WELL-POSEDNESS OF THE R13 EQUATIONS USING TENSOR-VALUED KORN INEQUALITIES*

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Abstract. In this paper, we finally catch up with proving the well-posedness of the linearized R13 moment model, which describes, e.g., rarefied gas flows. As an extension of the classical fluid equations, moment models are robust and have been frequently used, yet they are challenging to analyze due to their additional equations. By effectively grouping variables, we identify a 2-by-2 block structure, allowing the analysis of the well-posedness within the abstract LBB framework of saddle point problems. Due to the unique tensorial structure of the equations, in addition to an interesting combination of tools from Stokes' and linear elasticity theory, we also need new coercivity estimates for tensor fields. These Korn-type inequalities are established by analyzing the symbol map of the symmetric and trace-free part of tensor derivative fields. Together with the corresponding right inverse of the tensorial divergence, we obtain the existence and uniqueness of weak solutions.

Key words. Regularized 13-moment equations, well-posedness, Korn inequalities, coercivity estimates, ellipticity, saddle point problem

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1. Introduction. In this paper, we propose and analyze a mixed formulation for the linear regularized 13-moment equations (R13) [40] on bounded Lipschitz domains $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma \coloneqq \partial\Omega$, i.e., Ω is an open, non-empty, connected set whose boundary Γ can be locally expressed by the graph of a Lipschitz continuous function. The R13 equations are a system of partial differential equations that describe the evolution of the macroscopic quantities in, e.g., rarefied gases or microfluids. In these flows, the length of the mean free path between particle collisions becomes significant when compared to a reference length scale of the process. The ratio between these length scales is typically given by the *Knudsen number* $\mathrm{Kn} > 0$, which serves as a parameter in the equations. Let $m : \Omega \to \mathbb{R}$, $\boldsymbol{b} : \Omega \to \mathbb{R}^3$, and $r : \Omega \to \mathbb{R}$ be a mass source, a body force, and an energy source. Similar to the Stokes equations combined with Fourier's law of heat conduction, we are interested in the fields for the pressure $p : \Omega \to \mathbb{R}$, the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$, and the temperature $\theta : \Omega \to \mathbb{R}$, given by

$$(1.1)\qquad \begin{cases} \operatorname{div} \boldsymbol{u} = m, \\ \boldsymbol{\nabla} p + \operatorname{Div} \boldsymbol{\sigma} = \boldsymbol{b}, \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} + \operatorname{div} \boldsymbol{s} = r, \end{cases}$$

where we distinguish between $\operatorname{div} \boldsymbol{u} : \Omega \to \mathbb{R}$, $\operatorname{div} \boldsymbol{u} \coloneqq \sum_j \partial_j u_j$, and $\operatorname{Div} \boldsymbol{\sigma} : \Omega \to \mathbb{R}^3$, $(\operatorname{Div} \boldsymbol{\sigma})_i \coloneqq \sum_j \partial_j \sigma_{ij}$, acting on vectors and matrices (row-wise), respectively. A

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detailed explanation of the notation is provided in subsection 1.3. Instead of using closure relations for the heat flux vector $\boldsymbol{s} : \Omega \to \mathbb{R}^3$ and the symmetric and trace-free stress tensor $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{3\times 3}_{\mathrm{stf}} := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3} : \boldsymbol{A} = \boldsymbol{A}^\top, \operatorname{tr} \boldsymbol{A} = 0\}$ in (1.1), we solve their corresponding evolution equations as

$$
\left\{\begin{aligned} \tfrac{4}{5} \operatorname{stf} \mathrm{D}\boldsymbol{s} + 2 \operatorname{stf} \mathrm{D}\boldsymbol{u} + \operatorname{DIV} \boldsymbol{m} &= -\tfrac{1}{\mathrm{Kn}} \boldsymbol{\sigma}, \\ \tfrac{5}{2} \boldsymbol{\nabla} \theta + \operatorname{Div} \boldsymbol{\sigma} + \tfrac{1}{2} \operatorname{Div} \boldsymbol{R} + \tfrac{1}{6} \boldsymbol{\nabla} \triangle &= -\tfrac{1}{\mathrm{Kn}} \tfrac{2}{3} \boldsymbol{s}, \end{aligned}\right. \quad \text{in } \Omega. \tag{1.2}
$$

In (1.2), $\boldsymbol{m} : \Omega \to \mathbb{R}^{3\times 3\times 3}_{\mathrm{Stf}}$, $\boldsymbol{R} : \Omega \to \mathbb{R}^{3\times 3}_{\mathrm{stf}}$, and $\triangle : \Omega \to \mathbb{R}$ are the *highest-order moments*, $(\operatorname{DIV} \boldsymbol{m})_{ij} := \sum_k \partial_k m_{ijk}$ defines the matrix field $\operatorname{DIV} \boldsymbol{m} : \Omega \to \mathbb{R}^{3\times 3}$, and $\operatorname{stf} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\mathrm{stf}}$ with $\operatorname{stf} \boldsymbol{A} := \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^\top) - \frac{\operatorname{tr} \boldsymbol{A}}{3} \mathbf{1}_3$ returns the symmetric and trace-free part of a matrix. For the highest-order moments, we use the regularized closure relations given by

$$
\boldsymbol{m} = -2 \,\mathrm{Kn} \operatorname{Stf} \mathrm{D}\boldsymbol{\sigma}, \quad \boldsymbol{R} = -\tfrac{24}{5} \,\mathrm{Kn} \operatorname{stf} \mathrm{D}\boldsymbol{s}, \quad \triangle = -12 \,\mathrm{Kn} \operatorname{div} \boldsymbol{s}, \tag{1.3}
$$

where $\mathrm{D}\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{3\times 3\times 3}$ is defined by $(\mathrm{D}\boldsymbol{\sigma})_{ijk} := \partial_k \sigma_{ij}$, and $\operatorname{Stf} : \mathbb{R}^{3\times 3\times 3} \to \mathbb{R}^{3\times 3\times 3}_{\mathrm{Stf}}$ by $(\operatorname{Stf} \boldsymbol{T})_{ijk} := T_{(ijk)} - \frac{1}{5}(T_{(ill)}\delta_{jk} + T_{(ljl)}\delta_{ik} + T_{(llk)}\delta_{ij})$ [39] where, with S_3 denoting the symmetric group, $T_{(ijk)} := \frac{1}{6} \sum_{\pi \in S_3} T_{\pi(i)\pi(j)\pi(k)}$ defines the symmetric part of $\boldsymbol{T}$. The coefficients in (1.2) and (1.3), which may appear unconventional at first glance, result from the assumption of so-called *Maxwell molecules*[1] in the underlying kinetic derivation (see the next subsection 1.1) and are, therefore, part of the model. With the boundary-aligned moving 3-frame $(\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{n})$ (i.e., unit outer normal $\boldsymbol{n} \in \mathbb{R}^3$ and $\boldsymbol{t}_1, \boldsymbol{t}_2 \in \mathbb{R}^3$ spanning the tangential plane), let $u_n := \sum_i u_i n_i$, $R_{nn} := \sum_{i,j} R_{ij} n_i n_j$, $m_{nnn} := \sum_{i,j,k} m_{ijk} n_i n_j n_k$, and other components be defined analogously. Then, we use Onsager boundary conditions (BCs) (see [36] with the thermodynamic adaptation made in [45]) as

$$
\begin{aligned}
(u_n - u_n^{\mathrm{w}}) &= \epsilon^{\mathrm{w}} \tilde{\chi} \left((p - p^{\mathrm{w}}) + \sigma_{nn}\right), \\
\sigma_{nt_i} &= \tilde{\chi} \left((u_{t_i} - u_{t_i}^{\mathrm{w}}) + \tfrac{1}{5} s_{t_i} + m_{nnt_i}\right) \quad (i \in \{1,2\}), \\
R_{nt_i} &= \tilde{\chi} \left(-(u_{t_i} - u_{t_i}^{\mathrm{w}}) + \tfrac{11}{5} s_{t_i} - m_{nnt_i}\right) \quad (i \in \{1,2\}), \\
s_n &= \tilde{\chi} \left(2(\theta - \theta^{\mathrm{w}}) + \tfrac{1}{2} \sigma_{nn} + \tfrac{2}{5} R_{nn} + \tfrac{2}{15} \triangle\right), \\
m_{nnn} &= \tilde{\chi} \left(-\tfrac{2}{5}(\theta - \theta^{\mathrm{w}}) + \tfrac{7}{5} \sigma_{nn} - \tfrac{2}{25} R_{nn} - \tfrac{2}{75} \triangle\right), \\
\left(\tfrac{1}{2} m_{nnn} + m_{nt_1t_1}\right) &= \tilde{\chi} \left(\tfrac{1}{2} \sigma_{nn} + \sigma_{t_1t_1}\right), \\
m_{nt_1t_2} &= \tilde{\chi} \sigma_{nt_1t_2},
\end{aligned} \tag{1.4}
$$

in which $u_n^{\mathrm{w}}, u_{t_1}^{\mathrm{w}}, u_{t_2}^{\mathrm{w}}, p^{\mathrm{w}}, \theta^{\mathrm{w}} \in \mathbb{R}$ are the velocity components, the pressure, and the temperature on Γ, $\epsilon^{\mathrm{w}} \geq 0$ is a parameter controlling the velocity prescription strength, and $\tilde{\chi} > 0$ is the modified accommodation factor [47]. Eliminating the highest-order moments by inserting the closures (1.3) into (1.1) and (1.2) while prescribing (1.4) yields the R13 boundary value problem, which we analyze in this work.

1.1. State of the art, context, and our contribution. From a high level, the system (1.1)–(1.4) is the result of multiple modeling and approximation steps. Starting from kinetic theory in the 6-dimensional phase space, a Galerkin approximation into a lower-dimensional subspace leads to a set of nonlinearly coupled and time-dependent moment equations [1]. Using a Hermite basis [21] leads to such a

[1] See, e.g., [41] for other molecular potentials resulting in different coefficients.

model hierarchy (including, e.g., the R13, R26, and G45 models [35, 40, 23]) that allows for a systematic model improvement while still recovering the well-established models by Navier–Stokes and Fourier in the limit $\mathrm{Kn} \to 0$ [44, 7]. However, with an increased number of moments, the number of solution variables – and at the same time – the tensorial rank increases, and thus the complexity of the equations.

The R13 model is a good compromise between accuracy and complexity [44]. Exact solutions only exist in certain domains [28, 8]. Numerical solutions, on the other hand, have been obtained, e.g., by using finite difference (FD) schemes [34], finite volume (FV) methods [22], discontinuous Galerkin (DG) approaches [45], the method of fundamental solution (MFS) [27], and finite element methods (FEM) [46, 47, 43]. For the latter, there is an open-source solver available in [42] that is capable of solving the model equations of this work, i.e., the steady-state, linearized, and dimensionless case in three dimensions on arbitrary geometries. However, even with these simplifications, the mathematical analysis of the R13 equations still lags, and even fundamental questions remain unclear: *Do the R13 equations even have a weak solution?* If so, *is this weak solution unique and continuously dependent on the data?* In this work, we answer both questions in the affirmative, meaning that the equations are well-posed (see Theorem 4.9). The key idea is to consider the weak formulation within the abstract saddle point framework [6] of mixed formulations. This well-posedness result is not only interesting in itself, but it also forms the foundation for numerical efforts.

The analysis requires new theoretical tools due to the increased tensor rank of the equations. We combine estimates from the context of Stokes' equations, Poisson's equation, and – interestingly – also from linear elasticity. In particular, by applying the framework from Nečas [29], we will derive tensor-valued Korn inequalities that can be used for coercivity estimates in matrix equations (see our Lemmata 3.8 and 3.11). Indeed, Korn's inequalities are the key tool in (linear) elasticity or fluid mechanics since they provide the needed coercivity of the appearing bilinear forms and lead to well-posedness and regularity results; cf. [25, 24, 9, 11] for an incomplete list and the discussion in the subsection 3 below. Korn inequalities have been generalized to many different contexts, e.g., to the geometrically nonlinear counterpart [12], the case of non-constant coefficients [30, 33] or incompatible tensor fields [26, 20, 17, 18], cf. also with the references in [26, 17]. In particular, Korn–Maxwell–Sobolev, i.e., Korn inequalities for incompatible tensor fields, have been derived in the framework of gradient plasticity with plastic spin [13] or extended continuum type models like the relaxed micromorphic model [31], again to prove coercivity type estimates.

1.2. The connection to Stokes' and Poisson's problem. Since the R13 equations (1.1)–(1.3) seem quite complicated, we want to explain their structure further. In fact, the R13 equations can be seen as an extension to the classical Stokes and Poisson problems coupled together with additional moment terms that are higher-order in the Knudsen number Kn. To see that, we choose $m = 0$ and rewrite the R13 equations as

$$\begin{cases} \boldsymbol{\nabla} p + \operatorname{Div} \boldsymbol{\sigma} = \boldsymbol{b}, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \mathcal{O}(\mathrm{Kn}^2) + \boldsymbol{\sigma} = -2 \,\mathrm{Kn} \operatorname{stf} \mathrm{D}\boldsymbol{u}, \end{cases} \qquad \begin{cases} \operatorname{div} \boldsymbol{s} = r, \\ \mathcal{O}(\mathrm{Kn}^2) + \boldsymbol{s} = -\frac{15}{4} \,\mathrm{Kn}\, \boldsymbol{\nabla}\theta. \end{cases} \tag{1.5}$$

After neglecting the $\mathcal{O}\left(\mathrm{Kn}^2\right)$-terms in (1.5), we obtain the Stokes problem in first-order form with the Navier–Stokes law as closure for $\boldsymbol{\sigma}$ as well as the Poisson (or

Darcy) problem with Fourier's law as closure for $\boldsymbol{s}$. Note that for the Stokes problem, with $\operatorname{div} \boldsymbol{u} = 0$, we have $-2 \operatorname{Div} \operatorname{stf} \mathrm{D}\boldsymbol{u} = -\operatorname{Div} \mathrm{D}\boldsymbol{u} = -\Delta \boldsymbol{u}$, resembling the classical notation. On the boundary Γ, with $\tilde{\chi} = 1$ for simplicity, taking only a subset of the full BCs (1.4) as

$$(u_n - u_n^{\mathrm{w}}) = \epsilon^{\mathrm{w}} \left((p - p^{\mathrm{w}}) + \sigma_{nn} \right), \quad \text{and} \quad \sigma_{nt_i} = u_{t_i} - u_{t_i}^{\mathrm{w}} \quad (i \in \{1, 2\}), \tag{1.6}$$

for the Stokes problem yields a slip (or no-slip if $\epsilon^{\mathrm{w}} = 0$) condition for u_n and a (Navier- or friction-type) slip condition for the tangential stress. For the Poisson problem, the boundary condition subset

$$s_n = 2(\theta - \theta^{\mathrm{w}}), \tag{1.7}$$

resembles a Robin-type condition. In short, the R13 equations extend the classical Stokes and Poisson problems to cases where the $\mathcal{O}(\mathrm{Kn}^2)$-contributions are non-negligible and further coupling effects in Ω as well as on Γ play a role.

1.3. Additional notation. We will denote by $\cdot$, $:$, and $\because$ the scalar product of vectors, matrices, and 3-tensors, respectively. We annotate all scalar products and norms if they are not clear from the context (e.g., $\|a\|$ denotes the operator norm of the bilinear forms a). To emphasize the slightly different action of the divergence operator on vector, matrix, and 3-tensor fields, we will use the notations div, Div, and DIV, respectively. However, the gradient will always be denoted by $\mathrm{D}\cdot = \cdot \otimes \boldsymbol{\nabla}$. For a square matrix, $\boldsymbol{P} \in \mathbb{R}^{3\times3}$ we denote by $\boldsymbol{P}^\top$ its transpose, by $\operatorname{tr} \boldsymbol{P} \coloneqq \boldsymbol{P} : \mathbf{1}_3$ its trace, by $\operatorname{dev} \boldsymbol{P} \coloneqq \boldsymbol{P} - \frac{\operatorname{tr} \boldsymbol{P}}{3} \mathbf{1}_3$ its deviatoric or trace-free part, by $\operatorname{sym} \boldsymbol{P} \coloneqq \frac{1}{2}(\boldsymbol{P} + \boldsymbol{P}^\top)$ its symmetric part, and by $\operatorname{stf} \boldsymbol{P} \coloneqq \operatorname{dev} \operatorname{sym} \boldsymbol{P}$ its symmetric and trace-free part. Thus, in particular, we are interested in orthogonal projections $\mathscr{A} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$, e.g., $\mathscr{A} \in \{\operatorname{dev}, \operatorname{sym}, \operatorname{stf}\}$, and the corresponding subspaces $\mathbb{R}^{3\times3}_{\mathscr{A}} \coloneqq \{\boldsymbol{P} \in \mathbb{R}^{3\times3} : \mathscr{A}[\boldsymbol{P}] = \boldsymbol{P}\}$. Similarly, we introduce the deviatoric part and explicitly write out the previously mentioned symmetric part as well as the symmetric, trace-free part of the 3-tensor $\boldsymbol{P} = (P_{ijk})_{i,j,k=1}^3 \in \mathbb{R}^{3\times3\times3}$ in coordinate-wise notation as

$$(\operatorname{Dev} \boldsymbol{P})_{ijk} \coloneqq P_{ijk} - \tfrac{1}{5} \left(P_{ill}\delta_{jk} + P_{ljl}\delta_{ik} + P_{llk}\delta_{ij} \right), \tag{1.8a}$$

$$(\operatorname{Sym} \boldsymbol{P})_{ijk} \coloneqq P_{(ijk)} \coloneqq \tfrac{1}{6} \left(P_{ijk} + P_{jki} + P_{kij} + P_{jik} + P_{ikj} + P_{kji} \right), \tag{1.8b}$$

$$(\operatorname{Stf} \boldsymbol{P})_{ijk} \coloneqq P_{\langle ijk \rangle} \coloneqq P_{(ijk)} - \tfrac{1}{5} \left(P_{(ill)}\delta_{jk} + P_{(ljl)}\delta_{ik} + P_{(llk)}\delta_{ij} \right). \tag{1.8c}$$

2. A mixed formulation for the linear R13 equations. For the analysis, recognizing the saddle point structure is crucial. Therefore, let us first introduce the Hilbert spaces as

$$\mathrm{V} \coloneqq \mathrm{H}^1(\Omega; \mathbb{R}^{3\times3}_{\mathrm{stf}}) \times \mathrm{H}^1(\Omega; \mathbb{R}^3) \times \tilde{\mathrm{H}}^1(\Omega; \mathbb{R}), \quad \mathrm{Q} \coloneqq \mathrm{L}^2(\Omega; \mathbb{R}^3) \times \mathrm{L}^2(\Omega; \mathbb{R}), \tag{2.1}$$

together with their canonical norms $\|\cdot\|_{\mathrm{V}}$ and $\|\cdot\|_{\mathrm{Q}}$, whereas we distinguish between $\tilde{\mathrm{H}}^1(\Omega; \mathbb{R}) \coloneqq \left\{ p \in \mathrm{H}^1(\Omega; \mathbb{R}) : \int_\Omega p \,\mathrm{d}\boldsymbol{x} = 0 \right\}$ if $\epsilon^{\mathrm{w}} = 0$ and $\tilde{\mathrm{H}}^1(\Omega; \mathbb{R}) \coloneqq \mathrm{H}^1(\Omega; \mathbb{R})$ if $\epsilon^{\mathrm{w}} > 0$.

We can then introduce for all solution fields $(\boldsymbol{\sigma}, \boldsymbol{s}, p, \boldsymbol{u}, \theta)$ from section 1, corresponding test functions $(\boldsymbol{\psi}, \boldsymbol{r}, q, \boldsymbol{v}, \kappa)$, which we group into two sets of variables

$$\boldsymbol{\mathcal{U}} = (\boldsymbol{\sigma}, \boldsymbol{s}, p) \in \mathrm{V}, \quad \boldsymbol{\mathcal{P}} = (\boldsymbol{u}, \theta) \in \mathrm{Q}, \tag{2.2}$$

$$\boldsymbol{\mathcal{V}} = (\boldsymbol{\psi}, \boldsymbol{r}, q) \in \mathrm{V}, \quad \boldsymbol{\mathcal{Q}} = (\boldsymbol{v}, \kappa) \in \mathrm{Q}. \tag{2.3}$$

To derive the weak form, we integrate and test (1.1) and (1.2) with their corresponding test functions, apply integration by parts, insert the boundary conditions (1.4), and use the closure relations (1.3). As the calculations are very lengthy, we refer to [43] for the detailed derivation in the synthetic two-dimensional case (using a slab geometry) with a straightforward extension to three dimensions. We, thus, obtain the abstract mixed formulation: Given $\mathcal{F} \in \mathrm{V}', \mathcal{G} \in \mathrm{Q}'$, find $\boldsymbol{\mathcal{U}} \in \mathrm{V}, \boldsymbol{\mathcal{P}} \in \mathrm{Q}$ such that

$$\mathcal{A}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) + \mathcal{B}(\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{P}}) = \mathcal{F}(\boldsymbol{\mathcal{V}}) \quad \forall \boldsymbol{\mathcal{V}} \in \mathrm{V}, \tag{2.4}$$

$$\mathcal{B}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = \mathcal{G}(\boldsymbol{\mathcal{Q}}) \quad \forall \boldsymbol{\mathcal{Q}} \in \mathrm{Q}. \tag{2.5}$$

In (2.4) and (2.5), the grouped bilinear forms are given by

$$\mathcal{A}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) = a(\boldsymbol{s}, \boldsymbol{r}) + c(\boldsymbol{s}, \boldsymbol{\psi}) - c(\boldsymbol{r}, \boldsymbol{\sigma}) + d(\boldsymbol{\sigma}, \boldsymbol{\psi}) + f(p, \boldsymbol{\psi}) + f(q, \boldsymbol{\sigma}) + h(p, q), \tag{2.6}$$

$$\mathcal{B}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = -e(\boldsymbol{v}, \boldsymbol{\sigma}) - g(p, \boldsymbol{v}) - b(\kappa, \boldsymbol{s}), \tag{2.7}$$

which are defined by the bilinear forms

$$\begin{aligned} a(\boldsymbol{s}, \boldsymbol{r}) = {} & \frac{24}{25} \mathrm{Kn} \int_\Omega \operatorname{sym} \mathrm{D}\boldsymbol{s} : \operatorname{sym} \mathrm{D}\boldsymbol{r} \,\mathrm{d}\boldsymbol{x} + \frac{12}{25} \mathrm{Kn} \int_\Omega \operatorname{div}(\boldsymbol{s}) \operatorname{div}(\boldsymbol{r}) \,\mathrm{d}\boldsymbol{x} \\ & + \frac{4}{15} \frac{1}{\mathrm{Kn}} \int_\Omega \boldsymbol{s} \cdot \boldsymbol{r} \,\mathrm{d}\boldsymbol{x} + \frac{1}{2} \frac{1}{\tilde{\chi}} \int_\Gamma s_n r_n \,\mathrm{d}l + \frac{12}{25} \tilde{\chi} \sum_{i=1}^{2} \int_\Gamma s_{t_i} r_{t_i} \,\mathrm{d}l, \end{aligned} \tag{2.8a}$$

$$c(\boldsymbol{r}, \boldsymbol{\sigma}) = \frac{2}{5} \int_\Omega \boldsymbol{\sigma} : \mathrm{D}\boldsymbol{r} \,\mathrm{d}\boldsymbol{x} - \frac{3}{20} \int_\Gamma \sigma_{nn} r_n \,\mathrm{d}l - \frac{1}{5} \sum_{i=1}^{2} \int_\Gamma \sigma_{nt_i} r_{t_i} \,\mathrm{d}l, \tag{2.8b}$$

$$\begin{aligned} d(\boldsymbol{\sigma}, \boldsymbol{\psi}) = {} & \mathrm{Kn} \int_\Omega \operatorname{Stf} \mathrm{D}\boldsymbol{\sigma} \mathbin{\vdots} \operatorname{Stf} \mathrm{D}\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} + \frac{1}{2} \frac{1}{\mathrm{Kn}} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} \\ & + \frac{9}{8} \tilde{\chi} \int_\Gamma \sigma_{nn} \psi_{nn} \,\mathrm{d}l + \tilde{\chi} \int_\Gamma \left(\sigma_{t_1 t_1} + \frac{1}{2} \sigma_{nn}\right) \left(\psi_{t_1 t_1} + \frac{1}{2} \psi_{nn}\right) \mathrm{d}l \\ & + \tilde{\chi} \int_\Gamma \sigma_{t_1 t_2} \psi_{t_1 t_2} \,\mathrm{d}l + \frac{1}{\tilde{\chi}} \sum_{i=1}^{2} \int_\Gamma \sigma_{nt_i} \psi_{nt_i} \,\mathrm{d}l + \epsilon^{\mathrm{w}} \tilde{\chi} \int_\Gamma \sigma_{nn} \psi_{nn} \,\mathrm{d}l, \end{aligned} \tag{2.8c}$$

$$b(\theta, \boldsymbol{r}) = \int_\Omega \theta \operatorname{div}(\boldsymbol{r}) \,\mathrm{d}\boldsymbol{x}, \quad e(\boldsymbol{u}, \boldsymbol{\psi}) = \int_\Omega \operatorname{Div}(\boldsymbol{\psi}) \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}, \tag{2.8d}$$

$$f(p, \boldsymbol{\psi}) = \epsilon^{\mathrm{w}} \tilde{\chi} \int_\Gamma p \psi_{nn} \,\mathrm{d}l, \quad g(p, \boldsymbol{v}) = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla} p \,\mathrm{d}\boldsymbol{x}, \tag{2.8e}$$

$$h(p, q) = \epsilon^{\mathrm{w}} \tilde{\chi} \int_\Gamma p q \,\mathrm{d}l. \tag{2.8f}$$

The linear functionals in (2.4) and (2.5), on the other hand, read

$$\mathcal{F}(\boldsymbol{\mathcal{V}}) = l_1(\boldsymbol{r}) + l_3(\boldsymbol{\psi}) + l_5(q), \tag{2.9}$$

$$\mathcal{G}(\boldsymbol{\mathcal{Q}}) = -l_2(\kappa) - l_4(\boldsymbol{v}), \tag{2.10}$$

with the functionals

$$l_1(\boldsymbol{r}) = -\int_\Gamma \theta^{\mathrm{w}} r_n \,\mathrm{d}l, \quad l_2(\kappa) = \int_\Omega (r - m) \, \kappa \,\mathrm{d}\boldsymbol{x}, \tag{2.11a}$$

$$l_3(\boldsymbol{\psi}) = -\int_\Gamma \left(\sum_{i=1}^{2} u_{t_i}^{\mathrm{w}} \psi_{nt_i} + (u_n^{\mathrm{w}} - \epsilon^{\mathrm{w}} \tilde{\chi} p^{\mathrm{w}}) \, \psi_{nn} \right) \mathrm{d}l, \quad l_4(\boldsymbol{v}) = \int_\Omega \boldsymbol{b} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \tag{2.11b}$$

$$l_5(q) = \int_\Omega m q \,\mathrm{d}\boldsymbol{x} - \int_\Gamma (u_n^{\mathrm{w}} - \epsilon^{\mathrm{w}} \tilde{\chi} p^{\mathrm{w}}) \, q \,\mathrm{d}l. \tag{2.11c}$$

$$
\begin{array}{l}
\begin{array}{c}\text{heat}\\ \\ \\ \text{fluid}\\ \\ \end{array}
\left[\begin{array}{cc:ccc|l}
+a(\boldsymbol{s},\boldsymbol{r}) & -b(\theta,\boldsymbol{r}) & -c(\boldsymbol{r},\boldsymbol{\sigma}) & 0 & 0 & =l_1(\boldsymbol{r})\\
+b(\kappa,\boldsymbol{s}) & 0 & 0 & 0 & 0 & =l_2(\kappa)\\ \hdashline
+c(\boldsymbol{s},\boldsymbol{\psi}) & 0 & +d(\boldsymbol{\sigma},\boldsymbol{\psi}) & -e(\boldsymbol{u},\boldsymbol{\psi}) & +f(p,\boldsymbol{\psi}) & =l_3(\boldsymbol{\psi})\\
0 & 0 & +e(\boldsymbol{v},\boldsymbol{\sigma}) & 0 & +g(p,\boldsymbol{v}) & =l_4(\boldsymbol{v})\\
0 & 0 & +f(q,\boldsymbol{\sigma}) & -g(q,\boldsymbol{u}) & +h(p,q) & =l_5(q)
\end{array}\right]\\
\Leftrightarrow
\left[\begin{array}{ccc:cc|l}
+a(\boldsymbol{s},\boldsymbol{r}) & -c(\boldsymbol{r},\boldsymbol{\sigma}) & 0 & -b(\theta,\boldsymbol{r}) & 0 & =l_1(\boldsymbol{r})\\
+c(\boldsymbol{s},\boldsymbol{\psi}) & +d(\boldsymbol{\sigma},\boldsymbol{\psi}) & +f(p,\boldsymbol{\psi}) & 0 & -e(\boldsymbol{u},\boldsymbol{\psi}) & =l_3(\boldsymbol{\psi})\\
0 & +f(q,\boldsymbol{\sigma}) & +h(p,q) & 0 & -g(q,\boldsymbol{u}) & =l_5(q)\\ \hdashline
-b(\kappa,\boldsymbol{s}) & 0 & 0 & 0 & 0 & =-l_2(\kappa)\\
0 & -e(\boldsymbol{v},\boldsymbol{\sigma}) & -g(p,\boldsymbol{v}) & 0 & 0 & =-l_4(\boldsymbol{v})
\end{array}\right]
\end{array}
$$

FIG. 1. *Visualization of the weak equation structure, in which for the first system (top), the two equations of the heat system only couple through the bilinear form $c(\boldsymbol{r},\boldsymbol{\sigma})$ to the three fluid equations. A reordering according to trivial diagonal terms yields the saddle point structure in the second system (bottom).*

Remark 2.1. To simplify notation, later on, adding $d(\boldsymbol{\sigma},\boldsymbol{\psi})$, $f(p,\boldsymbol{\psi})$, $f(q,\boldsymbol{\sigma})$, and $h(p,q)$ from (2.8c), (2.8e), and (2.8f) allows us to define $\bar{d} : [\mathrm{H}^1(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{stf}}) \times \tilde{\mathrm{H}}^1(\Omega;\mathbb{R})] \times [\mathrm{H}^1(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{stf}}) \times \tilde{\mathrm{H}}^1(\Omega;\mathbb{R})] \to \mathbb{R}$ as

$$
\begin{aligned}
\bar{d}((\boldsymbol{\sigma},p),(\boldsymbol{\psi},q)) &= \mathrm{Kn}\int_\Omega \operatorname{Stf}\mathrm{D}\boldsymbol{\sigma} \mathbin{\vdots} \operatorname{Stf}\mathrm{D}\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\frac{1}{\mathrm{Kn}}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\\
&\quad + \frac{9}{8}\tilde{\chi}\int_\Gamma \sigma_{nn}\psi_{nn}\,\mathrm{d}l + \tilde{\chi}\int_\Gamma\left(\sigma_{t_1t_1}+\frac{1}{2}\sigma_{nn}\right)\left(\psi_{t_1t_1}+\frac{1}{2}\psi_{nn}\right)\mathrm{d}l\\
&\quad + \tilde{\chi}\int_\Gamma \sigma_{t_1t_2}\psi_{t_1t_2}\,\mathrm{d}l + \frac{1}{\tilde{\chi}}\sum_{i=1}^{2}\int_\Gamma \sigma_{nt_i}\psi_{nt_i}\,\mathrm{d}l + \epsilon^{\mathrm{w}}\tilde{\chi}\int_\Gamma (p+\sigma_{nn})(q+\psi_{nn})\,\mathrm{d}l.
\end{aligned}
\tag{2.12}
$$

Comparing $\bar{d}$ to d from (2.8c), only the last term $\epsilon^{\mathrm{w}}\tilde{\chi}\int_\Gamma(p+\sigma_{nn})(q+\psi_{nn})\,\mathrm{d}l$ differs and now contains the *total pressure* $p+\sigma_{nn}$. This term highlights the positive semidefiniteness of $\bar{d}$ on the $(\boldsymbol{\sigma},p)$-variables. Then, $\mathcal{A}$ from (2.6) is rewritten as

$$
\mathcal{A}(\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{V}}) = a(\boldsymbol{s},\boldsymbol{r}) + c(\boldsymbol{s},\boldsymbol{\psi}) - c(\boldsymbol{r},\boldsymbol{\sigma}) + \bar{d}((\boldsymbol{\sigma},p),(\boldsymbol{\psi},q)). \tag{2.13}
$$

Using the Cauchy–Bunyakovsky–Schwarz (CBS) inequality, we conclude that both bilinear forms $\mathcal{A}$ and $\mathcal{B}$ are continuous (see Appendix A). At the same time, $\mathcal{A}$ is not entirely symmetric as $\mathcal{A}(\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{V}}) = \mathcal{A}(\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{U}}) + 2(c(\boldsymbol{r},\boldsymbol{\sigma}) - c(\boldsymbol{s},\boldsymbol{\psi}))$, as only a and $\bar{d}$ are symmetric. We also induce corresponding linear continuous operators $A : \mathrm{V} \to \mathrm{V}'$, defined by $\langle A\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{V}}\rangle_{\mathrm{V}'\times\mathrm{V}} \coloneqq \mathcal{A}(\boldsymbol{\mathcal{U}},\mathbf{V}), \forall \boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{V}} \in \mathrm{V}$, and $B : \mathrm{V} \to \mathrm{Q}'$, defined by $\langle B\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{Q}}\rangle_{\mathrm{Q}'\times\mathrm{Q}} \coloneqq \mathcal{B}(\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{Q}}), \forall\boldsymbol{\mathcal{V}} \in \mathrm{V}, \forall \boldsymbol{\mathcal{Q}} \in \mathrm{Q}$, which are needed later on.

The choice of $\mathcal{A}$ and $\mathcal{B}$ reorders the physical heat $(\theta,\boldsymbol{s})$ and fluid $(p,\boldsymbol{u},\boldsymbol{\sigma})$ variables into a saddle point system; see Figure 1 for a sketch of the process. We chose this grouping to make the treatment of the system maximally condensed. Still, this may not be the only possible strategy, and the separate analysis of the heat and fluid equations, coupled only with the bilinear form c, might also be possible (see, e.g., the discussion in [3, Sec. 4.3.1]). Having established the saddle point formulation, we can now proceed with the well-posedness analysis. For that, the main two ingredients are:

1. the coercivity of $\mathcal{A}$ on $\ker B$ (see subsection 4.1);
2. and the inf–sup condition for $\mathcal{B}$ (see subsection 4.2).

Both of the above points rely on new ellipticity results for tensor fields, which we will derive in the next section.

3. Coercivity and Korn-type inequalities. Korn inequalities play a fundamental role in the study of variational principles in linear elasticity or fluid mechanics. In their most famous version, they say that the L^2-norm of a full gradient of a vector field can be controlled by the L^2-norm of only its symmetric part if suitable additional conditions are assumed, like zero boundary values (see, e.g., [9, 25, 32]). More precisely, let us denote by $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain, then there exists a constant $c = c(\Omega) > 0$ such that for all $u \in \mathrm{H}^1_0(\Omega;\mathbb{R}^d)$, we have

$$\|\mathrm{D}\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} \leq c\ \|\operatorname{sym}\mathrm{D}\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)}\,. \tag{Korn-1}$$

The complete classification of differential operators that might appear on the right-hand side of such types of inequalities is already contained in [2]. We will refer to such coercivity estimates when zero boundary conditions are present as *Korn inequalities of the first type* and include Aronszajn's characterization for completeness (only focusing on first-order differential operators):

LEMMA 3.1 (Aronszajn [2], Korn inequalities of the first type). *Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $q \in (1,\infty)$, V and $\tilde{V}$ two finite-dimensional real inner product spaces, and $\mathbb{A} \coloneqq \sum_{j=1}^d \mathbb{A}_j\partial_j$ a linear homogeneous differential operator of first order with constant coefficients $\mathbb{A}_j \in L(V,\tilde{V})$. Then the following are equivalent:*

1. *There exists a constant $c = c(\mathbb{A}, q, \Omega) > 0$ such that the inequality*

$$\|\mathrm{D}u\|_{\mathrm{L}^q(\Omega)} \leq c\ \|\mathbb{A}u\|_{\mathrm{L}^q(\Omega)}\,, \tag{K-1}$$

   holds for all $u \in \mathrm{W}^{1,q}_0(\Omega;V)$.
2. *$\mathbb{A}$ is an $\mathbb{R}$-elliptic differential operator, meaning that*

$$\forall \boldsymbol{\xi} \in \mathbb{R}^d \setminus \{\mathbf{0}\} : \left(\sum_{j=1}^d \xi_j\mathbb{A}_j v \overset{!}{=} 0 \ \Rightarrow\ v = 0\right). \tag{$\mathbb{R}$-ellipt.}$$

Example 3.2. Set $V = \mathbb{R}^d$, $\tilde{V} = \mathbb{R}^{d\times d}$ and consider the symmetrized gradient for vector fields, given by

$$\mathbb{A}\boldsymbol{u} \coloneqq \operatorname{sym}\mathrm{D}\boldsymbol{u}. \tag{3.1}$$

It is well known that the symmetrized gradient is $\mathbb{R}$-elliptic. Thus, by Lemma 3.1, we recover (Korn-1).

Example 3.3. In many applications, the differential operator of interest has more structure and is given by applying an orthogonal projection on the gradient of a vector field as

$$\mathbb{A}\boldsymbol{u} = \mathscr{A}[\mathrm{D}\boldsymbol{u}] \quad \text{with} \quad \mathscr{A} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d} \quad \text{such that} \quad \mathscr{A}^2 = \mathscr{A}. \tag{3.2}$$

Classical examples are the symmetric part $\mathscr{A} = \operatorname{sym}$, the deviatoric (trace-free) part $\mathscr{A} = \operatorname{dev}$, or the symmetric trace-free part $\mathscr{A} = \operatorname{stf} = \operatorname{sym}\operatorname{dev}$, which all give $\mathbb{R}$-elliptic operators.

Later in subsection 3.2, we also need the following connection to the classical elliptic regularity theory:

Remark 3.4 (Legendre–Hadamard ellipticity). The projection $\mathscr{A} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ induces an elliptic differential operator $\mathbb{A} := \mathscr{A}[\boldsymbol{\cdot} \otimes \boldsymbol{\nabla}]$ if and only if for all $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ the corresponding symbol map $\mathbb{A}[\boldsymbol{\xi}] := \mathscr{A}[\boldsymbol{\cdot} \otimes \boldsymbol{\xi}] : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is injective. In that case, since $\mathscr{A}$ is continuous and $\partial B_1(\boldsymbol{0}) \times \partial B_1(\boldsymbol{0})$ is compact, we find a $\lambda > 0$ such that for all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$, we have

$$\lambda \le \left|\mathscr{A}\left[\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|} \otimes \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right]\right| \quad \text{by just taking} \quad \lambda := \min_{\substack{|\boldsymbol{z}|=1\\|\boldsymbol{y}|=1}} |\mathscr{A}[\boldsymbol{z} \otimes \boldsymbol{y}]|, \tag{3.3}$$

while $\lambda > 0$ holds by the injectivity of $\mathscr{A}[\boldsymbol{\cdot} \otimes \boldsymbol{\xi}]$. Thus, by the linearity of $\mathscr{A}$, we conclude that for all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^d$:

$$\lambda |\boldsymbol{\eta}||\boldsymbol{\xi}| \le |\mathscr{A}[\boldsymbol{\eta} \otimes \boldsymbol{\xi}]|. \tag{3.4}$$

Since $\mathscr{A}$ is an orthogonal projection, we have

$$\mathscr{A}[\boldsymbol{\eta} \otimes \boldsymbol{\xi}] : \boldsymbol{\eta} \otimes \boldsymbol{\xi} = \mathscr{A}[\boldsymbol{\eta} \otimes \boldsymbol{\xi}] : \mathscr{A}[\boldsymbol{\eta} \otimes \boldsymbol{\xi}] = |\mathscr{A}[\boldsymbol{\eta} \otimes \boldsymbol{\xi}]|^2 \overset{(3.4)}{\ge} \lambda^2 |\boldsymbol{\eta}|^2 |\boldsymbol{\xi}|^2, \tag{3.5}$$

which is the Legendre–Hadamard ellipticity condition.

The well-posedness of our model strongly relies on a *Korn inequality of the second type*, i.e., when the boundary conditions are not *essentially* imposed in the function space but are instead *naturally* imposed in the weak form. We recall the following classification of differential operators:

LEMMA 3.5 (Nečas [29, Thm. 5], Korn inequalities of the second type). *Let $d \ge 2$, $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $q \in (1, \infty)$, V, and $\tilde{V}$ two finite-dimensional real inner product spaces, and $\mathbb{A} := \sum_{j=1}^d \mathbb{A}_j \partial_j$ a linear homogeneous differential operator of first order with constant coefficients $\mathbb{A}_j \in L(V, \tilde{V})$. Then the following are equivalent:*

1. *There exists a constant $c = c(\mathbb{A}, q, \Omega) > 0$ such that the inequality*

$$\|u\|_{\mathrm{W}^{1,q}(\Omega)} \le c\,(\|u\|_{\mathrm{L}^q(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^q(\Omega)}), \tag{K-2}$$

   holds for all $u \in \mathrm{W}^{1,q}(\Omega; V)$.

2. *$\mathbb{A}$ is a $\mathbb{C}$-elliptic differential operator, meaning that*

$$\forall\, \boldsymbol{\xi} \in \mathbb{C}^d \setminus \{\boldsymbol{0}\} : \left(\sum_{j=1}^d \xi_j \mathbb{A}_j v + \mathrm{i}\, \xi_j \mathbb{A}_j w \overset{!}{=} 0 \;\Rightarrow\; v, w = 0\right). \tag{$\mathbb{C}$-ellipt.}$$

A similar result is also valid for higher-order differential operators and norms of negative Sobolev spaces; see [38, Cor. 7.3]. It is worth noting that such results for $\mathbb{C}$-elliptic differential operators also exist on John domains (cf. [15]) that do not have to allow for a boundary operator.

Example 3.6. Set $V = \mathbb{R}^d$, $\tilde{V} = \mathbb{R}^{d\times d}$ and consider the symmetrized gradient for vector fields as

$$\mathbb{A}\boldsymbol{u} := \operatorname{sym} \mathrm{D}\boldsymbol{u}. \tag{3.6}$$

It is well known that the symmetrized gradient is $\mathbb{C}$-elliptic. Thus, by Lemma 3.5, we find a constant $c > 0$, such that for all $u \in \mathrm{H}^1(\Omega)$, we have

$$\|\boldsymbol{u}\|_{\mathrm{H}^1(\Omega)} \le c\,(\|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} + \|\operatorname{sym} \mathrm{D}\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)}), \tag{Korn-2}$$

which is the classical Korn inequality of the second type.

3.1. Tensor-valued Korn inequality. For the R13 model in the previous notation, we have the dimension $d = 3$, the integrability $q = 2$, $V = \mathbb{R}^{3\times 3}_{\mathrm{stf}} := \{\boldsymbol{P} \in \mathbb{R}^{3\times 3} : \boldsymbol{P} = \operatorname{dev}\operatorname{sym}\boldsymbol{P} = \operatorname{stf}\boldsymbol{P}\}$, $\tilde{V} = \mathbb{R}^{3\times 3\times 3}$, and consider the differential operator $\mathbb{A} := \operatorname{Stf}\circ \mathrm{D} = \operatorname{Stf}(\,\cdot \otimes \boldsymbol{\nabla})$, where we recall the symmetric and trace-free part of a 3-tensor $\boldsymbol{P} \in \mathbb{R}^{3\times 3\times 3}$, defined by

$$(\operatorname{Stf}\boldsymbol{P})_{ijk} = P_{\langle ijk\rangle} = P_{(ijk)} - \tfrac{1}{5}\left(P_{(ill)}\delta_{jk} + P_{(ljl)}\delta_{ik} + P_{(llk)}\delta_{ij}\right). \tag{3.7}$$

Thus, to check the $\mathbb{C}$-ellipticity of $\mathbb{A}$, we start with a symmetric and trace-free tensor $\boldsymbol{T}$ and consider the vanishing symbol of its symmetric, trace-free gradient (3.7), which simplifies with $\operatorname{tr}\boldsymbol{T} = 0$ to

$$\tfrac{1}{3}\left(T_{ij}\xi_k + T_{ik}\xi_j + T_{jk}\xi_i\right) - \tfrac{2}{15}\Big(\sum_l T_{lk}\xi_l\delta_{ij} + \sum_l T_{lj}\xi_l\delta_{ik} + \sum_l T_{li}\xi_l\delta_{jk}\Big) = 0, \tag{3.8}$$

with $\boldsymbol{\xi} \in \mathbb{C}^3 \setminus \{\boldsymbol{0}\}$ being arbitrary. Our goal is to show ($\mathbb{C}$-ellipt.), i.e., to conclude that this equation possesses only the trivial complex-valued solution $\boldsymbol{T} = 0$ for all $\boldsymbol{\xi} \in \mathbb{C}^3 \setminus \{\boldsymbol{0}\}$. A multiplication of (3.8) with ξ_k and summation over the index k yields

$$\begin{aligned}\tfrac{1}{3}\Big(T_{ij}\sum_k \xi_k\xi_k + \sum_k T_{ik}\xi_k\xi_j + \xi_i\sum_k T_{jk}\xi_k\Big)&\\ - \tfrac{2}{15}\Big(\delta_{ij}\sum_{l,k} T_{lk}\xi_l\xi_k + \xi_i\sum_l T_{lj}\xi_l + \sum_l T_{li}\xi_l\xi_j\Big) &= 0,\end{aligned} \tag{3.9}$$

which in coordinate-free notations reads as

$$\tfrac{1}{3}\left(\boldsymbol{T}(\boldsymbol{\xi}\cdot\boldsymbol{\xi}) + (\boldsymbol{T}\boldsymbol{\xi})\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\otimes(\boldsymbol{T}\boldsymbol{\xi})\right) - \tfrac{2}{15}\left(\boldsymbol{1}_3((\boldsymbol{T}\boldsymbol{\xi})\cdot\boldsymbol{\xi}) + \boldsymbol{\xi}\otimes(\boldsymbol{T}\boldsymbol{\xi}) + (\boldsymbol{T}\boldsymbol{\xi})\otimes\boldsymbol{\xi}\right) = 0. \tag{3.10}$$

Note that we use the notation $\boldsymbol{\xi}\cdot\boldsymbol{\zeta} := \sum_{j=1}^3 \xi_j\zeta_j$ for all $\boldsymbol{\xi},\boldsymbol{\zeta} \in \mathbb{C}^3$, which is different from the Hermitian scalar product $\sum_{j=1}^3 \xi_j\overline{\zeta_j}$, where $\overline{\cdot}$ denotes the complex conjugate. In particular, this $\cdot$ is *not* non-negative on $\mathbb{C}^3$. Thus, (3.10) simplifies to

$$\tfrac{1}{3}\boldsymbol{T}(\boldsymbol{\xi}\cdot\boldsymbol{\xi}) + \tfrac{1}{5}\left((\boldsymbol{T}\boldsymbol{\xi})\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\otimes(\boldsymbol{T}\boldsymbol{\xi})\right) - \tfrac{2}{15}\boldsymbol{1}_3((\boldsymbol{T}\boldsymbol{\xi})\cdot\boldsymbol{\xi}) = 0. \tag{3.11}$$

By a multiplication with $\boldsymbol{\xi}^\top$ from the left and $\boldsymbol{\xi}$ from the right, we obtain that

$$(\boldsymbol{\xi}\cdot\boldsymbol{\xi})(\boldsymbol{\xi}\cdot(\boldsymbol{T}\boldsymbol{\xi}))\left(\tfrac{1}{3} + \tfrac{2}{5} - \tfrac{2}{15}\right) = 0. \tag{3.12}$$

We distinguish two cases:

1. case ($\boldsymbol{\xi} \in \mathbb{C}^3 \setminus \{\boldsymbol{0}\}$ satisfies $\boldsymbol{\xi}\cdot\boldsymbol{\xi} \neq 0$): Then by (3.12), we have $\boldsymbol{\xi}\cdot(\boldsymbol{T}\boldsymbol{\xi}) = 0$, and multiplying (3.11) with $\boldsymbol{\xi}$ from the right yields

$$(\boldsymbol{\xi}\cdot\boldsymbol{\xi})\left(\tfrac{1}{3} + \tfrac{1}{5}\right)\boldsymbol{T}\boldsymbol{\xi} = 0 \overset{\boldsymbol{\xi}\cdot\boldsymbol{\xi}\neq 0}{\Rightarrow} \boldsymbol{T}\boldsymbol{\xi} = 0, \tag{3.13}$$

such that (3.11) simplifies to $\boldsymbol{T}(\boldsymbol{\xi}\cdot\boldsymbol{\xi}) = 0$, and since $\boldsymbol{\xi}\cdot\boldsymbol{\xi} \neq 0$, we conclude that $\boldsymbol{T} = \boldsymbol{0}$.

2. case ($\boldsymbol{\xi} \in \mathbb{C}^3 \setminus \{\boldsymbol{0}\}$ satisfies $\boldsymbol{\xi}\cdot\boldsymbol{\xi} = 0$ but $|\boldsymbol{\xi}|^2 \neq 0$): Multiplying (3.11) with $\boldsymbol{\xi}$ from the right, we obtain

$$\boldsymbol{\xi}((\boldsymbol{T}\boldsymbol{\xi})\cdot\boldsymbol{\xi})\left(\tfrac{1}{5} - \tfrac{2}{15}\right) = 0, \tag{3.14}$$

so that a multiplication with $\overline{\boldsymbol{\xi}}^\top$ from the left yields

$$|\boldsymbol{\xi}|^2(\boldsymbol{T}\boldsymbol{\xi})\cdot\boldsymbol{\xi}=0 \quad \overset{\boldsymbol{\xi}\neq\boldsymbol{0}}{\Rightarrow} \quad (\boldsymbol{T}\boldsymbol{\xi})\cdot\boldsymbol{\xi}=0. \tag{3.15}$$

Hence, (3.11) simplifies to

$$(\boldsymbol{T}\boldsymbol{\xi})\otimes\boldsymbol{\xi}+\boldsymbol{\xi}\otimes(\boldsymbol{T}\boldsymbol{\xi})=0. \tag{3.16}$$

Multiplying now with $\overline{\boldsymbol{\xi}}$ results in

$$|\boldsymbol{\xi}|^2\boldsymbol{T}\boldsymbol{\xi}+\boldsymbol{\xi}((\boldsymbol{T}\boldsymbol{\xi})\cdot\overline{\boldsymbol{\xi}})=0. \tag{3.17}$$

Thus, since $\boldsymbol{\xi}\neq\boldsymbol{0}$, we find an $\alpha\in\mathbb{C}$, such that

$$\boldsymbol{T}\boldsymbol{\xi}=\alpha\boldsymbol{\xi}. \tag{3.18}$$

Reinserting (3.18) in (3.16) results in $2\alpha\boldsymbol{\xi}\otimes\boldsymbol{\xi}=0$, and multiplication with $\overline{\boldsymbol{\xi}}^\top$ from the left and $\overline{\boldsymbol{\xi}}$ from the right yields

$$\alpha|\boldsymbol{\xi}|^4=0 \quad \overset{\boldsymbol{\xi}\neq\boldsymbol{0}}{\Rightarrow} \quad \alpha=0 \quad \overset{(3.18)}{\Rightarrow} \quad \boldsymbol{T}=0. \tag{3.19}$$

Summarizing, we have shown that for a symmetric trace-free tensor $\boldsymbol{T}$ that

$$\tfrac{1}{3}\left(T_{ij}\xi_k+T_{ik}\xi_j+T_{jk}\xi_i\right)-\tfrac{2}{15}\Big(\sum_l T_{lk}\xi_l\delta_{ij}+\sum_l T_{lj}\xi_l\delta_{ik}+\sum_l T_{li}\xi_l\delta_{jk}\Big)\overset{!}{=}0, \tag{3.20}$$

for $\boldsymbol{\xi}\in\mathbb{C}^3\setminus\{\boldsymbol{0}\}$, implies that $\boldsymbol{T}=\boldsymbol{0}$, meaning that the symmetric trace-free gradient is $\mathbb{C}$-elliptic on symmetric trace-free tensor fields.

Remark 3.7. Note that skew-symmetric matrices are in the kernel of the symbol of the symmetric trace-free gradient (since, by definition (3.7), we only consider the symmetric and trace-free parts) such that it is not even $\mathbb{R}$-elliptic on $\mathbb{R}^{3\times3}$. Also, for just symmetric but not trace-free matrices, we do not have $\mathbb{R}$-ellipticity. Consider, for example, $a\boldsymbol{1}_3\in\mathbb{R}^{3\times3}_{\text{sym}}$ ($a\in\mathbb{R}$) such that for $\boldsymbol{P}=a\boldsymbol{1}_3\otimes\boldsymbol{\xi}$ ($\boldsymbol{\xi}\in\mathbb{R}^3$), we have that $P_{(ill)}=\frac{1}{3}(a\operatorname{tr}\boldsymbol{1}_3+2a\boldsymbol{1}_3)\xi_i=\frac{5}{3}a\xi_i$. Then, $(\operatorname{Stf}(a\boldsymbol{1}_3\otimes\boldsymbol{\xi}))_{ijk}$ in (3.7) vanishes as

$$\tfrac{1}{3}a(\xi_k\delta_{ij}+\xi_j\delta_{ik}+\xi_i\delta_{jk})-\tfrac{1}{5}\left(\tfrac{5}{3}a\xi_i\delta_{jk}+\tfrac{5}{3}a\xi_j\delta_{ik}+\tfrac{5}{3}a\xi_k\delta_{ij}\right)=0 \quad \forall\,\boldsymbol{\xi}\in\mathbb{R}^3. \tag{3.21}$$

LEMMA 3.8. *The symmetric trace-free gradient on symmetric trace-free* 2*-tensor fields is* $\mathbb{C}$*-elliptic if and only if* $d\geq3$*.*

Proof. Besides the case of $d=3$, in which the previous calculations prove the result, we distinguish the following cases:

Case $d>3$: The prefactors have to be changed to

$$\tfrac{1}{5}\mapsto\tfrac{1}{d+2} \text{ in (3.7)},\quad \tfrac{2}{15}\mapsto\tfrac{2}{3(d+2)} \text{ in (3.8)},\quad \tfrac{1}{3}+\tfrac{2}{5}-\tfrac{2}{15}\mapsto\tfrac{d}{d+2} \text{ in (3.12)},$$
$$\tfrac{1}{3}+\tfrac{1}{5}\mapsto\tfrac{2(d+1)}{3(d+2)} \text{ in (3.13)},\quad \tfrac{1}{5}-\tfrac{2}{15}\mapsto\tfrac{d-2}{3(d+2)} \text{ in (3.14)}. \tag{3.22}$$

None of these prefactors vanishes for $d\geq3$, meaning that we can basically follow the arguments from the case of $d=3$ to conclude the $\mathbb{C}$-ellipticity of the symmetric trace-free gradient also on symmetric trace-free tensors with $d\geq3$.

Case $d = 2$: Here, the prefactor in the replacement of (3.14) would vanish. In fact, consider, for example, the choice of

$$\boldsymbol{T} = \begin{pmatrix} \mathrm{i} & 1 \\ 1 & -\mathrm{i} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\xi} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \tag{3.23}$$

for which we have, for all $i, j, k \in \{1, 2\}$, that

$$\tfrac{1}{3}\left(T_{ij}\xi_k + T_{ik}\xi_j + T_{jk}\xi_i\right) - \tfrac{1}{6}\Big(\sum_l T_{lk}\xi_l\delta_{ij} + \sum_l T_{lj}\xi_l\delta_{ik} + \sum_l T_{li}\xi_l\delta_{jk}\Big) = 0. \tag{3.24}$$

Thus, the symmetric trace-free gradient on symmetric trace-free tensors is not $\mathbb{C}$-elliptic in 2 dimensions. ◻

Remark 3.9. Note that the symmetric trace-free gradient on symmetric trace-free tensors is $\mathbb{R}$-elliptic in all dimensions $d \geq 2$. For the proof, we follow the arguments above and stop after the first case. The second case is not relevant since $\boldsymbol{\xi} \in \mathbb{R}^d$.

Remark 3.10. With the lack of $\mathbb{C}$-ellipticity for $d = 2$ in Lemma 3.8, we encounter a phenomenon similar to the known missing $\mathbb{C}$-ellipticity of the symmetric trace-free gradient for vector fields in two dimensions (cf. [37, 10, 19]), though at one tensor rank higher.

Lemma 3.11 (Tensor-valued Korn inequality). *Let $d \geq 3$, $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\boldsymbol{\sigma} \in \mathrm{H}^1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{stf}})$. There exists a constant $c > 0$ such that*

$$\|\boldsymbol{\sigma}\|_{\mathrm{H}^1(\Omega)} \leq c \left(\|\mathrm{Stf}\,\mathrm{D}\boldsymbol{\sigma}\|_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{\sigma}\|_{\mathrm{L}^2(\Omega)}\right). \tag{3.25}$$

Proof. In the preceding calculations, we have shown that the differential operator $\mathbb{A} \coloneqq \mathrm{Stf} \circ \mathrm{D}$ is $\mathbb{C}$-elliptic for symmetric trace-free tensor fields. Thus, the condition ($\mathbb{C}$-ellipt.) is satisfied, and we conclude by Lemma 3.5. ◻

With the above estimates, we already obtain the coercivity of the bilinear forms a and $\bar{d}$ from section 2 since

$$\begin{aligned} a(\boldsymbol{s}, \boldsymbol{s}) &= \tfrac{24}{25}\,\mathrm{Kn}\,\|\mathrm{sym}\,\mathrm{D}\boldsymbol{s}\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{12}{25}\,\mathrm{Kn}\,\|\mathrm{div}\,\boldsymbol{s}\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{4}{15}\tfrac{1}{\mathrm{Kn}}\,\|\boldsymbol{s}\|^2_{\mathrm{L}^2(\Omega)} \\ &\quad + \tfrac{1}{2}\tfrac{1}{\tilde{\chi}}\,\|s_n\|^2_{\mathrm{L}^2(\Gamma)} + \tfrac{12}{25}\tilde{\chi}\textstyle\sum_{i=1}^2 \|s_{t_i}\|^2_{\mathrm{L}^2(\Gamma)} \\ &\geq \min\left\{\tfrac{24}{25}\,\mathrm{Kn}, \tfrac{4}{15}\tfrac{1}{\mathrm{Kn}}\right\}\left(\|\mathrm{sym}\,\mathrm{D}\boldsymbol{s}\|^2_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{s}\|^2_{\mathrm{L}^2(\Omega)}\right) \\ &\overset{\text{(Korn-2)}}{\geq} c_0\,\|\boldsymbol{s}\|^2_{\mathrm{H}^1(\Omega)}, \end{aligned} \tag{3.26}$$

as well as

$$\begin{aligned} \bar{d}((\boldsymbol{\sigma}, p), (\boldsymbol{\sigma}, p)) &= \mathrm{Kn}\,\|\mathrm{Stf}\,\mathrm{D}\boldsymbol{\sigma}\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{1}{2}\tfrac{1}{\mathrm{Kn}}\,\|\boldsymbol{\sigma}\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{9}{8}\tilde{\chi}\,\|\sigma_{nn}\|^2_{\mathrm{L}^2(\Gamma)} \\ &\quad + \tilde{\chi}\left\|\sigma_{t_1t_1} + \tfrac{1}{2}\sigma_{nn}\right\|^2_{\mathrm{L}^2(\Gamma)} + \tilde{\chi}\,\|\sigma_{t_1t_2}\|^2_{\mathrm{L}^2(\Gamma)} \\ &\quad + \tfrac{1}{\tilde{\chi}}\textstyle\sum_{i=1}^2 \|\sigma_{nt_i}\|^2_{\mathrm{L}^2(\Gamma)} + \epsilon^w\tilde{\chi}\,\|p + \sigma_{nn}\|^2_{\mathrm{L}^2(\Gamma)} \\ &\geq \min\left\{\mathrm{Kn}, \tfrac{1}{2}\tfrac{1}{\mathrm{Kn}}\right\}\left(\|\mathrm{Stf}\,\mathrm{D}\boldsymbol{\sigma}\|^2_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{\sigma}\|^2_{\mathrm{L}^2(\Omega)}\right) \\ &\overset{\text{Lemma 3.11}}{\geq} c_1\,\|\boldsymbol{\sigma}\|^2_{\mathrm{H}^1(\Omega)}. \end{aligned} \tag{3.27}$$

3.2. Right-inverse of the matrix-valued divergence operator. The ellipticity property of the previous section also yields the second main theoretical tool regarding the inversion of the matrix-valued divergence. But first, we recall the classical result about vector fields (see, e.g., [5, Lem. 11.2.3] or [4, Satz 6.3]).

LEMMA 3.12. *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists a constant $c = c(\Omega) > 0$, such that for all $\kappa \in \mathrm{L}^2(\Omega;\mathbb{R})$, we find a $\boldsymbol{t} \in \mathrm{H}^1(\Omega;\mathbb{R}^d)$ satisfying*

$$-\operatorname{div} \boldsymbol{t} = \kappa \quad \text{and} \quad \|\boldsymbol{t}\|_{\mathrm{H}^1(\Omega)} \leq c\, \|\kappa\|_{\mathrm{L}^2(\Omega)}\,. \tag{3.28}$$

Now, we extend this result to higher-order tensors.

LEMMA 3.13. *Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and $\mathscr{A} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ be an orthogonal projection, such that $\mathbb{A} := \mathscr{A}[\boldsymbol{\cdot} \otimes \boldsymbol{\nabla}]$ is an elliptic differential operator from $\mathbb{R}^d$ to $\mathbb{R}^{d\times d}$. Let us further denote by $\mathbb{R}^{d\times d}_{\mathscr{A}} := \{\boldsymbol{\tau} \in \mathbb{R}^{d\times d} : \mathscr{A}[\boldsymbol{\tau}] = \boldsymbol{\tau}\}$. Then there exists a constant $c = c(\mathscr{A},\Omega) > 0$, such that for all $\boldsymbol{u} \in \mathrm{L}^2(\Omega;\mathbb{R}^d)$, we find a $\boldsymbol{\tau} \in \mathrm{H}^1(\Omega;\mathbb{R}^{d\times d}_{\mathscr{A}})$ satisfying*

$$-\operatorname{Div} \boldsymbol{\tau} = \boldsymbol{u} \quad \text{and} \quad \|\boldsymbol{\tau}\|_{\mathrm{H}^1(\Omega)} \leq c\, \|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)}\,. \tag{3.29}$$

Proof. Consider the bounded linear form

$$\ell(\boldsymbol{\varphi}) := \int_\Omega \boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}, \tag{3.30}$$

and the bounded and symmetric bilinear form

$$B(\boldsymbol{v},\boldsymbol{\varphi}) := \int_\Omega \mathscr{A}[\mathrm{D}\boldsymbol{v}] : \mathscr{A}[\mathrm{D}\boldsymbol{\varphi}] \,\mathrm{d}\boldsymbol{x}, \tag{3.31}$$

for $\boldsymbol{v},\boldsymbol{\varphi} \in \mathrm{H}^1_0(\Omega;\mathbb{R}^d)$. By the ellipticity of $\mathscr{A}[\boldsymbol{\cdot} \otimes \boldsymbol{\nabla}]$, we can apply Lemma 3.1 to conclude the coercivity of B. Using Lax–Milgram, we then find a unique solution $\boldsymbol{v} \in \mathrm{H}^1_0(\Omega;\mathbb{R}^d)$, such that

$$B(\boldsymbol{v},\boldsymbol{\varphi}) = \ell(\boldsymbol{\varphi}) \quad \forall\, \boldsymbol{\varphi} \in \mathrm{H}^1_0(\Omega;\mathbb{R}^d). \tag{3.32}$$

Thus, in particular, for all $\boldsymbol{\varphi} \in \mathrm{H}^1_0(\Omega;\mathbb{R}^d)$, we have that

$$\begin{aligned} \int_\Omega \boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathscr{A}[\mathrm{D}\boldsymbol{v}] : \mathscr{A}[\mathrm{D}\boldsymbol{\varphi}] \,\mathrm{d}\boldsymbol{x} &\overset{\mathscr{A}\text{o.p.}}{=} \int_\Omega \mathscr{A}[\mathrm{D}\boldsymbol{v}] : \mathrm{D}\boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} \\ &\overset{\boldsymbol{\varphi}\in\mathrm{H}^1_0}{=} -\int_\Omega \operatorname{Div} \mathscr{A}[\mathrm{D}\boldsymbol{v}] \boldsymbol{\cdot} \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{3.33}$$

meaning that $\boldsymbol{v} \in \mathrm{H}^1_0(\Omega;\mathbb{R}^d)$ is a weak solution of

$$\begin{cases} -\operatorname{Div} \mathscr{A}[\mathrm{D}\boldsymbol{v}] = \boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.34}$$

Since $\mathscr{A}[\boldsymbol{\cdot} \otimes \boldsymbol{\nabla}]$ is elliptic, we can apply classical elliptic regularity theory (cf. [14, Thm. 10.3, 3.18] with the Legendre–Hadamard condition from Remark 3.4 and also the literature mentioned after [16, Prop. 2.11]) to conclude that for the unique solution, we moreover have

$$\boldsymbol{v} \in \mathrm{H}^2(\Omega;\mathbb{R}^d) \quad \text{and} \quad \|\boldsymbol{v}\|_{\mathrm{H}^2(\Omega)} \leq \tilde{c}\,(\|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}). \tag{3.35}$$

Furthermore, by the Poincaré and Korn inequalities, we estimate that

$$\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \le c_1 \|\mathrm{D}\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \le c_2 \|\mathscr{A}[\mathrm{D}\boldsymbol{v}]\|_{\mathrm{L}^2(\Omega)} . \tag{3.36}$$

Now, we show that $\boldsymbol{\tau} := \mathscr{A}[\mathrm{D}\boldsymbol{v}]$ does the trick. Since $\boldsymbol{v} \in \mathrm{H}^2(\Omega;\mathbb{R}^d) \cap \mathrm{H}^1_0(\Omega;\mathbb{R})$ is the unique solution of (3.34), with this choice, we have $\boldsymbol{\tau} \in \mathrm{H}^1(\Omega;\mathbb{R}^{d\times d}_{\mathscr{A}})$ and $\operatorname{Div}\boldsymbol{\tau} = -\boldsymbol{u}$. Moreover, we have that

$$\begin{aligned} \|\boldsymbol{\tau}\|^2_{\mathrm{L}^2(\Omega)} = \|\mathscr{A}[\mathrm{D}\boldsymbol{v}]\|^2_{\mathrm{L}^2(\Omega)} &\overset{(3.32)}{=} \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \overset{\mathrm{CBS}}{\le} \|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \\ &\overset{(3.36)}{\le} c_2 \|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} \|\mathscr{A}[\mathrm{D}\boldsymbol{v}]\|_{\mathrm{L}^2(\Omega)} = c_2 \|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{\tau}\|_{\mathrm{L}^2(\Omega)} , \end{aligned} \tag{3.37}$$

such that

$$\|\boldsymbol{\tau}\|_{\mathrm{L}^2(\Omega)} \le c_2 \|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} , \tag{3.38}$$

and we can estimate the H(Div)-norm of $\boldsymbol{\tau}$ as

$$\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathrm{Div})(\Omega)} = \sqrt{\|\boldsymbol{\tau}\|^2_{\mathrm{L}^2(\Omega)} + \|\operatorname{Div}\boldsymbol{\tau}\|^2_{\mathrm{L}^2(\Omega)}} \le \sqrt{c_2^2+1}\, \|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} . \tag{3.39}$$

But we can also estimate the full H^1-norm using (3.38) and

$$\begin{aligned} \|\mathrm{D}\boldsymbol{\tau}\|_{\mathrm{L}^2(\Omega)} &\overset{\mathscr{A}\,\mathrm{o.p.}}{\le} \|\mathrm{D}^2\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \le \|\boldsymbol{v}\|_{\mathrm{H}^2(\Omega)} \overset{(3.35)}{\le} \tilde{c}\left(\|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\right) \\ &\overset{(3.36)}{\le} \tilde{c}\left(\|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} + c_2\|\boldsymbol{\tau}\|_{\mathrm{L}^2(\Omega)}\right) \overset{(3.38)}{\le} \tilde{c}\,(1+c_2^2)\,\|\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} . \end{aligned} \tag{3.40}$$

Hence, a combination of (3.38) and (3.40) concludes the proof. ∎

4. Existence of unique weak solutions for the linear R13 equations. We are now ready to prove the existence of a unique weak solution to the linear R13 equations. Namely, given some $(\mathcal{F},\mathcal{G}) \in \mathrm{V}'\times\mathrm{Q}'$, there exists a unique $(\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{P}}) \in \mathrm{V}\times\mathrm{Q}$ such that

$$\begin{cases} \mathcal{A}(\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{V}}) + \mathcal{B}(\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{P}}) = \mathcal{F}(\boldsymbol{\mathcal{V}}) & \forall\,\boldsymbol{\mathcal{V}}\in\mathrm{V}, \\ \mathcal{B}(\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{Q}}) = \mathcal{G}(\boldsymbol{\mathcal{Q}}) & \forall\,\boldsymbol{\mathcal{Q}}\in\mathrm{Q}. \end{cases} \tag{4.1}$$

To this end, we follow the classical strategy to prove the existence and uniqueness of our saddle point problem using the classical framework that we briefly recall.

THEOREM 4.1 ([6, Thm. 1.1]). *Let* V *and* Q *be two Hilbert spaces, and* $a(\cdot,\cdot) : \mathrm{V}\times\mathrm{V}\to\mathbb{R}$ *and* $b(\cdot,\cdot) : \mathrm{V}\times\mathrm{Q}\to\mathbb{R}$ *be two continuous linear forms. Let us suppose that the range of the operator* B*, associated with* $b(\cdot,\cdot)$*, is closed in* Q'*; that is, there exists a constant* $k_0 > 0$ *such that*

$$\sup_{v\in\mathrm{V}} \frac{b(v,q)}{\|v\|_{\mathrm{V}}} \ge k_0 \|q\|_{\mathrm{Q}/\ker B^\top} = k_0\Big(\inf_{q_0\in\ker B^\top} \|q+q_0\|_{\mathrm{Q}}\Big). \tag{4.2}$$

If, moreover, $a(\cdot,\cdot)$ *is invertible on* $\ker B$*, that is, there exists* $\alpha_0 > 0$*, such that*

$$\inf_{u_0\in\ker B}\sup_{v_0\in\ker B} \frac{a(u_0,v_0)}{\|u_0\|_{\mathrm{V}}\|v_0\|_{\mathrm{V}}} \ge \alpha_0 \quad \textit{and} \quad \inf_{v_0\in\ker B}\sup_{u_0\in\ker B} \frac{a(u_0,v_0)}{\|u_0\|_{\mathrm{V}}\|v_0\|_{\mathrm{V}}} \ge \alpha_0, \tag{4.3}$$

then there exists a solution to

$$\begin{cases} a(u,v) + b(v,p) = f(v) & \forall\, v \in \mathrm{V}, \\ \qquad\qquad b(u,q) = g(q) & \forall\, q \in \mathrm{Q}, \end{cases} \tag{4.4}$$

for any $f \in \mathrm{V}'$ *and any* $g \in \operatorname{im} B$. *The first component* u *is unique, and* p *is defined up to an element of* $\ker B^\top$. *Moreover, one has the bounds*

$$\|u\|_{\mathrm{V}} \leq \frac{1}{\alpha_0}\|f\|_{\mathrm{V}'} + \left(\frac{\|a\|}{\alpha_0} + 1\right)\frac{1}{k_0}\|g\|_{\mathrm{Q}'} \tag{4.5}$$

$$\|p\|_{\mathrm{Q}/\ker B^\top} \leq \frac{1}{k_0}\left(1 + \frac{\|a\|}{\alpha_0}\right)\|f\|_{\mathrm{V}'} + \frac{\|a\|}{k_0^2}\left(1 + \frac{\|a\|}{\alpha_0}\right)\|g\|_{\mathrm{Q}'}. \tag{4.6}$$

4.1. Coercivity on the kernel. To continue with proving the coercivity of $\mathcal{A}$ on the kernel of B (i.e., the linear map associated with $\mathcal{B}$; see section 2), we first need the following characterization.

Lemma 4.2 (Kernel of B). *We have*

$$\ker B = \{(\boldsymbol{\sigma}, \boldsymbol{s}, p) \in \mathrm{V} : \operatorname{Div}\boldsymbol{\sigma} = -\boldsymbol{\nabla} p, \operatorname{div}\boldsymbol{s} = 0\}\,. \tag{4.7}$$

Proof. By the definition of the kernel, we have for all $\boldsymbol{\mathcal{U}} \in \ker B$:

$$\mathcal{B}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = 0 \qquad \forall\, \boldsymbol{\mathcal{Q}} \in \mathrm{Q} \tag{4.8}$$

$$\Leftrightarrow -e(\boldsymbol{v}, \boldsymbol{\sigma}) - g(p, \boldsymbol{v}) - b(\kappa, \boldsymbol{s}) = 0 \quad \forall\, (\boldsymbol{v}, \kappa) \in \mathrm{Q}. \tag{4.9}$$

Since this holds for any $(\boldsymbol{v}, \kappa) \in \mathrm{Q}$, we extract two separated conditions, i.e., $e(\boldsymbol{v}, \boldsymbol{\sigma}) + g(p, \boldsymbol{v}) = 0$ and $b(\kappa, \boldsymbol{s}) = 0$. We insert the explicit expressions of these bilinear forms from section 2 and directly conclude that $\operatorname{Div}\boldsymbol{\sigma} = -\boldsymbol{\nabla} p$ and $\operatorname{div}\boldsymbol{s} = 0$ define the kernel of B. ∎

Lemma 4.3 (Coercivity of $\mathcal{A}$ on $\ker B$). *The bilinear form* $\mathcal{A}$ *is coercive on* $\ker B$, *i.e., there exists a constant* $\alpha_0 > 0$ *such that* $\mathcal{A}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{U}}) \geq \alpha_0 \|\boldsymbol{\mathcal{U}}\|_{\mathrm{V}}^2$ *for all* $\boldsymbol{\mathcal{U}} \in \ker B$.

Proof. By Lemma 4.2, with $\boldsymbol{\mathcal{U}} = (\boldsymbol{\sigma}, \boldsymbol{s}, p) \in \ker B$, we have $\operatorname{Div}\boldsymbol{\sigma} = -\boldsymbol{\nabla} p$. Thus, we estimate that

$$\|\boldsymbol{\sigma}\|_{\mathrm{H}^1(\Omega)}^2 \geq \|\mathrm{D}\boldsymbol{\sigma}\|_{\mathrm{L}^2(\Omega)}^2 \overset{(*)}{\geq} \|\operatorname{Div}\boldsymbol{\sigma}\|_{\mathrm{L}^2(\Omega)}^2 = \|\boldsymbol{\nabla} p\|_{\mathrm{L}^2(\Omega)}^2\,, \tag{4.10}$$

using, in the lower bound $(*)$, the CBS inequality for the Frobenius inner products in $\int_\Omega \|\operatorname{Div}\boldsymbol{\sigma}\|_{\mathbb{R}^3}^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega \|[\langle(\delta_{kl})_{k,l=1}^3, (\partial_k \sigma_{il})_{k,l=1}^3\rangle_{\mathrm{F}}]_{i=1}^3\|_{\mathbb{R}^3}^2\,\mathrm{d}\boldsymbol{x} \leq \|\mathbf{1}_3\|_{\mathrm{F}}^2 \int_\Omega \|\mathrm{D}\boldsymbol{\sigma}\|_{\mathrm{F}}^2\,\mathrm{d}\boldsymbol{x}$. Then, we conclude with

$$\begin{aligned} \mathcal{A}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{U}}) &= a(\boldsymbol{s}, \boldsymbol{s}) + \bar{d}((\boldsymbol{\sigma}, p), (\boldsymbol{\sigma}, p)) \\ &\overset{(3.26),(3.27)}{\geq} c_2\left(\|\boldsymbol{s}\|_{\mathrm{H}^1(\Omega)}^2 + \|\boldsymbol{\sigma}\|_{\mathrm{H}^1(\Omega)}^2\right) + \tilde{\chi}\epsilon^{\mathrm{w}}\|p\|_{\mathrm{L}^2(\Gamma)}^2 \\ &\overset{(4.10)}{\geq} \frac{c_2}{2}\left(\|\boldsymbol{s}\|_{\mathrm{H}^1(\Omega)}^2 + \|\boldsymbol{\sigma}\|_{\mathrm{H}^1(\Omega)}^2\right) + \frac{c_2}{2}\|\boldsymbol{\nabla} p\|_{\mathrm{L}^2(\Omega)}^2 + \tilde{\chi}\epsilon^{\mathrm{w}}\|p\|_{\mathrm{L}^2(\Gamma)}^2 \\ &\overset{(*)}{\geq} \alpha_0\left(\|\boldsymbol{\sigma}\|_{\mathrm{H}^1(\Omega)}^2 + \|\boldsymbol{s}\|_{\mathrm{H}^1(\Omega)}^2 + \|p\|_{\mathrm{H}^1(\Omega)}^2\right) = \alpha_0\|\boldsymbol{\mathcal{U}}\|_{\mathrm{V}}^2 \quad \forall \boldsymbol{\mathcal{U}} \in \ker B, \end{aligned} \tag{4.11}$$

where the step $(*)$ in (4.11), depending on the value of ϵ^{w}, either uses Poincaré's inequality $\|p\|_{\mathrm{L}^2(\Omega)} \leq c\,\|\boldsymbol{\nabla} p\|_{\mathrm{L}^2(\Omega)}$ for $p \in \mathrm{H}^1(\Omega;\mathbb{R})$ with $\int_\Omega p\,\mathrm{d}\boldsymbol{x} = 0$ ($\epsilon^{\mathrm{w}} = 0$) or Friedrich's inequality $\|p\|_{\mathrm{L}^2(\Omega)} \leq c\,(\|\boldsymbol{\nabla} p\|_{\mathrm{L}^2(\Omega)} + \|p\|_{\mathrm{L}^2(\Gamma)})$ for $p \in \mathrm{H}^1(\Omega;\mathbb{R})$ ($\epsilon^{\mathrm{w}} > 0$) in accordance with the definition of V in (2.1). ∎

Remark 4.4. Interestingly, for the coercivity of $\mathcal{A}$ in Lemma 4.3, we used the kernel property only to include the pressure p in the estimate by relating it to the stress $\boldsymbol{\sigma}$. It becomes clear that we must proceed in this manner when considering the case of $\epsilon^{\mathrm{w}} = 0$, in which $\mathcal{A}$ would not contain any p-terms at all. The property $\operatorname{div} \boldsymbol{s} = 0$, however, is not needed for coercivity.

4.2. Sup-condition. The existence of the right-inverse of the matrix divergence from Lemma 3.12, combined with results in the vectorial case, yields the following result:

LEMMA 4.5 (Sup-condition of $\mathcal{B}$). *There exists a constant $k_0 > 0$ such that for all $\boldsymbol{\mathcal{Q}} \in \mathrm{Q}$, it holds*

$$\sup_{\boldsymbol{\mathcal{V}} \in \mathrm{V}} \frac{\mathcal{B}(\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{Q}})}{\|\boldsymbol{\mathcal{V}}\|_{\mathrm{V}}} \geq k_0 \|\boldsymbol{\mathcal{Q}}\|_{\mathrm{Q}}. \tag{4.12}$$

Proof. For $\boldsymbol{\mathcal{V}} = (\boldsymbol{\psi}, \boldsymbol{r}, q) \in \mathrm{V}$ and $\boldsymbol{\mathcal{Q}} = (\boldsymbol{v}, \kappa) \in \mathrm{Q}$, the bilinear form $\mathcal{B}$ reads as

$$\begin{aligned} \mathcal{B}(\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{Q}}) &= -e(\boldsymbol{v}, \boldsymbol{\psi}) - g(q, \boldsymbol{v}) - b(\kappa, \boldsymbol{r}) \\ &= -\int_\Omega \left(\operatorname{Div} \boldsymbol{\psi} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} q + \kappa \operatorname{div} \boldsymbol{r}\right) \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.13}$$

Now, let $\boldsymbol{\mathcal{Q}} = (\boldsymbol{v}, \kappa) \in \mathrm{Q}$ be given. Then, by Lemmata 3.12 and 3.13, we find a $\boldsymbol{\tau} \in \mathrm{H}^1(\Omega; \mathbb{R}^{3\times3}_{\mathrm{stf}})$ and a $\boldsymbol{t} \in \mathrm{H}^1(\Omega; \mathbb{R}^3)$, such that

$$-\operatorname{Div} \boldsymbol{\tau} = \boldsymbol{v}, \qquad -\operatorname{div} \boldsymbol{t} = \kappa, \tag{4.14a}$$

$$\|\boldsymbol{\tau}\|_{\mathrm{H}^1(\Omega)} \leq c \|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}, \quad \|\boldsymbol{t}\|_{\mathrm{H}^1(\Omega)} \leq c \|\kappa\|_{\mathrm{L}^2(\Omega)}, \tag{4.14b}$$

with some constant $c > 0$. In particular, we choose $\tilde{\boldsymbol{\mathcal{V}}} \coloneqq (\boldsymbol{\tau}, \boldsymbol{t}, 0) \in \mathrm{V}$ and obtain

$$\begin{aligned} \sup_{\boldsymbol{\mathcal{V}} \in \mathrm{V}} \frac{\mathcal{B}(\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{Q}})}{\|\boldsymbol{\mathcal{V}}\|_{\mathrm{V}}} \geq \frac{\mathcal{B}(\tilde{\boldsymbol{\mathcal{V}}}, \boldsymbol{\mathcal{Q}})}{\|\tilde{\boldsymbol{\mathcal{V}}}\|_{\mathrm{V}}} &\overset{(4.13),(4.14a)}{=} \frac{\int_\Omega \left(\boldsymbol{v}^2 + \kappa^2\right) \mathrm{d}\boldsymbol{x}}{(\|\boldsymbol{\tau}\|^2_{\mathrm{H}^1} + \|\boldsymbol{t}\|^2_{\mathrm{H}^1})^{\frac{1}{2}}} \\ &\overset{(4.14b)}{\geq} \frac{\|\boldsymbol{v}\|^2_{\mathrm{L}^2} + \|\kappa\|^2_{\mathrm{L}^2}}{c(\|\boldsymbol{v}\|^2_{\mathrm{L}^2} + \|\kappa\|^2_{\mathrm{L}^2})^{\frac{1}{2}}} = \frac{1}{c}\left(\|\boldsymbol{v}\|^2_{\mathrm{L}^2} + \|\kappa\|^2_{\mathrm{L}^2}\right)^{\frac{1}{2}} = \frac{1}{c}\|\boldsymbol{\mathcal{Q}}\|_{\mathrm{Q}}. \end{aligned} \tag{4.15}$$ ∎

Remark 4.6. Remarkably, the choice of $\tilde{\mathcal{V}}$ had a trivial third component in the proof of Lemma 4.5, corresponding to the pressure p, which is not needed for the sup-condition.

COROLLARY 4.7. *The associated linear map $B : \mathrm{V} \to \mathrm{Q}'$ is an isomorphism.*

Proof. This property follows (see, e.g., [4, Satz 3.6]) since the corresponding bilinear form $\mathcal{B}$ is continuous, fulfills the sup-condition, and for all $\boldsymbol{\mathcal{Q}} \neq 0$, we find a $\tilde{\boldsymbol{\mathcal{V}}} \in \mathrm{V}$ such that $\mathcal{B}(\tilde{\boldsymbol{\mathcal{V}}}, \boldsymbol{\mathcal{Q}}) \neq 0$. For the latter, consider $\tilde{\boldsymbol{\mathcal{V}}}$ from the proof of Lemma 4.5, for which we showed that

$$\mathcal{B}(\tilde{\boldsymbol{\mathcal{V}}}, \boldsymbol{\mathcal{Q}}) = \|\boldsymbol{\mathcal{Q}}\|^2_{\mathrm{Q}} > 0. \tag{4.16}$$ ∎

COROLLARY 4.8. *For the transpose $B^\top : \mathrm{Q} \to \mathrm{V}'$, given via $\langle \boldsymbol{\mathcal{V}}, B^\top \boldsymbol{\mathcal{Q}} \rangle_{\mathrm{V}\times\mathrm{Q}'} \coloneqq \mathcal{B}(\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{Q}})$, it holds*

$$\ker B^\top = \{0\}. \tag{4.17}$$

4.3. Proof of the main result. We can now use the Lemmata 4.3 and 4.5 to obtain the main result.

THEOREM 4.9. *For any* $(\mathcal{F}, \mathcal{G}) \in \mathrm{V}' \times \mathrm{Q}'$*, there exists a unique solution* $(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{P}}) \in \mathrm{V} \times \mathrm{Q}$ *to the weak linear R13 equations* (4.1) *that fulfills*

$$\|\boldsymbol{\mathcal{U}}\|_{\mathrm{V}} \leq \frac{1}{\alpha_0}\|\mathcal{F}\|_{\mathrm{V}'} + \left(\frac{\|\mathcal{A}\|}{\alpha_0} + 1\right)\frac{1}{k_0}\|\mathcal{G}\|_{\mathrm{Q}'} \tag{4.18}$$

$$\|\boldsymbol{\mathcal{P}}\|_{\mathrm{Q}} \leq \frac{1}{k_0}\left(1 + \frac{\|\mathcal{A}\|}{\alpha_0}\right)\|\mathcal{F}\|_{\mathrm{V}'} + \frac{\|\mathcal{A}\|}{k_0^2}\left(1 + \frac{\|\mathcal{A}\|}{\alpha_0}\right)\|\mathcal{G}\|_{\mathrm{Q}'}, \tag{4.19}$$

with the constants $\alpha_0, k_0 > 0$ *from Lemmata* 4.3 *and* 4.5*.*

Proof. The existence follows directly from an application of Theorem 4.1 to our situation since $\mathcal{A}$ and $\mathcal{B}$ are both continuous (see Appendix A), $\mathcal{A}$ is coercive on $\ker B$ by Lemma 4.3, and $\mathcal{B}$ fulfills the sup-condition in Lemma 4.5. Moreover, we have $\operatorname{im} B = \mathrm{Q}'$ and $\ker B^\top = \{0\}$ by Corollary 4.8, such that both components are unique. ∎

Remark 4.10. When comparing the Theorem 4.9 to the corresponding result (see, e.g., [3, Thm. 8.2.1]) for Stokes' problem from subsection 1.2 – where the fluid subsystem depicted in Figure 1 serves as an extension – we, interestingly, obtain that the pressure is in $\mathrm{H}^1(\Omega)$ and the velocity is in $\mathrm{L}^2(\Omega)$. Conversely, when $\epsilon^w = 0$, the pressure is determined only up to a constant, as expected. The regularity for θ also does not follow the classical result for Poisson's problem, as we only have $\theta \in \mathrm{L}^2(\Omega)$ by Theorem 4.9.

5. Conclusion and future work. In this paper, we derived the well-posedness for the weak R13 equations. The first important step was to reformulate the problem as a grouped mixed problem within the abstract LBB framework. Due to the unique tensorial structure of the equations, which also involved matrix-valued differential equations, we derived new theoretical tools, such as a tensor-valued Korn inequality for symmetric and trace-free parts of matrix derivatives. Proving the ellipticity of such operators then led to the existence of a right inverse of the matrix-valued divergence as the second crucial ingredient needed to analyze the equations. These estimates could also be used in the context of linear elasticity if, e.g., nonstandard materials laws are involved, and we tried to present them in a general form.

Our analysis is the first step towards a numerical analysis of discretization schemes for the equations. For example, in the context of mixed finite element methods – although promising numerical progress has been made, e.g., using generalized Taylor–Hood elements in [43] – no known stable element pairing is available yet. In fact, as a mixed problem, continuous results do not directly transfer to the discrete case. Especially the discrete sup-condition and handling of the discrete kernel of B are unclear for now since the symmetry and trace-free constraints for the tensor $\boldsymbol{\sigma}$ are challenging; see, e.g., the discussion in [3, Sec. 9.1].

Further aspects include the extension to the time-dependent case, the construction of efficient preconditioned iterative solvers, and, eventually, the solution and analysis of the complete nonlinear system. On the other hand, from the theoretical side, reformulating the system as a variational optimization problem, alternative analyses using different grouping, the use of augmented formulations, and the extension to higher-order moment systems are a natural next step. Such systems will require coercivity estimates for symmetric and trace-free tensor derivatives of even higher order.

Appendix A. Continuity of the bilinear forms. The continuity of the bilinear form $\mathcal{B}$ is straightforward. For any $\boldsymbol{\mathcal{U}} = (\boldsymbol{\sigma}, \boldsymbol{s}, p) \in \mathrm{V}$ and any $\boldsymbol{\mathcal{Q}} = (\boldsymbol{v}, \kappa) \in \mathrm{Q}$, we have

$$
\begin{aligned}
|\mathcal{B}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}})| &= |\mathcal{B}((\boldsymbol{\sigma}, \boldsymbol{s}, p), (\boldsymbol{v}, \kappa))| \\
&\overset{\triangle\text{-in.}}{\leq} |e(\boldsymbol{v}, \boldsymbol{\sigma})| + |g(p, \boldsymbol{v})| + |b(\kappa, \boldsymbol{s})| \\
&= \left| \int_\Omega \operatorname{Div} \boldsymbol{\sigma} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \right| + \left| \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla} p \,\mathrm{d}\boldsymbol{x} \right| + \left| \int_\Omega \kappa \operatorname{div} \boldsymbol{s} \,\mathrm{d}\boldsymbol{x} \right| \\
&\overset{\text{CBS}}{\leq} \|\operatorname{Div} \boldsymbol{\sigma}\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{\nabla} p\|_{\mathrm{L}^2(\Omega)} + \|\kappa\|_{\mathrm{L}^2(\Omega)} \|\operatorname{div} \boldsymbol{s}\|_{\mathrm{L}^2(\Omega)} \\
&\leq \|\boldsymbol{\sigma}\|_{\mathrm{H}^1(\Omega)} \|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \|p\|_{\mathrm{H}^1(\Omega)} + \|\kappa\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{s}\|_{\mathrm{H}^1(\Omega)} \\
&\leq C \|\boldsymbol{\mathcal{U}}\|_{\mathrm{V}} \|\boldsymbol{\mathcal{Q}}\|_{\mathrm{Q}},
\end{aligned} \tag{A.1}
$$

where we used the norm equivalence on $\mathbb{R}^n$ in the last step. The continuity of $\mathcal{A}$ is obtained similarly, and we demonstrate it here, exemplary, for the following bilinear form a as

$$
\begin{aligned}
|a(\boldsymbol{s}, \boldsymbol{r})| \leq c \big\{ &\|\operatorname{sym} \mathrm{D}\boldsymbol{s}\|_{\mathrm{L}^2(\Omega)} \|\operatorname{sym} \mathrm{D}\boldsymbol{r}\|_{\mathrm{L}^2(\Omega)} + \|\operatorname{div} \boldsymbol{s}\|_{\mathrm{L}^2(\Omega)} \|\operatorname{div} \boldsymbol{r}\|_{\mathrm{L}^2(\Omega)} \\
&+ \|\boldsymbol{s}\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{r}\|_{\mathrm{L}^2(\Omega)} + \|s_n\|_{\mathrm{L}^2(\Gamma)} \|r_n\|_{\mathrm{L}^2(\Gamma)} + \textstyle\sum_{i=1}^2 \|s_{t_i}\|_{\mathrm{L}^2(\Gamma)} \|r_{t_i}\|_{\mathrm{L}^2(\Gamma)} \big\} \\
\leq c_1 \big\{ &\|\mathrm{D}\boldsymbol{s}\|_{\mathrm{L}^2(\Omega)} \|\mathrm{D}\boldsymbol{r}\|_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{s}\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{r}\|_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{s}\|_{\mathrm{L}^2(\Gamma)} \|\boldsymbol{r}\|_{\mathrm{L}^2(\Gamma)} \big\} \\
\leq c_2 \, &\|\boldsymbol{s}\|_{\mathrm{H}^1(\Omega)} \|\boldsymbol{r}\|_{\mathrm{H}^1(\Omega)},
\end{aligned} \tag{A.2}
$$

where the last step used the trace theorem [4, p. 42] to estimate the L^2-norm on Γ.

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