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Research paper

Optimal planar immersions of prescribed winding number and Arnold invariants

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ABSTRACT

Vladimir Arnold defined three invariants for generic planar immersions, i.e. planar curves whose self-intersections are all transverse double points. We use a variational approach to study these invariants by investigating a suitably truncated knot energy, the tangent-point energy. We prove existence of energy minimizers for each truncation parameter $\delta > 0$ in a class of immersions with prescribed winding number and Arnold invariants, and establish Gamma convergence of the truncated tangent-point energies to a limiting renormalized tangent-point energy as $\delta \to 0$. Moreover, we show that any sequence of minimizers subconverges in C^1, and the corresponding limit curve has the same topological invariants, self-intersects exclusively at right angles, and minimizes the renormalized tangent-point energy among all curves with right self-intersection angles. In addition, the limit curve is an almost-minimizer for all of the original truncated tangent-point energies as long as the truncation parameter δ is sufficiently small. Therefore, this limit curve serves as an "optimal" curve in the class of generic planar immersions with prescribed winding number and Arnold invariants.

1. Introduction

Planar closed immersed curves $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ can be classified in terms of their winding number $W(\gamma)$ which equals the degree of the map $s \mapsto \gamma'(s)/|\gamma'(s)|$. Roughly speaking, the winding number counts the number of turns of the tangent vector while traveling once along the curve. The Whitney-Graustein theorem [1, Theorem 1][1] states: *Two planar curves are regularly homotopic if and only if they have the same winding number.* V.I. Arnold proposed in [3,4] a much deeper plan of study and introduced three invariants J^+, J^- and St, which are locally constant on *generic* immersions, i.e. on curves where all self-intersections are transverse double points. It turns out that the combinatorics of generic loops is potentially as rich as that of knots in 3-space.

Our main motivation is to explore with variational tools – similarly in spirit as in geometric knot theory – the different connected components of generic planar immersions with prescribed Arnold invariants. One may ask, e.g., to what extent such components of generic curves are potential wells for some energy? One typically expects energy minimizers and critical points to be particularly interesting configurations in the given topological class. Among other things, one hopes for nice geometric properties, so that these minimizers can serve as optimal curves in the class of immersions with prescribed invariants.

In geometric knot theory one minimizes so-called *knot energies* to find optimal shapes in given knot classes. This is possible since knot energies blow up along sequences of curves that converge to a limit curve with self-intersections. Immersed planar curves with winding number different from ± 1, however, must have self-intersections, so we need to desingularize the knot energy near double points to obtain finite energy values.

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[1] For an alternative contact-geometric proof of the Whitney-Graustein theorem see H. Geiges [2].

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Since Arnold's invariants are well-defined on C^1-immersions we choose here the *tangent-point energy* TP_q for some $q > 2$ whose natural energy space is a fractional Sobolev space that embeds compactly into C^1. This energy is well-suited for our variational approach to study the space of generic immersions, since it is uniquely minimized by the (round) circle among all embedded closed space curves, and in every prescribed knot class there is a minimizing knot. Additional critical knots were found by symmetric criticality. Moreover, long-time existence for the (Banach-)gradient flow was established, and subconvergence of its solution to critical points holds for variants of the tangent-point energy whose energy space is a Hilbert space. In that case, the Palais–Smale condition holds, which opens up the possibility of Ljusternik–Schnirelman theory. In addition, the tangent-point energy induces a complete Riemannian metric on the open subset of embeddings in that Hilbert space, together with the existence of distance-minimizing geodesics connecting any given pair of knots in the same knot class; see Section 1.2 for more details and references.

To treat immersions with double points we cut out small δ-neighborhoods of the curves' self-intersections to obtain the *truncated* and therefore desingularized variant $\mathrm{TP}_{q,\delta}$ of the tangent-point energy. We prove for each $\delta > 0$ the existence of $\mathrm{TP}_{q,\delta}$-minimizers in a class of generic planar immersions with fixed winding number W and prescribed Arnold invariants J^+, J^-, and St. Rescaling the truncated energies with a suitable power of the truncation parameter δ adapted to the energy blow-up near self-intersections, and then sending δ to zero, we establish Gamma convergence of the rescaled energies to a limiting energy R_q, the *renormalized tangent-point energy*, that measures only the intersection angles between the tangent lines at the double points. The limiting process $\delta \to 0$ also yields a C^1-convergent subsequence of minimizers to a limit curve with the same topological invariants and with exclusively right angles at every self-intersection. Moreover, this limit curve minimizes the renormalized tangent-point energy R_q among all generic immersions of the given topological class with right self-intersection angles, and it almost-minimizes in addition the original truncated energies $\mathrm{TP}_{q,\delta}$ in the full topological class for all sufficiently small δ. Because of these minimizing properties these limiting immersions may be regarded as optimal configurations in their topological classes. The precise mathematical statements are formulated in Section 1.3, but for a first impression of the shapes of minimizing configurations for various prescribed values of winding number and Arnold invariants we refer the reader to the numerically computed minimizers depicted in Fig. 1.

1.1. Arnold invariants

If a regular homotopy between two generic planar immersions leaves the class of generic immersions, different degeneracies could happen: intermediate curves with self-tangencies or triple points, or more complicated self-intersections. Arnold [3,4] defined the *discriminant* Δ as the set of all non-generic immersions. It turns out that the discriminant contains a "good" part, which is a submanifold of codimension one in the space of all planar C^1-immersions. This good part consists of three different types of curves that are generic apart from exactly one degenerate self-intersection. The degeneration can manifest itself either as a direct self-tangency of multiplicity two, where the tangent vectors point in the same direction, an inverse self-tangency of multiplicity two, where the tangent vectors point in opposite directions, or a transverse triple point. We denote these submanifolds by Δ^d, Δ^i and Δ^t, respectively. To any transverse intersection of a path in the space of immersions with the good part $\Delta^d \cup \Delta^i \cup \Delta^t$ of the discriminant one can then assign a well-defined sign, e.g., an intersection with Δ^d or with Δ^i is positive if the number of double points increases along the homotopy, see Fig. 2.

The invariants are then defined as follows: For any winding number, fix a representative curve γ_R^ω; see Fig. 3.

Arnold prescribes values of the three invariants on the representative curves as follows:

$$\begin{aligned}
&J^+(\gamma_R^0) = J^+(\tilde{\gamma}_R^0) = 0, \qquad J^+(\gamma_R^\omega) = -2\,(|\omega| - 1) \ \text{ for } \omega \in \mathbb{Z} \setminus \{0\},\\
&J^-(\gamma_R^0) = J^-(\tilde{\gamma}_R^0) = -1, \quad J^-(\gamma_R^\omega) = -3\,(|\omega| - 1) \ \text{ for } \omega \in \mathbb{Z} \setminus \{0\},\\
&St(\gamma_R^0) = St\left(\tilde{\gamma}_R^0\right) = 0, \qquad St(\gamma_R^\omega) = |\omega| - 1 \text{ for } \omega \in \mathbb{Z} \setminus \{0\}.
\end{aligned}$$

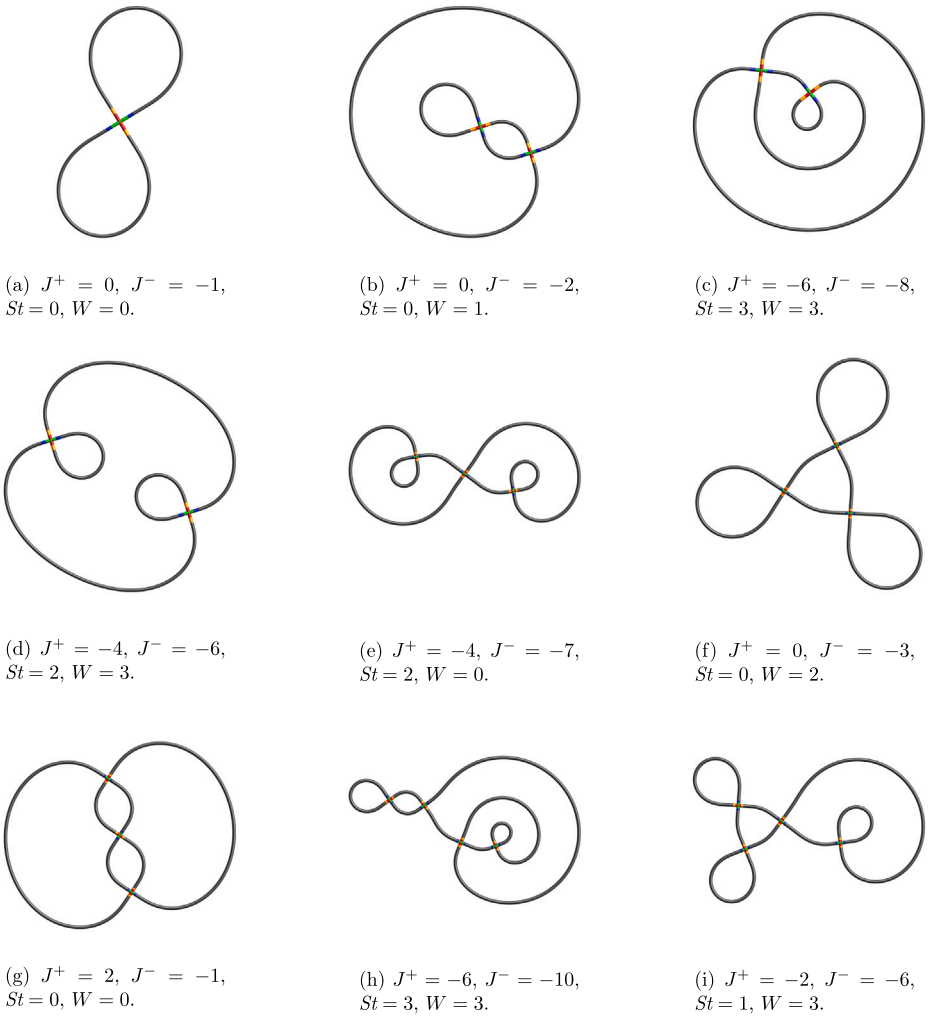
By the Whitney-Graustein theorem one can find a homotopy between any generic immersion and the corresponding representative curve with the same winding number. In fact, that path can be chosen in such a way that it intersects the discriminant Δ only in its good part $\Delta^d \cup \Delta^i \cup \Delta^t$ and only in finitely many points. Such a path will be called a *generic path*. For an illustration, see Fig. 4. Arnold then establishes rules on how the invariants change at intersections with the discriminant according to the given sign as follows:

(i) The invariant J^+ increases by 2 under a positive crossing of Δ^d and remains unchanged at crossings with Δ^i and Δ^t.
(ii) The invariant J^- increases by -2 under a positive crossing of Δ^i and remains unchanged at crossings with Δ^d and Δ^t.
(iii) The invariant St increases by 1 under a positive crossing of Δ^t and remains unchanged at crossings with Δ^d and Δ^i.

By construction, the following equality holds:

$$J^+(\gamma) - J^-(\gamma) = \text{number of self-intersections of } \gamma. \tag{1.1}$$

To prove existence and well-definedness of the Arnold invariants, one needs to show the existence of such a generic path intersecting the discriminant only in the codimension one part and at finitely many points. Furthermore, one has to verify that the definition of the invariants is independent of the chosen path. A detailed proof can be found in [5, Chapter 5], which fills some gaps in the original proof of Arnold. Explicit formulas to compute the Arnold invariants as well as sharp upper and lower bounds on them can be found, e.g., in [6–8]. Generalizations to planar immersions of the real line or to closed spherical or real algebraic curves were treated in [7,9,10], whereas the original Arnold invariants reappear as coefficients in the Taylor expansions of curvature



(a) $J^+ = 0,\ J^- = -1,$ $St = 0,\ W = 0.$

(b) $J^+ = 0,\ J^- = -2,$ $St = 0,\ W = 1.$

(c) $J^+ = -6,\ J^- = -8,$ $St = 3,\ W = 3.$

(d) $J^+ = -4,\ J^- = -6,$ $St = 2,\ W = 3.$

(e) $J^+ = -4,\ J^- = -7,$ $St = 2,\ W = 0.$

(f) $J^+ = 0,\ J^- = -3,$ $St = 0,\ W = 2.$

(g) $J^+ = 2,\ J^- = -1,$ $St = 0,\ W = 0.$

(h) $J^+ = -6,\ J^- = -10,$ $St = 3,\ W = 3.$

(i) $J^+ = -2,\ J^- = -6,$ $St = 1,\ W = 3.$

Fig. 1. Examples of numerically computed[2] minimizers of the energy $\mathrm{TP}_{q,\delta}$ in admissibility classes $\mathcal{F}(\eta, j_\pm, s, \omega)$ of curves with different prescribed Arnold invariants $J^+ = j_+$, $J^- = j_-$, $St = s$, and winding numbers $W = \omega$. All these minimizers seem to self-intersect exclusively in right-angles. The competing curves in $\mathcal{F}(\eta, j_\pm, s, \omega)$ are affine linear within arclength η around every self-intersection; see Definition 1.1. The ratio of the truncation parameter δ (green or red) to η (blue or yellow) is approximately $1/2$ in our computations. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

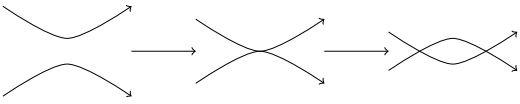


Fig. 2. Homotopy through a positive direct self-tangency.

integral formulas [7,11,12]. They also play a role in the analysis of periodic orbits of the restricted three-body problem [13–15], and there is a connection to Arnold-type invariants for so-called (weak) flat knot types of curves on surfaces studied in the context of curve-shortening or Finsler geodesic flows or, more generally, in Hamiltonian dynamics [16,17].

Because of various relations between the Arnold invariants and bounds on them (see, e.g., [3, Theorem p. 43] and Arnold's conjectures [3, pp. 60–61] proven in [6,7]) it is clear that there are combinations of values for the Arnold invariants that cannot

[2] We thank Henrik Schumacher who taught us how to use his code developed for knot energies on embedded space curves [18], and how to adapt it to the present situation of planar immersed curves.

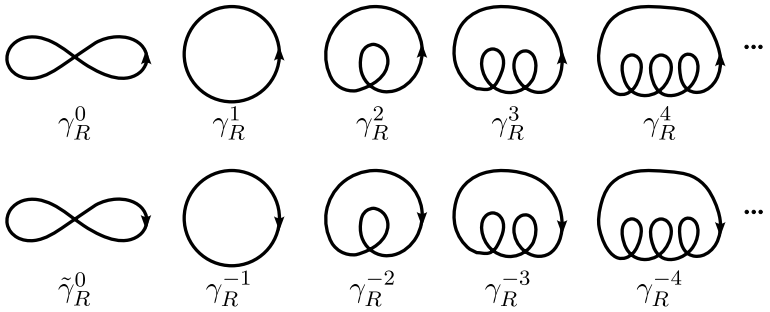


Fig. 3. Arnold's representative curves γ_R^ω for various winding numbers $\omega \in \mathbb{Z}$ ([3, cf. Figure 5]).

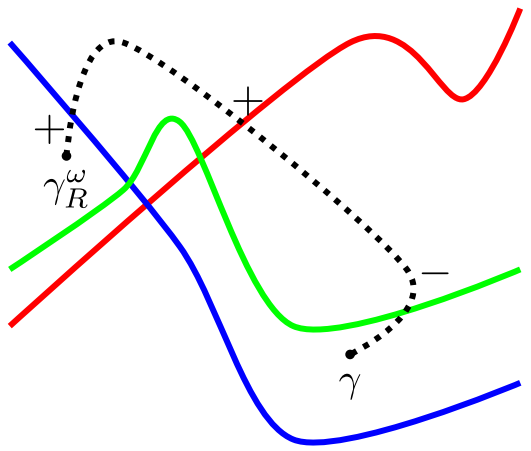


Fig. 4. The space of planar C^1-immersions with fixed winding number equal to ω. The three colored lines without their mutual intersections correspond to the good part $\Delta^d \cup \Delta^i \cup \Delta^t$ of the discriminant Δ. γ_R^ω is the representative curve with winding number ω, and the dotted line represents a generic path connecting γ_R^ω with a generic immersion γ of equal winding number. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

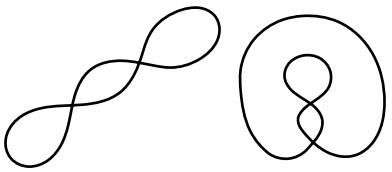


Fig. 5. Two curves with Arnold invariants $J^+ = 0$, $J^- = -2$, $St = 0$ and winding number $W = 1$.

be realized by any generic immersion. But even if there exist, say, two generic immersions with identical winding number and Arnold invariants it may happen that these curves cannot be deformed into each other by regular homotopy without leaving the class of generic immersions during that deformation; see Fig. 5. In other words, the set of all generic immersions γ with fixed Arnold invariants $J^+(\gamma) = j_+$, $J^-(\gamma) = j_-$, and $St(\gamma) = s$ and winding number $W(\gamma) = \omega$ splits into several open path-connected components with respect to regular homotopy. In the sequel we call any such component a $(j_\pm, s, \omega)$-*compartment* $C(j_\pm, s, \omega)$, and we are going to investigate such compartments with energy methods.

1.2. Desingularized knot energies

A variational approach to investigate knotted space curves is to consider knot energies as proposed by Fukuhara [19], and the first examples of such energies were introduced by O'Hara in [20]. Typically, knot energies model self-repulsive behavior and therefore provide infinite energy barriers between different knot classes since any curve with self-intersections has infinite energy. However, planar curves exhibit self-intersections unless they can be deformed into the circle. Hence, the value of any knot energy would be infinite on curves with self-intersections, which is why a modification of these energies is needed. Dunning investigated

in [21,22] minimizers of a truncated version of the Möbius energy[3] in the class of figure-eight shaped curves. Later on, Kube [24] closed several gaps in the work of Dunning and identified a renormalized version of the Möbius energy to characterize an optimally immersed figure-eight curve in terms of its single intersection angle.

We study a modified version of the so-called *tangent-point energy*

$$\mathrm{TP}_q(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^2} \frac{|\gamma'(s)||\gamma'(t)|}{\big(r_{\mathrm{tp}}[\gamma](\gamma(t),\gamma(s))\big)^q}\,ds\,dt = \iint_{(\mathbb{R}/\mathbb{Z})^2} \left(\frac{2\,\mathrm{dist}\,(\ell(t),\gamma(s))}{|\gamma(t)-\gamma(s)|^2}\right)^q |\gamma'(s)||\gamma'(t)|\,ds\,dt \tag{1.2}$$

for $q > 2$ and regular closed curves $\gamma \in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^d)$, $d \geq 2$. This energy was first investigated analytically by P. Strzelecki and H. von der Mosel in [25]. Here, $r_{\mathrm{tp}}[\gamma]\,(\gamma(t),\gamma(s))$ is the radius of the unique circle passing through $\gamma(s)$ and $\gamma(t)$ that is tangent to the curve at $\gamma(t)$, and this radius may be expressed as in (1.2) using the affine tangent line $\ell(t) := \gamma(t)+\mathbb{R}\gamma'(t)$ of γ at the curve point $\gamma(t)$. S. Blatt characterized the corresponding energy space and showed that a curve parametrized by arclength has finite tangent-point energy if and only if it is embedded and lies in the fractional Sobolev[4] space $W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^d)$; see [27, Theorem 1.1] or [28, Theorem 1.1]. This characterization opened the way towards critical point theory, either via symmetric criticality (see [29] for TP_q alone, or [30] where the tangent-point energy is added as a self-avoidance term to the bending energy), or by means of (Banach-)gradient flows in [31]. For variants of the tangent-point energy whose underlying energy space is Hilbert, very recently the Palais–Smale condition was shown [32], which leads to long-time existence and subconvergence to critical points of the gradient flow. In [33] this variant of the tangent-point energy was used to construct a complete Riemannian metric on knot space such that every pair of knots in a knot class can be connected by a distance-minimizing geodesic.

To prevent the blow-up of the energy on curves *with* self-intersections, we modify the integration domain and "cut out" the points in the preimage near the self-intersections. To make this more precise, let $d_\gamma(x,y)$ denote the intrinsic distance between the points $\gamma(x)$ and $\gamma(y)$ along the curve. Suppose that $\gamma \in C^1\left(\mathbb{R}/\mathbb{Z},\mathbb{R}^2\right)$ is a regular curve with finitely many self-intersections, i.e. the set

$$S(\gamma) := \big\{(u,v) \in (\mathbb{R}/\mathbb{Z})^2 \,:\, 0 \leq u < v < 1 \text{ and } \gamma(u) = \gamma(v)\big\} \tag{1.3}$$

is finite. Thus, $S(\gamma) = \{(u_1,v_1),\dots,(u_n,v_n)\}$ for some $n \in \mathbb{N}$. Define furthermore

$$T(\gamma) := \big\{u_1,v_1,\dots,u_n,v_n\big\} \quad\text{and}\quad \Lambda(\gamma) := \min\big\{d_\gamma(a,b) \mid a,b \in T(\gamma), a \neq b\big\}, \tag{1.4}$$

so that $\Lambda(\gamma)$ is the shortest intrinsic distance along the curve between any two self-intersections. For $\delta \in (0,\Lambda(\gamma))$ we define the set

$$\begin{aligned} Y_\delta(\gamma) :=& \big\{(x,y) \in (\mathbb{R}/\mathbb{Z})^2 \;:\; \big(d_\gamma(x,u_i) < \delta \text{ and } d_\gamma(y,v_i) < \delta \text{ for some } i = 1,\dots,n\big) \\ &\text{or } \big(d_\gamma(x,v_i) < \delta \text{ and } d_\gamma(y,u_i) < \delta \text{ for some } i = 1,\dots,n\big)\big\}. \end{aligned}$$

Sometimes, it is more suitable to consider tuples $(x,x+w) \in (\mathbb{R}/\mathbb{Z})^2$ for $x \in \mathbb{R}/\mathbb{Z}$ and $w \in [-\frac{1}{2},\frac{1}{2}]$, as this yields $|(x+w)-x|_{\mathbb{R}/\mathbb{Z}} = |w|_{\mathbb{R}/\mathbb{Z}} = |w|$. Here, $|\cdot|_{\mathbb{R}/\mathbb{Z}}$ denotes the periodic distance on $\mathbb{R}/\mathbb{Z}$ defined as

$$|x-y|_{\mathbb{R}/\mathbb{Z}} := \min_{k\in\mathbb{Z}} |x-y+k|.$$

Then we set

$$\hat{Y}_\delta(\gamma) := \big\{(x,w) \in \mathbb{R}/\mathbb{Z}\times[-\tfrac{1}{2},\tfrac{1}{2}] \;:\; (x,x+w) \in Y_\delta(\gamma)\big\}. \tag{1.5}$$

Define for $0 < \delta < \frac{\Lambda(\gamma)}{2}$ the *truncated tangent-point energy* as

$$\mathrm{TP}_{q,\delta}(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^2\setminus Y_\delta(\gamma)} \frac{|\gamma'(s)||\gamma'(t)|}{\big(r_{\mathrm{tp}}[\gamma](\gamma(t),\gamma(s))\big)^q}\,ds\,dt = \iint_{\mathbb{R}/\mathbb{Z}\times[-\frac{1}{2},\frac{1}{2}]\setminus\hat{Y}_\delta(\gamma)} \frac{|\gamma'(x)||\gamma'(x+w)|}{\big(r_{\mathrm{tp}}[\gamma](\gamma(x),\gamma(x+w))\big)^q}\,dw\,dx.$$

We mostly restrict $\mathrm{TP}_{q,\delta}$ to the following class of admissible curves with fixed winding number and Arnold invariants, that are affine linear near their self-intersections.

Definition 1.1. For $\eta \in (0,1)$ and $j_+,j_-,s,\omega \in \mathbb{Z}$, let $\mathcal{F}(\eta,j_\pm,s,\omega)$ be the set of all arclength parametrized generic immersions $\gamma \in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^2)$ with $J^+(\gamma) = j_+$, $J^-(\gamma) = j_-$, $St(\gamma) = s$, $W(\gamma) = \omega$, and $\Lambda(\gamma) \geq 2\eta$, such that $\gamma|_{B_\eta(u)}$ is affine linear for all $u \in T(\gamma)$.

Restricting to arclength parametrizations (and therefore to unit length curves) does not change or reduce the topological information, so there is no loss of generality in doing so. Notice that even if a suitable combination of integers j_+, j_-, s, ω permits a non-empty $(j_\pm,s,\omega)$-compartment $C(j_\pm,s,\omega)$ of generic immersions, the class $\mathcal{F}(\eta,j_\pm,s,\omega)$ is empty if $2(j_+ - j_-)\eta > 1$ by means of (1.1), since every self-intersection consumes 2η of the curve's unit length. We will show in Theorem 1.2, however, that in any compartment $C(j_\pm,s,\omega)$ we find for sufficiently small $\eta > 0$ a generic $C^{1,1}$-immersion contained in $\mathcal{F}(\eta,j_\pm,s,\omega)$. By definition, the sets $\mathcal{F}(\eta,j_\pm,s,\omega)$ are nested with respect to the parameter η, i.e.,

$$\mathcal{F}(\eta_2,j_\pm,s,\omega) \subset \mathcal{F}(\eta_1,j_\pm,s,\omega) \qquad \text{for all } \; 0 < \eta_1 \leq \eta_2. \tag{1.6}$$

[3] The Möbius energy was originally introduced by O'Hara in [20] and it owes its name to its invariance under Möbius transformations [23, Theorem 2.1].

[4] For the definition and a brief account on the basic properties of one-dimensional periodic fractional Sobolev spaces it suffices to consult, e.g., [26, Appendix A].

1.3. Main results

We now state the main results of this paper, starting with the existence of admissible $C^{1,1}$-curves in $\mathcal{F}(\eta, j_\pm, s, \omega)$.

Theorem 1.2 (*Existence of Admissible Curves*). *For every $(j_\pm, s, \omega)$-compartment $\mathcal{C} = \mathcal{C}(j_\pm, s, \omega)$ there exists $\eta_0 = \eta_0(\mathcal{C}) > 0$ such that $\mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C} \cap C^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \neq \emptyset$ for all $\eta \in (0, \eta_0]$. In addition, there is some $\eta_1 = \eta_1(\mathcal{C}) \in (0, \eta_0(\mathcal{C}))$ such that for every $\eta \in (0, \eta_1]$ there is a curve in $\mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C}$ of class $C^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ for which the intersection angle at every self-intersection equals $\frac{\pi}{2}$.*

From now on and throughout the paper we use the positive number $\eta_0(\mathcal{C})$ for a $(j_\pm, s, \omega)$-compartment $\mathcal{C} := \mathcal{C}(j_\pm, s, \omega)$ to formulate our results on the non-empty admissible classes $\mathcal{F}(\eta, j_\pm, s, \omega)$ for $\eta \in (0, \eta_0]$.

Theorem 1.3 (*Existence of Minimizers*). *For all compartments $\mathcal{C} = \mathcal{C}(j_\pm, s, \omega)$ and any $\eta \in (0, \eta_0(\mathcal{C})]$, $\delta \in (0, \frac{\eta}{2}]$ and $q > 2$ there exists an immersion $\gamma_\delta^\eta \in \mathcal{F}(\eta, j_\pm, s, \omega) \cap W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \cap \mathcal{C}$ such that*

$$\mathrm{TP}_{q,\delta}\left(\gamma_\delta^\eta\right) \leq \mathrm{TP}_{q,\delta}(\gamma) \text{ for all } \gamma \in \mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C}.$$

For small truncation parameters δ close to the threshold $\frac{\eta}{2}$ we have computed $\mathrm{TP}_{q,\delta}$-minimizing configurations numerically for various sets of prescribed winding numbers and Arnold invariants; see Fig. 1. As a predominant feature we observe in these numerical minimizers right angles at every self-intersection. To support this numerical evidence analytically we address the question whether it is possible to send the truncation parameter δ to zero so that the truncated energies $\mathrm{TP}_{q,\delta}$ see an increasingly larger portion of a curve γ. But the energy values $\mathrm{TP}_{q,\delta}(\gamma)$ would tend to infinity if γ self-intersects. To take this blow-up into account, we scale the energy by the correct blow-up rate δ^{q-2}, which will then allow for the limiting process. It turns out, that these scaled versions of the energies indeed Gamma-converge towards the *renormalized tangent-point energy* $R_q := \lim_{\delta\to 0} \delta^{q-2}\mathrm{TP}_{q,\delta}$, a functional which depends only on the intersection angles of the curves.

Theorem 1.4 (*Gamma Convergence as $\delta \to 0$*). *For all compartments $\mathcal{C} = \mathcal{C}(j_\pm, s, \omega)$ and any $\eta \in (0, \eta_0(\mathcal{C})]$ and $q > 2$ one has*

$$\delta^{q-2}\mathrm{TP}_{q,\delta} \xrightarrow{\Gamma} R_q \quad \text{as } \delta \to 0 \quad \text{on } \left(\mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C} \cap W^{2-\frac{1}{q},q}\left(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2\right), \|\cdot\|_{C^1}\right).$$

The renormalized tangent-point energy R_q depends only on the intersection angles of the curves and is minimized if and only if every intersection angle is a right angle.

Minimizing the renormalized energy R_q itself would lead to minimizing curves of arbitrarily complicated shapes as long as all self-intersection angles equal $\frac{\pi}{2}$. In search of representatives with an optimal shape we prove in addition to Gamma convergence, the convergence of a sequence of $\mathrm{TP}_{q,\delta}$-minimizers γ_δ^η to a limit curve Γ^η as $\delta \to 0$. This limit Γ^η has exclusively right self-intersection angles and can be seen as an optimal curve in the class $\mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C}$, because Γ^η is an *almost-minimizer* of all truncated energies for sufficiently small truncation parameters δ. To derive the necessary a priori estimates independent of δ we use comparison curves whose intersection angles equal $\frac{\pi}{2}$. That is why we use the threshold parameter $\eta_1(\mathcal{C})$ from Theorem 1.2 instead of $\eta_0(\mathcal{C})$ in the formulation of the following results.

Theorem 1.5 (*Limit Immersion is Almost-Minimizer*). *Let $\mathcal{C} = \mathcal{C}(j_\pm, s, \omega)$ be a compartment, $q > 2$ and $\eta \in (0, \eta_1(\mathcal{C})]$. Then every sequence $(\gamma_\delta^\eta)_\delta \subset \mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C}$ of $\mathrm{TP}_{q,\delta}$-minimizers subconverges (after suitable translations) in C^1 to some curve $\Gamma^\eta \in \mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C} \cap W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ as $\delta \to 0$. Moreover, all intersection angles of Γ^η equal $\frac{\pi}{2}$, and for any $\epsilon > 0$ there is $\hat{\delta} = \hat{\delta}(\epsilon) > 0$ such that*

$$\inf_{\mathcal{F}(\eta, j_\pm, s, \omega)\cap\mathcal{C}} \mathrm{TP}_{q,\delta}(\cdot) \leq \mathrm{TP}_{q,\delta}(\Gamma^\eta) < \inf_{\mathcal{F}(\eta, j_\pm, s, \omega)\cap\mathcal{C}} \mathrm{TP}_{q,\delta}(\cdot) + \epsilon \quad \text{for all } \delta \in (0, \hat{\delta}). \tag{1.7}$$

Notice that the minimal energy values on the left-hand side of (1.7) blow up as $\delta \to 0$, but the energy values $\mathrm{TP}_{q,\delta}(\Gamma^\eta)$ remain in the fixed ϵ-neighborhood of these minimal energies as $\delta \to 0$. The proof of Theorem 1.5 reveals even more energetic control: If $(\gamma_{\delta_k}^\eta)_k$ is a subsequence of (suitably translated) TP_{q,δ_k}-minimizers, converging in C^1 to Γ^η for $\delta_k \to 0$ as $k \to \infty$, then

$$R_q(\Gamma^\eta) = \lim_{k\to\infty} \delta_k^{q-2}\mathrm{TP}_{q,\delta_k}(\gamma_{\delta_k}^\eta). \tag{1.8}$$

Finally, it turns out that this optimal curve is a true minimizer for *all truncated energies* among all admissible curves whose intersection angles are right angles.

Corollary 1.6 (*Optimal Immersion Minimizes Among Curves with Intersection Angles $\frac{\pi}{2}$*). *Under the assumptions of Theorem 1.5 one has*

$$\mathrm{TP}_{q,\delta}(\Gamma^\eta) \leq \mathrm{TP}_{q,\delta}(\gamma) \quad \text{for all } \delta \in (0, \frac{\eta}{2}],\ \gamma \in \mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C}, \tag{1.9}$$

if all intersection angles of γ equal $\frac{\pi}{2}$.

Remarks. 1. At this point it is not clear if the limit curves Γ^η also minimize the truncated tangent-point energy in the full admissibility class $\mathcal{F}(\eta, j_\pm, s, \omega) \cap C$. By Arzela–Ascoli the Γ^η subconverge uniformly to a Lipschitz continuous limit curve Γ^0 as $\eta \to 0$, but we do not have any further information about Γ^0, in spite of the fact that the a priori energy bound on the minimizers γ_δ^η in Corollary 5.6 in Section 5 does *not* depend on η. In addition, the map $\eta \mapsto \mathrm{TP}_{q,\theta}(\Gamma^\eta)$ is non-decreasing on the interval $(2\theta, \eta_1(C))$; see Corollary 5.7, but uniform convergence does not suffice to prove lower semi-continuity of $\mathrm{TP}_{q,\theta}$. We do expect, however, that right intersection angles and straight segments near the self-intersections will remain prevalent in the limit $\eta \to 0$ since this local geometry contributes least to the energy. Therefore, we believe that the numerically computed minimizers in Fig. 1 are fairly close to the true minimizing immersions even without the restriction of being locally affine linear near the self-intersections.

2. There is an interesting connection to Legendrian knots in $\mathbb{R}^3$ equipped with the standard contact structure; see, e.g., [34, Chapter 3]. The Lagrangian projection, i.e., the projection onto the xy-plane of a Legendrian knot yields a planar immersed curve. On the other hand, any planar immersion that encloses zero area can be lifted to a closed Legendrian knot that is unique up to translation in the z-direction. An example of such a planar immersion is a point-symmetric figure-eight shaped curve with winding number $W = 0$ and Arnold invariants $J^+ = 0$, $J^- = -1$, and $St = 0$. In [5, Chapter 7] it was shown by means of Palais's principle of symmetric criticality that such curves arise as critical points of the energy $\mathrm{TP}_{q,\delta}$. A natural question to ask is whether lifting these symmetric critical points yields special Legendrian knots. At this point it is open, however, if *all* figure-eight shaped minimizers γ_δ^η obtained in Theorem 1.3 have this point symmetry.

The paper is structured as follows. In Section 2 we construct admissible curves in the class $\mathcal{F}(\eta, j_\pm, s, \omega)$ with sufficient regularity which will prove Theorem 1.2. Section 3 is devoted to the analysis of the truncated tangent-point energy, its regularizing effects (Lemma 3.5) and its uniform control on bilipschitz constants; see Proposition 3.7. The proof of the existence result (Theorem 1.3) in Section 4 rests on lower semi-continuity (Lemma 4.1) and compactness established in Theorem 4.2. To investigate the limit $\delta \to 0$ we introduce in Definition 5.1 the annular truncated tangent-point energy for which we prove in Proposition 5.2 a crucial representation that leads to the purely angle-dependent Gamma limit R_q proving Theorem 1.4. The results on the limit curve Γ^η require a priori estimates on the minimal energies $\mathrm{TP}_{q,\delta}(\gamma_\delta^\eta)$ established in Corollary 5.6. The appendix provides the details for a specific reparametrization of the immersions with local graph patches near self-intersections (Lemma A.1), and the analysis of the angle-dependent term in the annular energy of Definition 5.1 in Lemma A.2. Throughout the paper we use the notation $B_r(x)$ for open subintervals $(x - r, x + r) \subset \mathbb{R}/\mathbb{Z}$.

2. Existence of admissible curves

We devote this preparatory section to prove the existence of admissible curves in a given compartment $C(j_\pm, s, \omega)$.

Proof of Theorem 1.2. The idea is to deform a suitably reparametrized curve in $C(j_\pm, s, \omega)$ locally near every of its finitely many transverse double points to a pair of graphs which intersect in straight line segments. To obtain right angles at these self-intersections we rotate the inner quarter of one segment appropriately with a smooth cut-off to leave the rest of the curve unchanged. Here are the details presented in four steps.

Step 1. Any curve in the open compartment $C = C(j_\pm, s, \omega)$ can be smoothened, e.g. by convolution, since the winding number and Arnold invariants are locally constant. In addition, such a smooth representative can be rescaled to unit length and reparametrized to arclength to obtain a generic immersion $\gamma \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \cap C$ with winding number $W(\gamma) = \omega$ and Arnold invariants $J^+(\gamma) = j_+$, $J^-(\gamma) = j_-$, $St(\gamma) = s$. So, $|\gamma'| \equiv 1$ on $\mathbb{R}/\mathbb{Z}$, and γ has exactly $n := j_+ - j_-$ transverse self-intersections, each of multiplicity two, and no other self-intersections. In other words, there are mutually distinct arclength parameters $s_i, t_i \in \mathbb{R}/\mathbb{Z}$ with $\gamma(s_i) = \gamma(t_i)$ for $i = 1, \dots, n$. Abbreviate the unit tangents at these intersection points by $S_i := \gamma'(s_i)$ and $T_i := \gamma'(t_i)$ and define

$$d := \min_{i \in \{1,\dots,n\}} \big\{ |S_i - T_i| \big\}. \tag{2.1}$$

Note that $d \in (0, 2)$ by transversality.

By virtue of Lemma A.1 for $k := 2$ we can reparametrize γ preserving the orientation to obtain $\tilde\gamma \in C^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ which locally near every self-intersection coincides with a C^2-graph over the affine tangent plane $\gamma(s_i) + \mathbb{R}S_i$ and $\gamma(t_i) + \mathbb{R}T_i$, respectively. More precisely, $\tilde\gamma(\mathbb{R}/\mathbb{Z}) = \gamma(\mathbb{R}/\mathbb{Z})$, $|\tilde\gamma'| \geq \frac{1}{4}$ on $\mathbb{R}/\mathbb{Z}$, and there is some $r_0 > 0$ such that

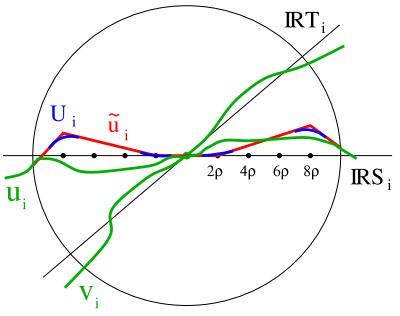
$$\tilde\gamma(x) = \begin{cases} \gamma(x) & \text{for } \ x \notin \bigcup_{i=1}^n B_{2r_0}(s_i) \cup B_{2r_0}(t_i) \\ \gamma(s_i) + (x - s_i)S_i + u_i(x - s_i)S_i^\perp & \text{for } \ x \in B_{r_0}(s_i) \\ \gamma(t_i) + (x - t_i)S_i + v_i(x - t_i)T_i^\perp & \text{for } \ x \in B_{r_0}(t_i). \end{cases} \tag{2.2}$$

Here, the subintervals $B_{3r_0}(s_i), B_{3r_0}(t_i) \subset \mathbb{R}/\mathbb{Z}$, $i = 1, \dots, n$, are mutually disjoint, and the intrinsic distance $d_{\tilde\gamma}(x, s_i) \geq r_0$ for all $x \notin B_{r_0}(s_i)$, so that

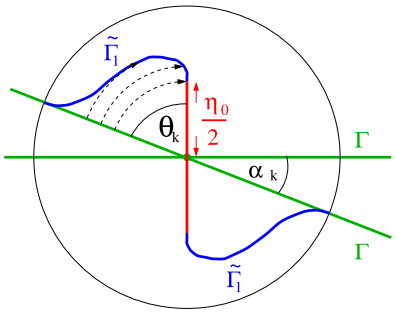
$$\Lambda(\tilde\gamma) \geq 2r_0, \tag{2.3}$$

and the graph functions $u_i, v_i \in C^2(\mathbb{R})$ satisfy $0 = u_i(0) = v_i(0) = u_i'(0) = v_i'(0)$ for all $i = 1, \dots, n$. In addition, we have used in (2.2) the notation $p^\perp := (-p_2, p_1)^T$ for a vector $p = (p_1, p_2)^T \in \mathbb{R}^2$. By continuity we can choose $r_1 \in (0, r_0)$ so small that

$$\max\big\{ \|u_i'\|_{C^0(B_{r_1}(0))}, \|v_i'\|_{C^0(B_{r_1}(0))} \big\} < \frac{d}{40} \quad \text{for all } \ i = 1, \dots, n. \tag{2.4}$$



(a) Step 2: The auxiliary piecewise linear function $\tilde{u}_i$ (in red) interpolates the point-tangent data of u_i at $\partial B_{10\rho}(0)$ and $\partial B_{2\rho}(0)$. Smoothing out the corners of $\tilde{u}_i$ with circular arcs (blue) yields the $C^{1,1}$-comparison function U_i.



(b) Step 4: Rotating one of the two intersecting linear segments to a perpendicular position (in red) and smoothly cutting off the rotational angle (blue) away from the self-intersection yields a comparison curve $\tilde{\Gamma}_1$ whose intersection angles equal $\frac{\pi}{2}$.

Fig. 6. For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.

Step 2. The quantity

$$D := \min_{i\in\{1,\dots,n\}} \min_{y\in(\mathbb{R}/\mathbb{Z})\setminus[B_{r_1}(s_i)\cup B_{r_1}(t_i)]} \big\{|\gamma(s_i)-\tilde{\gamma}(y)|\big\} \tag{2.5}$$

is strictly positive, since the reparametrization $\tilde{\gamma}$ has exactly the same self-intersection points as γ,

$$\tilde{\gamma}(s_i)=\gamma(s_i)=\gamma(t_i)=\tilde{\gamma}(t_i) \quad \text{for } i=1,\dots,n,$$

and not any other self-intersections. Now choose a smaller radius

$$\rho \in \big(0, \min\big\{\frac{r_1}{10}, \frac{D}{40}\big\}\big), \tag{2.6}$$

and define for each $i=1,\dots,n$ the piecewise linear auxiliary function $\tilde{u}_i : B_{r_1}(0)\to\mathbb{R}$ as

$$\tilde{u}_i(x) := \begin{cases} 0 & \text{for } x\in\overline{B_{2\rho}(0)} \\ u_i(-10\rho)+(x+10\rho)u_i'(-10\rho) & \text{for } x\in[-10\rho,-8\rho] \\ u_i(10\rho)+(x-10\rho)u_i'(10\rho) & \text{for } x\in[8\rho,10\rho] \\ \text{linearly interpolating} & \text{on } B_{8\rho}(0)\setminus\overline{B_{2\rho}(0)} \\ u_i(x) & \text{for } x\in B_{r_1}(0)\setminus B_{10\rho}(0); \end{cases}$$

see Fig. 6(a).

The slope of $\tilde{u}_i$ on $[-10\rho,-8\rho]$ and on $[8\rho,10\rho]$ is bounded by the norm $\|u_i'\|_{C^0(B_{r_1}(0))}$, whereas on $B_{8\rho}(0)\setminus B_{2\rho}(0)$ it is controlled by

$$\left|\frac{\tilde{u}_i(\pm 8\rho)}{6\rho}\right| = \left|\frac{u_i(\pm 10\rho)\mp 2\rho u_i'(\pm 10\rho)}{6\rho}\right| \le \frac{4}{3}\|u_i'\|_{C^0(B_{r_1}(0))}+\frac{1}{3}\|u_i'\|_{C^0(B_{r_1}(0))} < 2\|u_i'\|_{C^0(B_{r_1}(0))}.$$

Replacing the graph of $\tilde{u}_i$ on the intervals $[-9\rho,-7\rho]$, $[-3\rho,-\rho]$, $[\rho,3\rho]$, and on $[7\rho,9\rho]$ by the respective C^1-interpolating circular arcs (see Fig. 6(a)) we obtain a piecewise linear and circular function $U_i\in C^{1,1}(B_{r_1}(0))$ vanishing on $\overline{B_\rho(0)}$ and coinciding with u_i on $B_{r_1}(0)\setminus\overline{B_{10\rho}(0)}$, and satisfying

$$\|U_i'\|_{C^0(B_{r_1}(0))} < 2\|u_i'\|_{C^0(B_{r_1}(0))} \quad \text{for } i=1,\dots,n. \tag{2.7}$$

Analogously, we obtain a piecewise linear and circular function $V_i\in C^{1,1}(B_{r_1}(0))$ vanishing on $\overline{B_\rho(0)}$ and coinciding with v_i on $B_{r_1}(0)\setminus\overline{B_{10\rho}(0)}$, such that

$$\|V_i'\|_{C^0(B_{r_1}(0))} < 2\|v_i'\|_{C^0(B_{r_1}(0))} \quad \text{for } i=1,\dots,n. \tag{2.8}$$

Step 3. Consider the convex combinations $u_{i,\lambda} := (1-\lambda)u_i+\lambda U_i$ on $B_{r_1}(0)$ and $v_{i,\lambda} := (1-\lambda)v_i+\lambda V_i$ on $B_{r_1}(0)$ for $\lambda\in[0,1]$, $i=1,\dots,n$, satisfying

$$\max\big\{\|u_{i,\lambda}'\|_{C^0(B_{r_1}(0))},\|v_{i,\lambda}'\|_{C^0(B_{r_1}(0))}\big\} < 3\|u_i'\|_{C^0(B_{r_1}(0))} < \frac{3d}{40} \tag{2.9}$$

by means of (2.7) and (2.8) in combination with (2.4). Now define the 1-parameter family of curves

$$\tilde{\gamma}_\lambda(x) := \begin{cases} \tilde{\gamma}(x) & \text{for } x \notin \bigcup_{i=1}^n [B_{r_1}(s_i) \cup B_{r_1}(t_i)] \\ \gamma(s_i) + (x-s_i)S_i + u_{i,\lambda}(x-s_i)S_i^\perp & \text{for } x \in B_{r_1}(s_i) \\ \gamma(s_i) + (x-t_i)T_i + v_{i,\lambda}(x-t_i)T_i^\perp & \text{for } x \in B_{r_1}(t_i). \end{cases}$$

Claim: For each $\lambda \in [0,1]$ the curve $\tilde{\gamma}_\lambda$ has exactly the same self-intersection points as γ,

$$\tilde{\gamma}_\lambda(s_i) = \gamma(s_i) = \gamma(t_i) = \tilde{\gamma}_\lambda(t_i) \quad \textit{for } i = 1, \dots, n,$$

and no other self-intersections.

Indeed, with (2.6) and (2.9) we obtain for any $\lambda \in [0,1]$ and $x \in \overline{B_{10\rho}(s_i)}$

$$|\tilde{\gamma}_\lambda(x) - \gamma(s_i)|^2 = (x-s_i)^2 + u_{i,\lambda}^2(x-s_i) \le (1 + \|u'_{i,\lambda}\|^2_{C^0(B_{r_1}(0))})(10\rho)^2$$
$$\overset{(2.9)}{\le} (1 + \frac{9d^2}{1600})100\rho^2 < 200\rho^2 \overset{(2.6)}{<} 200\frac{D^2}{40^2} < \frac{D^2}{4},$$

which implies for every $\lambda \in [0,1]$ by definition of the quantity D in (2.5)

$$|\tilde{\gamma}_\lambda(x) - \tilde{\gamma}_\lambda(y)| > \frac{D}{2} \text{ for all } x \in \overline{B_{10\rho}(x_i)},\ y \notin B_{r_1}(s_i) \cup B_{r_1}(t_i).$$

For $x \in \overline{B_{10\rho}(s_i)}$ and $y \in B_{r_1}(s_i) \setminus \{x\}$ one has

$$|\tilde{\gamma}_\lambda(x) - \tilde{\gamma}_\lambda(y)|^2 = (x-y)^2 + (u_{i,\lambda}(x-s_i) - u_{i,\lambda}(y-s_i))^2 \ge (x-y)^2 > 0.$$

Finally, for $x \in \overline{B_{10\rho}(s_i)}$ and $y \in B_{r_1}(t_i)$ we first estimate the distance between the respective tangential projections of $\tilde{\gamma}_\lambda(x)$ and $\tilde{\gamma}_\lambda(y)$, i.e.,

$$\begin{aligned} f_i(x,y) &:= |\gamma(s_i) + (x-s_i)S_i - [\gamma(s_i) + (y-t_i)T_i]| \\ &\ge \max\{\operatorname{dist}((x-s_i)S_i, \mathbb{R}T_i), \operatorname{dist}((y-t_i)T_i, \mathbb{R}S_i)\} \\ &= \max\{|x-s_i|, |y-t_i|\} \sin\sphericalangle(S_i, T_i). \end{aligned} \tag{2.10}$$

With $\sin\sphericalangle(S_i,T_i) > \frac{1}{2}\sin\sphericalangle(S_i,T_i) = \frac{1}{2}|S_i - T_i| \ge \frac{d}{2} > 0$ we find from (2.10)

$$f_i(x,y) \ge \max\{|x-s_i|, |y-t_i|\} \cdot \frac{d}{2} \quad \text{for all } x \in \overline{B_{10\rho}(s_i)},\ y \in B_{r_1}(t_i),\ i = 1, \dots, n. \tag{2.11}$$

On the other hand, by (2.9)

$$|\tilde{\gamma}_\lambda(x) - (\gamma(s_i) + (x-s_i)S_i)|^2 = u_{i,\lambda}^2(x-s_i) \le \frac{9d^2}{1600}|x-s_i|^2$$

and, likewise,

$$|\tilde{\gamma}_\lambda(y) - (\gamma(s_i) + (y-t_i)T_i)|^2 = v_{i,\lambda}^2(y-t_i) \le \frac{9d^2}{1600}|y-t_i|^2,$$

so that we obtain with (2.11)

$$|\tilde{\gamma}_\lambda(x) - \tilde{\gamma}_\lambda(y)| \ge \max\{|x-s_i|, |y-t_i|\} \cdot \frac{7}{20}d \quad \text{for } x \in \overline{B_{10\rho}(s_i)},\ y \in B_{r_1}(t_i),$$

which proves the claim.

In addition, we observe that the map $\lambda \mapsto H(\cdot,\lambda) := \tilde{\gamma}_\lambda(\cdot)$ is of class $C^0([0,1], C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2))$, and

$$\tilde{\gamma}'_\lambda(x) = \begin{cases} \tilde{\gamma}'(x) & \text{for } x \notin \bigcup_{i=1}^n [B_{r_1}(s_i) \cup B_{r_1}(t_i)] \\ S_i + u'_{i,\lambda}(x-s_i)S_i^\perp & \text{for } x \in B_{r_1}(s_i) \\ T_i + v'_{i,\lambda}(x-t_i)T_i^\perp & \text{for } x \in B_{r_1}(t_i), \end{cases}$$

so that $|\tilde{\gamma}'_\lambda(x)| \ge \frac{1}{4}$ for all $x \in \mathbb{R}/\mathbb{Z}$. Consequently, H is a regular homotopy between $H(\cdot,0) = \tilde{\gamma}(\cdot)$ and $H(\cdot,1) = \tilde{\gamma}_1 \in C$ where $\tilde{\gamma}_1|_{B_\rho(s_i)}$ and $\tilde{\gamma}_1|_{B_\rho(t_i)}$ are affine linear, by definition of $u_{i,1} = U_i$ and $v_{i,1} = V_i$ for $i = 1, \dots, n$, so that $\Lambda(\tilde{\gamma}_1) > 20\rho$ by means of (2.3) and (2.6). Rescaling $\tilde{\gamma}_1$ by its length $L := \mathscr{L}(\tilde{\gamma}_1)$ and then reparametrizing $\tilde{\gamma}_1/L$ to arclength yields the desired curve $\Gamma \in C \cap C^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ which is linear within positive arclength $\eta_0(C) = \eta_0 := \frac{\rho}{L}$ around each self-intersection parameter, satisfying $\Lambda(\Gamma) > 20\eta_0 > 2\eta_0$, as required in the admissibility class $\mathcal{F}(\eta_0, j_\pm, s, \omega) \subset \mathcal{F}(\eta, j_\pm, s, \omega)$ for all $\eta \in (0, \eta_0]$ by means of (1.6).

Step 4. Let $\Gamma(\sigma_k) = \Gamma(\tau_k)$ for mutually distinct arclength parameters $\sigma_k, \tau_k \in \mathbb{R}/\mathbb{Z}$, $k = 1, \dots, n$ be the $n = j_+ - j_-$ double points of the curve $\Gamma \in C$ obtained in the previous step. Notice that our construction implies that Γ intersects the η_0-ball around each self-intersection exactly in the two self-intersecting straight line segments, i.e.,

$$\Gamma(\mathbb{R}/\mathbb{Z}) \cap B_{\eta_0}(\Gamma(\sigma_k)) = \Gamma(B_{\eta_0}(\sigma_k)) \cup \Gamma(B_{\eta_0}(\tau_k)) \quad \text{for } k = 1, \dots, n. \tag{2.12}$$

and our choice of ρ in (2.6). Denote the intersection angles of the tangent lines by $\alpha_k := \sphericalangle(\mathbb{R}\Gamma'(\sigma_k), \mathbb{R}\Gamma'(\tau_k)) \in (0, \frac{\pi}{2}]$ and set $\theta_k := \frac{\pi}{2} - \alpha_k \in [0, \frac{\pi}{2})$ for $k = 1, \dots, n$. Identifying $(p_1, p_2)^T \in \mathbb{R}^2$ with points $p_1 + ip_2$ in the complex plane $\mathbb{C}$ we can rotate any point

$\xi \in \mathbb{C} \simeq \mathbb{R}^2$ by the angle $\epsilon_k\theta_k$ by multiplying $e^{i\epsilon_k\theta_k} \in \mathbb{C}$ with ξ. Here, the sign $\epsilon_k \in \{1,-1\}$ is determined in such a way that the resulting angle

$$\sphericalangle\big(\mathbb{R}e^{i\epsilon_k\theta_k}\Gamma'(\sigma_k), \mathbb{R}\Gamma'(\tau_k)\big) = \frac{\pi}{2} \quad \text{for } k = 1,\dots,n. \tag{2.13}$$

Now choose a cut-off function $\phi_{\eta_0} \in C_0^\infty((-\eta_0,\eta_0))$ (extended to all of $\mathbb{R}$ by the value 0) such that $0 \le \phi_{\eta_0}(x) \le 1$ for all $x \in \mathbb{R}$, $\phi_{\eta_0}(x) = 1$ for all $x \in [-\frac{\eta_0}{2}, \frac{\eta_0}{2}]$. Define the deformed curves for $\lambda \in [0,1]$ (see Fig. 6(b) for $\lambda = 1$)

$$\tilde\Gamma_\lambda(t) := \begin{cases} e^{i\lambda\phi_{\eta_0}(t-\sigma_k)\epsilon_k\theta_k}(\Gamma(t)-\Gamma(\sigma_k)) + \Gamma(\sigma_k) & \text{for } t \in B_{\eta_0}(\sigma_k) \\ \Gamma(t) & \text{for } t \notin \bigcup_{k=1}^n B_{\eta_0}(\sigma_k), \end{cases}$$

so that $\tilde\Gamma_\lambda \in C^{1,1}(\mathbb{R}/\mathbb{Z},\mathbb{R}^2)$ has for each $\lambda \in [0,1]$ exactly the same double points as Γ, i.e.,

$$\tilde\Gamma_\lambda(\sigma_k) = \Gamma(\sigma_k) = \Gamma(\tau_k) = \tilde\Gamma_\lambda(\tau_k) \quad \text{for } k = 1,\dots,n,$$

and no additional self-intersections by means of (2.12). Since $\Gamma(t) - \Gamma(\sigma_k) = \pm\Gamma'(t)$ for $t \in B_{\eta_0}(\sigma_k)$ we find for the speed of $\tilde\Gamma_\lambda$

$$|\tilde\Gamma_\lambda'(t)| = |e^{i\lambda\phi_{\eta_0}(t-\sigma_k)\epsilon_k\theta_k}(1 \pm i\lambda\epsilon_k\theta_k\phi_{\eta_0}'(t-\sigma_k))\Gamma'(t)| \ge |\Gamma'(t)| = 1 \text{ for all } t \in B_{\eta_0}(\sigma_k),$$

and $|\tilde\Gamma_\lambda'(t)| = |\Gamma'(t)| = 1$ for $t \notin \bigcup_{k=1}^n B_{\eta_0}(\sigma_k)$, so that the map $\lambda \mapsto \tilde H(\cdot,\lambda) := \tilde\Gamma_\lambda(\cdot)$ is a regular homotopy between $\tilde H(\cdot,0) = \Gamma(\cdot)$ and $\tilde H(\cdot,1) = \tilde\Gamma_1$. With $\tilde\Gamma_1'(\sigma_k) = e^{i\epsilon_k\theta_k}\Gamma'(\sigma_k)$ and $\tilde\Gamma_1'(\tau_k) = \Gamma'(\tau_k)$ we obtain for the intersection angles by means of (2.13)

$$\sphericalangle\big(\mathbb{R}\tilde\Gamma_1'(\sigma_k), \mathbb{R}\tilde\Gamma_1'(\tau_k)\big) = \sphericalangle\big(e^{i\epsilon_k\theta_k}\mathbb{R}\Gamma'(\sigma_k), \mathbb{R}\Gamma'(\tau_k)\big) = \frac{\pi}{2} \text{ for all } k = 1,\dots,n.$$

Rescaling $\tilde\Gamma_1$ to unit length and then reparametrizing to arclength yields the desired curve of class $C^{1,1}(\mathbb{R}/\mathbb{Z},\mathbb{R}^2)$ in $C \cap \mathcal{F}(\eta_1, j_\pm, s, \omega)$ for some $\eta_1 \in (0,\eta_0)$ such that the straight line segments at each intersection point are perpendicular to each other. This shows again by (1.6) that for all $\eta \in (0,\eta_1]$ the admissibility class $C \cap \mathcal{F}(\eta, j_\pm, s, \omega)$ contains curves with right intersection angles. □

3. Energy space of the δ-renormalized tangent-point energy

In this section, we characterize the energy space of the energy $\mathrm{TP}_{q,\delta}$ following the ideas of Blatt and Reiter in [28, Section 2]. Crucial for our proofs are bilipschitz estimates for planar immersions away from their self-intersections. Notice that for any arclength parametrized curve $\gamma \in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^d)$ we have

$$\begin{aligned} |\gamma(s+w)-\gamma(s)| &= \Big|\int_s^{s+w}\gamma'(t^*)\,dt + \int_s^{s+w}\big(\gamma'(t)-\gamma'(t^*)\big)\,dt\Big| \\ &\ge |w|\big(1 - \max_{t\in[s,s+w]}|\gamma'(t)-\gamma'(t^*)|\big) \quad \text{for all } (s,w) \in \mathbb{R}/\mathbb{Z}\times[-\tfrac12,\tfrac12],\ t^* \in [s,s+w]. \end{aligned} \tag{3.1}$$

This inequality will be used repeatedly throughout the paper to obtain bilipschitz estimates, like in the following *individual global bilipschitz estimate* for planar immersed curves away from their self-intersections.

Lemma 3.1 (*Global Individual Bilipschitz Estimate*). *Let $\gamma \in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^2)$ be an immersion with finitely many self-intersections. Then for every $0 < \delta < \frac{\Lambda(\gamma)}{2}$ there exists a constant $c = c(\gamma,\delta) > 0$ such that*

$$|x-y|_{\mathbb{R}/\mathbb{Z}} \le c(\gamma,\delta)|\gamma(x)-\gamma(y)| \quad \textit{for all } (x,y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma). \tag{3.2}$$

Proof. Since γ' is uniformly continuous on $\mathbb{R}/\mathbb{Z}$ we find by means of (3.1) a constant $\tau_\gamma > 0$ such that

$$|\gamma(s+w)-\gamma(s)| \ge \frac12|w| \quad \text{for all } (s,w) \in \mathbb{R}/\mathbb{Z}\times[-\tau_\gamma,\tau_\gamma]. \tag{3.3}$$

The function $(s,w) \mapsto |\gamma(s+w)-\gamma(s)|$ is positive and uniformly continuous on the compact set $K(\delta,\tau_\gamma) := \{(s,w) \in (\mathbb{R}/\mathbb{Z}\times[-\frac12,\frac12]) \setminus \hat Y_\delta(\gamma) : |w| \ge \tau_\gamma\}$ so that there exists a constant $c_0(\gamma,\delta) > 0$ such that

$$|\gamma(s+w)-\gamma(s)| \ge c_0(\gamma,\delta) \quad \text{for all } (s,w) \in K(\delta,\tau_\gamma).$$

Combining this with (3.3) and the fact that $|w| \le \frac12$ yields the desired global bilipschitz estimate (3.2) with $c(\gamma,\delta) := \max\{2, (2c_0(\gamma,\delta))^{-1}\}$, replacing s by x and $s+w$ by y.

We prove most statements in this section in a larger class than the set $\mathcal{F}(\eta, j_\pm, s, \omega)$ introduced in Definition 1.1.

Definition 3.2. Let $n \in \mathbb{N}$. Define $\mathcal{F}_n \subset C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^2)$ to be the subset of all arclength parametrized immersions $\gamma \in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^2)$ that have exactly n transverse self-intersections of multiplicity two and no other self-intersections.

Note that $\mathcal{F}(\eta, j_\pm, s, \omega) \subset \mathcal{F}_{j_+-j_-}$ by virtue of (1.1). Restricting to arclength parametrized curves fixes the length which takes care of the missing scale-invariance of the energy. Indeed, one has

$$\mathrm{TP}_{q,\delta}(R\gamma) = R^{2-q}\mathrm{TP}_{q,\delta}(\gamma) \quad \text{for all } R > 0. \tag{3.4}$$

Observe that the truncated tangent-point energy can be rewritten as

$$\mathrm{TP}_{q,\delta}(\gamma) = \iint_{(\mathbb{R}/\mathbb{Z}\times[-\frac{1}{2},\frac{1}{2}])\setminus\hat{Y}_\delta(\gamma)} \left(\frac{2\,|P^\perp_{\gamma'(x)}(\gamma(x+w)-\gamma(x))|}{|\gamma(x+w)-\gamma(x)|^2}\right)^q |\gamma'(x+w)||\gamma'(x)|\,dw\,dx, \tag{3.5}$$

where for any unit vector $\nu \in \mathbb{S}^1 \subset \mathbb{R}^2$ the expression $P^\perp_\nu$ denotes the orthogonal projection onto its orthogonal complement $\nu^\perp$. The expression (3.5) is quite useful to relate the energy to a fractional Sobolev seminorm and will be used quite frequently from now on.

Lemma 3.3. *Let $q > 2$, $n \in \mathbb{N}$, $\gamma \in W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^2) \cap \mathcal{F}_n$ and $0 < \delta < \frac{\Lambda(\gamma)}{2}$. Then $\mathrm{TP}_{q,\delta}(\gamma) < \infty$.*

Proof. With the bilipschitz estimate (3.2) from Lemma 3.1 we can bound the integrand in (3.5) from above by

$$2^q c^{2q}|w|^{-2q}\left|P^\perp_{\gamma'(x)}\left(\int_0^1 \frac{d}{dt}\gamma(x+tw)\,dt\right)\right|^q = 2^q c^{2q}|w|^{-q}\left|\int_0^1 P^\perp_{\gamma'(x)}(\gamma'(x+tw)-\gamma'(x))\,dt\right|^q,$$

which is now integrable over all of $\mathbb{R}/\mathbb{Z}\times[-\frac{1}{2},\frac{1}{2}]$. Indeed, by Jensen's inequality, $\|P^\perp\| \le 1$, and Fubini, we estimate

$$\frac{1}{2^q}\mathrm{TP}_{q,\delta}(\gamma) \le c^{2q}\int_0^1\int_{\mathbb{R}/\mathbb{Z}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|\gamma'(x+tw)-\gamma'(x)|^q}{|w|^q}\,dw\,dx\,dt.$$

Now we change variables according to $\sigma(w) := tw$ to arrive at

$$\int_0^1\int_{\mathbb{R}/\mathbb{Z}}\int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{|\gamma'(x+\sigma)-\gamma'(x)|^q}{|\sigma|^q}t^{q-1}\,d\sigma\,dx\,dt \le \frac{1}{q}\int_{\mathbb{R}/\mathbb{Z}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|\gamma'(x+\sigma)-\gamma'(x)|^q}{|\sigma|^q}\,d\sigma\,dx = \frac{1}{q}\left[\gamma'\right]^q_{1-\frac{1}{q},q}.$$

as an upper bound for $(2c^2)^{-q}\mathrm{TP}_{q,\delta}$. Notice that the constant c depends on γ and δ so that this bound does not yield an a priori estimate on the truncated energy.

If we want to investigate, on the other hand, whether the truncated tangent point energy $\mathrm{TP}_{q,\delta}$ also regularizes the curve to belong to the fractional Sobolev space $W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^2)$, we need to first relate the minimal parameter distance of intersection pairs to the truncation parameter δ. Namely, for $n \in \mathbb{N}$ and $\gamma \in \mathcal{F}_n$ set

$$\lambda := \min_{(u,v)\in S(\gamma)} |u-v|_{\mathbb{R}/\mathbb{Z}}, \quad \text{with } S(\gamma) \text{ as in (1.3).}$$

It turns out that the truncated parts of the domain stay away from the diagonal, if δ is chosen sufficiently small.

Lemma 3.4. *For $\gamma \in \mathcal{F}_n$, $\delta \in (0, \frac{\Lambda(\gamma)}{4}]$, and any $\tau \in [0,\frac{1}{2}]$ such that $|\gamma'(s+w)-\gamma'(s)| \le \frac{1}{2}$ for all $s \in \mathbb{R}/\mathbb{Z}$ and $|w| \le \tau$, one has the inequality $\lambda - 2\delta > \tau$. In particular, $\hat{Y}_\delta(\gamma) \subset \mathbb{R}/\mathbb{Z}\times\{\tau < |w| \le \frac{1}{2}\}$.*

Proof. It suffices to show that $\tau < \frac{\lambda}{2}$ since then we find by means of $\delta \le \frac{\Lambda(\gamma)}{4} \le \frac{\lambda}{4}$ the desired inequality $\lambda - 2\delta \ge \frac{\lambda}{2} > \tau$. Take $(u,v) \in S(\gamma)$ with $0 \le u < v < 1$ and $|u-v|_{\mathbb{R}/\mathbb{Z}} = \lambda$, and assume to the contrary that $\tau \ge \frac{\lambda}{2}$. This implies $|t - \frac{1}{2}(u+v)|_{\mathbb{R}/\mathbb{Z}} \le \tau$ for all $t \in [u,v]$. Hence we infer from (3.1) for $s := u$, $w := v-u$ and $t^* := \frac{1}{2}(u+v)$ the contradiction

$$|\gamma(v)-\gamma(u)| \ge \frac{1}{2}|v-u|_{\mathbb{R}/\mathbb{Z}} > 0,$$

which proves the inequality. For the last statement take a point (x,w) in the closure of $\hat{Y}_\delta(\gamma)$ with $|x-u_i|_{\mathbb{R}/\mathbb{Z}} \le \delta$ and $|x+w-v_i|_{\mathbb{R}/\mathbb{Z}} \le \delta$ for some $(u_i,v_i) \in S(\gamma)$. Then estimate $\frac{1}{2} \ge |w| \ge |u_i-v_i|_{\mathbb{R}/\mathbb{Z}} - |v_i-(x+w)|_{\mathbb{R}/\mathbb{Z}} - |x-u_i|_{\mathbb{R}/\mathbb{Z}} \ge \lambda - 2\delta > \tau$.

The following a priori estimates for the seminorm and the oscillation of the tangent in terms of the truncated tangent point energy $\mathrm{TP}_{q,\delta}$ will be important in the proof of C^1-compactness of sublevel sets in $\mathcal{F}(\eta, j_\pm, s, \omega) \subset \mathcal{F}_n$; see Theorem 4.2.

Lemma 3.5 (*Energy Controls Seminorm and Oscillation of Tangents*). *Let $n \in \mathbb{N}$, $q > 2$, $\gamma \in \mathcal{F}_n$, and $\delta \in (0, \frac{\Lambda(\gamma)}{4}]$. If $\mathrm{TP}_{q,\delta}(\gamma) < \infty$, then there exist constants $c_1(q), c_2(q) > 0$ only depending on q such that*

$$[\gamma']^q_{1-\frac{1}{q},q} \le c_1(q)\left[\mathrm{TP}_{q,\delta}(\gamma) + \left(\mathrm{TP}_{q,\delta}(\gamma)\right)^{\frac{q-1}{q-2}}\right]. \tag{3.6}$$

Furthermore,

$$|\gamma'(x)-\gamma'(y)| \le c_2(q)\mathrm{TP}^{\frac{1}{q}}_{q,\delta}(\gamma)|x-y|^{1-\frac{2}{q}}_{\mathbb{R}/\mathbb{Z}} \quad \text{for all } x,y \in \mathbb{R}/\mathbb{Z}. \tag{3.7}$$

Proof. Choose $\tau \in [0,\frac{1}{2}]$ such that, on the one hand,

$$|\gamma'(s+w)-\gamma'(s)| \le \frac{1}{2} \quad \text{for all } s \in \mathbb{R}/\mathbb{Z},\ |w| \le \tau, \tag{3.8}$$

and such that there exist,[5] on the other hand, parameters $s_0 \in \mathbb{R}/\mathbb{Z}$ and $|w_0| \le \tau$ with

$$|\gamma'(s_0 + w_0) - \gamma'(s_0)| = \frac{1}{2}. \tag{3.9}$$

Now we proceed in two steps. We first establish a preliminary upper bound for the seminorm of γ' in terms of the truncated energy and the parameter τ that still depends on γ. To bound τ uniformly from below again in terms of $\mathrm{TP}_{q,\delta}$ we insert (3.9) into a Morrey-type inequality which we finally prove in the second step. This inequality also yields (3.7).

Step 1. Lemma 3.4 implies for the seminorm on the truncated region $\hat{Y}_\delta(\gamma)$

$$\iint_{\hat{Y}_\delta(\gamma)} \frac{|\gamma'(x+w) - \gamma'(x)|^q}{|w|^q}\,dw\,dx \le 2^q \iint_{\hat{Y}_\delta(\gamma)\cap\{|w|>\tau\}} \frac{1}{|w|^q}\,dw\,dx \le \frac{2^{q+1}}{q-1}\tau^{1-q}. \tag{3.10}$$

Similarly, we obtain on the domain of $\mathrm{TP}_{q,\delta}$ away from the diagonal

$$\iint_{\substack{(\mathbb{R}/\mathbb{Z}\times[-\frac{1}{2},\frac{1}{2}])\setminus\hat{Y}_\delta(\gamma)\\ \cap\{|w|>\tau\}}} \frac{|\gamma'(x+w) - \gamma'(x)|^q}{|w|^q}\,dx\,dw \le \frac{2^{q+1}}{q-1}\tau^{1-q}, \tag{3.11}$$

so that it remains to investigate the domain of $\mathrm{TP}_{q,\delta}$ near the diagonal. For that we change variables to split the truncated energy (3.5) into two equal summands as

$$\mathrm{TP}_{q,\delta}(\gamma) = \frac{1}{2}\iint_{(\mathbb{R}/\mathbb{Z}\times[-\frac{1}{2},\frac{1}{2}])\setminus\hat{Y}_\delta(\gamma)} \frac{|2P^\perp_{\gamma'(x+w)}(\gamma(x+w)-\gamma(x))|^q + |2P^\perp_{\gamma'(x)}(\gamma(x+w)-\gamma(x))|^q}{|\gamma(x+w)-\gamma(x)|^{2q}}\,dw\,dx, \tag{3.12}$$

and bound the resulting numerator (including the prefactor $\frac{1}{2}$) from below by the expression

$$N^{\frac{q}{2}} := \left|P^\perp_{\gamma'(x+w)}\big(\gamma(x+w)-\gamma(x)\big) - P^\perp_{\gamma'(x)}\big(\gamma(x+w)-\gamma(x)\big)\right|^q. \tag{3.13}$$

Abbreviating the difference $\triangle f \equiv \triangle_w f(x) := f(x+w) - f(x)$ for any function $f : \mathbb{R}/\mathbb{Z}\times[-\frac{1}{2},\frac{1}{2}]$ and using the explicit form $P^\perp_\nu(\xi) = \xi - \langle\xi,\nu\rangle\nu$ of the orthogonal projection onto the orthogonal complement of $\nu \in \mathbb{S}^1$ we can estimate N from below as

$$\begin{aligned} N &= \langle\triangle\gamma, \gamma'(x+w)-\gamma'(x)\rangle^2 + 2\langle\triangle\gamma,\gamma'(x+w)\rangle\langle\triangle\gamma,\gamma'(x)\rangle\underbrace{(1-\langle\gamma'(x+w),\gamma'(x)\rangle)}_{=\frac{1}{2}|\triangle\gamma'|^2} \\ &\ge \langle\triangle\gamma,\gamma'(x+w)\rangle\langle\triangle\gamma,\gamma'(x)\rangle|\triangle\gamma'|^2. \end{aligned}$$

With $\triangle\gamma = w\int_0^1 \gamma'(x+\sigma w)\,d\sigma$ one obtains

$$N \ge w^2\int_0^1 \langle\gamma'(x+\sigma w),\gamma'(x+w)\rangle\,d\sigma\int_0^1\langle\gamma'(x+\theta w),\gamma'(x+w)\rangle\,d\theta\cdot|\triangle\gamma'|^2,$$

and again by means of $\langle a,b\rangle = 1 - \frac{1}{2}|a-b|^2$ for $a,b\in\mathbb{S}^1$ applied to both integrands we finally obtain for $x\in\mathbb{R}/\mathbb{Z}$ and $w\in[-\tau,\tau]$ because of (3.8)

$$N^{\frac{q}{2}} \ge \left(\frac{7}{8}\right)^q |w|^q|\gamma'(x+w)-\gamma'(x)|^q. \tag{3.14}$$

Combining (3.12),(3.13), and (3.14) with the Lipschitz estimate $|\gamma(x+w)-\gamma(x)| \le |w|$ in the denominator we arrive at

$$\iint_{\substack{(\mathbb{R}/\mathbb{Z}\times[-\frac{1}{2},\frac{1}{2}])\setminus\hat{Y}_\delta(\gamma)\\ \cap\{|w|\le\tau\}}} \frac{|\gamma'(x+w)-\gamma'(x)|^q}{|w|^q}\,dw\,dx \le \left(\frac{8}{7}\right)^q \mathrm{TP}_{q,\delta}(\gamma). \tag{3.15}$$

Adding up (3.10), (3.11), and (3.15) yields

$$[\gamma']^q_{1,1-\frac{1}{q}} = \iint_{\mathbb{R}/\mathbb{Z}\times[-\frac{1}{2},\frac{1}{2}]} \frac{|\gamma'(x+w)-\gamma'(x)|^q}{|w|^q}\,dw\,dx \le \left(\frac{8}{7}\right)^q \mathrm{TP}_{q,\delta}(\gamma) + \frac{2^{q+2}}{q-1}\tau^{1-q}. \tag{3.16}$$

Note that τ still depends on the curve γ, so this is not yet the desired a priori estimate for the seminorm. Furthermore, the second term blows up if τ tends to zero.

We need to establish a uniform lower bound on τ in order to prove the a priori estimate (3.6). In step 2 we will show that there is a constant $c_M(q) > 0$ independent of γ such that

$$\sup_{x\in\mathbb{R}/\mathbb{Z}} |\gamma'(x+w)-\gamma'(x)| \le c_M(q)\mathrm{TP}^{\frac{1}{q}}_{q,\delta}(\gamma)|w|^{1-\frac{2}{q}} \quad \text{for all } |w| \le \frac{\tau}{4}. \tag{3.17}$$

Then the explicit choice of s_0 and w_0 satisfying (3.9) and the triangle inequality yield

$$\frac{1}{2} = |\gamma'(s_0+w_0)-\gamma'(s_0)| \overset{(3.17)}{\le} 4c_M(q)\mathrm{TP}^{\frac{1}{q}}_{q,\delta}(\gamma)\left|\frac{w_0}{4}\right|^{1-\frac{2}{q}} \le 4^{\frac{2}{q}}c_M(q)\mathrm{TP}^{\frac{1}{q}}_{q,\delta}(\gamma)\tau^{1-\frac{2}{q}}. \tag{3.18}$$

[5] Such parameters do exist since γ is a closed curve so that the unit tangent vectors cannot be contained in a cone with opening angle $\alpha \le \frac{\pi}{2}\sin\frac{\alpha}{2} = \frac{\pi}{4}|\gamma'(s+w)-\gamma'(s)| \le \frac{\pi}{8}$.

Hence, setting $\tilde{c}(q) := (2c_M(q))^{\frac{q(q-1)}{q-2}} 16^{\frac{q-1}{q-2}}$ leads to $\tau^{1-q} \le \tilde{c}(q)\left(\mathrm{TP}_{q,\delta}(\gamma)\right)^{\frac{q-1}{q-2}}$. Finally, setting $c_1(q) := \max\left\{\left(\frac{8}{7}\right)^q, \frac{2^{q+2}}{q-1}\tilde{c}(q)\right\}$ implies by virtue of (3.16) inequality (3.6). Moreover, (3.17) is also the essential local estimate which leads to the global oscillation bound (3.7) for the tangent. Indeed, for $|x-y|_{\mathbb{R}/\mathbb{Z}} > \frac{\tau}{4}$ one simply estimates $|\gamma'(x)-\gamma'(y)| \le 2 < 2(\frac{4}{\tau}|x-y|_{\mathbb{R}/\mathbb{Z}})^{1-\frac{2}{q}}$, which together with (3.18) leads to

$$|\gamma'(x)-\gamma'(y)| < 16c_M(q)\mathrm{TP}_{q,\delta}^{\frac{1}{q}}(\gamma)|x-y|_{\mathbb{R}/\mathbb{Z}}^{1-\frac{2}{q}} \quad \text{for all } |x-y|_{\mathbb{R}/\mathbb{Z}} > \frac{\tau}{4}.$$

Combining this with (3.17) yields (3.7) if we set $c_2(q) := 16c_M(q)$.

Step 2. To complete the proof, we need to show inequality (3.17). For $r \in (0, \frac{\tau}{2}]$ define the open neighborhood $M_r := \bigcup_{x\in\mathbb{R}/\mathbb{Z}}\left(B_r(x)\times B_r(x)\right)$ of the diagonal in $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$, where $B_r(x) := \{y\in\mathbb{R}/\mathbb{Z} : |y-x|_{\mathbb{R}/\mathbb{Z}} < r\}$. Lemma 3.4 implies that M_r is contained in $(\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma)$. Set $\gamma'_{B_r(x)} := \fint_{B_r(x)}\gamma'(z)\,dz \equiv \frac{1}{2r}\int_{B_r(x)}\gamma'(z)\,dz$ and apply Hölder's inequality to estimate for $x\in\mathbb{R}/\mathbb{Z}$

$$\begin{aligned}
&\frac{1}{2r}\int_{B_r(x)}|\gamma'(y)-\gamma'_{B_r(x)}|dy \le \frac{1}{4r^2}\iint_{B_r(x)\times B_r(x)}|\gamma'(y)-\gamma'(z)|dzdy\\
&\le \Big(\frac{1}{4r^2}\iint_{B_r(x)\times B_r(x)}|\gamma'(y)-\gamma'(z)|^q dzdy\Big)^{\frac{1}{q}} \le \Big((2r)^{q-2}\iint_{B_r(x)\times B_r(x)}\frac{|\gamma'(y)-\gamma'(z)|^q}{|y-z|^q}dzdy\Big)^{\frac{1}{q}}\\
&\le 2^{1-\frac{2}{q}}r^{1-\frac{2}{q}}\Big(\iint_{\substack{(\mathbb{R}/\mathbb{Z})^2\setminus Y_\delta(\gamma)\\ \cap\{|y-z|<2r\}}}\frac{|\gamma'(y)-\gamma'(z)|^q}{|y-z|^q}dzdy\Big)^{\frac{1}{q}} \overset{(3.15)}{\le} 2^{1-\frac{2}{q}}r^{1-\frac{2}{q}}\frac{8}{7}\mathrm{TP}_{q,\delta}^{\frac{1}{q}}(\gamma).
\end{aligned} \tag{3.19}$$

For two points $x,y\in\mathbb{R}/\mathbb{Z}$ with $r := |x-y| \in (0,\frac{\tau}{4}]$ one finds

$$\begin{aligned}
|\gamma'(x)-\gamma'(y)| &\le \limsup_{k\to\infty}|\gamma'_{B_{2^{-k}r}(x)}-\gamma'_{B_{2^{-k}r}(y)}| \qquad (3.20)\\
&\le \sum_{l=0}^{\infty}\left|\gamma'_{B_{2^{-l}r}(x)}-\gamma'_{B_{2^{-l+1}r}(x)}\right| + \left|\gamma'_{B_{2r}(x)}-\gamma'_{B_{2r}(y)}\right| + \sum_{l=0}^{\infty}\left|\gamma'_{B_{2^{-l}r}(y)}-\gamma'_{B_{2^{-l+1}r}(y)}\right|.
\end{aligned}$$

Averaging the inequality $|\gamma'_{B_{2r}(x)}-\gamma'_{B_{2r}(y)}| \le |\gamma'_{B_{2r}(x)}-\gamma'(z)| + |\gamma'(z)-\gamma'_{B_{2r}(y)}|$ over all $z\in B_{2r}(x)\cap B_{2r}(y)$ in combination with the inclusions $B_r(x), B_r(y) \subset B_{2r}(x)\cap B_{2r}(y) \subset B_{2r}(x), B_{2r}(y)$ bounds the middle summand in (3.20) as

$$\begin{aligned}
\left|\gamma'_{B_{2r}(x)}-\gamma'_{B_{2r}(y)}\right| &\le \frac{1}{|B_{2r}(x)\cap B_{2r}(y)|}\Big[\int_{B_{2r}(x)}|\gamma'_{B_{2r}(x)}-\gamma'(z)|dz + \int_{B_{2r}(y)}|\gamma'_{B_{2r}(y)}-\gamma'(z)|dz\Big]\\
&\le 2\Big[\fint_{B_{2r}(x)}|\gamma'_{B_{2r}(x)}-\gamma'(z)|dz + \fint_{B_{2r}(y)}|\gamma'_{B_{2r}(y)}-\gamma'(z)|dz\Big]\\
&\overset{(3.19)}{\le} \frac{16}{7}(4r)^{1-\frac{2}{q}}\mathrm{TP}_{q,\delta}^{\frac{1}{q}}(\gamma) =: \hat{c}_1(q)r^{1-\frac{2}{q}}\mathrm{TP}_{q,\delta}^{\frac{1}{q}}(\gamma).
\end{aligned} \tag{3.21}$$

For the remaining terms in the series, we use the equality $|B_{2R}(x)| = 2|B_R(x)|$ for any $R\in(0,\frac{\tau}{4}]$ to estimate again by means of (3.19)

$$\begin{aligned}
|\gamma'_{B_{2R}(x)}-\gamma'_{B_R(x)}| &\le \fint_{B_R(x)}|\gamma'_{B_{2R}(x)}-\gamma'(z)|dz + \fint_{B_R(x)}|\gamma'_{B_R(x)}-\gamma'(z)|dz\\
&\le 2\fint_{B_{2R}(x)}|\gamma'_{B_{2R}(x)}-\gamma'(z)|dz + \fint_{B_R(x)}|\gamma'_{B_R(x)}-\gamma'(z)|dz\\
&\overset{(3.19)}{\le} \frac{8}{7}R^{1-\frac{2}{q}}\mathrm{TP}_{q,\delta}^{\frac{1}{q}}(\gamma)\left(2\cdot 4^{1-\frac{2}{q}}+2^{1-\frac{2}{q}}\right) =: \hat{c}_2(q)R^{1-\frac{2}{q}}\mathrm{TP}_{q,\delta}^{\frac{1}{q}}(\gamma).
\end{aligned} \tag{3.22}$$

Inserting (3.21) and (3.22) for $R := 2^{-l}r$, $l = 0,1,2,\dots$ into (3.20) yields

$$\begin{aligned}
|\gamma'(x)-\gamma'(y)| &\le \max\{\hat{c}_1(q),\hat{c}_2(q)\}\mathrm{TP}_{q,\delta}^{\frac{1}{q}}(\gamma)r^{1-\frac{2}{q}}\Big(\sum_{l=0}^{\infty}2^{-l(1-\frac{2}{q})}+\sum_{k=0}^{\infty}2^{-k(1-\frac{2}{q})}+1\Big)\\
&=: c_M(q)\mathrm{TP}_{q,\delta}^{\frac{1}{q}}(\gamma)|x-y|^{1-\frac{2}{q}} \quad \text{for all } |x-y|_{\mathbb{R}/\mathbb{Z}} \le \frac{\tau}{4}. \quad \square
\end{aligned}$$

Our next goal is to prove a *global bilipschitz estimate* away from self-intersection points for curves of finite truncated tangent-point energy where the bilipschitz constant depends *only* on the energy bound and the intrinsic distance between any two self-intersections; see Proposition 3.7 below. It is crucial for the compactness of sublevel sets of $\mathrm{TP}_{q,\delta}$ in $\mathcal{F}_n(\eta, j_\pm, s, \omega)$ proven in Theorem 4.2 in Section 4 that the bilipschitz constant in Proposition 3.7 does *not* depend on the shape of the individual curve as it does in Lemma 3.1. To prepare this we bound $\mathrm{TP}_{q,\delta}$ uniformly from below for normalized curves that possess a secant perpendicular to the curve in one of its two endpoints. To make this precise, define for $d\ge 2$, $\alpha\in(0,1]$, $L>2$, and $H\in(0,\infty)$ the set[6] $\mathscr{L}(\alpha,H,L,d)$ to be the collection of all arclength parametrized loops $\Gamma\in C^1(\mathbb{R}/L\mathbb{Z},\mathbb{R}^d)$ such that

[6] Note that without the condition $L>2$ the set $\mathscr{L}(\alpha,H,L,d)$ might be empty because of condition ($\mathscr{L}$1).

($\mathscr{L}1$) there are $s,t \in \mathbb{R}/L\mathbb{Z}$ with $|\Gamma(s)-\Gamma(t)| = 1$;
($\mathscr{L}2$) $\Gamma'(s) \perp \big(\Gamma(s)-\Gamma(t)\big)$;
($\mathscr{L}3$) $|\Gamma'(\sigma)-\Gamma'(\tau)| \le H|\sigma-\tau|^{\alpha}_{\mathbb{R}/\mathbb{Z}}$ for all $\sigma,\tau \in \mathbb{R}/L\mathbb{Z}$.

Lemma 3.6 (*Least Energy Contribution*). *Let $q \in (0,\infty)$ and $\Gamma \in \mathscr{L}(\alpha,H,L,d)$. Then*

$$\int_{I_\epsilon(s)}\int_{I_\rho(t)} \frac{d\tau d\sigma}{(r_{tp}[\Gamma](\Gamma(\sigma),\Gamma(\tau)))^q} > \frac{1}{6^q}\epsilon\rho \tag{3.23}$$

for all $\rho \le \frac{1}{8}$, $\epsilon \le \epsilon_0(\alpha,H) := \min\{\frac{1}{8},(\frac{1}{8H})^{\frac{1}{\alpha}}\}$, where $I_\epsilon(s) \in \{(s-\epsilon,s),(s,s+\epsilon)\}$ and $I_\rho(t) \in \{(t-\rho,t),(t,t+\rho)\}$.

Proof. Fix an arbitrary loop $\Gamma \in \mathscr{L}(\alpha,H,L,d)$ with two parameters $s,t \in \mathbb{R}/L\mathbb{Z}$ satisfying conditions ($\mathscr{L}1$) and ($\mathscr{L}2$). In particular, ($\mathscr{L}2$) implies

$$r_{\mathrm{tp}}[\Gamma](\Gamma(s),\Gamma(t)) = \frac{1}{2}|\Gamma(s)-\Gamma(t)| = \frac{1}{2}. \tag{3.24}$$

Keep t fixed for a moment and abbreviate the numerator and the denominator of the tangent-point radius for any $\sigma \in \mathbb{R}/L\mathbb{Z}$ as

$$r_{\mathrm{tp}}[\Gamma](\Gamma(\sigma),\Gamma(t)) = \frac{|\Gamma(t)-\Gamma(\sigma)|^2}{2\mathrm{dist}\left(\Gamma(t),\Gamma(\sigma)+\mathbb{R}\Gamma'(\sigma)\right)} := \frac{N(\sigma)}{D(\sigma)},$$

and use the triangle inequality and (3.24) to find

$$\frac{N(\sigma)}{D(\sigma)} \le \frac{1}{2} + \left|\frac{N(\sigma)}{D(\sigma)} - \frac{N(s)}{D(s)}\right| \le \frac{1}{2} + \frac{1}{2D(\sigma)}\Big|2N(\sigma)-D(\sigma)\Big|,$$

where we also used condition ($\mathscr{L}1$) which implies $2N(s) = D(s) = 2$. If $N(\sigma) > N(s) = 1$ we estimate by means of the binomial identity and ($\mathscr{L}1$)

$$\begin{aligned}|N(\sigma)-N(s)| &= \big(|\Gamma(t)-\Gamma(\sigma)|+1\big)\cdot\big(|\Gamma(t)-\Gamma(\sigma)|-1\big)\\ &\le \big(2+|\Gamma(\sigma)-\Gamma(s)|\big)|\Gamma(s)-\Gamma(\sigma)| < 3|s-\sigma|\end{aligned} \tag{3.25}$$

for all $|s-\sigma| < \epsilon_1 := 1 < \frac{L}{2}$ by means of $L > 2$. In any case, $N(\sigma) < 2$ for $|s-\sigma| < \epsilon_2 := \frac{1}{3} < \epsilon_1$. With conditions ($\mathscr{L}2$) and ($\mathscr{L}3$) one finds

$$\begin{aligned}&\frac{1}{2}|D(\sigma)-D(s)| \le |\Gamma(t)-\Gamma(\sigma)-\langle\Gamma(t)-\Gamma(\sigma),\Gamma'(\sigma)\rangle\Gamma'(\sigma)-(\Gamma(t)-\Gamma(s))|\\ &= \Big|\Gamma(s)-\Gamma(\sigma)-\big[\langle\Gamma(t)-\Gamma(\sigma),\Gamma'(\sigma)-\Gamma'(s)\rangle+\langle\Gamma(s)-\Gamma(\sigma),\Gamma'(s)\rangle\big]\Gamma'(\sigma)\Big|\\ &\le 2|s-\sigma|+2H|s-\sigma|^{\alpha},\end{aligned}$$

where we also used again that Γ has Lipschitz constant 1, and therefore also $|\Gamma(t)-\Gamma(\sigma)| \le |\Gamma(t)-\Gamma(s)|+|\Gamma(s)-\Gamma(\sigma)| \le 1+|s-\sigma| < 2$ for all $|s-\sigma| < \epsilon_1$. This implies in particular that $D(\sigma) \in (1,3)$ for all $|s-\sigma| < \epsilon_0 := \min\{\frac{1}{8},\left(\frac{1}{8H}\right)^{\frac{1}{\alpha}}\} < \epsilon_2$ so that we infer $r_{\mathrm{tp}}[\Gamma](\Gamma(\sigma),\Gamma(t)) = \frac{N(\sigma)}{D(\sigma)} < 2$ for all $|s-\sigma| < \epsilon_0$. Now abbreviate $\frac{Z(\sigma,\tau)}{W(\sigma,\tau)} := r_{\mathrm{tp}}[\Gamma](\Gamma(\sigma),\Gamma(\tau))$. Similarly as in (3.25) we estimate in case $Z(\sigma,\tau) > N(\sigma)$

$$\begin{aligned}|Z(\sigma,\tau)-N(\sigma)| &= \big(|\Gamma(\tau)-\Gamma(\sigma)|+\sqrt{N(\sigma)}\big)\cdot\big(|\Gamma(\tau)-\Gamma(\sigma)|-\sqrt{N(\sigma)}\big)\\ &\le \big(2\sqrt{2}+|t-\tau|\big)|t-\tau| < 3|t-\tau|\end{aligned}$$

for all $|s-\sigma| < \epsilon_0$ and $|t-\tau| < \frac{1}{8} =: \rho_1$. In particular, $Z(\sigma,\tau) < 2+3|t-\tau| < 3$ for all $|t-\tau| < \rho_1$. Moreover,

$$\begin{aligned}\frac{1}{2}|W(\sigma,\tau)-D(\sigma)| &\le |\Gamma(\tau)-\Gamma(\sigma)-\langle\Gamma(\tau)-\Gamma(\sigma),\Gamma'(\sigma)\rangle\Gamma'(\sigma)-D(\sigma)|\\ &= |\Gamma(\tau)-\Gamma(t)-\langle\Gamma(\tau)-\Gamma(t),\Gamma'(\sigma)\rangle\Gamma'(\sigma)| \le 2|\tau-t| < \frac{1}{4},\end{aligned}$$

in particular, $W(\sigma,\tau) > \frac{1}{2}$ for $|s-\sigma| < \epsilon_0$ and $|t-\tau| < \rho_1$. Thus, we easily obtain the rough estimate $r_{\mathrm{tp}}[\Gamma](\Gamma(\sigma),\Gamma(\tau)) < 6$ for all $|s-\sigma| < \epsilon_0$ and $|t-\tau| < \rho_1$, from which we infer the claim by integrating over $I_\epsilon(s)\times I_\rho(t)$ for $\epsilon \in (0,\epsilon_0]$ and $\rho \le \rho_1 = \frac{1}{8}$.

Proposition 3.7 (*Global Uniform Bilipschitz Estimate*). *Let $q > 2$, $\eta \in (0,1)$, and $E > 0$. Then there exists a constant $c = c(q,E,\eta) > 0$ such that for all $\gamma \in \mathcal{F}(\eta, j_\pm, s, \omega)$ with $\mathrm{TP}_{q,\delta}(\gamma) \le E$*

$$|x-y|_{\mathbb{R}/\mathbb{Z}} \le c(q,E,\eta)|\gamma(x)-\gamma(y)|$$

for all $(x,y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma)$ and all $\delta \in \left(0,\frac{\eta}{2}\right]$.

Proof. Combining the basic inequality (3.1) with the uniform oscillation estimate (3.7) for the tangents γ' of any curve $\gamma \in \mathcal{F}(\eta, j_\pm, s, \omega) \subset \mathcal{F}_n$ for $n = j_+ - j_-$ we find a constant $\tau_1 = \tau_1(q, E) > 0$ such that

$$|\gamma(y) - \gamma(x)| \geq \frac{1}{2}|y - x|_{\mathbb{R}/\mathbb{Z}} \quad \text{for all } \ |y - x|_{\mathbb{R}/\mathbb{Z}} \leq \tau_1. \tag{3.26}$$

As in the proof[7] of Lemma 3.1 we find that

$$C := \inf\big\{|\gamma(x) - \gamma(y)| \ : \ (x, y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma) \text{ and } |x - y|_{\mathbb{R}/\mathbb{Z}} \geq \tau_1\big\}$$

is strictly positive and attained by at least one parameter pair (x_0, y_0) because of uniform continuity of the mapping $(x, y) \mapsto |\gamma(x) - \gamma(y)|^2$ on compact subsets of $(\mathbb{R}/\mathbb{Z})^2$. We can assume $C \in (0, \frac{1}{2})$ since otherwise we could conclude the proof with

$$|\gamma(x) - \gamma(y)| \geq \frac{1}{2} \geq |x - y|_{\mathbb{R}/\mathbb{Z}} \quad \text{for all } (x, y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma) \text{ with } |x - y|_{\mathbb{R}/\mathbb{Z}} \geq \tau_1.$$

If $|x_0 - y_0|_{\mathbb{R}/\mathbb{Z}} = \tau_1$, we estimate with (3.26) $2C = 2|\gamma(x_0) - \gamma(y_0)| \geq \tau_1$ and thus

$$|x - y|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{2} \leq \frac{C}{\tau_1} \leq \frac{|\gamma(x) - \gamma(y)|}{\tau_1} \quad \text{for all } (x, y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma), |x - y|_{\mathbb{R}/\mathbb{Z}} \geq \tau_1,$$

which finishes the proof in that case. Now, assume $|x_0 - y_0|_{\mathbb{R}/\mathbb{Z}} > \tau_1$. The goal is to derive a uniform lower bound on C in order to finish the proof. Define the intervals $I_r^+(x) := (x, x + r)$ and $I_r^-(x) := (x - r, x)$ for $x \in \mathbb{R}/\mathbb{Z}$ and $r > 0$. Then it is easy to check by means of $0 < \delta \leq \frac{\eta}{2} < \frac{\Lambda}{4}$ (cf. Definition 1.1) that at least one of the following two inclusions is true

$$I_\eta^+(x_0) \times I_\eta^+(y_0) \subset (\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma), \tag{3.27}$$

$$I_\eta^-(x_0) \times I_\eta^-(y_0) \subset (\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma). \tag{3.28}$$

If $(x_0, y_0) \in (\mathbb{R}/\mathbb{Z})^2 \setminus \overline{Y_\delta(\gamma)}$ then it realizes the infimum C as an interior point so that the gradient of the mapping $(x, y) \mapsto |\gamma(x) - \gamma(y)|^2$ vanishes at (x_0, y_0) which leads to $\gamma'(x_0) \perp (\gamma(x_0) - \gamma(y_0)) \perp \gamma'(y_0)$. But also if (x_0, y_0) lies in the boundary of $Y_\delta(\gamma)$, we can infer the orthogonality of the chord $\gamma(x_0) - \gamma(y_0)$ to one of the tangent vectors $\gamma'(x_0)$ or $\gamma'(y_0)$ since with $\delta < \eta$ both points are contained in straight segments of the curve γ near a self-intersection pair $(u, v) \in S(\gamma)$ (recall (1.3) and Definition 1.1). Indeed, at least one of the points, say $\gamma(x_0)$ lies on the intersection of one straight segment of γ with the circle of radius δ centered at the double point $\gamma(u) = \gamma(v)$. On the other hand, $\gamma(y_0)$ sits on the other straight segment of γ and may be closer to the double point. But since $|\gamma(x_0) - \gamma(y_0)| = C$ is minimal the chord $\gamma(x_0) - \gamma(y_0)$ must be perpendicular to γ at $\gamma(y_0)$. Summarizing we can say that in any case one has $\gamma'(y_0) \perp (\gamma(x_0) - \gamma(y_0))$. To bring Lemma 3.6 into play we first look at the reparametrization $\eta(x) := \gamma(Cx)$ for $x \in \mathbb{R}/\frac{1}{C}\mathbb{Z}$ and change variables according to $z(x, y) = (Cx, Cy)$ to compute

$$\mathrm{TP}_{q, \frac{\delta}{C}}(\eta) = \iint_{(\mathbb{R}/\frac{1}{C}\mathbb{Z})^2 \setminus Y_{\frac{\delta}{C}}(\eta)} \Big(\frac{2|P^\perp_{\eta'(x)}(\eta(y) - \eta(x))|}{|\eta(y) - \eta(x)|^2}\Big)^q |\eta'(x)||\eta'(y)| dx dy = \mathrm{TP}_{q,\delta}(\gamma).$$

Together with the scaling behavior (3.4) of the tangent-point energy we obtain for the rescaled curve $Y := \frac{1}{C}\eta$

$$\mathrm{TP}_{q, \frac{\delta}{C}}(Y) = \Big(\frac{1}{C}\Big)^{2-q} \mathrm{TP}_{q, \frac{\delta}{C}}(\eta) = \Big(\frac{1}{C}\Big)^{2-q} \mathrm{TP}_{q,\delta}(\gamma) \leq \Big(\frac{1}{C}\Big)^{2-q} E. \tag{3.29}$$

On the other hand, Y is contained in the set $\mathscr{L}(\alpha, H, L, d)$ referred to in Lemma 3.6 for $d := 2$, $L := \frac{1}{C} > 2$. $\alpha := 1 - \frac{2}{q}$, $H := (\frac{1}{2})^{1-\frac{2}{q}} c_2(q) E^{\frac{1}{q}}$, where $c_2(q)$ is the constant in the oscillation inequality (3.7) of Lemma 3.5. Properties ($\mathscr{L}1$) and ($\mathscr{L}2$) hold for $s := \frac{y_0}{C}$ and $t := \frac{x_0}{C}$. From (3.27) and (3.28) we infer that at least one of the following two inclusions holds true as well:

$$I^+_{\frac{\eta}{C}}\Big(\frac{x_0}{C}\Big) \times I^+_{\frac{\eta}{C}}\Big(\frac{y_0}{C}\Big) \subset (\mathbb{R}/\frac{1}{C}\mathbb{Z})^2 \setminus Y_{\frac{\delta}{C}}(Y), \tag{3.30}$$

$$I^-_{\frac{\eta}{C}}\Big(\frac{x_0}{C}\Big) \times I^-_{\frac{\eta}{C}}\Big(\frac{y_0}{C}\Big) \subset (\mathbb{R}/\frac{1}{C}\mathbb{Z})^2 \setminus Y_{\frac{\delta}{C}}(Y). \tag{3.31}$$

Assuming without loss of generality that inclusion (3.30) holds, we can combine the lower bound (3.23) of Lemma 3.6 for $\Gamma := Y$ with the energy estimate (3.29) to obtain

$$\begin{aligned} C^{q-2} E \geq \mathrm{TP}_{q, \frac{\delta}{C}}(Y) &\geq \int_{I^+_{\frac{\eta}{C}}(\frac{y_0}{C})} \int_{I^+_{\frac{\eta}{C}}(\frac{x_0}{C})} \frac{d\tau d\sigma}{(r_{\mathrm{tp}}[Y](Y(\sigma), Y(\tau)))^q} \\ &> \frac{1}{6^q} \min\Big\{\frac{\eta}{C}, \epsilon_0\Big\} \min\Big\{\frac{\eta}{C}, \frac{1}{8}\Big\} > \min\big\{2\eta, \epsilon_0\big\} \min\Big\{2\eta, \frac{1}{8}\Big\}, \end{aligned} \tag{3.32}$$

since $C \in (0, \frac{1}{2})$, where ϵ_0 is the constant from Lemma 3.6 depending on α and H, which themselves were identified above and depend only on q and on E. Since $q > 2$ the right-hand side of (3.32) gives us the desired lower bound for C solely in terms of η, q, and E.

[7] With the difference that τ_1 does *not* depend on the individual shape of the curve in contrast to τ_γ in (3.3) in the proof of Lemma 3.1.

4. Minimizing the δ-truncated energy

So far, we have proven an a priori estimate for the fractional Sobolev seminorm for curves in $\mathcal{F}_n$ (Lemma 3.5), and a uniform bilipschitz estimate for curves in the subclass $\mathcal{F}(\eta, j_\pm, s, \omega) \subset \mathcal{F}_{j_+-j_-}$ (Proposition 3.7), both under the assumption of uniformly bounded truncated energy $\mathrm{TP}_{q,\delta}$. A key ingredient in the direct method in the calculus of variations to prove existence of minimizers is the lower semi-continuity of the energy. The truncated tangent-point energy is in fact lower semi-continuous if the lengths of the subarcs between self-intersections of the converging curves are bounded from below.

Lemma 4.1. *Let $q, \Lambda_0 > 0$, $n \in \mathbb{N}$ and $\delta \in (0, \frac{\Lambda_0}{2})$. Furthermore, let $(\gamma_k)_{k\in\mathbb{N}} \subset \mathcal{F}_n$ be a sequence with $\Lambda(\gamma_k) \geq \Lambda_0$ and $\gamma_k \to \gamma \in \mathcal{F}_n$ in C^1 as $k \to \infty$. Then*

$$\mathrm{TP}_{q,\delta}(\gamma) \leq \liminf_{k\to\infty} \mathrm{TP}_{q,\delta}(\gamma_k). \tag{4.1}$$

Proof. We may assume that $\liminf_{k\to\infty} \mathrm{TP}_{q,\delta}(\gamma_k) < \infty$. The parameter pairs of self-intersection points are collected in the sets $S(\gamma_k) = \{(u_1^k, v_1^k), \dots, (u_n^k, v_n^k)\}$ and $S(\gamma) = \{(u_1, v_1), \dots, (u_n, v_n)\}$ as defined in (1.3). As the sequences $(u_i^k)_{k\in\mathbb{N}}$ and $(v_i^k)_{k\in\mathbb{N}}$ are bounded, we can assume that $u_i^k \to \tilde{u}_i$ and $v_i^k \to \tilde{v}_i$ for some $\tilde{u}_i, \tilde{v}_i \in \mathbb{R}/\mathbb{Z}$, $i = 1, \dots, n$, as $k \to \infty$. Since $\Lambda(\gamma_k) \geq \Lambda_0$ for all $k \in \mathbb{N}$ it is ensured that the set $\{\tilde{u}_i, \tilde{v}_i \mid i = 1, \dots, n\}$ has exactly $2n$ elements. The C^1-convergence then implies $\gamma(\tilde{u}_i) = \gamma(\tilde{v}_i)$ for all $i = 1, \dots, n$. Since γ has exactly n self-intersections, this implies that for each $i \in \{1, \dots, n\}$ there is exactly one $j = j(i) \in \{1, \dots, n\}$ such that $(\tilde{u}_i, \tilde{v}_i) = (u_j, v_j)$. Assume for a moment that the integrands converge pointwise, i.e.,

$$\left(\frac{2\mathrm{dist}\,(\ell_k(x), \gamma_k(y))}{|\gamma_k(x) - \gamma_k(y)|^2}\right)^q \chi_{(\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma_k)}(x, y) \to \left(\frac{2\mathrm{dist}\,(\ell(x), \gamma(y))}{|\gamma(x) - \gamma(y)|^2}\right)^q \chi_{(\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma)}(x, y) \tag{4.2}$$

for almost all $(x, y) \in (\mathbb{R}/\mathbb{Z})^2$ as $k \to \infty$. Fatou's lemma then implies (4.1).

Hence, it remains to prove the pointwise convergence in (4.2). If $(x, y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus \overline{Y_\delta(\gamma)}$ with $x \neq y$, then due to the convergence of the sequences $(u_i^k)_{k\in\mathbb{N}}$ and $(v_i^k)_{k\in\mathbb{N}}$ there exists an index $k_1 \in \mathbb{N}$ such that $(x, y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus \overline{Y_\delta(\gamma_k)}$ for every $k \geq k_1$. The C^1-convergence $\gamma_k \to \gamma$ as $k \to \infty$ then implies pointwise convergence in (4.2) for those tuples. In the same manner one can argue that if $(x, y) \in Y_\delta(\gamma)$, there exists a $k_2 \in \mathbb{N}$ such that $(x, y) \in Y_\delta(\gamma_k)$ for all $k \geq k_2$. Hence, the pointwise convergence in (4.2) is proven for all $(x, y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus \partial Y_\delta(\gamma)$, and the exceptional set $\partial Y_\delta(\gamma)$ has measure zero.

The following compactness theorem will be essential to show the existence of minimizers.

Theorem 4.2. *Let $q > 2$, $j_+, j_-, s, l \in \mathbb{Z}$, and $\eta \in (0, 1)$ such that $\mathcal{F}(\eta, j_\pm, s, \omega) \neq \emptyset$. If $\delta \in (0, \frac{\eta}{2}]$ then for every $E < \infty$ the set*

$$\mathcal{A}(\delta, E, \eta, j_\pm, s, \omega) := \left\{\gamma \in \mathcal{F}(\eta, j_\pm, s, \omega) : \mathrm{TP}_{q,\delta}(\gamma) \leq E \text{ and } 0 \in \gamma(\mathbb{R}/\mathbb{Z})\right\}$$

is sequentially compact with respect to the C^1-norm.

Proof. If $\mathcal{A}(\delta, E, \eta, j_\pm, s, \omega) = \emptyset$ there is nothing to prove, so let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence contained in $\mathcal{A}(\delta, E, \eta, j_\pm, s, \omega)$. Let $n := j_+ - j_-$ be the number of the exclusively transverse self-intersections of each curve γ_k and denote by (u_i^k, v_i^k) the distinct parameters with $\gamma_k(u_i^k) = \gamma_k(v_i^k)$ for every $k \in \mathbb{N}$ and $i = 1, \dots, n$. Since the origin is contained in the image of every curve γ_k and the curves are 1-lipschitz due to arclength parametrization, we have

$$|\gamma_k(x)| \leq \frac{1}{2} \quad \text{and} \quad |\gamma_k(x) - \gamma_k(y)| \leq |x - y|_{\mathbb{R}/\mathbb{Z}} \quad \text{for all } x, y \in \mathbb{R}/\mathbb{Z} \text{ and } k \in \mathbb{N}.$$

By Lemma 3.5, we obtain a uniform Hölder estimate

$$|\gamma_k'(x) - \gamma_k'(y)| \leq c(q, E)|x - y|_{\mathbb{R}/\mathbb{Z}}^{1-\frac{2}{q}} \quad \text{for all } x, y \in \mathbb{R}/\mathbb{Z} \text{ and } k \in \mathbb{N}.$$

By means of the Arzelà-Ascoli theorem we obtain a subsequence, again denoted by $(\gamma_k)_{k\in\mathbb{N}}$, and a curve $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ such that $\gamma_k \to \gamma$ in C^1 as $k \to \infty$. It remains to check that $\gamma \in \mathcal{A}(\delta, E, \eta, j_\pm, s, \omega)$. The lower-semicontinuity of $\mathrm{TP}_{q,\delta}$ established in Lemma 4.1 implies $\mathrm{TP}_{q,\delta}(\gamma) \leq E$. As a C^1-limit of the γ_k the curve γ satisfies $|\gamma'| = 1$ on $\mathbb{R}/\mathbb{Z}$ and $0 \in \gamma(\mathbb{R}/\mathbb{Z})$. Since the sequences $(u_i^k)_{k\in\mathbb{N}}$ and $(v_i^k)_{k\in\mathbb{N}}$ with $\gamma_k(u_i^k) = \gamma_k(u_i^k)$ are bounded in $\mathbb{R}/\mathbb{Z}$, there exist subsequences, again denoted with index k, such that $u_i^k \to u_i$ and $v_i^k \to v_i$ for some $u_i, v_i \in \mathbb{R}/\mathbb{Z}$ as $k \to \infty$. The C^1-convergence then implies $\gamma(u_i) = \gamma(v_i)$ for all $i = 1, \dots, n$. Recall from (1.4) that $T(\gamma_k) = \{u_1^k, v_1^k, \dots, u_n^k, v_n^k\}$. Define the set $\tilde{T} := \{u_1, v_1, \dots, u_n, v_n\}$ and observe

$$\begin{aligned} 2\eta \leq \Lambda(\gamma_k) &= \min\{|a - b|_{\mathbb{R}/\mathbb{Z}} \ : \ a, b \in T(\gamma_k), a \neq b\} \\ &\to \min\{|a - b|_{\mathbb{R}/\mathbb{Z}} \ : \ a, b \in \tilde{T}, a \neq b\} \quad \text{as } k \to \infty. \end{aligned} \tag{4.3}$$

Hence, the set $\tilde{T}$ contains $2n$ distinct elements and γ has at least n self-intersections. The affine linearity of $\gamma_k|_{B_\eta(u_i^k)}$ and of $\gamma_k|_{B_\eta(v_i^k)}$ transfers via the C^1-convergence as $k \to \infty$ to affine linearity of $\gamma|_{B_\eta(u_i)}$ and of $\gamma|_{B_\eta(v_i)}$ for $i = 1, \dots, n$. It still needs to be shown that γ has *exactly* n self-intersections and that every self-intersection is transverse. From Proposition 3.7, we obtain the uniform bilipschitz estimate $|x - y|_{\mathbb{R}/\mathbb{Z}} \leq c(q, E, \eta)|\gamma_k(x) - \gamma_k(y)|$ for all $(x, y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus Y_\delta(\gamma_k)$ and $k \in \mathbb{N}$. Passing to the limit $k \to \infty$ yields

$$|x - y|_{\mathbb{R}/\mathbb{Z}} \leq c(q, E, \eta)|\gamma(x) - \gamma(y)| \tag{4.4}$$

for all $(x,y) \notin \tilde{M} := \bigcup_{i=1}^{n} \big(B_\delta(u_i) \times B_\delta(v_i)\big) \cup \big(B_\delta(v_i) \times B_\delta(u_i)\big)$, where as before we denote $B_r(x) := \{y \in \mathbb{R}/\mathbb{Z} : |y-x|_{\mathbb{R}/\mathbb{Z}} < r\}$. Note that for $i = 1, \dots, n$ we have $|u_i - v_i|_{\mathbb{R}/\mathbb{Z}} = \lim_{k\to\infty} |u_i^k - v_i^k|_{\mathbb{R}/\mathbb{Z}} \geq \lim_{k\to\infty} \Lambda(\gamma_k) \geq 2\eta$ by definition of the set $\mathcal{F}(\eta, j_\pm, s, \omega)$. With the bilipschitz estimate (4.4) we infer

$$c(q,E,\eta)|\gamma(u_i+\delta) - \gamma(v_i+\delta)| \geq |u_i - v_i|_{\mathbb{R}/\mathbb{Z}} \geq 2\eta \quad \text{for} \quad i = 1, \dots, n.$$

Thus, since $\delta \leq \frac{\eta}{2}$ we can use the fact that γ is linear near each self-intersection point to estimate the intersection angle by

$$\begin{aligned} \measuredangle\big(\gamma'(u_i), \gamma'(v_i)\big) &= 2\arcsin\Big(\frac{|\gamma(u_i+\delta) - \gamma(v_i+\delta)|}{2\delta}\Big) \geq 2\arcsin\Big(\frac{\eta}{\delta c(q,E,\eta)}\Big) \\ &\geq 2\arcsin\Big(\frac{2}{c(q,E,\eta)}\Big) > 0. \end{aligned} \tag{4.5}$$

By means of (4.4) the only possible self-intersections of γ other than at the parameter pairs (u_i, v_i), $i = 1, \dots, n$, would be at $(x,y) \in \tilde{M}$. Without loss of generality we assume $(x,y) \in B_\delta(u_i) \times B_\delta(v_i) \setminus \{(u_i, v_i)\}$ for some $i = 1, \dots, n$, otherwise we exchange the roles of x and y. Due to linearity of γ near the self-intersection at (u_i, v_i), this is only possible if the two lines lie on one straight line. However, this is prevented by (4.5). Hence, γ has not more than and therefore exactly n self-intersections at (u_i, v_i) for $i = 1, \dots, n$, with minimal parameter distance $\Lambda(\gamma) \geq 2\eta$ according to (4.3), and all these self-intersections are transverse. In particular, γ is a generic curve and therefore its Arnold invariants are well-defined. Since the Arnold invariants are locally constant under C^1-convergence, we have $J^+(\gamma) = j_+$, $J^-(\gamma) = j_-$ and $St(\gamma) = s$, as these values were prescribed on the curves γ_k for each $k \in \mathbb{N}$. By continuity of the mapping degree we obtain for the winding number $W(\gamma) = \lim_{k\to\infty} W(\gamma_k) = \omega$. Thus, we have proven $\gamma \in \mathcal{F}(\eta, j_\pm, s, \omega)$.

We have now all tools together to prove existence of minimizers of the truncated tangent-point energy $\mathrm{TP}_{q,\delta}$ in the set of admissible curves.

Proof of Theorem 1.3. The infimum $\mathcal{I} := \inf_{\mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C}} \mathrm{TP}_{q,\delta}$ is non-negative, since the energy is. Moreover, $\mathcal{I}$ is finite, because the admissibility class $\mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C}$ contains a $C^{1,1}$-curve by Theorem 1.2, for which the energy is finite according to Lemma 3.3 since $C^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \subset W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$. Let $(\gamma_k)_{k\in\mathbb{N}} \subset \mathcal{F}(\eta, j_\pm, s, \omega) \cap \mathcal{C}$ be a minimizing sequence such that $\lim_{k\to\infty} \mathrm{TP}_{q,\delta}(\gamma_k) = \mathcal{I}$. Hence, there exists a constant $E > 0$ such that $\mathrm{TP}_{q,\delta}(\gamma_k) \leq E$ for all k sufficiently large. Since $\mathrm{TP}_{q,\delta}$ and the admissibility class are invariant under translations we may assume that $0 \in \gamma_k(\mathbb{R}/\mathbb{Z})$ for all $k \in \mathbb{N}$. By the compactness result of Theorem 4.2 there exists a subsequence, again denoted by $(\gamma_k)_{k\in\mathbb{N}}$, and a limit curve $\gamma_\delta^\eta \in \mathcal{F}(\eta, j_\pm, s, \omega)$ with $\gamma_k \to \gamma_\delta^\eta$ in C^1 as $k \to \infty$. In addition, γ_δ^η is a generic immersion contained in some $(j_\pm, s, \omega)$-compartment $\tilde{\mathcal{C}}$, which is open with respect to the C^1-topology, so that $\gamma_k \in \tilde{\mathcal{C}}$ for $k \gg 1$; hence $\tilde{\mathcal{C}} = \mathcal{C}$. Applying Lemma 4.1 for $\Lambda_0 := 2\eta \leq \Lambda(\gamma_k)$, we deduce $\mathcal{I} \leq \mathrm{TP}_{q,\delta}(\gamma_\delta^\eta) \leq \liminf_{k\to\infty} \mathrm{TP}_{q,\delta}(\gamma_k) = \mathcal{I}$. The $W^{2-\frac{1}{q},q}$-regularity of γ_δ^η now follows from Lemma 3.5. □

5. The limit $\delta \to 0$ and an optimally immersed curve

5.1. The renormalized energy and Gamma convergence

In general, the truncated energy $\mathrm{TP}_{q,\delta}$ does not allow for the limiting process δ to zero since the limit energy would be infinite due to the self-intersections. To take this blow-up into account, we scale the energy by a factor following an idea of Dennis Kube [24], who worked with a logarithmic rescaling of a suitably truncated Möbius energy on the space of figure-eight shaped curves with only one self-intersection and vanishing winding number. To deduce the correct scaling factor for the tangent-point energy we focus on the dominating interactions of different curve strands near self-intersection points to study the energy's blow-up behavior. For that we restrict the tangent-point energy to annular regions of arclength near self-intersections of curves in the class $\mathcal{F}_n$ defined in Definition 3.2.

Definition 5.1. Let $q > 0$, $n \in \mathbb{N}$, $0 < \delta < \theta < \eta$ and $\gamma \in \mathcal{F}_n$. Define the (θ, δ)-*annular tangent-point energy* $A_{q,\theta,\delta}$ of γ as

$$A_{q,\theta,\delta}(\gamma) := \iint_{Y_\theta(\gamma) \setminus Y_\delta(\gamma)} \left(\frac{2\mathrm{dist}\,(\ell(x), \gamma(y))}{|\gamma(x) - \gamma(y)|^2}\right)^q dxdy = \mathrm{TP}_{q,\delta}(\gamma) - \mathrm{TP}_{q,\theta}(\gamma). \tag{5.1}$$

The following representation of the annular tangent-point energy on curves in the subclass

$$\mathcal{F}_{n,\eta} := \{\gamma \in \mathcal{F}_n : \gamma \text{ is linear within arclength } \eta \text{ of each self-intersection}\} \tag{5.2}$$

is of central significance.

Proposition 5.2. *Let $q \geq 2$, $n \in \mathbb{N}$, $0 < \delta < \theta < \eta$, and $\gamma \in \mathcal{F}_{n,\eta}$. Then*

$$A_{q,\theta,\delta}(\gamma) = \frac{2^{q+2}}{q-2}\left(\delta^{2-q} - \theta^{2-q}\right) \sum_{i=1}^{n} F(\alpha_i(\gamma)), \tag{5.3}$$

where $\alpha_i(\gamma)$ is the angle between the two tangent lines of γ at the ith self-intersection for $i \in \{1, \dots, n\}$, and

$$F(\alpha) := \frac{1}{(\sin\alpha)^{q-1}} \Bigg[\int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} \Big(\big(\frac{1}{z^2+1}\big)^q + \big(\frac{z\sin\alpha + \cos\alpha}{z^2+1}\big)^q \Big) dz + \int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \Big(\big(\frac{1}{z^2+1}\big)^q + \big(\frac{z\sin\alpha - \cos\alpha}{z^2+1}\big)^q \Big) dz \Bigg] \quad \text{for } \alpha \in \Big(0, \frac{\pi}{2}\Big]. \tag{5.4}$$

Before proving this proposition let us draw some immediate conclusions. We show in Lemma A.2 in the appendix that the map F defined in (5.4) is strictly monotone decreasing and minimal for $\alpha = \frac{\pi}{2}$.

Corollary 5.3. *A curve $\gamma \in \mathcal{F}_{n,\eta}$ minimizes the annular tangent-point energy $A_{q,\theta,\delta}$ if and only if $\alpha_i(\gamma) = \frac{\pi}{2}$ for all $i = 1, \dots, n$.*

Observe as a direct consequence of the representation (5.3) the following expression for the limit of the rescaled annular tangent-point energy as $\delta \to 0$:

$$\lim_{\delta\to 0} \delta^{q-2} A_{q,\theta,\delta}(\gamma) = \frac{2^{q+2}}{q-2} \sum_{i=1}^{n} F(\alpha_i(\gamma)) \quad \text{for all } \gamma \in \mathcal{F}_{n,\eta}. \tag{5.5}$$

This motivates the renormalization of the truncated tangent-point energy as defined in the introduction.

Definition 5.4. For $q > 2$, $n \in \mathbb{N}$, $\eta \in (0,1)$, and $\gamma \in \mathcal{F}_{n,\eta}$ define the *renormalized tangent-point energy*

$$R_q(\gamma) := \lim_{\delta\to 0} \delta^{q-2} \mathrm{TP}_{q,\delta}(\gamma) \in [0, \infty]. \tag{5.6}$$

Note that the definition of the annular energy (5.1) implies the relation

$$\mathrm{TP}_{q,\theta}(\gamma) = \mathrm{TP}_{q,\delta}(\gamma) - A_{q,\theta,\delta}(\gamma) \text{ for } 0 < \delta < \theta < \eta,\ \gamma \in \mathcal{F}_{n,\eta}. \tag{5.7}$$

This together with the additional assumption that $\mathrm{TP}_{q,\frac{\eta}{2}}(\gamma) < \infty$ leads to

$$\begin{aligned} R_q(\gamma) &= \lim_{\delta\to 0} \delta^{q-2} \mathrm{TP}_{q,\delta}(\gamma) \overset{(5.7)}{=} \lim_{\delta\to 0} \delta^{q-2} (\mathrm{TP}_{q,\frac{\eta}{2}}(\gamma) + A_{q,\frac{\eta}{2},\delta}(\gamma)) \\ &= \lim_{\delta\to 0} \delta^{q-2} A_{q,\frac{\eta}{2},\delta}(\gamma) \overset{(5.5)}{=} \frac{2^{q+2}}{q-2} \sum_{i=1}^{n} F(\alpha_i(\gamma)). \end{aligned} \tag{5.8}$$

Thus, we have established an explicit formula for the renormalized tangent-point energy R_q in terms of the function F, if the curve γ has sufficient regularity as stated in the following lemma.

Lemma 5.5. *For $q > 2$, $n \in \mathbb{N}$, $\eta \in (0,1)$, and $\gamma \in W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \cap \mathcal{F}_{n,\eta}$ one has*

$$R_q(\gamma) = \frac{2^{q+2}}{q-2} \sum_{i=1}^{n} F(\alpha_i(\gamma)).$$

Moreover, $\gamma \in W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \cap \mathcal{F}_{n,\eta}$ is a global minimizer of R_q if and only if $\alpha_i(\gamma) = \frac{\pi}{2}$ for all $i = 1, \dots, n$.

Proof. By Lemma 3.3, the assumed fractional Sobolev regularity ensures that $\mathrm{TP}_{q,\frac{\eta}{2}}(\gamma)$ is finite, so we can apply (5.8). The last statement follows from Lemma A.2 in the appendix.

In Definition 5.4, the renormalized tangent-point energy R_q is defined as a pointwise limit. However, one can prove more: The energy is in fact the Γ-limit of the scaled truncated tangent-point energies $(\delta^{q-2}\mathrm{TP}_{q,\delta})_{\delta>0}$. In order to show this stronger convergence result stated in Theorem 1.4, we need to verify the $\liminf$ and $\limsup$ inequalities, see [35, Definition 1.5].

Proof of Theorem 1.4. We start with the $\liminf$-inequality. Let $(\xi_\delta)_\delta$ be a sequence of curves contained in the space $\mathcal{W} := \mathcal{F}(\eta, j_\pm, s, \omega) \cap W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \cap C$ such that $\xi_\delta \to \xi$ in C^1 as $\delta \to 0$, and with $\xi \in \mathcal{W}$ as well. The latter implies by means of Lemma 3.3 that $\mathrm{TP}_{q,\frac{\eta}{2}}(\xi) < \infty$. By virtue of (5.7) for $\theta := \frac{\eta}{2}$ one has

$$\mathrm{TP}_{q,\delta}(\xi_\delta) = \mathrm{TP}_{q,\frac{\eta}{2}}(\xi_\delta) + A_{q,\frac{\eta}{2},\delta}(\xi_\delta) \quad \text{for } 0 < \delta < \frac{\eta}{2}.$$

Then we obtain by (5.3) in Proposition 5.2

$$\delta^{q-2}\mathrm{TP}_{q,\delta}(\xi_\delta) \geq \delta^{q-2} A_{q,\frac{\eta}{2},\delta}(\xi_\delta) \overset{(5.3)}{=} \frac{2^{q+2}}{q-2}\Big(1 - \big(\frac{2\delta}{\eta}\big)^{q-2}\Big) \sum_{i=1}^{n} F\left(\alpha_i\left(\xi_\delta\right)\right),$$

where $n := j_+ - j_-$ by (1.1). Due to $\xi \in \mathcal{W}$ and the C^1-convergence $\xi_\delta \to \xi$ we have $\alpha_i(\xi_\delta) \to \alpha_i(\xi) > 0$ as $\delta \to 0$ for every $i = 1, \dots, n$. By the continuity of F on $(0, \frac{\pi}{2}]$ this leads to $F(\alpha_i(\xi_\delta)) \to F(\alpha_i(\xi))$ as $\delta \to 0$ for $i = 1, \dots, n$. Hence, applying Lemma 5.5 to $\xi \in \mathcal{W}$,

$$\liminf_{\delta \to 0} \delta^{q-2} \mathrm{TP}_{q,\delta}(\xi_\delta) \geq \frac{2^{q+2}}{q-2} \sum_{i=1}^{n} F(\alpha_i(\xi)) = R_q(\xi).$$

It remains to prove the lim sup-inequality. Let $\xi \in \mathcal{W}$. As a recovery sequence, we simply choose the constant sequence $\xi_\delta = \xi$ for every $\delta > 0$. Then by definition of R_q as a pointwise limit we infer $\lim_{\delta \to 0} \delta^{q-2} \mathrm{TP}_{q,\delta}(\xi_\delta) = \lim_{\delta \to 0} \delta^{q-2} \mathrm{TP}_{q,\delta}(\xi) = R_q(\xi)$. □

It remains to prove the central representation formula (5.3).

Proof of Proposition 5.2. Let $\gamma \in \mathcal{F}_{n,\eta}$ and let $u_i, v_i \in \mathbb{R}/\mathbb{Z}$ with $u_i \neq v_i$ be the distinct points with $\gamma(u_i) = \gamma(v_i)$ for every $i = 1, \dots, n$. Then define for $i = 1, \dots, n$ the intervals

$$\begin{aligned} J_1^i &= [u_i - \theta, u_i - \delta], & J_5^i &= [v_i - \theta, v_i - \delta], \\ J_2^i &= [u_i - \delta, u_i], & J_6^i &= [v_i - \delta, v_i], \\ J_3^i &= [u_i, u_i + \delta], & J_7^i &= [v_i, v_i + \delta], \\ J_4^i &= [u_i + \delta, u_i + \theta], & J_8^i &= [v_i + \delta, v_i + \theta]. \end{aligned}$$

Denote by $\mathcal{E}_{kl}^i$ the energy integral over $J_k^i \times J_l^i$ for $k, l \in \{1, \dots 8\}$. Notice that there is no interaction between different parts of a single straight segment, and that also the interactions $\mathcal{E}_{26}^i$ and $\mathcal{E}_{27}^i$ and its symmetric counterparts between the central portions at the self-intersections vanish for each $i = 1, \dots, n$ since the truncated annular energy does not see these portions by definition. Therefore, the truncated annular energy can be written as

$$\begin{aligned} A_{q,\theta,\delta}(\gamma) &= \sum_{i=1}^{n} \big(\mathcal{E}_{15}^i + \mathcal{E}_{16}^i + \mathcal{E}_{17}^i + \mathcal{E}_{18}^i + \mathcal{E}_{25}^i + \mathcal{E}_{28}^i + \mathcal{E}_{35}^i + \mathcal{E}_{38}^i + \mathcal{E}_{45}^i + \mathcal{E}_{46}^i + \mathcal{E}_{47}^i + \mathcal{E}_{48}^i \\ &\quad + \mathcal{E}_{51}^i + \mathcal{E}_{52}^i + \mathcal{E}_{53}^i + \mathcal{E}_{54}^i + \mathcal{E}_{61}^i + \mathcal{E}_{64}^i + \mathcal{E}_{71}^i + \mathcal{E}_{74}^i + \mathcal{E}_{81}^i + \mathcal{E}_{82}^i + \mathcal{E}_{83}^i + \mathcal{E}_{84}^i \big) \\ &= 4 \sum_{i=1}^{n} \big(\mathcal{E}_{15}^i + \mathcal{E}_{16}^i + \mathcal{E}_{25}^i + \mathcal{E}_{17}^i + \mathcal{E}_{18}^i + \mathcal{E}_{28}^i \big), \end{aligned}$$

where the last equality follows from the symmetric structure around each self-intersection. Since $A_{q,\theta,\delta}$ takes only the linear segments into account, we can rewrite it as a function that solely depends on the intersection angles $\alpha_i := \alpha_i(\gamma)$, $i = 1, \dots, n$. The local situation is shown in Fig. 7. Consider first the energy integrals $\mathcal{E}_{15}^i + \mathcal{E}_{16}^i$. For $(x, y) \in [u_i - \theta, u_i - \delta] \times [v_i - \theta, v_i]$ set

$$s(x) := |\gamma(x) - \gamma(u_i)| \quad \text{and} \quad t(y) := |\gamma(y) - \gamma(v_i)|, \tag{5.9}$$

and infer by basic trigonometry $\operatorname{dist}(\ell(x), \gamma(y)) = t(y) \sin \alpha_i$ and

$$|\gamma(x) - \gamma(y)|^2 = s^2(x) + t^2(y) - 2s(x)t(y)\cos(\pi - \alpha_i) = s^2(x) + t^2(y) - 2s(x)t(y)\cos\alpha_i.$$

Therefore,

$$\begin{aligned} \mathcal{E}_{15}^i + \mathcal{E}_{16}^i &= 2^q \int_{u_i - \theta}^{u_i - \delta} \int_{v_i - \theta}^{v_i} \Big(\frac{t(y) \sin \alpha_i}{s^2(x) + t^2(y) - 2s(x)t(y)\cos\alpha_i} \Big)^q \, dy dx \\ &= 2^q \int_{\delta}^{\theta} \int_{0}^{\theta} \Big(\frac{t \sin \alpha_i}{s^2 + t^2 + 2st\cos\alpha_i} \Big)^q \, dt ds, \end{aligned}$$

where we changed variables in both integrals using

$$t'(y) = \frac{\langle \gamma(y) - \gamma(v_i), \gamma'(y) \rangle}{|\gamma(y) - \gamma(v_i)|} = \Big\langle \frac{\gamma(y) - \gamma(v_i)}{|\gamma(y) - \gamma(v_i)|}, \gamma'(y) \Big\rangle = -\langle \gamma'(y), \gamma'(y) \rangle = -1,$$

and analogously, $s'(x) = -1$, since the vectors $\gamma(y) - \gamma(v_i)$ and $\gamma(x) - \gamma(u_i)$ are contained in their respective linear segment and point in the opposite direction of the tangent-vector. Furthermore, $t(v_i - \theta) = \theta$ and $t(v_i) = 0$, as well as $s(u_i - \theta) = \theta$ and $s(u_i - \delta) = \delta$, due to the linear structure of the curve γ and its arclength parametrization. For $\mathcal{E}_{25}^i$ we obtain in the same manner

$$\mathcal{E}_{25}^i = 2^q \int_{0}^{\delta} \int_{\delta}^{\theta} \Big(\frac{t \sin \alpha_i}{t^2 + s^2 + 2st\cos\alpha_i} \Big)^q dt ds.$$

We treat the remaining integrals similarly, where the angle α_i instead of $\pi - \alpha_i$ appears in the Euclidean distance $|\gamma(x) - \gamma(y)|$ for $y \in [v_i, v_i + \theta]$ and $x \in [u_i - \theta, u_i - \delta]$, or $x \in [u_i - \delta, u_i]$, respectively, to obtain

$$\mathcal{E}_{17}^i + \mathcal{E}_{18}^i + \mathcal{E}_{28}^i = 2^q \int_{\delta}^{\theta} \int_{0}^{\theta} \Big(\frac{t \sin \alpha_i}{t^2 + s^2 - 2st\cos\alpha_i} \Big)^q dt ds + 2^q \int_{0}^{\delta} \int_{\delta}^{\theta} \Big(\frac{t \sin \alpha_i}{t^2 + s^2 - 2st\cos\alpha_i} \Big)^q dt ds.$$

Abbreviating $h(\alpha, s, t) := (t \sin(\alpha))^q \big((t^2 + s^2 + 2st\cos\alpha)^{-q} + (t^2 + s^2 - 2st\cos\alpha)^{-q} \big)$ we can rewrite the truncated annular energy as

$$A_{q,\theta,\delta}(\gamma) = 2^{q+2} \sum_{i=1}^{n} \Big(\int_{0}^{\delta} \int_{\delta}^{\theta} h(\alpha_i, s, t) dt ds + \int_{\delta}^{\theta} \int_{0}^{\theta} h(\alpha_i, s, t) dt ds \Big)$$

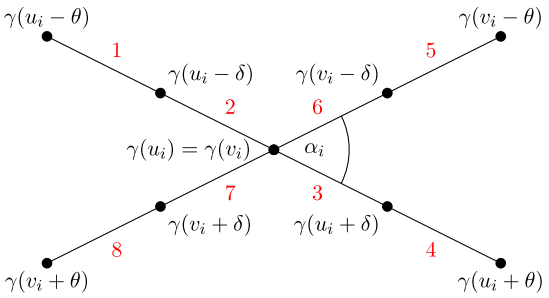


Fig. 7. The local situation around the ith self-intersection of γ: The numbers $k=1,\dots,8$ (in red) at the segments correspond to the set J_k^i. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$=2^{q+2}\sum_{i=1}^{n}\iint_{Q_{\theta,\delta}} h(\alpha_i,s,t)dtds,$$

where $Q_{\theta,\delta} := [0,\theta]^2 \setminus [0,\delta]^2$. Splitting the integration domain along the diagonal and using the transformation formula for the transformation $T : Q_{\theta,\delta}\cap\{t\geq s\} \to Q_{\theta,\delta}\cap\{t\leq s\}$ defined by $J(t,s) := (s,t)$ leads to

$$\begin{aligned} A_{q,\theta,\delta}(\gamma) &= 2^{q+2}\sum_{i=1}^{n}\iint_{Q_{\theta,\delta}\cap\{t\leq s\}} [h(\alpha_i,s,t)+h(\alpha_i,t,s)]dtds \\ &= 2^{q+2}\sum_{i=1}^{n}\int_{\delta}^{\theta}\int_{0}^{s}[h(\alpha_i,s,t)+h(\alpha_i,t,s)]dtds. \end{aligned}$$

To compute the inner integral consider the two functions

$$H_s(\alpha) := \int_0^s h(\alpha,t,s)dt, \quad G_s(\alpha) := \int_0^s h(\alpha,s,t)dt, \qquad \text{for } s\in[\delta,\theta].$$

Let us first rewrite H_s as

$$H_s(\alpha) = \frac{1}{(s\sin\alpha)^q}\Big(\int_0^s\Big[\Big(\frac{t+s\cos\alpha}{s\sin\alpha}\Big)^2+1\Big]^{-q}dt + \int_0^s\Big[\Big(\frac{t-s\cos\alpha}{s\sin\alpha}\Big)^2+1\Big]^{-q}dt\Big).$$

Substituting $z(t)=\frac{t+s\cos\alpha}{s\sin\alpha}$ in the first integral and $z(t)=\frac{t-s\cos\alpha}{s\sin\alpha}$ in the second one yields

$$H_s(\alpha) = \frac{1}{(s\sin\alpha)^{q-1}}\left(\int_{\cot\alpha}^{\cot\frac{\alpha}{2}}\Big(\frac{1}{z^2+1}\Big)^q dz + \int_{-\cot\alpha}^{\tan\frac{\alpha}{2}}\Big(\frac{1}{z^2+1}\Big)^q dz\right).$$

Similarly, one computes

$$G_s(\alpha) = \frac{1}{(s\sin\alpha)^{q-1}}\left(\int_{\cot\alpha}^{\cot\frac{\alpha}{2}}\Big(\frac{z\sin\alpha-\cos\alpha}{z^2+1}\Big)^q dz + \int_{-\cot\alpha}^{\tan\frac{\alpha}{2}}\Big(\frac{z\sin\alpha+\cos\alpha}{z^2+1}\Big)^q dz\right).$$

Together with the formula of F from (5.4) this yields

$$A_{q,\theta,\delta}(\gamma) = 2^{q+2}\sum_{i=1}^{n}\int_{\delta}^{\theta} s^{1-q}F(\alpha_i(\gamma))ds = \frac{2^{q+2}}{q-2}\left(\delta^{2-q}-\theta^{2-q}\right)\sum_{i=1}^{n}F(\alpha_i(\gamma)),$$

which proves (5.3). □

5.2. Optimally immersed curves with prescribed Arnold invariants

It remains to be investigated whether a sequence of minimizers $(\gamma_\delta^\eta)_{\delta>0}$ of the truncated energies $\mathrm{TP}_{q,\delta}$ converges in C^1 if δ is sent to zero. So far, we do not have a uniform bound on the energy values $\mathrm{TP}_{q,\delta}(\gamma_\delta^\eta)$ independent of δ. Hence, our compactness result Theorem 4.2 is not directly applicable. However, we will be able to deduce a uniform bound on $\mathrm{TP}_{q,\theta}(\gamma_\delta^\eta)$ independent of $\delta\in(0,\frac{\theta}{2}]$ and of η as long as $\theta\leq\eta$, see Corollary 5.6. This will allow us to obtain subconvergence of the minimizers γ_δ^η to a limit curve Γ^η as $\delta\to 0$. Properties of the curve Γ^η are then investigated at the end of this section.

To establish the uniform bound on $\mathrm{TP}_{q,\theta}(\gamma_\delta^\eta)$ we use identity (5.7). Since the curves γ_δ^η are minimizers, the energy value $\mathrm{TP}_{q,\delta}(\gamma_c)$ of any admissible comparison curve $\gamma_c\in\mathcal{F}(\eta,j_\pm,s,\omega)$ gives an upper bound for the first summand on the right-hand side of (5.7). In addition, Corollary 5.3 implies that any comparison curve $\gamma_c\in\mathcal{F}(\eta,j_\pm,s,\omega)\subset\mathcal{F}_{n,\eta}$ whose intersection angles $\alpha_i(\gamma_c)$ equal $\frac{\pi}{2}$ actually *minimizes* the second summand on the right-hand side of (5.7). Here is the precise statement.

Corollary 5.6. *For $j_+, j_-, s, \omega \in \mathbb{Z}$ consider the compartment $C = C(j_\pm, s, \omega)$, and fix any $\eta \in (0, \eta_1(C)]$. Then for any $q > 2$ and $\theta \in (0, \eta)$ there is a constant $c = c(\theta, q, j_+, j_-, s, \omega) > 0$ such that*

$$\mathrm{TP}_{q,\theta}(\gamma_\delta^\eta) \le c \quad \text{for all } 0 < \delta \le \frac{\theta}{2},$$

where $\eta_1(C)$ is the constant of Theorem 1.2, and the curves γ_δ^η are the minimizers obtained in Theorem 1.3.

Proof. Theorem 1.2 for $\eta := \eta_1(C)$ guarantees the existence of a comparison curve $\gamma_c \in C^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \cap \mathcal{F}(\eta_1, j_\pm, s, \omega) \cap C$ whose intersection angles equal $\frac{\pi}{2}$, and which is also contained in $\mathcal{F}(\eta, j_\pm, s, \omega)$ by means of (1.6). Set

$$c = c(\theta, q, j_+, j_-, s, \omega) := \mathrm{TP}_{q,\theta}(\gamma_c),$$

which by Lemma 3.3 is finite, since $C^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2) \subset W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$. (Notice that $\delta \le \frac{\theta}{2} < \frac{\eta}{2} \le \frac{\Lambda(\gamma_c)}{4}$ by definition of the class $\mathcal{F}(\eta, j_\pm, s, \omega)$ so that Lemma 3.3 is applicable.) By virtue of (5.7), the fact that γ_δ^η minimizes the truncated tangent-point energy $\mathrm{TP}_{q,\delta}$ in $\mathcal{F}(\eta, j_\pm, s, \omega) \cap C$, and by (5.3) in combination with Corollary 5.3 we infer

$$\mathrm{TP}_{q,\theta}(\gamma_\delta^\eta) \overset{(5.7)}{=} \mathrm{TP}_{q,\delta}(\gamma_\delta^\eta) - A_{q,\theta,\delta}(\gamma_\delta^\eta) \overset{(5.3)}{\le} \mathrm{TP}_{q,\delta}(\gamma_c) - A_{q,\theta,\delta}(\gamma_c) \overset{(5.7)}{=} c. \quad \square$$

With this uniform energy bound, we are now able to pass to the limit of a sequence of minimizers.

Proof of Theorem 1.5. Fix $\eta \in (0, \eta_1(C)]$, set $\delta_k := \frac{1}{k}$ and consider the subsequence of TP_{q,δ_k}-minimizing curves $\gamma_k^\eta := \gamma_{\delta_k}^\eta \in \mathcal{F}(\eta, j_\pm, s, \omega) \cap C$ for $k \in \mathbb{N}$ such that $k \ge \frac{4}{\eta}$, whose existence is guaranteed by Theorem 1.3. By translational invariance of the energy we may assume that $0 \in \gamma_k^\eta(\mathbb{R}/\mathbb{Z})$ for all $k \ge \frac{4}{\eta}$. By Corollary 5.6 for $\theta := \frac{\eta}{2}$ there exists a constant $c = c(\theta, q, j_+, j_-, s, \omega) > 0$ such that $\mathrm{TP}_{q,\frac{\eta}{2}}(\gamma_k^\eta) \le c$ for all $k \ge \frac{4}{\eta}$. Hence, we can apply the compactness result of Theorem 4.2 for $\delta := \frac{\eta}{2}$ to obtain a subsequence, again denoted by $(\gamma_k^\eta)_k$, and some curve $\Gamma^\eta \in \mathcal{F}(\eta, j_\pm, s, \omega)$, such that $\gamma_k^\eta \to \Gamma^\eta$ in C^1 as $k \to \infty$. Since Γ^η is a generic immersion in some open $(j_\pm, s, \omega)$-compartment $\tilde{C}$ we find $\gamma_k^\eta \in \tilde{C}$ for $k \ll 1$ implying $\tilde{C} = C$. The lower semi-continuity result Lemma 4.1 implies that $\mathrm{TP}_{q,\frac{\eta}{2}}(\Gamma^\eta) \le c$, which by means of Lemma 3.3 implies that $\Gamma^\eta \in W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$. The minimizers γ_k^η also minimize the scaled energies ${\delta_k}^{q-2}\mathrm{TP}_{q,\delta_k}$ in $\mathcal{F}(\eta, j_\pm, s, \omega) \cap C$. By [36, Corollary 7.20], the curve Γ^η is then a minimizer of the limit energy $R_q(\cdot) = \lim_{k\to\infty} \delta_k^{q-2}\mathrm{TP}_{q,\delta_k}(\cdot)$ introduced in Definition 5.4, and Γ^η satisfies (1.8) in addition. According to Lemma 5.5, the energy $R_q(\cdot)$ is minimized if and only if every intersection angle is equal to $\frac{\pi}{2}$. Hence, $\alpha_i(\Gamma^\eta) = \frac{\pi}{2}$ for every $i = 1, \dots, n$.

It remains to show (1.7). Fix $\varepsilon > 0$. As before, we use the relation (5.7) for $\theta := \frac{\eta}{2} > \delta$ to write

$$0 \le \mathrm{TP}_{q,\delta}(\Gamma^\eta) - \mathrm{TP}_{q,\delta}(\gamma_\delta^\eta) = \mathrm{TP}_{q,\frac{\eta}{2}}(\Gamma^\eta) + A_{q,\frac{\eta}{2},\delta}(\Gamma^\eta) - \mathrm{TP}_{q,\frac{\eta}{2}}(\gamma_\delta^\eta) - A_{q,\frac{\eta}{2},\delta}(\gamma_\delta^\eta).$$

Every self-intersection angle of Γ^η equals $\frac{\pi}{2}$ so that Γ^η minimizes $A_{q,\frac{\eta}{2},\delta}$ according to Corollary 5.3, so that $A_{q,\frac{\eta}{2},\delta}(\Gamma^\eta) \le A_{q,\frac{\eta}{2},\delta}(\gamma_\delta^\eta)$. We deduce

$$0 \le \mathrm{TP}_{q,\delta}(\Gamma^\eta) - \mathrm{TP}_{q,\delta}(\gamma_\delta^\eta) \le \mathrm{TP}_{q,\frac{\eta}{2}}(\Gamma^\eta) - \mathrm{TP}_{q,\frac{\eta}{2}}(\gamma_\delta^\eta), \tag{5.10}$$

and hence

$$\begin{aligned} 0 &\le \liminf_{\delta\to 0} \left(\mathrm{TP}_{q,\delta}(\Gamma^\eta) - \mathrm{TP}_{q,\delta}(\gamma_\delta^\eta)\right) \le \limsup_{\delta\to 0} \left(\mathrm{TP}_{q,\delta}(\Gamma^\eta) - \mathrm{TP}_{q,\delta}(\gamma_\delta^\eta)\right) \\ &\overset{(5.10)}{\le} \limsup_{\delta\to 0} \left(\mathrm{TP}_{q,\frac{\eta}{2}}(\Gamma^\eta) - \mathrm{TP}_{q,\frac{\eta}{2}}(\gamma_\delta^\eta)\right) = \mathrm{TP}_{q,\frac{\eta}{2}}(\Gamma^\eta) - \liminf_{\delta\to 0} \mathrm{TP}_{q,\frac{\eta}{2}}(\gamma_\delta^\eta) \le 0. \end{aligned}$$

The last inequality holds by the lower semi-continuity with respect to C^1-convergence, as shown in Lemma 4.1. Hence, $\lim_{\delta\to 0}(\mathrm{TP}_{q,\delta}(\Gamma^\eta) - \mathrm{TP}_{q,\delta}(\gamma_\delta^\eta)) = 0$, so there exists $\hat{\delta} = \hat{\delta}(\varepsilon) > 0$ such that for all $0 < \delta < \hat{\delta}$

$$\inf_{\mathcal{F}(\eta, j_\pm, s, \omega)\cap C} \mathrm{TP}_{q,\delta} \le \mathrm{TP}_{q,\delta}(\Gamma^\eta) < \mathrm{TP}_{q,\delta}(\gamma_\delta^\eta) + \varepsilon = \inf_{\mathcal{F}(\eta, j_\pm, s, \omega)\cap C} \mathrm{TP}_{q,\delta} + \varepsilon. \quad \square$$

If we initially restrict the minimizing process to curves in $\mathcal{F}(\eta, j_\pm, s, \omega) \cap C$ that have a right angle at each self-intersection, the limit curves Γ^η indeed minimize the truncated energy $\mathrm{TP}_{q,\delta}$ for every $\delta \le \frac{\eta}{2}$ among those curves as stated in Corollary 1.6.

Proof of Corollary 1.6. Fix any $0 < \delta \le \frac{\eta}{2}$ and take $\sigma \in (0, \delta)$. Let γ be a curve in $\mathcal{F}(\eta, j_\pm, s, \omega) \cap C$ whose intersection angles equal $\frac{\pi}{2}$. Denote by γ_σ^η a minimizer of $\mathrm{TP}_{q,\sigma}$ in $\mathcal{F}(\eta, j_\pm, s, \omega) \cap C$ whose existence is guaranteed by Theorem 1.3. Combining the fact that γ_σ^η is a minimizer with Corollary 5.3 yields by means of (5.7)

$$\mathrm{TP}_{q,\delta}(\gamma_\sigma^\eta) \overset{(5.7)}{=} \mathrm{TP}_{q,\sigma}(\gamma_\sigma^\eta) - A_{q,\sigma,\delta}(\gamma_\sigma^\eta) \le \mathrm{TP}_{q,\sigma}(\gamma) - A_{q,\sigma,\delta}(\gamma) \overset{(5.7)}{=} \mathrm{TP}_{q,\delta}(\gamma). \tag{5.11}$$

Furthermore, Theorem 1.5 guarantees the existence of a subsequence $(\gamma_{\sigma_k}^\eta)_k$ that converges in C^1 to Γ^η for $\sigma_k \to 0$ as $k \to \infty$. Using the lower semi-continuity in Lemma 4.1 and (5.11) then yields $\mathrm{TP}_{q,\delta}(\Gamma^\eta) \le \liminf_{k\to\infty} \mathrm{TP}_{q,\delta}(\gamma_{\sigma_k}^\eta) \le \mathrm{TP}_{q,\delta}(\gamma)$. $\square$

The same line of arguments also yields the monotonicity of the energy values $\mathrm{TP}_{q,\theta}(\Gamma^\eta)$ in the η-variable.

Corollary 5.7. *The map $\eta \mapsto \mathrm{TP}_{q,\theta}(\Gamma^\eta)$ is non-decreasing on the interval $(2\theta, \eta_1(C)]$. In particular, $\lim_{\eta\searrow 2\theta} \mathrm{TP}_{q,\theta}(\Gamma^\eta) \le \lim_{\eta\nearrow\eta_1(C)} \mathrm{TP}_{q,\theta}(\Gamma^\eta)$.*

Proof. The truncated energies can be split into two terms according to (5.7):

$$\begin{aligned}\mathrm{TP}_{q,\theta}(\gamma_\delta^\eta) &\overset{(5.7)}{=} \mathrm{TP}_{q,\delta}(\gamma_\delta^\eta) - A_{q,\theta,\delta}(\gamma_\delta^\eta) \\ &\leq \mathrm{TP}_{q,\delta}(\Gamma^{\eta_*}) - A_{q,\theta,\delta}(\Gamma^{\eta_*}) \overset{(5.7)}{=} \mathrm{TP}_{q,\theta}(\Gamma^{\eta_*}) \quad \text{for} \quad 2\theta < \eta \leq \eta_* \leq \eta_1(C),\end{aligned} \tag{5.12}$$

where we used the minimality of γ_δ^η in $\mathcal{F}(\eta, j_\pm, s, \omega) \cap C$ containing the set $\mathcal{F}(\eta_*, j_\pm, s, \omega) \cap C$ by means of (1.6), and Corollary 5.3, since Γ^{η_*} intersects exclusively in right angles according to Theorem 1.5. Combining (5.12) with the lower semicontinuity of $\mathrm{TP}_{q,\theta}$ (Lemma 4.1) yields for $2\theta < \eta \leq \eta_* \leq \eta_1(C)$

$$\mathrm{TP}_{q,\theta}(\Gamma^\eta) \leq \liminf_{\delta\to 0} \mathrm{TP}_{q,\theta}(\gamma_\delta^\eta) \overset{(5.12)}{\leq} \liminf_{\delta\to 0} \mathrm{TP}_{q,\theta}(\Gamma^{\eta_*}) = \mathrm{TP}_{q,\theta}(\Gamma^{\eta_*}). \quad \square$$

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Appendix. Auxiliary statements

Lemma A.1 (*Global Parametrization with Local Graphs*). *Let $k \geq 1$ and $\gamma \in C^k(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ with $|\gamma'| = 1$ on $\mathbb{R}/\mathbb{Z}$. Then for any mutually distinct arclength parameters $x_1, \dots, x_N \in \mathbb{R}/\mathbb{Z}$ there exists $r_0 > 0$ and a reparametrization $\tilde\gamma \in C^k(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ of γ with the same orientation as γ satisfying*

$$\tilde\gamma(x) = \begin{cases} \gamma(x) & \text{for} \quad x \notin \bigcup_{i=1}^N B_{2r_0}(x_i) \\ \gamma(x_i) + (x - x_i)\gamma'(x_i) + u_i(x - x_i)(\gamma'(x_i))^\perp & \text{for} \quad x \in B_{r_0}(x_i), i = 1, \dots, N, \end{cases} \tag{A.1}$$

where $u_i \in C^k(\mathbb{R})$ satisfies $0 = u_i(0) = u_i'(0)$ for $i = 1, \dots, N$. In addition, $|\tilde\gamma'(x)| \geq \frac{1}{4}$ for all $x \in \bigcup_{i=1}^N B_{2r_0}(x_i)$, and the balls $B_{3r_0}(x_i)$ are mutually disjoint.

Proof. Choose $r_0 > 0$ so small that the subintervals $B_{3r_0}(x_i) \subset \mathbb{R}/\mathbb{Z}$ are mutually disjoint for $i = 1, \dots, N$ and such that, by uniform continuity of the unit tangent,

$$|\gamma'(x) - \gamma'(x_i)| < \frac{1}{9} \quad \text{for all} \quad x \in B_{\frac{9}{4}r_0}(x_i) \quad \text{and} \quad i = 1, \dots, N. \tag{A.2}$$

Set $\epsilon := \frac{9}{4} r_0$ for brevity.

Step 1. For $i = 1$ take the rotation $R \in SO(2)$ satisfying $R\gamma'(x_1) = e_1 = (1, 0)^T \in \mathbb{R}^2$. The curve $\zeta(x) := R(\gamma(x + x_1) - \gamma(x_1))$ is of class $C^k(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ and satisfies $\zeta(0) = 0$, $\zeta'(0) = e_1$ and $(\zeta'(0))^\perp = e_2 = (0, 1)^T \in \mathbb{R}^2$. In addition, $|\zeta'| = 1$ on $\mathbb{R}/\mathbb{Z}$ and the derivative of the first component ζ_1 satisfies

$$1 \geq \zeta_1'(x) > \frac{8}{9} \quad \text{for all} \quad x \in B_\epsilon(0) \tag{A.3}$$

by means of (A.2), so that $\zeta_1|_{B_\epsilon(0)}$ is strictly monotonically increasing with value $\zeta_1(0) = 0$. Hence its inverse $g := (\zeta_1|_{B_\epsilon(0)})^{-1} : V_1 := \zeta_1(B_\epsilon(0)) \to B_\epsilon(0)$ with $\zeta_1(0) = 0 \in V_1$ exists and satisfies $g \in C^k(V_1)$, $g(0) = 0$, and $(g \circ \zeta_1)(x) = x$ for all $x \in B_\epsilon(0)$ and $(\zeta_1 \circ g)(y) = y$ for all $y \in V_1$. So, $\zeta(B_\epsilon(0))$ can be reparametrized in an orientation preserving way as a graph of the function $u_1 := \zeta_2 \circ g \in C^k(V_1)$ according to

$$V_1 \ni y \mapsto \begin{pmatrix} (\zeta_1 \circ g)(y) \\ (\zeta_2 \circ g)(y) \end{pmatrix} = \begin{pmatrix} y \\ (\zeta_2 \circ g)(y) \end{pmatrix} = \begin{pmatrix} y \\ u_1(y) \end{pmatrix}. \tag{A.4}$$

Notice that $B_{\frac{8}{9}\epsilon}(0) = B_{2r_0}(0) \subset V_1$ by virtue of (A.3).

Step 2. Choose a cut-off function $\varphi \in C_0^\infty(B_{2r_0}(0))$ (extended to all of $\mathbb{R}$ by zero), such that $0 \leq \varphi \leq 1$ on $\mathbb{R}$, $\varphi \equiv 1$ on $B_{r_0}(0)$ and $0 \leq \varphi' \leq \frac{2}{r_0}$ on $[-2r_0, -r_0]$ and $0 \geq \varphi' \geq -\frac{2}{r_0}$ on $[r_0, 2r_0]$. Now define the global orientation preserving reparametrization

$$\tilde\zeta(x) := \begin{cases} \zeta(x) & \text{for} \quad x \in \mathbb{R}/\mathbb{Z} \setminus B_{2r_0}(0) \\ \zeta\big((1 - \varphi(x))x + \varphi(x)g(x)\big) & \text{for} \quad x \in B_{2r_0}(0), \end{cases} \tag{A.5}$$

so that $\tilde\zeta'$ coincides with ζ' on $\mathbb{R}/\mathbb{Z} \setminus B_{2r_0}(0)$. Note that $\tilde\zeta$ is of class $C^k(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ since ζ, g are in C^k, and φ is smooth and has compact support in $B_{2r_0}(0)$. For $x \in B_{2r_0}(0)$ compute

$$\tilde\zeta'(x) = \zeta'\big((1 - \varphi(x))x + \varphi(x)g(x)\big) \cdot [\varphi'(x)(g(x) - x) + \varphi(x)(g'(x) - 1) + 1], \tag{A.6}$$

and notice that (A.3) implies

$$1 \le g'(x) = \frac{1}{\zeta_1'(g(x))} < \frac{9}{8} \quad \text{for } x \in B_{2r_0}(0), \tag{A.7}$$

and therefore $g' - 1 \ge 0$ on $B_{2r_0}(0)$, and $g(x) - x = g(x) - g(0) - x \in [0, \frac{1}{8}x)$ for $x \in [0, 2r_0)$ as well as $g(x) - x \in (\frac{1}{8}x, 0]$ for $x \in (-2r_0, 0]$. Hence we can bound the term in square brackets in (A.6) from below by

$$\varphi'(x)(g(x) - x) + 1 > -\frac{2}{r_0} \cdot \frac{1}{8}x + 1 > \frac{1}{2} \quad \text{for } x \in [0, 2r_0)$$

and by $\frac{2}{r_0} \cdot \frac{1}{8}x + 1 > \frac{1}{2}$ for $x \in (-2r_0, 0]$. This shows that $|\tilde{\zeta}'(x)| > \frac{1}{2}$ for all $x \in B_{r_0}(0)$.

Step 3. Define $\tilde{\gamma}(x) := \gamma(x_1) + R^{-1}\tilde{\zeta}(x - x_1)$, which is of class $C^k(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$, to find by definition of $\tilde{\zeta}$ in (A.5) $\tilde{\gamma}(x) = \gamma(x_1) + R^{-1}\tilde{\zeta}(x - x_1) = \gamma(x_1) + R^{-1}\zeta(x - x_1) = \gamma(x)$ for all $x \notin B_{2r_0}(x_1)$. For $x \in B_{r_0}(x_1)$, on the other hand, we have

$$\begin{aligned}\tilde{\gamma}(x) &= \gamma(x_1) + R^{-1}\big((x - x_1)e_1 + u_1(x - x_1)e_2\big)\\ &= \gamma(x_1) + (x - x_1)\gamma'(x_1) + u_1(x - x_1)(\gamma'(x_1))^{\perp}\end{aligned}$$

since $\varphi \equiv 1$ on $B_{r_0}(0)$ so that the local graph representation (A.4) can be used.

Step 4. Now repeat steps 1–3 for all $i = 2, \dots, N$ to obtain the full statement. Notice that the smallness condition on r_0 guarantees that the consecutive reparametrizations locally near x_i do not affect the previous ones near $x_1, \dots, x_{i-1}$.

Lemma A.2. *The map F defined in (5.4) is strictly monotone decreasing and minimized for $\alpha = \frac{\pi}{2}$.*

We decompose the function F given in (5.4), as $F(\alpha) = H(\alpha) + G(\alpha)$, where

$$H(\alpha) := \frac{1}{(\sin\alpha)^{q-1}}\Big[\int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} \big(\frac{1}{z^2+1}\big)^q dz + \int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \big(\frac{1}{z^2+1}\big)^q dz\Big], \tag{A.8}$$

$$G(\alpha) := \frac{1}{(\sin\alpha)^{q-1}}\Big[\int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} \big(\frac{z\sin\alpha + \cos\alpha}{z^2+1}\big)^q dz + \int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \big(\frac{z\sin\alpha - \cos\alpha}{z^2+1}\big)^q dz\Big]. \tag{A.9}$$

To show that F is strictly monotone decreasing we analyze the first derivatives of H and G. By means of the Leibniz rule for parameter integrals we compute

$$\begin{aligned}\frac{d}{d\alpha}H(\alpha) = &-(q-1)\frac{\cos\alpha}{(\sin\alpha)^q}\Big[\int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \big(\frac{1}{z^2+1}\big)^q dz + \int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} \big(\frac{1}{z^2+1}\big)^q dz\Big]\\ &+ \frac{1}{(\sin\alpha)^{q-1}}\Big[\big(\frac{1}{(\cot\frac{\alpha}{2})^2+1}\big)^q \frac{d}{d\alpha}\cot\frac{\alpha}{2} + \big(\frac{1}{(\tan\frac{\alpha}{2})^2+1}\big)^q \frac{d}{d\alpha}\tan\frac{\alpha}{2}\Big].\end{aligned} \tag{A.10}$$

With the identities $(\cot\frac{\alpha}{2})^2 + 1 = (\sin\frac{\alpha}{2})^{-2}$, $\frac{d}{d\alpha}\cot\frac{\alpha}{2} = -\frac{1}{2}(\sin\frac{\alpha}{2})^{-2}$, $(\tan\frac{\alpha}{2})^2 + 1 = (\cos\frac{\alpha}{2})^{-2}$, and $\frac{d}{d\alpha}\tan\frac{\alpha}{2} = \frac{1}{2}(\cos\frac{\alpha}{2})^{-2}$ we can replace the second line in (A.10) by the expression $\frac{1}{(\sin\alpha)^{q-1}}\big[\frac{1}{2}(\cos\frac{\alpha}{2})^{2q-2} - \frac{1}{2}(\sin\frac{\alpha}{2})^{2q-2}\big]$ to obtain

$$\begin{aligned}\frac{d}{d\alpha}H(\alpha) = \frac{1}{(\sin\alpha)^{q-1}}\Big[&-(q-1)(\cot\alpha)\Big(\int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \big(\frac{1}{z^2+1}\big)^q dz + \int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} \big(\frac{1}{z^2+1}\big)^q dz\Big)\\ &+ \frac{1}{2}(\cos\frac{\alpha}{2})^{2q-2} - \frac{1}{2}(\sin\frac{\alpha}{2})^{2q-2}\Big].\end{aligned}$$

We use the function $f(z) := \frac{1}{2}(z^2+1)^{1-q}$ with $f'(z) = (1-q)z(z^2+1)^{-q}$ to rewrite the last two summands in square brackets as integrals

$$\int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} f'(z)dz - \int_{\cot\alpha}^{\cot\frac{\alpha}{2}} f'(z)dz = (q-1)\left(\int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \frac{z}{(z^2+1)^q}dz - \int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} \frac{z}{(z^2+1)^q}dz\right),$$

so that we can deduce

$$\frac{d}{d\alpha}H(\alpha) = \frac{(q-1)}{(\sin\alpha)^{q-1}}\left(\int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \frac{z - \cot\alpha}{(z^2+1)^q}dz - \int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} \frac{z + \cot\alpha}{(z^2+1)^q}dz\right).$$

In the second integral we may substitute $y = z + 2\cot(\alpha)$, use the identity $\tan\frac{\alpha}{2} + 2\cot\alpha = \cot\frac{\alpha}{2}$, and finally replace the new integration variable y by z again to arrive at

$$\frac{d}{d\alpha}H(\alpha) = \frac{(q-1)}{(\sin\alpha)^{q-1}}\int_{\cot\alpha}^{\cot\frac{\alpha}{2}} (z - \cot\alpha)\big(\frac{1}{(z^2+1)^q} - \frac{1}{((z - 2\cot\alpha)^2+1)^q}\big)dz \tag{A.11}$$

and notice that the prefactor of the integral is strictly positive for $\alpha \in (0, \pi)$. For $0 < \alpha < \frac{\pi}{2}$ we have

$$(z^2+1)^{-q} < \big((z - 2\cot(\alpha))^2 + 1\big)^{-q} \iff -4z\cot(\alpha) + 4\cot(\alpha)^2 < 0 \iff z > \cot(\alpha),$$

so that the integrand $g(z) := (z - \cot(\alpha))\left([z^2+1]^{-q} - [(z - 2\cot(\alpha))^2 + 1]^{-q}\right)$ in (A.11) satisfies $g(z) < 0$ for $z > \cot(\alpha)$ and $0 < \alpha < \frac{\pi}{2}$, and $g(\cot(\alpha)) = 0$. Furthermore, $g(z) = 0$ for all $z \in \left[\cot(\alpha), \cot\left(\frac{\alpha}{2}\right)\right]$ if and only if $\alpha = \frac{\pi}{2}$ since $\cot\left(\frac{\pi}{2}\right) = 0$. Therefore,

$$\frac{d}{d\alpha} H(\alpha) \begin{cases} < 0, & \text{if } \alpha \in \left(0, \frac{\pi}{2}\right), \\ = 0, & \text{if } \alpha = \frac{\pi}{2}. \end{cases} \tag{A.12}$$

Thus, H is strictly monotone decreasing and has a global minimum at $\alpha = \frac{\pi}{2}$.

Next, we consider the function G defined in (A.9) and abbreviate the sum of integrals in square brackets as $\tilde{G}(\alpha)$ so that

$$\frac{d}{d\alpha} G(\alpha) = -(q-1)\frac{\cos\alpha}{(\sin\alpha)^q}\tilde{G}(\alpha) + \frac{1}{(\sin\alpha)^{q-1}}\frac{d}{d\alpha}\tilde{G}(\alpha). \tag{A.13}$$

Since $q > 2$ and $\alpha \in (0, \frac{\pi}{2}]$ we have $\tilde{G}(\alpha) > 0$, and according to (A.13) it suffices to prove that $\frac{d}{d\alpha}\tilde{G}(\alpha) \le 0$ to find

$$\frac{d}{d\alpha} G(\alpha) \begin{cases} < 0, & \text{if } \alpha \in \left(0, \frac{\pi}{2}\right), \\ \le 0, & \text{if } \alpha = \frac{\pi}{2}, \end{cases} \tag{A.14}$$

which together with (A.12) proves the lemma.

Applying the Leibniz rule for parameter integrals yields

$$\begin{aligned} \frac{d}{d\alpha}\tilde{G}(\alpha) = &\int_{\cot\alpha}^{\cot\frac{\alpha}{2}} q\frac{(z\sin\alpha - \cos\alpha)^{q-1}}{(z^2+1)^q}(z\cos\alpha + \sin\alpha)dz \\ &+ \int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} q\frac{(z\sin\alpha + \cos\alpha)^{q-1}}{(z^2+1)^q}(z\cos\alpha - \sin\alpha)dz \\ &+ \left(\frac{\cot\frac{\alpha}{2}\sin\alpha - \cos\alpha}{(\cot\frac{\alpha}{2})^2+1}\right)^q \frac{d}{d\alpha}\cot\frac{\alpha}{2} + \left(\frac{\tan\frac{\alpha}{2}\sin\alpha + \cos\alpha}{(\tan\frac{\alpha}{2})^2+1}\right)^q \frac{d}{d\alpha}\tan\frac{\alpha}{2}. \end{aligned}$$

Together with $\cot\frac{\alpha}{2}\sin\alpha - \cos\alpha = 1$ and $\tan\frac{\alpha}{2}\sin\alpha + \cos\alpha = 1$, we therefore obtain

$$\begin{aligned} \frac{d}{d\alpha}\tilde{G}(\alpha) = &\int_{\cot\alpha}^{\cot\frac{\alpha}{2}} q\frac{(z\sin\alpha - \cos\alpha)^{q-1}}{(z^2+1)^q}(z\cos\alpha + \sin\alpha)dz \\ &+ \int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} q\frac{(z\sin\alpha + \cos\alpha)^{q-1}}{(z^2+1)^q}(z\cos\alpha - \sin\alpha)dz + \frac{1}{2}(\cos\frac{\alpha}{2})^{2q-2} - \frac{1}{2}(\sin\frac{\alpha}{2})^{2q-2}. \end{aligned} \tag{A.15}$$

First, we express the last two summands by means of the functions $g_1(z) := \frac{(z\sin\alpha - \cos\alpha)^q}{(z^2+1)^{q-1}}$ for $z \in [\cot\alpha, \cot\frac{\alpha}{2}]$ and $g_2(z) := \frac{(z\sin\alpha + \cos\alpha)^q}{(z^2+1)^{q-1}}$ for $z \in [-\cot(\alpha), \tan\frac{\alpha}{2}]$ as

$$\frac{1}{2}(\cos\frac{\alpha}{2})^{2q-2} - \frac{1}{2}(\sin\frac{\alpha}{2})^{2q-2} = \frac{1}{2}\Big(\int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} g_2'(z)dz - \int_{\cot\alpha}^{\cot\frac{\alpha}{2}} g_1'(z)dz\Big). \tag{A.16}$$

The derivatives can be computed as

$$\begin{aligned} g_1'(z) &= q\sin\alpha\Big(\frac{z\sin\alpha - \cos\alpha}{z^2+1}\Big)^{q-1} - 2z(q-1)\Big(\frac{z\sin\alpha - \cos\alpha}{z^2+1}\Big)^q \\ &= \frac{(z\sin\alpha - \cos\alpha)^{q-1}}{(z^2+1)^q}\big(q\sin\alpha - (q-2)z^2\sin\alpha + 2z(q-1)\cos\alpha\big), \\ g_2'(z) &= \frac{(z\sin\alpha + \cos\alpha)^{q-1}}{(z^2+1)^q}\big(q\sin\alpha - (q-2)z^2\sin\alpha - 2z(q-1)\cos\alpha\big). \end{aligned}$$

With (A.16) we can express the derivative in (A.15) purely in terms of integrals. For their integrands we calculate

$$\begin{aligned} &q\frac{(z\sin\alpha - \cos\alpha)^{q-1}}{(z^2+1)^q}(z\cos\alpha + \sin\alpha) - \frac{1}{2}g_1'(z) \\ =&\frac{(z\sin\alpha - \cos\alpha)^{q-1}}{(z^2+1)^q}\Big(qz\cos\alpha + q\sin\alpha - \frac{q}{2}\sin\alpha + \frac{q-2}{2}z^2\sin\alpha - z(q-1)\cos\alpha\Big) \\ =&\frac{(z\sin\alpha - \cos\alpha)^{q-1}}{(z^2+1)^q}\Big(\frac{q}{2}\sin\alpha + \frac{q-2}{2}z^2\sin\alpha + z\cos\alpha\Big) \end{aligned}$$

and

$$\begin{aligned} &q\frac{(z\sin\alpha + \cos\alpha)^{q-1}}{(z^2+1)^q}(z\cos\alpha - \sin\alpha) + \frac{1}{2}g_2'(z) \\ =&\frac{(z\sin\alpha + \cos\alpha)^{q-1}}{(z^2+1)^q}\Big(-\frac{q}{2}\sin\alpha - \frac{q-2}{2}z^2\sin\alpha + z\cos\alpha\Big). \end{aligned}$$

Combining all calculations yields

$$\frac{d}{d\alpha}\tilde{G}(\alpha) = \int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \frac{(z\sin\alpha-\cos\alpha)^{q-1}}{(z^2+1)^q}\big(\frac{q}{2}\sin\alpha+\frac{q-2}{2}z^2\sin\alpha+z\cos\alpha\big)dz + \int_{-\cot\alpha}^{\tan\frac{\alpha}{2}} \frac{(z\sin\alpha+\cos\alpha)^{q-1}}{(z^2+1)^q}\big(-\frac{q}{2}\sin\alpha-\frac{q-2}{2}z^2\sin\alpha+z\cos\alpha\big)dz. \tag{A.17}$$

In the second integral, we substitute again $y = z + 2\cot(\alpha)$ to obtain for that integral

$$\int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \frac{(y\sin\alpha-\cos\alpha)^{q-1}}{((y-2\cot\alpha)^2+1)^q}\big(-\frac{q}{2}\sin\alpha-\frac{q-2}{2}(y-2\cot\alpha)^2\sin\alpha+(y-2\cot\alpha)\cos\alpha\big)dy$$
$$= \int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \frac{(y\sin\alpha-\cos\alpha)^{q-1}}{((y-2\cot\alpha)^2+1)^q}\big(-\frac{q}{2}\sin\alpha-\frac{q-2}{2}y^2\sin\alpha-y\cos\alpha+2(q-1)\cos\alpha(y-\cot\alpha)\big)dy.$$

Hence, replacing the y-variable by z and inserting this expression into (A.17) yields

$$\frac{d}{d\alpha}\tilde{G}(\alpha) = \int_{\cot\alpha}^{\cot\frac{\alpha}{2}} \Big[\frac{(z\sin\alpha-\cos\alpha)^{q-1}}{(z^2+1)^q}\big(\frac{q}{2}\sin\alpha+\frac{q-2}{2}z^2\sin\alpha+z\cos\alpha\big) + \frac{(z\sin\alpha-\cos\alpha)^{q-1}}{((z-2\cot\alpha)^2+1)^q}\big(-\frac{q}{2}\sin\alpha-\frac{q-2}{2}z^2\sin\alpha-z\cos\alpha+2(q-1)\cos\alpha(z-\cot\alpha)\big)\Big]dz.$$

To prove that this expression is non-positive we show as before that the integrand is non-positive for all $z \in [\cot\alpha, \cot\frac{\alpha}{2}]$, $\alpha \in (0, \frac{\pi}{2}]$. As $z\sin\alpha - \cos\alpha \geq 0$ for $z \geq \cot\alpha$ this is equivalent to showing

$$\frac{\frac{q}{2}\sin\alpha+\frac{q-2}{2}z^2\sin\alpha+z\cos\alpha}{(z^2+1)^q} \leq \frac{\frac{q}{2}\sin\alpha+\frac{q-2}{2}z^2\sin\alpha+z\cos\alpha-2(q-1)\cos\alpha(z-\cot\alpha)}{\big((z-2\cot\alpha)^2+1\big)^q}. \tag{A.18}$$

Note that for $z \geq \cot\alpha$, $0 < \alpha \leq \frac{\pi}{2}$, and $q \geq 2$ one has

$$h(\alpha) := \frac{q}{2}\sin\alpha+\frac{q-2}{2}z^2\sin\alpha+z\cos\alpha \geq \sin\alpha+\cot\alpha\cos\alpha = \frac{1}{\sin\alpha} > 0.$$

Hence, we can divide inequality (A.18) above by the factor $h(\alpha)$ for $\alpha \in (0, \frac{\pi}{2}]$ and multiply with $((z-2\cot\alpha)^2+1)^q$ to obtain the equivalent characterization of (A.18),

$$\Big(\frac{(z-2\cot\alpha)^2+1}{z^2+1}\Big)^q \leq \frac{q\sin\alpha+(q-2)z^2\sin\alpha+2z\cos\alpha-4(q-1)\cos\alpha(z-\cot\alpha)}{q\sin\alpha+(q-2)z^2\sin\alpha+2z\cos\alpha}$$
$$\Leftrightarrow \quad \Big(1-4\cot\alpha\,\frac{z-\cot\alpha}{z^2+1}\Big)^q \leq 1-4(q-1)\cot\alpha\,\frac{z-\cot\alpha}{q+(q-2)z^2+2z\cot\alpha}. \tag{A.19}$$

The maximal value of the function $f_\alpha(z) := \frac{z-\cot\alpha}{z^2+1}$ for $z \in [\cot\alpha, \cot\frac{\alpha}{2}]$ for fixed $\alpha \in (0, \frac{\pi}{2}]$ can be determined with the help of its derivative $f'_\alpha(z) = \frac{z^2+1-2z(z-\cot\alpha)}{(z^2+1)^2} = \frac{1-z^2+2z\cot\alpha}{(z^2+1)^2}$ whose zeroes are given by

$$z_1 = \cot\alpha - \frac{1}{\sin\alpha} = -\tan\frac{\alpha}{2} \quad\text{and}\quad z_2 = \cot\alpha + \frac{1}{\sin\alpha} = \cot\frac{\alpha}{2}. \tag{A.20}$$

Only z_2 is contained in the interval $[\cot\alpha, \cot\frac{\alpha}{2}]$, so that we conclude for $\alpha \in (0, \frac{\pi}{2}]$

$$f_\alpha(\cot\alpha) = 0 \quad\text{and}\quad f_\alpha(z_2) = \frac{\cot\frac{\alpha}{2}-\cot\alpha}{(\cot\frac{\alpha}{2})^2+1} = \frac{1}{2}\tan\frac{\alpha}{2} > 0.$$

Hence, f_α attains on $[\cot\alpha, \cot\frac{\alpha}{2}]$ its global maximum at z_2, which yields for all $\alpha \in (0, \frac{\pi}{2}]$ and $z \in [\cot\alpha, \cot\frac{\alpha}{2}]$

$$0 \leq 4\cot\alpha f_\alpha(z) \leq 4\cot\alpha f_\alpha(z_2) \overset{(A.20)}{=} 4\cot\alpha\,\frac{\cot\frac{\alpha}{2}-\cot\alpha}{(\cot\frac{\alpha}{2})^2+1} \overset{(A.20)}{=} \frac{2\cos(\alpha)}{1+\cos\alpha} \leq \frac{2\cos\alpha}{2\cos\alpha} = 1,$$

so that

$$\Big(1-4\cot\alpha\,\frac{z-\cot\alpha}{z^2+1}\Big)^q \leq 1-4\cot\alpha\,\frac{z-\cot\alpha}{z^2+1} \quad \text{for} \quad z \in [\cot\alpha, \cot\tfrac{\alpha}{2}],\ \alpha \in (0, \tfrac{\pi}{2}],$$

since $q > 2$. Therefore, in view of (A.19) above, it suffices to prove

$$4\cot\alpha\,\frac{z-\cot\alpha}{z^2+1} \geq 4(q-1)\cot\alpha\,\frac{z-\cot\alpha}{q+(q-2)z^2+2z\cot\alpha} \quad \text{for} \quad z \in [\cot\alpha, \cot\tfrac{\alpha}{2}],\ \alpha \in (0, \tfrac{\pi}{2}]. \tag{A.21}$$

For $\alpha = \frac{\pi}{2}$ both sides vanish, so we can restrict to $\alpha \in (0, \frac{\pi}{2})$ from now on. Also for $z = \cot\alpha$ both sides are zero. For $z \in (\cot\alpha, \cot\frac{\alpha}{2}]$ and $\alpha \in (0, \frac{\pi}{2})$ inequality (A.21) is equivalent to

$$\frac{1}{z^2+1} \geq \frac{q-1}{q+(q-2)z^2+2z\cot(\alpha)} \Leftrightarrow -z^2+2z\cot(\alpha)+1 \geq 0, \tag{A.22}$$

where we used that $q+(q-2)z^2+2z\cot\alpha = (q-1)(1+z^2)+1-z^2+2z\cot\alpha > 1-z^2+2z\cot\alpha \geq 0$ by means of (A.20). By (A.20), the last inequality in (A.22) holds for all $z \in (\cot\alpha, \cot\frac{\alpha}{2}]$ and we have proven $\frac{d}{d\alpha}\tilde{G}(\alpha) \leq 0$. □

Data availability

No data was used for the research described in the article.

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