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RESEARCH

Explicit representations of the norms of the Laguerre-Sobolev and Jacobi-Sobolev polynomials

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Abstract
This paper deals with discrete Sobolev orthogonal polynomials with respect to inner products built upon the classical Laguerre and Jacobi measures on the intervals $[0, \infty)$ and $[-1, 1]$, respectively. In addition, they are equipped with point masses at a finite endpoint of the interval involving the underlying functions and their derivatives of first or higher order. One of the intrinsic features of these polynomials are their L^2-norms in the corresponding inner product spaces. Their knowledge is essential to orthonormalize the polynomials and thus indispensable to treat the corresponding Fourier-Sobolev series and other topics, notably in approximation theory, spectral theory or mathematical physics. Proceeding from an appropriate representation of the Sobolev polynomials which reflect the influence of the point masses, we explicitly establish their squared norm in an efficient form. In each case, the value differs from the familiar squared norm of the Laguerre or Jacobi polynomials by a factor which itself is a product of two essentially identical terms. Surprisingly, each of these factors turns out to be the quotient of the leading coefficients of the Sobolev polynomial and its classical counterpart. Obviously, our results enable to determine the asymptotic behavior of the norms of the orthogonal polynomials considered for large n.

Keywords Sobolev-type inner products · Laguerre-Sobolev polynomials · Jacob-Sobolev polynomials · Norm evalutation

Mathematics Subject Classification (2010) 33C47 · 33C45 · 33F05 · 42C10

1 Introduction

Over nearly four decades, the theory of orthogonal polynomials with respect to inner products of Sobolev type has been attracting enormous interest with continuous progress, either from a general point of view or with regard to specific orthogonal

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polynomial systems. This is best witnessed by the very useful bibliography of more than 550 references collected and currently updated by F. Marcellán ('Orthogonal polynomials and Sobolev inner products', personal communication). For various topics and important developments in the field we refer, e.g., to the profound survey article by F. Marcellán and Y. Xu [1] as well as to other informative papers [2, 3].

In particular, many authors investigated the rich structure of discrete Sobolev (or Sobolev-type) orthogonal polynomials, where derivatives occur only at discrete mass points of the corresponding inner products. Among their many features discovered so far are algebraic properties, representations of different kind, spectral differential equations, recurrence relations, asymptotic properties, distribution of zeros, orthogonal expansions, or analytic aspects, see, in particular, the relevant papers [3–25] and the various contributions described in their titles and abstracts.

In general, the discrete Laguerre-Sobolev polynomials $\{L_n^{\alpha,\{S_r\}}(x)\}_{n=0}^{\infty}$ are defined for $\alpha > -1, 0 \leq x < \infty$, as the orthogonal polynomials with respect to the inner product of two functions f, g in the respective Sobolev space,

$$(f,g)_{w(\alpha,\{S_r\})} = (f,g)_{w(\alpha)} + \sum_{j=0}^{r} S_j f^{(j)}(0)g^{(j)}(0),\ \ S_j \geq 0,\ j=0,...,r,\ r \in \mathbb{N}. \tag{1.1}$$

Likewise, the system $\{P_n^{\alpha,\beta,\{S_r\}}(x)\}_{n=0}^{\infty}$ denotes the discrete Jacobi-Sobolev polynomials for $\alpha, \beta > -1, -1 \leq x \leq 1$, which are orthogonal with respect to the inner product

$$(f,g)_{w(\alpha,\beta,\{S_r\})} = (f,g)_{w(\alpha,\beta)} + \sum_{j=0}^{r} S_j f^{(j)}(1)g^{(j)}(1),\ \ S_j \geq 0,\ j=0,...,r,\ r \in \mathbb{N}. \tag{1.2}$$

If the point masses S_j in (1.1) or (1.2) vanish for all $0 \leq j \leq r$, the inner products reduce to the weighted scalar products of the 'classical' Laguerre and Jacobi polynomials, respectively, where the weight functions are normalized to induce a probability measure, i.e.,

$$(f,g)_{w(\alpha)} = \int_0^{\infty} f(x)g(x)w_\alpha(x)dx,\ \ w_\alpha(x) = h_\alpha^{-1}e^{-x}x^\alpha,\ \ h_\alpha = \Gamma(\alpha+1), \tag{1.3}$$

$$\begin{aligned}(f,g)_{w(\alpha,\beta)} &= \int_{-1}^{1} f(x)g(x)w_{\alpha,\beta}(x)dx,\\ w_{\alpha,\beta}(x) &= h_{\alpha,\beta}^{-1}(1-x)^\alpha(1+x)^\beta,\quad h_{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.\end{aligned} \tag{1.4}$$

The discrete Laguerre-Sobolev polynomials have been introduced and widely studied in a series of papers by R. Koekoek and H. G. Meijer [26–29]. Concerning both cases (1.1) and (1.2) we also refer to [1, Sec.7] and the papers cited there.

As is well known, the Laguerre and Jacobi polynomials are characterized by a number of essential features listed, e.g., in [30, 10.6] or [31, 18.3]. In particular, they are members of the renowned Askey scheme with hypergeometric representations

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n;\alpha+1;x),\ 0 \le x < \infty,$$
$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\Big(-n, n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\Big),\ -1 \le x \le 1, \tag{1.5}$$

for $n \in \mathbb{N}_0$. Moreover, they possess the orthonormalization constants h_n, $n \in \mathbb{N}_0$,

$$\begin{aligned} \|L_n^\alpha\|^2_{w(\alpha)} &= h_n^\alpha := \frac{(\alpha+1)_n}{n!}, \\ \|P_n^{\alpha,\beta}\|^2_{w(\alpha,\beta)} &= h_n^{\alpha,\beta} := \frac{(\alpha+1)_n(\beta+1)_n}{(2n+\alpha+\beta+1)\,n!\,(\alpha+\beta+2)_{n-1}}, \end{aligned} \tag{1.6}$$

while the first and second leading coefficients of their power series are typically denoted by

$$L_n^\alpha(x) = k_n^\alpha x^n + \bar{k}_n^\alpha x^{n-1} + q_{n-2},\ k_n^\alpha = \frac{(-1)^n}{n!},\ \frac{\bar{k}_n^\alpha}{k_n^\alpha} = -n(n+\alpha),$$
$$P_n^{\alpha,\beta}(x) = k_n^{\alpha,\beta} x^n + \bar{k}_n^{\alpha,\beta} x^{n-1} + q_{n-2},\ k_n^{\alpha,\beta} = \frac{(n+\alpha+\beta+1)_n}{2^n n!},\ \frac{\bar{k}_n^{\alpha,\beta}}{k_n^{\alpha,\beta}} = \frac{n(\alpha-\beta)}{2n+\alpha+\beta}. \tag{1.7}$$

But rather than the latter expansion, we will use the more appropriate version

$$\begin{aligned} P_n^{\alpha,\beta}(x) &= \kappa_n^{\alpha,\beta}\Big(\frac{x-1}{2}\Big)^n + \bar{\kappa}_n^{\alpha,\beta}\Big(\frac{x-1}{2}\Big)^{n-1} + q_{n-2}, \\ \kappa_n^{\alpha,\beta} &= \frac{(n+\alpha+\beta+1)_n}{n!}, \quad \frac{\bar{\kappa}_n^{\alpha,\beta}}{\kappa_n^{\alpha,\beta}} = \frac{n(n+\alpha)}{2n+\alpha+\beta}. \end{aligned} \tag{1.8}$$

Here and in the following, $(\alpha)_0 = 1,\ (\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1),\ \alpha \in \mathbb{C},\ m \in \mathbb{N}$, is the Pochhammer symbol, and $q_{n-2} \in \mathcal{P}_{n-2}$ denotes some unspecific algebraic polynomial of degree $n-2$.

In this paper we will focus on two prominent particular settings which reflect the principles of our approach in a comprehensible way. They are associated with the inner products (1.1–2) with either two point masses up to the first derivative, say $S_0 = N$, $S_1 = S$, or with one point mass S_r, $r \ge 2$, up to a quite high order. It is the purpose of the present paper to establish an exact relationship between the squared norms of the Laguerre-Sobolev and Jacobi-Sobolev orthogonal polynomials on the one hand, and the classical quantities stated in (1.6-8) on the other hand. In the Laguerre-Sobolev cases, we will use the notations

$$\begin{aligned} h_n^{\alpha,N,S} &= \|L_n^{\alpha,N,S}\|^2_{w(\alpha,N,S)}, \quad L_n^{\alpha,N,S}(x) = k_n^{\alpha,N,S}x^n + \bar{k}_n^{\alpha,N,S}x^{n-1} + q_{n-2}, \\ h_n^{\alpha,r} &= \|L_n^{\alpha,S_r}\|^2_{w(\alpha,S_r)}, \quad L_n^{\alpha,S_r}(x) = k_n^{\alpha,r}x^n + \bar{k}_n^{\alpha,r}x^{n-1} + q_{n-2}, \end{aligned} \tag{1.9}$$

while $h_n^{\alpha,\beta,N,S}$, $\kappa_n^{\alpha,\beta,N,S}$, $\bar{\kappa}_n^{\alpha,\beta,N,S}$ and $h_n^{\alpha,\beta,r}$, $\kappa_n^{\alpha,\beta,r}$, $\bar{\kappa}_n^{\alpha,\beta,r}$ stand for the respective values in the Jacobi-Sobolev cases.

In each of the four entries, we basically proceed as follows:

(1) We start off from an explicit representation of the Sobolev polynomials in terms of the classical Laguerre and Jacobi polynomials (1.5),

$$\sum_{j=0}^{r+1} B_{j,n} x^j L_{n-j}^{\alpha+2j}(x) \quad \text{and} \quad \sum_{j=0}^{r+1} B_{j,n} \left(\frac{1-x}{2}\right)^j P_{n-j}^{\alpha+2j,\beta}(x), \tag{1.10}$$

with certain coefficients $B_{j,n}$ depending on all the parameters, cf. [20, (1.8)].

(2) With some technical efforts we determine an expression Ω_n based on the index n and all other parameters, which coincides with the quotient of the leading coefficients of the Sobolev polynomial, say k_n^{Sob}, and its classical counterpart k_n^{class}, i.e.,

$$\Omega_n = k_n^{Sob}/k_n^{class}, \ n \in \mathbb{N}. \tag{1.11}$$

(3) Furthermore, the second leading coefficient $\bar{k}_{n+1}^{Sob}$ is shown to be equal to a certain sum involving the classical counterparts $\bar{k}_{n+1-j}^{class}$, $0 \le j \le r+1$, with appropriately shifted parameters.

(4) Combining the steps (1) to (3) then yields our final result

$$h_n^{Sob}/h_n^{class} = \Omega_n \cdot \Omega_{n+1}, \ n \in \mathbb{N}. \tag{1.12}$$

For the Sobolev polynomials $\{L_n^{\alpha,N,S}(x)\}_{n=0}^{\infty}$ and $\{P_n^{\alpha,\beta,N,S}(x)\}_{n=0}^{\infty}$, the identity (1.12) can also be proved directly via some lengthy calculations.

As a general property of any orthogonal polynomial system $\{p_n\}_{n=0}^{\infty}$ in a weighted Hilbert space $L_w^2(I)$, $I \subset \mathbb{R}$, its normalization constant h_n and the leading coefficients k_n, $\bar{k}_n$ naturally occur in the three-term recurrence relation (TTRR), see e.g. [31, 18.2.10-11],

$$\begin{aligned} &x\, p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \ \text{where} \\ &a_n = \frac{k_n}{k_{n+1}}, \ b_n = \frac{\bar{k}_n}{k_n} - \frac{\bar{k}_{n+1}}{k_{n+1}}, \ n \ge 0, \ c_n = a_{n-1}\frac{h_n}{h_{n-1}}, \ n \ge 1, \ a_{n-1}c_n > 0. \end{aligned} \tag{1.13}$$

This, in turn, gives rise to the well-known Christoffel-Darboux formula [31, 18.2.12]

$$K_n(x,y) = \sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{k_n}{k_{n+1}h_n} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}, \tag{1.14}$$

which considerably simplifies the reproducing kernel of the partial sum operator

$$S_n(f,x) = \int_I f(x)K_n(x,y)w(x)dx, \ \ f \in L_w^2, \tag{1.15}$$

and hence becomes a fundamental tool in approximation theory.

For discrete Sobolev orthogonal polynomials, however, it is clear that a three-term recurrence relation no longer holds, because the multiplication operator by x is not symmetric with respect to the inner product. Nevertheless, it has been shown in [32] that there is always a multiplication operator by a polynomial $h(x)$ of degree $r+1$, r being the order of the highest derivative in (1.1-2), which generates a $(2r+3)$-term recurrence relation for the Sobolev polynomials. This relation may be interpreted as a difference equation, which implies, together with the differential equation of the orthogonal polynomials, cf. [12, 21], their bispectrality. For more details and important aspects as, e.g., the relationship with matrix orthogonal polynomials see [33, Sec. 3], [34–38].

In the Laguerre-Sobolev case, the $(2r+3)$-term recurrence relation has been stated by R. Koekoek [26, (2.5.1-3)], [27, Sec. 7] as

$$\begin{aligned} x^{r+1} L_n^{\alpha,\{S_r\}}(x) &= \sum_{k=\max(0,n-r-1)}^{n+r+1} E_{k,n}^{\alpha,\{S_r\}} L_k^{\alpha,\{S_r\}}(x), \ n \in \mathbb{N}_0, \ \text{where} \\ h_k^{\alpha,\{S_r\}} E_{k,n}^{\alpha,\{S_r\}} &= \big(x^{r+1} L_n^{\alpha,\{S_r\}}(x), L_k^{\alpha,\{S_r\}}(x)\big)_{w(\alpha,\{S_r\})} \\ &= \big(L_n^{\alpha,\{S_r\}}(x), x^{r+1} L_k^{\alpha,\{S_r\}}(x)\big)_{w(\alpha,\{S_r\})}. \end{aligned} \tag{1.16}$$

Moreover, it has been shown in [26, (2.6.2)], [27, Sec.8], that a Christoffel-Darboux type formula can be derived by employing the identity (1.16) in the sum

$$(x^{r+1} - y^{r+1}) \sum_{k=0}^{n} \frac{L_k^{\alpha,\{S_r\}}(x) L_k^{\alpha,\{S_r\}}(y)}{h_k^{\alpha,\{S_r\}}}, \ n \in \mathbb{N}_0. \tag{1.17}$$

In case of the Laguerre-Sobolev polynomials $L_n^{\alpha,N,S}(x)$, we precisely determined the coefficients in the five-term recurrence relation (1.16) for $r = 2$ by (the parameters are dropped for simplicity)

$$\begin{aligned} E_{n+2,n} &= \frac{k_n}{k_{n+2}}, \quad E_{n+1,n} = \frac{k_n}{k_{n+1}} \Big[\frac{\bar{k}_n}{k_n} - \frac{\bar{k}_{n+2}}{k_{n+2}}\Big], \\ E_{n,n} &= \Big[1 - \frac{\bar{k}_{n+1}}{k_{n+1}}\Big]\Big[\frac{\bar{k}_n}{k_n} - \frac{\bar{\bar{k}}_{n+2}}{k_{n+2}}\Big], \\ E_{n-1,n} &= E_{n,n-1}\frac{h_n}{h_{n-1}}, \quad E_{n-2,n} = E_{n,n-2}\frac{h_n}{h_{n-2}}. \end{aligned} \tag{1.18}$$

Here, $\bar{\bar{k}}_n$ denotes the third leading coefficient of the Laguerre-Sobolev polynomial $L_n^{\alpha,N,S}(x)$, which can easily be deduced from its representation, cf.(2.2). Via (1.18) we accomplish the Christoffel-Darboux type formula (with $h_n = h_n^{\alpha,N,S}$, $k_n = k_n^{\alpha,N,S}$)

$$\sum_{k=0}^{n} \frac{L_k^{\alpha,N,S}(x)L_k^{\alpha,N,S}(y)}{h_k} = \frac{k_n}{k_{n+2}h_n} \frac{L_{n+2}^{\alpha,N,S}(x)L_n^{\alpha,N,S}(y) - L_n^{\alpha,N,S}(x)L_{n+2}^{\alpha,N,S}(y)}{x^2 - y^2}$$
$$+ \frac{k_n}{k_{n+1}h_n}\left[\frac{\bar{k}_n}{k_n} - \frac{\bar{k}_{n+2}}{k_{n+2}}\right] \frac{L_{n+1}^{\alpha,N,S}(x)L_n^{\alpha,N,S}(y) - L_n^{\alpha,N,S}(x)L_{n+1}^{\alpha,N,S}(y)}{x^2 - y^2}. \tag{1.19}$$

Certainly, such a formula holds for the Jacobi-Sobolev polynomials $P_n^{\alpha,\beta,N,S}(x)$, as well, and may be extended to even more general situations. Also notice that for Laguerre-Sobolev polynomials in a more general setting, the five-term recurrence relation has recently been determined in [14, Sec.4], [37, Sec.5] by investigating the TTRR of the corresponding, so-called 2-iterated kernel orthogonal polynomials.

Numerous authors have investigated the orthogonal expansions of functions into Sobolev orthogonal polynomials and dealt with questions of their convergence or summability, see in particular [7, 8, 16, 24, 25]. Needless to say, it is an important task here to treat the corresponding reproducing kernels $K_n^{Sob}(x, y)$, whence a detailed knowledge of the squared norm and the leading coefficients of the Sobolev polynomials is highly appreciated.

The paper is organized as follows. In Section 2 we provide the Laguerre-Sobolev polynomials $\{L_n^{\alpha,N,S}(x)\}_{n=0}^{\infty}$ in a form accessible for our purpose. After carrying out the two preliminary steps (Props.2.1-2), we obtain the desired representation of the squared norm (Thm.1.1). Once knowing this result, we are in a position to confirm it twice. In a second proof, we basically use an alternative representation of the Laguerre-Sobolev polynomials used in [27, 28]. Moreover, we add a third proof by directly evaluating the inner product of the polynomial with itself. The crucial part here is to show that all the occurring terms can actually be combined to yield the product of the Ω-factors as indicated in (1.12). To this end, we repeatedly make use of the symbolic computer program MAPLE.

Section 3 is devoted to the Sobolev-Laguerre polynomials $\{L_n^{\alpha,S_r}(x)\}_{n=0}^{\infty}$ associated with a higher derivative. Here our main approach furnishes a result of similar type (Thm.3.3). But other proofs as those in Section 2 would either be highly complicated or at least extremely tedious.

In Section 4 we simultaneously present the results for the Jacobi-Sobolev polynomials $\{P_n^{\alpha,\beta,N,S}(x)\}_{n=0}^{\infty}$ and $\{P_n^{\alpha,\beta,S_r}(x)\}_{n=0}^{\infty}$, see Thm.4.3. Due to the additional 'Jacobi' parameter β, the formulas look fairly elaborate and so we just point out the basic ideas. If necessary, various calculations have been checked and verified by invoking MAPLE again.

In Section 5, we close with some remarks how to extend our results to other Sobolev-type orthogonal polynomials.

2 The norm of the Laguerre-Sobolev polynomials associated with two perturbations

Following the second version of the Laguerre-Sobolev polynomials in [26, (4.2.7-8)], [28, (10.1-2)] or [21, (1.2-3)], they are related to the Laguerre polynomials (1.5) by

$$
\begin{aligned}
&L_n^{\alpha,N,S}(x) = \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} x^j L_{n-j}^{\alpha+2j}(x),\ 0 \le x \le \infty,\ n \in \mathbb{N}_0,\ \text{where}\\
&B_{0,n}^{\alpha,N,S} = \left[1 - S\, u_{0,n}^{\alpha}\right],\ B_{1,n}^{\alpha,N,S} = -\left[N\, t_n^{\alpha} + S\, u_{1,n}^{\alpha}\right],\ B_{2,n}^{\alpha,N,S} = \left[S\, u_{2,n}^{\alpha} + NS\, v_n^{\alpha}\right],\\
&\text{and } t_n^{\alpha} = \frac{(\alpha+2)_{n-1}}{n!},\ v_n^{\alpha} = \frac{(\alpha+2)_{n-1}(\alpha+3)_{n-1}}{(\alpha+1)_3\,(n-1)!\,n!},\\
&u_{0,n}^{\alpha} = \frac{(\alpha+2)_n}{(\alpha+1)_3\,(n-2)!},\ u_{1,n}^{\alpha} = \frac{(\alpha+2)_{n-1}}{(\alpha+1)(\alpha+3)(n-2)!},\ u_{2,n}^{\alpha} = \frac{(\alpha+2)_{n-1}}{(\alpha+1)_3\,(n-1)!}.
\end{aligned}
\tag{2.1}
$$

This representation has the advantage that the first and second leading coefficients in (1.9) can be derived from those in (1.7) by means of

$$
\begin{aligned}
k_n^{\alpha,N,S} &= \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} k_{n-j}^{\alpha+2j},\ \text{where } k_{n-j}^{\alpha+2j} = \frac{(-1)^{n-j}}{(n-j-2)!},\ j = 0, 1, 2,\\
\bar{k}_n^{\alpha,N,S} &= \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} \bar{k}_{n-j}^{\alpha+2j},\ \text{where } \bar{k}_{n-j}^{\alpha+2j} = \frac{(-1)^{n-j-1}(n+\alpha+j)}{(n-j-1)!},\ j = 0, 1, 2.
\end{aligned}
\tag{2.2}
$$

Proposition 2.1 *For $n \in \mathbb{N}_0$, $\alpha > -1$, and $N, S \ge 0$, there holds*

$$
\begin{aligned}
&\frac{k_n^{\alpha,N,S}}{k_n^{\alpha}} = \Omega_n^{\alpha,N,S},\ \textit{where}\ \ \Omega_n^{\alpha,N,S} = 1 + \phi_n^{\alpha} N + \chi_n^{\alpha} S + \psi_n^{\alpha} NS \ \ \textit{with}\\
&\phi_n^{\alpha} = \frac{(\alpha+2)_{n-1}}{(n-1)!},\ \chi_n^{\alpha} = \frac{(\alpha+2)_{n-1}}{(\alpha+1)_3\,(n-2)!}[(\alpha+2)(n-1)+1],\\
&\psi_n^{\alpha} = \frac{(\alpha+2)_{n-1}(\alpha+3)_{n-1}}{(\alpha+1)_3\,(n-2)!\,(n-1)!}.
\end{aligned}
\tag{2.3}
$$

Proof Separating the influence of the point masses N, S in (2.1), the first line of (2.2) yields

$$
\begin{aligned}
\frac{k_n^{\alpha,N,S}}{k_n^{\alpha}} &= \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} \frac{k_{n-j}^{\alpha+2j}}{k_n^{\alpha}}\\
&= 1 - t_n^{\alpha} \frac{k_{n-1}^{\alpha+2}}{k_n^{\alpha}} N - \left\{ u_{0,n}^{\alpha} + u_{1,n}^{\alpha} \frac{k_{n-1}^{\alpha+2}}{k_n^{\alpha}} - u_{2,n}^{\alpha} \frac{k_{n-2}^{\alpha+4}}{k_n^{\alpha}} \right\} S + v_n^{\alpha} \frac{k_{n-2}^{\alpha+4}}{k_n^{\alpha}} NS
\end{aligned}
$$

$$= 1 + t_n^\alpha n\, N - \left\{u_{0,n}^\alpha - u_{1,n}^\alpha n - u_{2,n}^\alpha (n-1)_2\right\} S + v_n^\alpha (n-1)_2 NS$$
$$= 1 + \phi_n^\alpha\, N + \chi_n^\alpha\, S + \psi_n^\alpha\, NS.$$

In fact, the last identity follows by observing that

$$t_n^\alpha\, n = \frac{(\alpha+2)_{n-1}\, n}{n!} = \frac{(\alpha+2)_{n-1}}{(n-1)!} = \phi_n^\alpha,$$
$$v_n^\alpha (n-1)_2 = \frac{(\alpha+2)_{n-1}(\alpha+3)_{n-1}(n-1)_2}{(\alpha+1)_3\,(n-1)!\,n!} = \psi_n^\alpha,$$

while the coefficient of S yields

$$-u_{0,n}^\alpha + u_{1,n}^\alpha n + u_{2,n}^\alpha (n-1)_2 = \frac{(\alpha+2)_{n-1}[-(n+\alpha+1)+(\alpha+2)n+n]}{(\alpha+1)_3\,(n-2)!}$$
$$= \frac{(\alpha+2)_{n-1}[(\alpha+2)(n-1)+1]}{(\alpha+1)_3\,(n-2)!} = \chi_n^\alpha.$$

□

Analogously, a representation of the second leading coefficient follows from the second line in (2.2). For our purpose, however, we need a more intricate relationship.

Proposition 2.2 *With the parameters stated in Prop.2.1,*

$$\frac{\bar{k}_{n+1}^{\alpha,N,S}}{\bar{k}_{n+1}^{\alpha}} = \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} \frac{(n+\alpha+1)_j}{(n-j+2)_j} \frac{\bar{k}_{n+1-j}^{\alpha+2j}}{\bar{k}_{n+1}^{\alpha}}. \tag{2.4}$$

Proof In view of (2.2),

$$\frac{\bar{k}_{n+1}^{\alpha,N,S}}{\bar{k}_{n+1}^{\alpha}} = \sum_{j=0}^{2} B_{j,n+1}^{\alpha,N,S} \frac{\bar{k}_{n+1-j}^{\alpha+2j}}{\bar{k}_{n+1}^{\alpha}} \quad \text{with} \quad \frac{\bar{k}_{n+1-j}^{\alpha+2j}}{\bar{k}_{n+1}^{\alpha}} = \frac{(n+\alpha+j+1)(-n)_j}{(n+\alpha+1)}, \quad j = 0, 1, 2.$$

Observing further that on the right-hand side of (2.4),

$$\frac{(n+\alpha+1)_j}{(n-j+2)_j} \frac{\bar{k}_{n+1-j}^{\alpha+2j}}{\bar{k}_{n+1}^{\alpha}} = \frac{(n-j+1)(n+\alpha+2)_j}{(-1)^j (n+1)},$$

we have to show that

$$\sum_{j=0}^{2} B_{j,n+1}^{\alpha,N,S} \frac{(n+\alpha+j+1)_j(-n)_j}{(n+\alpha+1)} = \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} \frac{(n-j+1)(n+\alpha+2)_j}{(-1)^j (n+1)}, \tag{2.5}$$

or, by (2.1),

$$\begin{aligned}\left[1 - S\,u^{\alpha}_{0,n+1}\right] + \left[N\,t^{\alpha}_{n+1} + S\,u^{\alpha}_{1,n+1}\right]\frac{(n+\alpha+2)n}{(n+\alpha+1)} +\\ + \left[S\,u^{\alpha}_{2,n+1} + NS\,v^{\alpha}_{n+1}\right]\frac{(n+\alpha+3)_j(-n)_2}{(n+\alpha+1)}\\ = \left[1 - Su^{\alpha}_{0,n}\right] + \left[Nt^{\alpha}_{n} + Su^{\alpha}_{1,n}\right]\frac{n(n+\alpha+2)}{(n+1)}\\ + \left[Su^{\alpha}_{2,n} + NSv^{\alpha}_{n}\right]\frac{(n-1)(n+\alpha+2)_2}{(n+1)}.\end{aligned}$$

Checking this equation separately for the coefficients of N and NS, we easily see that

$$\begin{aligned}\frac{t^{\alpha}_{n+1}}{(n+\alpha+1)} &= \frac{(\alpha+2)_{n-1}}{(n+1)!} = \frac{t^{\alpha}_{n}}{(n+1)},\ v^{\alpha}_{n+1}\frac{(n+\alpha+3)(-n)_2}{(n+\alpha+1)}\\ &= \frac{(\alpha+2)_n(\alpha+3)_n(n+\alpha+3)(-n)_2}{(\alpha+1)_3 n!(n+1)!(n+\alpha+1)} = v^{\alpha}_{n}\frac{(n-1)(n+\alpha+2)_2}{(n+1)}.\end{aligned}$$

Furthermore, some calculation shows that the required identity for the coefficient of S holds by appropriately combining the values of $u^{\alpha}_{j,n+1}$ and $u^{\alpha}_{j,n}$ for $j = 0, 1, 2$, i.e.,

$$\begin{aligned}&- u^{\alpha}_{0,n+1} + u^{\alpha}_{1,n+1}\frac{(n+\alpha+2)n}{(n+\alpha+1)} + u^{\alpha}_{2,n+1}\frac{(n+\alpha+3)(-n)_2}{(n+\alpha+1)}\\ &= -\frac{(\alpha+2)_{n+1}}{(\alpha+1)_3\,(n-1)!} + \frac{(\alpha+2)(\alpha+2)_n(n+\alpha+2)n}{(\alpha+1)_3(n-1)!(n+\alpha+1)}\\ &+ \frac{(\alpha+2)_n(n+\alpha+3)(-n)_2}{(\alpha+1)_3\,n!(n+\alpha+1)}\\ &= \frac{(\alpha+2)_{n-1}}{(\alpha+1)_3\,(n-2)!}[(n+\alpha+2)(\alpha+2)+1]\\ &= -u^{\alpha}_{0,n} + u^{\alpha}_{1,n}\frac{n(n+\alpha+2)}{(n+1)} + u^{\alpha}_{2,n}\frac{(n-1)(n+\alpha+2)_2}{(n+1)}.\end{aligned}$$

This settles the proof of (2.5) and hence that of Prop.2.2. □

Theorem 2.3 *For $n \in \mathbb{N}$, $\alpha > -1$, and $N, S \geq 0$, let k^{α}_n, $k^{\alpha,N,S}_n$ denote the leading coefficients of the Laguerre (-Sobolev) polynomials as in Prop.2.1, and let $\Omega^{\alpha,N,S}_n = 1 + \phi^{\alpha}_n N + \chi^{\alpha}_n S + \psi^{\alpha}_n NS$ be defined in (2.3). Then*

$$\frac{\|L^{\alpha,N,S}_n\|^2_{w(\alpha,N,S)}}{\|L^{\alpha}_n\|^2_{w(\alpha)}} = \frac{k^{\alpha,N,S}_n}{k^{\alpha}_n}\,\frac{k^{\alpha,N,S}_{n+1}}{k^{\alpha}_{n+1}} \equiv \Omega^{\alpha,N,S}_n\,\Omega^{\alpha,N,S}_{n+1},\ \ where\ \ \|L^{\alpha}_n\|^2_{w(\alpha)} = h^{\alpha}_n. \tag{2.6}$$

In particular, it follows that

$$\frac{\|L_n^{\alpha,N,0}\|^2_{w(\alpha,N,0)}}{\|L_n^{\alpha}\|^2_{w(\alpha)}} = \big[1+\phi_n^{\alpha} N\big]\big[1+\phi_{n+1}^{\alpha} N\big],\ \phi_n^{\alpha} = \frac{(\alpha+2)_{n-1}}{(n-1)!},$$
$$\frac{\|L_n^{\alpha,0,S}\|^2_{w(\alpha,0,S)}}{\|L_n^{\alpha}\|^2_{w(\alpha)}} = \big[1+\chi_n^{\alpha} S\big]\big[1+\chi_{n+1}^{\alpha} S\big],\ \chi_n^{\alpha} = \frac{(\alpha+2)_{n-1}[(\alpha+2)(n-1)+1]}{(\alpha+1)_3\,(n-2)!}. \tag{2.7}$$

First Proof By orthogonality of the Laguerre-Sobolev polynomials we get, in view of (1.9),

$$\begin{aligned} 0 &= (L_n^{\alpha,N,S}, L_{n+1}^{\alpha,N,S})_{w(\alpha,N,S)} = (L_n^{\alpha,N,S}, k_{n+1}^{\alpha,N,S} x^{n+1} + \bar{k}_{n+1}^{\alpha,N,S} x^n + q_{n-1})_{w(\alpha,N,S)} \\ &= (L_n^{\alpha,N,S}, k_{n+1}^{\alpha,N,S} x^{n+1})_{w(\alpha,N,S)} + \frac{\bar{k}_{n+1}^{\alpha,N,S}}{k_n^{\alpha,N,S}} h_n^{\alpha,N,S}. \end{aligned} \tag{2.8}$$

Obviously, the boundary terms in the latter inner product vanish, so that via (2.1),

$$\begin{aligned} (L_n^{\alpha,N,S}, k_{n+1}^{\alpha,N,S} x^{n+1})_{w(\alpha,N,S)} &= \frac{k_{n+1}^{\alpha,N,S}}{k_{n+1}^{\alpha}} \Big(\sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} x^j L_{n-j}^{\alpha+2j}(x), k_{n+1}^{\alpha} x^{n+1}\Big)_{w(\alpha)} \\ &= \frac{k_{n+1}^{\alpha,N,S}}{k_{n+1}^{\alpha}} \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} \frac{h_{\alpha+2j}}{h_{\alpha}} \frac{k_{n+1}^{\alpha}}{k_{n+1-j}^{\alpha+2j}} (L_{n-j}^{\alpha+2j}(x), k_{n+1-j}^{\alpha+2j} x^{n+1-j})_{w(\alpha+2j)}. \end{aligned} \tag{2.9}$$

Here we utilized that the Laguerre weight function in (1.3) satisfies

$$x^{2j}\, h_{\alpha}\, w_{\alpha}(x) = e^{-x} x^{\alpha+2j} = h_{\alpha+2j}\, w_{\alpha+2j}(x),\ \ j \in \mathbb{N}_0.$$

Since

$$k_{n+1-j}^{\alpha+2j} x^{n+1-j} = L_{n+1-j}^{\alpha+2j}(x) - \frac{\bar{k}_{n+1-j}^{\alpha+2j}}{k_{n-j}^{\alpha+2j}} L_{n-j}^{\alpha+2j}(x) - q_{n-1-j},$$

a combination of (2.8) and (2.9) yields

$$\begin{aligned} -\frac{\bar{k}_{n+1}^{\alpha,N,S}}{k_n^{\alpha,N,S}} h_n^{\alpha,N,S} &= (L_n^{\alpha,N,S}, k_{n+1}^{\alpha,N,S} x^{n+1})_{w(\alpha,N,S)} \\ &= -\frac{k_{n+1}^{\alpha,N,S}}{k_{n+1}^{\alpha}} \frac{h_n^{\alpha}}{k_n^{\alpha}} \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} \frac{h_{\alpha+2j}}{h_{\alpha}} \frac{k_{n+1}^{\alpha}}{k_{n+1-j}^{\alpha+2j}} \frac{k_n^{\alpha}}{k_{n-j}^{\alpha+2j}} \frac{h_{n-j}^{\alpha+2j}}{h_n^{\alpha}} \bar{k}_{n+1-j}^{\alpha+2j} \\ &= -\frac{k_{n+1}^{\alpha,N,S}}{k_{n+1}^{\alpha}} \frac{h_n^{\alpha}}{k_n^{\alpha}} \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} \frac{(n+\alpha+1)_j}{(n+2-j)_j} \bar{k}_{n+1-j}^{\alpha+2j}. \end{aligned}$$

Employing now Prop.2.2, we arrive at the required identity

$$\frac{h_n^{\alpha,N,S}}{h_n^{\alpha}} = \frac{k_n^{\alpha,N,S}}{k_n^{\alpha}} \frac{k_{n+1}^{\alpha,N,S}}{k_{n+1}^{\alpha}} \frac{1}{\bar{k}_{n+1}^{\alpha,N,S}} \sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} \frac{(n+\alpha+1)_j}{(n+2-j)_j} \bar{k}_{n+1-j}^{\alpha+2j} = \frac{k_n^{\alpha,N,S}}{k_n^{\alpha}} \frac{k_{n+1}^{\alpha,N,S}}{k_{n+1}^{\alpha}}.$$

The second identity in (2.6) then follows by Prop.2.1. □

Second Proof It has been shown in [28, (1.2-3)] that an alternative version of the Laguerre-Sobolev polynomials is given by

$$L_n^{\alpha,N,S}(x) = \sum_{j=0}^{2} A_{j,n}^{\alpha,N,S} \frac{d^j}{dx^j} L_n^{\alpha}(x), \quad \text{where} \quad A_0 = 1 + \frac{(\alpha+2)_{n-1}}{(n-1)!} N +$$
$$+ \frac{[n(\alpha+2)-(\alpha+1)](\alpha+3)_{n-2}}{(\alpha+1)(\alpha+3)(n-2)!} S + \frac{(\alpha+2)_{n-1}(\alpha+4)_{n-2}}{(\alpha+1)_2\,(n-1)!\,(n-2)!} NS,$$
$$A_1 = \frac{(\alpha+1)_n}{n!} N + \frac{(n-1)(\alpha+2)_{n-1}}{(\alpha+1)(n-1)!} S + \frac{2(\alpha+1)_n(\alpha+4)_{n-2}}{(\alpha+1)^2\, n!(n-2)!} NS,$$
$$A_2 = \frac{(\alpha+2)_{n-1}}{(\alpha+1)(n-1)!} S + \frac{(\alpha+1)_n(\alpha+3)_{n-1}}{(\alpha+1)^2\, n!\,(n-1)!} NS.$$

By invoking some known properties of the Laguerre polynomials, the authors then deduced the orthogonality of the Laguerre-Sobolev polynomials as well as its norm in the form [28, (4.4)]

$$h_n^{\alpha,N,S} = \binom{n+\alpha}{n} A_0\,[A_0 + A_1 + A_2]. \tag{2.10}$$

The factor A_0 coincides already with $\Omega_n^{\alpha,N,S} = 1 + \phi_n^{\alpha}\, N + \chi_n^{\alpha}\, S + \psi_n^{\alpha}\, NS$. Surprisingly, by adding the coefficients of N, S, and NS in A_0, A_1, A_2, separately, we find that

$$A_0 + A_1 + A_2 = 1 + \phi_{n+1}^{\alpha}\, N + \chi_{n+1}^{\alpha}\, S + \psi_{n+1}^{\alpha}\, NS = \Omega_{n+1}^{\alpha,N,S}.$$

Notice, for instance, that the coefficient of S yields

$$\frac{[n(\alpha+2)-(\alpha+1)](\alpha+3)_{n-2}}{(\alpha+1)(\alpha+3)(n-2)!} + \frac{(n-1)(\alpha+2)_{n-1}}{(\alpha+1)(n-1)!} + \frac{(\alpha+2)_{n-1}}{(\alpha+1)(n-1)!}$$
$$= \frac{(\alpha+2)_{n-1}(n+\alpha+1)}{(\alpha+1)_3\,(n-1)!}[(\alpha+2)n+1] = \chi_{n+1}^{\alpha}.$$

Hence, we arrive at the second identity on the right-hand side of (2.6).

The particular case $S = 0$ concerns Koornwinder's generalized Laguerre polynomials known in the literature as Bochner-Krall type. Here, the first corollary in (2.7) has been stated already in [26, (3.1.8)]. □

Third Proof Proceeding directly from the representation (2.1) of the Laguerre-Sobolev polynomials we have to evaluate

$$\|L_n^{\alpha,N,S}\|^2_{w(\alpha,N,S)} = \int_0^\infty \Big(\sum_{j=0}^{2} B_{j,n}^{\alpha,N,S} x^j L_{n-j}^{\alpha+2j}(x)\Big)^2 w_\alpha(x)dx \\ + N\big[L_n^{\alpha,N,S}(0)\big]^2 + S\big[(L_n^{\alpha,N,S})'(0)\big]^2.$$

Using that

$$L_n^\alpha(0) = \frac{(\alpha+1)_n}{n!}, \quad (L_n^\alpha)'(0) = -L_{n-1}^{\alpha+1}(0) = -\frac{(\alpha+2)_{n-1}}{(n-1)!},$$

we find that the respective values in the Laguerre-Sobolev case either depend on S or N, namely

$$L_n^{\alpha,N,S}(0) = B_{0,n}^{\alpha,N,S} L_n^\alpha(0) = (1 - S\, u_{0,n}^\alpha)\frac{(\alpha+1)_n}{n!},$$
$$(L_n^{\alpha,N,S})'(0) = -B_{0,n}^{\alpha,N,S} L_{n-1}^{\alpha+1}(0) + B_{1,n}^{\alpha,N,S} L_{n-1}^{\alpha+2}(0) \\ = -\frac{(\alpha+2)_{n-1}}{(n-1)!} - N\frac{(\alpha+2)_{n-1}}{n!}\frac{(\alpha+3)_{n-1}}{(n-1)!}.$$

Hence, the two boundary terms possess the representations

$$N\big[L_n^{\alpha,N,S}(0)\big]^2 = N\big[1 - S\, u_{0,n}^\alpha\big]^2 \frac{(\alpha+1)_n}{n!} h_n^\alpha,$$
$$S\big[(L_n^{\alpha,N,S})'(0)\big]^2 = S\big[1 + N\frac{(\alpha+3)_{n-1}}{n!}\big]^2 \frac{(\alpha+2)_{n-1}\, n}{(\alpha+1)(n-1)!} h_n^\alpha.$$

In the integral we separately carry out the six terms emerging from the square of the sum,

$$\|L_n^{\alpha,N,S}\|^2_{w(\alpha)} = E_0^2 + E_1^2 + E_2^2 + 2\,[E_0E_1 + E_0E_2 + E_1E_2],$$

where, after some lengthy calculations,

$$E_0^2 = B_{0,n}^2 \int_0^\infty [L_n^\alpha(x)]^2 w_\alpha(x)dx = \big[1 - S\, u_{0,n}^\alpha\big]^2 h_n^\alpha,$$
$$E_1^2 = B_{1,n}^2 \frac{h_{\alpha+2}}{h_\alpha}\int_0^\infty [L_{n-1}^{\alpha+2}(x)]^2 w_{\alpha+2}(x)dx = \big[N\, t_n^\alpha + S\, u_{1,n}^\alpha\big]^2 n(n+\alpha+1)h_n^\alpha,$$
$$E_2^2 = B_{2,n}^2 \frac{h_{\alpha+4}}{h_\alpha}\int_0^\infty [L_{n-2}^{\alpha+4}(x)]^2 w_{\alpha+4}(x)dx = \big[Su_{2,n}^\alpha + NSv_n^\alpha\big]^2 (n-1)_2(n+\alpha+1)_2 h_n^\alpha$$

and, by employing the leading coefficient of the Laguerre polynomial in (1.7),

$$\begin{aligned} 2\,E_0E_1 &= 2\,B_{0,n}B_{1,n}\int_0^\infty L_n^\alpha(x)L_{n-1}^{\alpha+2}(x)w_\alpha(x)dx \\ &= 2\,B_{0,n}B_{1,n}\frac{k_{n-1}^{\alpha+2}}{k_n^\alpha}h_n^\alpha = 2\,B_{0,n}B_{1,n}\,n\,h_n^\alpha, \end{aligned}$$

$$\begin{aligned} 2\,E_0E_2 &= 2\,B_{0,n}B_{2,n}\frac{k_{n-2}^{\alpha+4}}{k_n^\alpha}h_n^\alpha = 2\,B_{0,n}B_{2,n}\,(n-1)_2\,h_n^\alpha, \\ 2\,E_1E_2 &= 2\,B_{1,n}B_{2,n}\frac{k_{n-2}^{\alpha+4}}{k_{n-1}^{\alpha+2}}\frac{h_{\alpha+2}}{h_\alpha}h_{n-1}^{\alpha+2} = 2\,B_{1,n}B_{2,n}(n-1)_2(n+\alpha+1)h_n^\alpha. \end{aligned}$$

Next, we combine all parts and expand the resulting sum into terms of similar powers of N and S,

$$\begin{aligned} \|L_n^{\alpha,N,S}\|^2_{w(\alpha,N,S)}/h_n^\alpha =& 1+\theta_N N+\theta_{N^2}N^2+\theta_S S+\theta_{S^2}S^2+ \\ &+\theta_{NS}NS+\theta_{N^2S}N^2S+\theta_{NS^2}NS^2+\theta_{N^2S^2}N^2S^2. \end{aligned}$$

The crucial point now is to realize that the resulting coefficients all add up to

$$\begin{aligned} &\theta_N = \phi_n^\alpha+\phi_{n+1}^\alpha, \quad \theta_{N^2} = \phi_n^\alpha\phi_{n+1}^\alpha, \quad \theta_S = \chi_n^\alpha+\chi_{n+1}^\alpha, \quad \theta_{S^2} = \chi_n^\alpha\chi_{n+1}^\alpha, \\ &\theta_{NS} = \phi_n^\alpha\chi_{n+1}^\alpha+\chi_n^\alpha\phi_{n+1}^\alpha+\psi_n^\alpha+\psi_{n+1}^\alpha, \\ &\theta_{N^2S} = \phi_n^\alpha\psi_{n+1}^\alpha+\phi_{n+1}^\alpha\psi_n^\alpha, \quad \theta_{NS^2} = \chi_n^\alpha\psi_{n+1}^\alpha+\chi_{n+1}^\alpha\psi_n^\alpha, \quad \theta_{N^2S^2} = \psi_n^\alpha\psi_{n+1}^\alpha. \end{aligned} \tag{2.11}$$

This could indeed be proved by using MAPLE. Consequently,

$$\|L_n^{\alpha,N,S}\|^2_{w(\alpha,N,S)} = h_n^\alpha\big[1+\phi_n^\alpha N+\chi_n^\alpha S+\psi_n^\alpha NS\big]\big[1+\phi_{n+1}^\alpha N+\chi_{n+1}^\alpha S+\psi_{n+1}^\alpha NS\big].$$

□

Finally, we like to point out that, in a slightly more general setting, an explicit relationship between the squared norms of the monic Laguerre-Sobolev orthogonal polynomials and their classical counterparts can be found in [14, La.4.2] or [37, La.2]. When taking Prop.2.1 and the boundary values stated in the *Third Proof* into account, it seems to be feasible to derive a further proof of Thm.2.3 in this way. This as well as a possible extension to the Jacobi-Sobolev case is certainly worthwhile to be worked out in more detail.

3 The norm of the Laguerre-Sobolev polynomials associated with a higher derivative

In [20, Thm.2.2], we have explicitly derived a second version of the Laguerre-Sobolev polynomials in a more general form. For our purpose, it reduces to $L_n^{\alpha,S_r}(x) = L_n^\alpha(x)$

if $n \le r$, while for $n > r$,

$$L_n^{\alpha,S_r} = \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} x^j L_{n-j}^{\alpha+2j}(x), \;\; B_{j,n}^{\alpha,r} = \begin{cases} 1 - S_r \gamma_n^{\alpha,r} \sum_{s=1}^{r} \eta_{s,n}^{\alpha,r} \omega_{s,n}^{\alpha,r}, \;\; j = 0 \\ S_r \gamma_n^{\alpha,r} \omega_{j,n}^{\alpha,r}, \;\; 1 \le j \le r+1 \end{cases}, \text{ where}$$
$$\gamma_n^{\alpha,r} = \frac{(r+\alpha+1)_{n-r} r!}{(\alpha+1)_{2r+1}(n-r-1)!}, \;\; \eta_{s,n}^{\alpha,r} = \frac{(-r)_s (n+\alpha+1)_s}{(r+\alpha+1)_s},$$
$$\omega_{s,n}^{\alpha,r} = \frac{(-1)^s (r+\alpha+1)_{r+1} (n-s)!\,(\alpha+2s)}{(r+\alpha+1)_{s+1}(n-r)!\,(r+1-s)!}. \tag{3.1}$$

With the first and second leading coefficients of the Laguerre (-Sobolev) polynomials as in (1.7), (1.9), we readily find that

$$\begin{aligned} \frac{k_n^{\alpha,r}}{k_n^{\alpha}} &= \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} \frac{k_{n-j}^{\alpha+2j}}{k_n^{\alpha}} = \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} (-n)_j, \\ \frac{\bar{k}_n^{\alpha,r}}{\bar{k}_n^{\alpha}} &= \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} \frac{\bar{k}_{n-j}^{\alpha+2j}}{\bar{k}_n^{\alpha}} = \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} \frac{(n+j+\alpha)(1-n)_j}{(n+\alpha)}. \end{aligned} \tag{3.2}$$

Proposition 3.1 *For $n > r$ and $1 \le r \le 20$,*

$$\begin{aligned} \frac{k_n^{\alpha,r}}{k_n^{\alpha}} &= 1 + \chi_n^{\alpha,r} S_r =: \Omega_n^{\alpha,r}, \quad \textit{where} \\ \chi_n^{\alpha,r} &= \frac{(\alpha+r+1)_{n-r}}{(\alpha+1)_{2r+1}(n-1-r)!} \sum_{j=0}^{r} \left(\frac{r!}{j!}\right)^2 (r+\alpha+1)_j (n-r)_j. \end{aligned} \tag{3.3}$$

Proof For $r = 1$, it has been shown in Prop.2.1 already that

$$\chi_n^{\alpha,1} = \frac{(\alpha+2)_{n-1}}{(\alpha+1)_3 (n-2)!}[(\alpha+2)(n-1)+1], \;\; n > 1,$$

and for $r = 2$, we find that in view of (3.1) and (3.2) and after some tedious calculations,

$$\frac{k_n^{\alpha,2}}{k_n^{\alpha}} = \sum_{j=0}^{3} B_{j,n}^{\alpha,2}(-n)_j = \frac{(\alpha+3)_{n-2}\big[(\alpha+3)_2(n-2)_2 + 4(\alpha+3)(n-2) + 4\big]}{(\alpha+1)_5 (n-3)!}, \;\; n > 2.$$

These first two cases already suggest the general representation stated in (3.3). In fact, via the use of MAPLE, it was not hard to verify the result up to $r = 20$. But most probably, it holds for even higher values of r. □

The analogue of Prop.2.2 is exactly of the same structure.

Proposition 3.2 *For $n > r$ and $1 \leq r \leq 20$,*

$$\frac{\bar{k}_{n+1}^{\alpha,r}}{\bar{k}_{n+1}^{\alpha}} = \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} \frac{(n+\alpha+1)_j}{(n-j+2)_j} \frac{\bar{k}_{n+1-j}^{\alpha+2j}}{\bar{k}_{n+1}^{\alpha}}. \tag{3.4}$$

Proof Analogously to (2.5), we have to show that

$$\sum_{j=0}^{r+1} B_{j,n+1}^{\alpha,r} \frac{(n+\alpha+j+1)(-n)_j}{(n+\alpha+1)} = \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} \frac{(n-j+1)(n+\alpha+2)_j}{(-1)^j(n+1)}. \tag{3.5}$$

Identity (3.5), in turn, is equivalent to $1 + \Phi_n^{\alpha,r} S_r = 1 + \Psi_n^{\alpha,r} S_r$, where

$$\begin{aligned}\Phi_n^{\alpha,r} &= \gamma_{n+1}^{\alpha,r} \left\{ \begin{array}{l} \sum_{j=1}^{r} \omega_{j,n+1}^{\alpha,r} \left[\frac{(n+j+\alpha+1)(-n)_j}{(n+\alpha+1)} - \eta_{j,n+1}^{\alpha,r} \right] + \\ + \omega_{r+1,n+1}^{\alpha,r} \frac{(n+r+\alpha+2)(-n)_{r+1}}{(n+\alpha+1)} \end{array} \right\} \\ &= \gamma_n^{\alpha,r} \left\{ \begin{array}{l} \sum_{j=1}^{r} \frac{(-1)^j (r+\alpha+1)_{r+1} (n+1-j)!\,(\alpha+2j)}{(r+\alpha+1)_{j+1}(n+2-r)!\,(r+1-j)!} \left[\frac{(n+j+\alpha+1)(-n)_j}{(n-r)} - \right. \\ \left. - \frac{(-r)_j (n+\alpha+1)_{j+1}}{(n-r)(r+\alpha+1)_j} \right] + \frac{(n+r+\alpha+2)(n-r)_{r+1}}{(n-r)_2} \end{array} \right\}\end{aligned}$$

and

$$\begin{aligned}\Psi_n^{\alpha,r} &= \gamma_n^{\alpha,r} \left\{ \begin{array}{l} \sum_{j=1}^{r} \omega_{j,n}^{\alpha,r} \left[\frac{(n-j+1)(n+\alpha+2)_j}{(-1)^j(n+1)} - \eta_{j,n}^{\alpha,r} \right] + \\ + \omega_{r+1,n}^{\alpha,r} \frac{(n-r)(n+\alpha+2)_{r+1}}{(-1)^{r+1}(n+1)} \end{array} \right\} \\ &= \gamma_n^{\alpha,r} \left\{ \begin{array}{l} \sum_{j=1}^{r} \frac{(-1)^j (r+\alpha+1)_{r+1} (n-j)!\,(\alpha+2j)}{(r+\alpha+1)_{j+1}(n-r)!\,(r+1-j)!} \cdot \\ \cdot \left[\frac{(n-j+1)(n+\alpha+2)_j}{(-1)^j(n+1)} - \frac{(-r)_j(n+\alpha+1)_j}{(r+\alpha+1)_j} \right] + \frac{(n+\alpha+2)_{r+1}}{(n+1)} \end{array} \right\}.\end{aligned}$$

In fact, we succeeded to verify that $\Phi_n^{\alpha,r} = \Psi_n^{\alpha,r}$ for at least $1 \leq r \leq 20$. For $r = 1$, we checked this identity by hand, and otherwise, we employed MAPLE again. □

Theorem 3.3 *For $\alpha > -1$, $1 \leq r \leq 20$ and $n > r$, let the Laguerre-Sobolev polynomials $L_n^{\alpha,S_r}(x)$ and their leading coefficients $k_n^{\alpha,r}$ be given as in (3.1-2) and let $\Omega_n^{\alpha,r} = 1 + \chi_n^{\alpha,r} S_r$ be defined in (3.3). Then*

$$\frac{\|L_n^{\alpha,S_r}\|_{w(\alpha,S_r)}^2}{\|L_n^{\alpha}\|_{w(\alpha)}^2} = \frac{k_n^{\alpha,r}}{k_n^{\alpha}} \frac{k_{n+1}^{\alpha,r}}{k_{n+1}^{\alpha}} \equiv \Omega_n^{\alpha,r}\, \Omega_{n+1}^{\alpha,r}, \quad \textit{where} \quad \|L_n^{\alpha}\|_{w(\alpha)}^2 = h_n^{\alpha}. \tag{3.6}$$

Proof We can widely adopt the first proof of Thm.2.3, where $h_n^{\alpha,r}$ denotes the squared norm of the Laguerre-Sobolev polynomial in (1.9). Analogously to (2.8-9) there holds

$$\begin{aligned}
-\frac{\bar{k}_{n+1}^{\alpha,r}}{k_n^{\alpha,r}}\, h_n^{\alpha,r} &= (L_n^{\alpha,S_r}, k_{n+1}^{\alpha,r} x^{n+1})_{w(\alpha,S_r)} \\
&= \frac{k_{n+1}^{\alpha,r}}{k_{n+1}^{\alpha}} \Big(\sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} x^j L_{n-j}^{\alpha+2j}(x), k_{n+1}^{\alpha} x^{n+1}\Big)_{w(\alpha)} \\
&= \frac{k_{n+1}^{\alpha,r}}{k_{n+1}^{\alpha}} \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} \frac{h_{\alpha+2j}}{h_\alpha} \frac{k_{n+1}^{\alpha}}{k_{n+1-j}^{\alpha+2j}} (L_{n-j}^{\alpha+2j}(x), k_{n+1-j}^{\alpha+2j}\, x^{n+1-j})_{w(\alpha+2j)} \\
&= -\frac{k_{n+1}^{\alpha,r}}{k_{n+1}^{\alpha}} \frac{h_n^{\alpha}}{k_n^{\alpha}} \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} \frac{(n+\alpha+1)_j}{(n+2-j)_j}\, \bar{k}_{n+1-j}^{\alpha+2j}.
\end{aligned} \tag{3.7}$$

But this identity yields, in combination with Prop.3.3,

$$\frac{h_n^{\alpha,r}}{h_n^{\alpha}} = \frac{k_n^{\alpha,r}}{k_n^{\alpha}} \frac{k_{n+1}^{\alpha,r}}{k_{n+1}^{\alpha}} \frac{1}{\bar{k}_{n+1}^{\alpha,r}} \sum_{j=0}^{r+1} B_{j,n}^{\alpha,r} \frac{(n+\alpha+1)_j}{(n+2-j)_j}\, \bar{k}_{n+1-j}^{\alpha+2j} = \frac{k_n^{\alpha,r}}{k_n^{\alpha}} \frac{k_{n+1}^{\alpha,r}}{k_{n+1}^{\alpha}}.$$

In view of Prop.3.1, the right-hand side equals $\Omega_n^{\alpha,r}\, \Omega_{n+1}^{\alpha,r}$ as required. □

Notice that the restriction of Thm.3.3 to $r \le 20$ is due to the fact that (3.3) and (3.4) have been verified up to this order. But undoubtedly, the result should hold for even higher values of r.

4 The norm of the Jacobi-Sobolev polynomials

For any $\alpha, \beta > -1$ the two classes of the Jacobi-Sobolev polynomials to be considered here, have been explicitly determined in [20, Sec.3] as particular cases of the polynomials $\{P_n^{\alpha,\beta,S_R,S_T}(x)\}_{n=0}^{\infty}$, which are orthogonal on $-1 \le x \le 1$ with respect to (1.2) with two point masses S_R, S_T involving derivatives of any order $0 \le R < T \in \mathbb{N}$. In terms of the classical Jacobi polynomials, the two classes are given by, cf. (1.10),

$$\begin{aligned}
P_n^{\alpha,\beta,N,S}(x) &= \sum_{j=0}^{2} B_{j,n}^{\alpha,\beta,N,S} \Big(\frac{1-x}{2}\Big)^j P_{n-j}^{\alpha+2j,\beta}(x), \text{ where } B_{0,n}^{\alpha,\beta,N,S} = \big[1 - S u_{0,n}^{\alpha,\beta}\big], \\
B_{1,n}^{\alpha,\beta,N,S} &= -\big[N\, t_n^{\alpha,\beta} + S u_{1,n}^{\alpha,\beta}\big],\ B_{2,n}^{\alpha,\beta,N,S} = \big[S u_{2,n}^{\alpha,\beta} + N S v_n^{\alpha,\beta}\big], \text{ with} \\
t_n^{\alpha,\beta} &= \frac{(\alpha+2)_{n-1}(\alpha+\beta+2)_n}{n!\,(\beta+1)_{n-1}},\ u_{0,n}^{\alpha,\beta} = \frac{(\alpha+2)_n(\alpha+\beta+2)_{n+1}}{4(\alpha+1)_3\,(n-2)!\,(\beta+1)_{n-1}},
\end{aligned}$$

$$
\begin{aligned}
u_{1,n}^{\alpha,\beta} &= \frac{(\alpha+2)_{n-1}(n+\alpha+\beta+1)(\alpha+\beta+2)_{n+1}}{4(\alpha+1)(\alpha+3)(n-2)!\,(\beta+1)_{n-1}},\\
u_{2,n}^{\alpha,\beta} &= \frac{(\alpha+2)_{n-1}(n+\alpha+\beta+1)(\alpha+\beta+2)_{n+1}}{4(\alpha+1)_3\,(n-1)!\,(\beta+1)_{n-2}},\\
v_{n}^{\alpha,\beta} &= \frac{(\alpha+2)_{n-1}(\alpha+3)_{n-1}(\alpha+\beta+2)_{n}(\alpha+\beta+2)_{n+1}}{4(\alpha+1)_3\,(n-1)!\,n!\,(\beta+1)_{n-2}(\beta+1)_{n-1}},
\end{aligned} \tag{4.1}
$$

and, for any $r \geq 2$,

$$
\begin{aligned}
&P_n^{\alpha,\beta,S_r}(x) = \sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,r}\big(\frac{1-x}{2}\big)^j P_{n-j}^{\alpha+2j,\beta}(x), \quad \text{where}\\
B_{j,n}^{\alpha,\beta,r} &= \begin{cases} 1 - S_r\gamma_n^{\alpha,\beta,r}\displaystyle\sum_{s=1}^{r}\eta_{s,n}^{\alpha,\beta,r}\omega_{s,n}^{\alpha,\beta,r}, & j=0\\ S_r\gamma_n^{\alpha,\beta,r}\omega_{j,n}^{\alpha,\beta,r}, & 1 \leq j \leq r+1 \end{cases}, \quad \text{and}\\
\gamma_n^{\alpha,\beta,r} &= \frac{(r+\alpha+1)_{n-r}r!\,(n+\alpha+\beta+1)_{r+1}(\alpha+\beta+2)_{n+r-1}}{2^{2r}(\alpha+1)_{2r+1}(n-r-1)!},\\
\eta_{s,n}^{\alpha,\beta,r} &= \frac{(-r)_s(n+\alpha+1)_s}{(r+\alpha+1)_s(n+\alpha+\beta+1)_s},\\
\omega_{s,n}^{\alpha,\beta,r} &= \frac{(-1)^s(r+\alpha+1)_{r+1}(n-s)!\,(\alpha+2s)}{(r+\alpha+1)_{s+1}(n-r)!\,(r+1-s)!\,(\beta+1)_{n-s}}.
\end{aligned} \tag{4.2}
$$

In both situations and in analogy to the Laguerre-Sobolev cases (2.2) and (3.2), the first and second leading coefficients of the Jacobi-Sobolev polynomials are related to their classical counterparts defined in (1.8) by

$$
\begin{aligned}
\frac{\kappa_n^{\alpha,\beta,\{S_r\}}}{\kappa_n^{\alpha,\beta}} &= \sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,\{S_r\}}\frac{\kappa_{n-j}^{\alpha+2j,\beta}}{\kappa_n^{\alpha,\beta}} = \sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,\{S_r\}}\frac{(-n)_j}{(n+\alpha+\beta+1)_j},\\
\frac{\bar\kappa_n^{\alpha,\beta,\{S_r\}}}{\bar\kappa_n^{\alpha,\beta}} &= \sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,\{S_r\}}\frac{\bar\kappa_{n-j}^{\alpha+2j,\beta}}{\bar\kappa_n^{\alpha,\beta}} = \sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,\{S_r\}}\frac{(n+j+\alpha)(1-n)_j}{(n+\alpha)(n+\alpha+\beta+1)_j}.
\end{aligned} \tag{4.3}
$$

Proposition 4.1 *For $\alpha, \beta > -1$ and $n \in \mathbb{N}$, the following identities hold.*

(a) $\dfrac{\kappa_n^{\alpha,\beta,N,S}}{\kappa_n^{\alpha,\beta}} = \Omega_n^{\alpha,\beta,N,S}$, *where* $\Omega_n^{\alpha,\beta,N,S} = 1+\phi_n^{\alpha,\beta}N+\chi_n^{\alpha,\beta}S+\psi_n^{\alpha,\beta}NS$ *with*

$$
\phi_n^{\alpha,\beta} = \frac{(\alpha+2)_{n-1}(\alpha+\beta+2)_{n-1}}{((n-1)!\,(\beta+1)_{n-1}},
$$

$$\chi_n^{\alpha,\beta} = \frac{(\alpha+2)_{n-1}(\alpha+\beta+2)_n}{4(\alpha+1)_3\,(n-2)!\,(\beta+1)_{n-1}}[(\alpha+2)(n-1)(n+\alpha+\beta+1)+\beta],$$
$$\psi_n^{\alpha,\beta} = \frac{(\alpha+2)_{n-1}(\alpha+3)_{n-1}(\alpha+\beta+2)_{n-1}(\alpha+\beta+2)_n}{4(\alpha+1)_3\,(n-2)!\,(n-1)!\,(\beta+1)_{n-2}(\beta+1)_{n-1}}.$$

(b) $$\frac{\bar{\kappa}_{n+1}^{\alpha,\beta,N,S}}{\bar{\kappa}_{n+1}^{\alpha,\beta}} = \sum_{j=0}^{2} B_{j,n}^{\alpha,\beta,N,S}\frac{(n+\alpha+1)_j(n+\alpha+\beta+2)_j}{(n-j+2)_j(n-j+\beta+1)_j}\frac{\bar{\kappa}_{n+1-j}^{\alpha+2j,\beta}}{\bar{\kappa}_{n+1}^{\alpha,\beta}}.$$

Proof **(a)** In view of (4.3), it follows by definition of $B_{j,n}^{\alpha,\beta,N,S}$ that

$$\frac{\kappa_n^{\alpha,\beta,N,S}}{\kappa_n^{\alpha,\beta}} = 1+\frac{t_n^{\alpha,\beta}\,n}{(n+\alpha+\beta+1)}N-[u_{0,n}^{\alpha,\beta}-\frac{u_{1,n}^{\alpha,\beta}\,n}{(n+\alpha+\beta+1)}-\frac{u_{2,n}^{\alpha,\beta}\,(n-1)_2}{(n+\alpha+\beta+1)_2}]\,S$$
$$+\frac{v_n^{\alpha,\beta}\,(n-1)_2}{(n+\alpha+\beta+1)_2}NS.$$

Some calculations then show that the coefficients of N, S, and NS are equal to $\phi_n^{\alpha,\beta}$, $\chi_n^{\alpha,\beta}$, and $\psi_n^{\alpha,\beta}$, respectively.

(b) We note that the identity is equivalent to

$$\sum_{j=0}^{2} B_{j,n+1}^{\alpha,\beta,N,S}\frac{(n+j+\alpha+1)(-n)_j}{(n+\alpha+1)(n+\alpha+\beta+2)_j} = \sum_{j=0}^{2} B_{j,n}^{\alpha,\beta,N,S}\frac{(n-j+1)(n+\alpha+2)_j}{(n+1)(-n-\beta)_j}.$$

With the knowledge of the coefficients and some efforts, this can indeed be verified. □

In case of an inner product with a single point mass S_r related to a derivative of order r, we obtain the following.

Proposition 4.2 *For $\alpha,\beta>-1$, $n>r$ and $1\le r\le 10$, the following identities hold.*

(a) $$\frac{\kappa_n^{\alpha,\beta,r}}{\kappa_n^{\alpha,\beta}} = \Omega_n^{\alpha,\beta,r},\quad \textit{where}\quad \Omega_n^{\alpha,\beta,r} = 1+\chi_n^{\alpha,\beta,r}\,S_r\quad \textit{with}$$
$$\chi_n^{\alpha,\beta,r} = \delta_n^{\alpha,\beta,r}\sum_{j=0}^{r}\left(\frac{r!}{j!}\right)^2(r+\alpha+1)_j(n-r)_j(n+r+\alpha+\beta+1-j)_j(\beta)_{r-j},$$
$$\textit{and}\quad \delta_n^{\alpha,\beta,r} = \frac{(\alpha+r+1)_{n-r}(\alpha+\beta+2)_{n-1+r}}{(\alpha+1)_{2r+1}(n-1-r)!\,(\beta+1)_{n-1}}.$$

(b) $$\frac{\bar{\kappa}_{n+1}^{\alpha,\beta,r}}{\bar{\kappa}_{n+1}^{\alpha,\beta}} = \sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,r}\frac{(n+\alpha+1)_j(n+\alpha+\beta+2)_j}{(n-j+2)_j(n-j+\beta+1)_j}\frac{\bar{\kappa}_{n+1-j}^{\alpha+2j,\beta}}{\bar{\kappa}_{n+1}^{\alpha,\beta}}.$$

Proof **(a)** Notice that $\chi_n^{\alpha,\beta,1}$ is equal to $\chi_n^{\alpha,\beta}$ defined in Prop.4.1(a), while

$$\chi_n^{\alpha,\beta,2} = \frac{(\alpha+3)_{n-2}(\alpha+\beta+2)_{n+1}}{2^4(\alpha+1)_5(n-3)!\,(\beta+1)_{n-1}} \left\{ \begin{array}{l} (\alpha+3)_2(n-2)_2(n+\alpha+\beta+1)_2 \\ +\,4\beta\,(\alpha+3)(n-2)(n+\alpha+\beta+2) \\ +\,4\beta(\beta+1) \end{array} \right\}.$$

For higher values up to $r = 10$, the formula has been verified by MAPLE.

(b) To prove this identity, we made use of MAPLE, as well. □

When compared with Prop.3.1 and Prop.3.2 in the Laguerre-Sobolev case, the even higher complexity of the two identities in Prop.4.2 is responsible for the stronger limitation to $r \leq 10$. Again, there is no reason to believe that this restriction is essential.

Theorem 4.3 *For $\alpha, \beta > -1$ and $N, S \geq 0$, $S_r > 0$, let the Jacobi-Sobolev polynomials $P_n^{\alpha,\beta,N,S}(x)$ and $P_n^{\alpha,\beta,S_r}(x)$ be given in (4.1-2), and let their first and second leading coefficients be denoted by $\kappa_n^{\alpha,\beta,N,S}$, $\bar{\kappa}_n^{\alpha,\beta,N,S}$ and $\kappa_n^{\alpha,\beta,r}$, $\bar{\kappa}_n^{\alpha,\beta,r}$, respectively. We also recall that $\| P_n^{\alpha,\beta} \|_{w(\alpha,\beta)}^2 = h_n^{\alpha,\beta}$ as stated in (1.6).*

(a) *Let $\Omega_n^{\alpha,\beta,N,S} = 1 + \phi_n^{\alpha,\beta}\,N + \chi_n^{\alpha,\beta}\,S + \psi_n^{\alpha,\beta}\,NS$, $n \in \mathbb{N}$, be given as in Prop.4.1.(a). Then*

$$\frac{\| P_n^{\alpha,\beta,N,S} \|_{w(\alpha,\beta,N,S)}^2}{\| P_n^{\alpha,\beta} \|_{w(\alpha,\beta)}^2} = \frac{\kappa_n^{\alpha,\beta,N,S}}{\kappa_n^{\alpha,\beta}} \, \frac{\kappa_{n+1}^{\alpha,\beta,N,S}}{\kappa_{n+1}^{\alpha,\beta}} \equiv \Omega_n^{\alpha,\beta,N,S}\, \Omega_{n+1}^{\alpha,\beta,N,S}. \tag{4.4}$$

In particular, it follows that

$$\frac{\| P_n^{\alpha,\beta,N,0} \|_{w(\alpha,\beta,N,0)}^2}{\| P_n^{\alpha,\beta} \|_{w(\alpha,\beta)}^2} = \left[1 + \phi_n^{\alpha,\beta} N\right]\left[1 + \phi_{n+1}^{\alpha,\beta} N\right],$$
$$\frac{\| P_n^{\alpha,\beta,0,S} \|_{w(\alpha,\beta,0,S)}^2}{\| P_n^{\alpha,\beta} \|_{w(\alpha,\beta)}^2} = \left[1 + \chi_n^{\alpha,\beta} S\right]\left[1 + \chi_{n+1}^{\alpha,\beta} S\right].$$

(b) *Let $\Omega_n^{\alpha,\beta,r} = 1 + \chi_n^{\alpha,\beta,r}\,S_r$, $n > r$, $1 \leq r \leq 10$, be given as in Prop.4.2(a). Then*

$$\frac{\| P_n^{\alpha,\beta,S_r} \|_{w(\alpha,\beta,S_r)}^2}{\| P_n^{\alpha,\beta} \|_{w(\alpha,\beta)}^2} = \frac{\kappa_n^{\alpha,\beta,r}}{\kappa_n^{\alpha,\beta}} \, \frac{\kappa_{n+1}^{\alpha,\beta,r}}{\kappa_{n+1}^{\alpha,\beta}} \equiv \Omega_n^{\alpha,\beta,r}\, \Omega_{n+1}^{\alpha,\beta,r}. \tag{4.5}$$

Proof The method we used to prove Thm.2.3 and Thm.3.3 in the Laguerre-Sobolev cases is applicable here, as well. Again, the basic ingredients are provided in Props.4.1-2. Notice that the decisive transition from the Jacobi-Sobolev inner product to the Jacobi scalar product holds true, since the Sobolev polynomials are expanded into powers of $(1-x)/2$ which vanish at the boundary $x=1$. To begin with part (b), the identities (3.7) correspond to

$$\begin{aligned}
-\frac{\bar{\kappa}_{n+1}^{\alpha,\beta,r}}{\kappa_n^{\alpha,\beta,r}}\, h_n^{\alpha,\beta,r} &= (P_n^{\alpha,\beta,S_r}, \kappa_{n+1}^{\alpha,\beta,r}\big(\frac{1-x}{2}\big)^{n+1})_{w(\alpha,\beta,S_r)}\\
&= \frac{\kappa_{n+1}^{\alpha,\beta,r}}{\kappa_{n+1}^{\alpha,\beta}}\big(\sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,r}\big(\frac{1-x}{2}\big)^j P_{n-j}^{\alpha+2j,\beta}(x), \kappa_{n+1}^{\alpha,\beta}\big(\frac{1-x}{2}\big)^{n+1}\big)_{w(\alpha,\beta)}\\
&= \frac{\kappa_{n+1}^{\alpha,\beta,r}}{\kappa_{n+1}^{\alpha,\beta}}\sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,r}\frac{h_{\alpha+2j,\beta}}{2^{2j}h_{\alpha,\beta}}\frac{\kappa_{n+1}^{\alpha,\beta}}{\kappa_{n+1-j}^{\alpha+2j,\beta}}\big(P_{n-j}^{\alpha+2j,\beta}(x), \kappa_{n+1-j}^{\alpha+2j,\beta}\big(\frac{1-x}{2}\big)^{n+1-j}\big)_{w(\alpha+2j,\beta)} \qquad (4.6)\\
&= -\frac{\kappa_{n+1}^{\alpha,\beta,r}}{\kappa_{n+1}^{\alpha,\beta}}\sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,r}\frac{h_{\alpha+2j,\beta}}{2^{2j}h_{\alpha,\beta}}\frac{\kappa_{n+1}^{\alpha,\beta}}{\kappa_{n+1-j}^{\alpha+2j,\beta}}\frac{\bar{\kappa}_{n+1-j}^{\alpha+2j,\beta}}{\kappa_{n-j}^{\alpha+2j,\beta}}\, h_{n-j}^{\alpha+2j,\beta}\\
&= -\frac{\kappa_{n+1}^{\alpha,\beta,r}}{\kappa_{n+1}^{\alpha,\beta}}\frac{h_n^{\alpha,\beta}}{\kappa_n^{\alpha,\beta}}\sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,r}\frac{(n+\alpha+1)_j(n+\alpha+\beta+2)_j}{(n-j+2)_j(n-j+\beta+1)_j}\,\bar{\kappa}_{n+1-j}^{\alpha+2j,\beta}.
\end{aligned}$$

Here, the shift of the parameter α in the Jacobi weight function (1.4) was justified in view of

$$\big(\frac{1-x}{2}\big)^{2j} h_{\alpha,\beta}\, w_{\alpha,\beta}(x) = \frac{1}{2^{2j}}(1-x)^{\alpha}(1+x)^{\beta} = \frac{1}{2^{2j}} h_{\alpha+2j,\beta}\, w_{\alpha+2j,\beta}(x),\ \ j\in\mathbb{N}_0,$$

while the last identity in (4.6) follows in view of

$$\frac{h_{\alpha+2j,\beta}}{2^{2j}h_{\alpha,\beta}}\frac{\kappa_{n+1}^{\alpha,\beta}}{\kappa_{n+1-j}^{\alpha+2j,\beta}}\frac{\kappa_n^{\alpha,\beta}}{\kappa_{n-j}^{\alpha+2j,\beta}}\frac{h_{n-j}^{\alpha+2j,\beta}}{h_n^{\alpha,\beta}} = \frac{(n+\alpha+1)_j(n+\alpha+\beta+2)_j}{(n-j+2)_j(n-j+\beta+1)_j}.$$

By converting (4.6) we get

$$\frac{h_n^{\alpha,\beta,r}}{h_n^{\alpha,\beta}} = \frac{\kappa_n^{\alpha,\beta,r}}{\kappa_n^{\alpha,\beta}}\frac{\kappa_{n+1}^{\alpha,\beta,r}}{\kappa_{n+1}^{\alpha,\beta}}\frac{1}{\bar{\kappa}_{n+1}^{\alpha,\beta,r}}\sum_{j=0}^{r+1} B_{j,n}^{\alpha,\beta,r}\frac{(n+\alpha+1)_j(n+\alpha+\beta+2)_j}{(n-j+2)_j(n-j+\beta+1)_j}\,\bar{\kappa}_{n+1-j}^{\alpha+2j,\beta}.$$

Applying now both parts (b) and (a) of Prop.4.2 then yields the result (4.5).

To verify (4.4), we proceed from the definition (4.1) of the Jacobi-Sobolev polynomials $P_n^{\alpha,\beta,N,S}(x)$ with coefficients $B_{j,n}^{\alpha,\beta,N,S}$, $j=0,1,2$. Essentially, the procedure (4.6) is independent of the choice of the coefficients, so that

$$\frac{h_n^{\alpha,\beta,N,S}}{h_n^{\alpha,\beta}} = \frac{\kappa_n^{\alpha,\beta,N,S}\kappa_{n+1}^{\alpha,\beta,N,S}}{\kappa_n^{\alpha,\beta}\kappa_{n+1}^{\alpha,\beta}\bar{\kappa}_{n+1}^{\alpha,\beta,N,S}}\sum_{j=0}^{2} B_{j,n}^{\alpha,\beta,N,S}\frac{(n+\alpha+1)_j(n+\alpha+\beta+2)_j}{(n-j+2)_j(n-j+\beta+1)_j}\,\bar{\kappa}_{n+1-j}^{\alpha+2j,\beta}.$$

By invoking Prop.4.1, the proof is accomplished. □

The results stated above have the following interesting consequence in common.

Corollary 4.4 *Let $\{P_n^{Sob}\}_{n=0}^{\infty}$ denote any of the polynomial systems treated in Thms.2.3, 3.3 and 4.3 and let Ω_n^{Sob} be the corresponding quantity defined in (2.6), (3.6), (4.4) and (4.5), respectively. Then the L^2-norms of the Sobolev polynomials and their classical counterparts are asymptotically related to each other by*

$$\|P_n^{Sob}\| \simeq \|P_n^{class}\| \cdot \Omega_n^{Sob} \quad (n \to \infty).$$

Remark 4.5 We are not aware of an identity analogous to (2.10) in case of the Jacobi-Sobolev polynomials $P_n^{\alpha,\beta,N,S}(x)$, which could be utilized to give another proof of Thm.4.4(a). We found, however, a direct proof similar to the third one of Thm.2.3. Again, it required to split up the representation of the squared norm

$$\begin{aligned}\|P_n^{\alpha,\beta,N,S}\|^2_{w(\alpha,\beta,N,S)} &= \int_{-1}^{1}\Big(\sum_{j=0}^{2} B_{j,n}^{\alpha,\beta,N,S}\big(\frac{1-x}{2}\big)^j P_{n-j}^{\alpha+2j,\beta}(x)\Big)^2 w_{\alpha,\beta}(x))dx \\ &\quad + N\big[P_n^{\alpha,\beta,N,S}(1)\big]^2 + S\big[(P_n^{\alpha,\beta,N,S})'(1)\big]^2\end{aligned}$$

into eight pieces and to evaluate each of them, separately. After expanding their sum into similar powers of N and S, it was really amazing to realize that, similarly as in (2.11) and again by the use of MAPLE, all the resulting coefficients can be written in terms of $\phi_n^{\alpha,\beta}$, $\chi_n^{\alpha,\beta}$, $\psi_n^{\alpha,\beta}$ in Prop.4.1(a) in such way that they factorize into the second product on the right-hand side of (4.4).

Remark 4.6 Comparing the results in Thm.4.3 with those in the Laguerre-Sobolev cases one encounters a striking similarity. This is not surprising when taking into account that the Jacobi- and Laguerre-Sobolev polynomials are linked to each other via a limit process which extends the well-known relation $\lim_{\beta\to\infty} P_n^{\alpha,\beta}(1-2x/\beta) = L_n^{\alpha}(x)$. In fact, it has been shown in [21, Prop.2.2] and, more generally, in [20, Cor.3.1] that, after replacing the point mass S_r by $S_r(\beta) = (2/\beta)^{2r} S_r$, there holds

$$\lim_{\beta\to\infty} P_n^{\alpha,\beta,N,S_1(\beta)}\big(1-\frac{2x}{\beta}\big) = L_n^{\alpha,N,S_1}(x),\ \lim_{\beta\to\infty} P_n^{\alpha,\beta,S_r(\beta)}\big(1-\frac{2x}{\beta}\big) = L_n^{\alpha,S_r}(x),\ \ r\in\mathbb{N}.$$

Furthermore, we find that

$$\begin{aligned}&\big(-\frac{2}{\beta}D_\xi\big)^r P_n^{\alpha,\beta,N,S_1(\beta)}(\xi)\big|_{\xi=1} \xrightarrow[\beta\to\infty]{} (L_n^{\alpha,S_r})^{(r)}(0),\ \ r\in\mathbb{N}.\\ &\frac{2}{\beta} w_{\alpha,\beta}\big(1-\frac{2x}{\beta}\big) \xrightarrow[\beta\to\infty]{} w_\alpha(x),\ \ h_n^{\alpha,\beta} \xrightarrow[\beta\to\infty]{} h_n^{\alpha}.\end{aligned}$$

Hence, the quotients of the squared norms in (4.4) and (4.5) tend in the limit $\beta \to \infty$ to the quotients of the respective norms of the Laguerre-Sobolev polynomials in (2.6) and (3.6). The identities stated there then follow by observing that

$$\Omega_n^{\alpha,\beta,N,S(\beta)} = 1 + \phi_n^{\alpha,\beta}\, N + \chi_n^{\alpha,\beta}\, \frac{4S}{\beta^2} + \psi_n^{\alpha,\beta}\, N \frac{4S}{\beta^2}$$
$$\xrightarrow[\beta\to\infty]{} 1 + \phi_n^{\alpha}\, N + \chi_n^{\alpha}\, S + \psi_n^{\alpha}\, NS = \Omega_n^{\alpha,N,S}$$

and

$$\Omega_n^{\alpha,\beta,S_r(\beta)} = 1 + \chi_n^{\alpha,\beta,r}\Big(\frac{2}{\beta}\Big)^{2r} S_r$$
$$= 1+\Big(\frac{2}{\beta}\Big)^{2r} \delta_n^{\alpha,\beta,r} \sum_{j=0}^{r} \Big(\frac{r!}{j!}\Big)^2 (r+\alpha+1)_j (n-r)_j (n+r+\alpha+\beta+1-j)_j (\beta)_{r-j} S_r$$
$$\xrightarrow[\beta\to\infty]{} 1+\Big[\frac{(r+\alpha+1)_{n-r}}{(\alpha+1)_{2r+1}(n-1-r)!} \sum_{j=0}^{r} \Big(\frac{r!}{j!}\Big)^2 (r+\alpha+1)_j (n-r)_j\Big] S_r = \Omega_n^{\alpha,S_r}.$$

5 Concluding remarks and perspectives

There are further systems of Sobolev-type orthogonal polynomials with important properties one would like to know, notably their representations and the values of their norms. For instance, one may think of the Jacobi-Sobolev polynomials with point masses at both sides of the interval $[-1, 1]$, say $\{P_n^{\alpha,\beta,M,N,T,S}(x)\}_{n=0}^{\infty}$, being orthogonal with respect to the inner product, cf. [6, Sec.3], [24],

$$(f, g)_{w(\alpha,\beta,M,N,T,S)} = (f, g)_{(w\alpha,\beta)} + \begin{cases} M\, f(-1)g(-1) + N\, f(1)g(1) + \\ T\, f'(-1)g'(-1) + S\, f'(1)g'(1). \end{cases} \tag{5.1}$$

In three particular situations, the norm values can be treated by our methods.

1. First of all, we note that our results in Section 4 immediately carry over to the Jacobi-Sobolev polynomials with only point masses at $x = -1$, i.e., in case $N = S = 0$ of (5.1). This is due to the symmetry relation of the Jacobi-Sobolev polynomials $P_n^{\alpha,\beta,M,N,T,S}(-x)=(-1)^n P_n^{\beta,\alpha,N,M,S,T}(x)$ [6, Prop.1], [8, Sec.3], combined with a corresponding relationship of the inner products.
2. In the particular case $T = S = 0$, (5.1) reduces to the scalar product for the generalized Jacobi polynomials of Bochner-Krall type, which have been introduced by J. and R. Koekoek [39] and studied by many authors. Here, the squared norm has been given already in a form analogous to Thm.4.1(a) [40]. The proof is based on the four-term representation of the polynomials $P_n^{\alpha,\beta,M,N,0,0}(x)$ and requires an elaborate evaluation of the numerous pieces of the squared norm similarly to the third proof of Thm.2.3.

3. In the literature, there are also many valuable contributions in the symmetric case $M = N$, $T = S$. The corresponding polynomials are known as the (symmetric) Gegenbauer-Sobolev polynomials as given in [41, Sec.2], see also [21, Sec.3]. By applying our new approach described in Section 1, we recently determined their squared norms in a form similar to the Jacobi-Sobolev case treated in Thm.4.1(a). But here the indices of the two resulting $\Omega-$factors differ by 2. We intent to publish these and related results with further relevant references in due course.

To conclude: We are confident that the disclosed norm values of the considered Sobolev orthogonal polynomials are useful to further explore their features and to gain deeper insight into their nature. Moreover, there is a fair chance that the presented concept will open promising new perspectives.

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