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A simplified analysis of the ascending auction to sell a matroid base

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ABSTRACT

We give a simpler analysis of the ascending auction of Bikhchandani, de Vries, Schummer, and Vohra (OR 2011) to sell a welfare-maximizing base of a matroid at Vickrey prices. The new proofs for economic efficiency and the charge of Vickrey prices only require a few matroid folklore lemmas, therefore shortening the analysis of the design goals of the auction significantly.

1. Introduction

We consider a market consisting of a set of m indivisible distinguishable items E and a set of n buyers N, where each buyer $i \in N$ is interested in a subset $E_i \subseteq E$. Each buyer $i \in N$ has a valuation $v_i(e) \in \mathbb{Z}_+$ for every $e \in E_i$ which may be interpreted as buyer i's maximum willingness to pay for item e. We may also write $v_i(X) = \sum_{e \in X} v_i(e)$ for any subset $X \subseteq E_i$ to denote the additive valuation of any buyer. We consider the particular case where the auctioneer is constrained to sell a base of a matroid (see Section 2 for basics on matroid theory).

This model has been studied by Bikhchandani, de Vries, Schummer, and Vohra in the paper "An Ascending Vickrey Auction for Selling Bases of Matroids" presented at SODA in 2008 [1] and published in the journal OR in 2011 [2]. In the sequel, we only reference the journal version. [2] present an ascending auction for selling a base of a matroid, and prove that the auction fulfills the following three important mechanism design goals: (i) signaling truthfully is an ex-post equilibrium, (ii) the truthful equilibrium results in an efficient (i.e. welfare-maximizing) outcome, and (iii) the auction runs in strongly polynomial time.The main building block for showing that the ascending auction (described in Section 3 along with some basics on auction theory) fulfills properties (i)-(iii) is Theorem 13 in [2] (or Theorem 4 below). It states that the auction computes a welfare-maximizing base in strongly polynomial time (which yields (ii) and (iii) immediately) and charges Vickrey prices (which is the key for proving (i)). This paper provides a simpler proof of the theorem using only some matroid folklore lemmas.

While at first sight the restriction to auctions in which the auctioneer is constrained to sell a base of a matroid seems to be an abstract model in the mechanism design literature, it turns out to contain various interesting special cases. As described in [2], examples include scheduling matroids (cf. [3]), the allocation of homogeneous objects (cf. [4]), pairwise kidney exchange (cf. [5]), spatially distributed markets (cf. [6]), bandwidth markets (cf. [7]), and multi-class queueing systems (cf. [8]). Some of the examples require a generalization of the model to the setting where the auctioneer is constrained to sell a base of an integral polymatroid, and the buyers $i \in N$ have non-decreasing, concave, and piece-wise-linear valuations with integral breakpoints. However, as also observed in [2], this more general model can easily be reduced to our setting via Helgason's pseudo-polynomial reduction (cf. [9]). More recently, an extension of this auction that sells a base of a polymatroid which runs in polynomial time has been studied in [10]. Its analysis is still grounded on the original work in [2]. In particular, the analysis involves Theorem 4 to show that the updated auction computes VCG prices and its truthfulness also follows from the original paper.

It is well-known that there exists a *sealed-bid* auction for our model which fulfills the three mechanism design goals (i)-(iii), namely the VCG mechanism [11–13]. The VCG mechanism operates by first asking the buyers for their valuations for the items. Based on the reported valuations, which may or may not coincide with the true valuations, the VCG mechanism computes a welfare maximizing base, and charges Vickrey prices to every buyer. Due to the fact that the auctioneer is constrained to sell a base of a matroid, the computation of the welfare maximizing base and the Vickrey prices can be done in strongly polynomial time with the matroid greedy algorithm. There are, however, various reasons for preferring an ascending auction over a sealed-bid auction.

In general, in an ascending auction, the auctioneer announces prices, and the buyers report their demands at announced prices. The auction ends once the demands are feasibly satisfied. Otherwise, the auctioneer increases the prices. The here presented auction follows this outline and we describe it more precisely in Section 3. Typically ascending auctions have the drawback that they are slower than sealed-bid auctions since

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they require multiple rounds of communication as bid elicitation process. However, in many settings ascending auctions are still preferred over sealed-bid auctions as they are more transparent, require less information to be revealed by the buyers, and less bits of communication. In particular, in ascending auctions cognitively limited agents can recognize an optimal action more easily [14,15] and the auctioneer has less room for strategic behavior (to increase the revenue, for instance) [16,17]. For an extended discussion of advantages of ascending auctions over sealed-bid auctions see [4] and [18].

1.1. Related literature

Since there is an extensive literature on ascending auctions, we restrict in this paper only to those references that are closest to our model and our results, and refer to the references in e.g. Chapter 11 in [19] or [20,21] for a broader overview on the theory of ascending auctions.

Matroids play an important role in mechanism design theory, and particularly in the analysis of ascending auctions. To mention a prominent example, consider the general setting of auctions where buyers are interested in buying subsets (*bundles*) of items and have private valuations over the bundles. Walrasian equilibrium prices (cf. [22]) are guaranteed to exist if the valuation functions satisfy the *gross substitutes* condition [23,24]. In fact, by the Maximal Domain Theorem in [24], gross substitute valuations are the largest class containing unit-demand valuations that guarantee existence of a Walrasian equilibrium. The gross substitutes condition is closely related to matroids in the sense that the gross substitutes condition is equivalent to $M^\natural$-concavity [25,26], implying that the inclusion-wise minimal preferred bundles form the base set of a matroid for any given price vector (see e.g. [27]). The rich theory of matroids, polymatroids, and discrete convexity turned out to be extremely helpful for understanding the computational complexity of iterative auctions with gross substitutes valuations (see e.g. the survey [28] and [26,29,30]).

The preferences in our model also satisfy the gross substitutes condition, so that our model fits into the frameworks analyzed in [31–35]. In all of these frameworks, efficient outcomes are realized at equilibrium. However, these more general frameworks rely on an exponential (in the number of items) number of the price-raising steps. In contrast, the ascending auction for our model runs in strongly polynomial time without the necessity to run several dummy auctions (like in [31]), and without relying on a proxy bidding scheme (like in [33] and [36]). See [2] for a more detailed comparison with the above mentioned frameworks and also [10] for the extension to the polymatroid case.

1.2. Contribution and overview

We provide a simplified proof that the ascending auction described in [2] returns a welfare-maximizing base and charges Vickrey prices. The original proof requires a very microscopic analysis of the auction: In every step, one has to keep track of deleted items and cocircuits of the matroid (these are subsets of items from which at least one has to be sold) that act as a witness for price-setting items. This constructs an auction history, which is called *VCG sequence* in [2]. In contrast, our proof does not require this step-by-step view of the auction and instead only uses three matroid folklore lemmas which reduces the length of the proof significantly and avoids overly technical notation. The paper as a whole, however, is still self-contained: We introduce all terminology needed from matroid theory in Section 2, give a slightly amended description of the auction with examples in Section 3, and the aforementioned simplified analysis in Section 4.

2. Basics on matroid theory

We consider a finite ground set E, which in the sequel will be the set of items. For the sake of convenience, we may write $X+e \coloneqq X\cup\{e\}$, and $X-e \coloneqq X\setminus\{e\}$, for any subset $X\subseteq E$ and any element $e\in E$. Moreover, given a function $w\colon E\to\mathbb{R}$, we may write $w(X)\coloneqq\sum_{e\in X} w(e)$.

This section contains some basic definitions and facts from the theory of matroids, which originated in [37]. A modern textbook on matroid theory is the monograph [38]. Our notation is mainly based on Chapter 39 in the book [39]. Readers familiar with matroid theory are encouraged to skip this section.

We say a pair $M=(E,\mathcal{I})$ consisting of a finite *ground set* E and a family of *independent sets* $\mathcal{I}\subseteq 2^E$ is a *matroid* if (I1) $\emptyset\in\mathcal{I}$, (I2) for all $I_1\in\mathcal{I}$ and $I_2\subseteq I_1$, it holds that $I_2\in\mathcal{I}$, and (I3) for all $I_1,I_2\in\mathcal{I}$ with $|I_1|>|I_2|$, there exists $e\in I_1\setminus I_2$, such that $I_2+e\in\mathcal{I}$.

An inclusion-wise maximal independent subset of E is called a *base* and we denote the set of bases of a matroid by $\mathcal{B}$. For example, if $G=(V,E)$ is a connected undirected graph, the collection of all cycle-free edge sets forms the independence system of a matroid (called a *graphic matroid*), whose bases are exactly the spanning trees of G. It follows from (I3) that all bases have the same cardinality. A set X that is not independent (i.e. $X\notin\mathcal{I}$) is called *dependent*. An inclusion-wise minimal dependent set is called a *circuit*. A circuit of cardinality 1 is also called a *loop*. Two elements $e,f\in E$ are called *parallel* if $\{e,f\}$ is a circuit. In graphic matroids the terms circuit, loop, and parallel elements align completely with their graph theory counter parts, i.e. cycle, self-loop, and parallel edges, respectively. The *rank* $\rho(X)$ of a set $X\subseteq E$ is the size of any maximal independent set contained in X, i.e. $\rho(X)=\max\{|I|: I\subseteq X, I\in\mathcal{I}\}$. We may also refer to the rank $\rho(B)$ of any base $B\in\mathcal{B}$, as the *rank of the matroid* M, abbreviated with $\rho(M)$.

An inclusion-wise minimal set that has a non-empty intersection with every base of M is called a *cocircuit* of M. Notice that cocircuits correspond to inclusion-wise minimal cuts in the graphic matroid. A cocircuit of cardinality 1 (i.e. an element that is contained in every base) is called *coloop*. We denote by $\mathcal{C}^*$ the collection of all cocircuits of matroid M. Cocircuits (and also circuits) of matroids also obey the following cocircuit exchange property. For all $C_1^*,C_2^*\in\mathcal{C}^*$ with $e\in C_1^*\cap C_2^*$, there exists $C_3^*\in\mathcal{C}^*$, such that $C_3^*\subseteq(C_1^*\cup C_2^*)-e$ (Proposition 1.4.12 in [38]).

Given a matroid $M=(E,\mathcal{I})$ and a subset $Z\subseteq E$, we can derive two new matroids on ground set $E\setminus Z$ by *deleting* and *contracting* the elements in Z. The deletion operation results in the matroid $M\setminus Z\coloneqq(E\setminus Z,\mathcal{I}\setminus Z)$ with independent sets $\mathcal{I}\setminus Z\coloneqq\{I\subseteq E\setminus Z\mid I\in\mathcal{I}\}$. The contraction operation results in the matroid $M/Z\coloneqq(E\setminus Z,\mathcal{I}/Z)$ with independent sets $\mathcal{I}/Z\coloneqq\{I\subseteq E\setminus Z\mid I\cup Z^{\mathcal{B}}\in\mathcal{I}\}$, where $Z^{\mathcal{B}}$ denotes an arbitrary base of $M\setminus(E\setminus Z)$. It is well-known that each of the two *minors* $M\setminus Z$ and M/Z is again a matroid (see Section 39.3 in [39]). For the sake of convenience, we may also write $\mathcal{S}\setminus Z$ and $\mathcal{S}/Z$ for $\mathcal{S}\in\{\mathcal{B},\mathcal{C},\mathcal{C}^*\}$ for the set of bases, circuits, cocircuits of $M\setminus Z$ and M/Z, respectively. If Z is a singleton $\{e\}$, we may also write $M\setminus e$ and M/e instead of $M\setminus\{e\}$ and $M/\{e\}$, respectively. Also note that a series of deletions and contractions can be performed in any order of the elements, the resulting minor will always be the same (Proposition 3.1.25 in [38]). Hence, it is valid to write $M\setminus D/F$ to denote the minor of M after deleting D and contracting F in any particular order.

Given a matroid $M=(E,\mathcal{I})$ with base set $\mathcal{B}$, and a weight function $w\colon E\to\mathbb{R}_+$, a base $\hat{B}\in\mathcal{B}$ of maximal weight $w(\hat{B})=\sum_{e\in\hat{B}} w(e)$ can be computed with the *matroid greedy algorithm* which starts with the initially empty independent set $I\coloneqq\emptyset$, and iteratively, in the order of non-increasing weights $w(e_1)\geq\ldots\geq w(e_m)$, adds the next element e_t to I in iteration $t\in[m]$ if (and only if) $I+e_t\in\mathcal{I}$. For details see Theorem 40.1 in [39]. Hence, with the greedy algorithm, it is easy to determine whether I is a base or whether $E\setminus X$ for some X still contains a base. Those kind of queries will become important in the auction presented in the next section.

3. Description of the ascending auction

Recall that we consider an auction in which an auctioneer is selling a finite set of distinguishable, indivisible items E to a finite set of buyers N. We assume that each buyer $i\in N$ is interested in buying only

the items in a subset $E_i \subseteq E$, and that each buyer $i \in N$ has an additive valuation function. That is, they have a non-negative integral valuation $v_i(e) \in \mathbb{Z}_+$ for each item $e \in E_i$ and for each subset $X \subseteq E_i$, we write $v_i(X) = \sum_{e \in X} v_i(e)$. In the auction we consider, the auctioneer is constrained to sell a base of a given matroid $M = (E, \mathcal{I})$ on E, where $\mathcal{I}$ denotes the collection of independent sets.

We may assume that E only contains those items for which at least one buyer is interested in buying them. Notice that we can assume w.l.o.g. that no two players are interested in buying the same item. If not, say if $e \in E_i \cap E_j$ for some item $e \in E$ and two buyers $i \neq j$, we can replace e by two parallel copies e^i and e^j. More concretely, we replace $M = (E, \mathcal{I})$ by $M' = (E', \mathcal{I}')$ with ground set $E' = E - e \cup \{e^i, e^j\}$ (and $E_i' = E_i - e + e^i, E_j' = E_j - e + e^j$) and independence system $\mathcal{I}'$, where $\mathcal{I}' = \{I \in \mathcal{I} \mid e \notin I\} \cup \{I - e + e^i, I - e + e^j : I \in \mathcal{I}, e \in I\}$. This construction extends naturally to the case when even more than two buyers are interested in the same item. In other words, we can assume that $\{E_i\}_{i \in N}$ is a partition of E.

Thus, an instance of our auction is defined by a matroid $M = (E, \mathcal{I})$ on a finite set E, a partition of E into $E = \bigcup_{i \in N} E_i$, and a valuation function $v\colon E \to \mathbb{R}_+$ with $v(e) = v_i(e)$ for the (unique) buyer $i \in N$ with $e \in E_i$. We furthermore assume that none of the buyers initially is a monopsonist in the sense that no cocircuit of M is contained in any E_i. Otherwise, we could just resolve the monopsonies by selling an item from the monopsony (buyer's choice) for price 0 and perform a contraction on the matroid with that item. We resolve monopsonies that occur later in the auction in a similar way (see below).

Before we consider the ascending auction, we remark that there exists a *sealed-bid* Vickrey auction which runs in strongly polynomial time for this model: the auctioneer starts by collecting the bids $b(e) \in \mathbb{Z}_+$ from every buyer $i \in N$ for every item $e \in E_i$, which may or may not be equal to her true valuation $v_i(e)$. Based on the resulting bid vector $b = (b(e))_{e \in E}$, the auctioneer first computes a base $\hat{B} \in \mathcal{B}$ of maximum b-weight. To compute $\hat{B}$, the auctioneer might use the matroid greedy algorithm as described above. Afterwards, the auctioneer computes for every buyer $i \in N$ the Vickrey price, which incentivizes truthful bidding (cf. [11–13]). In our setting the Vickrey price of buyer $i \in N$ is

$$\hat{p}_{\hat{B}}(i) = \max_{B \in \mathcal{B} \setminus E_i} b(B) - b(\hat{B} \setminus E_i).$$

Since for every $i \in N$ a base of maximum b-weight in $M \setminus E_i$ can be computed with the matroid greedy algorithm, the auction runs in strongly polynomial time. As already argued in the introduction, we prefer an ascending auction over a sealed-bid auction, since an ascending auction is more transparent and requires less information from the buyers.

The paper [2] describes and analyzes an ascending auction (more precisely a *clock auction*) that meets the three mechanism design goals to (i) incentivize truthful signals, (ii) compute a welfare-maximizing base, and (iii) run in strongly polynomial time. In this section, we give a slightly modified but equivalent presentation of the auction introduced in [2]. An analysis of the auction, which is significantly simpler than the one in [2], is presented in Section 4.

Sketch of the ascending auction. At any point of the auction, we can partition the items in E into three parts: the set of deleted items D, the set of contracted items I, and all remaining items $E \setminus (D \cup I)$. Intuitively, items in D will remain unsold, the items in I are already sold, and for all remaining items are still active in the auction, which may or may not be sold later. A key ingredient of the auction is that the auctioneer has to keep track of the cocircuits of the currently observed matroid $M \setminus D / I$ (a minor of the original matroid M). As cocircuits are the inclusion-wise minimal sets that have a non-empty intersection with every base, the auctioneer has to make sure that he sells at least one item of each cocircuit in order to sell a base. The auctioneer starts at all zero prices and increases the price for all items simultaneously. At any time, a buyer j has the option to announce that there is an item $f \in E_j$ that she would not buy if the price (currently p) would increase further. She may also announce this for multiple items of her set and other buyers may do so at the same time. In the sequel, we refer to those items as *critical* at price p. Then, the auctioneer removes critical items from the matroid one after another. This removal will be carried out as a *deletion* in the matroid. Now as every item occurs in some cocircuit (unless it is a loop) and the cocircuits of the restricted matroid are the same up to removal of the deleted item (see Lemma 2 below), some cocircuits decrease in size. At some point there will be a cocircuit C^* that is completely contained in some buyer's item set E_i; in that case, we call C^* a *monopsony*. Note that in [2], they refer to such a cocircuit as a *monopoly*, which is just as accurate in case of a reverse auction or procurement. Whenever a monopsony $C^* \subseteq E_i$ occurs, the auctioneer asks the buyer i (the monopsonist) to name her most valuable item e from C^*. This announced item e will then immediately be sold to buyer i at price p (in the words of Ausubel [4], buyer i has *clinched* an item). After item e is sold, the auctioneer removes e from the matroid. This removal, however, will be performed as a *contraction*. Note that a deletion of an item f might create multiple monopsonies (up to $n - 1$) which will all be resolved with the aforementioned procedure. After that the auctioneer resumes to deleting critical items until he either recognizes another monopsony (he then proceeds as above) or he runs out of critical items (he then increases the price of all remaining items by 1). Note that it might very well happen that an item that is announced as critical at price p still gets sold at price p if there are multiple items of the same cocircuit announced as critical at the same time. In that case this item will just be skipped by the auctioneer in the deletion loop.

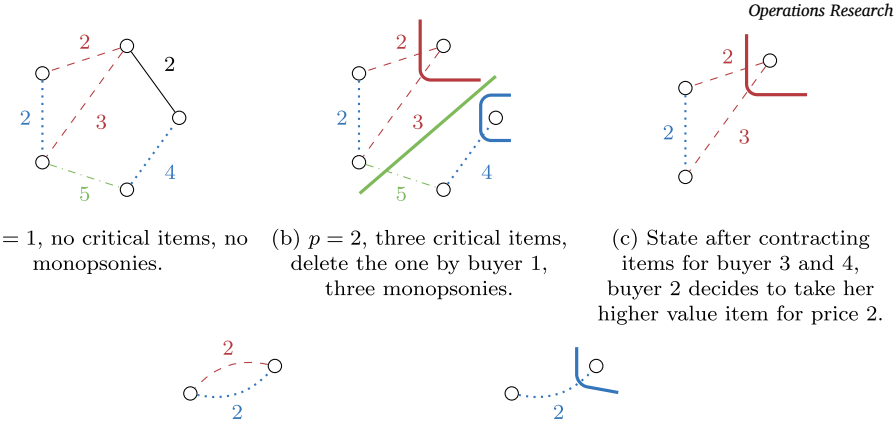
The pseudocode Ascending Matroid Auction below gives a precise description of the auction. We amended the original pseudocode of [2] only in the sense that we left out the tie-breaking (which is arbitrary anyway) and perform the deletion steps right away instead of after resolving all monopsonies that occur.

Initially, the auctioneer only knows the matroid $M = (E, \mathcal{I})$ and the partition $\{E_i\}_{i \in N}$. For all further information, the auctioneer has to receive *signals* from the buyers, i.e. answers to queries about their valuation functions (see paragraph on truthful signaling below). As usual in matroid optimization, we assume that the auctioneer has access to an independence oracle. That is, he might query for any set $X \subseteq E$, whether $X \in \mathcal{I}$ in time $\mathcal{O}(1)$. Note that, using an independence oracle, the auctioneer can efficiently check whether the conditions in the two **while**-loops are satisfied.

Ascending Matroid Auction: $M = (E, \mathcal{I})$ monopsony-free, partition $\{E_i\}_{i \in N}$ of E.

```
1  I := ∅, p := 0, p̂(e) := 0 for all e ∈ E
2  while I ∉ B do
3      p := p + 1
4      R(p) := {f ∈ E | f announced as critical at price p}
5      for f ∈ R(p) do
6          if f ∉ I then
7              M := M \ f
8              while M has monopsony C* ⊆ E_i of some buyer i do
9                  e := item from C* that buyer i chooses at price p
10                 I := I + e
11                 p̂(e) := p
12                 M := M / e
13 return B̂ := I and p̂
```

Truthful signaling. We emphasize that the communication between auctioneer and buyers takes place only in lines 4 and 9. We say that buyer i is *signaling truthfully* if her signals to the auctioneer are consistent with v_i in the following sense: If buyer i is asked to report on her critical items in line 4, she truly announces those items e as critical for which $v_i(e) = p$. Furthermore, whenever the auctioneer asks buyer i for her



(a) $p = 1$, no critical items, no monopsonies.

(b) $p = 2$, three critical items, delete the one by buyer 1, three monopsonies.

(c) State after contracting items for buyer 3 and 4, buyer 2 decides to take her higher value item for price 2.

(d) Delete critical item of buyer 2.

(e) Buyer 3 has now a monopsony, sell her the item.

Fig. 1. Example iteration on a graphic matroid with four buyers 1 (black, solid), 2 (red, dashed), 3 (blue, dotted), 4 (green, dotdashed): For $p = 0$ and $p = 1$ there are no critical items. When p increases to 2 multiple items get critical and trigger some sales, including some of critical items. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

most-valued item in her monopsony $C^* \subseteq E_i$ in line 9, she truly selects the item of largest v-value in her monopsony C^*.

Example. Consider the graphic matroid in Fig. 1 with edges labeled by its buyer's valuation. At price $p = 1$, there are no critical items (Fig. 1a). Hence, the auctioneer raises the price of all items to $p = 2$ at which multiple items become critical. After deleting one of those critical items, the auctioneer observes that three buyers now hold a monopsony (Fig. 1b). While two of those monopsonies are singletons, one buyer still has to signal to the auctioneer which item in her cocircuit she values most (Fig. 1c). Note that in this example untruthful signaling would be a strategy that is inconsistent with every possible valuation function since one of the two items was announced as critical while the other one was not). After contracting all best elements from each monopsony, the auctioneer is left with two more parallel elements, both of which are critical, so he performs another deletion (Fig. 1d) resulting in another singleton monopsony (Fig. 1e) which can be contracted.

4. Analysis of the auction

Recall the three mechanism design goals (i)-(iii) described in the Introduction. Computational efficiency, i.e. property (iii), is clearly fulfilled: The running time of the auction scales polynomially in the number of items and buyers. It even runs in strongly polynomial time by using a long-step variant that only requires a reasonable consistency check on the signals (see [2] and [10] for the polymatroid case). That is, by replacing line 3 in Ascending Matroid Auction with

3' | $p := \min_{i \in N} \{q_i \mid \text{smallest } q_i > p \text{ for which } i \text{ has a critical item}\}$

and the understanding that in line 4 every buyer who minimized the term in line 3' has to announce at least one item as critical, we obtain a long-step auction that results in the same outcome.

The keys to show that the Ascending Matroid Auction yields a welfare maximizing allocation (under truthful signals) and assigns Vickrey prices are a few statements about cocircuits and how they evolve by taking minors of the matroid.

In essence, we will need the following three matroid folklore lemmas.

Lemma 1 *(Cocircuits after deletion, [38, Exercise 2, Section 3.1]). Let C^* be a cocircuit of M and $e \in E$. Then $C^* - e$ is a union of cocircuits of $M \setminus e$.*

Lemma 2 *(Cocircuits before deletion, [39, End of Section 39.3]). Let C^* be a cocircuit of $M \setminus e$. Then C^* or $C^* + e$ is a cocircuit of M.*

Lemma 3 *(Augmentation by cocircuit, [40, Theorem 1]). Let I be an independent set of a matroid M that can be extended to a maximum weight base. Let C^* be any cocircuit of M / I and $e \in C^*$ its maximum weight element. Then $I + e$ is independent and can be extended to a maximum weight base.*

Now let us state and prove the main theorem of [2].

Theorem 4 *(Theorem 13 in [2]). The Ascending Matroid Auction returns a welfare-maximizing allocation and assigns Vickrey prices.*

Proof. For the proof, we assume truthful signals from the buyers. The discussion in Section 4.1 shows why this is justified.

First, we show that the resulting allocation $\hat{B}$ is welfare-maximizing. For that it suffices to show that I, the set of sold items in the Ascending Matroid Auction, is max-weight base extendable throughout the whole auction. Since we start with $I = \emptyset$, the statement is obviously true in the beginning. Given some stage of the auction with sold items I and deleted items D, we consider the minor $M \setminus D / I$. Say $e \in E \setminus (D \cup I)$ is the next item to be added to I. This means e is an item of maximum value in some monopsony C^*, i.e. a cocircuit of $M \setminus D / I$. Thus, by applying Lemma 2 repeatedly, we have that $\tilde{C}^* = C^* \cup D'$ is a cocircuit of M / I for some $D' \subseteq D$. As $v(e) \geq v(f)$ for all $f \in D$ (and in particular D'), e has also maximum value in $\tilde{C}^*$. Hence, by Lemma 3, $I + e$ is max-weight base extendable.

Now let us show that the auction also charges Vickrey prices to the buyers. For each $e \in \hat{B}$, let f_e denote the item which is deleted before e was added to I, i.e. f_e is the price-setting item for e and we have $p(e) = v(f_e)$. We need to show that for all $i \in N$

$$\hat{B}' = \hat{B} \setminus E_i \cup \{f_e : e \in \hat{B} \cap E_i\}$$

is a max-weight base of $M \setminus E_i$. Note that a deletion of an item f can create at most one monopsony per buyer: If C_1^*, C_2^* are both monopsonies of buyer j, then $C_1^* + f$ and $C_2^* + f$ are both cocircuits in the matroid

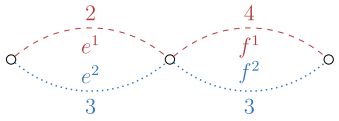


Fig. 2. Graphic matroid with two buyers 1 (red, dashed) and 2 (blue, dotted) where signaling truthfully is not a dominant strategy for buyer 1.

before deleting f and hence, by the cocircuit exchange property, there exists a cocircuit $C_3^* \subseteq (C_1^* \cup C_2^*) - f$ which would be a monopsony of j before deleting f. That is, $\hat{B}'$ has the correct cardinality to be a base of $M \setminus E_i$ (recall that initially there is no monopsony in M). Thus, we only need to show that $\hat{B}'$ is indeed welfare-maximizing and independent $M \setminus E_i$. We again use Lemma 3 to show that all elements $e \in \hat{B} \setminus E_i$ are still suitable for the new base and that for all $e \in \hat{B} \cap E_i$, its corresponding f_e is a suitable replacement. Consider some $e \in \hat{B}$ and the set I before e was added. Then e was an item of maximum value in some monopsony C^* of $M \setminus D / I$. We also have $v(e) \geq v(f)$ for all $f \in D$. Thus, e also has maximum value in $\tilde{C}^* = C^* \cup D'$, a cocircuit of M / I with $D' \subseteq D$ by applying Lemma 2. Moreover $f_e \in D'$ as otherwise C^* would have been a monopsony before f_e was deleted. From here, we make a case distinction:

Case 1: $e \notin E_i$. By applying Lemma 1 on M / I w.r.t. all the items in E_i, we also have that e has maximum value in some cocircuit of $M / I \setminus E_i = M \setminus E_i / I$ and hence, by Lemma 3, $I + e$ is max-weight base extendable for $M \setminus E_i$.

Case 2: $e \in E_i$. Deleting E_i (including e) yields the minor $M / I \setminus E_i = M \setminus E_i / I$. Note that $v(f_e) \geq v(f)$ for all $f \in D$ as f_e was deleted last. Applying Lemma 1 on M / I w.r.t. all items in E_i yields that there is a cocircuit $\check{C}^* \subseteq \tilde{C}^*$ of $M \setminus E_i / I$ with f_e as item of maximum value. So by Lemma 3, we have that $I + f_e$ is max-weight base extendable in $M \setminus E_i$. □

4.1. From Theorem 4 to truthful equilibrium

Note that one cannot simply apply the VCG Theorem [11–13] to claim that the auction incentivizes truthful signaling just because the prices match Vickrey prices. The reason is that, in contrast to a sealed-bid auction, in an ascending auction buyers gain information about the other buyers' valuations as the auction progresses. Indeed, one can show that truthful signaling is not necessarily a dominant strategy in this auction (and in fact, no dominant strategy need to exist).

Example. Consider the graphic matroid in Fig. 2 and two buyers $N = \{1,2\}$, where $E_i = \{e^i, f^i\}$ for both $i \in N$. Let $v_1 = (2,4)$ and $v_2 = (3,3)$ (where $v_i = (v_i(e^i), v_i(f^i))$). Consider two different types of buyer 2: cooperative or spiteful. Let us say in both types, buyer 2 signals truthfully if buyer 1 does. However, if buyer 1 plays altruistically, and announces e^1 as critical already at price 1 (giving buyer 2 more payoff), then the cooperative version of buyer 2 also announces f^2 as critical at an earlier point, say at $p = 2$. On the other hand, the spiteful version of buyer 2 might only announce at price 4 that f^2 is critical, lowering buyer 1's utility to 0. If buyer 2 is cooperative, then the (non-truthful) altruistic signaling is better than signaling truthfully; however, against a spiteful buyer 2, truthful signaling is better than altruistic signaling, which makes neither strategy dominant.

However, the following theorem can be shown.

Theorem 5 *(Theorem 17 in [2]). Truthful signaling is an ex-post equilibrium of the Ascending Matroid Auction.*

The proof is a bit tedious but straight-forward: A signaling strategy that is inconsistent with any valuation function can always be replaced with a consistent signaling strategy without changing the outcome for any buyer, provided that all other buyers signal truthfully. However, among all consistent strategies, a truthful one is best by the VCG Theorem. We refer to [2] for proof details.

This justifies that (here and in [2]) Theorem 4 is proven under the assumption that all buyers are signaling truthfully.

CRediT authorship contribution statement

Britta Peis: Writing – review & editing. **Niklas Rieken:** Writing – review & editing, Writing – original draft.

Data availability

No data was used for the research described in the article.

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