and $n=1$; or for all primes $\mathfrak{p}$ of R, we have $\nu_{\mathfrak{p}}(\mathfrak{d}) = 2\nu_{\mathfrak{p}}(\mathfrak{c})+\nu_{\mathfrak{p}}(\delta)=0$, i.e. $\nu_{\mathfrak{p}}(\delta)\in 2\mathbb{Z}$, for all $\mathfrak{p}$ and $\mathfrak{c}=\prod_{\mathfrak{p}}\mathfrak{p}^{-\frac{\nu_{\mathfrak{p}}(d)}{2}}$.*

Proof. The claim $\mathcal{C}_0(E)\cong\mathcal{X}$ is immediate from Lemma 3.4.15 (ii). If $\mathcal{Y}:=\mathcal{C}_0(E_1)\otimes\mathcal{C}({}^{\mathfrak{d}}E_2)$, then $\mathcal{Y}\subseteq\mathcal{X}$ follows from an analogous argument as was used in the proof of Theorem 3.4.5. Thus, $\mathcal{C}_0(E)\cong\mathcal{Y}$, if and only if $\mathcal{Y}=\mathcal{X}$ and this can be checked locally, so the proof of the local version of this theorem suffices. $\square$

The orthogonal decomposition for odd rank - local version

Suppose that R is a complete discrete valuation ring with uniformiser π and field of fractions K of characteristic zero. Then $\mathcal{Z}=R1\oplus Rz$, with $z\in\mathcal{C}_1(E_1)$ and $\delta := z^2\in R$, so we identify $\operatorname{disq}(E_1)$ with the R-square class $\delta(R^\times)^2$, as in Remark 3.2.3. In particular, $\delta(K^\times)^2=\operatorname{disc}'(V_1,q_1)$ holds. Choose R-bases $(e_i \mid i\in\underline{2m+1})$ of E_1 and $(f_j\mid j\in\underline{n})$ of E_2, so that $(e_I\mid I\subseteq\underline{2m+1})$ and $({}^{-\delta}f_J\mid J\subseteq\underline{n})$ are R-bases of $\mathcal{C}(E_1)$ and $\mathcal{C}({}^{-\delta}E_2)$ respectively. Also note that $\mathcal{C}({}^{-\delta}E_2)$ is an R-order inside $\mathcal{C}({}^{-\delta}V_2)$.

Lemma 3.4.18. *$z\mathcal{C}_1(E_1)\le\mathcal{C}_0(E_1)$ is an R-sublattice and $[\mathcal{C}_0(E_1):z\mathcal{C}_1(E_1)]=\pi^s$, where $s=\nu(\delta)2^{2m-1}$, if $m\ge 1$; and $s=\nu(\delta)$, if $m=0$. More concisely, $s=\nu(\delta)2^{\max\{2m-1,0\}}$.*

Proof. By Proposition 3.2.8 (ii), we have $z \in \mathcal{C}_1(E_1)$, so $z\mathcal{C}_1(E_1)$ is a sublattice of $\mathcal{C}_0(E_1)$ as claimed. It remains compute the index $[\mathcal{C}_0(E_1) : z\mathcal{C}_1(E_1)]$. The case $m = 0$ is trivial, as then $\mathcal{C}_1(E_1) = E_1 = Rz$, so $z\mathcal{C}_1(E_1) = R\delta$ and $\mathcal{C}_0(E_1) = R1$. Now, let $m \geq 1$ and put $e := e_{2m+1}$. Without loss of generality, write $E_1 = \bigoplus_{i=1}^{2m} Re_i \perp Re =: (E', q') \perp Re$. Put $a := q(e) \in R - \{0\}$, so that $\delta = \operatorname{disq}(E_1) = \operatorname{disq}(E')\operatorname{disq}([a]) = a \cdot \operatorname{disq}(E')$, by Theorem 3.2.11. Hence, the element $z' := a^{-1}ez = a^{-1}ze \in \mathcal{C}_0(E') \subseteq \mathcal{C}_0(E_1)$ satisfies $(z')^2 = a^{-1}\delta$, so $(1, z')$ is an R-basis of $\mathcal{Z}(E', q')^o$. Using this, we compute

$$\begin{aligned} z\mathcal{C}_1(E_1) &= \bigoplus_{\substack{I \subseteq \underline{2m} \\ |I| \text{ odd}}} zRe_I \oplus \bigoplus_{\substack{I \subseteq \underline{2m} \\ |I| \text{ even}}} zRe_I e = \bigoplus_{|I| \text{ odd}} z'eRe_I \oplus \bigoplus_{|I| \text{ even}} z'eRe_I e \\ &= \bigoplus_{|I| \text{ odd}} z'Re_I e \oplus \bigoplus_{|I| \text{ even}} z'Rae_I = z' \left(\bigoplus_{|I| \text{ odd}} Re_I e \oplus \bigoplus_{|I| \text{ even}} Rae_I \right). \end{aligned}$$

This implies that there is an inclusion chain of R-lattices

$$z\mathcal{C}_1(E_1) \leq z'\mathcal{C}_0(E_1) \leq \mathcal{C}_0(E_1)$$

and $[z'\mathcal{C}_0(E_1) : z\mathcal{C}_1(E_1)] = a^{2^{2m-1}}$. Finally, left multiplication with z' is an R-module endomorphism of $\mathcal{C}_0(E_1)$ with image $z'\mathcal{C}_0(E_1)$. Thus, using $\operatorname{rank}(\mathcal{C}_0(E_1)) = 2^{2m}$ and $(z')^2 = a^{-1}\delta$, we have $[\mathcal{C}_0(E_1) : z'\mathcal{C}_0(E_1)] = (a^{-1}\delta)^{2^{2m-1}}$. Now the claim follows from

$$[\mathcal{C}_0(E_1) : z\mathcal{C}_1(E_1)] = [\mathcal{C}_0(E_1) : z'\mathcal{C}_0(E_1)] \cdot [z'\mathcal{C}_0(E_1) : z\mathcal{C}_1(E_1)] = \delta^{2^{2m-1}}. \qquad \square$$

Theorem 3.4.19. *Keep the notation from above and let $\mathcal{X}$ denote the R-order in $\mathcal{C}_0(V_1) \otimes_K \mathcal{C}({}^{-\delta}V_2)$ with basis*

$$\left((-\delta)^{-|J|/2} e_I \otimes {}^{-\delta}f_J \mid |I| \equiv_2 0\right) \cup \left((-\delta)^{-(|J|+1)/2} z e_I \otimes {}^{-\delta}f_J \mid |I| \equiv_2 1\right),$$

where (I, J) runs through the pairs of subsets of $\underline{2m+1}$ and $\underline{n}$, that satisfy $|I| \equiv_2 |J|$. Then $\mathcal{C}_0(E) \cong \mathcal{X}$ as R-algebras, so $\mathcal{Y} := \mathcal{C}_0(E_1) \otimes \mathcal{C}({}^{-\delta}E_2)$ is an R-suborder of $\mathcal{X}$. Moreover, the index of $\mathcal{Y}$ in $\mathcal{X}$ is $[\mathcal{X} : \mathcal{Y}] = \pi^s$, where $s = \nu(\delta)2^{2(m-1)+n}n$, if $m \geq 1$; and $s = \nu(\delta)2^{n-2}(n-1)$, if $m = 0$. In particular, $\mathcal{X} = \mathcal{Y}$, i.e. $\mathcal{C}_0(E) \cong \mathcal{C}_0(E_1) \otimes \mathcal{C}({}^{-\delta}E_2)$, if and only if $\delta \in R^\times$ is a unit; or both $m = 0$ and $n = 1$ hold.

Proof. By Theorem 3.4.17, we have $\mathcal{C}_0(E) \cong X$ and $\mathcal{Y} \subseteq \mathcal{X}$. Consider the following three full R-lattices inside $\mathcal{C}_0(V_1) \otimes_K \mathcal{C}({}^{-\delta}V_2)$, where I runs through the subsets of $\underline{2m+1}$ and J runs through the subsets of $\underline{n}$:

1) $\bigoplus_{|I|\equiv|J|\equiv 0} R(-\delta)^{-|J|/2}(e_I \otimes {}^{-\delta}f_J) \oplus \bigoplus_{|I|\equiv|J|\equiv 1} R(-\delta)^{-(|J|+1)/2}(ze_I \otimes {}^{-\delta}f_J)$

2) $\bigoplus_{|I|\equiv|J|\equiv 0} R(e_I \otimes {}^{-\delta}f_J) \oplus \bigoplus_{|I|\equiv 0, |J|\equiv 1} R(e_I \otimes {}^{-\delta}f_J)$

3) $\bigoplus_{|I|\equiv|J|\equiv 0} R(e_I \otimes {}^{-\delta}f_J) \oplus \bigoplus_{|I|\equiv|J|\equiv 1} R(ze_I \otimes {}^{-\delta}f_J)$

Note that the first lattice is $\mathcal{X}$, the second one is $\mathcal{Y}$ and the third one, call it L, satisfies $L \subseteq \mathcal{Y}$. Then, a computation analogous to the one in the proof of Theorem 3.4.12 yields $[\mathcal{X} : L] = \pi^{s_1}$, with $s_1 = \nu(\delta)2^{2(m-1)+n}(n+1)$. Furthermore, using Lemma 3.4.18 and $\operatorname{rank}(\mathcal{C}(E_2)) = 2^n$, we have $[\mathcal{Y} : L] = \pi^{s_2}$ with $s_2 = \nu(\delta)2^{\max\{2(m-1)+n,\, n-1\}}$. Thus, $[\mathcal{X} : \mathcal{Y}] = [\mathcal{X} : L] \cdot [\mathcal{Y} : L]^{-1} = \pi^{s_1 - s_2}$. We compute

$$s_1 - s_2 = \nu(\delta)\left(2^{2(m-1)+n}(n+1) - 2^{\max\{2(m-1)+m,\, n-1\}}\right) = \begin{cases} 2^{2(m-1)+n}n, & m \geq 1 \\ 2^{n-2}(n-1), & m = 0 \end{cases},$$

so we are done. □

Remark 3.4.20. For $m = 0$, we regain the statement of Proposition 2.3.7 (ii) in the special case that R is a complete discrete valuation ring.

3.5 Centroids of the root lattices

In this section we conduct a case study on the centroids of the so-called root lattices. We prepare it with a brief introduction to the theory.

3.5.1 Root lattices

For the theory presented in this subsection, we mainly follow [Ebe02, Section 1.4].

Definition 3.5.1. Let $\mathbb{R}^n = (\mathbb{R}^n, \Phi)$ be the euclidean n-space with the standard scalar product Φ and $(L, q) \subset \mathbb{R}^n$ be an integral quadratic $\mathbb{Z}$-lattice. The set of *roots* of L is $\mathfrak{R} := \{x \in L \mid q(x) = 1\}$. L is called a **root lattice**, if and only if L is generated by $\mathfrak{R}$.

Remark 3.5.2 ([Ebe02] Theorem 1.1)**.** Let (L, q) be a root lattice. Then, with respect to a suitable basis $(e_1, \ldots, e_n)$, we have $q(e_i) = 1$, for $i = 1, \ldots, n$ and $\Phi(e_i, e_j) \in \{0, -1\}$. This means that the Gram matrix with respect to this basis has twos on its main diagonal and all other entries are 0 or -1. In particular, a root lattice is always even, so it has a Clifford order.

Definition 3.5.3. A lattice $L \subset \mathbb{R}^n$ is called **reducible** if there are non-zero lattices $L_1 \subset \mathbb{R}^{n_1}, L_2 \subset \mathbb{R}^{n_2}$ with $L = L_1 \perp L_2$. Otherwise, it is called **irreducible**.

One can assign an undirected graph to a root lattice, the so-called *Coxeter-Dynkin* diagram. It has the vertex set $\{e_1, \ldots, e_n\}$ and two vertices are connected by an edge, if and only if $\Phi(e_i, e_j) = e_i e_j + e_j e_i = -1 \in \mathcal{C}(L)$.

Theorem 3.5.4 ([Ebe02] Theorem 1.2)**.** *Every root lattice is the orthogonal direct sum of the irreducible root lattices with the Coxeter-Dynkin diagrams of types* $A_n, D_n\, (n \geq 3), E_6, E_7$ *and* E_8*. Here,* $n \geq 1$*, for type A; and* $n \geq 3$*, for type D.*

We will denote the corresponding irreducible root lattices by $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$. For detailed constructions we refer to [Ebe02]. Instead, we provide a brief summary of their construction, as well as a Gram matrix and the discriminant for each of these lattices. In the following, let $\mathbb{I}_n = (\mathbb{Z}^n, \Phi)$ denote the *standard lattice*, given by the orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$.

The lattices $\mathbb{A}_n$

These lattices can be constructed as a sublattice of $\mathbb{I}_{n+1}$, namely it is the orthogonal complement of the vector $e = (1, \ldots, 1)^{\mathrm{tr}} \in \mathbb{I}_{n+1}$. Then $\mathbb{A}_n$ has the basis $(e_1, \ldots, e_n)$ with $e_i = \varepsilon_i - \varepsilon_{i+1}$ and the associated Gram matrix is given by

$$\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}. \tag{3.3}$$

Its determinant is $n+1$, so $\mathrm{disc}(\mathbb{A}_n) = (-1)^{\binom{n}{2}}(n+1) \in \mathbb{Z}$, under the identification $\mathbb{Z}/(\mathbb{Z}^\times)^2 \cong \mathbb{Z}$.

The lattices $\mathbb{D}_n\ (n \geq 3)$

For a fixed $n \geq 3$, the lattice $\mathbb{D}_n$ is the so-called *even sublattice* of $\mathbb{I}_n$, that is

$$\mathbb{D}_n = \{x \in \mathbb{I}_n \mid q(x) \in 2\mathbb{Z}\} = \{(x_1, \ldots, x_n)^{\mathrm{tr}} \in \mathbb{I}_n \mid \sum_i x_i \in 2\mathbb{Z}\}.$$

Thus, a basis is given by $(e_1, \ldots, e_n)$, with $e_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \ldots, n-1$ and $e_n = \varepsilon_{n-1} + \varepsilon_n$. The associated Gram matrix is

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & -1 \\ & & -1 & 2 & 0 \\ & & -1 & 0 & 2 \end{pmatrix}. \tag{3.4}$$

Its determinant is 4, because $[\mathbb{I}_n : \mathbb{D}_n] = 2$. Hence, $\mathrm{disc}(\mathbb{D}_n) = (-1)^{\binom{n}{2}} 4 \in \mathbb{Z}$. Finally, note that $\mathbb{D}_3 \cong \mathbb{A}_3$ by swapping the first two basis elements of $\mathbb{D}_3$. For $n \geq 4$, the lattices $\mathbb{A}_n$ and $\mathbb{D}_n$ are not isometric.

The lattices $\mathbb{E}_n$ $(n = 6, 7, 8)$ We start by considering $\mathbb{E}_8$. This lattice is known to be even and unimodular, so $\mathrm{disc}(\mathbb{E}_8) = 1$. It can be constructed from the so-called *extended Hamming code of length 8* (cf. [Ebe02] 1.3). The lattices $\mathbb{E}_7$ and $\mathbb{E}_6$ can be constructed as the orthogonal complements of a one-dimensional and two-dimensional subspace of $\mathbb{E}_8$ respectively. We represent the root lattices $\mathbb{E}_n$ by a basis $(e_1, \ldots, e_n)$ $n = 6, 7, 8$ with Gram matrix

$$\begin{pmatrix} G_{n-4} & -1 & & & \\ -1 & 2 & -1 & 0 & -1 \\ & -1 & 2 & -1 & 0 \\ & 0 & -1 & 2 & 0 \\ & -1 & 0 & 0 & 2 \end{pmatrix} \in \mathbb{Z}^{n \times n}, \tag{3.5}$$

where $G_{n-4} \in \mathbb{Z}^{(n-4)\times(n-4)}$ is the Gram matrix of the lattice $\mathbb{A}_{n-4}$ from above. One has $\mathrm{disc}(\mathbb{E}_6) = -3$, $\mathrm{disc}(\mathbb{E}_7) = -2$.

Remark 3.5.5. While one does not obtain a root lattice anymore, one can define the bilinear $\mathbb{Z}$-lattice $\mathbb{E}_n$, with $n \geq 10$ as the non-degenerate even $\mathbb{Z}$-lattice of rank n with Gram matrix 3.5. This lattice is not positive definite, so it cannot be regarded as a sublattice of $(\mathbb{R}^n, \Phi)$. Its determinant is $9 - n$, so $\mathrm{disc}(\mathbb{E}_n) = (-1)^{\binom{n}{2}}(9 - n)$.

Remark 3.5.6. Not only does the Gram matrix of $\mathbb{A}_{n-4}$ occur as a submatrix of the Gram matrix of $\mathbb{E}_n$, but also the Gram matrix of $\mathbb{A}_{n-1}$ occurs as a submatrix of the Gram matrix of $\mathbb{D}_n$. This is a key observation for the construction of the bases of the centroids in the next subsection.

3.5.2 Computing the centroids

Let (L, q) be a root lattice with quadratic discriminant $\delta := \mathrm{disq}(L) \in \mathbb{Z}/(\mathbb{Z}^\times)^2 \cong \mathbb{Z}$. As $\mathbb{Z}$ is a principal ideal domain, each $\mathbb{Z}$-order is free as $\mathbb{Z}$-lattice, so $\mathcal{Z} := \mathcal{Z}(L, q)$ has a $\mathbb{Z}$-basis $(1, x)$, where $x \in \mathcal{Z}$ is integral as an element of the ambient space $\mathcal{Z}(\mathbb{Q}L, q) = \mathbb{Q}[X]/(X^2 - \delta)$. The general strategy for finding such an element x is to first find an element $x' \in \mathcal{Z}$ such that $(1, x')$ is a $\mathbb{Z}$-basis of the maximal orthogonal suborder $\mathcal{Z}^o$. Then, either $x = x'$, so $\mathcal{Z} = \mathcal{Z}^o$ is orthogonal, or $x = \frac{1+x'}{2}$ and then $\mathcal{Z} = \mathbb{Z}[X]/(X^2 - X + \frac{1-\delta}{4})$. The latter happens, if and only if $\frac{1+x'}{2} \in \mathcal{C}(L)$ (this requires $\mathrm{rank}(L) \in 2\mathbb{Z}$ by Proposition 3.2.8 (ii)). In the following, we call x' a *maximal orthogonal element* of (L, q). It is unique up to conjugation with the unique non-trivial automorphism β of $\mathcal{Z}$, i.e. up to its sign. It satisfies $(x')^2 = \mathrm{disq}(L, q) = \delta$.

By Theorem 3.5.4, every root lattice is the orthogonal direct sum of irreducible ones, the latter being the lattices of types $\mathbb{A}_n, \mathbb{D}_n$ and $\mathbb{E}_n$ with their respective allowed values for n. Thus, in view of Theorem 3.2.11, it is sufficient to compute the centroids of said irreducible root lattices. Alternatively, if we have the centroids of the irreducible root lattices available, we can apply Theorem 3.2.14. We illustrate this.

Example 3.5.7. In Theorem 3.5.10 we show that $\mathrm{disq}(\mathbb{A}_5) = 3$, $\mathrm{disq}(\mathbb{A}_6) = -7$ and give explicit formulas for the maximal orthogonal elements of these lattices.

(i) We have $\mathrm{disq}(\mathbb{A}_5 \perp \mathbb{A}_5) = 9$, so $\mathcal{Z}(\mathbb{A}_5 \perp \mathbb{A}_5)^o \cong \mathbb{Z}[X]/(X^2 - 9)$ and, due to Theorem 3.2.14, the centroid is orthogonal, whence $\mathcal{Z}(\mathbb{A}_5 \perp \mathbb{A}_5) = \mathcal{Z}(\mathbb{A}_5 \perp \mathbb{A}_5)^o$.

(ii) We have $\mathrm{disq}(\mathbb{A}_5 \perp \mathbb{A}_6) = -21$, so $\mathcal{Z}(\mathbb{A}_5 \perp \mathbb{A}_6)^o \cong \mathbb{Z}[X]/(X^2 + 21)$ and since the rank of this lattice is odd, also $\mathcal{Z}(\mathbb{A}_5 \perp \mathbb{A}_6) = \mathcal{Z}(\mathbb{A}_5 \perp \mathbb{A}_6)^o$. We could, of course, have used Theorem 3.2.14 once again.

(iii) We have $\mathrm{disq}(\mathbb{A}_6 \perp \mathbb{A}_6) = 49$, so $\mathcal{Z}(\mathbb{A}_6 \perp \mathbb{A}_6)^o \cong \mathbb{Z}[X]/(X^2 - 49)$. This time, the centroid is not orthogonal, by Theorem 3.2.14. This implies $\mathcal{Z}(\mathbb{A}_6 \perp \mathbb{A}_6) \cong \mathbb{Z}[X]/(X^2 - X - 12)$.

If one is interested in formulas for the maximal orthogonal elements of these direct sums, one can use Theorem 3.2.11.

The lattices $\mathbb{A}_n$

The Gram matrix 3.3 provides a natural embedding $\mathcal{C}(\mathbb{A}_n) \subset \mathcal{C}(\mathbb{A}_{n+1})$, for all $n \geq 1$. Thus, we can consider all the following computations inside $\mathcal{C}(\mathbb{A}_m)$ for some sufficiently large $m \in \mathbb{N}$.
Let $(e_1, \ldots, e_m)$ be the basis of $\mathbb{A}_m$ with the Gram matrix from 3.3 and define the sequence $(a_i)_{i \geq 0}$ by

$$a_0 = 1, a_1 = e_1 \text{ and } a_i = \varepsilon_i a_{i-1} e_i + a_{i-2}, \text{ with } \varepsilon_i = \begin{cases} 2, & i \text{ even} \\ 1, & i \text{ odd} \end{cases}.$$

If necessary, we put $a_i = 0$, for $i < 0$.

Example 3.5.8. The first elements of this sequence are

$$a_0 = 1,\ a_1 = e_1,\ a_2 = 1 + 2e_1e_2,\ a_3 = e_1 + e_3 + 2e_1e_2e_3.$$

Our goal is to show that a_n is a maximal orthogonal element of $\mathbb{A}_n$ for $n \geq 1$. To do so, we need a number of technical results on this sequence.

Lemma 3.5.9. *(i) For $i \geq 1$, we have* $a_i \in \begin{cases} \mathcal{C}_0(\mathbb{A}_i) \subset \mathcal{C}_0(\mathbb{A}_m), & i \text{ even} \\ \mathcal{C}_1(\mathbb{A}_i) \subset \mathcal{C}_1(\mathbb{A}_m), & i \text{ odd} \end{cases}$.

(ii) If the coefficient of the basis vector $e_{j_1} \dots e_{j_r} \in \mathcal{C}(\mathbb{A}_m)$ as a summand of a_i is non-zero, then it is $2^{\lfloor \frac{r}{2} \rfloor}$. In particular, the greatest common divisor of all these coefficients is equal to 1 and $\frac{1+a_n}{2} \in \mathcal{C}(\mathbb{A}_m)$, if and only if n is even.

(iii) For all $0 \leq i \leq j-2 \leq m$, we have $e_j a_i = (-1)^i a_i e_j$.

(iv) For all $i \geq 0$: $e_{i+1}a_i = \begin{cases} 2a_{i-1} + a_i e_{i+1} = a_{i+1} + a_{i-1}, & i \text{ even} \\ -a_{i-1} - a_i e_{i+1} = -a_{i+1} + a_i e_{i+1}, & i \text{ odd} \end{cases}$.

In particular, $e_{i+1}a_i - (-1)^i a_i e_{i+1} = (-1)^i \varepsilon_i a_{i+1}$.

(v) For all $i \geq 1$: $e_i a_i = (-1)^{i-1} a_i e_i$ and $e_{i-1} a_i = (-1)^{i-1} a_i e_{i-1}$.

Proof. Assertion (i) is an easy induction and (iii) is an immediate consequence of (i) as e_j does not occur as a factor in any summand of a_i, if $i \leq j-2$. To see (ii), note that a_i and a_{i+1} do not share any common summands, as one of these is even and the other one is odd by (i). Now (ii) is an easy induction. Assertion (iv) is obvious, for $i = 0, 1$, so let $i \geq 2$. Then

$$e_{i+1}a_i \overset{(iii)}{=} \begin{cases} -2a_{i-1}e_{i+1}e_i + a_{i-2}e_{i+1}, & i \text{ even} \\ a_{i-1}e_{i+1}e_i - a_{i-2}e_{i+1}, & i \text{ odd} \end{cases}$$
$$= \begin{cases} -2a_{i-1}(-1 - e_i e_{i+1}) + a_{i-2}e_{i+1}, & i \text{ even} \\ a_{i-1}(-1 - e_i e_{i+1}) - a_{i-2}e_{i+1}, & i \text{ odd} \end{cases}$$
$$= \begin{cases} 2a_{i-1} + a_i e_{i+1}, & i \text{ even} \\ -a_{i-1} - a_i e_{i+1}, & i \text{ odd} \end{cases},$$

which proves (iv). Finally, consider (v). The statements hold for $i = 1, 2$ respectively, so let $i \geq 3$. Then

$$e_i a_i \overset{(iii)}{=} \begin{cases} 2e_i a_{i-1} e_i + a_{i-2} e_i, & i \text{ even} \\ e_i a_{i-1} e_i - a_{i-2} e_i, & i \text{ odd} \end{cases}$$

$$\overset{(iv)}{=} \begin{cases} 2(-a_i + a_{i-1} e_i) + a_{i-2} e_i, & i \text{ even} \\ (a_i + a_{i-2}) e_i - a_{i-2} e_i, & i \text{ odd} \end{cases}$$

$$= \begin{cases} -a_i e_i, & i \text{ even} \\ a_i e_i, & i \text{ odd} \end{cases},$$

where the last equation in the even case holds due to $e_i^2 = 1$. This proves the first part of (v). Using it yields

$$e_{i-1} a_i + (-1)^i a_i e_{i-1} = \begin{cases} 2a_{i-1}(e_{i-1} e_i + e_i e_{i-1}) + e_{i-1} a_{i-2} + a_{i-2} e_{i-1}, & i \text{ even} \\ -a_{i-1}(e_{i-1} e_i + e_i e_{i-1}) + e_{i-1} a_{i-2} - a_{i-2} e_{i-1}, & i \text{ odd} \end{cases}$$

$$\overset{(iv)}{=} \begin{cases} -2a_{i-1} + 2a_{i-1} = 0, & i \text{ even} \\ a_{i-1} - a_{i-1} = 0, & i \text{ odd} \end{cases},$$

so the proof is finished. □

Theorem 3.5.10. *(i) For all $n \in \mathbb{N}$ and $1 \leq j \leq n$, the equality*

$$e_j a_n = (-1)^{n-1} a_n e_j$$

holds. In particular, $(1, a_n)$ is a $\mathbb{Q}$-basis of $\mathcal{Z}(\mathbb{Q}\mathbb{A}_n)$.

(ii) For all $n \geq 0$, we have $a_n^2 = (-1)^{\binom{n}{2}} \cdot \begin{cases} n+1, & n \text{ even} \\ \frac{n+1}{2}, & n \text{ odd} \end{cases}$.

(iii) a_n is a maximal orthogonal element for $\mathbb{A}_n$, for all $n \in \mathbb{N}$.

(iv) Put $\delta_n := a_n^2 = \operatorname{disq}(\mathbb{A}_n)$. Then, for all $n \in \mathbb{N}$, we have an isomorphism of $\mathbb{Z}$-orders

$$\mathcal{Z}(\mathbb{A}_n) \cong \begin{cases} \mathbb{Z}[X]/(X^2 - X + \frac{1-\delta_n}{4}), & n \text{ even} \\ \mathbb{Z}[X]/(X^2 - \delta_n), & n \text{ odd} \end{cases}.$$

Proof. Assertion (i) is clearly true for $i = 1$, so suppose that it already holds up to some fixed $n \in \mathbb{N}$. If $1 \le j \le n-1$, using this hypothesis, we find

$$\begin{aligned} e_j a_{n+1} &= e_j a_n e_{n+1} + e_j a_{n-1} = (-1)^{n-1} a_n e_j e_{n+1} + (-1)^{n-2} a_{n-1} e_j \\ &= (-1)^{n-1} a_n(-e_{n+1}e_j) + (-1)^n a_{n-1} e_j = (-1)^n a_{n+1} e_j. \end{aligned}$$

The remaining cases $j = n$ and $j = n+1$ follow from Lemma 3.5.9 (v). Assertion (ii) holds for $n = 0, 1$, so let $n \ge 2$ and note that assertion (i) implies

$$a_{n-1}a_{n-2} = (-1)^n a_{n-2}a_{n-1}.$$

Using this, we find

$$\begin{aligned} a_n^2 &= \begin{cases} 4a_{n-1}e_n a_{n-1}e_n + 2(a_{n-1}e_n a_{n-2} + a_{n-2}a_{n-1}e_n) + a_{n-2}^2, & n \text{ even} \\ a_{n-1}e_n a_{n-1}e_n + a_{n-1}e_n a_{n-2} + a_{n-2}a_{n-1}e_n + a_{n-2}^2, & n \text{ odd} \end{cases} \\ &= \begin{cases} 4a_{n-1}(-a_{n-2} - a_{n-1}e_n)e_n + 4a_{n-2}a_{n-1}e_n + a_{n-2}^2, & n \text{ even} \\ a_{n-1}(2a_{n-2} + a_{n-1}e_n)e_n + 2a_{n-2}a_{n-1}e_n + a_{n-2}^2, & n \text{ odd} \end{cases} \\ &= \begin{cases} -4a_{n-1}^2 + a_{n-2}^2, & n \text{ even} \\ a_{n-1}^2 + a_{n-2}^2, & n \text{ odd} \end{cases} \\ &= \begin{cases} (-1)^{\binom{n}{2}}(2n - (n-1)) = (-1)^{\binom{n}{2}}(n+1), & n \text{ even} \\ (-1)^{\binom{n}{2}}(n - \frac{n-1}{2}) = (-1)^{\binom{n}{2}}\frac{n+1}{2}, & n \text{ odd} \end{cases}, \end{aligned}$$

which proves (ii). Now (iii) follows from (i), (ii) and Lemma 3.5.9 (ii). Finally, (iv) follows from (iii). The respective isomorphism is given by sending a_n or $\frac{1+a_n}{2}$ to the residue class of X. □

Corollary 3.5.11. *$\mathcal{Z}(\mathbb{A}_n)$ is the maximal order in $\mathcal{Z}(\mathbb{Q}\mathbb{A}_n)$, if and only if $n+1$ is squarefree, for even n; and $\frac{n+1}{2}$ is squarefree, for odd n. Moreover, $\mathcal{Z}(\mathbb{A}_n) = \mathcal{Z}(\mathbb{A}_n)^o$ is orthogonal, if and only if n is odd.*

The lattices $\mathbb{D}_n$ $(n \ge 3)$

For $n \ge 3$, we fix a basis $(e_1, \dots, e_n)$ of $\mathbb{D}_n$, with associated Gram matrix 3.4. Then we have an embedding $\mathcal{C}(\mathbb{A}_{n-1}) \subset \mathcal{C}(\mathbb{D}_n)$ because the $\mathbb{Z}$-subalgebra generated by $(e_1, \dots, e_{n-1})$ is naturally isomorphic to $\mathcal{C}(\mathbb{A}_{n-1})$. Thus, the finite subsequence a_i, $i = 0, \dots, n-1$ which is defined as for the lattices $\mathbb{A}_{n-1}$ makes sense inside of $\mathcal{C}(\mathbb{D}_n)$

and for $n \geq 3$, we can define

$$d_n := a_{n-1}e_n + a_{n-3}e_{n-1} \in \mathcal{C}(\mathbb{D}_n).$$

We claim that d_n is a maximal orthogonal element for $\mathbb{D}_n$. As for the lattices of type A_n, we will again need a number of technical results. Note, that we have the results from Lemma 3.5.9 and Theorem 3.5.10 available, as long as all terms involved do lie in the subalgebra $\mathcal{C}(\mathbb{A}_{n-1})$.

Lemma 3.5.12. *Let $n \geq 3$. The following relations hold inside of $\mathcal{C}(\mathbb{D}_n)$.*

(i) For all $0 \leq i \leq n-3$, we have $e_n a_i = (-1)^i a_i e_n$.

(ii) $e_n a_{n-2} = (-1)^n(a_{n-2}e_n + \varepsilon_n a_{n-3}) = \begin{cases} 2a_{n-3} + a_{n-2}e_n, & n \text{ even} \\ -a_{n-3} - a_{n-2}e_n, & n \text{ odd} \end{cases}.$

(iii) $e_n a_{n-1} = (-1)^{n-1}(a_{n-1}e_n - 2a_{n-3}e_{n-1})$.

(iv) $a_{n-1} + a_{n-3}e_{n-1}e_{n-2} + (-1)^n e_{n-2}a_{n-3}e_{n-1} = 0$.

Proof. Assertion (i) follows from Lemma 3.5.9 (iii), while (ii) is pretty much just (iv) from said lemma, but in the current situation. Its proof is the same calculation, using $e_{n-2}e_n = -e_n e_{n-2}$. For assertion (iii), we compute

$$\begin{aligned} e_n a_{n-1} &= e_n(\varepsilon_{n-1}a_{n-2}e_{n-1} + a_{n-3}) \\ &\overset{(i),(ii)}{=} (-1)^n(\varepsilon_{n-1}a_{n-2}(-e_{n-1}e_n) + 2a_{n-3}e_{n-1}) + (-1)^{n-1}a_{n-3}e_n \\ &= (-1)^{n-1}(\varepsilon_{n-1}a_{n-2}e_{n-1} + a_{n-3})e_n + (-1)^n 2a_{n-3}e_{n-1} \\ &= (-1)^{n-1}(a_{n-1}e_n - 2a_{n-3}e_{n-1}). \end{aligned}$$

Finally, consider (iv). Using the recursive definition of a_{n-1} twice yields

$$a_{n-1} = 2a_{n-3}e_{n-2}e_{n-1} + \varepsilon_{n-1}a_{n-4}e_{n-1} + a_{n-3}.$$

Using this and $e_{n-1}e_{n-2} = -1 - e_{n-2}e_{n-1}$, we find the identity

$$a_{n-1} + a_{n-3}e_{n-1}e_{n-2} = (a_{n-3}e_{n-2} + \varepsilon_{n-1}a_{n-4})e_{n-1}.$$

Now, by Lemma 3.5.9 (iv), the equation

$$(-1)^n e_{n-2}a_{n-3}e_{n-1} = -(a_{n-3}e_{n-2} + \varepsilon_{n-1}a_{n-4})e_{n-1}$$

holds, whence the proof is finished. ☐

Theorem 3.5.13. *(i) For all $n \geq 3$ and $1 \leq j \leq n$, the equality*

$$e_j d_n = (-1)^{n-1} d_n e_j$$

holds. In particular, $(1, d_n)$ is a $\mathbb{Q}$-basis of $\mathcal{Z}(\mathbb{Q}\mathbb{D}_n)$.

(ii) For all $n \geq 0$, we have $d_n^2 = (-1)^{\binom{n}{2}} \cdot \begin{cases} 1, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}$.

(iii) d_n is a maximal orthogonal element for $\mathbb{D}_n$, for $n \geq 3$.

(iv) Put $\delta_n \coloneqq d_n^2 = \operatorname{disq}(\mathbb{D}_n)$. Then, for all $n \geq 3$, we have an isomorphism of $\mathbb{Z}$-orders $\mathcal{Z}(\mathbb{D}_n) \cong \mathbb{Z}[X]/(X^2 - \delta_n)$. More precisely,

$$\mathcal{Z}(\mathbb{D}_n) \cong \begin{cases} \mathbb{Z}[X]/(X^2-1), & n \equiv_4 0 \\ \mathbb{Z}[X]/(X^2-2), & n \equiv_4 1 \\ \mathbb{Z}[X]/(X^2+1), & n \equiv_4 2 \\ \mathbb{Z}[X]/(X^2+2), & n \equiv_4 3 \end{cases}.$$

Proof. Beginning with (i), we have $e_j d_n = e_j(a_{n-1}e_n - a_{n-3}e_j)$, so the claimed equality is easy to see for all $j \neq n-2, n$. We first consider $j = n$. Then, using Lemma 3.5.12 (iii), we compute

$$\begin{aligned} e_n d_n &= (-1)^{n-1}(a_{n-1}e_n + 2a_{n-3}e_{n-1})e_n - (-1)^{n-1}a_{n-3}e_n e_{n-1} \\ &= (-1)^{n-1}(a_{n-1} + 2a_{n-3}e_{n-1}e_n - a_{n-3}e_{n-1}e_n) \\ &= (-1)^{n-1}(a_{n-1}e_n + a_{n-3}e_{n-1})e_n = (-1)^{n-1}d_n e_n. \end{aligned}$$

Now let $j = n-2$. Similarly, we compute

$$\begin{aligned} e_{n-2}d_n &= (-1)^n(a_{n-1}e_{n-2}e_n) - e_{n-2}a_{n-3}e_{n-1} \\ &= (-1)^{n-1}(a_{n-1} + a_{n-1}e_n e_{n-2}) - e_{n-2}a_{n-3}e_{n-1} \\ &= (-1)^{n-1}(d_n e_{n-2} + a_{n-1} + a_{n-3}e_{n-1}e_{n-2} + (-1)^n e_{n-2}a_{n-3}e_{n-1}) \\ &= (-1)^{n-1}d_n e_{n-2}, \end{aligned}$$

where in the last step, Lemma 3.5.12 (iv) was used. This proves (i). For assertion (ii) we have

$$d_n^2 = (a_{n-1}e_{n-1})^2 - (a_{n-1}e_n a_{n-3}e_{n-1} + a_{n-3}e_{n-1}a_{n-1}e_n) + (a_{n-3}e_{n-1})^2.$$

Using $a_{n-1}a_{n-3} = a_{n-3}a_{n-1}$, for all $n \geq 3$ and Lemma 3.5.12 (i), the summand in the middle simplifies to

$$-(a_{n-1}e_n a_{n-3}e_{n-1} + a_{n-3}e_{n-1}a_{n-1}e_n) = (-1)^{n-1}2a_{n-1}a_{n-3}e_{n-1}e_n.$$

The same argument applied to the last summand yields

$$(a_{n-3}e_{n-1})^2 = (-1)^{n-3}a_{n-3}^2 e_{n-1}^2 = (-1)^{n-1}a_{n-3}^2.$$

Lastly, using Lemma 3.5.12 (iii), the first summand becomes

$$(a_{n-1}e_{n-1})^2 = (-1)^{n-1}(a_{n-1}^2 - 2a_{n-1}a_{n-3}e_{n-1}e_n).$$

After substituting the equalities for the three summands and using the identities for a_i^2 from Theorem 3.5.10, we obtain

$$d_n^2 = (-1)^{n-1}(a_{n-1}^2 + a_{n-3}^2) = (-1)^{\binom{n}{2}} \cdot \begin{cases} \frac{n}{2} - \frac{n-2}{2} = 1, & n \text{ even} \\ n - (n-2) = 2, & n \text{ odd} \end{cases}$$

whence (ii). Assertion (iii) follows from (i) and (ii), together with Lemma 3.5.9 (ii): By the definition of d_n, the coefficient as a summand of d_n of the basis element e_{n-1}, if n is odd; or of the basis element e_1e_{n-1}, otherwise, is -1 respectively. Finally, because the coefficient of $1 \in \mathcal{C}(\mathbb{D}_n)$ is zero, $\frac{1+d_n}{2} \notin \mathcal{C}(\mathbb{D}_n)$, so (iv) follows immediately from (iii). □

Corollary 3.5.14. *$\mathcal{Z}(\mathbb{D}_n)$ is the maximal order in $\mathcal{Z}(\mathbb{Q}\mathbb{D}_n)$, if and only if n is not divisible by 4. If $n \equiv_4 0$, $\mathcal{Z}(\mathbb{D}_n)$ is isomorphic to the unique index-two suborder of the maximal order $\mathbb{Z}[X]/(X^2 - X)$. In all cases, $\mathcal{Z}(\mathbb{D}_n) = \mathcal{Z}(\mathbb{D}_n)^o$ is orthogonal.*

The lattices $\mathbb{E}_n$ $(n = 6, 7, 8)$

For these lattices of exceptional type, finding a maximal orthogonal element, call it $x_n \in \mathcal{C}(\mathbb{E}_n)$, can be solved algorithmically; see Algorithm 2. Let $(e_1, \ldots, e_n)$ be a basis of $\mathbb{E}_n$ with associated Gram matrix 3.5. Then, as in the case of the lattices $\mathbb{D}_n$, we have a natural inclusion $\mathcal{C}(\mathbb{A}_{n-4}) \subset \mathcal{C}(\mathbb{E}_n)$ and the elements of the sequence a_i make sense inside $\mathcal{C}(E_n)$, for $i = 0, \ldots, n-4$.

Theorem 3.5.15. *Let $n = 6, 7, 8$.*

(i) The following elements x_n are maximal orthogonal for $\mathbb{E}_n$:

$$x_6 = a_2(1 + 2e_4e_5) + 2a_5e_6,$$
$$x_7 = a_3(1 + 2e_5e_6) + a_6e_7,$$
$$x_8 = a_4(1 + 2e_6e_7) + 2a_7e_8.$$

Moreover, the equalities $x_6^2 = -3,\ x_7^2 = -1,\ x_8^2 = 1$ *hold, so* $\operatorname{disq}(E_6) = -3$, $\operatorname{disq}(E_7) = -1$ *and* $\operatorname{disq}(E_8) = 1$.

(ii) We have the isomorphisms of $\mathbb{Z}$-orders

$$\mathcal{Z}(\mathbb{E}_6) \cong \mathbb{Z}[X]/(X^2 - X + 1),$$
$$\mathcal{Z}(\mathbb{E}_7) \cong \mathbb{Z}[X]/(X^2 + 1),$$
$$\mathcal{Z}(\mathbb{E}_8) \cong \mathbb{Z}[X]/(X^2 - X),$$

so $\mathcal{Z}(\mathbb{E}_n)$ is the unique maximal order in $\mathcal{Z}(\mathbb{Q}\mathbb{E}_n)$ in all cases.

Proof. Assertion (i) is an explicit computation, while (ii) follows from (i), and Lemma 3.5.9 (ii). For $n = 6, 8$ the isomorphism is given by sending $\frac{1+x_n}{2}$, for $n = 7$ by sending x_7 to the respective residue class of X. □

Remark 3.5.16. More generally, if one considers the lattices $\mathbb{E}_n$ with $n \neq 9$ and $n \geq 6$ that were described in Remark 3.5.5 then one can show that

$$x_n = \begin{cases} a_{n-4}(1 + 2e_{n-2}e_{n-1}) + 2a_{n-1}e_n, & n \text{ even} \\ a_{n-4}(1 + 2e_{n-2}e_{n-1}) + a_{n-1}e_n, & n \text{ odd} \end{cases}$$

is a maximal orthogonal element for $\mathbb{E}_n$ and that

$$x_n^2 = (-1)^{\binom{n}{2}} \cdot \begin{cases} 9 - n, & n \text{ even} \\ \frac{9-n}{2} & n \text{ odd} \end{cases}$$

holds. Hence, putting $\delta_n \coloneqq x_n^2$, we have the isomorphisms of $\mathbb{Z}$-orders

$$\mathcal{Z}(\mathbb{E}_n) \cong \begin{cases} \mathbb{Z}[X]/(X^2 - X + \frac{1-\delta_n}{4}), & n \text{ even} \\ \mathbb{Z}[X]/(X^2 - \delta_n), & n \text{ odd} \end{cases}.$$

In particular, for even n, $\mathcal{Z}(\mathbb{E}_n)$ is the maximal R-order in $\mathcal{Z}(\mathbb{Q}\mathbb{E}_n)$, if and only if $9-n$ is squarefree. For odd n, it is the maximal R-order, if and only if $\frac{9-n}{2}$ is squarefree.

3.6 Practical considerations

In this final section, we present how the centroid and the quadratic discriminant can be computed effectively over an arbitrary Dedekind domain R. This is done in Algorithm 1 and Algorithm 2 below. The idea of this algorithm is heavily based on the short discussion at the beginning of Subsection 3.5.2, as will become apparent in the proof of Theorem 3.6.2. Let (L,q) be non-degenerate of rank n with pseudo-basis $(e_i,\mathfrak{a}_i)_{i\in\underline{n}}$. For better readability, we identify 0^{-1} with K. This is effectively

Algorithm 1 Quadratic discriminant and centroid

Input: The Clifford order $\mathcal{C}(L)$ with pseudo-basis $\big((e_I,\mathfrak{a}_I)\mid I\subseteq\underline{n}\big)$

Output: Pseudo-elements $(\mathfrak{z},\mathfrak{b})$ and $(z,\mathfrak{a})$, such that $\operatorname{disq}(L)=(\mathfrak{b}^2\mathfrak{z}^2,\mathfrak{z}^2(K^\times)^2)$ and $\mathcal{Z}(L,q)=R1\oplus\mathfrak{a}z$

1: Compute an orthogonal basis $(x_1,\ldots,x_n)$ of (KL,b_q)

2: Put $\mathfrak{z}\leftarrow x_1\ldots x_n=\sum_{I\subseteq\underline{n}}\lambda_I e_I\in\mathcal{C}(KL)$, $\mathfrak{c}\leftarrow\bigcap_{\emptyset\neq I\subseteq\underline{n}}\lambda_I^{-1}\mathfrak{a}_I$ ▷ Identifying 0^{-1} with K

3: $\mathfrak{b}\leftarrow\lambda_\emptyset^{-1}R\cap\mathfrak{c}$, $z\leftarrow\lambda_\emptyset+\mathfrak{z}$ and $\mathfrak{a}\leftarrow(2\lambda_\emptyset)^{-1}R\cap\mathfrak{c}$ ▷ Identifying 0^{-1} with K

4: **return** $(\mathfrak{z},\mathfrak{b})$ and $(z,\mathfrak{a})$

the same as omitting the intersections with the ideals $\lambda_I^{-1}\mathfrak{a}$, if $I\subseteq\underline{n}$ with $\lambda_I=0$.

Remark 3.6.1. Let $\mathcal{Z}:=\mathcal{Z}(L,q)$.

(i) If $\lambda_\emptyset=0$, e.g. if n is odd (see Proposition 3.2.8), then $\mathcal{Z}=\mathcal{Z}^o$.

(ii) Algorithm 1 returns $\mathcal{Z}=\Lambda(\mathfrak{a},2\lambda_\emptyset,\lambda_\emptyset^2-\mathfrak{z}^2)$. If $\mathcal{Z}\supsetneq\mathcal{Z}^o$ then $\lambda_\emptyset\neq 0$ and we have

$$\Lambda(\mathfrak{a},2\lambda_\emptyset,\lambda_\emptyset^2-\mathfrak{z}^2)=\Lambda\left(2\lambda_\emptyset\mathfrak{a},1,\frac{\lambda_\emptyset^2-\mathfrak{z}^2}{4\lambda_\emptyset^2}\right).$$

Here the right-hand is as in Remark 3.1.6. It is obtained by adding a normalisation step to Algorithm 1. This in turns leads to the normalised algorithm Algorithm 2 further below.

Theorem 3.6.2. *Algorithm 1 is correct.*

Proof. Let $\mathcal{Z}:=\mathcal{Z}(L,q)$. We first focus on the quadratic discriminant. From Theorem 3.1.12 it follows that $\mathcal{Z}^o=R1\oplus\mathcal{Z}^o_-$, the latter summand being the elements in $\mathcal{Z}$ with vanishing trace. Now $\mathcal{Z}^o_-$ has rank one and from [KS02, (7.9) Satz], we obtain that $K\mathcal{Z}^o_-=K\mathfrak{z}$, with $\mathfrak{z}$ as defined in Algorithm 1. Thus, $\mathcal{Z}^o_-=\mathfrak{b}\mathfrak{z}$, with $\mathfrak{b}$ the largest ideal of R such that $\mathfrak{b}\mathfrak{z}\subseteq\mathcal{C}(L)$. Since $\big((e_I,\mathfrak{a}_I)\mid I\subseteq\underline{n}\big)$ is a pseudo-basis of $\mathcal{C}(L)$ and $\mathfrak{z} = \sum_{I \subseteq \underline{n}} \lambda_I e_I \in \mathcal{C}(KL)$, with unique $\lambda_I \in K$, this implies $\mathfrak{b}\lambda_I \subseteq \mathfrak{a}_I$, for all $I \subseteq \underline{n}$. Now,

$$\mathfrak{b} = \bigcap_{\substack{I \subseteq \underline{n}, \\ \lambda_I \neq 0}} \lambda_I^{-1}\mathfrak{a}_I$$

satisfies this and, by its definition, is the largest such ideal. Note that $\mathfrak{b} = \lambda_\emptyset^{-1} R \cap \mathfrak{c}$, if $\lambda_\emptyset \neq 0$; and $\mathfrak{b} = \mathfrak{c}$, otherwise. Thus, Algorithm 1 computes $\operatorname{disq}(L)$ correctly. With that it remains to consider $\mathcal{Z}$. Due to Proposition 2.1.8 and Theorem 3.1.12 (iii), there are fractional ideals $\mathfrak{b}'$, $R \subseteq \mathcal{J} \subseteq \frac{1}{2}R$ and some z, such that $\mathcal{Z}^o = R1 \oplus \mathfrak{b}'z$ and $\mathcal{Z} = R1 \oplus \mathcal{J}\mathfrak{b}'z$ (compatible pseudo-bases). Using Proposition 2.1.7 (i), we may assume that $\mathfrak{b}' = \mathfrak{b}$ and $z = \lambda + \mathfrak{z}$, with some $\lambda \in \mathfrak{b}^{-1}$. Clearly, λ must be chosen, such that $R \subseteq \mathcal{J} \subseteq \frac{1}{2}R$ is as large as possible, with $\mathcal{J}\mathfrak{b}z \subseteq \mathcal{C}(L)$. Now, $\mathcal{J}\mathfrak{b}z \subseteq \mathcal{C}(L)$ is equivalent to $\mathcal{J} \subseteq \mathfrak{b}^{-1}(\lambda_I^{-1}\mathfrak{a}_I)$, for all $\emptyset \neq I \subseteq \underline{n}$, with $\lambda_I \neq 0$; and $(\lambda + \lambda_\emptyset)\mathcal{J}\mathfrak{b} \subseteq \mathfrak{a}_\emptyset = R$. For short, the two conditions

$$\mathcal{J} \subseteq \mathfrak{b}^{-1}\mathfrak{c} \ \text{ and } \ (\lambda + \lambda_\emptyset)\mathcal{J}\mathfrak{b} \subseteq R$$

must hold. Note that only the second condition depends on λ and is always satisfied for $\lambda := \lambda_\emptyset \in \mathfrak{b}^{-1}$, due to $\mathcal{J} \subseteq \frac{1}{2}R$. Thus, $z := \lambda_\emptyset + \mathfrak{z}$ and since $R \subseteq \mathcal{J} \subseteq \frac{1}{2}R$ must be as large as possible with $\mathcal{J} \subseteq \mathfrak{b}^{-1}\mathfrak{c}$, we have $\mathcal{J} := \frac{1}{2}R \cap \mathfrak{b}^{-1}\mathfrak{c}$. Thus, $\mathfrak{a} := \mathcal{J}\mathfrak{b} = \frac{1}{2}\mathfrak{b} \cap \mathfrak{c}$ satisfies $\mathcal{Z} = R1 \oplus \mathfrak{a}z$. Finally, using $\mathfrak{b} = \lambda_\emptyset^{-1} R \cap \mathfrak{c}$, we obtain

$$\mathfrak{a} = \frac{1}{2}\mathfrak{b} \cap \mathfrak{c} = (2\lambda_\emptyset)^{-1} R \cap \frac{1}{2}\mathfrak{c} \cap \mathfrak{c} = (2\lambda_\emptyset)^{-1} R \cap \mathfrak{c},$$

so Algorithm 1 is correct. □

Remark 3.6.3. The algorithm itself does not really need to compute the maximal orthogonal suborder $\mathcal{Z}(L,q)^o$ before computing $\mathcal{Z}(L,q)$, even though this order of operations is crucial to the proof. Note that, in view of the discussion at the beginning of Subsection 3.5.2, we could call $\mathfrak{b}\mathfrak{z}$ the *maximal orthogonal pseudo-element* of (L,q). It is uniquely determined by (L,q).

As a part of this thesis, Algorithm 2 was implemented for the OSCAR project [Osc24].

Example 3.6.4. Put $\mathcal{Z} := \mathcal{Z}(L,q)$, $K = \mathbb{Q}(\sqrt{-5})$ and $R = \mathbb{Z}[\sqrt{-5}]$. Consider the R-lattice $(L,q) = Re_1 \oplus \mathfrak{a}'e_2$ with $q(e_1) = q(e_2) = 0$, $b_q(e_1,e_2) = -2$ and $R \subseteq \mathfrak{a}' \subseteq \frac{1}{2}R$. Clearly, $(KL,q) \cong \mathbb{H}(K)$ and, by Remark 3.2.4 (ii), $\mathcal{Z} = \Lambda(\mathfrak{a}', -2, 0)$ and $\mathcal{Z}^o = \Lambda(\frac{1}{2}R, 0, -4)$. We want to verify this again, using Algorithm 2, while also obtaining the canonical form of $\mathcal{Z}$ as described in Remark 3.1.6.

First, we compute the orthogonal basis (x_1, x_2) of KL, where $x_1 = e_1 - \frac{1}{4}e_2$, $x_2 =$

Algorithm 2 Quadratic discriminant and centroid (normalised)

Input: The Clifford order $\mathcal{C}(L)$ with pseudo-basis $\big((e_I, \mathfrak{a}_I) \mid I \subseteq \underline{n}\big)$

Output: Pseudo-elements $(\mathfrak{z}, \mathfrak{b})$ and $(z, \mathfrak{a})$, such that $\operatorname{disq}(L) = (\mathfrak{b}^2\mathfrak{z}^2, \mathfrak{z}^2(K^\times)^2)$ and $\mathcal{Z}(L,q) = R1 \oplus \mathfrak{a}z = \Lambda(\mathfrak{a}, \mathfrak{t}, \mathfrak{n})$. If $\mathfrak{t} \neq 0$, then $\mathfrak{t} = 1$ and $\mathfrak{a}$ is integral

Compute an orthogonal basis $(x_1, \dots, x_n)$ of (KL, b_q)

Put $\mathfrak{z} \leftarrow x_1 \dots x_n = \sum_{I \subseteq \underline{n}} \lambda_I e_I \in \mathcal{C}(KL)$, $\mathfrak{c} \leftarrow \bigcap_{\emptyset \neq I \subseteq \underline{n}} \lambda_I^{-1} \mathfrak{a}_I$ ▷ Identifying 0^{-1} with K

if $\lambda_\emptyset = 0$ **then** ▷ Here, $\mathcal{Z}(L,q) = \mathcal{Z}(L,q)^o$

  $\mathfrak{b} \leftarrow \mathfrak{c}$, $\mathfrak{a} \leftarrow \mathfrak{b}$ and $z \leftarrow \mathfrak{z}$

else ▷ Here, $\mathcal{Z}(L,q) \supsetneq \mathcal{Z}(L,q)^o$

  $\mathfrak{z} \leftarrow \frac{1}{2\lambda_\emptyset}\mathfrak{z}$ and $\mathfrak{c} \leftarrow 2\lambda_\emptyset \mathfrak{c}$ ▷ normalisation

  $\mathfrak{b} \leftarrow 2R \cap \mathfrak{c}$, $z \leftarrow \frac{1}{2} + \mathfrak{z}$ and $\mathfrak{a} \leftarrow R \cap \mathfrak{c}$

end if

return $(\mathfrak{z}, \mathfrak{b})$ and $(z, \mathfrak{a})$

$2e_1 + \frac{1}{2}e_2$, using a Gram-Schmidt method from [Osc24]. We find that $\mathfrak{z} \leftarrow x_1x_2 = 1 + e_1e_2$, so $\mathfrak{c} \leftarrow 1^{-1}\mathfrak{a}' = \mathfrak{a}'$ and $\lambda_\emptyset = 1 \neq 0$. Thus, we perform the normalisation step $\mathfrak{z} \leftarrow \frac{1}{2}\mathfrak{z} = \frac{1}{2}(1 + e_1e_2)$, $\mathfrak{c} \leftarrow 2\mathfrak{c} = 2\mathfrak{a}'$. Finally, $\mathfrak{b} \leftarrow 2R \cap \mathfrak{c} = 2R \cap 2\mathfrak{a}' = 2R$, $z \leftarrow \frac{1}{2} + \mathfrak{z} = 1 + \frac{1}{2}e_1e_2$ and $\mathfrak{a} \leftarrow R \cap \mathfrak{c} = R \cap 2\mathfrak{a}' = 2\mathfrak{a}'$. To summarise, the algorithm returns that $\mathcal{Z} = \Lambda(2\mathfrak{a}', 1, 0)$ and $\mathcal{Z}^o = \Lambda(2R, 0, -\frac{1}{4})$. Note that $2\mathfrak{a}'$ is indeed integral.

Remark 3.6.5. Note that $\Lambda(R, 0, -1) = \Lambda(2R, 0, -\frac{1}{4})$ in Example 3.6.4. In general there does not seem to be an obvious way to describe the maximal orthogonal suborder $\mathcal{Z}^o = \Lambda(\mathfrak{b}, 0, -\mathfrak{z}^2)$ canonically, given no further information about $\mathfrak{b}$. An exception to this is if $\mathcal{Z} \supsetneq \mathcal{Z}^o$, because the canonical form of $\mathcal{Z}$ also yields a description for $\mathcal{Z}^o$: In the notation of Algorithm 2, if $\mathcal{Z} = \Lambda(\mathfrak{a}, 1, \mathfrak{n})$, then $\mathcal{Z}^o = \Lambda\left(2R \cap \mathfrak{a}, 0, -\frac{1}{4}(1 - 4\mathfrak{n})\right)$.

4 Clifford orders over complete discrete valuation rings

The final chapter of this thesis concerns the study of Clifford orders in the following setting. Following [Neu92] II. §5, K is a local field of characteristic zero with normalised valuation ν. Let $R = \{x \in K \mid \nu(x) \geq 0\}$ be the valuation ring and π be a fixed uniformiser of R, that is $\nu(\pi) = 1$. Moreover, let $k = R/\pi R$ denote the finite residue field of positive characteristic p. Equivalently, K is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$, with $[K : \mathbb{Q}_p] = \nu(p) \cdot [k : \mathbb{F}_p]$, see [Neu92] II. (5.2) and (6.8). We call such a finite extension of $\mathbb{Q}_p$ a *p-adic number field*.

In the first part of this chapter, we provide some basic facts about quaternion algebras, because they will be used frequently throughout this chapter. After this, we record some basic facts about maximal lattices over complete discrete valuation rings. In addition to that, we revisit quadratic orders, mainly because their description is significantly easier and more specific over complete discrete valuation rings. Building on this, we provide a full classification of the centroids and the (even) Clifford orders of the maximal lattices. The main results of this chapter are Theorem 4.3.11, Theorem 4.4.6 and Theorem 4.4.7.

4.1 Quaternion algebras over p-adic number fields

Quaternion algebras occur naturally in the context of quadratic forms, because they are precisely the Clifford algebras of the two-dimensional regular quadratic K-spaces. In this section, we summarise the most important properties of quaternion algebras. The results in this chapter are taken from [GS17] and [Voi21].

Definition 4.1.1. A **(generalised) quaternion algebra** over K is a K-algebra that has a basis $(1, x, y, xy)$ with defining relations $x^2 = a, y^2 = b, xy = -yx$, for some $a, b \in K^\times$. This algebra is denoted by $(a, b)_K$ and a basis of this form is called a **quaternion basis** of $(a, b)_K$. The K-linear antiautomorphism

$$\overline{} : Q \to Q,\ \overline{\lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 xy} := \lambda_1 - \lambda_2 x - \lambda_3 y - \lambda_4 xy$$

is called the **canonical involution** on Q.

Proposition 4.1.2. *The canonical involution on Q has the following properties.*

(i) $N(w) := w\overline{w} \in K$ *and* $T(w) := w + \overline{w} \in K$*, for all* $w \in Q$*.*

(ii) $w^2 - T(w)w + N(w) = 0$*, for all* $w \in Q$*.*

(iii) $N(w_1 + w_2) = N(w_1) + N(w_2) + T(w_1\overline{w_2})$*, for all* $w_1, w_2 \in Q$*, so* N *is a quadratic form on* Q *with polarisation* $b_N(w_1, w_2) = T(w_1\overline{w_2})$*, called the* ***(reduced) norm form*** *on* Q*.*

Example 4.1.3. The canonical involution on $K^{2\times 2} = (1,1)_K$ is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, so the maps T and N are just the usual trace and determinant of 2×2 matrices.

Proposition 4.1.4 (cf. [Voi21] Theorem 5.4.4.)**.** *Let* $a, b \in K^\times$ *and put* $Q := (a,b)_K$*. The following statements are equivalent:*

(i) Q *is split, i.e.* $Q \cong K^{2\times 2}$ *as* K*-algebras.*

(ii) Q *is not a division algebra.*

(iii) The norm form (Q, N) *is isotropic.*

(iv) The norm form of the field extension $K(\sqrt{a})/K$*, that is,* $[1, -a]$ *represents* b*.*

Remark 4.1.5. Proposition 4.1.4 states that a quaternion algebra is either isomorphic to $K^{2\times 2}$, or isomorphic to a central K-division algebra in dimension four. The converse of this statement holds as well; see [GS17] Proposition 1.2.1. Thus, one can equivalently define a quaternion algebra to be a four-dimensional central simple K-algebra. The reduced norm and the reduced trace (see [Rei03] 9a) of such a central simple algebra Q coincide with the maps $N, T : \to K$ from Proposition 4.1.2, by [Voi21] 3.3.

Definition 4.1.6. Call two central simple K-algebras A, A' **Brauer equivalent**, if they are isomorphic to full matrix rings over the same K-division algebra. The **Brauer group** $\mathrm{Br}(K)$ is the abelian group of Brauer equivalence classes of central simple K-algebras with the tensor product as group operation.

The multiplication in $\mathrm{Br}(K)$ is well-defined, because the tensor product of central simple K-algebras is again a central simple, by [Sch85] §8, Theorem 3.2. The neutral element is the class of K. In abuse of notation, we write $(a,b)_K$ or just (a,b) to refer to the Brauer class of the quaternion algebra $(a,b)_K$. By [KS02] (11.10), the Brauer class of the tensor product of two quaternion algebras is again represented by a quaternion algebra. Moreover, $(a,b)^2 = 1_{\mathrm{Br}(K)}$, so the Brauer classes of quaternion algebras generate an elementary abelian 2-subgroup of $\mathrm{Br}(K)$. We denote it by $\mathrm{Br}_2(K)$.

4.1.1 Division algebras over p-adic number fields

Up until this point, we did not use the fact that K is a local field. In fact, all of the results in this section hold over any field of characteristic zero. However, over the local field K, the central division algebras are easily classified, immediately clarifying the structure of $\mathrm{Br}_2(K)$. In the following, we summarise the main results of chapter three in [Rei03].

Let D be a central K-division algebra. Then $\dim_K(D) = n^2$, for some $n \in \mathbb{N}$, called the *Schur index* of D, see [Rei03] (7.15). Similarly to extensions of p-adic number fields, there is a unique discrete valuation w on D that extends ν and is explicitly given by

$$w(a) := \frac{1}{n^2}\nu(N_{D/K}(a)),\ a \in D,$$

where $N_{D/K}$ denotes the usual norm map. It holds that $a \in D$ is integral over R, if and only if $N_{D/K}(a) \in R$, if and only if $w(a) \geq 0$. Thus, if one puts

$$\Delta := \{a \in D \mid w(a) \geq 0\} = \{a \in D \mid N_{D/K}(a) \in R\},$$

the *valuation ring* of D with respect to w, then Δ is the integral closure of R in D. One can show that Δ is a local ring, i.e. the set of non-units in Δ forms a two-sided ideal $\mathfrak{p} = \pi_D\Delta$. Here $\pi_D \in \Delta$ is any uniformiser of Δ, i.e. an element with minimal strictly positive valuation, which is of the form $\frac{1}{e}$, for some $e \in \mathbb{N}$. Further the residue class ring $\Delta/\mathfrak{p}$ is a skew field over the finite field k, i.e. a finite field extension of k. Put $f := f(D/K) := [\Delta/\mathfrak{p} : k]$. For finite k, one can show that $e = f = n$, so if we put $l := |k|$, there are $(l^n - 1)$-th roots of unity in D, due to Hensel's Lemma and n is maximal with that property. The following two theorems completely describe the structure of a central division algebra over the local field K of characteristic zero.

Theorem 4.1.7 ([Rei03] (14.5) Theorem)**.** *Let $\omega \in D$ be a primitive $(l^n - 1)$-th root of unity and $\pi \in R$ be a uniformiser. There is a uniformiser $\pi_D \in \Delta$ with*

$$\pi_D^n = \pi,\ \ \pi_D\omega\pi_D^{-1} = \omega^{l^r},$$

*where r is a positive integer, such that $1 \leq r \leq n$ and $\gcd(r, n) = 1$. The integer r is uniquely determined by the division algebra D and does not depend on the choice of ω or π. The fraction $\frac{r}{n}$ is called the **Hasse invariant** of D.*

Theorem 4.1.8 ([Rei03] (14.6) Theorem)**.** *Let $1 \leq r \leq n$ and $\gcd(r, n) = 1$. Then there exists a central K-division algebra D with index n and Hasse invariant $\frac{r}{n}$.*

Returning to quaternion algebras, we have $n = 2$. This implies that $\mathrm{Br}_2(K) \cong C_2$, so there is a unique four-dimensional K-division algebra. We denote it by $\mathcal{Q} = \mathcal{Q}_K$. Reiner also gives an explicit construction for the division algebra in Theorem 4.1.8.

Corollary 4.1.9. *Let Q be a quaternion algebra over K.*

(i) Either $Q \cong K^{2\times 2}$ or $Q \cong \mathcal{Q}_K =: \mathcal{Q}$ is isomorphic to the division algebra

$$\mathcal{Q} := \langle 1, \omega^*, \pi_{\mathcal{Q}}^*, \omega^*\pi_{\mathcal{Q}}^* \mid (\omega^*)^{l^2-1} = 1,\, (\pi_{\mathcal{Q}}^*)^2 = \pi,\, \pi_{\mathcal{Q}}^*\omega^* = (\omega^*)^l\pi_{\mathcal{Q}}^*\rangle_K.$$

(ii) Let ω be a (l^2-1)-th root of unity in the algebraic closure $\overline{K}$ and put $F := K(\omega)$. Then F is isomorphic to the unique unramified degree-two extension of K and a reduced representation of $\mathcal{Q}$ is given by

$$\varphi:\ F \otimes_K \mathcal{Q} \xrightarrow{\sim} F^{2\times 2}, 1 \otimes \omega^* \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^l \end{pmatrix}, 1 \otimes \pi_{\mathcal{Q}}^* \mapsto \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix}.$$

Remark 4.1.10. Using the explicit reduced representation of $\mathcal{Q}$ over F and Example 4.1.3, the norm form on $F \otimes_K \mathcal{Q} \cong F^{2\times 2}$ with respect to the basis $(1 \otimes 1, 1 \otimes \omega^*, 1 \otimes \pi_{\mathcal{Q}}^*, 1 \otimes \omega^*\pi_{\mathcal{Q}}^*)$ is given by

$$(F \otimes_K \mathcal{Q}, N) = \begin{bmatrix} 1 & \mathfrak{t} \\ & \mathfrak{n} \end{bmatrix} \perp \begin{bmatrix} -\pi & -\pi\mathfrak{t} \\ & -\pi\mathfrak{n} \end{bmatrix}.$$

Here $\mathfrak{t} := \omega^* + (\omega^*)^l$ is the *reduced trace* and $\mathfrak{n} := (\omega^*)^{l+1}$ is the *reduced norm* of ω^* and both quantities are units in R, because they lie in R and do not vanish mod π. In particular, the norm form on $\mathcal{Q}$ only takes values in K, thus defining the quadratic K-space $(\mathcal{Q}, N_{\text{red}})$. The quadratic form N_{red} is called the reduced norm of $\mathcal{Q}$. It is regular with discriminant $1(K^\times)^2$ and anisotropic, due to Proposition 4.1.4.

Notation 4.1.11. Put $\Delta := \langle 1, \omega^*, \pi_{\mathcal{Q}}^*, \omega^*\pi_{\mathcal{Q}}^* \mid (\omega^*)^{l^2-1} = 1,\, (\pi_{\mathcal{Q}}^*)^2 = \pi,\, \pi_{\mathcal{Q}}^*\omega^* = (\omega^*)^l\pi_{\mathcal{Q}}^*\rangle_R$. This is an R-order in $\mathcal{Q}$, so by Remark 4.1.10, (Δ, N_{red}) is an even R-lattice. Its discriminant is $(t^2-4n)^2\pi^2(R^\times)^2 = \pi^2(R^\times)^2$, due to $t^2 - 4n \in R^\times$. Then it follows from [Rei03] (14.9) and Theorem 2.2.13 that Δ is the unique maximal order in $\mathcal{Q}$.

4.2 Lattices over complete discrete valuation rings

We begin this section with a fundamental result that connects even unimodular R-lattices with regular quadratic k-spaces. Given a quadratic R-lattice (L, q), denote its reduction mod π by $(\overline{L}, \overline{q})$, the latter being a quadratic k-space.

Theorem 4.2.1 ([KS02] Satz (15.6))**.** *Let (E, q) be a quadratic R-lattice and (F, q') be a regular quadratic R-module. If $u : E \to F$ induces an isometry via $\overline{u} : (\overline{E}, \overline{q}) \to$*

$(\overline{F}, \overline{q'}), \overline{x} \mapsto \overline{u(x)}$, then there is an isometry $\widetilde{u} : E \to F$ with $\widetilde{u}(x) \equiv u(x) \bmod \pi$. In particular, if $\overline{E} \cong \overline{F}$ as quadratic k-spaces, then $E \cong F$ as quadratic R-lattices.

Note that the regular quadratic R-modules are precisely the even unimodular R-lattices, by Corollary 2.1.29.

Corollary 4.2.2. *Let (U, q) be an even unimodular R-lattice in an anisotropic quadratic K-space. Then $n = 2$ and U is isometric to the* ***norm form*** *$N(R)$. It is the valuation ring of the unique unramified degree-two extension of K, equipped with the usual relative field norm. In addition, if $2 \in R^\times$, then also $n = 1$ is possible, with $U \cong [1]$ or $U \cong [\varepsilon]$, where $\overline{\varepsilon} \in k^\times$ is any non-square.*

Proof. This follows from Theorem 4.2.1, $\overline{N(R)} = N(k)$ and Proposition 1.1.10. □

Remark 4.2.3. The set of elements represented by $N(R)$ is given by

$$\bigcup_{i=0}^{\infty} \pi^{2i} R^\times \cup \{0\} = \{x \in R \mid \nu(x) \equiv_2 0\} \cup \{0\}.$$

We denote the ambient space of $N(R)$ by $N(K)$. Thus, $N(K)$ is the unique unramified degree-two extension of K, equipped with the relative field norm. Moreover, $N(K)$ represents precisely the elements of K with even valuation and zero.

4.2.1 Maximal lattices

In Corollary 3.2.17, we already described the general structure of a maximal lattice over a Dedekind domain. Interpreting this result in the more specific setting of this chapter, we obtain that an arbitrary maximal R-lattice (E, q) with ambient space (V, q) decomposes as

$$(E, q) = (E', q') \perp \mathop{\perp}_{i=1}^{k} \mathbb{H}(R).$$

Here, E' denotes a maximal R-lattice inside the anisotropic kernel V' of V. It is well known that E' is uniquely determined. Put $E_i(V', q') := \{x \in V' \mid q'(x) \in \pi^i R\}$, for $i \in \mathbb{N}_0$.

Proposition 4.2.4 ([KS02] (16.1))**.** *$E_i(V', q')$ is an R-submodule of V', so it is the unique π^i-maximal lattice in (V', q'). In particular,*

$$(E, q) = E_0(V', q') \perp \mathop{\perp}_{i=1}^{k} \mathbb{H}(R)$$

is the unique maximal R-lattice in (V, q). We call $E_0(V', q')$ the ***anisotropic kernel*** *of (E, q).*

Thus, in order to classify all maximal R-lattices, it is necessary to classify the anisotropic quadratic K-spaces first. We summarise the known theory.

Proposition 4.2.5. *Let (V, q) be an anisotropic quadratic K-space.*

(i) $n := \dim(V) \leq 4$.

(ii) V *is determined up to isometry by the three invariants* $\dim(V) \bmod 2$, $\operatorname{disq}(V)$ *and the Clifford invariant* $\mathfrak{c}(V) := [\mathcal{C}(V)] \in \operatorname{Br}_2(K)$. *Any such triple occurs as invariants of an anisotropic quadratic K-space.*

(iii) *If* $\dim(V) = 4$, *then* $(V, q) = N(K) \perp {}^{\pi}N(K)$ *and V is universal. We denote this space by* $\mathcal{U}_K$.

Proof. This is the content of [KS02] (16.3) - (16.9). Just note that by definition, $\operatorname{disq}(V) = \operatorname{disc}(V)$, if $\dim(V)$ is even; and $\operatorname{disq}(V) = \operatorname{disc}'(V)$, if $\dim(V)$ is odd. □

Remark 4.2.6. Using $\operatorname{Br}_2(K) \cong C_2$ and Corollary 4.1.9, we make the identification $\mathfrak{c}(V) = 1$, if $\mathcal{C}(V)$ is a full matrix ring over K; and $\mathfrak{c}(V) = -1$, if $\mathcal{C}(V)$ is a full matrix ring over $\mathcal{Q}$.

Proposition 4.2.7. $\mathcal{U}_K \cong (\mathcal{Q}, N_{\text{red}})$ *and* $E_0(\mathcal{U}_K) \cong (\Delta, N_{\text{red}}) \cong N(R) \perp {}^{\pi}N(R)$.

Proof. By Remark 4.1.10, $(\mathcal{Q}, N_{\text{red}})$ is anisotropic and four-dimensional, so the first claim follows from Proposition 4.2.5 (iii). The second claim follows from the first one and Notation 4.1.11, because Δ is the unique maximal order in $\mathcal{Q}$. □

Classification of maximal anisotropic lattices

In the following, (V, q) denotes an anisotropic quadratic K-space of dimension n. Put $E := E_0(V, q)$. Following Proposition 4.2.5 (i), we go through the possible dimensions $n \in \{0, 1, 2, 3, 4\}$. The classification presented in Theorem 4.2.19 is taken from [Eic74] II. Satz 9.7.

We agree on the following additional notation. Put $e := \nu(2)$, which is zero unless $p = 2$. Moreover, let δ denote a fixed representative of the square class $\operatorname{disq}(V)$, with $\nu(\delta) \in \{0, 1\}$. For simplicity, we write $\delta = \operatorname{disq}(V)$ instead of $\delta(K^\times)^2 = \operatorname{disq}(V)$ and $\delta = \operatorname{disq}(E)$ instead of $\delta(R^\times)^2 = \operatorname{disq}(E)$. Finally, call the K-square class $\delta(K^\times)^2$ *even*, if $\nu(\delta) = 0$; and *odd* otherwise.

Remark 4.2.8. By Proposition 4.2.5 (ii), there are precisely $4|K^\times/(K^\times)^2|$ isometry classes of anisotropic quadratic K-spaces. If $p \neq 2$, then a transversal of $K^\times/(K^\times)^2$ is $\{1, \varepsilon, \pi, \varepsilon\}$, with $\varepsilon \in R^\times$, such that $\overline{\varepsilon} \notin (k^\times)^2$. If $p = 2$, we have $|K^\times/(K^\times)^2| = 2^{d+2}$ with $d = [K : \mathbb{Q}_2]$, by [Neu92] II (5.8).

To summarise, there exist precisely 16 isometry classes of anisotropic quadratic K-spaces in the non-dyadic case and 2^{d+4} of such isometry classes in the dyadic case.

<u>n = 0:</u> Trivially, $V = E = \{0\}$. For consistency reasons, we put $\mathrm{disq}(V) := 1$ and similarly $\mathrm{disq}(E) := 1$.

<u>n = 1:</u> Here, $V = [\delta]$ as quadratic K-space and, due to $\nu(\delta) \in \{0, 1\}$, also $E = [\delta]$ is the maximal lattice.

<u>n = 2:</u> Due to Proposition 4.2.5 (ii), there are at most two isometry classes of anisotropic quadratic K-spaces with $\mathrm{disq}(V) = \delta$.

Remark 4.2.9. The representative δ is a non-square. Let $L := K(\sqrt{\delta})$, $N = N_{L/K}$ the relative field norm and $\eta \in K^\times$, with $\eta \notin N(L)$. Then $(V, q) \cong (L, N)$, if $\mathfrak{c}(V) = 1$; and $(V, q) \cong (L, {}^\eta N)$, if $\mathfrak{c}(V) = -1$. Thus, the maximal lattice (E, q) is obtained by restricting N or ${}^\eta N$ (for a suitable choice of η) to the valuation ring of L.

Proof. $\{0\}$ and $\mathcal{U}_K$ exhaust the even-dimensional spaces with $\mathrm{disq}(V) = 1(K^\times)^2$. We have $\mathfrak{c}(L, N) = (1, -\delta)_K = 1$ and $\mathfrak{c}(L, {}^\eta N) = (\eta, -\eta\delta)_K = (\eta, -\eta\delta) = (\delta, \eta) = -1$, using Proposition 4.1.4 (iv) and [KS02] (11.10). Clearly, both spaces have quadratic discriminant δ. □

Proposition 4.2.10. *Keep the notation from Remark 4.2.9 and denote the valuation ring of L by S.*

(i) If the square class of δ is odd, an R-basis of S is given by $(1, \sqrt{\delta})$. Thus,

$$(S, N) \cong [1, -\delta] \quad \textit{and} \quad (S, {}^\eta N) \cong [\eta, -\eta\delta],$$

with $\eta \in R^\times$.

(ii) If the square class of δ is even, an R-basis of S is given by $(1, \pi^{-g}(u + \sqrt{\delta}))$, where $0 \leq g \leq e$ is the maximal integer such that $\delta \equiv u^2 \bmod \pi^{2g}$, for some $u \in R^\times$. Thus,

$$(S, N) \cong \begin{bmatrix} 1 & 2\pi^{-g}u \\ & \pi^{-2g}(u^2 - \delta) \end{bmatrix} \quad \textit{and} \quad (S, {}^\eta N) \cong \begin{bmatrix} \eta & 2\pi^{-g}u\eta \\ & \pi^{-2g}(u^2 - \delta)\eta \end{bmatrix},$$

with $\eta \in R^\times$, if $K(\sqrt{\delta})$ is ramified; and $\eta = \pi$, if $K(\sqrt{\delta})$ is unramified.

Proof. In [Eic74] II 6.1, the R-bases of S are computed. The isometries of R-lattices then follow from $(L, N) = [1, -\delta]$, with respect to the K-basis $(1, -\delta)$ of L. □

The integer g in assertion (ii) is called the *quadratic defect*. In the remainder of this thesis it will occur frequently, so we define it separately.

Definition 4.2.11. Let $x \in K^\times$, $\nu(x) \in \{0,1\}$. The **quadratic defect** of x in K is

$$\operatorname{defq}(x) := \operatorname{defq}_K(x) := \begin{cases} e, & x \text{ is a square} \\ \max\{n \in \mathbb{N}_0 \mid \exists u \in R^\times : x \equiv_{\pi^{2n}} u^2\}, & \text{otherwise} \end{cases}.$$

Proposition 4.2.12. *Let $x \in K^\times$ with $\nu(x) \in \{0,1\}$.*

(i) If $\nu(x) = 0$ or $p \neq 2$, then $\operatorname{defq}(x) = 0$.

(ii) One has $0 \leq \operatorname{defq}(x) \leq e$.

Proof. Assertion (i) is obvious. For assertion (ii), the case $p = 2$ and x even is proven in [Eic74, II. Satz 6.1]. The remaining cases follow from (i). □

Remark 4.2.13. In [OMe00] §63A, the quadratic defect of an arbitrary element x of $K^\times$ is defined as a certain ideal $I_x \trianglelefteq R$. For $x \in K^\times$ with $\nu(x) \in \{0,1\}$, it compares to our definition as follows. The element x is a square, if and only if $I_x = \{0\}$. If $\nu(x) = 0$, then $I_x = \pi^{\operatorname{defq}(x)}R$. If $\nu(x) = 1$, then $I_x = xR = \pi R$.

Our definition of the quadratic defect has the advantage, that we can describe the maximal anisotropic R-lattices, and later on their Clifford orders, independent of the residue characteristic p. For now, we collect some further properties of the quadratic defect in the context of the two-dimensional lattices.

Remark 4.2.14. Keep the notations from Proposition 4.2.10.

(i) After replacing δ by a suitable representative, we may assume $u = 1$. There is a unit $a \in R^\times$ with $2 = a\pi^e$, so

$$\begin{bmatrix} 1 & 2\pi^{-g}u \\ & \pi^{-2g}(u^2 - \delta) \end{bmatrix} \cong \begin{bmatrix} 1 & \pi^{e-g} \\ & \pi^{2(e-g)}(\frac{1-\delta}{4}) \end{bmatrix}.$$

(ii) If $g = 0$, then

$$\begin{bmatrix} 1 & 2\pi^{-g}u \\ & \pi^{-2g}(u^2 - \delta) \end{bmatrix} \cong [1, -\delta].$$

In particular, (S, N) admits an orthogonal basis, if and only if $g = 0$.

This description allows us to conclude the following classical result about the unramified degree-two extension of a 2-adic number field, see e.g. [DV18] Proposition 4.8.

Proposition 4.2.15. *Let $p = 2$ and $d \in R^\times$. Then $K(\sqrt{d})$ is unramified, if and only if d has maximal quadratic defect* $\operatorname{defq}(d) = e > 0$*, if and only if d is a square mod* 4.

Proof. If $d \in (R^\times)^2$ is a square, the claim is obvious, so suppose d that is a non-square. The even R-lattice $(E, q) = \begin{bmatrix} 1 & \pi^{e-g} \\ & \pi^{2(e-g)}(\frac{1-d}{4}) \end{bmatrix}$ with $\operatorname{disq}(E) = d$ and $g = \operatorname{defq}(d)$ is unimodular, if and only if $g = e$, if and only if $d \equiv_4 1$. This implies $(E, q) \cong N(R)$, by Corollary 2.1.29, Theorem 4.2.1 and using the fact that (E, q) is anisotropic. Now, by Corollary 4.2.2, $N(R)$ is the norm form of the unique unramified degree-two extension of K. □

Corollary 4.2.16. *Let $\varepsilon \in R$. Then $K(\sqrt{\varepsilon})$ is the unique unramified degree-two field extension of K, if and only if* $\operatorname{disq}(N(R)) = \varepsilon(R^\times)^2$.

Proof. For $p = 2$, this follows from the proof Proposition 4.2.15. For $p \neq 2$, this follows from $N(R) = [1, -\operatorname{disq}(N(R))]$ and Corollary 4.2.2. □

Corollary 4.2.17. *Let $\varepsilon \in R^\times$ such that $K(\sqrt{\varepsilon})$ is unramified of degree two over K. We have $(\varepsilon, \delta) = -1$, if and only if $\nu(\delta) \equiv_2 1$. In particular, $\varepsilon \notin N_{K(\sqrt{\delta})/K}\Big(K(\sqrt{\delta})\Big)$, if and only if $\nu(\delta) \equiv_2 1$.*

Proof. By Proposition 4.1.4 (iv) and Corollary 4.2.16, we have $(\varepsilon, \delta) = -1$, if and only if δ is not represented by $N(R)$. This is equivalent to $\nu(\delta) \equiv_2 1$, by Remark 4.2.3. □

As a consequence, we always choose $\eta = \varepsilon$ in Proposition 4.2.10 (i).

<u>**n = 3:**</u> In dimension three, there exists exactly one anisotropic quadratic K-space (V, q), for each possible choice of δ. They all satisfy $\mathfrak{c}(V) = -1$. From [Eic74] II. Satz 9.7, we obtain the maximal lattices as follows.

Proposition 4.2.18. *Let $\varepsilon \in R^\times$, with $K(\sqrt{\varepsilon})$ unramified of degree two over K.*

(i) If the square class of δ is odd, the maximal lattice in (V, q) is given by

$$(E, q) \cong N(R) \perp [\varepsilon\delta].$$

(ii) If the square class of δ is even, the maximal lattice in (V, q) is given by

$$(E, q) \cong {}^{\pi}N(R) \perp [\varepsilon\delta].$$

n = 4: This case is the content of Proposition 4.2.7.

Theorem 4.2.19. *For each one of the* $4|K/(K^\times)^2|$ *distinct isometry classes of anisotropic regular quadratic* K*-spaces, the maximal lattice* (E, q) *in* (V, q) *is given by the table below. Here,* δ *runs through all square classes of* K*, satisfies* $\nu(\delta) \in \{0, 1\}$ *and* $\delta \equiv_{\pi^{2g}} 1$*, with* $g := \mathrm{defq}(\delta)$ *the quadratic defect. The last column in the table denotes the number of isometry classes for the respective row.*

dim(V)	disq(V, q)	$\mathfrak{c}(V, q)$	max. lattice in (V, q)	No. of spaces
0	1	1	$\{0\}$	1
1	δ	1	$[\delta]$	$\lvert K/(K^\times)^2\rvert$
2	δ odd	1	$[1, -\delta]$	$\frac{1}{2}\lvert K/(K^\times)^2\rvert$
2	$1 \neq \delta$ even	1	$\begin{bmatrix} 1 & \pi^{e-g} \\ & \pi^{2(e-g)}\left(\frac{1-\delta}{4}\right) \end{bmatrix}$	$\frac{1}{2}\lvert K/(K^\times)^2\rvert - 1$
2	δ odd	-1	$[\varepsilon, -\varepsilon\delta]$	$\frac{1}{2}\lvert K/(K^\times)^2\rvert$
2	$1 \neq \delta$ even	-1	$\begin{bmatrix} \eta & \pi^{e-g}\eta \\ & \pi^{2(e-g)}\left(\frac{1-\delta}{4}\right)\eta \end{bmatrix}$	$\frac{1}{2}\lvert K/(K^\times)^2\rvert - 1$
3	δ odd	-1	$N(R) \perp [\varepsilon\delta]$	$\frac{1}{2}\lvert K/(K^\times)^2\rvert$
3	δ even	-1	${}^{\pi}N(R) \perp [\varepsilon\delta]$	$\frac{1}{2}\lvert K/(K^\times)^2\rvert$
4	1	-1	$N(R) \perp {}^{\pi}N(R)$	1

Theorem 4.2.20. *In the non-dyadic case, for each isometry class of anisotropic regular quadratic* K*-spaces, the maximal lattice* (E, q) *in* (V, q) *admits the orthogonal basis given in the table below, where* $\delta \in \{1, \varepsilon, \pi, \varepsilon\pi\}$*.*

dim(V)	disq(V, q)	$\mathfrak{c}(V, q)$	max. lattice in (V, q)	No. of spaces
0	1	1	$\{0\}$	1
1	δ	1	$[\delta]$	4
2	$1 \neq \delta$	1	$[1, -\delta]$	3
2	ε	-1	$[\pi, -\varepsilon\pi]$	1
2	π	-1	$[\varepsilon, -\varepsilon\pi]$	1
2	$\varepsilon\pi$	-1	$[\varepsilon, -\pi]$	1
3	1	-1	$[\pi, -\varepsilon\pi, \varepsilon]$	1
3	ε	-1	$[\pi, -\varepsilon\pi, 1]$	1
3	π	-1	$[1, -\varepsilon, \varepsilon\pi]$	1
3	$\varepsilon\pi$	-1	$[1, -\varepsilon, \pi]$	1
4	1	-1	$[1, -\varepsilon, \pi, -\varepsilon\pi]$	1

4.3 Centroids over complete discrete valuation rings

In this section, we consider the centroids of Clifford orders over local fields. First, we provide an easier description of quadratic R-orders, using the fact that any R-lattice is free. After this, we state local versions of the more important theorems from the previous chapter; see, e.g. Theorem 4.3.8. Finally, in Theorem 4.3.11, we present the classification of the centroids of the maximal quadratic R-lattices.

4.3.1 Quadratic orders over local fields

Put $e := \nu(2)$ and let $d \in K^\times$ with $\nu(d) \in \{0,1\}$. Further, let Λ be an R-order in the étale algebra $A := K[X]/(X^2 - d)$. Then, because Λ is free, we can write $\Lambda = \Lambda(R,t,n) = R[z]$ using Notation 3.1.3. Here, $z^2 - tz + n = 0$, where $t = z + \beta(z)$ and $n = z\beta(z)$ as in Remark 3.1.2 and both lie in R. By Theorem 3.1.12, this lattice admits an orthogonal basis, if and only if $t \in 2R$. This is equivalent to $\nu(t) \geq e$.

Notation 4.3.1. One either has $t = 0$, or after replacing z by $z+1$ and thus t by $t+2$, we can assume that $0 \leq \nu(t) \leq \nu(2)$. Hence, we can assume $t \in R^\times$, if $p \neq 2$; and $2t^{-1} \in R$, if $p = 2$. In these cases, we put $\Lambda(t,n) := \Lambda$.

We put the results of Theorem 3.1.12 in the new context. The quadratic order $\Lambda := \Lambda(t,n)$ has the basis $(1,z)$, so if $p \neq 2$, then $(1, t-2z)$ is another basis, so $\Lambda = \Lambda^o = \Lambda(0, -(t^2-4n))$.

If instead $p = 2$ and $t \neq 0$, then $(1, 1-2t^{-1}z)$ is an R-basis of $\Lambda^o = \Lambda(0, -(1 - 4t^{-2}n))$, so $[\Lambda : \Lambda^o] = \pi^{e-\nu(t)}$. Note that $\Lambda^o \cong R[X]/(X^2 - \pi^{2i}d)$, for some suitable $i \in \mathbb{N}_0$. Hence, there is an $a \in R^\times$ with

$$\pi^{2i}d = a^2(1 - 4t^{-2}n).$$

Comparing the valuations on both sides, either $\nu(t) = e$ and then $\Lambda = \Lambda^o = \Lambda(0, -(1-4t^{-2}n))$, or the right-hand side of this equation is a unit, which implies $i = 0$ and $\nu(d) = 0$. In the latter case, put $g := \operatorname{defq}(d)$. Then $e - \nu(t) \leq g$, by Proposition 4.2.10, yielding an upper bound for the index $[\Lambda : R[X]/(X^2-d)] = [\Lambda : \Lambda^o]$. Conversely, write

$$d = 1 + \pi^{2g}u = 1 - 4\pi^{2(g-e)}u'$$

with suitable $u, u' \in R^\times$. Then $\Lambda(\pi^{e-g}, u')$ is an R-order in A, which contains the suborder $R[X]/(X^2-d)$ of index π^g. Thus, the upper bound is achieved and

$$\Lambda(\pi^{e-g}, u') = \Lambda\left(\pi^{e-g}, \pi^{2(e-g)}\tfrac{1-d}{4}\right) \cong \begin{bmatrix} 1 & \pi^{e-g} \\ & \pi^{2(e-g)}\left(\frac{1-d}{4}\right) \end{bmatrix}$$

is the unique maximal R-order in A. With that we have a complete overview of the R-orders in A.

Notation 4.3.2. Let $A = K[X]/(X^2 - d)$ and $g := \text{defq}(d)$. For $j \in \underline{g}$, put $\Lambda_{-j}(d) := \Lambda\left(\pi^{e-j}, \pi^{2(e-j)}\frac{1-d}{4}\right)$. For $j \geq 0$, put $\Lambda_j(d) := \Lambda(0, -\pi^{2j}d)$.

Theorem 4.3.3. *Let $A = K[X]/(X^2 - d)$ and $g := \text{defq}(d)$. The orders in A are linearly ordered,*

$$\Lambda_{-g}(d) \supset \Lambda_{-g+1}(d) \supset \ldots \supset \Lambda_0(d) \supset \Lambda_1(d) \supset \ldots,$$

so $\Lambda_{-g}(d)$ is the unique maximal order in A. In this inclusion chain, each order has index π in the previous one. Finally, for $j \geq 0$, $\Lambda_{-j}(d)^o = \Lambda_0(d)$ and $\Lambda_j(d)^o = \Lambda_j(d)$.

Remark 4.3.4. (i) If d is a non-square, $(\Lambda_{-g}(d), q)$ is the unique maximal anisotropic R-lattice in the two-dimensional K-space with discriminant d and Clifford invariant 1.

(ii) If d is a square then $\Lambda_{-e}(d) \cong \mathbb{H}(R)$.

Proof. Assertion (i) is immediate from Theorem 4.2.19. For (ii) we may without loss of generality assume $d = 1$, so one calculates

$$(\Lambda_{-e}(1), q) = (\Lambda(1, 0), q) \cong \begin{bmatrix} 1 & 1 \\ & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \cong \mathbb{H}(R). \qquad \square$$

We also obtain another representation of the norm form $N(R)$.

Remark 4.3.5. Let $\overline{K}$ be the algebraic closure of the p-adic number field K. Put $l := |k|$, let $\omega \in \overline{K}$ be an $(l^2 - 1)$-th root of unity, so that $L := K(\omega)$ is the unique unramified degree-two field extension of K, by Corollary 4.1.9. Further, put $\mathfrak{t} := N_{L/K}(\omega) = \omega + \omega^l$ and $\mathfrak{n} := N(\omega) := \omega^{l+1}$. Then

$$(R[\omega], N_{L/K}) = \begin{bmatrix} 1 & \mathfrak{t} \\ & \mathfrak{n} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ & \mathfrak{t}^{-2}\mathfrak{n} \end{bmatrix} = N(R).$$

In particular, $R[\omega]$ is the valuation ring of L.

Proof. Only the last equality needs a proof. Put $\Lambda := (R[\omega], N_{L/K})$ and a unit $\varepsilon \in R^\times$ with $\text{disq}(\Lambda) = \varepsilon$. Then $\Lambda = \Lambda_i(\varepsilon)$, with some $i \geq -e$, by Theorem 4.3.3. An easy computation yields that $\Lambda^o = \Lambda\left(0, 1 - 4\mathfrak{t}^{-2}\mathfrak{n}\right) = \Lambda_0(\varepsilon)$ is the index π^e sublattice of Λ. Thus, $i = -e$ and $R[\omega]$ is the unique maximal order in L, i.e. its valuation ring. $\square$

Using Theorem 4.3.3, we collect some properties of the quadratic discriminant of a quadratic order.

Remark 4.3.6. Let $\Lambda = \Lambda_i(d)$, with $i \geq -\mathrm{defq}(d)$. Then $\mathrm{disq}(\Lambda) = \pi^{2\max\{0,i\}}d(R^\times)^2$ and $\Lambda^o = R[X]/(X^2 - \mathrm{disq}(\Lambda))$.

Proposition 4.3.7. *(i) If* $j \in \underline{\mathrm{defq}(d)}$ *then* $\mathrm{disc}(\Lambda_{-j}(d)) = \pi^{2(e-j)}\mathrm{disq}(\Lambda_{-j}(d)) = \pi^{2(e-j)}d(R^\times)^2$.

(ii) If $j \geq 0$ *then* $\mathrm{disc}(\Lambda_j(d)) = 4\mathrm{disq}(\Lambda_j(d)) = 4\pi^{2j}d(R^\times)^2$.

(iii) If Λ *is a quadratic order then* $\mathrm{disq}(\Lambda)$ *is a square class of units, if and only if* d *is even and* $\Lambda = \Lambda_{-i}(d)$*, for some* $0 \leq i \leq \mathrm{defq}(d)$.

The centroid of an orthogonal direct sum

Using Notation 4.3.2, we provide the local version of Theorem 3.2.11.

Theorem 4.3.8. *Suppose that the non-degenerate even R-lattice (L,q) decomposes as the orthogonal direct sum $(L,q) = (L_1,q_1) \perp (L_2,q_2)$ of two non-zero lattices and put $s := \mathrm{rank}(L_1)\mathrm{rank}(L_2)$. Write $\mathcal{Z}(L_i,q_i) = \Lambda_{j_i}(d_i)$, with $j_i \geq -\mathrm{defq}(d_i)$. Here, d_i is a representative of* $\mathrm{disq}(KL_i)$*, with $\nu(d_i) \in \{0,1\}$. Then*

$$\mathcal{Z}(L,q) \cong \begin{cases} \Lambda_{j_1+j_2}((-1)^s d_1 d_2), & j_1, j_2 \geq 0 \text{ and } \nu(d_1d_2) \in \{0,1\} \\ \Lambda_{j_1+j_2+1}((-1)^s \pi^{-2} d_1 d_2), & j_1, j_2 \geq 0 \text{ and } \nu(d_1d_2) = 2 \\ \Lambda_{\max\{j_1,j_2\}}(d_1d_2), & \text{else} \end{cases}.$$

In particular, $\mathcal{Z}(L,q)^o = \mathcal{Z}(L,q)$, if and only if $j_1 \geq 0$ or $j_2 \geq 0$. Otherwise, if both $j_1, j_2 < 0$, the ranks of L_1 and L_2 are even and $\mathcal{Z}(L,q)^o = \Lambda_0(d_1d_2) \subsetneq \Lambda_{\max\{j_1,j_2\}}(d_1d_2) = \mathcal{Z}(L,q)$. In particular, $\mathrm{disq}(L) = (-1)^s\mathrm{disq}(L_1)\mathrm{disq}(L_2)$.

We also present a result for the special case where the lattice (L,q) admits an orthogonal basis. Although this is a consequence of Theorem 4.3.8, we choose to prove it directly instead.

Theorem 4.3.9. *Suppose that the non-degenerate even R-lattice (L,q) of rank n admits an orthogonal basis $(l_1,\dots,l_n)$ and put $\delta := (-1)^{\binom{r}{2}}q(l_1)\dots q(l_r)$. Then $\mathcal{Z}(L,q) = \mathcal{Z}(L,q)^o \cong R[X]/(X^2-\delta)$, so* $\mathrm{disq}(L) = \delta(R^\times)^2$*. Moreover, if $\delta = \pi^{2i}d$, with some $i \geq 0$ and $\nu(d) \in \{0,1\}$, then $\mathcal{Z}(L,q) = \Lambda_i(d)$.*

Proof. The result holds for $\mathcal{Z}(V,q)$ by [KS02] II Satz (7.9). Now, $(l_I \mid I \subseteq \underline{n})$ is an R-basis of $\mathcal{C}(L)$, by Example 2.3.4. Thus, using $\mathcal{Z}(L,q) = \mathcal{Z}(V,q) \cap \mathcal{C}(L)$, the result follows. □

4.3.2 Classification of the centroids of maximal lattices

The key ingredients for the classification of the centroids of the maximal lattices are Corollary 3.2.17 and Theorem 4.2.19. The corollary states that the centroid of a maximal lattice equals the centroid of its anisotropic kernel and the theorem provides a full list of the latter. Thus, in the following, we go through the table in Theorem 4.2.19 and explicitly compute their centroids. Note that given these centroids, we easily obtain their quadratic discriminants from Proposition 4.3.7.

We use the notation from Theorem 4.2.19. Thus, (V, q) is an anisotropic quadratic K-space of dimension $n \in \{0, 1, 2, 3, 4\}$ and $E := E_0(V, q)$ is its unique maximal sublattice. Further, let δ be a representative of $\operatorname{disq}(V)$ with $\nu(\delta) \in \{0, 1\}$ and $\delta \equiv_{\pi^{2g}} 1$, where $g := \operatorname{defq}(\delta)$. For simplicity, we write $\delta = \operatorname{disq}(V)$ instead of $\delta(K^\times)^2 = \operatorname{disq}(V)$ and $\delta = \operatorname{disq}(E)$ instead of $\delta(R^\times)^2 = \operatorname{disq}(E)$

n = 0: Clearly, $E = \{0\}$ implies $\mathcal{Z}(E, q) = R$.

n = 1: Here, $(E, q) = [\delta]$ and $\mathcal{Z}(E, q) = \Lambda_0(\delta)$, by Theorem 4.3.9. Thus, the centroid has index π^g in the unique maximal order of $\mathcal{Z}(V, q)$ and $\operatorname{disq}(E) = \delta$.

n = 2: By Theorem 4.2.19 and Theorem 4.3.9, it is enough to consider the cases where δ is a non-square and represents an even square class. Then, depending on the Clifford invariant of (V, q), we have

$$(E, q) \cong \begin{bmatrix} 1 & \pi^{e-g} \\ & \pi^{2(e-g)}(\frac{1-\delta}{4}) \end{bmatrix} \text{ or } (E, q) \cong \begin{bmatrix} \eta & \pi^{e-g}\eta \\ & \pi^{2(e-g)}(\frac{1-\delta}{4})\eta \end{bmatrix},$$

with η not a norm in $K(\sqrt{\delta})$. Thus, by Remark 1.2.19, we find that

$$\mathcal{Z}(E, q) = \Lambda\left(\pi^{e-g}, \pi^{2(e-g)}(\tfrac{1-\delta}{4})\right) \text{ or } \mathcal{Z}(E, q) = \Lambda\left(\pi^{e-g}\eta, \pi^{2(e-g)}(\tfrac{1-\delta}{4})\eta^2\right).$$

Now, if $\mathfrak{c}(V) = -1$, then, by Proposition 4.2.10 (ii), either $(E, q) \cong {}^{\eta}N(R)$, so we may choose $\eta = \pi$; or, in any other case, $\eta \in R^\times$ is a unit. Theorem 4.3.3 implies

$$\mathcal{Z}(E, q) = \begin{cases} \Lambda_{-e+1}(\varepsilon), & (E, q) \cong {}^{\pi}N(R) \\ \Lambda_{-g}(\delta), & \text{else} \end{cases},$$

where $\varepsilon \in R^\times$ is a fixed element such that $K(\sqrt{\varepsilon})$ is unramified of degree two over K. These orders are maximal in $\mathcal{Z}(V, q)$, unless if $(E, q) \cong {}^{\pi}N(R)$, in which case it has index π in the maximal order. Finally, we have $\operatorname{disq}({}^{\pi}N(R)) = \pi^{2t}\delta$ and, in all other cases, $\operatorname{disq}(E) = \delta$. Here, $t := \max\{0, -e+1\}$, so $t = 0$, if $p = 2$; and $t = 1$, if $p \neq 2$.

Remark 4.3.10. If $(E, q) \cong N(R)$ or $(E, q) \cong {}^{\pi}N(R)$, then $\varepsilon(K^\times)^2 = \delta(K^\times)^2$, so we may assume $\varepsilon = \delta$.

<u>**n = 3:**</u> By Theorem 4.2.19, we have $(E,q) \cong N(R) \perp [\varepsilon\delta]$, if the square class of δ is odd; and $(E,q) \cong {}^{\pi}N(R) \perp [\varepsilon\delta]$, otherwise. We already know $\mathcal{Z}([\varepsilon\delta]) = \Lambda_0(\varepsilon\delta)$, $\mathcal{Z}(N(R)) = \Lambda_{-e}(\varepsilon)$ and $\mathcal{Z}({}^{\pi}N(R)) = \Lambda_{-e+1}(\varepsilon)$ from the cases $n = 1$ and $n = 2$. Hence, using Theorem 4.3.8, we find that

$$\mathcal{Z}(E,q) \cong \Lambda_{\max\{0,-e\}}(\varepsilon^2\delta) = \Lambda_0(\delta),$$

if the square class of δ is odd, so this is the unique maximal order in $\mathcal{Z}(V,q)$ and $\operatorname{disq}(E,q) = \delta$. In the remaining cases, we find

$$\mathcal{Z}(E,q) = \Lambda_{\max\{0,-e+1\}}(\varepsilon^2\delta) = \Lambda_t(\delta),$$

again writing $t \coloneqq \max\{0,-e+1\}$. Thus, $\mathcal{Z}(E,q)$ is the index π^{t+g} suborder of the maximal order in $\mathcal{Z}(V,q)$ and $\operatorname{disq}(E,q) = \pi^{2t}\delta$.

<u>**n = 4:**</u> Here, we only have $(V,q) = \mathcal{U}_K$, with $(E,q) = N(R) \perp {}^{\pi}N(R)$. From the case $n = 2$, we know that $\mathcal{Z}(N(R)) = \Lambda_{-e}(\varepsilon)$ and $\mathcal{Z}({}^{\pi}N(R)) = \Lambda_{-e+1}(\varepsilon)$. Thus, by Theorem 4.3.8, we have $\mathcal{Z}(E,q) = \Lambda_{\max\{-e,-e+1\}}(\varepsilon^2) = \Lambda_{-e+1}(1)$, so it is the unique suborder of index π in the maximal order $\Lambda_{-e}(1) = \Lambda(1,0) = R \oplus R$, whence $\mathcal{Z}(E,q) = \langle (1,1),(0,\pi) \rangle$. Finally, $\operatorname{disq}(E,q) = \pi^{2t}\delta$, with t as above.

Summary In Theorem 4.3.11 below, we provide the classification of the centroids and the quadratic discriminants of the maximal anisotropic quadratic R-lattices (E,q) in form of a table. It contains five columns, the second being the most notable one. It is labelled 'Cond.' and has the following purpose: In Theorem 4.2.19, we describe the isometry class of the maximal anisotropic lattices (E,q) in terms of $\delta = \operatorname{disq}(V,q)$. There, as well as in the preceding discussion, we noticed that we sometimes have to infer multiple conditions on δ; see especially the case $n = 2$ above. Thus, the column 'Cond.' contains abbreviations for the many possible conditions that we may need to infer. Below is an exhaustive overview of all sets of conditions that may occur either in Theorem 4.3.11, but also later on in Theorem 4.4.6 and Theorem 4.4.7, where these abbreviations are used again.

Cond.	Explanation
$-$	p is arbitrary and δ can only take one value.
$p2-$	$p = 2$ and δ can only take one value.
$np2-$	$p \neq 2$ and δ can only take one value.
a	Both p and δ are arbitrary.
νi	p is arbitrary and $\nu(\delta) = i \in \{0,1\}$.
$\nu 0 n1$	p is arbitrary and $\nu(\delta) = 0$ with $\delta \neq 1$.
$\nu 0 n1\varepsilon$	$\nu(\delta) = 0$ with $\delta \neq 1, \varepsilon$ (this implies $p = 2$).
$p2\nu 0 n1$	$p = 2$ and $\nu(\delta) = 0$ with $\delta \neq 1$.

Beyond this, the first column describes the isometry type of the lattice (E,q), while the third and fourth contain information about the quadratic discriminant and the centroid of (E, q), respectively. The last column contains the index of $\mathcal{Z}(E, q)$ in the unique maximal order of its ambient algebra $\mathcal{Z}(V, q)$.

Theorem 4.3.11. *An overview of the centroids and quadratic discriminants of the maximal anisotropic quadratic R-lattices over the p-adic valuation ring R is given by the table below. Here, $\delta = \operatorname{disq}(V)$ runs through the square classes of K with $\nu(\delta) \in \{0, 1\}$ and $\delta \equiv_{\pi^{2g}} 1$, where $g := \operatorname{defq}(\delta)$. Moreover, δ is subject to the respective condition in the column 'Cond.'. Finally, $t := \max\{0, -e+1\}$, so $t = 0$ in the dyadic case; and $t = 1$ in the non-dyadic case.*

Max. lattice in (V, q)	Cond.	$\operatorname{disq}(E)$	$\mathcal{Z}(E, q)$	Index
$\{0\}$	–	1	R	1
$[\delta]$	a	δ	$\Lambda_0(\delta)$	π^g
$[1, -\delta]$	$\nu 1$	δ	$\Lambda_0(\delta)$	π^g
$\begin{bmatrix} 1 & \pi^{e-g} \\ & \pi^{2(e-g)}\left(\frac{1-\delta}{4}\right) \end{bmatrix}$	$\nu 0 n 1$	δ	$\Lambda_{-g}(\delta)$	1
$[\varepsilon, -\varepsilon\delta]$	$\nu 1$	δ	$\Lambda_0(\delta)$	1
${}^{\pi}N(R)$	–	$\pi^{2t}\varepsilon$	$\Lambda_{-e+1}(\varepsilon)$	π
$\begin{bmatrix} \eta & \pi^{e-g}\eta \\ & \pi^{2(e-g)}\left(\frac{1-\delta}{4}\right)\eta \end{bmatrix}$	$\nu 0 n 1 \varepsilon$	δ	$\Lambda_{-g}(\delta)$	π^g
$N(R) \perp [\varepsilon\delta]$	$\nu 1$	δ	$\Lambda_0(\delta)$	1
${}^{\pi}N(R) \perp [\varepsilon\delta]$	$\nu 0$	$\pi^{2t}\delta$	$\Lambda_t(\delta)$	π^{t+g}
$N(R) \perp {}^{\pi}N(R)$	–	π^{2t}	$\Lambda_{-e+1}(1)$	π

4.4 Clifford orders of maximal lattices

In this section we provide a complete overview of both the Clifford orders and the even Clifford orders of the maximal R-lattices (E, q). By Corollary 3.4.9,

$$\mathcal{C}(E) \cong \mathcal{C}(E')^{2^k \times 2^k},$$

as graded R-algebras, with $k = \operatorname{ind}(KE)$ and E' the anisotropic kernel of E. Thus, it is again sufficient to consider the anisotropic maximal R-lattices and determine their Clifford orders and even Clifford orders, respectively. The results can be found in Theorem 4.4.6 and Theorem 4.4.7.

Throughout this section, we use the following, mostly familiar notation. Let (V, q) be an n-dimensional anisotropic quadratic K-space with $\delta = \operatorname{disq}(V)$, such that $\nu(\delta) \in \{0,1\}$. Moreover, we assume $\delta \equiv_{\pi^{2g}} 1$, with $g := \mathrm{defq}(\delta)$, the quadratic defect from Definition 4.2.11. Further, let (E,q) be the unique maximal R-lattice in (V,q).

We make two specific choices for δ, namely $\delta = 1$ to represent $1(K^\times)^2$ and $\delta = \varepsilon \in R^\times$ a unit with $L := K(\sqrt{\varepsilon})$ the unique unramified degree-two extension of K. We further specify this ε: Recall that, by Corollary 4.1.9 (i), the unique four-dimensional K-division algebra $\mathcal{Q} := \mathcal{Q}_K$ has the presentation

$$\mathcal{Q} := \langle 1, \omega^*, \pi_{\mathcal{Q}}^*, \omega^*\pi_{\mathcal{Q}}^* \mid (\omega^*)^{l^2-1} = 1,\, (\pi_{\mathcal{Q}}^*)^2 = \pi,\, \pi_{\mathcal{Q}}^*\omega^* = (\omega^*)^l \pi_{\mathcal{Q}}^* \rangle_K,$$

with $l = |k|$. For simplicity, we usually omit the $*$ from now on and regard ω^* as an element of L, so that $L = K(\omega)$. Now, if we put $\mathfrak{t} := \omega + \omega^l$, $\mathfrak{n} := \omega^{l+1}$, the reduced trace and reduced norm of ω, then $R[\omega]$ is the valuation ring of L, by Remark 4.3.5 and

$$(R[\omega], N_{L/K}) = \begin{bmatrix} 1 & \mathfrak{t} \\ & \mathfrak{n} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ & \mathfrak{t}^{-2}\mathfrak{n} \end{bmatrix} = N(R).$$

Thus, comparing discriminants, $\varepsilon' := \mathfrak{t}^2 - 4\mathfrak{n} = (\omega - \omega^l)^2 \in R^\times$ is a non-square with $K(\sqrt{\varepsilon'}) = K(\omega)$. Using Proposition 4.2.15, we conclude that ε' has maximal quadratic defect $\mathrm{defq}(\varepsilon') = e$. Now, recall that our choice for ε must satisfy $\varepsilon \equiv_{\pi^{2e}} 1$, but as of yet $\varepsilon' \equiv_{\pi^{2e}} \mathfrak{t}^2$. Note that $\mathfrak{t} \in R^\times$ as its reduction mod π does not vanish, so we choose $\varepsilon := \mathfrak{t}^{-2}\varepsilon' = (1 - 2\mathfrak{t}^{-1}\omega)^2 = 1 - 4\mathfrak{t}^{-2}\mathfrak{n} \in R^\times$ as our representative. In particular, this implies $\mathfrak{n} = \frac{\mathfrak{t}^2}{4}(1-\varepsilon)$, so the discriminant of the quadratic space

$$N(R) = \begin{bmatrix} 1 & 1 \\ & \mathfrak{t}^{-2}\mathfrak{n} \end{bmatrix}$$

is exactly ε. For later use, we also fix the root $\sqrt{\varepsilon} := \mathfrak{t}^{-1}(\omega^l - \omega) = 1 - 2\mathfrak{t}^{-1}\omega$, regarding it as an element of $R[\omega]$.

4.4.1 Clifford orders of anisotropic maximal lattices - computations

In this subsection we conduct a case study on the (even) Clifford orders of maximal anisotropic lattices. We abbreviate $\otimes_R$ by $\otimes$, as we will use it constantly throughout this chapter. We start by noting that all occurring (even) Clifford orders do have a reduced matrix representation as a suborder of either $K^{2^n \times 2^n}$ or $K(\omega)^{2^n \times 2^n}$ and that the first one is possible, if and only if $\mathfrak{c}(V,q) = 1$. We call representations of the first type R-representations and of the second type $R[\omega]$-representations. Clearly, given an $R[\omega]$-representation, it is easy to produce an R-representation.

Proposition 4.4.1. *Let $\mathfrak{t} = \omega + \omega^l$, $\mathfrak{n} = \omega^{l+1}$, so that $\omega^2 - \mathfrak{t}\omega + \mathfrak{n} = 0$. Then*

$$\iota : R[\omega] \hookrightarrow R^{2\times 2},\ 1 \mapsto I_2,\ \omega \mapsto \begin{pmatrix} 0 & -\mathfrak{n} \\ 1 & \mathfrak{t} \end{pmatrix}$$

induces a monomorphism of R-algebras, by R-linear extension. It respects the $\mathbb{Z}/2\mathbb{Z}$-gradings of the respective ambient algebras. Here, we have the gradings $K(\omega) = K1 \oplus K\sqrt{\varepsilon}$ and for $K^{2\times 2}$, we use the grading from Definition 3.4.8.

Proof. This is obvious because the image of ω under this map is just the companion matrix of ω. □

Example 4.4.2. From Corollary 4.1.9 we obtain an $R[\omega]$-representation of the unique maximal R-order Δ in $\mathcal{Q}$ (see Notation 4.1.11). This representation is given by

$$\Delta \hookrightarrow R[\omega]^{2\times 2},\ \omega^* \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \mathfrak{t} - \omega \end{pmatrix},\ \pi_{\mathcal{Q}}^* \mapsto \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix}.$$

Then, applying ι from Proposition 4.4.1 entrywise, we obtain the R-representation

$$\Delta \hookrightarrow R^{4\times 4},\ \omega^* \mapsto \left(\begin{array}{c|c} \begin{smallmatrix} 0 & -\mathfrak{n} \\ 1 & \mathfrak{t} \end{smallmatrix} & \mathbf{0} \\ \hline \mathbf{0} & \begin{smallmatrix} -\mathfrak{t} & \mathfrak{n} \\ -1 & 0 \end{smallmatrix} \end{array}\right),\ \pi_{\mathcal{Q}} \mapsto \left(\begin{array}{c|c} \mathbf{0} & \begin{smallmatrix} \pi & 0 \\ 0 & \pi \end{smallmatrix} \\ \hline \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \mathbf{0} \end{array}\right).$$

More generally, given an $R[\omega]$-representation as matrices in $R[\omega]^{n\times n}$, we obtain an R-representation as matrices in $R^{2n\times 2n}$, by applying ι entry-wise. Keeping this in mind, we start with the classification of the Clifford orders of the maximal R-lattices. We go through the isometry classes of the lattices listed in Theorem 4.2.19.

n = 0: Here, $V = E = \{0\}$ as sets, so $\mathcal{C}(E) = \mathcal{C}_0(E) = R$ is the unique maximal order in $\mathcal{C}(V) = \mathcal{C}_0(V) = K$.

n = 1: $\operatorname{disq}(V,q) = \delta$ forces $(E,q) = [\delta]$. Thus, $\mathcal{C}(E) \cong R[X]/(X^2 - \delta) = \Lambda_0(\delta)$ as an R-algebra, due to $\nu(\delta) \in \{0,1\}$. Thus, $\mathcal{C}(E)$ is the unique index π^g suborder of the unique maximal R-order $\Lambda_{-g}(\delta)$ in $\mathcal{C}(V)$. Recall that we have $g = 0$, whenever $p \neq 2$; or $p = 2$ and $\nu(\delta) = 1$, by Proposition 4.2.12. Hence, in these cases, $\mathcal{C}(E)$ is the unique maximal R-order. In terms of R-representations, we have

$$\Lambda_0(\delta) = \left\langle I_2,\ \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} \right\rangle_R \quad\text{and}\quad \Lambda_{-g}(\delta) = \left\langle I_2,\ \pi^{-g}\begin{pmatrix} 1 & \delta \\ 1 & 1 \end{pmatrix} \right\rangle_R.$$

Finally, it is clear that $\mathcal{C}_0(E) = R$ is the unique maximal order in $\mathcal{C}_0(V) = K$.

n = 2: We make two preliminary remarks that should seem familiar. Firstly, by Remark 3.2.4, the even Clifford order $\mathcal{C}_0(E)$ coincides with the centroid $\mathcal{Z}(E,q)$ and we already computed these Theorem 4.3.11. Secondly, there is no maximal anisotropic R-lattice with quadratic discriminant $\delta = 1$ in dimension two. Keeping this in mind, it remains to compute $\mathcal{C}(E)$ in four cases. These arise from the possibilities for the values $\nu(\delta) \in \{0,1\}$ and $\mathfrak{c}(V,q) \in \{1,-1\}$. Specifically, in the non-dyadic case there are six lattices to consider: For a fixed Clifford invariant, there are two lattices with $\nu(\delta) = 1$ respectively, namely $[1, -\pi], [1, -\varepsilon\pi]$ and $[\varepsilon, -\varepsilon\pi], [\varepsilon, -\pi]$. Similarly, for $\nu(\delta) = 0$, the lattices $N(R)$ and ${}^{\pi}N(R)$ are to be considered.

Notation 4.4.3. Since it will occur frequently in the following, we extend Definition 4.1.1 to valuation rings. Thus, given the local field X with valuation ring S and $a, b \in S - \{0\}$, the symbol $(a, b)_S$ denotes the S-algebra with basis $(1, x, y, xy)$ and the relations $x^2 = a$, $y^2 = b$ and $xy = -yx$. We call such a basis a *quaternion basis* of $(a, b)_S$.

Clearly, $(a, b)_S$ is an S-order in the quaternion algebra $(a, b)_X$ and any non-degenerate even S-lattice that admits an orthogonal basis has a Clifford order of the form $(a, b)_S$, for some $a, b \in S - \{0\}$.

We start by considering the cases $\mathfrak{c}(V, q) = 1$, i.e. $\mathcal{C}(V) \cong K^{2\times 2}$. First, let $\nu(\delta) = 1$. By Theorem 4.2.19, we have $(E, q) = [1, -\delta]$, i.e. $\mathcal{C}(E) = (1, -\delta)_R$. Let $(1, x, y, xy)$ be a quaternion basis of $\mathcal{C}(E)$. There is a monomorphism of R-algebras $\varphi : \mathcal{C}(E) \to R^{2\times 2}$ given by

$$\varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \varphi(y) = \begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}.$$

A simple computation shows

$$\varphi\left(\frac{1}{2}(1 + x)\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \varphi\left(\frac{1}{2}(1 - x)\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\varphi\left(\frac{1}{2}(y - xy)\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \varphi\left(-\frac{1}{2\delta}(y + xy)\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and these four matrices form an R-basis of the maximal R-order $R^{2\times 2}$ in $K^{2\times 2}$. Clearly, $[R^{2\times 2} : \varphi(\mathcal{C}(E))] = 4\delta$. Now $\nu(4\delta) = 2e + 1$, due to $\nu(\delta) = 1$. Thus, $\mathcal{C}(E)$ is a suborder of index π^{2e+1} in $R^{2\times 2}$. More precisely, as a suborder of $R^{2\times 2}$, we have

$$\mathcal{C}(E) \cong \left\{ \begin{pmatrix} x_1 + x_2 & -\delta(x_3 + x_4) \\ x_3 - x_4 & x_1 - x_2 \end{pmatrix} \middle| \ x_1, x_2, x_3, x_4 \in R \right\}$$
$$= \left\{ \begin{pmatrix} x_1 + x_2 & -\pi(x_3 + x_4) \\ x_3 - x_4 & x_1 - x_2 \end{pmatrix} \middle| \ x_1, x_2, x_3, x_4 \in R \right\}.$$

In particular, independent of the quadratic discriminant δ, all the Clifford orders $\mathcal{C}(E)$ are isomorphic as R-algebras in this case, but not as graded algebras. The grading of the ambient algebra $K^{2\times 2}$, of course, depends only on δ and so does the grading of $\mathcal{C}(E)$. If $p \neq 2$ then (E, q) is either equal to $[1, -\pi]$ or $[1, -\varepsilon\pi]$ and in both cases the algebra above simplifies to the index-π suborder

$$\mathcal{C}(E) \cong \begin{pmatrix} R & \pi R \\ R & R \end{pmatrix} \leq R^{2\times 2}.$$

Next, let $\nu(\delta) = 0$. Then, by Remark 4.2.14 (i) and Theorem 4.2.19, we have

$$(E, q) = \begin{bmatrix} 1 & 2\pi^{-g} \\ & \pi^{-2g}(1-\delta) \end{bmatrix} \cong \begin{bmatrix} 1 & \pi^{e-g} \\ & \pi^{2(e-g)}\frac{(1-\delta)}{4} \end{bmatrix}.$$

We use the first description to avoid introducing the unit $a \in R^\times$ with $2 = a\pi^e$; the multiplication of the second basis element with it induces the isometry above. By Proposition 4.2.10, the description on the left-hand side is obtained as the norm form of $K(\sqrt{\delta})$, restricted to its valuation ring, call it R_δ, with respect to the R-basis $B = (1, \pi^{-g}(1 + \sqrt{\delta}))$ of R_δ. Put $E' := (R_\delta)^o$, the maximal orthogonal sublattice. Then $B' = (1, \sqrt{\delta})$ is an R-basis of E', so $E' = [1, -\delta]$ and the change of basis from B to B' is given by the matrix

$$T := \begin{pmatrix} 1 & -1 \\ 0 & \pi^g \end{pmatrix} \in R^{2\times 2}.$$

Thus, the Clifford order $\mathcal{C}(E)$ contains $\mathcal{C}(E') = (1, -\delta)_R$ as an R-suborder. The change of basis from the former order to the latter one is given by the matrix

$$\begin{pmatrix} 1 & & -1 \\ & T & \\ & & \pi^g \end{pmatrix} \in R^{4\times 4}.$$

Thus, $[\mathcal{C}(E) : \mathcal{C}(E')] = \pi^{2g}$. More precisely, if $(1, x, y', xy')$ is a quaternion basis of $(1, -\delta)_R$, then $(1, x, y, xy)$ is an R-basis of $\mathcal{C}(E)$, where $y := \pi^{-g}(x + y')$. In particular, an R-representation of $\mathcal{C}(E)$ is given by

$$\varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \varphi(y) = \pi^{-g}\begin{pmatrix} 1 & -\delta \\ 1 & -1 \end{pmatrix}.$$

Hence, the index of $\mathcal{C}(E)$ in any maximal order of $\mathcal{C}(V) \cong K^{2\times 2}$ is $\pi^{2(e-g)}$. This can either be verified by direct computation, or by noticing that $[R^{2\times 2} : \mathcal{C}(E')] = \pi^{2e}$. In particular, $C(E)$ is a maximal order, if and only if $K(\sqrt{\delta})$ is unramified of degree two over K and this equivalent to $(E, q) = N(R)$ and $\delta = \varepsilon$, by Corollary 4.2.16. Thus, we have $\mathcal{C}(N(R)) \cong R^{2\times 2} \cong \mathcal{C}(\mathbb{H}(R))$ as R-algebras, but not as graded R-algebras. Finally, if $p \neq 2$ then $(E, q) = N(R)$ is the only option in the case we just considered.

Only the quadratic spaces with $\mathfrak{c}(V,q) = -1$ remain. Thus, the Clifford order $\mathcal{C}(E)$ is an R-order in unique four-dimensional K-division algebra $\mathcal{Q} := \mathcal{Q}_K$ from Corollary 4.1.9 (i). Recall that $\mathcal{Q}$ contains the unique maximal R-order

$$\Delta = \langle 1, \omega, \pi_{\mathcal{Q}}, \omega\pi_{\mathcal{Q}} \mid \omega^{l^2-1} = 1,\ \pi_{\mathcal{Q}}^2 = \pi,\ \pi_{\mathcal{Q}}\omega = \omega^l \pi_{\mathcal{Q}} \rangle_R,$$

so we try to describe $\mathcal{C}(E)$ in terms of Δ.

Let $\nu(\delta) = 1$. Then Theorem 4.2.19 yields $(E,q) = [\varepsilon, -\varepsilon\delta]$, so $\mathcal{C}(E) = (\varepsilon, -\varepsilon\delta)_R$. Thus, we must find elements $x, y \in \Delta$ that satisfy the relations

$$x^2 = \varepsilon,\ y^2 = -\varepsilon\delta \text{ and } xy = -yx.$$

To construct x, we compare the R-order $R[\sqrt{\varepsilon}]$ to the valuation ring of the unramified extension $K(\sqrt{\varepsilon}) = K(\omega)$, which is $R[\omega]$, by Remark 4.3.5. Using the identity $\omega^2 - \mathfrak{t}\omega + \mathfrak{n} = 0$, we have

$$(\mathfrak{t}^{-1}(\mathfrak{t} - 2\omega))^2 = \mathfrak{t}^{-2}(\mathfrak{t}^2 - 4\omega\mathfrak{t} + 4(\omega\mathfrak{t} - \mathfrak{n})) = \mathfrak{t}^{-2}(\mathfrak{t}^2 - 4\mathfrak{n}) = \varepsilon,$$

by the definition of ε. Thus, $R[\sqrt{\varepsilon}]$ is an R-suborder of $R[\omega]$, the base change being given by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -2t^{-1} \end{smallmatrix}\right) \in R^{2\times 2}$. It is invertible, if and only if $p \neq 2$. Either way, using Corollary 4.1.9, as a first step to construct an $R[\omega]$-representation of $\mathcal{C}(E)$, we put

$$x = \begin{pmatrix} 1 - 2t^{-1}\omega & 0 \\ 0 & -(1 - 2t^{-1}\omega) \end{pmatrix} = \begin{pmatrix} \sqrt{\varepsilon} & 0 \\ 0 & -\sqrt{\varepsilon} \end{pmatrix}.$$

If we want to make a choice for y that is consistent with said corollary, we have to find $\lambda, \lambda' \in R$ such that

$$y = \lambda \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix} + \lambda' \begin{pmatrix} 0 & \pi\omega \\ \omega^q & 0 \end{pmatrix}.$$

This is always possible since on one hand, arbitrary linear combinations of this form anti-commute with x and on the other hand, the norm form on the R-lattice generated by these two matrices is isometric to ${}^{\pi}N(R)$, by Remark 4.1.10. By Remark 4.2.3, the lattice ${}^{\pi}N(R)$ represents exactly those elements of R that have odd valuation. Now $\nu(y^2) = \nu(-\varepsilon\delta) = 1$ is odd and $-N(y) = y^2$.

The advantage of being consistent with Corollary 4.1.9 is that it is easy to determine the index $[\Delta : \mathcal{C}(E)]$. The base change matrix from Δ to the basis $(1, x, y, xy)$ of $\mathcal{C}(E)$, with the above choices for x, y, is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2t^{-1} & 0 & 0 \\ 0 & 0 & \lambda & (\lambda + 2t^{-1}\lambda' n) \\ 0 & 0 & \lambda' & -(2t^{-1}\lambda + \lambda') \end{pmatrix} \in R^{4\times 4}.$$

Its determinant is $4t^{-2}(\lambda^2 + \lambda\lambda' t + (\lambda')^2 n) = 4t^{-2}(\pi^{-1}y^2)$, where this equality holds due to

$$-\varepsilon\delta = y^2 = \pi(\lambda^2 + \lambda\lambda' t + (\lambda')^2 n).$$

Thus, the determinant is 4 up to units in R and $[\Delta : \mathcal{C}(E)] = \pi^{2e}$, so $\Delta = \mathcal{C}(E)$ is maximal, if and only if $p \neq 2$.

If we do not wish to be consistent with Corollary 4.1.9, the much simpler and straightforward choice is

$$y = \begin{pmatrix} 0 & -\varepsilon\delta \\ 1 & 0 \end{pmatrix}$$

and satisfies the desired relations. In order to obtain an R-representation of $\mathcal{C}(E)$ from the $R[\omega]$-representations above, we can simply invoke Proposition 4.4.1.

Finally, let $\nu(\delta) = 0$. Using Remark 4.2.14 and Theorem 4.2.19, we have

$$(E, q) = \begin{bmatrix} \eta & 2\pi^{-g}\eta \\ & \pi^{-2g}(1-\delta)\eta \end{bmatrix} \cong \begin{bmatrix} \eta & \pi^{e-g}\eta \\ & \pi^{2(e-g)}(\frac{1-\delta}{4})\eta \end{bmatrix}$$

and, similar to the case with trivial Clifford invariant, we prefer the first description. Thus, we are looking for elements $x, y \in \Delta$, such that the relations

$$x^2 = \eta,\ y^2 = \pi^{-2g}(1-\delta)\eta \ \text{ and } \ xy + yx = 2\pi^{-g}\eta$$

hold, because then $(1, x, y, xy)$ is an R-basis of $\mathcal{C}(E)$. The problem that we encounter is that there does not seem to be a canonical choice for the non-norm η. However, we can still compute the index of $\mathcal{C}(E)$ in the maximal order Δ, by using the inclusion chain of R-lattices $\mathcal{C}(E) \subseteq \Delta \subseteq \Delta^\# \subseteq \mathcal{C}(E)^\#$, where the dual is taken with respect to the reduced trace form T_{red} on $\mathcal{Q}$. Then the Gram matrix of T_{red} with respect to the R-basis $(1, x, y, xy)$ of $\mathcal{C}(E)$ is

$$\begin{pmatrix} 2 & 0 & 0 & 2\pi^{-g}\eta \\ 0 & 2\eta & 2\pi^{-g}\eta & 0 \\ 0 & 2\pi^{-g}\eta & 2\pi^{-2g}(1-\delta)\eta & 0 \\ 2\pi^{-g}\eta & 0 & 0 & 2\pi^{-2g}(1-\delta)\eta^2 \end{pmatrix} \in R^{4\times 4}.$$

The determinant of this matrix is $(4\pi^{-2g}\delta\eta^2)^2$, so its valuation is $4(e - g + \nu(\eta))$. By Notation 4.1.11, $[\Delta^\# : \Delta] = \pi^2$, so $[\Delta : \mathcal{C}(E)] = \pi^{2(e-g+\nu(\eta))-1}$. Recall that η is a unit, whenever $K(\sqrt{\delta})/K$ is ramified and that we may choose $\eta = \pi$ in the unramified case. Thus, if $\delta \neq \varepsilon$, the index is $\pi^{2(e-g)-1}$; and, if instead $\delta = \varepsilon$, the index is π.

In addition, if $\delta = \varepsilon$, we can construct the Clifford order explicitly in terms of Δ. This is useful for the classification of the Clifford orders in dimension three. For said construction, recall that

$$\Delta = \langle 1, \omega, \pi_{\mathcal{Q}}, \omega\pi_{\mathcal{Q}} \mid \omega^{l^2-1} = 1,\, \pi_{\mathcal{Q}}^2 = \pi,\, \pi_{\mathcal{Q}}\omega = \omega^l \pi_{\mathcal{Q}} \rangle_R$$

and also note that, due to $\delta = \varepsilon$,

$$(E, q) = {}^{\pi}N(R) = \begin{bmatrix} \pi & \pi \\ & \pi t^{-2} n \end{bmatrix} = \begin{bmatrix} \pi & \pi t \\ & \pi n \end{bmatrix}.$$

Now, if (x, y) is an R-basis of the right-most description of ${}^{\pi}N(R)$, i.e. $x^2 = \pi$, $y^2 = \pi n, xy + yx = \pi t$ in $\mathcal{C}({}^{\pi}N(R))$, the map defined by

$$x \mapsto \pi_{\mathcal{Q}},\;\; y \mapsto \pi_{\mathcal{Q}}\omega = (t - \omega)\pi_{\mathcal{Q}},\;\; xy \mapsto \pi\omega$$

induces an isomorphism of R-algebras. This can be verified by checking the respective relations. As a consequence, $\mathcal{C}(E)$ has index π inside Δ, as we already established earlier. In addition to that, we obtain the $R[\omega]$-representation

$$\varphi : R[\omega] \otimes \mathcal{C}(E) \to R[\omega]^{2\times 2},\, \varphi(x) = \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix},\, \varphi(y) = \begin{pmatrix} 0 & \pi\omega^q \\ \omega & 0 \end{pmatrix},$$

from Corollary 4.1.9. Thus, we identify $\mathcal{C}(E)$ with the R-subalgebra $\mathcal{X}_R$ of $R^{4\times 4}$, which one obtains after applying Proposition 4.4.1 to the $R[\omega]$-algebra

$$\mathcal{Y} := \left\{ \begin{pmatrix} a + b\pi\omega & \pi(c + d\omega^q) \\ c + d\omega & a + b\pi\omega^q \end{pmatrix} \,\middle|\, a, b, c, d \in R \right\} \le R[\omega]^{2\times 2}.$$

More precisely, we have

$$\mathcal{X}_R := \left\{ \left(\begin{array}{cc|cc} a & -\pi b n & \pi(c + dt) & \pi d n \\ \pi b & a + \pi b t & -\pi d & \pi c \\ \hline c & -dn & a + \pi b t & \pi b n \\ d & c + dt & -\pi b & a \end{array}\right) \,\middle|\, a, b, c, d \in R \right\} \le R^{4\times 4}$$

and, in particular, $R[\omega] \otimes \mathcal{X}_R \cong \mathcal{Y}$ as $R[\omega]$-algebras and R-algebras.

We return to the case $\mathfrak{c}(V, q) = -1$, with $\nu(\delta) = 0$, but $\delta \neq \varepsilon$. Thus,

$$(E, q) = \begin{bmatrix} \eta & 2\pi^{-g}\eta \\ & \pi^{-2g}(1 - \delta)\eta \end{bmatrix} \cong \begin{bmatrix} \eta & \pi^{e-g}\eta \\ & \pi^{2(e-g)}(\frac{1-\delta}{4})\eta \end{bmatrix}$$

with $\eta \in R^{\times}$. We describe $\mathcal{C}(E)$ but not in terms of Δ. Instead, we use the fact that $K(\sqrt{\eta})/K$ is a degree-two field extension, i.e. it is a splitting field for $\mathcal{Q}$, by [Rei03] (31.10). Thus, $K(\sqrt{\eta}) \otimes_K \mathcal{Q} \cong K(\sqrt{\eta})^{2\times 2}$. Now, we proceed as in the case

$\nu(\delta) = 0$, with trivial Clifford invariant.

Consider the orthogonal sublattice $E' = [\eta, -\eta\delta]$ of E, which has the Clifford order $\mathcal{C}(E') = (\eta, -\eta\delta)_{R_\eta}$, where $R_\eta = R[\pi^{-g}(1+\sqrt{\eta})]$ the valuation ring of $K(\sqrt{\eta})$. If $(1, x, y', xy')$ is a suitable quaternion basis of this algebra, then $(1, x, y, xy)$ is an R-basis of $\mathcal{C}(E)$, where $y \coloneqq \pi^{-g}(x + y')$. Consequently, an R_η-representation of $\mathcal{C}(E)$ is given by

$$\varphi(x) = \sqrt{\eta}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \varphi(y) = \pi^{-g}\sqrt{\eta}\begin{pmatrix} 1 & -\delta \\ 1 & -1 \end{pmatrix}.$$

Moreover, this implies that $\mathcal{C}(E)$ has index $\pi^{2(e-g)-1}$ in any maximal order of $\mathcal{C}(V) \cong \mathcal{Q}$ because the maximal order Δ is mapped to the index-π suborder $\left(\begin{smallmatrix} R_\eta & \pi R_\eta \\ R_\eta & R_\eta \end{smallmatrix}\right)$ of the maximal order $R_\eta^{2\times 2}$.

<u>$\mathsf{n = 3}$:</u> From Theorem 4.2.19, we know that

$$(E, q) = \begin{cases} N(R) \perp [\varepsilon\delta], & \nu(\delta) = 1 \\ {}^{\pi}N(R) \perp [\varepsilon\delta], & \nu(\delta) = 0 \end{cases}.$$

In view of Theorem 3.4.12 and Theorem 3.4.19, we first consider the quadratic discriminants of $N(R)$ and ${}^{\pi}N(R)$. From Theorem 4.3.11 we obtain $\operatorname{disq}(N(R)) = \varepsilon$ and $\operatorname{disq}({}^{\pi}N(R)) = \pi^{2t}\varepsilon$, with $t = 0$, if $p = 2$; and $t = 1$, if $p \neq 2$. Thus, if $\nu(\delta) = 1$, whether $\operatorname{disq}({}^{\pi}N(R))$ is a unit depends on p; therefore, we must distinguish between the dyadic and non-dyadic case.

We start with $\nu(\delta) = 1$. Then we have

$$\mathcal{C}(N(R) \perp [\varepsilon\delta]) \cong \mathcal{C}(N(R)) \otimes \mathcal{C}([\varepsilon^2\delta]) \cong R^{2\times 2} \otimes R[\sqrt{\delta}] \cong R[\sqrt{\delta}]^{2\times 2}.$$

as R-algebras, by Theorem 3.4.12, because $\varepsilon \in R^\times$. Since $\operatorname{defq}(\delta) = 0$, we know that $R[\sqrt{\delta}]$ is the unique maximal R-order in $K(\sqrt{\delta})$, so using Example 2.2.6, we conclude that $R[\sqrt{\delta}]^{2\times 2} \cong \mathcal{C}(E)$ is a maximal R-order in $\mathcal{C}(V)$. Beyond this, Theorem 3.4.12 yields that the eight matrices below form an R-basis of $\mathcal{C}(N(R) \otimes [\varepsilon\delta])$. These are obtained by using the Kronecker product of matrices and the matrix representations of $\mathcal{C}(N(R))$ and $\mathcal{C}([\varepsilon\delta])$, which we discussed in the lower-dimensional cases.

$$I_4, \left(\begin{array}{cc|cc} 0 & \varepsilon\delta & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\varepsilon\delta \\ 0 & 0 & -1 & 0 \end{array}\right), \ \pi^{-e}\left(\begin{array}{c|c} I_2 & -\varepsilon I_2 \\ \hline -I_2 & I_2 \end{array}\right), \ \pi^{-e}\left(\begin{array}{cc|cc} 0 & \varepsilon\delta & 0 & -\varepsilon^2\delta \\ 1 & 0 & -\varepsilon & 0 \\ \hline 0 & \varepsilon\delta & 0 & -\varepsilon\delta \\ 1 & 0 & -1 & 0 \end{array}\right),$$

$$\left(\begin{array}{c|c} I_2 & \mathbf{0} \\ \hline \mathbf{0} & -I_2 \end{array}\right), \ \pi^{-e}\left(\begin{array}{c|c} I_2 & -\varepsilon I_2 \\ \hline I_2 & -I_2 \end{array}\right), \ \left(\begin{array}{cc|cc} 0 & \varepsilon\delta & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \varepsilon\delta \\ 0 & 0 & 1 & 0 \end{array}\right), \ \pi^{-e}\left(\begin{array}{cc|cc} 0 & \varepsilon\delta & 0 & -\varepsilon^2\delta \\ 1 & 0 & -\varepsilon & 0 \\ \hline 0 & -\varepsilon\delta & 0 & \varepsilon\delta \\ -1 & 0 & 1 & 0 \end{array}\right).$$

Moreover, the matrices are ordered such that the four matrices in the first row form an R-basis of $\mathcal{C}_0(E)$. As $\mathfrak{c}(V, q) = -1$, the K-span of these four matrices is isomorphic to $\mathcal{Q}$ as K-algebra and $\mathcal{C}_0(E)$ is an R-order inside it.

Using the fact that $\nu(-\varepsilon\delta) = 1$ and applying Proposition 2.3.7 (ii), we conclude that $\mathcal{C}_0(E)$ contains the suborder $\mathcal{C}({}^{-\varepsilon\delta}N(R)) \cong \mathcal{C}({}^{\pi}N(R))$ of index π. However, we already know from the two-dimensional case that $\mathcal{C}({}^{\pi}N(R))$ is an index-π suborder of the maximal R-order Δ in $\mathcal{Q}$. Since Δ is the unique maximal R-order inside $\mathcal{Q}$, this implies $\mathcal{C}_0(E) \cong \Delta$.

To summarise the case $\nu(\delta) = 1$, both the Clifford order $\mathcal{C}(E)$ and the even Clifford order $\mathcal{C}_0(E)$ are maximal R-orders in the Clifford algebra $\mathcal{C}(V)$ and the even Clifford algebra $\mathcal{C}_0(V)$, respectively.

Next, let $\nu(\delta) = 0$ and consider the dyadic case first. Then, by Theorem 3.4.12, the Clifford order $\mathcal{C}(E)$ is isomorphic to

$$\mathcal{C}(E) \cong \mathcal{C}({}^{\pi}N(R)) \otimes \mathcal{C}([\varepsilon^2\delta]) \cong \mathcal{X}_R \otimes R[X]/(X^2 - \delta)$$

as an R-algebra. Now, if $\delta = 1$, then $R[X]/(X^2 - 1) = \Lambda_0(1)$ is the unique index π^e suborder of $\Lambda_{-e}(1) \cong R \oplus R$. In this case, $\mathcal{C}(E)$ has index $\pi^2 \cdot \pi^{4e} = \pi^{4e+2}$ in the maximal R-order

$$\Delta \otimes R[X]/(X^2 - X) \cong \Delta \otimes (R \oplus R) \cong \Delta \oplus \Delta.$$

of $\mathcal{Q} \oplus \mathcal{Q}$. To be precise,

$$\mathcal{C}(E) \cong \langle (a, a), (a, -a) \mid a \in \mathcal{X}_R \rangle =: \langle (\mathcal{X}_R, \mathcal{X}_R), (\mathcal{X}_R, -\mathcal{X}_R) \rangle \leq \Delta \oplus \Delta.$$

Otherwise, $\delta \neq 1$ and then $\mathcal{C}(E) \cong \mathcal{X}_R \otimes R[\sqrt{\delta}]$ as R-algebra. From this we obtain an R-representation as 8×8 matrices, either by using the Kronecker product and the R-representations of $\mathcal{X}_R$ and $R[\sqrt{\delta}]$ that we discussed earlier, or by noticing that

$$R[\omega] \otimes (\mathcal{X}_R \otimes R[\sqrt{\delta}]) \cong (R[\omega] \otimes \mathcal{X}_R) \otimes R[\sqrt{\delta}] \cong \mathcal{Y} \otimes R[\sqrt{\delta}]$$

as an $R[\omega]$-algebra. Then we can again use the Kronecker product of matrices to obtain an $R[\omega]$-representation as 4×4 matrices and use Proposition 4.4.1 to get the desired R-representation as 8×8 matrices.

We determine its index in a maximal order in $\mathcal{C}(V)$. Note that $\mathcal{C}(V) \cong \mathcal{Q} \otimes_K K(\sqrt{\delta}) \cong K(\sqrt{\delta})^{2\times 2}$, because, by Corollary (31.10) in [Rei03], any degree-two field extension of K is a splitting field for the K-division algebra $\mathcal{Q}$. Thus, $R_\delta^{2\times 2}$ is a maximal R-order in $\mathcal{C}(V)$ and it is easily seen that Λ is mapped to

$$\begin{pmatrix} R_\delta & \pi R_\delta \\ R_\delta & R_\delta \end{pmatrix}$$

under the isomorphism $\mathcal{Q} \otimes_K K(\sqrt{\delta}) \cong K(\sqrt{\delta})^{2\times 2}$. Hence, the isomorphic image of Λ has index π^2 in $R_\delta^{2\times 2}$ and, in total, we find that $\mathcal{C}(E) \cong \mathcal{X}_R \otimes R[\sqrt{\delta}]$ has index $\pi^{4g+2} \cdot \pi^2 = \pi^{4(g+1)}$ in any maximal R-order of $\mathcal{C}(V)$.

In contrast, the isomorphism type and the index of the even Clifford order $\mathcal{C}_0(E)$ in the maximal R-order Δ are easier to determine. As a reminder, we have $\nu(\delta) = 0$ and this time we also allow $\delta = 1$. By Proposition 2.3.7 (ii), we have the isomorphism of R-algebras

$$\mathcal{C}_0(E) \cong \mathcal{C}({}^{-\varepsilon\delta\pi}N(R)) \cong \mathcal{C}({}^{\pi}N(R)) = \mathcal{X}_R,$$

because $\varepsilon\delta \in R^\times$ and, by Remark 4.2.3, the norm form $N(R)$ represents every unit of R. We have already seen that $[\Delta : \mathcal{X}_R] = \pi$, when we considered $n = 2$.

With that, only the non-dyadic cases with $\nu(\delta) = 0$ remain and actually, the above result regarding the isomorphism type and the index in Δ of $\mathcal{C}_0(E)$ carry over to the non-dyadic case. This is because in order to apply Proposition 2.3.7 (ii), we only require $\varepsilon\delta \in R^\times$. This holds independent of p.

As we already discussed, we have $\operatorname{disq}({}^{\pi}N(R)) = \pi^2\varepsilon$. Thus, by Theorem 3.4.12, the R-order $\mathcal{Y} := \mathcal{C}({}^{\pi}N(R)) \otimes \mathcal{C}([\pi^2\delta])$ has index π^4 in the Clifford order $\mathcal{C}(E)$. On the other hand, $\mathcal{Y}$ also has index π^4 inside the R-order $\Gamma := \mathcal{C}({}^{\pi}N(R)) \otimes \mathcal{C}([\delta])$. Hence, the index of $\mathcal{C}(E)$ in a maximal order coincides with the index of $\Gamma \cong \mathcal{X}_R \otimes R[X]/(X^2 - \delta)$ in a maximal order, so we are essentially back in the dyadic case. However, this time all quadratic defects are zero, so we conclude that the index of $\mathcal{C}(E)$ in a maximal order is π^2, if $\delta = 1$; and it is π^4, if $\delta = \varepsilon$.

We can also consider the R-orders above in more detail. Recall that in the non-dyadic case, ${}^{\pi}N(R) = [\pi, -\varepsilon\pi]$ and that, if (e_1, e_2) is a suitable orthogonal R-basis of ${}^{\pi}N(R)$, then $\mathcal{Z}({}^{\pi}N(R)) = \langle 1, z\rangle_R$, with $z := e_1e_2$ satisfying $z^2 = \pi^2\varepsilon = \operatorname{disq}({}^{\pi}N(R))$. Write $E = [\pi, -\varepsilon\pi] \perp [\varepsilon\delta]$ and let f be a generator of the second summand. Then $\mathcal{C}(E) = \langle e_1, e_2, f\rangle$ as R-algebra and we have the K-algebra isomorphism

$$\varphi : \mathcal{C}(KE) \to K\Gamma = \mathcal{C}(K[\pi, -\varepsilon\pi]) \otimes_K \mathcal{C}(K[\delta]),\ e_i \mapsto e_i \otimes 1,\ f \mapsto \pi^{-1}z \otimes x,$$

where x is a generator of $[\delta]$ with $q(x) = \delta$. Doing the computation, we find

$$\varphi(f) = \pi^{-1}e_1e_2 \otimes x,\ \varphi(e_1f) = e_2 \otimes x,\ \varphi(e_2f) = \varepsilon e_1 \otimes x,\ \varphi(e_1e_2f) = \varepsilon\pi \otimes x$$

and since $\varepsilon \in R^\times$ is a unit, we reobtain the claim about the indices of $\mathcal{C}(E)$ and Γ in a maximal order. Moreover, the R-suborder of $\mathcal{Q} \otimes_K K[X]/(X^2 - \delta)$ that is generated by the images $\varphi(e_1), \varphi(e_2), \varphi(f)$ is isomorphic to $\mathcal{C}(E)$, so we can obtain a matrix representation of $\mathcal{C}(E)$ from this.

n = 4: The unique maximal anisotropic R-lattice of rank four is $(E, q) = N(R) \perp {}^{\pi}N(R)$ with ambient space $(V, q) = \mathcal{U}_K = N(K) \perp {}^{\pi}N(K)$. Independent of p, we have $\operatorname{disq}(N(R)) = \varepsilon$, so Theorem 3.4.12 yields

$$\mathcal{C}(E) \cong \mathcal{C}(N(R)) \otimes \mathcal{C}({}^{\varepsilon\pi}N(R)) \cong R^{2\times 2} \otimes \mathcal{C}({}^{\pi}N(R)) \cong R^{2\times 2} \otimes \mathcal{X}_R \cong \mathcal{X}_R^{2\times 2}$$

as R-orders. Using the determinant formula for the Kronecker product of matrices, this is a suborder of the maximal order $\Delta^{2\times 2}$ of index π^4, and an R-representation of $\mathcal{C}(E)$ is easily obtained from the R-representation of $\mathcal{X}_R$, which we discussed in the two-dimensional case.

Finally, we consider the even Clifford order $\mathcal{C}_0(E)$. Recalling $\omega^2 - \mathfrak{t}\omega + \mathfrak{n} = 0$, we have

$$N(R) = \begin{bmatrix} 1 & \mathfrak{t} \\ & \mathfrak{n} \end{bmatrix} \quad \text{and} \quad {}^{\pi}N(R) = \begin{bmatrix} \pi & \pi\mathfrak{t} \\ & \pi\mathfrak{n} \end{bmatrix}.$$

Let (x, y) and (x', y') be suitable bases of these R-lattices respectively, i.e. $x^2 = 1$, $y^2 = n$, $xy + yx = t \in \mathcal{C}(N(R))$, and similarly for ${}^{\pi}N(R)$. Then we get the isomorphism of R-algebras

$$\mathcal{C}_0(E) \cong \mathcal{C}_0(N(R)) \otimes \mathcal{C}_0({}^{\pi}N(R)) \oplus \mathcal{C}_1(N(R)) \otimes \mathcal{C}_1({}^{\pi}N(R)) =: \mathcal{A}$$

and, after a suitable permutation, the R-algebra $\mathcal{A}$ has the R-basis

$$(1 \otimes 1, x \otimes x', xy \otimes 1, y \otimes x') \cup (xy \otimes x'y', y \otimes y', x \otimes y', 1 \otimes x'y').$$

From now on, for simplicity, we write 1 instead of $1 \otimes 1$ and put

$$e := x \otimes x', \ f := xy \otimes 1, \ \text{so also} \ ef = y \otimes x'.$$

Then $e^2 = \pi$, $f^2 - tf + n = 0$, $(ef)^2 = (y \otimes x)^2 = \pi n$ and $ef + fe = te$ hold. Hence, because the relations are satisfied, the R-algebra generated by $(1, e, f, ef)$ is isomorphic to the unique maximal R-order Δ in $\mathcal{Q}$, via

$$e \mapsto \pi_{\mathcal{Q}}, \quad f \mapsto \omega, \quad ef \mapsto \pi_{\mathcal{Q}}\omega = (t - \omega)\pi_{\mathcal{Q}}.$$

Moreover, if we put $\mathfrak{z} := (\varepsilon\pi)^{-1}\big(x(2t^{-1}y - x) \otimes x'(2t^{-1}y' - x')\big)$, then the identities $e\mathfrak{z} = \mathfrak{z}e$, $f\mathfrak{z} = \mathfrak{z}f$ and $\mathfrak{z}^2 = 1$ hold. Now the R-algebra with basis

$$(1, e, f, ef) \cup (\mathfrak{z}, e\mathfrak{z}, f\mathfrak{z}, ef\mathfrak{z}),$$

call it $\mathcal{B}$, is also an R-order in the K-algebra $K\mathcal{A}$ and it satisfies $\mathcal{B} \cong \Delta \otimes R[X]/(X^2 - 1)$. Consequently, we get the isomorphism of K-algebras

$$\mathcal{C}_0(V) \cong K\mathcal{A} = K\mathcal{B} \cong \mathcal{Q} \otimes_K K[X]/(X^2 - 1) \cong \mathcal{Q} \oplus \mathcal{Q}$$

for the ambient space and, because $\mathcal{Q} \oplus \mathcal{Q}$ contains a unique maximal R-order, namely $\Delta \oplus \Delta$, so does $\mathcal{C}_0(V)$. Moreover, with respect to the two bases above, the base change matrix from $\mathcal{A}$ to $\mathcal{B}$ is $T = \left(\begin{smallmatrix} I_4 & W \\ 0 & U \end{smallmatrix}\right) \in R^{8 \times 8}$ with

$$U = (\varepsilon\mathfrak{t})^{-1} \begin{pmatrix} 4(\pi\mathfrak{t})^{-1} & 0 & 2\pi^{-1} & 0 \\ 0 & 4\mathfrak{t}^{-1} & 0 & 2 \\ 0 & -2 & 0 & -4\mathfrak{t}^{-1}\mathfrak{n} \\ -2\pi^{-1} & 0 & -4(\pi\mathfrak{t})^{-1}\mathfrak{n} & 0 \end{pmatrix}, \ W = \varepsilon^{-1} \begin{pmatrix} 1 & 0 & 2\mathfrak{t}^{-1}\mathfrak{n} & 0 \\ 0 & 1 & 0 & 2\mathfrak{t}^{-1}\mathfrak{n} \\ -2\mathfrak{t}^{-1} & 0 & -1 & 0 \\ 0 & -2\mathfrak{t}^{-1} & 0 & -1 \end{pmatrix}$$

and its determinant is $\det(T) = \det(U) = -16(\mathfrak{t}^2\varepsilon\pi)^{-2}$ whence $[\mathcal{A} : \mathcal{B}] = \pi^{4e-2}$. Now $R[X]/(X^2-1)$ has index π^e in the maximal order $R[X]/(X^2-X)$ of $K[X]/(X^2-1)$, so using the determinant formula for the Kronecker product of matrices and $\Delta\oplus\Delta \cong \Delta\otimes R[X]/(X^2-X)$, we conclude that $\mathcal{C}_0(E)$ has index $\pi^{4e-(4e-2)} = \pi^2$ in the unique maximal order of $\mathcal{C}_0(V)$. In order to precisely determine the isomorphic image of $\mathcal{C}_0(E)$ as a suborder of $\mathcal{Q}\oplus\mathcal{Q}$, we first note that on one hand

$$T^{-1} = \begin{pmatrix} I_4 & -WU^{-1} \\ \mathbf{0} & U^{-1} \end{pmatrix}, U^{-1} = \begin{pmatrix} -\pi n & 0 & 0 & -\frac{1}{2}\pi t \\ 0 & -n & -\frac{1}{2}t & 0 \\ \frac{1}{2}\pi t & 0 & 0 & \pi \\ 0 & \frac{1}{2}t & 1 & 0 \end{pmatrix}, WU^{-1} = -\frac{1}{2}t \begin{pmatrix} 0&0&0&\pi \\ 0&0&1&0 \\ \pi&0&0&0 \\ 0&1&0&0 \end{pmatrix}$$

and on the other hand the maximal order $\mathfrak{m}$ inside $K\mathcal{A}$ has the R-basis

$$(\mathfrak{z}_+, e\mathfrak{z}_+, f\mathfrak{z}_+, ef\mathfrak{z}_+) \cup (\mathfrak{z}_-, e\mathfrak{z}_-, f\mathfrak{z}_-, ef\mathfrak{z}_-)$$

with $\mathfrak{z}_+ := \frac{1}{2}(1+\mathfrak{z}), \mathfrak{z}_- := \frac{1}{2}(1-\mathfrak{z}) \in K\mathcal{A}$, the centrally primitive idempotents. With respect to these bases, the base change matrix from $\mathfrak{m}$ to $\mathcal{B}$ is $M = \left(\begin{smallmatrix} I_4 & I_4 \\ I_4 & -I_4 \end{smallmatrix}\right) \in R^{8\times 8}$, because $1 = \mathfrak{z}_+ + \mathfrak{z}_-$ and $\mathfrak{z} = \mathfrak{z}_+ - \mathfrak{z}_-$. Hence, the base change matrix from $\mathfrak{m}$ to $\mathcal{A}$, which is the one of interest, is given by

$$V := MT^{-1} = \begin{pmatrix} I_4 & I_4 \\ I_4 & -I_4 \end{pmatrix} \begin{pmatrix} I_4 & -WU^{-1} \\ 0 & U^{-1} \end{pmatrix} = \begin{pmatrix} I_4 & U^{-1} - WU^{-1} \\ I_4 & -(U^{-1} + WU^{-1}) \end{pmatrix}.$$

There is a suitable $E \in \mathrm{GL}_8(R)$ with $XE = \left(\begin{smallmatrix} I_4 & 0 \\ I_4 & D \end{smallmatrix}\right)$ with $D = \mathrm{diag}(\pi, 1, \pi, 1)$, as can be verified by using Gaussian elimination on the columns of X. This right-multiplication corresponds to a change of basis of the R-lattice $\mathcal{A}$, so

$$(1, e, f, ef) \cup (\pi\mathfrak{z}_-, e\mathfrak{z}_-, \pi f\mathfrak{z}_-, ef\mathfrak{z}_-)$$

is an R-basis of $\mathcal{A} \cong \mathcal{C}_0(E)$ that is compatible with the R-basis

$$(1, e, f, ef) \cup (\mathfrak{z}_-, e\mathfrak{z}_-, f\mathfrak{z}_-, ef\mathfrak{z}_-)$$

of the maximal order $\mathfrak{m}$. Inside the K-algebra $\mathcal{Q}\oplus\mathcal{Q}$, which is isomorphic to $\mathcal{C}_0(V)$, we identify these two R-orders, by mapping $\mathfrak{z}_+ \mapsto (1,0)$ and $\mathfrak{z}_- \mapsto (0,1)$.

Then $\Delta\oplus\Delta = \Delta(1,1)\oplus\Delta(0,1) \cong \mathfrak{m}$ and $\Delta(1,1)\oplus\widetilde{\Delta}(0,1) \cong \mathcal{A} \cong \mathcal{C}_0(E)$, where $\widetilde{\Delta}$ is the sublattice of Δ of index π^2 with the R-basis $(\pi, e, \pi f, ef)$.

4.4.2 Clifford orders of anisotropic maximal lattices - results

We collect our results regarding the isomorphism types of the Clifford orders and the even Clifford orders considered above. As before, we let $E = E_0(V,q)$ be the unique maximal R-lattice in the anisotropic quadratic K-space (V,q). The tables in Theorem 4.4.6 and Theorem 4.4.7 below should be interpreted as follows: The first column lists the maximal R-lattices (E, q) in roughly the same order as in Theorem 4.2.19. We add some rows, since some results in the table only hold under extra conditions on the prime p and the quadratic discriminant δ of (E, q). Thus, the second column contains abbreviations for said conditions. We repeat the already familiar abbreviations.

Cond.	Explanation
$-$	p is arbitrary and δ can only take one value.
$p2-$	$p = 2$ and δ can only take one value.
$np2-$	$p \neq 2$ and δ can only take one value.
a	Both p and δ are arbitrary.
νi	p is arbitrary and $\nu(\delta) = i \in \{0, 1\}$.
$\nu 0 n 1$	p is arbitrary and $\nu(\delta) = 0$ with $\delta \neq 1$.
$\nu 0 n 1 \varepsilon$	$\nu(\delta) = 0$ with $\delta \neq 1, \varepsilon$ (this implies $p = 2$).
$p 2 \nu 0 n 1$	$p = 2$ and $\nu(\delta) = 0$ with $\delta \neq 1$.

The third column is reserved for the isomorphism type of $\mathcal{C}(E)$ or $\mathcal{C}_0(E)$ as R-algebra. Usually, it contains a matrix representation of it. The last column contains the index of $\mathcal{C}(E)$ or $\mathcal{C}_0(E)$ in a maximal order of $\mathcal{C}(V)$ or $\mathcal{C}_0(V)$ as R-order.

Notation 4.4.4. Recall that the K-algebras $K \oplus K$ and $K(\omega)$ both contain a unique maximal order $\mathfrak{m}$, by Remark 3.1.1. In the first case, we have $\mathfrak{m} = R \oplus R$ and in the second case, we have $\mathfrak{m} = R[\omega]$. Let σ denote the unique non-trivial automorphism of $\mathfrak{m}$, so that in the first case $\sigma(x_1 + x_2) = x_1 - x_2 \in R \oplus R$ and $\sigma(x_1 + x_2\omega) = x_1 + x_2\omega^q \in R[\omega]$ in the second case. Since all orders of $K \oplus K$ and $K(\omega)$ are linearly ordered, we have $\Gamma^\sigma := \sigma(\Gamma) = \Gamma$ for any suborder $\Gamma \subseteq \mathfrak{m}$. Now, given suborders $\Gamma_1, \Gamma_2 \subseteq \mathfrak{m}$ and scalars $a, b, c, d \in R$, we put

$$\begin{pmatrix} a\Gamma_1 & b\Gamma_2^\sigma \\ c\Gamma_2 & d\Gamma_1^\sigma \end{pmatrix} := \left\{ \begin{pmatrix} ax_1 & b\sigma(x_2) \\ cx_2 & d\sigma(x_1) \end{pmatrix} \;\middle|\; x_1 \in \Gamma_1, x_2 \in \Gamma_2 \right\} \leq \mathfrak{m}^{2\times 2}.$$

To better illustrate this new notation, we use it to describe some of the orders, which we encountered in this section.

Example 4.4.5. (i) The Clifford order of ${}^{\pi}N(R)$ is given by

$$\begin{aligned} \mathcal{C}({}^{\pi}N(R)) = \mathcal{X}_R &\cong \left\{ \begin{pmatrix} a + b\pi\omega & \pi(c + d\omega^q) \\ c + d\omega & a + b\pi\omega^q \end{pmatrix} \;\middle|\; a, b, c, d \in R \right\} \\ &= \begin{pmatrix} R[\omega] & \pi R[\omega]^\sigma \\ R[\omega] & R[\omega]^\sigma \end{pmatrix}. \end{aligned}$$

(ii) The lattice $[1,-\delta]$ with $\nu(\delta)=1$ has the Clifford order

$$\mathcal{C}([1,-\delta]) \cong \left\{ \begin{pmatrix} x_1+x_2 & -\delta(x_3-x_4) \\ x_3+x_4 & x_1-x_2 \end{pmatrix} \;\middle|\; x_1,x_2,x_3,x_4 \in R \right\}$$
$$= \begin{pmatrix} R\oplus R & -\delta(R\oplus R)^\sigma \\ R\oplus R & (R\oplus R)^\sigma \end{pmatrix}.$$

Using this notation, we present the two main results of this chapter, the classification of the Clifford orders and of the even Clifford orders of the maximal anisotropic R-lattices. Keep in mind that this information is sufficient to classify the Clifford orders of all maximal R-lattices, by Corollary 3.4.9.

Theorem 4.4.6. *An overview of the Clifford orders of the maximal anisotropic R-lattices over the p-adic valuation ring R is given by the table below.*

Max. lattice in (V,q)	Cond.	Isomorphism type of $\mathcal{C}(E)$	Index
$\{0\}$	–	R	1
$[\delta]$	a	$\Lambda_0(\delta)$	π^g
$[1,-\delta]$	$\nu 1$	$\begin{pmatrix} R\oplus R & -\delta(R\oplus R)^\sigma \\ R\oplus R & (R\oplus R)^\sigma \end{pmatrix}$	π^{2e+1}
$\begin{bmatrix} 1 & \pi^{e-g} \\ & \pi^{2(e-g)}\frac{1-\delta}{4} \end{bmatrix}$	$\nu 0 n 1$	$\mathcal{A}_\delta$	$\pi^{2(e-g)}$
$[\varepsilon,-\varepsilon\delta]$	$\nu 1$	$\begin{pmatrix} R[\sqrt{\varepsilon}] & -\varepsilon\delta R[\sqrt{\varepsilon}]^\sigma \\ R[\sqrt{\varepsilon}] & R[\sqrt{\varepsilon}]^\sigma \end{pmatrix}$	π^{2e}
${}^{\pi}N(R)$	–	$\mathcal{X}_R \cong \begin{pmatrix} R[\omega] & \pi R[\omega]^\sigma \\ R[\omega] & R[\omega]^\sigma \end{pmatrix}$	π
$\begin{bmatrix} \eta & \pi^{e-g}\eta \\ & \pi^{2(e-g)}\frac{1-\delta}{4}\eta \end{bmatrix}$	$\nu 0 n 1 \varepsilon$	$\mathcal{B}_\delta$	$\pi^{2(e-g)-1}$
$N(R)\perp[\varepsilon\delta]$	$\nu 1$	$R[\sqrt{\delta}]^{2\times 2}$	1
${}^{\pi}N(R)\perp[\varepsilon]$	$p2-$	$\langle(\mathcal{X}_R,\mathcal{X}_R),(\mathcal{X}_R,-\mathcal{X}_R)\rangle$	π^{4e+2}
$[\pi,-\varepsilon\pi,\varepsilon]$	$np2-$	$*$	π^2
${}^{\pi}N(R)\perp[\varepsilon\delta]$	$p2\nu 0 n1$	$R[\sqrt{\delta}]\otimes\mathcal{X}_R$	$\pi^{4(g+1)}$
$[\pi,-\varepsilon\pi,1]$	$np2-$	$*$	π^4
$N(R)\perp{}^{\pi}N(R)$	–	$\mathcal{X}_R^{2\times 2}$	π^4

Here, let $\mathcal{X}_R$ be as in Example 4.4.5 (i), and regard $\langle(\mathcal{X}_R,\mathcal{X}_R),(\mathcal{X}_R,-\mathcal{X}_R)\rangle \le \Delta\oplus\Delta$

as a suborder of $\mathcal{Q} \oplus \mathcal{Q}$*. Moreover, put*

$$\mathcal{A}_\delta = \left\{ \begin{pmatrix} x_1 + x_2 + \pi^{-g}(x_3 + x_4) & -\delta\pi^{-g}(x_3 + x_4) \\ \pi^{-g}(x_3 - x_4) & x_1 - x_2 + \pi^{-g}(x_4 - x_3) \end{pmatrix} \right\},$$

$$\mathcal{B}_\delta = \left\{ \begin{pmatrix} x_1 + \sqrt{\eta}x_2 + \pi^{-g}(\sqrt{\eta}x_3 + \eta x_4) & -\delta\pi^{-g}(\sqrt{\eta}x_3 + \eta x_4) \\ \pi^{-g}(\sqrt{\eta}x_3 - \eta x_4) & x_1 - \sqrt{\eta}x_2 + \pi^{-g}(\eta x_4 - \sqrt{\eta}x_3) \end{pmatrix} \right\}$$

where all $x_i \in R$ *respectively.*

The two $*$*-symbols can be replaced by a (left) regular representation of the Clifford order in question, or, by the representation of it as a suborder of* $\mathcal{Q} \otimes_K K[X]/(X^2 - \delta)$ *(then, as* $p \neq 2$, $\delta \in \{1, \varepsilon\}$*).*

Theorem 4.4.7. *In the following, the expressions* $\Lambda_i(\cdot)$ *with* $i \in \mathbb{Z}$ *are as in Notation 4.3.2. Moreover, regard* $\Delta(1,1) \oplus \widetilde{\Delta}(0,1) \leq \Delta \oplus \Delta$ *as a suborder of* $\mathcal{Q} \oplus \mathcal{Q}$ *with* $\widetilde{\Delta}$ *the* R*-sublattice of* Δ *with basis* $(\pi, \pi_{\mathcal{Q}}, \pi\omega, \pi_{\mathcal{Q}}\omega)$ *(recall that* $(1, \pi_{\mathcal{Q}}, \omega, \pi_{\mathcal{Q}}\omega)$ *is an* R*-basis of* Δ*). Then, an overview of the even Clifford orders of the maximal anisotropic quadratic* R*-lattices over the* p*-adic valuation ring* R *is given by the table below.*

Max. lattice in (V, q)	Cond.	Isomorphism type of $\mathcal{C}_0(E)$	Index
$\{0\}$	–	R	1
$[\delta]$	a	R	1
$[1, -\delta]$	$\nu 1$	$\Lambda_0(\delta)$	1
$\begin{bmatrix} 1 & \pi^{e-g} \\ & \pi^{2(e-g)}\frac{1-\delta}{4} \end{bmatrix}$	$\nu 0 n 1$	$\Lambda_{-g}(\delta)$	1
$[\varepsilon, -\varepsilon\delta]$	$\nu 1$	$\Lambda_0(\delta)$	1
${}^{\pi}N(R)$	–	$\Lambda_{-e+1}(\varepsilon)$	π
$\begin{bmatrix} \eta & \pi^{e-g}\eta \\ & \pi^{2(e-g)}\frac{1-\delta}{4}\eta \end{bmatrix}$	$\nu 0 n 1 \varepsilon$	$\Lambda_{-g}(\delta)$	1
$N(R) \perp [\varepsilon\delta]$	$\nu 1$	Δ	1
${}^{\pi}N(R) \perp [\varepsilon\delta]$	$\nu 0$	$\mathcal{X}_R \cong \begin{pmatrix} R[\omega] & \pi R[\omega]^\sigma \\ R[\omega] & R[\omega]^\sigma \end{pmatrix}$	π
$N(R) \perp {}^{\pi}N(R)$	–	$\Delta(1,1) \oplus \widetilde{\Delta}(0,1)$	π^2

Bibliography

[Bra16] O. Braun. "Orthogonal Representations of Finite Groups". PhD thesis. RWTH Aachen University, Oct. 2016.

[Bra29] R. Brauer. "Über Systeme hyperkomplexer Zahlen". In: *Math. Z.* 30.1 (1929), pp. 79–107. ISSN: 0025-5874,1432-1823. DOI: 10.1007/BF01187754. URL: https://doi.org/10.1007/BF01187754.

[Car81] Élie Cartan. *The Theory of Spinors*. Dover Books on Advanced Mathematics. With a foreword by Raymond Streater, A reprint of the 1966 English translation. Dover Publications, Inc., New York, 1981, pp. v+157. ISBN: 0-486-64070-1.

[Che54] Claude C. Chevalley. *The Algebraic Theory of Spinors*. Columbia University Press, New York, 1954, pp. viii+131.

[Cli78] W. K. Clifford. "Applications of Grassmann's Extensive Algebra". In: *Amer. J. Math.* 1.4 (1878), pp. 350–358. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2369379. URL: https://doi.org/10.2307/2369379.

[Coh00] H. Cohen. *Advanced Topics in Computational Number Theory*. Springer, 2000.

[DV18] D. S. Dummit and J. Voight. "The 2-Selmer group of a number field and heuristics for narrow class groups and signature ranks of units". In: *Proc. Lond. Math. Soc. (3)* 117.4 (2018), pp. 682–726. ISSN: 0024-6115. DOI: 10.1112/plms.12143. URL: https://doi.org/10.1112/plms.12143.

[Ebe02] W. Ebeling. *Lattices and Codes*. revised. Advanced Lectures in Mathematics. A course partially based on lectures by F. Hirzebruch. Friedr. Vieweg & Sohn, Braunschweig, 2002, pp. xviii+188. ISBN: 3-528-16497-2. DOI: 10.1007/978-3-322-90014-2. URL: https://doi.org/10.1007/978-3-322-90014-2.

[Eic74] M. Eichler. *Quadratische Formen und orthogonale Gruppen*. Die Grundlehren der mathematischen Wissenschaften, Band 63. Zweite Auflage. Springer-Verlag, Berlin-New York, 1974, pp. xii+222.

[GS17] P. Gille and T. Szamuely. *Central Simple Algebras and Galois Cohomology*. Vol. 165. Cambridge Studies in Advanced Mathematics. Second edition of [MR2266528]. Cambridge University Press, Cambridge, 2017, pp. xi+417. ISBN: 978-1-107-15637-1.

[Hah94] A. J. Hahn. *Quadratic algebras, Clifford algebras, and arithmetic Witt groups*. Universitext. Springer-Verlag, New York, 1994, pp. xii+286. ISBN: 0-387-94110-X. DOI: `10.1007/978-1-4684-6311-8`. URL: `https://doi.org/10.1007/978-1-4684-6311-8`.

[Has24] H. Hasse. "Äquivalenz quadratischer Formen in einem beliebigen algebraischen Zahlkörper." In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 153 (1924), pp. 158–162. DOI: `10.1515/crll.1924.153.158`.

[HO89] Alexander J. Hahn and O. Timothy O'Meara. *The Classical Groups and K-Theory*. Vol. 291. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With a foreword by J. Dieudonné. Springer-Verlag, Berlin, 1989, pp. xvi+576. ISBN: 3-540-17758-2. DOI: `10.1007/978-3-662-13152-7`. URL: `https://doi.org/10.1007/978-3-662-13152-7`.

[Kit73] K. Kitamura. "On the free quadratic extensions of a commutative ring". In: *Osaka Math. J.* 10 (1973), pp. 15–20. ISSN: 0388-0699. URL: `http://projecteuclid.org/euclid.ojm/1200694120`.

[Knu+98] M. Knus, A. Merkurjev, M. Rost, and J. Tignol. *The book of involutions*. Vol. 44. American Mathematical Society Colloquium Publications. With a preface in French by J. Tits. American Mathematical Society, Providence, RI, 1998, pp. xxii+593. ISBN: 0-8218-0904-0. DOI: `10.1090/coll/044`. URL: `https://doi.org/10.1090/coll/044`.

[KS02] M. Kneser and R. Scharlau. *Quadratische Formen*. Springer Berlin Heidelberg, 2002. DOI: `https://doi.org/10.1007%2F978-3-642-56380-5`.

[Lan02] S. Lang. *Algebra*. third. Vol. 211. Graduate Texts in Mathematics. Springer-Verlag, New York, 2002, pp. xvi+914. ISBN: 0-387-95385-X. DOI: `10.1007/978-1-4613-0041-0`. URL: `https://doi.org/10.1007/978-1-4613-0041-0`.

[NC10] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, Cambridge, 2010. ISBN: 978-1-107-00217-3.

[Neu92] J. Neukirch. *Algebraische Zahlentheorie*. Springer-Verlag, Berlin, 1992, pp. xiii+595. ISBN: 3-540-54273-6. DOI: 10.1007/978-3-540-37663-7. URL: https://doi.org/10.1007/978-3-540-37663-7.

[OMe00] O. T. O'Meara. *Introduction to Quadratic Forms*. Classics in Mathematics. Reprint of the 1973 edition. Springer-Verlag, Berlin, 2000, pp. xiv+342. ISBN: 3-540-66564-1.

[Osc24] *OSCAR – Open Source Computer Algebra Research system, Version 1.2.0-DEV*. The OSCAR Team, 2024. URL: https://www.oscar-system.org.

[Rei03] I. Reiner. *Maximal Orders*. Vol. 28. London Mathematical Society Monographs. New Series. Corrected reprint of the 1975 original, With a foreword by M. J. Taylor. The Clarendon Press, Oxford University Press, Oxford, 2003, pp. xiv+395. ISBN: 0-19-852673-3.

[Sch85] W. Scharlau. *Quadratic and Hermitian forms*. Vol. 270. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1985, pp. x+421. ISBN: 3-540-13724-6. DOI: 10.1007/978-3-642-69971-9. URL: https://doi.org/10.1007/978-3-642-69971-9.

[Shi10] G. Shimura. *Arithmetic of quadratic forms*. Springer Monographs in Mathematics. Springer, New York, 2010, pp. xii+237. ISBN: 978-1-4419-1731-7. DOI: 10.1007/978-1-4419-1732-4. URL: https://doi.org/10.1007/978-1-4419-1732-4.

[Voi21] J. Voight. *Quaternion Algebras*. Vol. 288. Graduate Texts in Mathematics. Springer, Cham, 2021, pp. xxiii+885. ISBN: 978-3-030-56692-0. DOI: 10.1007/978-3-030-56694-4. URL: https://doi.org/10.1007/978-3-030-56694-4.

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