

Constructing symmetric simplicial surfaces

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Abstract

This thesis studies triangulated 2-dimensional manifolds from a combinatorial perspective. We explore the properties of such triangulations with the help of simplicial surfaces. Here, a simplicial surface encodes the incidence relations between the vertices, edges and triangles of a triangulated 2-dimensional manifold, e.g., a disc, a sphere or a torus. For instance, some of the Platonic solids, namely the tetrahedron, the octahedron and the icosahedron, can be understood as triangulations of the 2-sphere and hence can be translated into simplicial surfaces. Since each triangle (face) of a simplicial surface is incident to three edges, these combinatorial structures can be linked to cubic graphs. While constructing a cubic graph from a given simplicial surface is computationally easy, deciding whether a given cubic graph is the graph describing the incidences between the faces and edges of some simplicial surface is quite challenging; even more, determining the existence of such a simplicial surface for an arbitrary cubic graph is still an open problem in graph-theoretical research.

The goal of this thesis is to study simplicial surfaces and their connections to cubic graphs. Moreover, we provide construction methods for cubic graphs and simplicial surfaces arising from cubic graphs. To achieve this goal, we make use of group-theoretical approaches.

First, we examine homomorphisms between simplicial surfaces. Of particular interest are simplicial surfaces that have no proper epimorphic images. That means we study simplicial surfaces, where every epimorphism from such a simplicial surface onto another simplicial surface is already an isomorphism. We establish the existence of infinitely many simplicial surfaces that satisfy the above property and further elaborate on special homomorphisms, namely endomorphisms and butterfly-friendly homomorphisms between simplicial surfaces.

As another main result, we modify a well-known cubic graph construction, presented by Frucht in 1949, and exploit our modification to establish the existence of simplicial surfaces having automorphism groups that are isomorphic to arbitrary finite groups. For certain groups, we further associate the arising simplicial surfaces to polyhedra in Euclidean 3-space such that the simplicial surfaces and the corresponding polyhedra have isomorphic automorphism and symmetry groups.

Next, we exploit highly symmetric cubic graphs to construct simplicial surfaces with large automorphism groups. More precisely, we demonstrate that the automorphism group of a vertex- or edge-transitive cubic graph can be used to construct symmetric simplicial surfaces such as closed face-transitive or closed edge-transitive surfaces. These constructions are achieved by providing suitable cycle double covers of given vertex- or edge-transitive cubic graphs. Our theoretical results allow us to compile two censuses: one containing closed face-transitive surfaces with at most 1280 faces, derived from cubic vertex-transitive graphs with automorphism groups of order at most 10^{11} , and one containing closed edge-transitive surfaces with at most 5000 faces, constructed from cubic edge-transitive graphs. Finally, we classify the sets of open face-transitive surfaces and open edge-transitive surfaces.

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Authorship Declaration

This thesis contains work that has been or will be submitted to a journal.

- **Details of the work:** [2] Reymond Akpanya and Tom Goertzen. *Surfaces with given Automorphism Group*. 2023. [arXiv:2307.12681](#) [math.GR].

Student contribution to work: The author of this thesis contributed fully to all mathematical ideas. The content of this work can be found in Section 7 of this thesis. The author of this thesis was responsible for writing [2, Sections 4, 7], and the mathematical ideas of these sections were developed in joint work with Goertzen. [2, Sections 1, 2] were written in collaboration by both authors. Goertzen was responsible for writing [2, Sections 3, 5, 6], and the mathematical ideas of these sections were developed in joint work with the author of this thesis. The implementations developed for this work are available in [72].

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Student contribution to work: This work emerged as the result of a research visit at the University of Sydney. The author of this thesis contributed fully to all mathematical ideas. The content of this work can be found in Sections 3 and 5 of this thesis. The author of this thesis was responsible for writing [5, Sections 2, 3, 4, 6], and all mathematical ideas were developed in joint work with Spreer. Further, [5, Section 1] was written in collaboration by both authors. Spreer was responsible for writing [5, Sections 5, 7], and the mathematical ideas were developed in joint work with the author of this thesis. The implementations developed for this work were implemented by the author of this thesis and are contained in [6].

Introduction

Combinatorial structures have attracted significant interest across diverse fields of mathematical research as powerful tools that generate profound theoretical insights which enable wide-ranging applications. These structures play an important role in numerous mathematical disciplines, such as graph theory, topology, and geometry, while also proving to be valuable in fields such as computer science, computer graphics, and civil engineering. Moreover, combinatorial methods offer innovative solutions to complex problems, making them essential for both, theoretical studies and practical applications, see [53, 67, 82, 83] to name only a few.

The combinatorial objects that are the focus of this thesis are called simplicial surfaces. Loosely speaking, a simplicial surface is a combinatorial structure that describes the incidence relations between the vertices, edges and faces of a triangulated 2-dimensional manifold, e.g., a disc, a sphere or a torus. We refer the reader to [74] for the theoretical foundation and to [14, 23, 55, 86] for some extensive studies of simplicial surfaces.

At RWTH Aachen University, the investigation of simplicial surfaces was initially motivated by a research question posed by Prof. Dr. Martin Trautz, see [23]. He asked whether surfaces in Euclidean 3-space can be approximated using triangulated surfaces composed of triangles belonging to only a few congruence classes. Reducing this problem to the approximation via triangulated surfaces that consist of triangles of a single congruence class admits a purely combinatorial investigation. This approach leads to a rich theory for such triangulated surfaces, namely the theory of simplicial surfaces. In recent years, the study of simplicial surfaces has been partly motivated by real-life applications such as the challenge of reducing greenhouse gas emissions, see [92]. This challenge, particularly relevant to the building sector, is one of the many reasons why the framework CRC/TRR 280 (Project-ID: 417002380) is interested in creating innovative design strategies for material minimized carbon-reinforced concrete structures and hence explores new construction methods for 3D-geometries. In that context simplicial surfaces have found many applications, as they facilitate the construction of 3D-geometries with interesting properties, e.g., convexity or interlocking properties, see [49, 50, 87] for some references.

In this work we study simplicial surfaces and their relationship to cubic graphs from an algebraic perspective. To build an initial intuition for the concept of a simplicial surface, let us begin by considering polyhedra in Euclidean 3-space, as some of these structures can be interpreted as triangulations of the 2-sphere. In the literature, polyhedra are described as three-dimensional bodies whose boundaries are composed of points (vertices), straight line segments (edges), and flat polygons (faces). There are various definitions of polyhedra, see [15, 30, 32, 34, 66] to name only a few. While some studies require polyhedra to be convex, we also regard non-convex or self-intersecting three-dimensional bodies with surfaces made up of vertices, edges and faces as polyhedra, as described in [22, 31, 93] for instance. As an example of a convex polyhedron we consider the *tetrahedron*, i.e. the Platonic solid whose surface consists of four vertices, six edges and four triangular faces, see Figure Ia. As described above, we can translate the tetrahedron into a simplicial surface by labelling its vertices by $v_1, \dots, v_4$, its edges by $e_1, \dots, e_6$, its faces by $F_1, \dots, F_4$ and recording the incidences between the different elements. The arising simplicial surface can be visualised in a folding plan, see Figure Ib. A folding plan of a simplicial surface allows us to create a paper model of a polyhedron that corresponds to the given simplicial surface. In order to create such a model, the folding plan has to be cut out from a piece of paper, and vertices and edges with the same labels have to be identified (glued together).

So, we observe that translating a given polyhedron with triangular faces into a simplicial surface is straightforward. Now, we could ask for an even more concise description of a polyhedron consisting of triangular faces by just recording a partial incidence structure of the given polyhedron.

For instance, instead of denoting all incidences of the elements of a polyhedron, we might only be interested in the incidences between the faces and edges. Since each face of a polyhedron with triangular faces is incident to exactly three edges, we obtain a cubic graph called the *face graph*. Figure 1c illustrates the face graph of the simplicial surface describing the tetrahedron.

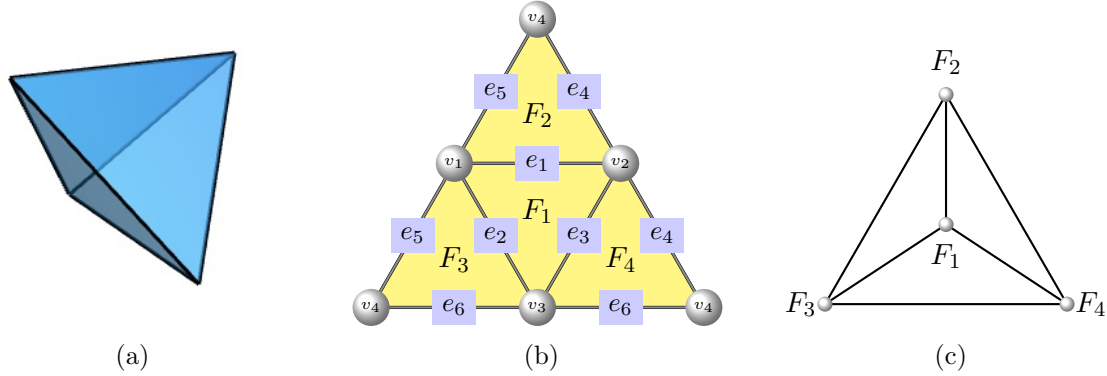


Figure 1: The tetrahedron (a), a corresponding simplicial surface (b), and a corresponding face graph (c)

By Steinitz (see [84]) a cubic graph that is constructed from a polyhedron, forming a triangulation of the 2-sphere, as described above is a 3-connected simple planar graph. Hence, in this case many of the properties of given simplicial surfaces and corresponding polyhedra can be derived from the study of planar graphs. For instance, the famous Four Colour Theorem implies that if a cubic graph encodes a partial incidence structure of a polyhedron forming a triangulation of the 2-sphere, then it is edge-3-colourable and vertex-4-colourable, see [10]. Such colourings of a given cubic graph can be translated into colourings of a corresponding simplicial surface to enrich the combinatorial structure of the simplicial surface. Additionally, Fowler characterises the 3-connected cubic planar graphs with unique edge-3-colourings, see [41]. In particular, these are exactly the cubic graphs that describe partial incidence structures of so-called stacked polyhedra which are closely related to Apollonian networks, see [9] and [15] for instance.

Even in the general case, triangulated 2-manifolds can be represented by simplicial surfaces, which can be translated into corresponding face graphs that form cubic graphs. This relation emphasises the importance of studying cubic graphs, as a means to gain a deeper understanding of simplicial surfaces, see [39, 60, 90] for some related studies. Thus, databases containing different classes of cubic graphs are essential for our investigation of simplicial surfaces, see [25, 76] for instance.

The fact that every simplicial surface can be translated into a cubic graph raises the question whether every cubic graph can be interpreted as the graph describing the incidence relations between the edges and faces of a triangulated 2-manifold. This question can be summarised as follows:

Question. *Does every cubic graph arise as the face graph of a simplicial surface?*

In order to construct a simplicial surface that has a given cubic graph as face graph, we have to construct a cycle double cover of the given graph, i.e. a collection of cycles such that every edge of the cubic graph is contained in exactly two cycles, see [88]. It is still an open problem whether every (cubic) graph contains such a collection of cycles. This instance is formulated in the *cycle double cover conjecture*. In particular, this long-standing open problem addresses the following question:

Question. *Does every (cubic) graph have a cycle double cover?*

This conjecture was posed independently by Seymour for bridgeless graphs in [81] and Szekeres for bridgeless cubic graphs in [88]. We refer the reader to [59] for a survey of open problems that can be related to the cycle double cover conjecture.

For cubic graphs with certain properties the existence of corresponding cycle double covers has been established. For instance, it is well known that if a given bridgeless cubic graph is planar, then it contains a cycle double cover. Moreover, every bridgeless cubic graph that has an edge-3-colouring contains a cycle double cover. This result has been established by Szekeres in [88]. Note, since the cycle double cover conjecture remains unproven, it is unknown how to construct a cycle double cover of an arbitrary cubic graph efficiently.

As mentioned above, our study of simplicial surfaces was originally motivated by a research question posed by Prof. Dr. Martin Trautz. The connection between this research question and the study of simplicial surfaces is outlined by the following question:

Question. *Given a simplicial surface, can we construct a corresponding triangulated surface whose faces all form congruent triangles?*

To solve this problem for a given simplicial surface, we first have to colour the edges of the simplicial surface such that every face of the simplicial surface is incident to the same number of edges coloured in a certain colour. There are exactly three ways to achieve this, namely colouring the edges of a simplicial surface such that (1) all edges are coloured in the same colour, (2) two edges of any face are coloured in a first colour and the remaining edge of any face is coloured in a second colour and (3) the edges of any face are coloured in three different colours. The colouring described in (2) is referred to as an isosceles-colouring (see [74]), while the colouring in (3) is known as a Grünbaum-colouring (see [54]). Once one of these colourings of the edges of a given simplicial surface has been established, the next step is to determine 3D-coordinates for the vertices of the simplicial surface such that edges of the same colour all have the same edge length. Solving the arising system of multivariate quadratic equations yields a triangulated surface in Euclidean 3-space whose triangles all have the same congruence type. The complexity of solving such systems of polynomial equations that arise for the simplicial surface describing the combinatorial structure of the regular icosahedron is discussed in [22] and [67]. For more studies on the problem of embedding simplicial surfaces into Euclidean 3-space as polyhedra consisting of congruent triangles, we refer to [2, 31, 40, 52, 68, 93].

We note that simplicial surfaces can also be examined in a topological setting. More precisely, in certain cases they can be interpreted as simplicial complexes and maps on 2-manifolds, see [26, 29]. Here, we say that a simplicial surface can be understood or interpreted as a simplicial complex, if the incidences between the vertices and faces of the simplicial surface are sufficient to determine the whole incidence structure of the simplicial surface. However, as the primary focus of this thesis is the combinatorial study of simplicial surfaces, we provide relevant references [24, 28, 29, 58, 75, 94] and proceed by outlining the structure of this thesis.

Structure of this thesis

The goal of this thesis is to explore simplicial surfaces and their combinatorial properties. In particular, we analyse the connection of simplicial surfaces to cubic graphs and triangulated manifolds in Euclidean 3-space. Our investigations aim to contribute to the understanding of these combinatorial objects. Here, the questions outlined in the above section serve as a source of inspiration for our studies. In particular, we address special cases of these open problems and hence derive construction methods for cubic graphs, simplicial surfaces and polyhedra in Euclidean-3-space. Throughout this work, group-theoretical approaches play a central role in achieving our objectives.

Moreover, this thesis demonstrates that finite groups can be employed to construct cubic graphs, simplicial surfaces and polyhedra with prescribed symmetry properties.

In order to illustrate our different results, we start this thesis by giving some preliminary notes on simplicial surfaces in Section 1. We formally introduce the definition of a simplicial surface and elaborate on various properties that can be derived from given simplicial surfaces, see Section 1.1. Of particular interest for this thesis is the definition of the face graph of a simplicial surface (see Section 1.2). Furthermore, we introduce colourings of the vertices and edges of a given simplicial surface that can be used to enrich its combinatorial structure (see Section 1.3). In Section 2 we introduce families of simplicial surfaces that are of fundamental importance for this thesis.

Section 3 examines different group actions that can be employed to analyse the combinatorial structures of given simplicial surfaces and cubic graphs. This section lays the foundation for our investigations in Sections 5, 6 and 7, and further introduces the notion of automorphism induced cycles of cubic graphs.

In Section 4 we study homomorphisms between different simplicial surfaces. Here, our focus lies on simple and endomorphism-simple simplicial surfaces. Loosely speaking, a simplicial surface is said to be *simple*, if every epimorphism of the given simplicial surface onto another simplicial surface is already an isomorphism. Here, we prove that the simplicial tetrahedron, see Figure Ib, is the only simple simplicial sphere, i.e. a simplicial surface that describes the incidence structure of a triangulated 2-sphere. Furthermore, a simplicial surface is said to be *endomorphism-simple*, if every endomorphism of the given simplicial surface is already an automorphism of the simplicial surface. In this context, we show that every closed connected simplicial surface with all vertex-degrees being odd is endomorphism-simple. Finally, we establish the existence of infinitely many simple and infinitely many endomorphism-simple but not simple simplicial surfaces.

In Section 5, we examine simplicial surfaces that are face-transitive. Loosely speaking, a simplicial surface is said to be *face-transitive*, if it corresponds to a simplicial complex and its automorphism group acts transitively on the faces of the simplicial surface. We aim to analyse this class of highly symmetric simplicial surfaces by distinguishing between closed and open simplicial surfaces. As a main result, we establish that there are exactly five classes of open face-transitive surfaces. Furthermore, we present a distinguishing invariant that helps us to examine closed face-transitive surfaces more closely. Based on this invariant, we show that there are exactly 13 types of closed face-transitive surfaces. We use our theoretical results to construct a census of closed face-transitive surfaces with at most 1280 faces, derived from cubic graphs with automorphism groups of order at most 10^{11} . We achieve this goal by constructing cycle double covers of given vertex-transitive cubic graphs. In particular, these cycle double covers are constructed by exploiting the automorphism groups of the corresponding vertex-transitive cubic graphs.

Next, we study edge-transitive surfaces, see Section 6. Loosely speaking, we refer to a simplicial surface as *edge-transitive*, if it can be interpreted as a simplicial complex and its automorphism group acts transitively on the edges of the simplicial surface. First, we establish that there is exactly one open connected edge-transitive surface. Similarly as presented in Section 5, we aim to classify the set of closed edge-transitive surfaces by exploiting suitable group actions on edge-transitive cubic graphs. As a main result, we show that there are exactly five types of closed edge-transitive surfaces. We exploit this result to construct a complete census of closed edge-transitive surfaces with at most 5000 faces by providing suitable cycle double covers of cubic edge-transitive graphs. Again, this construction is achieved by making use of the automorphism groups of corresponding cubic edge-transitive graphs.

Section 7 considers simplicial surfaces whose automorphism groups are isomorphic to given finite groups. For this purpose, we exploit a well established cubic graph construction introduced by Frucht, see [42]. In his work, Frucht starts with an arbitrary finite group and constructs a cubic

graph with the goal that the automorphism group of the resulting cubic graph is isomorphic to the given finite group. Frucht's construction modifies a Cayley graph of the given finite group to obtain a suitable cubic graph. In his construction, Frucht distinguishes between groups that are generated by one element, by two elements or by more than two elements. First, we simplify Frucht's construction for groups that are generated by two elements and consequently obtain smaller cubic graphs that have given finite groups as automorphism groups. In the general case, we address a slight oversight in Frucht's proposed construction from [42]. We illustrate that the cubic graphs resulting from the construction introduced by Frucht do not have automorphism groups that are isomorphic to the given groups. Hence, in case a finite group is generated by more than two elements, we present a slight modification of Frucht's construction to correct the proof presented in [42]. Based on our modified constructions, we prove that every finite group can be realised as the automorphism group of some simplicial surface. We achieve this by equipping cubic graphs resulting from our modified constructions with suitable cycle double covers. For some classes of finite groups, we provide alternative constructions of simplicial spheres. Additionally, we construct infinite families of simplicial surfaces that can be realised in Euclidean 3-space as polyhedra consisting of congruent equilateral triangles as faces such that the automorphism groups of the simplicial surfaces and the symmetry groups of the corresponding polyhedra are isomorphic.

In our investigations we have employed various computer algebra systems to accomplish our objectives, including GAP [44], Magma [21] and Maple [63]. In particular, we make use of the GAP-packages `simpcomp` [37] and `SimplicialSurfaces` [72] to examine the combinatorial structures of simplicial surfaces. The GAP-packages `Digraphs` [18] and `GraphSym` [76] are used to handle cubic graphs. Furthermore, we have used the GAP-package `GAPic` [71] to create illustrations of cubic graphs and different polyhedra. The folding plans of simplicial surfaces presented in this thesis have been created using the functionalities provided in `SimplicialSurfaces` [72]. Here, Magma is exploited to examine graphs and to deal with large automorphism groups of simplicial surfaces and cubic graphs. Furthermore, we have exploited the programming language Julia [19] to construct numerical approximations of polyhedra. Lastly, we have used the Maple-package `SimplicialSurfaceEmbeddings` [80] to construct polyhedra corresponding to given simplicial spheres. The implementations and functionalities, that are relevant to this thesis and have been created during our investigations, are available in [1, 3, 6, 8, 71, 72].

Notations

In this section, we introduce some notations that are used throughout this thesis. For more details, we refer to [74] and the content of this thesis.

Notation	Description
$ M $	The cardinality of a set M
$\text{Pow}(M)$	The power set of a set M
$M \uplus N$	The disjoint union of two sets M and N
$M \Delta N$	The symmetric difference of two sets M and N
$\lfloor x \rfloor$	The floor function of $x \in \mathbb{R}$
I_n	The identity matrix in $\mathbb{R}^{n \times n}$
x^t	The transpose of a vector $x \in \mathbb{R}^n$
$\ x\ $	The Euclidean norm of a vector $x \in \mathbb{R}^n$
$\text{gcd}(n, k)$	The greatest common divisor of natural numbers n and k
$\text{conv}(M)$	The convex set of a subset $M \subseteq \mathbb{R}^n$
$H \leq G$	A group H is a subgroup of a group G
$ g $	The order of an element g of a group G
$G \times H$	The direct product of two groups G and H
$N \rtimes H$	The semi-direct product of two groups N and H
$\text{orb}_G(x)$	The orbit of x under the group G
$\text{Stab}_G(x)$	The stabiliser of x in G
$\mathcal{C}_{G,S}$	The Cayley graph of a group G with generating set S
$\chi(X)$	The Euler characteristic of a simplicial surface X
$g(X)$	The genus of a simplicial surface X
Λ	The set of all $(\ell_1, \ell_2, \ell_3) \in \mathbb{R}_{>0}^3$ satisfying the triangle inequality
$\sim$	An admissible relation on a simplicial surface
$(X/\sim)$	The simplicial surface obtained via an admissible relation $\sim$
$u(v)$	The umbrella of an inner vertex v of a simplicial surface
$\Theta(X)$	The set of vertices of faces of a simplicial surface X
$\mathcal{V}(X)$	The vertex-edge graph of a simplicial surface X
λ_X	An embedding of $\text{Aut}(X)$ into $\text{Aut}(\mathcal{F}(X))$
$X(F)$	The sub-surface of X induced by a face F
$T^F(X)$	The simplicial surface obtained by attaching a tetrahedron to X
$T_v(X)$	The simplicial surface obtained by removing a tetrahedron from X
$\alpha(\sigma, F_1, \dots, F_n)$	The σ -induced α -cycle
$(c_1, \dots, c_k)$ -cycle	A cycle with edges coloured in $c_1, \dots, c_k$

For more information on the groups used in our investigations, we refer the reader to [16, 17, 44].

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1 Preliminaries

The objects that form the main interest of this thesis are called *simplicial surfaces*. This section outlines the fundamental definitions and concepts that are essential to study these combinatorial triangulations and are therefore used throughout this work. Unless stated otherwise, these notions are taken from [74]. As [2, 4, 5] are also based on [74], these notions can be also found in the mentioned works.

1.1 Simplicial surfaces

A simplicial surface describes the incidence relations between the vertices, edges and faces of a triangulated 2-manifold. Below, we give a formal definition of these combinatorial triangulations.

Definition 1.1.1

A **simplicial surface** $(X, <)$ is a countable set X partitioned into three non-empty sets X_0 , X_1 and X_2 such that the relation $<$, called the **incidence**, is a subset of the union $(X_0 \times X_1) \uplus (X_1 \times X_2) \uplus (X_0 \times X_2)$ satisfying the conditions (1.) - (5.) given below. Here, we refer to the elements of X_0 , X_1 and X_2 as **vertices**, **edges** and **faces**, respectively.

1. For each edge $e \in X_1$ there are exactly two vertices $v \in X_0$ with $v < e$.
2. For each face $F \in X_2$ there are exactly three edges $e \in X_1$ with $e < F$ and exactly three vertices $v \in X_0$ with $v < F$. Moreover, any of these three vertices is incident to exactly two of these three edges.
3. For each edge $e \in X_1$ there are either exactly two faces $F_1, F_2 \in X_2$ with $e < F_1$ and $e < F_2$, or exactly one face $F \in X_2$ with $e < F$. In the first case, e is called an **inner edge**. In the latter case, we say that e is a **boundary edge**.
4. For each vertex $v \in X_0$ the **vertex-degree** of v , denoted by $\deg(v)$, is the number of faces that are incident to v . This number satisfies $1 \leq \deg(v) < \infty$.
5. **Umbrella condition:** For each vertex $v \in X_0$ with $n := \deg(v)$ there exists a sequence $(F_1, \dots, F_n)$, where $F_1, \dots, F_n$ are pairwise distinct faces of X with $v < F_i$ for $i = 1, \dots, n$, such that F_j and F_{j+1} share a common edge $e_j \in X_1$ with $v < e_j$ for $j = 1, \dots, n-1$. In the case that F_n and F_1 share a further edge $e_n \in X_1$ with $v < e_n$ and $e_n \notin \{e_1, \dots, e_{n-1}\}$, the vertex v is called an **inner vertex**. In this case, the sequence can be viewed as a cycle $(F_1, \dots, F_n)$, called the **umbrella** of v and denoted by $u(v)$. In the case, where F_1 and F_n do not share such an edge, v is called a **boundary vertex**, and the sequence of faces is called the **half-umbrella** of v .

If every $e \in X_1$ is an inner edge, we say that X is a **closed simplicial surface**. Otherwise, we refer to X as an **open simplicial surface**. A set $Y = Y_0 \uplus Y_1 \uplus Y_2$ with $Y_i \subseteq X_i$ for $i = 1, 2, 3$ is called a **sub-surface** of $(X, <)$, if $(Y, <)$ satisfies the conditions (1.)-(5.).

In this thesis, all simplicial surfaces are assumed to be finite, i.e. if X is a simplicial surface, then $|X| < \infty$. The above definition yields a huge variety of interesting combinatorial structures. For instance, up to relabelling, there exists exactly one simplicial surface consisting of three vertices,

three edges and one face, and exactly one closed simplicial surface consisting of three vertices, three edges and two faces. These simplicial surfaces, namely the **simplicial One-Face** and the **simplicial Janus-Head** are illustrated in Figure 1.1.

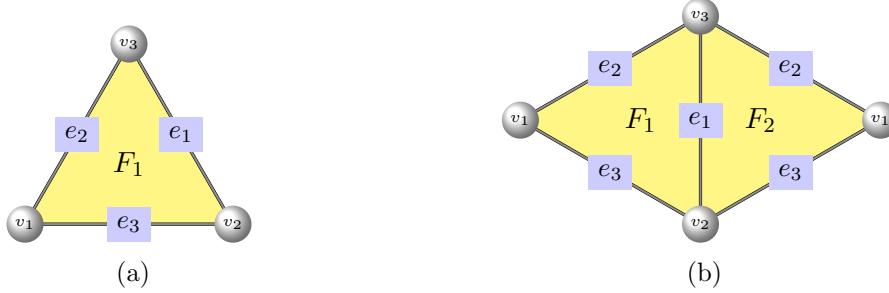


Figure 1.1: The simplicial One-Face (a) and the simplicial Janus-Head (b)

Furthermore, up to relabelling, there exist exactly three open simplicial surfaces that consist of two faces. One such simplicial surface can be constructed by the disjoint union of two simplicial One-Faces. The other two simplicial surfaces, called the **simplicial Butterfly** and the **simplicial Open-Bag**, are shown in Figure 1.2.

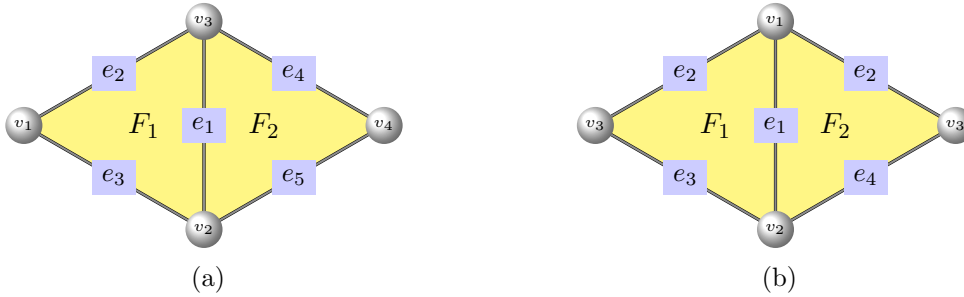


Figure 1.2: The simplicial Butterfly (a) and the simplicial Open-Bag (b)

The simplicial surfaces in Figures 1.1 and 1.2 are first examples of so-called *connected* simplicial surfaces.

Definition 1.1.2

Let $(X, <)$ be a simplicial surface. We say that $(X, <)$ is **connected** if any pair of distinct faces $F, F' \in X_2$ satisfies one of the following conditions:

1. there exists an edge $e \in X_1$ with $e < F$ and $e < F'$, or
2. there exist faces $F_1, \dots, F_n \in X_2$ and edges $e_1, \dots, e_{n+1} \in X_1$, where $n \geq 1$, such that $e_1 < F$, $e_{n+1} < F'$, and $e_i < F_i$ and $e_{i+1} < F_i$ for all $i = 1, \dots, n$.

We call a connected sub-surface $(Y, <)$ of $(X, <)$ a **connected component** of $(X, <)$, if there exists no connected sub-surface $(Z, <)$ of $(X, <)$ with $Y \subsetneq Z$.

In Section 2 we present further examples of connected simplicial surfaces. For the remainder of this section, we assume, if not stated otherwise, that the discussed simplicial surfaces are connected.

A combinatorial invariant that helps us to analyse simplicial surfaces is the **vertex-counter**.

This invariant counts the degrees of the vertices of a given simplicial surface X and is defined as a polynomial in the indeterminates $\{V_n \mid n \geq 1\}$, i.e.

$$\prod_{v \in X_0} V_{\deg(v)}.$$

The vertex-counters of the simplicial One-Face and the simplicial Butterfly are given by V_1^3 and $V_1^2 \cdot V_2^2$, respectively. Another invariant that is essential in the study of simplicial surfaces is the *Euler characteristic*, which allows us to understand simplicial surfaces in a topological setting.

Definition 1.1.3

Let $(X, <)$ be a simplicial surface. We define the **Euler characteristic** $\chi(X)$ of X as

$$\chi(X) := |X_0| - |X_1| + |X_2|.$$

For instance, the Euler characteristic of the simplicial One-Face is equal to 1 and the simplicial Janus-Head has Euler characteristic 2. There are many properties that could be introduced in order to investigate simplicial surfaces. For simplicity, we define properties such as the orientability etc. as usual and refer the reader to [74] for more details. Together with the orientability of simplicial surfaces we can define the **genus** $g(X)$ of a closed simplicial surface X as shown below:

$$g(X) := \begin{cases} \frac{2-\chi(X)}{2}, & \text{if } X \text{ is orientable,} \\ 2 - \chi(X), & \text{if } X \text{ is non-orientable.} \end{cases}$$

We call a closed simplicial surface with Euler characteristic 2 (genus 0) a **simplicial sphere**. Further, we refer to a closed simplicial surface of genus 1 as a **simplicial projective plane** if it is non-orientable and as a **simplicial torus** if it is orientable. Additionally, a closed simplicial surface that is non-orientable and has genus 2 is called a **simplicial Klein bottle**. Note that if a given simplicial surface X is closed, then the faces and edges of X satisfy the equality

$$3 \cdot |X_2| = 2 \cdot |X_1|.$$

Next, we introduce a notion that allows us to describe incidences between certain elements of a given simplicial surface more precisely.

Definition 1.1.4

Let $(X, <)$ be a simplicial surface and $i, j \in \{0, 1, 2\}$ with $i \neq j$. For an element $x \in X_i$ we define the set $X_j(x)$ as

$$X_j(x) := \begin{cases} \{y \in X_j \mid x < y\}, & \text{if } i < j, \\ \{y \in X_j \mid y < x\}, & \text{if } i > j. \end{cases}$$

Furthermore, for a subset $S \subseteq X_i$, we define

$$X_j(S) := \{X_j(x) \mid x \in S\}.$$

From now on we identify a simplicial surface by its underlying set of vertices, edges and faces. Thus, whenever it is clear from the context, we simply denote a simplicial surface $(X, <)$ by X .

For instance, we see that the set $X_0(F_1)$, where X is the simplicial Butterfly illustrated in Figure 1.2a, is given by $X_0(F_1) = \{v_1, v_2, v_3\}$. The above notion allows us to introduce the sub-surface that is defined by a face of a given simplicial surface. Hence, if $F \in X_2$ is a face of a simplicial surface X , we define $X(F)$ to be the sub-surface of X given by the set

$$X(F) := X_0(F) \cup X_1(F) \cup \{F\}. \quad (*)$$

Furthermore, the notion presented in Definition 1.1.4 enables us to define so-called *vertex-faithful* simplicial surfaces. For such a simplicial surface it is sufficient to examine a partial incidence structure to determine the entire simplicial surface, up to isomorphism. This is detailed below.

Definition 1.1.5

Let X be a simplicial surface. If the map

$$f : X_1 \cup X_2 \rightarrow \text{Pow}(X_0), x \mapsto X_0(x)$$

is injective, we say that X is a **vertex-faithful** simplicial surface.

Note that vertex-faithful simplicial surfaces can be understood as simplicial complexes. Thus, the entire incidence structure of a given vertex-faithful simplicial surface is determined by its **vertices of faces**, i.e. the incidences between the vertices and faces of the simplicial surface. We observe that the simplicial One-Face is vertex-faithful, whereas the simplicial Janus-Head is not. Another example of a vertex-faithful simplicial surface is given by the **simplicial tetrahedron** $\mathcal{T}$. As the name indicates, this simplicial surface describes the incidence structure of a regular tetrahedron, see Figure 1.3a, and hence consists of four vertices, six edges and four faces. We determine the incidence structure of $\mathcal{T}$ by giving its set of vertices of faces. If the vertices of $\mathcal{T}$ are denoted by v_1, v_2, v_3, v_4 , then the vertices of faces, i.e. the set $\mathcal{T}_0(\mathcal{T}_2) = \{\mathcal{T}_0(F) \mid F \in \mathcal{T}_2\}$, describing the simplicial tetrahedron, is given by

$$\mathcal{T}_0(\mathcal{T}_2) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}.$$

A folding plan of the simplicial tetrahedron is shown in Figure 1.3b.

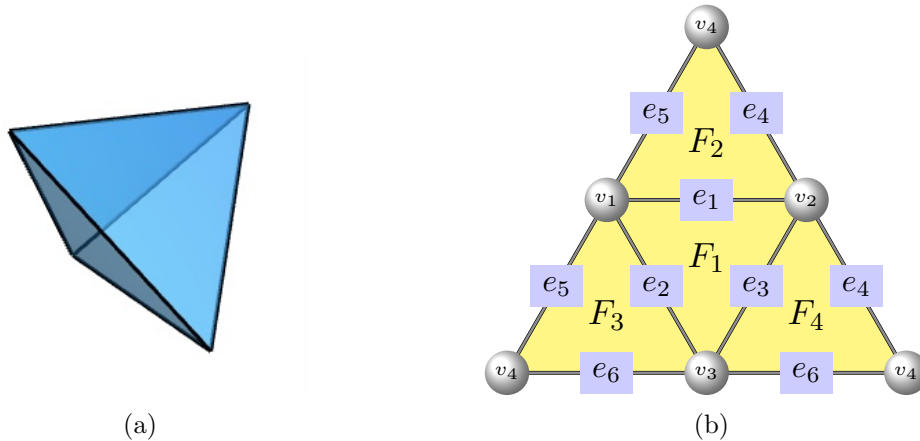


Figure 1.3: A regular tetrahedron (a) and the simplicial tetrahedron (b)

Since we constantly construct and analyse simplicial surfaces by using their corresponding vertices of faces, we introduce a notation for the vertices of faces of a given simplicial surface. In particular, if X is a simplicial surface, then we denote the vertices of faces $X_0(X_2)$ of X by $\Theta(X)$.

A way to construct new examples of vertex-faithful simplicial surfaces is to manipulate the incidence structure of given simplicial surfaces. In the following, we present two such modifications. We therefore denote the symmetric difference of two sets M, N by $M \Delta N := (M \cup N) \setminus (M \cap N)$.

Definition 1.1.6

Let X be a vertex-faithful simplicial surface, M a set with $X \subset M$ and $F \in X_2$ a face with $X_0(F) = \{v_1, v_2, v_3\}$. Furthermore, let $v \in M \setminus X$ be an element. Then, we define the vertex-faithful simplicial surface $Y = T^F(X)$ by the following vertices of faces:

$$\Theta(Y) := \Theta(X) \Delta \{\{v_1, v_2, v_3\}, \{v_1, v_2, v\}, \{v_1, v_3, v\}, \{v_2, v_3, v\}\}.$$

We say that Y is constructed by **attaching a tetrahedron** onto X .

Definition 1.1.7

Let X be a vertex-faithful simplicial surface. Further, let $v \in X_0$ be an inner vertex satisfying $\deg(v) = 3$ and $X_2(v) = \{F_1, F_2, F_3\}$. Since the degree of v is 3, there exist vertices $v_1, v_2, v_3 \in X_0$ with

$$(X_0(F_1), X_0(F_2), X_0(F_3)) = (\{v_2, v_3, v\}, \{v_1, v_3, v\}, \{v_1, v_2, v\}).$$

If $\{v_1, v_2, v_3\} \notin \Theta(X)$, then the following vertices of faces yield the vertex-faithful simplicial surface $Y = T_v(X)$:

$$\Theta(Y) := \Theta(X) \Delta \{\{v_2, v_3, v\}, \{v_1, v_3, v\}, \{v_1, v_2, v\}, \{v_1, v_2, v_3\}\}.$$

We say that Y is constructed by **removing a tetrahedron** from X .

In [74] the operations introduced in Definitions 1.1.6 and 1.1.7 (illustrated in Figure 1.4), are referred to as **tetrahedral extension** and **tetrahedral reduction**, respectively. We note that it is possible to define the above operations for simplicial surfaces that are not vertex-faithful in a similar way.

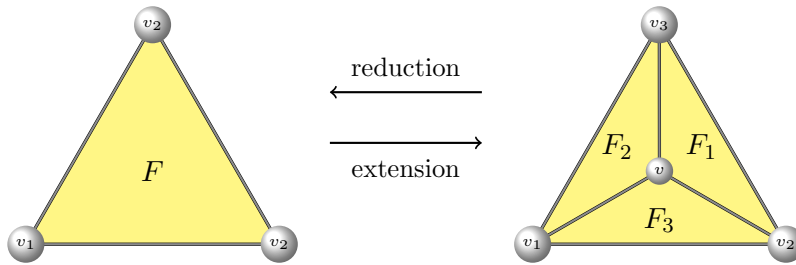


Figure 1.4: The tetrahedral extension and reduction of a simplicial surface

An interesting class of simplicial surfaces that can be constructed by using the above operations is the family of multi-tetrahedral spheres, see [74]. Here, a multi-tetrahedral sphere is a simplicial surface that can be constructed from recursively applying tetrahedral extensions to the simplicial tetrahedron. Since these simplicial surfaces correspond exactly to Apollonian networks (see [9]) and stacked 2-spheres (see [26]) their properties are well understood. For instance, the work of Fowler (see [41]) implies that multi-tetrahedral spheres are exactly the simplicial spheres that have

unique vertex-4-colourings. Here, a simplicial surface has a vertex-4-colouring if its vertices can be coloured with exactly four colours such that neighbouring vertices are coloured differently. We refer the reader to [4, 15, 20, 26, 52, 66] for more studies related to multi-tetrahedral spheres.

Next, we give the notion of homomorphism between simplicial surfaces.

Definition 1.1.8

Let $(X, <)$ and $(Y, <)$ be simplicial surfaces. A map $\phi : X \rightarrow Y$ is called a **homomorphism** if it satisfies the following two conditions:

1. for all $x, x' \in X$ the incidence $x < x'$ in X implies $\phi(x) < \phi(x')$ in Y ,
2. for every face $F \in X_2$ the restriction of ϕ to the vertices and edges incident to F is a bijection onto the set of vertices and edges that are incident to $\phi(F)$, respectively. This means $\phi(x) \in Y_i$ for all $x \in X_i$ and $i = 1, 2, 3$.

We define iso-, mono- and epimorphisms in the usual way. Furthermore, we say that simplicial surfaces X and Y are **isomorphic**, if there exists an isomorphism $\phi : X \rightarrow Y$.

Definition 1.1.8 enables us to define the automorphism group of a simplicial surface. In this work, we will see that the automorphism group of a simplicial surface provides valuable insights into the properties of the corresponding simplicial surface, see Sections 5, 6 and 7.

Definition 1.1.9

Let X be a simplicial surface. We call an isomorphism $\phi : X \rightarrow X$ an **automorphism** of X . We define the set of all automorphisms of X as the **automorphism group** of X and denote it by $\text{Aut}(X)$.

It is easy to see that the automorphism group $\text{Aut}(X)$ of a simplicial surface X indeed forms a group with the composition of maps as group multiplication. For instance, the automorphism group of the simplicial tetrahedron is isomorphic to the symmetric group S_4 . In Section 3 we will explore various group actions of an automorphism group on the corresponding simplicial surface.

1.2 Face graphs of simplicial surfaces

By considering partial incidence structures of a given simplicial surface, we can associate the simplicial surface to different graphs which allows us to examine the simplicial surface from a graph-theoretical perspective. In this section we focus on the *face graph* of a simplicial surface, see Definition 1.2.2. This graph results from recording the incidences between the faces and edges of a given simplicial surface. Note, in Section 1.3 we will also discuss the *vertex-edge graph* of a simplicial surface which is the graph that describes the incidences between the vertices and edges of a given simplicial surface. However, before introducing the definition of a face graph, we give the following remark.

Remark 1.2.1 *In this work, we follow the standard terminology of (topological) graph theory as given in [51, 70] for instance. We note that although the graphs in this thesis are undirected, we refer to the elements of a given graph as nodes and arcs to distinguish the elements of given graphs from those of given simplicial surfaces. Thus, let $\mathcal{G} = (V, E)$ be a graph. We define the automorphism group $\text{Aut}(\mathcal{G})$ of $\mathcal{G}$ as the set of all automorphisms of $\mathcal{G}$, i.e. the set of all bijective functions $\sigma : V \rightarrow V$ that preserve the incidences between the nodes and arcs of $\mathcal{G}$. Further, for*

an arc $e = \{v_1, v_2\} \in E$ we define $\sigma(e) := \{\sigma(v_1), \sigma(v_2)\}$. The group $\text{Aut}(\mathcal{G})$ acts on the nodes and arcs of $\mathcal{G}$ via

$$\text{Aut}(\mathcal{G}) \times (V \cup E) \rightarrow (V \cup E), (\sigma, x) \mapsto \sigma(x).$$

We denote the orbit of an $x \in V \cup E$ under a subgroup $H \leq \text{Aut}(\mathcal{G})$ by $\text{orb}_H(x)$. Furthermore, for a subset $S \subseteq V \cup E$ we define $\text{orb}_H(S) := \{\text{orb}_H(x) \mid x \in S\}$.

Definition 1.2.2

Let X be a simplicial surface. The **face graph** $\mathcal{F}(X)$ of X is defined as the graph $\mathcal{F}(X) = (V, E)$ with nodes $V = X_2$ and arcs $E = X_1$. Two nodes $F_1, F_2 \in V$ are connected in $\mathcal{F}(X)$ via an arc $e \in E$ if and only if F_1, F_2 interpreted as faces and e interpreted as an edge of X satisfy $e \in X_1(F_1) \cap X_1(F_2)$.

Since simplicial surfaces consist of faces that are incident to three edges, the resulting face graphs are cubic. We see that boundary edges of a simplicial surfaces translate into loops of the corresponding face graph. Hence, the face graph $\mathcal{F}(X)$ of a simplicial surface X is loop-less if and only if X is closed. Additionally, $\mathcal{F}(X)$ has no multi-arcs if and only if X has no vertices of degree 1 or inner vertices of degree 2. These observations establish that $\mathcal{F}(X)$ is simple if and only if X is closed and all vertex-degrees are greater than 2. The face graphs of the simplicial One-Face, simplicial Janus-Head and simplicial tetrahedron are illustrated in Figure 1.5.

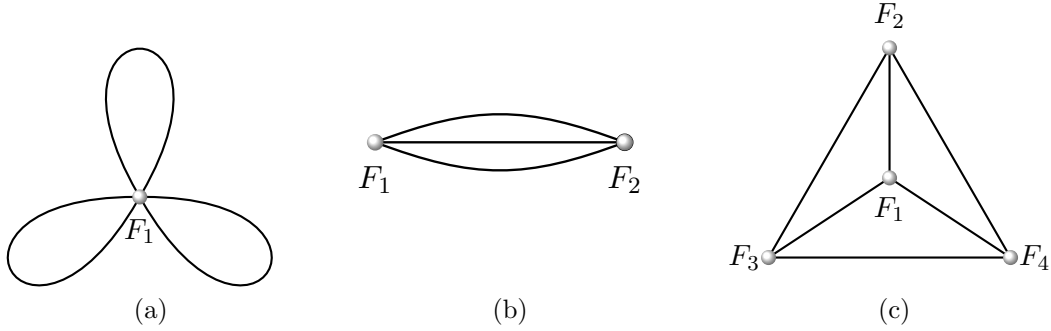


Figure 1.5: The face graphs of the simplicial One-Face (a), the simplicial Janus-Head (b) and the simplicial tetrahedron (c)

We refer the reader to [13, 39, 95] for some works that can be related to the study of face graphs of simplicial surfaces. Since we are mainly interested in the face graphs of closed simplicial surfaces that do not contain vertices of degree 2, we make the following assumption.

Remark 1.2.3 If $\mathcal{G}$ is an arbitrary cubic graph that is not associated to a simplicial surface, i.e. it is not precisely stated that $\mathcal{G}$ forms the face graph of some simplicial surface, then we assume $\mathcal{G}$ to be a simple graph. However, if $\mathcal{G}$ is known to be the face graph of some simplicial surface X , then X determines whether $\mathcal{G}$ is simple or not.

An essential tool for studying cubic graphs is an arc-colouring of a cubic graph.

Definition 1.2.4

Let $\mathcal{G} = (V, E)$ be a cubic graph. We call a map $\kappa : E \rightarrow \{1, 2, 3\}$ an **arc-colouring** of $\mathcal{G}$. Furthermore, we call κ an **arc-3-colouring**, if for every node $v \in V$ the arcs that are incident to v are all coloured differently.

As described above, the face graph of a simplicial surface represents a partial incidence structure of the corresponding simplicial surface: The faces and edges of a simplicial surface X translate directly into the nodes and arcs of the corresponding face graph $\mathcal{F}(X)$. For the translation of the vertices of the simplicial surface additional data is necessary. When dealing with a closed simplicial surface, the vertices of the simplicial surface or more precisely their umbrellas can be realised in the corresponding face graph as a *cycle double cover*.

Definition 1.2.5 [88]

Let $\mathcal{G} = (V, E)$ be a cubic graph and $\mathcal{C}$ a set of cycles of $\mathcal{G}$. We call $\mathcal{C}$ a **cycle double cover** of $\mathcal{G}$, if every arc in $\mathcal{G}$ is contained in exactly two cycles in $\mathcal{C}$. We say that $\mathcal{C}$ is **vertex-faithful** if any two cycles of $\mathcal{C}$ intersect in at most one arc.

Hence, in order to associate a cubic graph to a (vertex-faithful) simplicial surface, we have to construct a (vertex-faithful) cycle double cover of the given graph. Note that the notion of a cycle double cover of a (cubic) graph has been introduced in [81, 88].

If X is a closed simplicial surface and $\mathcal{F}(X)$ its face graph, we call a cycle C in $\mathcal{F}(X)$ that describes an umbrella of a vertex of X a **vertex-defining cycle** of X . Thus, a simplicial surface X is uniquely determined by $\mathcal{F}(X)$ together with the cycle double cover that contains all vertex-defining cycles of X .

We notice that a given cubic graph has to be bridgeless in order to have a cycle double cover. However, even for bridgeless cubic graphs the existence of cycle double covers is not established. This instance is formulated in [81, 88] as the **cycle double cover conjecture**.

Conjecture 1.2.6 [81, 88]

Every bridgeless (cubic) graph has a cycle double cover.

There are many conjectures that arise from altering the above conjecture. An overview of open problems that are related to the cycle double cover conjecture can be found in [59, 96]. For more investigations dealing with cycle double covers of cubic graphs, we refer the reader to [39, 47, 64, 69, 95].

To exemplify the definition of a cycle double cover, we examine the face graph of the simplicial tetrahedron $\mathcal{T}$, see Figure 1.5c. Here, we see that the graph

$$\mathcal{F}(\mathcal{T}) = (\{F_1, F_2, F_3, F_4\}, \{\{F_1, F_2\}, \{F_1, F_3\}, \{F_1, F_4\}, \{F_2, F_3\}, \{F_2, F_4\}, \{F_3, F_4\}\})$$

has exactly two cycle double covers. These cycle double covers, see Figure 1.6, are given by

$$\begin{aligned} \mathcal{C}_1 &:= \{(F_1, F_2, F_3), (F_1, F_2, F_4), (F_1, F_3, F_4), (F_2, F_3, F_4)\}, \\ \mathcal{C}_2 &:= \{(F_1, F_2, F_3, F_4), (F_1, F_3, F_2, F_4), (F_1, F_2, F_4, F_3)\}. \end{aligned}$$

Here, the cycle double cover $\mathcal{C}_1$ describes the vertex-defining cycles of the simplicial tetrahedron $\mathcal{T}$. Further, the cycle double cover $\mathcal{C}_2$ gives rise to a closed simplicial surface X with vertex counter V_4^3 and Euler characteristic 1. It can be verified that X is not orientable and thus X forms a simplicial projective plane. Since $\mathcal{C}_2$ contains pairs of cycles that intersect in more than one arc, we notice that X is not vertex-faithful.

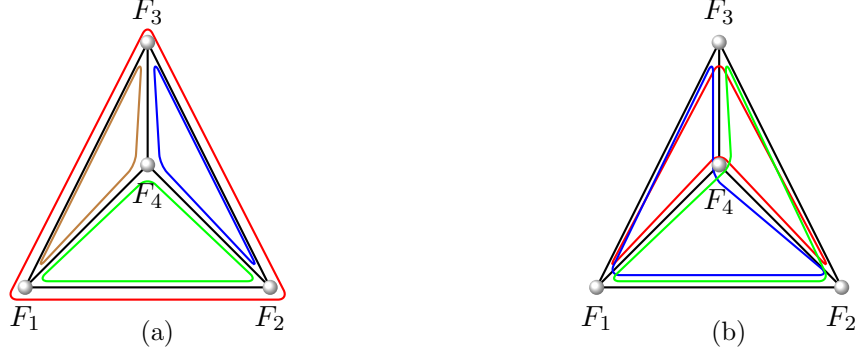


Figure 1.6: Two different cycle double covers of the face graph $\mathcal{F}(\mathcal{T})$

In the following we comment on the existence of vertex-faithful cycle double covers of cubic graphs. In particular, we notice that there are bridgeless cubic graphs that do not give rise to vertex-faithful simplicial surfaces. The cubic graph $\mathcal{G}$ (shown in Figure 1.7) is an example of such a graph. We notice that this graph is the generalised Petersen graph $G(9, 3)$, see [45]. By using the functionalities provided in [72], we have verified that this cubic graph has exactly 140 cycle double covers.

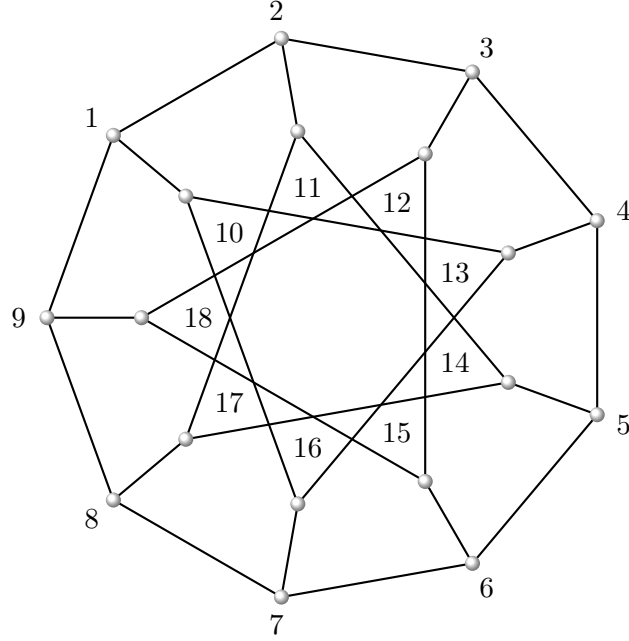


Figure 1.7: The generalised Petersen graph $G(9, 3)$

Moreover, if $\mathcal{C}$ is a cycle double cover of $\mathcal{G}$, then $\mathcal{C}$ satisfies $4 \leq |\mathcal{C}| \leq 8$. Below, we give an example of a cycle double cover $\mathcal{C}'$ of $\mathcal{G}$ consisting of four cycles.

$$\begin{aligned} \mathcal{C}' = \{ & (1, 9, 8, 7, 6, 5, 4, 3, 2), (1, 10, 13, 16, 7, 6, 15, 12, 18, 9, 8, 17, 14, 11, 2), \\ & (3, 12, 18, 15, 6, 5, 14, 11, 17, 8, 7, 16, 10, 13, 4), \\ & (1, 10, 16, 13, 4, 5, 14, 17, 11, 2, 3, 12, 15, 18, 9) \}. \end{aligned}$$

Take a moment to verify that any two cycles of the cycle double cover $\mathcal{C}'$ intersect in more than

one arc. The simplicial surface X with $\mathcal{F}(X) = \mathcal{G}$ that arises from the cycle double cover $\mathcal{C}'$ has 4 vertices, 27 edges and 18 faces. Thus, X satisfies $\chi(X) = -5$. With GAP, we have verified that X is non-orientable with

$$S_3 \cong \text{Aut}(X) \hookrightarrow \text{Aut}(\mathcal{F}(X)) \cong D_{2,9}.$$

We conclude this section with a note on the construction of cycle double covers for given cubic graphs.

Remark 1.2.7 *Since the existence of a cycle double cover of an arbitrary bridgeless cubic graph $\mathcal{G}$ is still an open problem, see Conjecture 1.2.6, in general it is not clear how to construct such a collection of cycles covering every arc of $\mathcal{G}$ exactly twice. Enforcing some properties on $\mathcal{G}$ leads to results on the construction of cycle double covers of $\mathcal{G}$. For instance, Szekeres shows in [88] that if $\mathcal{G}$ has an arc-3-colouring, then a cycle double cover of $\mathcal{G}$ is given by the set of bi-coloured cycles, see Figure 1.8 for illustration.*

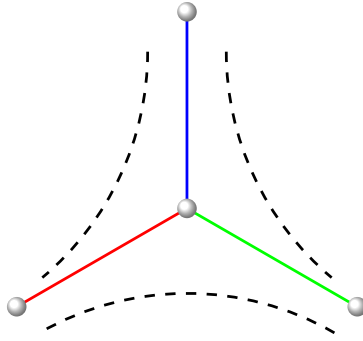


Figure 1.8: A subgraph of an arc-3-coloured cubic graph with bi-coloured cycles depicted by dotted lines

As an example, we consider the face graph of the simplicial tetrahedron $\mathcal{T}$ illustrated in Figure 1.5c. As described above, $\mathcal{F}(\mathcal{T})$ is determined by

$$\mathcal{F}(\mathcal{T}) = (V, E) = (\{F_1, F_2, F_3, F_4\}, \{\{F_1, F_2\}, \{F_1, F_3\}, \{F_1, F_4\}, \{F_2, F_3\}, \{F_2, F_4\}, \{F_3, F_4\}\}).$$

Thus, the arc-3-colouring $\kappa : E \rightarrow \{1, 2, 3\}$ satisfying

$$\begin{aligned} \kappa(\{F_1, F_2\}) &= \kappa(\{F_3, F_4\}) = 1, \\ \kappa(\{F_1, F_3\}) &= \kappa(\{F_2, F_4\}) = 2, \\ \kappa(\{F_1, F_4\}) &= \kappa(\{F_2, F_3\}) = 3 \end{aligned}$$

gives rise to the cycle double cover $\mathcal{C}_2$ of $\mathcal{F}(\mathcal{T})$ illustrated in Figure 1.6b. Further, if $\mathcal{G}$ is a planar graph, then a cycle double cover of $\mathcal{G}$ can be constructed from the faces of $\mathcal{G}$ resulting from a planar embedding of $\mathcal{G}$. For instance, the cycle double cover $\mathcal{C}_1$ of the face graph $\mathcal{F}(\mathcal{T})$ (see Figure 1.6a) is derived by computing the faces of the illustrated planar embedding of $\mathcal{F}(\mathcal{T})$. Moreover, Tarsi and Goddyn make use of Hamiltonian paths in bridgeless (multi) graphs to construct corresponding cycle double covers consisting of six cycles, see [48, 89].

In Sections 5 and 6 we demonstrate that if $\mathcal{G}$ is assumed to be highly symmetric, then the automorphism group of $\mathcal{G}$ can be helpful for the construction of a corresponding cycle double cover.

1.3 Colourings of simplicial surfaces

We present colourings of a simplicial surface that can be used to enhance its combinatorial structure. In particular, we present vertex- k -colourings, where $k \in \mathbb{N}$, and Grünbaum-colourings of simplicial surfaces, see Definitions 1.3.1 and 1.3.4. These colourings (for $k = 3$) form maps on the vertices and edges of a simplicial surface whose restrictions to the vertices and edges of any face yield bijections.

Definition 1.3.1

Let X be a simplicial surface and $k \in \mathbb{N}$ a natural number. A map $\kappa : X_0 \rightarrow \{1, \dots, k\}$ is called a **vertex-colouring** of X . Furthermore, κ is called a **vertex- k -colouring** of X if κ is surjective and every edge $e \in X_1$ with $X_0(e) = \{v, v'\}$ satisfies $\kappa(v) \neq \kappa(v')$. We say that κ is **proper**, if $k \leq k'$ holds for every vertex- k' -colouring of X .

It is easy to see that every simplicial surface X has a vertex- $|X_0|$ -colouring. Furthermore, we observe that a vertex- k -colouring of a simplicial surface can be translated into a node-colouring of the corresponding *vertex-edge graph*, i.e. the graph that arises from the incidences between the vertices and edges of X .

Definition 1.3.2

Let X be a simplicial surface. The **vertex-edge graph** $\mathcal{V}(X)$ of X is defined as the graph $\mathcal{V}(X) = (V, E)$ with nodes $V = X_0$ and arcs $E = X_1$. Two nodes $v_1, v_2 \in V$ are connected in $\mathcal{V}(X)$ via an arc $e \in E$ if and only if v_1, v_2 interpreted as vertices and e interpreted as an edge of X satisfy $X_0(e) = \{v_1, v_2\}$.

Figure 1.9 illustrates the resulting vertex-edge graphs of the simplicial One-Face (see Figure 1.1a) and the simplicial tetrahedron (see Figure 1.3b).



Figure 1.9: The vertex-edge graphs of the simplicial One-Face (a) and the simplicial tetrahedron (b)

Note that the graphs $\mathcal{V}(\mathcal{T})$ and $\mathcal{F}(\mathcal{T})$, where $\mathcal{T}$ is the simplicial tetrahedron, form complete graphs on four nodes and hence are isomorphic. Moreover, we know that the vertex-edge graph of a simplicial surface X is simple if and only if X is vertex-faithful.

We observe that a face of a simplicial surface X induces a clique of size 3 in the corresponding *vertex-edge graph*. Hence, for every $k \in \mathbb{N}$ with the property that there exists a vertex- k -colouring of X , we know that $k \geq 3$ follows. Thus, if X has a vertex-3-colouring, then this colouring must be proper. In Section 4 we will illustrate various examples of simplicial surfaces that have vertex-3-colourings. In general, it is a computationally hard task to find a proper vertex- k -colouring of a given simplicial surface (see [46]). In the case that such a proper vertex- k -colouring of a simplicial surface X exists, the number k corresponds to the chromatic number of the vertex-edge graph of X .

Although it is challenging to compute the chromatic number of the vertex-edge graph of X , various upper bounds for this number have been established in graph-theoretical studies. In Remark 1.3.3 we address some of these upper bounds. We see that such bounds for the chromatic number of the vertex-edge graph of X can be translated into upper bounds for the number of colours of a proper vertex- k -colouring of X .

Remark 1.3.3 *Let X be a closed simplicial surface and κ a proper vertex- k -colouring of X . If X is a simplicial sphere, the vertex-edge graph of X is planar and we deduce by the Four-Colour-Theorem (see [10]) that $k = 3$ or $k = 4$. Furthermore, if X is a simplicial torus, we know by [79] that a proper vertex- k -colouring of X consists of at most 7 colours, i.e. $k \leq 7$. In particular, the Ringel-Young Theorem established in [79] (which proves the Heawood-conjecture formulated in 1890) gives us an upper bound for the number k with respect to the Euler characteristic of X . By this theorem we know that the following inequality holds:*

$$k \leq \left\lfloor \frac{7 + \sqrt{49 - 24\chi(X)}}{2} \right\rfloor.$$

In Section 4.1 we use the above facts to prove that there is exactly one simplicial sphere X with the property that every simplicial surface that forms an epimorphic image of X is already isomorphic to X , namely the simplicial tetrahedron $\mathcal{T}$. Now, we present the notion of a Grünbaum-colouring.

Definition 1.3.4 [54]

Let X be a simplicial surface. A map $\omega : X_1 \rightarrow \{1, 2, 3\}$ is called an **edge-colouring**. We say that ω is a **Grünbaum-colouring** of X , if the restriction of ω to the edges of any face of X yields a bijection.

Grünbaum-colourings were first introduced in [54]. In [74], Niemeyer et al. refer to a colouring, as described in Definition 1.3.4, as a *wild-colouring*. In their work, these wild-colourings are extensively studied in a group-theoretical setting. Moreover, we observe that a Grünbaum-colouring of a closed simplicial surface X can be translated into an arc-3-colouring of the underlying face graph $\mathcal{F}(X)$. This allows us to study Grünbaum-coloured simplicial surfaces from a graph-theoretical perspective.

For simplicity, we restrict ourselves to defining colourings of simplicial surfaces by illustrating the corresponding colours in figures. In particular, we represent the colours 1, 2, 3 of a Grünbaum-coloured and the colours 1, ..., 4 of a vertex-4-coloured simplicial surface with the colours red, green, blue and orange, respectively. In order to illustrate Definitions 1.3.1 and 1.3.4, we consider the simplicial tetrahedron $\mathcal{T}$ and construct corresponding colourings.

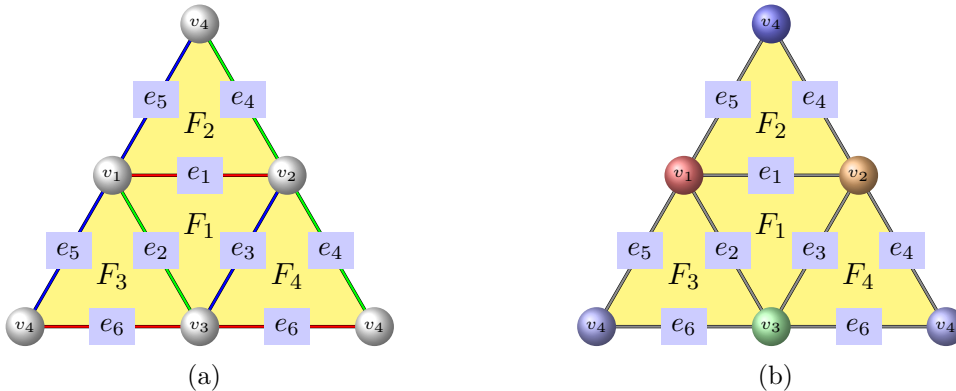


Figure 1.10: A Grünbaum-colouring (a) and a vertex-4-colouring (b) of the simplicial tetrahedron $\mathcal{T}$

We obtain a Grünbaum-colouring of $\mathcal{T}$ by colouring the edges of $\mathcal{T}$ as illustrated in Figure 1.10a. Moreover, a proper vertex-4-colouring of $\mathcal{T}$ is given in Figure 1.10b. Up to permutation of colours, the simplicial tetrahedron $\mathcal{T}$ has exactly one Grünbaum-colouring and exactly one vertex-4-colouring. This is due to the fact that the vertex-edge graph $\mathcal{V}(\mathcal{T})$ and the face graph $\mathcal{F}(\mathcal{T})$ form complete graphs on four nodes.

A Grünbaum-colouring of a simplicial surface gives rise to exactly two different types of edges, namely *mirror* and *rotational* edges. These edges, arising from a Grünbaum-colouring, are described in the following definition.

Definition 1.3.5

Let X be a simplicial surface, $\omega : X_1 \rightarrow \{1, 2, 3\}$ a Grünbaum-colouring and $e \in X_1$ an inner edge with $X_2(e) = \{F_1, F_2\}$. Further, let $e_1, e_2 \in X_1$ be edges with $e_1 \in X_1(F_1) \setminus \{e\}$, $e_2 \in X_1(F_2) \setminus \{e\}$ and $X_0(e) \cap X_0(e_1) \cap X_0(e_2) \neq \emptyset$. If $\omega(e_1) \neq \omega(e_2)$, then we say that e is a **rotational edge**, see Figure 1.11a. Otherwise, we define e to be a **mirror edge**, see Figure 1.11b.



Figure 1.11: A rotational edge (a) and a mirror edge (b) of a Grünbaum-coloured simplicial surface

Although a Grünbaum-colouring of a simplicial surface forms a combinatorial property of the simplicial surface, it can be interpreted geometrically. We comment on this in the following. For this, we define Λ as the set of all triplets of positive real numbers satisfying the triangle inequalities, i.e. if $(\ell_1, \ell_2, \ell_3) \in \Lambda$, then there exists a triangle with edge lengths ℓ_1 , ℓ_2 , and ℓ_3 , see [4].

Let X be a vertex-faithful simplicial surface that is equipped with a Grünbaum-colouring ω . A natural question that arises is whether X can be translated into a triangulated surface in Euclidean 3-space whose faces all form congruent triangles. To answer this question, we have to assign lengths to the edges of X such that all edges with the same colour under ω have the same edge length. Thus, we have to compute a (ℓ_1, ℓ_2, ℓ_3) -embedding of X with $(\ell_1, \ell_2, \ell_3) \in \Lambda$.

Definition 1.3.6 (see [4, 74])

Let X be a vertex-faithful simplicial surface and $\omega : X_1 \rightarrow \{1, 2, 3\}$ a Grünbaum-colouring of X . An (ℓ_1, ℓ_2, ℓ_3) -**embedding** of X with $(\ell_1, \ell_2, \ell_3) \in \Lambda$ is a map $\phi : X_0 \rightarrow \mathbb{R}^3$ such that any pair of vertices $v_1, v_2 \in X_0$ with $\{v_1, v_2\} \in X_0(X_1)$ satisfies the distance equation

$$\|\phi(v_1) - \phi(v_2)\| = \ell_i,$$

where the edge $e \in X_1$ with $X_0(e) = \{v_1, v_2\}$ satisfies $\omega(e) = i$. We denote the three-dimensional body $\bigcup_{m \in \Theta(X)} \text{conv}(\phi(m))$ that results from X together with ϕ by $\phi(X)$ and call it the **embedded simplicial surface**.

If X is a vertex-faithful simplicial sphere and ϕ an (ℓ_1, ℓ_2, ℓ_3) -embedding of X , then the embedded simplicial surface $\phi(X)$ can be interpreted as a polyhedron in Euclidean 3-space consisting of triangles that all have edge lengths $(\ell_1, \ell_2, \ell_3) \in \Lambda$.

In general, one has to solve a system of multivariate quadratic equations to construct an embedding of a given Grünbaum-coloured simplicial surface. Hence, constructing embeddings of given simplicial surfaces by finding solutions for such systems can be a difficult task. In [22], Brakhage et al. investigate such a system of polynomial equations and construct all injective $(1, 1, 1)$ -embeddings of the *simplicial icosahedron*, i.e. the simplicial surface that describes the incidence structure of the regular icosahedron (see Section 2), such that the symmetry groups (see Definition 1.3.7) of the resulting polyhedra are non-trivial.

Definition 1.3.7

Let X be a vertex-faithful simplicial surface and $\phi : X_0 \rightarrow \mathbb{R}^3$ an (ℓ_1, ℓ_2, ℓ_3) -embedding of X with $(\ell_1, \ell_2, \ell_3) \in \Lambda$. The **symmetry group** $\text{Aut}(\phi(X))$ is defined as the group of all isometries $\pi \in \text{E}(3)$ that leave $\phi(X)$ invariant, i.e. $\pi(\phi(X)) = \phi(X)$ for $\pi \in \text{Aut}(\phi(X))$.

If X is a vertex-faithful simplicial surface and ϕ an (ℓ_1, ℓ_2, ℓ_3) -embedding of X , then

$$\text{Aut}(\phi(X)) \hookrightarrow \text{Aut}(X) \hookrightarrow \text{Aut}(\mathcal{F}(X)).$$

Moreover, if $\sum_{v \in X_0} \phi(v) = (0, 0, 0)^t$, then $\text{Aut}(\phi(X))$ is a subgroup of the orthogonal group $\text{O}(3)$. Figure 1.12 shows various views of three different polyhedra that result from constructing injective $(1, 1, 1)$ -embeddings of the simplicial icosahedron.

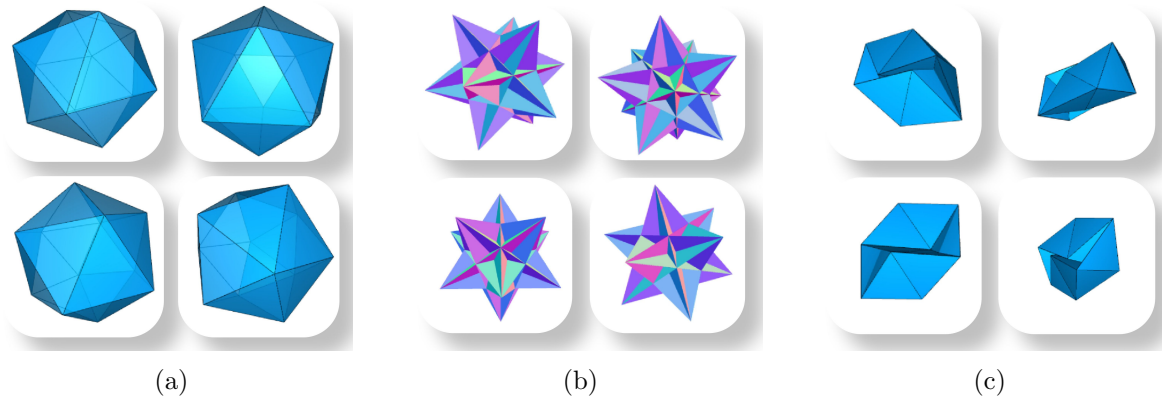


Figure 1.12: Various views of different polyhedra corresponding to the simplicial icosahedron: the regular icosahedron (a), the great icosahedron [33] (b) and an icosahedron with symmetry group of order 2 [22] (c); see [2]

By Miller, the simplicial icosahedron has exactly one Grünbaum-colouring, see [67]. Furthermore, Miller classifies the (ℓ_1, ℓ_2, ℓ_3) -embeddings of the Grünbaum-coloured simplicial icosahedron, where $(\ell_1, \ell_2, \ell_3) \in \Lambda$ form the edge lengths of a scalene triangle. We refer the reader to [2, 4, 22, 31, 40, 65, 68, 93] for more studies on embeddings of simplicial surfaces.

Due to Niemeyer et al. we have the following conjecture:

Conjecture 1.3.8 (see [73])

For every simplicial sphere there exists an injective $(1, 1, 1)$ -embedding.

A natural question that arises is whether the above conjecture can be formulated for closed simplicial surfaces of arbitrary genus. This generalisation turns out to be false. Simplicial surfaces that do not have injective $(1, 1, 1)$ -embeddings can be derived from [35, 38, 65, 85] for instance. In [2] this question is answered by stating that a simplicial surface whose vertex-edge graph contains a clique of size greater than four cannot be embedded into $\mathbb{R}^3$ via an injective $(1, 1, 1)$ -embedding.

Remark 1.3.9 (see [2]) *Let X be a simplicial surface such that its vertex-edge graph $\mathcal{V}(X) = (V, E)$ contains a clique $C \subseteq V = X_0$ of size $n \geq 5$. In this case there exists no injective $(1, 1, 1)$ -embedding of X . This can be verified as described in the following: Let us assume that there exists an injective $(1, 1, 1)$ -embedding ϕ of X . Furthermore, let $v_1, \dots, v_5 \in X_0$ be vertices such that $v_1, \dots, v_5$ interpreted as nodes of $\mathcal{V}(X)$ satisfy $v_1, \dots, v_5 \in C$. Since $\phi(v_1), \phi(v_2), \phi(v_3)$ satisfy*

$$\|\phi(v_1) - \phi(v_2)\| = \|\phi(v_1) - \phi(v_3)\| = \|\phi(v_2) - \phi(v_3)\| = 1,$$

these coordinates give rise to an equilateral triangle with edge lengths 1 in $\mathbb{R}^3$. Without loss of generality, we assume that this triangle is given by

$$(\phi(v_1), \phi(v_2), \phi(v_3)) = \left((0, 0, 0)^t, (1, 0, 0)^t, \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)^t \right).$$

Now, the points $\phi(v_4)$ and $\phi(v_5)$ have to be on the intersection of the spheres $S_1, S_2, S_3 \subset \mathbb{R}^3$ with

$$\begin{aligned} S_1 &:= \{x \in \mathbb{R}^3 \mid \|x - \phi(v_1)\| = 1\}, \\ S_2 &:= \{x \in \mathbb{R}^3 \mid \|x - \phi(v_2)\| = 1\}, \\ S_3 &:= \{x \in \mathbb{R}^3 \mid \|x - \phi(v_3)\| = 1\}. \end{aligned}$$

Since the intersection $S_1 \cap S_2 \cap S_3$ equals

$$\left\{ \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}} \right)^t, \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, -\sqrt{\frac{2}{3}} \right)^t \right\}$$

and $\phi(v_4) \neq \phi(v_5)$ (ϕ is injective), we can assign $\phi(v_4)$ to the first and $\phi(v_5)$ to the second coordinate in the above set. This implies $\|\phi(v_4) - \phi(v_5)\| = 2\sqrt{\frac{2}{3}}$ which contradicts ϕ being a $(1, 1, 1)$ -embedding.

As an example of a simplicial surface that does not have an injective $(1, 1, 1)$ -embedding, we consider the simplicial surface X that is described by the following vertices of faces:

$$\begin{aligned} \Theta(X) &:= \{\{v_3, v_5, v_6\}, \{v_3, v_6, v_7\}, \{v_3, v_4, v_7\}, \{v_2, v_5, v_7\}, \{v_2, v_3, v_5\}, \{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}, \\ &\quad \{v_1, v_4, v_5\}, \{v_4, v_5, v_6\}, \{v_2, v_4, v_7\}, \{v_2, v_4, v_6\}, \{v_1, v_2, v_6\}, \{v_1, v_6, v_7\}, \{v_1, v_5, v_7\}\}. \end{aligned}$$

We see that X consists of 7 vertices, 21 edges and 14 faces and hence satisfies $\chi(X) = 0$. Since X is closed, connected and orientable, it forms an example of a simplicial torus. In fact, in the literature, this simplicial surface is also known as the Möbius torus, i.e. the minimal triangulation of the torus, see [34]. We observe $\deg(v) = 6$ for all $v \in X_0$ and hence $\mathcal{V}(X)$ is isomorphic to the complete graph on 7 nodes. Although this simplicial surface cannot be embedded into Euclidean 3-space via an injective $(1, 1, 1)$ -embedding, its combinatorial structure can be visualised in a folding plan, see Figure 1.13.

Since the vertex-edge graph of a simplicial sphere X cannot contain a clique of size 5 or greater ($\mathcal{V}(X)$ is planar), Conjecture 1.3.8 remains open.

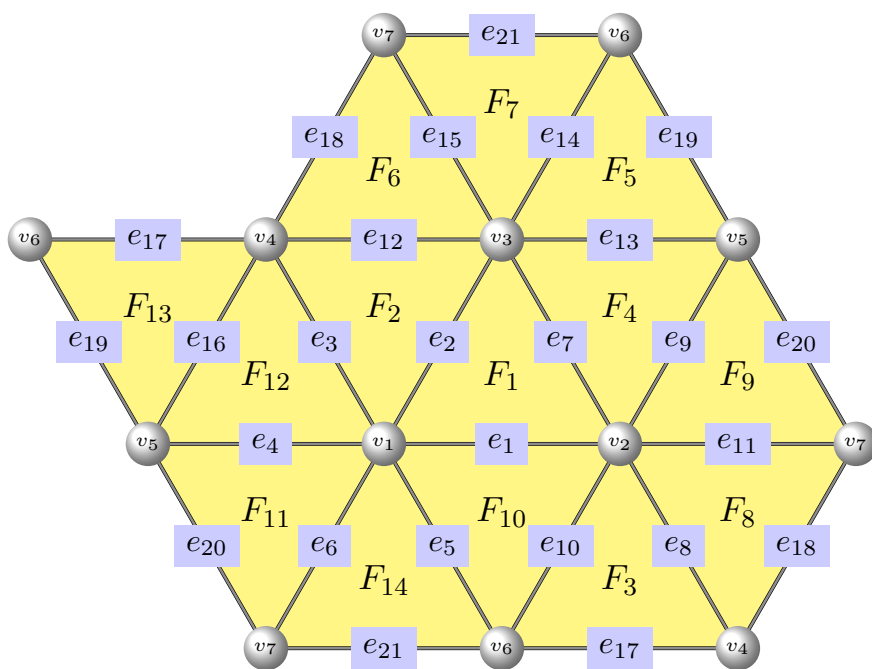


Figure 1.13: The simplicial surface X forming a minimal triangulation of the torus [34]

2 Families of simplicial surfaces

Here, we present examples of simplicial surfaces that are particularly relevant to this thesis, to enhance the understanding of these combinatorial structures. The examples presented in this section include simplicial surfaces derived from well-known polyhedra such as the tetrahedron, octahedron, and icosahedron. Additionally, we introduce more complex and less familiar simplicial surfaces to demonstrate the richness of the theory of simplicial surfaces. The simplicial surfaces presented in this section can be found in [74].

2.1 Closed simplicial surfaces

We start this subsection by introducing examples of closed simplicial surfaces. Here, we illustrate that the Platonic solids can be exploited to construct closed vertex-faithful simplicial surfaces, see Section 2.1.1. Moreover, we present the simplicial double-umbrellas and the generalised Janus-Heads as further examples, see Sections 2.1.2 and 2.1.3.

2.1.1 The Platonic solids

The Platonic solids, named after the ancient Greek philosopher Plato, have fascinated mathematicians and artists for centuries due to their high symmetry and aesthetic beauty. In this work, these three-dimensional shapes also play a significant role and hence appear in various contexts. There are exactly five such solids: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron, see Figure 2.1. These are exactly the convex polyhedra in $\mathbb{R}^3$ whose faces all form congruent regular polygons.

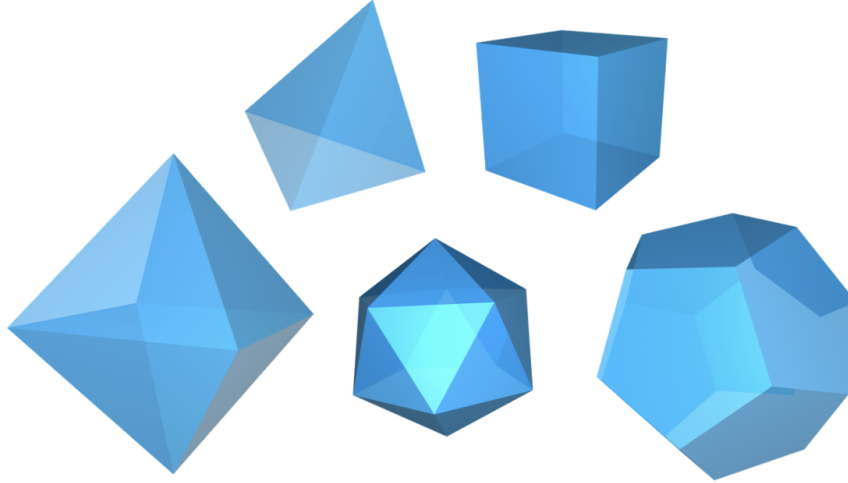


Figure 2.1: The five Platonic Solids, see [7]

Three of these solids, namely the tetrahedron, the octahedron, and the icosahedron, consist of equilateral triangles as faces and can therefore directly be translated into simplicial surfaces that describe their incidence geometry. By subdividing the faces of the dodecahedron and the cube into congruent triangles, we obtain simplicial surfaces that allow us to analyse the incidence structure of these polyhedra. The simplicial surfaces arising from the Platonic solids are detailed below.

The simplicial tetrahedron

The *simplicial tetrahedron* $\mathcal{T}$ has been introduced in Section 1. Since every vertex of $\mathcal{T}$ has vertex-degree 3, the vertex-counter of $\mathcal{T}$ is given by V_3^4 . Moreover, the automorphism groups of $\mathcal{T}$ and its face graph satisfy $\text{Aut}(\mathcal{T}) \cong \text{Aut}(\mathcal{F}(\mathcal{T})) \cong S_4$.

The simplicial octahedron

Next, we introduce the *simplicial octahedron* $\mathcal{O}$, which describes the incidence structure of a regular octahedron. Thus, $\mathcal{O}$ is vertex-faithful and consists of 6 vertices, 12 edges and 8 faces. If the vertices of the simplicial surface $\mathcal{O}$ are given by $v_1, \dots, v_6$, then the entire incidence structure of $\mathcal{O}$ is determined by the following vertices of faces:

$$\Theta(\mathcal{O}) := \{\{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}, \{v_1, v_4, v_5\}, \{v_1, v_2, v_5\}, \\ \{v_2, v_3, v_6\}, \{v_2, v_5, v_6\}, \{v_3, v_4, v_6\}, \{v_4, v_5, v_6\}\}.$$

In Figure 2.2a we present a folding plan of the simplicial surface $\mathcal{O}$. Moreover, Figure 2.2b shows the corresponding face graph $\mathcal{F}(\mathcal{O})$.

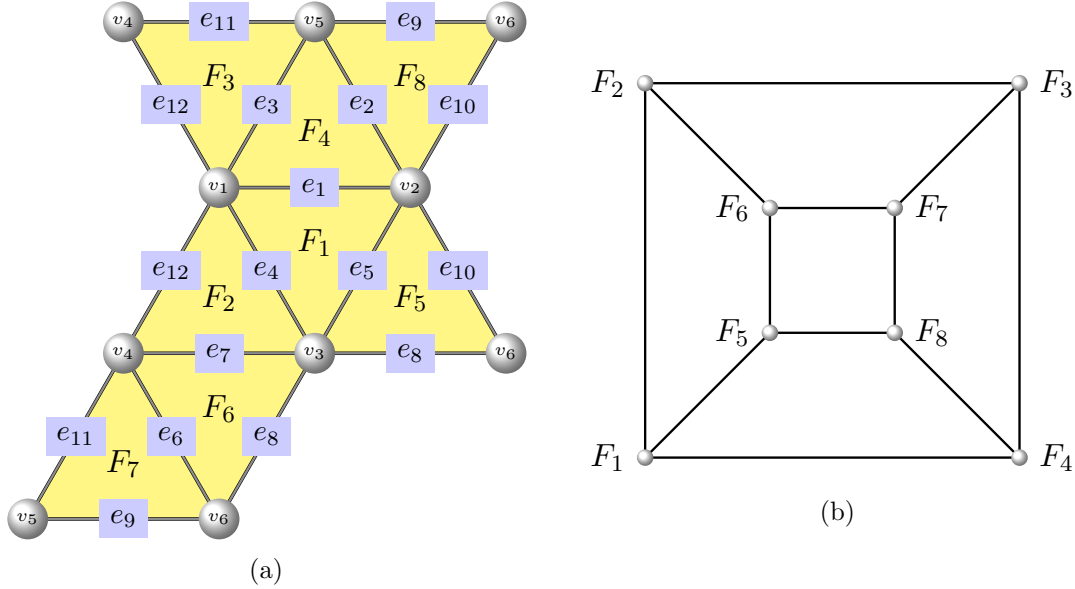


Figure 2.2: The simplicial octahedron $\mathcal{O}$ (a) and its face graph $\mathcal{F}(\mathcal{O})$ (b)

Since every vertex of the simplicial octahedron is incident to exactly four faces, we know that the vertex-counter of $\mathcal{O}$ is given by V_4^6 . Furthermore, this simplicial surface satisfies the statement $\text{Aut}(\mathcal{O}) \cong \text{Aut}(\mathcal{F}(\mathcal{O})) \cong C_2 \times S_4$.

2.1.1.1 The simplicial icosahedron

The *simplicial icosahedron* $\mathcal{I}$ is the simplicial surface that describes the incidence structure of the vertices, edges and triangular faces of a regular icosahedron. Hence, this simplicial surface is vertex-faithful and has 12 vertices that all have vertex-degree 5. The vertex-counter of $\mathcal{I}$ is given by V_5^{12} . Because of $\mathcal{I}$ being vertex-faithful, the entire incidence structure of this simplicial surface is determined by the vertices of faces

$$\begin{aligned}\Theta(\mathcal{I}) := & \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_4, v_5\}, \{v_1, v_5, v_6\}, \{v_1, v_3, v_6\}, \\ & \{v_2, v_3, v_7\}, \{v_2, v_4, v_8\}, \{v_4, v_5, v_9\}, \{v_5, v_6, v_{10}\}, \{v_3, v_6, v_{11}\}, \\ & \{v_2, v_7, v_8\}, \{v_4, v_8, v_9\}, \{v_5, v_9, v_{10}\}, \{v_6, v_{10}, v_{11}\}, \{v_3, v_7, v_{11}\}, \\ & \{v_7, v_8, v_{12}\}, \{v_8, v_9, v_{12}\}, \{v_9, v_{10}, v_{12}\}, \{v_{10}, v_{11}, v_{12}\}, \{v_7, v_{11}, v_{12}\}\}.\end{aligned}$$

A folding plan illustrating the incidence structure of the simplicial icosahedron is presented in Figure 2.3. In this figure we have omitted the information on the edges for simplicity.

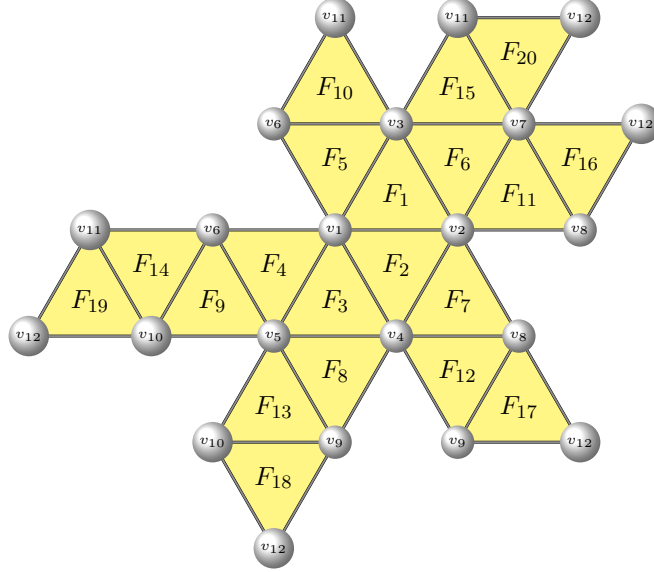


Figure 2.3: The simplicial icosahedron $\mathcal{I}$

The face graph of the simplicial icosahedron is a cubic graph that consists of 20 nodes and 30 arcs, see Figure 2.4. We notice that the automorphism group of the simplicial surface $\mathcal{I}$ and its face graph $\mathcal{F}(\mathcal{I})$ are isomorphic with the property that $\text{Aut}(\mathcal{I}) \cong \text{Aut}(\mathcal{F}(\mathcal{I})) \cong C_2 \times A_5$.

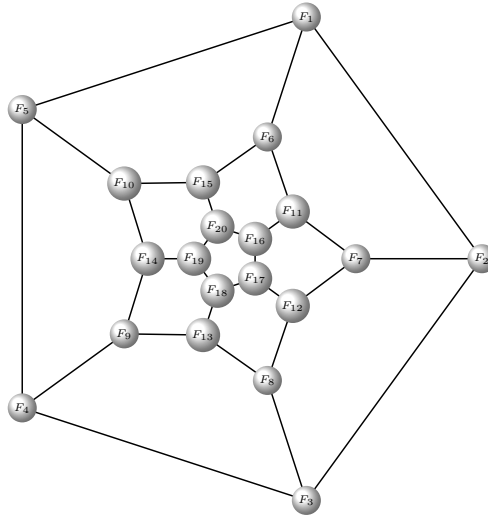


Figure 2.4: The face graph $\mathcal{F}(\mathcal{I})$ of the simplicial icosahedron $\mathcal{I}$

The simplicial cube

The cube is the Platonic solid that consists of 6 congruent squares as faces. Hence, this polyhedron does not directly yield a simplicial surface. However, we can modify this polyhedron to obtain a related polyhedron that consists of triangular faces and has the same symmetry group as the cube. This is achieved by subdividing each face of the cube into 4 triangles as indicated in Figure 2.5.

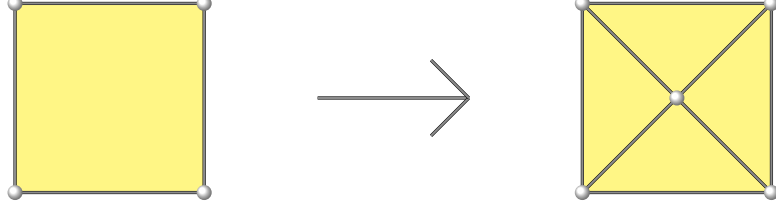


Figure 2.5: Subdividing a square into four triangles

Applying the above modification to the faces of the cube gives rise to the *simplicial cube*, i.e. the simplicial surface X given by the following vertices of faces:

$$\begin{aligned} \Theta(X) := & \{ \{v_1, v_2, v_9\}, \{v_2, v_3, v_9\}, \{v_3, v_4, v_9\}, \{v_1, v_4, v_9\}, \{v_1, v_2, v_{10}\}, \{v_2, v_6, v_{10}\}, \\ & \{v_1, v_5, v_{10}\}, \{v_5, v_6, v_{10}\}, \{v_2, v_3, v_{11}\}, \{v_2, v_6, v_{11}\}, \{v_3, v_7, v_{11}\}, \{v_6, v_7, v_{11}\}, \\ & \{v_1, v_4, v_{12}\}, \{v_4, v_8, v_{12}\}, \{v_1, v_5, v_{12}\}, \{v_5, v_8, v_{12}\}, \{v_3, v_4, v_{13}\}, \{v_3, v_7, v_{13}\}, \\ & \{v_4, v_8, v_{13}\}, \{v_7, v_8, v_{13}\}, \{v_6, v_7, v_{14}\}, \{v_7, v_8, v_{14}\}, \{v_5, v_8, v_{14}\}, \{v_5, v_6, v_{14}\} \}. \end{aligned}$$

It can be observed that this simplicial surface consists of $14(= 8 + 6)$ vertices, $36(= 12 + 4 \cdot 6)$ edges and $24(= 6 \cdot 4)$ faces. Thus, this simplicial surface forms a simplicial sphere that satisfies $\text{Aut}(X) \cong \text{Aut}(\mathcal{F}(X)) \cong C_2 \times S_4$. Further, we know that the vertex-counter of X is given by $V_4^6 V_6^8$. In the literature the operation of replacing every face of polyhedron by triangles as indicated in Figure 2.5 is known as the *kis operation*, see [30].

The simplicial dodecahedron

Finally, we present a simplicial surface that can be obtained by manipulating the incidence structure of a regular dodecahedron. The dodecahedron forms the dual of the icosahedron and hence consists of 20 vertices, 30 edges and 12 faces forming pentagons. As presented above, we can subdivide the faces of the dodecahedron by applying the kis operator to obtain a triangulation of this polyhedron that consists of congruent triangular faces, see Figure 2.6 for illustration.

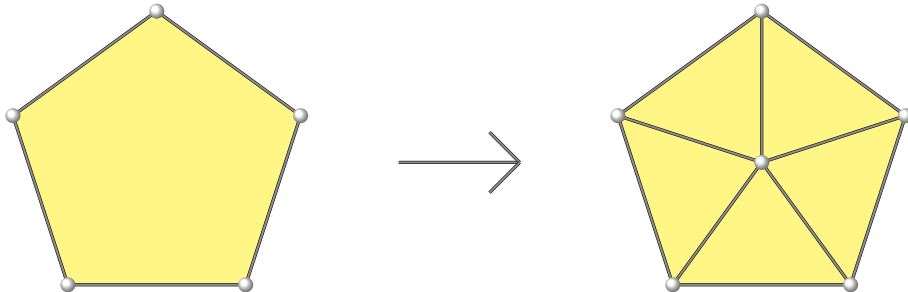


Figure 2.6: Subdividing a pentagon into five triangles

Hence, the simplicial surface that arises from this construction has $32(= 20 + 12)$ vertices, $90(= 30 + 12 \cdot 5)$ edges and $60(= 12 \cdot 5)$ faces. We refer to this simplicial surface as the ***simplicial dodecahedron***. In the following we define the incidence structure of this simplicial surface by a suitable group action. Let therefore $G \leq S_{32}$ be the finite group that is given by

$$G := \langle (3, 6)(4, 7)(5, 8)(11, 14)(12, 15)(13, 16)(17, 19)(18, 20)(21, 22)(23, 24)(25, 27)(26, 28), \\ (2, 4)(3, 5)(6, 19)(8, 17)(9, 16)(11, 18)(12, 20)(14, 15)(22, 29)(23, 32)(25, 30)(28, 31), \\ (1, 2)(4, 5)(7, 8)(9, 10)(12, 13)(15, 16)(17, 18)(19, 20)(25, 26)(27, 28)(29, 30)(31, 32) \rangle.$$

By exploiting this group, we obtain the set $M := \{\{\pi(1), \pi(2), \pi(21)\} \mid \pi \in G\}$ which enables us to construct the vertices of faces of the simplicial dodecahedron X via

$$\Theta(X) := \{\{v_i, v_j, v_k\} \mid \{i, j, k\} \in M\}.$$

Note that the vertex-counter of the simplicial dodecahedron is given by $V_5^{12}V_6^{20}$ and the statement $\text{Aut}(X) \cong \text{Aut}(\mathcal{F}(X)) \cong C_2 \times A_5$ holds.

2.1.2 The simplicial double-umbrellas

Next, we present the class of simplicial double-umbrellas. For $n \geq 3$, we introduce the ***simplicial double-umbrella*** $(n)^2$ as the simplicial surface that describes the incidence geometry of the n -gonal bi-pyramid, i.e. the polyhedron that results from combining two n -gonal pyramids such that the bases of both pyramids coincide, see Figure 2.7 for illustration.

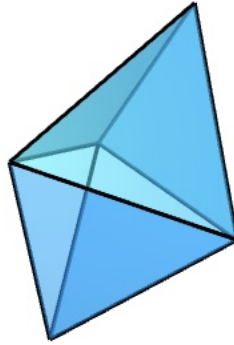


Figure 2.7: Bi-pyramid constructed from two triangular pyramids

Hence, the simplicial double-umbrella $X := (n)^2$ consists of $n + 2$ vertices, $3n$ edges and $2n$ faces. We define $X_0 := \{v, v', v_1, \dots, v_n\}$ and determine X by the following vertices of faces:

$$\Theta(X) := \{\{v, v_i, v_{i+1}\} \mid i = 1, \dots, n\} \uplus \{\{v', v_i, v_{i+1}\} \mid i = 1, \dots, n\},$$

where we read the subscripts modulo n . In Figure 2.8 we show the folding plans of the simplicial double-umbrellas $(3)^2$ and $(5)^2$. In these folding plans we have omitted the information on the corresponding edges for simplicity. Figure 2.9 illustrates the face graphs $\mathcal{F}((3)^2)$ and $\mathcal{F}((5)^2)$.

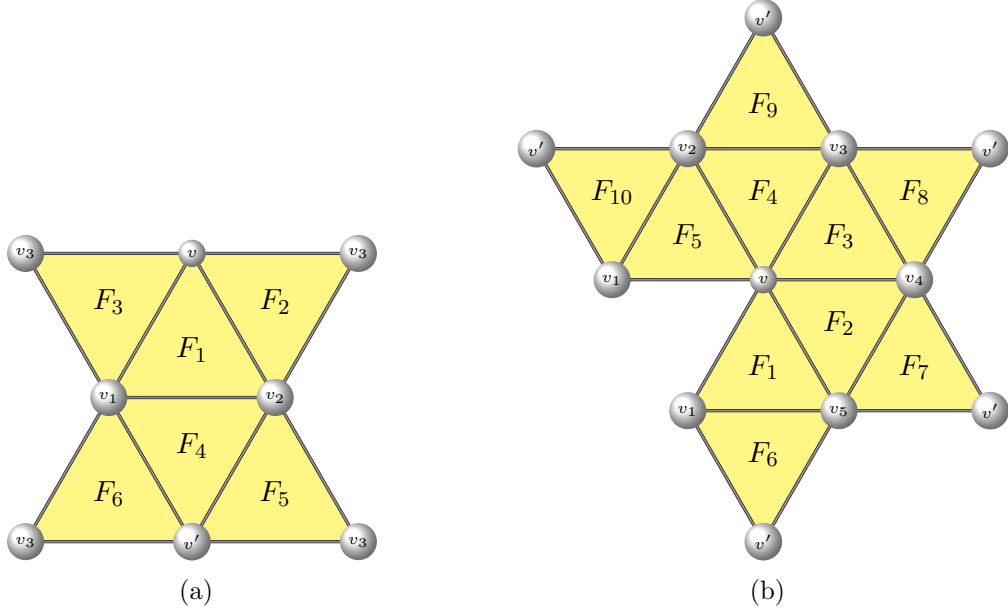


Figure 2.8: The simplicial double-umbrella $(n)^2$ for $n = 3$ (a) and $n = 5$ (b)

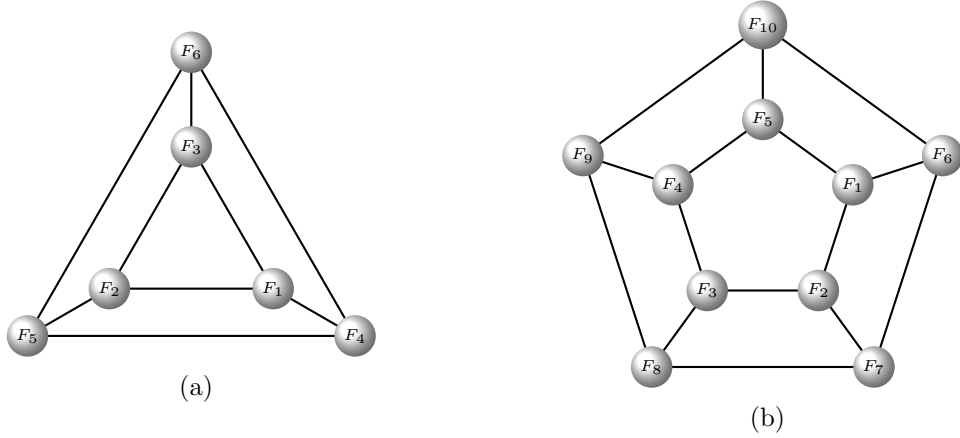


Figure 2.9: The face graph $\mathcal{F}((n)^2)$ for $n = 3$ (a) and $n = 5$ (b)

By observing the different vertex-degrees we see that the vertex-counter of $(n)^2$ is given by V_4^6 for $n = 4$ and $V_4^n V_n^2$ otherwise. Moreover, for $n = 4$ the simplicial surface $(n)^2$ is isomorphic to the simplicial octahedron. Hence, it can be shown that for every $n \neq 4$ the automorphism group of the simplicial surface $(n)^2$ is isomorphic to the direct product $C_2 \times D_{2n}$.

2.1.3 The generalised Janus Heads

In Section 1.1 we have presented an example of a closed simplicial surface that is not vertex-faithful and consists of two faces, namely the simplicial Janus-Head. Motivated by this example we introduce the family of generalised Janus-Heads.

For $n \in \mathbb{N}$ we denote the **generalised Janus-Head** consisting of $2n$ faces by $\mathcal{J}_n$. In the case that $n = 1$ we define the simplicial surface $\mathcal{J}_n$ as the simplicial Janus-Head. For $n > 1$ we determine

$\mathcal{J}_n$ by the following incidences: First, let the vertices, edges and faces of $X := \mathcal{J}_n$ be given by

$$\begin{aligned} X_0 &:= \{v, v'\} \uplus \{v_1, \dots, v_n\}, \\ X_1 &:= \{e_1, \dots, e_n\} \uplus \{e'_1, \dots, e'_n\} \uplus \{\bar{e}_1, \dots, \bar{e}_n\}, \\ X_2 &:= \{F_1, \dots, F_n\} \uplus \{F'_1, \dots, F'_n\}, \end{aligned}$$

respectively. We use these elements to describe the incidence structure of X . This incidence structure is given as follows:

$$\begin{aligned} X_0(e_i) &= \{v, v_i\}, X_0(e'_i) = \{v', v_i\}, X_0(\bar{e}_i) = \{v, v'\}, \\ X_0(F_i) &= \{e_i, e'_i, \bar{e}_i\}, X_0(F'_i) = \{e_i, e'_i, \bar{e}_{i+1}\}, \end{aligned}$$

where $i = 1, \dots, n$ and the subscripts are read modulo n . Hence, $\mathcal{J}_n$ forms a non-vertex-faithful simplicial sphere with $\text{Aut}(\mathcal{J}_n) \cong \text{Aut}(\mathcal{F}(\mathcal{J}_n)) \cong C_2 \times D_{2n}$. We observe that the vertex-counter of $\mathcal{J}_n$ is given by V_2^3 if $n = 1$ and by $V_2^n V_{2n}^2$ if $n \neq 1$. Figure 2.10 illustrates the simplicial surface $\mathcal{J}_3$ and its face graph forming a non-simple graph.

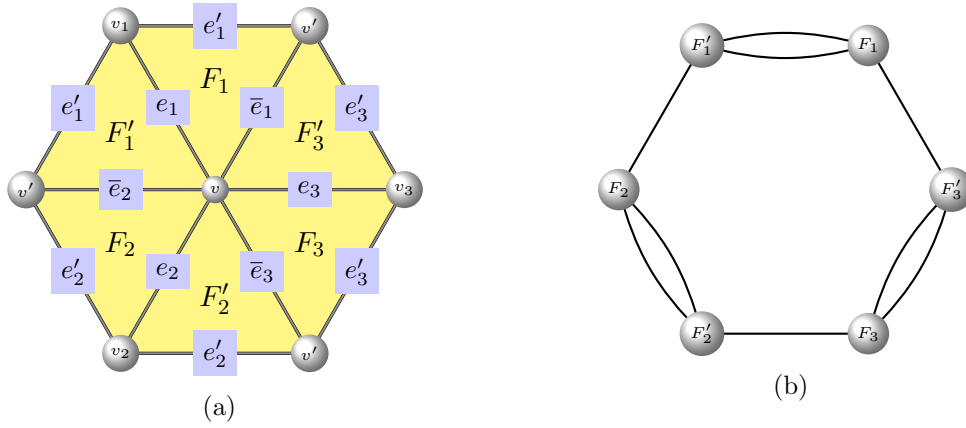


Figure 2.10: The generalised Janus-Head $\mathcal{J}_3$ (a) and its face graph $\mathcal{F}(\mathcal{J}_3)$ (b)

2.2 Open simplicial surfaces

Definition 1.1.1 admits various examples of open simplicial surfaces. Here, we present some of these simplicial surfaces.

2.2.1 The simplicial ripped umbrellas

First, we present the family of simplicial ripped umbrellas. For $n \in \mathbb{N}$ we define the **simplicial ripped umbrella** $\mathcal{R}_n$ as follows: Let the vertices of $\mathcal{R}_n$ be given by $v, v_1, \dots, v_{n+1}$. We set $\mathcal{R}_n$ to be the vertex-faithful simplicial surface that is determined by the following vertices of faces:

$$\Theta(\mathcal{R}_n) := \{\{v, v_i, v_{i+1}\} \mid i = 1, \dots, n\}.$$

Hence, the simplicial surface $\mathcal{R}_n$ is connected and consists of $n + 2$ vertices, $2n + 1$ edges and n faces. Thus, the Euler characteristic of $\mathcal{R}_n$ is given by $\chi(\mathcal{R}_n) = n + 2 - (2n + 1) + n = 1$. As an example we show the folding plan of the simplicial surface $\mathcal{R}_5$.

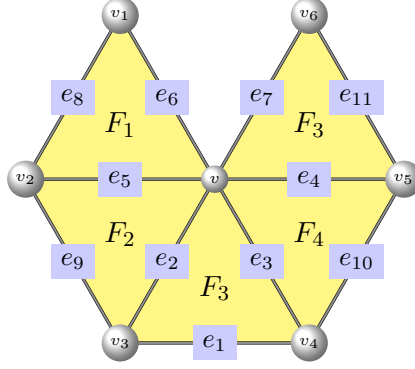


Figure 2.11: The simplicial surface $\mathcal{R}_5$

Moreover, we observe that the vertex-counter of $\mathcal{R}_n$ is given by V_1^3 if $n = 1$, by $V_1^2 V_2^2$ if $n = 2$ and by $V_1^2 V_2^{n-1} V_n^1$ otherwise. It can be shown that the automorphism group $\text{Aut}(\mathcal{R}_n)$ is isomorphic to the group $D_{2.3}$ if $n = 1$ and isomorphic to the group $C_2 \times C_2$ if $n = 2$. Furthermore, we know that for $n \geq 3$ the automorphism group of $\mathcal{R}_n$ satisfies $\text{Aut}(\mathcal{R}_n) = C_2$.

2.2.2 The simplicial umbrellas

Next, we introduce the simplicial umbrellas. For a natural number $n > 1$ we denote the **simplicial umbrella** consisting of n faces by $\mathcal{U}_n$. We define $\mathcal{U}_n$ to be a simplicial surface consisting of exactly one inner vertex of degree n , and n boundary vertices of degree 2. For simplicity, we define $\mathcal{U}_2$ to be the simplicial surface drawn in Figure 2.12a.

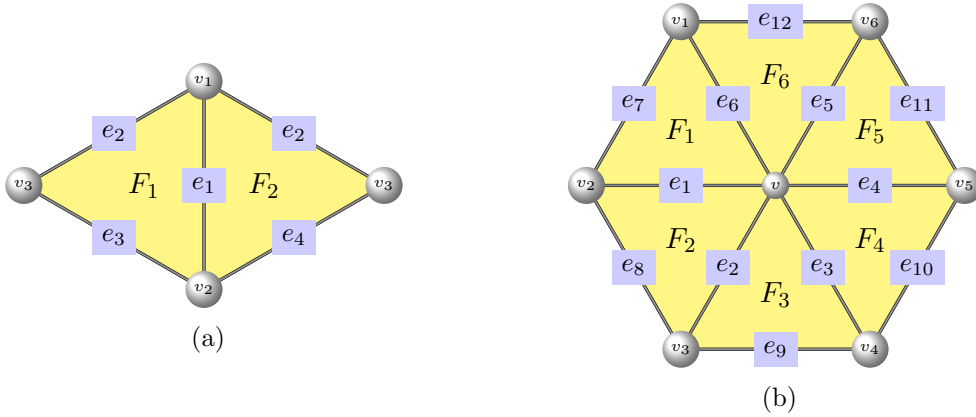


Figure 2.12: The simplicial umbrella $\mathcal{U}_n$ for $n = 2$ (a) and $n = 6$ (b)

Note that the simplicial surface $\mathcal{U}_2$ is isomorphic to the simplicial Open-Bag, see Figure 1.2b. For $n \geq 3$ we want the simplicial surface $X := \mathcal{U}_n$ to be vertex-faithful and hence defined by the following vertices of faces:

$$\Theta(X) := \{\{v, v_i, v_{i+1}\} \mid i = 1, \dots, n\},$$

where we read the subscripts modulo n . It can be observed that the automorphism group of the simplicial surface $\mathcal{U}_n$ is isomorphic to the dihedral group $D_{2.n}$ for $n \neq 2$ and isomorphic to $C_2 \times C_2$ for $n = 2$. As an example we illustrate the folding plan of the simplicial 6-umbrella in Figure 2.12b.

2.2.3 The simplicial 3-strips

The simplicial umbrellas and the simplicial ripped umbrellas form examples of open simplicial surfaces that have Euler characteristic 1. In the following we present examples of open simplicial surfaces with Euler characteristic 0, namely the *simplicial 3-strip*. More precisely, let $n \geq 5$ be a natural number. We denote the simplicial 3-strip of length n by $\mathcal{S}(3, n)$ and define this simplicial surface as follows:

If n is even, we define $k := \frac{n}{2}$ and the vertices of faces of the simplicial surface $\mathcal{S}(3, n)$ as

$$\Theta(\mathcal{S}(3, n)) := \{\{v_i, v_{i+1}, v_{k+i}\}, \{v_{i+1}, v_{k+i}, v_{k+i+1}\} \mid i = 1, \dots, k-1\} \cup \{\{v_1, v_k, v_n\}, \{v_1, v_{k+1}, v_n\}\}.$$

This incidence structure is illustrated in Figure 2.13.

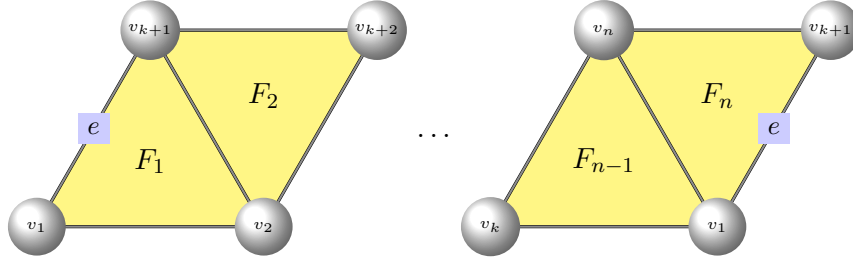


Figure 2.13: The simplicial 3-strip $\mathcal{S}(3, n)$ for an even $n \geq 5$

For $n = 6$ we obtain the simplicial 3-strip $\mathcal{S}(3, 6)$ shown in the below figure.

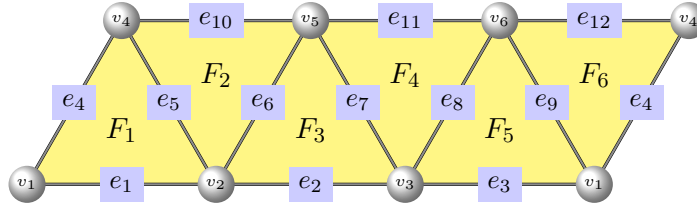


Figure 2.14: The simplicial 3-strip $\mathcal{S}(3, 6)$

In case n is odd, k is defined as $k := \frac{n-1}{2}$ and the vertices of faces of $\mathcal{S}(3, n)$ (illustrated in Figure 2.15) are given as follows:

$$\Theta(\mathcal{S}(3, n)) := \{\{v_i, v_{i+1}, v_{k+i+1}\}, \{v_i, v_{k+i}, v_{k+i+1}\} \mid i = 1, \dots, k\} \cup \{\{v_1, v_{k+1}, v_n\}\}.$$

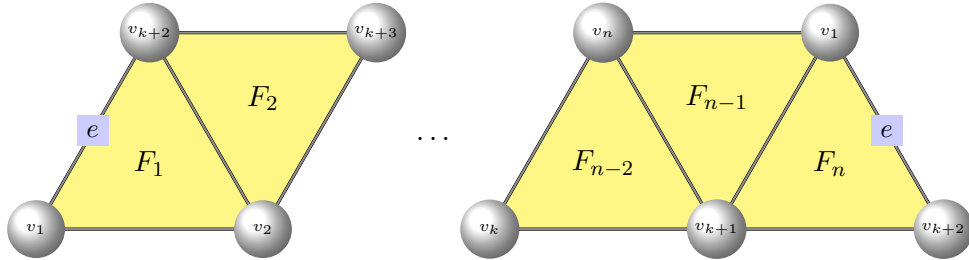


Figure 2.15: The simplicial 3-strip $\mathcal{S}(3, n)$ for an odd $n \geq 5$

In Figure 2.16 we show the simplicial surface $\mathcal{S}(3, 5)$.

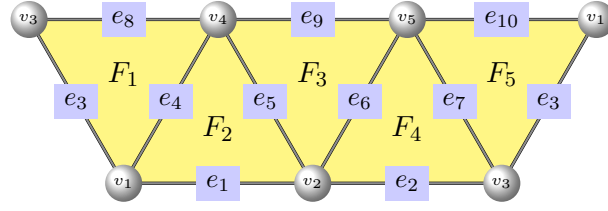


Figure 2.16: The simplicial 3-strip $\mathcal{S}(3, 5)$

Hence, the simplicial surface $\mathcal{S}(3, n)$ consists of n vertices, $2n$ edges and n faces and thus $\chi(\mathcal{S}(3, n)) = 0$ holds. Moreover, we know that $\mathcal{S}(3, n)$ is orientable if and only if n is even.

2.2.4 The simplicial $(4, 2)$ -strips

Lastly, we introduce the family of simplicial $(4, 2)$ -strips. These simplicial surfaces form further examples of simplicial surfaces with Euler characteristic 0. For a natural number $n > 2$ we denote the **simplicial** $(4, 2)$ -**strip** consisting of $2n$ faces by $\mathcal{S}(4, 2n)$. For simplicity, if $n = 3$ and $n = 4$, we define the simplicial $(4, 2)$ -strips $\mathcal{S}(4, 6)$ and $\mathcal{S}(4, 8)$ as illustrated in Figures 2.17 and 2.18, respectively.

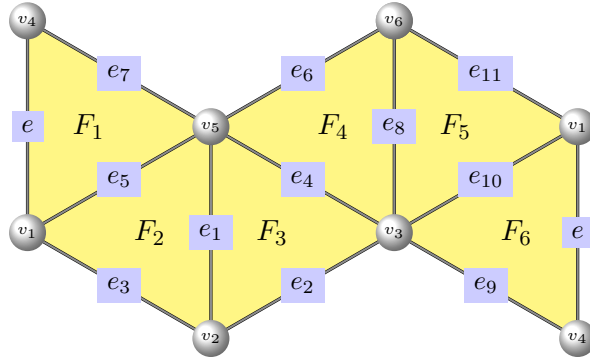


Figure 2.17: The simplicial surface $\mathcal{S}(4, 6)$

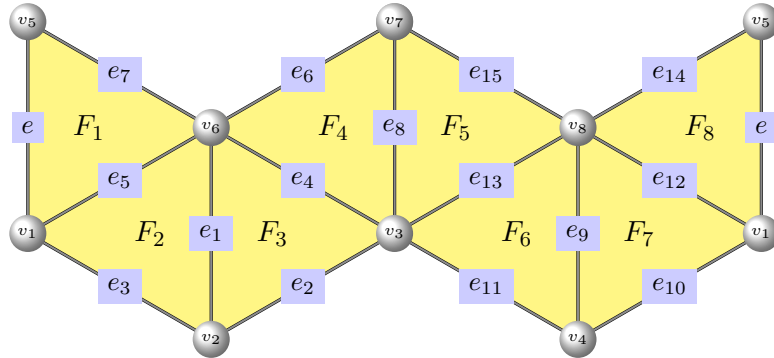


Figure 2.18: The simplicial surface $\mathcal{S}(4, 8)$

For $n > 4$ we define $\mathcal{S}(4, 2n)$ to be the vertex-faithful simplicial surface that is constructed

from the simplicial 3-strip $\mathcal{S}(3, n)$ by subdividing each face of $\mathcal{S}(3, n)$ into two faces such that every boundary edge of $\mathcal{S}(3, n)$ is subdivided into two boundary edges, see Figure 2.19 for illustration.

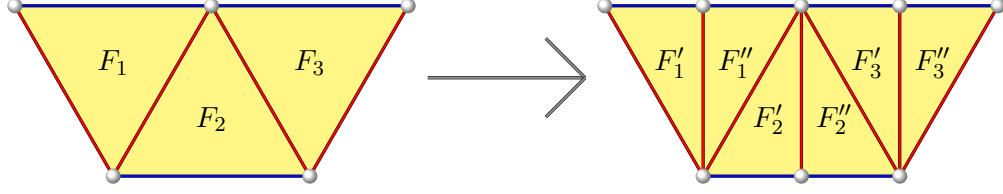


Figure 2.19: Subdividing the faces of the simplicial surface $\mathcal{S}(3, n)$, where boundary edges are illustrated in blue and inner edges in red

Figures 2.20 and 2.21 shows illustrations of the simplicial surfaces that arise from the above construction.

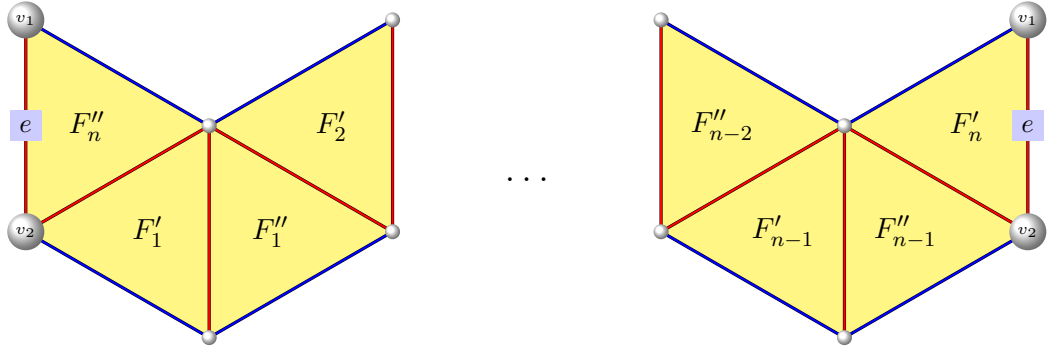


Figure 2.20: The simplicial surface $\mathcal{S}(4, 2n)$ for an even $n > 4$, where boundary edges are illustrated in blue and inner edges in red

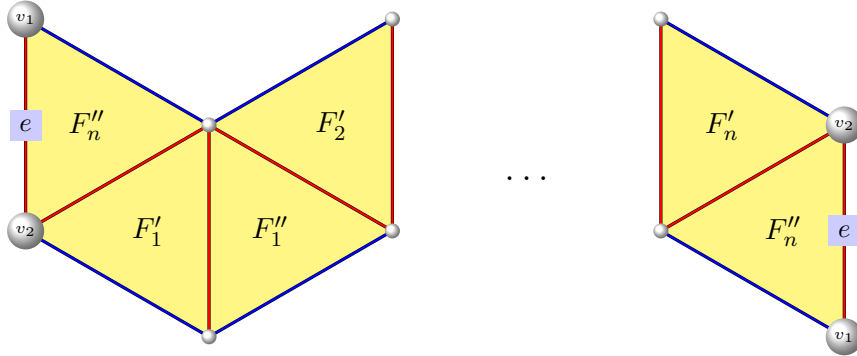


Figure 2.21: The simplicial surface $\mathcal{S}(4, 2n)$ for an odd $n > 4$, where boundary edges are illustrated in blue and inner edges in red

Hence, we see that the simplicial surface $\mathcal{S}(4, 2n)$ consists of $2n$ vertices, $4n$ edges and $2n$ faces. This results in the Euler characteristic of $\mathcal{S}(4, 2n)$ being 0. Finally, we know that $\mathcal{S}(4, 2n)$ is orientable if and only if n is even.

3 Group actions

In this section, we study group actions that can be used to investigate simplicial surfaces and their face graphs. Note that these actions have also been discussed in [2, 74] for instance. The statements derived in this section are essential for our study of edge- and face-transitive surfaces in Sections 5 and 6. For simplicity, we assume that the simplicial surfaces in this section are always connected. In this work we follow the standard group-theoretical terminology as given in [27, 36] for instance.

3.1 Group actions on simplicial surfaces and their face graphs

Understanding the action of automorphism groups on the corresponding simplicial surfaces allows us to explore the symmetries of simplicial surfaces and their associated face graphs in much more detail. Thus, we begin this section by describing the action of $\text{Aut}(X)$ on a simplicial surface X . We then link this group action to an action of $\text{Aut}(\mathcal{F}(X))$ on $\mathcal{F}(X)$.

Remark 3.1.1 *Let X be a simplicial surface. We observe that $\text{Aut}(X)$ acts on X via*

$$\text{Aut}(X) \times X \rightarrow X, (\phi, x) \mapsto \phi(x).$$

We denote the orbit of an $x \in X$ under a subgroup $H \leq \text{Aut}(X)$ by $\text{orb}_H(x)$. Furthermore, for a subset $S \subseteq X$ we define $\text{orb}_H(S) := \{\text{orb}_H(x) \mid x \in S\}$. Similarly, $\text{Aut}(\mathcal{F}(X))$ acts on $\mathcal{F}(X)$ via

$$\text{Aut}(\mathcal{F}(X)) \times \mathcal{F}(X) \rightarrow \mathcal{F}(X), (\sigma, x) \mapsto \sigma(x).$$

Since an automorphism $\phi \in \text{Aut}(X)$ satisfies $\phi(x) \in X_i$, for all $x \in X_i$ and $i \in \{0, 1, 2\}$, the introduced action of $\text{Aut}(X)$ on X can be restricted to an action $\text{Aut}(X) \times X_i \rightarrow X_i$. In particular, the action of $\text{Aut}(X)$ on the faces X_2 induces a homomorphism

$$\text{Aut}(X) \rightarrow \text{Sym}(X_2)$$

that maps an automorphism $\phi \in \text{Aut}(X)$ onto the permutation $\pi \in \text{Sym}(X_2)$ of faces in X_2 it effects. Since the automorphism ϕ respects the incidences between the faces and edges of X , the permutation π can be understood as an automorphism of $\mathcal{F}(X)$. This instance yields the homomorphism

$$\lambda_X : \text{Aut}(X) \rightarrow \text{Aut}(\mathcal{F}(X)). \tag{1}$$

With Proposition 3.1.3 it follows that, if X is a vertex-faithful simplicial surface, then the homomorphism λ_X is a monomorphism. Thus, if X is vertex-faithful, λ_X allows us to embed $\text{Aut}(X)$ into $\text{Aut}(\mathcal{F}(X))$.

We proceed by examining the action of $\text{Aut}(X)$ on X . First, we show that, if X is not isomorphic to the simplicial Janus-Head, then an automorphism $\phi \in \text{Aut}(X)$ is uniquely determined by the images of the edges of any face of X .

Proposition 3.1.2

Let X be a connected simplicial surface that is not isomorphic to the simplicial Janus-Head, $F \in X_2$ a face with $X_1(F) = \{e_1, e_2, e_3\}$ and $\phi \in \text{Aut}(X)$. If there exists an automorphism $\psi \in \text{Aut}(X)$ satisfying $\phi(e_i) = \psi(e_i)$ for $i = 1, 2, 3$, then $\phi = \psi$ follows.

This section is based on research originally published in [5].

Proof. If X is isomorphic to the simplicial One-Face, it is easy to verify that the above statement holds. Thus, we assume that X is not isomorphic to the simplicial One-Face. We first notice that $\phi(e_i) = \psi(e_i)$ for $i = 1, 2, 3$ implies $\phi(x) = \psi(x)$ for every $x \in \{F\} \cup X_1(F) \cup X_0(F)$. Hence, we prove the proposition by showing that if $\phi(x) = \psi(x)$ holds for all $x \in \{F_1\} \cup X_1(F_1) \cup X_0(F_1)$, where F_1 is an arbitrary face, then $\phi(y) = \psi(y)$ holds for all $y \in \{F_2\} \cup X_1(F_2) \cup X_0(F_2)$ with F_2 being a face of X with $X_1(F_1) \cap X_1(F_2) \neq \emptyset$. The result then follows from the connectivity of X .

Thus, let $F_1 \in X_2$ be an arbitrary face with $X_1(F_1) = \{e, e', e''\}$ and $\phi(x) = \psi(x)$ for all $x \in \{F_1\} \cup X_1(F_1) \cup X_0(F_1)$. Since X is connected and not isomorphic to the simplicial One-Face, there exists a face $F_2 \in X_2$ that satisfies $X_1(F_1) \cap X_1(F_2) \neq \emptyset$. Without loss of generality, we assume that e is the edge that is incident to F_1 and F_2 , and that the edges incident to F_2 are given by $X_1(F_2) = \{e, \bar{e}, \bar{e}'\}$. Here, we prove the equality of $\phi(F_2) = \psi(F_2)$. The reasoning of $\phi(y) = \psi(y)$ for all $y \in X_1(F_2) \cup X_0(F_2)$ can be achieved by an analogous argument.

Because ϕ and ψ respect the incidences of X , the edge $\phi(e)$ has to be incident to $\phi(F_1) = \psi(F_1)$, to $\psi(F_2)$ and to $\phi(F_2)$. Now, we know that ϕ and ψ are both bijective and that $\phi(e)$ is incident to exactly two faces which yields $\phi(F_2) = \psi(F_2)$. This concludes the result. $\square$

With a similar strategy as presented in the above proof, we can show the following statement.

Proposition 3.1.3

Let X be a vertex-faithful simplicial surface, $F \in X_2$ a face with $X_0(F) = \{v_1, v_2, v_3\}$ and $\phi \in \text{Aut}(X)$. If there exists an automorphism $\psi \in \text{Aut}(X)$ with $\phi(v_i) = \psi(v_i)$ for $i = 1, 2, 3$, then $\phi = \psi$ follows.

We notice that the above statement does not hold if the simplicial surface X is not connected. Further, the simplicial Janus-Head $\mathcal{J}$ is a non-vertex-faithful simplicial surface with the property that for every $\phi \in \text{Aut}(\mathcal{J})$ there exists a distinct automorphism $\psi \in \text{Aut}(\mathcal{J})$ with $\phi(v) = \psi(v)$ for every $v \in \mathcal{J}_0$. Thus, it is necessary that a given simplicial surface is vertex-faithful to ensure the correctness of the above proposition.

Moreover, Proposition 3.1.3 implies that in case X is a vertex-faithful simplicial surface, it is sufficient to only examine the images of the vertices of X under an automorphism in order to determine the given automorphism fully. This instance is summarised in the following remark.

Remark 3.1.4 *Let X be a vertex-faithful simplicial surface and $X_0 = \{v_1, \dots, v_n\}$ its set of vertices. By Proposition 3.1.3 we know that an automorphism $\phi \in \text{Aut}(X)$ is uniquely determined by the images $\phi(v_1), \dots, \phi(v_n)$. Hence, it suffices to determine the permutation $\pi \in \text{Sym}(\{v_1, \dots, v_n\})$ that satisfies $\pi(v_i) = \phi(v_i)$, for all $1 \leq i \leq n$, in order to determine the automorphism ϕ fully.*

Next, we discuss the stabilisers of certain elements of a simplicial surface. If G is a group acting on a set M and $x \in M$ is an element, then we denote the stabiliser of x in G by $\text{Stab}_G(x)$. We observe that the stabilisers of edges and faces of a vertex-faithful simplicial surface can be embedded into finite groups of small orders, see Lemmas 3.1.5 and 3.1.6.

Lemma 3.1.5

Let X be a vertex-faithful simplicial surface and $F \in X_2$ a face. Then the stabiliser $\text{Stab}_{\text{Aut}(X)}(F)$ can be embedded into the group $\text{Sym}(X_0(F))$.

Proof. Let $X_0(F)$ be given by $\{v_1, v_2, v_3\}$. Since an automorphism $\phi \in \text{Stab}_{\text{Aut}(X)}(F)$ respects the incidence structure of X , the vertices $\phi(v_1), \phi(v_2)$ and $\phi(v_3)$ have to be incident to the face $\phi(F) = F$. Thus, the vertices v_1, v_2, v_3 are permuted. With Proposition 3.1.3 the automorphism ϕ is uniquely determined by the permutation of the vertices v_1, v_2, v_3 that is effected by ϕ . Thus, we conclude that $\text{Stab}_{\text{Aut}(X)}(F)$ is isomorphic to a subgroup of $\text{Sym}(X_0(F))$. $\square$

Lemma 3.1.6

Let X be a vertex-faithful simplicial surface and $e \in X_1$ an inner edge. Then, the stabiliser $\text{Stab}_{\text{Aut}(X)}(e)$ can be embedded into the group $C_2 \times C_2$.

Proof. First, let $v_1, v_2 \in X_0$ be vertices with $X_0(e) = \{v_1, v_2\}$, and $F_1, F_2 \in X_1$ faces satisfying $X_2(e) = \{F_1, F_2\}$. For every $i = 1, 2$ let $w_i \in X_0$ be the vertex such that $w_i \in X_0(F_i) \setminus \{v_1, v_2\}$. Every automorphism $\phi \in \text{Aut}(X)$ that stabilises e has to permute the two vertices v_1 and v_2 , and also w_1 and w_2 . With Proposition 3.1.3 the automorphism ϕ is uniquely identified by the images of v_1, v_2, w_1, w_2 under ϕ . This means that ϕ can be identified with a permutation

$$\pi \in \{(), (v_1, v_2), (w_1, w_2), (v_1, v_2)(w_1, w_2)\} = \langle (v_1, v_2), (w_1, w_2) \rangle$$

which implies that $\text{Stab}_{\text{Aut}(X)}(e)$ is isomorphic to a subgroup of the group $C_2 \times C_2$. $\square$

With similar arguments, we can show the following statement.

Corollary 3.1.7 *Let X be a vertex-faithful simplicial surface and $e \in X_1$ a boundary edge. Then, the stabiliser $\text{Stab}_{\text{Aut}(X)}(e)$ can be embedded into the group C_2 .*

Let X be a vertex-faithful simplicial surface and $\text{Aut}(X)$ its automorphism group. If $e \in X_1$ is an edge and $F \in X_2$ is a face, then $|\text{Stab}_{\text{Aut}(X)}(F)| \in \{1, 2, 3, 6\}$ and $|\text{Stab}_{\text{Aut}(X)}(e)| \in \{1, 2, 4\}$ follows. Hence, $|\text{Aut}(X)| \leq 6|X_2|$ which is the well-known upper bound on the number of automorphisms of a triangulated surface, closely related to Hurwitz's automorphism theorem, see [57].

It is well known that if G is a group acting on a set M , then elements in M that are contained in the same G -orbit have stabilisers that are conjugate in G . In the following, we reformulate this well-known result to exploit this fact in this thesis.

Lemma 3.1.8

Let X be a simplicial surface and $x_1, x_2 \in X$ two elements of X . If x_1 and x_2 are contained in the same orbit under $\text{Aut}(X)$, then the stabilisers of x_1 and x_2 are conjugate in $\text{Aut}(X)$.

Proof. For every $i = 1, 2$, let $S_i = \text{Stab}_{\text{Aut}(X)}(x_i)$ be the stabiliser of the element x_i in $\text{Aut}(X)$. Per assumption, there exists an automorphism $\phi \in \text{Aut}(X)$ with $\phi(x_1) = x_2$. Thus, the desired result follows by showing that

$$f : S_1 \rightarrow S_2, \psi \mapsto \phi \circ \psi \circ \phi^{-1}$$

is a group isomorphism. First, we see that $f(\psi)(x_2) = x_2$ holds for all $\psi \in S_1$ which establishes that the map f is well-defined. Furthermore, with

$$f^{-1} : S_2 \rightarrow S_1, \psi \mapsto \phi^{-1} \circ \psi \circ \phi$$

and $f(\psi_1 \circ \psi_2) = \phi \circ \psi_1 \circ \phi^{-1} \circ \phi \circ \psi_2 \circ \phi^{-1} = f(\psi_1) \circ f(\psi_2)$ for all $\psi_1, \psi_2 \in S_1$ it follows that f is a group isomorphism. We obtain $f(S_1) = S_2$ and thus the stabilisers S_1 and S_2 are conjugate. $\square$

Moreover, if X is a simplicial surface, then the order of the stabiliser of an element $x \in X$ tells us how many automorphisms map x onto a given element $y \in \text{orb}_{\text{Aut}(X)}(x)$. This result follows directly from the orbit-stabiliser theorem. Again, we reformulate this fact to exploit it in our studies.

Proposition 3.1.9

Let X be a simplicial surface and x_1, x_2 two elements of X that are contained in the same orbit under $\text{Aut}(X)$. Then there exist exactly $|\text{Stab}_{\text{Aut}(X)}(x_1)| = |\text{Stab}_{\text{Aut}(X)}(x_2)|$ automorphisms of X that map x_1 onto x_2 .

Proof. Let M be the set of all automorphisms that map x_1 onto x_2 . Since x_1 and x_2 are contained in the same $\text{Aut}(X)$ -orbit, M is non-empty. Thus, let $\phi \in M$ be an automorphism. We observe that $f : \text{Stab}_{\text{Aut}(X)}(x_2) \rightarrow M, \psi \mapsto \psi \circ \phi$ is well-defined. With $f^{-1} : M \rightarrow \text{Stab}_{\text{Aut}(X)}(x_2), \psi' \mapsto \psi' \circ \phi^{-1}$, we know that the map f is bijective. Hence, the equality $|M| = |\text{Stab}_{\text{Aut}(X)}(x_2)|$ follows. $\square$

By closely analysing the map f that has been constructed in the proof of Proposition 3.1.9, we obtain the following corollary.

Corollary 3.1.10 *Let X be a simplicial surface, $x_1, x_2 \in X$ two elements and $\phi \in \text{Aut}(X)$ an automorphism that maps x_1 onto x_2 . Then for every $\phi' \in \text{Aut}(X)$ that maps x_1 onto x_2 there exists an automorphism $\psi \in \text{Stab}_{\text{Aut}(X)}(x_2)$ such that $\phi' = \psi \circ \phi$.*

3.2 Group actions on vertex-defining cycles and umbrellas

In Sections 5 and 6 we aim to construct vertex-faithful simplicial surfaces by constructing vertex-faithful cycle double covers of cubic graphs with the help of the automorphism group of the given graph. For this purpose, we introduce an action of the automorphism group of a cubic graph on its cycles and an action of the automorphism group of a simplicial surface on its umbrellas.

Remark 3.2.1 *Let $\mathcal{G} = (V, E)$ be a cubic graph and M be the set of all cycles in $\mathcal{G}$. The group action $\text{Aut}(\mathcal{G}) \times V \rightarrow V, (\sigma, x) \mapsto \sigma(x)$ yields a well-defined group action of $\text{Aut}(\mathcal{G})$ on M via:*

$$\text{Aut}(\mathcal{G}) \times M \rightarrow M, \quad (\sigma, (v_1, \dots, v_n)) \mapsto \sigma((v_1, \dots, v_n)) := (\sigma(v_1), \dots, \sigma(v_n)).$$

Moreover, let X be a closed simplicial surface. In the following we denote the umbrella of a vertex $v \in X_0$ by $u(v)$. Thus, if $u(X) := \{u(v) \mid v \in X_0\}$ is the set containing all umbrellas of X , then the action of $\text{Aut}(X)$ on the faces X_2 can be extended to an action of $\text{Aut}(X)$ on $u(X)$ via

$$\text{Aut}(X) \times u(X) \rightarrow u(X), \quad (\phi, (F_1, \dots, F_n)) \mapsto \phi((F_1, \dots, F_n)) := (\phi(F_1), \dots, \phi(F_n)).$$

The set $u(X)$ can also be interpreted as the set of vertex-defining cycles of X in $\mathcal{F}(X)$. Thus, the group $H := \lambda_X(\text{Aut}(X))$ (see (1) in Remark 3.1.1) is maximal in $\text{Aut}(\mathcal{F}(X))$ with the property that the action of H on the cycles of $\mathcal{F}(X)$ leaves the cycle double cover $u(X)$ invariant. Hence, $\text{Aut}(X) \cong \text{Aut}(\mathcal{F}(X))$ is equivalent to saying that for every vertex-defining cycle $C \in u(X)$ and every $\sigma \in \text{Aut}(\mathcal{F}(X))$ the image $\sigma(C)$ is again a vertex-defining cycle of X , see [2]. Finally, we observe that

$$\bigcup_{v \in X_0} u(v)^H$$

describes the cycle double cover of $\mathcal{F}(X)$ consisting of the vertex-defining cycles of X .

Lastly, we introduce the definition of an automorphism-induced α -cycle. In Sections 5 and 6 this definition is used for the construction of vertex-faithful cycle double covers. Loosely speaking, the main idea of this definition is to construct a cycle of a cubic graph $\mathcal{G}$ by first constructing a path P in $\mathcal{G}$ and then let a cycle in $\mathcal{G}$ arise by applying different powers of an automorphism $\sigma \in \text{Aut}(\mathcal{G})$ to P .

Definition 3.2.2

Let $\mathcal{G} = (V, E)$ be a cubic graph and $F_1, \dots, F_n \in V$ nodes such that $\{F_i, F_{i+1}\} \in E$ for $i = 1, \dots, n-1$. Furthermore, let $\sigma \in \text{Aut}(\mathcal{G})$ be an automorphism of order ℓ satisfying $\sigma(F_1) = F_n$ and $\Sigma := \{\sigma^i(F_j) \mid i = 1, \dots, \ell, j = 1, \dots, n-1\}$. We define a **σ -induced α -cycle** as the cycle

$$\alpha(\sigma, F_1, \dots, F_n) := (\sigma(F_1), \dots, \sigma(F_{n-1}), \dots, \sigma^\ell(F_1), \dots, \sigma^\ell(F_{n-1}))$$

if $|\Sigma| = (n-1)\ell$, and as the empty cycle $()$ otherwise.

We exemplify the above definition by considering the cubic graph $\mathcal{G} = (V, E)$ illustrated in Figure 3.1.

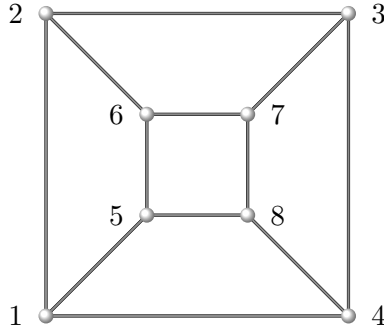


Figure 3.1: A cubic graph $\mathcal{G}$ with 8 nodes

The nodes 1 and 2 of $\mathcal{G}$ are connected by an arc in $\mathcal{G}$. Furthermore, we see that the automorphism group $\text{Aut}(\mathcal{G}) = \langle (3, 6)(4, 5), (2, 4)(6, 8), (1, 2)(3, 4)(5, 6)(7, 8) \rangle$ contains the automorphism σ defined by $\sigma := (1, 2, 3, 4)(5, 6, 7, 8)$. Since σ has order 4 and we observe

$$|\{\sigma^i(1) \mid i = 1, \dots, 4\}| = |\{1, 2, 3, 4\}| = 4 = (2-1) \cdot 4,$$

the σ induced α -cycle $\alpha(\sigma, 1, 2)$ is given by

$$\alpha(\sigma, 1, 2) = (\sigma(1), \sigma^2(1), \sigma^3(1), \sigma^4(1)) = (2, 3, 4, 1) = (1, 2, 3, 4).$$

We observe that the orbit

$$\begin{aligned} \text{orb}_{\text{Aut}(\mathcal{G})}(\alpha(\sigma, 1, 2)) &= \text{orb}_{\text{Aut}(\mathcal{G})}((1, 2, 3, 4)) \\ &= \{(1, 2, 3, 4), (1, 2, 6, 5), (1, 4, 8, 5), (2, 3, 7, 6), (3, 4, 8, 7), (5, 6, 7, 8)\} \end{aligned}$$

is a vertex-faithful cycle double cover that forms the vertex-defining cycles of a simplicial octahedron, see Section 2.

4 Homomorphisms of simplicial surfaces

This section deals with the study of homomorphisms of simplicial surfaces (see Definition 1.1.8). Of particular interest in this section are homomorphisms that form epimorphisms and endomorphisms (Definition 4.2.2) of simplicial surfaces. We use these special homomorphisms to introduce simple and endomorphism-simple simplicial surfaces (Definitions 4.1.1 and 4.2.3). Here, we investigate these classes of simplicial surfaces and describe their properties. Loosely speaking, we refer to a simplicial surface X as *simple* if every simplicial surface that forms an epimorphic image of X is already isomorphic to X . We establish the result that a simple simplicial sphere has to be isomorphic to the simplicial tetrahedron. Moreover, X is said to be *endomorphism-simple* if every endomorphism $\phi : X \rightarrow X$ is an automorphism of X . In this context, we show that every closed connected simplicial surface with all vertex-degrees being odd is endomorphism-simple. Finally, we establish the existence of infinitely many simple and infinitely many endomorphism-simple but not simple simplicial surfaces.

4.1 Simple simplicial surfaces

In this subsection we study simple simplicial surfaces. A natural question that arises in this context is whether the existence of certain sub-surfaces of a simplicial surface implies that the given simplicial surface cannot be simple. Here, we present the fact that closed simplicial surfaces with vertices of degree 2 and closed vertex-faithful simplicial surfaces with two neighbouring vertices of degree 4 that are not contained in a so-called *3-waist* (see Definition 4.1.8) cannot be simple. Further, we examine different classes of simplicial surfaces and determine whether these classes contain examples of simple simplicial surfaces. For instance, we show that a simplicial surface that forms a simplicial umbrella or a simplicial ripped umbrella is not simple. Theorem 4.1.23 forms the main result of this subsection. This theorem establishes that the only simple simplicial sphere is the simplicial tetrahedron. We prove this result by equipping simplicial spheres with vertex-colourings. Therefore, we state different facts about simplicial surfaces whose vertex-edge graphs can be coloured with 3 or 4 colours. By using an algorithm implemented in GAP and available in [72], we construct a new example of a simple simplicial surface and exploit it to deduce the existence of infinitely many simple simplicial surfaces.

First, we present the definition of a simple simplicial surface.

Definition 4.1.1

*A simplicial surface X is called **simple**, if every simplicial surface Y with the property that there exists an epimorphism $\psi : X \rightarrow Y$ is already isomorphic to X .*

This definition differs from the definition that has been presented in [74]. In this section we show that Definition 4.1.1 and the definition introduced by Niemeyer et al. are equivalent. We proceed by giving some examples of simple simplicial surfaces to enhance clarity of the definition.

Example 4.1.2 • *Let X be the simplicial One-Face. Then X is simple. This follows directly from the fact that the simplicial One-Face is the smallest simplicial surface.*

- *Let $\mathcal{T}$ be the simplicial tetrahedron. Then it follows that $\mathcal{T}$ is simple. This can be seen by the below fact: Let Y be a simplicial surface such that there exists an epimorphism $\phi : \mathcal{T} \rightarrow Y$. If Y is not isomorphic to $\mathcal{T}$, then there exist two distinct vertices $v_1, v_2 \in \mathcal{T}_0$ with $\phi(v_1) = \phi(v_2)$. We notice that the vertex-edge graph of $\mathcal{T}$ is the complete graph on four nodes. Thus, there*

exists an edge $e \in \mathcal{T}_1$ with $\mathcal{T}_0(e) = \{v_1, v_2\}$. Further, for a face $F \in \mathcal{T}_2(e)$, we observe that the map ϕ is not bijective on the sub-surface $\mathcal{T}(F)$ (see (*) in Section 1). This results in ϕ not being a well-defined homomorphism which is a contradiction. We therefore conclude that ϕ is injective on the vertices of $\mathcal{T}$. Because of $\mathcal{T}$ being vertex-faithful, this directly implies that the map ϕ is injective, which means that the epimorphism ϕ is an isomorphism. With similar arguments as presented above it can be shown that the simplicial umbrella $\mathcal{U}_3$ is simple.

In general, simple simplicial surfaces do not have to be connected. This instance is described in Example 4.1.28. Next, we define the notion of an *admissible relation* of a simplicial surface. This notion allows us to decide whether a given simplicial surface is simple by constructing suitable partitions of the vertices, edges and faces of the corresponding simplicial surface.

Definition 4.1.3 (see [74])

Let $(X, <)$ be a simplicial surface and $\sim$ an equivalence relation on X with corresponding partition $\mathcal{P}$ of X into equivalence classes. Furthermore, let $\lesssim$ be the relation on $\mathcal{P}$ that is induced by the relation $<$ as follows: Two classes $\mathcal{C}, \mathcal{C}' \in \mathcal{P}$ satisfy $\mathcal{C} \lesssim \mathcal{C}'$ if and only if for every $y \in \mathcal{C}'$ there exists an element $x \in \mathcal{C}$ such that $x < y$ holds in X . We call $\sim$ an **admissible relation** of X , if the following properties are satisfied:

1. For each class $\mathcal{C} \in \mathcal{P}$ there exists an $i \in \{1, 2, 3\}$ with $\mathcal{C} \subseteq X_i$. That means that the set of equivalence classes $\mathcal{P}$ can be partitioned as follows:

$$\mathcal{P} = (\mathcal{P} \cap \text{Pow}(X_0)) \uplus (\mathcal{P} \cap \text{Pow}(X_1)) \uplus (\mathcal{P} \cap \text{Pow}(X_2)).$$

2. The tuple $(\mathcal{P}, \lesssim)$ forms a well-defined simplicial surface, i.e. $(\mathcal{P}, \lesssim)$ satisfies the axioms formulated in Definition 1.1.1.

If the equivalence relation $\sim$ on X satisfies the above properties, we denote the simplicial surface $(\mathcal{P}, \lesssim)$ by $(X/\sim)$.

An admissible relation of a simplicial surface requires the corresponding partition of the simplicial surface that is equipped with an incidence relation to form a well-defined simplicial surface. It can be observed that every simplicial surface has an admissible relation. In particular, if X is a simplicial surface, then the set $\{\{x\} \mid x \in X\}$ corresponds to a set of equivalence classes of an admissible relation $\sim$ of X . The simplicial surface arising from this relation is isomorphic to X . Here, we refer to this admissible relation as the *trivial admissible relation* of X . Below, we give some examples of non-trivial admissible relations.

Example 4.1.4 Let X be the simplicial Butterfly with vertices, edges and faces, and corresponding incidences as illustrated in Figure 4.1.

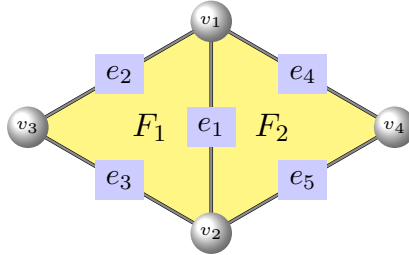


Figure 4.1: The simplicial Butterfly X

We observe that, up to isomorphism, X has exactly four admissible relations. These admissible relations $\sim_1, \dots, \sim_4$ are identified by their corresponding partition sets $\mathcal{P}_1, \dots, \mathcal{P}_4$ given as follows:

$$\begin{aligned}\mathcal{P}_1 &:= \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\} \uplus \{\{e_1\}, \{e_2\}, \{e_4\}, \{e_3\}, \{e_5\}\} \uplus \{\{F_1\}, \{F_2\}\}, \\ \mathcal{P}_2 &:= \{\{v_1\}, \{v_2\}, \{v_3, v_4\}\} \uplus \{\{e_1\}, \{e_2\}, \{e_4\}, \{e_3, e_5\}\} \uplus \{\{F_1\}, \{F_2\}\}, \\ \mathcal{P}_3 &:= \{\{v_1\}, \{v_2\}, \{v_3, v_4\}\} \uplus \{\{e_1\}, \{e_2, e_4\}, \{e_3, e_5\}\} \uplus \{\{F_1\}, \{F_2\}\}, \\ \mathcal{P}_4 &:= \{\{v_1\}, \{v_2\}, \{v_3, v_4\}\} \uplus \{\{e_1\}, \{e_2, e_4\}, \{e_3, e_5\}\} \uplus \{\{F_1, F_2\}\}.\end{aligned}$$

The simplicial surfaces corresponding to the partitions $\mathcal{P}_1, \dots, \mathcal{P}_4$ are shown in Figures 4.2a to 4.2d, respectively.

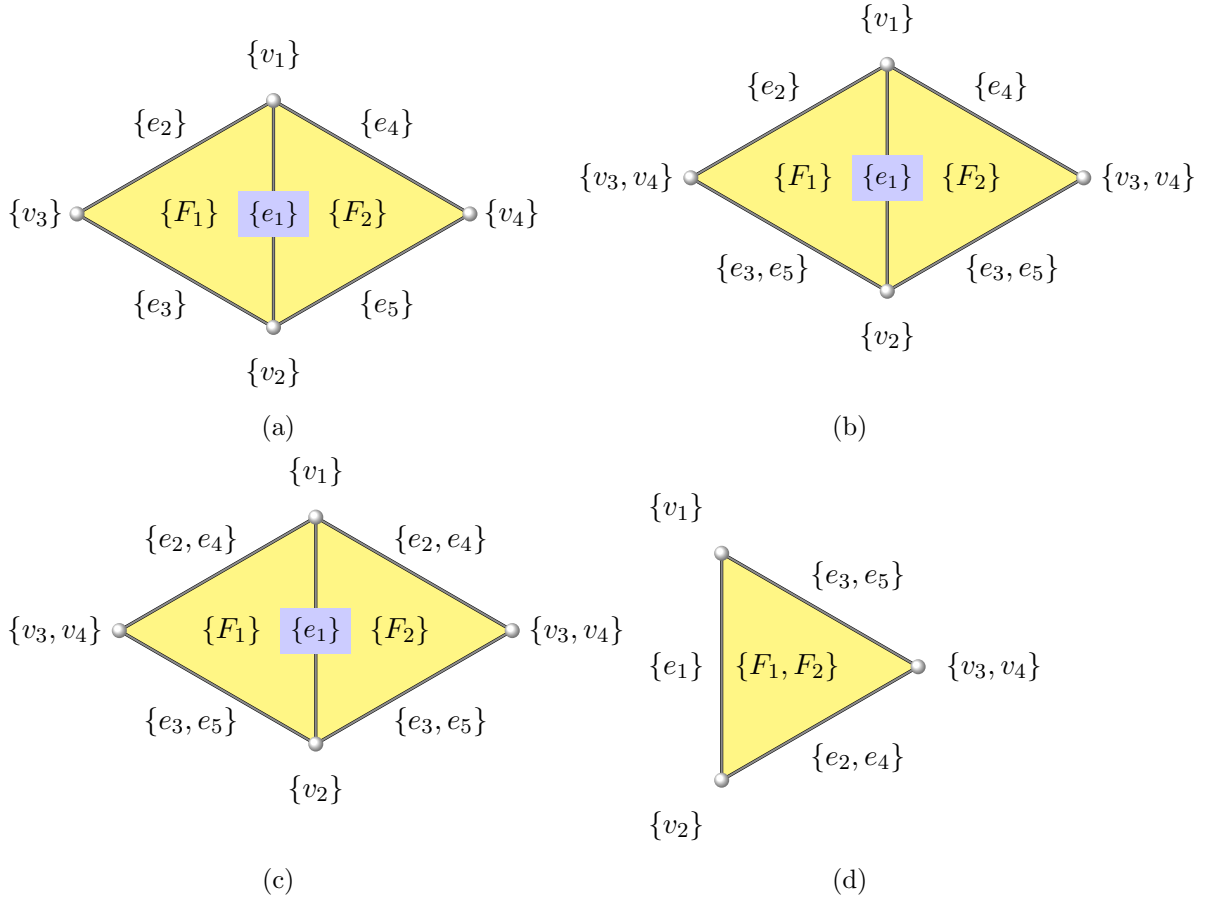


Figure 4.2: All admissible relations of the simplicial Butterfly X

Since the admissible relation $\sim_1$ is trivial, the simplicial surface $(X/\sim_1)$ is isomorphic to X . The simplicial surface $(X/\sim_2)$ consists of three vertices, four edges and two faces. Hence, this simplicial surface is isomorphic to the simplicial Open-Bag. Moreover, we notice that the simplicial surfaces $(X/\sim_3)$ and $(X/\sim_4)$ are isomorphic to the simplicial Janus-Head and the simplicial One-Face, respectively.

In the following, we highlight the advantages of introducing the notion of admissible relations. It can be observed that a simplicial surface Y that is constructed from an admissible relation of a given simplicial surface X forms an epimorphic image of X . A more general form of this statement

is presented in [74] as the Homomorphism Theorem of simplicial surfaces. We introduce a version of this theorem that is tailored to our needs.

Proposition 4.1.5

Let X and Y be simplicial surfaces. Then there exists an epimorphism $\phi : X \rightarrow Y$ if and only if there exists an admissible relation $\sim$ of X such that $(X/\sim)$ and Y are isomorphic.

Proof. First, we assume that there exists an epimorphism $\phi : X \rightarrow Y$. In this case, we can define an equivalence relation $\sim$ by the partition $\mathcal{P}$ that arises from the pre-images of the elements of Y under ϕ , i.e. $\mathcal{P} := \phi^{-1}(Y) = \{\phi^{-1}(y) \mid y \in Y\}$. We show that the equivalence relation $\sim$ is an admissible relation of X . Since the map ϕ is a well-defined homomorphism, we obtain

$$\mathcal{P} = (\mathcal{P} \cap \text{Pow}(X_0)) \uplus (\mathcal{P} \cap \text{Pow}(X_1)) \uplus (\mathcal{P} \cap \text{Pow}(X_2)).$$

Thus, we obtain an incidence relation $\lesssim$ that arises from $\sim$ as follows: Two equivalence classes $\mathcal{C}, \mathcal{C}' \in \mathcal{P}$ satisfy $\mathcal{C} \lesssim \mathcal{C}'$ if and only if for every $y \in \mathcal{C}'$ there exists an element $x \in \mathcal{C}$ such that $\phi(x)$ is incident to $\phi(y)$ in Y . Since the different equivalence classes of $\mathcal{P}$ form sets of pre-images under the epimorphism ϕ , we know that equipping $\mathcal{P}$ with the relation $\lesssim$ results in a well-defined simplicial surface $(X/\sim) = (\mathcal{P}, \lesssim)$.

Thus, the equivalence relation $\sim$ is indeed an admissible relation of X . It remains to show that $(X/\sim)$ and Y are isomorphic. This can be seen by constructing the map $\phi' : (X/\sim) \rightarrow Y$ defined by $\phi'(\mathcal{C}) := \phi(x)$, where x is an element in $\mathcal{C}$. Since the equivalence classes of $\mathcal{P}$ are sets of pre-images, we know that the equality $\phi(x) = \phi(x')$ holds for all $x, x' \in \mathcal{C}$. Hence, the map ϕ' is independent of the choice of a representative of an equivalence class and therefore well-defined. It is easy to see that ϕ' forms an isomorphism from $(X/\sim)$ to Y .

Now, we assume that there exists an admissible relation $\sim$ of X with $(X/\sim)$ being isomorphic to Y . Therefore, there exists an isomorphism $\psi_1 : (X/\sim) \rightarrow Y$. Further, we know that for every $x \in X$ there exists a unique equivalence class $M_x \in (X/\sim)$ such that x is contained in M_x . Thus, the map $\psi_2 : X \rightarrow (X/\sim), x \mapsto M_x$ forms a well-defined epimorphism from X onto $(X/\sim)$. Hence, $\psi_1 \circ \psi_2$ is an epimorphism from X onto Y . $\square$

Inspecting the proof of Proposition 4.1.5 in more detail gives us the following result.

Corollary 4.1.6 *Let X and Y be simplicial surfaces and $\phi : X \rightarrow Y$ an epimorphism. Then the partition $\{\phi^{-1}(y) \mid y \in Y\}$ yields an admissible relation $\sim$ of X with $(X/\sim) \cong Y$.*

In [74] simple simplicial surfaces have been introduced as simplicial surfaces that have exactly one admissible relation, namely the trivial admissible relation. By combining the results that are presented in Proposition 4.1.5 and Corollary 4.1.6, we see that the definitions of simple simplicial surfaces presented in this work and in [74] are equivalent. We make use of the above results in order to determine whether a given simplicial surface X is simple. Instead of considering all possible epimorphisms ϕ of X and verifying that all surfaces $\phi(X)$ are isomorphic to X , Proposition 4.1.5 allows us to prove that X is simple by showing that the trivial admissible relation is the only admissible relation of X .

In general, it is not clear whether the existence of a certain sub-surface of a simplicial surface X implies that X is simple. On the other hand, we are able to deduce that a closed simplicial surface is not simple if it contains vertices of degree 2.

Proposition 4.1.7

Let X be a closed simplicial surface and $v \in X_0$ with $\deg(v) = 2$. Then X is not simple.

Proof. Without loss of generality we choose X to be connected. If the simplicial surface X consists of exactly two faces, then X is isomorphic to the simplicial Janus-Head. Since the two faces of X are incident to the same three edges and three vertices, it is easy to see that X can be mapped onto the simplicial One-Face and that the resulting map forms an epimorphism. Now, let us assume $|X_2| \geq 4$. Since $\deg(v) = 2$, there exist distinct vertices v_1, v_2, v_3 , edges $e_1, \dots, e_6$ and faces F_1, F_2, F_3 of X such that the following incidences are satisfied:

$$\begin{aligned} X_0(e_1) &= \{v, v_1\}, X_0(e_2) = \{v, v_2\}, X_0(e_3) = \{v_1, v_2\}, X_0(e_4) = \{v_1, v_2\}, \\ X_0(e_5) &= \{v_2, v_3\}, X_0(e_6) = \{v_1, v_3\}, X_1(F_1) = \{e_1, e_2, e_3\}, \\ X_1(F_2) &= \{e_1, e_2, e_4\}, X_1(F_3) = \{e_3, e_5, e_6\}. \end{aligned}$$

The sub-surface of X that contains the above vertices, edges and faces is illustrated in Figure 4.3 on the left. With this information we can construct an admissible relation $\sim$ of X whose corresponding equivalence classes are given by

$$\{\{v_1\}, \{v_2\}, \{v, v_3\}, \{e_1, e_6\}, \{e_2, e_5\}, \{e_3, e_4\}, \{F_1, F_2, F_3\}\} \uplus \{\{x\} \mid x \notin X \setminus M\},$$

where $M := \{v, v_1, v_2, v_3\} \uplus \{e_1, \dots, e_6\} \uplus \{F_1, F_2, F_3\}$. The sub-surface of $Y := (X/\sim)$ that contains all non-trivial equivalence classes of $\sim$, i.e. $Y(\{F_1, F_2, F_3\})$, where $\{F_1, F_2, F_3\}$ forms a face of Y (see (*) in Section 1), is illustrated in Figure 4.3 on the right.

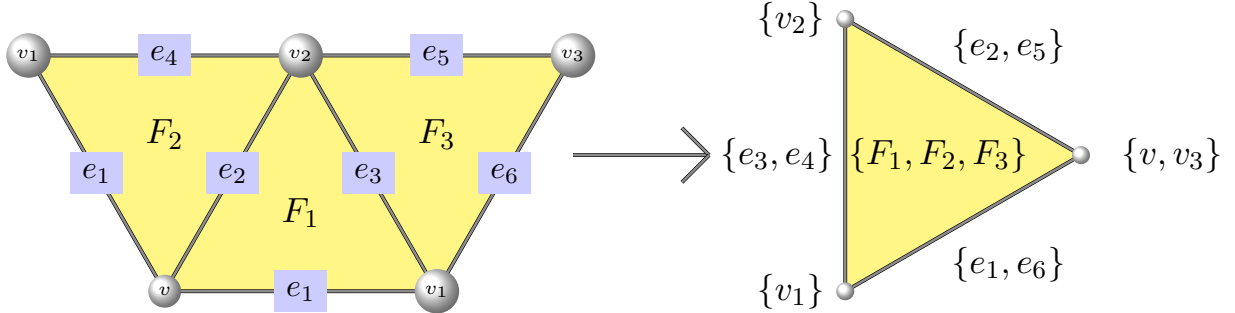


Figure 4.3: Constructing an admissible relation of the simplicial surface X

Since the number of faces of Y satisfies $|Y_2| = |X_2| - 2 < |X_2|$, we know that X and Y are not isomorphic. This implies that $\sim$ is not trivial and X can therefore not be simple. $\square$

The above result implies that for every $n \in \mathbb{N}$ the simplicial n -fold Janus-Head (see Section 2) is not simple. In particular, it can be shown that for every $n > 1$ there exist exactly n admissible relations of the simplicial n -fold Janus-Head such that the resulting simplicial surfaces are isomorphic to the simplicial $(n-1)$ -fold Janus-Head. Moreover, we observe that the simplicial n -fold Janus-Head can be mapped onto the simplicial One-Face via a corresponding epimorphism. Note that by using similar arguments as presented in the proof of Proposition 4.1.7 it can be verified that open simplicial surfaces with vertices of degree 2 are not simple.

Next, we examine simplicial surfaces that do not contain so-called 3-waists.

Definition 4.1.8 (see [74])

Let X be a closed simplicial surface. Furthermore, for $i = 1, 2, 3$ let v_i be a vertex and e_i an edge of X . We call a tuple $(v_1, e_1, v_2, e_2, v_3, e_3)$ a **3-waist** of X , if

- $X_0(e_i) = \{v_i, v_{i+1}\}$ for $i = 1, 2, 3$, where we read the subscripts modulo 3 and
- $X_2(e_i) \cap X_2(e_{i+1}) = \emptyset$ for $i = 1, 2, 3$, where we read the subscripts modulo 3.

In particular, we show that closed vertex-faithful simplicial surfaces containing two neighbouring vertices of degree 4 that do not lie on a 3-waist cannot be simple.

Proposition 4.1.9

Let X be a closed vertex-faithful simplicial surface. Furthermore, let v and v' be two neighbouring vertices with $\deg(v) = \deg(v') = 4$. If v and v' are not contained in a 3-waist of X , then X is not simple.

Proof. Since v and v' are vertices of X with $\deg(v) = \deg(v') = 4$ that do not lie on a 3-waist and X is vertex-faithful, we know that there exist distinct vertices $v_1, \dots, v_4$, edges $e_1, \dots, e_{11}$ and faces $F_1, \dots, F_6$ of X such that the following incidences are satisfied:

$$\begin{aligned} X_0(e_1) &= \{v_1, v_4\}, X_0(e_2) = \{v, v_1\}, X_0(e_3) = \{v', v_1\}, X_0(e_4) = \{v_1, v_2\}, \\ X_0(e_5) &= \{v, v_4\}, X_0(e_6) = \{v, v'\}, X_0(e_7) = \{v', v_2\}, X_0(e_8) = \{v_3, v_4\}, \\ X_0(e_9) &= \{v, v_3\}, X_0(e_{10}) = \{v', v_3\}, X_0(e_{11}) = \{v_2, v_3\}, \\ X_1(F_1) &= \{e_1, e_2, e_5\}, X_1(F_2) = \{e_2, e_3, e_6\}, X_1(F_3) = \{e_3, e_4, e_7\}, \\ X_1(F_4) &= \{e_5, e_8, e_9\}, X_1(F_5) = \{e_6, e_9, e_{10}\}, X_1(F_6) = \{e_7, e_{10}, e_{11}\}. \end{aligned}$$

The sub-surface consisting of the above vertices, edges and faces of X is illustrated in Figure 4.4 on the left.

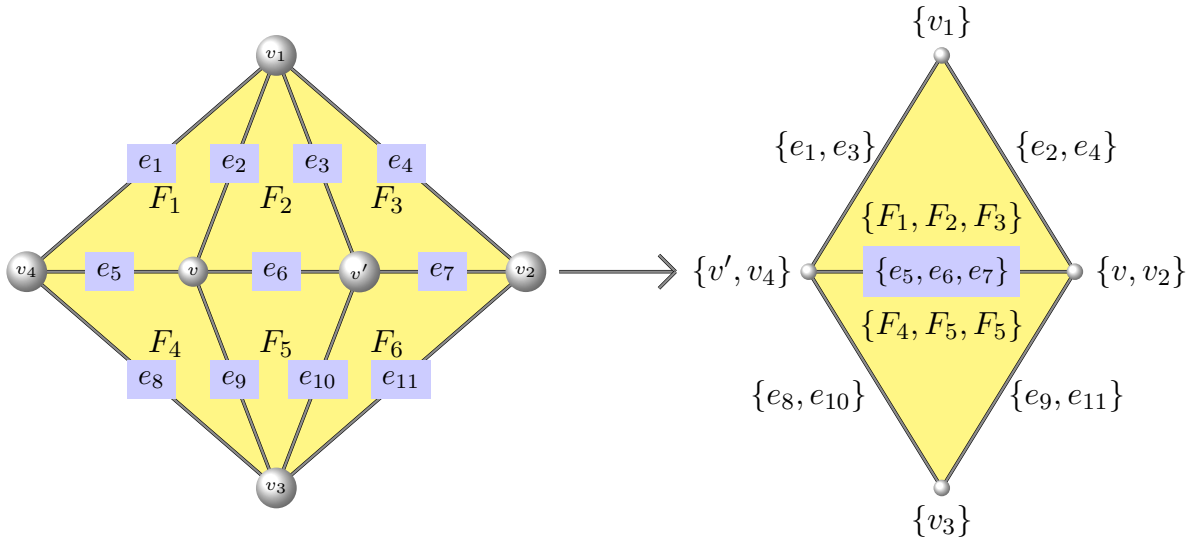


Figure 4.4: Constructing an admissible relation of the simplicial surface X

We prove the statement by giving a suitable admissible relation of X . We define an admissible relation $\sim$ of X by constructing the following partition of X into equivalence classes (see Figure 4.4 on the right):

$$\begin{aligned} & \{\{v', v_4\}, \{v, v_2\}\} \uplus \{\{\hat{v}\} \mid \hat{v} \in X_0 \setminus \{v, v', v_2, v_4\}\} \uplus \\ & \{\{e_1, e_3\}, \{e_2, e_4\}, \{e_5, e_6, e_7\}, \{e_8, e_{10}\}, \{e_9, e_{11}\}\} \uplus \{\{e\} \mid e \in X_1 \setminus \{e_1, \dots, e_{11}\}\} \uplus \\ & \{\{F_1, F_2, F_3\}, \{F_4, F_5, F_6\}\} \uplus \{\{F\} \mid F \in X_2 \setminus \{F_1, \dots, F_6\}\}. \end{aligned}$$

The simplicial surface $Y := (X/\sim)$ satisfies $|Y_2| = |X_2| - 4 < |X_2|$. This implies that X and Y are not isomorphic and hence X is not simple by Proposition 4.1.5. $\square$

By exploiting the above construction, we are able to give the following corollary.

Corollary 4.1.10 *Let $n \geq 4$ be a natural number. Then there exists an epimorphism from the simplicial double-umbrella $(n)^2$ onto the simplicial double-umbrella $(n-2)^2$.*

We observe that the simplicial double-umbrella $(2)^2$ is not simple by Proposition 4.1.7. Moreover, it can be observed that simplicial umbrella $(3)^3$ can be mapped onto the simplicial umbrella $\mathcal{U}_3$ via an epimorphism. Therefore, the class of simplicial double-umbrellas does not yield examples of simple simplicial surfaces.

With the simplicial One-Face and the simplicial umbrella $\mathcal{U}_3$ we have already seen two examples of open simplicial surfaces that are simple. The *boundary cycles* of both simplicial surfaces have length 3.

Definition 4.1.11 (see [74])

Let X be a simplicial surface. We call $(v_1, e_1, \dots, v_n, e_n)$ a **boundary cycle**, if

1. v_i is a boundary vertex and e_i a boundary edge of X for all $1 \leq i \leq n$,
2. $X_0(e_i) = \{v_i, v_{i+1}\}$ for all $1 \leq i \leq n$, where we read the subscripts modulo n .

Here, we describe that the above statement holds in general, i.e. boundary cycles of open simple simplicial surfaces always have length 3.

Proposition 4.1.12

Let X be an open simplicial surface, $n \in \mathbb{N}$ a natural number and $(v_1, e_1, \dots, v_n, e_n)$ a boundary cycle of X . If X is simple, then the equality $n = 3$ holds.

Proof. We prove the statement by contradiction and assume that X is an open simple simplicial surface containing a boundary cycle $P = (v_1, e_1, \dots, v_n, e_n)$ of length $n \neq 3$. We aim to conclude the result by giving a case distinction with respect to the length n .

Case $n = 2$: If $n = 2$, we know that X yields an admissible relation $\sim$ that corresponds to the partition $\{\{e_1, e_2\}\} \uplus \{\{x\} \mid x \in X \setminus \{e_1, e_2\}\}$. We notice that $(X/\sim)$ has more inner edges than the simplicial surface X . Hence, $(X/\sim)$ is not isomorphic to X and the existence of the admissible relation $\sim$ establishes that X is not simple. This is a contradiction.

Case $n > 3$: If $n > 3$, then, the vertices v_1, v_3 and the edges e_1, e_2 are pairwise distinct. Because $n > 3$, we can assume that these vertices and edges are not incident to the same face in X . Hence, the partition $\{\{v_1, v_3\}, \{e_1, e_2\}\} \uplus \{\{x\} \mid x \in X \setminus \{v_1, v_3, e_1, e_2\}\}$ gives rise to an admissible relation $\sim$ of X . As described in the previous case, we observe that the simplicial surface $(X/\sim)$ is not

isomorphic to X and hence X is not simple. Since this is a contradiction and the desired result is established. $\square$

Using the terminology from [74], we obtain a different perspective on the simplicial surfaces that are constructed in the above proof. In particular, these simplicial surfaces can be seen as simplicial surfaces that arise from applying *crater* and *rip mender* to the simplicial surface X .

A useful approach to show that a simplicial surface is not simple is to colour the vertices of the given simplicial surface and exploit this colouring to derive a corresponding admissible relation, see Definition 1.3.1. We use these vertex- k -colourings to show that the only simple simplicial surface that has a vertex-3-colouring is the simplicial One-Face, see Corollary 4.1.14.

First, we show that simplicial surfaces with vertex-3-colourings admit admissible relations that result in the simplicial One-Face.

Lemma 4.1.13

Let X be a simplicial surface and $\kappa : X_0 \rightarrow \{1, 2, 3\}$ a vertex-3-colouring of X . Then there exists an admissible relation $\sim$ of X such that the resulting simplicial surface $(X/\sim)$ is isomorphic to the simplicial One-Face.

Proof. In order to prove the above statement, we make use of the vertex-3-colouring κ and construct a Grünbaum-colouring ω of X which yields an admissible relation. We proceed as follows: For an arbitrary edge $e \in X_1$ with $X_0(e) = \{v, v'\}$ we define the colour c_e of e as $c_e := \{\kappa(v), \kappa(v')\}$. Thus, we observe that the vertex-3-colouring κ gives rise to a partition of X_1 consisting of exactly three partition sets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$. These sets are given by

$$\mathcal{E}_i := \{e \in X_1 \mid c_e = \{1, 2, 3\} \setminus \{i\}\}$$

for $i = 1, 2, 3$. Since for every face $F \in X_2$ and every $i = 1, 2, 3$ the statement $X_1(F) \cap \mathcal{E}_i \neq \emptyset$ holds, the three partition sets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ can be interpreted as the colour classes of a Grünbaum-colouring. Hence, the set

$$\mathcal{P} := \{\kappa^{-1}(1), \kappa^{-1}(2), \kappa^{-1}(3)\} \uplus \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\} \uplus \{X_2\}$$

corresponds to a set of equivalence classes of an admissible relation $\sim$ of X . The simplicial surface $(X/\sim)$ arising from the admissible relation $\sim$ has exactly three vertices, three edges and one face. We therefore deduce that this simplicial surface is isomorphic to the simplicial One-Face. $\square$

Corollary 4.1.14 is a direct consequence of the above lemma.

Corollary 4.1.14 *Let X be a simplicial surface and $\kappa : X_0 \rightarrow \{1, 2, 3\}$ a vertex-3-colouring of X . The simplicial surface X is simple if and only if X is isomorphic to the simplicial One-Face.*

By observing the partition sets that are constructed in the proof of Lemma 4.1.13, we are able to give the following remark.

Remark 4.1.15 *We consider the simplicial surface X , the vertex-3-colouring κ and the Grünbaum-colouring ω of X as in the setting of Lemma 4.1.13. Furthermore, we assume that $|X_2| \geq 2$. In the proof of Lemma 4.1.13 we have used the vertex-3-colouring κ and the partition sets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ to construct an admissible relation $\sim$ of X that yields the simplicial surface $(X/\sim)$ that is isomorphic to the simplicial One-Face. Here, we describe that by using the same vertex-3-colouring κ and the same partition sets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, we are able to construct admissible relations of X such that the resulting simplicial surfaces are isomorphic to the simplicial Janus-Head. Let therefore $\emptyset \neq \mathcal{F} \subsetneq X_2$ be an*

arbitrary subset. With this subset we can give a partition of X that corresponds to an admissible relation $\sim'$. This partition is given as follows:

$$\{\kappa^{-1}(1), \kappa^{-1}(2), \kappa^{-1}(3)\} \uplus \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\} \uplus \{\mathcal{F}, X_2 \setminus \mathcal{F}\}.$$

The equivalence relation $\sim'$ is indeed an admissible relation of X . We observe that the resulting simplicial surface has exactly three vertices, three edges and two faces. Hence, $(X/\sim')$ is isomorphic to the simplicial Janus-Head. We see that every non-empty subset $\mathcal{F}$ that does not contain all faces of X yields an admissible relation of X whose corresponding simplicial surface is isomorphic to the simplicial Janus-Head. Thus, by defining $n := |X_2|$ we observe that the above construction yields exactly

$$\frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} = 2^{n-1} - 1$$

admissible relations of X that result in simplicial surfaces that are isomorphic to the simplicial Janus-Head. With similar constructions we can show that there are exactly $3(2^{n-1} - 1)$ admissible relations of X that give rise to simplicial surfaces that are isomorphic to the simplicial Open-Bag and exactly $3(2^{n-1} - 1)$ relations yielding the simplicial Butterfly.

Lemma 4.1.13 forms a powerful tool to examine simplicial surfaces with respect to them being simple. For instance, it allows us to prove that simplicial ripped umbrellas are not simple by constructing corresponding epimorphisms onto the simplicial One-Face.

Proposition 4.1.16

Let $n \in \mathbb{N}$ be a natural number with $n > 1$. Then there exists an epimorphism of $\mathcal{R}_n$ onto the simplicial One-Face.

Proof. The simplicial ripped umbrella $X := \mathcal{R}_n$ contains distinct vertices $v, v_1, \dots, v_{n+1}$ such that the vertices of faces determining the incidence structure of X are given by

$$\Theta(X) = \{\{v, v_i, v_{i+1}\} \mid i = 1, \dots, n\}.$$

With this incidences, we are able to construct a vertex-3-colouring of X . More precisely, the map $\kappa : X_0 \rightarrow \{1, 2, 3\}$ defined by $\kappa(v) := 3$ and $\kappa(v_i) := (i \bmod 2) + 1$ for all $1 \leq i \leq n + 1$ forms a vertex-3-colouring of X . By Lemma 4.1.13, we know that X can be mapped onto the simplicial One-Face. $\square$

With similar arguments we can determine which simplicial umbrellas can be mapped onto the simplicial One-Face.

Proposition 4.1.17

Let $n \in \mathbb{N}$ be a natural number with $n > 1$. Then the simplicial umbrella $\mathcal{U}_n$ has a vertex-3-colouring if and only if n is even.

Proof. If $n = 2$, then $\mathcal{U}_n$ has exactly three vertices. Hence, it is easy to see that $\mathcal{U}_n$ has a vertex-3-colouring. So, let us assume $n \geq 3$. Then the simplicial surface $\mathcal{U}_n$ is vertex-faithful. Thus, $\mathcal{U}_n$ has vertices $v, v_1, \dots, v_n$ such that the entire incidence structure of the simplicial surface is determined by the following set of vertices of faces:

$$\Theta(\mathcal{U}_n) = \{\{v, v_i, v_{i+1}\} \mid i = 1, \dots, n\},$$

where we read the subscripts modulo n . If n is even, we obtain a vertex-3-colouring of $X := \mathcal{U}_n$ by the map $\kappa : X_0 \rightarrow \{1, 2, 3\}$ defined by $\kappa(v) := 3$ and $\kappa(v_i) = (i \bmod 2) + 1$ for all $1 \leq i \leq n$.

Now, let $\kappa : X_0 \rightarrow \{1, 2, 3\}$ be a vertex-3-colouring of X . Further, we assume that the colour of v is given by $\kappa(v) = 3$. Since for all $1 \leq i \leq n$ the vertex v is adjacent to the vertex v_i , the colour of the vertex v_i satisfies $\kappa(v_i) \in \{1, 2\}$. Moreover, since the vertices v_i and v_{i+1} are adjacent for all $1 \leq i \leq n$, we know that $\kappa(v_i) \neq \kappa(v_{i+1})$ holds. So, the colours of the vertices $v_1, \dots, v_n$ alternate between 1 and 2, i.e. $(\kappa(v_1), \dots, \kappa(v_n)) \in \{(1, 2, \dots, 1, 2), (2, 1, \dots, 2, 1)\}$. Thus, n is even. $\square$

The umbrella of a vertex of a given vertex-faithful simplicial surface gives rise to a sub-surface that is isomorphic to a simplicial umbrella. For simplicial surfaces that are not vertex-faithful, this statement is not true in general. However, we can still make use of this observation to obtain the following corollary as a direct consequence of the above proposition.

Corollary 4.1.18 *Let X be a closed simplicial surface. If X has a vertex-3-colouring, then $\deg(v)$ is even for every $v \in X_0$.*

In Remark 4.1.20 we record the fact that the contrary does not hold. In fact, there are simplicial surfaces with only even vertex-degrees that do not have corresponding vertex-3-colourings. Before proceeding to the mentioned remark, we observe that the simplicial surfaces that have a vertex-3-colouring are exactly the simplicial surfaces that have a Grünbaum-colouring with edges being mirror edges.

Proposition 4.1.19

Let X be a closed simplicial surface. Then X can be Grünbaum-coloured such that all edges form mirror edges if and only if there exists a vertex-3-colouring of X .

Proof. First, let $\omega : X_1 \rightarrow \{1, 2, 3\}$ be a Grünbaum-colouring of X such that all edges of X form mirror edges with respect to ω . Furthermore, let $v \in X_0$ be an arbitrary vertex and $(F_1, \dots, F_n)$ the umbrella of v in X . For every $1 \leq i \leq n$ there exists an edge $e_i \in X_1(v)$ with $X_2(e_i) = \{F_i, F_{i+1}\}$. Here, we read the subscripts modulo n . Note that the edges $e_1, \dots, e_n$ are exactly the edges that are incident to the vertex v . We therefore define the colour c_v of the umbrella of v under ω as $c_v := (\omega(e_1), \dots, \omega(e_n))$. Since every edge of X is a mirror edge with respect to ω by our hypothesis, we know that $\omega(e_i) = \omega(e_{i+2})$ for all $i = 1, \dots, n$. This implies (up to cyclic shifts of entries of tuples) that the colour of the umbrella of v satisfies

$$c_v = (\omega(e_1), \dots, \omega(e_n)) \in \{(1, 2, \dots, 1, 2), (1, 3, \dots, 1, 3), (2, 3, \dots, 2, 3)\}.$$

Thus, we obtain a partition of the set X_0 via the sets $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ that are given by

$$\begin{aligned} \mathcal{V}_1 &:= \{v' \in X_0 \mid c_{v'} = (1, 2, \dots, 1, 2)\}, \\ \mathcal{V}_2 &:= \{v' \in X_0 \mid c_{v'} = (1, 3, \dots, 1, 3)\}, \\ \mathcal{V}_3 &:= \{v' \in X_0 \mid c_{v'} = (2, 3, \dots, 2, 3)\}. \end{aligned}$$

Since all edges of X are mirror edges with respect to ω , we know that every face of X is incident to exactly one vertex of each of the above partition sets. That means that the partition sets $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ can be interpreted as the colour classes of a vertex-3-colouring of X .

Now, we assume that X has a vertex-3-colouring $\kappa : X_0 \rightarrow \{1, 2, 3\}$. We prove the statement by constructing a colouring of the edges and showing that this map is a Grünbaum-colouring that enforces every edge of X to be a mirror edge. We define the map $\omega : X_1 \rightarrow \{1, 2, 3\}$ as follows: For

an edge $e \in X_1$ with $X_0(e) = \{v, v'\}$ we define the colour of e under ω as $\omega(e) := i$, where i satisfies $i \in \{1, 2, 3\} \setminus \{\kappa(v), \kappa(v')\}$. Since the map κ forms a vertex-3-colouring of X , we observe that the edges X_1 can be partitioned into three sets. These sets partitioning the edges of X are given by

$$\omega^{-1}(i) = \{e \in X_1 \mid \kappa(X_0(e)) = \{1, 2, 3\} \setminus \{i\}\}$$

for $i = 1, 2, 3$. Since every face $F \in X_2$ is incident to exactly one edge of each of the above partition sets, the map ω is a Grünbaum-colouring of X . It remains to show that every edge $e \in X_1$ is a mirror edge with respect to the Grünbaum-colouring ω . Let therefore e be an arbitrary edge of X . Since e is an inner edge, e is incident to two faces $F_1, F_2 \in X_2$. Thus, let $v_1, \dots, v_4 \in X_0(F_1) \cup X_0(F_2)$ be vertices and $e_1, \dots, e_4 \in X_1(F_1) \cup X_1(F_2)$ edges such that the incidences

$$X_0(e) = \{v_1, v_2\}, X_0(e_1) = \{v_1, v_3\}, X_0(e_2) = \{v_2, v_3\}, X_0(e_3) = \{v_1, v_4\}, X_0(e_4) = \{v_2, v_4\}$$

hold. We assume that e, e_1, e_2 and e, e_3, e_4 are incident to the same face, respectively (see Figure 4.5). If $\kappa(v_1) = 1$ and $\kappa(v_2) = 2$, then $\kappa(v_3) = \kappa(v_4) = 3$ follows. We therefore obtain

$$\omega(e) = 3, \omega(e_1) = 2, \omega(e_2) = 1, \omega(e_3) = 2, \omega(e_4) = 1.$$

The colours 1, 2 and 3 of the vertex colouring κ and the Grünbaum-colouring ω are illustrated in Figure 4.5 with the colours red, green and blue, respectively. Hence, the edge e is a mirror edge of X with respect to ω and the result follows.

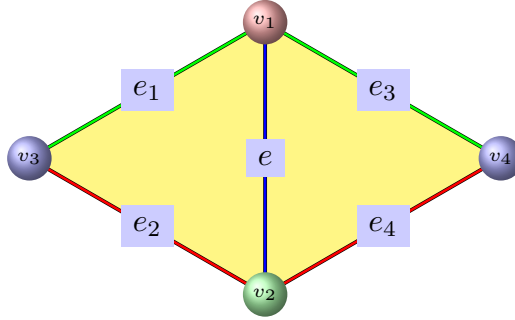


Figure 4.5: The sub-surface that is induced by an edge of the simplicial surface X with corresponding vertices and edges coloured in red, green and blue

□

The set of simplicial surfaces that can be Grünbaum-coloured such that all edges are mirror edges forms an interesting class of simplicial surfaces and therefore occurs in many different situations. For instance, in Section 5 we link these simplicial surfaces to face-transitive surfaces. In the literature these simplicial surfaces are also referred to as *balanced triangulations*, see [91] for further details. In particular, this class of simplicial surfaces have been largely analysed in [74]. Here, we give some additional comments.

Remark 4.1.20 *The simplicial surfaces that can be Grünbaum-coloured such that all edges form mirror edges have been studied in [74]. In their work, Niemeyer et al. have established the result that a simplicial sphere X has only even vertex-degrees if and only if X has a Grünbaum-colouring with all edges being mirror edges. A natural question that arises is whether this statement holds for closed simplicial surfaces in general. Unfortunately, the generalisation of this statement is not true.*

As a counterexample we consider the vertex-faithful simplicial surface X consisting of 9 vertices $v_1, \dots, v_9$ such that the entire incidence structure of X is determined by the vertices of faces below:

$$\begin{aligned} \Theta(X) = \{ & \{v_7, v_8, v_9\}, \{v_6, v_7, v_8\}, \{v_3, v_4, v_5\}, \{v_4, v_8, v_9\}, \{v_2, v_3, v_8\}, \{v_3, v_5, v_7\}, \\ & \{v_1, v_2, v_3\}, \{v_3, v_4, v_8\}, \{v_1, v_2, v_5\}, \{v_4, v_6, v_9\}, \{v_2, v_6, v_8\}, \{v_1, v_6, v_7\}, \\ & \{v_1, v_3, v_7\}, \{v_2, v_4, v_6\}, \{v_2, v_4, v_5\}, \{v_1, v_5, v_9\}, \{v_1, v_6, v_9\}, \{v_5, v_7, v_9\} \}. \end{aligned}$$

Figure 4.6 shows a folding plan of the corresponding simplicial surface. Since X is vertex-faithful, we omit the information on the edges of X in the figure below for simplicity. The simplicial surface X consists of 9 vertices, 27 edges, and 18 faces and therefore has Euler characteristic $\chi(X) = 0$. Since this simplicial surface is orientable, it forms an example of a simplicial torus. This simplicial surface satisfies $\deg(v) = 6$ for all $v \in X_0$. Up to permutation of colours, X has exactly one Grünbaum-colouring ω . This Grünbaum-colouring ω enforces every edge of the simplicial surface X to be a rotational edge.

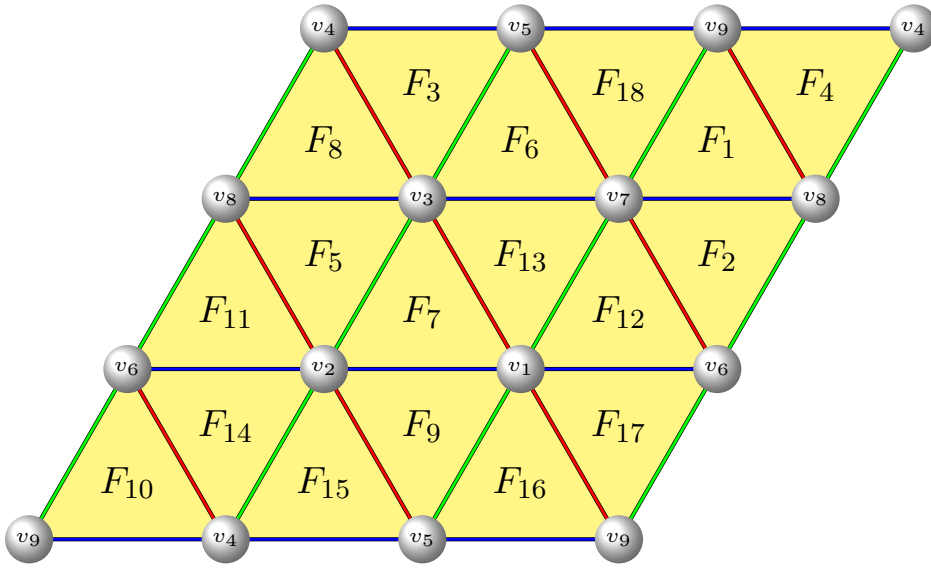


Figure 4.6: A simplicial surface with all vertex-degrees being even that has no Grünbaum-colouring enforcing every edge to be a mirror edge

Additionally, one might ask whether every simplicial surface with only even vertex-degrees has a Grünbaum-colouring that either enforces every edge to be a mirror edge or every edge to be a rotational edge. This statement turns out to be false. As an example, we consider the simplicial surface Y that is determined by

$$\begin{aligned} \Theta(Y) = \{ & \{v_4, v_5, v_6\}, \{v_2, v_3, v_7\}, \{v_1, v_2, v_7\}, \{v_1, v_2, v_8\}, \{v_1, v_3, v_6\}, \\ & \{v_2, v_4, v_9\}, \{v_3, v_5, v_9\}, \{v_3, v_4, v_7\}, \{v_1, v_5, v_{10}\}, \{v_3, v_4, v_6\}, \\ & \{v_2, v_4, v_{10}\}, \{v_2, v_5, v_8\}, \{v_4, v_5, v_9\}, \{v_1, v_3, v_8\}, \{v_1, v_4, v_{10}\}, \\ & \{v_2, v_5, v_{10}\}, \{v_1, v_4, v_7\}, \{v_2, v_3, v_9\}, \{v_1, v_5, v_6\}, \{v_3, v_5, v_8\} \}. \end{aligned}$$

This simplicial surface consisting of 10 vertices, 30 edges and 20 faces has vertex-counter $V_4^5 V_8^5$. Moreover, it can be observed that Y has exactly one Grünbaum-colouring which enforces 20 edges to be mirror edges and 10 edges to be rotational edges.

By Lemma 4.1.13 simplicial surfaces that have vertex-3-colourings and that are not isomorphic to the simplicial One-Face cannot be simple. We extend this fact and prove that a simplicial surface with a vertex-4-colouring is simple if and only if the given simplicial surface is either isomorphic to the simplicial tetrahedron or the simplicial umbrella $\mathcal{U}_3$. For this purpose we introduce the following lemma.

Lemma 4.1.21

Let X be a simplicial surface. Furthermore, let $\kappa : X \rightarrow \{1, \dots, 4\}$ be a proper vertex-4-colouring of X . Then there exists an admissible relation $\sim$ of X such that the resulting simplicial surface $(X/\sim)$ is either isomorphic to the simplicial tetrahedron or the simplicial umbrella $\mathcal{U}_3$.

Proof. We prove the above statement by constructing an admissible relation in a similar way as described in the proof of Lemma 4.1.13. We start by defining a map f that maps an edge of X onto a set of vertex-colours. This map f can be obtained as follows: For an arbitrary edge $e \in X_1$ with $X_0(e) = \{v, v'\}$ we define $f(e)$ to be given as $f(e) := \{\kappa(v), \kappa(v')\}$. We observe that the four colour classes of the vertex-colouring κ give rise to six sets that partition the set of edges X_1 , namely the sets $\mathcal{E}_1, \dots, \mathcal{E}_6$ given by

$$\begin{aligned}\mathcal{E}_1 &:= \{e \in X_1 \mid f(e) = \{1, 2\}\}, \mathcal{E}_2 := \{e \in X_1 \mid f(e) = \{1, 3\}\}, \\ \mathcal{E}_3 &:= \{e \in X_1 \mid f(e) = \{1, 4\}\}, \mathcal{E}_4 := \{e \in X_1 \mid f(e) = \{2, 3\}\}, \\ \mathcal{E}_5 &:= \{e \in X_1 \mid f(e) = \{2, 4\}\}, \mathcal{E}_6 := \{e \in X_1 \mid f(e) = \{3, 4\}\}.\end{aligned}$$

We claim that the sets $\mathcal{E}_1, \dots, \mathcal{E}_6$ are all non-empty. We assume the contrary and without loss of generality consider the set $\mathcal{E}_3$ to be empty. Then there exists no edge $e \in X_1$ that satisfies $f(e) = \{1, 4\}$. This implies $\{\kappa(v') \mid \{v, v'\} \in X_0(X_1(v))\} = \{2, 3\}$ for all $v \in X_0$ with $\kappa(v) \in \{1, 4\}$. This helps us to construct a vertex-3-colouring of X . This vertex-3-colouring κ' of X is achieved by $\kappa'(v') := 1$ for $v' \in \kappa^{-1}(1) \cup \kappa^{-1}(4)$ and $\kappa'(v') = i$ for $v' \in \kappa^{-1}(i)$, where $i = 2, 3$. Hence, κ' is a vertex-3-colouring which contradicts κ being proper. Thus, the sets $\mathcal{E}_1, \dots, \mathcal{E}_6$ are all non-empty.

Now, we construct a map g that assigns a face to a set of vertex-colours with respect to κ and use this map to construct an admissible relation of X . For an arbitrary face $F \in X_2$ with $X_0(F) = \{v_1, v_2, v_3\}$ we define $g(F)$ as $g(F) := \{\kappa(v_1), \kappa(v_2), \kappa(v_3)\}$. Moreover, for every $1 \leq i \leq 4$ we define the set $\mathcal{F}_i$ as the set

$$\mathcal{F}_i := \{F \in X_2 \mid g(F) = \{1, \dots, 4\} \setminus \{i\}\}.$$

Since every face of X lies in exactly one of the above sets, X_2 can be written as $X_2 = \bigsqcup_{i=1}^4 \mathcal{F}_i$. We claim that there exists at most one $1 \leq i \leq 4$ such that $\mathcal{F}_i$ is empty. We therefore assume the contrary. Without loss of generality, we assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are empty. This directly implies that there exists no edge $e \in X_1$ with $f(e) = \{3, 4\}$ which is a contradiction to $\mathcal{E}_6$ being non-empty. Thus, the set

$$\mathcal{P} := \{\kappa^{-1}(i) \mid i = 1, \dots, 4\} \sqcup \{\mathcal{E}_i \mid i = 1, \dots, 6\} \sqcup \{\mathcal{F}_i \mid \mathcal{F}_i \neq \emptyset, i = 1, \dots, 4\}$$

forms a partition of X that corresponds to an admissible relation $\sim$ of X . We observe that if the sets $\mathcal{F}_1, \dots, \mathcal{F}_4$ are all non-empty, then the admissible relation $\sim$ gives rise to a simplicial surface that is isomorphic to the simplicial tetrahedron. Further, if one of the sets $\mathcal{F}_1, \dots, \mathcal{F}_4$ is empty, then the simplicial surface $(X/\sim)$ is isomorphic to the simplicial umbrella $\mathcal{U}_3$. $\square$

The below Corollary follows directly from Lemma 4.1.21.

Corollary 4.1.22 *Let X be a simplicial surface and $\kappa : X_0 \rightarrow \{1, \dots, 4\}$ a proper vertex-4-colouring of X . The simplicial surface X is simple if and only if X is either isomorphic to the simplicial tetrahedron or the simplicial umbrella $\mathcal{U}_3$.*

By using the construction presented in the proof of Lemma 4.1.21, we can show that there exists an admissible relation of the simplicial umbrella $\mathcal{U}_n$, where $n \geq 3$ is odd, such that the simplicial surface that arises from this relation is isomorphic to the simplicial umbrella $\mathcal{U}_3$.

Now, we present Theorem 4.1.23. This theorem establishes that simplicial spheres that are not isomorphic to the simplicial tetrahedron cannot be simple. We deduce this result by exploiting Lemmas 4.1.13 and 4.1.21.

Theorem 4.1.23

Let X be a simplicial sphere. If X is simple, then X is isomorphic to the simplicial tetrahedron.

Proof. We know that the vertex-edge graph of X is planar (see [74] for instance). Thus, the Four-Colour-Theorem (see [10]) implies that the nodes of the vertex-edge graph can be coloured in at most four colours such that adjacent nodes are coloured differently. Such a colouring of the vertex-edge graph translates directly into a vertex- k -colouring κ of X with $k \leq 4$. Since a face of X induces a clique of size 3 in the vertex-edge graph, the number k is either 3 or 4. If $k = 3$, then Lemma 4.1.13 implies that X can be mapped onto the simplicial One-Face via an epimorphism. Hence, X is not simple. Furthermore, if $k = 4$ and there exists no vertex-3-colouring of X , then by Lemma 4.1.21 we know that there exists an epimorphism that maps X onto the simplicial tetrahedron or the simplicial umbrella $\mathcal{U}_3$. Thus, if X is not isomorphic to the simplicial tetrahedron, X cannot be simple. $\square$

As a consequence of the above result, we can decide whether a given open simplicial surface with Euler characteristic 1 is simple. This fact is addressed in the proposition below. The idea to prove this result is to construct a simplicial sphere Z that contains a given open simplicial surface X with $\chi(X) = 1$ as a sub-surface and then exploit Theorem 4.1.23.

Proposition 4.1.24

Let X be a connected open simplicial surface with $\chi(X) = 1$. Then X is simple if and only if X is isomorphic to the simplicial One-Face or the simplicial umbrella $\mathcal{U}_3$.

Proof. In Example 4.1.2, we have established that the simplicial One-Face and the simplicial umbrella $\mathcal{U}_3$ are simple. Now, we assume that X is not isomorphic to one of these two simplicial surfaces. Furthermore, let X' be a simplicial surface that forms an isomorphic copy of X , i.e. $X \cap X' = \emptyset$ and $\phi(X) = X'$, where $\phi : X \rightarrow X'$ is an isomorphism. With these information, we define the simplicial surface Y to be given as the disjoint union of X and X' . Further, we define M as the set that contains all boundary vertices and edges of X . Thus, the set that contains all the boundary vertices and edges of Y is given by $M' := M \cup \phi(M)$. We exploit this fact to construct an admissible relation $\sim$ of Y such that $(Y/\sim)$ is a simplicial sphere. This is achieved by the following partition set:

$$\mathcal{P} := \{\{x, \phi(x)\} \mid x \in M\} \uplus \{\{y\} \mid y \in Y \setminus M'\}.$$

We observe that the equivalence relation $\sim$ that corresponds to the partition $\mathcal{P}$ is indeed an admis-

sible relation of Y with $Z := (Y/\sim)$ being closed and connected. In other words Z is the simplicial surface obtained by glueing X and X' along their boundaries. Moreover, we notice that the number of boundary vertices in M is equal to the number of boundary edges in M . Thus, the simplicial surface Z satisfies

$$\chi(Z) = |Z_0| - |Z_1| + |Z_2| = (2|X_0| - |M|) - (2|X_1| - |M|) + 2|X_2| = 2\chi(X) = 2.$$

Hence, Z forms a simplicial sphere. By Theorem 4.1.23 we know that there exists an admissible relation $\sim_Z$ of Z (with corresponding partition $\mathcal{P}_Z$ of Z) such that the simplicial surface $(Z/\sim_Z)$ is isomorphic to the simplicial One-Face, the simplicial umbrella $\mathcal{U}_3$ or the simplicial tetrahedron. For every $\mathcal{C} \in \mathcal{P}_Z$ we define $\mathcal{C}_X := \bigcup_{m \in \mathcal{C}} (m \cap X)$ and see that

$$\{\mathcal{C}_X \mid \mathcal{C} \in \mathcal{P}_Z\} \setminus \{\emptyset\}$$

forms a partition of X that corresponds to an equivalence relation $\sim_X$ on X . In particular, this relation forms an admissible relation of X such that the simplicial surface $(X/\sim_X)$ is isomorphic to a sub-surface of the simplicial surface $(Z/\sim_Z)$. Hence, $(X/\sim_X)$ is either isomorphic to the simplicial One-Face, the simplicial Butterfly, the simplicial umbrella $\mathcal{U}_3$ or the simplicial tetrahedron. By our assumption, X is not isomorphic to the simplicial umbrella $\mathcal{U}_3$ or the simplicial One-Face. Hence, if $(X/\sim_X)$ is isomorphic to the simplicial One-Face or the simplicial umbrella $\mathcal{U}_3$, then X is not simple. If $(X/\sim_X)$ is isomorphic to the simplicial Butterfly, X can be mapped onto the simplicial One-Face via a corresponding epimorphism which means that X is not simple. Finally, if $(X/\sim_X)$ is isomorphic to the simplicial tetrahedron, then X and $(X/\sim_X)$ cannot be isomorphic (X is open), and again X is not simple. $\square$

With similar arguments, we can establish the result that every open simplicial surface that forms a sub-surface of a simplicial sphere is not simple.

Corollary 4.1.25 *Let X be a connected open simplicial surface containing exactly n boundary cycles with $n > 1$. If $\chi(X) = 2 - n$ and X is orientable, then X is not simple.*

We have observed that the key to proving Theorem 4.1.23 and Proposition 4.1.24 is that a simplicial sphere has a vertex-4-colouring that allows us to partition the vertices, edges and faces of a given simplicial surface in order to obtain an admissible relation. In general, a proper vertex- k -colouring ($k > 4$) of a given simplicial surface that yields an admissible relation derived from the vertex-colouring does not necessarily exist.

So far we have only seen the simplicial One-Face, the simplicial umbrella $\mathcal{U}_3$ and the simplicial tetrahedron as examples of simple simplicial surfaces. The question arises whether these simplicial surfaces are the only examples of simple simplicial surfaces.

Remark 4.1.26 *With our computations in GAP we are able to verify the existence of a simple simplicial surface consisting of 10 faces. We obtain this simplicial surface by the following observation: By implementing the algorithm that is presented in Section 4.3 in GAP, we see that up to isomorphism the simplicial icosahedron can be mapped onto exactly two simplicial surfaces via a butterfly-friendly epimorphism (see Definition 4.3.1). One of these two simplicial surfaces is the simplicial tetrahedron. The other one is a closed simplicial surface X that consists of 6 vertices, 15 edges and 10 faces. This simplicial surface contains six vertices $v_1, \dots, v_6$ such that the incidence structure of X is determined by the following vertices of faces (see Figure 4.7):*

$$\begin{aligned} \Theta(X) = \{ & \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_6\}, \{v_1, v_4, v_5\}, \{v_1, v_5, v_6\}, \\ & \{v_2, v_3, v_5\}, \{v_2, v_4, v_6\}, \{v_2, v_5, v_6\}, \{v_3, v_4, v_6\}, \{v_3, v_4, v_5\} \}. \end{aligned}$$

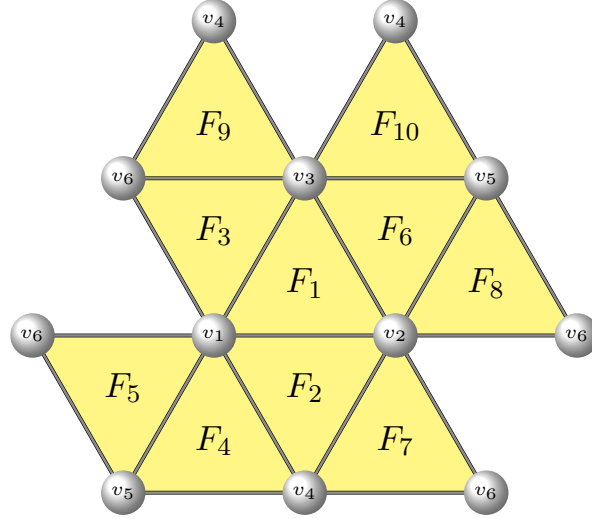


Figure 4.7: A simple simplicial surface X with 10 faces

This simplicial surface satisfies $\chi(X) = 1$ and is not orientable. We notice $\deg(v_i) = 5$ for all $1 \leq i \leq 6$. Even more, X is the famous 6-vertex triangulation of the real projective plane, see [61]. Because X is vertex-faithful, its vertex-edge graph is isomorphic to the complete graph on 6 nodes. We therefore know that no pair of vertices can be identified under an admissible relation of X . We further elaborate on this argument in Lemma 4.1.27 which implies that X is simple.

The examples of simple simplicial surfaces that have been presented in this section all have the property in common that the corresponding vertex-edge graphs are complete graphs. We notice that this observation can be generalised.

Lemma 4.1.27

Let X be a simplicial surface such that the vertex-edge graph of X is isomorphic to the complete graph with n nodes, where $n \in \mathbb{N}$. Then X is simple.

Proof. We show the statement by contradiction and assume that the simplicial surface X is not simple. Then there exists an admissible relation $\sim$ of X such that the arising simplicial surface $Y = (X/\sim)$ is not isomorphic to X . We define the map $\phi : X \rightarrow Y$ by $\phi(x) := \mathcal{C}_x$, where $\mathcal{C}_x$ is the unique equivalence class with respect to $\sim$ that contains the element $x \in X$. Since Y arises from an admissible relation of X , we know that the map ϕ is a well-defined epimorphism. Because X is not isomorphic to Y , there exists at least one equivalence class $\mathcal{V} \subseteq X_0$ of the relation $\sim$ that satisfies $|\mathcal{V}| \geq 2$. Since the vertex-edge graph of X is the complete graph on n nodes, we know that there exists an $e \in X_1$ with $X_0(e) \subseteq \mathcal{V}$. Thus, the epimorphism ϕ is not bijective on the vertices $X_0(e)$ which means that the epimorphism is not well-defined. $\square$

We present a construction that results in examples of simple simplicial surfaces that are not connected.

Example 4.1.28 *Let X and Y be closed simplicial surfaces such that the vertex-edge graphs of X and Y are isomorphic to the complete graphs with n and m nodes, respectively. By Lemma 4.1.27 we know that X and Y are simple. If $n \neq m$, then the disjoint union of X and Y forms a simple simplicial surface.*

Finally, we exploit the above lemma to establish the existence of infinitely many simple simplicial surfaces.

Proposition 4.1.29

There exist infinitely many connected simple simplicial surfaces.

Proof. With [78, Theorem 10.1] we know that there exist infinitely many simplicial surfaces such that the corresponding vertex-edge graphs are complete graphs. By Lemma 4.1.27 these simplicial surfaces are simple. Thus, the result follows. $\square$

4.2 Endomorphism-simple simplicial surfaces

In this subsection, we examine the set of endomorphism-simple simplicial surfaces. We investigate certain classes of simplicial surfaces and analyse whether they contain examples of endomorphism-simple simplicial surfaces. We see that there are indeed various examples of such simplicial surfaces. For instance, we prove that the simplicial umbrella $\mathcal{U}_n$ is endomorphism-simple if and only if n is odd, see Proposition 4.2.10. Moreover, we establish the existence of simplicial spheres that are endomorphism-simple. We conclude this fact by showing that closed simplicial surfaces with all vertex-degrees being odd are endomorphism simple, see Theorem 4.2.12. Finally, we exploit this theorem to show that there exist infinitely many simplicial spheres that are endomorphism-simple.

First, we present the definition of a strong homomorphism of a simplicial surface.

Definition 4.2.1

*Let X and Y be simplicial surfaces. We call a homomorphism $\phi : X \rightarrow Y$ **strong**, if the image $\phi(X)$ is a simplicial surface.*

In general, the image of a homomorphism of simplicial surfaces does not have to be a simplicial surface. Next, we introduce the definition of an endomorphism of a given simplicial surface.

Definition 4.2.2

*Let X be a simplicial surface. We call a strong homomorphism $\phi : X \rightarrow X$ an **endomorphism**.*

Note that the image of a simplicial surface X under an endomorphism forms a sub-surface of X . With the above definition of an endomorphism of a simplicial surface, we are able to present endomorphism-simple simplicial surfaces.

Definition 4.2.3

*Let X be a simplicial surface. We call X **endomorphism-simple**, if every endomorphism $\phi : X \rightarrow X$ is already an automorphism of X .*

The investigation of mathematical objects that do not admit any non-trivial endomorphisms is common in graph-theoretical studies.

Remark 4.2.4 *For instance, in [45] the authors examine generalised Petersen graphs to determine cubic graphs that are cores. In that context, a graph is referred to as a core if every endomorphism of the graph is an automorphism. In their work, García-Marco et al. classify the class of generalised*

Petersen graphs $G(n, k)$ forming cores based on the natural numbers n and k .

Since every endomorphism $\phi : X \rightarrow X$ of a simplicial surface X gives rise to an epimorphism $\phi' : X \rightarrow \phi(X)$, we directly conclude the following proposition.

Proposition 4.2.5

Let X be a simplicial surface. If X is simple, then X is endomorphism-simple.

Therefore, the simplicial One-Face, the simplicial tetrahedron and the simplicial umbrella $\mathcal{U}_3$ are endomorphism-simple. Next, we give an example of a simplicial surface that is endomorphism-simple, but not simple.

Example 4.2.6 *Let X be a simplicial surface with vertices $v_1, \dots, v_8$ such that the entire incidence structure is determined by the vertices of faces below:*

$$\Theta(X) = \{\{v_1, v_7, v_8\}, \{v_5, v_7, v_8\}, \{v_1, v_6, v_7\}, \{v_1, v_6, v_8\}, \{v_2, v_3, v_5\}, \{v_2, v_3, v_4\}, \\ \{v_2, v_4, v_5\}, \{v_3, v_4, v_6\}, \{v_3, v_5, v_7\}, \{v_3, v_6, v_7\}, \{v_4, v_5, v_8\}, \{v_4, v_6, v_8\}\}.$$

This simplicial surface can be constructed by manipulating the incidence structure of the simplicial octahedron $\mathcal{O}$. In particular, the simplicial surface X arises from the simplicial octahedron by applying tetrahedral extensions to two faces $F_1, F_2 \in \mathcal{O}_2$ with $\mathcal{O}_0(F_1) \cap \mathcal{O}_0(F_2) = \emptyset$. That means that X is isomorphic to the simplicial surface $T_{F_1}(T_{F_2}(\mathcal{O}))$. A folding plan of X is shown in Figure 4.8.

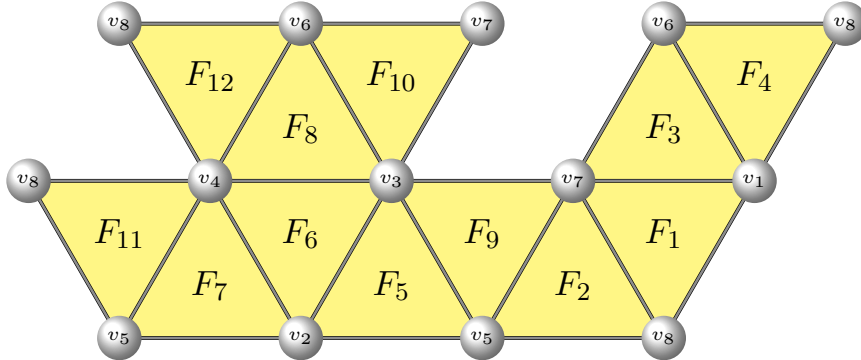


Figure 4.8: An endomorphism-simple simplicial sphere X with 12 faces

Thus, X forms a simplicial sphere with 12 faces and Theorem 4.1.23 implies that X is not simple. At this point we do not have the tools to prove that X is endomorphism-simple. The theorem that yields this result, Theorem 4.2.12, is presented in this section. We elaborate further on X being endomorphism-simple after presenting the proof of the mentioned theorem, see Remark 4.2.13.

We know that there exists an endomorphism of a given simplicial surface X , if there exists an admissible relation of X such that the resulting simplicial surface is isomorphic to a sub-surface of X . In this subsection we make use of this characterisation. In particular, we construct suitable admissible relations to show that a given simplicial surface is not endomorphism-simple. We proceed by observing that the simplicial double-umbrellas are not endomorphism-simple. We conclude this statement by showing that the simplicial double-umbrellas can be mapped onto simplicial umbrellas via corresponding endomorphisms.

Proposition 4.2.7

Let X be the simplicial double-umbrella with $2n$ faces, where $n \geq 2$ is a natural number. Then X is not endomorphism-simple.

Proof. Since X is the simplicial double-umbrella with $2n$ faces, we know that X consists of $n + 2$ vertices, $3n$ edges and $2n$ faces. Up to isomorphism, the sets X_0, X_1 and X_2 are given by

$$\begin{aligned} X_0 &= \{v_1, \dots, v_n\} \uplus \{v, v'\}, \\ X_1 &= \{e_1, \dots, e_n\} \uplus \{e'_1, \dots, e'_n\} \uplus \{\hat{e}_1, \dots, \hat{e}_n\}, \\ X_2 &= \{F_1, \dots, F_n\} \uplus \{F'_1, \dots, F'_n\}, \end{aligned}$$

respectively. By reading the following subscripts modulo n , we are able to describe the incidence structure of X fully. In particular, for $1 \leq i \leq n$ the incidences of the simplicial surface X are given by:

$$\begin{aligned} X_0(e_i) &= \{v, v_i\}, \quad X_0(e'_i) = \{v', v_i\}, \quad X_0(\hat{e}_i) = \{v_i, v_{i+1}\}, \\ X_1(F_i) &= \{e_i, e_{i+1}, \hat{e}_i\}, \quad X_1(F'_i) = \{e'_i, e'_{i+1}, \hat{e}_i\}. \end{aligned}$$

The simplicial surface X yielding the above incidences is illustrated in Figure 4.9.

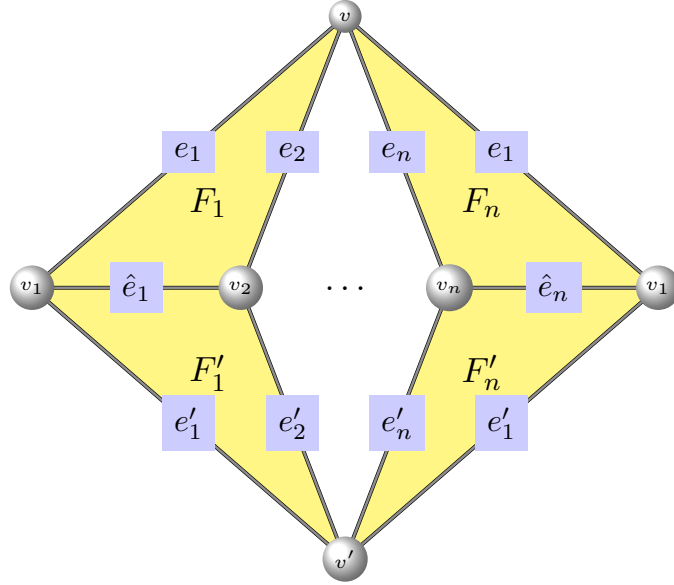


Figure 4.9: The simplicial double-umbrella X with $2n$ faces

With this information we define an admissible relation $\sim$ of X that corresponds to an endomorphism of X that is not an automorphism of X . We construct this admissible relation as follows: We define $\sim$ by the following partition of X into equivalence classes:

$$\begin{aligned} &\{\{v, v'\}\} \uplus \{\{v_i\} \mid i = 1, \dots, n\} \uplus \{\{\hat{e}_i\} \mid i = 1, \dots, n\} \uplus \\ &\{\{e_i, e'_i\} \mid i = 1, \dots, n\} \uplus \{\{F_i, F'_i\} \mid i = 1, \dots, n\}. \end{aligned}$$

The simplicial surface $(X/\sim)$ that arises from the admissible relation $\sim$ is isomorphic to the simplicial umbrella $\mathcal{U}_n$.

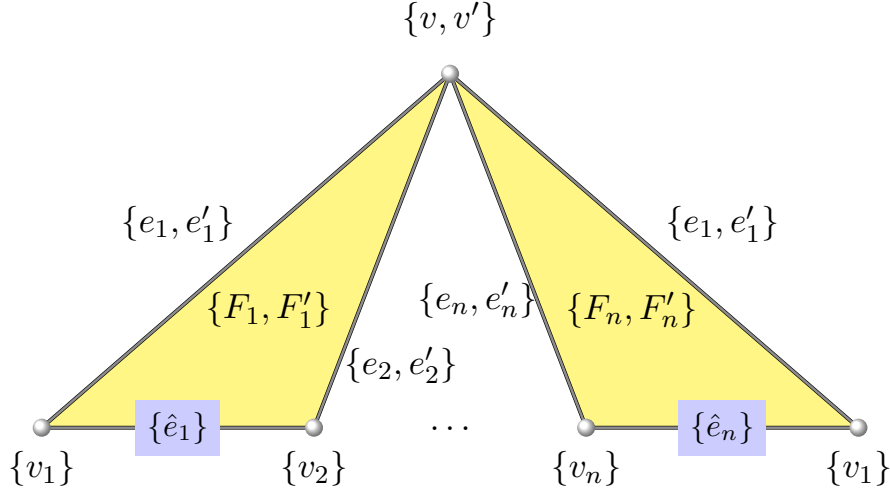


Figure 4.10: The simplicial surface $(X/\sim)$ that arises from an admissible relation of X

Since there exists a sub-surface of X that is isomorphic to the simplicial umbrella $\mathcal{U}_n$, we know that X is not endomorphism-simple. $\square$

Additionally, we present the fact that simplicial surfaces that have vertex-3-colourings cannot be endomorphism-simple.

Proposition 4.2.8

Let X be a simplicial surface and $\kappa : X_0 \rightarrow \{1, 2, 3\}$ a vertex-3-colouring of X . If X is not isomorphic to the simplicial One-Face, then X is not endomorphism-simple

Proof. If X has a vertex-3-colouring, then we can construct an admissible relation $\sim$ of X such that $(X/\sim)$ is isomorphic to the simplicial One-Face, see Lemma 4.1.13, which is a subsurface of every simplicial surface. Hence, every simplicial surface that has a vertex-3-colouring can be mapped onto the simplicial One-Face via an endomorphism. If the given simplicial surface X is not isomorphic to the simplicial One-Face, then the corresponding endomorphism is not bijective. $\square$

Next, we introduce the lemma that helps us to prove that simplicial surfaces whose vertex-degrees are all odd are endomorphism-simple. This lemma states that a homomorphism of a given simplicial surface maps inner vertices with odd vertex-degrees onto inner vertices with odd vertex-degrees.

Lemma 4.2.9

Let $n \geq 3$ be an odd natural number and $X := \mathcal{U}_n$ a simplicial umbrella. Furthermore, let Y be a simplicial surface such that there exists a strong homomorphism $\phi : \mathcal{U}_n \rightarrow Y$. If $v \in X_0$ is the inner vertex satisfying $\deg_X(v) = n$, then the vertex $\phi(v)$ is an inner vertex of Y with $\deg_Y(\phi(v))$ being odd.

Proof. In order to prove the statement we assume that the simplicial surface X consists of vertices $v, v_1, \dots, v_n$ such that the entire incidence structure of X is determined by the vertices of faces

$$\Theta(X) := \{\{v, v_i, v_{i+1}\} \mid i = 1, \dots, n\},$$

where we read the subscripts modulo n . Since the image of a simplicial surface under a strong homomorphism is again a simplicial surface, we can assume without loss of generality that the strong homomorphism $\phi : X \rightarrow Y$ is surjective.

First, let the equality $n = 3$ hold. Since the simplicial surface $\mathcal{U}_3$ is simple, every epimorphism ϕ of X onto another simplicial surface has to be an isomorphism. It therefore follows that $\phi(v)$ is inner and that $\deg_Y(\phi(v)) = 3$ is odd.

Next, let $n \geq 5$ be an odd natural number and d defined as $d := \deg_Y(\phi(v))$. Since n is odd, X cannot be mapped onto the simplicial One-Face via an epimorphism which implies $d \neq 1$. If $d = n$, then we conclude that $\phi(v)$ is an inner vertex with the vertex-degree $d = n$ being odd. Thus, we examine the case $d < n$. By Proposition 4.1.5, there exists an admissible relation $\sim$ such that $(X/\sim)$ is isomorphic to Y . For simplicity, we identify the simplicial surface Y with the simplicial surface $(X/\sim)$, i.e. $Y = (X/\sim)$.

Case $\phi(v)$ is boundary vertex: First, we assume that $\phi(v)$ is a boundary vertex. Since the simplicial surface Y could have inner edges that are not incident to $\phi(v)$, we exploit the admissible relation $\sim$ to construct another admissible relation $\sim'$ of X such that the simplicial surface $(X/\sim')$ is isomorphic to the simplicial ripped umbrella $\mathcal{R}_d$ with the equivalence class $\{v\}$ forming a vertex of $(X/\sim')$ with vertex-degree d . We achieve this as follows: Since the vertex v of X is adjacent to the vertices $v_1, \dots, v_n$, the set $\{v\}$ is an equivalence class with respect to the admissible relation $\sim$ that forms a vertex of Y . Since we have identified $(X/\sim)$ and Y , this means that $\phi(v) = \{v\}$. Furthermore, since $\phi(v)$ is a boundary vertex of degree d , the vertex $\phi(v)$ is incident to exactly $d+1$ edges. Hence, there exist equivalence classes $\mathcal{E}_1, \dots, \mathcal{E}_{d+1}$ such that these $d+1$ classes are exactly the edges that are incident to the vertex $\{v\}$ in Y . We use these edges to construct a partition of the vertices of X . More precisely, for every $1 \leq i \leq d+1$, we define the set $\mathcal{V}_i$ as $\mathcal{V}_i := \bigcup_{e \in \mathcal{E}_i} X_0(e) \setminus \{v\}$. It can be observed that the set of vertices can be partitioned into $X_0 = \{v\} \uplus \mathcal{V}_1 \uplus \dots \uplus \mathcal{V}_{d+1}$. In the following we exploit this partition of the vertices X_0 to construct a partition of the edges of X . For simplicity, we denote the class $\mathcal{V} \in \{\mathcal{V}_1, \dots, \mathcal{V}_{d+1}\}$ that contains a given vertex $v' \in X_0 \setminus \{v\}$ by $\mathcal{V}(v')$. Thus, let $e_1, \dots, e_n \in X_1$ be the edges satisfying $X_0(e_i) = \{v_i, v_{i+1}\}$ for $1 \leq i \leq n$, where we read the subscripts modulo n . Hence, these edges are not incident to v in X . Therefore, for every $1 \leq i \leq n$ we define the set $\mathcal{E}'_i$ as the set given by $\mathcal{E}'_i := \{e \in X_1 \mid X_0(e) \subseteq \mathcal{V}(v_i) \cup \mathcal{V}(v_{i+1})\}$. Then, for $1 \leq i < j \leq n$ two sets $\mathcal{E}'_i$ and $\mathcal{E}'_j$ have a non-trivial intersection if and only if $\mathcal{E}'_i = \mathcal{E}'_j$. We therefore know that there exist d indices $1 \leq i_1 < \dots < i_d \leq n$ such that the sets $\mathcal{E}'_{i_1}, \dots, \mathcal{E}'_{i_d}$ form a partition of the set $\{e_1, \dots, e_n\}$. Now, if $\mathcal{F}_1, \dots, \mathcal{F}_d$ are the equivalence classes that form exactly the faces of Y , then

$$\{\{v\}, \mathcal{V}_1, \dots, \mathcal{V}_{d+1}\} \uplus \{\mathcal{E}_1, \dots, \mathcal{E}_{d+1}\} \uplus \{\mathcal{E}'_{i_1}, \dots, \mathcal{E}'_{i_d}\} \uplus \{\mathcal{F}_1, \dots, \mathcal{F}_d\}$$

forms a partition of X into equivalence classes yielding an admissible relation $\sim'$. We notice that the simplicial surface $(X/\sim')$ is connected with $\{v\}$ being a vertex of degree d . We see that $(X/\sim')$ has one vertex of degree d , two vertices of degree 1 and $d-1$ vertices of degree 2 if $d \neq 2$ and two vertices of degree d and two vertices of degree 1 if $d = 2$. This implies that $(X/\sim') \cong \mathcal{R}_d$ and hence there exists an epimorphism ϕ' of X onto the simplicial surface $\mathcal{R}_d$, see Proposition 4.1.5. By Proposition 4.1.16 there exists an epimorphism ψ of $\mathcal{R}_d$ onto the simplicial One-Face. Thus, $\psi \circ \phi'$ is an epimorphism of X onto the simplicial One-Face. That means that X has a vertex-3-colouring. This is a contradiction to the fact that the number n is odd. Hence, $\phi(v)$ has to be an inner vertex.

Case d is even: Now, we assume that the vertex $\phi(v)$ is inner with the degree d of $\phi(v)$ being even. Similarly as presented in the case of $\phi(v)$ being a boundary vertex, we can exploit the admissible relation $\sim$ to derive an admissible relation $\approx$ of X such that the arising simplicial surface $(X/\approx)$ is isomorphic to the simplicial umbrella $\mathcal{U}_d$. By Proposition 4.1.17 we know that $\mathcal{U}_d$

has a vertex-3-colouring. Furthermore, Lemma 4.1.13 implies that there exists an epimorphism $\bar{\phi}$ that maps $\mathcal{U}_d$ onto the simplicial One-Face. Finally, Proposition 4.1.5 establishes that there exists an epimorphism $\psi : X \rightarrow (X/\sim)$. We therefore obtain a well-defined epimorphism that maps X onto the simplicial One-Face via $\bar{\phi} \circ \psi$. Again, this means that X has a vertex-3-colouring which contradicts n being odd. Hence, the result follows. $\square$

As a direct consequence of the above lemma we present the following proposition. We show that the simplicial umbrella $\mathcal{U}_n$ is endomorphism-simple if and only if n is odd.

Proposition 4.2.10

Let $n \in \mathbb{N}$ be a natural number. Then the simplicial umbrella $\mathcal{U}_n$ is endomorphism-simple if and only if n is odd.

Proof. If n is even, then there exists an endomorphism from $\mathcal{U}_n$ onto a sub-surface of $\mathcal{U}_n$ that is isomorphic to the simplicial One-Face, see Lemma 4.1.13. Hence, $\mathcal{U}_n$ is not endomorphism-simple. If n is odd, then we conclude the statement as follows: Let therefore $\phi : \mathcal{U}_n \rightarrow \mathcal{U}_n$ be an endomorphism of $\mathcal{U}_n$ with $Y := \phi(\mathcal{U}_n)$ and $v \in \mathcal{U}_n$ the inner vertex of degree n . We know that the image of v under ϕ has to be an inner vertex, by Lemma 4.2.9. Since v is the only inner vertex of $\mathcal{U}_n$, we obtain $\deg_Y(\phi(v)) = n$. Because $\mathcal{U}_n$ is the only sub-surface of $\mathcal{U}_n$ that contains an inner vertex of degree n , we know that the endomorphism ϕ is an automorphism of $\mathcal{U}_n$. Hence, the simplicial surface $\mathcal{U}_n$ is endomorphism-simple. $\square$

Hence, there exist infinitely many connected open endomorphism-simple simplicial surfaces that are not simple. Since a vertex of a vertex-faithful simplicial surface induces a sub-surface that is isomorphic to a simplicial umbrella, we deduce that a closed vertex-faithful simplicial surface with all vertex-degrees being odd can only have a closed simplicial surface as an epimorphic image.

Proposition 4.2.11

Let X be a closed vertex-faithful simplicial surface and Y a simplicial surface such that an epimorphism $\phi : X \rightarrow Y$ exists. If $\deg(v)$ is odd for all $v \in X_0$, then Y is closed.

Proof. With Lemma 4.2.9 we know that for every $v \in X_0$ the vertex $\phi(v)$ is an inner vertex. Thus, it follows that that $Y = \phi(X)$ is closed. $\square$

The above proposition gives an alternative method to prove that the simplicial tetrahedron is simple: Since the simplicial tetrahedron is closed with all vertex-degrees being odd, an epimorphic image of the simplicial tetrahedron must have the same properties. We observe that there is no smaller simplicial surface than the simplicial tetrahedron that is closed and has only odd vertex-degrees and thus we rediscover the fact that the simplicial tetrahedron is simple.

With Proposition 4.2.11 we can prove the theorem establishing the result that a simplicial surface with all vertex-degrees being odd is endomorphism-simple.

Theorem 4.2.12

Let X be a closed connected vertex-faithful simplicial surface such that the vertex-degree $\deg(v)$ is odd for all $v \in X_0$. Then X is endomorphism-simple.

Proof. Let $\phi : X \rightarrow X$ be an endomorphism of X . By Proposition 4.2.11 we know that the sub-surface $\phi(X)$ has to be closed. Since X is connected, X is the only sub-surface of X that is closed. Thus, we conclude the equality $\phi(X) = X$. Therefore, the endomorphism ϕ is an automorphism of X and the result follows. $\square$

Now, we are able to explain that the simplicial surface presented in Example 4.2.6 is indeed endomorphism-simple.

Remark 4.2.13 *We notice that the vertices of the simplicial surface X presented in Example 4.2.6 either have degree 3 or 5. Hence, all vertex-degrees are odd and Theorem 4.2.12 implies that X is endomorphism-simple.*

The observation that simplicial surfaces whose vertex-degrees are all odd form examples of endomorphism-simple simplicial surfaces helps us to construct an infinite family of endomorphism-simple simplicial spheres.

Proposition 4.2.14

There exist infinitely many endomorphism-simple simplicial spheres.

Proof. Let $n > 2$ be an odd natural number. In the following we read all subscripts modulo n . We define the vertex-faithful simplicial surface X^n as follows: Let the vertices of the simplicial surface X^n be given by $v, v_1, \dots, v_n$ and $v', v'_1, \dots, v'_n$. Hence, we determine the incidence structure of X^n by the below vertices of faces:

$$\Theta(X^n) = \{\{v, v_i, v_{i+1}\} \mid i = 1, \dots, n\} \uplus \{\{v_i, v_{i+1}, v'_{i+1}\} \mid i = 1, \dots, n\} \uplus \{\{v_i, v'_i, v'_{i+1}\} \mid i = 1, \dots, n\} \uplus \{\{v', v'_i, v'_{i+1}\} \mid i = 1, \dots, n\}.$$

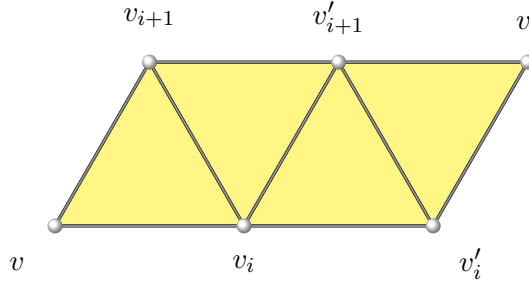


Figure 4.11: A sub-surface of X^n consisting of four faces

By construction, every edge of the simplicial surface X^n is incident to exactly two faces, hence X^n is closed. Thus, the Euler characteristic of X^n is given by

$$\chi(X^n) = |(X^n)_0| - |(X^n)_1| + |(X^n)_2| = |(X^n)_0| - \frac{1}{2}|(X^n)_2| = 2n + 2 - \frac{1}{2} \cdot 4n = 2.$$

Since the simplicial surface X^n contains $4n \geq 12$ faces, it is a simplicial sphere that is not isomorphic to the simplicial tetrahedron. Hence, X^n is not simple. By examining the degrees of the different vertices, we observe that $\deg(v) = \deg(v') = n$ and $\deg(w) = 5$ for all $w \in X_0 \setminus \{v, v'\}$ hold. Thus, by Theorem 4.2.12 the simplicial surface X^n is endomorphism-simple. $\square$

The simplicial surface X^3 that results from the construction in the above proof is isomorphic to the simplicial surface that is presented in Example 4.2.6. Moreover, the simplicial surface X^5 is isomorphic to the simplicial icosahedron. Additionally, we know that for every odd $n \in \mathbb{N} \setminus \{1\}$ the vertex counter of X^n is $V_5^{2n}V_n^2$ if $n \neq 5$ and V_5^{12} if $n = 5$. Lastly, we give a remark on simplicial surfaces whose vertex-degrees are all odd.

Remark 4.2.15 • *Let X be a closed simplicial surface. If all vertex-degrees of X are odd then $|X_0|$ is even. This can be seen by the following fact: We assume that the number of vertices of X is odd. Since every face is incident to exactly three vertices, we know that the following equality holds:*

$$3|X_2| = \sum_{v \in X_0} \deg(v).$$

The sum of the vertex-degrees on the right hand side of the equation is odd. But since X is closed, the number of faces is even, which implies that $3|X_2|$ is even. Thus, the above equation results in a contradiction.

- *By using plantri [25] or the simplicial surfaces library contained in [72] it can be verified that there exist exactly three vertex-faithful simplicial spheres with all vertex-degrees being odd that do not have 3-waists and that have at most 28 faces. Two of those are well known examples of spheres, namely the simplicial tetrahedron and the simplicial icosahedron. The last one is a simplicial sphere with 28 faces that has vertex counter $V_5^{14}V_7^2$. This simplicial surface is isomorphic to the simplicial surface X^7 that forms a member of the family constructed in the proof of Proposition 4.2.14.*

4.3 Computing butterfly-friendly homomorphism

This subsection deals with the investigation of homomorphisms of simplicial surfaces from a computational point of view. As described above, up to isomorphism, we can determine an epimorphism of a given simplicial surface by constructing a suitable admissible relation. Thus, the computer algebra system GAP can be employed to compute epimorphisms of a simplicial surface by constructing partitions of the corresponding vertices, edges and faces that result in admissible relations. In experiments with GAP, we have observed that constructing all admissible relations of a given simplicial surface turns out to be computationally infeasible. In general, there is more hope to compute all butterfly-friendly epimorphisms of a given simplicial surface (see Definition 4.3.1).

Definition 4.3.1 (see [74])

*Let X and Y be simplicial surfaces. A homomorphism $\phi : X \rightarrow Y$ is called **butterfly-friendly**, if for each inner edge $e \in X_1$ the restriction of ϕ to $X_2(e)$ is a surjection onto the set $Y_2(\phi(e))$.*

In this section, we present the details of the algorithm that computes all butterfly-friendly epimorphisms of a closed simplicial surface and therefore allows us to study epimorphisms of given simplicial surfaces in GAP. The corresponding function is available in [72].

Let X and Y be simplicial surfaces such that there exists a homomorphism $\phi : X \rightarrow Y$. Enforcing ϕ to be butterfly-friendly is equivalent to saying that for every inner edge $e \in X_1$ with $X_2(e) = \{F_1, F_2\}$ the image $\phi(e)$ is a boundary edge if and only if $\phi(F_1) = \phi(F_2)$. For instance, the homomorphisms that have been presented in the proof of Proposition 4.2.7 are butterfly-friendly. We proceed by giving an example of a homomorphism that is not butterfly-friendly.

Example 4.3.2 Let X be the disjoint union of two simplicial butterflies with elements and incidences as illustrated in Figure 4.12.

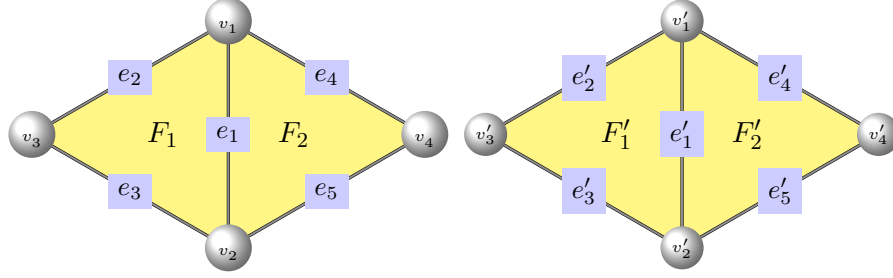


Figure 4.12: The simplicial surface X that arises from the disjoint union of two simplicial butterflies

Thus, the partition set

$$\{\{v_1, v'_1\}, \{v_2, v'_2\}, \{v_3, v_4\}, \{v'_3, v'_4\}\} \uplus \{\{e_1, e'_1\}, \{e_2, e_4\}, \{e'_2, e'_4\}, \{e_3, e_5\}, \{e'_3, e'_5\}\} \uplus \{\{F_1, F_2\}, \{F'_1, F'_2\}\}$$

corresponds to an admissible relation $\sim$ of X . The resulting simplicial surface $(X/\sim)$ is isomorphic to the simplicial Butterfly, see Figure 4.13.

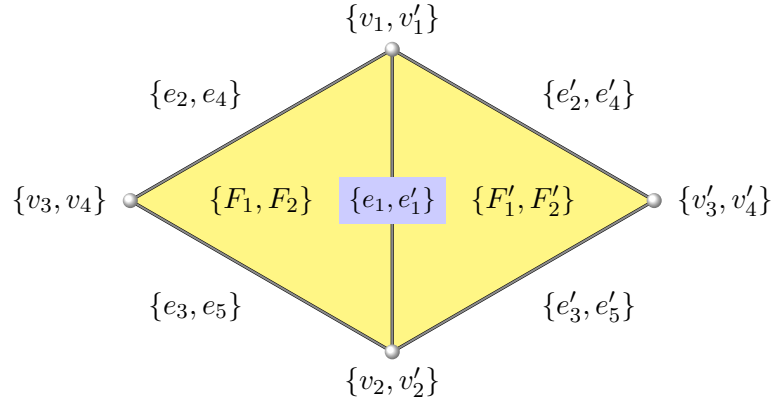


Figure 4.13: The simplicial surface $(X/\sim)$ that arises from an admissible relation of X

We know that for every $x \in X$ there exists a unique equivalence class $M_x \in (X/\sim)$ such that the statement $x \in M_x$ holds. Thus, the map ϕ that is defined by $\phi(x) := M_x$ for an element $x \in X$ forms an epimorphism of X onto $(X/\sim)$ that is not butterfly-friendly. This can be seen by the fact that the restriction of ϕ to $X_2(e_1)$ does not form a surjection onto the faces that are incident to the edge $\phi(e_1) = \{e_1, e'_1\}$, namely $\{\{F_1, F_2\}, \{F'_1, F'_2\}\}$.

In order to establish an algorithm that computes all admissible relations of a closed simplicial surface that correspond to butterfly-friendly epimorphisms, we modify the definition of an admissible relation of a simplicial surface slightly to develop the concept of a *weak admissible relation*.

Definition 4.3.3

Let $(X, <)$ be a simplicial surface and $\sim$ an equivalence relation on a subset $S \subseteq X$ with corresponding partition $\mathcal{P}$ of S into equivalence classes. Furthermore, let $\lesssim$ be the relation on $\mathcal{P}$ that is induced by the relation $<$ as follows: Two elements $\mathcal{C}, \mathcal{C}' \in \mathcal{P}$ satisfy $\mathcal{C} \lesssim \mathcal{C}'$ if and only if for every $y \in \mathcal{C}'$ there exists an element $x \in \mathcal{C}$ such that the statement $x < y$ in X holds. We call $\sim$ a **weak admissible relation** on X , if the following properties are satisfied:

1. For each class $\mathcal{C} \in \mathcal{P}$ there exists $1 \leq i \leq 3$ with $\mathcal{C} \subseteq X_i$, i.e. $\mathcal{P}$ is partitioned into

$$\mathcal{P} = (\mathcal{P} \cap \text{Pow}(X_0)) \uplus (\mathcal{P} \cap \text{Pow}(X_1)) \uplus (\mathcal{P} \cap \text{Pow}(X_2)).$$

2. The tuple $(\mathcal{P}, \lesssim)$ satisfies the axioms (1) - (4) formulated in Definition 1.1.1.

As an example we consider the simplicial ripped umbrella that consists of three faces.

Example 4.3.4 Let $X = \mathcal{R}_3$ be a simplicial ripped umbrella. Then we know that X consists of vertices $X_0 = \{v, v_1, \dots, v_4\}$, edges $X_1 = \{e_1, \dots, e_4\} \uplus \{e'_1, e'_2, e'_3\}$ and faces $X_2 = \{F_1, F_2, F_3\}$ that are given in such a way that the incidence structure of X is determined by $X_0(e_i) = \{v, v_i\}$, for $i = 1, \dots, 4$, and $X_1(F_j) = \{e_j, e_{j+1}, e'_j\}$ and $X_0(e'_j) = \{v_j, v_{j+1}\}$ for $j = 1, 2, 3$. Hence, we observe that the set $\{\{x\} \mid x \in X \setminus \{F_2, e'_2\}\}$ forms a set of equivalence classes that corresponds to a weak admissible relation $\sim'$.

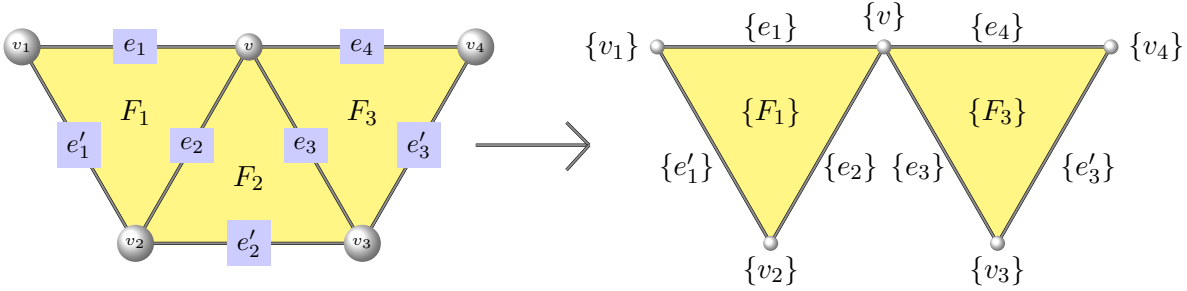


Figure 4.14: Constructing a weak admissible relation of the ripped umbrella $\mathcal{R}_3$

It can be observed that every admissible relation of a given simplicial surface is also a weak admissible relation. Moreover, if X is a simplicial surface and $F \in X_2$ is a face, then the set

$$\{\{x\} \mid x \in X_0(F) \cup X_1(F) \cup \{F\}\}$$

forms a partition set that corresponds to a weak admissible relation of X . We refer to such a weak admissible relation as trivial.

In the following remark we demonstrate how a given admissible relation can be exploited to construct different weak admissible relations of a given simplicial surface.

Remark 4.3.5 Let X be a simplicial surface and $\sim$ an admissible relation of X with corresponding partition set $\mathcal{P}$. Furthermore, let the faces of X be given by $X_2 = \{F_1, \dots, F_n\}$. Here, we show that the admissible relation $\sim$ can be constructed by modifying weak admissible relations step by step: For $i = 1, \dots, n$, we define $\mathcal{F}_i := \{F_1, \dots, F_i\}$. With these sets of faces we can construct an

equivalence relation $\sim_i$, where $1 \leq i \leq n$, that corresponds to the following set of equivalence classes

$$\mathcal{P}_i := \{\mathcal{C} \cap \bigcup_{F \in \mathcal{F}_i} (X_0(F) \cup X_1(F) \cup \{F\}) \mid \mathcal{C} \in \mathcal{P}\}.$$

Hence, we see that the relation $\sim_i$ is a weak admissible relation satisfying $\mathcal{F}_i \subseteq \bigcup_{\mathcal{C} \in \mathcal{P}_i} \mathcal{C}$. After n steps, we finally obtain the admissible relation $\sim_n$ that corresponds exactly to the admissible relation $\sim$.

The above remark gives us an algorithm to construct an admissible relation of a given simplicial surface X with $|X_2| = n$. The idea of this algorithm is to construct a sequence of distinct weak admissible relations $(\sim_1, \dots, \sim_n)$ such that:

1. the weak admissible relation $\sim_1$ is trivial,
2. (loosely speaking) for $i = 1, \dots, n-1$ the relation $\sim_{i+1}$ contains the relation $\sim_i$, (This is described in more detail below.)
3. the relation $\sim_n$ is an admissible relation of X .

Condition 2. means that if $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ are the partitions that correspond to the weak admissible relations $\sim_i$ and $\sim_{i+1}$ respectively, then we know that for every class $\mathcal{C} \in \mathcal{P}_i$ there exists exactly one class $\mathcal{C}' \in \mathcal{P}_{i+1}$ that satisfies $\mathcal{C} \subseteq \mathcal{C}'$. In this case we know that for $i = 1, \dots, n-1$ the statement

$$\bigcup_{\mathcal{C} \in \mathcal{P}_i} \mathcal{C} \subseteq \bigcup_{\mathcal{C}' \in \mathcal{P}_{i+1}} \mathcal{C}'.$$

holds. Here, we aim to construct the weak admissible relation $\sim_{i+1}$ from the relation $\sim_i$ by finding a face $F \in X_2$ that is not contained in any equivalence class of $\mathcal{P}_i$ and modifying these equivalence classes in such a way that the face F , the edges $X_1(F)$ and the vertices $X_0(F)$ are contained in equivalence classes of the arising weak admissible relation $\sim_{i+1}$. There are exactly two cases that can occur by pursuing the above approach, namely we modify the classes of the relation $\sim_i$ in such a way that

1. the set $\{F\}$ forms an equivalence class with respect to the weak admissible relation $\sim_{i+1}$ or
2. there exists an equivalence class $\mathcal{C} \in \mathcal{P}_i$ that yields $\mathcal{C} \cup \{F\}$ being an equivalence class in the partition $\mathcal{P}_{i+1}$

In the first case the partition $\mathcal{P}_{i+1}$ arises from the partition $\mathcal{P}_i$ via

$$\mathcal{P}_{i+1} = \mathcal{P}_i \uplus \{\{x\} \mid x \in (X_0(F) \cup X_1(F) \cup \{F\}) \setminus (\bigcup_{\mathcal{C} \in \mathcal{P}_i} \mathcal{C})\}.$$

This can be achieved as indicated in the pseudo-code formulated in Algorithm 1.

In the second case, the classes of the partition $\mathcal{P}_i$ can be exploited to construct the classes of $\mathcal{P}_{i+1}$. In particular, if $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \in (\mathcal{P}_i \cap \text{Pow}(X_1))$ and $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in (\mathcal{P}_i \cap \text{Pow}(X_0))$ are exactly the equivalence classes that contain the vertices and edges that are incident to the faces in $\mathcal{C}$, then we know that there exist sets $M_1 \subseteq \{1, 2, 3\} \times X_0(F)$ and $M_2 \subseteq \{1, 2, 3\} \times X_1(F)$ with $|M_1| = |M_2| = 3$ such that the equivalence classes of $\mathcal{P}_{i+1}$ are given by

$$\begin{aligned} \mathcal{P}_{i+1} = & (\mathcal{P}_i \setminus \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{C}\}) \uplus \{\mathcal{V}_j \cup \{v\} \mid (j, v) \in M_1\} \uplus \\ & \{\mathcal{E}_j \cup \{e\} \mid (j, e) \in M_2\} \uplus \{\mathcal{C} \cup \{F\}\}. \end{aligned}$$

This is detailed in the pseudo-code presented in Algorithm 2.

Algorithm 1: Extend weak admissible relation 1 (Extend1)

Input: A simplicial surface X , a partition $\mathcal{P}$ corresponding to a weak admissible relation $\sim$ of X ; a face $F \in X_2$ with $F \notin \bigcup_{\mathcal{C} \in \mathcal{P}} \mathcal{C}$

Output: A set of weak admissible relations of X that all contain the equivalence class $\{F\}$

```

1  $M \leftarrow X_0(F) \cup X_1(F) \cup \{F\};$ 
2  $\mathcal{P}' \leftarrow \mathcal{P};$ 
3 candidates  $\leftarrow \{\};$ 
4 for  $x$  in  $M$  do
5   if there exists no class  $\mathcal{C}$  in  $\mathcal{P}$  with  $x \in \mathcal{C}$  then
6      $\mathcal{P}' \leftarrow \mathcal{P}' \cup \{\{x\}\};$ 
7 if  $\mathcal{P}'$  is a weak admissible relation of  $X$  then
8   candidates  $\leftarrow$  candidates  $\cup \{\mathcal{P}'\};$ 
9 return candidates;

```

Algorithm 2: Extend weak admissible relation 2 (Extend2)

Input: A simplicial surface X ; a partition $\mathcal{P}$ corresponding to a weak admissible relation $\sim$ of X ; a class $\mathcal{C} \in \mathcal{P} \cap X_2$; distinct classes $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \in \mathcal{P} \cap X_1$ with $\mathcal{E}_i \lesssim \mathcal{C}$ for $i = 1, 2, 3$; a face $F \in X_2$ with $F \notin \bigcup_{\mathcal{C}' \in \mathcal{P}} \mathcal{C}'$; distinct edges $e_1, e_2, e_3 \in X_1(F)$

Output: Set of weak admissible relations of X that all contain the equivalence class $\mathcal{C} \cup \{F\}$

```

1 candidates  $\leftarrow \{\};$ 
2 for  $i = [1, 2, 3], [3, 1, 2], [2, 3, 1], [3, 2, 1], [1, 3, 2], [2, 1, 3]$  do
3   for  $j = [1, 2, 3], [3, 1, 2], [2, 3, 1], [3, 2, 1], [1, 3, 2], [2, 1, 3]$  do
4     for  $l = 1, 2, 3$  do
5        $[i_1, i_2] \leftarrow [m, n]$  with  $\{i[l], m, n\} = \{1, 2, 3\};$ 
6        $[j_1, j_2] \leftarrow [m, n]$  with  $\{j[l], m, n\} = \{1, 2, 3\};$ 
7        $\mathcal{E}'_{j[l]} \leftarrow \mathcal{E}_{j[l]} \cup \{e_{i[l]}\};$ 
8        $v_{i[l]} \leftarrow v \in \{v' \mid v' \in X_0(e_{i_1}) \cap X_0(e_{i_2})\};$ 
9        $\mathcal{V}_{j[l]} \leftarrow \mathcal{V} \in \{\mathcal{V}' \mid \mathcal{V}' \lesssim \mathcal{E}_{j_1}, \mathcal{E}_{j_2}\};$ 
10       $\mathcal{V}'_{j[l]} \leftarrow \mathcal{V}_{j[l]} \cup \{v_{i[l]}\};$ 
11       $\bar{\mathcal{V}} \leftarrow \{\mathcal{V}'_1, \mathcal{V}'_2, \mathcal{V}'_3\};$ 
12       $\bar{\mathcal{E}} \leftarrow \{\mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3\};$ 
13       $\bar{\mathcal{C}} \leftarrow \mathcal{C} \cup \{F\};$ 
14       $\bar{\mathcal{P}} \leftarrow \mathcal{P} \setminus (\{x \in \mathcal{P} \mid x \lesssim \mathcal{C}\} \cup \{\mathcal{C}\});$ 
15       $\bar{\mathcal{P}} \leftarrow \bar{\mathcal{P}} \cup (\bar{\mathcal{V}} \cup \bar{\mathcal{E}} \cup \{\bar{\mathcal{C}}\});$ 
16      if  $\bar{\mathcal{P}}$  is a weak admissible relation of  $X$  then
17        candidates  $\leftarrow$  candidates  $\cup \{\bar{\mathcal{P}}\}$ 
18 return candidates;

```

We use the above algorithms to describe the algorithm that computes all admissible relations of a given simplicial surface. In experiments we have observed that the ordering of the faces that is used to build up the admissible relation from weak admissible relations has a great impact on the runtime of our implemented algorithm. We have therefore implemented the algorithm in such

a way that we first add the faces that are incident to a given vertex to an already existing weak admissible relation. If all faces of the given vertex are included into the weak admissible relation, we proceed with a vertex that is contained in the resulting weak admissible relation and that is incident to faces that are not yet contained in the weak admissible relation. The details of this algorithm are formulated in Algorithm 3.

Algorithm 3: Algorithm to construct all admissible relations that correspond to butterfly-friendly epimorphism

Input: A simplicial surface X
Output: The set of all admissible relations corresponding to butterfly-friendly epimorphisms

```

1  $v \leftarrow v' \in X_0$ ;
2  $F \leftarrow F' \in X_2(v)$ ;
3  $\text{candidates} \leftarrow \{ \{x\} \mid x \in (X_0(F) \cup X_1(F) \cup \{F\}) \}$ ;
4  $\text{visitedVertices} \leftarrow \{v\}$ ;
5  $\text{visitedFaces} \leftarrow \{F\}$ ;
6 while  $\text{visitedVertices} \neq X_0$  and  $\text{visitedFaces} \neq X_2$  do
7    $\text{umbOfV} \leftarrow [F_1, \dots, F_n]$ , where  $(F_1, \dots, F_n)$  is the (half-)umbrella of  $v$  and  $F_1$  in
    $\text{visitedFaces}$ ;
8    $\text{newcandidates} \leftarrow \{\}$ ;
9   for  $i = 2, \dots, n$  do
10    if  $F_i$  not in  $\text{visitedFaces}$  then
11       $e_1, e_2, e_3 \leftarrow$  edges in  $X_1$  with  $X_1(F_i) = \{e_1, e_2, e_3\}$ ;
12      for  $\mathcal{P}$  in  $\text{candidates}$  do
13         $\text{newcandidates} \leftarrow \text{newcandidates} \cup \text{Extend1}(\mathcal{P}, F_i)$ ;
14        for  $\mathcal{C}$  in  $\mathcal{P} \cap X_2$  do
15           $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \leftarrow$  classes in  $\mathcal{P} \cap X_1$  that are incident to  $\mathcal{C}$ ;
16           $\text{newP} \leftarrow \text{Extend2}(\mathcal{P}, \mathcal{C}, F_i, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, e_1, e_2, e_3)$ ;
17           $\text{newcandidates} \leftarrow \text{newcandidates} \cup \text{newP}$ ;
18       $\text{visitedFaces} \leftarrow \text{visitedFaces} \cup \{F_i\}$ ;
19    $\text{visitedVertices} \leftarrow \text{visitedVertices} \cup \{v\}$ ;
20   if  $(X_0 \setminus \text{visitedvertices}) \neq \emptyset$  then
21      $v \leftarrow v' \in X_0 \setminus \text{visitedvertices}$ ;
22   if  $\text{newcandidates} \neq \emptyset$  then
23      $\text{candidates} \leftarrow \text{newcandidates}$ ;
24   else
25     return  $\{\}$ ;
26  $\text{result} \leftarrow \{\}$ ;
27 for  $\mathcal{P}$  in  $\text{candidates}$  do
28   if  $\mathcal{P}$  corresponds to a Butterfly-friendly epimorphism then
29      $\text{result} \leftarrow \text{result} \cup \{\mathcal{P}\}$ ;
30 return  $\text{result}$  ;

```

By implementing the above algorithm in GAP, we are able to observe the following.

Example 4.3.6 *Here, we present some of the results that we obtain by running Algorithm 3 in*

GAP to compute epimorphic images of different simplicial surfaces.

1. Let $\mathcal{O}$ be the simplicial octahedron. Then, up to isomorphism, there exists exactly 8 simplicial surfaces that form epimorphic images of the simplicial surface $\mathcal{O}$, where the resulting epimorphism is butterfly-friendly, namely the simplicial One-Face, the simplicial Butterfly, the simplicial Open-Bag, the simplicial Janus-Head, the simplicial umbrella $\mathcal{U}_2$, the simplicial umbrella $\mathcal{U}_4$, the simplicial double-umbrella $(2)^2$ and the simplicial octahedron $\mathcal{O}$.
2. Next, we discuss the simplicial double-umbrella $(5)^2$. Since this simplicial surface contains two inner vertices of degree 5, we know by Lemma 4.2.9 that an epimorphic image of $(5)^2$ contains inner vertices of degree 3 or 5. With GAP we verify that $(5)^2$ has exactly 4 epimorphic images, where the corresponding epimorphism is butterfly-friendly, namely the simplicial double-umbrellas $(5)^2$ and $(3)^2$, and the simplicial ripped umbrella $\mathcal{R}_3$ and $\mathcal{R}_5$, up to isomorphism.
3. We know that, up to isomorphism, there exist exactly two vertex-faithful simplicial spheres with 12 faces without 3-waists. One of these simplicial spheres is the simplicial double-umbrella $(6)^2$. By using our derived algorithm in GAP, we see that, up to isomorphism, $(6)^2$ has exactly 18 epimorphic images, where the corresponding epimorphisms are butterfly-friendly. The other simplicial sphere is the simplicial surface X that is described by the vertices of faces

$$\Theta(X) := \{\{v_1, v_4, v_6\}, \{v_1, v_4, v_8\}, \{v_1, v_6, v_7\}, \{v_1, v_7, v_8\}, \{v_2, v_3, v_5\}, \{v_2, v_3, v_6\}, \\ \{v_2, v_4, v_5\}, \{v_2, v_4, v_6\}, \{v_3, v_5, v_7\}, \{v_3, v_6, v_7\}, \{v_4, v_5, v_8\}, \{v_5, v_7, v_8\}\}.$$

The vertex-counter of X is given by $V_4^4 V_5^4$. Again, by Lemma 4.2.9 we know that an epimorphic image of X must contain inner vertices of odd degree. We observe that, up to isomorphism, there are exactly four epimorphic images of X with corresponding butterfly-friendly epimorphism, namely the simplicial surface X , the simplicial tetrahedron, the simplicial surface Y with vertex-counter $V_3^2 V_4^2 V_5^2$ defined by the vertices of faces

$$\Theta(Y) := \{\{v_1, v_3, v_5\}, \{v_1, v_3, v_6\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_6\}, \\ \{v_2, v_3, v_5\}, \{v_2, v_3, v_4\}, \{v_3, v_4, v_6\}, \{v_2, v_4, v_6\}\}$$

and the simplicial surface Z illustrated in the below figure.

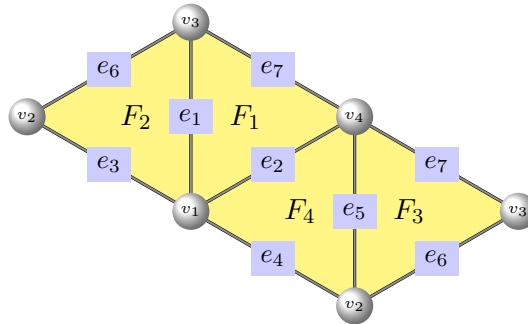


Figure 4.15: An epimorphic image Z of the simplicial surface X

4. Up to isomorphism, there are exactly four vertex-faithful simplicial spheres with 14 faces without 3-waists. These simplicial surfaces can be distinguished by their vertex-counter. We

see that these simplicial surfaces with vertex-counter $V_4^5V_5^2V_6^2$, $V_4^7V_7^2$, $V_4^4V_5^4V_6$ and $V_4^3V_5^6$ have exactly 11, 7, 5 and 2 epimorphic images, where the corresponding epimorphisms are butterfly-friendly, respectively (up to isomorphism).

5 Face-transitive surfaces

In this section we investigate a special class of simplicial surfaces, namely the class of *face-transitive surfaces*. We refer to a simplicial surface as *face-transitive*, if the simplicial surface is vertex-faithful and its automorphism group acts transitively on the faces of the simplicial surface. We aim to analyse this class by distinguishing between closed and open simplicial surfaces. For open face-transitive surfaces we make the following observation:

Observation. *There are exactly five classes of open face-transitive surfaces, see Theorem 5.2.6.*

In that context, we give various examples of open face-transitive surfaces. The key to classifying the set of open face-transitive surfaces is to examine the boundary edges of given face-transitive surfaces.

As a first step in the classification of closed face-transitive surfaces, we present the vertex-face type (see Definition 5.3.2). This distinguishing invariant helps us to establish the following observation.

Observation. *There exist exactly seven types of closed face-transitive surfaces, see Theorem 5.3.3.*

These types of closed face-transitive surfaces correspond exactly to the different vertex-face types. Moreover, we show that these types can be further subdivided into 13 sub-types. We demonstrate that for every sub-type there exists a corresponding closed face-transitive surface, see Section 5.4. Additionally, we present procedures that enable us to reconstruct given face-transitive surfaces from node-transitive cubic graphs. Our investigations lead to the following observation.

Observation. *A face-transitive surface X can be reconstructed from its face graph $\mathcal{F}(X)$ and a certain subgroup of the automorphism group $\text{Aut}(\mathcal{F}(X))$, see Section 5.4.*

We follow the below methodology to construct closed face-transitive surfaces of a specific type: Let $\mathcal{G}$ be a cubic node-transitive graph and $\text{Aut}(\mathcal{G})$ its automorphism group. Our aim is to construct a closed face-transitive surface of a certain vertex-face type having $\mathcal{G}$ as its face graph. For this purpose, we compute a suitable subgroup $H \leq \text{Aut}(\mathcal{G})$ and exploit its properties to derive a cycle double cover that satisfies the necessary and sufficient conditions for it to be the cycle double cover of a face-transitive surface of a prescribed vertex-face type. We address the conditions and the suitable choice of a subgroup H depending on the vertex-face type of the examined face-transitive surface in Section 5.4.

Our investigation is based on the census of cubic node-transitive graphs with at most 1280 nodes addressed in [77] and available in [76]. Based on this database, we compile a census of closed face-transitive surfaces that have at most 1280 faces and whose underlying face graphs have automorphism groups of order at most 10^{11} .

5.1 Properties of face-transitive surfaces

First, we give the formal definition of a face-transitive surface.

Definition 5.1.1

*A simplicial surface X is called **face-transitive**, if it is vertex-faithful and $\text{Aut}(X)$ acts transitively on X_2 .*

This section is based on research originally published in [5].

This definition yields a wide range of interesting simplicial surfaces. We have already seen examples of face-transitive surfaces. The simplicial surfaces that are derived from the Platonic solids in Section 2 are all closed face-transitive surfaces. Additionally, the simplicial One-Face and the simplicial Butterfly are examples of open face-transitive surfaces.

First, we note that if the automorphism group of a simplicial surface acts transitively on the faces of the simplicial surface, then the automorphism group of the corresponding face graph has to be node-transitive.

Lemma 5.1.2

Let X be a face-transitive surface. Then $\mathcal{F}(X)$ is node-transitive.

Proof. We have to show that for any pair of nodes F_1, F_2 of $\mathcal{F}(X)$ there exists an automorphism $\sigma \in \text{Aut}(\mathcal{F}(X))$ that maps F_1 onto F_2 . The nodes F_1 and F_2 can be interpreted as faces of the face-transitive surface X and hence there exists an automorphism $\phi \in \text{Aut}(X)$ that satisfies $\phi(F_1) = F_2$. Since ϕ respects the incidences between the faces and edges of X , we obtain $\phi|_{X_2} \in \text{Aut}(\mathcal{F}(X))$. $\square$

In general, a face-transitive surface does not have to be connected. For instance, the disjoint union of a finite number of simplicial One-Faces forms a face-transitive surface. In the following proposition we establish that the connected components of a face-transitive surface have to be face-transitive surfaces themselves.

Proposition 5.1.3

Let X be a face-transitive surface and Y a connected component of X . Then Y is a face-transitive surface.

Proof. Per assumption, X is face-transitive. Thus, for arbitrary faces $F_1, F_2 \in Y_2 \subseteq X_2$ there exists an automorphism $\phi \in \text{Aut}(X)$ that maps F_1 onto F_2 . Since ϕ respects the incidences of X , every $y_1 \in Y$ has to be mapped onto an element $y_2 \in Y$ under ϕ . This implies $\phi(Y) = Y$ and hence the map $\psi : Y \rightarrow Y, y \mapsto \phi|_Y(y)$ is an automorphism of Y satisfying $\psi(F_1) = F_2$. Since F_1 and F_2 are arbitrary faces of Y , the simplicial surface Y is face-transitive. $\square$

Next, we show that the connected components of a face-transitive surface are all isomorphic.

Proposition 5.1.4

Let X be a face-transitive surface and Y, Z two connected components of X . Then Y and Z are isomorphic.

Proof. Let $F_1 \in Y_2$ and $F_2 \in Z_2$ be arbitrary faces. Since X is face-transitive, there exists an automorphism $\phi \in \text{Aut}(X)$ with $\phi(F_1) = F_2$. Because ϕ respects the incidences of X , we know that every $y \in Y$ has to be mapped onto an element $z \in Z$, i.e. $\phi(y) \in Z$. This implies $\phi(Y) = Z$. Hence, $\phi|_Y$ is injective with $\phi|_Y(Y) = Z$, and therefore Y and Z are isomorphic simplicial surfaces. $\square$

Hence, we are able to restrict our investigations to connected face-transitive surfaces only.

Remark 5.1.5 *Since the connected components of a face-transitive surface are all isomorphic, we assume that the simplicial surfaces in the remainder of this section are all connected.*

We know that the action of the automorphism group of a face-transitive surface X on X_2 results in exactly one orbit. This yields the following question:

Question. Can we determine $|\text{orb}_{\text{Aut}(X)}(X_0)|$ and $|\text{orb}_{\text{Aut}(X)}(X_1)|$?

This question is addressed in detail in Lemma 5.1.6 and Corollary 5.1.7.

Lemma 5.1.6

Let X be a face-transitive surface. Then $|\text{orb}_{\text{Aut}(X)}(X_0)| \leq 3$ holds.

Proof. Let $F \in X_2$ be a face with $X_0(F) = \{v_1, v_2, v_3\}$ and $v \in X_0$ an arbitrary vertex. We know that there exists a face $F' \in X_2$ satisfying $v \in X_0(F')$. Because X is face-transitive, there exists an automorphism $\phi \in \text{Aut}(X)$ mapping F onto F' . Since ϕ respects the incidences of X , the vertices $\phi(v_1), \phi(v_2), \phi(v_3)$ have to be incident to $\phi(F) = F'$. So, there exists a $1 \leq i \leq 3$ with $\phi(v_i) = v$. That means that $v \in \text{orb}_{\text{Aut}(X)}(v_i)$ holds and thus the result follows. $\square$

Corollary 5.1.7 Let X be a face-transitive surface. Then $|\text{orb}_{\text{Aut}(X)}(X_1)| \leq 3$ holds.

Due to the orbit-stabiliser theorem, we know that the orbit of an edge $e \in X_1$ of a closed face-transitive surface X satisfies

$$|\text{orb}_{\text{Aut}(X)}(e)| \in \{\frac{1}{2}|X_2|, |X_2|, \frac{3}{2}|X_2|\}.$$

Moreover, two vertices $v_1, v_2 \in X_0$ that are contained in the same $\text{Aut}(X)$ -orbit must have the same vertex-degree, i.e. $\deg(v_1) = \deg(v_2)$. We exploit this observation to give the following remark.

Remark 5.1.8 Let X be a closed face-transitive surface and $\text{Aut}(X)$ its automorphism group. Furthermore, let $n = |\text{orb}_{\text{Aut}(X)}(X_0)|$ be the number of $\text{Aut}(X)$ -orbits on X_0 . With Lemma 5.1.6, we know that $1 \leq n \leq 3$ holds.

If $n = 3$, then $\text{Aut}(X)$ has exactly three orbits on the vertices of X , namely V_1, V_2, V_3 satisfying $|V_1| = k, |V_2| = l$ and $|V_3| = m$. Thus, for $v_1 \in V_1, v_2 \in V_2$ and $v_3 \in V_3$ we deduce

$$3|X_2| = \sum_{v \in X_0} \deg(v) = \deg(v_1)k + \deg(v_2)l + \deg(v_3)m.$$

In the case that we have exactly two orbits on the vertices of X , namely V_1 of length k and V_2 of length l , we obtain

$$3|X_2| = \sum_{v \in X_0} \deg(v) = \deg(v_1)k + \deg(v_2)l,$$

where $v_1 \in V_1$ and $v_2 \in V_2$. Finally, if $\text{Aut}(X)$ has exactly one orbit on X_0 , we conclude

$$3|X_2| = \sum_{v \in X_0} \deg(v) = \deg(v)|X_0|,$$

where $v \in X_0$. Thus, if all vertices are contained in one $\text{Aut}(X)$ -orbit, either the degree of a vertex or the number of vertices is divisible by 3.

The number of orbits on the vertices of a face-transitive surface can give insights into the structure of the given simplicial surface. For instance, let X be a closed face-transitive surface with exactly one $\text{Aut}(X)$ -orbit on X_0 , and $v \in X_0$ a vertex with $\deg(v) = d$. Then by Remark 5.1.8 the Euler characteristic of X is given by

$$\chi(X) = |X_0| - |X_1| + |X_2| = |X_0| - \frac{1}{2}|X_2| = |X_0| - \frac{d}{6}|X_0| = \frac{6-d}{6}|X_0|.$$

This implies that a face-transitive surface X whose automorphism group acts transitively on the vertices X_0 is a simplicial torus or a simplicial Klein-Bottle if and only if $\deg(v) = 6$. Note that the result that the vertex-degrees of a vertex-transitive simplicial torus all have to be 6 has already been established in [24].

5.2 Open face-transitive surfaces

In the following, we examine the set of open face-transitive surfaces. As a main result, we present Theorem 5.2.6 which establishes that there are exactly five classes of open face-transitive surfaces.

First, we establish the existence of infinitely many open face-transitive surfaces.

Proposition 5.2.1

Let $n \geq 3$ be a natural number. Then the simplicial umbrella $\mathcal{U}_n$ is face-transitive.

Proof. The simplicial surface $X := \mathcal{U}_n, n \geq 3$, is vertex-faithful. Thus, X contains vertices $v, v_1, \dots, v_n$ such that the vertices of faces of X are given by

$$\Theta(X) = \{\{v, v_i, v_{i+1}\} \mid i = 1, \dots, n\},$$

where we read the subscripts modulo n . Note that a permutation of the vertices that leaves $\Theta(X)$ invariant corresponds to an automorphism of X , see Remark 3.1.4. Hence, we construct corresponding permutations of the vertices $v, v_1, \dots, v_n$ to show that all faces of X are contained in the same $\text{Aut}(X)$ -orbit. Let therefore F_1 and F_2 be arbitrary faces of X . Without loss of generality, we assume $X_0(F_1) = \{v, v_1, v_2\}$. Moreover, there exists a $1 \leq i \leq n$ such that $X_0(F_2)$ is given by $X_0(F_2) = \{v, v_i, v_{i+1}\}$. Thus, we see that the permutation $(v_1, \dots, v_n)^{i-1}$ corresponds to an automorphism $\phi \in \text{Aut}(X)$ that maps F_1 onto F_2 . Hence, the result follows. $\square$

We notice that the automorphism group of the simplicial umbrella with n faces is isomorphic to the group $D_{2 \cdot n}$ if $n \geq 3$ and $C_2 \times C_2$ if $n = 2$.

In order to prove the main theorem of this section, Theorem 5.2.6, we have to examine the action of the automorphism group on a given simplicial surface in more detail. First, we notice that an automorphism of a simplicial surface maps boundary edges onto boundary edges and inner edges onto inner edges.

Lemma 5.2.2

Let X be a simplicial surface, $\phi \in \text{Aut}(X)$ an automorphism and $e \in X_1$ an edge. Then e is an inner edge if and only if $\phi(e)$ is an inner edge.

Proof. Let e be an inner edge with $X_2(e) = \{F_1, F_2\}$. Since ϕ is an automorphism of X , i.e. ϕ is bijective and respects the incidences of X , it follows that $\phi(F_1)$ and $\phi(F_2)$ are exactly the faces that are incident to $\phi(e)$. Hence, $\phi(e)$ is an inner edge. If $\phi(e)$ is an inner edge we conclude the proof with the same argument by exploiting ϕ^{-1} as an automorphism that maps $\phi(e)$ onto e . $\square$

Thus, we show that automorphisms of a simplicial surface map inner vertices onto inner vertices.

Lemma 5.2.3

Let X be a simplicial surface, $\phi \in \text{Aut}(X)$ an automorphism and $v \in X_0$ a vertex. Then v is an inner vertex if and only if $\phi(v)$ is an inner vertex.

Proof. Since v is an inner vertex, the edges in $X_1(v)$ are all inner edges. Further, the edges in $X_1(\phi(v))$ are all images of edges in $X_1(v)$ under ϕ . Thus, Lemma 5.2.2 implies that the edges in $X_1(\phi(v))$ are all inner edges and $\phi(v)$ is therefore an inner vertex. By exploiting ϕ^{-1} , we can reason by the same argument that $\phi(v)$ being an inner vertex implies that v is an inner vertex. $\square$

Next, we examine the inner vertices of an open face-transitive surface. With Lemma 5.2.2, we are able to establish that an open face-transitive surface contains at most one inner vertex.

Lemma 5.2.4

If X is an open face-transitive surface, then there exists at most one inner vertex in X .

Proof. We prove the statement by contradiction and assume that X contains two distinct inner vertices v and v' . Since X is open, there exists a boundary edge $e \in X_1$ with $X_0(e) = \{v_1, v_2\}$. Note that v_1 and v_2 are boundary vertices and thus $\{v, v'\} \cap \{v_1, v_2\} = \emptyset$. Furthermore, we know that there exists a face $F \in X_2$ that is incident to e , i.e. $X_2(F) = \{v_1, v_2, v_3\}$ for a vertex $v_3 \in X_0$. If v_3 is a boundary vertex, then all vertices of F are boundary vertices and thus there exists no automorphism $\phi \in \text{Aut}(X)$ that maps F onto a face $F' \in X_2(v)$, see Lemma 5.2.3. Thus, v_3 has to be an inner vertex and we can assume $v = v_3$. We see that the edges e' and e'' with $X_2(e') = \{v, v_1\}$ and $X_2(e'') = \{v, v_2\}$ are both inner edges. Since X is face-transitive, every face in X is incident to exactly two inner edges and one boundary edge by Lemma 5.2.2. This implies that the umbrella of v is a connected component of X , see Figure 5.1 for illustration.

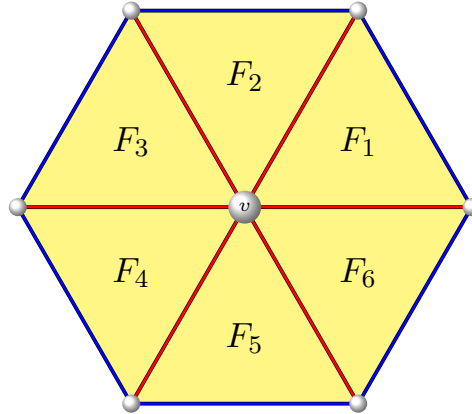


Figure 5.1: The umbrella of v with boundary edges coloured in blue and inner edges coloured in red

Since v' is an inner vertex that is not contained in the described connected component and X is connected, we obtain the desired contradiction. $\square$

As a direct consequence of the above lemma, we obtain the following result.

Corollary 5.2.5 *Let X be a face-transitive surface with exactly one inner vertex. Then X is isomorphic to a simplicial umbrella $\mathcal{U}_n$ with $n \geq 3$.*

The above lemma allows us to prove the main result of this subsection. In particular, we show that an open face-transitive surface is contained in one of five families of simplicial surfaces. We refer the reader to Section 2 for a detailed description of these families.

Theorem 5.2.6

Let X be an open face-transitive surface. Then X is isomorphic to one of the following simplicial surfaces:

1. the simplicial One-Face,
2. the simplicial Butterfly,
3. the simplicial umbrella $\mathcal{U}_n$ with $n \geq 3$,
4. the simplicial strip $\mathcal{S}(3, n)$ with $n \geq 5$,
5. the simplicial strip $\mathcal{S}(4, 2 \cdot n)$ with $n \geq 3$.

Proof. Let $F \in X_2$ be a face and n_F the number of inner edges of F . Since F is incident to exactly three edges, we know that $0 \leq n_F \leq 3$. Further, because of X being face-transitive, every face of X is incident to exactly n_F inner edges. Thus, we prove the above statement by giving a case distinction with respect to the value of n_F .

Case 1 ($n_F = 0$): If $n_F = 0$, then F is incident to exactly three boundary edges. Since X is connected, we know that $X_2 = \{F\}$ and hence X is isomorphic to the simplicial One-Face.

Case 2 ($n_F = 1$): Next, we examine the case $n_F = 1$. Let therefore $e \in X_1$ be the inner edge with $X_2(e) = \{F, F'\}$. The face F' is also incident to exactly one inner edge, namely e , and two boundary edges. Because of X being connected, we know that $X_2 = \{F, F'\}$ and X is therefore isomorphic to the simplicial Butterfly.

Case 3 ($n_F = 3$): If $n_F = 3$, then every edge in X is an inner edge. This implies that X is a closed simplicial surface which is a contradiction.

Case 4 ($n_F = 2$): Finally, we examine $n_F = 2$. Therefore, let $v, v_1, v_2 \in V_0$ be vertices and $e_1, e_2, e'_1 \in X_1$ edges that are incident to the face F . In particular, we assume that e_1 is the boundary edge incident to F and that

$$X_0(e_1) = \{v, v_1\}, X_0(e_2) = \{v, v_2\}, X_0(e'_1) = \{v_1, v_2\}, X_1(F) = \{e_1, e_2, e'_1\}.$$

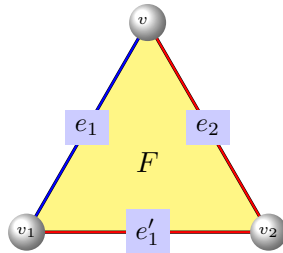


Figure 5.2: The face F of X with boundary edges coloured in blue and inner edges in red

If one of the vertices v, v_1, v_2 is an inner vertex, then it is the only inner vertex of X and thus X is isomorphic to a simplicial umbrella by Corollary 5.2.5. It remains to consider the case that all vertices of X are boundary vertices. We first show that the degrees of v, v_1, v_2 satisfy $\deg(v), \deg(v_1), \deg(v_2) \leq 4$. We verify this statement by contradiction and assume $d := \deg(v) > 4$. Recall that the edge e_1 is the boundary edge that is incident to $F_1 := F$. Let $(F_1, \dots, F_d)$ be the half-umbrella of v with $X_0(F_i) = \{v, v_i, v_{i+1}\}$ for $1 \leq i \leq d$. Further, for every $1 \leq i \leq d$ let e'_i be an edge of X with $X_0(e'_i) = \{v_i, v_{i+1}\}$. Finally, for every $1 \leq i \leq d+1$ let e_i be an edge with $X_0(e_i) = \{v, v_i\}$, see Figure 5.3 for illustration.

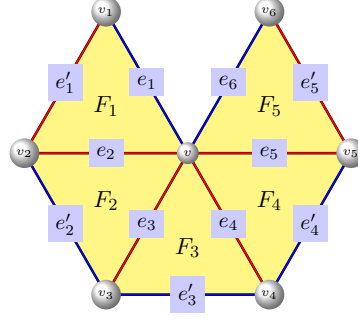


Figure 5.3: The half-umbrella of the vertex v with boundary edges coloured in blue and inner edges coloured in red

Since F_1 has to be incident to exactly two inner edges, we know that e'_1 is an inner edge and the vertex-degree of v_2 therefore satisfies $\deg(v_2) \geq 3$. Moreover, due to e'_2, e'_3, e'_4 being boundary edges, we know that $\deg(v_3) = \deg(v_4) = 2$. Thus, F_3 is incident to two vertices of degree 2 and one vertex of degree d , i.e. $(\deg(v), \deg(v_3), \deg(v_4)) = (d, 2, 2)$. Since an automorphism of X has to map a vertex onto a vertex with the same vertex-degree, there exist no automorphism of X that maps F_1 onto F_3 . Therefore, X is not face-transitive, which is a contradiction.

Hence, $\deg(v), \deg(v_1), \deg(v_2) \leq 4$ follows. We recall that v and v_1 are the vertices of F_1 that are incident to the boundary edge e_1 . Since v_2 is incident to two inner edges, the vertex-degrees satisfy $\deg(v_2) \geq 3$ and $2 \leq \deg(v), \deg(v_1) \leq 4$. Further, because v_2 is a boundary vertex and therefore incident to a boundary edge $e \in X_1$ with $X_2(e) = \{F'\}$, we know that an automorphism that maps $F = F_1$ onto F' has to map the vertex v onto v_2 or the vertex v_1 onto v_2 . We assume that v_1 gets mapped onto v_2 which implies $\deg(v_1) = \deg(v_2)$ (the other case follows with a similar argument). Now, we give a case distinction with respect to the possibilities of forming a tuple $(\deg(v), \deg(v_1), \deg(v_2))$. By our above argumentation we know that $2 \leq \deg(v) \leq 4$ and $3 \leq \deg(v_1) = \deg(v_2) \leq 4$ holds. Thus, there are exactly $3 \cdot 2 = 6$ possibilities to form such a tuple, namely

$$(\deg(v), \deg(v_1), \deg(v_2)) \in \{(2, 3, 3), (3, 3, 3), (4, 3, 3), (2, 4, 4), (3, 4, 4), (4, 4, 4)\}. \quad (**)$$

Since every face of X is incident to exactly two inner edges and one boundary edge, we see that $\deg(v_1) = \deg(v_2) = 4$ holds if and only if $\deg(v) = 2$. Furthermore, $\deg(v) = 4$ implies that a neighbour vertex of v must have vertex-degree 2. This yields $\deg(v_1) = \deg(v_2) = 2$ which is not possible, see (**). Hence, $(\deg(v), \deg(v_1), \deg(v_2))$ is either $(3, 3, 3)$ or $(2, 4, 4)$.

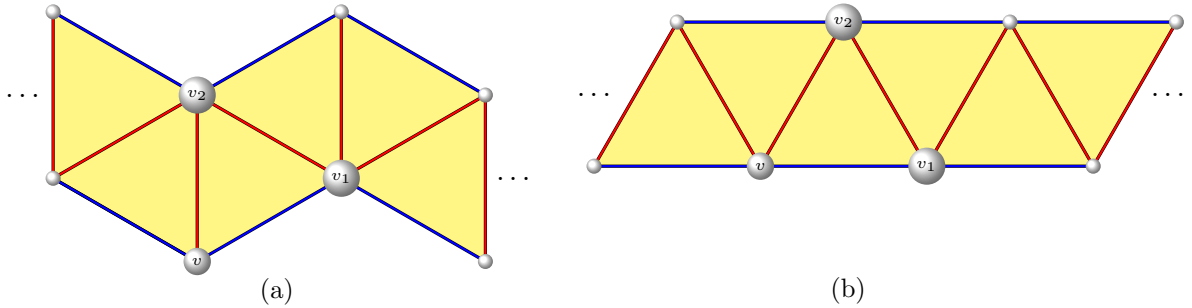


Figure 5.4: Sub-surfaces of a simplicial $(4, 2)$ -strip (a) and a simplicial 3-strip (b), both with boundary edges coloured in blue and inner edges coloured in red

If $(\deg(v), \deg(v_1), \deg(v_2)) = (3, 3, 3)$, then there exists an $n \geq 5$ such that X is isomorphic to the simplicial 3-strip, see Figure 5.4b. Furthermore, if $(\deg(v), \deg(v_1), \deg(v_2)) = (2, 4, 4)$, then there exists an $n \geq 3$ such that X is isomorphic to the simplicial $(4, 2)$ -strip $\mathcal{S}(4, 2 \cdot n)$, see Figures 5.4a. In Propositions 5.2.7 and 5.2.8 we show that if X is isomorphic to one of the above simplicial surfaces, then X is face-transitive, which concludes the proof. $\square$

To prove Theorem 5.2.6 it remains to show that the simplicial strips defined in Section 2 are indeed face-transitive surfaces in order to verify the above theorem.

Proposition 5.2.7

The simplicial strip $\mathcal{S}(3, n)$ with $n \geq 5$ is an open face-transitive surface.

Proof. Let $X := \mathcal{S}(3, n)$ be a simplicial 3-strip with $n \geq 5$. We prove that X is face-transitive by constructing permutations of X_0 that leave $\Theta(X)$ invariant and therefore correspond to automorphisms of X that map arbitrary faces onto each other, see Remark 3.1.4.

Case (n is even): If n is even, we define $k := \frac{n}{2}$ and hence $\Theta(X)$ is given by

$$\Theta(X) = \{\{v_i, v_{i+1}, v_{k+i}\}, \{v_{i+1}, v_{k+i}, v_{k+i+1}\} \mid i = 1, \dots, k-1\} \cup \{\{v_1, v_k, v_n\}, \{v_1, v_{k+1}, v_n\}\}.$$

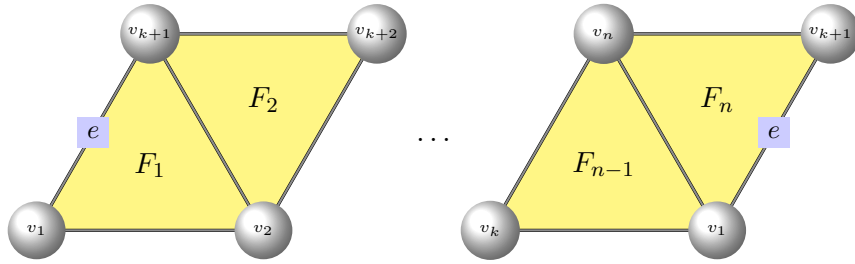


Figure 5.5: The simplicial 3-strip $\mathcal{S}(3, n)$ for an even $n \geq 5$

Furthermore, let $F_1, F_2 \in X_2$ be arbitrary faces and $\pi := (v_1, v_{k+1}, v_2, v_{k+2}, \dots, v_k, v_n)$ a permutation of the vertices. Without loss of generality, we can assume that $X_0(F_1) = \{v_1, v_2, v_{k+1}\}$. Thus, if $X_0(F_2) = \{v_i, v_{i+1}, v_{k+i}\}$ for some $1 \leq i \leq k-1$, then the permutation

$$\pi^{2(i-1)} = ((v_1, \dots, v_k)(v_{k+1}, \dots, v_n))^{(i-1)}$$

corresponds to an automorphism $\phi \in \text{Aut}(X)$ that maps F_1 onto F_2 . If $X_0(F_2) = \{v_1, v_k, v_n\}$, then π^{-2} yields the desired automorphism mapping F_1 onto F_2 . We proceed by examining the case $X_0(F_2) = \{v_{i+1}, v_{k+i}, v_{k+i+1}\}$ for $1 \leq i \leq k-1$. Here, $\pi^{2(i-1)+1}$ is a permutation yielding an automorphism that maps F_1 onto F_2 . Furthermore, if $X_0(F_2) = \{v_1, v_{k+1}, v_n\}$, we obtain a desired permutation of the vertices by π^{-1} . Hence, X is face-transitive.

Case (n is odd): In the case that n is odd, k is defined as $k := \frac{n-1}{2}$ and the vertices of faces of X are given as follows (Figure 5.6):

$$\Theta(X) = \{\{v_i, v_{i+1}, v_{k+i+1}\}, \{v_i, v_{k+i}, v_{k+i+1}\} \mid i = 1, \dots, k\} \cup \{\{v_1, v_{k+1}, v_n\}\}.$$

Again, we construct corresponding permutations of vertices $v, v_1, \dots, v_n$ to show that any two faces of X are contained in the same $\text{Aut}(X)$ -orbit. By defining π as

$$\pi := (v_1, v_{k+2}, v_2, v_{k+3}, \dots, v_k, v_n, v_{k+1}),$$

we can reason that X is face-transitive as shown in the case of n being even. Hence, the result follows.

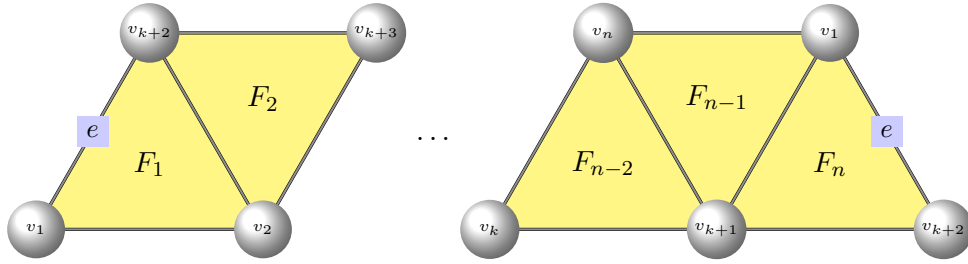


Figure 5.6: The simplicial 3-strip $\mathcal{S}(3, n)$ for an odd $n \geq 5$

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The fact that $\mathcal{S}(4, 2 \cdot n)$ forms a face-transitive surface can be shown with similar arguments, i.e. by constructing permutations of the vertices of $\mathcal{S}(4, 2 \cdot n)$ that correspond to automorphisms that map arbitrary faces onto each other.

Proposition 5.2.8

The simplicial strip $\mathcal{S}(4, 2 \cdot n)$ with $n \geq 3$ is an open face-transitive surface.

By examining the automorphism groups of open face-transitive surfaces, we obtain the following corollary.

Corollary 5.2.9 *Let X be an open face-transitive surface. Then $\text{Aut}(X)$ is isomorphic to*

1. the direct product $C_2 \times C_2$ if X is isomorphic to the simplicial Butterfly,
2. a dihedral group if X is not isomorphic to the simplicial Butterfly.

5.3 Closed face-transitive surfaces

We proceed by investigating closed face-transitive surfaces. First, we recall that the simplicial surfaces in this section are all considered to be connected. Additionally, we assume all following simplicial surfaces to be closed and vertex-faithful. A natural question that arises in this study is whether the set of closed face-transitive surfaces is finite. This question is answered by establishing that the simplicial double umbrellas form examples of face-transitive surfaces.

Proposition 5.3.1

Let $n \geq 3$ be a natural number. Then the simplicial double umbrella $X := (n)^2$ is a face-transitive surface.

Proof. Since X is vertex-faithful, we can determine X by its vertices of faces, namely

$$\Theta(X) = \{\{v, v_i, v_{i+1}\} \mid i = 1, \dots, n\} \uplus \{\{v', v_i, v_{i+1}\} \mid i = 1, \dots, n\},$$

where we read the subscripts modulo n , see Section 2. Hence, v and v' are vertices of degree n and $v_1, \dots, v_n$ are vertices of degree 4.

By Remark 3.1.4, we know that a permutation of X_0 that leaves $\Theta(X)$ invariant corresponds to an automorphism of X . Hence, we construct corresponding permutations to show that X is face-transitive. Thus, let F_1 and F_2 be arbitrary faces of X . Without loss of generality, we can assume $X_0(F_1) = \{v, v_1, v_2\}$. If there exists a $1 \leq i \leq n$ with $X_0(F_2) = \{v, v_i, v_{i+1}\}$, then the permutation $(v_1, \dots, v_n)^{i-1}$ gives us an automorphism of X that maps F_1 onto F_2 . Additionally, if there exists a $1 \leq i \leq n$ with $X_0(F_2) = \{v', v_i, v_{i+1}\}$, we see that the permutation $(v, v')(v_1, \dots, v_n)^{i-1}$ yields an automorphism that maps F_1 onto F_2 , which concludes the proof. $\square$

Next, we present the vertex-face type of a face-transitive surface which forms an invariant that is fundamental for our desired classification of face-transitive surfaces. This invariant helps us to determine the structure of a given face-transitive surface and is the key to the construction methods described in Section 5.4.

Definition 5.3.2

Let X be a face-transitive surface and $F \in X_2$ a face. Further, let $S = \text{Stab}_{\text{Aut}(X)}(F)$ be the stabiliser of F in $\text{Aut}(X)$. Hence, we define the **vertex-face type** $\text{vf}(X)$ of X as

$$\text{vf}(X) := (|\text{orb}_{\text{Aut}(X)}(X_0)|, |S|).$$

Since all face-stabilisers of a face-transitive surface are conjugate, see Lemma 3.1.8, they all have the same order. Thus, the vertex-face type of a closed face-transitive surface X does not depend on the choice of a face $F \in X_2$ and is therefore well-defined.

By examining the possible orders of a face-stabiliser of a closed face-transitive surface X (see Lemma 3.1.5) and the possible number of orbits of the action of $\text{Aut}(X)$ on X_0 (see Lemma 5.1.6), we deduce that there are at most $4 \cdot 3 = 12$ possible vertex-face types for X . In the following theorem, we prove that out of these 12 possibilities only 7 can really occur as the vertex-face type of a given face-transitive surface.

Theorem 5.3.3

Let X be a face-transitive surface. Then the vertex-face type of X satisfies

$$\text{vf}(X) \in \{(3, 1), (2, 1), (2, 2), (1, 1), (1, 2), (1, 3), (1, 6)\}.$$

Proof. Let $F \in X_2$ be a face with $X_0(F) = \{v_1, v_2, v_3\}$. We give a case distinction on the values of $n := |\text{orb}_{\text{Aut}(X)}(X_0)|$, i.e. the number of $\text{Aut}(X)$ -orbits on X_0 . By Lemma 5.1.6, we know that $1 \leq n \leq 3$ holds.

Case $n = 3$: If $n = 3$, then v_1, v_2, v_3 are contained in three pairwise disjoint $\text{Aut}(X)$ -orbits. Hence, an automorphism $\phi \in \text{Stab}_{\text{Aut}(X)}(F)$ has to satisfy $\phi(v_i) = v_i$ for $i = 1, 2, 3$. With Proposition 3.1.3 we know that ϕ must be trivial. So, $|\text{Stab}_{\text{Aut}(X)}(F)| = 1$ and $\text{vf}(X) = (3, 1)$ follow.

Case $n = 2$: We have to show that $\text{vf}(X) \in \{(2, 3), (2, 6)\}$ is not possible. If $|\text{Stab}_{\text{Aut}(X)}(F)| = 3$, then $\text{Stab}_{\text{Aut}(X)}(F)$ is cyclic. Thus, $\text{Stab}_{\text{Aut}(X)}(F)$ contains an automorphism ϕ satisfying

$$\phi|_{\{v_1, v_2, v_3\}} = (v_1, v_2, v_3) \in \text{Sym}(\{v_1, v_2, v_3\}).$$

Thus, v_1, v_2, v_3 are all contained in the same orbit which implies $|\text{orb}_{\text{Aut}(X)}(X_0)| = 1$. This contradicts $n = 2$. With similar arguments we can exclude $|\text{Stab}_{\text{Aut}(X)}(F)| = 6$ which concludes the proof. $\square$

Note that we obtain the following equality for the automorphism group of a face-transitive surface X :

$$\begin{aligned} |\text{Aut}(X)| &= |\text{Stab}_{\text{Aut}(X)}(v)| \cdot |\text{orb}_{\text{Aut}(X)}(v)| \\ &= |\text{Stab}_{\text{Aut}(X)}(e)| \cdot |\text{orb}_{\text{Aut}(X)}(e)| \\ &= |\text{Stab}_{\text{Aut}(X)}(F)| \cdot |X_2|, \end{aligned}$$

where $v \in X_0$ is a vertex, $e \in X_1$ is an edge and $F \in X_2$ is a face. The above equalities enable us to establish the following proposition.

Proposition 5.3.4

Let X be a face-transitive surface with $\chi(X) \in \{1, 2\}$. Then $\text{vf}(X) \neq (1, 1)$.

Proof. We prove the statement by contradiction and assume that X is a face-transitive surface with $\text{vf}(X) = (1, 1)$ and $\chi(X) \in \{1, 2\}$. Hence, if $v \in X_0$ is a vertex, then the order of the automorphism group $\text{Aut}(X)$ satisfies $|\text{Stab}_{\text{Aut}(X)}(v)| \cdot |X_0| = |\text{Aut}(X)| = 1 \cdot |X_2|$. This statement is equivalent to

$$|\text{Stab}_{\text{Aut}(X)}(v)| = \frac{|X_2|}{|X_0|} = \frac{|X_2|}{\chi(X) + \frac{|X_2|}{2}}.$$

Since the order of $\text{Stab}_{\text{Aut}(X)}(v)$ has to be a natural number, the above equation implies that $\chi(X) + \frac{1}{2}|X_2|$ divides $|X_2|$. Because of $\chi(X) + \frac{1}{2}|X_2| > \frac{1}{2}|X_2|$, we conclude $\chi(X) + \frac{1}{2}|X_2| = |X_2|$. This results in $\chi(X) = 2$ and $|X_2| = 4$. So, X is isomorphic to the simplicial tetrahedron $\mathcal{T}$ with $\text{vf}(\mathcal{T}) = (1, 6) \neq (1, 1)$, which is a contradiction. $\square$

5.4 Construction of face-transitive surfaces

Now, we describe our procedures to compute face-transitive surfaces. In particular, we aim to construct these simplicial surfaces by providing suitable cycle double covers of cubic node-transitive graphs. Theorem 5.3.3 forms one of our key ingredients to achieve this goal. For this purpose, we recall that the automorphism group of a simplicial surface X can be embedded into the automorphism group $\text{Aut}(\mathcal{F}(X))$ via the map λ_X , see Remark 3.1.1. Furthermore, we refer the reader to the definition of an automorphism induced α -cycle (see Definition 3.2.2) as this definition is of fundamental importance for the following constructions. We pursue the following approach:

We analyse face-transitive surfaces and their structure based on their vertex-face types, and find strategies to reconstruct the umbrellas of these simplicial surfaces from their face graphs. That means, we describe the corresponding vertex-faithful cycle double covers of the face graphs of given face-transitive surfaces. The results of this section can be summarised in the following observation.

Observation. *A face-transitive surface X can be reconstructed from its face-graph $\mathcal{F}(X)$ with the help of $\lambda_X(\text{Aut}(X))$ acting on $\mathcal{F}(X)$.*

With these information, we approach arbitrary cubic node-transitive graphs and aim to construct closed face-transitive surfaces that have the given cubic graph as face graph. In particular, we start with a cubic node-transitive graph $\mathcal{G} = (V, E)$ and compute subgroups $H \leq \text{Aut}(\mathcal{G})$ satisfying $|H| = s|V|$, where $s \in \{1, 2, 3, 6\}$. Based on the order of a derived subgroup, we use our theoretical results and aim to construct vertex-faithful cycle double covers of $\mathcal{G}$ yielding face-transitive surfaces with prescribed vertex-face types.

To achieve our constructions, we recall that the umbrella of a vertex $v \in X_0$ of a simplicial surface X is denoted by $u(v)$. Moreover, we introduce the definition of a coloured umbrella of X .

Definition 5.4.1

Let X be a simplicial surface, $\omega : X_1 \rightarrow \{1, 2, 3\}$ an edge-colouring and $u(v) = (F_1, \dots, F_n)$ the umbrella of a vertex $v \in X_0$. In the following, we read all subscripts modulo n . Furthermore, let $e_1, \dots, e_n$ be the edges in X with $X_2(e_i) = \{F_i, F_{i+1}\}$.

- The umbrella $u(v)$ is called **mono-coloured**, if $|\{\omega(e_i) \mid i = 1, \dots, n\}| = 1$.
- We say that $u(v)$ is **bi-coloured**, if $|\{\omega(e_i) \mid i = 1, \dots, n\}| = 2$ and the colouring satisfies $\omega(e_i) \neq \omega(e_{i+1})$ for $i = 1, \dots, n$.
- We refer to a cycle as **tri-coloured** if $|\{\omega(e_i) \mid i = 1, \dots, n\}| = 3$ and the colouring satisfies $\omega(e_i) \neq \omega(e_{i+1})$ for $i = 1, \dots, n$.

We define the **colour of the umbrella of v** by $\omega(v) := \omega(u(v)) := (\omega(e_1), \dots, \omega(e_n))$. Furthermore, we call $u(v)$ a $(c_1, \dots, c_k)$ -**umbrella** with respect to ω with $c_i \in \{1, 2, 3\}$, if n is divisible by k and $\omega(e_{i+k}) = \omega(e_i)$ for all $i \in 1 \leq i \leq n$.

Note, an edge-colouring ω of a simplicial surface X can be translated into an arc-colouring κ of the corresponding face graph $\mathcal{F}(X)$ via $\kappa(\{F, F'\}) := \omega(e)$ for $e \in X_1$ with $X_2(e) = \{F, F'\}$. Hence, we can define mono-, bi- and tri-coloured cycles of given cubic graphs in a similar way. Further, we recall that the colours 1, 2 and 3 of given edge-coloured simplicial surface and of given arc-coloured cubic graphs are illustrated in figures with the colours red, green and blue. Before proceeding to the different constructions of face-transitive surfaces, we establish a relation between the $\text{Aut}(X)$ -orbits on the vertices of a simplicial surface X and the $\lambda_X(\text{Aut}(X))$ -orbits on the vertex-defining cycles of X .

Lemma 5.4.2

Let X be a simplicial surface. Furthermore, let $v_1, v_2 \in X_0$ be vertices with corresponding umbrellas $u(v_1)$ and $u(v_2)$, and vertex-defining cycles C_1 and C_2 in $\mathcal{F}(X)$, respectively. Then the following statements are equivalent:

1. $v_1 \in \text{orb}_{\text{Aut}(X)}(v_2)$,
2. $u(v_1) \in \text{orb}_{\text{Aut}(X)}(u(v_2))$,
3. $C_1 \in \text{orb}_{\lambda_X(\text{Aut}(X))}(C_2)$.

Proof. The result follows from observing that automorphisms of a simplicial surface as well as automorphisms of its face graph preserve the face-edge incidences and the node-arc incidences. $\square$

5.4.1 Face-transitive surfaces with vertex-face type (3,1)

In this section, we examine face-transitive surfaces whose automorphism groups yield exactly three orbits on the vertices of the corresponding simplicial surfaces. Here, we show that the umbrellas representing the different vertices of such a simplicial surface are all bi-coloured. In order to describe the structure of a face-transitive surface X with $\text{vf}(X) = (3, 1)$, we give the following definition.

Definition 5.4.3

Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the nodes V with $|H| = |V|$, and
2. the action of H on E results in exactly three orbits E_1, E_2 and E_3 .

The partition of $E = E_1 \uplus E_2 \uplus E_3$ induces an arc-colouring $\kappa : E \rightarrow \{1, 2, 3\}$ with $\kappa(\{F, F'\}) := i$ for $\{F, F'\} \in E_i$. If the set of all cycles that are bi-coloured with respect to κ is a vertex-faithful cycle double cover of $\mathcal{G}$, we call H a **(3,1)-group** of $\mathcal{F}(X)$ and we denote the above cycle double cover by $\mathcal{C}^{(3,1)}(H)$.

The following theorem gives the desired description of a face-transitive surface X of vertex-face type (3, 1).

Theorem 5.4.4

A simplicial surface X is a face-transitive surface with $\text{vf}(X) = (3, 1)$ if and only if $H := \lambda_X(\text{Aut}(X))$ is a (3,1)-group of $\mathcal{F}(X)$ and the cycle double cover $\mathcal{C}^{(3,1)}(H)$ contains exactly the vertex-defining cycles of X .

Proof. Let X be a face-transitive surface with $\text{vf}(X) = (3, 1)$. Furthermore, let $X_0 = V_1 \uplus V_2 \uplus V_3$ be the three $\text{Aut}(X)$ -orbits on X_0 . These orbits define a partition of X_1 into $X_1 = \mathcal{E}_1 \uplus \mathcal{E}_2 \uplus \mathcal{E}_3$ with

$$\begin{aligned}\mathcal{E}_1 &:= \{e \in X_1 \mid X_0(e) \subseteq V_2 \cup V_3\}, \\ \mathcal{E}_2 &:= \{e \in X_1 \mid X_0(e) \subseteq V_1 \cup V_3\}, \\ \mathcal{E}_3 &:= \{e \in X_1 \mid X_0(e) \subseteq V_1 \cup V_2\}.\end{aligned}$$

Note, these partition sets form $\text{Aut}(X)$ -orbits on X_1 . Since every face of X is incident to exactly one edge of each class, the above partition defines a Grünbaum-colouring of X . In particular, the Grünbaum-colouring $\omega : X_1 \rightarrow \{1, 2, 3\}$ defined by $\omega(e) := i$ for $e \in \mathcal{E}_i$ enforces every edge of X to be a mirror edge, see Definition 1.3.5. Thus, all umbrellas of X must be bi-coloured, see Figure 5.7a.

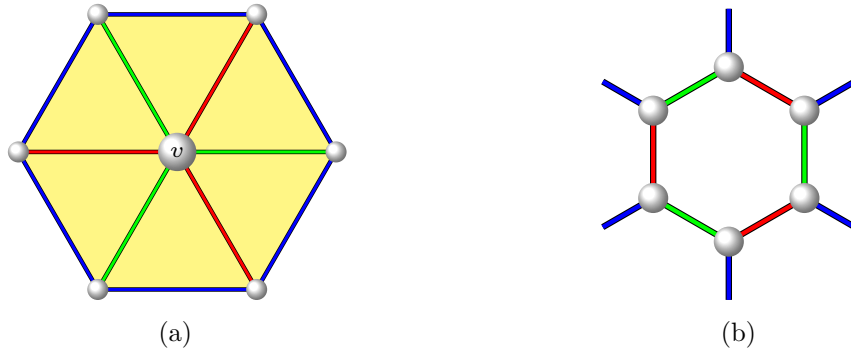


Figure 5.7: An edge-coloured umbrella of the simplicial surface X (a) and the corresponding arc-coloured subgraph in $\mathcal{F}(X)$

The Grünbaum-colouring ω corresponds to an arc-colouring κ of $\mathcal{F}(X)$ via $\kappa(\{F, F'\}) := \omega(e)$ where $e \in X_1$ is an edge with $X_2(e) = \{F, F'\}$. Note that the colour classes of κ are exactly the

H -orbits of the arcs of $\mathcal{F}(X)$. Thus, the bi-coloured umbrellas of X (with respect to ω) directly translate into bi-coloured cycles of $\mathcal{F}(X)$ (with respect to κ), see Figure 5.7b. Moreover, since the vertex-defining cycles of X are exactly the bi-coloured cycles in $\mathcal{F}(X)$ with respect to κ , we deduce that H is a $(3,1)$ -group of $\mathcal{F}(X)$ and $\mathcal{C}^{(3,1)}(H) = \{u(v) \mid v \in X_0\}$, where we interpret an umbrella of a vertex as a cycle in the face graph $\mathcal{F}(X)$.

Now, we assume that $H = \lambda_X(\text{Aut}(X))$ is a $(3,1)$ -group of $\mathcal{F}(X)$. Thus, let κ be the arc-colouring of $\mathcal{F}(X)$ resulting in the cycle double cover $\mathcal{C} := \mathcal{C}^{(3,1)}(H)$. Since H acts transitively on the nodes of $\mathcal{F}(X)$ with trivial node-stabiliser (according to the orbit-stabiliser theorem), X is face-transitive with trivial face-stabilisers. Moreover, Lemma 5.4.2 implies that vertices v_1, v_2 of X satisfy $v_1 \in \text{orb}_{\text{Aut}(X)}(v_2)$ if and only if the corresponding vertex-defining cycles C_1, C_2 in $\mathcal{C}$ satisfy $C_1 \in \text{orb}_H(C_2)$. Since colour classes of κ coincide with H -orbits on the arcs of $\mathcal{F}(X)$, we know that $C_1 \in \text{orb}_H(C_2)$ is equivalent to $\kappa(C_1) = \kappa(C_2)$. This implies that the action of $\text{Aut}(X)$ on the vertices X_0 produces three orbits, and X is therefore a face-transitive surface with $\text{vf}(X) = (3,1)$. $\square$

A direct consequence of Theorem 5.4.4 is stated below.

Corollary 5.4.5 *Let $\mathcal{G}$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ a $(3,1)$ -group of $\mathcal{G}$, then $\mathcal{G}$ is the face graph of a face-transitive surface.*

The cubic graph $\mathcal{G}$, illustrated in Figure 5.8, is an example of a node-transitive graph that yields a face-transitive surface X with $\text{vf}(X) = (3,1)$. The group $H = \text{Aut}(\mathcal{G})$ yields the following orbits on the arcs of $\mathcal{G}$:

$$\begin{aligned} E_1 &:= \{ \{1, 14\}, \{22, 24\}, \{13, 15\}, \{18, 23\}, \{3, 5\}, \{16, 17\}, \\ &\quad \{6, 8\}, \{19, 20\}, \{9, 11\}, \{4, 7\}, \{2, 21\}, \{10, 12\} \}, \\ E_2 &:= \{ \{1, 15\}, \{23, 24\}, \{14, 17\}, \{3, 6\}, \{11, 22\}, \{7, 13\}, \\ &\quad \{5, 21\}, \{2, 20\}, \{12, 18\}, \{4, 16\}, \{8, 19\}, \{9, 10\} \}, \\ E_3 &:= \{ \{1, 24\}, \{2, 7\}, \{15, 23\}, \{3, 14\}, \{6, 17\}, \{5, 22\}, \\ &\quad \{13, 20\}, \{11, 21\}, \{10, 16\}, \{8, 12\}, \{4, 9\}, \{18, 19\} \}. \end{aligned}$$

In Figure 5.8 the H -orbits E_1, E_2 and E_3 are coloured in red, green and blue, respectively.

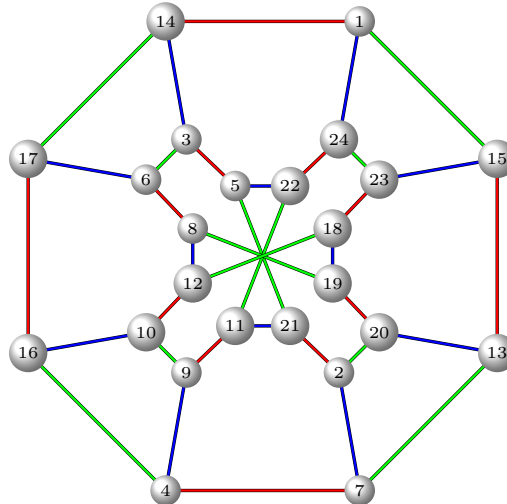


Figure 5.8: The cubic node-transitive graph $\mathcal{G}$ with $\text{Aut}(\mathcal{G})$ -orbits on its arcs coloured in red, green and blue

Thus, the set $\mathcal{C}^{(3,1)}(H)$ is given by

$$\begin{aligned} & \{(1, 15, 23, 24), (1, 14, 3, 5, 22, 24), (1, 14, 17, 16, 4, 7, 13, 15), \\ & (2, 20, 13, 7), (2, 21, 5, 3, 6, 8, 19, 20), (2, 7, 4, 9, 11, 21), \\ & (3, 6, 17, 14), (4, 9, 10, 16), (5, 21, 11, 22), (6, 17, 16, 10, 12, 8), \\ & (8, 12, 18, 19), (9, 11, 22, 24, 23, 18, 12, 10), (13, 20, 19, 18, 23, 15)\}. \end{aligned}$$

Hence, $\mathcal{C}^{(3,1)}(H)$ forms a vertex-faithful cycle double cover of $\mathcal{G}$. The resulting face-transitive surface X with $\mathcal{F}(X) \cong \mathcal{G}$ and $\text{vf}(X) = (3, 1)$ can be described by the following vertices of faces:

$$\begin{aligned} \Theta(X) = \{ & \{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \{v_2, v_5, v_7\}, \{v_3, v_6, v_8\}, \{v_2, v_5, v_9\}, \\ & \{v_5, v_7, v_{10}\}, \{v_3, v_4, v_6\}, \{v_5, v_{10}, v_{11}\}, \{v_6, v_8, v_{12}\}, \{v_8, v_{10}, v_{12}\}, \\ & \{v_6, v_9, v_{12}\}, \{v_{10}, v_{11}, v_{12}\}, \{v_3, v_4, v_{13}\}, \{v_2, v_3, v_7\}, \{v_1, v_3, v_{13}\}, \\ & \{v_3, v_8, v_{10}\}, \{v_3, v_7, v_{10}\}, \{v_{11}, v_{12}, v_{13}\}, \{v_5, v_{11}, v_{13}\}, \{v_4, v_5, v_{13}\}, \\ & \{v_5, v_6, v_9\}, \{v_2, v_9, v_{12}\}, \{v_1, v_{12}, v_{13}\}, \{v_1, v_2, v_{12}\}\}. \end{aligned}$$

By observing the above vertices of faces, we see that X has 13 vertices, 36 edges and 24 faces, and hence has Euler characteristic -1 . Further, the simplicial surface X is non-orientable and satisfies $\text{Aut}(X) \cong \text{Aut}(\mathcal{F}(X)) \cong S_4$.

Note, in Theorem 5.4.4 we fix the automorphism group of a given face-transitive surface X to be isomorphic to the $(3, 1)$ -group H of $\mathcal{F}(X)$. In general, a $(3, 1)$ -group can construct a simplicial surface with a larger automorphism group and hence a different vertex-face type. That means, a simplicial surface X that results from a cubic node-transitive graph from constructing a vertex-faithful cycle double cover consisting of bi-coloured cycles obtained as described in Corollary 5.4.5, is not necessarily a face-transitive surface of vertex-face type $\text{vf}(X) = (3, 1)$. As an example, we consider the simplicial octahedron $\mathcal{O}$ and its face graph $\mathcal{F}(\mathcal{O}) = (V, E)$ illustrated in Figure 5.9.

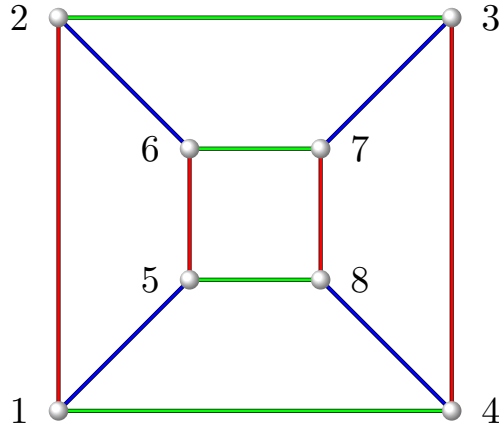


Figure 5.9: The face graph $\mathcal{F}(\mathcal{O})$ with H -orbits on its arcs coloured in red, green and blue

We see that

$$H = \langle (1, 6)(2, 5)(3, 8)(4, 7), (1, 3)(2, 4)(5, 7)(6, 8), (1, 2)(3, 4)(5, 6)(7, 8) \rangle$$

forms a subgroup of $\text{Aut}(\mathcal{F}(\mathcal{O}))$ that is a $(3, 1)$ -group of $\mathcal{F}(\mathcal{O})$. More precisely, H is a subgroup of $\text{Aut}(\mathcal{F}(\mathcal{O}))$ of order $1 \cdot 8$ that acts transitively on the set $V = \mathcal{O}_2$. Furthermore, the H -orbits on the arcs of $\mathcal{F}(\mathcal{O})$ are given by

$$\begin{aligned} \text{orb}_H(E) = \{ & \{ \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\} \}, \\ & \{ \{1, 4\}, \{2, 3\}, \{5, 8\}, \{6, 7\} \}, \\ & \{ \{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\} \} \} \end{aligned}$$

and hence $|\text{orb}_H(E)| = 3$. Thus, the corresponding set of bi-coloured cycles is given by

$$\mathcal{C}' = \{(1, 2, 3, 4), (1, 2, 6, 5), (2, 3, 7, 6), (3, 4, 8, 7), (1, 4, 5, 8), (5, 6, 7, 8)\}$$

which forms a vertex-faithful cycle double cover of $\mathcal{F}(\mathcal{O})$. We observe that the simplicial surface that arises from this vertex-faithful cycle double cover is the simplicial octahedron $\mathcal{O}$ which implies $\text{vf}(\mathcal{O}) = (1, 6) \neq (3, 1)$.

We see that a face-transitive surface X with $\text{vf}(X) = (3, 1)$ enforces certain restrictions to the vertex-degrees of vertices of the corresponding simplicial surface. These restrictions are addressed in the following.

Lemma 5.4.6

Let X be a face-transitive surface with $\text{vf}(X) = (3, 1)$. Then for all $v \in X_0$ the degree of v , namely $\deg(v)$, is even.

Proof. This follows directly from the fact, that the umbrellas of vertices of X are bi-coloured. Such a bi-coloured umbrella exists, if and only if the umbrella contains an even number of faces. $\square$

Next, we count the automorphisms of a face-transitive surface X that stabilise an arbitrary vertex $v \in X_0$.

Lemma 5.4.7

Let X be a face-transitive surface with $\text{vf}(X) = (3, 1)$. Then $|\text{Stab}_{\text{Aut}(X)}(v)| = \deg(v)$ holds for every vertex $v \in X_0$.

Proof. Let $v \in X_0$ be a vertex and $F \in X_2(v)$ a face. Since the automorphisms of X respect the three $\text{Aut}(X)$ -orbits on X_0 , we know that an automorphism $\phi \in \text{Aut}(X)$ mapping F onto an arbitrary face $F' \in X_2(v)$ has to satisfy $\phi(v) = v$. This implies $\phi \in \text{Stab}_{\text{Aut}(X)}(v)$. Because of the face-stabilisers of F and F' in $\text{Aut}(X)$ being trivial, we further know that there exists exactly one automorphism mapping F onto F' . We therefore obtain

$$|\text{Stab}_{\text{Aut}(X)}(v)| = |\{\phi \in \text{Aut}(X) \mid \phi(F) \in X_2(v)\}| = \deg(v),$$

which concludes the proof. $\square$

Now, we describe how we use Theorem 5.4.4 to implement an algorithm in GAP that computes all face-transitive surfaces of vertex-face type $(3, 1)$ of a given node-transitive graph.

Algorithm 4: Compute all face-transitive surfaces with vertex-face type $(3, 1)$ of a given cubic node-transitive graph

Input: A cubic node-transitive graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(3, 1)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 1 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 3\}$ ;
3 for  $H$  in candidates do
4   red  $\leftarrow$  first  $H$ -orbit on  $E$  // coloured in 1;
5   blue  $\leftarrow$  second  $H$ -orbit on  $E$  // coloured in 2;
6   green  $\leftarrow$  third  $H$ -orbit on  $E$  // coloured in 3;
7   cdc  $\leftarrow \{C \mid C \text{ is a } (1, 2)\text{-cycle}\}$ ;
8   cdc  $\leftarrow \text{cdc} \cup \{C \mid C \text{ is a } (1, 3)\text{-cycle}\}$ ;
9   cdc  $\leftarrow \text{cdc} \cup \{C \mid C \text{ is a } (2, 3)\text{-cycle}\}$ ;
10  if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
11     $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  with vertices corresponding to cdc;
12    if  $|\text{Aut}(X)| = 1 \cdot |X_2|$  then
13       $\text{Add}(\text{surfaces}, X)$ ;
14 return surfaces;

```

5.4.2 Face-transitive surfaces with vertex-face type $(2, 2)$

In this subsection, we analyse face-transitive surfaces with vertex-face type $(2, 2)$. In order to determine such a simplicial surface with the help of its face graph, we introduce $(2, 2)$ -groups of given cubic node-transitive graphs.

Definition 5.4.8

Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the nodes V with $|H| = 2 \cdot |V|$,
2. E partitions into two H -orbits E_1 and E_2 with $|E_1| = |V|$ and $|E_2| = \frac{1}{2}|V|$, and
3. there exist nodes $F_1, F_2, F_3 \in V$ satisfying $\{\{F_1, F_2\}, \{F_2, F_3\}\} \not\subseteq E_1$, and an automorphism $\sigma \in H$ satisfying $\sigma(F_1) = F_3$.

The partition $E = E_1 \uplus E_2$ defines an arc-colouring $\kappa : E \rightarrow \{1, 2\}$ of $\mathcal{G}$, by setting $\kappa(\{F, F'\}) := i$ for $\{F, F'\} \in E_i$. Let u be a mono-coloured cycle in $\mathcal{G}$ with respect to κ . If $\text{orb}_H(u) \cup \text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ (cf. Remark 3.2.1, Definitions 3.2.2 and 5.4.1) forms a vertex-faithful cycle double cover of $\mathcal{G}$, we say that H is a $(2, 2)$ -group of $\mathcal{G}$. We denote the above cycle double cover by $\mathcal{C}^{(2,2)}(H)$.

With Definition 5.4.8 we can describe the face-transitive surfaces of vertex-face type $(2, 2)$.

Theorem 5.4.9

A simplicial surface X is a face-transitive surface with $\text{vf}(X) = (2, 2)$ if and only if $H := \lambda_X(\text{Aut}(X))$ is a $(2, 2)$ -group of $\mathcal{F}(X)$ and the cycle double cover $\mathcal{C}^{(2,2)}(H)$ contains exactly the vertex-defining cycles of X .

Proof. Let X be a closed face-transitive surface satisfying $\text{vf}(X) = (2, 2)$. Then, we know that $|\text{Aut}(X)| = 2 \cdot |X_2|$. Further, X_0 is partitioned into two $\text{Aut}(X)$ -orbits, namely V_1 and V_2 . Without loss of generality, we assume that every face of X is incident to exactly one vertex in V_1 and exactly two vertices in V_2 . Hence, this defines a partition of X_1 into

$$\mathcal{E}_1 := \{e \in X_1 \mid X_0(e) \cap V_1 \neq \emptyset\} \text{ and } \mathcal{E}_2 := \{e \in X_1 \mid X_0(e) \cap V_2 = \emptyset\}.$$

Since the face-stabilisers of X are all of order two, $\mathcal{E}_1$ and $\mathcal{E}_2$ are precisely the $\text{Aut}(X)$ -orbits on X_1 with $|\mathcal{E}_1| = |X_2|$ and $|\mathcal{E}_2| = \frac{1}{2}|X_2|$. We denote the edge-colouring arising from this orbits by ω . Hence, $\omega(e) = i$ for $e \in \mathcal{E}_i$ with $i = 1, 2$. In figures, we illustrate edges in $\mathcal{E}_1$ with the colour red and edges in $\mathcal{E}_2$ with green. For $i = 1, 2$ let $v_i \in V_i$ be a vertex, and $u(v_1) = (F_1, \dots, F_n)$ and $u(v_2) = (F'_1, \dots, F'_m)$ umbrellas in X . Hence, the edges in $u(v_1)$ are all contained in $\mathcal{E}_1$ and the umbrella $u(v_1)$ is mono-coloured with respect to ω , see Figure 5.10a.

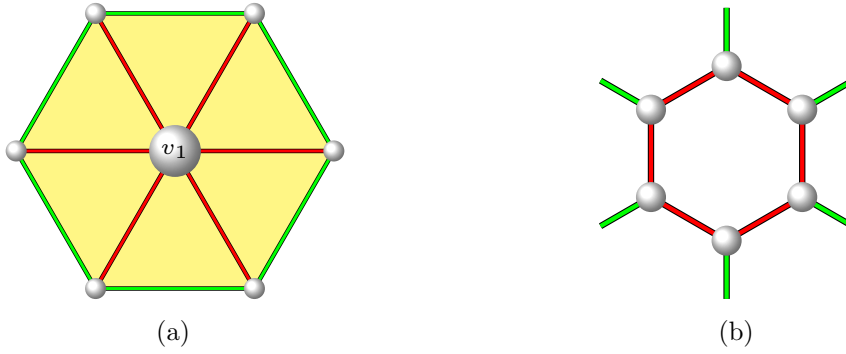


Figure 5.10: The edge-coloured umbrella $u(v_1)$ of the simplicial surface X (a) and the corresponding arc-coloured subgraph in $\mathcal{F}(X)$ (b)

Further, the edges in $u(v_2)$ alternate between $\mathcal{E}_1$ and $\mathcal{E}_2$, and thus the umbrella $u(v_2)$ is bi-coloured with respect to ω , see Figure 5.11a.

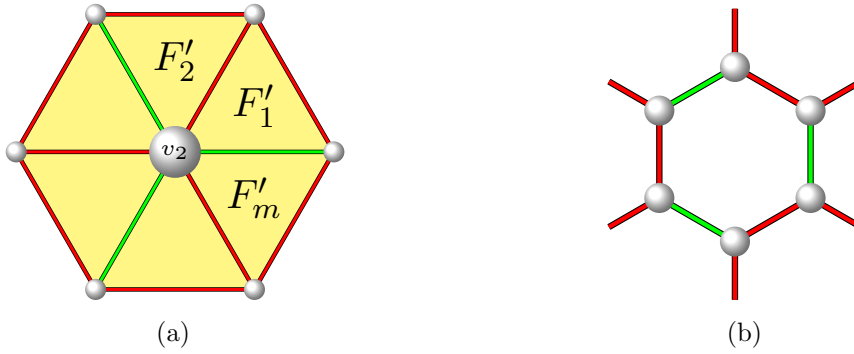


Figure 5.11: The edge-coloured umbrella $u(v_2)$ of the simplicial surface X (a) and the corresponding arc-coloured subgraph in $\mathcal{F}(X)$ (b)

We notice that the edge-colouring ω yields an arc-colouring κ of $\mathcal{F}(X)$. In particular, this colouring is defined by $\kappa(\{F, F'\}) := \omega(e)$, where $e \in X_1$ is an edge with $X_2(e) = \{F, F'\}$. Hence, the colour classes of κ form H -orbits on the arcs of $\mathcal{F}(X)$. Since $\kappa^{-1}(2) = \{X_2(e') \mid e' \in \mathcal{E}_2\}$ forms a perfect matching of $\mathcal{F}(X)$, the deletion of these arcs results in a disjoint union of (mono-coloured) cycles in $\mathcal{F}(X)$ corresponding to the vertex-defining umbrellas in $\text{orb}_{\text{Aut}(X)}(v_1)$.

Further, in order to describe the cycles in $\mathcal{F}(X)$ corresponding to the umbrellas of $\text{orb}_{\text{Aut}(X)}(v_2)$, we assume without loss that $e \in X_1$ with $X_2(e) = \{F'_m, F'_1\}$ is in $\mathcal{E}_2$. Since X is face-transitive, there exists an automorphism $\phi \in \text{Aut}(X)$ mapping F'_m onto F'_2 . Moreover, the stabiliser of the face F'_2 in $\text{Aut}(X)$ has order two (due to the orbit-stabiliser theorem). Thus, there exists a non-trivial $\psi \in \text{Stab}_{\text{Aut}(X)}(F'_2)$ that interchanges the vertices that are incident to the edge $e' := \phi(e)$ with $X_2(e') = \{\phi(F'_m), \phi(F'_1)\} = \{F'_2, F'_3\}$. We observe that either ϕ or $\psi \circ \phi$ fixes v_2 , cf. the green edges in Figure 5.11a. For simplicity, we assume it is ϕ . Thus, we conclude $\phi(F'_i) = F'_{i+2}$ for every $i = 1, \dots, m$, where we read the subscripts modulo m . By defining ℓ as the order of ϕ , we see that the umbrella of v_2 can be written as

$$(F'_1, \dots, F'_m) = (\phi(F'_m), \phi(F'_1), \dots, \phi^\ell(F'_m), \phi^\ell(F'_1)).$$

Let $\sigma := \lambda_X(\phi)$ be an automorphism of $\mathcal{F}(X)$. Then, we see that the cycle

$$(F'_1, \dots, F'_m) = (\sigma(F'_m), \sigma(F'_1), \dots, \sigma^\ell(F'_m), \sigma^\ell(F'_1)) = \alpha(\sigma, F'_m, F'_1, F'_2)$$

is a σ -induced α -cycle in $\mathcal{F}(X)$. This implies that every vertex-defining cycle in the $\text{Aut}(X)$ -orbit of v_2 is an automorphism induced α -cycle. Altogether, the set of vertex-defining cycles of X is given by

$$\mathcal{C} := \text{orb}_H((F_1, \dots, F_n)) \cup \text{orb}_H(\alpha(\sigma, F'_m, F'_1, F'_2)).$$

Since H acts transitively on the nodes X_2 with $|H| = |\text{Aut}(X)| = 2 \cdot |X_2|$, we conclude that H is a $(2, 2)$ -group of $\mathcal{F}(X)$.

Now, let H be a $(2, 2)$ -group of $\mathcal{F}(X) = (V, E)$ with vertex-faithful cycle double cover $\mathcal{C}^{(2,2)}(H)$. The arc-colouring κ of $\mathcal{F}(X)$, whose colour classes are given by the H -orbits on E , corresponds to an edge-colouring ω of X via $\omega(e) := \kappa(\{F, F'\})$, where $e \in X_1$ is the edge with $X_2(e) = \{F, F'\}$. Since H acts transitively on the nodes of $\mathcal{F}(X)$ with $|H| = 2 \cdot |V|$, we know that X is face-transitive with face-stabilisers of order two (due to the orbit-stabiliser theorem). Moreover, by Lemma 5.4.2, two vertices v_1 and v_2 of X lie in the same $\text{Aut}(X)$ -orbit if and only if the corresponding vertex-defining cycles $C_1, C_2 \in \mathcal{C}^{(2,2)}(H)$ are contained in the same H -orbit. This is equivalent to saying the cycles C_1, C_2 have the same colour with respect to κ . Thus, there exist exactly two $\text{Aut}(X)$ -orbits on the vertices, namely the set of vertices with mono-coloured umbrellas and the set of vertices with bi-coloured umbrellas, each with respect to ω . Thus, $\text{vf}(X) = (2, 2)$ follows. $\square$

Theorem 5.4.9 yields the following immediate corollary.

Corollary 5.4.10 *Let $\mathcal{G}$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ a $(2, 2)$ -group of $\mathcal{G}$, then $\mathcal{G}$ is the face graph of a face-transitive surface.*

An example of a cubic node-transitive graph that is isomorphic to the face graph of a face-transitive surface X with $\text{vf}(X) = (2, 2)$ is the graph $\mathcal{G}$ shown in Figure 5.12. The arcs of $\mathcal{G}$ can be subdivided into two H -orbits with $H := \text{Aut}(\mathcal{G})$, namely:

$$\begin{aligned} E_1 &:= \{\{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 8\}, \{5, 9\}, \{3, 9\}, \{6, 10\}, \{4, 10\}, \{1, 7\}, \{2, 8\}\}, \\ E_2 &:= \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\}. \end{aligned}$$

By defining $\sigma := (1, 4)(2, 3)(5, 6)(7, 10)(8, 9) \in H$, we obtain the following vertex-faithful cycle double cover of $\mathcal{G}$:

$$\begin{aligned} &\text{orb}_H(\alpha(\sigma, 4, 2, 1)) \cup \text{orb}_H((1, 7, 5, 9, 3)) \\ &= \{(1, 3, 4, 2), (5, 7, 8, 6), (5, 9, 10, 6), (1, 7, 8, 2), (3, 9, 10, 4), (1, 7, 5, 9, 3), (2, 8, 6, 10, 4)\}. \end{aligned}$$

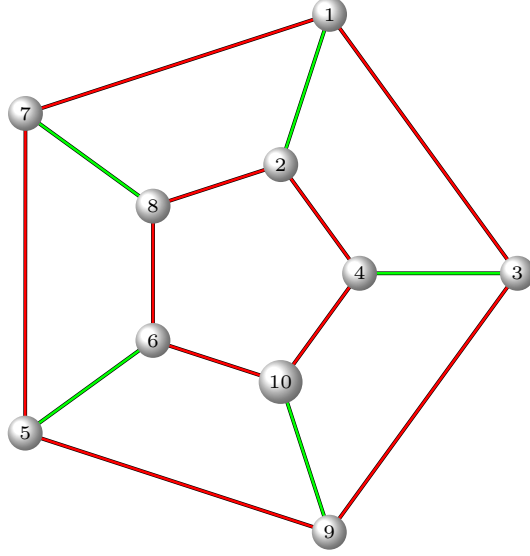


Figure 5.12: A cubic node-transitive graph $\mathcal{G}$ with $\text{Aut}(\mathcal{G})$ -orbits on its arcs coloured in red and green

The resulting simplicial surface X that is described by the vertices of faces

$$\Theta(X) = \{\{v_1, v_4, v_5\}, \{v_1, v_4, v_6\}, \{v_1, v_5, v_7\}, \{v_1, v_6, v_7\}, \{v_2, v_3, v_5\}, \\ \{v_2, v_3, v_6\}, \{v_2, v_4, v_5\}, \{v_2, v_4, v_6\}, \{v_3, v_5, v_7\}, \{v_3, v_6, v_7\}\}$$

satisfies $\mathcal{F}(X) \cong \mathcal{G}$ and $\text{vf}(X) = (2, 2)$. Moreover, we observe that $\text{Aut}(X)$ and $\text{Aut}(\mathcal{G})$ are both isomorphic to the dihedral group $D_{2 \cdot 10}$.

However, a face-transitive surface X that is constructed by using Corollary 5.4.10 does not necessarily satisfy $\text{vf}(X) = (2, 2)$. Again, we consider the simplicial octahedron $\mathcal{O}$ and the corresponding face graph illustrated in Figure 5.13 as an example. The group

$$H' := \langle (1, 2, 3, 4)(5, 6, 7, 8), (1, 2)(3, 4)(5, 6)(7, 8), (1, 5)(2, 6)(3, 7)(4, 8) \rangle$$

forms a subgroup of the automorphism group $\text{Aut}(\mathcal{F}(\mathcal{O}))$ of order $16 = 2 \cdot 8$. We show that H' forms a $(2, 2)$ -group of $\mathcal{F}(\mathcal{O})$, where the resulting cycle double cover $\mathcal{C}'$ corresponds to the umbrellas of the simplicial octahedron. The H' -orbits on the arcs of $\mathcal{F}(\mathcal{O})$ are illustrated in different colours in the below figure.

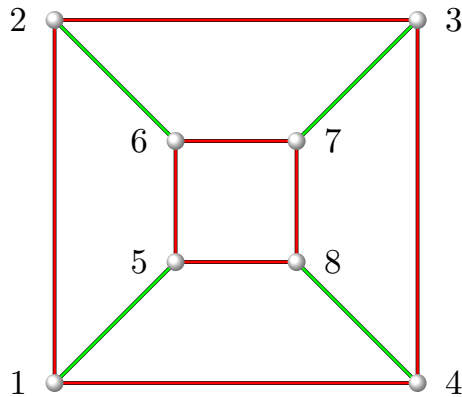


Figure 5.13: The face graph $\mathcal{F}(\mathcal{O})$ with H' -orbits on its arcs coloured in red and green

With $\sigma' := (1, 6)(2, 5)(3, 8)(4, 7) \in H'$ and $(1, 2, 3, 4)$ being a mono-coloured cycle in $\mathcal{F}(\mathcal{O})$, we observe that the set

$$\begin{aligned} \mathcal{C}' &= \text{orb}_{H'}(\alpha(\sigma', 1, 5, 6)) \cup \text{orb}_{H'}((1, 2, 3, 4)) \\ &= \{(1, 2, 6, 5), (1, 4, 8, 5), (2, 3, 7, 6), (3, 4, 8, 7)\} \cup \{(1, 2, 3, 4), (5, 6, 7, 8)\} \\ &= \{(1, 2, 3, 4), (1, 2, 6, 5), (2, 3, 7, 6), (3, 4, 8, 7), (1, 4, 5, 8), (5, 6, 7, 8)\} \end{aligned}$$

is a vertex-faithful cycle double cover whose cycles correspond to the vertex-defining cycles of the simplicial octahedron.

Note, a face-transitive surface X with $\text{vf}(X) = (2, 2)$ contains vertices with even vertex-degree.

Proposition 5.4.11

Let X be a face-transitive surface with $\text{vf}(X) = (2, 2)$. Furthermore, let V_1 and V_2 be the $\text{Aut}(X)$ -orbits on X_0 with $|V_1| < |V_2|$. Then for all $v \in V_2$ the degree $\deg(v)$ is even.

We conclude this section by translating our results into an algorithm for our investigations in GAP.

Algorithm 5: Compute all face-transitive surfaces with vertex-face type $(2, 2)$ of a given cubic node-transitive graph

Input: A cubic node-transitive graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(2, 2)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 2 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 2\}$ ;
3 for  $H$  in candidates do
4   red  $\leftarrow$  first  $H$ -orbit on  $E$  with  $|\text{red}| = |E|$  // coloured in 1;
5   green  $\leftarrow$  second  $H$ -orbit on  $E$  with  $|\text{green}| = \frac{1}{2}|E|$  // coloured in 2;
6    $[F_1, F_2, F_3] \leftarrow$  a path of length 3 in  $\mathcal{G}$  with  $\{F_1, F_2\}$  and  $\{F_1, F_2\}$  coloured differently;
7   automorphisms  $\leftarrow \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_3\}$ ;
8   for  $\sigma$  in automorphisms do
9     cdc  $\leftarrow \text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ ;
10    cdc  $\leftarrow \text{cdc} \cup \{C \mid C \text{ is a monocoloured cycle of colour 1}\}$ ;
11    if cdc is a vertex-faithful cycle double cover then
12       $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
13      if  $|\text{Aut}(X)| = 2 \cdot |X_2|$  then
14        Add(surfaces,  $X$ );
15 return surfaces;
```

5.4.3 Face-transitive surfaces with vertex-face type $(1, 6)$

Here, we discuss the structure of a face-transitive surface with vertex-face type $(1, 6)$. We show that the vertex-defining cycles of such a face-transitive surface X form cycles that are induced by automorphisms of $\mathcal{F}(X)$. In order to characterise face-transitive surfaces, where all face-stabilisers have order 6, we give the definition of a $(1, 6)$ -group of a cubic node-transitive graph.

Definition 5.4.12

Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the nodes V and arcs E with $|H| = 6 \cdot |V|$, and
2. for $F_1, F_2 \in V$, $\{F_1, F_2\} \in E$, there exists an automorphism $\sigma \in H$ such that $\sigma(F_1) = F_2$.

If the orbit $\text{orb}_H(\alpha(\sigma, F_1, F_2))$ (Remark 3.2.1 and Definition 3.2.2) is a vertex-faithful cycle double cover of $\mathcal{G}$, then we say that H is a $(1, 6)$ -group of $\mathcal{G}$. We denote the above cycle double cover by $\mathcal{C}^{(1,6)}(H)$.

Theorem 5.4.13

A simplicial surface X is a face-transitive surface with $\text{vf}(X) = (1, 6)$ if and only if $H := \lambda_X(\text{Aut}(X))$ is a $(1, 6)$ -group of $\mathcal{F}(X)$ and the cycle double cover $\mathcal{C} := \mathcal{C}^{(1,6)}(H)$ contains exactly the vertex-defining cycles of X .

Proof. Let X be a face-transitive surface with $\text{vf}(X) = (1, 6)$. Since all face-stabilisers of X have order 6, we know that the automorphism group $\text{Aut}(X)$ acts transitively on the edges and vertices of X . Additionally, if $e \in X_1$ is an edge, then the orbit-stabiliser theorem implies that the stabiliser of e satisfies $|\text{Stab}_{\text{Aut}(X)}(e)| = 4$. These facts allow us to determine the vertices of X with the help of the face graph $\mathcal{F}(X)$ and the group H .

Thus, let $v \in X_0$ be a vertex and $u(v) = (F_1, \dots, F_n)$ the umbrella of v . Moreover, for $1 \leq i \leq n$ let $F_i \in X_2$ be a face with $X_0(F_i) = \{v, v_i, v_{i+1}\}$, where we read the subscripts modulo n .

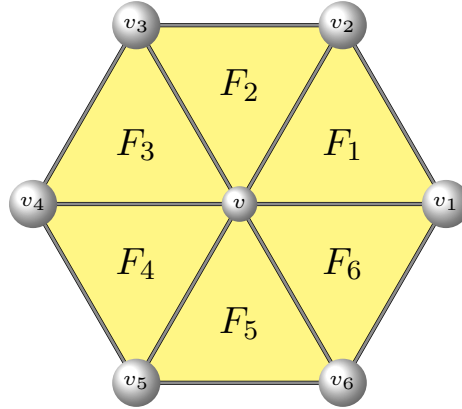


Figure 5.14: The umbrella $u(v)$ of the simplicial surface X

Since the stabiliser of the edge $e' \in X_1$ with $X_0(e') = \{v, v_2\}$ has order 4, there exists a unique automorphism $\psi \in \text{Stab}_{\text{Aut}(X)}(e')$ that satisfies

$$(\psi(v), \psi(v_1), \psi(v_2), \psi(v_3)) = (v_2, v_3, v, v_1).$$

Additionally, we know that the stabiliser $\text{Stab}_{\text{Aut}(X)}(F_2)$ contains a unique automorphism ϕ' satisfying $(\phi'(v), \phi'(v_2), \phi'(v_3)) = (v_3, v, v_2)$. Hence, $\phi := \phi' \circ \psi$ satisfies $\phi(v) = v$ and $\phi(F_i) = F_{i+1}$ for $i = 1, \dots, n$. We conclude that the order of ϕ is n and the umbrella of v can be written as

$$(F_1, \dots, F_n) = (\phi(F_1), \dots, \phi^n(F_1)).$$

By interpreting the umbrella $(F_1, \dots, F_n)$ as a cycle of the $\mathcal{F}(X)$ and defining $\sigma := \lambda_X(\phi)$ we observe that $(F_1, \dots, F_n) = \alpha(\sigma, F_1, F_2)$ is a σ -induced α -cycle. Since all the vertices lie in exactly one $\text{Aut}(X)$ -orbit, the cycle double cover consisting of all vertex-defining cycles of X is given by $\text{orb}_H(\alpha(\sigma, F_1, F_2))$. Thus, H forms a $(1, 6)$ -group of $\mathcal{F}(X)$.

To complete the proof, we assume that H is a $(1, 6)$ -group of $\mathcal{F}(X)$. Let $F \in X_2$ be an arbitrary face. Since $\text{Aut}(X) \cong H$, it follows that X is face-transitive and the stabiliser $\text{Stab}_H(F)$ of F interpreted as a node of $\mathcal{F}(X)$ is isomorphic to $\text{Stab}_{\text{Aut}(X)}(F)$ with $|\text{Stab}_H(F)| = 6$. We therefore conclude that the stabiliser $\text{Stab}_{\text{Aut}(X)}(F)$ is transitive on the set $X_0(F)$ which implies that the automorphism group $\text{Aut}(X)$ is transitive on X_0 . That means that X is a face-transitive surface with $\text{vf}(X) = (1, 6)$. $\square$

As a direct consequence of the above theorem, we obtain the following corollary.

Corollary 5.4.14 *Let $\mathcal{G}$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ a $(1, 6)$ -group of $\mathcal{G}$, then $\mathcal{G}$ is the face graph of a face-transitive surface.*

An example of a cubic node-transitive graph $\mathcal{G}$ that gives rise to a simplicial surface of vertex-face type $(1, 6)$ is the Petersen graph shown in Figure 5.15.

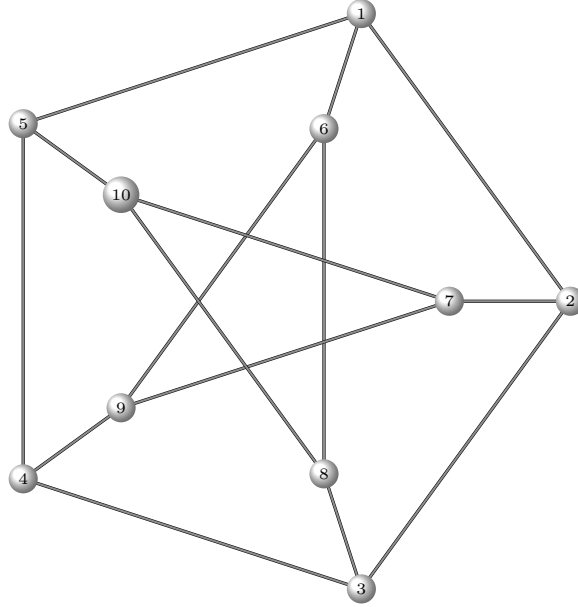


Figure 5.15: The Petersen graph

We construct a face-transitive surface from $\mathcal{G}$ with Corollary 5.4.14 by defining

$$H := \langle (1, 2)(3, 5)(6, 7)(8, 10), (2, 5)(3, 4)(7, 10)(8, 9), (3, 7)(4, 9)(5, 6)(8, 10) \rangle \leq \text{Aut}(\mathcal{G}).$$

By choosing σ as the automorphism $(1, 2, 3, 4, 5)(6, 7, 8, 9, 10) \in H$, we obtain the following vertex-faithful cycle double cover of $\mathcal{G}$:

$$\begin{aligned} \text{orb}_H(\alpha(\sigma, 1, 2)) = \{ & (1, 2, 3, 4, 5), (1, 6, 9, 7, 2), (4, 9, 7, 10, 5), \\ & (3, 8, 6, 9, 4), (1, 6, 8, 10, 5), (2, 7, 10, 8, 3) \}. \end{aligned}$$

Since $|H| = 60 = 6 \cdot 10$ and H acts transitively on the nodes and arcs of $\mathcal{G}$, the subgroup H is a $(1, 6)$ -group of $\mathcal{G}$. The resulting simplicial surface X forms a face-transitive surface of vertex-face

type $(1, 6)$ that can be described by the following vertices of faces:

$$\Theta(X) = \{\{v_1, v_2, v_5\}, \{v_1, v_2, v_6\}, \{v_1, v_4, v_6\}, \{v_1, v_3, v_4\}, \{v_1, v_3, v_5\}, \\ \{v_2, v_4, v_5\}, \{v_2, v_3, v_6\}, \{v_4, v_5, v_6\}, \{v_2, v_3, v_4\}, \{v_3, v_5, v_6\}\}.$$

We observe that the simplicial surface X is non-orientable and has 6 vertices, 15 edges and 10 faces. Furthermore, the automorphism group of X satisfies $A_5 \cong \text{Aut}(X) \hookrightarrow \text{Aut}(\mathcal{F}(X)) \cong S_5$.

Note, in the literature, a face-transitive surface X satisfying $\text{vf}(X) = (1, 6)$ is often referred to as a regular map. For more details on the study of regular maps we refer to [29].

We conclude this subsection by giving an algorithm that computes face-transitive surfaces of vertex-face type $(1, 6)$ from given node-transitive graphs.

Algorithm 6: Compute all face-transitive surfaces with vertex-face type $(1, 6)$ of a given cubic node-transitive graph

Input: A cubic node-transitive graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(1, 6)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 6 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 1\}$ ;
3  $[F_1, F_2] \leftarrow \{F_1, F_2\} \in E$ ;
4 for  $H$  in candidates do
5   automorphisms  $\leftarrow \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_2\}$ ;
6   for  $\sigma$  in automorphisms do
7     cdc  $\leftarrow \text{orb}_H(\alpha(\sigma, F_1, F_2))$ ;
8     if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
9        $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
10      Add(surfaces,  $X$ );
11 return surfaces;
```

5.4.4 Face-transitive surfaces with vertex-face type $(1, 2)$

In this section, we focus on face-transitive surfaces with vertex-face type $(1, 2)$. Similarly to the previous case, we show that the vertex-defining cycles of a face-transitive surface with vertex-face type $(1, 2)$ are given by the orbit of an automorphism induced α -cycles. For this, we introduce $(1, 2)$ -groups of given cubic node-transitive graphs.

Definition 5.4.15

Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the nodes V with $|H| = 2 \cdot |V|$,
2. E partitions into two H -orbits E_1 and E_2 with $|E_1| = |V|$ and $|E_2| = \frac{1}{2}|V|$, and
3. for $F_1, \dots, F_4 \in V$ with $\{F_1, F_2\} \in E_2$ and $\{F_2, F_3\}, \{F_3, F_4\} \in E_1$, there exists an automorphism $\sigma \in \text{Aut}(\mathcal{G})$ such that $\sigma(F_1) = F_4$.

If $\text{orb}_H(\alpha(\sigma, F_1, F_2, F_3, F_4))$ (see Remark 3.2.1 and Definition 3.2.2) forms a vertex-faithful cycle double cover of $\mathcal{G}$, then H is said to be a $(1, 2)$ -group of $\mathcal{G}$. We denote the above cycle double cover by $\mathcal{C}^{(1,2)}(H)$.

The following theorem characterises face-transitive surfaces X with $\text{vf}(X) = (1, 2)$.

Theorem 5.4.16

A simplicial surface X is a face-transitive surface with $\text{vf}(X) = (1, 2)$ if and only if $H := \lambda_X(\text{Aut}(X))$ is a $(1, 2)$ -group of $\mathcal{F}(X)$ and the cycle double cover $\mathcal{C}^{(1,2)}(H)$ contains the vertex-defining cycles of X .

Proof. Let X be a face-transitive surface with $\text{vf}(X) = (1, 2)$. Due to the orbit-stabiliser theorem, we know that an edge $e \in X_1$ satisfies

$$|\text{Aut}(X)| = 2 \cdot |X_2| = |\text{Stab}_{\text{Aut}(X)}(e)| \cdot |\text{orb}_{\text{Aut}(X)}(e)|.$$

Hence, an $\text{Aut}(X)$ -orbit on X_1 must either be of length $\frac{1}{2}|X_2|$ or $|X_2|$. Because the face-stabilisers of X have order 2, we know that there exists exactly two $\text{Aut}(X)$ -orbits on X_1 , namely $\mathcal{E}_1$ and $\mathcal{E}_2$ of cardinality $|X_2|$ and $\frac{1}{2}|X_2|$, respectively.

Thus, we obtain an edge-colouring ω of X via $\omega(e) := i$ for $e \in \mathcal{E}_i$ with $i = 1, 2$. This edge-colouring naturally yields an arc-colouring κ of $\mathcal{F}(X)$ via $\kappa(\{F, F'\}) := \omega(e)$, where $e \in X_1$ is an edge with $X_2(e) = \{F, F'\}$. As a first step, we describe the different vertices of X by determining the colours of the corresponding umbrellas with respect to the edge-colouring ω . We then conclude that H is a $(1, 2)$ -group of $\mathcal{F}(X)$ by translating the colour classes of ω and the umbrellas of X into H -orbits on the arcs of $\mathcal{F}(X)$ and vertex-defining cycles of X .

Let therefore $v \in X_0$ be a vertex and $u(v) = (F_1, \dots, F_n)$ its umbrella. Further, for $1 \leq i \leq n$ let $F_i \in X_2$ be a face with $X_0(F_i) = \{v, v_i, v_{i+1}\}$, where we read the subscripts modulo n . Moreover, let $e_1, e_2, e_3 \in X_1$ be edges with $X_0(e_1) := \{v, v_1\}$, $X_0(e_2) = \{v, v_2\}$, $X_0(e_3) = \{v_1, v_2\}$. Without loss of generality, we assume $e_1, e_3 \in \mathcal{E}_1$, $e_2 \in \mathcal{E}_2$. Note that $|\text{Stab}_{\text{Aut}(X)}(e_1)| = 2$ and $|\text{Stab}_{\text{Aut}(X)}(e_2)| = 4$.

First, we assume that the non-trivial element $\phi \in \text{Stab}_{\text{Aut}(X)}(e_1)$ fixes v_1 . Since the edge-stabilisers of the edges in $\mathcal{E}_1$ are all conjugate in $\text{Aut}(X)$ (see Lemma 3.1.8), this implies that if $\psi \in \text{Stab}_{\text{Aut}(X)}(e)$ is the non-trivial automorphism stabilising an arbitrary edge $e \in \mathcal{E}_1$, then $\psi(v') = v'$ for all $v' \in X_0(e)$. Since v_1 is incident to both $e_1, e_3 \in \mathcal{E}_1$, the above argument enforces the umbrella of v_1 to be mono-coloured with $\omega(u(v_1)) = (1, \dots, 1)$. This means that there exist no automorphism $\phi' \in \text{Aut}(X)$ mapping v onto v_1 , which contradicts $\text{vf}(X) = (1, 2)$.

Thus, ϕ interchanges the vertices v and v_1 , i.e. $\phi(v) = v_1$ and $\phi(v_1) = v$. Therefore, we obtain $\omega(u(v)) = (1, 1, 2, \dots, 1, 1, 2)$ and hence $u(v)$ is an umbrella a $(1, 1, 2)$ -umbrella, see Definition 5.4.1.

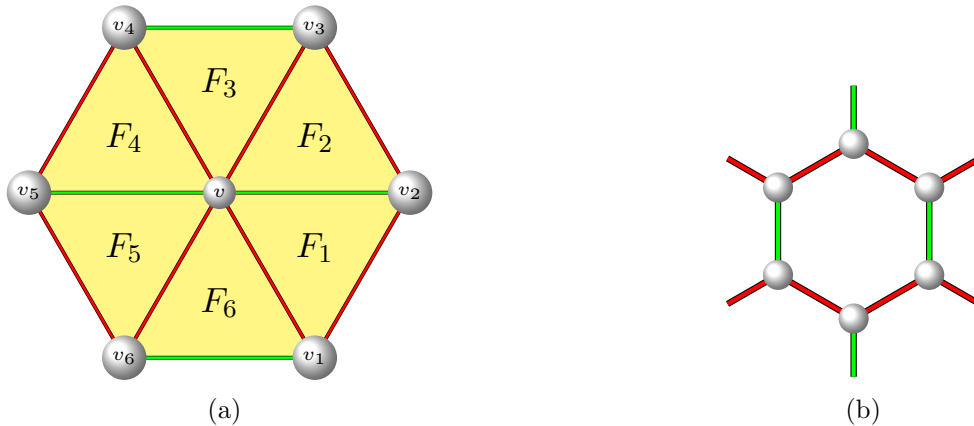


Figure 5.16: The edge-coloured umbrella $u(v)$ of the simplicial surface X (a) and the corresponding arc-coloured subgraph in $\mathcal{F}(X)$ (b)

Since $\text{Aut}(X)$ acts transitively on the vertices of X , every umbrella of X forms a $(1, 1, 2)$ -umbrella and every vertex-degree must be a multiple of 3 but not equal to 3. Because, if the degree of a vertex is 3, then all vertex-degrees are equal to 3 and X must be isomorphic to the simplicial tetrahedron with $\text{vf}(X) = (1, 6)$, which is a contradiction.

In the following, we describe how these $(1, 1, 2)$ -umbrellas of X can be computed by exploiting different automorphisms of $\text{Aut}(X)$. For this, we recall that $X_2(v) = \{F_1, \dots, F_n\}$ with $n > 3$. Thus, let ϕ_1 be the automorphism mapping F_1 to F_4 . Since the stabiliser of the edge $e_5 \in \mathcal{E}_2$ with $X_0(e_5) = \{v, v_5\}$ has order 4 (see Figure 5.16), there exists an automorphism $\phi_2 \in \text{Stab}_{\text{Aut}(X)}(e_5)$ such that $\bar{\phi} = \phi_2 \circ \phi_1$ stabilises the vertex v and satisfies $\bar{\phi}(F_1) = F_4$. This results in $\bar{\phi}(F_i) = F_{i+3}$ for all $1 \leq i \leq n$, where the subscripts are read modulo n . Thus, the umbrella $u(v)$ is given by

$$u(v) = (F_1, \dots, F_n) = (\bar{\phi}(F_1), \bar{\phi}(F_2), \bar{\phi}(F_3), \dots, \bar{\phi}^\ell(F_1), \bar{\phi}^\ell(F_2), \bar{\phi}^\ell(F_3)),$$

where ℓ denotes the order of $\bar{\phi}$. Since the faces $F_1, \dots, F_n$ can also be seen as nodes of $\mathcal{F}(X)$, the automorphism $\sigma := \lambda_X(\bar{\phi})$ enforces the cycle $(F_1, \dots, F_n) = \alpha(\sigma, F_1, F_2, F_3, F_4)$ to be a σ -induced α -cycle of $\mathcal{F}(X)$. Since the set of vertex-defining cycles of X can be obtained from $\text{orb}_H(\alpha(F_1, F_2, F_3, F_4))$, it follows that H forms a $(1, 2)$ -group of $\mathcal{F}(X)$.

Now, let H be a $(1, 2)$ -group of the face graph $\mathcal{F}(X)$. We know that $\text{Aut}(X)$ is isomorphic to H . Thus, $|\text{Aut}(X)| = |H| = 2 \cdot |X_2|$ and the node-transitivity of H translates into the face-transitivity of $\text{Aut}(X)$. That means that X is face-transitive, with face-stabilisers of order 2 (due the orbit-stabiliser theorem). Moreover, by the construction of $\mathcal{C}^{(1,2)}(H)$, it directly follows that H acts transitively on the vertex-defining cycles of X which is equivalent to $\text{Aut}(X)$ acting transitively on X_0 . Hence, H and $\mathcal{C}^{(1,2)}(H)$ enforce X to be a face-transitive surface with $\text{vf}(X) = (1, 2)$. $\square$

As usual we obtain the following corollary.

Corollary 5.4.17 *Let $\mathcal{G}$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ a subgroup. If H is a $(1, 2)$ -group of $\mathcal{G}$, then $\mathcal{G}$ is the face graph of a face-transitive surface.*

An example of a cubic node-transitive graph that gives rise to a face-transitive surface of vertex-face type $(1, 2)$ is the graph $\mathcal{G} = (V, E)$ illustrated in Figure 5.17.

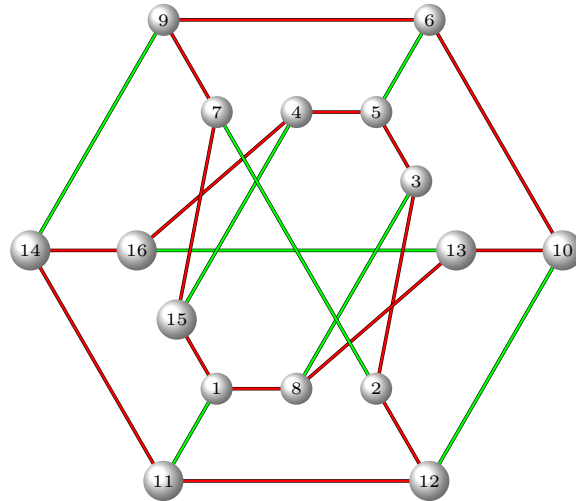


Figure 5.17: The cubic node-transitive graph $\mathcal{G}$ with H -orbits on its arcs coloured in red and green

The group H defined by

$$\begin{aligned} H := \langle & (2, 16)(3, 4)(7, 13)(8, 15)(9, 10)(12, 14), \\ & (1, 5)(2, 7)(3, 15)(4, 8)(6, 11)(9, 12)(10, 14)(13, 16), \\ & (1, 15)(2, 3)(4, 11)(5, 12)(6, 10)(7, 8)(9, 13)(14, 16) \rangle \leq \text{Aut}(\mathcal{G}), \end{aligned}$$

satisfies $|H| = 32$ and $|\text{orb}_H(V)| = 1$. Furthermore, we obtain the below H -orbits on the arcs of $\mathcal{G}$:

$$\begin{aligned} E_1 := & \{\{6, 10\}, \{6, 9\}, \{11, 14\}, \{11, 12\}, \{10, 13\}, \{4, 16\}, \{4, 5\}, \{7, 9\}, \\ & \{14, 16\}, \{2, 3\}, \{8, 13\}, \{3, 5\}, \{1, 8\}, \{2, 12\}, \{7, 15\}, \{1, 15\}\}, \\ E_2 := & \{\{9, 14\}, \{10, 12\}, \{13, 16\}, \{5, 6\}, \{2, 7\}, \{1, 11\}, \{3, 8\}, \{4, 15\}\}. \end{aligned}$$

In Figure 5.17, the arcs contained in the orbits E_1 and E_2 are coloured in red and green, respectively. As described in Corollary 5.4.17, the automorphism

$$\sigma := (1, 2)(3, 8)(4, 10)(5, 13)(6, 16)(7, 11)(9, 14)(12, 15) \in H$$

gives rise to the following vertex-faithful cycle double cover:

$$\begin{aligned} \text{orb}_H(\alpha(\sigma, 1, 11, 12, 2)) = & \{(6, 9, 14, 11, 12, 10), (4, 16, 13, 10, 6, 5), (4, 16, 14, 9, 7, 15), \\ & (2, 12, 10, 13, 8, 3), (2, 7, 9, 6, 5, 3), (1, 8, 3, 5, 4, 15), \\ & (1, 8, 13, 16, 14, 11), (1, 15, 7, 2, 12, 11)\}. \end{aligned}$$

Note that the arising face-transitive surface X with $\mathcal{F}(X) \cong \mathcal{G}$ satisfies $\text{vf}(X) = (1, 2)$ and can be described by the below vertices of faces:

$$\begin{aligned} \Theta(X) = & \{\{v_6, v_7, v_8\}, \{v_4, v_5, v_8\}, \{v_4, v_5, v_6\}, \{v_2, v_3, v_6\}, \{v_2, v_5, v_6\}, \{v_1, v_2, v_5\}, \\ & \{v_3, v_5, v_8\}, \{v_4, v_6, v_7\}, \{v_1, v_3, v_5\}, \{v_1, v_2, v_4\}, \{v_1, v_7, v_8\}, \{v_1, v_4, v_8\}, \\ & \{v_2, v_4, v_7\}, \{v_1, v_3, v_7\}, \{v_3, v_6, v_8\}, \{v_2, v_3, v_7\}\}. \end{aligned}$$

Hence, X is a simplicial torus consisting of 8 vertices, 24 edges and 16 faces, and an automorphism group satisfying

$$C_8 \rtimes (C_2 \times C_2) \cong \text{Aut}(X) \hookrightarrow \text{Aut}(\mathcal{F}(X)) \cong \text{GL}(2, 3) \rtimes C_2.$$

Next, we describe that a face-transitive surface X that is constructed with Corollary 5.4.17 does not have to satisfy $\text{vf}(X) = (1, 2)$. For this purpose, we consider the graph $\mathcal{G}'$ illustrated in Figure 5.18.

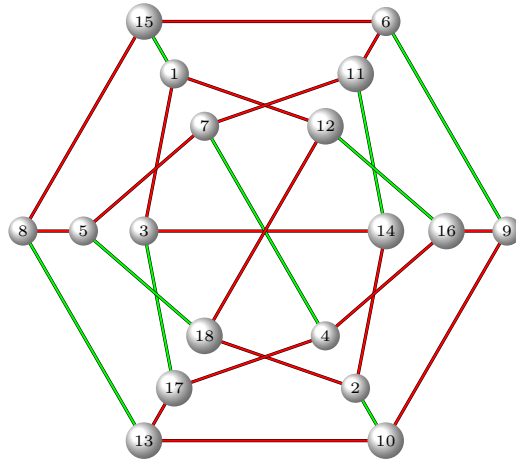


Figure 5.18: The cubic node-transitive graph $\mathcal{G}'$ with H' -orbits on its arcs coloured in red and green

We choose H' to be the subgroup of $\text{Aut}(\mathcal{G}')$ defined by

$$\begin{aligned} H' := & \langle (3, 12)(5, 11)(6, 8)(9, 13)(14, 18)(16, 17), \\ & (1, 15)(2, 7)(3, 8)(4, 10)(5, 14)(6, 12)(9, 16)(11, 18)(13, 17), \\ & (1, 17, 8)(2, 9, 11)(3, 13, 15)(4, 5, 12)(6, 14, 10)(7, 18, 16), \\ & (1, 7, 10)(2, 15, 4)(3, 5, 9)(6, 17, 18)(8, 16, 14)(11, 13, 12) \rangle. \end{aligned}$$

Then H' has order $2 \cdot 18$ and acts transitively on the nodes of $\mathcal{G}'$. Furthermore, H' gives rise to the following orbits on the arcs of $\mathcal{G}'$:

$$\begin{aligned} E_1 := & \{\{1, 3\}, \{2, 14\}, \{12, 18\}, \{1, 12\}, \{2, 18\}, \{3, 14\}, \{9, 10\}, \{4, 16\}, \{13, 17\}, \\ & \{10, 13\}, \{4, 17\}, \{9, 16\}, \{5, 7\}, \{8, 15\}, \{6, 11\}, \{7, 11\}, \{6, 15\}, \{5, 8\}\}, \\ E_2 := & \{\{1, 15\}, \{11, 14\}, \{5, 18\}, \{2, 10\}, \{12, 16\}, \{3, 17\}, \{4, 7\}, \{8, 13\}, \{6, 9\}\}. \end{aligned}$$

Hence, defining $\sigma' := (1, 4)(2, 10)(3, 16)(5, 8)(6, 11)(7, 15)(9, 14)(12, 17)(13, 18) \in H'$ results in the following cycle double cover of $\mathcal{G}'$:

$$\begin{aligned} \text{orb}_{H'}(\alpha(\sigma', 1, 12, 16, 4)) = & \{(6, 15, 8, 13, 10, 9), (4, 16, 9, 6, 11, 7), (4, 17, 13, 8, 5, 7), \\ & (2, 18, 12, 16, 9, 10), (2, 14, 3, 17, 13, 10), (2, 18, 5, 7, 11, 14), \\ & (1, 15, 6, 11, 14, 3), (1, 12, 16, 4, 17, 3), (1, 15, 8, 5, 18, 12)\}. \end{aligned}$$

The resulting simplicial surface X' satisfies $\text{vf}(X') = (1, 6)$ and is described by the vertices of faces:

$$\begin{aligned} \Theta(X') = & \{\{v_7, v_8, v_9\}, \{v_4, v_5, v_6\}, \{v_5, v_7, v_8\}, \{v_2, v_3, v_8\}, \{v_3, v_6, v_9\}, \{v_1, v_2, v_7\}, \\ & \{v_2, v_3, v_6\}, \{v_1, v_3, v_9\}, \{v_1, v_2, v_4\}, \{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_4, v_8, v_9\}, \\ & \{v_1, v_3, v_5\}, \{v_5, v_6, v_7\}, \{v_1, v_7, v_9\}, \{v_2, v_4, v_8\}, \{v_3, v_5, v_8\}, \{v_4, v_6, v_9\}\}. \end{aligned}$$

In the proof of Theorem 5.4.16 we have established the following result.

Proposition 5.4.18

Let X be a face-transitive surface with $\text{vf}(X) = (1, 2)$. Then for all $v \in X_0$ we know that the degree $\deg(v)$ is divisible by 3.

Lastly, we present a pseudocode that describes how we construct of face-transitive surface with vertex-face type $(1, 2)$ in GAP.

Algorithm 7: Compute all face-transitive surfaces with vertex-face type $(1, 2)$ of a given cubic node-transitive graph

Input: A cubic node-transitive graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(1, 2)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 2 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 2\}$ ;
3 for  $H$  in candidates do
4   red  $\leftarrow$  first  $H$ -orbit on  $E$  with  $|\text{red}| = |E|$  // coloured in 1;
5   green  $\leftarrow$  second  $H$ -orbit on  $E$  with  $|\text{green}| = \frac{1}{2}|E|$  // coloured in 2;
6   paths  $\leftarrow \{[F_1, \dots, F_4] \mid [F_1, \dots, F_4] \text{ is a path of length 4 with } \{F_1, F_2\} \text{ in green and } \{F_2, F_3\}, \{F_3, F_4\} \text{ in red}\}$ ;
7   for  $[F_1, \dots, F_4]$  in paths do
8     automorphisms  $\leftarrow \{\sigma' \in \text{Aut}(\mathcal{G}) \mid \sigma'(F_1) = F_4\}$ ;
9     for  $\sigma$  in automorphisms do
10      cdc  $\leftarrow \text{orb}_H(\alpha(\sigma, F_1, \dots, F_4))$ ;
11      if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
12         $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
13        if  $|\text{Aut}(X)| = 2 \cdot |X_2|$  then
14          Add(surfaces,  $X$ );
15 return surfaces;
```

5.4.5 Face-transitive surfaces with vertex-face type $(1, 3)$

In this section, we investigate face-transitive surfaces of vertex-face type $(1, 3)$. We show that the automorphism group of such a simplicial surface falls into one of two types. Moreover, we describe that the vertex-defining cycles of a face-transitive surface X with $\text{vf}(X) = (1, 3)$ form automorphism induced α -cycles of the face graph $\mathcal{F}(X)$.

First, we establish the definition of a $(1, 3)$ -group of a given cubic node-transitive graph.

Definition 5.4.19

Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the nodes V and arcs E with $|H| = 3 \cdot |V|$,
2. for $F_1, F_2, F_3 \in V$ with $\{F_1, F_2\}, \{F_2, F_3\} \in E$, there exists an automorphism $\sigma \in H$ such that $\sigma(F_1) = F_3$.

Furthermore, let $\text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ (cf. Remark 3.2.1 and Definition 3.2.2) be a vertex-faithful cycle double cover of $\mathcal{G}$. We say that H is a **$(1, 3)$ -group of $\mathcal{G}$ of type 1** if there exists an automorphism $\pi \in \text{Aut}(\mathcal{G})$ with $\alpha(\pi, F_1, F_2) = \alpha(\sigma, F_1, F_2, F_3)$, we denote the above cycle double cover by $\mathcal{C}_1^{(1,3)}(H)$. If there exists no such π , then we say that H is a **$(1, 3)$ -group of $\mathcal{G}$ of type 2** and denote the above cycle double cover by $\mathcal{C}_2^{(1,3)}(H)$.

We call a subgroup H of the automorphism group of a node-transitive cubic graph a $(1, 3)$ -group, if H is a $(1, 3)$ -group of any type. In this case, we denote the corresponding cycle double cover by $\mathcal{C}^{(1,3)}(H)$.

Theorem 5.4.20

A simplicial surface X is a face-transitive surface with $\text{vf}(X) = (1, 3)$ if and only if $H := \lambda_X(\text{Aut}(X))$ is a $(1, 3)$ -group of $\mathcal{F}(X)$ and the cycle double cover $\mathcal{C}^{(1,3)}(H)$ contains the vertex-defining cycles of X .

Proof. If X is a face-transitive surface with $\text{vf}(X) = (1, 3)$, then the automorphism group $\text{Aut}(X)$ has order $|\text{Aut}(X)| = 3 \cdot |X_2|$, and all face-stabilisers are cyclic of order 3. In particular, for every $F \in X_2$, the stabiliser $\text{Stab}_{\text{Aut}(X)}(F)$ acts transitively on $X_1(F)$. This implies that $\text{Aut}(X)$ acts transitively on X_1 and all edge-stabilisers are therefore conjugate in $\text{Aut}(X)$, see Lemma 3.1.8.

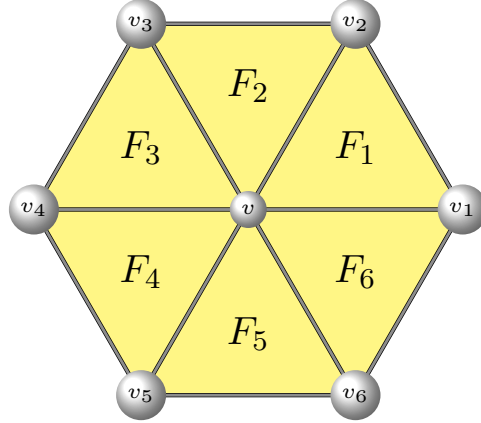


Figure 5.19: An umbrella of a face-transitive surface X

We exploit the above facts to show that the subgroup H of the automorphism group $\text{Aut}(\mathcal{F}(X))$ forms a $(1, 3)$ -group of $\mathcal{F}(X)$. Let $v \in X_0$ be a vertex of X with umbrella $u(v) = (F_1, \dots, F_n)$. Moreover, for $1 \leq i \leq n$ let F_i be a face with $X_0(F_i) = \{v, v_i, v_{i+1}\}$ and $e_i \in X_1$ an edge with $X_0(e) = \{v, v_i\}$, where the subscripts are read modulo n . Note that $|\text{Stab}_{\text{Aut}(X)}(e_i)| = 2$ for all $i = 1, \dots, n$ by the orbit-stabiliser theorem. Thus, for every $1 \leq i \leq n$ let $\phi_i \in \text{Stab}_{\text{Aut}(X)}(e_i)$ be the non-trivial automorphism stabilising the edge e_i . It follows, that either **(1)** $\phi_i(v) = v_i$ and $\phi_i(v_i) = v$ for all $1 \leq i \leq n$, or **(2)** $\phi_i(v) = v$ and $\phi_i(v_i) = v_i$ for all $1 \leq i \leq n$.

Case 1: We observe that because of the stabiliser of F_1 being cyclic of order 3, the automorphism ϕ_2 cannot fix F_1 . We therefore know $\phi_2(v_1) = v_3$ and $\phi_2(v_3) = v_1$. This enables us to compute an automorphism of X that applies a cyclic shift to the faces F_i incident to v , where $1 \leq i \leq n$. Let $\psi \in \text{Stab}_{\text{Aut}(X)}(F_2)$ be the unique automorphism satisfying $\psi(v) = v_3$, $\psi(v_2) = v$ and $\psi(v_3) = v_2$. Then $\phi := \psi \circ \phi_2$ satisfies $\phi(v) = v$, $\phi(v_1) = v_2$ and $\phi(v_2) = v_3$. Since ϕ respects the incidences of X , we know that ϕ is an automorphism stabilising v and mapping F_i onto F_{i+1} , where subscripts are read modulo n . Hence, the umbrella of v can be written as

$$u(v) = (F_1, \dots, F_n) = (\phi(F_1), \dots, \phi^n(F_1)).$$

By considering $F_1, \dots, F_n$ as nodes of the face graph $\mathcal{F}(X)$ and defining $\sigma := \lambda_X(\phi)$ we observe that $(F_1, \dots, F_n) = \alpha(\sigma, F_1, F_2) = \alpha(\sigma^2, F_1, F_2, F_3)$ is a σ -induced α -cycle of $\mathcal{F}(X)$. Moreover, all vertices of X lie in the same $\text{Aut}(X)$ -orbit and the vertex-defining cycles of X are contained in the orbit $\text{orb}_H(\alpha(\sigma, F_1, F_2))$. This results in H being a $(1, 3)$ -group of $\mathcal{F}(X)$ of type 1.

Case 2: Since the face-stabilisers are cyclic and of order 3, we know that ϕ_2 interchanges the faces F_1 and F_2 and ϕ_3 interchanges the faces F_2 and F_3 . Thus, we define $\phi' := \phi_3 \circ \phi_2$ and obtain

$\phi(F_i) = F_{i+2}$ and $\phi'(v) = v$. Thus, the umbrella of v can be written as

$$u(v) = (F_1, \dots, F_n) = (\phi'(F_1), \phi'(F_2), \dots, \phi'^\ell(F_1), \phi'^\ell(F_2)),$$

where ℓ denotes the order of ϕ' . Thus, by interpreting $F_1, \dots, F_n$ as nodes of the face graph and considering $\sigma := \lambda_X(\phi')$, we see that $(F_1, \dots, F_n) = \alpha(\sigma, F_1, F_2, F_3)$ is a σ -induced α -cycle of $\mathcal{F}(X)$.

It remains to show that there exists no $\pi \in \text{Aut}(\mathcal{F}(X))$ with $\alpha(\pi, F_1, F_2) = \alpha(\sigma, F_1, F_2, F_3)$. We assume that there is such an automorphism π . This automorphism π must be the image of an automorphism $\psi \in \text{Aut}(X)$ under λ_X satisfying $(F_1, \dots, F_n) = (\psi(F_1), \dots, \psi^n(F_1))$ in X . Hence, ψ maps F_1 either onto F_2 or F_n . Without loss of generality, we assume $\psi(F_1) = F_2$. This means $\psi(v) = v, \psi(v_1) = v_2$ and $\psi(v_2) = v_3$. Since the stabiliser of F_2 is cyclic of order 3, there exists a unique automorphism $\psi' \in \text{Stab}_{\text{Aut}(X)}(F_2)$ satisfying $\psi'(v) = v_2, \psi'(v_2) = v_3$ and $\psi'(v_3) = v$. Hence, the automorphism $\gamma := \psi' \circ \psi$ satisfies $\gamma(v) = v_2$ and $\gamma(v_2) = v$. That means $\gamma \in \text{Stab}_{\text{Aut}(X)}(e_2)$. Because of γ being non-trivial and $\gamma \notin \text{Stab}_{\text{Aut}(X)}(v)$, we know $\gamma \neq \phi_2$. Hence, the stabiliser of e_2 contains $\{id, \phi_2, \gamma\}$ and therefore has order 4, a contradiction because $\text{vf}(X) = (1, 3)$ implies $|\text{Stab}_{\text{Aut}(X)}(e)| = 2$ for all $e \in X_1$. Since all the vertices of X lie in one $\text{Aut}(X)$ -orbit, the vertex-defining umbrellas of X can be translated into the orbit $\text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ and hence H forms a $(1, 3)$ -group of $\mathcal{F}(X)$ of type 2.

To complete the proof, let H be a $(1, 3)$ -group of $\mathcal{F}(X)$. Let $F \in X_2$ be an arbitrary face. Since $\text{Aut}(X) \cong H$ we know that X is face-transitive and the stabiliser $\text{Stab}_H(F)$ of the node F is isomorphic to $\text{Stab}_{\text{Aut}(X)}(F)$ of the face F , hence, cyclic of order 3. We conclude that $\text{Stab}_{\text{Aut}(X)}(F)$ is transitive on $X_0(F)$, $\text{Aut}(X)$ is transitive on the vertices of X , and hence X is a face-transitive surface with $\text{vf}(X) = (1, 3)$. □

Corollary 5.4.21 *Let $\mathcal{G}$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ a subgroup. If H is a $(1, 3)$ -group of $\mathcal{G}$, then $\mathcal{G}$ is the face graph of a face-transitive surface.*

An example of a cubic node-transitive graph that gives rise to a face-transitive surface X with $\text{vf}(X) = (1, 3)$ is the graph $\mathcal{G}$ illustrated in Figure 5.20.

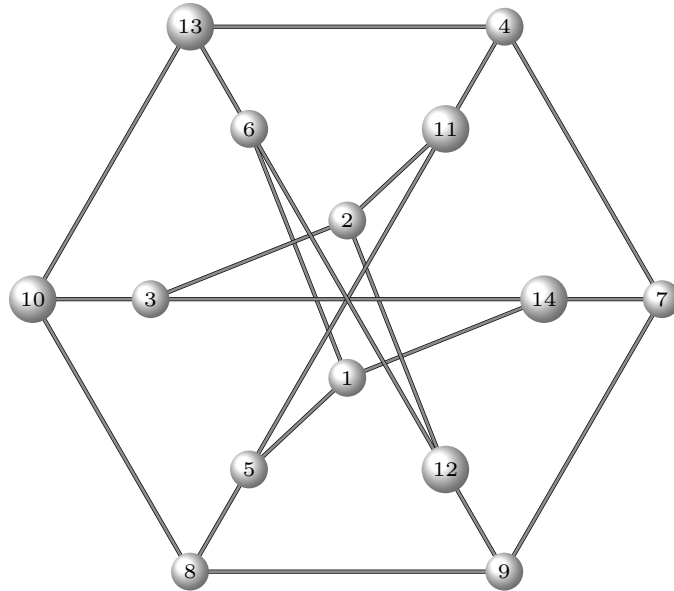


Figure 5.20: A cubic node-transitive graph $\mathcal{G}$

The group H defined by

$$H := \langle (1, 2)(3, 5, 12, 14, 11, 6)(4, 13, 10, 8, 9, 7), (1, 5)(2, 7)(3, 4)(6, 8)(9, 12)(10, 13)(11, 14) \rangle$$

forms a subgroup of $\text{Aut}(\mathcal{G})$ that has order $3 \cdot 14$, and acts transitively on the nodes and arcs of $\mathcal{G}$. Furthermore, H contains the automorphism $\sigma := (1, 5, 11, 4, 13, 6)(2, 7, 10, 12, 14, 8)(3, 9)$. Hence, by exploiting Corollary 5.4.21, we obtain the following vertex-faithful cycle double cover of $\mathcal{G}$:

$$\begin{aligned} \text{orb}_H(\alpha(\sigma, 1, 5)) = \{ & (4, 13, 10, 8, 9, 7), (2, 12, 6, 13, 10, 3), (2, 11, 4, 7, 14, 3), (2, 12, 9, 8, 5, 11), \\ & (1, 6, 13, 4, 11, 5), (1, 14, 3, 10, 8, 5), (1, 14, 7, 9, 12, 6) \}. \end{aligned}$$

Thus, H is a $(1, 3)$ -group of $\mathcal{G}$ of type 1 and the above cycle double cover gives rise to the face-transitive surface X that can be described by the following vertices of faces:

$$\begin{aligned} \Theta(X) = \{ & \{v_5, v_6, v_7\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_6\}, \{v_1, v_3, v_5\}, \{v_4, v_5, v_6\}, \{v_2, v_5, v_7\}, \{v_1, v_3, v_7\}, \\ & \{v_1, v_4, v_6\}, \{v_1, v_4, v_7\}, \{v_1, v_2, v_6\}, \{v_3, v_4, v_5\}, \{v_2, v_4, v_7\}, \{v_1, v_2, v_5\}, \{v_3, v_6, v_7\} \}. \end{aligned}$$

This simplicial surface X forms a minimal triangulation of the torus (see [34]) with

$$C_7 \rtimes C_6 \cong \text{Aut}(X) \hookrightarrow \text{Aut}(\mathcal{F}(X)) \cong \text{PSL}(3, 2) \rtimes C_2.$$

As described in the previous cases, a face-transitive surface X that results from the procedure established in Corollary 5.4.21 does not have to satisfy $\text{vf}(X) = (1, 3)$. An example of this observation is the simplicial tetrahedron $\mathcal{T}$ and its face graph. The vertex-defining cycles of the simplicial tetrahedron $\mathcal{T}$ can be reconstructed from its face graph $\mathcal{F}(\mathcal{T})$ with Corollary 5.4.21 by exploiting $H' := \langle (1, 2)(3, 4), (1, 2, 3) \rangle \leq \text{Aut}(\mathcal{F}(\mathcal{T}))$ as a $(1, 3)$ -group of type 1 of $\mathcal{F}(\mathcal{T})$ and the automorphism $\sigma' := (1, 2, 3) \in H'$.

Note, in the literature, a face-transitive surface whose automorphism group yields a $(1, 3)$ -group of type 1 of the corresponding face graph is also referred to as a chiral map, see [29] for more details. With our computations in GAP, we have been able to verify the existence of a face-transitive surface Y , where the group $\lambda_Y(\text{Aut}(Y))$ forms a $(1, 3)$ -group of type 2 of the face graph $\mathcal{F}(Y)$. The smallest example of such a face-transitive surface with vertex-face-type $(1, 3)$ is the simplicial surface Y described by the below vertices of faces. This simplicial surface is orientable and consists of 36 vertices, 216 edges and 144 faces. Hence, the Euler characteristic of Y is $\chi(Y) = -36$. Note that its automorphism group satisfies

$$\text{Aut}(Y) \cong \text{Aut}(\mathcal{F}(Y)) \cong (((C_3 \times C_3) \rtimes Q_8) \rtimes C_3) \rtimes C_2.$$

The vertices of faces of Y are given by

$$\begin{aligned} \Theta(Y) = \{ & \{v_{34}, v_{35}, v_{36}\}, \{v_{31}, v_{32}, v_{33}\}, \{v_{28}, v_{29}, v_{30}\}, \{v_{25}, v_{26}, v_{27}\}, \{v_{24}, v_{32}, v_{35}\}, \{v_{32}, v_{34}, v_{35}\}, \\ & \{v_{23}, v_{24}, v_{32}\}, \{v_{22}, v_{29}, v_{33}\}, \{v_{20}, v_{21}, v_{27}\}, \{v_{18}, v_{19}, v_{20}\}, \{v_{17}, v_{28}, v_{29}\}, \{v_{15}, v_{16}, v_{22}\}, \\ & \{v_{14}, v_{23}, v_{26}\}, \{v_{13}, v_{15}, v_{22}\}, \{v_{11}, v_{12}, v_{36}\}, \{v_{10}, v_{21}, v_{33}\}, \{v_8, v_9, v_{19}\}, \{v_{21}, v_{35}, v_{36}\}, \\ & \{v_9, v_{21}, v_{23}\}, \{v_{19}, v_{29}, v_{30}\}, \{v_8, v_{24}, v_{35}\}, \{v_7, v_{11}, v_{29}\}, \{v_{21}, v_{29}, v_{35}\}, \{v_6, v_{23}, v_{28}\}, \\ & \{v_{15}, v_{19}, v_{36}\}, \{v_{20}, v_{22}, v_{34}\}, \{v_4, v_5, v_{24}\}, \{v_{18}, v_{19}, v_{31}\}, \{v_3, v_{10}, v_{21}\}, \{v_5, v_{28}, v_{30}\}, \\ & \{v_8, v_9, v_{13}\}, \{v_{21}, v_{29}, v_{33}\}, \{v_9, v_{12}, v_{17}\}, \{v_{16}, v_{28}, v_{31}\}, \{v_7, v_{25}, v_{29}\}, \{v_8, v_{16}, v_{28}\}, \\ & \{v_2, v_3, v_{17}\}, \{v_2, v_5, v_{15}\}, \{v_6, v_{12}, v_{25}\}, \{v_{17}, v_{27}, v_{28}\}, \{v_{11}, v_{13}, v_{18}\}, \{v_1, v_3, v_{16}\}, \\ & \{v_3, v_{10}, v_{24}\}, \{v_6, v_{10}, v_{28}\}, \{v_6, v_{25}, v_{35}\}, \{v_{14}, v_{17}, v_{18}\}, \{v_{19}, v_{31}, v_{36}\}, \{v_{23}, v_{31}, v_{32}\}, \end{aligned}$$

$\{v_1, v_{11}, v_{29}\}, \{v_3, v_{16}, v_{31}\}, \{v_{12}, v_{16}, v_{20}\}, \{v_4, v_7, v_{31}\}, \{v_{10}, v_{15}, v_{19}\}, \{v_1, v_{32}, v_{34}\},$
 $\{v_3, v_{25}, v_{26}\}, \{v_1, v_{19}, v_{29}\}, \{v_5, v_9, v_{25}\}, \{v_3, v_{17}, v_{24}\}, \{v_{14}, v_{26}, v_{30}\}, \{v_{18}, v_{31}, v_{33}\},$
 $\{v_{13}, v_{30}, v_{32}\}, \{v_4, v_{14}, v_{30}\}, \{v_6, v_{10}, v_{12}\}, \{v_3, v_{25}, v_{31}\}, \{v_8, v_{19}, v_{24}\}, \{v_{17}, v_{22}, v_{24}\},$
 $\{v_3, v_{26}, v_{30}\}, \{v_7, v_{25}, v_{31}\}, \{v_{10}, v_{19}, v_{24}\}, \{v_8, v_{16}, v_{35}\}, \{v_{17}, v_{22}, v_{29}\}, \{v_{16}, v_{20}, v_{22}\},$
 $\{v_9, v_{12}, v_{25}\}, \{v_{18}, v_{20}, v_{27}\}, \{v_{15}, v_{27}, v_{32}\}, \{v_{10}, v_{14}, v_{15}\}, \{v_4, v_5, v_{30}\}, \{v_3, v_{13}, v_{30}\},$
 $\{v_7, v_{11}, v_{13}\}, \{v_{11}, v_{26}, v_{36}\}, \{v_6, v_{13}, v_{22}\}, \{v_{11}, v_{23}, v_{26}\}, \{v_{12}, v_{32}, v_{33}\}, \{v_{23}, v_{28}, v_{31}\},$
 $\{v_5, v_9, v_{34}\}, \{v_4, v_{21}, v_{36}\}, \{v_{11}, v_{12}, v_{16}\}, \{v_2, v_7, v_8\}, \{v_1, v_2, v_3\}, \{v_4, v_{21}, v_{27}\},$
 $\{v_{22}, v_{26}, v_{34}\}, \{v_{20}, v_{21}, v_{23}\}, \{v_2, v_6, v_{23}\}, \{v_{26}, v_{34}, v_{36}\}, \{v_9, v_{14}, v_{17}\}, \{v_3, v_{13}, v_{21}\},$
 $\{v_2, v_{17}, v_{36}\}, \{v_8, v_{26}, v_{33}\}, \{v_{12}, v_{30}, v_{32}\}, \{v_{12}, v_{20}, v_{30}\}, \{v_7, v_{10}, v_{14}\}, \{v_{15}, v_{25}, v_{27}\},$
 $\{v_5, v_{28}, v_{34}\}, \{v_{14}, v_{15}, v_{16}\}, \{v_{10}, v_{28}, v_{34}\}, \{v_2, v_{15}, v_{36}\}, \{v_{17}, v_{18}, v_{27}\}, \{v_{14}, v_{16}, v_{35}\},$
 $\{v_6, v_{13}, v_{18}\}, \{v_{13}, v_{15}, v_{32}\}, \{v_{19}, v_{20}, v_{30}\}, \{v_{22}, v_{26}, v_{33}\}, \{v_4, v_{22}, v_{24}\}, \{v_7, v_8, v_{13}\},$
 $\{v_1, v_9, v_{34}\}, \{v_1, v_2, v_6\}, \{v_2, v_5, v_{33}\}, \{v_1, v_9, v_{19}\}, \{v_7, v_{20}, v_{34}\}, \{v_8, v_{27}, v_{28}\},$
 $\{v_5, v_{18}, v_{33}\}, \{v_1, v_{11}, v_{16}\}, \{v_8, v_{26}, v_{27}\}, \{v_7, v_{10}, v_{34}\}, \{v_2, v_7, v_{20}\}, \{v_{14}, v_{18}, v_{35}\},$
 $\{v_2, v_{20}, v_{23}\}, \{v_9, v_{13}, v_{21}\}, \{v_{25}, v_{29}, v_{35}\}, \{v_1, v_4, v_{27}\}, \{v_9, v_{14}, v_{23}\}, \{v_{12}, v_{17}, v_{36}\},$
 $\{v_5, v_{11}, v_{18}\}, \{v_4, v_{31}, v_{36}\}, \{v_1, v_4, v_6\}, \{v_{11}, v_{23}, v_{24}\}, \{v_2, v_8, v_{33}\}, \{v_1, v_{27}, v_{32}\},$
 $\{v_5, v_{15}, v_{25}\}, \{v_5, v_{11}, v_{24}\}, \{v_4, v_7, v_{14}\}, \{v_6, v_{18}, v_{35}\}, \{v_4, v_6, v_{22}\}, \{v_{10}, v_{12}, v_{33}\}.$

The above example establishes that a face-transitive surface satisfying $|\text{Aut}(X)| = 3|X_2|$ does not have to be a chiral map.

Lastly, we present algorithms that construct the face-transitive surfaces discussed in this subsection in GAP.

Algorithm 8: Compute all face-transitive surfaces with vertex-face type $(1, 3)$ of a given cubic node-transitive graph (Type 1)

Input: A cubic node-transitive graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(1, 3)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 3 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 1\}$ ;
3  $[F_1, F_2] \leftarrow \{F_1, F_2\} \in E$ ;
4 for  $H$  in candidates do
5   automorphisms  $\leftarrow \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_2\}$ ;
6   for  $\sigma$  in automorphisms do
7     cdc  $\leftarrow \text{orb}_H(\alpha(\sigma, F_1, F_2))$ ;
8     if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
9        $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
10      if  $|\text{Aut}(X)| = 3 \cdot |X_2|$  then
11        Add(surfaces,  $X$ );
12 return surfaces;
```

Algorithm 9: Compute all face-transitive surfaces with vertex-face type $(1, 3)$ of a given cubic node-transitive graph (Type 2)

Input: A node-transitive cubic graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(1, 3)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 3 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 1\}$ ;
3  $[F_1, F_2, F_3] \leftarrow [F_1, F_2, F_3]$  is a path of length 3 in  $\mathcal{G}$ ;
4 for  $H$  in candidates do
5     automorphisms2  $\leftarrow \{\sigma^2 \mid \sigma \in \text{Aut}(\mathcal{G}), \sigma(F_1) = F_2\}$ ;
6     automorphisms  $\leftarrow \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_3, \sigma \notin \text{automorphisms2}\}$ ;
7     for  $\sigma$  in automorphisms do
8         cdc  $\leftarrow \text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ ;
9         if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
10              $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
11             if  $|\text{Aut}(X)| = 3 \cdot |X_2|$  then
12                 Add(surfaces,  $X$ );
13 return surfaces;
```

5.4.6 Face-transitive surfaces with vertex-face type $(2, 1)$

In this section, we investigate face-transitive surfaces with trivial face-stabilisers and automorphism groups yielding exactly two vertex-orbits. Here, we show that the umbrellas of a face-transitive surface X with $\text{vf}(X) = (2, 1)$ are either (1) mono-coloured and bi-coloured umbrellas that are induced by automorphisms or (2) correspond to tri-coloured cycles of the face graph $\mathcal{F}(X)$. We note that in the case of $\text{vf}(X) = (2, 1)$, the face stabilisers are trivial, and we need to be more careful in constructing the corresponding vertex-defining cycles. We start this section by giving the different definitions of $(2, 1)$ -groups of a given node-transitive cubic graph.

Definition 5.4.22

Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the nodes V with $|H| = 1 \cdot |V|$,
2. E partitions into two H -orbits E_1 and E_2 with $|E_1| = |V|$ and $|E_2| = \frac{1}{2}|V|$, and
3. there exist $F_1, \dots, F_5 \in V$ with $\{F_2, F_3\}, \{F_4, F_5\} \in E_1$ and $\{F_1, F_2\}, \{F_3, F_4\} \in E_2$, and an automorphism $\sigma \in \text{Aut}(\mathcal{G})$ such that $\sigma(F_1) = F_5$.

The partition $E = E_1 \uplus E_2$ gives rise to an arc-colouring $\kappa : E \rightarrow \{1, 2\}$ of $\mathcal{G}$ by $\kappa(\{F, F'\}) := i$ for $\{F, F'\} \in E_i$. Let u be a mono-coloured cycle in $\mathcal{G}$ with respect to κ and $\text{orb}_H(u) \cup \text{orb}_H(\alpha(\sigma, F_1, \dots, F_5))$ (cf. Remark 3.2.1, Definition 3.2.2 and Definition 5.4.1) a vertex-faithful cycle double cover of $\mathcal{G}$. We say that H is a **$(2, 1)$ -group of $\mathcal{G}$ of type 1**, if there exists an automorphism $\pi \in \text{Aut}(\mathcal{G})$ with $\alpha(\pi, F_1, F_2, F_3) = \alpha(\sigma, F_1, \dots, F_5)$. We denote the above cycle double cover by $\mathcal{C}_1^{(2,1)}(H)$. If there exists no such π , then we say that H is a **$(2, 1)$ -group of $\mathcal{G}$ of type 2** and denote the above cycle double cover by $\mathcal{C}_2^{(2,1)}(H)$.

Definition 5.4.23

Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the nodes V with $|H| = 1 \cdot |V|$, and
2. E partitions into three H -orbits E_1, E_2 , and E_3 , all of cardinality $\frac{1}{2}|V|$.

The partition $E = E_1 \uplus E_2 \uplus E_3$ yields the arc-colouring $\kappa : E \rightarrow \{1, 2, 3\}$, defined by $\kappa(\{F, F'\}) := i$ for $\{F, F'\} \in E_i$. If the (unique) set containing all the $(1, 2)$ - and $(3, 1, 3, 2)$ -cycles with respect to κ (cf. Definition 5.4.1) is a vertex-faithful cycle double cover of $\mathcal{G}$, then we say that H is a **$(2, 1)$ -group of $\mathcal{G}$ of type 3**, with the cycle double cover denoted by $C_3^{(2,1)}(H)$.

We call a subgroup H of the automorphism group of a node-transitive cubic graph a $(2, 1)$ -group, if H is a $(2, 1)$ -group of any type and simply denote the corresponding cycle double cover by $C^{(2,1)}(H)$. We proceed by formulating the theorem that describes the structure of a face-transitive surface of vertex-face type $(2, 1)$.

Theorem 5.4.24

A simplicial surface X is a face-transitive surface with vertex-face type $\text{vf}(X) = (2, 1)$ if and only if $H := \lambda_X(\text{Aut}(X))$ is a $(2, 1)$ -group of $\mathcal{F}(X)$ and the cycle double cover $C^{(2,1)}(H)$ contains exactly the vertex-defining cycles of X .

Proof. Let X be a face-transitive surface with $\text{vf}(X) = (2, 1)$. Then, X satisfies $|\text{Aut}(X)| = 1 \cdot |X_2|$. Furthermore, let V_1 and V_2 be the two $\text{Aut}(X)$ -orbits on X_0 . Without loss of generality, we assume that every face of X is incident to exactly one vertex in V_1 and exactly two vertices in V_2 . With the orbit-stabiliser theorem, we see that the number of $\text{Aut}(X)$ -orbits on X_1 satisfies $|\text{orb}_{\text{Aut}(X)}(X_1)| \in \{2, 3\}$. We therefore conduct a case distinction to determine the structure of the umbrellas of X with respect to the number of orbits on the edges.

Case 1: We examine the case $|\text{orb}_{\text{Aut}(X)}(X_1)| = 2$. The orbits V_1, V_2 induce a partition of X_1 by

$$\mathcal{E}_1 := \{e \in X_1 \mid X_0(e) \cap V_1 \neq X_0(e)\}, \mathcal{E}_2 := \{e \in X_1 \mid X_0(e) \cap V_2 = X_0(e)\}.$$

We notice that $|\mathcal{E}_1| = |X_2|$ and $|\mathcal{E}_2| = \frac{1}{2}|X_2|$. Since $|\text{orb}_{\text{Aut}(X)}(X_1)| = 2$, the partition sets $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\text{Aut}(X)$ -orbits. Hence, the map $\omega : X_1 \rightarrow \{1, 2\}$ defined by $\omega(e) := i$ for $e \in \mathcal{E}_i$ with $i = 1, 2$ forms an edge-colouring of X which can be translated into an arc-colouring κ of $\mathcal{F}(X)$ via $\kappa(\{F, F'\}) := \omega(e)$ for $X_2(e) = \{F, F'\}$.

Further, let $v_1 \in V_1, v_2 \in V_2$ be vertices with corresponding umbrellas $u(v_1) = (F_1, \dots, F_n)$ and $u(v_2) = (F'_1, \dots, F'_m)$. The umbrellas of the vertices in $\text{orb}_{\text{Aut}(X)}(v_1)$ are mono-coloured and the umbrellas of the vertices in $\text{orb}_{\text{Aut}(X)}(v_2)$ are bi-coloured umbrellas with respect to ω . The mono-coloured cycles representing the vertices $\text{orb}_{\text{Aut}(X)}(v_1)$ are directly obtained from deleting the edges $\{X_2(e) \mid e \in \mathcal{E}_2\}$ from $\mathcal{F}(X)$ as also described in the proof of Theorem 5.4.9. Hence, these cycles describe the umbrellas in $\text{orb}_{\text{Aut}(X)}(u(v_1))$.

Next, we describe the umbrellas of the vertices in the $\text{Aut}(X)$ -orbit of v_2 . First, note that $u(v_2)$ is bi-coloured, and hence m is even. Thus, we define $k := \frac{m}{2}$. Moreover, let $e_1, \dots, e_m$ be the edges with $X_2(e_1) = \{F'_1, F'_2\}, \dots, X_2(e_m) = \{F'_m, F'_1\}$, respectively. Without loss of generality, we assume that $e_{2i} \in \mathcal{E}_2, i = 1, \dots, k$, see Figure 5.21 for an illustration.

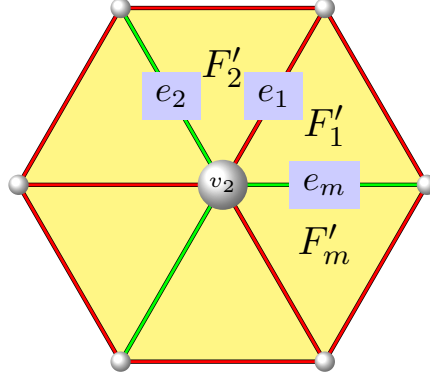


Figure 5.21: The edge-coloured umbrella $u(v_2)$ of the simplicial surface X

By Lemma 3.1.8 the edge-stabilisers of $e_2, e_4, \dots, e_m$ are all conjugate in $\text{Aut}(X)$ and have order two. Thus, for $1 \leq i \leq k$ let ϕ_{2i} be the non-trivial automorphism in $\text{Stab}_{\text{Aut}(X)}(e_{2i})$. We have the following two cases:

- (a) $\phi_{2i}(v_2) = v_2$ for all $1 \leq i \leq k$, that is, these automorphisms stabilise the vertex v_2 ; or
- (b) $\phi_{2i}(v_2) \neq v_2$ for all $1 \leq i \leq k$, that is, the vertices of the corresponding edges are interchanged.

Case (1a): If all $\phi_2, \phi_4, \dots, \phi_m$ stabilise v_2 , then the automorphism $\phi := \phi_2 \circ \phi_m$ of order ℓ satisfies $\phi(F'_i) = F'_{i+4}$, where all subscripts are read modulo n . Hence, the umbrella of v_2 can be written as

$$(F'_1, \dots, F'_m) = (\phi(F'_1), \phi(F'_2), \phi(F'_3), \phi(F'_4), \dots, \phi^\ell(F'_1), \phi^\ell(F'_2), \phi^\ell(F'_3), \phi^\ell(F'_4)).$$

By defining $\sigma := \lambda_X(\phi)$ we see that $u(v_2) = \alpha(\sigma, F'_1, \dots, F'_5)$ is a σ -induced α -cycle in $\mathcal{F}(X)$ which implies that the cycle double cover containing all the vertex-defining cycles of X is given by $\text{orb}_H((F_1, \dots, F_n)) \cup \text{orb}_H(\alpha(\sigma, F'_1, \dots, F'_5))$. If there exists an automorphism $\pi \in \text{Aut}(\mathcal{F}(X))$ satisfying $\alpha(\pi, F'_1, F'_2, F'_3) = \alpha(\sigma, F'_1, \dots, F'_5)$ then without loss of generality π maps F'_1 onto F'_3 . Thus, the automorphism $\sigma' := \pi \circ \lambda_X(\phi_m)$ is a non-trivial element in the stabiliser of e_1 , a contradiction to $|\text{Stab}_{\text{Aut}(X)}(e_1)| = 1$. Thus, H is a $(2, 1)$ -group of $\mathcal{F}(X)$ of type 2.

Case (1b): Now, we assume that the automorphisms $\phi_2, \phi_4, \dots, \phi_m$ do not stabilise the vertex v_2 . Hence, let $\psi \in \text{Aut}(X)$ be the automorphism that maps F'_1 onto F'_2 . Since $|\mathcal{E}_1| = |X_2|$, the stabiliser $\text{Stab}_{\text{Aut}(X)}(e_1)$ is trivial and the automorphism ψ cannot stabilise e_1 . Hence, we know that $\psi(v_2) \neq v_2$ and $\psi(e_m) = e_2$. If we define $\phi' := \phi_2 \circ \psi \in \text{Aut}(X)$, we observe $\phi'(v_2) = v_2$ and hence $\phi' \in \text{Stab}_{\text{Aut}(X)}(v_2)$. Furthermore, we know that $\phi'(F'_1) = \phi_2(\psi(F'_1)) = \phi_2(F'_2) = F'_3$ and $\phi'(F'_m) = \phi_2(\psi(F'_m)) = \phi_2(F'_3) = F'_2$. We therefore conclude $\phi'(F'_i) = F'_{i+2}$. If ℓ is the order of ϕ' , then the umbrella of v_2 can be written as $(F'_1, F'_2, \dots, F'_m) = (\phi'(F'_1), \phi'(F'_2), \dots, \phi'^\ell(F'_1), \phi'^\ell(F'_2))$. By defining $\sigma := \lambda_X(\phi')$ we observe that $u(v_2) = \alpha(\sigma, F'_1, F'_2, F'_3)$ is a σ -induced α -cycle in $\mathcal{F}(X)$. Thus, we obtain the cycle double cover containing all the vertex-defining cycles of X as the set $\text{orb}_H((F_1, \dots, F_n)) \cup \text{orb}_H(\alpha(\sigma, F'_1, F'_2, F'_3))$. Together with $\alpha(\sigma, F'_1, F'_2, F'_3) = \alpha(\sigma^2, F'_1, \dots, F'_5)$ it follows that H is a $(2, 1)$ -group of $\mathcal{F}(X)$ of type 1.

Case 2: Here, we assume that the $\text{Aut}(X)$ -orbits of X_1 are given by $\mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3$ with $|\mathcal{E}'_i| = \frac{1}{2}|X_2|$ for $i = 1, 2, 3$. We define an edge-colouring ω' of X via $\omega'(e) := i$ for $e \in \mathcal{E}'_i$ and $i = 1, 2, 3$ which can be translated into a corresponding arc-colouring κ of $\mathcal{F}(X)$ as usual. Note that we still obtain a partition of X_1 via

$$\mathcal{P}_1 := \{e \in X_1 \mid X_0(e) \cap V_1 \neq X_0(e)\}, \mathcal{P}_2 := \{e \in X_1 \mid X_0(e) \cap V_2 = X_0(e)\}.$$

Since $|\mathcal{P}_2| = \frac{1}{2}|X_2|$, the set $\mathcal{P}_2$ must be an $\text{Aut}(X)$ -orbit. Thus, without loss of generality, we can assume $\mathcal{P}_2 = \mathcal{E}'_3$. This implies $\mathcal{P}_1 = \mathcal{E}'_1 \cup \mathcal{E}'_2$ and the umbrellas of vertices in V_1 must be bi-coloured. This implies that the edges in $\mathcal{E}'_1 \cup \mathcal{E}'_2$ are all mirror edges with respect to ω . Hence, the following cases remain to be checked:

- (a) all the edges of X are mirror edges,
- (b) the edges contained in $\mathcal{E}'_1 \cup \mathcal{E}'_2$ are mirror edges and the edges in $\mathcal{E}'_3$ are rotational edges.

Case (2a): If all edges are mirror edges, the automorphism group $\text{Aut}(X)$ has three vertex-orbits (as described in the proof of Theorem 5.4.4), a contradiction to $\text{vf}(X) = (2, 1)$.

Case (2b): In this case, we observe that all umbrellas in $\text{orb}_{\text{Aut}(X)}(v_1)$ are $(1, 2, \dots, 1, 2)$ -coloured and all umbrellas in $\text{orb}_{\text{Aut}(X)}(v_2)$ are $(3, 1, 3, 2)$ -coloured. These umbrellas can directly be translated into $(1, 2)$ - and $(3, 1, 3, 2)$ -coloured cycles of the face graph $\mathcal{F}(X)$ with respect to κ . Hence, we obtain a vertex-faithful cycle double cover of $\mathcal{F}(X)$ consisting of only $(1, 2)$ - and $(3, 1, 3, 2)$ -coloured cycles and H is a $(2, 1)$ -group of $\mathcal{F}(X)$ of type 3.

Next, we assume that H forms a $(2, 1)$ -group of $\mathcal{F}(X)$. The arc-colouring of $\mathcal{F}(X)$ defines an edge-colouring ω of X by $\omega(e) := \kappa(\{F, F'\})$, where $e \in X_1$ and $X_2(e) = \{F, F'\}$. Since $H = \lambda_X(\text{Aut}(X))$, and H acts transitively on the nodes of $\mathcal{F}(X)$ with $|H| = 1 \cdot |V|$, we have that X is face-transitive with trivial face stabiliser. Due to Lemma 5.4.2, two vertices v_1 and v_2 of X lie in the same $\text{Aut}(X)$ -orbit if and only if the corresponding cycles $C_1, C_2 \in \mathcal{C}^{(2,1)}(H)$ in $\mathcal{F}(X)$ are contained in the same H -orbit. This is equivalent to saying that the corresponding vertex-defining cycles of X have the same colour with respect to κ . Thus, there exist exactly two $\text{Aut}(X)$ -orbits on the vertices, and we obtain $\text{vf}(X) = (2, 1)$. $\square$

The below corollary is a direct consequence of the above result.

Corollary 5.4.25 *Let $\mathcal{G}$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ a subgroup. If H is a $(2, 1)$ -group of $\mathcal{G}$, then $\mathcal{G}$ is the face graph of a face-transitive surface.*

The cubic node-transitive graph $\mathcal{G} = (V, E)$ that is illustrated in Figure 5.22 admits the construction of a face-transitive surface with vertex-face type $(2, 1)$. Here, the group H defined by

$$\begin{aligned} H := & \langle (1, 3, 6)(2, 4, 5)(7, 12, 10)(8, 11, 9)(13, 23, 15)(14, 24, 16)(17, 30, 33)(18, 29, 34) \\ & (19, 26, 36)(20, 25, 35)(21, 28, 31)(22, 27, 32)(37, 39, 41)(38, 40, 42), \\ & (1, 8, 3, 11, 6, 9)(2, 7, 4, 12, 5, 10)(13, 39, 23, 41, 15, 37)(14, 40, 24, 42, 16, 38) \\ & (17, 26, 30, 36, 33, 19)(18, 25, 29, 35, 34, 20)(21, 32, 28, 22, 31, 27), \\ & (1, 2)(3, 14)(4, 13)(5, 21)(6, 22)(7, 19)(8, 20)(9, 17)(10, 18)(11, 15)(12, 16)(23, 29) \\ & (24, 30)(25, 27)(26, 28)(31, 41)(32, 42)(33, 39)(34, 40)(35, 36)(37, 38) \rangle \end{aligned}$$

forms a subgroup $\text{Aut}(\mathcal{G})$ that has order $42 = 1 \cdot |V|$ and acts transitively on the nodes of $\mathcal{G}$. We observe that the H -orbits on E (illustrated in red and green) are given by

$$\begin{aligned} E_1 := & \{ \{1, 10\}, \{3, 7\}, \{2, 8\}, \{2, 18\}, \{6, 12\}, \{4, 11\}, \{14, 19\}, \{1, 20\}, \{4, 29\}, \{7, 25\}, \\ & \{5, 9\}, \{16, 22\}, \{13, 15\}, \{24, 26\}, \{17, 40\}, \{3, 25\}, \{8, 18\}, \{5, 34\}, \{12, 35\}, \\ & \{13, 23\}, \{19, 27\}, \{17, 21\}, \{14, 27\}, \{31, 38\}, \{37, 39\}, \{16, 36\}, \{30, 42\}, \{28, 30\}, \\ & \{9, 34\}, \{6, 35\}, \{11, 29\}, \{10, 20\}, \{21, 40\}, \{15, 23\}, \{39, 41\}, \{26, 32\}, \{24, 32\}, \\ & \{37, 41\}, \{33, 38\}, \{31, 33\}, \{22, 36\}, \{28, 42\} \}, \end{aligned}$$

$$E_2 := \{\{1, 2\}, \{3, 4\}, \{7, 8\}, \{5, 6\}, \{11, 12\}, \{13, 14\}, \{19, 20\}, \{9, 10\}, \{21, 22\}, \{15, 16\}, \\ \{23, 24\}, \{39, 40\}, \{25, 26\}, \{17, 18\}, \{27, 28\}, \{31, 32\}, \{37, 38\}, \{41, 42\}, \{29, 30\}, \\ \{33, 34\}, \{35, 36\}\}.$$

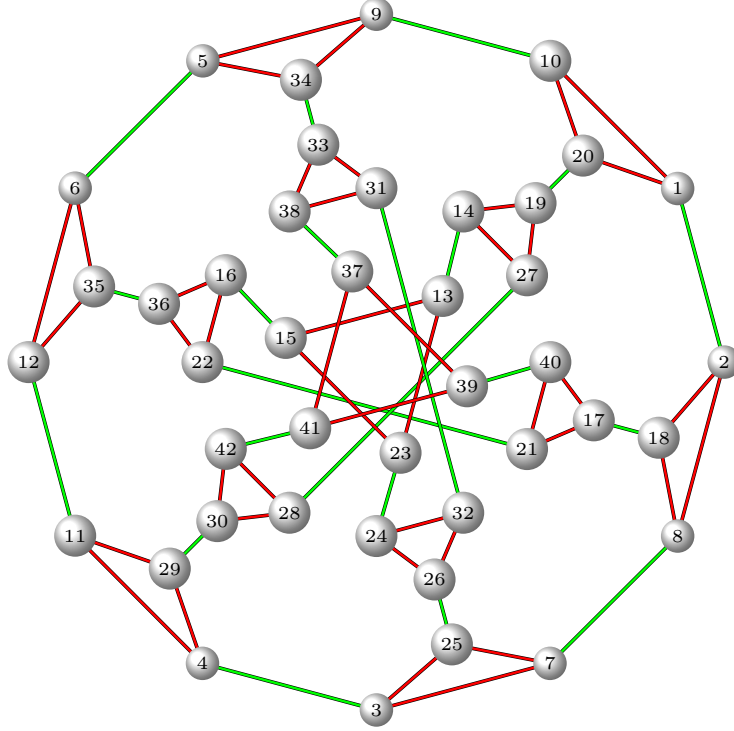


Figure 5.22: The cubic node-transitive graph $\mathcal{G}$ with H -orbits on its arcs coloured in red and green

By choosing σ defined as

$$\sigma := (1, 9, 6, 11, 3, 8)(2, 10, 5, 12, 4, 7)(13, 37, 15, 41, 23, 39)(14, 38, 16, 42, 24, 40) \\ (17, 19, 33, 36, 30, 26)(18, 20, 34, 35, 29, 25)(21, 27, 31, 22, 28, 32) \in H,$$

we construct the following cycle double cover with Corollary 5.4.25:

$$\text{orb}_H(\alpha(\sigma, 1, 10, 9)) \cup \text{orb}_H((1, 10, 20)) \\ = \{(1, 10, 9, 5, 6, 12, 11, 4, 3, 7, 8, 2), (1, 20, 19, 14, 13, 15, 16, 22, 21, 17, 18, 2), \\ (3, 25, 26, 24, 23, 13, 14, 27, 28, 30, 29, 4), (5, 34, 33, 31, 32, 24, 23, 15, 16, 36, 35, 6), \\ (7, 25, 26, 32, 31, 38, 37, 39, 40, 17, 18, 8), (9, 34, 33, 38, 37, 41, 42, 28, 27, 19, 20, 10), \\ (11, 29, 30, 42, 41, 39, 40, 21, 22, 36, 35, 12), (37, 41, 39), (13, 23, 15), (14, 27, 19), \\ (3, 25, 7), (31, 38, 33), (17, 40, 21), (4, 29, 11), (5, 34, 9), (6, 35, 12), (1, 20, 10), \\ (24, 32, 26), (16, 36, 22), (2, 18, 8), (28, 42, 30)\}.$$

Since the above cycle double cover is vertex-faithful, we know that H is a $(2, 1)$ -group of $\mathcal{G}$ of type 1. This cycle double cover gives rise to the face-transitive surface X satisfying $\text{vf}(X) = (2, 1)$ and $\mathcal{F}(X) \cong \mathcal{G}$. With GAP we have been able to verify that X is orientable with $C_7 \rtimes C_6 \cong \text{Aut}(X)$ and $\text{Aut}(\mathcal{F}(X)) \cong \text{PSL}(3, 2) \rtimes C_2$.

Furthermore, the simplicial surface Y described by the below vertices of faces forms a face-transitive surface with $\lambda_Y(\text{Aut}(Y))$ being a $(2, 1)$ -group of type 2 :

$$\begin{aligned} \Theta(Y) = \{ & \{v_{11}, v_{12}, v_{13}\}, \{v_9, v_{10}, v_{14}\}, \{v_9, v_{11}, v_{14}\}, \{v_7, v_8, v_{15}\}, \{v_6, v_{11}, v_{13}\}, \{v_8, v_9, v_{13}\}, \\ & \{v_5, v_{11}, v_{16}\}, \{v_4, v_7, v_{17}\}, \{v_5, v_8, v_{16}\}, \{v_4, v_9, v_{18}\}, \{v_3, v_{10}, v_{19}\}, \{v_4, v_9, v_{17}\}, \\ & \{v_7, v_8, v_{20}\}, \{v_2, v_9, v_{17}\}, \{v_{11}, v_{12}, v_{20}\}, \{v_3, v_{10}, v_{18}\}, \{v_9, v_{10}, v_{20}\}, \{v_4, v_{12}, v_{21}\}, \\ & \{v_8, v_{12}, v_{13}\}, \{v_2, v_3, v_{16}\}, \{v_6, v_{11}, v_{16}\}, \{v_3, v_{11}, v_{21}\}, \{v_8, v_{12}, v_{17}\}, \{v_4, v_6, v_{16}\}, \\ & \{v_3, v_8, v_{20}\}, \{v_3, v_4, v_{21}\}, \{v_6, v_8, v_{19}\}, \{v_1, v_7, v_{17}\}, \{v_1, v_3, v_{19}\}, \{v_1, v_5, v_{13}\}, \\ & \{v_1, v_7, v_{15}\}, \{v_2, v_{12}, v_{16}\}, \{v_8, v_9, v_{15}\}, \{v_7, v_{10}, v_{19}\}, \{v_2, v_6, v_{19}\}, \{v_2, v_7, v_{19}\}, \\ & \{v_3, v_8, v_{16}\}, \{v_5, v_{10}, v_{13}\}, \{v_7, v_{11}, v_{14}\}, \{v_4, v_6, v_{14}\}, \{v_2, v_5, v_{18}\}, \{v_7, v_{11}, v_{21}\}, \\ & \{v_4, v_7, v_{14}\}, \{v_4, v_5, v_{15}\}, \{v_2, v_7, v_{20}\}, \{v_1, v_{12}, v_{21}\}, \{v_5, v_{10}, v_{14}\}, \{v_1, v_{12}, v_{18}\}, \\ & \{v_9, v_{11}, v_{15}\}, \{v_1, v_9, v_{18}\}, \{v_1, v_9, v_{13}\}, \{v_1, v_5, v_{19}\}, \{v_5, v_8, v_{19}\}, \{v_6, v_{10}, v_{21}\}, \\ & \{v_7, v_{10}, v_{21}\}, \{v_3, v_{11}, v_{20}\}, \{v_1, v_3, v_{15}\}, \{v_1, v_6, v_{17}\}, \{v_4, v_5, v_{18}\}, \{v_2, v_{12}, v_{17}\}, \\ & \{v_6, v_{10}, v_{13}\}, \{v_6, v_8, v_{17}\}, \{v_2, v_9, v_{20}\}, \{v_2, v_5, v_{14}\}, \{v_{10}, v_{12}, v_{20}\}, \{v_3, v_4, v_{15}\}, \\ & \{v_5, v_{11}, v_{15}\}, \{v_{10}, v_{12}, v_{18}\}, \{v_1, v_6, v_{21}\}, \{v_4, v_{12}, v_{16}\}, \{v_2, v_6, v_{14}\}, \{v_2, v_3, v_{18}\} \}. \end{aligned}$$

We observe that the simplicial surface Y is non-orientable and consists of 21 vertices, 108 edges and 72 faces and therefore forms an example of a simplicial surface with Euler characteristic -15 . It can be shown that Y satisfies $\text{Aut}(Y) \cong \text{Aut}(\mathcal{F}(Y)) \cong (C_3 \times C_3) \rtimes C_8$.

Lastly, the simplicial surface Z , determined by the following vertices of faces, forms a face-transitive surface with $\text{vf}(Z) = (2, 1)$ and $|\text{orb}_{\text{Aut}(Z)}(Z_1)| = 3$:

$$\begin{aligned} \Theta(Z) = \{ & \{v_{14}, v_{15}, v_{16}\}, \{v_{12}, v_{13}, v_{17}\}, \{v_{10}, v_{11}, v_{18}\}, \{v_8, v_9, v_{19}\}, \{v_6, v_7, v_{20}\}, \{v_5, v_6, v_{21}\}, \\ & \{v_8, v_{14}, v_{21}\}, \{v_{11}, v_{14}, v_{22}\}, \{v_4, v_{12}, v_{23}\}, \{v_3, v_{11}, v_{17}\}, \{v_2, v_{13}, v_{23}\}, \{v_9, v_{10}, v_{24}\}, \\ & \{v_5, v_6, v_{25}\}, \{v_3, v_6, v_{20}\}, \{v_5, v_{14}, v_{26}\}, \{v_3, v_6, v_{25}\}, \{v_{10}, v_{15}, v_{27}\}, \{v_7, v_{15}, v_{27}\}, \\ & \{v_4, v_5, v_{26}\}, \{v_{10}, v_{11}, v_{16}\}, \{v_3, v_{11}, v_{18}\}, \{v_3, v_{12}, v_{17}\}, \{v_4, v_{15}, v_{26}\}, \{v_{11}, v_{13}, v_{17}\}, \\ & \{v_2, v_4, v_{28}\}, \{v_6, v_8, v_{21}\}, \{v_7, v_9, v_{29}\}, \{v_8, v_{13}, v_{19}\}, \{v_1, v_2, v_{28}\}, \{v_7, v_{15}, v_{30}\}, \\ & \{v_{11}, v_{14}, v_{16}\}, \{v_{14}, v_{15}, v_{26}\}, \{v_7, v_9, v_{27}\}, \{v_4, v_{15}, v_{30}\}, \{v_{12}, v_{13}, v_{23}\}, \{v_6, v_8, v_{29}\}, \\ & \{v_2, v_{13}, v_{19}\}, \{v_7, v_{12}, v_{30}\}, \{v_2, v_9, v_{24}\}, \{v_{10}, v_{15}, v_{16}\}, \{v_1, v_3, v_{18}\}, \{v_1, v_5, v_{28}\}, \\ & \{v_8, v_9, v_{29}\}, \{v_1, v_3, v_{25}\}, \{v_1, v_{10}, v_{18}\}, \{v_4, v_{12}, v_{30}\}, \{v_1, v_5, v_{25}\}, \{v_2, v_9, v_{19}\}, \\ & \{v_8, v_{14}, v_{22}\}, \{v_2, v_4, v_{23}\}, \{v_7, v_{12}, v_{20}\}, \{v_6, v_7, v_{29}\}, \{v_4, v_5, v_{28}\}, \{v_1, v_2, v_{24}\}, \\ & \{v_{11}, v_{13}, v_{22}\}, \{v_3, v_{12}, v_{20}\}, \{v_9, v_{10}, v_{27}\}, \{v_1, v_{10}, v_{24}\}, \{v_5, v_{14}, v_{21}\}, \{v_8, v_{13}, v_{22}\} \}. \end{aligned}$$

This orientable simplicial surface has 30 vertices, 90 edges and 60 faces. We furthermore see that Z satisfies $\text{Aut}(Z) \cong \text{Aut}(\mathcal{F}(Z)) \cong S_3 \times D_{2 \cdot 10}$.

A face-transitive surface X that is constructed with the help of Corollary 5.4.25 does not have to satisfy $\text{vf}(X) = (2, 1)$. This could be verified by examining the simplicial octahedron $\mathcal{O}$ together with the $(2, 1)$ -group H of $\mathcal{F}(\mathcal{O})$ of type 1:

$$H := \langle (1, 5)(2, 8)(3, 7)(4, 6), (1, 6)(2, 5)(3, 8)(4, 7), (1, 8)(2, 7)(3, 6)(4, 5) \rangle.$$

The proof of Theorem 5.4.24 implies that a face-transitive surface of vertex-face type $(2, 1)$ contains vertices of even vertex-degree.

Proposition 5.4.26

Let X be a face-transitive surface with $\text{vf}(X) = (2, 1)$. Furthermore, let V_1 and V_2 be the orbits on the vertices of X with $|V_1| < |V_2|$. Then for all $v \in V_2$ the degree $\deg(v)$ is even.

We conclude this section by stating how we construct face-transitive surfaces of vertex-face-type $(2, 1)$ in GAP.

Algorithm 10: Compute all face-transitive surfaces with vertex-face type $(2, 1)$ of a given node-transitive graph (Type 1)

Input: A node-transitive cubic graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(2, 1)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 1 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 2\}$ ;
3 for  $H$  in candidates do
4   red  $\leftarrow H$ -orbit on  $E$  of size  $|V|$  // coloured in 1;
5   green  $\leftarrow H$ -orbit on  $E$  of size  $\frac{1}{2}|V|$  // coloured in 2;
6    $[F_1, F_2, F_3] \leftarrow [F_1, F_2, F_3]$  is a path of length 3 in  $\mathcal{G}$  with  $\{\{F_1, F_2\}, \{F_2, F_3\}\} \not\subseteq \text{red}$ ;
7    $\sigma \leftarrow \sigma' \in \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_3\}$ ;
8   cdc  $\leftarrow \text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ ;
9   cdc  $\leftarrow \text{cdc} \cup \{C \mid C \text{ is a mono-coloured of } \mathcal{G} \text{ of colour } 1\}$ ;
10  if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
11     $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
12    if  $|\text{Aut}(X)| = 1 \cdot |X_2|$  then
13      Add(surfaces,  $X$ );
14 return surfaces;
```

Algorithm 11: Compute all face-transitive surfaces with vertex-face type $(2, 1)$ of a given cubic node-transitive graph (Type 2)

Input: A node-transitive cubic graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(2, 1)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 1 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 2\}$ ;
3 paths  $\leftarrow \{[F_1, \dots, F_5] \mid [F_1, \dots, F_5] \text{ is a path of length 5 in } \mathcal{G}\}$ ;
4 for  $H$  in candidates do
5   red  $\leftarrow H$ -orbit on  $E$  of size  $|V|$  // coloured in 1;
6   green  $\leftarrow H$ -orbit on  $E$  of size  $\frac{1}{2}|V|$  // coloured in 2;
7   for  $[F_1, \dots, F_5]$  in paths do
8     if  $\{F_1, F_2\}, \{F_3, F_4\}$  in red and  $\{F_2, F_3\}, \{F_4, F_5\}$  in green then
9        $\sigma_1 \leftarrow \sigma' \in \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_3\}$ ;
10       $\sigma_2 \leftarrow \sigma' \in \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_5\}$ ;
11      if  $\sigma_1^2 \neq \sigma_2$  then
12        cdc  $\leftarrow \text{orb}_H(\alpha(\sigma_2, F_1, \dots, F_5))$ ;
13        cdc  $\leftarrow \text{cdc} \cup \{C \mid C \text{ is a mono-coloured of } \mathcal{G} \text{ of colour 1}\}$ ;
14        if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
15           $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
16          if  $|\text{Aut}(X)| = 1 \cdot |X_2|$  then
17            Add(surfaces,  $X$ );
18 return surfaces;
```

Algorithm 12: Compute all face-transitive surfaces with vertex-face type $(2, 1)$ of a given cubic node-transitive graph (Type 3)

Input: A node-transitive cubic graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(2, 1)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 1 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 3\}$ ;
3 for  $H$  in candidates do
4   for  $l$  in  $[[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]$  do
5     red  $\leftarrow l[1]$ -th  $H$ -orbit on  $E$  // edges coloured in 1;
6     blue  $\leftarrow l[2]$ -th  $H$ -orbit on  $E$  // edges coloured in 2;
7     green  $\leftarrow l[3]$ -th  $H$ -orbit on  $E$  // edges coloured in 3;
8     cdc  $\leftarrow \{C \mid C \text{ is a } (1, 2)\text{-cycle}\}$ ;
9     cdc  $\leftarrow \text{cdc} \cup \{C \mid C \text{ is a } (3, 1, 3, 2)\text{-cycle}\}$ ;
10    if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
11       $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  with vertices corresponding to cdc;
12      if  $|\text{Aut}(X)| = 1 \cdot |X_2|$  then
13        Add(surfaces,  $X$ );
14 return surfaces;
```

5.4.7 Face-transitive surfaces with vertex-face type (1,1)

Lastly, we deal with the face-transitive surfaces with vertex-face type (1,1). We prove that the umbrellas representing the vertices of such a simplicial surface are either tri-coloured or correspond to automorphism induced α -cycles. For this purpose, we introduce different notions of (1,1)-groups of given node-transitive cubic graphs.

Definition 5.4.27

Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the nodes V with $|H| = 1 \cdot |V|$, and
2. E partitions into three H -orbits E_1, E_2 , and E_3 , all of cardinality $\frac{1}{2}|V|$.

The partition $E = E_1 \uplus E_2 \uplus E_3$ gives rise to the arc-colouring $\kappa : E \rightarrow \{1, 2, 3\}$ via $\kappa(\{F, F'\}) := i$ for $\{F, F'\} \in E_i$. If the (unique) set consisting of all the (1,2,3)-cycles with respect to κ forms a vertex-faithful cycle double cover of $\mathcal{G}$, then we say that H is a **(1,1)-group of $\mathcal{G}$ of type 1**, with the cycle double cover denoted by $\mathcal{C}_1^{(1,1)}(H)$. If the (unique) set consisting of all the (1,2,3,1,3,2)-cycles with respect to κ forms a vertex-faithful cycle double cover of $\mathcal{G}$, then we say that H is a **(1,1)-group of $\mathcal{G}$ of type 2**, with the cycle double cover denoted by $\mathcal{C}_2^{(1,1)}(H)$.

Definition 5.4.28

Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the nodes V with $|H| = 1 \cdot |V|$,
2. E partitions into two H -orbits E_1 and E_2 with $|E_1| = |V|$ and $|E_2| = \frac{1}{2}|V|$, and
3. there exist nodes $F_1, \dots, F_7 \in V$ satisfying $\{F_2, F_3\}, \{F_3, F_4\}, \{F_5, F_6\}, \{F_6, F_7\} \in E_1$ and $\{F_1, F_2\}, \{F_4, F_5\} \in E_2$, and $\sigma \in \text{Aut}(\mathcal{G})$ such that $\sigma(F_1) = F_7$.

The partition $E = E_1 \uplus E_2$ gives rise to an arc-colouring $\kappa : E \rightarrow \{1, 2\}$ of $\mathcal{G}$ via $\kappa(\{F, F'\}) := i$ for $\{F, F'\} \in E_i$. Further, let $\text{orb}_H(\alpha(\sigma, F_1, \dots, F_7))$ be a vertex-faithful cycle double cover of $\mathcal{G}$. We say that H is a **(1,1)-group of $\mathcal{G}$ of type 3**, if there exists an automorphism $\pi \in \text{Aut}(\mathcal{G})$ with $\alpha(\pi, F_1, \dots, F_4) = \alpha(\sigma, F_1, \dots, F_7)$. We denote the above cycle double cover by $\mathcal{C}_3^{(1,1)}(H)$. If there exists no such automorphism π , then we say that H is a **(1,1)-group of $\mathcal{G}$ of type 4** and denote the above cycle double cover by $\mathcal{C}_4^{(1,1)}(H)$.

In this section, we refer to a subgroup $H \leq \text{Aut}(\mathcal{G})$ as a (1,1)-group, if H is a (1,1)-group of any type. Additionally, we denote the corresponding cycle double cover by $\mathcal{C}^{(1,1)}(H)$. Now, we formulate the main result of this subsection.

Theorem 5.4.29

A simplicial surface X is a face-transitive surface with $\text{vf}(X) = (1,1)$ if and only if $H := \lambda_X(\text{Aut}(X))$ is a (1,1)-group of $\mathcal{F}(X)$ and the cycle double cover $\mathcal{C}^{(1,1)}(H)$ contains exactly the vertex-defining cycles of X .

Proof. Let X be a face-transitive surface satisfying $\text{vf}(X) = (1, 1)$. Then the automorphism group $\text{Aut}(X)$ has exactly $1 \cdot |X_2|$ elements. By the orbit-stabiliser theorem, there are either **(1)** exactly three edge-orbits or **(2)** exactly two edge-orbits of $\text{Aut}(X)$ acting on X_1 . We therefore give a case distinction with respect to the number of $\text{Aut}(X)$ -orbits on X_1 to obtain the desired result.

Case 1:

Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ with $|\mathcal{E}_1| = |\mathcal{E}_2| = |\mathcal{E}_3| = \frac{1}{2}|X_2|$ be the $\text{Aut}(X)$ -orbits on X_1 . These edge-orbits naturally yield a Grünbaum-colouring ω of X via $\omega(e) := i$ for $e \in \mathcal{E}_i$. This Grünbaum-colouring can be translated into an arc-colouring κ of $\mathcal{F}(X)$ via $\kappa(\{F_1, F_2\}) := \omega(e)$, where $e \in X_1$ is an edge with $X_2(e) = \{F_1, F_2\}$. Since $\text{Aut}(X)$ and H are isomorphic, the colour classes of κ are exactly the H -orbits on the arcs. By Lemma 3.1.8, all edges that are contained in the same $\text{Aut}(X)$ -orbit have the same type with respect to ω , i.e. the edges in an edge-orbit $\mathcal{E}_i$ for an $i \in \{1, 2, 3\}$ are either all rotational edges or all mirror edges with respect to ω . This can be verified by examining the non-trivial automorphism of an edge-stabiliser. To describe the vertex-defining umbrellas of X , we give a case distinction with respect to the types of the edges contained in the sets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$. Without loss of generality, this case distinction results in the following four cases:

- (a) all the edges of X are mirror edges,
- (b) the edges in $\mathcal{E}_1 \cup \mathcal{E}_2$ are mirror edges and the edges in $\mathcal{E}_3$ are rotational edges,
- (c) the edges in $\mathcal{E}_1$ are mirror edges and the edges in $\mathcal{E}_2 \cup \mathcal{E}_3$ are rotational edges,
- (d) all the edges of X are rotational edges.

In the following we will verify that only **(c)** and **(d)** occur as possible structures of the Grünbaum-colouring ω of X .

Case 1(a): If all edges of X are mirror edges, then the umbrellas of X are all bi-coloured with respect to ω . More precisely, let $F \in X_2$ be a face with $X_0(F) = \{v_1, v_2, v_3\} \in X_2$ and e_1, e_2, e_3 edges with $X_0(e_i) = \{v_1, v_2, v_3\} \setminus \{v_i\}$. Furthermore, we assume that for $i = 1, 2, 3$ the edge e_i is contained in the edge-orbit $\mathcal{E}_i$. Since all edges of X are mirror edges, we know that the colours of the umbrellas $u(v_1)$, $u(v_2)$ and $u(v_3)$ are given by $\omega(u(v_1)) = (2, 3, \dots, 2, 3)$, $\omega(u(v_2)) = (1, 3, \dots, 1, 3)$ and $\omega(u(v_3)) = (1, 2, \dots, 1, 2)$. In particular, v_1 is not incident to any edge in $\mathcal{E}_1$, but v_2 is. Since the colour classes of ω are $\text{Aut}(X)$ -orbits on the edges, it follows that no automorphism in $\text{Aut}(X)$ can map v_1 onto v_2 . This contradicts $\text{vf}(X) = (1, 1)$.

Case 1(b): Let F be a face, e_1, e_2, e_3 edges and v_1, v_2, v_3 vertices with incidences as described in the previous case. Since e_1 and e_2 are mirror edges with respect to ω , the umbrella of v_3 must be bi-coloured with $\omega(u(v_3)) = (1, 2, \dots, 1, 2)$. Hence, we see that v_3 is not incident to any edge of colour 3. Since the colour classes of ω are $\text{Aut}(X)$ -orbits on X_1 , there exists no automorphism of X mapping v_3 onto v_2 or v_1 . Again, this contradicts the fact that $\text{vf}(X) = (1, 1)$.

Case 1(c): Here, we describe that this case leads to tri-coloured umbrellas corresponding to the vertices of X , as defined in Definition 5.4.1: For a vertex $v \in X_0$ the colour of the corresponding umbrella $u(v) = (F_1, \dots, F_n)$ is given by

$$\omega(u(v)) = (1, 2, 3, 1, 3, 2, \dots, 1, 2, 3, 1, 3, 2),$$

see Figure 5.23. Take a moment to verify that, no matter the starting configuration, the colour of the umbrella is determined. In addition to that, the vertex-degrees must be multiples of 6.

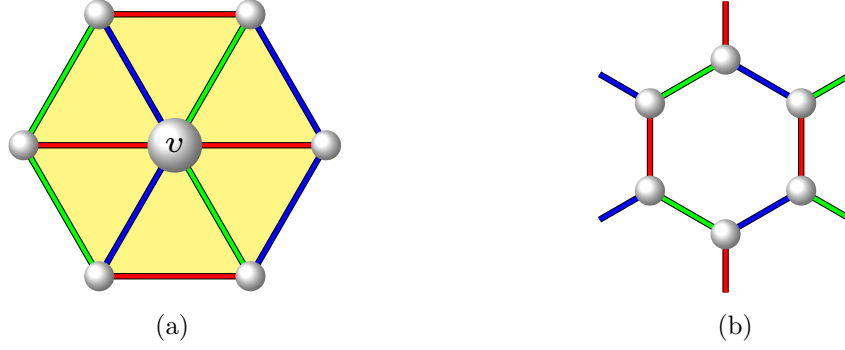


Figure 5.23: The edge-coloured umbrella $u(v)$ of the simplicial surface (a) and the corresponding arc-coloured subgraph in $\mathcal{F}(X)$ (b)

Since $\text{Aut}(X)$ acts transitively on the vertices of X , we conclude that all vertex-defining umbrellas of X are $(1, 2, 3, 1, 3, 2)$ -coloured. Hence, we obtain a vertex-faithful cycle double cover of $\mathcal{F}(X)$ consisting of only $(1, 2, 3, 1, 3, 2)$ -cycles and H is a $(1, 1)$ -group of $\mathcal{F}(X)$ of type 2.

Case 1(d): Here, we conclude that, for all $v \in X_0$, the colour of the corresponding umbrella $u(v) = (F_1, \dots, F_n)$ is given by $\omega(u(v)) = (1, 2, 3, \dots, 1, 2, 3)$. In Figure 5.24a we give an illustration of such an umbrella.

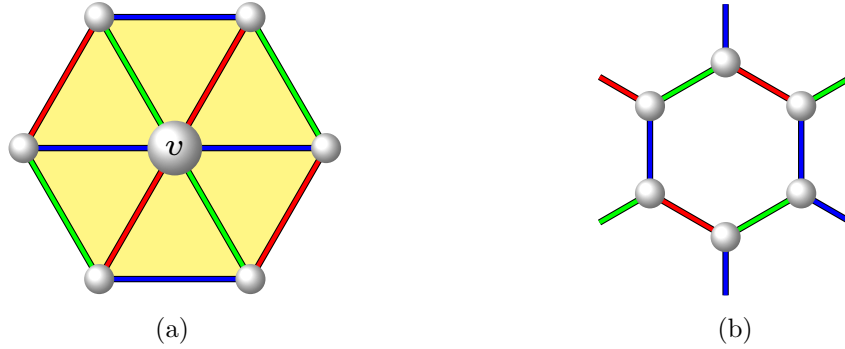


Figure 5.24: The edge-coloured umbrella $u(v)$ of the simplicial surface X (a) and the corresponding arc-coloured subgraph in $\mathcal{F}(X)$ (b)

As before, these $(1, 2, 3)$ -umbrellas are translated into $(1, 2, 3)$ -cycles of the underlying face graph $\mathcal{F}(X)$, see Figure 5.24b. This results in H being a $(1, 1)$ -group of type 1.

Case 2:

We assume that the $\text{Aut}(X)$ -orbits on X_1 are given by $\mathcal{E}_1, \mathcal{E}_2$ with $|\mathcal{E}_1| = |X_2|$ and $|\mathcal{E}_2| = \frac{1}{2}|X_2|$. Hence, the map $\omega : X_1 \rightarrow \{1, 2\}$ defined by $\omega(e) := i$ for $i = 1, 2$ forms an edge-colouring of X leading to an arc-colouring κ of $\mathcal{F}(X)$ via $\kappa(\{F, F'\}) := \omega(e)$ for $X_2(e) = \{F, F'\}$. This enables us to determine the umbrellas of X as follows: Let $v \in X_0$ be a vertex with corresponding umbrella $u(v) = (F_1, \dots, F_n)$. As described in the proof of Theorem 5.4.16, the umbrella of the vertex v forms a $(1, 1, 2)$ -umbrella with respect to ω . Since $\text{Aut}(X)$ acts transitively on the vertices of X , all the umbrellas of X are $(1, 1, 2)$ -umbrellas and must be of the same vertex-degree divisible by 3. If the vertex-degree of any vertex of X is 3, then X must be isomorphic to the simplicial tetrahedron which results in $\text{vf}(X) = (1, 6) \neq (1, 1)$. Hence, all vertex-degrees of X are at least 6.

Next, we construct automorphisms of X to show that the umbrella $u(v)$ is induced by an automorphism. Therefore, for $i = 1, \dots, n$ let F_i be a face with $X_0(F_i) = \{v, v_i, v_{i+1}\}$ and $e_i \in X_i$ an edge with $X_0(e_i) = \{v, v_i\}$ (subscripts read modulo n). Without loss of generality, we assume $e_1 \in \mathcal{E}_1$ and $e_2 \in \mathcal{E}_2$.

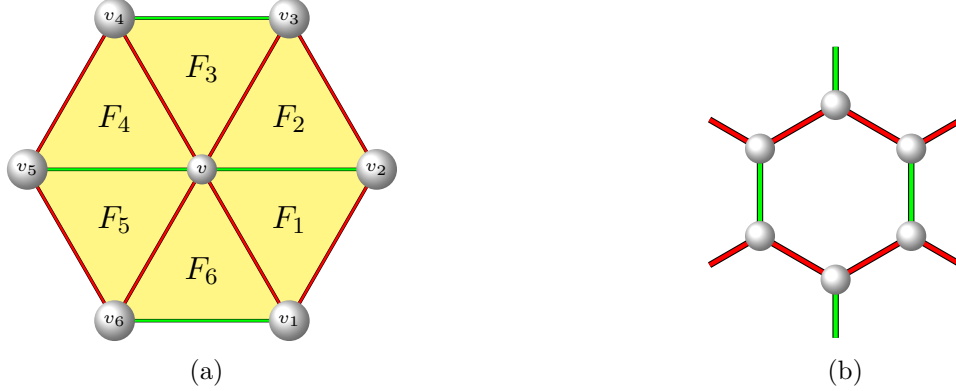


Figure 5.25: The edge-coloured umbrella $u(v)$ of the simplicial surface X (a) and the corresponding arc-coloured subgraph in $\mathcal{F}(X)$ (b)

Furthermore, let $\phi_1, \phi_2 \in \text{Aut}(X)$ be the unique automorphism that satisfy $\phi_1(F_1) = F_4$ and $\phi_2(F_4) = F_7$, where we read the subscripts modulo $\deg(v)$. With these maps we define the automorphisms $\sigma_1 := \lambda_X(\phi_1)$ and $\sigma_2 := \lambda_X(\phi_2)$. This results in two cases, namely

- (a) $(F_1, \dots, F_n) = \alpha(\sigma_1, F_1, \dots, F_4)$, and
- (b) $(F_1, \dots, F_n) \neq \alpha(\sigma, F_1, \dots, F_4)$.

Case 2(a): We directly conclude $\sigma_1 = \sigma_2$ and $\alpha(\sigma_1, F_1, \dots, F_4) = \alpha(\sigma_1^2, F_1, \dots, F_7)$. Since the vertex-defining cycles of X are given by $\text{orb}_H(\alpha(\sigma_1, F_1, \dots, F_7))$, we see that H is a $(1, 1)$ -group of type 3.

Case 2(b): Note that in this case the automorphisms ϕ_1 and ϕ_2 both do not stabilise the vertex v . However, we can make use of these automorphisms to construct an automorphism that does. By defining $\phi' := \phi_2 \circ \phi_1$, we observe that $\phi'(F_i) = F_{i+6}$, for all $1 \leq i \leq n$. Hence, we define $\gamma := \lambda_X(\phi')$ and obtain the vertex-defining cycle of v via $(F_1, \dots, F_n) = \alpha(\gamma, F_1, \dots, F_7)$. Since there exists no $\pi \in \text{Aut}(\mathcal{F}(X))$ with $\alpha(\pi, F_1, \dots, F_4) = \alpha(\sigma, F_1, \dots, F_7)$, the subgroup H is a $(1, 1)$ -group of type 4.

Now, let H be a $(1, 1)$ -group of $\mathcal{F}(X)$ with cycle double cover $\mathcal{C}^{(1,1)}(H)$. Further, let $F \in X_2$ be an arbitrary face. Since $\text{Aut}(X) \cong H$, the simplicial surface X is face-transitive and the stabiliser $\text{Stab}_H(F)$ of the node F is isomorphic to $\text{Stab}_{\text{Aut}(X)}(F)$ of the face F with $|\text{Stab}_{\text{Aut}(X)}(F)| = 1$. Additionally, Lemma 5.4.2 implies that $\text{Aut}(X)$ acts transitively on the vertices of X , i.e. X is a face-transitive surface satisfying $\text{vf}(X) = (1, 1)$. $\square$

The above theorem allows us to formulate the following corollary.

Corollary 5.4.30 *Let $\mathcal{G}$ be a cubic node-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ a subgroup. If H is a $(1, 1)$ -group of $\mathcal{G}$, then $\mathcal{G}$ is the face graph of a face-transitive surface.*

An example of a face-transitive surface X with exactly one $\text{Aut}(X)$ -orbit on X_0 and trivial face-stabilisers is determined by the following vertices of faces. We observe that X contains 9 vertices, 27 edges and 18 faces. Thus, the Euler-characteristic of X is given by $\chi(X) = 0$. Since X is orientable, X is a simplicial torus:

$$\begin{aligned}\Theta(X) = \{ & \{v_7, v_8, v_9\}, \{v_6, v_7, v_8\}, \{v_3, v_4, v_5\}, \{v_4, v_8, v_9\}, \{v_2, v_3, v_8\}, \{v_3, v_5, v_7\}, \\ & \{v_1, v_2, v_3\}, \{v_3, v_4, v_8\}, \{v_1, v_2, v_5\}, \{v_4, v_6, v_9\}, \{v_2, v_6, v_8\}, \{v_1, v_6, v_7\}, \\ & \{v_1, v_3, v_7\}, \{v_2, v_4, v_6\}, \{v_2, v_4, v_5\}, \{v_1, v_5, v_9\}, \{v_1, v_6, v_9\}, \{v_5, v_7, v_9\}\}.\end{aligned}$$

This face-transitive surface can be reconstructed from its corresponding face graph $\mathcal{F}(X)$, see Figure 5.26, and the automorphism group

$$\begin{aligned}H := \text{Aut}(\mathcal{F}(X)) = \langle & (1, 3)(2, 15)(4, 8)(5, 10)(6, 18)(7, 17)(9, 12)(11, 14)(13, 16), \\ & (1, 4)(2, 10)(3, 6)(5, 16)(7, 9)(8, 18)(11, 17)(12, 14)(13, 15), \\ & (1, 16, 7, 8, 10, 12, 6, 15, 11)(2, 18, 9, 5, 4, 17, 13, 3, 14) \rangle.\end{aligned}$$

The action of H on the arcs of $\mathcal{F}(X)$ yields the following three orbits:

$$\begin{aligned}E_1 &:= \{\{1, 2\}, \{3, 15\}, \{4, 10\}, \{16, 18\}, \{6, 13\}, \{11, 14\}, \{5, 8\}, \{12, 17\}, \{7, 9\}\}, \\ E_2 &:= \{\{1, 4\}, \{3, 8\}, \{16, 17\}, \{6, 18\}, \{10, 14\}, \{7, 13\}, \{5, 11\}, \{9, 15\}, \{2, 12\}\}, \\ E_3 &:= \{\{1, 18\}, \{3, 6\}, \{4, 8\}, \{9, 16\}, \{14, 15\}, \{10, 17\}, \{12, 13\}, \{5, 7\}, \{2, 11\}\}.\end{aligned}$$

Figure 5.26 shows an illustration of the face graph $\mathcal{F}(X)$ with the partition sets E_1, E_2, E_3 coloured in red, green and blue, respectively.

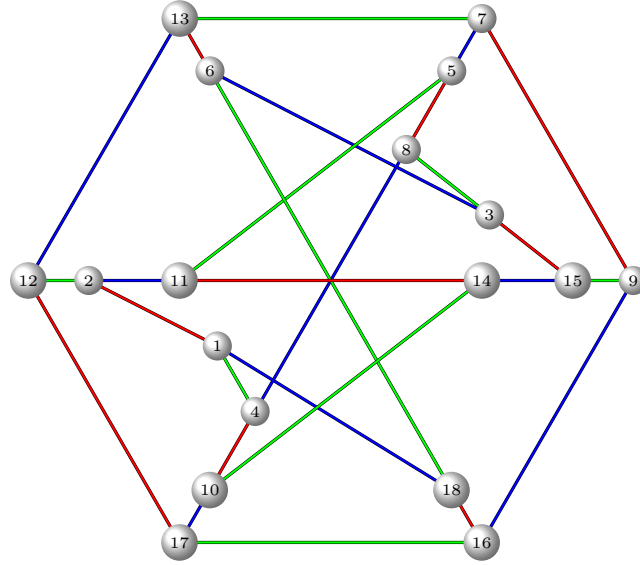


Figure 5.26: The face graph $\mathcal{F}(X)$ with H -orbits on its arcs coloured in red, green and blue

Hence, we obtain the following set of $(1, 2, 3)$ -cycles and notice that this set yields the vertices of the simplicial surface X :

$$\begin{aligned}\{ & (7, 13, 12, 17, 16, 9), (5, 11, 14, 15, 9, 7), (3, 8, 5, 7, 13, 6), \\ & (3, 15, 14, 10, 4, 8), (3, 15, 9, 16, 18, 6), (2, 12, 17, 10, 14, 11), \\ & (1, 18, 6, 13, 12, 2), (1, 4, 8, 5, 11, 2), (1, 18, 16, 17, 10, 4) \}.\end{aligned}$$

Since the above cycle double cover is vertex-faithful, H forms a $(1, 1)$ -group of $\mathcal{F}(X)$ of type 1. Note that $\text{Aut}(X) \cong \text{Aut}(\mathcal{F}(X)) \cong D_{2 \cdot 9}$ holds. With Y (corresponding vertices of faces given

below), we obtain an example of a face-transitive surface with $\text{vf}(Y) = (1, 1)$, where $\lambda_Y(\text{Aut}(Y))$ is a $(1, 1)$ -group of type 2 of $\mathcal{F}(Y)$:

$$\begin{aligned} \Theta(Y) = \{ & \{v_{16}, v_{17}, v_{18}\}, \{v_{13}, v_{14}, v_{15}\}, \{v_{11}, v_{12}, v_{14}\}, \{v_8, v_9, v_{10}\}, \{v_6, v_7, v_9\}, \{v_5, v_8, v_{17}\}, \\ & \{v_4, v_{12}, v_{13}\}, \{v_3, v_{15}, v_{16}\}, \{v_5, v_{10}, v_{18}\}, \{v_2, v_7, v_8\}, \{v_2, v_9, v_{17}\}, \{v_7, v_{10}, v_{11}\}, \\ & \{v_3, v_4, v_{13}\}, \{v_1, v_{11}, v_{16}\}, \{v_6, v_{12}, v_{17}\}, \{v_{13}, v_{16}, v_{17}\}, \{v_2, v_3, v_{14}\}, \{v_6, v_{13}, v_{18}\}, \\ & \{v_2, v_6, v_{11}\}, \{v_1, v_8, v_9\}, \{v_3, v_5, v_8\}, \{v_{12}, v_{13}, v_{18}\}, \{v_{14}, v_{16}, v_{18}\}, \{v_1, v_5, v_7\}, \\ & \{v_4, v_9, v_{17}\}, \{v_2, v_{11}, v_{15}\}, \{v_6, v_{12}, v_{16}\}, \{v_1, v_4, v_{11}\}, \{v_6, v_{10}, v_{15}\}, \{v_3, v_{13}, v_{16}\}, \\ & \{v_8, v_{12}, v_{17}\}, \{v_1, v_{14}, v_{18}\}, \{v_5, v_{11}, v_{14}\}, \{v_2, v_7, v_{13}\}, \{v_2, v_{10}, v_{16}\}, \{v_4, v_8, v_{12}\}, \\ & \{v_4, v_5, v_{14}\}, \{v_7, v_8, v_{18}\}, \{v_1, v_5, v_{15}\}, \{v_4, v_{11}, v_{15}\}, \{v_1, v_{12}, v_{16}\}, \{v_5, v_6, v_7\}, \\ & \{v_3, v_9, v_{12}\}, \{v_8, v_{10}, v_{15}\}, \{v_2, v_3, v_8\}, \{v_1, v_7, v_{17}\}, \{v_5, v_6, v_{11}\}, \{v_9, v_{13}, v_{14}\}, \\ & \{v_2, v_{13}, v_{15}\}, \{v_3, v_4, v_{10}\}, \{v_2, v_9, v_{10}\}, \{v_{11}, v_{12}, v_{18}\}, \{v_5, v_{17}, v_{18}\}, \{v_1, v_{14}, v_{15}\}, \\ & \{v_3, v_{12}, v_{14}\}, \{v_6, v_{10}, v_{18}\}, \{v_1, v_9, v_{12}\}, \{v_6, v_{15}, v_{16}\}, \{v_2, v_6, v_{17}\}, \{v_3, v_7, v_9\}, \\ & \{v_4, v_5, v_{10}\}, \{v_3, v_5, v_{15}\}, \{v_1, v_8, v_{18}\}, \{v_2, v_{14}, v_{16}\}, \{v_7, v_{13}, v_{17}\}, \{v_6, v_9, v_{13}\}, \\ & \{v_3, v_7, v_{10}\}, \{v_4, v_8, v_{15}\}, \{v_1, v_4, v_{17}\}, \{v_{10}, v_{11}, v_{16}\}, \{v_4, v_9, v_{14}\}, \{v_7, v_{11}, v_{18}\} \}. \end{aligned}$$

Note that Y is orientable and satisfies $|Y_0| = 18, |Y_1| = 108, |Y_2| = 72$. With GAP we can verify that $\text{Aut}(Y) \cong \text{Aut}(\mathcal{F}(Y)) \cong (S_3 \times S_3) \rtimes C_2$. Next, we present the simplicial surface Z that is described by the following vertices of faces:

$$\begin{aligned} \Theta(Z) = \{ & \{v_{18}, v_{19}, v_{20}\}, \{v_{15}, v_{16}, v_{17}\}, \{v_{12}, v_{13}, v_{14}\}, \{v_{10}, v_{11}, v_{13}\}, \{v_9, v_{12}, v_{19}\}, \{v_8, v_{12}, v_{13}\}, \\ & \{v_7, v_{16}, v_{18}\}, \{v_6, v_{10}, v_{16}\}, \{v_5, v_7, v_{15}\}, \{v_9, v_{14}, v_{16}\}, \{v_3, v_4, v_{10}\}, \{v_6, v_{13}, v_{20}\}, \\ & \{v_4, v_{12}, v_{17}\}, \{v_2, v_3, v_{19}\}, \{v_1, v_2, v_{13}\}, \{v_2, v_9, v_{15}\}, \{v_5, v_{14}, v_{17}\}, \{v_4, v_{13}, v_{15}\}, \\ & \{v_8, v_{14}, v_{17}\}, \{v_{12}, v_{14}, v_{19}\}, \{v_2, v_6, v_{17}\}, \{v_4, v_7, v_{20}\}, \{v_{10}, v_{17}, v_{19}\}, \{v_3, v_{11}, v_{14}\}, \\ & \{v_6, v_9, v_{16}\}, \{v_2, v_7, v_{10}\}, \{v_1, v_{11}, v_{17}\}, \{v_3, v_4, v_{16}\}, \{v_{13}, v_{18}, v_{19}\}, \{v_{10}, v_{13}, v_{16}\}, \\ & \{v_5, v_6, v_{15}\}, \{v_{13}, v_{16}, v_{19}\}, \{v_6, v_{13}, v_{14}\}, \{v_2, v_6, v_{14}\}, \{v_5, v_{10}, v_{11}\}, \{v_2, v_{11}, v_{13}\}, \\ & \{v_1, v_6, v_{20}\}, \{v_1, v_4, v_7\}, \{v_9, v_{10}, v_{20}\}, \{v_9, v_{11}, v_{18}\}, \{v_9, v_{15}, v_{19}\}, \{v_9, v_{11}, v_{20}\}, \\ & \{v_7, v_{11}, v_{16}\}, \{v_4, v_5, v_{13}\}, \{v_7, v_8, v_{18}\}, \{v_3, v_5, v_6\}, \{v_2, v_{10}, v_{19}\}, \{v_2, v_3, v_{14}\}, \\ & \{v_{15}, v_{16}, v_{19}\}, \{v_6, v_7, v_{10}\}, \{v_3, v_6, v_{18}\}, \{v_{13}, v_{15}, v_{20}\}, \{v_1, v_5, v_9\}, \{v_3, v_{18}, v_{20}\}, \\ & \{v_6, v_{12}, v_{18}\}, \{v_1, v_8, v_{16}\}, \{v_2, v_8, v_{15}\}, \{v_2, v_7, v_8\}, \{v_1, v_7, v_{12}\}, \{v_4, v_5, v_{14}\}, \\ & \{v_1, v_8, v_{13}\}, \{v_1, v_{11}, v_{12}\}, \{v_8, v_{14}, v_{20}\}, \{v_1, v_5, v_{19}\}, \{v_2, v_{17}, v_{18}\}, \{v_1, v_2, v_4\}, \\ & \{v_4, v_{17}, v_{18}\}, \{v_1, v_3, v_{19}\}, \{v_8, v_9, v_{10}\}, \{v_3, v_{11}, v_{12}\}, \{v_6, v_{15}, v_{17}\}, \{v_1, v_6, v_9\}, \\ & \{v_4, v_{12}, v_{15}\}, \{v_3, v_{12}, v_{15}\}, \{v_8, v_{10}, v_{12}\}, \{v_5, v_7, v_{19}\}, \{v_4, v_{10}, v_{20}\}, \{v_4, v_9, v_{14}\}, \\ & \{v_{10}, v_{12}, v_{17}\}, \{v_3, v_5, v_{10}\}, \{v_5, v_8, v_{18}\}, \{v_2, v_4, v_9\}, \{v_7, v_{11}, v_{15}\}, \{v_3, v_8, v_{20}\}, \\ & \{v_7, v_{14}, v_{20}\}, \{v_7, v_{14}, v_{19}\}, \{v_8, v_{16}, v_{17}\}, \{v_5, v_{11}, v_{17}\}, \{v_5, v_8, v_9\}, \{v_{11}, v_{15}, v_{20}\}, \\ & \{v_9, v_{12}, v_{18}\}, \{v_1, v_3, v_{16}\}, \{v_3, v_8, v_{15}\}, \{v_2, v_{11}, v_{18}\}, \{v_1, v_{17}, v_{20}\}, \{v_{11}, v_{14}, v_{16}\}, \\ & \{v_4, v_{16}, v_{18}\}, \{v_6, v_7, v_{12}\}, \{v_5, v_{13}, v_{18}\}, \{v_{17}, v_{19}, v_{20}\} \}. \end{aligned}$$

We observe that Z is a face-transitive surface with vertex-face type $(1, 1)$. Furthermore, we see that Z is orientable and has exactly 20 vertices, 150 edges and 100 faces. Additionally, the automorphism groups of Z and its face graph $\mathcal{F}(Z)$ are isomorphic with $\text{Aut}(Z) \cong \text{Aut}(\mathcal{F}(Z)) \cong C_5 \times (C_5 \rtimes C_4)$. It can be verified that the group $\lambda_Z(\text{Aut}(Z))$ is a $(1, 1)$ -group of type 3 of $\mathcal{F}(Z)$.

Lastly, we present a non-orientable surface Z' that is an example of a face-transitive surface with $\text{vf}(Z') = (1, 1)$ and $\lambda_{Z'}(\text{Aut}(Z'))$ forming a $(1, 1)$ -group of $\mathcal{F}(Z')$ of type 4. The incidence structure of this simplicial surface is fully determined by the following vertices of faces:

$$\begin{aligned} \Theta(Z') = \{ & \{v_{26}, v_{27}, v_{28}\}, \{v_{24}, v_{25}, v_{27}\}, \{v_{21}, v_{22}, v_{23}\}, \{v_{19}, v_{20}, v_{22}\}, \{v_{18}, v_{19}, v_{26}\}, \{v_{16}, v_{17}, v_{20}\}, \\ & \{v_{14}, v_{15}, v_{24}\}, \{v_{12}, v_{13}, v_{15}\}, \{v_{10}, v_{11}, v_{22}\}, \{v_{15}, v_{21}, v_{28}\}, \{v_9, v_{13}, v_{14}\}, \{v_7, v_8, v_{28}\}, \\ & \{v_6, v_{14}, v_{18}\}, \{v_8, v_{13}, v_{26}\}, \{v_4, v_5, v_{26}\}, \{v_{11}, v_{15}, v_{19}\}, \{v_3, v_{18}, v_{25}\}, \{v_{12}, v_{21}, v_{22}\}, \\ & \{v_2, v_4, v_{28}\}, \{v_9, v_{24}, v_{27}\}, \{v_3, v_{24}, v_{26}\}, \{v_4, v_9, v_{13}\}, \{v_{18}, v_{26}, v_{27}\}, \{v_6, v_{17}, v_{21}\}, \\ & \{v_1, v_6, v_{11}\}, \{v_6, v_7, v_{17}\}, \{v_{11}, v_{16}, v_{19}\}, \{v_{20}, v_{23}, v_{27}\}, \{v_7, v_{22}, v_{23}\}, \{v_2, v_{18}, v_{27}\}, \\ & \{v_7, v_{12}, v_{16}\}, \{v_5, v_{19}, v_{25}\}, \{v_{10}, v_{15}, v_{20}\}, \{v_7, v_{15}, v_{28}\}, \{v_{10}, v_{13}, v_{16}\}, \{v_7, v_8, v_{23}\}, \\ & \{v_4, v_{17}, v_{26}\}, \{v_{13}, v_{14}, v_{16}\}, \{v_{10}, v_{16}, v_{20}\}, \{v_2, v_8, v_{17}\}, \{v_2, v_{14}, v_{18}\}, \{v_7, v_{17}, v_{27}\}, \\ & \{v_3, v_{17}, v_{26}\}, \{v_7, v_{12}, v_{15}\}, \{v_2, v_{14}, v_{24}\}, \{v_1, v_4, v_{21}\}, \{v_{12}, v_{18}, v_{22}\}, \{v_4, v_{11}, v_{23}\}, \\ & \{v_1, v_7, v_{27}\}, \{v_{16}, v_{19}, v_{24}\}, \{v_5, v_6, v_{12}\}, \{v_{13}, v_{15}, v_{19}\}, \{v_2, v_{10}, v_{13}\}, \{v_6, v_7, v_{14}\}, \\ & \{v_2, v_4, v_5\}, \{v_5, v_{12}, v_{26}\}, \{v_5, v_{14}, v_{22}\}, \{v_{12}, v_{24}, v_{25}\}, \{v_3, v_{11}, v_{17}\}, \{v_8, v_9, v_{23}\}, \\ & \{v_4, v_{23}, v_{25}\}, \{v_5, v_{10}, v_{22}\}, \{v_{18}, v_{19}, v_{22}\}, \{v_{19}, v_{24}, v_{26}\}, \{v_{10}, v_{15}, v_{24}\}, \{v_{13}, v_{19}, v_{25}\}, \\ & \{v_1, v_3, v_9\}, \{v_4, v_{11}, v_{28}\}, \{v_2, v_8, v_{13}\}, \{v_{20}, v_{27}, v_{28}\}, \{v_1, v_8, v_{25}\}, \{v_2, v_{17}, v_{27}\}, \\ & \{v_7, v_{14}, v_{22}\}, \{v_5, v_8, v_{25}\}, \{v_6, v_{10}, v_{11}\}, \{v_{14}, v_{16}, v_{21}\}, \{v_3, v_9, v_{18}\}, \{v_8, v_{17}, v_{20}\}, \\ & \{v_5, v_8, v_9\}, \{v_{11}, v_{16}, v_{17}\}, \{v_2, v_5, v_{10}\}, \{v_5, v_9, v_{14}\}, \{v_9, v_{23}, v_{27}\}, \{v_8, v_{26}, v_{28}\}, \\ & \{v_1, v_{16}, v_{21}\}, \{v_4, v_{17}, v_{21}\}, \{v_{12}, v_{16}, v_{24}\}, \{v_{20}, v_{22}, v_{28}\}, \{v_{12}, v_{18}, v_{25}\}, \{v_{11}, v_{22}, v_{28}\}, \\ & \{v_5, v_6, v_{19}\}, \{v_6, v_{19}, v_{20}\}, \{v_1, v_7, v_{16}\}, \{v_6, v_{10}, v_{18}\}, \{v_{12}, v_{13}, v_{26}\}, \{v_{11}, v_{15}, v_{23}\}, \\ & \{v_1, v_4, v_9\}, \{v_3, v_{23}, v_{25}\}, \{v_1, v_{25}, v_{27}\}, \{v_9, v_{10}, v_{18}\}, \{v_4, v_{13}, v_{25}\}, \{v_{14}, v_{15}, v_{21}\}, \\ & \{v_2, v_3, v_{28}\}, \{v_9, v_{10}, v_{24}\}, \{v_3, v_{21}, v_{28}\}, \{v_{15}, v_{20}, v_{23}\}, \{v_1, v_6, v_{20}\}, \{v_6, v_{12}, v_{21}\}, \\ & \{v_1, v_8, v_{20}\}, \{v_3, v_{21}, v_{23}\}, \{v_2, v_3, v_{24}\}, \{v_1, v_3, v_{11}\} \}. \end{aligned}$$

Hence, Z' satisfies $(|Z'_0|, |Z'_1|, |Z'_2|) = (28, 168, 112)$ and

$$\text{Aut}(Z') \cong \text{Aut}(\mathcal{F}(Z')) \cong C_2 \times ((C_2 \times C_2 \times C_2) \rtimes C_7).$$

Next, we demonstrate that a simplicial surface that is obtained from Corollary 5.4.30 does not have to be a face-transitive surface satisfying $\text{vf}(X) = (1, 1)$. We therefore consider the simplicial tetrahedron $\mathcal{T}$ and its face graph illustrated in Figure 5.27. Furthermore, we define the group H' as $H' := \langle (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3) \rangle \leq \text{Aut}(\mathcal{F}(\mathcal{T}))$. We observe that $|H'| = 4$ and the H' -orbits on the arcs of $\mathcal{F}(\mathcal{T}) = (V, E)$ are given by

$$\text{orb}_{H'}(E) = \{ \{ \{1, 2\}, \{3, 4\} \}, \{ \{1, 3\}, \{2, 4\} \}, \{ \{1, 4\}, \{2, 3\} \} \}.$$

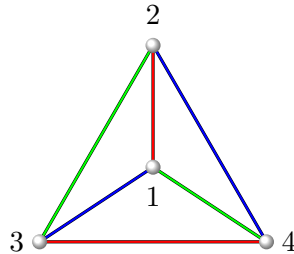


Figure 5.27: The face graph $\mathcal{F}(\mathcal{T})$ with H' -orbits on its arcs coloured in red, green and blue

Hence, the set $\{(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)\}$ forms the cycle double cover consisting of the resulting $(1, 2, 3)$ -cycles. These cycles are exactly the vertex-defining cycles of the simplicial tetrahedron $\mathcal{T}$ and hence the corresponding face-stabilisers are not trivial.

By examining the proof of Theorem 5.4.29 more closely, we obtain the following result.

Proposition 5.4.31

Let X be a face-transitive surface with $\text{vf}(X) = (1, 1)$. Then for all $v \in X_0$ the degree $\deg(v)$ is divisible by 3.

Finally, we present the algorithms that enable us to construct face-transitive surfaces with vertex-face type $(1, 1)$ in GAP.

Algorithm 13: Compute all face-transitive surfaces with vertex-face type $(1, 1)$ of a given cubic node-transitive graph (Type 1)

Input: A node-transitive cubic graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(1, 1)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 1 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 3\}$ ;
3 for  $H$  in candidates do
4   red  $\leftarrow$  first  $H$ -orbit on  $E$  // edges coloured in 1;
5   blue  $\leftarrow$  second  $H$ -orbit on  $E$  // edges coloured in 2;
6   green  $\leftarrow$  third  $H$ -orbit on  $E$  // edges coloured in 3;
7   cdc  $\leftarrow \{C \mid C \text{ is a } (1, 2, 3)\text{-cycle}\}$ ;
8   if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
9      $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
10    if  $|\text{Aut}(X)| = 1 \cdot |X_2|$  then
11      Add(surfaces,  $X$ );
12 return surfaces;

```

Algorithm 14: Compute all face-transitive surfaces with vertex-face type $(1, 1)$ of a given cubic node-transitive graph (Type 2)

Input: A node-transitive cubic graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(1, 1)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 1 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 3\}$ ;
3 for  $H$  in candidates do
4   for 1 in  $[[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]$  do
5     red  $\leftarrow$  1[1]-th  $H$ -orbit on  $E$  // edges coloured in 1 ;
6     blue  $\leftarrow$  1[2]-th  $H$ -orbit on  $E$  // edges coloured in 2;
7     green  $\leftarrow$  1[3]-th  $H$ -orbit on  $E$  // edges coloured in 3;
8     cdc  $\leftarrow \{C \mid C \text{ is a } (1, 2, 3, 1, 3, 2)\text{-cycle}\}$ ;
9     if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
10       $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
11      if  $|\text{Aut}(X)| = 1 \cdot |X_2|$  then
12        Add(surfaces,  $X$ );
13 return surfaces;
```

Algorithm 15: Compute all face-transitive surfaces with vertex-face type $(1, 1)$ of a given cubic node-transitive graph (Type 3)

Input: A node-transitive cubic graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(1, 1)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 1 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 2\}$ ;
3 paths  $\leftarrow \{[F_1, \dots, F_4] \mid [F_1, \dots, F_4] \text{ is a path of length 4}\}$ ;
4 for  $H$  in candidates do
5   for  $[F_1, \dots, F_4]$  in paths do
6      $\sigma \leftarrow \sigma' \in \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_4\}$ ;
7     cdc  $\leftarrow \text{orb}_H(\alpha(\sigma, F_1, \dots, F_4))$ ;
8     if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
9        $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
10      if  $|\text{Aut}(X)| = 1 \cdot |X_2|$  then
11        Add(surfaces,  $X$ );
12 return surfaces;
```

Algorithm 16: Compute all face-transitive surfaces with vertex-face type $(1, 1)$ of a given cubic node-transitive graph (Type 4)

Input: A node-transitive cubic graph $\mathcal{G} = (V, E)$

Output: A set of face-transitive surfaces with vertex-face type $(1, 1)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 1 \cdot |V|, |\text{orb}_H(V)| = 1 \text{ and } |\text{orb}_H(E)| = 2\}$ ;
3 paths  $\leftarrow \{[F_1, \dots, F_7] \mid [F_1, \dots, F_7] \text{ is a path of length 7}\}$ ;
4 for  $H$  in candidates do
5   for  $[F_1, \dots, F_7]$  in paths do
6      $\sigma_1 \leftarrow \sigma' \in \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_4\}$ ;
7      $\sigma_2 \leftarrow \sigma' \in \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_7\}$ ;
8     cdc  $\leftarrow \text{orb}_H(\alpha(\sigma_2, F_1, \dots, F_7))$ ;
9     if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  and  $\sigma_1^2 \neq \sigma_2$  then
10       $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
11      if  $|\text{Aut}(X)| = 1 \cdot |X_2|$  then
12        Add(surfaces,  $X$ );
13 return surfaces;
```

5.5 Notes on implementations

In this section we briefly describe how we use our theoretical results to compute a (not complete) census of face-transitive surfaces. In particular, we present the face-transitive surfaces whose face graphs have at most 1280 nodes and an automorphism group of order at most 10^{11} . Our implementations and the corresponding database of face-transitive surfaces are available in [6]. For this construction we make use of GAP[44] and Magma [21]. First, we exploit the GAP-packages `SimplicialSurfaces` [72], `Simpcomp` [37] and `Digraphs` [18] to handle the combinatorial structure of the different simplicial surfaces and their face graphs. More precisely, for a given cubic node-transitive graph $\mathcal{G} = (V, E)$ with corresponding automorphism group $\text{Aut}(\mathcal{G})$, we apply the following steps to achieve our goal:

1. We compute the set $\mathcal{H}$ containing all subgroups $H \leq \text{Aut}(\mathcal{G})$ satisfying $|H| = s|V|$, where $s \in \{1, 2, 3, 6\}$,
2. for every $H \in \mathcal{H}$ we compute all face-transitive surfaces that have $\mathcal{G}$ as a face graph and an automorphism group isomorphic to H .

(1) Computing all subgroups of a given automorphism group with desired orders is a task of high complexity. Therefore, we fasten up our computation by restricting ourselves to computing all subgroups with prescribed orders up to conjugation. This restriction is justified, since conjugate subgroups of $\text{Aut}(\mathcal{G})$ give rise to the same set of face-transitive surfaces, up to isomorphism. In particular, given a cubic node-transitive graph $\mathcal{G} = (V, E)$, we compute the desired subgroups of $\text{Aut}(\mathcal{G})$ by constructing chains of maximal subgroups and check along the way whether the computed subgroups are still transitive on the nodes of $\mathcal{G}$ and whether the computed subgroups contain $s|V|$ elements, where $s \in \{1, 2, 3, 6\}$. By aiming to achieve this computation solely in GAP, we encountered some problems. In particular, the data base provided in [76] contains cubic graphs $\mathcal{G} = (V, E)$ with $|V| \leq 1280$ and $|\text{Aut}(\mathcal{G})| > 10^6$. Starting from such a large automorphism group and computing subgroups that have at most $6|V| \leq 6 \cdot 1280 = 7680$ elements by computing chains

of maximal subgroups could not always be achieved in GAP. We therefore made use of Magma to compute chains of maximal subgroups which has allowed us to construct all face-transitive surfaces, where the automorphism groups of the corresponding face graphs have order at most 10^{11} . Out of the 111360 cubic node-transitive graphs that are contained in [76], only 480 have automorphism groups with order greater than 10^{11} . Based on experiments that we conducted with these cubic graphs in GAP, we formulate the following conjecture for the remaining 480 cubic graphs with large automorphism groups.

Conjecture 5.5.1. *Let $\mathcal{G} = (V, E)$ be a cubic node-transitive graph satisfying $|V| \leq 1280$ and $\text{Aut}(\mathcal{G}) > 10^{11}$, then there exists no face-transitive surface X with $\mathcal{F}(X) \cong \mathcal{G}$.*

(2) Depending on the order of a derived subgroup $H \leq \text{Aut}(\mathcal{G})$ of a cubic node-transitive graph $\mathcal{G}$, we exploit the different results established in Section 5.4 to construct cycle double covers of $\mathcal{G}$. Since not every cycle double cover of $\mathcal{G}$ gives rise to a vertex-faithful simplicial surface, we also check whether the constructed cycle double cover is vertex-faithful. To speed up the construction of the corresponding cycle double covers we only construct one, two or three cycles of a given cycle double cover and let the other cycles be computed by the action of the subgroup H on these computed cycles.

The results of our computations are summarised in the following table. In this table, we provide the numbers of different face-transitive surfaces that we have computed in our investigation. For this purpose, we denote the set of all face-transitive surfaces with a certain vertex-face type $\text{vf}(X)$, where X is a face-transitive surface, that (1) have at most 1280 faces and (2) have face graphs whose automorphism groups are of order at most 10^{11} , by $M(\text{vf}(X))$.

$\text{vf}(X)$	$ M(\text{vf}(X)) $	Type 1	Type 2	Type 3	Type 4
(3, 1)	6645	×	×	×	×
(2, 2)	4373	×	×	×	×
(2, 1)	10491	1160	528	8803	×
(1, 1)	60733	56592	3376	555	210
(1, 2)	4098	×	×	×	×
(1, 3)	223	210	13	×	×
(1, 6)	239	×	×	×	×

Table 1: Numbers of computed face-transitive surfaces with at most 1280 faces with respect to the different vertex-face types.

Here, an entry "×" denotes that the corresponding type of face-transitive surfaces is not defined. Hence, we see that, up to isomorphism, there are exactly 86802 face-transitive surfaces with at most 1280 faces whose face graphs have automorphism groups of order at most 10^{11} . Out of these face-transitive surfaces exactly 80243 are orientable and 6559 are non-orientable. In Section 8.1, we provide more information on the different numbers of face-transitive surfaces with respect to their Euler characteristics and their vertex-face types. Note, in the tables contained in Section 8.1 and the tables contained in this section we write $(v, s).i$ if the corresponding column or entry labelled by $(v, s).i$ contains information about face-transitive surfaces X with vertex-face type (v, s) , where the groups $\lambda_X(\text{Aut}(X))$ are (v, s) -groups of type i . Similarly, we write (v, s) if a column or entry labelled by (v, s) contains information about face-transitive surfaces with vertex-face type (v, s) .

The following tables contains the number $n(\text{vf}(X))$ that describes the number of faces of a minimal example of a face-transitive surface with a certain vertex-face type. Here, we consider a face-transitive surface X to be minimal, if for every face-transitive surface Y , where (1) X and Y have the same vertex-face type and (2) $\lambda_X(\text{Aut}(X))$ and $\lambda_Y(\text{Aut}(Y))$ are both (v, s) -groups of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ of the same type (see Section 5.4), the inequality $|X_2| \leq |Y_2|$ holds.

vf(X)	(3, 1)	(2, 2)	(2, 1).1	(2, 1).2	(2, 1).3
$n(\text{vf}(X))$	24	6	20	56	32
vf(X)	(1, 1).1	(1, 1).2	(1, 1).3	(1, 1).4	(1, 2)
$n(\text{vf}(X))$	18	72	72	112	24
vf(X)	(1, 3).1	(1, 3).2	(1, 6)		
$n(\text{vf}(X))$	14	144	4		

Table 2: Minimal number of faces of a face-transitive surface with a certain vertex-face type

We have designed several tests to ensure that the census of face-transitive surfaces provided in [6] is correct. In particular, we have conducted the following tests:

1. For every node-transitive cubic graph $\mathcal{G}$ with fewer than 50 vertices, we constructed all face-transitive surfaces by computing all possible cycle double covers of $\mathcal{G}$ using the function `AllSimplicialSurfacesOfDigraph` provided in [72]. We then verified that applying our algorithms to $\mathcal{G}$ produced the same set of face-transitive surfaces, up to isomorphism.
2. For every node-transitive cubic graph $\mathcal{G} = (V, E)$ with $|\text{Aut}(\mathcal{G})| \leq 4000$ and $|V| \leq 600$, we computed all subgroups $H \leq \text{Aut}(\mathcal{G})$ of order $|H| = s|V|$, where $s = 1, 2, 3, 6$, using the function `ConjugacyClassesSubgroups` from [44]. We then compared these results to the candidate subgroups computed via our algorithm, which is based on computing chains of maximal subgroups, ensuring consistency up to conjugacy.
3. For every node-transitive cubic graph $\mathcal{G}$ with $|\text{Aut}(X)| \leq 200000$ and $|V| \leq 800$ we have computed a copy $\mathcal{G}'$ of $\mathcal{G}$ by randomly relabelling the nodes of $\mathcal{G}$ with the labels $1, \dots, |V|$ and checked whether the set of face-transitive surfaces with $\mathcal{G}$ as face graph is equal to the set of face-transitive surfaces with $\mathcal{G}'$ as face graph, up to isomorphism.
4. Finally, we have taken 200 face-transitive surfaces from [6, 37, 94] and checked whether we can reconstruct them, by first computing their face graphs and then applying our algorithms. We have further constructed random copies of these simplicial surfaces constructed the face graphs of the original simplicial surfaces and the copies, and checked whether the sets of candidate groups and the sets of face-transitive surfaces are the same, up to isomorphism.

As described above, we have thoroughly tested the implemented code to ensure its functionality and reliability. However, we acknowledge that errors may still occur despite these efforts.

6 Edge-transitive surfaces

In this section, we study the class of edge-transitive surfaces. Here, we refer to a vertex-faithful simplicial surface X as *edge-transitive*, if there exists exactly one $\text{Aut}(X)$ -orbit on X_1 , see Definition 6.1.1. We demonstrate that this class includes examples of both, open and closed simplicial surfaces. When dealing with open edge-transitive surfaces, we make the following observation.

Observation. *Up to isomorphism, there is exactly one open connected edge-transitive surface, namely the simplicial One-Face, see Theorem 6.1.5.*

Similarly as presented in Section 5, we aim to classify the set of closed edge-transitive surfaces by exploiting suitable group actions on given simplicial surfaces and their face graphs. As a main result, we classify the set of edge-transitive surfaces and make the following observations:

Observation. *There are exactly four types of closed edge-transitive surfaces, see Theorem 6.1.10.*

We demonstrate that these types can be further subdivided into five sub-types, see Section 6.2. We make use of the following approach to construct edge-transitive surfaces: Let $\mathcal{G}$ be a given cubic arc-transitive graph and $\text{Aut}(\mathcal{G})$ its automorphism group. Our goal is to construct an edge-transitive surface of a certain type that has $\mathcal{G}$ as its face graph. For this purpose, we compute a suitable subgroup $H \leq \text{Aut}(\mathcal{G})$ and make use of its properties to construct a vertex-faithful cycle double cover that yields a desired edge-transitive surface. We address the conditions and the suitable choice of a subgroup H , depending on an invariant called the face-edge type (Definition 6.1.9) in Section 6.2. In order to achieve the described constructions, we exploit the census of cubic arc-transitive graphs contained in [76]. As a result, we compute a complete census of all edge-transitive surfaces with at most 5000 faces.

6.1 Properties of edge-transitive surfaces

We begin this subsection by introducing the formal definition of an edge-transitive surface.

Definition 6.1.1

*Let X be a simplicial surface. We say that X is **edge-transitive**, if X is vertex-faithful and the automorphism group of X acts transitively on X_1 .*

The simplicial One-Face is the smallest example of an open edge-transitive surface. Additionally, some of the face-transitive surfaces, illustrated in Section 5, are also examples of edge-transitive surfaces. For instance, the simplicial tetrahedron, the simplicial octahedron and the simplicial icosahedron (see Section 2) are all face- and edge-transitive. Furthermore, we observe that the disjoint union of a finite number of simplicial One-Faces forms an edge-transitive surface. This implies that edge-transitive surfaces do not have to be connected. In the following proposition we comment on edge-transitive surfaces that are not connected by showing that their corresponding connected components form edge-transitive surfaces.

Proposition 6.1.2

Let X be an edge-transitive surface. If Y forms a connected component of X , then Y is an edge-transitive surface.

Proof. Per definition, we know that for arbitrary edges $e_1, e_2 \in Y_1 \subseteq X_1$ there exists an automorphism $\phi \in \text{Aut}(X)$ with $\phi(e_1) = e_2$. As ϕ respects the incidence structure of X , we see that ϕ has to map every $y_1 \in Y$ onto another element $y_2 \in Y$. This implies $\phi(Y) = Y$. Hence, $\psi : Y \rightarrow Y, x \mapsto \phi|_Y(x)$ is an automorphism of Y satisfying $\psi(e_1) = \phi|_Y(e_1) = e_2$. Hence, the connected component Y is edge-transitive which concludes the result. $\square$

Moreover, we can show that the connected components of an edge-transitive surface are all isomorphic.

Proposition 6.1.3

Let X be an edge-transitive surface. If Y, Z are connected components of X , then Y and Z are isomorphic.

Proof. Let $e_1 \in Y$ and $e_2 \in Z$ be arbitrary edges. Since X is edge-transitive, there exists an automorphism $\phi \in \text{Aut}(X)$ that maps e_1 onto e_2 . Because ϕ respects the incidences of X , the equality $\phi(Y) = Z$ follows. Thus, $\phi|_Y$ is an isomorphism that maps Y onto Z . $\square$

Proposition 6.1.3 justifies the following restriction.

Remark 6.1.4 *Since the connected components of an edge-transitive surface are all isomorphic, we assume that the simplicial surfaces in the remainder of this section are all connected.*

We have introduced the simplicial One-Face as an example of an open edge-transitive surface. Now, we show that every open edge-transitive surface has to be isomorphic to the simplicial One-Face.

Theorem 6.1.5

An open simplicial surface X is edge-transitive if and only if X is isomorphic to the simplicial One-Face.

Proof. It suffices to show that an open edge-transitive surface is isomorphic to the simplicial One-Face. Let therefore $e \in X_1$ be a boundary edge. By Lemma 5.2.2 we know that the edges in $\text{orb}_{\text{Aut}(X)}(e) = X_1$ are all boundary edges. Since X is connected, $|X_1| = 3$ follows. Hence, X has to be isomorphic to the simplicial One-Face. $\square$

Therefore, we make the following assumption.

Remark 6.1.6 *We assume that the simplicial surfaces in the remaining section are all closed.*

Next, we study different orbits that arise from the action of an automorphism group $\text{Aut}(X)$ on a given edge-transitive surface X . Per definition, $\text{Aut}(X)$ is transitive on the edges X_1 . A natural question that arises is:

Question. *How many orbits does $\text{Aut}(X)$ induce on X_0 and X_2 ?*

This question is addressed in detail in Propositions 6.1.7 and 6.1.8.

Proposition 6.1.7

Let X be an edge-transitive surface. Then $\text{Aut}(X)$ has at most two orbits on X_2 .

Proof. Let $e \in X_1$ be an edge with $X_2(e) = \{F_1, F_2\}$. We show that an arbitrary face $F \in X_2$ lies in the $\text{Aut}(X)$ -orbit of F_1 or F_2 . For this purpose, let $e' \in X_1(F)$ be an edge. Since X is edge-transitive, there is an automorphism $\phi \in \text{Aut}(X)$ satisfying $\phi(e) = e'$. Since ϕ respects the incidences of X , we know that either F_1 or F_2 has to be mapped onto F under ϕ . $\square$

Thus, if F is a face of an edge-transitive surface X , then $|\text{orb}_{\text{Aut}(X)}(F)| \in \{\frac{1}{2}|X_2|, |X_2|\}$. Next, we show that the automorphism group of an edge-transitive surface acts transitively on the corresponding vertices.

Proposition 6.1.8

Let X be an edge-transitive surface. Then $\text{Aut}(X)$ is transitive on X_0 .

Proof. With similar arguments as presented in the proof of Proposition 6.1.7, we can show that the action of $\text{Aut}(X)$ on X_0 yields at most two vertex-orbits. So, let us assume that there are exactly two $\text{Aut}(X)$ -orbits on X_0 , namely V_1 and V_2 . If there exists an edge $e \in X_1$ and an $i \in \{1, 2\}$ such that e is incident to two vertices in V_i , then $|\text{orb}_{\text{Aut}(X)}(X_0)| = 1$ and $V_i = X_0$ follows by the edge-transitivity of X . This contradicts our assumption. So, every edge in X is incident to a vertex in V_1 and to a vertex in V_2 . Now, let $F \in X_2$ be a face with $X_1(F) = \{e_1, e_2, e_3\}$ and $X_0(F) = \{v_1, v_2, v_3\}$, where $v_i \notin X_0(e_i)$ for $i = 1, 2, 3$ (see Figure 6.1 for illustration).

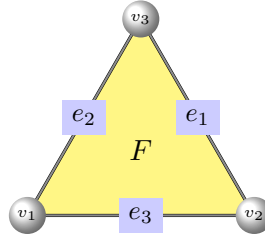


Figure 6.1: A face F of the edge-transitive surface X

The edges e_1 and e_2 are both incident to vertices that lie in distinct $\text{Aut}(X)$ -orbits. Without loss of generality, we assume $v_3 \in V_1$ and $v_1, v_2 \in V_2$. This implies that e_3 is incident to two vertices in V_2 , namely v_1 and v_2 . This is a contradiction and hence we conclude the result. $\square$

Similarly to our investigations in Section 5, we introduce an invariant that will enable us to construct edge-transitive surfaces from given cubic arc-transitive graphs. Here, this invariant is given by the *face-edge type* of a given edge-transitive surface.

Definition 6.1.9

*Let X be an edge-transitive surface and $e \in X_1$ an edge. Further, let $S = \text{Stab}_{\text{Aut}(X)}(e)$ be the stabiliser of e in $\text{Aut}(X)$. We define the **face-edge type** $\text{fe}(X)$ of X as the tuple*

$$\text{fe}(X) := (|\text{orb}_{\text{Aut}(X)}(X_2)|, |S|).$$

Because of Lemma 3.1.8 the edge-stabilisers of an edge-transitive surface X are all conjugate in $\text{Aut}(X)$. This implies that the above invariant is well-defined and not dependant on the edge-choice.

In Section 6.2, we make use of the face-edge type to achieve our desired classification of edge-transitive surfaces with at most 5000 faces. For this purpose, we need to examine the different values taken by the entries of the face-edge type. Because of Proposition 6.1.7 we know that the first entry of a face-edge type is either one or two. Together with Lemma 3.1.6, we are able to deduce that the face-edge type of an edge-transitive surface X satisfies the following theorem.

Theorem 6.1.10

Let X be an edge-transitive surface. Then the face-edge type of X satisfies

$$\text{fe}(X) \in \{(1, 2), (1, 4), (2, 1), (2, 2)\}.$$

Proof. By Lemmas 6.1.7 and 3.1.6 we know that the face-edge type of X satisfies

$$\text{fe}(X) \in \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4)\}.$$

Thus, we have to show that $\text{fe}(X) \in \{(1, 1), (2, 4)\}$ is not possible. We prove this statement by contradiction. First, we assume $\text{fe}(X) = (2, 4)$. In this case, $\text{Aut}(X)$ has exactly two orbits on X_2 , where each orbit is of size $\frac{|X_2|}{2}$. By the orbit-stabiliser theorem, we then obtain

$$|\text{Stab}_{\text{Aut}(X)}(F)| \cdot \frac{|X_2|}{2} = |\text{Aut}(X)| = 4 \cdot |X_1| = 4 \cdot \frac{3}{2} \cdot |X_2| = 6 \cdot |X_2|,$$

where $F \in X_2$ is a face. This implies $|\text{Stab}_{\text{Aut}(X)}(F)| = 12$ and thus contradicts that the stabiliser $\text{Stab}_{\text{Aut}(X)}(F)$ can be embedded into a symmetric group of order 6, see Lemma 3.1.5.

Now, we examine the case $\text{fe}(X) = (1, 1)$. That means that X is face-transitive and all edge-stabilisers are trivial. Hence, the orbit-stabiliser theorem implies

$$|\text{Stab}_{\text{Aut}(X)}(F)| \cdot |X_2| = |\text{Aut}(X)| = 1 \cdot |X_1| = 1 \cdot \frac{3}{2} \cdot |X_2| = \frac{3}{2} \cdot |X_2|,$$

where $F \in X_2$ is a face. Thus, $|\text{Stab}_{\text{Aut}(X)}| = \frac{3}{2}$ follows. Since this is a contradiction, the result follows. $\square$

6.2 Construction of edge-transitive surfaces

In this section we focus on the construction of edge-transitive surfaces. Here, Theorem 6.1.10 allows us to analyse edge-transitive surfaces with different face-edge types case by case and propose methods to reconstruct these simplicial surfaces from their face graphs. Then, these construction methods are used to compute edge-transitive surfaces from given cubic graphs by providing suitable cycle double covers. Since $\lambda_X : \text{Aut}(X) \rightarrow \text{Aut}(\mathcal{F}(X))$ is a monomorphism (see Equation (1) in Remark 3.1.1), we know that a cubic graph giving rise to an edge-transitive surface has to be arc-transitive.

Lemma 6.2.1

The face graph $\mathcal{F}(X)$ of an edge-transitive surface X is arc-transitive.

The results of this section can be summarised as follows.

Observation. *An edge-transitive surface X can be reconstructed by exploiting its face-graph $\mathcal{F}(X)$ and the action of the group $\lambda_X(\text{Aut}(X)) \leq \text{Aut}(\mathcal{F}(X))$ on $\mathcal{F}(X)$.*

An essential tool that allows us to achieve the desired constructions is the notion of a face-colouring of a simplicial surface.

Definition 6.2.2

Let X be a simplicial surface. A map $f : X_2 \rightarrow \{1, 2\}$ is called a **face-2-colouring** of X , if every pair of faces $F_1, F_2 \in X_2$ with $X_1(F_1) \cap X_1(F_2) \neq \emptyset$ satisfies $f(F_1) \neq f(F_2)$.

Note that a simplicial surface X has a face-2-colouring if and only if $\mathcal{F}(X)$ is bipartite. This provides us with the foundation to conduct the described case-by-case analysis.

6.2.1 Edge-transitive surfaces with face-edge type (1,4)

First, we discuss the structure of an edge-transitive surface with face-edge type $(1, 4)$. Per definition, such a simplicial surface is face-transitive. Thus, we obtain the desired description of these edge-transitive surfaces by determining the corresponding vertex-face types, see Definition 5.3.2.

Theorem 6.2.3

A simplicial surface X is edge-transitive with $\text{fe}(X) = (1, 4)$ if and only if X is face-transitive with $\text{vf}(X) = (1, 6)$.

Proof. If X is edge-transitive with $\text{fe}(X) = (1, 4)$, then we know that X is face-transitive. We compute the order of a face-stabiliser with the orbit-stabiliser theorem. In particular, if $F \in X_2$ is a face, then

$$|\text{Stab}_{\text{Aut}(X)}(F)| \cdot |X_2| = |\text{Aut}(X)| = 4 \cdot |X_1| = 4 \cdot \frac{3}{2} \cdot |X_2| = 6 \cdot |X_2|.$$

Thus, $|\text{Stab}_{\text{Aut}(X)}(F)| = 6$ and $\text{vf}(X) = (1, 6)$ follow. Furthermore, if X is face-transitive surface with $\text{vf}(X) = (1, 6)$, then the stabiliser of a face $F \in X_2$ is transitive on the edges $X_1(F)$. Hence, we conclude that $\text{Aut}(X)$ is transitive on X_1 . For an edge $e \in X_1$ we obtain

$$|\text{Stab}_{\text{Aut}(X)}(e)| \cdot |X_1| = |\text{Aut}(X)| = 6 \cdot |X_2| = 6 \cdot \frac{2}{3} \cdot |X_1| = 4 \cdot |X_1|,$$

which implies $\text{fe}(X) = (1, 4)$. □

Theorem 6.2.3 establishes that if X is an edge-transitive surface with $\text{fe}(X) = (1, 4)$, then the group $\lambda_X(\text{Aut}(X))$ forms a $(1, 6)$ -group of the face graph $\mathcal{F}(X)$ and hence the corresponding vertex-defining cycles of X can be described as the $\lambda_X(\text{Aut}(X))$ -orbit of a σ -induced α -cycle $\alpha(\sigma, F_1, F_2)$, where F_1, F_2 are suitable nodes in $\mathcal{F}(X)$ and $\sigma \in \lambda_X(\text{Aut}(X))$ is a suitable automorphism, see Definition 5.4.12. Thus, we refer the reader to Section 5.4.3 for examples of such edge-transitive surfaces.

6.2.2 Edge-transitive surfaces with face-edge type (1,2)

Next, we examine edge-transitive surfaces with face-edge type $(1, 2)$. Since these simplicial surfaces are face-transitive per definition, we determine the corresponding vertex-face types in order to obtain our desired description.

Theorem 6.2.4

A simplicial surface X is edge-transitive with $\text{fe}(X) = (1, 2)$ if and only if X is face-transitive with $\text{vf}(X) = (1, 3)$.

Proof. Let X be edge-transitive with $\text{fe}(X) = (1, 2)$. That means X is face-transitive and thus it remains to show that the face-stabilisers of X all have order 3. By observing

$$|\text{Stab}_{\text{Aut}(X)}(F)| \cdot |X_2| = |\text{Aut}(X)| = 2 \cdot |X_1| = 2 \cdot \frac{3}{2} \cdot |X_2| = 3 \cdot |X_2|,$$

where $F \in X_2$ is an arbitrary face, we conclude $\text{vf}(X) = (1, 3)$. Now, let X be a face-transitive surface with $\text{vf}(X) = (1, 3)$. In this case, the stabiliser of a face $F \in X_2$ is cyclic and hence transitive on the edges $X_1(F)$. This implies that $\text{Aut}(X)$ is transitive on X_1 . By the orbit-stabiliser theorem, we obtain

$$|\text{Stab}_{\text{Aut}(X)}(e)| \cdot |X_1| = |\text{Aut}(X)| = 3 \cdot |X_2| = 3 \cdot \frac{2}{3} \cdot |X_1| = 2 \cdot |X_1|$$

for an edge $e \in X_1$. Thus, $\text{fe}(X) = (1, 2)$ follows. $\square$

Theorem 6.2.4 demonstrates that the vertex-defining cycles of an edge-transitive surface X with $\text{fe}(X) = (1, 2)$ can be described as the $\lambda_X((\text{Aut}(X))$ -orbit of a suitable automorphism-induced α -cycle. Moreover, the above result establishes that there are exactly two types of edge-transitive surfaces, namely edge-transitive surfaces X with $\lambda_X(\text{Aut}(X))$ forming a $(1, 3)$ -group of $\mathcal{F}(X)$ of type 1 or type 2. For examples of such edge-transitive surfaces, we refer the reader to Section 5.4.5.

6.2.3 Edge-transitive surfaces with face-edge type $(2, 2)$

We proceed by focusing on edge-transitive surfaces with face-edge types equal to $(2, 2)$. In order to describe the vertex-defining cycles in the face graphs of these simplicial surfaces, we introduce the definition of a $(2, 2)$ -group of a given cubic arc-transitive graph.

Definition 6.2.5

Let $\mathcal{G} = (V, E)$ be a cubic arc-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the arcs E with $|H| = 2 \cdot |E|$,
2. the action of H on V yields exactly two H -orbits,
3. for $F_1, F_2, F_3 \in V$ with $\{F_1, F_2\}, \{F_2, F_3\} \in E$, there exists an automorphism $\sigma \in H$ that maps F_1 onto F_3 .

If $\text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ is a vertex-faithful cycle double cover of $\mathcal{G}$ (see Remark 3.2.1 and Definition 3.2.2), then we say that H is a $(2, 2)$ -group of $\mathcal{G}$. We denote the above cycle double cover by $\mathcal{C}^{(2,2)}(H)$.

The above definition allows us to characterise the desired edge-transitive surfaces.

Theorem 6.2.6

A simplicial surface X is an edge-transitive surface with $\text{fe}(X) = (2, 2)$ if and only if $H := \lambda_X(\text{Aut}(X))$ is a $(2, 2)$ -group of $\mathcal{F}(X)$ and the cycle double cover $\mathcal{C}^{(2,2)}(H)$ contains exactly the vertex-defining cycles of X .

Proof. First, let X be an edge-transitive surface with $\text{fe}(X) = (2, 2)$. Thus, $|\text{Stab}_{\text{Aut}(X)}(e)| = 2$ holds for all $e \in X_1$ and $\text{Aut}(X)$ has exactly two orbits on X_2 . Moreover, if $F \in X_2$ is a face, then $|\text{Stab}_{\text{Aut}(X)}(F)| = 6$ by the orbit-stabiliser theorem. In order to prove the result, let $e \in X_1$ be an

edge, $\phi \in \text{Stab}_{\text{Aut}(X)}(e)$ a non-trivial automorphism and $F \in X_2(e)$ a face. First, we argue that ϕ stabilises F . To prove this statement, we assume the contrary, namely $\phi(F) \neq F$. In this case, we know that X is face-transitive. This can be verified as follows: Let $F' \in X_2$ be an arbitrary face and $e' \in X_1(F')$ an edge. Since X is edge-transitive, there exists an automorphism $\psi \in \text{Aut}(X)$ that maps e onto e' . By Lemma 3.1.8 the non-trivial automorphism $\phi' \in \text{Stab}_{\text{Aut}(X)}(e')$ does not stabilise F' . Thus, either ψ or $\phi' \circ \psi$ maps F onto F' . Since F' was arbitrary, X is face-transitive, which contradicts $\text{fe}(X) = (2, 2)$.

Hence, ϕ stabilises F . Because $\text{Aut}(X)$ has exactly two orbits on X_2 , there exists a face-2-colouring of X with colour classes corresponding to the $\text{Aut}(X)$ -orbits on X_2 . This helps us to prove that $H := \lambda_X(\text{Aut}(X))$ is a $(2, 2)$ -group of $\mathcal{F}(X)$ as follows: Let $v \in X_0$ be a vertex and $n := \deg(v)$. Furthermore, for every $1 \leq i \leq n$ let F_i be a face and e_i an edge with $v \in X_0(F_i) \cap X_0(e_i)$. More precisely, for every $1 \leq i \leq n$ we assume $X_2(e_i) = \{F_i, F_{i+1}\}$, where we read the subscripts modulo n , see Figure 6.2 for illustration.

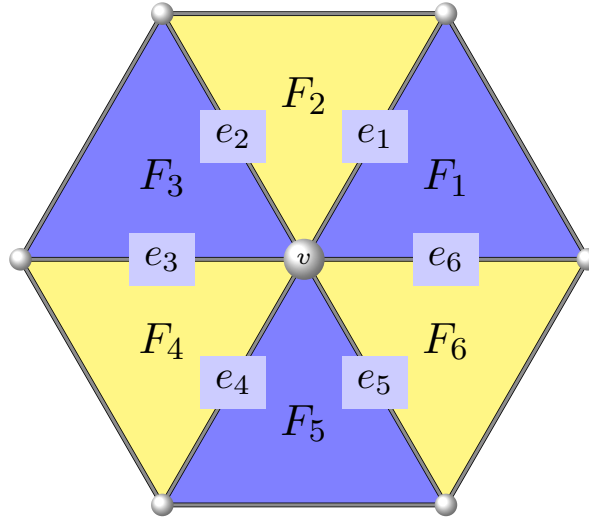


Figure 6.2: The umbrella $u(v)$ of the simplicial surface X with $\text{Aut}(X)$ -orbits on X_2 coloured in blue and yellow

Thus, the umbrella of v is given by $u(v) = (F_1, \dots, F_n)$. We know that n is even and hence we define $k := \frac{n}{2}$. In the following we aim to construct an automorphism that applies a cyclic shift to the faces that are incident to v . For this, let ψ_1 be an automorphism that maps e_1 onto e_3 and $\psi_2 \in \text{Stab}_{\text{Aut}(X)}(e_3)$ non-trivial. Note that ψ_1 maps F_1 onto F_3 . Further, we observe that either ψ_1 or $\psi_2 \circ \psi_1$ stabilises the vertex v . For simplicity, we assume that ψ_1 stabilises v . Because of $\psi_1(F_n) = F_2$ and $\psi_1(F_1) = F_3$, we obtain

$$(F_1, \dots, F_n) = (\psi_1(F_1), \psi_1(F_2), \dots, \psi_1^k(F_1), \psi_1^k(F_2)).$$

Interpreting $(F_1, \dots, F_n)$ as a cycle of $\mathcal{F}(X)$ and defining the automorphism $\sigma := \lambda_X(\psi_1)$ yields $(F_1, \dots, F_n) = \alpha(\sigma, F_1, F_2, F_3)$ being a σ -induced α -cycle. Since all the vertices lie in exactly one $\text{Aut}(X)$ -orbit (see Proposition 6.1.8), the cycle double cover consisting of all vertex-defining cycles of X is given by $\text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ and therefore H forms a $(2, 2)$ -group of $\mathcal{F}(X)$.

Next, let $H := \lambda_X(\text{Aut}(X))$ be a $(2, 2)$ -group of $\mathcal{F}(X) = (V, E)$ and $e \in X_1$ is an arbitrary edge. The statement $\text{Aut}(X) \cong H$ implies that X is edge-transitive and that the stabiliser $\text{Stab}_H(e)$ of e interpreted as an arc of $\mathcal{F}(X)$ is isomorphic to the stabiliser $\text{Stab}_{\text{Aut}(X)}(e)$ of e interpreted as an

edge of X with $|\text{Stab}_{\text{Aut}(X)}(e)| = 2$. Moreover, H having exactly two orbits on $V = X_2$ translates directly into $\text{Aut}(X)$ having exactly two orbits on X_2 . This implies that X is an edge-transitive surface with $\text{vf}(X) = (2, 2)$. $\square$

Corollary 6.2.7 *Let $\mathcal{G}$ be a cubic arc-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ a $(2, 2)$ -group of $\mathcal{G}$, then $\mathcal{G}$ is the face graph of an edge-transitive surface.*

By using Corollary 6.2.7, we are able to construct a simplicial surface X with $\text{fe}(X) = (2, 2)$. This simplicial surface X consists of 36 vertices (all of vertex-degree 12), 216 edges, and 144 faces. The corresponding vertices of faces of X are given by:

$$\begin{aligned} \Theta(X) = \{ & \{v_{34}, v_{35}, v_{36}\}, \{v_{32}, v_{33}, v_{35}\}, \{v_{31}, v_{32}, v_{34}\}, \{v_{29}, v_{30}, v_{34}\}, \{v_{28}, v_{29}, v_{36}\}, \{v_{26}, v_{27}, v_{35}\}, \\ & \{v_{25}, v_{26}, v_{36}\}, \{v_{23}, v_{24}, v_{35}\}, \{v_{22}, v_{23}, v_{33}\}, \{v_{20}, v_{21}, v_{32}\}, \{v_{20}, v_{28}, v_{33}\}, \{v_{18}, v_{19}, v_{34}\}, \\ & \{v_{17}, v_{24}, v_{34}\}, \{v_{15}, v_{16}, v_{32}\}, \{v_{18}, v_{22}, v_{31}\}, \{v_{14}, v_{17}, v_{30}\}, \{v_{15}, v_{25}, v_{31}\}, \{v_{12}, v_{13}, v_{29}\}, \\ & \{v_{12}, v_{27}, v_{30}\}, \{v_{19}, v_{23}, v_{36}\}, \{v_{17}, v_{19}, v_{35}\}, \{v_{18}, v_{24}, v_{36}\}, \{v_{11}, v_{16}, v_{35}\}, \{v_{10}, v_{20}, v_{29}\}, \\ & \{v_{14}, v_{23}, v_{28}\}, \{v_9, v_{17}, v_{27}\}, \{v_9, v_{18}, v_{25}\}, \{v_8, v_{11}, v_{24}\}, \{v_7, v_{12}, v_{26}\}, \{v_{11}, v_{15}, v_{26}\}, \\ & \{v_5, v_6, v_{23}\}, \{v_5, v_{21}, v_{24}\}, \{v_{15}, v_{27}, v_{33}\}, \{v_{11}, v_{27}, v_{32}\}, \{v_{16}, v_{26}, v_{33}\}, \{v_{10}, v_{21}, v_{34}\}, \\ & \{v_{10}, v_{30}, v_{32}\}, \{v_{17}, v_{18}, v_{23}\}, \{v_8, v_{15}, v_{22}\}, \{v_{11}, v_{13}, v_{21}\}, \{v_{13}, v_{26}, v_{28}\}, \{v_6, v_{10}, v_{19}\}, \\ & \{v_7, v_{10}, v_{16}\}, \{v_5, v_{14}, v_{20}\}, \{v_4, v_8, v_{18}\}, \{v_2, v_3, v_{17}\}, \{v_4, v_9, v_{15}\}, \{v_4, v_{16}, v_{19}\}, \\ & \{v_2, v_{13}, v_{24}\}, \{v_{20}, v_{30}, v_{31}\}, \{v_{21}, v_{29}, v_{31}\}, \{v_7, v_{13}, v_{36}\}, \{v_7, v_{25}, v_{29}\}, \{v_6, v_{20}, v_{22}\}, \\ & \{v_3, v_7, v_{19}\}, \{v_2, v_9, v_{12}\}, \{v_1, v_2, v_{11}\}, \{v_1, v_6, v_{16}\}, \{v_{12}, v_{25}, v_{28}\}, \{v_6, v_{14}, v_{33}\}, \\ & \{v_3, v_{12}, v_{14}\}, \{v_2, v_{14}, v_{27}\}, \{v_1, v_5, v_{13}\}, \{v_1, v_3, v_{10}\}, \{v_1, v_8, v_{21}\}, \{v_3, v_9, v_{30}\}, \\ & \{v_5, v_{22}, v_{28}\}, \{v_8, v_9, v_{31}\}, \{v_2, v_5, v_8\}, \{v_1, v_4, v_7\}, \{v_3, v_4, v_6\}, \{v_4, v_{22}, v_{25}\}, \\ & \{v_{32}, v_{34}, v_{35}\}, \{v_{29}, v_{34}, v_{36}\}, \{v_{26}, v_{35}, v_{36}\}, \{v_{23}, v_{33}, v_{35}\}, \{v_{20}, v_{32}, v_{33}\}, \{v_{18}, v_{31}, v_{34}\}, \\ & \{v_{17}, v_{30}, v_{34}\}, \{v_{15}, v_{31}, v_{32}\}, \{v_{12}, v_{29}, v_{30}\}, \{v_{23}, v_{28}, v_{36}\}, \{v_{17}, v_{27}, v_{35}\}, \{v_{18}, v_{25}, v_{36}\}, \\ & \{v_{11}, v_{24}, v_{35}\}, \{v_{20}, v_{28}, v_{29}\}, \{v_{12}, v_{26}, v_{27}\}, \{v_{15}, v_{25}, v_{26}\}, \{v_5, v_{23}, v_{24}\}, \{v_{15}, v_{22}, v_{33}\}, \\ & \{v_{11}, v_{21}, v_{32}\}, \{v_{26}, v_{28}, v_{33}\}, \{v_{10}, v_{19}, v_{34}\}, \{v_{21}, v_{24}, v_{34}\}, \{v_{10}, v_{16}, v_{32}\}, \{v_{18}, v_{22}, v_{23}\}, \\ & \{v_5, v_{20}, v_{21}\}, \{v_4, v_{18}, v_{19}\}, \{v_2, v_{17}, v_{24}\}, \{v_4, v_{15}, v_{16}\}, \{v_{20}, v_{22}, v_{31}\}, \{v_{14}, v_{20}, v_{30}\}, \\ & \{v_{25}, v_{29}, v_{31}\}, \{v_{13}, v_{21}, v_{29}\}, \{v_{27}, v_{30}, v_{32}\}, \{v_7, v_{19}, v_{36}\}, \{v_{16}, v_{19}, v_{35}\}, \{v_{13}, v_{24}, v_{36}\}, \\ & \{v_7, v_{10}, v_{29}\}, \{v_{14}, v_{17}, v_{23}\}, \{v_2, v_{12}, v_{13}\}, \{v_6, v_{19}, v_{23}\}, \{v_3, v_{17}, v_{19}\}, \{v_8, v_{18}, v_{24}\}, \\ & \{v_1, v_{11}, v_{16}\}, \{v_6, v_{10}, v_{20}\}, \{v_{12}, v_{14}, v_{28}\}, \{v_9, v_{15}, v_{27}\}, \{v_9, v_{12}, v_{25}\}, \{v_7, v_{16}, v_{26}\}, \\ & \{v_{11}, v_{13}, v_{26}\}, \{v_{14}, v_{27}, v_{33}\}, \{v_6, v_{16}, v_{33}\}, \{v_9, v_{17}, v_{18}\}, \{v_8, v_{11}, v_{15}\}, \{v_3, v_7, v_{12}\}, \\ & \{v_1, v_5, v_6\}, \{v_2, v_{11}, v_{27}\}, \{v_1, v_{10}, v_{21}\}, \{v_3, v_{10}, v_{30}\}, \{v_5, v_8, v_{22}\}, \{v_5, v_{13}, v_{28}\}, \\ & \{v_9, v_{30}, v_{31}\}, \{v_8, v_{21}, v_{31}\}, \{v_2, v_5, v_{14}\}, \{v_1, v_4, v_8\}, \{v_1, v_2, v_3\}, \{v_3, v_4, v_9\}, \\ & \{v_1, v_7, v_{13}\}, \{v_4, v_7, v_{25}\}, \{v_4, v_6, v_{22}\}, \{v_{22}, v_{25}, v_{28}\}, \{v_2, v_8, v_9\}, \{v_3, v_6, v_{14}\} \}. \end{aligned}$$

We observe that X is orientable and satisfies $\chi(X) = -36$. With GAP we are able to verify that

$$\text{Aut}(X) \cong \text{Aut}(\mathcal{F}(X)) \cong (((C_3 \times C_3) \rtimes Q_8) \rtimes C_3) \rtimes C_2.$$

Note that X is minimal with the property $\text{fe}(X) = (2, 2)$ with respect to the number of faces. Next, we present how we construct edge-transitive surfaces with face-edge type equal to $(2, 2)$ in GAP.

Algorithm 17: Compute all edge-transitive surfaces with face-edge type $(2, 2)$ of a given cubic arc-transitive graph

Input: A cubic arc-transitive graph $\mathcal{G} = (V, E)$

Output: A set of edge-transitive surfaces with face-edge type $(2, 2)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 2 \cdot |E|, |\text{orb}_H(V)| = 2 \text{ and } |\text{orb}_H(E)| = 1\}$ ;
3  $[F_1, F_2, F_3] \leftarrow [F_1, F_2, F_3]$  is a path of length 3 in  $\mathcal{G}$ ;
4 for  $H$  in candidates do
5     automorphisms  $\leftarrow \{\sigma' \in \text{Aut}(\mathcal{G}) \mid \sigma'(F_1) = F_3\}$ ;
6     for  $\sigma$  in automorphisms do
7         cdc  $\leftarrow \text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ ;
8         if cdc is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
9              $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to cdc;
10            if  $|\text{Aut}(X)| = 2 \cdot |X_1|$  then
11                Add(surfaces,  $X$ );
12 return surfaces;
```

6.2.4 Edge-transitive surfaces with face-edge type $(2, 1)$

Finally, we examine the structure of an edge-transitive surface with face-edge type $(2, 1)$. We prove that the vertex-defining cycles corresponding to the vertices of such a simplicial surface form automorphism-induced α -cycles. Before proceeding further, we give the definition of a $(2, 1)$ -group of a given cubic arc-transitive graph.

Definition 6.2.8

Let $\mathcal{G} = (V, E)$ be a cubic arc-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ such that

1. H acts transitively on the arcs E with $|H| = 1 \cdot |E|$,
2. the action of H on V yields exactly two orbits V_1, V_2 ,
3. for $F_1, F_2, F_3 \in V$ with $\{F_i, F_{i+1}\} \in E$ for $i = 1, 2$ there exists an automorphism $\sigma \in H$ that maps F_1 onto F_3 .

If the orbit $\text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ (Remark 3.2.1 and Definition 3.2.2) is a vertex-faithful cycle double cover of $\mathcal{G}$, then we say that H is a $(2, 1)$ -group of $\mathcal{G}$. We denote the above cycle double cover by $\mathcal{C}^{(2,1)}(H)$.

This enables us to formulate the desired description of an edge-transitive surface X satisfying $\text{fe}(X) = (2, 1)$.

Theorem 6.2.9

A simplicial surface X is an edge-transitive surface with $\text{fe}(X) = (2, 1)$ if and only if $H := \lambda_X(\text{Aut}(X))$ is a $(2, 1)$ -group of $\mathcal{F}(X)$ and the cycle double cover $\mathcal{C}^{(2,1)}(H)$ contains exactly the vertex-defining cycles of X .

Proof. First, let X be an edge-transitive surface that satisfies $\text{fe}(X) = (2, 1)$, i.e. $|\text{Stab}_{\text{Aut}(X)}(e)| = 1$ for all $e \in X_1$ and $|\text{orb}_{\text{Aut}(X)}(X_2)| = 2$. Furthermore, let $v \in X_0$ be a vertex and $n := \deg(v)$. Since $\text{Aut}(X)$ has exactly two orbits on X_2 , the vertex-degree of v is even and we can define $k := \frac{n}{2}$. Further, for every $1 \leq i \leq n$ let $F_i \in X_2$ be a face and $e_i \in X_1$ an edge with $v \in X_0(F_i) \cap X_0(e_i)$. More precisely, for every $1 \leq i \leq n$ we set $X_0(F_i) = \{v, v_i, v_{i+1}\}$ and $X_0(e_i) = \{v, v_i\}$, where we read the subscripts modulo n , see Figure 6.3.

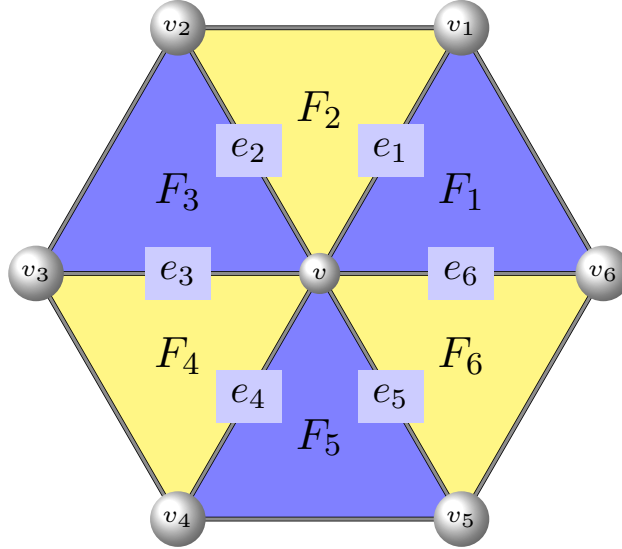


Figure 6.3: The umbrella $u(v)$ of the simplicial surface X with $\text{Aut}(X)$ -orbits on the faces coloured in blue and yellow

Thus, $u(v) = (F_1, \dots, F_n)$ forms the umbrella of v . As shown in the proof of Theorem 6.2.6, we seek an automorphism of X that applies a cyclic shift to the faces $F_1, \dots, F_n$. Let therefore ϕ_1 be an automorphism that maps e_1 onto e_2 and $\phi_2 \in \text{Aut}(X)$ an automorphism that maps e_2 onto e_3 . Since $\text{fe}(X) = (2, 1)$, we know that $\phi_1(F_2) \neq F_3$ holds. That means that $\phi_1(F_2) = F_2$. Moreover, since the edge-stabiliser of the edge $e \in X_1$ with $X_0(e) = \{v_1, v_2\}$ is trivial, ϕ_1 cannot stabilise the vertex v . Hence, $\phi_1(v) = v_2, \phi_1(v_1) = v$ and $\phi_1(v_2) = v_1$ hold. Additionally, we know that the edge $e' \in X_1$ with $X_0(e') = \{v_2, v_3\}$ satisfies $|\text{Stab}_{\text{Aut}(X)}(e')| = 1$. Hence, the automorphism ϕ_2 can also not stabilise v , and $\phi_2(v) = v_3, \phi_2(v_3) = v_2$ and $\phi_2(v_2) = v$ follows. This implies that $\phi := \phi_2 \circ \phi_1$ is an automorphism of X satisfying $\phi(v) = v, \phi(v_1) = v_3$ and $\phi(F_1) = F_3$. By observing the incidence structure of X , we obtain $\phi(F_i) = F_{i+2}$ for all $i = 1, \dots, n$, where we read the subscripts modulo n . Thus, the umbrella of v can be written as

$$(F_1, \dots, F_n) = (\phi(F_1), \phi(F_2), \dots, \phi^k(F_1), \phi^k(F_2)).$$

If we interpret the umbrella $(F_1, \dots, F_n)$ as a cycle of $\mathcal{F}(X)$ and define $\sigma := \lambda_X(\phi)$, we observe that $(F_1, \dots, F_n) = \alpha(\sigma, F_1, F_2, F_3)$ is a σ -induced α -cycle. Since all the vertices lie in exactly one $\text{Aut}(X)$ -orbit (see Proposition 6.1.8), the cycle double cover consisting of all vertex-defining cycles of X is given by $\text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ and H hence forms a $(2, 1)$ -group of $\mathcal{F}(X)$.

It remains to show that if the subgroup H is a $(2, 1)$ -group of $\mathcal{F}(X) = (V, E)$, then X is edge-transitive with $\text{fe}(X) = (2, 1)$. Therefore, let H be such a group and $e \in X_1$ an arbitrary edge. From $\text{Aut}(X) \cong H$ we deduce that X is edge-transitive and that the stabiliser $\text{Stab}_H(e)$ of e interpreted as an arc of $\mathcal{F}(X)$ is isomorphic to the stabiliser of e interpreted as an edge of X such that $|\text{Stab}_{\text{Aut}(X)}(e)| = 1$. Moreover, H having exactly two orbits on V translates directly into

$\text{Aut}(X)$ having exactly two orbits on X_2 . Hence, we conclude that X is an edge-transitive surface with $\text{fe}(X) = (2, 1)$. $\square$

Corollary 6.2.10 *Let $\mathcal{G}$ be a cubic arc-transitive graph and $H \leq \text{Aut}(\mathcal{G})$ a $(2, 1)$ -group of $\mathcal{G}$, then $\mathcal{G}$ is the face graph of an edge-transitive surface.*

With Corollary 6.2.10, we can construct a simplicial surface X with $\text{fe}(X) = (2, 1)$. As an example we give the simplicial surface X that is described by the below vertices of faces:

$$\begin{aligned} \Theta(X) = \{ & \{v_{26}, v_{27}, v_{28}\}, \{v_{24}, v_{25}, v_{27}\}, \{v_{22}, v_{23}, v_{26}\}, \{v_{21}, v_{24}, v_{26}\}, \{v_{19}, v_{20}, v_{28}\}, \{v_{17}, v_{18}, v_{27}\}, \\ & \{v_{16}, v_{22}, v_{28}\}, \{v_{14}, v_{15}, v_{25}\}, \{v_{13}, v_{18}, v_{26}\}, \{v_{11}, v_{12}, v_{24}\}, \{v_{11}, v_{19}, v_{27}\}, \{v_{10}, v_{17}, v_{25}\}, \\ & \{v_8, v_9, v_{23}\}, \{v_6, v_7, v_{21}\}, \{v_{13}, v_{17}, v_{28}\}, \{v_5, v_{12}, v_{27}\}, \{v_4, v_{21}, v_{22}\}, \{v_{14}, v_{22}, v_{24}\}, \\ & \{v_{12}, v_{18}, v_{23}\}, \{v_3, v_{11}, v_{21}\}, \{v_2, v_5, v_{20}\}, \{v_1, v_9, v_{18}\}, \{v_3, v_6, v_{16}\}, \{v_5, v_{11}, v_{25}\}, \\ & \{v_4, v_{14}, v_{26}\}, \{v_4, v_{23}, v_{24}\}, \{v_{10}, v_{16}, v_{19}\}, \{v_8, v_{19}, v_{22}\}, \{v_6, v_{17}, v_{26}\}, \{v_4, v_{13}, v_{20}\}, \\ & \{v_1, v_{16}, v_{21}\}, \{v_2, v_4, v_{15}\}, \{v_2, v_7, v_{13}\}, \{v_1, v_8, v_{12}\}, \{v_1, v_6, v_{11}\}, \{v_{14}, v_{21}, v_{23}\}, \\ & \{v_8, v_{10}, v_{28}\}, \{v_6, v_{13}, v_{27}\}, \{v_{10}, v_{20}, v_{22}\}, \{v_3, v_{10}, v_{14}\}, \{v_5, v_{19}, v_{24}\}, \{v_9, v_{14}, v_{17}\}, \\ & \{v_6, v_{18}, v_{28}\}, \{v_5, v_{13}, v_{15}\}, \{v_7, v_{11}, v_{16}\}, \{v_9, v_{10}, v_{15}\}, \{v_3, v_{15}, v_{17}\}, \{v_4, v_5, v_7\}, \\ & \{v_8, v_{16}, v_{20}\}, \{v_3, v_9, v_{25}\}, \{v_{12}, v_{19}, v_{25}\}, \{v_2, v_8, v_{18}\}, \{v_2, v_9, v_{12}\}, \{v_1, v_3, v_7\}, \\ & \{v_1, v_2, v_{23}\}, \{v_7, v_{15}, v_{20}\}, \{v_{24}, v_{26}, v_{27}\}, \{v_{22}, v_{26}, v_{28}\}, \{v_{19}, v_{27}, v_{28}\}, \{v_{17}, v_{25}, v_{27}\}, \\ & \{v_{14}, v_{24}, v_{25}\}, \{v_{18}, v_{23}, v_{26}\}, \{v_{11}, v_{21}, v_{24}\}, \{v_8, v_{22}, v_{23}\}, \{v_6, v_{21}, v_{26}\}, \{v_{13}, v_{20}, v_{28}\}, \\ & \{v_{12}, v_{18}, v_{27}\}, \{v_{16}, v_{21}, v_{22}\}, \{v_5, v_{19}, v_{20}\}, \{v_9, v_{17}, v_{18}\}, \{v_6, v_{16}, v_{28}\}, \{v_5, v_{15}, v_{25}\}, \\ & \{v_4, v_{13}, v_{26}\}, \{v_{12}, v_{23}, v_{24}\}, \{v_{11}, v_{16}, v_{19}\}, \{v_{10}, v_{17}, v_{28}\}, \{v_4, v_{14}, v_{15}\}, \{v_2, v_{13}, v_{18}\}, \\ & \{v_1, v_{11}, v_{12}\}, \{v_6, v_{11}, v_{27}\}, \{v_{10}, v_{19}, v_{25}\}, \{v_9, v_{14}, v_{23}\}, \{v_4, v_7, v_{21}\}, \{v_5, v_{13}, v_{27}\}, \\ & \{v_4, v_{20}, v_{22}\}, \{v_{10}, v_{14}, v_{22}\}, \{v_3, v_{11}, v_{25}\}, \{v_8, v_9, v_{10}\}, \{v_6, v_7, v_{13}\}, \{v_{13}, v_{15}, v_{17}\}, \\ & \{v_2, v_5, v_{12}\}, \{v_{19}, v_{22}, v_{24}\}, \{v_3, v_{14}, v_{21}\}, \{v_2, v_8, v_{20}\}, \{v_1, v_6, v_{18}\}, \{v_3, v_{10}, v_{16}\}, \\ & \{v_{14}, v_{17}, v_{26}\}, \{v_4, v_5, v_{24}\}, \{v_8, v_{12}, v_{19}\}, \{v_3, v_6, v_{17}\}, \{v_1, v_{21}, v_{23}\}, \{v_1, v_3, v_9\}, \\ & \{v_5, v_7, v_{11}\}, \{v_2, v_4, v_{23}\}, \{v_1, v_8, v_{16}\}, \{v_2, v_9, v_{15}\}, \{v_8, v_{18}, v_{28}\}, \{v_7, v_{16}, v_{20}\}, \\ & \{v_1, v_2, v_7\}, \{v_{10}, v_{15}, v_{20}\}, \{v_9, v_{12}, v_{25}\}, \{v_3, v_7, v_{15}\} \}. \end{aligned}$$

We observe that X satisfies $(|X_0|, |X_1|, |X_2|) = (28, 168, 112)$ and $\chi(X) = -28$. With GAP we can verify that

$$\text{Aut}(X) \cong \text{Aut}(\mathcal{F}(X)) \cong (C_2 \times C_2 \times C_2) \rtimes (C_7 \rtimes C_3).$$

Note that X is minimal with the property $\text{fe}(X) = (2, 2)$ with respect to the number of faces. We conclude this section by presenting an algorithm that allows the construction of edge-transitive surfaces with face-edge type $(2, 1)$ in GAP.

Algorithm 18: Compute all edge-transitive surfaces with face-edge type $(2, 1)$ of a given cubic arc-transitive graph

Input: An cubic arc-transitive cubic graph $\mathcal{G} = (V, E)$

Output: A set of edge-transitive surfaces with face-edge type $(2, 1)$

```

1 surfaces  $\leftarrow \{\}$ ;
2 candidates  $\leftarrow \{H \leq \text{Aut}(\mathcal{G}) \mid |H| = 1 \cdot |E|, |\text{orb}_H(V)| = 2 \text{ and } |\text{orb}_H(E)| = 1\}$ ;
3  $[F_1, F_2, F_3] \leftarrow [F_1, F_2, F_3]$  is a path of length 3 in  $\mathcal{G}$ ;
4 for  $H$  in candidates do
5    $\sigma \leftarrow \sigma' \in \{\sigma \in \text{Aut}(\mathcal{G}) \mid \sigma(F_1) = F_3\}$ ;
6    $\text{cdc} \leftarrow \text{orb}_H(\alpha(\sigma, F_1, F_2, F_3))$ ;
7   if  $\text{cdc}$  is a vertex-faithful cycle double cover of  $\mathcal{G}$  then
8      $X \leftarrow$  simplicial surface with  $\mathcal{F}(X) = \mathcal{G}$  and vertices corresponding to  $\text{cdc}$ ;
9     if  $|\text{Aut}(X)| = 1 \cdot |X_1|$  then
10       $\text{Add}(\text{surfaces}, X)$ ;
11 return surfaces;

```

6.3 Notes on implementations

In this section we describe how we use our theoretical results to compute the complete census of edge-transitive surfaces with at most 5000 faces. Since our approach to construct edge-transitive surfaces is similar to our construction of face-transitive surface, we refer to Section 5.4 and [5] for more technical details on our constructions. Our implementations and the resulting database of edge-transitive surfaces are available in [1]. Similar to our investigation of face-transitive surfaces in Section 5.5, we exploit GAP and Magma to compute the desired simplicial surfaces. First, we analyse edge-transitive surfaces and their face graphs with the GAP-packages `SimplicialSurfaces` [72], `Simpcomp` [37] and `Digraphs` [18]. More precisely, for a given cubic arc-transitive graph $\mathcal{G} = (V, E)$ with corresponding automorphism group $\text{Aut}(\mathcal{G})$, we apply the following steps to achieve our desired goal:

1. We compute the set $\mathcal{H}$ containing all subgroups $H \leq \text{Aut}(\mathcal{G})$ that act transitively on the arcs of $\mathcal{G}$ and satisfy $|H| = s|V|$, where $s \in \{1, 2, 4\}$,
2. for every $H \in \mathcal{H}$ we compute all edge-transitive surfaces that have $\mathcal{G}$ as a face graph and an automorphism group isomorphic to H .

We refer the reader to Section 5.5 for more details on the above two steps. Note that there are exactly 2389 cubic arc-transitive with at most 5000 nodes, see [76]. With GAP we have verified that a cubic arc-transitive graph $\mathcal{G} = (V, E)$ with $|V| \leq 5000$ satisfies $|\text{Aut}(\mathcal{G})| \leq 225792$. Thus, Step 1. turns out to be easier than the task of computing all subgroups $H' \leq \text{Aut}(\mathcal{G}')$ of a cubic node-transitive graph $\mathcal{G}' = (V', E')$ satisfying $|H'| = s'|V'|$, where $s' \in \{1, 2, 3, 6\}$, as described in Section 5.5.

The results of our computations are summarised in the following table. In this table, we provide the numbers of different edge-transitive surfaces that are computed with the algorithms described in Section 6.2. Here, we denote the set of all edge-transitive surfaces with at most 5000 faces that have a certain face-edge type $\text{fe}(X)$, where X is an edge-transitive surface, by $M(\text{fe}(X))$.

$\text{fe}(X)$	$ M(\text{fe}(X)) $	Type 1	Type 2
(1,4)	790	×	×
(1,2)	1040	958	82
(2,2)	119	×	×
(2,1)	236	×	×

Table 3: Numbers of edge-transitive surfaces with at most 5000 faces with respect to the different face-edge types

Again, we write "×" if the type of edge-transitive surfaces described by the corresponding entry is not defined. Our results establish that there are exactly $790 + 958 + 82 + 119 + 236 = 2185$ edge-transitive surfaces with at most 5000 faces, up to isomorphism. Out of these edge-transitive surfaces exactly 2002 are orientable and 183 are non-orientable. In Section 8.2, we give more detailed information on the different numbers of edge-transitive surfaces with respect to their Euler characteristics and their face-edge types. Note, in the tables in Section 8.2 and also the tables in this section we write $(f, s).i$ if the corresponding column or entry labelled by $(f, s).i$ contains information about edge-transitive surfaces X with face-edge type (f, s) , where the groups $\lambda_X(\text{Aut}(X))$ form (f, s) -groups of type i of $\mathcal{F}(X)$. Similarly, we make use of the notation (f, s) . The following table contains the numbers $n(\text{fe}(X))$ that describe the number of faces of minimal examples of edge-transitive surfaces with face-edge types $\text{fe}(X)$ (with respect to the number of faces).

$\text{fe}(X)$	(1,4)	(1,2).1	(1,2).2	(2,2)	(2,1)
$n(\text{fe}(X))$	4	14	144	144	112

Table 4: Minimal number of faces of a edge-transitive surface with a certain face-edge type

We have implemented several tests to ensure that the derived census of edge-transitive surfaces contained in [6] is correct. In particular, we have conducted the following tests:

1. For every arc-transitive cubic graph $\mathcal{G}$ with at most 56 nodes, we have constructed all edge-transitive surfaces by computing all possible cycle double covers of $\mathcal{G}$ with the help of the implementations in [72]. We then verified that applying our algorithms to $\mathcal{G}$ produced the same set of edge-transitive surfaces, up to isomorphism.
2. For every cubic arc-transitive graph $\mathcal{G} = (V, E)$ with $|\text{Aut}(\mathcal{G})| \leq 4000$, we computed all subgroups $H \leq \text{Aut}(\mathcal{G})$ of order $|H| = s|V|$, where $s = 1, 2, 4$, using the GAP-function `ConjugacyClassesSubgroups`, see [44]. We then checked whether the candidate subgroups computed via our algorithm, which is based on computing chains of maximal subgroups is equal to the above set of subgroups, up to conjugacy.
3. For every cubic arc-transitive graph $\mathcal{G} = (V, E)$ with $|V| \leq 5000$ we have computed a copy $\mathcal{G}'$ of $\mathcal{G}$ by randomly relabelling the nodes of $\mathcal{G}$ with the labels $1, \dots, |V|$. We then checked whether the set of edge-transitive surfaces with $\mathcal{G}$ as face graph is equal to the set of edge-transitive surfaces with $\mathcal{G}'$ as face graph, up to isomorphism.

Again, we are aware of the fact that errors may still occur despite these efforts of testing.

7 Surfaces with given automorphism groups

In this section, we examine simplicial surfaces that have automorphism groups that are isomorphic to given finite groups. In particular, we show that for every finite group G , there exists a simplicial surface that has an automorphism group that is isomorphic to G , see Theorem 7.3.1 and Theorem 7.3.5. In order to achieve this result, we make use of a cubic graph construction that has been introduced by Frucht in [42]. In his work, Frucht exploits a finite group G and a corresponding generating set to construct a cubic graph Γ and then claims that Γ has an automorphism group isomorphic to G . First, we show that the construction introduced by Frucht can be simplified for finite groups that are generated by two elements, i.e. if G is a group with a generating set of size two, then we can construct a cubic graph Γ with $\text{Aut}(\Gamma) \cong G$ that has less nodes than the graph that arises from Frucht's original construction. In the general case, we fix a slight oversight by Frucht by modifying his construction to obtain cubic graphs that actually have given groups as automorphism groups. These constructions are essential to prove Theorem 7.3.5 as described in the following:

We show that the cubic graphs that result from our modified constructions, see Remark 7.1.10, can be equipped with cycle double covers such that the arising simplicial surfaces have automorphism groups that are isomorphic to the automorphism groups of the underlying cubic graphs and hence isomorphic to the prescribed finite groups. In the general cases C_n, D_n, A_5 for $n \geq 4$, we provide alternative constructions of simplicial surfaces that form simplicial spheres. Furthermore, in these cases we construct $(1, 1, 1)$ -embeddings of the corresponding simplicial surfaces such that the automorphism groups of the simplicial surfaces, the underlying cubic graphs and the symmetry groups of the arising polyhedra are all isomorphic.

Note that Frucht appeared to be aware of this minor oversight, see [12, 43]. Furthermore, alternative constructions of cubic graphs with given finite groups as automorphism groups, but significantly more nodes, can be found in [12, 62]. Before proceeding further, we make the following assumptions.

- Remark.**
1. In contrast to the previous sections, where D_n denoted the dihedral group of order n , we now choose D_n to be the dihedral group acting on n points. That means that $|D_n| = 2n$.
 2. We assume all simplicial surfaces in this section to be closed and connected.
 3. If G is a finite group and S a generating set, then we choose S to be a minimal generating set satisfying $1_G \notin S$. Hence, if $|S| \geq 2$, then G is assumed to be non-cyclic.

7.1 Frucht's cubic graphs

In this subsection, we slightly modify the cubic graph construction that has been introduced by Frucht in [42]. Note that, for an arbitrary finite group G generated by n elements, the cubic graph Γ that results from Frucht's construction has $2(n+2)|G|$ nodes. In the case $n = 2$, we modify Frucht's cubic graph construction by contracting 3-cycles in Γ (see Figure 7.1 for illustration) such that the cubic graph Γ' resulting from this modification has $2(2+1)|G| = 6|G|$ nodes and still satisfies $\text{Aut}(\Gamma') \cong G$.

This section is based on research originally published in [2].

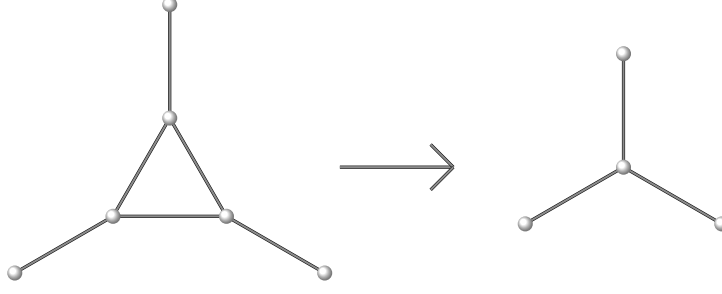


Figure 7.1: Contracting a 3-cycle in a cubic graph

Furthermore, in the general case we demonstrate that the construction introduced by Frucht does not yield cubic graphs whose automorphism groups are isomorphic to given finite groups, see Remark 7.1.8. Hence, we modify his construction for groups that are generated by more than two elements. Moreover, we show that the cubic graphs that arise from our modification have arc-3-colourings and hence yield cycle double covers, as described in Remark 1.2.7. For this purpose, we recall Frucht's construction in this section, see Remark 7.1.6. Note, the case that G is a cyclic group is discussed in Section 7.2.

Since Frucht's construction is based on Cayley graphs, we introduce the formal definition of these directed graphs.

Definition 7.1.1

Let G be a finite group with generating set $S = \{g_1, \dots, g_n\}$. We define the **Cayley graph** $\mathcal{C}_{G,S} = (N, A)$ as the directed graph with $N = G$ as nodes and $A = \{(g, gs) \mid g \in G, s \in S\}$ as arcs.

In Figure 7.2, we see a subgraph that corresponds to an undirected version of a subgraph of a Cayley graph with n generators.

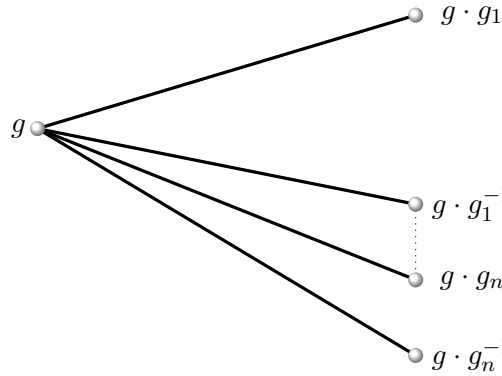


Figure 7.2: Undirected Cayley graph of a group G with n generators

Note that if S is an inverse-closed generating set of a group G , then the Cayley graph $\mathcal{C}_{G,S}$ can be interpreted as an undirected graph. Moreover, we see that the Cayley graph $\mathcal{C}_{G,S}$ of a group G with generating set $S = \{g_1, \dots, g_n\}$ can be equipped with an arc- n -colouring κ that is induced by S as follows:

$$\kappa : \{(g, g \cdot s) \mid g \in G, s \in S\} \rightarrow \{1, \dots, n\}, (g, g \cdot g_i) \mapsto i.$$

Furthermore, the automorphism group of the Cayley graph $\mathcal{C}_{G,S}$ that is equipped with the arc-colouring κ corresponds to the left-action of the group G on itself.

Lemma 7.1.2

Let G be a finite group and $S = \{g_1, \dots, g_n\}$ a generating set of G . The automorphism group of the Cayley graph $\mathcal{C}_{G,S}$ respecting its S -induced arc- n -colouring is given by $\text{Aut}(\mathcal{C}_{G,S}) = \{\pi_g: G \rightarrow G, h \mapsto g \cdot h \mid g \in G\}$. Moreover, $\text{Aut}(\mathcal{C}_{G,S}) \cong G$ holds.

In this work, Frucht makes use of *quadratic forms* to define cubic graphs, see [42]. These polynomials or more precisely the corresponding monomials describe the incidences of nodes of the arising cubic graphs. This is detailed in the below definition.

Definition 7.1.3

*Let $\Gamma = (V, E)$ be a graph with $V = \{F_1, \dots, F_n\}$ and $x_1, \dots, x_n$ formal indeterminates such that F_i corresponds to x_i for all $1 \leq i \leq n$. Furthermore, for $1 \leq i, j \leq n$ we define $a_{i,j}$ as 1 if $\{F_i, F_j\} \in E$ and 0 otherwise. Thus, we define the **quadratic form** Q_Γ of Γ as*

$$Q_\Gamma := \sum_{i,j=1}^n a_{i,j} x_i x_j.$$

It can be observed that the quadratic form of a given graph can be related the adjacency matrix of the graph as described below.

Remark 7.1.4 *Let $\Gamma = (V, E)$ be a graph and $V = \{F_1, \dots, F_n\}$ the nodes of Γ . Furthermore, let $x = (x_1, \dots, x_n)$ be formal indeterminates such that x_i corresponds to the node F_i for all $1 \leq i \leq n$, and $Q := Q_\Gamma$ the quadratic form of Γ . If $A \in \{0, 1\}^{n \times n}$ denotes the adjacency matrix of Γ , then*

$$Q = \frac{1}{2} x A x^t.$$

We observe that each monomial $x_i x_j$ of Q corresponds to an arc $\{F_i, F_j\} \in E$ and thus the adjacency matrix A can be recovered from Q . Moreover, we know that $\text{Aut}(\Gamma)$ is isomorphic to the group consisting of the permutations of the columns and rows of A that leave the adjacency matrix A invariant. Thus, the automorphism group of Γ is isomorphic to the group of permutations of $x = (x_1, \dots, x_n)$ that preserve the quadratic form Q .

With Remark 7.1.4 it suffices to determine a quadratic form in order to define a cubic graph.

Remark 7.1.5 *Let $\Gamma = (V, E)$ be a cubic graph and Q its quadratic form with corresponding indeterminates $x_1, \dots, x_n$. Since Q determines Γ and also $\text{Aut}(\Gamma)$, up to isomorphism, we identify V with the set of formal indeterminates i.e. $V = \{x_1, \dots, x_n\}$.*

Now, we are able to recall Frucht's graph construction from [42].

Remark 7.1.6 *Let G be a finite group and $S = \{g_1, \dots, g_n\}$ be a generating set of G . In [42] Frucht constructs a cubic graph Γ based on G and S by splitting each node of the Cayley graph $\mathcal{C}_{G,S}$, see Figure 7.3a, into several nodes as shown in Figure 7.3b. This construction is formalised below by presenting a suitable quadratic form.*

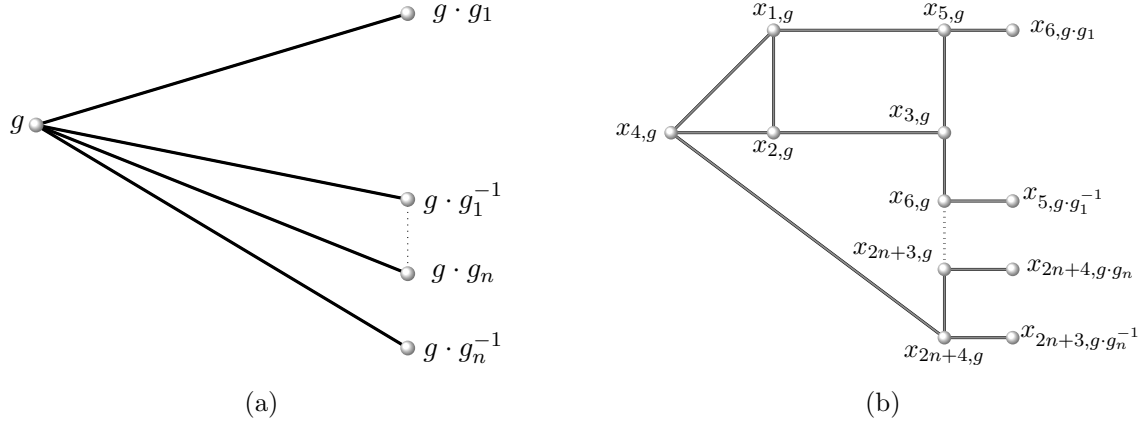


Figure 7.3: Undirected Cayley graph (a) and part of Frucht's original cubic graph construction (b)

In order to give the quadratic form describing the desired cubic graph Γ , we assume that $|G| = h$ and that G is given by

$$G = \{\underbrace{g_1, \dots, g_n}_{\in S}, g_{n+1}, \dots, g_h\}.$$

Note, every node of the Cayley graph $\mathcal{C}_{G,S}$ corresponds to a group element of G . Thus, for every $g \in G$ Frucht presents $2n + 4$ indeterminates, namely $x_{1,g}, \dots, x_{2n+4,g}$. Furthermore, for $i \neq j$ Frucht introduces a quadratic form $Q_{i,j}$ via

$$Q_{i,j} := \sum_{k=1}^h x_{i,g_k} x_{j,g_k}.$$

Based on the quadratic forms Q_{ij} , Frucht defines the quadratic form Q_F that yields the cubic graph Γ as follows:

$$\begin{aligned} Q_F = & Q_{1,2} + Q_{1,4} + Q_{1,5} + Q_{2,3} + Q_{2,4} + Q_{3,5} + Q_{3,6} + Q_{4,2n+4} \\ & + \sum_{i=2}^n Q_{2i+3,2i+4} + \sum_{i=2}^n Q_{2i+2,2i+3} + R, \end{aligned}$$

where R is given by

$$R = \sum_{k=1}^h \sum_{j=1}^n x_{2j+3,g_k} x_{2j+4,g_j \cdot g_k}.$$

We observe that the resulting graph Γ is indeed cubic and has exactly $(2n + 4)|G| = 2(n + 2)|G|$ nodes.

Next, we show that the cubic graphs that result from Frucht's cubic graph construction can be arc-3-coloured.

Proposition 7.1.7

Let G be a finite group and $S = \{g_1, \dots, g_n\}$ be a generating set of G with $n \geq 2$. Then the cubic graph Γ with quadratic form Q_F as given in Remark 7.1.6 has an arc-3-colouring.

Proof. Based on the quadratic form Q_F , we can construct an arc-3-colouring of $\Gamma = (V, E)$ by presenting a suitable partition of the arcs E . For this purpose, we define $E(q)$ as the set of arcs in E that correspond to the monomials that are summands of q , where q is a subsum of the quadratic form Q_F . This notion helps us to partition the arcs E into the sets E_1, E_2, E_3 with

$$\begin{aligned} E_1 &:= E(Q_{1,4} + Q_{2,3} + R), \\ E_2 &:= E(Q_{1,5} + Q_{2,4} + Q_{3,6} + \sum_{i=2}^n Q_{2i+3,2i+4}), \\ E_3 &:= E(Q_{1,2} + Q_{3,5} + Q_{4,2n+4} + \sum_{i=2}^n Q_{2i+2,2i+3}). \end{aligned}$$

Thus, $\kappa : E \rightarrow \{1, 2, 3\}$ defined by $\kappa(e) := i$ for $e \in E_i$ with $i = 1, 2, 3$ forms an arc-colouring of Γ . The arc-colouring κ is shown in Figure 7.4, where arcs coloured in 1, 2 and 3 are illustrated with the colours red, green and blue, respectively. Hence, it is straightforward to see that κ is indeed an arc-3-colouring of Γ .

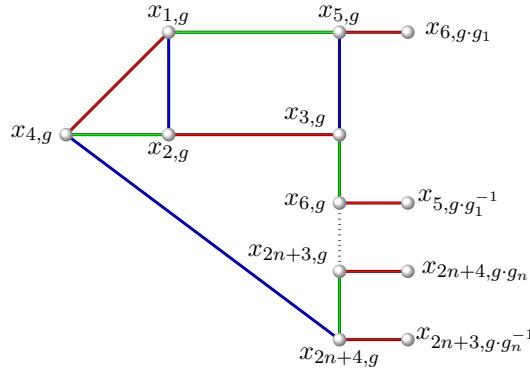


Figure 7.4: Part of Frucht's original cubic graph construction equipped with an arc-3-colouring

□

We proceed by demonstrating that in general Frucht's cubic graph construction (see Remark 7.1.6) does not yield cubic graphs with given finite groups as automorphism groups.

Remark 7.1.8 For a group G that is generated by two elements $S = \{g_1, g_2\}$, Frucht proves in [42] that the cubic graph Γ with quadratic form Q_F as described in Remark 7.1.6 satisfies $\text{Aut}(\Gamma) \cong G$. However, in the case that G is generated by more than two elements, this statement does not hold in general. As an example, we consider the group $G = A_5$ with the generating set

$$S = \{g_1 = (1, 5)(2, 4), g_2 = (1, 2, 4, 3, 5), g_3 = (2, 5, 3)\}.$$

With GAP, we have been able to verify that the automorphism group of the cubic graph Γ resulting from Frucht's construction based on G and S satisfies $\text{Aut}(\Gamma) \cong C_2 \times A_5 \not\cong A_5$. Hence, in this case Frucht's construction does not lead to a cubic graph with A_5 as automorphism group. Furthermore, we have observed that in Frucht's construction the order of the generators is important. That means by reordering the generators in S , we can obtain non-isomorphic cubic graphs. This instance is also described in [43].

The example presented in Remark 7.1.8 raises the following question.

Question. Why does Frucht's reasoning fail in the case $n > 2$?

This question is addressed in the following: In [42] Frucht computes the automorphism group of a cubic graph Γ that is constructed via Remark 7.1.6 by exploiting the idea that an automorphism of Γ maps cycles in Γ onto cycles with the same length. He then uses these cycle lengths to argue that an automorphism of Γ maps nodes onto nodes with the same *cycle triplets*, see Definition 7.1.9. However, for $n > 2$ generators, his argumentation fails as he missed pairs of nodes with the same cycle triplet that can be mapped onto each other with automorphisms that do not correspond to elements of a given group G , namely the nodes $x_{1,g}$ and $x_{2,g}$ illustrated in Figure 7.3b. This instance leads to an automorphism group $\text{Aut}(\Gamma)$ that is larger than G .

Definition 7.1.9

Let $\Gamma = (V, E)$ be a cubic graph and $e_1, e_2, e_3 \in E$ arcs that are all incident to the same node $v \in V$. For all $1 \leq i < j \leq 3$ let $c_{i,j}$ be the length of a minimal cycle passing through the arcs e_i, e_j . We define the **cycle triplet** of v as the tuple $(c_{1,2}, c_{1,3}, c_{2,3})$.

Now, we present our modified construction with the aim of filling the gap in his proof.

Remark 7.1.10 Let $G = \{g_1, \dots, g_h\}$ be a finite group with h elements and $S = \{g_1, \dots, g_n\}$ a generating set of G . For $n = 2$ and $n > 2$ we can construct cubic graphs by substituting each node of the Cayley graph $\mathcal{C}_{G,S}$ with components as illustrated in Figure 7.5a and Figure 7.5b, respectively. This constructions are formalised below by presenting suitable quadratic forms.

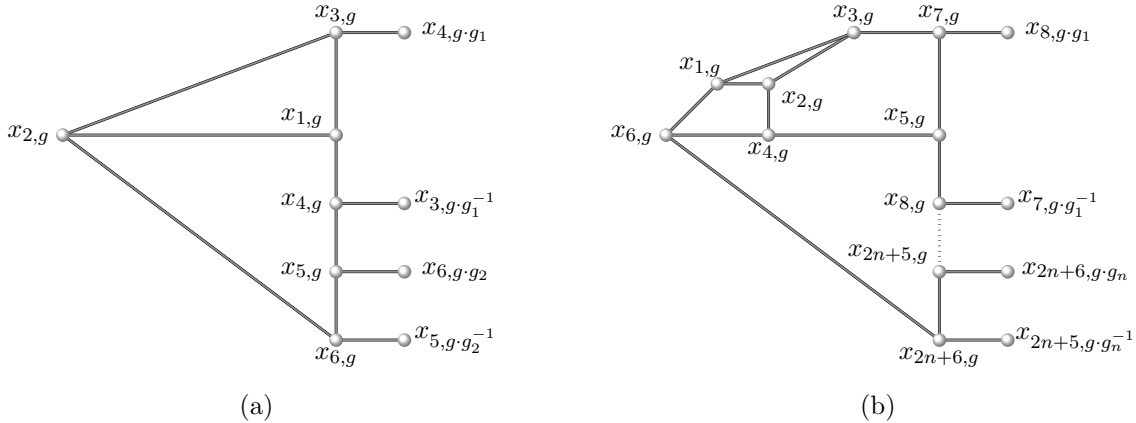


Figure 7.5: Simplified construction for two generators (a) and modified construction for more than two generators (b)

Again, for $i \neq j$ we use the quadratic form $Q_{i,j}$ given by

$$Q_{i,j} := \sum_{k=1}^h x_{i,g_k} x_{j,g_k}.$$

This allows us to define the desired quadratic forms. For $n = 2$, let x_{i,g_j} be formal indeterminates, where $i = 1, \dots, 6$ and $j = 1, \dots, h$. Thus, we define the graph Γ_{simp} by the quadratic form

$$Q_{\text{simp}} := Q_{1,2} + Q_{1,3} + Q_{1,4} + Q_{2,3} + Q_{2,6} + Q_{4,5} + Q_{5,6} + R_{\text{simp}},$$

where R_{simp} is given by

$$R_{simp} := \sum_{k=1}^h x_{3,g_k} x_{4,g_k \cdot g_1} + x_{5,g_k} x_{6,g_k \cdot g_2}.$$

For $n > 2$, the graph Γ_{mod} is defined by the following quadratic form:

$$\begin{aligned} Q_{mod} := & Q_{1,2} + Q_{1,3} + Q_{1,6} + Q_{2,3} + Q_{2,4} + Q_{3,7} + Q_{4,5} + Q_{4,6} + Q_{5,7} + Q_{5,8} + Q_{6,2n+6} \\ & + \sum_{i=2}^n Q_{2i+5,2i+6} + \sum_{i=2}^n Q_{2i+4,2i+5} + R_{mod}, \end{aligned}$$

with R_{mod} given by

$$R_{mod} := \sum_{k=1}^h \sum_{j=1}^n x_{2j+5,g_k} x_{2j+6,g_k \cdot g_j}.$$

We observe that in both cases the resulting graphs are indeed cubic. For $n = 2$ and $n > 2$ the arising cubic graphs have exactly $6|G|$ and $(2n + 6)|G|$ nodes, respectively.

In Figure 7.6a we illustrate an example of a cubic graph that results from Frucht's original construction for the group Q_8 with two generators. The cubic graph shown in Figure 7.6a has 8 cycles of length 3. Note that contracting these 3-cycles gives rise to the cubic graph that results from our simplification of Frucht's construction for groups generated by two elements, see Figure 7.6b and Remark 7.1.10. The cubic graph illustrated in Figure 7.6b has $64 - 2 \cdot 8 = 48$ nodes and an automorphism group that is isomorphic to Q_8 . Iterating the process of contracting 3-cycles leads to a cubic graph with larger automorphism group on 32 nodes. By Proposition 7.1.7, we know that the cubic graph in Figure 7.6a is arc-3-colourable. Figure 7.6b illustrates an arc-3-colouring of the cubic graph resulting from our simplified construction. In Proposition 7.1.12, we will show that a cubic graph resulting from one of our modifications in Remark 7.1.10 can always be equipped with an arc-3-colouring.

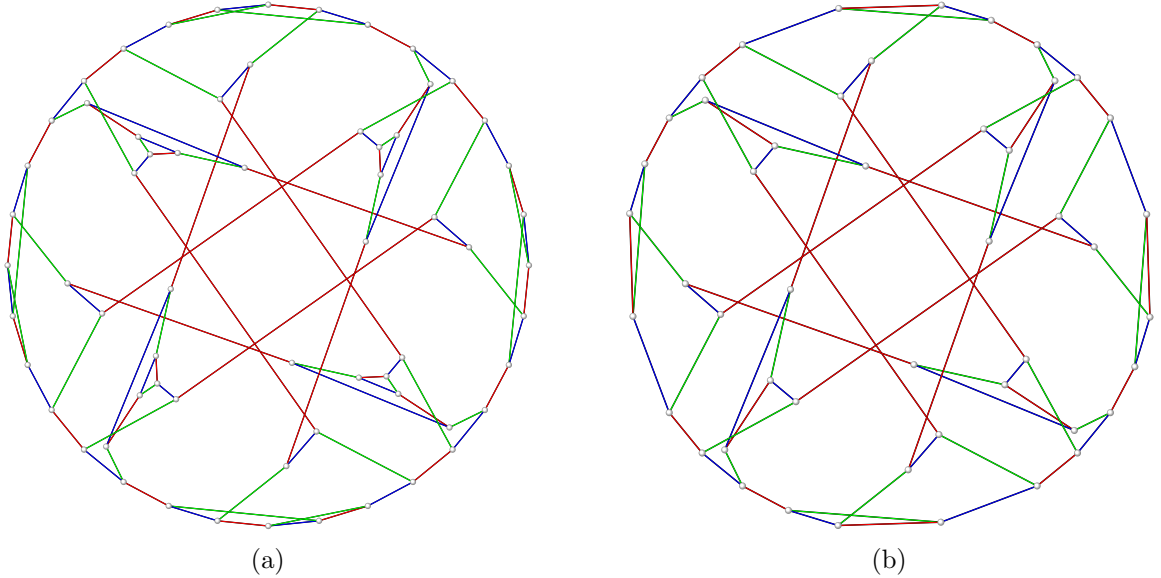


Figure 7.6: Cubic graphs with automorphism groups isomorphic to Q_8 : Frucht's original construction on 64 nodes (a) and our modified construction on 48 nodes (b)

We proceed by showing that our modifications in Remark 7.1.10 lead to cubic graphs with given groups as automorphism groups. We prove this statement by making use of cycle triplets of given cubic graphs, see Definition 7.1.9. For simplicity, we assume that the entries of a cycle triplet of a node of a cubic graph are ordered lexicographically.

Theorem 7.1.11

Let G be a finite group with generating set $S = \{g_1, \dots, g_n\}$, where $n \geq 2$. Then there exists a cubic graph Γ with $\text{Aut}(\Gamma) \cong G$ on $6|G|$ nodes for $n = 2$ and $(2n + 6)|G|$ nodes for $n > 2$.

Proof. Let Γ_{mod} and Γ_{simp} be the cubic graphs with quadratic forms Q_{mod} and Q_{simp} as given in Remark 7.1.10, respectively. In order to prove the result we exploit the incidence structures of Γ_{mod} and Γ_{simp} as illustrated in Figure 7.5. First, we show that automorphisms of Γ_{mod} and Γ_{simp} have to map components illustrated in Figure 7.5a and Figure 7.5b onto other such components, respectively. Hence, we give a case distinction with respect to (1) $n = 2$ and (2) $n > 2$.

Case (1): For $n = 2$ and some $g \in G$, the nodes $x_{1,g}$ and $x_{2,g}$ (shown in Figure 7.5a) have the same cycle triplet, namely $(3, 5, 6)$. However, the node $x_{4,g}$ is incident to two nodes that lie on a 3-cycle, i.e. $x_{1,g}$ and $x_{3,g \cdot g_1^{-1}}$, but $x_{6,g}$ is incident to exactly one node that lies on a 3-cycle, i.e. $x_{2,g}$. Thus, an automorphism $\pi \in \text{Aut}(\Gamma)$ has to map a node $x_{i,g}$ onto a node $x_{i,g'}$ for some $g' \in G$, i.e. $\pi(x_{i,g}) = x_{i,g'}$. Hence, components illustrated in Figure 7.5a are mapped onto other such components.

Case (2): Similarly, for $n > 2$ and some $g \in G$ the node $x_{4,g}$ (see Figure 7.5b) is the only node in the illustrated component that has cycle triplet $(4, 5, 2 + 2n)$. Thus, the incidence structure of the underlying cubic graph enforces illustrated components to be mapped onto other such components.

The remaining part of this proof is analogous to the proof given by Frucht in [42]. In particular, the remaining part is based on showing that the automorphism group of the cubic graphs Γ_{simp} and Γ_{mod} are isomorphic to the automorphism groups of the corresponding arc- n -coloured Cayley graphs, where the arc- n -colouring is induced by S for $n = 2$ and $n > 2$, respectively. Hence, we obtain $\text{Aut}(\Gamma_{simp}) \cong G$ for $n = 2$ and $\text{Aut}(\Gamma_{mod}) \cong G$ for $n > 2$ by Lemma 7.1.2 $\square$

Thus, our proposed modification indeed fills the gap in Frucht's proof for his desired cubic graph construction. Hence, we obtain Frucht's result that for every finite non-cyclic group G , there exists a cubic graph Γ satisfying $\text{Aut}(\Gamma) \cong G$. Note that Frucht gives two cubic graphs (see Fig. 1 and Fig. 2 in [42]) with automorphism groups of order 1 and 2, respectively. Further, the case that G is cyclic is handled in Theorem 7.2.2.

We conclude this section by establishing that the cubic graphs resulting from Remark 7.1.10 can always be arc-3-coloured.

Proposition 7.1.12

Let G be a finite group and $S = \{g_1, \dots, g_n\}$ a generating set of G with $n \geq 2$. Then the cubic graphs Γ_{simp} and Γ_{mod} with quadratic forms Q_{simp} and Q_{mod} from Remark 7.1.10 can be arc-3-coloured.

Proof. We prove the above statement by introducing suitable arc-3-colourings of $\Gamma_{simp} = (V, E)$ for $n = 2$ and $\Gamma_{mod} = (V', E')$ for $n > 2$. For this purpose, if $(E'', Q) \in \{(E, Q_{simp}), (E', Q_{mod})\}$ is one of the above quadratic forms, then we define $E''(q)$, where q is a subsum of the quadratic form Q , as the set of arcs in E'' whose corresponding monomials are summands of q .

Case $n = 2$: We make use of the quadratic form Q_{simp} as given in Remark 7.1.10 to construct an arc-3-colouring of Γ_{simp} . We observe that E can be partitioned into the sets E_1, E_2, E_3 with

$$\begin{aligned} E_1 &:= E(Q_{1,2} + R_{simp}), \\ E_2 &:= E(Q_{1,4} + Q_{2,3} + Q_{5,6}), \\ E_3 &:= E(Q_{1,3} + Q_{2,6} + Q_{4,5}). \end{aligned}$$

Thus, $\kappa_{simp} : E \rightarrow \{1, 2, 3\}$ with $\kappa_{simp}(e) := i$ for $e \in E_i, i = 1, 2, 3$ is an arc-colouring of Γ_{simp} , see Figure 7.7a for an illustration of κ_{simp} with arcs coloured in 1, 2 and 3 visualised with red, green and blue, respectively. Hence, we conclude that κ_{simp} is indeed an arc-3-colouring of Γ_{simp} .

Case $n > 2$: Here, we exploit the quadratic form Q_{mod} as given in Remark 7.1.10. In this case the arcs of the cubic graph Γ_{mod} can be partitioned into the three sets E'_1, E'_2, E'_3 with

$$\begin{aligned} E'_1 &:= E'(Q_{1,6} + Q_{2,3} + Q_{4,5} + R_{mod}), \\ E'_2 &:= E'(Q_{1,2} + Q_{3,7} + Q_{4,6} + Q_{5,8} + \sum_{i=2}^n Q_{2i+5, 2i+6}), \\ E'_3 &:= E'(Q_{1,3} + Q_{2,4} + Q_{5,7} + Q_{6, 2n+6} + \sum_{i=2}^n Q_{2i+4, 2i+5}). \end{aligned}$$

Hence, we see that the map $\kappa_{mod} : E' \rightarrow \{1, 2, 3\}$ with $\kappa_{mod}(e) := i$ for $e \in E'_i, i = 1, 2, 3$ is an arc-colouring of Γ_{mod} . This arc-colouring is shown in Figure 7.7b with arcs coloured in 1, 2 and 3 visualised with red, green and blue, respectively. Thus, κ_{mod} is an arc-3-colouring of Γ_{mod} and we conclude the result.

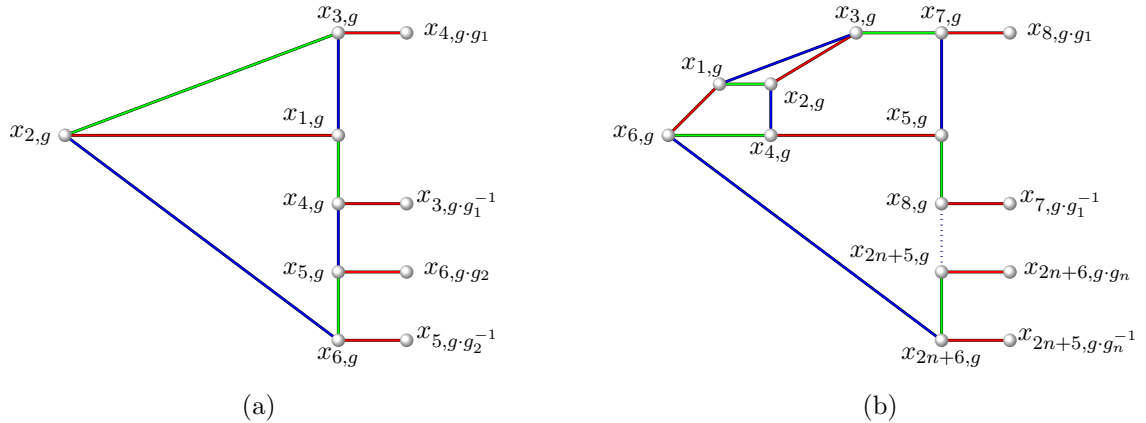


Figure 7.7: Cubic graphs equipped with arc-3-colourings: Simplified construction for two generators (a) and modified construction for more than two generators (b)

□

In Section 7.3 we exploit the arc-colourings κ_{mod} and κ_{simp} that are constructed in the above proof to show that a given finite group can be interpreted as the automorphism group of a simplicial surface, see Theorem 7.3.5.

7.2 Cubic graphs with cyclic or dihedral automorphism groups

Frucht's cubic graph construction for cyclic groups in [42] differs from his construction for non-cyclic groups (see Remark 7.1.6). We first recall his construction for cyclic groups and then introduce

a new family of cubic graphs whose automorphism groups are dihedral. For both, the family of cubic graphs with cyclic automorphism groups and the newly introduced family with dihedral automorphism groups, we construct corresponding planar straight-line embeddings.

Remark 7.2.1 For a cyclic group C_n with $n > 2$, Frucht constructs a cubic graph Γ_{C_n} consisting of $6n$ nodes such that the corresponding automorphism group is cyclic of order n . The quadratic form Q_{C_n} that uniquely determines the cubic graph Γ_{C_n} is given as follows: If the indeterminates corresponding to the nodes of the graph Γ_{C_n} are given by $a_i, b_i, c_i, d_i, e_i, f_i$ for $i = 1, \dots, n$, then Frucht defines the quadratic form Q_{C_n} via:

$$Q_{C_n} := \sum_{i=1}^n (a_i b_i + a_i e_i + a_i f_i + b_i f_i + c_i d_i + c_i f_i + c_i e_i) + \sum_{j=1}^{n-1} (b_{j+1} e_j + d_j d_{j+1}) + b_1 e_n + d_1 d_n.$$

In Figures 7.8a and 7.8b we show illustrations of the graph Γ_{C_n} for $n = 5$ and $n = 6$, respectively.

With these quadratic forms, we are able to formulate the following result by Frucht.

Theorem 7.2.2 ([42] Frucht)

If $n > 2$ is a natural number, then the cubic graph Γ_{C_n} from Remark 7.2.1 satisfies $\text{Aut}(\Gamma_{C_n}) \cong C_n$.

Next, we compute a planar straight-line embedding of the cubic graph Γ_{C_n} , where $n > 2$. We construct such an embedding by exploiting the following remark.

Remark 7.2.3 Let $n > 2$ be a natural number. Then Γ_{C_n} is a well-defined cubic graph with nodes $a_i, b_i, c_i, d_i, e_i, f_i$, where $i = 1, \dots, n$. By defining $\pi := (a_1, \dots, a_n) \cdots (f_1, \dots, f_n)$ we see that $\text{Aut}(\Gamma_{C_n}) = \langle \pi \rangle$. Thus, $\text{Aut}(\Gamma_{C_n})$ can be embedded into $\text{GL}_2(\mathbb{R})$ via $\Psi : \text{Aut}(\Gamma_{C_n}) \rightarrow \text{GL}_2(\mathbb{R})$ defined by $\Psi(\pi^j) := (M_\alpha)^j$ for all $1 \leq j \leq n$, where $\alpha := \frac{2\pi}{n}$ and

$$M_\alpha := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Hence, $\text{Aut}(\Gamma_{C_n}) \cong \langle M_\alpha \rangle$ follows.

We can construct the desired planar straight-line embedding of Γ_{C_n} by exploiting the above matrix M_α to assign 2D-coordinates to the nodes of Γ_{C_n} . Note that by just assigning 2D-coordinates to the different nodes of Γ_{C_n} the corresponding straight-line embedding is determined fully.

Proposition 7.2.4

Let $n > 2$ be a natural number and Γ_{C_n} the cubic graph from Remark 7.2.1. Then Γ_{C_n} is planar.

Proof. We prove this statement by constructing a planar straight line embedding of the cubic graph $\Gamma_{C_n} = (V, E)$. Let therefore α be defined by $\alpha := \frac{2\pi}{n}$ and $v, w_1, w_2 \in \mathbb{R}^2$ defined by

$$v := (1, 0)^t, w_1 := (\cos(\alpha) - 1, \sin(\alpha))^t, w_2 := (\sin(\alpha), 1 - \cos(\alpha))^t.$$

Furthermore, let $\phi: V \rightarrow \mathbb{R}^2$ be given as follows: We define the images of $a_1, \dots, f_1$ under ϕ as

$$\begin{aligned}\phi(a_1) &:= v + \frac{1}{2}w_1, \phi(b_1) := v + \frac{1}{4}w_1, \\ \phi(c_1) &:= v + \frac{1}{2}w_1 + 2w_2, \phi(d_1) := v + \frac{1}{2}w_1 + 3w_2, \\ \phi(e_1) &:= v + \frac{3}{4}w_1, \phi(f_1) := v + \frac{1}{2}w_1 + w_2.\end{aligned}$$

With this information we are able to determine the map ϕ fully. More precisely, for all $1 \leq i \leq n-1$ we define

$$\begin{aligned}\phi(a_{i+1}) &:= (M_\alpha)^i \phi(a_1), \phi(b_{i+1}) := (M_\alpha)^i \phi(b_1), \\ \phi(c_{i+1}) &:= (M_\alpha)^i \phi(c_1), \phi(d_{i+1}) := (M_\alpha)^i \phi(d_1), \\ \phi(e_{i+1}) &:= (M_\alpha)^i \phi(e_1), \phi(f_{i+1}) := (M_\alpha)^i \phi(f_1).\end{aligned}$$

We observe that ϕ is a planar straight line embedding of Γ_{C_n} . □

In Section 7.3 we give an alternative argument to prove that the graph Γ_{C_n} is planar. Furthermore, Figure 7.8a and 7.8b illustrate the above planar embedding for $n = 5$ and $n = 6$, respectively.

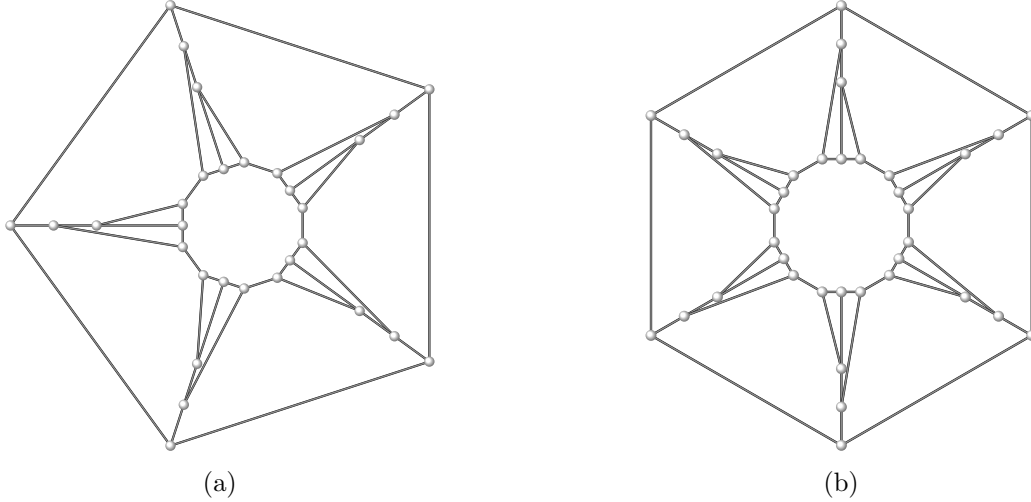


Figure 7.8: Planar straight-line embeddings of Γ_{C_n} for $n = 5$ (a) and $n = 6$ (b)

Next, we introduce a new infinite family of cubic graphs that all have dihedral automorphism groups with less nodes than the cubic graphs that result from the cubic graph constructions in the previous section.

Remark 7.2.5 For $n > 3$ the dihedral group D_n of order $2n$ can be generated by 2 elements. Thus, constructing a cubic graph with an automorphism group that is isomorphic to D_n is covered by Frucht's general construction in Remark 7.1.6 and our simplification of Frucht's construction in Remark 7.1.10. However, for $n > 3$, we can define a cubic graph Γ_{D_n} with $4n$ instead of $(2 \cdot 2 + 4) \cdot |D_n| = 16n$ or $6 \cdot |D_n| = 12n$ nodes that still has a dihedral automorphism group as described in the following: Let a_i, b_i, c_i, d_i for $i = 1, \dots, n$ be indeterminates corresponding to the nodes of the graph $\Gamma_{D_n} = (V, E)$. Hence, we define Γ_{D_n} by the quadratic form Q_{D_n} with

$$Q_{D_n} := \sum_{i=1}^n (a_i b_i + a_i c_i + c_i d_i + b_i c_i) + \sum_{j=1}^{n-1} (a_j b_{j+1} + d_j d_{j+1}) + a_n b_1 + d_1 d_n.$$

Figures 7.9a and 7.9b illustrate the graph Γ_{D_n} for $n = 5$ and for $n = 6$, respectively.

First, we show that the cubic graph Γ_{D_n} for $n > 3$ indeed has an automorphism group that forms a dihedral group of order $2n$.

Theorem 7.2.6

If $n > 3$ is a natural number, then the cubic graph Γ_{D_n} from Remark 7.2.5 satisfies $\text{Aut}(\Gamma_{D_n}) \cong D_n$.

Proof. By examining the quadratic form Q_{D_n} , we see that Γ_{D_n} has exactly n cycles of length 3, namely

$$(a_1, b_1, c_1), \dots, (a_n, b_n, c_n).$$

We prove the above statement by examining the action of an automorphism of Γ_{D_n} on these cycles. Hence, let $\pi \in \text{Aut}(\Gamma_{D_n})$ be an automorphism. Since π has to map a cycle of length 3 onto a cycle of length 3, we know that π permutes the cycles $(a_1, b_1, c_1), \dots, (a_n, b_n, c_n)$. Thus, there exists a $1 \leq i \leq n$ with $\pi((a_1, b_1, c_1)) = (a_i, b_i, c_i)$. Because of π being an automorphism and therefore respecting the incidences of Γ_{D_n} , this implies $\pi(d_1) = d_i$. Moreover, π has to permute the nodes $d_1, \dots, d_n$ such that $\pi((d_1, \dots, d_n)) = (d_1, \dots, d_n)$. That means that either **(1)** $\pi(d_{1+j}) = d_{i+j}$ for all $1 \leq j \leq n$ or **(2)** $\pi(d_{1+j}) = d_{i-j}$ for all $1 \leq j \leq n$, where the subscripts are read modulo n .

Case (1): If $\pi(d_{1+j}) = d_{i+j}$ for all $1 \leq j \leq n$, then $\pi(c_{1+j}) = c_{i+j}$ for all $1 \leq j \leq n$ follows. Moreover, π has to map the cycle $(d_1, c_1, a_1, b_2, c_2, d_2)$ onto a cycle of the same length. This instance implies $\pi(a_{1+j}) = a_{i+j}$ and $\pi(b_{1+j}) = b_{i+j}$ for all $1 \leq j \leq n$. Therefore, π can be written as $\pi = \sigma^{i-1}$, where σ is defined as $\sigma := (a_1, \dots, a_n) \cdots (d_1, \dots, d_n)$.

Case (2): If $\pi(d_{1+j}) = d_{i-j}$ for all $1 \leq j \leq n$, we deduce that π maps c_{1+j} onto c_{i-j} for all $1 \leq j \leq n$, which implies $\pi(a_{1+j}) = b_{i-j}$ and $\pi(b_{1+j}) = a_{i-j}$ for all $1 \leq j \leq n$ by the same argument as presented in the previous case. Thus, π is given by $\pi = \sigma^{i-1}s$, where s is defined by

$$s := (a_1, b_1) \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} (a_{1-j}, b_{1+j})(a_{1+j}, b_{1-j})(d_{1-j}, d_{1+j})(c_{1-j}, c_{1+j}),$$

with the transpositions $(c_k, c_k), (d_k, d_k)$ for $k \in \{1, \lfloor \frac{n}{2} \rfloor\}$ defined as the identity. Hence, we conclude $\text{Aut}(\Gamma_{D_n}) = \langle \sigma, s \rangle \cong D_n$ and the result follows. $\square$

We can exploit the orthogonal matrix M_α , see Remark 7.2.3, to construct planar straight-line embeddings of the graphs Γ_{D_n} for $n > 3$ as seen in the cyclic case in Proposition 7.2.4.

Proposition 7.2.7

Let $n > 3$ be a natural number and Γ_{D_n} the cubic graph from Remark 7.2.5. Then Γ_{D_n} is planar.

Proof. We proceed in the same way as presented in the proof of Proposition 7.2.4 and construct a planar straight line embedding of $\Gamma_{D_n} = (V, E)$ as follows: Let α be defined by $\alpha := \frac{2\pi}{n}$ and $v, w_1, w_2 \in \mathbb{R}^2$ defined by

$$v := (1, 0)^t, w_1 := (\cos(\alpha) - 1, \sin(\alpha))^t, w_2 := (\sin(\alpha), 1 - \cos(\alpha))^t.$$

Furthermore, let $\phi: V \rightarrow \mathbb{R}^2$ be a map, where the images of a_1, b_1, c_1, d_1 under ϕ are defined by

$$\phi(a_1) := v + \frac{3}{4}w_1, \phi(b_1) := v + \frac{1}{4}w_1,$$

$$\phi(c_1) := v + \frac{1}{2}w_1 + w_2, \phi(d_1) := v + \frac{1}{2}w_1 + 2w_2.$$

Thus, for all $1 \leq i \leq n-1$ we determine the map ϕ via

$$\begin{aligned} \phi(a_{i+1}) &:= (M_\alpha)^i \phi(a_1), \phi(b_{i+1}) := (M_\alpha)^i \phi(b_1), \\ \phi(c_{i+1}) &:= (M_\alpha)^i \phi(c_1), \phi(d_{i+1}) := (M_\alpha)^i \phi(d_1). \end{aligned}$$

Thus, it is straightforward to see that ϕ yields a planar straight-line embedding of Γ_{D_n} . $\square$

Figure 7.9a shows the constructed planar embedding for $n = 5$ and Figure 7.9b for $n = 6$.

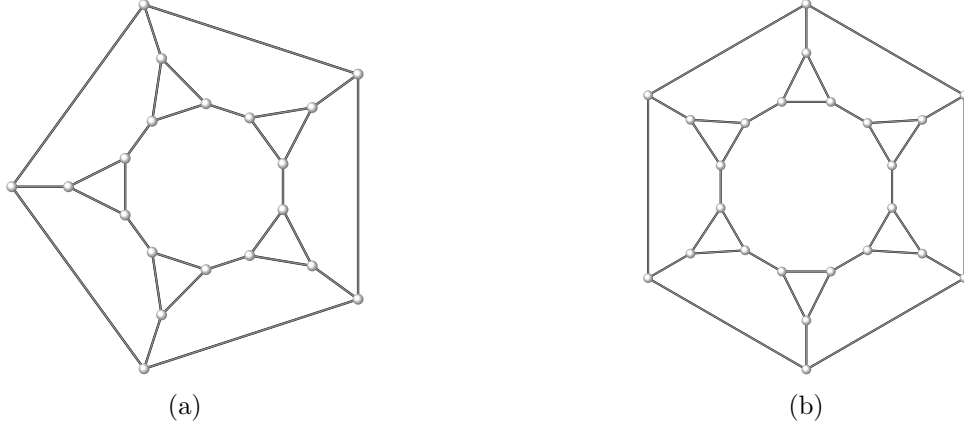


Figure 7.9: Planar straight-line embedding of Γ_{D_n} for $n = 5$ (a) and $n = 6$ (b)

7.3 Cycle double covers and simplicial surface constructions

In Sections 7.1 and 7.2 we have exploited Frucht's cubic graph construction to establish that every finite group can be interpreted as the automorphism group of a cubic graph. Based on this result, we show that every finite group G can be realised as the automorphism group of a simplicial surface. We achieve this by computing an $\text{Aut}(\Gamma)$ -invariant cycle double cover of a cubic graph Γ that arises from one of the constructions presented in Remarks 7.1.10 and 7.2.1. Since Γ satisfies $\text{Aut}(\Gamma) \cong G$, we obtain a simplicial surface with prescribed automorphism group (This instance is detailed in Remark 3.2.1). We therefore follow the below methodology: In [42] Frucht conducts a case distinction to construct cubic graphs based on a finite group G , namely **(1)** $|G| \leq 2$, **(2)** $G \cong C_n$ for $n > 2$ and **(3)** $G \not\cong C_n$ for $n \geq 1$. Based on this case distinction we construct simplicial surfaces with prescribed automorphism groups. **(1)** For $|G| \leq 2$ Frucht gives two cubic graphs (see Fig. 1 and Fig. 2 in [42]) with automorphism groups of order 1 and 2 which yield simplicial spheres with automorphism groups of order 1 and 2, respectively. **(2)** In Theorem 7.3.1 we show that in case $G \cong C_n$ with $n > 2$ the cubic graph Γ_{C_n} constructed in Remark 7.2.1 can be equipped with an $\text{Aut}(\Gamma_{C_n})$ -invariant cycle double cover. **(3)** The general case, namely $G \not\cong C_n$ for $n \in \mathbb{N}$ is covered by Theorem 7.3.1. Here, we use Remark 1.2.7 and construct cycle double covers of given cubic graphs by exploiting corresponding arc-3-colourings as described by Szekeres in [88]. Hence, if Γ is a cubic graph resulting from our modification of Frucht's construction and X a simplicial surface with $\mathcal{F}(X) = \Gamma$, then $|X_2|$ equals the number of nodes of Γ , i.e. $6|G|$ for $n = 2$ and $(2n+6)|G|$ for $n > 2$. Additionally, if $G = D_n$ with $n > 3$, then we construct an $\text{Aut}(\Gamma_{D_n})$ -invariant cycle double cover of the cubic graph Γ_{D_n} constructed in Remark 7.2.5, see Theorem 7.3.3. Finally, we present some examples of cubic graphs and corresponding simplicial surfaces with automorphism groups that are isomorphic to $G = A_5$.

We start this section by showing that the cubic graph Γ_{C_n} with $n > 2$ yields a vertex-faithful simplicial surface with cyclic automorphism group of order n .

Theorem 7.3.1

Let Γ_{C_n} with $n > 2$ be the cubic graph from Remark 7.2.1. Then there exists a vertex-faithful simplicial sphere X satisfying $\mathcal{F}(X) = \Gamma_{C_n}$ and $\text{Aut}(X) \cong C_n$.

Proof. In the following, we read all subscripts modulo n . First, we recall that the quadratic form of the cubic graph Γ_{C_n} is given by

$$Q_{C_n} = \sum_{i=1}^n (a_i b_i + a_i e_i + a_i f_i + b_i f_i + c_i d_i + c_i f_i + c_i e_i) \\ + \sum_{j=1}^{n-1} (b_{j+1} e_j + d_j d_{j+1}) + b_1 e_n + d_1 d_n.$$

We define a cycle double cover of the graph Γ_{C_n} as follows:

$$\mathcal{C} = \{(a_i, b_i, f_i) \mid i = 1, \dots, n\} \cup \{(a_i, f_i, c_i, e_i) \mid i = 1, \dots, n\} \cup \\ \{(e_i, c_i, d_i, d_{i+1}, c_{i+1}, f_{i+1}, b_{i+1}) \mid i = 1, \dots, n\} \cup \\ \{(d_1, \dots, d_n), (b_1, a_1, e_1, \dots, b_n, a_n, e_n)\}.$$

It is straightforward to see that $\mathcal{C}$ forms a vertex-faithful cycle double cover of Γ_{C_n} . We observe that $\text{Aut}(\Gamma_{C_n})$ leaves the above cycle cover $\mathcal{C}$ invariant, i.e. $\pi(\mathcal{C}) = \mathcal{C}$ for every $\pi \in \text{Aut}(\Gamma_{C_n})$. Thus, the resulting simplicial surface X with $\mathcal{F}(X) = \Gamma_{C_n}$ and vertices corresponding to $\mathcal{C}$ satisfies

$$\text{Aut}(X) \cong \text{Aut}(\mathcal{F}(X)) \cong C_n.$$

It remains to show that X is a simplicial sphere. This can be shown by examining the vertex-counter of X . Since the different vertex-degrees correspond to the lengths of the cycles in $\mathcal{C}$, we see that the vertex-counter of X is given by $V_3^n V_4^n V_7^n V_n V_{3n}$. Thus, $\chi(X) = (n + n + n + 2) - 9n + 6n = 2$, which concludes the proof. $\square$

As a direct consequence, we obtain the following proposition.

Proposition 7.3.2

Let Γ_{C_n} with $n > 2$ be the cubic graph constructed in Remark 7.2.1. Then Γ_{C_n} is a 3-connected planar graph.

Proof. By Theorem 7.3.1 the graph Γ_{C_n} is the face graph of a vertex-faithful simplicial sphere X . By Steinitz, see [84], this is the case if and only if Γ_{C_n} is planar and 3-connected. $\square$

For $n > 3$ we give an analogous result for the dihedral group D_n .

Theorem 7.3.3

Let Γ_{D_n} with $n > 3$ be the cubic graph from Remark 7.2.5. Then there exists a vertex-faithful simplicial sphere X satisfying $\mathcal{F}(X) = \Gamma_{D_n}$ and $\text{Aut}(X) \cong D_n$.

Proof. Here, we read all subscripts in this proof modulo n . We know that the quadratic form Q_{D_n} is given by

$$Q_{D_n} = \sum_{i=1}^n (a_i b_i + a_i c_i + c_i d_i + b_i c_i) + \sum_{j=1}^{n-1} (a_j b_{j+1} + d_j d_{j+1}) + a_n b_1 + d_1 d_n.$$

Hence, we construct a cycle double cover of Γ_{D_n} via

$$\begin{aligned} \mathcal{C} = & \{(a_i, b_i, c_i) \mid i = 1, \dots, n\} \cup \\ & \{(a_i, c_i, d_i, d_{i+1}, c_{i+1}, b_{i+1}) \mid i = 1, \dots, n\} \cup \\ & \{(d_1, \dots, d_n), (a_1, b_1, a_2, b_2, \dots, a_n, b_n)\}. \end{aligned}$$

The above set forms a vertex-faithful cycle double cover of Γ_{D_n} . Moreover, $\text{Aut}(\Gamma_{D_n})$ leaves the above cycle double cover $\mathcal{C}$ invariant. Thus, the resulting simplicial surface X with $\mathcal{F}(X) = \Gamma_{D_n}$ and vertices corresponding to $\mathcal{C}$ satisfies

$$\text{Aut}(X) \cong \text{Aut}(\mathcal{F}(\Gamma_{D_n})) \cong D_n.$$

Since the vertex-counter of X is given by $V_3^n V_6^n V_n V_{2n}$, we know that $\chi(X) = (n+n+2) - 6n + 4n = 2$. Hence, X is vertex-faithful simplicial sphere with a dihedral automorphism group. $\square$

The above theorem directly implies the below result.

Proposition 7.3.4

Let Γ_{D_n} with $n > 3$ be the cubic graph constructed in Remark 7.2.5. Then Γ_{D_n} is a 3-connected planar graph.

Proof. Theorem 7.3.3 establishes that Γ_{D_n} is the face graph of a vertex-faithful simplicial sphere. Hence, Γ_{D_n} is planar and 3-connected by Steinitz [84]. $\square$

Next, we show that the cubic graphs that arise from our modification of Frucht's cubic graph construction are the face graphs of simplicial surfaces.

Theorem 7.3.5

Let G be a finite non-cyclic group with generating set $S = \{g_1, \dots, g_n\}$, where $n \geq 2$. Then there exists a simplicial surface X such that $\text{Aut}(X) \cong G$.

Proof. We give a case distinction with respect to (1) $n > 2$ and (2) $n = 2$.

Case (1): First, we assume $n > 2$. By Theorem 7.1.11 there exists a cubic graph $\Gamma = (V, E)$ that satisfies $\text{Aut}(\Gamma) \cong G$. We choose Γ to be the cubic graph that is described by the quadratic form Q_{mod} in Remark 7.1.10, i.e. $\Gamma := \Gamma_{mod}$. Furthermore, we define $E(q)$, where q is a subsum of the quadratic form Q_{mod} , as the set of arcs that corresponds to the monomials that are summands of q . In the proof of Proposition 7.1.12, we have constructed an arc-3-colouring κ_{mod} of Γ , see Figure 7.10b for illustration. We exploit this arc-3-colouring to construct the cycle double cover $\mathcal{C}_{mod}$ of Γ consisting of all bi-coloured cycles of Γ as described in Remark 1.2.7. The resulting simplicial surface X with $\mathcal{F}(X) = \Gamma$ and vertices corresponding to $\mathcal{C}_{mod}$ satisfies $\text{Aut}(X) \leq \text{Aut}(\Gamma)$. Thus, it remains to show that every automorphism of $\text{Aut}(\Gamma)$ leaves the cycle double cover $\mathcal{C}_{mod}$ invariant. If we denote G by $G = \{g_1, \dots, g_n\}$, as in Remark 7.1.10, we observe that for every

$1 \leq m \leq h = |G|$ the element $g_m \in G$ induces an automorphism σ_m of Γ via $\sigma_m(x_{i,g_k}) := x_{i,g_m g_k}$ for all $1 \leq i \leq 2n+6$ and $1 \leq k \leq h$. Since $\text{Aut}(\Gamma) \cong G$ and $\sigma_m \neq \sigma_{m'}$ for $m \neq m'$, we know that $\text{Aut}(\Gamma) = \{\sigma_1, \dots, \sigma_h\}$. Moreover, we observe $\sigma_m(E(Q_{i,j})) = E(Q_{i,j})$ and $\sigma_m(E(R_{mod})) = E(R_{mod})$ for all $1 \leq m \leq h$. Hence, $\text{Aut}(\Gamma)$ respects the arc-3-colouring κ_{mod} . We therefore conclude that the cycle double cover $\mathcal{C}_{mod}$ is invariant under $\text{Aut}(\Gamma)$ which means $\text{Aut}(X) \cong \text{Aut}(\Gamma) \cong G$.

Case (2): Similarly, if $n = 2$, we can exploit the arc-3-colouring κ_{simp} (Figure 7.10a) of the cubic graph $\Gamma' := \Gamma_{simp}$, see Remark 7.1.10, to construct a cycle double cover $\mathcal{C}_{simp}$ consisting of all the bicoloured cycles with respect to κ_{simp} . We can verify that the resulting simplicial surface X' with $\mathcal{F}(X') = \Gamma'$ and vertices corresponding to $\mathcal{C}_{simp}$ satisfies $\text{Aut}(X') \cong \text{Aut}(\Gamma') \cong G$ analogously to the case $n > 2$. Hence, the result follows. $\square$

In the following we comment on the vertex-degrees of the simplicial surfaces constructed in the proof of Theorem 7.3.5. Again, note that the construction of these simplicial surfaces is based on the arc-3-colourings of the cubic graphs illustrated in Figures 7.10a and 7.10b.

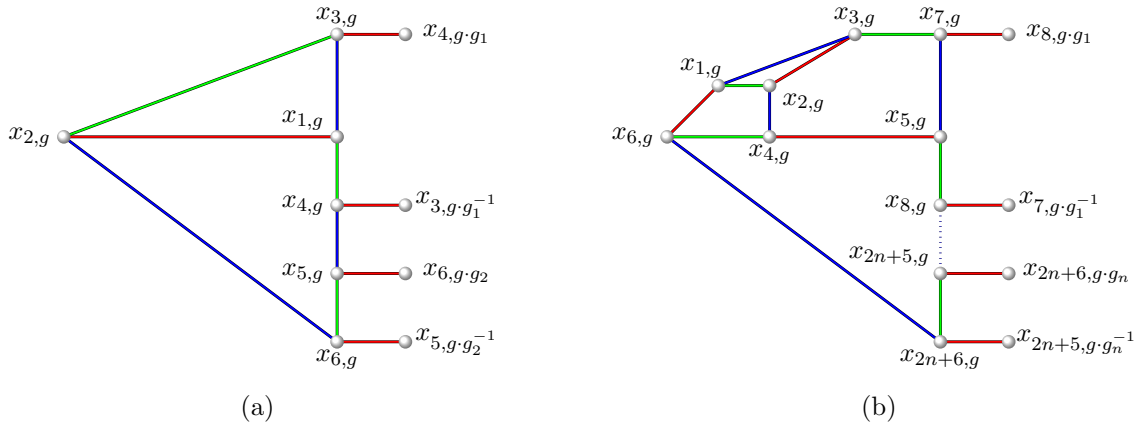


Figure 7.10: Cubic graphs equipped with arc-3-colourings: Simplified construction for two generators (a) and modified construction for more than two generators (b)

Remark 7.3.6 Let G be a finite group and $S = \{g_1, \dots, g_n\}$ a generating set of G . In the cases (1) $|S| = 2$ and (2) $|S| > 2$, we can determine the lengths of the cycles in the cycle double covers constructed in the proof of Theorem 7.3.5 and hence compute the vertex-counter of the arising simplicial surfaces. This allows us to compute the Euler characteristics of these simplicial surfaces.

Case (1): Let $\mathcal{C}_{simp}$ be the cycle double cover that has been established in the proof of Theorem 7.3.5 based on the arc-3-colouring κ_{simp} , and X' the corresponding simplicial surface. In this case, we know that $\mathcal{C}_{simp}$ yields $|G|$ blue-green cycles of length 6. Further, we have $|G|/|g_1 \cdot g_2|$ blue-red cycles of length $6 \cdot |g_1 \cdot g_2|$. Finally, we obtain $|G|/|g_2|$ red-green cycles of length $2|g_2|$ and $|G|/|g_1|$ red-green cycles of length $4|g_1|$. This implies that the vertex-counter of X' is given by

$$V_6^{|G|} \cdot V_{6|g_1 \cdot g_2|}^{|G|/|g_1 \cdot g_2|} \cdot V_{4|g_1|}^{|G|/|g_1|} \cdot V_{2|g_2|}^{|G|/|g_2|}.$$

Since the simplicial surface X' has $|G|(1 + \frac{1}{|g_1 \cdot g_2|} + \frac{1}{|g_1|} + \frac{1}{|g_2|})$ vertices, $9|G|$ edges and $6|G|$ faces, the Euler characteristic of X' is given by

$$\chi(X') = |G| \cdot \left(\frac{1}{|g_1 \cdot g_2|} + \frac{1}{|g_1|} + \frac{1}{|g_2|} - 2 \right).$$

Case (2): Next, let $\mathcal{C}_{\text{mod}}$ be the cycle double cover, κ_{mod} the arc-3-colouring and X the corresponding simplicial surface from the proof of Theorem 7.3.5. Hence, we observe that $\mathcal{C}_{\text{mod}}$ contains $|G|$ blue-green cycles of length $2n+6$. We have $|G|/|g_1 \cdots g_n|$ blue-red cycles of length $6 \cdot |g_1 \cdots g_n|$. Additionally, for $i = 2, \dots, n$ we obtain $|G|/|g_i|$ green-red cycles of length $2|g_i|$ and $|G|/|g_1|$ green-red cycles of length $4|g_1|$. In total, the vertex counter of X is given by

$$V_{2n+6}^{|G|} \cdot V_{6|g_1 \cdots g_n|}^{|G|/|g_1 \cdots g_n|} \cdot V_{4|g_1|}^{|G|/|g_1|} \cdot \prod_{i=2}^n V_{2|g_i|}^{|G|/|g_i|}.$$

Since we know $|X_0|$, it is straightforward to compute the Euler characteristic of X as follows:

$$\chi(X) = |G| \cdot \left(\frac{1}{|g_1 \cdots g_n|} + \sum_{i=1}^n \frac{1}{|g_i|} - n - 2 \right).$$

Note, in both cases we can find pairs of blue-red and green-red cycles that share more than one arc, hence the simplicial surfaces X and X' are not vertex-faithful.

For some finite groups, we can give alternative cycle double covers of the cubic graphs constructed in Remark 7.1.10 that yield vertex-faithful simplicial surfaces. As an example, we consider the quaternion group $Q_8 = \langle i, j | i^4 = j^4 = 1, i^2 = j^2, ij = j^{-1}i \rangle$ with generators $g_1 := i, g_2 := j$. The resulting cubic graph is isomorphic to the cubic graph shown in Figure 7.6b. Below, we describe a cycle double cover of this cubic graph yielding a vertex-faithful simplicial surface of Euler characteristic 0. This cycle double cover consists of the cycles:

$$(x_{1,g}, x_{2,g}, x_{3,g}), (x_{1,g}, x_{2,g}, x_{6,g}, x_{5,g}, x_{4,g}),$$

for all $g \in Q_8$ and

$$(x_{3,g}, x_{1,g}, x_{4,g}, x_{3,g \cdot i^{-1}}, x_{2,g \cdot i^{-1}}, x_{6,g \cdot i^{-1}}, x_{5,g \cdot i^{-1} \cdot j^{-1}}, x_{6,g \cdot i^{-1} \cdot j^{-1}}, x_{5,g \cdot i^{-1} \cdot j^{-1} \cdot j^{-1}}, x_{4,g \cdot i^{-1} \cdot j^{-1} \cdot j^{-1}}),$$

for all $g \in Q_8$ (some elements $g \in Q_8$ yield the same cycle). Further, we note that no graph with automorphism group Q_8 can be embedded onto a sphere, see [11]. Hence, if X is a simplicial surface with $\text{Aut}(\mathcal{F}(X)) \cong Q_8$, then $\chi(X) \neq 2$.

With Theorem 7.3.5 we have demonstrated that our cubic graphs derived from the constructions inspired by Frucht yield simplicial surfaces. The number of faces of such a simplicial surface is significantly larger than the order of the corresponding automorphism group. Hence, we do not necessarily obtain highly symmetric simplicial surfaces, e.g., simplicial surfaces whose automorphism groups act transitively on the corresponding faces.

However, if a cubic graph forms an undirected Cayley graph of a given group G , then we can associate a simplicial surface to the given graph that possesses a large automorphism group. In this case, the construction of cycle double covers of the cubic Cayley graph can be linked to arc-3-colourings of the graph as described in the following.

Remark 7.3.7 Let G be a finite group with generators S and $\mathcal{C}_{G,S}$ an undirected cubic Cayley graph. Then G satisfies one of the following statements:

1. The group G is generated by three distinct involutions $S = \{s_1, s_2, s_3\}$ and hence the graph $\mathcal{C}_{G,S}$ can be arc-3-coloured (each involution corresponds to a colour on the arcs). We can obtain a cycle double cover of $\mathcal{C}_{G,S}$ by the set of all bi-coloured cycles of $\mathcal{C}_{G,S}$. Note that a cycle that is constructed by the generators s_i and s_j with $1 \leq i < j \leq 3$ has length $2 \cdot |s_i \cdot s_j|$.

2. The group G is generated by one involution s and another element x with $|x| > 2$. If $|x|$ is even, we can construct an arc-3-colouring of $\mathcal{C}_{G,S}$ by colouring the arcs corresponding to the involution s in one colour and the arcs obtained by the cycles arising from powers of x in alternating colours. Hence, if G is a group generated by $S = \{s, x, x^{-1}\}$ with $|s| = 2$ and $|x| = 2n$ with $n > 1$, then the cycles of the resulting cycle double cover either have length $|x|$, $2 \cdot |s \cdot x|$ or $2 \cdot |s \cdot x^{-1}|$.

Even if $|x|$ is not even, we can always obtain a cycle double cover via

$$\{(g, g \cdot x, \dots, g \cdot x^{|x|-1}) \mid g \in G\} \cup \{(g, g \cdot s, g \cdot sx, \dots, g \cdot (sx)^{|sx|-1}, g \cdot (sx)^{|sx|-1} \cdot s) \mid g \in G\}.$$

In all the above cases the proposed cycle double covers are invariant under the action of a subgroup $H \leq \text{Aut}(\mathcal{C}_{G,S})$ that is isomorphic to G . Furthermore, if $\text{Aut}(\mathcal{C}_{G,S}) \cong G$, then we obtain a simplicial surface X with $|G|$ faces and automorphism group isomorphic to G .

Note that [56] establishes that every undirected Cayley graph can be equipped with a cycle double cover.

Lastly, we present some cubic graphs that have automorphism groups that are isomorphic to A_5 and further demonstrate that the presented graphs can be equipped with cycle double covers. The case $G = A_5$ turns out to be interesting, as there is a cubic graph Γ with $\text{Aut}(\Gamma) \cong G$ such that a corresponding simplicial surface can be equipped with a $(1, 1, 1)$ -embedding. As first example, we consider the cubic graph Γ illustrated in Figure 7.11.

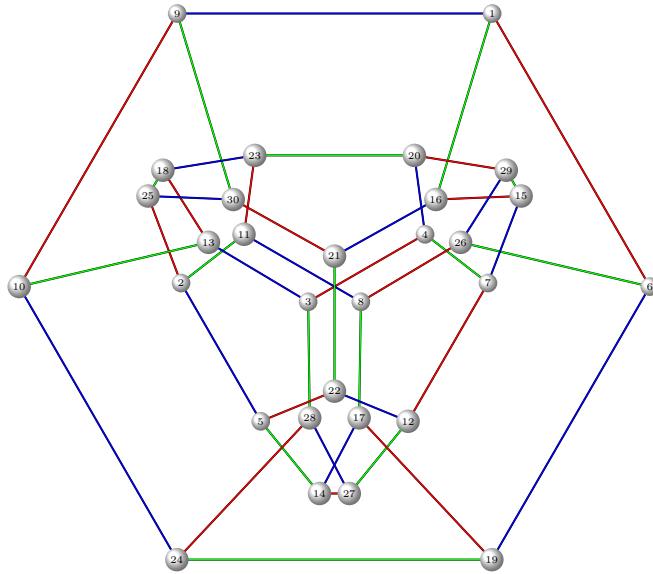


Figure 7.11: A node-transitive cubic graph with automorphism group isomorphic to A_5

The graph Γ has 30 nodes and forms the smallest node-transitive graph with an automorphism group that is isomorphic to A_5 . We observe that an $\text{Aut}(\Gamma)$ -invariant cycle double cover consisting of 10 cycles of length 6 and 6 cycles of length 5 is given by

$$\begin{aligned} &\{(14, 27, 28, 24, 19, 17), (9, 30, 25, 18, 13, 10), (8, 26, 29, 20, 23, 11), (7, 15, 16, 21, 22, 12), \\ &(3, 28, 27, 12, 7, 4), (3, 13, 18, 23, 20, 4), (2, 11, 8, 17, 14, 5), (2, 25, 30, 21, 22, 5), \\ &(1, 9, 10, 24, 19, 6), (1, 16, 15, 29, 26, 6), (3, 28, 24, 10, 13), (2, 25, 18, 23, 11), \\ &(1, 16, 21, 30, 9), (6, 26, 8, 17, 19), (5, 22, 12, 27, 14), (4, 20, 29, 15, 7)\}. \end{aligned}$$

Hence, the simplicial surface X with $\mathcal{F}(X) = \Gamma$ and vertices corresponding to the above cycle double cover has 16 vertices, 45 edges and 30 faces. We observe that X is a non-orientable simplicial surface satisfying $\chi(X) = 1$.

Next, we introduce a simplicial sphere Y with $\text{Aut}(Y) \cong A_5$ that can be embedded into Euclidean 3-space via a $(1, 1, 1)$ -embedding ϕ such that $A_5 \cong \text{Aut}(\phi(Y))$. For this purpose, we consider the snub dodecahedron, see Figure 7.12a. The snub dodecahedron forms an Archimedean solid with an automorphism group that is isomorphic to A_5 . Moreover, it can be shown that the graph that describes the incidences between its vertices and edges is isomorphic to a Cayley graph of A_5 . In particular, this Cayley graph can be obtained via the generators $S = \{(1, 2)(3, 4), (1, 2, 3, 4, 5), (1, 3, 5)\}$. Thus, by replacing every pentagonal face of the snub dodecahedron with 5 triangles via the kis operator, see [30], we obtain a triangulated sphere with automorphism group isomorphic to A_5 , called the pentakis snub dodecahedron, see Figure 7.12b. Figure 7.13 shows the face graph of the simplicial surface Y corresponding to the pentakis snub dodecahedron. Moreover, this face graph forms a planar cubic graph with automorphism group isomorphic to A_5 .

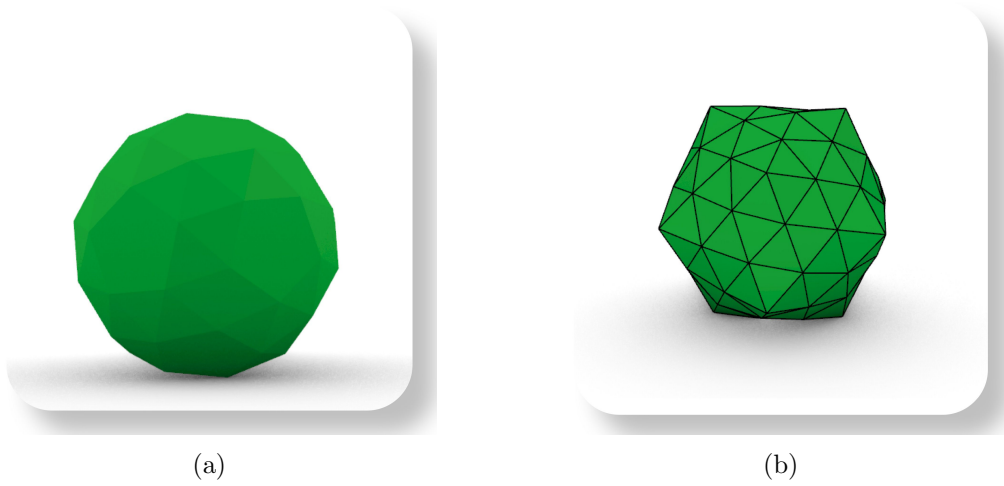


Figure 7.12: Snub dodecahedron and pentakis snub dodecahedron

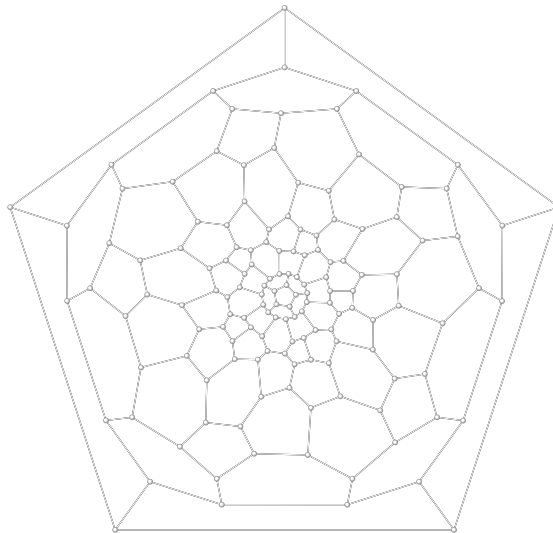


Figure 7.13: Face graph of the simplicial surface corresponding to the pentakis snub dodecahedron

7.4 Embeddings of simplicial surfaces

Inspired by the simplicial surfaces with cyclic and dihedral automorphism groups constructed in Theorems 7.3.1 and 7.3.3, we now establish the existence of infinite families of simplicial surfaces with cyclic or dihedral automorphism groups. In particular, we construct these families such that a corresponding simplicial surface can be equipped with a $(1, 1, 1)$ -embedding such that the simplicial surface and the arising polyhedron both either have dihedral or cyclic automorphism and symmetry groups. Hence, we establish the following result.

Theorem 7.4.1

If $G = C_n$ with $n > 2$ or $G = D_n$ with $n > 3$, then there exists a simplicial surface X and a corresponding $(1, 1, 1)$ -embedding ϕ of X such that $\text{Aut}(X) \cong \text{Aut}(\phi(X)) \cong G$.

We achieve this goal as described in the following: For every $n > 2$ we construct an infinite family $(X^{(n,k)})_{k \in \mathbb{N}_0}$ of vertex-faithful simplicial surfaces such that for every $k \in \mathbb{N}_0$ there exists a $(1, 1, 1)$ -embedding ϕ of $X^{(n,k)}$ with

$$C_n \cong \text{Aut}(\phi(X^{(n,k)})) \cong \text{Aut}(X^{(n,k)}).$$

Further, for every $n > 3$ we construct an infinite family of vertex-faithful simplicial surfaces $(Y^{(n,k)})_{k \in \mathbb{N}_0}$ such that for every $k \in \mathbb{N}_0$ there exists a $(1, 1, 1)$ -embedding ψ of $Y^{(n,k)}$ satisfying

$$D_n \cong \text{Aut}(\psi(Y^{(n,k)})) \cong \text{Aut}(Y^{(n,k)}).$$

The idea behind the construction is to combine tetrahedra, n -pyramids and n -strips (see Figures 7.14a and 7.14b) in such a way that they preserve the desired cyclic or dihedral symmetry.

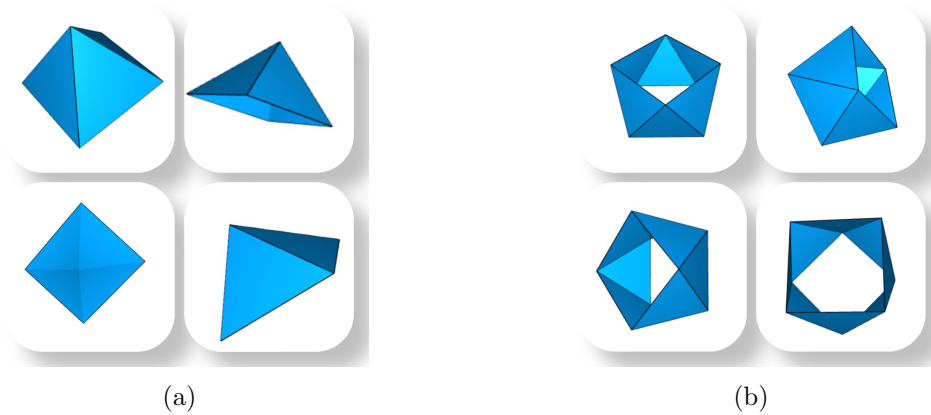


Figure 7.14: A 4-pyramid (a) and a 4-strip (b)

Since we aim to construct vertex-faithful simplicial surfaces, we can define a simplicial surface X by constructing a subset of the vertices of faces $\Theta(X)$ and then let the remaining vertices of faces be computed by a suitable group action on X_0 . For example, the group

$$G := \langle (v_2, v_3, v_4, v_5) \rangle \leq \text{Sym}(\{v_1, \dots, v_6\})$$

yields the vertices of faces of the simplicial octahedron $\mathcal{O}$ via

$$\Theta(\mathcal{O}) = \text{orb}_G(\{v_1, v_2, v_3\}) \cup \text{orb}_G(\{v_2, v_3, v_6\})$$

$$= \{\{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}, \{v_1, v_4, v_5\}, \{v_1, v_2, v_5\}, \\ \{v_2, v_3, v_6\}, \{v_3, v_4, v_6\}, \{v_4, v_5, v_6\}, \{v_2, v_5, v_6\}\},$$

where $g \in G$ acts on a subset $\{v, v', v''\}$ via $g(\{v, v', v''\}) = \{g(v), g(v'), g(v'')\}$.

Remark 7.4.2 For simplicity, we identify a face F of a vertex-faithful simplicial surface X with its set of incident vertices, i.e. we write $F = \{v_1, v_2, v_3\}$, if $v_1, v_2, v_3 \in X_0$ are the three vertices that are incident to F .

7.4.1 Construction of cyclic families

In this subsection, we give a detailed construction of the families $(X^{(n,k)})_{k \in \mathbb{N}_0}$ for $n > 2$. We define these simplicial surfaces by computing the corresponding vertices of faces. Furthermore, we present the desired $(1, 1, 1)$ -embeddings of these simplicial surfaces and prove that the arising automorphism and symmetry groups are all cyclic. Note that we are inspired by the quadratic form presented in Remark 7.2.1 and therefore denote the faces of $X^{(n,k)}$ by $F_{a_i}, F_{b_i}, F_{c_i}, F_{d_i}, F_{e_i}, F_{f_i}$ and $F_{i,j}, \overline{F_{i,j}}$ for suitable i, j .

Remark 7.4.3 Let $n > 2$ be a natural number and $k \in \mathbb{N}_0$. We define the vertex-faithful simplicial surface $X^{(n,k)}$ as follows: Let the vertices of $X^{(n,k)}$ be given by $v_1, \dots, v_{(k+3)n+2}$. Here, we make use of the cyclic group

$$G := \langle \underbrace{(v_1, \dots, v_n) \cdots (v_{(k+2)n+1}, \dots, v_{(k+3)n})}_{g:=} \rangle$$

to compute the vertices of faces $\Theta(X^{(n,k)})$. In particular, we define $\Theta(X^{(n,k)})$ by

$$\Theta(X^{(n,k)}) := \{F_{a_i}, F_{b_i}, F_{c_i}, F_{d_i}, F_{e_i}, F_{f_i} \mid i = 1, \dots, n\} \cup \{F_{i,j}, \overline{F_{i,j}} \mid i = 1, \dots, n, j = 1, \dots, k\},$$

where the different faces are specified below. First, we define the vertices that are incident to the faces $F_{a_1}, F_{b_1}, F_{c_1}, F_{d_1}, F_{e_1}, F_{f_1}$ by

$$\begin{aligned} F_{a_1} &:= \{v_{(k+1)n+1}, v_{(k+2)n+1}, v_{(k+3)n+2}\}, \\ F_{b_1} &:= \{v_{kn+1}, v_{(k+2)n+1}, v_{(k+3)n+2}\}, \\ F_{c_1} &:= \{v_{kn+1}, v_{kn+2}, v_{(k+1)n+1}\}, \\ F_{d_1} &:= \{v_1, v_2, v_{(k+3)n+1}\}, \\ F_{e_1} &:= \{v_{kn+2}, v_{(k+1)n+1}, v_{(k+3)n+2}\}, \\ F_{f_1} &:= \{v_{kn+1}, v_{(k+1)n+1}, v_{(k+2)n+1}\}. \end{aligned}$$

If $k > 0$, then the vertices of faces of $F_{1,j}$ and $\overline{F_{1,j}}$ for $0 < j \leq k$ are defined by

$$F_{1,j} := \{v_{(j-1)n+1}, v_{(j-1)n+2}, v_{jn+1}\} \text{ and } \overline{F_{1,j}} := \{v_{(j-1)n+2}, v_{jn+1}, v_{jn+2}\}.$$

This allows us to construct the remaining vertices of faces by the action of G on the above faces. In particular, the vertices of the faces $F_{a_{i+1}}, \dots, F_{f_{i+1}}$ for $i = 1, \dots, n-1$ are given by $g^i(F_{a_1}), \dots, g^i(F_{f_1})$, respectively. In the case that $k > 0$, we define the vertices of the faces $F_{i+1,j}$ and $\overline{F_{i+1,j}}$ for $i = 1, \dots, n-1$ and $j = 1, \dots, k$ by $g^i(F_{1,j})$ and $g^i(\overline{F_{1,j}})$, respectively.

For $n > 2$ and $k \in \mathbb{N}_0$ the simplicial surface $X^{(n,k)}$ has Euler characteristic 2 and hence forms a simplicial sphere. Furthermore, the simplicial surface $X^{(n,0)}$ is isomorphic to the simplicial surface constructed in the proof of Theorem 7.3.1. For example, for $n = 4$ and $k = 0$, we obtain the simplicial surface $X^{(4,0)}$ consisting of 14 vertices, 36 edges and 24 faces that is described by the following vertices of faces:

$$\begin{aligned} \Theta(X^{(4,0)}) = \{ & \{v_1, v_2, v_6\}, \{v_1, v_2, v_{13}\}, \{v_1, v_4, v_6\}, \{v_1, v_4, v_7\}, \{v_1, v_5, v_8\}, \{v_1, v_5, v_{13}\}, \\ & \{v_1, v_7, v_8\}, \{v_2, v_3, v_6\}, \{v_2, v_3, v_{11}\}, \{v_2, v_5, v_{11}\}, \{v_2, v_5, v_{14}\}, \{v_2, v_{13}, v_{14}\}, \\ & \{v_3, v_4, v_6\}, \{v_3, v_4, v_9\}, \{v_3, v_5, v_9\}, \{v_3, v_5, v_{12}\}, \{v_3, v_{11}, v_{12}\}, \{v_4, v_5, v_7\}, \\ & \{v_4, v_5, v_{10}\}, \{v_4, v_9, v_{10}\}, \{v_5, v_7, v_8\}, \{v_5, v_9, v_{10}\}, \{v_5, v_{11}, v_{12}\}, \{v_5, v_{13}, v_{14}\} \}. \end{aligned}$$

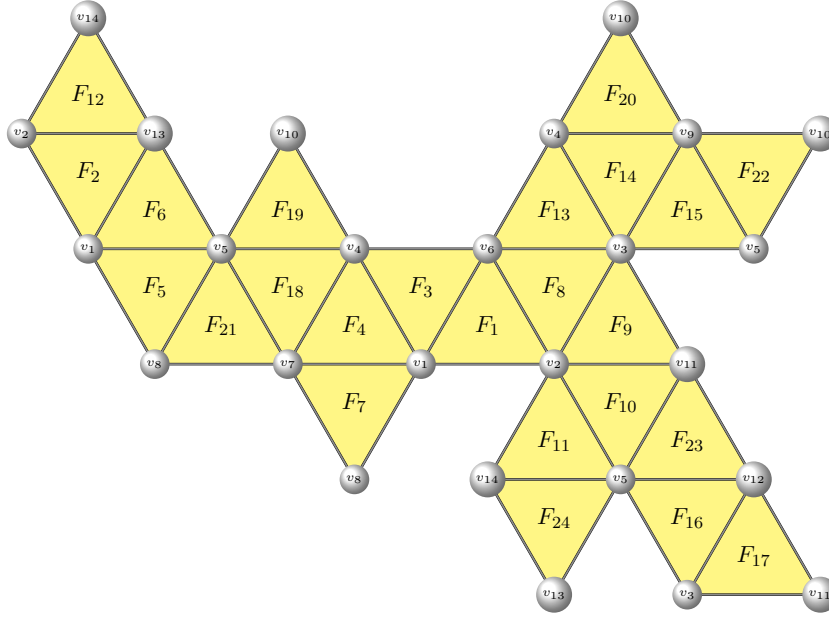


Figure 7.15: The simplicial surface $X^{(4,0)}$

Next, we seek to construct $(1,1,1)$ -embeddings of the simplicial surfaces constructed in Remark 7.4.3. To achieve this, we assign 3D-coordinates to the vertices of $X^{(n,k)}$ as follows.

Remark 7.4.4 Let $n > 2$ be a natural number and $k \in \mathbb{N}_0$. Further, let $l \in \mathbb{N}$ be a natural number satisfying $\gcd(n, l) = 1$ and $\cos(\alpha) \leq \frac{1}{2}$, where $\alpha := \frac{2\pi l}{n}$. Thus, we define scalars ρ and h as

$$\begin{aligned} \rho &:= \left(\left\| \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} - \begin{pmatrix} \cos(2\alpha) \\ \sin(2\alpha) \end{pmatrix} \right\| \right)^{-1} = \frac{1}{\sqrt{2(1 - \cos(\alpha))}}, \\ h &:= \sqrt{1 - \rho \left\| \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} - \begin{pmatrix} \cos(\frac{\alpha}{2}) \\ \sin(\frac{\alpha}{2}) \end{pmatrix} \right\|^2} = \sqrt{1 - 2\rho(1 - \cos(\frac{\alpha}{2}))}. \end{aligned}$$

Inspired by the construction of the vertices of faces $\Theta(X^{(n,k)})$ in Remark 7.4.3, we define the 3D-coordinates of some of the vertices of $X^{(n,k)}$ and obtain the remaining 3D-coordinates by introducing suitable group actions. Hence, a $(1,1,1)$ -embedding ϕ of X is given by

$$\phi(v_{jn+1}) := -jh \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t + \rho \begin{pmatrix} \cos(j\frac{\alpha}{2}) & \sin(j\frac{\alpha}{2}) & 0 \end{pmatrix}^t,$$

$$\phi(v_{jn+2}) := -jh \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t + \rho \begin{pmatrix} \cos((j+1)\frac{\alpha}{2}) & \sin((j+1)\frac{\alpha}{2}) & 0 \end{pmatrix}^t,$$

for $j = 0, \dots, k$ and

$$\phi(v_{(k+3)n+1}) := \sqrt{1-\rho^2} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t,$$

$$\phi(v_{(k+3)n+2}) := -(kh + \sqrt{1-\rho^2}) \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t,$$

$$\begin{aligned} \phi(v_{(k+1)n+1}) &:= \frac{1}{3}(\phi(v_{kn+1}) + \phi(v_{kn+2}) + \phi(v_{(k+3)n+2})) \\ &\quad + \frac{(\phi(v_{(k+3)n+2}) - \phi(v_{kn+1})) \times (\phi(v_{kn+2}) - \phi(v_{(k+3)n+2}))}{\|(\phi(v_{(k+3)n+2}) - \phi(v_{kn+1})) \times (\phi(v_{kn+2}) - \phi(v_{(k+3)n+2}))\|}, \end{aligned}$$

$$\begin{aligned} \phi(v_{(k+2)n+1}) &:= \frac{1}{3}(\phi(v_{kn+1}) + \phi(v_{(k+1)n+1}) + \phi(v_{(k+3)n+2})) \\ &\quad + \frac{(\phi(v_{kn+1}) - \phi(v_{(k+3)n+2})) \times (\phi(v_{kn+1}) - \phi(v_{(k+1)n+1}))}{\|(\phi(v_{kn+1}) - \phi(v_{(k+3)n+2})) \times (\phi(v_{kn+1}) - \phi(v_{(k+1)n+1}))\|}. \end{aligned}$$

For every $i = 1, \dots, n-1$ and every $j = 0, \dots, k$ we set that ϕ maps the vertex $v_{jn+(i+1)}$ of $X^{(n,k)}$ onto

$$\phi(v_{jn+(i+1)}) = \phi(g^i(v_{jn+1})) := (M_\alpha)^i \phi(v_{jn+1}),$$

with $g := (v_1, \dots, v_n) \cdots (v_{(k+2)n+1}, \dots, v_{(k+3)n})$ and M_α defined as

$$M_\alpha := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

We observe that the edges that are incident to the faces $F_{a_1}, \dots, F_{f_1}, F_{1,j}, \overline{F_{1,j}}$ all have length 1 by construction. Since the Euclidean norm is invariant under the multiplication with orthogonal matrices, ϕ is indeed a $(1,1,1)$ -embedding of $X^{(n,k)}$. For simplicity, we denote this embedding of $X^{(n,k)}$ by $\phi_l^{(n,k)}$.

Figure 7.16 shows the polyhedra that arise from the $(1,1,1)$ -embeddings $\phi_1^{(4,0)}, \phi_1^{(5,0)}$ and $\phi_2^{(7,0)}$ of the simplicial surfaces $X^{(4,0)}, X^{(5,0)}$ and $X^{(7,0)}$, respectively.

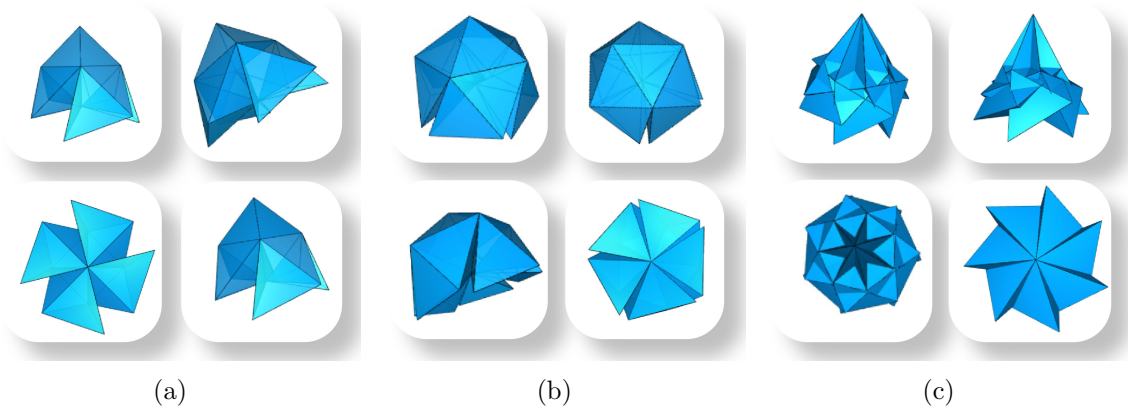


Figure 7.16: Polyhedra that result from embedding the simplicial surfaces $X^{(4,0)}$ (a), $X^{(5,0)}$ (b) and $X^{(7,0)}$ (c) into $\mathbb{R}^3$ with corresponding $(1,1,1)$ -embeddings

For $n > 2$ and $k \in \mathbb{N}_0$ the above procedure yields

$$|\{l \mid 1 \leq l \leq \frac{n}{2}, \gcd(n, l) = 1, \cos(\frac{2\pi l}{n}) \leq \frac{1}{2}\}|$$

different polyhedra corresponding to $X^{(n,k)}$ that cannot be transformed into each other by using rigid Euclidean motions only. In Figure 7.17, the construction of the family $(X^{(4,k)})_{k \in \mathbb{N}_0}$ is illustrated by showing the polyhedra that result from the $(1, 1, 1)$ -embeddings $\phi_1^{(4,1)}$, $\phi_1^{(4,2)}$ and $\phi_1^{(4,3)}$ of the simplicial surfaces $X^{(4,1)}$, $X^{(4,2)}$ and $X^{(4,3)}$, respectively.

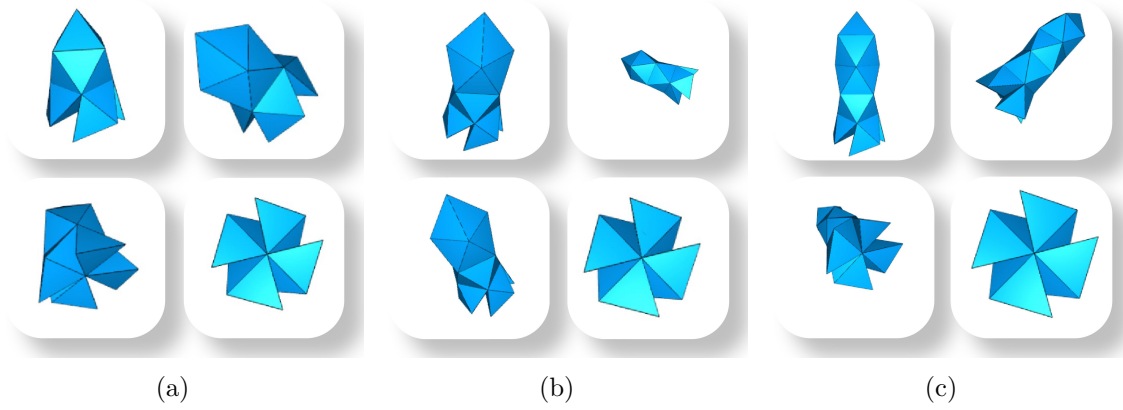


Figure 7.17: Polyhedra that result from embedding the simplicial surfaces $X^{(4,1)}$ (a), $X^{(4,2)}$ (b) and $X^{(4,3)}$ (c) into $\mathbb{R}^3$ with corresponding $(1, 1, 1)$ -embeddings

Next, we analyse the automorphism groups of the simplicial surfaces contained in $(X^{(n,k)})_{k \in \mathbb{N}_0}$.

Proposition 7.4.5

Let $n > 2$ be a natural number and $k \in \mathbb{N}_0$. Further, let $l \in \mathbb{N}$ be a natural number with $\gcd(n, l) = 1$ and $\cos(\frac{2\pi l}{n}) \leq \frac{1}{2}$. Then the $(1, 1, 1)$ -embedding $\phi_l^{(n,k)}$ from Remark 7.4.4 satisfies

$$\text{Aut}(\phi_l^{(n,k)}(X^{(n,k)})) \cong \text{Aut}(X^{(n,k)}) \cong C_n.$$

Proof. By construction $\langle M_\alpha \rangle$, with M_α given as in (2) in Remark 7.4.4, is a subgroup of the symmetry group $\text{Aut}(\phi_l^{(n,k)}(X^{(n,k)}))$. Since $\langle M_\alpha \rangle \cong C_n$, it suffices to show $\text{Aut}(X^{(n,k)}) \cong C_n$. Thus, let ψ be an automorphism of $X^{(n,k)}$. We observe that $v_{(k+2)n+1}, \dots, v_{(k+3)n}$ are exactly the vertices of degree 3 in $X^{(n,k)}$ and that $v_{(k+3)n+2}$ is the only with vertex-degree $3n$. Since ψ has to map vertices of $X^{(n,k)}$ onto vertices with the same vertex-degree, we conclude

$$\psi(v_{(k+2)n+1}) = v_{(k+2)n+i}$$

for some $i \in \{1, \dots, n\}$ and

$$\psi(v_{(k+3)n+1}) = v_{(k+3)n+1}.$$

Since $F = \{v_{(k+1)n+1}, v_{(k+2)n+1}, v_{(k+3)n+1}\}$ is a face of $X^{(n,k)}$, the same holds for its image under ψ , namely

$$F' := \psi(F) = \{\psi(v_{(k+1)n+1}), v_{(k+2)n+i}, v_{(k+3)n+1}\}.$$

Hence, $\psi(v_{(k+1)n+1})$ has to be a vertex with $\deg(\psi(v_{(k+1)n+1})) = \deg(v_{(k+1)n+1}) = 4$ such that F' forms a face of $X^{(n,k)}$. This leads to $\psi(v_{(k+1)n+1}) = v_{(k+1)n+i}$. Thus, the incidence structure of $X^{(n,k)}$ enforces that the restriction of ψ to the vertices of $X^{(n,k)}$ is given by

$$((v_1, \dots, v_n) \cdots (v_{(k+2)n+1}, \dots, v_{(k+3)n}))^{i-1}.$$

and by Remark 3.1.3 the automorphism group $\text{Aut}(X^{(n,k)})$ is cyclic. Thus, we obtain

$$C_n \cong \langle M_\alpha \rangle \leq \text{Aut}(\phi_l^{(n,k)}(X^{(n,k)})) \hookrightarrow \text{Aut}(X^{(n,k)}) \cong C_n.$$

□

7.4.2 Construction of dihedral families

Next, we present the construction of the simplicial surface $Y^{(n,k)}$ for $n > 3$ and $k \in \mathbb{N}_0$. We pursue a similar approach as presented in Section 7.4.1 to compute the incidence structure of the desired simplicial surfaces and the corresponding $(1, 1, 1)$ -embeddings. In Proposition 7.4.8 we prove that for $n > 3$ and $k \in \mathbb{N}_0$ the simplicial surface $Y^{(n,k)}$ and the polyhedra arising from the constructed $(1, 1, 1)$ -embeddings both have automorphism and symmetry groups that are isomorphic to the dihedral group of order $2n$.

First, we introduce the incidence structure of $Y^{(n,k)}$. Again, we are inspired by the quadratic form and the corresponding cubic graph established in Remark 7.2.5 and denote the faces of the desired simplicial surface $Y^{(n,k)}$ by $F_{a_i}, F_{b_i}, F_{c_i}, F_{d_i}$ and $F_{i,j}, \overline{F_{i,j}}$ for suitable i, j .

Remark 7.4.6 *Let $n > 3$ be a natural number and $k \in \mathbb{N}_0$. We follow the approach in Remark 7.4.3 and construct the vertex-faithful simplicial surface $Y^{(n,k)}$ as follows: Let the vertices of $Y^{(n,k)}$ be given by $v_1, \dots, v_{(k+2)n+2}$. We define $Y^{(n,k)}$ by its vertices of faces:*

$$\Theta(Y^{(n,k)}) := \{F_{a_i}, F_{b_i}, F_{c_i}, F_{d_i} \mid i = 1, \dots, n\} \cup \{F_{i,j}, \overline{F_{i,j}} \mid i = 1, \dots, n, j = 1, \dots, k\},$$

where the different faces are specified below. First, let the faces of $F_{a_1}, \dots, F_{d_1}$ be defined by

$$\begin{aligned} F_{a_1} &:= \{v_{kn+1}, v_{(k+1)n+1}, v_{(k+2)n+2}\}, \\ F_{b_1} &:= \{v_{kn+2}, v_{(k+1)n+1}, v_{(k+2)n+2}\}, \\ F_{c_1} &:= \{v_{kn+1}, v_{kn+2}, v_{(k+1)n+1}\}, \\ F_{d_1} &:= \{v_1, v_2, v_{(k+2)n+1}\}. \end{aligned}$$

If $k > 0$, then we define the vertices of the faces $F_{1,j}$ and $\overline{F_{1,j}}$ for $0 < j \leq k$ by

$$F_{1,j} := \{v_{(j-1)n+1}, v_{(j-1)n+2}, v_{jn+1}\} \text{ and } \overline{F_{1,j}} := \{v_{(j-1)n+2}, v_{jn+1}, v_{jn+2}\}.$$

Here, the permutation $h := (v_1, \dots, v_n) \cdots (v_{(k+1)n+1}, \dots, v_{(k+2)n})$ gives rise to the vertices of the remaining faces of $Y^{(n,k)}$. More precisely, for $i = 1, \dots, n-1$ the vertices of the faces $F_{a_{i+1}}, \dots, F_{d_{i+1}}$, are given by $h^i(f_{a_1}), \dots, h^i(f_{d_1})$, respectively. If $k > 0$, then we define the vertices of the faces $F_{i+1,j}$ and $\overline{F_{i+1,j}}$ for $i = 1, \dots, n-1$ and $j = 1, \dots, k$ by $h^i(F_{1,j})$ and $h^i(\overline{F_{1,j}})$, respectively.

For $n > 3$ and $k \in \mathbb{N}_0$ the simplicial surface $Y^{(n,k)}$ has Euler characteristic 2 and hence forms a simplicial sphere. Furthermore, the simplicial surface $Y^{(n,0)}$ is isomorphic to the simplicial surface constructed in the proof of Theorem 7.3.3. For $n = 4$ and $k = 0$, we obtain the simplicial surface

$X^{(4,0)}$ consisting of 10 vertices 24 edges and 16 faces that is described by the following vertices of faces:

$$\begin{aligned}\Theta(Y^{(4,0)}) = & \{ \{v_1, v_5, v_{10}\}, \{v_2, v_6, v_{10}\}, \{v_3, v_7, v_{10}\}, \{v_4, v_8, v_{10}\}, \\ & \{v_2, v_5, v_{10}\}, \{v_3, v_6, v_{10}\}, \{v_4, v_7, v_{10}\}, \{v_1, v_8, v_{10}\}, \\ & \{v_1, v_2, v_5\}, \{v_2, v_3, v_6\}, \{v_3, v_4, v_7\}, \{v_1, v_4, v_8\}, \\ & \{v_1, v_2, v_9\}, \{v_2, v_3, v_9\}, \{v_3, v_4, v_9\}, \{v_1, v_4, v_9\} \}.\end{aligned}$$

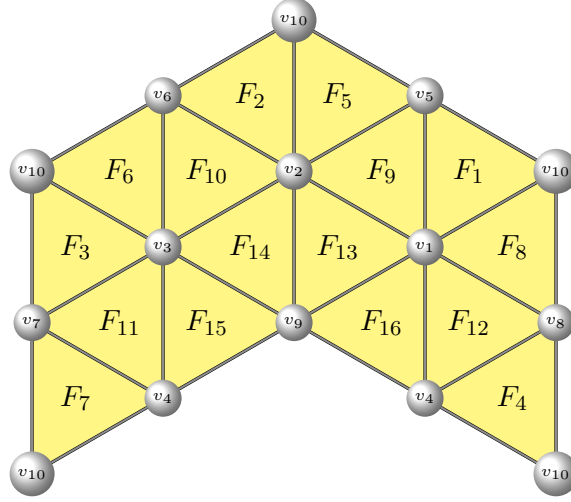


Figure 7.18: The simplicial surface $Y^{(4,0)}$

We proceed by presenting the desired $(1, 1, 1)$ -embedding of the simplicial surface $Y^{(n,k)}$.

Remark 7.4.7 Let $n > 3$ be a natural number and $k \in \mathbb{N}_0$. Further, let $l \in \mathbb{N}$ be a natural number satisfying $\gcd(n, l) = 1$ and $\cos(\alpha) \leq \frac{1}{2}$, where $\alpha := \frac{2\pi l}{n}$. We define the scalars ρ and h as in Remark 7.4.4. Hence, a $(1, 1, 1)$ -embedding ψ of the simplicial surface $Y^{(n,k)}$ is given as follows: For $0 \leq j \leq k$ we define the following images:

$$\begin{aligned}\psi(v_{jn+1}) &:= -jh \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t + \rho \begin{pmatrix} \cos(j\frac{\alpha}{2}) & \sin(j\frac{\alpha}{2}) & 0 \end{pmatrix}^t, \\ \psi(v_{jn+2}) &:= -jh \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t + \rho \begin{pmatrix} \cos((j+1)\frac{\alpha}{2}) & \sin((j+1)\frac{\alpha}{2}) & 0 \end{pmatrix}^t.\end{aligned}$$

Furthermore, we set

$$\begin{aligned}\psi(v_{(k+2)n+1}) &:= \sqrt{1-\rho^2} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t, \\ \psi(v_{(k+2)n+2}) &:= -(kh + \sqrt{1-\rho^2}) \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t, \\ \psi(v_{(k+1)n+1}) &:= \frac{1}{3}(\psi(v_{kn+1}) + \psi(v_{kn+2}) + \psi(v_{(k+2)n+2})) \\ &\quad + \frac{(\psi(v_{(k+2)n+2}) - \psi(v_{kn+1})) \times (\psi(v_{kn+2}) - \psi(v_{(k+2)n+2}))}{\|(\psi(v_{(k+2)n+2}) - \psi(v_{kn+1})) \times (\psi(v_{kn+2}) - \psi(v_{(k+2)n+2}))\|}.\end{aligned}$$

Hence, the images of the remaining vertices can be defined by

$$\psi(v_{jn+(i+1)}) = \psi(g^i(v_{jn+1})) := M_\alpha^i \psi(v_{in+1})$$

with $1 \leq i \leq n-1$ and $0 \leq j \leq k$, where M_α is given as in (2). Again, we see that the edges that are incident to the faces $F_{a_1}, F_{b_1}, F_{c_1}, F_{d_1}, F_{1,j}, \overline{F_{1,j}}$ all have length 1 by construction. Since the Euclidean norm is invariant under the multiplication with orthogonal matrices, ψ is indeed a $(1, 1, 1)$ -embedding of $Y^{(n,k)}$. For simplicity, we denote the $(1, 1, 1)$ -embedding ψ of the simplicial surface $Y^{(n,k)}$ constructed above by $\psi_l^{(n,k)}$.

Figure 7.19 illustrates the polyhedra that arise from the $(1, 1, 1)$ -embeddings $\psi_1^{(4,0)}, \psi_1^{(5,0)}$ and $\psi_2^{(7,0)}$ of the simplicial surfaces $X^{(4,0)}, X^{(5,0)}$ and $X^{(7,0)}$, respectively.

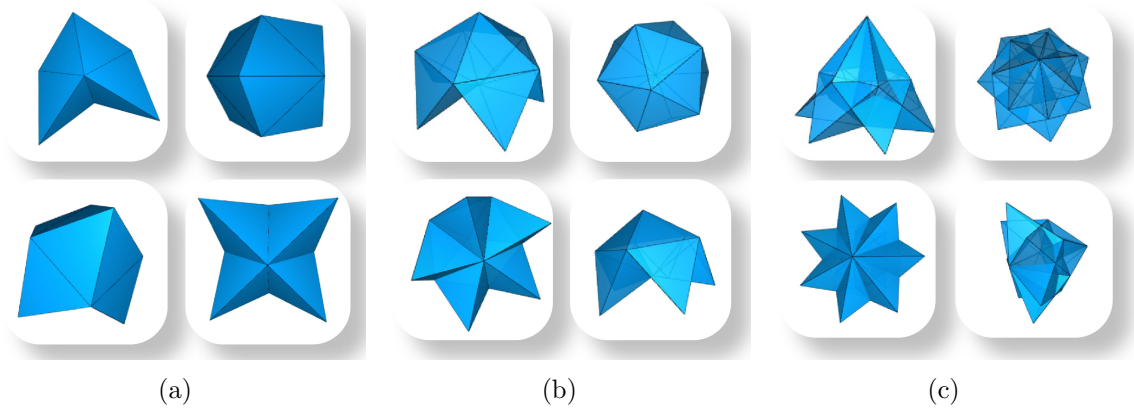


Figure 7.19: Polyhedra that result from embedding the simplicial surfaces $Y^{(4,0)}$ (a), $Y^{(5,0)}$ (b) and $Y^{(7,0)}$ (c) into $\mathbb{R}^3$ with corresponding $(1, 1, 1)$ -embeddings

Note, as presented in the cyclic case the above construction gives rise to

$$|\{l \mid 1 \leq l \leq \frac{n}{2}, \gcd(n, l) = 1, \cos(\frac{2\pi l}{n}) \leq \frac{1}{2}\}|$$

different polyhedra corresponding to $Y^{(n,k)}$ that can not be transformed into each other by using rigid Euclidean motions. In Figure 7.20, we illustrate the construction of the family $(Y^{(4,k)})_{k \in \mathbb{N}_0}$ by presenting the polyhedra that result from the $(1, 1, 1)$ -embeddings $\psi_1^{(4,1)}, \psi_1^{(4,2)}$ and $\psi_1^{(4,3)}$ of the simplicial surfaces $Y^{(4,1)}, Y^{(4,2)}$ and $Y^{(4,3)}$, respectively.

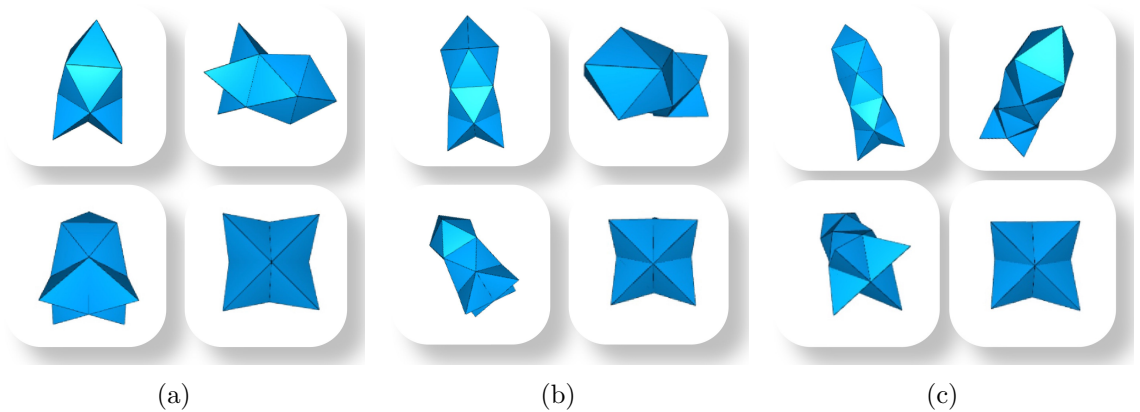


Figure 7.20: Polyhedra that result from embedding the simplicial surfaces $Y^{(4,1)}$ (a), $Y^{(4,2)}$ (b) and $Y^{(4,3)}$ (c) into $\mathbb{R}^3$ with corresponding $(1, 1, 1)$ -embeddings

Lastly, we show that the simplicial surfaces constructed in Remark 7.4.6 and the polyhedra arising from their above $(1, 1, 1)$ -embeddings all have dihedral symmetry.

Proposition 7.4.8

Let $n > 3$ be a natural number and $k \in \mathbb{N}_0$. Further, let $l \in \mathbb{N}$ be a natural number with $\gcd(n, l) = 1$ and $\cos(\frac{2\pi l}{n}) \leq \frac{1}{2}$. Then the $(1, 1, 1)$ -embedding $\psi_l^{(n,k)}$ from Remark 7.4.7 satisfies

$$\text{Aut}(\psi_l^{(n,k)}(Y^{(n,k)})) \cong \text{Aut}(Y^{(n,k)}) \cong D_n.$$

Proof. Let α and M_α be defined as in (2) in Remark 7.4.4. Further, let S be defined as

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We observe that M_α and S define automorphisms of the polyhedron $\psi_l^{(n,k)}(Y^{(n,k)})$ satisfying $M_\alpha^n = S^2 = I_3$. Since $\langle M_\alpha, S \rangle \cong D_n$ is a subgroup of $\text{Aut}(\psi_l^{(n,k)}(Y^{(n,k)}))$, it remains to show that $\text{Aut}(Y^{(n,k)}) \cong D_n$. Let therefore $\phi \in \text{Aut}(Y^{(n,k)})$ be an automorphism. By our construction in Remark 7.4.6, $v_{(k+1)n+1}, \dots, v_{(k+2)n}$ are exactly the vertices of degree 3 in $Y^{(n,k)}$. Since ϕ has to map vertices onto vertices with the same degree, ϕ permutes the vertices of degree 3. By examining the incidence structure of $Y^{(n,k)}$, we see that there exists a $1 \leq i \leq n$ with $\phi(v_{(k+1)n+1}) = v_{(k+1)n+i}$. Furthermore, the image of the face $F = \{v_{kn+1}, v_{kn+2}, v_{(k+1)n+1}\}$ under ϕ , namely $F' := \phi(F) = \{\phi(v_{kn+1}), \phi(v_{kn+2}), v_{(k+1)n+i}\}$, must again be a face of the simplicial surface $Y^{(n,k)}$. Examining the vertex-degrees of the simplicial surface $Y^{(n,k)}$ leads to the following two cases: (1) $\phi(v_{kn+1}) = v_{kn+i}$ or (2) $\phi(v_{kn+1}) = v_{kn+i+1}$.

Case (1): If $\phi(v_{kn+1}) = v_{kn+i}$, then $\phi(v_{kn+2}) = v_{kn+i+1}$ follows. Thus, the restriction of ϕ to the vertices is given by

$$((v_1, \dots, v_n) \cdots (v_{(k+1)n+1}, \dots, v_{(k+2)n}))^{i-1}.$$

Case (2): If $\phi(v_{kn+1}) = v_{kn+i+1}$, then we know that $\phi(v_{kn+2}) = v_{kn+i}$. Hence, the restriction of ϕ to the vertices of the simplicial surface is given by

$$((v_1, \dots, v_n) \cdots (v_{(k+1)n+1}, \dots, v_{(k+2)n}))^{i-1} \cdot \phi_S,$$

whereby ϕ_S is a suitable involution that arises from S by embedding $\text{Aut}(\psi_l^{(n,k)}(Y^{(n,k)}))$ into $\text{Aut}(Y^{(n,k)})$. Thus, by Remark 3.1.4 we obtain

$$D_n \cong \langle S, M_\alpha \rangle \leq \text{Aut}(\psi_l^{(n,k)}(Y^{(n,k)})) \hookrightarrow \text{Aut}(Y^{(n,k)}) \cong D_n.$$

□

8 Appendix

8.1 Data: Face-transitive surfaces with respect to vertex-face types and Euler characteristics

This section contains some information about the census of face-transitive surfaces contained in [6]. We refer the reader to Section 5.4 for a description of how to read the below tables.

8.1.1 Data of orientable face-transitive surfaces with at most 1280 faces

$\chi(X)$	2	0	-2	-4	-6	-8	-10	-12
(3,1)	2	14	2	6	5	8	6	6
(2,2)	642	45	1	6	6	9	5	4
(2,1).1	0	164	0	0	0	0	0	2
(2,1).2	0	0	0	0	0	0	0	1
(2,1).3	0	565	0	0	1	1	2	3
(1,1).1	0	54013	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	2393	0	0	0	1	2	0
(1,3).1	0	175	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	3	36	0	2	0	1	1	1

Table 5: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-14	-16	-18	-20	-22	-24	-26	-28
(3,1)	4	15	18	5	4	21	3	13
(2,2)	4	22	19	5	1	8	6	9
(2,1).1	1	0	4	3	4	0	1	1
(2,1).2	0	0	1	0	0	0	0	0
(2,1).3	0	7	8	4	0	18	0	3
(1,1).1	0	0	1	2	0	2	0	0
(1,1).2	0	0	1	0	0	0	0	0
(1,1).3	0	0	1	0	0	0	2	0
(1,1).4	0	0	0	0	0	0	0	1
(1,2)	0	2	2	2	0	1	0	0
(1,3).1	0	0	1	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	2	0	2	0	0	2	3	2

Table 6: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-30	-32	-34	-36	-38	-40	-42	-44
(3,1)	11	26	4	23	4	25	7	6
(2,2)	8	29	1	19	0	21	8	1
(2,1).1	1	4	2	3	1	6	6	0
(2,1).2	2	1	0	3	0	3	0	0
(2,1).3	6	25	0	14	0	15	3	2
(1,1).1	2	6	0	4	0	1	0	0
(1,1).2	1	4	0	1	0	0	0	0
(1,1).3	1	0	2	1	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	3	3	0	1	0	3	1	0
(1,3).1	0	1	0	0	0	0	1	0
(1,3).2	0	0	0	1	0	0	0	0
(1,6)	1	1	0	0	0	1	2	0

Table 7: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-46	-48	-50	-52	-54	-56	-58	-60
(3,1)	9	46	5	7	28	32	6	14
(2,2)	1	24	6	3	23	12	1	11
(2,1).1	0	3	1	6	4	2	0	3
(2,1).2	0	0	3	0	0	0	0	0
(2,1).3	0	67	3	2	25	13	0	20
(1,1).1	0	4	4	0	4	4	0	12
(1,1).2	0	0	3	0	1	0	0	6
(1,1).3	0	1	2	0	0	4	0	0
(1,1).4	0	0	0	0	0	1	0	0
(1,2)	0	6	0	0	4	4	0	6
(1,3).1	0	0	0	0	2	0	0	0
(1,3).2	0	0	1	0	0	0	0	0
(1,6)	0	2	2	0	3	0	0	0

Table 8: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-62	-64	-66	-68	-70	-72	-74	-76
(3,1)	3	65	5	5	16	51	3	7
(2,2)	0	60	3	2	15	34	0	3
(2,1).1	5	8	4	5	0	2	1	1
(2,1).2	0	2	0	1	1	2	0	0
(2,1).3	0	83	4	2	5	82	0	2
(1,1).1	0	14	1	0	0	18	0	0
(1,1).2	0	29	0	0	0	10	0	0
(1,1).3	0	1	0	2	0	3	2	0
(1,1).4	0	0	0	0	0	1	0	0
(1,2)	0	15	1	0	0	11	0	0
(1,3).1	0	1	0	0	0	1	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	3	0	0	1	2	0	0

Table 9: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-78	-80	-82	-84	-86	-88	-90	-92
(3,1)	15	58	5	15	4	37	36	7
(2,2)	7	44	1	12	0	12	24	1
(2,1).1	6	21	6	6	1	2	7	0
(2,1).2	1	7	0	0	0	1	2	0
(2,1).3	3	50	0	15	0	11	21	2
(1,1).1	1	7	0	7	0	0	3	0
(1,1).2	0	2	0	0	0	0	0	0
(1,1).3	0	1	2	2	0	0	0	0
(1,1).4	0	0	0	2	0	0	0	0
(1,2)	1	5	0	4	0	0	5	0
(1,3).1	1	0	0	1	0	0	1	0
(1,3).2	0	1	0	0	0	0	0	0
(1,6)	0	2	0	1	0	3	2	0

Table 10: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-94	-96	-98	-100	-102	-104	-106	-108
(3,1)	6	107	6	17	7	27	5	45
(2,2)	1	62	5	13	1	6	1	33
(2,1).1	0	8	7	15	5	8	0	7
(2,1).2	0	5	1	2	0	0	0	3
(2,1).3	0	204	6	11	5	7	0	50
(1,1).1	0	33	5	3	2	0	0	25
(1,1).2	0	32	5	4	0	0	0	10
(1,1).3	0	5	1	2	4	6	0	3
(1,1).4	0	3	0	0	0	0	0	1
(1,2)	0	26	2	0	1	0	0	11
(1,3).1	0	0	0	1	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	1
(1,6)	0	3	1	0	2	0	0	1

Table 11: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-110	-112	-114	-116	-118	-120	-122	-124
(3,1)	12	78	7	9	7	38	4	14
(2,2)	3	28	2	1	1	18	0	8
(2,1).1	6	2	10	9	0	10	5	0
(2,1).2	2	6	0	0	0	0	0	1
(2,1).3	1	41	3	2	0	109	0	2
(1,1).1	0	11	2	0	0	32	0	0
(1,1).2	0	0	0	0	0	20	0	0
(1,1).3	0	8	0	0	0	0	2	0
(1,1).4	0	3	0	0	0	2	0	0
(1,2)	0	6	1	0	0	21	0	0
(1,3).1	0	0	1	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	2	0	0	0	0	0	0	0

Table 12: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-126	-128	-130	-132	-134	-136	-138	-140
(3,1)	43	145	9	19	5	31	8	5
(2,2)	19	87	5	5	0	12	4	11
(2,1).1	15	19	0	0	1	6	1	4
(2,1).2	4	13	0	0	0	1	0	0
(2,1).3	24	271	0	17	0	12	4	17
(1,1).1	7	76	0	5	0	0	3	0
(1,1).2	0	175	0	0	0	0	0	0
(1,1).3	0	6	0	0	0	2	0	2
(1,1).4	0	3	0	0	0	0	0	2
(1,2)	3	44	0	1	0	0	1	0
(1,3).1	1	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	3	1	0	0	3	0	0

Table 13: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-142	-144	-146	-148	-150	-152	-154	-156
(3,1)	8	166	4	16	31	30	14	25
(2,2)	1	73	0	5	15	9	6	3
(2,1).1	4	10	3	5	0	2	4	9
(2,1).2	0	9	0	0	3	1	4	0
(2,1).3	0	285	0	2	28	10	2	16
(1,1).1	0	61	0	0	9	0	0	6
(1,1).2	0	69	0	0	0	0	0	0
(1,1).3	0	15	4	0	1	4	0	2
(1,1).4	0	8	0	0	0	0	0	0
(1,2)	0	43	0	0	5	0	0	1
(1,3).1	0	1	0	0	0	0	0	0
(1,3).2	0	1	0	0	0	0	0	0
(1,6)	0	5	0	0	2	0	1	0

Table 14: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-158	-160	-162	-164	-166	-168	-170	-172
(3,1)	4	148	43	6	5	56	5	4
(2,2)	0	93	44	2	1	31	3	2
(2,1).1	1	36	13	3	0	7	0	6
(2,1).2	0	13	2	0	0	1	0	0
(2,1).3	0	187	57	0	0	61	0	0
(1,1).1	0	30	36	0	0	24	0	0
(1,1).2	0	19	10	0	0	7	0	0
(1,1).3	0	0	1	2	0	10	0	0
(1,1).4	0	1	1	0	0	2	0	0
(1,2)	0	13	20	0	0	18	0	0
(1,3).1	0	0	0	0	0	1	0	0
(1,3).2	0	1	0	0	0	0	0	0
(1,6)	0	2	5	0	0	1	0	0

Table 15: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-174	-176	-178	-180	-182	-184	-186	-188
(3,1)	9	135	5	47	3	33	11	4
(2,2)	1	21	1	20	1	8	3	1
(2,1).1	1	2	2	11	10	11	6	0
(2,1).2	1	2	0	3	0	1	0	0
(2,1).3	1	23	0	67	0	5	0	0
(1,1).1	4	4	0	51	6	0	4	0
(1,1).2	0	0	0	12	0	0	0	0
(1,1).3	0	8	2	3	0	0	0	0
(1,1).4	0	0	0	2	0	0	0	0
(1,2)	1	0	0	22	4	0	1	0
(1,3).1	0	0	0	0	0	0	1	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	2	0	1	2	0	0	0

Table 16: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-190	-192	-194	-196	-198	-200	-202	-204
(3,1)	18	188	2	8	31	21	3	18
(2,2)	4	93	0	11	30	19	1	5
(2,1).1	2	14	3	1	4	24	4	7
(2,1).2	2	9	0	2	0	2	0	1
(2,1).3	2	483	0	10	20	57	0	8
(1,1).1	0	148	0	7	7	23	0	8
(1,1).2	0	217	0	7	0	16	0	0
(1,1).3	0	13	4	4	0	6	0	8
(1,1).4	0	7	0	2	0	1	0	0
(1,2)	0	94	0	2	9	8	0	1
(1,3).1	0	1	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	3	3	0	1	0	0	0	0

Table 17: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-206	-208	-210	-212	-214	-216	-218	-220
(3,1)	3	120	1	4	6	100	3	6
(2,2)	0	35	2	3	1	40	0	3
(2,1).1	1	13	9	1	0	3	0	3
(2,1).2	0	8	0	0	0	3	0	0
(2,1).3	0	17	3	0	0	134	0	4
(1,1).1	0	5	6	0	0	92	0	1
(1,1).2	0	0	0	0	0	57	0	0
(1,1).3	0	10	0	0	0	8	0	0
(1,1).4	0	0	0	0	0	3	0	0
(1,2)	0	0	3	0	0	59	0	2
(1,3).1	0	0	0	0	0	1	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	4	0	0	0	4	0	0

Table 18: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-222	-224	-226	-228	-230	-232	-234	-236
(3,1)	8	182	3	9	6	33	11	2
(2,2)	3	67	1	5	2	10	20	1
(2,1).1	7	15	2	5	0	1	8	0
(2,1).2	1	17	0	0	1	1	3	1
(2,1).3	0	102	0	12	1	2	6	0
(1,1).1	5	49	0	9	0	0	7	0
(1,1).2	0	17	0	0	0	0	0	0
(1,1).3	0	16	2	2	0	0	0	0
(1,1).4	0	7	0	0	0	0	2	0
(1,2)	1	9	0	1	0	0	3	0
(1,3).1	1	1	0	0	0	0	1	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	2	0	0	0	0

Table 19: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-238	-240	-242	-244	-246	-248	-250	-252
(3,1)	11	105	0	6	2	23	13	25
(2,2)	7	24	4	2	2	6	12	20
(2,1).1	0	20	13	2	4	1	3	18
(2,1).2	2	2	0	0	0	0	3	6
(2,1).3	6	301	4	0	1	1	0	27
(1,1).1	0	98	5	0	6	0	2	48
(1,1).2	0	65	9	0	0	0	3	16
(1,1).3	0	4	1	0	4	4	3	14
(1,1).4	0	3	0	0	0	0	0	5
(1,2)	0	49	0	0	1	0	0	5
(1,3).1	0	0	0	0	0	0	0	1
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	1	2	0	0	0	0	0	0

Table 20: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-254	-256	-258	-260	-262	-264	-266	-268
(3,1)	2	341	4	6	1	55	1	7
(2,2)	0	126	1	5	1	13	1	2
(2,1).1	0	12	1	13	0	4	7	0
(2,1).2	0	35	0	2	0	1	0	0
(2,1).3	0	704	0	5	0	52	0	0
(1,1).1	0	302	6	0	0	11	0	0
(1,1).2	0	781	0	0	0	0	0	0
(1,1).3	0	18	0	2	0	4	0	0
(1,1).4	0	11	0	0	0	0	0	0
(1,2)	0	143	1	0	0	5	0	0
(1,3).1	0	2	1	0	0	0	0	0
(1,3).2	0	1	0	0	0	0	0	0
(1,6)	0	7	0	0	0	1	0	0

Table 21: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-270	-272	-274	-276	-278	-280	-282	-284
(3,1)	40	61	0	5	1	13	7	6
(2,2)	38	20	1	5	0	13	0	1
(2,1).1	10	12	2	0	0	14	1	6
(2,1).2	8	2	0	0	0	0	0	0
(2,1).3	26	14	0	2	0	43	1	0
(1,1).1	26	7	0	11	0	7	7	0
(1,1).2	3	0	0	0	0	0	0	0
(1,1).3	0	6	2	0	0	10	0	0
(1,1).4	0	0	0	0	0	2	0	0
(1,2)	16	0	0	1	0	4	1	0
(1,3).1	1	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	1	0	0
(1,6)	1	0	0	0	0	0	0	0

Table 22: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-286	-288	-290	-292	-294	-296	-298	-300
(3,1)	0	365	5	11	1	21	0	13
(2,2)	1	97	0	4	8	14	1	4
(2,1).1	2	16	5	4	13	3	0	6
(2,1).2	0	13	0	1	1	0	0	0
(2,1).3	0	337	0	0	6	5	0	16
(1,1).1	0	278	0	0	14	0	0	57
(1,1).2	0	355	0	0	6	0	0	13
(1,1).3	1	37	6	2	2	4	0	8
(1,1).4	0	36	0	0	0	0	0	0
(1,2)	0	130	0	0	1	0	0	9
(1,3).1	0	0	0	0	1	0	0	1
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	7	0	0	1	0	0	0

Table 23: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-304	-306	-308	-310	-312	-314	-316	-318
(3,1)	54	2	1	10	105	1	3	3
(2,2)	20	16	5	4	8	0	1	1
(2,1).1	1	5	1	2	10	0	0	1
(2,1).2	0	0	2	1	0	0	0	1
(2,1).3	13	7	0	1	12	0	0	1
(1,1).1	8	10	0	0	13	0	0	8
(1,1).2	0	0	0	0	0	0	0	0
(1,1).3	8	4	2	0	12	0	0	0
(1,1).4	0	0	2	0	0	0	0	0
(1,2)	0	3	0	0	5	0	0	1
(1,3).1	0	0	0	0	1	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	1	0	0	0	0	0	0	0

Table 24: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-320	-322	-324	-328	-330	-332	-336	-338
(3,1)	267	1	21	25	3	5	121	1
(2,2)	91	1	16	6	4	0	39	0
(2,1).1	40	0	6	4	10	0	12	2
(2,1).2	47	0	0	2	0	0	1	0
(2,1).3	474	2	33	0	0	0	93	0
(1,1).1	143	0	116	0	0	0	23	0
(1,1).2	125	0	25	0	0	0	0	0
(1,1).3	6	0	14	0	0	0	22	0
(1,1).4	3	0	7	0	0	0	2	0
(1,2)	71	0	37	0	0	0	7	0
(1,3).1	0	0	0	0	0	0	1	0
(1,3).2	1	0	0	0	0	0	1	0
(1,6)	2	0	2	0	0	0	2	0

Table 25: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-340	-342	-344	-348	-350	-352	-354	-356
(3,1)	7	12	19	2	8	218	0	1
(2,2)	7	9	6	1	13	26	0	1
(2,1).1	0	9	0	7	2	3	1	0
(2,1).2	1	1	0	0	8	19	0	0
(2,1).3	14	3	4	1	17	41	1	0
(1,1).1	0	0	0	0	0	14	0	0
(1,1).2	0	0	0	0	0	0	0	0
(1,1).3	0	0	4	0	0	16	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	0	2	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	1	0	0	0	2	0	0	0

Table 26: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-360	-364	-366	-368	-370	-372	-374	-376
(3,1)	133	4	5	37	7	5	0	2
(2,2)	19	2	2	11	3	0	0	1
(2,1).1	14	0	2	2	0	9	2	1
(2,1).2	10	1	1	3	0	0	0	0
(2,1).3	103	0	0	2	0	1	0	5
(1,1).1	81	5	0	10	0	0	0	0
(1,1).2	11	0	0	0	0	0	0	0
(1,1).3	11	8	4	0	0	2	0	0
(1,1).4	1	0	0	0	0	0	0	0
(1,2)	29	6	0	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 27: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-378	-380	-384	-388	-390	-392	-394	-396
(3,1)	0	8	144	0	4	5	1	12
(2,2)	10	0	39	1	1	2	0	1
(2,1).1	10	1	0	2	5	2	0	2
(2,1).2	5	0	8	0	0	0	0	0
(2,1).3	0	0	208	0	5	1	0	1
(1,1).1	16	0	173	0	12	22	0	6
(1,1).2	0	0	265	0	0	3	0	0
(1,1).3	0	2	16	0	0	21	0	1
(1,1).4	0	0	9	0	0	1	0	0
(1,2)	12	0	41	0	10	4	0	2
(1,3).1	3	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	1	0	0	0	1	0	0	0

Table 28: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-400	-402	-404	-406	-408	-410	-414	-416
(3,1)	5	0	1	6	52	11	0	15
(2,2)	12	1	2	4	7	0	1	12
(2,1).1	20	0	0	0	6	2	8	8
(2,1).2	4	0	0	5	3	0	0	5
(2,1).3	37	0	0	2	21	0	3	13
(1,1).1	56	0	0	0	0	0	0	17
(1,1).2	35	0	0	0	0	0	0	0
(1,1).3	6	0	0	0	4	0	0	21
(1,1).4	4	0	0	0	0	0	0	0
(1,2)	14	0	0	0	0	0	0	2
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	1
(1,6)	0	0	0	0	0	0	0	2

Table 29: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-418	-420	-424	-426	-428	-430	-432	-434
(3,1)	1	3	5	0	1	6	24	0
(2,2)	0	1	2	0	0	4	17	0
(2,1).1	0	2	0	1	0	3	6	4
(2,1).2	0	0	1	0	0	5	2	0
(2,1).3	4	6	0	0	0	1	30	0
(1,1).1	0	0	0	0	0	0	26	0
(1,1).2	0	0	0	0	0	0	11	0
(1,1).3	0	7	0	0	0	0	2	0
(1,1).4	0	0	0	0	0	0	12	0
(1,2)	0	1	0	0	0	0	25	0
(1,3).1	0	1	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	1	0	0	0	0	0	2	0

Table 30: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-436	-440	-442	-444	-448	-450	-456	-460
(3,1)	0	1	0	2	19	0	16	0
(2,2)	1	2	0	1	15	5	4	0
(2,1).1	0	4	0	0	0	0	0	0
(2,1).2	0	0	1	1	1	2	1	0
(2,1).3	0	15	0	1	44	0	3	0
(1,1).1	0	0	0	0	3	0	0	0
(1,1).2	0	0	0	0	0	0	0	0
(1,1).3	0	4	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	1	0	0
(1,2)	0	0	0	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	1

Table 31: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-462	-464	-468	-470	-472	-480	-488	-490
(3,1)	0	6	0	1	0	6	4	1
(2,2)	0	4	1	1	0	6	0	3
(2,1).1	2	4	5	0	1	3	4	4
(2,1).2	0	2	0	1	0	0	0	3
(2,1).3	0	2	0	1	1	17	0	2
(1,1).1	3	0	1	0	0	11	0	0
(1,1).2	0	0	0	0	0	3	0	0
(1,1).3	0	0	0	0	0	3	0	2
(1,1).4	0	0	0	0	0	7	0	0
(1,2)	0	0	1	0	0	12	0	0
(1,3).1	0	0	1	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 32: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-496	-500	-504	-510	-512	-520	-528	-550
(3,1)	0	0	0	0	0	0	0	0
(2,2)	0	2	1	0	0	0	0	0
(2,1).1	0	2	0	0	1	0	0	2
(2,1).2	0	0	1	0	0	0	2	0
(2,1).3	0	1	0	0	0	0	0	0
(1,1).1	0	10	0	0	4	0	0	0
(1,1).2	0	0	0	0	8	0	0	0
(1,1).3	8	2	1	1	0	1	0	0
(1,1).4	0	0	6	0	2	0	0	0
(1,2)	0	5	3	0	2	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	2	0	0	0	0	1

Table 33: Numbers of computed orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

8.1.2 Data of non-orientable face-transitive surfaces with at most 1280 faces

$\chi(X)$	1	-2	-3	-4	-5	-6	-7	-8
(3,1)	2	2	2	3	2	1	2	6
(2,2)	3	1	4	4	2	2	4	7
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	1	0	0
(2,1).3	0	0	0	0	0	1	0	1
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	1	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	1	0	0	0	0	1	2	0

Table 34: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-9	-10	-11	-12	-13	-14	-15	-16
(3,1)	1	2	1	3	1	4	4	0
(2,2)	3	4	0	3	2	5	3	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	1	0
(2,1).3	1	1	0	2	0	1	2	1
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	2	0	0	0	0	1	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	1	1	1	1	0

Table 35: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-17	-18	-20	-21	-23	-24	-25	-26
(3,1)	1	3	4	3	2	4	0	2
(2,2)	0	3	4	5	0	4	0	1
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	2	0	0	0	0	0
(2,1).3	0	6	6	0	0	11	1	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	1	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	1	0	1	0	0

Table 36: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-27	-28	-29	-30	-32	-33	-34	-35
(3,1)	5	12	1	2	7	0	1	5
(2,2)	5	9	0	2	4	5	1	6
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	1	0
(2,1).3	1	7	0	6	2	0	0	3
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	5	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	1	0	0	0	0	0	0
(1,2)	0	4	0	4	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	1	0	0	0	0	0	0	0

Table 37: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-36	-38	-39	-40	-41	-42	-44	-45
(3,1)	4	2	4	4	1	6	5	6
(2,2)	4	1	4	4	0	9	8	9
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	1	1
(2,1).3	10	0	0	11	0	5	0	3
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	2	0	3	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	2	1

Table 38: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-47	-48	-49	-50	-51	-53	-54	-55
(3,1)	1	7	1	9	1	1	3	1
(2,2)	0	13	2	4	1	0	10	7
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	3	0	0	0	0	0	1
(2,1).3	0	30	1	1	0	0	12	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	8	0	1	0	0	6	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	5	0	0	0	0	2	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	1	0	0	0	0	2

Table 39: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-56	-58	-59	-60	-62	-63	-64	-65
(3,1)	11	4	1	5	5	6	10	1
(2,2)	8	0	0	9	7	5	6	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	1	1	0	0
(2,1).3	9	0	0	33	0	4	2	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	6	0	0	1	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	1	0	0	1	0	0	0	0
(1,2)	1	0	0	7	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 40: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-66	-68	-70	-71	-72	-74	-75	-76
(3,1)	1	6	11	1	2	2	3	6
(2,2)	1	5	13	0	3	1	4	4
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	0	2
(2,1).3	5	4	11	0	22	1	5	2
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	1	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	1	0	0	1	0	0	0

Table 41: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-77	-78	-80	-81	-82	-83	-84	-86
(3,1)	3	1	21	5	1	1	8	3
(2,2)	2	0	13	5	1	0	21	1
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	1	0	1	0	0	0	0	0
(2,1).3	0	4	22	3	0	0	22	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	1	0	0	0	6	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	1	0
(1,2)	0	0	6	2	0	0	9	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	2	0

Table 42: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-88	-89	-90	-91	-92	-95	-96	-98
(3,1)	12	1	5	0	4	7	11	3
(2,2)	5	0	8	6	2	1	15	2
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	1	0	1	0	0	1	0	0
(2,1).3	4	0	24	2	0	0	54	10
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	2	0	0	0	24	1
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	2	0
(1,2)	0	0	8	2	0	0	12	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	2	0	2	0	0

Table 43: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-99	-100	-101	-102	-104	-105	-106	-107
(3,1)	13	3	1	6	14	0	0	1
(2,2)	14	1	0	4	16	2	0	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	4	0	0	0
(2,1).3	4	10	0	3	4	2	1	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	3	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	3	2	0	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	3	0	0	0

Table 44: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-108	-110	-111	-112	-113	-114	-116	-117
(3,1)	4	2	0	10	1	8	9	4
(2,2)	10	11	0	6	0	2	1	11
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	1	2	0	0	0	0
(2,1).3	38	0	0	33	0	5	2	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	10	0	0	6	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	1	0	0	0	0
(1,2)	2	5	0	4	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	1	0	0

Table 45: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-118	-119	-120	-121	-122	-123	-124	-125
(3,1)	0	4	12	0	2	1	1	1
(2,2)	0	4	7	0	1	1	1	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	1	0	0	0	0	0	0	0
(2,1).3	0	2	73	1	0	0	0	1
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	10	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	2	0	0	0	0	0
(1,2)	0	0	11	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 46: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-126	-128	-130	-131	-132	-134	-135	-136
(3,1)	6	10	2	1	5	2	8	4
(2,2)	2	14	2	0	11	1	13	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	2	0	0	0	2	0	2	1
(2,1).3	26	21	0	0	9	0	10	1
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	13	16	0	0	0	0	2	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	0	0	2	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	2	0	0	0

Table 47: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-137	-138	-139	-140	-142	-143	-144	-145
(3,1)	1	5	1	11	2	3	20	1
(2,2)	0	0	0	18	0	8	13	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	0	0
(2,1).3	0	2	0	26	0	0	55	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	11	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	1	0	0	0	0
(1,2)	0	0	0	2	0	0	6	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	1	0	0	0	0
(1,6)	0	0	0	0	0	0	1	0

Table 48: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-146	-147	-148	-149	-150	-152	-153	-154
(3,1)	3	2	9	1	8	12	9	5
(2,2)	2	2	4	0	6	9	10	7
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	1	1	0	0	1	0	0	0
(2,1).3	0	1	0	0	22	2	3	4
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	3	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	3	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 49: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-155	-156	-158	-159	-160	-162	-164	-165
(3,1)	3	8	0	3	20	3	6	2
(2,2)	1	2	1	0	21	10	3	6
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	1	3	1	1	0
(2,1).3	0	3	0	0	38	25	2	5
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	8	15	0	2
(1,1).3	0	0	0	0	1	1	0	0
(1,1).4	0	0	0	0	0	1	0	0
(1,2)	0	0	0	0	8	13	0	7
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 50: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-168	-169	-170	-171	-172	-174	-175	-176
(3,1)	11	0	3	6	4	3	5	24
(2,2)	21	0	6	3	4	0	7	7
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	1	1	2
(2,1).3	60	1	5	0	0	1	6	12
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	14	0	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	2	0	0	0	0	0	0	0
(1,2)	5	0	0	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	1	0	0	0	0	0	0	0
(1,6)	2	0	0	0	0	0	1	0

Table 51: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-178	-180	-182	-183	-184	-186	-188	-189
(3,1)	1	8	7	1	4	0	0	7
(2,2)	1	11	7	0	2	0	1	16
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	1	0	0	0	0	2	4
(2,1).3	0	75	6	0	2	1	3	9
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	12	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	1	0	0	0	0	0	0
(1,2)	0	14	10	0	0	0	0	2
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 52: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-190	-192	-194	-195	-196	-198	-200	-202
(3,1)	1	5	1	3	6	5	5	0
(2,2)	1	6	2	8	3	20	9	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	1	0	0	0	0	0	1
(2,1).3	0	84	0	6	19	26	32	1
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	44	0	0	3	1	1	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	4	0	0	1	0	0	0
(1,2)	0	29	0	5	0	12	5	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 53: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-203	-204	-206	-207	-208	-209	-210	-212
(3,1)	3	9	1	13	27	0	4	0
(2,2)	1	6	1	0	19	9	12	2
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	1	0	0	4	1	0	0	0
(2,1).3	0	13	0	0	6	3	45	2
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	1	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	4	0	0	0	0	3	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	1	0	0	0	0	1	0

Table 54: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-214	-215	-216	-218	-219	-220	-222	-224
(3,1)	9	2	6	0	0	7	1	12
(2,2)	0	3	9	1	1	5	0	37
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	3	5	0	0	0	0	7
(2,1).3	0	0	63	0	0	8	2	41
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	23	0	0	1	0	24
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	2
(1,2)	0	0	3	0	0	12	0	13
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	2	0	0

Table 55: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-225	-226	-227	-228	-230	-231	-232	-234
(3,1)	1	1	1	4	0	2	3	3
(2,2)	2	1	0	12	1	0	0	8
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	1	0	0	0
(2,1).3	9	0	0	16	0	3	3	23
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	0	1
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	2
(1,2)	0	0	0	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	2	2	0	0	0

Table 56: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-236	-238	-240	-242	-243	-244	-245	-246
(3,1)	0	0	8	0	4	6	0	0
(2,2)	2	0	9	2	10	2	1	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	2	0	0
(2,1).3	0	6	125	0	7	0	2	7
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	68	1	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	3	0	0	0	0	0
(1,2)	0	0	25	0	2	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 57: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-248	-250	-252	-253	-254	-255	-256	-258
(3,1)	2	1	11	0	1	2	8	0
(2,2)	3	1	5	0	1	2	5	1
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	1	6	1
(2,1).3	1	2	28	0	0	0	38	6
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	13	54	0	0	0	67	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	3	0	1	0	0	0	0	0
(1,2)	0	2	11	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	1	0	4	0	0

Table 58: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-259	-260	-261	-264	-266	-267	-268	-270
(3,1)	0	7	0	4	0	1	3	0
(2,2)	1	2	3	4	1	1	2	2
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	0	0
(2,1).3	0	4	3	35	1	0	1	18
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	1	0	0	0	30
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	2	0	4	0	0	0	13
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 59: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-271	-272	-273	-274	-275	-276	-278	-279
(3,1)	1	11	1	0	1	0	1	0
(2,2)	0	11	8	1	6	5	1	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	5	0	0	1	0	0	0
(2,1).3	0	31	5	0	10	0	0	3
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	5	0	2	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	9	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	3	1	0	0	0	0	0

Table 60: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-280	-282	-284	-285	-286	-287	-288	-290
(3,1)	0	2	1	1	0	4	5	3
(2,2)	9	0	2	0	9	0	8	2
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	1	0	1	2	1	0
(2,1).3	28	0	0	0	12	0	97	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	55	0
(1,1).3	0	0	0	0	0	0	4	0
(1,1).4	1	0	0	0	0	0	3	0
(1,2)	7	0	0	0	0	0	41	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 61: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-291	-292	-294	-296	-297	-298	-299	-300
(3,1)	0	13	0	2	15	0	2	0
(2,2)	3	1	6	2	9	0	13	8
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	1	2
(2,1).3	0	0	22	2	2	1	0	21
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	2	0	0	0	0	23
(1,1).3	0	0	2	0	0	0	0	0
(1,1).4	0	0	1	0	0	0	0	1
(1,2)	0	0	0	0	0	0	0	7
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 62: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-302	-303	-304	-306	-308	-310	-312	-314
(3,1)	0	0	2	0	0	1	5	1
(2,2)	1	0	8	2	4	0	20	1
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	1	1	0	0	0	0	0
(2,1).3	0	0	7	6	4	0	13	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	1	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	1	0	0	0
(1,2)	0	0	0	2	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	2	0

Table 63: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-315	-316	-318	-319	-320	-322	-323	-324
(3,1)	8	2	1	4	2	0	0	2
(2,2)	6	3	0	0	1	5	7	7
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	1	0	0	0	1
(2,1).3	2	1	0	0	28	4	0	34
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	18	0	0	49
(1,1).3	0	0	0	0	1	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	2	0	0	0	17	0	0	7
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 64: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-325	-328	-329	-330	-331	-332	-333	-335
(3,1)	0	8	0	5	1	5	4	7
(2,2)	0	1	1	1	0	0	4	2
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	1	0	0	0	1	1
(2,1).3	1	1	0	6	0	0	0	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 65: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-336	-338	-339	-340	-342	-344	-348	-350
(3,1)	3	1	0	4	1	4	6	0
(2,2)	4	1	1	6	1	4	8	2
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	2	0	3
(2,1).3	55	0	0	22	2	3	10	14
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	11	1	0	0	0	0	0	1
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	1	0	0	0	0	0	0	0
(1,2)	11	0	0	1	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 66: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-351	-352	-354	-357	-359	-360	-362	-363
(3,1)	11	0	1	0	1	20	0	0
(2,2)	14	0	5	0	0	3	0	1
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	7	0	0	0	0	0	0	0
(2,1).3	1	3	2	5	0	55	1	4
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	1	0	0	0	6	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	0	9	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 67: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-364	-366	-368	-369	-370	-371	-372	-375
(3,1)	0	1	1	0	5	0	0	5
(2,2)	6	0	4	4	3	5	5	2
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	1	0	0	0	0	1
(2,1).3	2	1	4	3	1	0	2	2
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	5	0	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	14	0	0	0	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 68: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-376	-377	-378	-380	-384	-385	-386	-387
(3,1)	0	0	0	7	2	0	0	1
(2,2)	0	10	0	3	14	1	1	6
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	2	0	0	0	0
(2,1).3	7	9	9	2	86	0	0	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	7	0	107	0	0	0
(1,1).3	0	0	2	0	2	0	0	0
(1,1).4	0	0	0	0	1	0	0	0
(1,2)	0	0	11	1	10	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	3	0	0	0	0

Table 69: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-388	-390	-392	-394	-395	-396	-399	-400
(3,1)	0	0	0	0	1	0	0	2
(2,2)	1	10	0	0	1	1	1	2
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	1	1
(2,1).3	0	14	1	1	0	4	0	37
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	1	0	0	0	0	8
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	1
(1,2)	0	18	2	0	0	0	0	4
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	4	0

Table 70: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-402	-403	-404	-405	-408	-410	-411	-413
(3,1)	0	1	0	0	2	6	0	0
(2,2)	2	0	7	1	2	0	1	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	0	0
(2,1).3	2	0	0	0	19	0	0	4
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	2	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	8	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 71: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-414	-415	-416	-418	-420	-426	-429	-430
(3,1)	0	0	2	0	3	0	0	0
(2,2)	1	1	0	1	7	0	4	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	1	0	4	0
(2,1).3	0	0	12	0	21	9	0	4
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	1	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	0	0	3	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	1	0	0	0

Table 72: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-432	-435	-436	-438	-440	-441	-444	-447
(3,1)	4	1	0	0	2	0	0	1
(2,2)	2	0	1	0	3	0	2	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	1	0	0	0	1	0	0	0
(2,1).3	24	0	0	1	8	0	0	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	7	0	0	0	0	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	5	0	0	0	0	2	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 73: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-448	-450	-452	-455	-456	-459	-460	-468
(3,1)	4	0	0	1	0	1	0	0
(2,2)	0	0	1	2	0	0	4	1
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	2	0	0	0	0
(2,1).3	0	11	0	0	10	0	10	1
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	6	0	0	0	4
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	1	0	0	0	0	0	0
(1,2)	4	0	0	3	0	2	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0

Table 74: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-470	-475	-476	-480	-484	-487	-492	-496
(3,1)	0	0	0	0	0	1	0	0
(2,2)	0	5	0	0	0	0	1	0
(2,1).1	0	0	0	0	0	0	0	0
(2,1).2	0	0	0	0	0	0	0	4
(2,1).3	4	10	0	1	0	0	0	0
(1,1).1	0	0	0	0	0	0	0	0
(1,1).2	0	0	0	39	4	0	0	0
(1,1).3	0	0	0	0	0	0	0	0
(1,1).4	0	0	0	0	0	0	0	0
(1,2)	0	0	6	1	0	0	0	0
(1,3).1	0	0	0	0	0	0	0	0
(1,3).2	0	0	0	0	0	0	0	0
(1,6)	0	1	0	0	0	0	0	0

Table 75: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types

$\chi(X)$	-500	-512
(3,1)	0	0
(2,2)	0	0
(2,1).1	0	0
(2,1).2	0	0
(2,1).3	0	0
(1,1).1	0	0
(1,1).2	10	8
(1,1).3	2	0
(1,1).4	0	0
(1,2)	1	0
(1,3).1	0	0
(1,3).2	0	0
(1,6)	0	0

Table 76: Numbers of computed non-orientable face-transitive surfaces with at most 1280 faces with respect to their Euler characteristics and vertex-face types.

8.2 Data: Edge-transitive surfaces with respect to face-edge types and Euler characteristics

This section contains some information about the census of edge-transitive surfaces contained in [1]. We refer the reader to Section 6.2 for a description of how to read the below tables.

8.2.1 Data of orientable edge-transitive surfaces with at most 5000 faces

$\chi(X)$	2	0	-4	-8	-10	-12	-14	-18
(1,4)	3	75	2	1	1	1	2	2
(1,2).1	0	716	0	0	0	0	0	1
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	1
(2,1)	0	0	0	0	0	0	0	0

Table 77: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-24	-26	-28	-30	-32	-36	-40	-42
(1,4)	2	3	2	1	1	0	1	2
(1,2).1	0	0	0	0	1	0	0	1
(1,2).2	0	0	0	0	0	1	0	0
(2,2)	0	0	0	0	0	1	0	0
(2,1)	0	0	1	0	1	0	0	0

Table 78: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-48	-50	-54	-64	-70	-72	-78	-80
(1,4)	2	2	3	3	1	2	0	2
(1,2).1	0	0	2	1	0	1	1	0
(1,2).2	0	1	0	0	0	0	0	1
(2,2)	0	0	0	0	0	1	0	0
(2,1)	0	1	0	0	0	0	0	0

Table 79: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-84	-88	-90	-96	-98	-100	-102	-108
(1,4)	1	3	2	3	1	0	2	1
(1,2).1	1	0	1	0	0	1	0	0
(1,2).2	0	0	0	0	0	0	0	1
(2,2)	0	0	0	1	0	0	0	1
(2,1)	1	0	0	0	2	1	0	0

Table 80: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-110	-112	-114	-126	-128	-130	-136	-144
(1,4)	2	0	0	0	3	1	3	5
(1,2).1	0	0	1	1	0	0	0	1
(1,2).2	0	0	0	0	0	0	0	1
(2,2)	0	0	0	0	0	0	0	1
(2,1)	0	1	0	0	0	0	0	0

Table 81: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-150	-154	-160	-162	-168	-176	-180	-182
(1,4)	2	1	2	6	3	2	1	2
(1,2).1	0	0	0	1	1	0	0	0
(1,2).2	0	0	1	0	0	0	0	0
(2,2)	0	0	0	1	0	0	0	0
(2,1)	0	0	0	2	0	1	0	0

Table 82: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-186	-190	-192	-196	-200	-204	-208	-216
(1,4)	0	3	3	1	0	2	4	4
(1,2).1	1	0	1	0	0	0	0	2
(1,2).2	0	0	0	0	1	0	0	0
(2,2)	0	0	1	0	0	0	0	0
(2,1)	0	0	0	4	1	0	0	0

Table 83: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-222	-224	-228	-234	-238	-240	-242	-252
(1,4)	0	0	2	1	1	2	0	0
(1,2).1	1	1	0	1	0	0	1	1
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	1	0
(2,1)	0	2	0	2	0	0	2	5

Table 84: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-256	-258	-264	-270	-280	-288	-290	-294
(1,4)	11	0	3	2	0	9	3	1
(1,2).1	4	1	0	1	0	2	0	1
(1,2).2	1	0	0	0	1	0	0	0
(2,2)	1	0	0	0	1	2	0	0
(2,1)	3	0	0	2	0	0	0	0

Table 85: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-300	-304	-312	-320	-324	-330	-336	-338
(1,4)	0	1	0	2	3	4	6	0
(1,2).1	1	0	1	0	0	0	3	0
(1,2).2	0	0	0	1	2	0	1	1
(2,2)	0	0	0	0	2	2	1	0
(2,1)	1	0	0	0	2	0	6	3

Table 86: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-340	-342	-350	-360	-364	-366	-378	-384
(1,4)	1	0	2	1	0	0	1	5
(1,2).1	0	1	0	0	0	1	4	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	1
(2,1)	0	0	0	0	2	0	0	0

Table 87: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-390	-392	-400	-402	-408	-416	-418	-420
(1,4)	1	0	0	0	4	5	1	1
(1,2).1	0	0	1	1	0	0	0	1
(1,2).2	0	0	0	0	0	1	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	2	1	0	0	0	0	0

Table 88: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-432	-438	-440	-448	-450	-456	-460	-468
(1,4)	4	0	2	0	1	2	6	1
(1,2).1	0	1	0	4	0	1	0	2
(1,2).2	2	0	0	0	0	0	0	0
(2,2)	2	0	0	0	0	0	0	0
(2,1)	1	0	0	2	6	0	0	2

Table 89: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-474	-480	-484	-486	-490	-500	-504	-506
(1,4)	0	2	0	10	1	0	6	4
(1,2).1	1	0	0	9	0	2	3	0
(1,2).2	0	0	1	1	0	0	0	0
(2,2)	0	1	1	2	0	0	0	0
(2,1)	0	0	2	2	0	0	0	0

Table 90: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-510	-512	-528	-532	-540	-544	-546	-550
(1,4)	4	8	3	0	1	3	4	1
(1,2).1	0	3	0	0	0	0	2	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	2	0
(2,1)	0	1	0	2	0	0	0	0

Table 91: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-558	-560	-570	-576	-578	-582	-588	-598
(1,4)	0	0	3	6	0	0	0	1
(1,2).1	1	0	0	2	1	1	0	0
(1,2).2	0	1	0	3	1	0	0	0
(2,2)	0	1	0	6	1	0	0	0
(2,1)	0	0	0	3	3	0	5	0

Table 92: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-600	-612	-618	-620	-624	-640	-648	-650
(1,4)	2	2	0	2	6	6	10	1
(1,2).1	0	0	1	0	2	1	2	0
(1,2).2	0	0	0	0	0	3	0	0
(2,2)	0	0	0	0	0	1	1	0
(2,1)	0	0	0	0	0	4	0	0

Table 93: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-654	-666	-672	-676	-680	-684	-700	-702
(1,4)	0	0	1	0	1	0	4	0
(1,2).1	1	1	2	1	0	1	3	4
(1,2).2	0	0	1	0	0	0	1	0
(2,2)	0	0	0	0	0	0	1	0
(2,1)	0	0	3	3	0	2	3	2

Table 94: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-720	-726	-744	-748	-754	-756	-760	-762
(1,4)	2	1	0	1	1	0	3	0
(1,2).1	2	0	1	0	0	1	0	1
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	2	0	0	0	0	0	0	0
(2,1)	2	0	0	0	0	5	0	0

Table 95: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-768	-774	-780	-784	-798	-800	-810	-816
(1,4)	14	0	0	0	4	0	4	1
(1,2).1	5	1	1	0	2	0	0	0
(1,2).2	2	0	0	0	0	0	0	0
(2,2)	5	0	0	0	0	0	0	0
(2,1)	5	0	0	4	0	1	0	0

Table 96: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-832	-834	-840	-864	-868	-882	-888	-896
(1,4)	4	0	1	9	1	3	0	0
(1,2).1	4	1	0	1	0	4	1	6
(1,2).2	2	0	1	0	0	0	0	0
(2,2)	0	0	1	1	0	0	0	0
(2,1)	2	0	0	0	2	2	0	4

Table 97: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-900	-906	-912	-928	-936	-942	-950	-960
(1,4)	2	0	2	1	3	0	1	1
(1,2).1	1	1	2	0	4	1	0	0
(1,2).2	1	0	0	0	0	0	0	0
(2,2)	1	0	0	0	0	0	0	0
(2,1)	7	0	0	0	2	0	0	0

Table 98: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-968	-972	-978	-990	-1008	-1012	-1014	-1024
(1,4)	1	4	0	3	6	2	1	7
(1,2).1	0	1	1	0	7	0	1	4
(1,2).2	0	3	0	0	2	0	0	1
(2,2)	0	8	0	0	0	0	0	0
(2,1)	0	6	0	0	10	0	0	2

Table 99: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1026	-1032	-1036	-1050	-1054	-1056	-1078	-1080
(1,4)	0	0	0	0	1	1	2	5
(1,2).1	4	1	0	1	0	0	0	1
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	2	0	0	0	0	0

Table 100: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1086	-1088	-1092	-1098	-1116	-1120	-1134	-1140
(1,4)	0	3	0	0	0	1	6	0
(1,2).1	1	0	0	1	1	0	7	1
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	2	0	2	0	1	0

Table 101: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1152	-1156	-1158	-1160	-1170	-1176	-1188	-1194
(1,4)	15	0	0	3	0	1	1	0
(1,2).1	2	1	1	0	0	2	1	1
(1,2).2	1	1	0	0	1	0	0	0
(2,2)	14	1	0	0	2	0	0	0
(2,1)	6	3	0	0	1	0	0	0

Table 102: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1200	-1204	-1206	-1216	-1218	-1224	-1232	-1242
(1,4)	4	0	0	0	4	1	0	0
(1,2).1	2	0	1	4	0	0	0	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	1	0	0	0	0	0	0	0
(2,1)	2	2	0	0	0	0	1	2

Table 103: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1248	-1258	-1260	-1280	-1296	-1320	-1330	-1332
(1,4)	3	1	0	4	3	1	1	0
(1,2).1	1	0	1	1	1	0	0	1
(1,2).2	0	0	1	4	0	0	0	0
(2,2)	0	0	2	3	1	0	0	0
(2,1)	1	0	1	5	1	0	0	0

Table 104: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1344	-1350	-1368	-1372	-1392	-1400	-1404	-1408
(1,4)	4	1	0	0	1	1	1	0
(1,2).1	1	1	2	0	0	0	1	0
(1,2).2	1	1	0	1	0	0	0	4
(2,2)	0	0	0	0	0	0	0	0
(2,1)	15	1	0	0	0	0	0	15

Table 105: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1440	-1458	-1472	-1480	-1488	-1500	-1512	-1518
(1,4)	0	9	5	1	4	1	5	4
(1,2).1	2	9	0	0	1	0	5	0
(1,2).2	1	1	0	0	0	0	1	0
(2,2)	5	15	0	0	0	0	2	0
(2,1)	2	7	0	0	0	0	0	0

Table 106: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1520	-1530	-1536	-1548	-1550	-1558	-1560	-1584
(1,4)	3	4	1	0	4	1	3	1
(1,2).1	0	1	0	1	0	0	0	0
(1,2).2	0	0	4	0	0	0	0	0
(2,2)	0	1	4	0	0	0	0	0
(2,1)	0	0	2	0	0	0	0	0

Table 107: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1600	-1610	-1620	-1632	-1638	-1664	-1680	-1690
(1,4)	1	1	1	12	1	6	1	1
(1,2).1	0	0	0	0	3	0	1	0
(1,2).2	1	0	1	0	0	3	1	0
(2,2)	1	0	1	6	0	0	0	0
(2,1)	1	0	0	0	4	0	0	0

Table 108: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1700	-1720	-1728	-1764	-1776	-1800	-1804	-1848
(1,4)	2	1	7	1	0	1	1	1
(1,2).1	0	0	0	1	1	0	0	0
(1,2).2	0	0	1	0	0	1	0	0
(2,2)	1	0	1	0	0	0	0	0
(2,1)	0	0	1	0	0	1	0	0

Table 109: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1856	-1860	-1872	-1890	-1944	-1978	-2000	-2016
(1,4)	2	0	1	1	4	1	1	5
(1,2).1	0	1	3	5	3	0	0	0
(1,2).2	1	0	0	0	3	0	0	1
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	3	0	3	0	0	0

Table 110: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-2058	-2068	-2160	-2184	-2254	-2350
(1,4)	0	1	2	1	1	1
(1,2).1	7	0	0	1	0	0
(1,2).2	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0

Table 111: Numbers of orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

8.2.2 Data of non-orientable edge-transitive surfaces with at most 5000 faces

$\chi(X)$	1	-6	-7	-12	-13	-14	-15	-21
(1,4)	1	1	2	1	1	1	1	1
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 112: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-24	-27	-44	-45	-49	-55	-68	-72
(1,4)	1	1	2	1	1	2	1	1
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 113: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-84	-91	-95	-104	-114	-132	-140	-144
(1,4)	2	2	2	3	1	2	0	1
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	0	0	0	0	1	0
(2,2)	0	0	0	0	0	0	1	0
(2,1)	0	0	0	0	0	0	0	0

Table 114: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-145	-168	-175	-204	-210	-220	-228	-230
(1,4)	1	2	1	1	1	2	2	2
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	1	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 115: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-234	-253	-255	-272	-273	-310	-312	-324
(1,4)	1	3	4	3	1	1	2	1
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 116: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-325	-350	-380	-399	-420	-468	-475	-504
(1,4)	1	1	3	4	1	2	1	1
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	0	0	1	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 117: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-506	-539	-580	-609	-620	-630	-684	-703
(1,4)	2	1	1	2	2	1	1	1
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 118: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-729	-736	-744	-759	-765	-775	-792	-805
(1,4)	1	5	2	4	2	1	1	1
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 119: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-812	-816	-840	-924	-972	-975	-992	-1000
(1,4)	2	3	1	1	2	2	2	3
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	1	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 120: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1008	-1092	-1120	-1140	-1160	-1197	-1215	-1260
(1,4)	1	1	2	3	2	2	3	2
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 121: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1287	-1344	-1406	-1428	-1443	-1456	-1550	-1595
(1,4)	2	0	1	2	5	0	2	6
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	1	0	0	0	1	0	0
(2,2)	0	0	0	0	0	1	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 122: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

$\chi(X)$	-1624	-1632	-1680	-1716	-1767	-1932	-1984	-2015
(1,4)	4	1	2	2	1	1	4	8
(1,2).1	0	0	0	0	0	0	0	0
(1,2).2	0	0	0	0	0	0	0	0
(2,2)	0	0	0	0	0	0	0	0
(2,1)	0	0	0	0	0	0	0	0

Table 123: Numbers of non-orientable edge-transitive surfaces with at most 5000 faces with respect to their Euler characteristics and face-edge types

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References

- [1] R. Akpanya. A database of edge-transitive surfaces. <https://github.com/ReymondAkpanya/Edgetransitivesurfaces>, 2025.
- [2] R. Akpanya and T. Goertzen. Surfaces with given Automorphism Group. *arXiv e-prints*, page arXiv:2307.12681, July 2023.
- [3] R. Akpanya and V. Kirekod. Embeddings-of-wild-coloured-surfaces. <https://github.com/ReymondAkpanya/Embeddings-of-wild-coloured-surfaces.git>, 2024.
- [4] R. Akpanya, V. K. Kirekod, A. C. Niemeyer, and D. Robertz. Construction of toroidal polyhedra corresponding to perfect chains of wild tetrahedra. *arXiv preprint arXiv:2411.14924*, 2024.
- [5] R. Akpanya and J. Spreer. Classification of face-transitive surfaces, 2025. In preparation.
- [6] R. Akpanya and J. Spreer. A database of face-transitive surfaces. <https://github.com/jspreer/faceTransitiveSurfaces>, 2025.
- [7] R. Akpanya and S. Stüttgen. Talk: GeoCombSurfX - exploring properties of polygonal complexes. <https://pretalx.com/juliacon2024/talk/YDS7HF/>, 2024. Accessed: 2024-11-16, Presented at JuliaCon 2024.
- [8] R. Akpanya and S. Stüttgen. GeoCombSurfX, Version 0.1, 2025. <https://github.com/Saschobolt/GeoCombSurfX>.
- [9] J. S. Andrade, H. J. Herrmann, R. F. S. Andrade, and L. R. da Silva. Apollonian Networks: Simultaneously Scale-Free, Small World, Euclidean, Space Filling, and with Matching Graphs. *Phys. Rev. Lett.*, 94:018702, Jan 2005.
- [10] K. Appel and W. Haken. Every planar map is four colorable. *Bull. Amer. Math. Soc.*, 82(5):711–712, 1976.
- [11] L. Babai. Automorphism groups of planar graphs. I. *Discrete Math.*, 2(4):295–307, 1972.
- [12] L. Babai. On the abstract group of automorphisms. In *Combinatorics (Swansea, 1981)*, volume 52 of *London Math. Soc. Lecture Note Ser.*, pages 1–40. Cambridge Univ. Press, Cambridge-New York, 1981.
- [13] L. Babai. Vertex-transitive graphs and vertex-transitive maps. *J. Graph Theory*, 15(6):587–627, 1991.
- [14] M. Baumeister. *Regularity aspects for combinatorial simplicial surfaces*. Dissertation, RWTH Aachen University, Aachen, 2020. Veröffentlicht auf dem Publikationsserver der RWTH Aachen University.
- [15] L. W. Beineke and R. E. Pippert. Enumerating dissectible polyhedra by their automorphism groups. *Canadian J. Math.*, 26:50–67, 1974.
- [16] H. U. Besche, B. Eick, E. O’Brien, and M. Horn. SmallGrp, the gap small groups library, Version 1.5.3. <https://gap-packages.github.io/smallgrp/>, May 2023. GAP package.

- [17] H. U. BESCHE, B. EICK, and E. A. O'BRIEN. A millennium project: Constructing small groups. *International Journal of Algebra and Computation*, 12(05):623–644, 2002.
- [18] J. D. Beule, J. Jonušas, J. D. Mitchell, M. Torpey, M. Tsalakou, and W. A. Wilson. Digraphs - GAP package, version 1.9.0, Sep 2024.
- [19] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah. Julia: A fresh approach to numerical computing. *SIAM review*, 59(1):65–98, 2017.
- [20] G. D. Birkhoff. On the Number of Ways of Colouring a Map. *Proceedings of the Edinburgh Mathematical Society*, 2(2):83–91, 1930.
- [21] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).
- [22] K.-H. Brakhage, A. C. Niemeyer, W. Plesken, D. Robertz, and A. Strzelczyk. The icosahedra of edge length 1. *J. Algebra*, 545:4–26, 2020.
- [23] K.-H. Brakhage, A. C. Niemeyer, W. Plesken, and A. Strzelczyk. Simplicial surfaces controlled by one triangle. *J. Geom. Graph.*, 21(2):141–152, 2017.
- [24] U. Brehm and W. Kühnel. Equivelar maps on the torus. *European Journal of Combinatorics*, 29(8):1843–1861, 2008. In honor of Ludwig Danzer's 80th birthday.
- [25] G. Brinkmann and B. D. McKay. Fast generation of planar graphs. *MATCH Commun. Math. Comput. Chem.*, 58(2):323–357, 2007.
- [26] B. A. Burton, B. Datta, and J. Spreer. Flip graphs of stacked and flag triangulations of the 2-sphere. *Electron. J. Combin.*, 29(2):Paper No. 2.6, 30, 2022.
- [27] P. J. Cameron. *Permutation groups*, volume 45 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.
- [28] M. Conder. Arc-transitive trivalent Cayley graphs. *J. Algebra*, 610:896–910, 2022.
- [29] M. Conder and P. Dobcsányi. Determination of all regular maps of small genus. *J. Combin. Theory Ser. B*, 81(2):224–242, 2001.
- [30] J. H. Conway, H. Burgiel, and C. Goodman-Strauss. *The symmetries of things*. A K Peters, Ltd., Wellesley, MA, 2008.
- [31] K. Coolsaet and S. Schein. Some New Symmetric Equilateral Embeddings of Platonic and Archimedean Polyhedra. *Symmetry*, 10(9), 2018.
- [32] H. S. M. Coxeter. *Regular complex polytopes*. Cambridge University Press, London-New York, 1974.
- [33] H. S. M. Coxeter, P. Du Val, H. T. Flather, and J. F. Petrie. *The fifty-nine icosahedra*. Springer-Verlag, New York-Berlin, 1982. Reprint of the 1938 original.
- [34] A. Császár. A polyhedron without diagonals. *Acta Univ. Szeged. Sect. Sci. Math.*, 13:140–142, 1949.

- [35] T. J. Dekker. On reflections in Euclidean spaces generating free products. *Nieuw Arch. Wisk.* (3), 7:57–60, 1959.
- [36] J. D. Dixon and B. Mortimer. *Permutation groups*, volume 163 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996.
- [37] F. Effenberger and J. Spreer. **simpcomp** - a GAP toolkit for simplicial complexes, Version 2.1.14, 2022.
- [38] M. Elgersma and S. Wagon. Closing a platonic gap. *Math. Intelligencer*, 37(1):54–61, 2015.
- [39] K. Enami. Embeddings of 3-connected 3-regular planar graphs on surfaces of non-negative Euler characteristic. *Discrete Math. Theor. Comput. Sci.*, 21(4):Paper No. 11, 19, 2019.
- [40] D. Eppstein. On polyhedral realization with isosceles triangles. *Graphs Combin.*, 37(4):1247–1269, 2021.
- [41] T. G. Fowler. *Unique coloring of planar graphs*. ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)–Georgia Institute of Technology.
- [42] R. Frucht. Graphs of degree three with a given abstract group. *Canad. J. Math.*, 1:365–378, 1949.
- [43] R. W. Frucht. How I became interested in graphs and groups. *J. Graph Theory*, 6(2):101–104, 1982.
- [44] The GAP Group. *GAP – Groups, Algorithms, and Programming*, Version 4.14.0, 2024.
- [45] I. García-Marco and K. Knauer. Beyond symmetry in generalized Petersen graphs. *J. Algebraic Combin.*, 59(2):331–357, 2024.
- [46] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified np-complete problems. In *Proceedings of the Sixth Annual ACM Symposium on Theory of Computing*, STOC ’74, page 47–63, New York, NY, USA, 1974. Association for Computing Machinery.
- [47] B. Ghanbari and R. Šámal. Approximate cycle double cover. In *Combinatorial algorithms*, volume 14764 of *Lecture Notes in Comput. Sci.*, pages 421–432. Springer, Cham, [2024] ©2024.
- [48] L. A. Goddyn. Cycle double covers of graphs with Hamilton paths. *J. Combin. Theory Ser. B*, 46(2):253–254, 1989.
- [49] T. Goertzen, D. Macek, L. Schnelle, M. Weiß, S. Reese, H. Holthusen, and A. C. Niemeyer. Influence of block arrangement on mechanical performance in topological interlocking assemblies: A study of the versatile block. *International Journal of Solids and Structures*, 306:113102, 2025.
- [50] T. Goertzen, A. Niemeyer, and W. Plesken. Topological Interlocking via Symmetry. In *Proc. of the 6th fib International Congress 2022*. Novus Press, Oslo, Norway, 2022.
- [51] J. L. Gross and T. W. Tucker. *Topological graph theory*. Dover Publications, Inc., Mineola, NY, 2001. Reprint of the 1987 original [Wiley, New York; MR0898434 (88h:05034)] with a new preface and supplementary bibliography.
- [52] B. Grünbaum. A convex polyhedron which is not equifacetable. *Geombinatorics*, 10(4):165–171, 2001.

- [53] B. Grünbaum and G. C. Shephard. Satins and twills: an introduction to the geometry of fabrics. *Math. Mag.*, 53(3):139–161, 1980.
- [54] B. Grünbaum. Conjecture 6 in recent progress in combinatorics. In W. T. Tutte, editor, *Proceedings of the Third Waterloo Conference on Combinatorics, May 1968*, page 343. Academic Press, New York-London, 1969.
- [55] T. F. Görtzen. *Constructing simplicial surfaces with given geometric constraints*. Dissertation, RWTH Aachen University, Aachen, 2024. Veröffentlicht auf dem Publikationsserver der RWTH Aachen University.
- [56] F. Hoffman, S. C. Locke, and A. D. Meyerowitz. A note on cycle double covers in Cayley graphs. *Math. Pannon.*, 2(1):63–66, 1991.
- [57] A. Hurwitz. Ueber algebraische Gebilde mit eindeutigen Transformationen in sich. *Math. Ann.*, 41(3):403–442, 1892.
- [58] I. Izmistiev, S. Klee, and I. Novik. Simplicial moves on balanced complexes. *Adv. Math.*, 320:82–114, 2017.
- [59] F. Jaeger. A survey of the cycle double cover conjecture. In B. Alspach and C. Godsil, editors, *Annals of Discrete Mathematics (27): Cycles in Graphs*, volume 115 of *North-Holland Mathematics Studies*, pages 1–12. North-Holland, 1985.
- [60] A. Kotzig. Regularly connected trivalent graphs without non-trivial cuts of cardinality 3. *Acta Fac. Rerum Natur. Univ. Comenian. Math.*, 21:1–14, 1968.
- [61] W. Kühnel and T. F. Banchoff. The 9-vertex complex projective plane. *Math. Intelligencer*, 5(3):11–22, 1983.
- [62] L. Lovász. *Combinatorial problems and exercises*. AMS Chelsea Publishing, Providence, RI, second edition, 2007. Page: 642.
- [63] Maple User Manual. *Maplesoft, a division of Waterloo Maple Inc.* Waterloo, Ontario, Canada, 1996-2025.
- [64] K. Markström. Even cycle decompositions of 4-regular graphs and line graphs. *Discrete Math.*, 312(17):2676–2681, 2012.
- [65] J. H. Mason. Can regular tetrahedra be glued together face to face to form a ring? *The Mathematical Gazette*, 56(397):194–197, 1972.
- [66] E. Miller, V. Reiner, and B. Sturmfels, editors. *Geometric combinatorics*, volume 13 of *IAS/-Park City Mathematics Series*. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2007. Lectures from the Graduate Summer School held in Park City, UT, 2004.
- [67] E. N. Miller. Icosahedra constructed from congruent triangles. *Discrete Comput. Geom.*, 24(2-3):437–451, 2000. The Branko Grünbaum birthday issue.
- [68] MIT CompGeom Group, H. A. Akitaya, E. D. Demaine, A. Hesterberg, A. Lubiw, J. Lynch, J. O’Rourke, F. Stock, and J. Tkadlec. Deltahedral Domes over Equiangular Polygons. *arXiv e-prints*, page arXiv:2408.04687, 2024.

- [69] B. Mohar. Strong embeddings of minimum genus. *Discrete Math.*, 310(20):2595–2599, 2010.
- [70] B. Mohar and C. Thomassen. *Graphs on surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.
- [71] A. C. Niemeyer, R. Akpanya, T. Goertzen, M. Weiß, L. Schnelle, and F. Fehre. GAPic, Version 0.1. <https://github.com/GAP-ART-RWTH/GAPic>, 2025.
- [72] A. C. Niemeyer, M. Baumeister, R. Akpanya, T. Goertzen, M. Weiß, and L. Schnelle. SimplicialSurfaces, Version 0.6. <https://github.com/gap-packages/SimplicialSurfaces>, 2025.
- [73] A. C. Niemeyer, W. Plesken, and D. Robertz. Conjecture on (1,1,1)-embeddings of simplicial spheres, 2022. Personal communication.
- [74] A. C. Niemeyer, W. Plesken, and D. Robertz. Simplicial surfaces of congruent triangles. *In Preparation*, 2025.
- [75] U. Pachner. Konstruktionsmethoden und das kombinatorische Homöomorphieproblem für Triangulationen kompakter semilinearer Mannigfaltigkeiten. *Abh. Math. Sem. Univ. Hamburg*, 57:69–86, 1987.
- [76] P. Potočník and G. Potočník. Datasets of highly symmetric objects. <https://graphsymb.net>, 2024. Accessed: 2024-12-23.
- [77] P. Potočník, P. Spiga, and G. Verret. Cubic vertex-transitive graphs on up to 1280 vertices. *J. Symbolic Comput.*, 50:465–477, 2013.
- [78] G. Ringel. *Map color theorem*, volume Band 209 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York-Heidelberg, 1974.
- [79] G. Ringel and J. W. T. Youngs. Solution of the Heawood map-coloring problem. *Proc. Nat. Acad. Sci. U.S.A.*, 60:438–445, 1968.
- [80] D. Robertz. *SimplicialSurfaceEmbeddings*: a Maple package for the analysis and construction of embedded simplicial surfaces.
- [81] P. D. Seymour. Sums of circuits. In *Graph theory and related topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977)*, pages 341–355. Academic Press, New York-London, 1979.
- [82] D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss. An aperiodic monotile. *Comb. Theory*, 4(1):Paper No. 6, 91, 2024.
- [83] D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss. A chiral aperiodic monotile. *Comb. Theory*, 4(2):Paper No. 13, 25, 2024.
- [84] E. Steinitz and H. Rademacher. *Vorlesungen über die Theorie der Polyeder unter Einschluss der Elemente der Topologie*, volume No. 41 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1976. Reprint der 1934 Auflage.
- [85] I. Stewart. Tetrahedral chains and a curious semigroup. *Extracta Math.*, 34(1):99–122, 2019.
- [86] A. W. Strzelczyk. *Simpliziale Flächen aus kongruenten Dreiecken : kombinatorische Grundlagen und geometrische Beispiele*. Dissertation, RWTH Aachen University, Aachen, 2019. Veröffentlicht auf dem Publikationsserver der RWTH Aachen University.

- [87] S. Stüttgen, R. Akpanya, B. Beckmann, R. Chudoba, D. Robertz, and A. C. Niemeyer. Modular Construction of Topological Interlocking Blocks—An Algebraic Approach for Resource-Efficient Carbon-Reinforced Concrete Structures. *Buildings*, 13(10), 2023.
- [88] G. Szekeres. Polyhedral decompositions of cubic graphs. *Bull. Austral. Math. Soc.*, 8:367–387, 1973.
- [89] M. Tarsi. Semiduality and the cycle double cover conjecture. *J. Combin. Theory Ser. B*, 41(3):332–340, 1986.
- [90] C. Thomassen. The genus problem for cubic graphs. *J. Combin. Theory Ser. B*, 69(1):52–58, 1997.
- [91] L. Venturello. Balanced triangulations on few vertices and an implementation of cross-flips. *Electron. J. Combin.*, 26(3):Paper No. 3.61, 24, 2019.
- [92] T. Watari, N. Yamashita, and A. C. Serrenho. Net-Zero Embodied Carbon in Buildings with Today’s Available Technologies. *Environmental Science & Technology*, 58(4):1793–1801, Jan. 2024. Publisher: American Chemical Society.
- [93] W. T. Webber. Monohedral idemvalent polyhedra that are toroids. *Geom. Dedicata*, 67(1):31–44, 1997.
- [94] N. S. Wedd. Weddlist, 2025. Accessed: 2025-01-22.
- [95] M. Weiß and A. C. Niemeyer. Interplay of cubic graphs and simplicial surfaces. *arXiv e-prints*, page arXiv:2410.22137, 2024.
- [96] C.-Q. Zhang. Circuit double covers of graphs. In *Graph theory—favorite conjectures and open problems. 1*, Probl. Books in Math., pages 273–291. Springer, [Cham], 2016.