Heat Flow Methods and Saddle Configurations for Spherical Magnets

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Abstract

In the present thesis, we investigate a geometric model of a micromagnetic system on a curved surface described by maps between 2-spheres. This system is modeled by an energy functional that consists of exchange energy and easy-normal anisotropy. Critical points of this energy functional are called *skyrmions* and exist because of the stabilizing effect of the curvature. Despite its apparent simplicity, this toy model already exhibits interesting features. The main goal of this thesis is to prove the existence of skyrmions that are saddle points of the energy functional. The existence of minimizers with additional conditions on mapping degree and/or symmetries was shown in previous works [50, 15, 59].

Since the corresponding Euler–Lagrange equation is semilinear in nature, it is difficult to find solutions other than the ones obtained through the direct method in the calculus of variations. To overcome this difficulty, we use a heat flow method by transforming the elliptic equation into a parabolic one. This is inspired by the work of Eells and Sampson [20] who used this approach in the study of harmonic maps, to which our problem is closely related. In a first result, we prove the existence of a weak solution to the heat flow equation away from finitely many points in space-time at which the solution blows up. This is a result analogous to that of Struwe [62] for the harmonic map heat flow. Our proof closely follows Struwe's as the additional terms from the anisotropy are of lower order.

The long time limit of solutions to the heat flow equation is a critical points of the energy functional. In a second result, we show that for a special class of axisymmetric maps no blowup can occur, meaning that full regularity is retained in the limit of time going to infinity.

Then in a third step, we show that the skyrmion solutions obtained in this way are saddle points of the energy functional in a specific parameter range of the model. This is done by showing that the critical point is neither a local minimizer nor a local maximizer of the energy functional. The proof is based on the observation that these maps possess an additional symmetry, which we then break manually by a small perturbation to construct a map with lower energy.

Finally, we carry out numerical simulations of the heat flow equation in the axisymmetric setting to visualize the skyrmion solutions and to obtain a better understanding of their properties.

Zusammenfassung

In der vorliegenden Arbeit untersuchen wir ein geometrisches Modell eines mikromagnetischen Systems auf einer gekrümmten Oberfläche, das durch Abbildungen zwischen 2-Sphären beschrieben wird. Dieses System wird durch ein Energiefunktional modelliert, das aus Austauschenergie und Easy-Normal-Anisotropie besteht. Kritische Punkte dieser Energie werden als *Skyrmionen* bezeichnet und existieren aufgrund des stabilisierenden Effekts der Krümmung. Trotz seiner scheinbaren Einfachheit birgt das Modell interessante Eigenschaften. Hauptziel dieser Arbeit ist es, die Existenz von Skyrmionen nachzuweisen, die Sattelpunkte des Energiefunktionals sind. Die Existenz von Minimierern unter zusätzlichen Bedingungen bezüglich Abbildungsgrads und/oder Symmetrien wurde in früheren Arbeiten gezeigt [50, 15, 59].

Da die zugehörige Euler–Lagrange-Gleichung semilinearer Natur ist, ist es schwierig, andere Lösungen als die durch die direkte Methode der Variationsrechnung erhaltenen zu finden. Aus diesem Grund verwenden wir eine Wärmefluss-Methode, bei der wir die elliptische Gleichung in eine parabolische umwandeln. Dies ist angelehnt an die Ergebnisse von Eells und Sampson [20], die diesen Ansatz im Bereich der Untersuchung harmonischer Abbildungen verwendeten, mit denen unser Problem eng verwandt ist. In einem ersten Ergebnis beweisen wir die Existenz einer schwachen Lösung der Wärmefluss-Gleichung außerhalb endlich vieler Punkte in Raumzeit, an denen die Lösungen singulär werden. Dies ist ein analoges Resultat zu dem von Struwe [62] für den harmonischen Wärmefluss. Der Beweis folgt dem von Struwe eng, da unsere zusätzlichen Terme aus der Anisotropie niedrigerer Ordnung sind.

Der Limes für große Zeiten der Lösungen der Wärmefluss-Gleichung ist dann ein kritischer Punkt der Energie. Wir zeigen zweitens, dass für eine spezielle Klasse von achsensymmetrischen Abbildungen kein Blowup auftreten kann, sodass auch im Limes die volle Regularität erhalten bleibt.

In einem dritten Schritt beweisen wir, dass die auf diese Weise erhaltenen Skyrmionen Sattelpunkte des Energiefunktionals für bestimmte Modellparameter sind. Dazu zeigen wir, dass der kritische Punkt weder ein lokaler Minimierer noch Maximierer ist. Der Beweis basiert auf der Beobachtung, dass die Abbildungen zu einer speziellen Klasse von achsensymmetrischen Abbildungen mit einer zusätzlichen Symmetrie gehören, die wir dann durch eine Störung manuell brechen, um eine Abbildung mit niedrigerer Energie zu konstruieren.

Abschließend führen wir numerische Simulationen der Wärmefluss-Gleichung im achsensymmetrischen Setting durch, um die Skyrmion-Lösungen zu visualisieren und ihre Eigenschaften besser zu verstehen.

You live and learn. At any rate, you live.

— DOUGLAS ADAMS

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A manuscript like the one in front of you is seldom the work of a single individual although only one author is listed on the title page. Before writing down even one word, there are discussions, lessons, and uncountable sessions in front of a black board; to form ideas, understand concepts, and verify proofs. Especially this dissertation would not have seen the light of day without the help and encouragement of numerous people for which I am deeply grateful.

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Glück auf

Aachen, November 2025 Daniel Meinert

Contents

1. Introduction and Main Results

Magnetic skyrmions are topologically nontrivial configurations in magnetization fields that have attracted much attention in both applied and mathematical physics—thanks to their promising applications in data storage technologies and rich mathematical structure [24, 45]. In both cases, the stability of skyrmions is the chief characteristic. Skyrmions were first theoretically described by their eponym Skyrme in the year 1961 in the context of particle physics [61]. The term skyrmion was later adopted in condensed matter physics and refers to stable magnetic configurations that behave like localized quasi-particles.

In order to enable the formation of skyrmions, a micromagnetic system must exhibit some form of symmetry breaking, which allows energetically and structurally stable configurations to exist. This is most commonly achieved by introducing a so-called Dzyaloshinskii–Moriya interaction (DMI) which is an antisymmetric exchange interaction arising from spin-orbit coupling and lack of inversion symmetry [18, 51]. This approach was proposed by Bogdanov and Yablonskii in [10]. As observed in various experiments, skyrmionic structures emerge both in bulk materials and in thin films of ferromagnetic materials [52, 8, 31]. For planar films, the DMI can be generated by the crystalline structure of the material [54] or by interfacial effects at the layer boundary [41], causing the system to lose its inversion symmetry. This constitutes the necessary stabilizing factor and induces the emergence of topological patterns [48]. Solitons that form under this mechanism are called chiral skyrmions.

In recent years, advancements in the production of curved nanomagnets have led to an increased interest in the investigation of skyrmionic structures in curved geometries, see e.g., [35, 23]. These studies showed that the magnetization field couples with the curvature of the underlying manifold leading to the desired stabilizing effect [40, 17].

Mathematically, such micromagnetic systems are modeled by maps between manifolds. For typical conditions below the Curie temperature of the material, we can assume that the modulus of the magnetization field $\boldsymbol{m}$ is constant and hence can be normalized to 1 [14]. This means that the target manifold is the unit sphere $\mathbb{S}^2$. The domain manifold depends on the geometry of the underlying material modeled. The stable field configurations are then critical points of an energy functional—typically referred to as *the energy*—on some function space of admissible maps.

The planar case has been studied in the mathematics literature with care, see e.g., [48, 45, 27]. A common energy functional for fields $\boldsymbol{m}\colon \mathbb{R}^2 \to \mathbb{S}^2$ is

$$\mathcal{E}_{\text{planar}}(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \boldsymbol{m}|^2 + 2\langle \boldsymbol{m}, \nabla \times \boldsymbol{m}\rangle + \alpha(1 - m_3^2) + h(1 - m_3)\,\mathrm{d}x\,,$$

where the second, lower order term is responsible for the DMI effect and sufficient for the emergence of the stabilization of skyrmionic structures [9, 48]. The third term represents anisotropy, describing the material's preferred direction, while the fourth term accounts for the Zeeman effect, reflecting the influence of an external magnetic field. The strengths of these terms are given by the real parameters α and h.

In the present thesis, we study a spherical thin ferromagnet with perpendicular anisotropy modeled by a magnetization field $\boldsymbol{m} : \mathbb{S}^2 \to \mathbb{S}^2$ and energy functional

$$\mathcal{E}(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{S}^2} |\nabla \boldsymbol{m}|^2 + \kappa(1 - \langle \boldsymbol{m}, \boldsymbol{\nu}\rangle^2)\,\mathrm{d}\sigma\,,$$

where $\boldsymbol{\nu}(x) \in \mathbb{S}^2$ is the outer unit normal at $x \in \mathbb{S}^2$. This geometric model was first proposed for spherical shells by Kravchuk et al. in [40], where the authors found that the curvature of the underlying manifold induces an interaction similar to that of DMI, the so-called "curvature induced DMI". This can be demonstrated using stereographic coordinates on $\mathbb{S}^2$ and a co-rotating frame to express the coordinates of $\boldsymbol{m} \in \mathbb{S}^2 \subset \mathbb{R}^3$. In doing so, terms like the ones in $\mathcal{E}_{\text{planar}}$ emerge from the energy $\mathcal{E}$ which explains the phrasing "curvature induced DMI" [50]. This model combines aspects of energetic, topological, and geometrical nature. In fact, this energy is not invariant under individual rotations of domain or target, which is the key feature in stabilizing skyrmionic structures.

The first term of the integral represents the short-range exchange interactions among neighboring spins. As in the planar case we call it the Dirichlet energy. This part favors uniform fields as they are of lower energy. The second term is called the anisotropy term. We consider $\kappa > 0$ which corresponds to an *easy-normal anisotropy*, i.e., fields following the normal of the sphere are energetically preferred. It is apparent that both terms have a competing effect on the magnetization configuration.

For (sufficiently regular) fields $\boldsymbol{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ we define the mapping degree, or Skyrmion number

$$Q(\boldsymbol{m}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \boldsymbol{m}^* \mathrm{d}s\,,$$

where by $\boldsymbol{m}^*\mathrm{d}s$ we denote the pullback of the area element on $\mathbb{S}^2$ by the map $\boldsymbol{m}$. It is a classical result in differential topology that $Q(\boldsymbol{m}) \in \mathbb{Z}$ which results in a separation of the maps $\mathbb{S}^2 \to \mathbb{S}^2$ into different topological sectors [44]. Accordingly, maps with different degrees cannot be deformed into each other by homotopy. The mapping degree can also be defined for maps from $\mathbb{R}^2$, provided that the maps satisfy a certain decay condition at infinity. In this case, we can pull back the map from $\mathbb{R}^2 \cup \{\infty\}$ to $\mathbb{S}^2$ and use the mapping degree defined above. In the study of skyrmions in the plane, maps with nontrivial mapping degree are of particular interest due to their stability as they cannot decay continuously into the homogeneous state, which for the planar energy marks the ground state. In the spherical case, however, the situation is different as the constant homogeneous state is not critical. Therefore, we rather consider the outer normal field $\boldsymbol{\nu}$ and its inversion $-\boldsymbol{\nu}$ as the trivial fields which have mapping degree $Q(\pm\boldsymbol{\nu}) = \pm 1$ and are critical points of the energy $\mathcal{E}$ as can be verified easily. In 2019, Di Fratta et al. proved in [15] that the normal field is the global minimizing configuration, provided $\kappa > 4$. In the present thesis, maps with mapping degree $Q(\boldsymbol{m}) = 0$ are of interest. Even if they are trivial in the sense of homotopy (i.e., homotopic to the constant state), they are in a homotopy class distinct from the one of the normal.

In [14], Di Fratta derived the present model from three dimensional micromagnetism models in magnetic crystals via Γ-convergence. From a mathematical point of view, the model obtained falls into the field of nonlinear σ-models, which is closely related to the study of harmonic maps between manifolds. Indeed, the model can be understood as a minimal extension of harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$, especially in the limit $\kappa \to 0$. The theory of (weakly) harmonic maps between the 2-spheres is completely understood. They consist of all rational functions $\mathbb{C} \to \mathbb{C}$, where we identify $\mathbb{C}$ with $\mathbb{S}^2$ via stereographic coordinates [19, cf. 11.6]. Moreover, all harmonic maps are local minimizers of the Dirichlet energy with their energy equal to $4\pi Q(\boldsymbol{m})$. In particular, this means that no saddle points of the Dirichlet energy exist. Our goal is to expand the knowledge of critical points of the extended model, namely Dirichlet energy plus the anisotropy term.

In the recent literature, the functional $\mathcal{E}$ has been examined with respect to the existence of (local) minimizers with constraints on the mapping degree and further symmetries. In [50], Melcher and Sakellaris showed the attainment of

$$\inf\Big\{\mathcal{E}(\boldsymbol{m}) : \boldsymbol{m} \in H^1(\mathbb{S}^2,\mathbb{S}^2),\, Q(\boldsymbol{m}) = 0\Big\}$$

for any $\kappa > 0$. Here, $H^1(\mathbb{S}^2,\mathbb{S}^2)$ is the subspace of maps with modulus 1 almost everywhere in the usual Sobolev space $H^1(\mathbb{S}^2,\mathbb{R}^3)$, see Section 2.1. These local minimizers are maps that interpolate between the outer normal $m = \boldsymbol{\nu}$ near the south pole and its strongly localized inversion near the north pole for large κ.

Furthermore, in [59] it was shown by Schroeder that

$$\inf\Big\{\mathcal{E}(\boldsymbol{m}) : \boldsymbol{m} \in H^1(\mathbb{S}^2,\mathbb{S}^2),\, \boldsymbol{m} \text{ is axisymmetric}\Big\}$$

is attained, where axisymmetry is defined as in Definition 2.11. Finally, it was demonstrated that these constrained minimizers are local minimizers in $H^1(\mathbb{S}^2,\mathbb{S}^2)$ if $\kappa > 24$. These maps show the same interpolation behavior between the normal field and its inversion close to one pole.

The question that naturally arises is whether other critical points of $\mathcal{E}$ exist and of which type they are. With the present work, we give a positive answer. We prove the existence of critical points of $\mathcal{E}$ and classify them as saddle points. The key point for this identification is that these critical points exhibit a high degree of symmetry.

Theorem 1. *There exists $\kappa_0 \geq 4$ such that for all $\kappa \geq \kappa_0$ there exists a critical point $\boldsymbol{m}_\kappa \in H^1(\mathbb{S}^2,\mathbb{S}^2)$ of $\mathcal{E}$ with $Q(\boldsymbol{m}_\kappa) = 0$ that is a saddle point in the sense that it is neither a local minimizer nor a local maximizer of $\mathcal{E}$.*

Theorem 2. *There exists $0 < \kappa_1 < 4$ such that for all $\kappa > \kappa_1$ there exists a critical point $\boldsymbol{m}_\kappa \in H^1(\mathbb{S}^2,\mathbb{S}^2)$ of $\mathcal{E}$ with $Q(\boldsymbol{m}_\kappa) = 0$ that is a saddle point in the above sense and that is distinct from the (possible) saddle point from Theorem 1.*

We call the critical points from Theorem 1 saddle points of first type, and those from Theorem 2 saddle points of second type. We have no theoretical upper bound for κ_0 but numerical simulations we have conducted suggest that $\kappa_0 < 6.7$ and, in fact, we conjecture that $\kappa_0 = 4$. Furthermore, again supported by numerical evidence, we conjecture that for $\kappa \leq 4$ the saddle point of first type transforms into the global minimizer of $\mathcal{E}$, making $\kappa = 4$ a bifurcation point.

Remark 1.1 (Symmetry of the saddle points)**.** The saddle points of both theorems belong to a special class of maps as long as $\kappa \geq 4$, the so-called *hemispheric* *maps*, see Definition 4.3. In particular, this means that the maps are axisymmetric around a given axis and invariant under the antipodal map $\boldsymbol{m}(x) \mapsto \boldsymbol{m}(-x)$. $\boxdot$

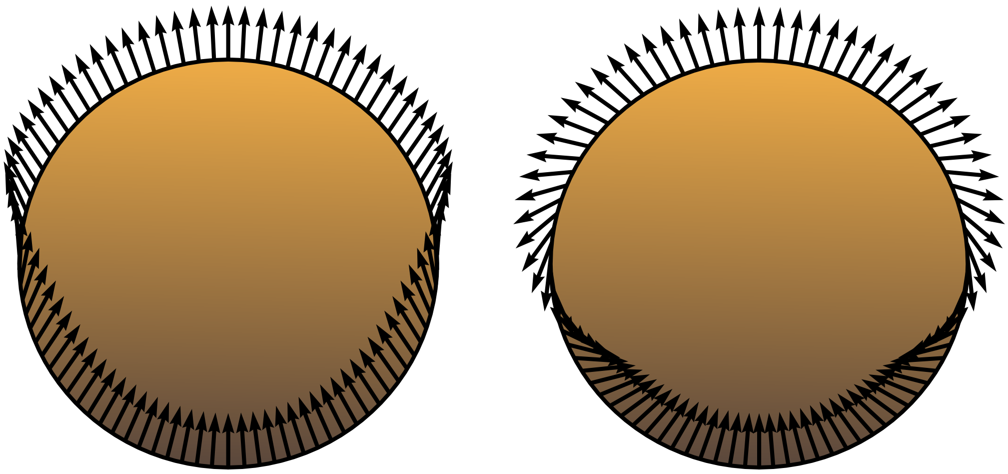

Figure 1.1.: Great circle cross-section plot of the saddle points of first type (left) and of second type (right) for $\kappa = 10$ obtained by numerical simulations (cf. Chapter 5).

Figure 1.1 depicts the two types of saddle points corresponding to Theorems 1 and 2, respectively, for an exemplary value of $\kappa = 10$. The plots show a cross-section of the sphere along a great circle and the field $\boldsymbol{m}$ along that cross-section. For the full picture on $\mathbb{S}^2$, each image needs to be rotated around its vertical axis. We can infer the geometric structure of the saddle points from the figure. Both fields follow the outer normal field $\boldsymbol{\nu}$ on the upper hemisphere and then switch at the equator to the inner normal field $-\boldsymbol{\nu}$ on the lower hemisphere.

We can also infer the structural difference between the two saddle points from Figure 1.1: the arrows of the saddle of the first type all point roughly in the same direction, whereas those of the second type undergo a full rotation from the north pole to the south pole. More precisely, for the saddle of the first type we find a homotopy in the class of axisymmetric maps to the constant field, which implies $Q(\boldsymbol{m}) = 0$ for this map. For the saddle of second type this is not possible within said class. Here we observe that the field $\boldsymbol{m}$ covers $\mathbb{S}^2$ once on each hemisphere but with opposite orientation, which is the reason why this map has mapping degree $Q(\boldsymbol{m}) = 0$. This observation is reminiscent of the so-called *skyrmionium*, a structure consisting of two nested skyrmions with opposite orientation in the plane [25, 38, 39]. In conclusion, we obtain that the two maps belong to different topological sectors within the class of axisymmetric maps, even though they both belong to the class of maps with mapping degree $Q(\boldsymbol{m}) = 0$.

In order to prove these existence statements, we introduce a parabolic equation similar to that of the harmonic map heat flow. We now let the field $\boldsymbol{m}$ also depend on time and let it evolve according to the initial value problem

$$\begin{aligned} \boldsymbol{m}_t &= \Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu} + \Big(|\nabla \boldsymbol{m}|^2 - \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle^2 \Big) \boldsymbol{m} \\ \boldsymbol{m}|_{t=0} &= \boldsymbol{m}_0 , \end{aligned}$$

where $\boldsymbol{m}_0 \colon \mathbb{S}^2 \to \mathbb{S}^2$ is a given initial data. We call this equation the *skyrmionic map heat flow* (SMHF). Eventually, we derive a method on how to obtain stationary solutions of this equation which will then be the critical points named in the two theorems above.

The SMHF is a semilinear parabolic partial differential equation and has properties similar to that of a gradient flow of the energy functional $\mathcal{E}$. This equation does not describe the dynamical behavior of magnetization states as this one is governed by the Landau–Lifshitz–Gilbert equation [42, 26, 49] which reads

$$\boldsymbol{m}_t = \boldsymbol{m} \times (\Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}) - \lambda \boldsymbol{m} \times \boldsymbol{m} \times (\Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}) ,$$

where $\lambda > 0$ is the so-called Gilbert damping parameter and $\times$ denotes the vector cross product in $\mathbb{R}^3$. Noticing that

$$-\boldsymbol{m} \times \boldsymbol{m} \times (\Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}) = \Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu} + \boldsymbol{m} \Big(|\nabla \boldsymbol{m}|^2 - \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle^2 \Big) ,$$

we see that the SMHF describes the dynamics with only the damping term present. Physically, as the name suggests, it means that the energy of the system is dissipated. This is a crucial property we will make use of later. Conversely, for a system without Gilbert damping, the energy is conserved.

Nonetheless, despite its “unphysicality”, the heat flow serves as an efficient method to study $\mathcal{E}$, as stationary solutions of it are critical points of $\mathcal{E}$. The approach here is analogous to the introduction of the harmonic map heat flow (HMHF) by Eells and Sampson in [20] in order to find solutions of the harmonic map equation between Riemannian manifolds. In fact, since the functional $\mathcal{E}$ only differs from the Dirichlet energy by a term of lower-order, we can transfer many results from the study of the harmonic map heat flow to the skyrmionic map heat flow. The main one is an existence result of the flow away from finitely many blowup points. Qualitatively, the flow exhibits the same behavior as the harmonic map heat flow: the flow is smooth up to singular blowup points in space-time. At these points, harmonic 2-spheres separate and take away a multiple of 4π of energy. This was found by Struwe in [62]. The difference in our case is that the limit map is a solution to (EL). The literature on the HMHF is vast and rich in results, see [33, 63, 47] and references therein for details. For the present work, we use these results as starting points and inspiration for the treatment of our model.

Our main result regarding the SMHF is the global existence of a weak solution. We adapt the theorem from [63, Thm. 6.6] regarding the HMHF with the addition of the blowup tree results as in [55, 46, 69] to our case.

Theorem 3. *For any $\kappa \geq 0$ and initial data $\boldsymbol{m}_0 \in H^1(\mathbb{S}^2, \mathbb{S}^2)$ there exists a weak solution $\boldsymbol{m}\colon \mathbb{S}^2 \times (0, \infty) \to \mathbb{S}^2$ of (SMHF) which is smooth on $\mathbb{S}^2 \times (0, \infty)$ away from at most finitely many points $(\bar{x}_k, \bar{t}_k) \in \mathbb{S}^2 \times (0, \infty)$, $1 \leq k \leq \bar{K}$, which satisfies the energy inequality $\mathcal{E}(\boldsymbol{m}(t)) \leq \mathcal{E}(\boldsymbol{m}(s))$ for all $0 \leq s \leq t$, and which assumes its initial data continuously in $H^1(\mathbb{S}^2, \mathbb{S}^2)$. The solution $\boldsymbol{m}$ is unique in this class.*

At a singularity $(\bar{x}, \bar{t})$, smooth harmonic maps separate in the following sense. For any $R_0 > 0$ such that $B_{2R_0}(\bar{x}) \times \{\bar{t}\}$ contains no other singularities, there exists $t_j \nearrow \bar{t}$ and

(i) finitely many harmonic maps $\omega_1, \dots, \omega_p$ with $\omega_i\colon \mathbb{S}^2 \to \mathbb{S}^2$ for all $1 \leq i \leq p$,

(ii) sequences of points $(a_j^1), \dots, (a_j^p) \subset \mathbb{R}^2$ with $\lim_{j\to\infty} a_j^i = 0$ for all $1 \leq i \leq p$,

(iii) sequences of scales $(\lambda_j^1), \dots, (\lambda_j^p) \subset \mathbb{R}_+$ with $\lim_{j\to\infty} \lambda_j^i = 0$ for all $1 \leq i \leq p$,

such that for $i \neq \ell$ it holds

$$\frac{\lambda_j^\ell}{\lambda_j^i} + \frac{\lambda_j^i}{\lambda_j^\ell} + \frac{|a_j^\ell - a_j^i|^2}{\lambda_j^\ell \lambda_j^i} \longrightarrow \infty\,, \text{ as } j \to \infty\,, \tag{1.1}$$

and

$$\Big\| \boldsymbol{m}(t_j) - \boldsymbol{m}(\bar{t}) - \sum_{i=1}^{p} \omega_j^i \Big\|_{H^1(B_{R_0}, \mathbb{R}^3)} \longrightarrow 0\,, \text{ as } j \to \infty\,, \tag{1.2}$$

where ω_j^i is defined in stereographic coordinates centered at $\bar{x}$ by

$$\omega_j^i(x) = \omega_i\left(\boldsymbol{\Phi}\left(\frac{x - a_j^i}{\lambda_j^i}\right)\right) - \omega_i(-\hat{e}_3)\,,$$

where $\mathbf{\Phi}\colon \mathbb{R}^2 \to \mathbb{S}^2$ *is the stereographic projection centered at* $\hat{e}_3$. *In particular,*

$$\lim_{j\to\infty} \mathcal{E}(\boldsymbol{m}(t_j), B_{R_0}) = \mathcal{E}(\boldsymbol{m}(\bar{t}), B_{R_0}) + \sum_{i=1}^{p} \int_{\mathbb{S}^2} \frac{|\nabla \omega_i|^2}{2} \, \mathrm{d}\sigma\,.$$

For the asymptotic behavior, as $t_j \to \infty$ *suitably, the sequence of maps* $\boldsymbol{m}(t_j)$ *converges weakly in* $H^1(\mathbb{S}^2, \mathbb{S}^2)$ *to a smooth solution* $\boldsymbol{m}_\infty\colon \mathbb{S}^2 \to \mathbb{S}^2$ *of* (EL), *and smoothly away from finitely many points* $\tilde{x}_k \in \mathbb{S}^2, 1 \le k \le \tilde{K}$, *where again harmonic spheres separate: there exist*

(iv) finitely many harmonic maps $\omega_1, \ldots, \omega_p$ *with* $\omega_i\colon \mathbb{S}^2 \to \mathbb{S}^2$ *for all* $1 \le i \le p$,

(v) sequences of points $(a_j^1), \ldots, (a_j^p) \subset \mathbb{R}^2$ *with* $\lim_{j\to\infty} a_j^i = \mathbf{\Phi}^{-1}(\tilde{x}_{k(i)})$ *for some* $1 \le k(i) \le \tilde{K}$ *for all* $1 \le i \le p$,

(vi) sequences of scales $(\lambda_j^1), \ldots, (\lambda_j^p) \subset \mathbb{R}_+$ *with* $\lim_{j\to\infty} \lambda_j^i = 0$ *for all* $1 \le i \le p$,

such that for $i \neq \ell$ *it holds*

$$\frac{\lambda_j^\ell}{\lambda_j^i} + \frac{\lambda_j^i}{\lambda_j^\ell} + \frac{|a_j^\ell - a_j^i|^2}{\lambda_j^\ell \lambda_j^i} \longrightarrow \infty\,, \text{ as } j \to \infty\,, \tag{1.3}$$

and

$$\Big\| \boldsymbol{m}(t_j) - \boldsymbol{m}_\infty - \sum_{i=1}^{p} \omega_j^i \Big\|_{H^1(\mathbb{S}^2, \mathbb{R}^3)} \longrightarrow 0\,, \text{ as } j \to \infty\,, \tag{1.4}$$

where ω_j^i *is defined in stereographic coordinates centered at* $\hat{e}_3$ *by*

$$\omega_j^i(x) = \omega_i\left(\mathbf{\Phi}\left(\frac{x - a_j^i}{\lambda_j^i}\right)\right) - \omega_i(-\hat{e}_3)\,.$$

In particular,

$$\lim_{j\to\infty} \mathcal{E}(\boldsymbol{m}(t_j)) = \mathcal{E}(\boldsymbol{m}_\infty) + \sum_{i=1}^{p} \int_{\mathbb{S}^2} \frac{|\nabla \omega_i|^2}{2} \, \mathrm{d}\sigma\,.$$

Moreover, the total number of singularities of $\boldsymbol{m}$, $K = \overline{K} + \tilde{K}$ *is a priori bounded,* $K \le \frac{1}{4\pi} E(\boldsymbol{m}_0)$.

Remark 1.2 (Energy concentration at blowup points)**.** The blowup points $(\bar{x}, \bar{t})$ can be characterized by

$$\limsup_{t \nearrow \bar{t}} \int_{B_r(\bar{x})} |\nabla \boldsymbol{m}|^2 \, \mathrm{d}\sigma > E_{\mathrm{b}}$$

for every $0 < r < \mathrm{inj}(\mathbb{S}^2)$, where $E_\mathrm{b} > 0$ is a universal constant and $\mathrm{inj}(\mathbb{S}^2)$ is the injectivity radius of $\mathbb{S}^2$. This means that at these points the energy concentrates. Conversely, if the energy does not concentrate at a space-time point during the flow, it follows that no blowup can occur. Specifically, if the Frobenius norm of the gradient is bounded pointwise locally in space-time, no blowup is present. ⊡

Remark 1.3 (Mapping degree and blowup)**.** Since the flow $\boldsymbol{m}(t)$ can be seen as homotopy in $H^1(\mathbb{S}^2, \mathbb{S}^2)$ until the first blowup at a time $T > 0$ and since the mapping degree Q is continuous with respect to the H^1-norm, we deduce that

$$Q(\boldsymbol{m}(t)) = Q(\boldsymbol{m}_0)$$

for all $t \in (0, T)$. Thus, if no blowup is present, i.e., $T = \infty$ and $K = 0$, then the solution $\boldsymbol{m}_\infty$ of (EL) satisfies

$$Q(\boldsymbol{m}_\infty) = Q(\boldsymbol{m}_0)\,.$$

If a blowup occurs, then the degree changes according to the degree of the harmonic maps separating. In view of the strong H^1-convergence during the blowup, we assert that at a blowup time $T \leq \infty$, where p harmonic maps $\omega_1, \ldots, \omega_p$ separate along a sequence $t_i \nearrow T$, not necessarily at the same base point, we gain

$$\lim_{i\to\infty} Q(\boldsymbol{m}(t)) = Q(\boldsymbol{m}(T)) + \sum_{k=1}^{p} Q(\omega_k)\,.$$ ⊡

With this existence result at hand, the proofs of Theorems 1 and 2 require two further steps. First, we show that there exist initial maps $\boldsymbol{m}_0$ such that the solution to the heat flow equation cannot blow up at any point in time, $T = \infty$ included. This step is inspired by a global existence result of Chang and Ding [12]. Then the limit function $\boldsymbol{m}_\infty$ for $t \to \infty$ of the flow is a critical point of $\mathcal{E}$ with same degree as $\boldsymbol{m}_0$. In the final step, we show that $\boldsymbol{m}_\infty$ is indeed a saddle point. The crucial element here is that $\boldsymbol{m}_\infty$ obeys the hemispheric symmetry. This implies that on the equator the maps point into a tangential direction, which is energetically unfavorable due to the anisotropy term in the energy which prefers alignment with the normal direction. By breaking this symmetry manually, we can construct maps with lower energy than $\boldsymbol{m}_\infty$ which allows us to conclude the saddle point property.

The present work is structured in the following way: in the forthcoming chapter we state some preliminaries. In particular, we first introduce Sobolev spaces for maps between spheres which are the function spaces on which we then define our functionals. Then, we deal with a special class of functions, so-called axisymmetric maps that obey a rotational and inversion symmetry. In the third chapter, we introduce the skyrmionic map heat flow equation and prove the existence of a weak solution to it, i.e., we prove Theorem 3. Then, we apply these results in the axisymmetric setting. The fourth chapter is the core of the thesis. Here we prove the existence of the saddle points for the energy functional $\mathcal{E}$. In the fifth chapter, we present several numerical simulations of the skyrmionic map heat flow and how they relate to the theoretical results. Moreover, they give us hints on possible further research directions. These will be collected and briefly discussed in the last chapter together with other possible paths to continue.

2. Preliminaries

We begin with three sections in which we elucidate the mathematical foundations for this thesis. In the first section we introduce the Sobolev spaces of maps between 2-spheres as the function spaces under considerations in this work. In the second section we define the functionals already mentioned in the introduction in a rigorous manner and collect properties already derived in previous works and some preliminary results. Finally, in the third section we study the class of axisymmetric maps from the sphere to the sphere and derive some basic properties as these types of maps will be crucial in the course of this thesis. Moreover, we adapt the skyrmion model to this class of maps and prove first results.

2.1. Sobolev spaces of maps $\mathbb{S}^2 \to \mathbb{S}^2$

In the literature, Sobolev spaces of maps between manifolds have been widely studied. There can be intrinsic and extrinsic definitions of these spaces. We use an extrinsic definition, that is based on the embedding of the manifold into Euclidean space, which allows us to use the spaces dealt with in Section A.4. Moreover, we mostly restrict ourselves to the case where both domain and target of the maps are the two-sphere $\mathbb{S}^2$, which is a compact Riemannian manifold. Therefore, the Sobolev spaces of interest in this thesis are of type $W^{k,p}(\mathbb{S}^2, \mathbb{R}^3)$. We choose as metric g the one induced by the embedding $\imath\colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ from the Euclidean metric g_E on $\mathbb{R}^3$, i.e., $g = \imath^* g_E$. We denote the corresponding volume form $\mathrm{d}v(g)$ by $\mathrm{d}\sigma$.

For $k \in \mathbb{N}_0$ and $p \geq 1$ we define the Sobolev spaces of $\mathbb{S}^2$-valued maps as

$$W^{k,p}(\mathbb{S}^2, \mathbb{S}^2) = \left\{ \boldsymbol{m} \in W^{k,p}(\mathbb{S}^2, \mathbb{R}^3) : \boldsymbol{m}(x) \in \imath(\mathbb{S}^2) \text{ for almost all } x \in \mathbb{S}^2 \right\}.$$

As usual, when $p = 2$, we write $H^k(\mathbb{S}^2, \mathbb{S}^2) = W^{k,2}(\mathbb{S}^2, \mathbb{S}^2)$. These spaces are closed subspaces of $W^{k,p}(\mathbb{S}^2, \mathbb{R}^3)$, which follows from the fact that the function

$$\boldsymbol{u} \mapsto \int_{\mathbb{S}^2} |1 - |\boldsymbol{u}|^p| \, \mathrm{d}\sigma .$$

is continuous on $W^{k,p}(\mathbb{S}^2, \mathbb{R}^3)$ and the preimage of $\{0\}$ under this function is $W^{k,p}(\mathbb{S}^2, \mathbb{S}^2)$. Therefore, letting the space $W^{k,p}(\mathbb{S}^2, \mathbb{S}^2)$ inherit a metric in terms of the $W^{k,p}(\mathbb{S}^2, \mathbb{R}^3)$-norm, we conclude that $W^{k,p}(\mathbb{S}^2, \mathbb{S}^2)$ is a complete metric space. However, these spaces lose the structure of a vector space, as the sum of two elements in $W^{k,p}(\mathbb{S}^2, \mathbb{S}^2)$ is not $\mathbb{S}^2$-valued in general, in fact, not even the 0 function is included.

Moreover, although $W^{k,p}(\mathbb{S}^2, \mathbb{R}^3)$ has the set of smooth functions as a dense subset, this is not the case for $W^{k,p}(\mathbb{S}^2, \mathbb{S}^2)$, in general. For instance, the results in [57, 7] show that $C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ is dense in $W^{1,p}(\mathbb{S}^2, \mathbb{S}^2)$ as long as $p \geq 2$, but for $p < 2$ this is not true.

The space $H^1(\mathbb{S}^2, \mathbb{S}^2)$ is of particular interest as it is the critical Sobolev space, which also becomes apparent by the former observation. Moreover, it is the first space in terms of p and k for which the embedding $W^{k,p}(\mathbb{S}^2, \mathbb{S}^2) \subset L^\infty(\mathbb{S}^2, \mathbb{S}^2)$ fails. Therefore, we can only define it with its elements having norm pointwise equal to 1 almost everywhere. If we increase the regularity to $H^k(\mathbb{S}^2, \mathbb{S}^2)$ with $k \geq 2$, we gain a lot of structure. The key aspect here is what we call the "algebra property" (see also Definition A.27): By virtue of Definition A.31 and the Sobolev imbedding $H^k(\mathbb{S}^2, \mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{S}^2, \mathbb{R}^3)$ we conclude that for all $\boldsymbol{u}, \boldsymbol{v} \in H^k(\mathbb{S}^2, \mathbb{R}^3)$ and $f \in H^k(\mathbb{S}^2, \mathbb{R})$ we have

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathbb{R}^3} \in H^k(\mathbb{S}^2, \mathbb{R}) \quad \text{and} \quad f\boldsymbol{u} \in H^k(\mathbb{S}^2, \mathbb{R}^3) .$$

With this at hand we can prove the following proposition.

Proposition 2.1. *Let $k \geq 2$. Then $H^k(\mathbb{S}^2, \mathbb{S}^2)$ is a C^∞-submanifold of $H^k(\mathbb{S}^2, \mathbb{R}^3)$ which is also a Hilbert manifold with Riemannian metric induced by the $H^k(\mathbb{S}^2, \mathbb{R}^3)$ inner product. The tangent space at $\boldsymbol{m} \in H^k(\mathbb{S}^2, \mathbb{S}^2)$ is given by*

$$T_{\boldsymbol{m}} H^k(\mathbb{S}^2, \mathbb{S}^2) = \left\{ \boldsymbol{v} \in H^k(\mathbb{S}^2, \mathbb{R}^3) : \langle \boldsymbol{m}(x), \boldsymbol{v}(x) \rangle = 0 \text{ for a. e. } x \in \mathbb{S}^2 \right\} .$$

Proof. We want to apply the submersion theorem. To this end, we define the map

$$\phi : H^k(\mathbb{S}^2, \mathbb{R}^3) \to H^k(\mathbb{S}^2, \mathbb{R}), \quad \boldsymbol{m} \mapsto |\boldsymbol{m}|^2 - 1 .$$

This map is C^2: For $\boldsymbol{m} \in H^k(\mathbb{S}^2, \mathbb{R}^3)$ and $\boldsymbol{v}, \boldsymbol{w} \in T_{\boldsymbol{m}} H^k(\mathbb{S}^2, \mathbb{R}^3) = H^k(\mathbb{S}^2, \mathbb{R}^3)$ we have $\mathrm{d}\phi_{\boldsymbol{m}} : T_{\boldsymbol{m}} H^k(\mathbb{S}^2, \mathbb{R}^3) \to H^k(\mathbb{S}^2, \mathbb{R})$ with

$$\mathrm{d}\phi_{\boldsymbol{m}}(\boldsymbol{v})(x) = 2\langle \boldsymbol{m}(x), \boldsymbol{v}(x) \rangle \tag{2.1}$$

for all $x \in \mathbb{R}^2$, and $\mathrm{d}^2\phi_{\boldsymbol{m}} : T_{\boldsymbol{m}} H^k(\mathbb{S}^2, \mathbb{R}^3) \times T_{\boldsymbol{m}} H^k(\mathbb{S}^2, \mathbb{R}^3) \to H^2(\mathbb{S}^2, \mathbb{R})$ with

$$\mathrm{d}^2\phi_{\boldsymbol{m}}(\boldsymbol{v})(\boldsymbol{w})(x) = 2\langle \boldsymbol{v}(x), \boldsymbol{w}(x) \rangle$$

for all $x \in \mathbb{R}^2$. In fact, all further derivatives vanish, making ϕ even $\mathcal{C}^\infty$. By the "algebra property", we find that ϕ is indeed well-defined, i.e., $\phi(\boldsymbol{m}) \in H^k(\mathbb{S}^2, \mathbb{R})$. Also, it is clear that the preimage of the 0 function in $H^k(\mathbb{S}^2, \mathbb{R})$ is $H^k(\mathbb{S}^2, \mathbb{S}^2)$. To apply the submersion theorem, we need to show that 0 is a regular value of ϕ. To this end, let $\boldsymbol{m} \in H^k(\mathbb{S}^2, \mathbb{S}^2)$ and we assert that $\mathrm{d}\phi_{\boldsymbol{m}}$ is surjective and $\ker \mathrm{d}\phi_{\boldsymbol{m}}$ splits $T_{\boldsymbol{m}} H^k(\mathbb{S}^2, \mathbb{R}^3) = H^k(\mathbb{S}^2, \mathbb{R}^3)$. For $f \in H^k(\mathbb{S}^2, \mathbb{R})$ given, we set

$$\boldsymbol{v}_f = \frac{f}{2}\boldsymbol{m},$$

which by the "algebra property" is in $H^k(\mathbb{S}^2, \mathbb{R}^3)$, and easily verify $\mathrm{d}\phi_{\boldsymbol{m}}(\boldsymbol{v}_f) = f$. Thus, $\mathrm{d}\phi_{\boldsymbol{m}}$ is onto.

Using (2.1), we see that

$$\boldsymbol{v} \in \ker \mathrm{d}\phi_{\boldsymbol{m}} \Leftrightarrow \boldsymbol{v}(x) \perp \boldsymbol{m}(x) \text{ for almost all } x \in \mathbb{S}^2 .$$

We define the linear projection operator $\Pi_{\boldsymbol{m}} : H^k(\mathbb{R}^2, \mathbb{R}^3) \to H^k(\mathbb{R}^2, \mathbb{R}^3)$ via

$$(\Pi_{\boldsymbol{m}} \boldsymbol{v})(x) = \boldsymbol{v}(x) - \langle \boldsymbol{m}(x), \boldsymbol{v}(x) \rangle \boldsymbol{m}(x) . \tag{2.2}$$

By virtue of the "algebra property", this operator is well-defined and bounded, thus continuous. Therefore, as $\operatorname{Im} \Pi_m = \ker \mathrm{d}\phi_m$, we conclude that the splitting property is fulfilled. As a result, applying the regular value theorem [70, Thm. 73.C] we conclude that $H^k(\mathbb{S}^2, \mathbb{S}^2)$ is a submanifold of $H^k(\mathbb{S}^2, \mathbb{R}^3)$, and by identification we find for $\boldsymbol{m} \in H^k(\mathbb{S}^2, \mathbb{S}^2)$

$$T_{\boldsymbol{m}} H^k(\mathbb{S}^2, \mathbb{S}^2) = \ker \mathrm{d}\phi_{\boldsymbol{m}} = \left\{ \boldsymbol{v} \in H^k(\mathbb{S}^2, \mathbb{R}^3) : \langle \boldsymbol{m}(x), \boldsymbol{v}(x) \rangle = 0 \text{ for a. e. } x \in \mathbb{S}^2 \right\} .$$

□

Generally, we will consider two different coordinates, i.e., parametrizations in the sense of differential geometry, on the sphere: spherical coordinates and stereographic coordinates. These are both coordinates on the whole sphere apart

from two and one poles, respectively. Hence, strictly speaking, each on their own do not define a complete atlas of $\mathbb{S}^2$. However, since the Sobolev spaces are defined through integral quantities and as leaving out sets of measure zero while integrating does not affect the values of the integrals, these coordinates will serve to be useful when working with $W^{k,p}(\mathbb{S}^2,\mathbb{R})$.
Spherical coordinates on the sphere with azimuthal angle around the $\hat{e}_3$-axis are defined as

$$\begin{aligned} \hat{\mathbf{\Upsilon}}\colon (0,\pi)\times(0,2\pi) &\to \mathbb{S}^2 \\ (\theta,\varphi) &\mapsto (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)^T \,. \end{aligned}$$

Note that we define $\hat{\mathbf{\Upsilon}}$ via the embedding of $\mathbb{S}^2$ into $\mathbb{R}^3$ (see the remark below). The components of the metric tensor in these coordinates are

$$g_{\varphi\varphi} = \sin^2\theta, \quad g_{\theta\varphi} = g_{\varphi\theta} = 0, \quad g_{\theta\theta} = 1\,,$$

and hence the volume form is $\mathrm{d}v(g) = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi$. Consequently, the $H^1(\mathbb{S}^2,\mathbb{R})$ norm becomes

$$\|\boldsymbol{u}\|_{H^1(\mathbb{S}^2,\mathbb{R}^3)} = \left(\int_0^{2\pi}\int_0^{\pi} \sin\theta|\partial_\theta\boldsymbol{u}|^2 + \frac{1}{\sin\theta}|\partial_\varphi\boldsymbol{u}|^2 + \sin\theta|\boldsymbol{u}|^2\,\mathrm{d}\theta\mathrm{d}\varphi\right)^{1/2}.$$

The spherical coordinates are not suited for when we want to examine maps at one of the poles. In these cases we will use stereographic coordinates centered at the respective pole. For the center at the north pole these read

$$\begin{aligned} \mathbf{\Phi}\colon \mathbb{R}^2 &\to \mathbb{S}^2 \\ (x_1,x_2) &\mapsto \frac{2}{1+|x|^2}\left(x_1, x_2, \frac{1-|x|^2}{2}\right)^T \,. \end{aligned}$$

We introduce for the prefactor, also called conformal factor, the notion

$$\lambda(x) = \frac{2}{1+|x|^2}\,.$$

Then, clearly $\mathbf{\Phi}(0,0) = \hat{e}_3$, which is why these coordinates are called centered at the north pole.

In these coordinates the Riemannian metric becomes $\lambda^2(x)\mathrm{Id}$ and thus the volume form is $\mathrm{d}v(g) = \lambda^2 \mathrm{d}x$. The $H^1(\mathbb{S}^2, \mathbb{R}^3)$ norm in these coordinates reads

$$\|\boldsymbol{u}\|_{H^1(\mathbb{S}^2,\mathbb{R}^3)} = \left(\int_{\mathbb{R}^2} |D\boldsymbol{u}|^2 + \lambda^2 |\boldsymbol{u}|^2 \,\mathrm{d}x \right)^{1/2}, \tag{2.3}$$

where $|D\boldsymbol{u}|$ is the Frobenius norm of the Jacobian of $\boldsymbol{u}$ seen as a map $\mathbb{R}^2 \to \mathbb{R}^3$, i.e., the standard norm of the gradient in $\mathbb{R}^2$.
We can also define stereographic coordinates centered at an arbitrary point $x_0 \in \mathbb{S}^2$. To this end, let $R \in SO(3)$ be the rotation matrix that maps the north pole to x_0. Then the stereographic coordinates centered at x_0 are given by $\boldsymbol{\Phi}_{x_0} = R\boldsymbol{\Phi}$. Then clearly $\boldsymbol{\Phi}_{x_0}(0,0) = x_0$. The metric in these coordinates then still is $\lambda^2(x)\mathrm{Id}$. Similarly, we can also define spherical coordinates with respect to an arbitrary axis.

Remark 2.2. Note that both coordinates are defined by a map $U \to \mathbb{R}^3$ such that the range of them is a subset of $\imath(\mathbb{S}^2) \subset \mathbb{R}^3$, where $\imath\colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is the embedding of the sphere into $\mathbb{R}^3$. That means, to be more precise in the language of differential manifolds, the correct parametrizations should be, in fact, $(\theta, \varphi) \mapsto \imath^{-1}\hat{\boldsymbol{\Upsilon}}(\theta, \varphi)$ and $(x_1, x_2) \mapsto \imath^{-1}\boldsymbol{\Phi}(x_1, x_2)$ However, when using these coordinates, we implicitly use them in this embedded sense without writing it down. $\boxdot$

Remark 2.3. Let $R < \mathrm{inj}(\mathbb{S}^2)$, where $\mathrm{inj}(\mathbb{S}^2)$ is the injectivity radius of $\mathbb{S}^2$. Then the geodesic ball $B_R(\hat{e}_3)$ is defined as the set of all points $x \in \mathbb{S}^2$ such that the geodesic connecting x and $\hat{e}_3$ has length less than R. Since on the sphere the geodesics are great circles, the geodesic ball has the form of a spherical cap. In stereographic coordinates centered at the north pole the geodesic ball $B_R(\hat{e}_3)$ is parametrized by the ball $B_{\tilde{R}}(0) \subset \mathbb{R}^2$, where $\tilde{R} = \tan(R/2)$. Care must be taken when considering integrals over such balls, either in a coordinate free form or with the parametrization applied, especially since we denote both types of balls in the same manner. We try to avoid this confusion by using the correct volume form in the integral and changing the notion of the differential, as done in (2.3). To illustrate, let $f\colon \mathbb{S}^2 \to \mathbb{R}$ be a differentiable function. Then it holds

$$\int_{B_R(\hat{e}_3)} f \,\mathrm{d}\sigma = \int_{B_{\tilde{R}}(0)} f\lambda^2 \,\mathrm{d}x\,,$$
$$\int_{B_R(\hat{e}_3)} |\nabla f|^2 \,\mathrm{d}\sigma = \int_{B_{\tilde{R}}(0)} |Df|^2 \,\mathrm{d}x\,. \qquad \boxdot$$

For spherical coordinates, since they are not defined at the poles, we can only represent $B_R(\hat{e}_3) \setminus \{\hat{e}_3\}$ in these coordinates. This set is parametrized by $(0, R) \times (0, 2\pi]$. However, as before, since the point has measure zero, we can equate

$$\int_{B_R(\hat{e}_3)} f \,\mathrm{d}\sigma = \int_0^{2\pi} \int_0^R f \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \,.$$

We can also use the above considerations when we use the stereographic coordinates centered at the north pole combined with polar coordinates in the plane. That is we define

$$\begin{aligned} \tilde{\mathbf{\Phi}} \colon (0, \infty) \times (0, 2\pi] &\to \mathbb{S}^2 \\ (r, \varphi) &\mapsto \mathbf{\Phi}(r \cos\varphi, r \sin\varphi) \,. \end{aligned}$$

We call these coordinates polar stereographic coordinates. Then by the above argument, we find that the geodesic distance to the north pole of $\tilde{\mathbf{\Phi}}(r, \varphi)$ is given by $\theta(r) = 2 \arctan(r)$. Moreover, there is a simple connection between the spherical and stereographic coordinates expressed in polar coordinates given through

$$\mathbf{\Psi}(\theta(r), \varphi) = \tilde{\mathbf{\Phi}}(r, \varphi) \,.$$

2.2. General definitions

After having defined and characterized the functions spaces considered in this thesis, we can now introduce the functionals and equations already stated in the introduction more properly.

Definition 2.4. We define for $\kappa > 0$ the functional $\mathcal{E} \colon H^1(\mathbb{S}^2, \mathbb{S}^2) \to \mathbb{R}$ via

$$\mathcal{E}(\boldsymbol{m}) = \frac{1}{2} \int_{\mathbb{S}^2} |\nabla \boldsymbol{m}|^2 + \kappa(1 - \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle^2) \,\mathrm{d}\sigma \,,$$

where $\boldsymbol{\nu}(x) \in \mathbb{R}^3$ is the outer unit normal to $\mathbb{S}^2$ at $x \in \mathbb{S}^2$ and the scalar product is seen with $\boldsymbol{m}$ embedded into $\mathbb{R}^3$. We define the integrand of $\mathcal{E}$ as

$$e(\boldsymbol{m}) = \frac{1}{2} \Big(|\nabla \boldsymbol{m}|^2 + \kappa(1 - \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle^2) \Big) \,. \qquad \boxdot$$

Likewise, we define the mapping degree for maps in $H^1(\mathbb{S}^2, \mathbb{S}^2)$

Definition 2.5. The mapping degree or skyrmion number is the functional $Q\colon C^\infty(\mathbb{S}^2,\mathbb{S}^2) \to \mathbb{Z}$ via

$$Q(\boldsymbol{m}) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \boldsymbol{m}^* \mathrm{d}s\,,$$

where by $\boldsymbol{m}^*\mathrm{d}s$ we denote the pullback of the area form on $\mathbb{S}^2$ by $\boldsymbol{m}$. ⊡

By density of $C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ in $H^1(\mathbb{S}^2,\mathbb{S}^2)$ this map extends continuously to $H^1(\mathbb{S}^2,\mathbb{S}^2)$. Critical points of $\mathcal{E}$ are maps $\boldsymbol{m} \in H^1(\mathbb{S}^2,\mathbb{S}^2)$ such that if we define for $\boldsymbol{\varphi} \in H^1 \cap L^\infty(\mathbb{S}^2,\mathbb{R}^3)$ and $\varepsilon < \|\boldsymbol{\varphi}\|_{L^\infty}$ the map

$$\boldsymbol{m}_\varepsilon = \frac{\boldsymbol{m}+\varepsilon\boldsymbol{\varphi}}{|\boldsymbol{m}+\varepsilon\boldsymbol{\varphi}|}\,,$$

then $\boldsymbol{m}_\varepsilon$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}(\boldsymbol{m}_\varepsilon)\Big|_{\varepsilon=0} = 0\,.$$

Note that since $H^1(\mathbb{S}^2,\mathbb{S}^2)$ does not have the structure of a Banach manifold, we have to define criticality in this sense and not in the sense of the derivative of $\mathcal{E}$ being zero as we have no notion of a derivative of $\mathcal{E}$ in $H^1(\mathbb{S}^2,\mathbb{S}^2)$.
However, if we consider $H^1(\mathbb{S}^2,\mathbb{S}^2)$ formally a Banach manifold as in Definition 2.1, we can state the derivative of $\mathcal{E}$ at $\boldsymbol{m}$ in direction $\boldsymbol{\psi} \in T_{\boldsymbol{m}}H^1(\mathbb{S}^2,\mathbb{S}^2)$ as

$$\mathrm{d}\mathcal{E}_m(\boldsymbol{\psi}) = \int_{\mathbb{S}^2} \langle \nabla\boldsymbol{m}, \nabla\boldsymbol{\psi}\rangle - \kappa\langle \boldsymbol{m},\boldsymbol{\nu}\rangle\langle\boldsymbol{\nu},\boldsymbol{\psi}\rangle\,. \tag{2.4}$$

As will become clear from the following exposition we see that the above definition of a critical point is compatible with this interpretation.
In order to do the calculation, we first compute under usage of $|\boldsymbol{m}| \equiv 1$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\boldsymbol{m}_\varepsilon\Big|_{\varepsilon=0} &= \left(-\frac{\boldsymbol{m}+\varepsilon\boldsymbol{\varphi}}{|\boldsymbol{m}+\varepsilon\boldsymbol{\varphi}|^3}\langle \boldsymbol{m}+\varepsilon\boldsymbol{\varphi},\boldsymbol{\varphi}\rangle + \frac{\boldsymbol{\varphi}}{|\boldsymbol{m}+\varepsilon\boldsymbol{\varphi}|}\right)\Big|_{\varepsilon=0}\\ &= \boldsymbol{\varphi} - \langle\boldsymbol{m},\boldsymbol{\varphi}\rangle\boldsymbol{\varphi} = \Pi_{\boldsymbol{m}}\boldsymbol{\varphi}\,,\end{aligned}$$

where $\Pi_{\boldsymbol{v}}$ is the pointwise orthogonal projection onto the tangent space of $\mathbb{S}^2$ at $\boldsymbol{v}$ embedded into $\mathbb{R}^3$ defined as

$$\Pi_{\boldsymbol{v}}\boldsymbol{w} = \boldsymbol{w} - \langle\boldsymbol{v},\boldsymbol{w}\rangle\boldsymbol{v}$$

for $\boldsymbol{v} \in \mathbb{S}^2$ and $\boldsymbol{w} \in \mathbb{R}^3$. We can quickly verify that $\Pi_{\boldsymbol{v}}$ is self-adjoint with respect to the scalar product in $\mathbb{R}^3$. For $\boldsymbol{u},\boldsymbol{w} \in \mathbb{R}^3$ we assert

$$\langle\Pi_{\boldsymbol{v}}\boldsymbol{w},\boldsymbol{u}\rangle = \langle\boldsymbol{w} - \langle\boldsymbol{v},\boldsymbol{w}\rangle\boldsymbol{v},\boldsymbol{u}\rangle = \langle\boldsymbol{w},\boldsymbol{u}\rangle - \langle\boldsymbol{v},\boldsymbol{w}\rangle\langle\boldsymbol{v},\boldsymbol{u}\rangle = \langle\boldsymbol{w},\Pi_{\boldsymbol{v}}\boldsymbol{u}\rangle\,. \tag{2.5}$$

Moreover, we get

$$\begin{aligned}\langle \nabla \boldsymbol{m}, \nabla \Pi_m \boldsymbol{\varphi} \rangle &= \langle \nabla \boldsymbol{m}, \nabla \boldsymbol{\varphi} - \langle \nabla \boldsymbol{m}, \boldsymbol{\varphi} \rangle_{\mathbb{R}^3} \boldsymbol{m} - \langle \boldsymbol{m}, \nabla \boldsymbol{\varphi} \rangle_{\mathbb{R}^3} \boldsymbol{m} - \langle \boldsymbol{m}, \boldsymbol{\varphi} \rangle \nabla \boldsymbol{m} \rangle \\ &= \langle \nabla \boldsymbol{m}, \nabla \boldsymbol{\varphi} \rangle - \left\langle |\nabla \boldsymbol{m}|^2 \boldsymbol{m}, \boldsymbol{\varphi} \right\rangle,\end{aligned}$$

where we used that $\langle \boldsymbol{m}, \nabla \boldsymbol{m} \rangle_{\mathbb{R}^3} = 0$, since $|\boldsymbol{m}| \equiv 1$.
Therefore, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{E}(\boldsymbol{m}_\varepsilon) \Big|_{\varepsilon=0} &= \int_{\mathbb{S}^2} \langle \nabla \boldsymbol{m}, \nabla \partial_\varepsilon \boldsymbol{m}_\varepsilon \rangle - \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \langle \boldsymbol{\nu}, \partial_\varepsilon \boldsymbol{m}_\varepsilon \rangle \Big|_{\varepsilon=0} \mathrm{d}\sigma \\ &= \int_{\mathbb{S}^2} \langle \nabla \boldsymbol{m}_\varepsilon, \nabla \Pi_{\boldsymbol{m}} \boldsymbol{\varphi} \rangle - \kappa \langle \boldsymbol{m}_\varepsilon, \boldsymbol{\nu} \rangle \langle \boldsymbol{\nu}, \Pi_{\boldsymbol{m}} \boldsymbol{\varphi} \rangle \, \mathrm{d}\sigma \\ &= \int_{\mathbb{S}^2} \langle \nabla \boldsymbol{m}, \nabla \boldsymbol{\varphi} \rangle - \left\langle |\nabla \boldsymbol{m}|^2 \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \Pi_{\boldsymbol{m}} \boldsymbol{\nu}, \boldsymbol{\varphi} \right\rangle \mathrm{d}\sigma \qquad (2.6)\end{aligned}$$

Noting that $\Pi_{\boldsymbol{m}} \boldsymbol{\varphi}$ is orthogonal to $\boldsymbol{m}$ at every point and thus in the formal tangent space $T_{\boldsymbol{m}} H^1(\mathbb{S}^2, \mathbb{S}^2)$, we see that we can indeed identify in (2.6) the formal derivative of $\mathcal{E}$ as in (2.4) in the second line.
Consequently, $\boldsymbol{m}$ is a critical point of $\mathcal{E}$ if and only if $\boldsymbol{m}$ solves the following equation in a weak sense in $H^1(\mathbb{S}^2, \mathbb{R}^3)$ with test functions in the space $H^1 \cap L^\infty(\mathbb{S}^2, \mathbb{R}^3)$

$$0 = \Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu} + \boldsymbol{m} \Big(|\nabla \boldsymbol{m}|^2 - \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle^2 \Big) \qquad \text{(EL)}$$

Formally, we can restate this equation as

$$0 = \Pi_{\boldsymbol{m}} (\Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}) \, ,$$

where the second line should be understood in the sense that it implies having act Π_m onto the test functions according to (2.5) which recovers the second line in (2.6). Additionally, for functions with $|\boldsymbol{m}| \equiv 1$ we can assert the following identity,

$$\langle \boldsymbol{m}, \Delta \boldsymbol{m} \rangle = -|\nabla \boldsymbol{m}|^2 \, , \qquad (2.7)$$

which follows from taking the divergence of $\langle \boldsymbol{m}, \nabla \boldsymbol{m} \rangle = 0$. Hence, for functions in $C^2(\mathbb{S}^2, \mathbb{S}^2)$ this equivalence of the equations is indeed justified. Furthermore, we note that for $\boldsymbol{v} \in \mathbb{S}^2$ and $\boldsymbol{w} \in \mathbb{R}^3$ we have

$$|\boldsymbol{v} \times \boldsymbol{w}|^2 = |w|^2 - \langle \boldsymbol{v}, \boldsymbol{w} \rangle^2 = |\Pi_{\boldsymbol{v}} \boldsymbol{w}|^2 \, .$$

Hence, another equivalent formulation of (EL) is

$$0 = \boldsymbol{m} \times (\Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}) \, ,$$

where make use of the standard cross product on $\mathbb{R}^3$. Here, for the translation into the weak formulation we have to use the Jacobi identity for the scalar triple product:

$$\langle \boldsymbol{m} \times (\Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}), \boldsymbol{\varphi} \rangle = \langle -\Delta \boldsymbol{m} - \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}, \boldsymbol{m} \times \boldsymbol{\varphi} \rangle .$$

We call (EL) the Euler–Lagrange equation of $\mathcal{E}$. It is a semilinear elliptic PDE and is the main object of study in this thesis, i.e., we are looking for a weak solution of (EL) which belongs to $H^1(\mathbb{S}^2, \mathbb{S}^2)$. Solutions with $Q(\boldsymbol{m}) = 0$ are called skyrmions. The "trivial" solution for all possible values of κ is $\pm\boldsymbol{\nu}$. This map has $Q(\pm\boldsymbol{\nu}) = \pm 1$. For $\kappa > 4$ it was proved that it is the global minimizer in the class $H^1(\mathbb{S}^2, \mathbb{S}^2)$ [15]. An important regularity result is the following.

Theorem 2.6 ([59, Thm. 2])**.** *Let $\boldsymbol{m} \in H^1(\mathbb{S}^2, \mathbb{S}^2)$ be a weak solution of* (EL). *Then $\boldsymbol{m}$ is smooth in the sense that there exists a smooth representative of $\boldsymbol{m}$ in $H^1(\mathbb{S}^2, \mathbb{S}^2)$.*

The argument to prove this is similar to the one used for the regularity of weakly harmonic maps between spheres, see e.g., in [32]. In particular, the statement can be localized which means that if $\boldsymbol{m}$ solves (EL) on an open subset $U \subset \mathbb{S}^2$ weakly in $H^1(U, \mathbb{R}^3)$ with test functions in $H^1_0 \cap L^\infty(U, \mathbb{R}^3)$, then $\boldsymbol{m}$ is smooth.
We can use this result to obtain the following statement that is once again an adaption of the properties of harmonic maps, the removability of point singularities of solutions of (EL). For harmonic maps this was first proved in [56] in a much more sophisticated way. With the regularity result at hand, the proof is much more straightforward.

Proposition 2.7 (Removability of point singularities)**.** *Let $\boldsymbol{m} \in H^1(\mathbb{S}^2, \mathbb{S}^2)$ and let $\{z_1, \dots, z_k\}$ be a finite set such that $\boldsymbol{m}$ is a weak solution of* (EL) *on $\mathbb{S}^2 \setminus \{z_1, \dots, z_k\}$. Then $\boldsymbol{m}$ extends to a smooth map on $\mathbb{S}^2$ which is a solution of* (EL).

Proof. Since the statement is local, we can restrict ourselves to the case of one singularity at the north pole and look at the problem on the set $B_1(0) \subset \mathbb{R}^2$ in stereographic coordinates centered at the north pole. Thus, the coordinate representation of $\boldsymbol{m}$ solves weakly in $H^1(B_1(0) \setminus \{0\}, \mathbb{R}^3)$ the equation

$$\Delta \boldsymbol{m} + |D\boldsymbol{m}|^2 \boldsymbol{m} + \kappa \lambda^2(x) \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \Pi_{\boldsymbol{m}} \boldsymbol{\nu} = 0 , \tag{2.8}$$

which is (EL) in stereographic coordinates. The Laplacian here is the standard one on $\mathbb{R}^2$.

We define the map $\phi\colon B_1(0) \to \mathbb{R}$ via

$$\phi(x) = \begin{cases} \ln\ln\left(\frac{1}{|x|}\right) & \text{for } |x| < \frac{1}{e} \\ 0 & \text{else}. \end{cases}$$

Then ϕ is differentiable almost everywhere. We compute in polar coordinates

$$\|\phi\|_{L^2(B_1(0))} = \int_0^{1/e} \left(\ln\ln\left(\frac{1}{r}\right)\right)^2 r \,\mathrm{d}r < \infty\,,$$

since $\ln^2\ln(1/r)r \to 0$ as $r \to 0$ such that the integrand is continuous on $[0, \frac{1}{e}]$. Moreover, we have

$$\begin{aligned} \|\nabla\phi\|_{L^2}^2 &= 2\pi \int_0^{1/e} |\partial_r\phi|^2 r \,\mathrm{d}r \\ &= 2\pi \int_0^{1/e} \frac{1}{\ln^2(r)r} \,\mathrm{d}r \\ &= 2\pi\,. \end{aligned}$$

Consequently, we obtain $\phi \in H_0^1(B_1(0), \mathbb{R})$. For $k \in \mathbb{N}$ we set $r_k := e^{-e^k}$. Then (r_k) is a strictly monotonic decreasing sequence and for all $x \in B_k := B_{r_k}(0)$ we have $\phi(x) \geq k$ with equality at the boundary. We can now define the map $\rho_k\colon B_1(0) \to \mathbb{R}$ via

$$\rho_k(x) = \begin{cases} 0 & \text{for } x \in B_{k+1}\,, \\ k + 1 - \phi(x) & \text{for } x \in B_k \setminus B_{k+1}\,, \\ 1 & \text{else}. \end{cases}$$

It follows that ρ_k is continuous and $\rho_k \to 1$ pointwise in $B_1(0)$. Additionally, we compute in polar coordinates

$$\|\nabla\rho_k\|_{L^2(B_1(0))} = \|\nabla\phi\|_{L^2(B_k \setminus B_{k+1})} \longrightarrow 0\,, \text{ as } k \to \infty\,,$$

by the absolute continuity of the Lebesgue integral.

Next, let $\boldsymbol{\varphi} \in H_0^1 \cap L^\infty(B_1(0), \mathbb{R}^3)$ be a test function. We multiply (2.8) by $\rho_k\boldsymbol{\varphi}$ which gives us a valid test function in $H_0^1 \cap L^\infty(B_1(0) \setminus \{0\}, \mathbb{R}^3)$ for all k. Since $\boldsymbol{m}$ is a weak solution of (EL) in $B_1(0) \setminus \{0\}$, this yields

$$0 = \int_{B_1(0)} \left(\Delta\boldsymbol{m} + |\nabla\boldsymbol{m}|^2\boldsymbol{m} + \kappa\lambda^2\langle\boldsymbol{m}, \boldsymbol{\nu}\rangle\Pi_{\boldsymbol{m}}\boldsymbol{\nu}\right)\rho_k\boldsymbol{\varphi} \,\mathrm{d}x$$

$$= \int_{B_1(0)} \Big(-\langle \nabla \boldsymbol{m}, \nabla \boldsymbol{\varphi} \rangle + |\nabla \boldsymbol{m}|^2 \langle \boldsymbol{m}, \boldsymbol{\varphi} \rangle + \kappa \lambda^2 \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \langle \Pi_{\boldsymbol{m}} \boldsymbol{\nu}, \boldsymbol{\varphi} \rangle \Big) \rho_k \, \mathrm{d}x$$
$$- \int_{B_1(0)} \langle \nabla \boldsymbol{m}, \boldsymbol{\varphi} \nabla \rho_k \rangle \, \mathrm{d}x \, .$$

For the second integral on the right-hand side we compute using the Cauchy–Schwarz inequality

$$| \int_{B_1(0)} \langle \nabla \boldsymbol{m}, \boldsymbol{\varphi} \nabla \rho_k \rangle \, \mathrm{d}x | \leq \|\boldsymbol{\varphi}\|_{L^\infty} \|\nabla \boldsymbol{m}\|_{L^2} \|\nabla \rho_k\|_{L^2} \longrightarrow 0 \, , \text{ as } k \to \infty \, .$$

Consequently, using the dominated convergence theorem when passing to the limit for the first integral, we obtain

$$0 = \int_{B_1(0)} -\langle \nabla \boldsymbol{m}, \nabla \boldsymbol{\varphi} \rangle + |\nabla \boldsymbol{m}|^2 \langle \boldsymbol{m}, \boldsymbol{\varphi} \rangle + \kappa \lambda^2 \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \langle \Pi_{\boldsymbol{m}} \boldsymbol{\nu}, \boldsymbol{\varphi} \rangle \, \mathrm{d}x \, ,$$

which means that $\boldsymbol{m}$ is a weak solution of (EL) in $B_1(0)$. Using the localized version of Definition 2.6, we conclude that $\boldsymbol{m}$ is smooth in $B_1(0)$. This finishes the proof. □

2.3. Axisymmetric maps

The energy $\mathcal{E}$ is invariant under joint rotations in the domain and target. For $R \in O(3)$ we define $\boldsymbol{m} \mapsto \boldsymbol{m}_R$, where

$$\boldsymbol{m}_R(x) = R^{-1} \boldsymbol{m}(R.x) \, . \tag{2.9}$$

Note that this definition uses two notations of action of a matrix. The outside action $R^{-1}\boldsymbol{m}$ is the standard matrix-vector multiplication in $\mathbb{R}^3$ if we consider $\boldsymbol{m}$ as already embedded into $\mathbb{R}^3$ (as we always do). The action $R.x$ for $x \in \mathbb{S}^2$ should be interpreted in the same way: $R.x = \imath^{-1} R \imath x$, where $\imath : \mathbb{S}^2 \to \mathbb{R}^3$ is again the standard embedding of the sphere into $\mathbb{R}^3$. We write this as a map $\phi_R \colon \mathbb{S}^2 \to \mathbb{S}^2, x \mapsto R.x$. We can immediately check that for $Q, R \in O(3)$ we have for all $x \in \mathbb{S}^2$ that

$$(QR).x = \imath^{-1} QR \imath x = \imath^{-1} Q \imath \imath^{-1} R \imath x = Q.(R.x) \, .$$

These joint rotations define a right group action of $O(3)$ on the space of functions $\mathbb{S}^2 \to \mathbb{S}^2$. To show this, we verify the two needed properties of a right group

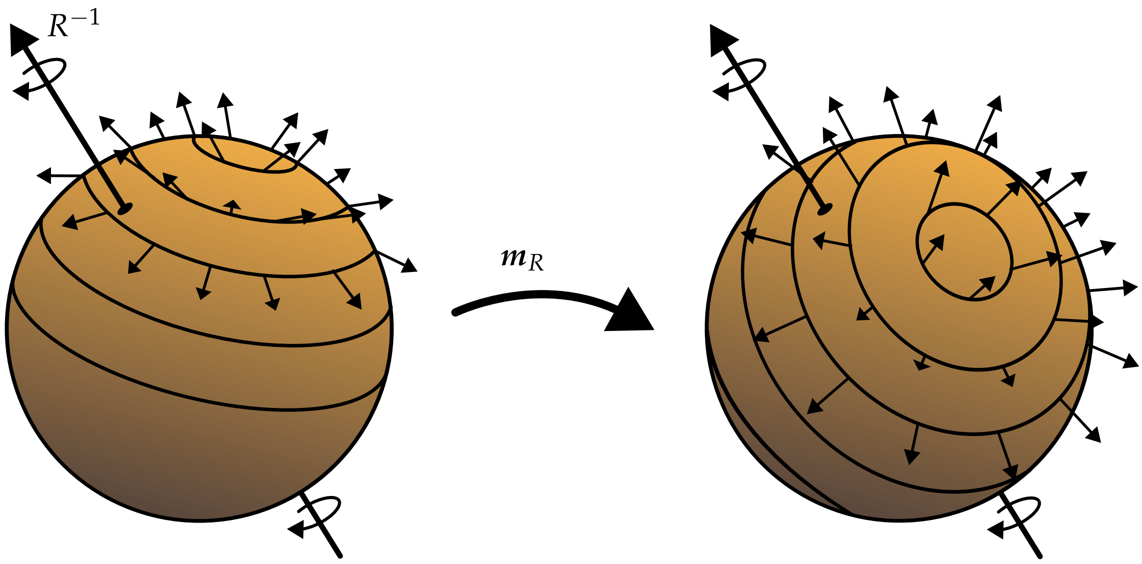

Figure 2.1.: The action of $\boldsymbol{m}_R$ can be interpreted as the action of R to the sphere with the vector field $\boldsymbol{m}$ attached to it at its base points.

action. First, we assert that the identity acts trivially:

$$\boldsymbol{m}_{\mathrm{Id}}(x) = \mathrm{Id}^{-1}\boldsymbol{m}(\mathrm{Id}.x) = \boldsymbol{m}(x)\,.$$

Second, we check that the action is compatible with the group operation of $O(3)$. Let $Q, R \in O(3)$. Then we compute

$$\begin{aligned}\boldsymbol{m}_{QR}(x) &= (QR)^{-1}\boldsymbol{m}((QR).x) = R^{-1}Q^{-1}\boldsymbol{m}(Q.(R.x)) \\ &= R^{-1}\boldsymbol{m}_Q(R.x) = (\boldsymbol{m}_Q)_R(x)\,.\end{aligned}$$

The action is also linear with respect to pointwise addition and scalar multiplication, i.e., for $\boldsymbol{u}, \boldsymbol{v} \colon \mathbb{S}^2 \to \mathbb{S}^2$ and $\alpha \in \mathbb{R}$ we have for all $R \in O(3)$ and $x \in \mathbb{S}^2$

$$(\alpha\boldsymbol{u} + \boldsymbol{v})_R(x) = R^{-1}(\alpha\boldsymbol{u}(R.x) + \boldsymbol{v}(R.x)) = \alpha\boldsymbol{u}(x) + \boldsymbol{v}_R(x)\,.$$

In the next two results we show that the energy $\mathcal{E}$ is invariant under this action and that solutions of (EL) remain solutions under this action. Moreover, we also deduce that the group actions induce linear isometries on the space $H^1(\mathbb{S}^2, \mathbb{S}^2)$.

Lemma 2.8. *Let $\boldsymbol{m} \in C^1(\mathbb{S}^2, \mathbb{S}^2)$. Then $\mathcal{E}(\boldsymbol{m}_R) = \mathcal{E}(\boldsymbol{m})$ for all $R \in O(3)$.*

Proof. We fix $R \in O(3)$ and compute the individual terms of $e(\boldsymbol{m}_R)(x)$ for $x \in \mathbb{S}^2$. We fix coordinates around x. Then for $|\nabla \boldsymbol{m}_R|^2(x)$ we first note that

$$\begin{aligned} |\nabla \boldsymbol{m}_R|^2 &= g^{\mu\nu}(x)\partial_\mu(R^{-1}_{ij} m_j \circ \phi_R)(x)\partial_\nu(R^{-1}_{ik} m_k \circ \phi_R)(x) \\ &= g^{\mu\nu}(x)\partial_\mu(m_i \circ \phi_R)(x)\partial_\nu(m_i \circ \phi_R)(x) \\ &= \sum_{i=1}^{3} |\nabla m_i \circ \phi_R|^2(x)\,, \end{aligned}$$

as R is an orthogonal matrix. From this it also follows by virtue of Definition A.6 that ϕ_R is an isometry from $\mathbb{S}^2$ to $\mathbb{S}^2$ with the induced metric because it is a composition of isometries, since $\imath$ is an isometry by definition of the induced metric and $v \mapsto Rv$ for $v \in \mathbb{R}^3$ is an isometry in $(\mathbb{R}^3, g_E)$ by the orthogonality of R. Since $m_i \circ \phi_R$ is a map from $\mathbb{S}^2$ to $\mathbb{R}$, we can interpret the covariant derivative $\nabla(m_i \circ \phi_R)$ as the differential $\mathrm{d}(m_i \circ \phi_R)$. Moreover, since pullback and differential commute [68, Prop. 17.10] we have

$$\nabla(m_i \circ \phi_R) = \mathrm{d}({\phi_R}^* m_i) = {\phi_R}^* \mathrm{d}m_i\,.$$

Therefore, as ϕ_R is an isometry, invoking Definition A.7 we conclude that

$$\begin{aligned} |\nabla(m_i \circ \phi_R)|(x)^2 &= \hat{g}_x({\phi_R}^*\mathrm{d}m_i, {\phi_R}^*\mathrm{d}m_i) \\ &= ((\phi_R)_*\hat{g})_{\phi_R(x)}(\mathrm{d}m_i, \mathrm{d}m_i) \\ &= \hat{g}_{R.x}(\mathrm{d}m_i, \mathrm{d}m_i) = |\nabla m_i|^2(R.x)\,. \end{aligned} \tag{2.10}$$

As a result, we obtain

$$|\nabla \boldsymbol{m}_R|^2(x) = |\nabla \boldsymbol{m}|^2(R.x) = |\nabla \boldsymbol{m}|^2 \circ \phi_R(x)\,. \tag{2.11}$$

Now we inspect the anisotropy term of $e(\boldsymbol{m}_R)$. For this we note that $\boldsymbol{\nu}(x) = \imath x$ and hence

$$\boldsymbol{\nu}(R.x) = \imath(\imath^{-1}R\imath x) = R\imath x = R\boldsymbol{\nu}(x)\,,$$

from which also follows $\boldsymbol{\nu}(x) = R^{-1}\boldsymbol{\nu}(R.x)$ as $x = R^{-1}.(R.x)$. Therefore, we gain

$$\langle \boldsymbol{m}_R(x), \boldsymbol{\nu}(x)\rangle = \left\langle R^{-1}\boldsymbol{m}(R.x), R^{-1}\boldsymbol{\nu}(R.x)\right\rangle = \langle \boldsymbol{m}, \boldsymbol{\nu}\rangle(R.x) = \langle \boldsymbol{m}, \boldsymbol{\nu}\rangle \circ \phi_R(x)\,. \tag{2.12}$$

Putting these two results together we thus obtain

$$\mathcal{E}(\boldsymbol{m}_R) = \int_{\mathbb{S}^2} e(\boldsymbol{m}_R)\,\mathrm{d}\sigma = \int_{\mathbb{S}^2} e(\boldsymbol{m}) \circ \phi_R\,\mathrm{d}\sigma = \mathcal{E}(\boldsymbol{m})\,,$$

where in the last step we used again that ϕ_R is an isometry and hence the integral is invariant under composition with it according to Definition A.8. □

In the proof for the invariance of the Dirichlet energy under R we have not used that $\boldsymbol{m}$ is $\mathbb{S}^2$-valued. Thus, since the Dirichlet energy of a general function $\boldsymbol{u} \in H^1(\mathbb{S}^2, \mathbb{R}^3)$ is the L^2-norm of the first derivative and since $|\boldsymbol{u}_R|(x) = |\boldsymbol{u}|(R.x)$ for all $x \in \mathbb{S}^2$ by the orthogonality of R we can conclude the following corollary.

Corollary 2.9. *The action $\boldsymbol{u} \mapsto \boldsymbol{u}_R$ for $R \in O(3)$ is a linear isometry (in the sense of Banach spaces) of $H^1(\mathbb{S}^2, \mathbb{R}^3)$. In particular, it holds*

$$\langle \boldsymbol{u}_R, \boldsymbol{v}_R \rangle_{H^1(\mathbb{S}^2,\mathbb{R}^3)} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{H^1(\mathbb{S}^2,\mathbb{R}^3)}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in H^1(\mathbb{S}^2, \mathbb{R}^3)$ and $R \in O(3)$.

Proof. It only remains to show the invariance of the H^1-scalar product. For $\langle \boldsymbol{u}_R, \boldsymbol{v}_R \rangle_{L^2(\mathbb{S}^2,\mathbb{R}^3)}$ this follows from the invariance of the integral under the isometry ϕ_R and the orthogonality of R. For the scalar product of the gradients we can adapt (2.10) to gain

$$\begin{aligned} \langle \nabla \boldsymbol{u}_R, \nabla \boldsymbol{v}_R \rangle(x) &= \sum_i \hat{g}_x({\phi_R}^* \mathrm{d}u_i, {\phi_R}^* \mathrm{d}v_i) \\ &= \sum_i ((\phi_R)_* \hat{g})_{\phi_R(x)}(\mathrm{d}u_i, \mathrm{d}v_i) \\ &= \sum_i \hat{g}_{R.x}(\mathrm{d}u_i, \mathrm{d}v_i) = \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v} \rangle(R.x)\,. \end{aligned}$$

Integrating this over $\mathbb{S}^2$ gives us the desired invariance of the H^1-scalar product, again by the invariance of the integral under the isometry ϕ_R according to Definition A.8. □

Lemma 2.10. *Let $\boldsymbol{m} \colon \mathbb{S}^2 \to \mathbb{S}^2$ be a critical point of $\mathcal{E}$. Then for any $R \in O(3)$ the map $\boldsymbol{m}_R$ is also a critical point of $\mathcal{E}$.*

Proof. Let $R \in O(3)$ be arbitrary. Let $\boldsymbol{\varphi} \in H^1 \cap L^\infty(\mathbb{S}^2, \mathbb{R}^3)$ be a test function and let $\varepsilon < \|\boldsymbol{\varphi}\|_{L^\infty}$. We define the map

$$\boldsymbol{m}_{R,\varepsilon} = \frac{\boldsymbol{m}_R + \varepsilon \boldsymbol{\varphi}}{|\boldsymbol{m}_R + \varepsilon \boldsymbol{\varphi}|}\,.$$

Using the aforementioned properties of the group action, we compute

$$(\boldsymbol{m}_{R,\varepsilon})_{R^{-1}}(x) = \frac{(\boldsymbol{m}_R + \varepsilon \boldsymbol{\varphi})_{R^{-1}}(x)}{|\boldsymbol{m}_R(R^{-1}.x) + \varepsilon \boldsymbol{\varphi}(R^{-1}.x)|} = \frac{\boldsymbol{m} + \varepsilon \boldsymbol{\varphi}_{R^{-1}}(x)}{|R^{-1}\boldsymbol{m}(R.(R^{-1}.x)) + \varepsilon \boldsymbol{\varphi}(R^{-1}.x)|}$$

$$= \frac{\boldsymbol{m} + \varepsilon \boldsymbol{\varphi}_{R^{-1}}(x)}{|\boldsymbol{m}(x) + \varepsilon R \boldsymbol{\varphi}(R^{-1}.x)|} = \frac{\boldsymbol{m} + \varepsilon \boldsymbol{\varphi}_{R^{-1}}}{|\boldsymbol{m} + \varepsilon \boldsymbol{\varphi}_{R^{-1}}|}(x)\,.$$

In the second to last step we used the orthogonality of R which keeps the norm in the denominator invariant. This also implies in combination with the previous Corollary that $\boldsymbol{\varphi}_{R^{-1}} \in H^1 \cap L^\infty(\mathbb{S}^2, \mathbb{R}^3)$. Thus, applying the invariance of $\mathcal{E}$ under the action of R^{-1} and the fact that $\boldsymbol{m}$ is a critical point of $\mathcal{E}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}(\boldsymbol{m}_{R,\varepsilon})\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}((\boldsymbol{m}_{R,\varepsilon})_{R^{-1}})\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}\left(\frac{\boldsymbol{m} + \varepsilon \boldsymbol{\varphi}_{R^{-1}}}{|\boldsymbol{m} + \varepsilon \boldsymbol{\varphi}_{R^{-1}}|}\right)\Big|_{\varepsilon=0} = 0\,.$$

Consequently, $\boldsymbol{m}_R$ is a critical point of $\mathcal{E}$ as well. □

This observation in combination with the fact that $O(3)$ is a Lie group shows us that critical points of the energy functional $\mathcal{E}$ cannot be isolated in the H^1-topology unless they are invariant under all elements of $O(3)$. However, this only applies to the trivial solutions $\pm\boldsymbol{\nu}$. We now define the class of *axisymmetric maps* which are invariant under a subgroup of $O(3)$.

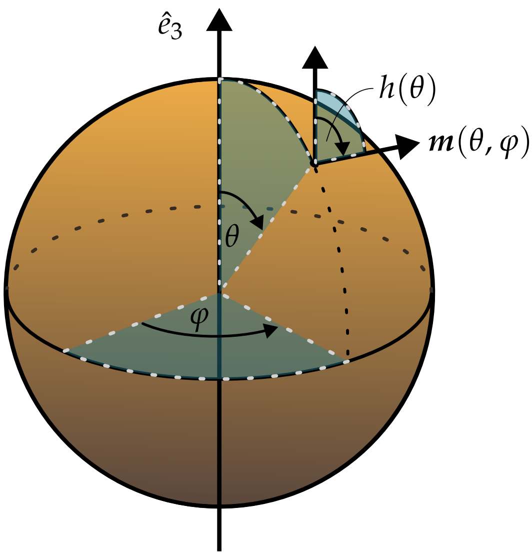

Figure 2.2.: Parametrization of an axisymmetric map $\boldsymbol{m}$ in spherical coordinates.

Definition 2.11. We call a map $\boldsymbol{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ axisymmetric if and only if $\boldsymbol{m}_R = \boldsymbol{m}$ for all $R \in O(3)_{\hat{e}_3}$, the stabilizer subgroup of $O(3)$ with respect to $\hat{e}_3$, that is rotations around the $\hat{e}_3$-axis and reflections in the $\hat{e}_1\hat{e}_2$-subspace. ⊡

The choice of the $\hat{e}_3$-axis is arbitrary; we could have chosen any other axis. However, by the above invariance of the energy and the H^1-topology under

rotations, we can always rotate any axisymmetric map with respect to any axis to the $\hat{e}_3$-axis and gain an equivalent axisymmetric map in the sense of Definition 2.11.
For the further study of these types of maps we will use the two kinds of coordinates for the domain introduced in Section 2.1: spherical coordinates $\hat{\mathbf{\Psi}}$ and stereographic coordinates $\mathbf{\Phi}$.
The following lemma tells us that all information of an axisymmetric map $\boldsymbol{m}$ is contained in a so-called profile function h that depends only on the polar angle. The proof can be found in [59].

Lemma 2.12 ([59, Lemma 2.5]). *A continuous map $\boldsymbol{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ is axisymmetric if and only if there exists a continuous map $h\colon [0,\pi] \to \mathbb{R}$ such that*

$$\boldsymbol{m}(\theta,\varphi) = \mathbf{\Psi}(h(\theta),\varphi) \quad \text{for all} \quad (\theta,\varphi) \in (0,\pi)\times(0,2\pi)\,, \tag{2.13}$$

where $\mathbf{\Psi}$ is the continuation of $\hat{\mathbf{\Psi}}$ from $(0,\pi)\times(0,2\pi)$ to $\mathbb{R}\times[0,2\pi)$. We then call h the profile *of $\boldsymbol{m}$. Note that by the symmetry condition $\boldsymbol{m}(\pm\hat{e}_3) = \pm\hat{e}_3$ we can continuously extend h to 0 and π by $h(0) = m\pi$, $h(\pi) = n\pi$ with $m,n \in \mathbb{Z}$.*

Even though the profile function h depends on the polar angle from the spherical coordinates, it also reappears when we express the axisymmetric map in (polar) stereographic coordinates
When
nates we do so by

$$\boldsymbol{m}(x_1,x_2) = \left(\frac{x_1}{|x|}\sin(h(\theta(|x|))), \frac{x_2}{|x|}\sin(h(\theta(|x|))), \cos(h(\theta(|x|)))\right)^T, \tag{2.14}$$

where the polar angle is given by

$$\theta(r) = 2\arctan(r)\,.$$

Setting $\boldsymbol{m}(0,0) = (0,0,\cos(m\pi))$, this map is certainly well-defined and continuous for continuous h since for $i = 1,2$ we check

$$\left|\frac{x_i}{|x|}\sin(h(\theta(|x|)))\right| \le |\sin(h(\theta(|x|)))| \longrightarrow \sin(h(\theta(0))) = 0\,, \text{ as } |x| \to 0\,.$$

In case we want to use polar stereographic coordinates, we can write

$$\boldsymbol{m}(r,\varphi) = \boldsymbol{m}(\theta(r),\varphi) = \mathbf{\Psi}(h(\theta(r)),\varphi)\,.$$

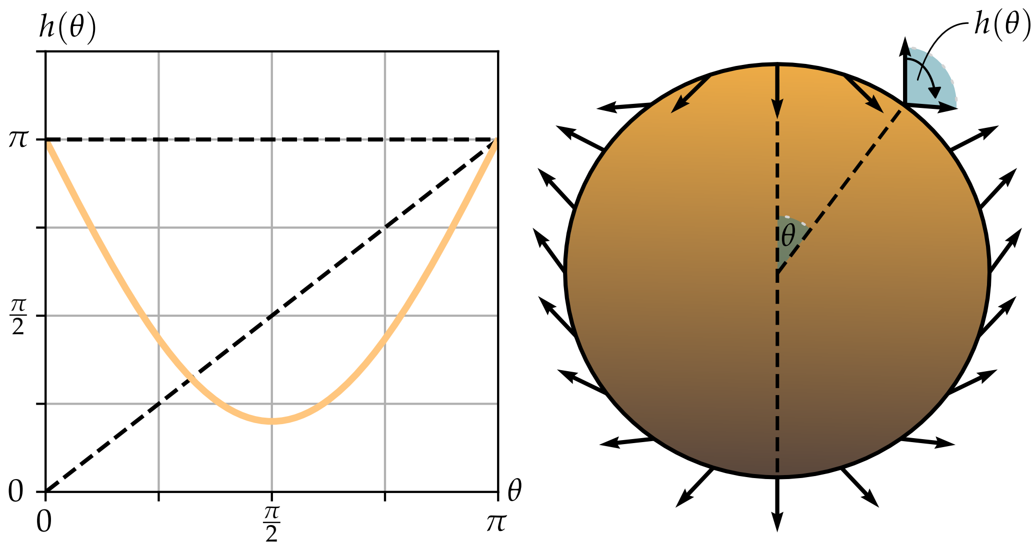

Figure 2.3.: We can plot an axisymmetric map $\boldsymbol{m}$ by using a cross section at a full circle in the x_1x_3-plane. The figure displays an example for a profile that is a parabola. The dashed diagonal line corresponds to the profile of the outer normal $\boldsymbol{\nu}$.

As for the spherical coordinates, because of the symmetry we can in fact continuously extend the map to $r = 0$, which in this case naturally happens since $\theta(0) = 0$.
We define for $k \in \mathbb{N}$ and $0 < \alpha < 1$

$$\mathcal{F}^{k,\alpha} := \{h \in C^{k,\alpha}([0,\pi]) : h(0) = m\pi,\, h(\pi) = n\pi,\ \ m,n \in \mathbb{Z}\}\,,$$

and likewise

$$\mathcal{F}^{\infty} := \{h \in C^{\infty}([0,\pi]) : h(0) = m\pi,\, h(\pi) = n\pi,\ \ m,n \in \mathbb{Z}\}\,,$$

and finally

$$\mathcal{PF}^{1} := \{h \in PC^{1}([0,\pi]) : h(0) = m\pi,\, h(\pi) = n\pi,\ \ m,n \in \mathbb{Z}\}\,,$$

where $PC^1([0,\pi])$ is the space of continuous and piecewise continuously differentiable functions on $[0,\pi]$, i.e., functions that are continuous and have a finite number of points of discontinuity of the first derivative. In all three cases, the existence of $h'(0)$ and $h'(\pi)$ is implied.

Lemma 2.13. *Let $\boldsymbol{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ be axisymmetric with profile h. Then for all $0 < \alpha < 1$ we have $\boldsymbol{m} \in C^{1,\alpha}(\mathbb{S}^2, \mathbb{S}^2)$ if $h \in \mathcal{F}^{1,\alpha}$. Conversely, for all $k \in \mathbb{N}$ and $0 < \alpha < 1$ we get $h \in \mathcal{F}^{k,\alpha}$ if $\boldsymbol{m} \in C^{k,\alpha}(\mathbb{S}^2, \mathbb{S}^2)$.*

Remark 2.14. The weaker first statement results from the fact that higher order derivatives of $\boldsymbol{m}$ at the poles in φ-direction cannot necessarily be bounded without more restrictions on h. ⊡

Proof. Let $h \in \mathcal{F}^{1,\alpha}$. Since $\boldsymbol{\Psi}$ is smooth, $\boldsymbol{m}$ is continuously differentiable on $\mathbb{S}^2 \setminus \{\pm e_3\}$, and in fact by Definition A.2, the derivative $\nabla \boldsymbol{m}$ is Hölder continuous with exponent α on this set. We need to check the poles and restrict the analysis to the north pole as for the south pole we can proceed analogously. We choose stereographic coordinates centered at the north pole such that we can write $\boldsymbol{m}$ as in (2.14). First we check differentiability. For this, it suffices if we check the components individually and since the first and second component only differ by the index we can just consider for $j = 1, 2$

$$m_j(x) = x_j \frac{\sin(h(\theta(|x|)))}{|x|}\,.$$

To this end, we define

$$v(r) = \frac{\sin(h(\theta(r)))}{r}$$

and compute

$$v'(r) = -\frac{v(r)}{r} + \frac{2}{r}\cos(h(\theta(r)))\frac{h'(\theta(r))}{1+r^2}\,.$$

Moreover, invoking that by **??** A.2**??** A.3 the function $r \mapsto \sin(h(\theta(r)))$ is $C^{1,\alpha}$ on any compact subset of $[0, \infty)$, we write using Taylor's theorem

$$v(r) = \frac{1}{r}\big(\cos(m\pi)h'(0)\theta'(0)r + g(r)r\big) = 2\cos(m\pi)h'(0) + g(r)\,. \tag{2.15}$$

with $|g(r)| \leq Cr^\alpha$ by Definition A.4. Using this, we find

$$|rv'(r)| = |-v(r) + 2\cos(h(\theta(r)))\frac{h'(\theta(r))}{1+r^2}| \leq Cr^\alpha\,,$$

since the function $r \mapsto 2\cos(h(\theta(r)))\frac{h'(\theta(r))}{1+r^2}$ is Hölder continuous by assumption and **??** A.2**??** A.3. Thus, $\lim_{r\to 0} |rv'(r)| = 0$.
Now, for $(x_1, x_2) \neq (0, 0)$ we can derive for $i = 1, 2$

$$\partial_{x_i} m_j = \delta_{ij} v(|x|) + \frac{x_i x_j}{|x|} v'(|x|)\,,$$

using the chain rule and the fact that $\partial_{x_i}|x| = x_i/|x|$. Using the limits from above and the fact that $x_i x_j / r^2$ is bounded we conclude

$$\lim_{|x|\to 0} \partial_{x_i} m_j = 2\delta_{ij} \cos(m\pi) h'(0) \,.$$

For the partial derivative of m_j in x_i direction at $(0,0)$ we recall $m_j(0,0) = 0$ and hence

$$\partial_{x_i} m_j|_{(x_1,x_2)=(0,0)} = \lim_{t\to 0} \frac{\delta_{ij} t v(t)}{t} = \lim_{t\to 0} \delta_{ij} v(t) = 2\delta_{ij}\cos(m\pi)h'(0) \,.$$

Consequently, $\partial_{x_i} m_j$ is continuous in $(0,0)$ for $i,j \in \{1,2\}$ and thus m_j is totally differentiable on $\mathbb{S}^2$. For $m_3 = \cos(h(\theta(|x|)))$ we compute for $i = 1,2$

$$\partial_{x_i} \cos(h(\theta(|x|))) = -2\sin(h(\theta(|x|)))\frac{h'(\theta(|x|))}{1+|x|^2}\frac{x_i}{|x|} \longrightarrow 0\,, \text{ as } |x| \to 0\,,$$

since the two fractions are bounded and the sine converges to 0. This coincides with the partial derivative in $(0,0)$, since m_3 only depends on $|x|$:

$$\partial_{x_i} m_3|_{(x_1,x_2)=(0,0)} = \lim_{t\to 0} \frac{1-\cos(h(\theta(t)))}{t} = 0\,.$$

As a result, $\boldsymbol{m}$ is continuously differentiable on $\mathbb{S}^2$. We need to check the Hölder condition for $\nabla \boldsymbol{m}$ in a neighborhood of $(0,0)$. For simplicity, we choose $B_1(0)$. Since the Hölder condition is already established for all points but the poles, we only need to check it for $(x_1,x_2) = (0,0)$ and $(y_1,y_2) \in B_1(0)$. Moreover, if all components of $\nabla \boldsymbol{m}$ satisfy the condition individually, then so does $\nabla \boldsymbol{m}$ by the definition of the Frobenius norm. For $i,j \in \{1,2\}$ we find

$$\begin{aligned} |(\nabla \boldsymbol{m}(y) - \nabla \boldsymbol{m}(0))_{ij}| &= \lambda^{-1}(y)|\delta_{ij}(v(|y|) - 2\cos(m\pi)h'(0)) + \frac{y_i y_j}{|y|} v'(|y|)| \\ &\le |v(y) - 2\cos(m\pi)h'(0)| + |y||v'(|y|)| \le C|y|^\alpha \end{aligned}$$

as we have seen above already. Consequently, $\boldsymbol{m} \in C^{1,\alpha}(\mathbb{S}^2, \mathbb{S}^2)$.
Now let $\boldsymbol{m} \in C^{k,\alpha}(\mathbb{S}^2,\mathbb{S}^2)$. We fix $\varphi = 0$, and find $m_1(\theta) = \sin h(\theta)$ and $m_3(\theta) = \cos h(\theta)$. We define for $l \in \mathbb{Z}$ the sets S_l and C_l as

$$S_l = \left\{ \theta \in [0,\pi] : \left(l - \frac{2}{3}\right)\pi < h(\theta) < \left(l + \frac{2}{3}\right)\pi \right\}$$

and

$$C_l = \left\{ \theta \in [0, \pi] : \left(l + \frac{1}{6} \right) \pi < h(\theta) < \left(l + \frac{5}{6} \right) \pi \right\}.$$

Since h is continuous and bounded the collection of all S_l and C_l is an open cover of $[0, \pi]$ where only finitely many are non-empty. Now, on S_l for $l \in \mathbb{Z}$ fixed the profile h maps into the injectivity set of sin and hence we can write for

$$h(\theta) = \arcsin m_1(\theta) + l\pi \text{ for all } \theta \in S_l\,,$$

where by arcsin we denote the principal branch of the inverse sine function with $\arcsin(0) = 0$. Likewise, we write

$$h(\theta) = \arccos m_3(\theta) + l\pi \text{ for all } \theta \in C_l\,,$$

where arccos denotes the principle branch of the inverse cosine function with $\arccos(0) = \frac{\pi}{2}$. Then, by Definition A.2 we deduce that h is $C^{k,\alpha}$ on S_l and C_l. Since we can cover $[0, \pi]$ with finitely many of these sets, we therefore find a global Hölder bound, hence $h \in C^{k,\alpha}([0, \pi], \mathbb{R})$. □

Also in the piecewise continuously differentiable case we can show an embedding of the corresponding axisymmetric map.

Lemma 2.15. *Let $h \in \mathcal{PF}^1$ and $\boldsymbol{m}$ be axisymmetric with profile h. Then $\boldsymbol{m} \in H^1(\mathbb{S}^2, \mathbb{S}^2)$.*

Proof. We can repeat the proof of the previous lemma with the only difference that the Taylor remainder in (2.15) cannot be bounded by Cr^α but it still goes to 0 as $r \to 0$. Hence, the conclusion that $\boldsymbol{m}$ is continuously differentiable in a neighborhood of $\pm\hat{e}_3$ still holds. For the parts of $\mathbb{S}^2$ away from the poles we note that $\boldsymbol{m}$ is continuously differentiable on slices $(\theta_i, \theta_{i+1}) \times [0, 2\pi)$ for $\theta_i \in (0, \pi)$ and i in a finite index set. Consequently, we can define the weak derivative $\nabla \boldsymbol{m}$ as these derivatives on the slices. As $\boldsymbol{m}$ is continuous on the whole sphere, we have therefore found a valid weak derivative of $\boldsymbol{m}$. Integrability of $|\nabla \boldsymbol{m}|^2$ is straightforward by the compactness of $\mathbb{S}^2$. □

We want to find a way to bound the difference in H^1-norm of two axisymmetric maps $\boldsymbol{m}_1, \boldsymbol{m}_2$ in terms of norms on their respective profiles h_1 and h_2.

We recall the form of $\boldsymbol{m}_i$ as

$$\boldsymbol{m}_i(\theta,\varphi) = \begin{pmatrix} \cos\varphi \sin h_i(\theta) \\ \sin\varphi \sin h_i(\theta) \\ \cos h_i(\theta) \end{pmatrix}.$$

Hence, we get

$$\begin{aligned} |\boldsymbol{m}_1 - \boldsymbol{m}_2|^2 &= \left| \begin{pmatrix} \cos\varphi(\sin h_1 - \sin h_2) \\ \sin\varphi(\sin h_1 - \sin h_2) \\ \cos h_1 - \cos h_2 \end{pmatrix} \right|^2 \\ &= 2(1 - (\sin h_1 \sin h_2 + \cos h_1 \cos h_2)) \\ &= 2(1 - \cos(h_1 - h_2)) \\ &\le |h_1 - h_2|^2, \end{aligned}$$

since $1 - \cos x \le x^2/2$ for all $x \in \mathbb{R}$. For the gradient norm we split it into

$$|\nabla \boldsymbol{m}_1 - \nabla \boldsymbol{m}_2|^2 = |\partial_\theta(\boldsymbol{m}_1 - \boldsymbol{m}_2)|^2 + \frac{1}{\sin^2\theta}\left|\partial_\varphi(\boldsymbol{m}_1 - \boldsymbol{m}_2)\right|^2$$

and compute the terms individually:

$$\begin{aligned} |\partial_\theta(\boldsymbol{m}_1 - \boldsymbol{m}_2)|^2 &= \left| \begin{pmatrix} \cos\varphi(h_1' \cos h_1 - h_2' \cos h_2) \\ \sin\varphi(h_1' \cos h_1 - h_2' \cos h_2) \\ -h_1' \sin h_1 + h_2' \sin h_2 \end{pmatrix} \right|^2 \\ &= \left(h_1' \cos h_1 - h_2' \cos h_2\right)^2 + \left(-h_1' \sin h_1 + h_2' \sin h_2\right)^2 \\ &= h_1'^2 + h_2'^2 - 2h_1' h_2' \cos(h_1 - h_2) \\ &= \left|h_1' - h_2'\right|^2 + 2h_1' h_2'(1 - \cos(h_1 - h_2)) \\ &\le \left|h_1' - h_2'\right|^2 + |h_1' h_2'||h_1 - h_2|^2. \end{aligned}$$

And

$$\begin{aligned} \frac{1}{\sin^2\theta}\left|\,\partial_\varphi(\boldsymbol{m}_1 - \boldsymbol{m}_2)\right|^2 &= \frac{1}{\sin^2\theta}\left| \begin{pmatrix} -\sin\varphi(\sin h_1 - \sin h_2) \\ \cos\varphi(\sin h_1 - \sin h_2) \\ 0 \end{pmatrix} \right|^2 \\ &= \frac{1}{\sin^2\theta}(\sin h_1 - \sin h_2)^2 \end{aligned}$$

Combining these, we gain

$$
\begin{aligned}
&\|\boldsymbol{m}_1-\boldsymbol{m}_2\|_{H^1}^2 = \int_{\mathbb{S}^2} |\nabla\boldsymbol{m}_1-\nabla\boldsymbol{m}_2|^2+|\boldsymbol{m}_1-\boldsymbol{m}_2|^2\,\mathrm{d}\sigma \\
&\leq 2\pi\int_0^\pi \left(|h_1'-h_2'|^2+|h_1-h_2|^2\right)\sin\theta\mathrm{d}\theta \\
&\quad +2\pi\int_0^\pi |h_1-h_2|^2|h_1'h_2'|\sin\theta+\frac{(\sin h_1-\sin h_2)^2}{\sin\theta}\mathrm{d}\theta \\
&\leq 2\pi\|h_1-h_2\|_{H^1([0,\pi])}^2+2\pi\int_0^\pi |h_1-h_2|^2|h_1'h_2'|\sin\theta+\frac{(\sin h_1-\sin h_2)^2}{\sin\theta}\mathrm{d}\theta
\end{aligned}
\tag{2.16}
$$

We note that instead of the H^1-norm we could also use $\|h_1-h_2\|_{C^1([0,\pi])}^2$ in the first term. The second integral can be bounded by $C\|h_1-h_2\|_{C^0([0,\pi])}$, where C depends on h_1 and h_2, though.
For axisymmetric maps expressed in spherical coordinates the energy reduces to the following expression

$$
\mathcal{E}(\boldsymbol{m}) = \pi\int_0^\pi h'^2\sin\theta+\frac{\sin^2 h}{\sin\theta}+\kappa\sin^2(h-\theta)\sin\theta\,\mathrm{d}\theta =: 2\pi E(h)\,, \tag{2.17}
$$

which follows from inserting (2.13) into $\mathcal{E}$ and computing the individual terms

$$
\begin{aligned}
|\nabla\boldsymbol{m}|^2 &= |\partial_\theta\boldsymbol{m}|^2+\frac{1}{\sin^2\theta}|\partial_\varphi\boldsymbol{m}|^2 = h'^2+\frac{\sin^2 h}{\sin^2\theta} \\
1-\langle\boldsymbol{m},\boldsymbol{\nu}\rangle^2 &= 1-(\sin h\sin\theta+\cos h\cos\theta)^2 \\
&= 1-\cos^2(h-\theta) = \sin^2(h-\theta)\,,
\end{aligned}
$$

and $\mathrm{d}\sigma = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi$. Note that $\boldsymbol{\nu}(\theta,\varphi) = \boldsymbol{\Psi}(\theta,\varphi)$. Since the integrand does not depend on φ we can integrate out that part. This was already shown in [59, Sec. 2.2.2] (note that they use θ for the profile function and x for the polar angle).

Lemma 2.16. *For axisymmetric $\boldsymbol{m}$ with profile h the mapping degree becomes*

$$
Q(\boldsymbol{m}) = \frac{1}{2}(\cos(h(0))-\cos(h(\pi)))\,.
$$

This means that if

$$
\frac{h(\pi)-h(0)}{\pi}
$$

is even, then $Q(\boldsymbol{m})=0$, if it is odd, then $Q(\boldsymbol{m})=\pm1$, where the sign is positive or negative whenever $h(0)/\pi$ is even or odd, respectively.

Proof. We compute the degree via the pullback using the explicit formula for spherical coordinates

$$\boldsymbol{m}^* \mathrm{d}\sigma = \boldsymbol{m} \cdot \left(\frac{\partial \boldsymbol{m}}{\partial \theta} \times \frac{\partial \boldsymbol{m}}{\partial \varphi} \mathrm{d}\theta \, \mathrm{d}\varphi \right),$$

yielding

$$\begin{aligned} Q(\boldsymbol{m}) &= \frac{1}{4\pi} \int_{\mathbb{S}^2} \boldsymbol{m}^* \mathrm{d}\sigma \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \boldsymbol{m} \cdot \left(\frac{\partial \boldsymbol{m}}{\partial \theta} \times \frac{\partial \boldsymbol{m}}{\partial \varphi} \right) \mathrm{d}\theta \, \mathrm{d}\varphi \\ &= \frac{1}{2} \int_0^{\pi} h' \sin h \, \mathrm{d}\theta = \frac{1}{2} (\cos(h(0)) - \cos(h(\pi))) \, . \end{aligned}$$

□

In [59, p. 24] the following lemma was shown which shows a correspondence between axisymmetric solutions of (EL) and solutions of a differential equation for the profile. Note that there was an error and the term with the κ was missing a factor $1/2$.

Lemma 2.17. *Let* $\boldsymbol{m}$ *be axisymmetric with profile* h*. Then* $\boldsymbol{m}$ *solves the Euler–Lagrange equation* (EL) *if and only if* h *solves*

$$0 = h'' + \frac{\cos\theta}{\sin\theta} h' - \frac{\sin 2h}{2\sin^2\theta} - \frac{\kappa}{2} \sin(2h - 2\theta) \, . \tag{2.18}$$

on $(0, \pi)$.

Proof. We first compute all the individual terms appearing in the Euler–Lagrange equation when plugging in the axisymmetric ansatz:

$$\Delta \boldsymbol{m} = \left(h'' + \frac{\cos\theta}{\sin\theta} h' \right) \begin{pmatrix} \cos\varphi \cos h \\ \sin\varphi \cos h \\ -\sin h \end{pmatrix} - h'^2 \begin{pmatrix} \cos\varphi \sin h \\ \sin\varphi \sin h \\ \cos h \end{pmatrix} - \frac{1}{\sin^2\theta} \begin{pmatrix} \cos\varphi \sin h \\ \sin\varphi \sin h \\ 0 \end{pmatrix}$$

$$\begin{aligned} |\nabla \boldsymbol{m}|^2 \boldsymbol{m} &= \left(h'^2 + \frac{\sin^2 h}{\sin^2\theta} \right) \begin{pmatrix} \cos\varphi \sin h \\ \sin\varphi \sin h \\ \cos h \end{pmatrix} \\ &= h'^2 \begin{pmatrix} \cos\varphi \sin h \\ \sin\varphi \sin h \\ \cos h \end{pmatrix} - \frac{\sin h \cos h}{\sin^2\theta} \begin{pmatrix} \cos\varphi \cos h \\ \sin\varphi \cos h \\ -\sin h \end{pmatrix} + \frac{1}{\sin^2\theta} \begin{pmatrix} \cos\varphi \sin h \\ \sin\varphi \sin h \\ 0 \end{pmatrix}, \end{aligned}$$

$$\langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \Pi_{\boldsymbol{m}} \boldsymbol{\nu} = \cos(h-\theta) \begin{pmatrix} \cos\varphi(\sin\theta - \cos(h-\theta)\sin h) \\ \sin\varphi(\sin\theta - \cos(h-\theta)\sin h) \\ \cos\theta - \cos(h-\theta)\cos h \end{pmatrix}.$$

We can write

$$\begin{aligned} &\cos(h-\theta)(\sin\theta - \cos(h-\theta)\sin h) \\ &= \cos(h-\theta)(\sin\theta - (\sin h\sin\theta + \cos h\cos\theta)\sin h) \\ &= \cos(h-\theta)\Big(\sin\theta(1-\sin^2 h) - \cos h\cos\theta\cos h\Big) \\ &= \cos(h-\theta)\sin(h-\theta)\cos h \\ &= \frac{1}{2}\sin(2h-2\theta)\cos h\,. \end{aligned}$$

Analogously, we find

$$\cos(h-\theta)(\cos\theta - \cos(h-\theta)\cos h) = -\frac{1}{2}\sin(2h-2\theta)\sin h$$

and thus

$$\langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \Pi_{\boldsymbol{m}} \boldsymbol{\nu} = \frac{1}{2}\sin(2h-2\theta) \begin{pmatrix} \cos\varphi\cos h \\ \sin\varphi\cos h \\ -\sin h \end{pmatrix}.$$

Inserting all these into the Euler–Lagrange equation we find

$$\begin{aligned} &\Delta \boldsymbol{m} + |\nabla \boldsymbol{m}|^2 \boldsymbol{m} - \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \Pi_{\boldsymbol{m}} \boldsymbol{\nu} \\ &= \left(h'' + \frac{\cos\theta}{\sin\theta} h' - \frac{\sin h\cos h}{\sin^2\theta} - \frac{\kappa}{2}\sin(2h-2\theta) \right) \begin{pmatrix} \cos\varphi\cos h \\ \sin\varphi\cos h \\ -\sin h \end{pmatrix}. \end{aligned}$$

Therefore, the left-hand side is zero for all $(\theta, \varphi) \in (0, \pi) \times [0, 2\pi)$ if and only if the coefficient of the vector on the right-hand side is zero, which is equivalent to (2.18). Hence, if $\boldsymbol{m}$ solves (EL) on $\mathbb{S}^2$, then h solves (2.18) on $(0, \pi)$. Conversely, if h solves (2.18) on $(0, \pi)$, then $\boldsymbol{m}$ solves (EL) on $\mathbb{S}^2 \setminus \{\pm \hat{e}_3\}$ and by Definition 2.7 on the whole sphere. □

Corollary 2.18. *Let $h \in \mathcal{PF}^1$ be a solution to* (2.18). *Then $h \in \mathcal{F}^\infty$.*

Proof. We define the map $\boldsymbol{m}$ as in (2.13) and conclude by means of Definition 2.15 that $\boldsymbol{m} \in H^1(\mathbb{S}^2, \mathbb{S}^2)$, and with Definition 2.17 we find that $\boldsymbol{m}$ solves (EL) on $\mathbb{S}^2$. Hence, by Definition 2.6 we have $\boldsymbol{m} \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ and thus again by Definition 2.13 we find $h \in \mathcal{F}^\infty$. □

This result shows the advantage of the axisymmetric ansatz: we can reduce the problem of finding solutions to the Euler–Lagrange equation (EL) on a two-dimensional manifold to finding solutions to a one-dimensional, albeit still nonlinear, differential equation (2.18) on an interval.
Interestingly, the equation (2.18) is in fact the Euler–Lagrange equation of the reduced energy functional. Computing the first variation of the reduced energy functional E at h for a function $g \in C^1([0,\pi])$ with $g(0) = g(\pi) = 0$, we find

$$\begin{aligned}\delta E[h](g) &= \frac{\mathrm{d}}{\mathrm{d}t} E(h+tg)\Big|_{t=0} \\ &= \int_0^\pi h'g'\sin\theta + \frac{\sin h \cos h}{\sin\theta} g + \kappa \sin(h-\theta)\cos(h-\theta) g \sin\theta \,\mathrm{d}\theta \\ &= \int_0^\pi \left(-h'' - \frac{\cos\theta}{\sin\theta} h' + \frac{\sin 2h}{2\sin^2\theta} + \frac{\kappa}{2}\sin(2h-2\theta) \right) g \sin\theta \,\mathrm{d}\theta \,.\end{aligned}$$

Consequently, we identify in (2.18) the Euler–Lagrange equation for the reduced energy functional E. Thus, we will occasionally refer to (2.18) as exactly that.
As $\boldsymbol{\nu}$ is a solution to (EL) for any value of κ, its profile $h(\theta) = \theta$ is a solution to (2.18), which can easily be verified. For a special value of κ, we can give another analytic solution to (2.18).

Proposition 2.19. *Let $\kappa = 4$. Then $h(\theta) = 2\theta$ is a solution to* (2.18)*.*

Proof. We insert the map into the right-hand side of (2.18) with $h'' = 0$ and $h' = 2$ to find

$$\begin{aligned}&h'' + \frac{\cos\theta}{\sin\theta} h' - \frac{\sin 2h}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2h-2\theta) \\ &= 0 + 2\,\frac{\cos\theta}{\sin\theta} - \frac{\sin 4\theta}{2\sin^2\theta} - \frac{\kappa}{2}\sin 2\theta \\ &= 4\,\frac{\cos\theta}{\sin\theta}\left(1-\cos^2\theta\right) - \frac{\kappa}{2}\sin 2\theta \\ &= \left(2 - \frac{\kappa}{2}\right)\sin 2\theta \,,\end{aligned}$$

making use of the trigonometric identities

$$\sin 4\theta = 8\cos^3\theta\sin\theta - 4\cos\theta\sin\theta$$

and

$$2\frac{\cos\theta}{\sin\theta}\left(1-\cos^2\theta\right) = 2\cos\theta\sin\theta = \sin 2\theta \,.$$

Therefore, if $\kappa = 4$, then $h(\theta) = 2\theta$ is a solution to (2.18). □

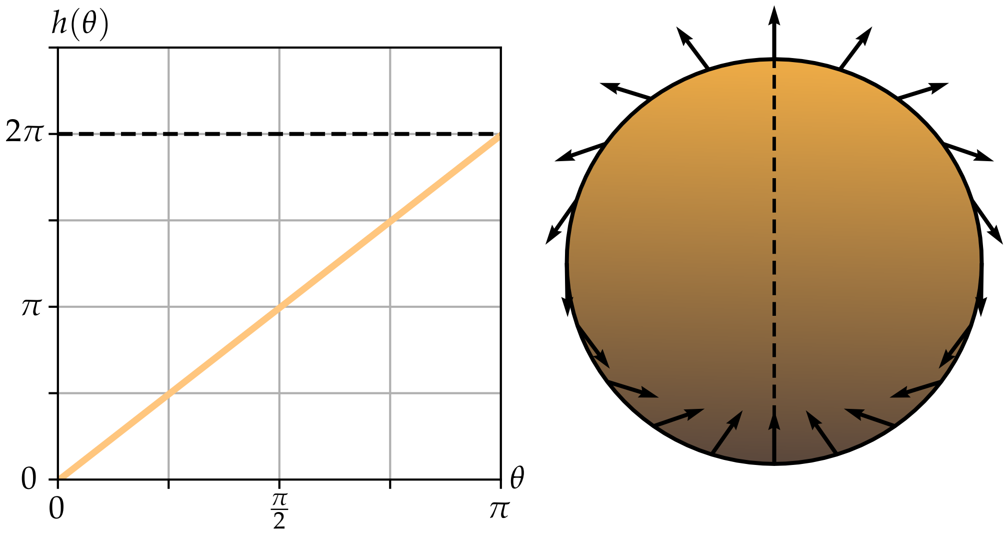

Figure 2.4.: Cross section plot of the solution $h(\theta) = 2\theta$ for $\kappa = 4$.

We also see that for $\kappa < 4$ the function $h(\theta) = 2\theta$ is a sub-solution to (2.18) and for $\kappa > 4$ it is a super-solution. In Chapter 4 we will see that the function $\boldsymbol{m}$ corresponding to this profile is in fact a saddle point of the energy functional $\mathcal{E}$ for $\kappa = 4$.

One can consider the above solution a "lucky coincidence" in which the anisotropy term exactly cancels out the other terms and thus conjecture that there might also exist other combinations of κ and a straight line profile $\theta \mapsto m\theta$ that solves the respective equation (2.18). Since the boundary values have to be multiples of π, m can only be an integer. However, apart for the already dealt with cases $m = 1$, $\kappa > 0$ arbitrary, and $m = 2$, $\kappa = 4$, there do not exist other such combinations as proved in the following proposition.

Proposition 2.20. *Let $m \in \mathbb{Z} \setminus \{1\}$ and $\kappa > 0$. Then the profile h defined by $h(\theta) = m\theta$ is a solution to* (2.18) *if and only if $m = 2$ and $\kappa = 4$.*

Proof. By virtue of the previous lemma, we only need to show the only-if-part. To this end, let $h(\theta) = m\theta$ be a solution of (2.18) for some $\kappa > 0$. We can immediately rule out the case $m = 0$ since the constant state is not a solution for $\kappa > 0$. Thus, we have the cases $m > 1$ and $m < 0$. In the first case, this yields

$$\begin{aligned} 0 &= h'' + \frac{\cos\theta}{\sin\theta}h' - \frac{\sin 2h}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2h - 2\theta) \\ &= m\frac{\cos(\theta)}{\sin(\theta)} - \frac{\sin(2m\theta)}{2\sin^2(\theta)} - \frac{\kappa}{2}\sin((2m-2)\theta) \end{aligned}$$

$$= \frac{1}{2\sin^2\theta}\Big(2m\sin\theta\cos\theta - \sin(2m\theta) - \kappa\sin^2\theta\sin((2m-2)\theta)\Big)\,.$$

We set $n = 2m$ and use the definition of the Chebyshev polynomials of the second kind U_k which read

$$\sin(k\theta) = U_{k-1}(\cos\theta)\sin\theta$$

for $k \in \mathbb{N}$ such that we gain

$$\frac{1}{2\sin\theta}\Big(n\cos\theta - U_{n-1}(\cos\theta) - \kappa\sin^2\theta U_{n-3}(\cos\theta)\Big) = 0\,.$$

Since this holds for all $\theta \in (0, \pi)$, the expression in the parentheses has to vanish. Thus, substituting $x = \cos\theta$, the following must hold for all $x \in (-1, 1)$:

$$nx - U_{n-1}(x) - \kappa(1-x^2)U_{n-3}(x) = 0\,.$$

This is a polynomial equation of degree $n-1$ in x. By the linear independence of the monomials x^k we deduce that the coefficients of the monomials must vanish individually. It holds that the coefficient of x^k in U_k is 2^k and hence we find for the coefficient of x^{n-1} stemming from U_{n-1} and x^2U_{n-3} that the following must hold

$$2^{n-1} - \kappa 2^{n-3} = 0\,.$$

Thus, necessarily $\kappa = 4$. Now we consider the lowest order term in x which is always of first order, since $n-1$ and $n-3$ are both odd. It holds that for odd k the coefficient of x in U_k is $(-1)^{(k-1)/2}(k+1)$. Consequently, it must hold under resubstituting $n = 2m$ that

$$m - (-1)^{m-1}m - \kappa(-1)^m(m-1) = 0\,,$$

and therefore,

$$\kappa = (1 + (-1)^m)(-1)^m\frac{m}{m-1}\,.$$

If m is odd, this implies $\kappa = 0$, contradicting the above. If m is even, then

$$4 = \kappa = \frac{2m}{m-1}\,,$$

which is only solved by $m = 2$, making $m = 2$, $\kappa = 4$ the necessary condition of h being a solution.

If $m < 0$, we can use the same argumentation with $n = -2m$ and find that the equation for $x = \cos\theta$ reduces to

$$-nx + U_{n-1}(x) + \kappa(1 - x^2)U_{n+1}(x) = 0\,,$$

by first pulling the factor -1 out of the sines. The left-hand side is now a polynomial of degree $n + 3$ in x which implies that $\kappa = 0$. This is a contradiction to the assumption that $\kappa > 0$. □

Remark 2.21. We note that maps $\boldsymbol{m}_h$ and $\boldsymbol{m}_g$ for profiles h and $g = h + 2\pi$ are identical by the 2π-periodicity of $\boldsymbol{\Psi}$ in the second argument. Hence, by introducing an equivalence relation $\sim$ by $h \sim g$ if $h - g = 2\pi n$ for $n \in \mathbb{Z}$, we can identify the set of profiles with the set of axisymmetric maps in a one-to-one manner. For the remainder of this thesis, we will implicitly use this equivalence relation. In particular, the observation $h(0) = m\pi$ for $m \in \mathbb{Z}$ reduces to $m \in \{0, 1\}$. The value of $h(\pi)$ can still be any integer multiple of π. ⊡

Remark 2.22. We also note that $h \in \mathcal{F}^\infty$ solves (2.18) if and only if $h + \pi$ solves it by the invariance of (2.18) under this transformation. This corresponds to the transformation $\boldsymbol{m} \mapsto -\boldsymbol{m}$, which is also clearly a symmetry of (EL). Therefore, whenever we find a solution h to (2.18), we immediately obtain a second, distinct solution with the same energy. When proving existence or nonexistence of solutions of (2.18), we can therefore restrict ourselves to one of the cases $h(0) = 0$ or $h(0) = \pi$. ⊡

3. The Skyrmionic Map Heat Flow

This chapter is devoted to the proof of Theorem 3 in all detail. We show that the skyrmionic map heat flow (SMHF) is well-defined for initial data $\boldsymbol{m}_0$ in a suitable Sobolev space and that it has similar properties to that of a gradient flow. Afterward, we analyze its properties in the axisymmetric setting which will give rise to a parabolic differential equation for the profile function.

3.1. Proof of existence of the flow: Theorem 3

In order to find solutions for (EL), we study the *skyrmionic map heat flow*, i.e., for a given function $\boldsymbol{m}_0 \colon \mathbb{S}^2 \to \mathbb{S}^2$ we consider the parabolic problem

$$\begin{aligned} \boldsymbol{m}_t &= \Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu} + \left(|\nabla \boldsymbol{m}|^2 - \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle^2 \right) \boldsymbol{m} \\ \boldsymbol{m}|_{t=0} &= \boldsymbol{m}_0 \,, \end{aligned} \tag{SMHF}$$

where Δ is the Laplace–Beltrami operator on the 2-sphere $\mathbb{S}^2$ and by $\boldsymbol{m}_t$ we denote the time derivative of $\boldsymbol{m}$. To define weak solutions of this equation we use the notion of space-time Sobolev spaces as dealt with in [22, Sec. 5.9] (see the next remark). We use the definition in accordance with [22, Sec. 7.1.1b].

Definition 3.1 (Weak solutions of SMHF)**.** For any $T > 0$, we say a map $\boldsymbol{m} \in L^2([0,T], H^1(\mathbb{S}^2, \mathbb{S}^2))$ satisfying $\boldsymbol{m}_t \in L^2([0,T], H^{-1}(\mathbb{S}^2, \mathbb{R}^3))$ is a weak solution of (SMHF) for initial data $\boldsymbol{m}_0 \in H^1(\mathbb{S}^2, \mathbb{S}^2)$ if for all $\boldsymbol{v} \in H^1 \cap L^\infty(\mathbb{S}^2, \mathbb{R}^3)$ and for almost all $t \in (0,T)$ it holds

$$\langle \boldsymbol{m}_t, \boldsymbol{v} \rangle_{H^{-1},H^1} + \int_{\mathbb{S}^2} \langle \nabla \boldsymbol{m}, \nabla \boldsymbol{v} \rangle - \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \langle \boldsymbol{\nu}, \boldsymbol{v} \rangle - \left(|\nabla \boldsymbol{m}|^2 - \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle^2 \right) \langle \boldsymbol{m}, \boldsymbol{v} \rangle \mathrm{d}\sigma = 0 \tag{3.1}$$

and $\boldsymbol{m}$ admits $\boldsymbol{m}_0$ for $t \to 0$ continuously in $H^1(\mathbb{S}^2, \mathbb{R}^3)$. Representing $\boldsymbol{m}_t$ as a function in $L^2(\mathbb{S}^2, \mathbb{R}^3) \hookrightarrow H^{-1}(\mathbb{S}^2, \mathbb{R}^3)$, the pairing $\langle \boldsymbol{m}_t, \boldsymbol{v} \rangle_{H^{-1},H^1}$ becomes the L^2-scalar product of the two functions. ⊡

Remark 3.2 (Notation of space-time Sobolev spaces (Bochner spaces))**.** If the space-time domains are clear—usually for the space coordinate x we have $\mathbb{S}^2$ and for the time coordinate t we have $[0,T]$ for some $T > 0$—we use the notation $L^p_t H^k_x$ for the Bochner space $L^p([0,T], H^k(\mathbb{S}^2, \mathbb{R}^3))$. Similarly, we denote by $C^0_t H^k_x$ the space of functions $t \mapsto \boldsymbol{m}(t) \in H^k(\mathbb{S}^2, \mathbb{R}^3)$ continuous in t, equipped with the uniform norm. By an abuse of notation we denote by $H^1_t L^2_x$ the space of functions $t \mapsto \boldsymbol{m}(t)$ such that

$$\int_0^T \|\boldsymbol{m}(t)\|^2_{L^2(\mathbb{S}^2,\mathbb{S}^2)} + \|\boldsymbol{m}_t(t)\|^2_{L^2(\mathbb{S}^2,\mathbb{R}^3)} \, \mathrm{d}t < \infty \,,$$

where $\boldsymbol{m}_t$ is the weak derivative of $\boldsymbol{m}$ with respect to the time variable [22, cf. 5.9.2]. Finally, by $H^k_{(x,t)}$ we denote the regular Sobolev space of functions $H^k(\mathbb{S}^2 \times [0,T], \mathbb{R}^3)$ on the Riemannian manifold $\mathbb{S}^2 \times [0,T]$. ⊡

We would like to stress that we do not expect weak solutions of (SMHF) to be unique. This is a well-studied feature of the harmonic map heat flow [13, 67, 6]. We have no reason to believe that the additional term would prevent this from happening in our case since the uniqueness gets lost at the blowup of the solution during which the anisotropy term is still well-behaved. However, in Theorem 3 we will show uniqueness of weak solutions in a special class, i.e., functions that are nonincreasing in energy along the flow.
In order to prove Theorem 3, we need some auxiliary results. In general we will follow the exposition of [63, Sec. 3.6] with slight adjustments from more recent literature. First, we establish short time existence of solutions for initial data with higher regularity. The proof is analogous to the one in [49].

Lemma 3.3. *For $\boldsymbol{m}_0 \in H^k(\mathbb{S}^2, \mathbb{S}^2)$ with $k \in \mathbb{N}$, $k > 2$ there exists $0 < T \leq \infty$ maximal such that* (SMHF) *admits a unique solution $\boldsymbol{m} \in C([0,T), H^k(\mathbb{S}^2, \mathbb{S}^2)) \cap C^\infty(\mathbb{S}^2 \times (0,T), \mathbb{S}^2)$. Moreover, if $T < \infty$, then*

$$\lim_{t \nearrow T} \|\nabla \boldsymbol{m}(\cdot, t)\|_{C^0} = \infty \,.$$

Proof. First we consider the equation as a partial differential equation in the unconstrained Hilbert space $H^k(\mathbb{S}^2, \mathbb{R}^3)$. Then by Proposition 7.7 in [65, Chapter 15] we deduce existence of a solution $\boldsymbol{m} \in C([0,T), H^k(\mathbb{S}^2, \mathbb{R}^3)) \cap C^\infty(\mathbb{S}^2 \times (0,T), \mathbb{R}^3)$ and the desired characterization of the maximal time T. We need to show that $\boldsymbol{m}$ is $\mathbb{S}^2$ valued. To this end, consider the map

$$\varphi(t) = \int_{\mathbb{S}^2} (1 - |\boldsymbol{m}(t)|^2)^2 \, \mathrm{d}\sigma \,.$$

This map exists for all $t \in (0, T)$ since $\boldsymbol{m}(t)$ is smooth on $\mathbb{S}^2$. Then, by using (SMHF) and integration by parts we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) &= -4\int_{\mathbb{S}^2}(1-|\boldsymbol{m}|^2)\langle \boldsymbol{m}_t, \boldsymbol{m}\rangle \,\mathrm{d}\sigma \\ &= -4\int_{\mathbb{S}^2}(1-|\boldsymbol{m}|^2)\big(\langle \Delta\boldsymbol{m}, \boldsymbol{m}\rangle + |\boldsymbol{m}|^2|\nabla\boldsymbol{m}|^2 + \kappa(1-|\boldsymbol{m}|^2)\langle \boldsymbol{m}, \nu\rangle^2\big)\mathrm{d}\sigma \\ &= 4\int_{\mathbb{S}^2}(1-|\boldsymbol{m}|^2)^2\Big(|\nabla\boldsymbol{m}|^2 - \kappa\langle \boldsymbol{m}, \nu\rangle^2\Big)\mathrm{d}\sigma + \int_{\mathbb{S}^2} g^{\mu\nu}\partial_\mu(1-|\boldsymbol{m}|^2)\langle \boldsymbol{m}, \partial_\nu\boldsymbol{m}\rangle \,\mathrm{d}\sigma \\ &\le C\|\boldsymbol{m}\|_{C^1}^2\varphi(t)\,,\end{aligned}$$

where we have used that

$$\int_{\mathbb{S}^2} g^{\mu\nu}\partial_\mu(1-|\boldsymbol{m}|^2)\langle \boldsymbol{m}, \partial_\nu\boldsymbol{m}\rangle \,\mathrm{d}\sigma = -2\int_{\mathbb{S}^2} g^{\mu\nu}\langle \boldsymbol{m}, \partial_\mu\boldsymbol{m}\rangle\langle \boldsymbol{m}, \partial_\nu\boldsymbol{m}\rangle \,\mathrm{d}\sigma \le 0\,.$$

Consequently, invoking Grönwall's lemma [22, cf. B.2.j] we deduce $\varphi(t) \le e^{-Ct}\varphi(0)$ and since $\boldsymbol{m}_0$ is $\mathbb{S}^2$-valued almost everywhere and $\varphi \ge 0$ we conclude $\varphi(t) \equiv 0$ and hence $\boldsymbol{m}(x,t) \in \mathbb{S}^2$ for almost all $x \in \mathbb{S}^2$ and all times $t \in [0, T)$.
Finally, we want to show uniqueness of the solution. For this purpose we assume that $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are two solutions to the same initial data $\boldsymbol{m}_0$ with maximal existence times T_1 and T_2, respectively. Let $T = \min\{T_1, T_2\}$ and $\boldsymbol{\phi} = \boldsymbol{m}_1 - \boldsymbol{m}_2$. Then $\boldsymbol{\phi}$ solves

$$\begin{aligned}\boldsymbol{\phi}_t =& \Delta\boldsymbol{\phi} + \boldsymbol{\phi}|\nabla\boldsymbol{m}_1|^2 + \langle \nabla\boldsymbol{\phi}, \nabla\boldsymbol{m}_1 + \nabla\boldsymbol{m}_2\rangle \boldsymbol{m}_2 \\ &+ \kappa\langle \boldsymbol{\phi}, \boldsymbol{\nu}\rangle(\boldsymbol{\nu} - \langle \boldsymbol{m}_1 + \boldsymbol{m}_2, \boldsymbol{\nu}\rangle \boldsymbol{m}_2) - \kappa\langle \boldsymbol{m}_1, \boldsymbol{\nu}\rangle^2\boldsymbol{\phi}\,.\end{aligned}$$

Multiplying with $2\boldsymbol{\phi}$ and integrating over $\mathbb{S}^2$ we obtain by using the already established fact $|\boldsymbol{m}_1| = |\boldsymbol{m}_2| = 1$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\phi}\|_{L^2}^2 \\ &\le 2\int_{\mathbb{S}^2} -|\nabla\boldsymbol{\phi}|^2 + |\nabla\boldsymbol{m}_1|^2|\boldsymbol{\phi}|^2 + |\nabla\boldsymbol{m}_1 + \nabla\boldsymbol{m}_2||\boldsymbol{\phi}||\nabla\boldsymbol{\phi}| + C\kappa|\boldsymbol{\phi}|^2 \,\mathrm{d}\sigma \\ &\le \alpha(\nabla\boldsymbol{m}_1, \nabla\boldsymbol{m}_2, \kappa)\|\boldsymbol{\phi}\|_{L^2}^2 - \|\nabla\boldsymbol{\phi}\|_{L^2}^2 \\ &\le \alpha(\nabla\boldsymbol{m}_1, \nabla\boldsymbol{m}_2, \kappa)\|\boldsymbol{\phi}\|_{L^2}^2\,,\end{aligned}$$

where we applied Young's inequality to absorb a positive term of $\|\nabla\boldsymbol{\phi}\|_{L^2}$ stemming from the third term in the integral in the second line. The function α depends on $\|\nabla\boldsymbol{m}_1\|_{L^\infty}$ and $\|\nabla\boldsymbol{m}_2\|_{L^\infty}$ which certainly are continuous for all times $0 < t < T$ as $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are smooth. Therefore, α is continuous in t. As a result, again by Grönwall's lemma, we find $\|\boldsymbol{\phi}(t)\|_{L^2}^2 = 0$ for all $t \in [0, T]$ and thus $\boldsymbol{m}_1 = \boldsymbol{m}_2$. □

Next, we show the following energy estimate that justifies why we can interpret the skyrmionic map heat flow as L^2-gradient flow of $\mathcal{E}$.

Lemma 3.4. *For any* $0 < T \leq \infty$, *if* $\boldsymbol{m} \in C^\infty(\mathbb{S}^2 \times (0, T), \mathbb{S}^2)$ *solves* (SMHF), *then*

$$\int_0^t \int_{\mathbb{S}^2} |\boldsymbol{m}_t|^2 \mathrm{d}\sigma \mathrm{d}t + \mathcal{E}(\boldsymbol{m}(t)) = \mathcal{E}(\boldsymbol{m}_0), \ \textit{for all } t \in [0, T) \,. \tag{3.2}$$

In particular, the energy is nonincreasing along the flow.

Proof. As $\boldsymbol{m} \perp T_{\boldsymbol{m}}\mathbb{S}^2$, when multiplying (SMHF) by $\boldsymbol{m}_t \in T_{\boldsymbol{m}}\mathbb{S}^2$ and integrating over $\mathbb{S}^2$, we obtain

$$\int_{\mathbb{S}^2} |\boldsymbol{m}_t|^2 d\sigma + \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}(\boldsymbol{m}(t))) = 0. \,,$$

since

$$\langle \boldsymbol{m}_t, \Delta \boldsymbol{m} + \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu} \rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(|\nabla \boldsymbol{m}|^2 + \kappa(1 - \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle^2) \Big) \,.$$

Integration over $[0, t]$ leads to (3.2). □

For $0 < R < \mathrm{inj}(\mathbb{S}^2)$ and $x_0 \in \mathbb{S}^2$ we define the local energy of geodesic balls as

$$\mathcal{E}(\boldsymbol{m}, B_R(x_0)) = \int_{B_R(x_0)} e(\boldsymbol{m}) \mathrm{d}\sigma \,.$$

With this definition we get the following local energy estimate which is the analog of Lemma 6.9 in [63]. It gives a bound on how much the energy can grow locally in a given amount of time and will be crucial for preserving local energy smallness for small time intervals.

Lemma 3.5. *Let* $\boldsymbol{m} \in C^\infty(\mathbb{S}^2 \times (0, T], \mathbb{S}^2)$ *solve* (SMHF). *For* $x_0 \in \mathbb{S}^2$ *and* $R < \frac{1}{2}\mathrm{inj}(\mathbb{S}^2)$ *it holds*

$$\mathcal{E}(\boldsymbol{m}(T), B_R(x_0)) \leq \mathcal{E}(\boldsymbol{m}_0, B_{2R}(x_0)) + C\frac{T}{R^2}\mathcal{E}(\boldsymbol{m}_0)$$

for some $C > 0$ *independent of* $\boldsymbol{m}$.

Proof. The proof is exactly the same as in [63]. Just note that for ϕ as in that proof we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\kappa}{2}\Big(1 - \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle^2\Big)\phi^2 = -\Big\langle \kappa \langle \boldsymbol{m}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}, \boldsymbol{m}_t \phi^2 \Big\rangle \,,$$

such that when multiplying (SMHF) by $\boldsymbol{m}_t\phi^2$ and identifying $\frac{\mathrm{d}}{\mathrm{d}t}e(\boldsymbol{m})$, we do not gain any additional terms by the anisotropy term. □

By adjusting the test function in the proof, we can also obtain the more general estimate

Corollary 3.6. *Let $\boldsymbol{m} \in C^\infty(\mathbb{S}^2 \times (0,T], \mathbb{S}^2)$ solve* (SMHF)*. For $x_0 \in \mathbb{S}^2$, $r < R < \operatorname{inj}(\mathbb{S}^2)$, and $0 \le t_1 < t_2 \le T$ we have*

$$\mathcal{E}(\boldsymbol{m}(t_2), B_r(x_0)) \le \mathcal{E}(\boldsymbol{m}(t_1), B_R(x_0)) + C\frac{t_2 - t_1}{(R-r)^2}\,\mathcal{E}(\boldsymbol{m}_0)\,.$$

Now we show a lemma that is the analog of Lemma 6.10 in [63]. It in fact shows that as long as the energy is locally small enough that the we gain higher regularity of the solution with a uniform bound.

Lemma 3.7. *Let $x_0 \in \mathbb{S}^2$ and $0 < R < \operatorname{inj}(\mathbb{S}^2)$ and set $B_r := B_r(x_0)$. There exists $E_V > 0$ with the following property. If $\boldsymbol{m} \in C^2\big(B_{2R} \times [0,T), \mathbb{S}^2\big)$ solves* (SMHF) *on $B_{2R} \times [0,T)$ for some $R \in (0, \frac{1}{2}\operatorname{inj}(\mathbb{S}^2))$, and if*

$$\sup_{0 \le t \le T} \mathcal{E}(\boldsymbol{m}(t), B_{2R}) < E_V\,,$$

then we have for some constant $C_\kappa = C(\kappa) > 0$

$$\int_0^T \int_{B_R} |\nabla^2 \boldsymbol{m}|^2 \mathrm{d}\sigma \mathrm{d}t \le C\mathcal{E}(\boldsymbol{m}_0)\left(1 + \frac{T}{R^2} + C_\kappa R^2 T\right).$$

Proof. We only state where we need to adjust the proof of Lemma 6.10 in [63] because of the additional terms. When multiplying (SMHF) with $\Delta\boldsymbol{m}\phi^2$ and integrating, we get the following terms in addition to the ones in the cited proof

$$I_1 = \kappa \int_0^T \int_{B_{2R}} \Big(\langle \boldsymbol{m}, \boldsymbol{\nu}\rangle\langle \boldsymbol{\nu}, \Delta\boldsymbol{m}\rangle - \langle \boldsymbol{m}, \boldsymbol{\nu}\rangle^2 \langle \boldsymbol{m}, \Delta\boldsymbol{m}\rangle\Big)\phi^2 \mathrm{d}\sigma \mathrm{d}t\,.$$

Additionally, we now have $\langle \nabla\boldsymbol{m}, \boldsymbol{m}_t\rangle = \langle \nabla\boldsymbol{m}, \Delta\boldsymbol{m}\rangle + \kappa\langle \boldsymbol{m}, \boldsymbol{\nu}\rangle\langle \boldsymbol{\nu}, \nabla\boldsymbol{m}\rangle$, which creates another term

$$I_2 = -2\kappa \int_0^T \int_{B_{2R}} \langle \boldsymbol{m}, \boldsymbol{\nu}\rangle\langle \boldsymbol{\nu}, \nabla\boldsymbol{m}\rangle \nabla\phi\phi\,.$$

Both terms we can estimate with Young's inequality by

$$|I_1 + I_2| \le \delta \int_0^T \int_{B_{2R}} |\Delta\boldsymbol{m}|^2\phi^2 + |\nabla\boldsymbol{m}|^2|\nabla\phi|^2 \mathrm{d}\sigma\mathrm{d}t + C\frac{\kappa^2}{\delta}\int_0^T \int_{B_{2R}} \phi^2 \mathrm{d}\sigma\mathrm{d}t\,,$$

where we now can choose $\delta > 0$ small enough such that the first integral can be absorbed into the other ones during the proof (which has an insignificant effect on the constants). Therefore, only the additional term $C_\kappa R^2 T$ remains and has to be added to the final estimate. $\square$

As explained in [47, after Lemma 6.2.5], by compactness of $\mathbb{S}^2$, we find for $\boldsymbol{m}_0 \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ a maximal $R_1 > 0$ such that

$$\sup_{x \in \mathbb{S}^2} \mathcal{E}(\boldsymbol{m}_0, B_{2R_1}(x)) \leq \frac{E_V}{2},$$

and then by virtue of Definition 3.5, setting

$$T_1 = \frac{E_V R_1^2}{C\mathcal{E}(\boldsymbol{m}_0)},$$

we obtain that the solution $\boldsymbol{m}$ to (SMHF) satisfies

$$\sup_{(x,t) \in \mathbb{S}^2 \times [0,T_1]} \mathcal{E}(\boldsymbol{m}, B_{R_1}(x)) < E_V. \tag{3.3}$$

Therefore, combining Definition 3.4 and Definition 3.7 we obtain by the fact that we can cover $\mathbb{S}^2$ with finitely many balls of radius R_1 the bound

$$\|\boldsymbol{m}\|^2_{V^{T_1}} \leq C\mathcal{E}(\boldsymbol{m}_0)\left(1 + \frac{T_1}{R_1^2} + C_\kappa R_1^2 T_1\right), \tag{3.4}$$

where the norm is defined as

$$\|\boldsymbol{m}\|^2_{V^{T_1}} = \sup_{t \in [0,T_1]} \mathcal{E}(\boldsymbol{m}(t)) + \int_0^{T_1} \int_{\mathbb{S}^2} |\nabla^2 \boldsymbol{m}|^2 + |\boldsymbol{m}_t|^2 \, \mathrm{d}\sigma \mathrm{d}t.$$

From the definition it follows immediately that if $\|\boldsymbol{m}\|_{V^{T_0}} < \infty$ then

$$\boldsymbol{m} \in L^2([0,T_0], H^2(\mathbb{S}^2, \mathbb{R}^3)) \cap L^\infty([0,T_0], H^1(\mathbb{S}^2, \mathbb{S}^2))$$

and

$$\boldsymbol{m}_t \in L^2([0,T_0], H^{-1}(\mathbb{S}^2, \mathbb{R}^3)).$$

Next, we prove local higher order energy estimates for solutions of the SMHF in the style of [16, Lemma 4] and [49, Lemma 1].

Lemma 3.8. *Let $x_0 \in \mathbb{S}^2$ and $0 < R < \operatorname{inj}(\mathbb{S}^2)$ and set $B_r := B_r(x_0)$. Suppose $k \in \mathbb{N}$ and $\boldsymbol{m} \in C([0,T_0), H^k(\mathbb{S}^2, \mathbb{S}^2)) \cap C^\infty(\mathbb{S}^2 \times (0,T_0), \mathbb{S}^2)$ is a solution of* (SMHF). *Then* *for all* $0 < T < T_0$ *it holds*

$$
\begin{aligned}
&\|\nabla \boldsymbol{m}(T)\|^2_{H^{k-1}(B_R)} + \int_{\frac{3}{4}T}^{T} \|\nabla \boldsymbol{m}(t)\|^2_{H^k(B_R)} \mathrm{d}t \\
\leq\ & C\left(\frac{1}{R^2} + \frac{1}{T} + \mathcal{P}\Big(\|\nabla \boldsymbol{m}\|_{L^\infty_{(x,t)}(B_{2R}\times[0,T])}\Big) + \kappa\right) \int_0^T \|\nabla \boldsymbol{m}(t)\|^2_{H^{k-1}(B_{2R})} \mathrm{d}t + C\kappa T R^2 ,
\end{aligned}
$$

where $\mathcal{P}$ *is a polynomial of degree 4, and* $C > 0$ *is some constant, which is independent of* $\boldsymbol{m}$ *but depends on* $\mathbb{S}^2$ *and* k*.*

Proof. We work with the total covariant derivative. The idea is to apply the k-th covariant derivative to (SMHF), which transforms the vector valued equation into an equation of elements in $(\mathfrak{T}^{(0,k)}T\mathbb{S}^2)^3$ (cf. Section A.4.3) and then take the tensor inner product with $\Phi^2\nabla^k \boldsymbol{m}$, where $\Phi(x,t) = \phi(x)\eta(t)$ is a cut-off function with $\phi \in C_0^\infty(B_{2R}, [0,1])$, $\phi|_{B_R} \equiv 1$, and $|\nabla\phi| < C/R$ and $\eta \in C^\infty([0,T],[0,1])$ with $\eta|_{[0,\frac{T}{2}]} \equiv 0$, $\eta|_{[\frac{3T}{4},T]} \equiv 1$, and $|\partial_t \eta| < C/T$. Then we integrate over $B_{2R} \times [0,T]$. We state the equation that arises from this procedure and estimate the individual terms afterward:

$$
\begin{aligned}
&\int_0^T \int_{B_{2R}} \Phi^2 \Big\langle \nabla^k \boldsymbol{m}_t, \nabla^k \boldsymbol{m} \Big\rangle \mathrm{d}\sigma \mathrm{d}t \\
=& \int_0^T \int_{B_{2R}} \Phi^2 \Big\langle \nabla^k \Delta \boldsymbol{m}, \nabla^k \boldsymbol{m} \Big\rangle \mathrm{d}\sigma \mathrm{d}t + \int_0^T \int_{B_{2R}} \Phi^2 \Big\langle \nabla^k (|\nabla \boldsymbol{m}|^2 \boldsymbol{m}), \nabla^k \boldsymbol{m} \Big\rangle \mathrm{d}\sigma \mathrm{d}t \\
&+ \kappa \int_0^T \int_{B_{2R}} \Phi^2 \Big\langle \nabla^k (\langle \boldsymbol{m}, \boldsymbol{\nu}\rangle \boldsymbol{\nu}) - \nabla^k (\langle \boldsymbol{m}, \boldsymbol{\nu}\rangle^2 \boldsymbol{m}), \nabla^k \boldsymbol{m} \Big\rangle \mathrm{d}\sigma \mathrm{d}t
\end{aligned}
\tag{3.5}
$$

We label the integrals with I_1 for the one on the left-hand side, and I_2, I_3 and I_4 for the three integrals on the right hand side, in appearing order. For I_1 we use that $\boldsymbol{m}$ is smooth and hence we can compute by virtue of Fubini's theorem

$$
\begin{aligned}
I_1 &= \int_0^T \int_{B_{2R}} \Phi^2 \Big\langle \nabla^k \boldsymbol{m}_t, \nabla^k \boldsymbol{m} \Big\rangle \,\mathrm{d}\sigma \mathrm{d}t \\
&= \int_0^T \int_{B_{2R}} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^2 |\nabla^k \boldsymbol{m}|^2) - \phi^2 \eta \eta_t |\nabla^k \boldsymbol{m}|^2 \mathrm{d}\sigma \mathrm{d}t \\
&= \frac{1}{2}\Big(\| \underbrace{\Phi(T)}_{=\phi^2} \nabla^k \boldsymbol{m}(T)\|^2_{L^2(B_{2R})} - \| \underbrace{\Phi(0)}_{=0} \nabla^k \boldsymbol{m}(0)\|^2_{L^2(B_{2R})} \Big) \\
&\quad - \int_0^T \int_{B_{2R}} \phi^2 \eta \eta_t |\nabla^k \boldsymbol{m}|^2 \mathrm{d}\sigma \mathrm{d}t \\
&= \frac{1}{2} \|\phi^2 \nabla^k \boldsymbol{m}(T)\|^2_{L^2(B_{2R})} - \int_0^T \int_{B_{2R}} \phi^2 \eta \eta_t |\nabla^k \boldsymbol{m}|^2 \mathrm{d}\sigma \mathrm{d}t \,.
\end{aligned}
$$

The final integral can be estimated by

$$\Big|\int_0^T\int_{B_{2R}}\phi^2\eta\eta_t|\nabla^k\boldsymbol{m}|^2\mathrm{d}\sigma\mathrm{d}t\Big|\le\frac{C}{T}\int_0^T\|\nabla^k\boldsymbol{m}\|^2_{L^2(B_{2R})}\,\mathrm{d}t\,.$$

Next, we turn our attention to I_2. This integral carries the highest order derivatives. We commute the Laplacian with k-th order covariant derivative using (A.10) and then integrate by parts. Note here that B_{2R} is a compact manifold with boundary but because of the cutoff function we do not gain a boundary term. Applying the $*$-notation, we obtain

$$\begin{aligned}
I_2 &= \int_0^T\int_{B_{2R}}\Phi^2\Big\langle\nabla^k\Delta\boldsymbol{m},\nabla^k\boldsymbol{m}\Big\rangle\mathrm{d}\sigma\mathrm{d}t\\
&= \int_0^T\int_{B_{2R}}\Phi^2\Big\langle\Delta\nabla^k\boldsymbol{m},\nabla^k\boldsymbol{m}\Big\rangle\,\mathrm{d}\sigma\mathrm{d}t+\int_0^T\int_{B_{2R}}\Phi^2\nabla^k(\mathrm{Rm}*\boldsymbol{m})*\nabla^k\boldsymbol{m}\,\mathrm{d}\sigma\mathrm{d}t\\
&= -\int_0^T\int_{B_{2R}}\Phi^2|\nabla^{k+1}\boldsymbol{m}|^2\mathrm{d}\sigma\mathrm{d}t+2\int_0^T\int_{B_{2R}}\phi\eta^2\nabla^{k+1}\boldsymbol{m}*\nabla\phi*\nabla^k\boldsymbol{m}\,\mathrm{d}\sigma\mathrm{d}t\\
&\quad+\int_0^T\int_{B_{2R}}\Phi^2\sum_{j=0}^k\nabla^{k-j}\mathrm{Rm}*\nabla^j\boldsymbol{m}*\nabla^k\boldsymbol{m}\,\mathrm{d}\sigma\mathrm{d}t\,,
\end{aligned}$$

where the second term stems from when the derivative acts on Φ^2 during the integration by parts. In the third term we applied (A.6). Now, since $\mathbb{S}^2$ is smooth and compact, all derivatives $\nabla^{k-j}\mathrm{Rm}$ are bounded and by Young's inequality we gain for the second and third integral the bounds

$$\begin{aligned}
&2\Big|\int_0^T\int_{B_{2R}}\phi\eta^2\nabla^{k+1}\boldsymbol{m}*\nabla\phi*\nabla^k\boldsymbol{m}\,\mathrm{d}\sigma\mathrm{d}t\Big|\\
&\le 2\int_0^T\int_{B_{2R}}\phi\eta^2|\nabla^{k+1}\boldsymbol{m}||\nabla\phi||\nabla^k\boldsymbol{m}|\,\mathrm{d}\sigma\mathrm{d}t\\
&\le\frac{1}{2}\int_0^T\int_{B_{2R}}\Phi^2|\nabla^{k+1}\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t+\frac{C}{R^2}\int_0^T\int_{B_{2R}}|\nabla^k\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_0^T\int_{B_{2R}}\Phi^2\sum_{j=0}^k\nabla^{k-j}\mathrm{Rm}*\nabla^j\boldsymbol{m}*\nabla^k\boldsymbol{m}\,\mathrm{d}\sigma\mathrm{d}t\Big|\\
&\le C\sum_{j=0}^k\int_0^T\int_{B_{2R}}|\nabla^j\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t+\int_0^T\int_{B_{2R}}|\nabla^k\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}&\le C\sum_{j=0}^{k-1}\int_0^T\int_{B_{2R}}|\nabla^{j+1}\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t + C\int_0^T\int_{B_{2R}}|\nabla^0\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t\\ &\le C\int_0^T\|\nabla\boldsymbol{m}\|^2_{H^{k-1}(B_{2R})}\mathrm{d}t + CTR^2\end{aligned}$$

Absorbing now the positive term with the $(k+1)$-th derivative, we arrive at the estimate

$$I_2 \le -\frac{1}{2}\int_0^T\int_{B_{2R}}\Phi^2|\nabla^{k+1}\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t + C\Big(1+\frac{1}{R^2}\Big)\int_0^T\|\nabla\boldsymbol{m}\|^2_{H^{k-1}(B_{2R})}\mathrm{d}t + CTR^2\,.$$

For the term I_3 we first integrate once by parts to obtain

$$\begin{aligned}|I_3| &= \Big|\int_0^T\int_{B_{2R}}\Big\langle\nabla^{k-1}(|\nabla\boldsymbol{m}|^2\boldsymbol{m}),\mathrm{div}(\Phi^2\nabla^k\boldsymbol{m})\Big\rangle\mathrm{d}\sigma\mathrm{d}t\Big|\\ &\le \int_0^T\int_{B_{2R}}|\nabla^{k-1}(|\nabla\boldsymbol{m}|^2\boldsymbol{m})|\,\Phi^2|\nabla^{k+1}\boldsymbol{m}|\mathrm{d}\sigma\mathrm{d}t\\ &\quad + 2\int_0^T\int_{B_{2R}}|\nabla^{k-1}(|\nabla\boldsymbol{m}|^2\boldsymbol{m})||\nabla\Phi||\Phi||\nabla^k\boldsymbol{m}|\mathrm{d}\sigma\mathrm{d}t\\ &\le \int_0^T\|\nabla^{k-1}(|\nabla\boldsymbol{m}|^2\boldsymbol{m})\|_{L^2(B_{2R})}\|\Phi\nabla^{k+1}\boldsymbol{m}\|_{L^2(B_{2R})}\mathrm{d}t\\ &\quad + \frac{C}{R}\int_0^T\|\nabla^{k-1}(|\nabla\boldsymbol{m}|^2\boldsymbol{m})\|_{L^2(B_{2R})}\|\nabla^k\boldsymbol{m}\|_{L^2(B_{2R})}\mathrm{d}t\,,\end{aligned}$$

where we used $|\operatorname{div} F| \le C|\nabla F|$ for a tensor F of any kind.
Now we compute by virtue of (A.7)

$$\nabla^{k-1}(|\nabla\boldsymbol{m}|^2\boldsymbol{m}) = \boldsymbol{m}*\nabla^{k-1}(\nabla\boldsymbol{m}*\nabla\boldsymbol{m}) + \nabla^{k-2}(\nabla\boldsymbol{m}*\nabla\boldsymbol{m}*\nabla\boldsymbol{m})\,,$$

Thus, we can estimate by exploiting Definition A.31 twice

$$\begin{aligned}\|\nabla^{k-1}(|\nabla\boldsymbol{m}|^2\boldsymbol{m})\|_{L^2} &\le C\|\boldsymbol{m}\|_{L^\infty}\|\nabla\boldsymbol{m}\|_{L^\infty}\|\nabla\boldsymbol{m}\|_{H^{k-1}} + \|\nabla\boldsymbol{m}\|^2_{L^\infty}\|\nabla\boldsymbol{m}\|_{H^{k-2}}\\ &\le C\Big(1+\|\nabla\boldsymbol{m}\|^2_{L^\infty}\Big)\|\nabla\boldsymbol{m}\|_{H^{k-1}}\,,\end{aligned}$$

since $\|\nabla\boldsymbol{m}\|_{H^{k-2}} \le \|\nabla\boldsymbol{m}\|_{H^{k-1}}$, where all the norms are on B_{2R}.
Putting these estimates together, we consequently obtain for I_3 by terms of Young's inequality

$$\begin{aligned}|I_3| &\le \frac{1}{4}\int_0^T\|\Phi\nabla^{k+1}\boldsymbol{m}(t)\|^2_{L^2(B_{2R})}\mathrm{d}t\\ &\quad + C\bigg(\frac{1}{R^2} + \mathcal{P}\Big(\|\nabla\boldsymbol{m}(t)\|_{L^\infty_{(x,t)}(B_{2R}\times[0,T])}\Big)\bigg)\int_0^T\|\nabla\boldsymbol{m}(t)\|^2_{H^{k-1}(B_{2R})}\mathrm{d}t\end{aligned}$$

with

$$\mathcal{P}\Big(\|\nabla \boldsymbol{m}(t)\|_{L^\infty_{(x,t)}(B_{2R}\times[0,T])}\Big) = \Big(1 + \|\nabla \boldsymbol{m}(t)\|^2_{L^\infty_{(x,t)}(B_{2R}\times[0,T])}\Big)^2 .$$

Finally, we turn our attention to I_4. First we estimate the L^2-norms of the terms in the scalar product. Those yield again under usage of Definition A.31 and taking the norms on B_{2R}

$$\begin{aligned}
\|\nabla^k(\langle \boldsymbol{m},\boldsymbol{\nu}\rangle\boldsymbol{\nu})\|_{L^2} &= \|\nabla^k(\boldsymbol{m} * \boldsymbol{\nu}^{*2})\|_{L^2} \\
&\leq C\|\boldsymbol{m}\|_{H^k}\|\boldsymbol{\nu}\|^2_{L^\infty} + \|\boldsymbol{m}\|_{L^\infty}\|\boldsymbol{\nu}\|_{L^\infty}\|\boldsymbol{\nu}\|_{H^k} \\
&\leq C\|\nabla \boldsymbol{m}\|_{H^{k-1}} + CR ,
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla^k(\langle \boldsymbol{m},\boldsymbol{\nu}\rangle^2\boldsymbol{m})\|_{L^2} &= \|\nabla^k(\boldsymbol{m}^{*3} * \boldsymbol{\nu}^{*2})\|_{L^2} \\
&\leq C\|\boldsymbol{m}\|^2_{L^\infty}\|\boldsymbol{\nu}\|_{L^\infty}(\|\boldsymbol{m}\|_{H^k}\|\boldsymbol{\nu}\|_{L^\infty} + \|\boldsymbol{m}\|_{L^\infty}\|\boldsymbol{\nu}\|_{H^k}) \\
&\leq C\|\nabla \boldsymbol{m}\|_{H^{k-1}} + CR
\end{aligned}$$

where we used that $\boldsymbol{\nu}$ is a smooth function on $\mathbb{S}^2$ and hence on the relatively compact set B_{2R} we have $\|\boldsymbol{\nu}\|_{H^k(B_{2R})} \leq C|B_{2R}|^{1/2} = CR$. Similarly, we got

$$\|\boldsymbol{m}\|_{H^k(B_{2R})} \leq C(\|\nabla \boldsymbol{m}\|_{H^{k-1}(B_{2R})} + \|\boldsymbol{m}\|_{L^2(B_{2R})}) = C\|\nabla \boldsymbol{m}\|_{H^{k-1}(B_{2R})} + |B_{2R}|^{1/2} ,$$

since $|\boldsymbol{m}| \equiv 1$. Therefore, we arrive at the estimate

$$\begin{aligned}
|I_4| &\leq C\kappa \int_0^T \Big(\|\nabla \boldsymbol{m}\|_{H^{k-1}(B_{2R})} + R\Big)\|\nabla^k \boldsymbol{m}\|_{L^2(B_{2R})}\mathrm{d}t \\
&\leq C\kappa \int_0^T \|\nabla \boldsymbol{m}\|^2_{H^{k-1}(B_{2R})}\mathrm{d}t + C\kappa T R^2 .
\end{aligned}$$

Combining all estimates and inserting them into (3.5), we obtain the inequality

$$\begin{aligned}
&\frac{1}{2}\|\phi^2\nabla^k \boldsymbol{m}(T)\|^2_{L^2(B_{2R})} + \frac{1}{4}\int_0^T \int_{B_{2R}} \Phi^2|\nabla^{k+1}\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t \\
\leq\; & C\left(\frac{1}{R^2} + \frac{1}{T} + \mathcal{P}\left(\sup_{t\in[0,T]} \|\nabla \boldsymbol{m}\|_{L^\infty(B_{2R})}\right)\right)\int_0^T \|\nabla \boldsymbol{m}(t)\|^2_{H^{k-1}(B_{2R})}\mathrm{d}t + C\kappa T R^2 .
\end{aligned}$$

Using the fact that $\phi \equiv 1$ on B_R and $\Phi \equiv 1$ on $B_R \times [3T/4, T]$, we find that four times the left-hand side dominates

$$\|\nabla^k \boldsymbol{m}(T)\|^2_{L^2(B_R)} + \int_{\frac{3}{4}T}^T \|\nabla^{k+1}\boldsymbol{m}(t)\|^2_{L^2(B_R)}\mathrm{d}t .$$

Hence, since this estimate holds for all $k \in \mathbb{N}$, we can now sum over all $0 < k' \leq k$ to obtain the estimate on the respective H^k-norms on the left-hand side which is the statement of this lemma. The right hand sides can always be absorbed into the highest order one by the fact that $\|\nabla \boldsymbol{m}\|_{H^{k'}} \leq \|\nabla \boldsymbol{m}\|_{H^k}$. □

We can adapt the proof in order to show the following more "loose" version of the previous lemma, which is needed when we slightly change the underlying differential equation.

Corollary 3.9. *Let $x_0 \in \mathbb{S}^2$ and $0 < R < \operatorname{inj}(\mathbb{S}^2)$ and set $B_r := B_r(x_0)$. Suppose $k \in \mathbb{N}$ and $\boldsymbol{m} \in C([0,T_0), H^k(B_{2R}, \mathbb{S}^2)) \cap C^\infty(B_{2R} \times (0,T_0), \mathbb{S}^2)$ is a solution of*

$$\boldsymbol{m}_t = \mu\,(\Delta \boldsymbol{m} + |\nabla \boldsymbol{m}|^2 \boldsymbol{m}) + \kappa \boldsymbol{F}(\boldsymbol{m})\,,$$

where $\mu \in C^\infty(\overline{B_{2R}}, \mathbb{R})$ with $\delta < \mu$ for some $\delta > 0$ and $\boldsymbol{F}(\boldsymbol{m})$ is a function of type $F(\boldsymbol{m}) = \boldsymbol{m}^{*l} * \boldsymbol{f}$ for some $l \in \mathbb{N}$, with $\boldsymbol{f}$ a smooth vector valued function on $\overline{B_{2R}}$. Then for all $0 < T < T_0$ it holds

$$\begin{aligned} &\|\nabla \boldsymbol{m}(T)\|^2_{H^{k-1}(B_R)} + \int_{\frac{3}{4}T}^{T} \|\nabla \boldsymbol{m}(t)\|^2_{H^k(B_R)} \mathrm{d}t \\ \leq\; & C\left(\frac{1}{R^2} + \frac{1}{T} + \mathcal{P}\Big(\|\nabla \boldsymbol{m}\|_{L^\infty_{(x,t)}(B_{2R}\times[0,T])}\Big) + \kappa\right) \int_0^T \|\nabla \boldsymbol{m}(t)\|^2_{H^{k-1}(B_{2R})} \mathrm{d}t + C\kappa T R^2\,, \end{aligned}$$

where $\mathcal{P}$ is a polynomial of degree 4, and $C > 0$ is some constant, which is independent of $\boldsymbol{m}$ but depends on $\mathbb{S}^2$, k, μ and $\boldsymbol{f}$.

Proof. We can repeat the proof of Definition 3.8 with the only difference that when estimating the terms where now μ appears, we redefine the spatial cut-off function to be $\tilde{\phi} = \phi\mu$, for which now holds $|\tilde{\phi}| < \|\mu\|_{L^\infty}$ and $\delta < |\tilde{\phi}|$ on B_R, and still $|\nabla\tilde{\phi}| \leq C/R$, since $\nabla\mu$ is bounded. For the term with $\boldsymbol{f}$ which corresponds with I_4 in the proof of Definition 3.8, we get the same result by reasserting in view of Definition A.31 the estimate

$$\begin{aligned} \|\nabla^k(\boldsymbol{m}^{*l} * \boldsymbol{f})\|_{L^2(B_{2R})} &\leq C\|\boldsymbol{m}\|^{l-1}_{L^\infty}\Big(\|\boldsymbol{m}\|_{H^k(B_{2R})}\|\boldsymbol{f}\|_{L^\infty(B_{2R})} + \|\boldsymbol{m}\|_{L^\infty(B_{2R})}\|\boldsymbol{f}\|_{H^k(B_{2R})}\Big) \\ &\leq C\|\boldsymbol{m}\|_{H^k(B_{2R})} + CR\,, \end{aligned}$$

where we used that $\|\boldsymbol{f}\|_{H^k(B_{2R})} \leq C|B_{2R}|^{1/2} = CR$.
Hence, all the estimates go through and just the constants involved change and now depend on μ and $\boldsymbol{f}$. □

Remark 3.10. We gain an equation of the form in the corollary when we do a transformation of variables in the equation (SMHF). To illustrate, we consider a case which we will use later. Let $\boldsymbol{m}$ be a smooth solution of (SMHF). In stereographic coordinates centered at $x_0 \in \mathbb{S}^2$ this means that $\boldsymbol{m}$ solves

$$\boldsymbol{m}_t = \lambda^{-2}(x)(\Delta_{\mathbb{R}^2}\boldsymbol{m} + |D\boldsymbol{m}|^2\boldsymbol{m}) + \kappa\langle \boldsymbol{m}, \boldsymbol{\nu}\rangle \Pi_{\boldsymbol{m}}\boldsymbol{\nu}$$

on $\mathbb{R}^2$, where $\Delta_{\mathbb{R}^2}$ denotes the standard Laplacian on $\mathbb{R}^2$ and $D\boldsymbol{m}$ is the classic Jacobian (see also Definition 2.3).
Now for $a > 0$ consider the rescaled function

$$\tilde{\boldsymbol{m}}(x,t) = \boldsymbol{m}(ax, a^2 t)\,.$$

Then by a quick computation we obtain that $\tilde{\boldsymbol{m}}$ solves

$$\tilde{\boldsymbol{m}}_t = \mu(\Delta\tilde{\boldsymbol{m}} + |\nabla\tilde{\boldsymbol{m}}|^2\tilde{\boldsymbol{m}}) + \kappa a^2\langle \tilde{\boldsymbol{m}}, \tilde{\boldsymbol{\nu}}\rangle \Pi_{\tilde{\boldsymbol{m}}}\tilde{\boldsymbol{\nu}}\,,$$

on $\mathbb{S}^2 \setminus \{-x_0\}$, where $\mu(x) = \lambda^2(x)/\lambda^2(ax)$ and $\tilde{\boldsymbol{\nu}}(x) = \boldsymbol{\nu}(ax)$ in stereographic coordinates. The functions μ and $\boldsymbol{F}(\boldsymbol{m}) = a^2\langle \tilde{\boldsymbol{m}}, \tilde{\boldsymbol{\nu}}\rangle \Pi_{\tilde{\boldsymbol{m}}}\tilde{\boldsymbol{\nu}}$ then satisfy the conditions of the corollary. ⊡

Remark 3.11. If $\boldsymbol{m}$ is smooth up to T_0, i.e., $\boldsymbol{m} \in C^\infty(\mathbb{S}^2 \times (0, T_0])$, then of course both previous inequalities also hold for $T = T_0$. ⊡

The following result is the crucial estimate to show that if energy does not concentrate at a point then the solution has bounded gradient. Later we then infer from this that the solution smoothly extends to the terminal time in a neighborhood of that point. The lemma is an analog of Proposition 6.2.6 in [47] or Lemma 7 in [16]. The idea of the proof was first introduced in [58] and is a standard technique for harmonic maps. For $(x_0, t_0) \in \mathbb{S}^2 \times (0, \infty)$ and $0 < r < \min\{\mathrm{inj}(\mathbb{S}^2), \sqrt{t_0}\}$ we define the parabolic cylinder

$$\begin{aligned} P_R((x_0,t_0)) &= \Big\{(x,t) \in \mathbb{S}^2 \times (0,\infty) : d(x,x_0) \le r, t_0 - r^2 \le t \le t_0\Big\} \\ &= \overline{B_r(x_0)} \times [t_0 - r^2, t_0]\,, \end{aligned}$$

where by $d\colon \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ we denote the geodesic distance on $\mathbb{S}^2$.

Lemma 3.12. *There exist $E_{\mathrm{s}}, R_{\mathrm{s}} > 0$ depending only κ, such that if for $r < R_{\mathrm{s}}$ the function $\boldsymbol{m} \in C^\infty(P_r((x_0,t_0)))$ solves* (SMHF) *and satisfies*

$$\sup_{t\in[t_0-r^2,t_0]} \int_{B_r(x_0)} |\nabla \boldsymbol{m}(t)|^2 \,\mathrm{d}\sigma \le E_{\mathrm{s}}\,,$$

then

$$\sup_{(x,t)\in P_{\frac{r}{2}}((x_0,t_0))} |\nabla \boldsymbol{m}(x,t)| \leq \frac{2}{r}\,.$$

Proof. Since the equation is invariant with respect to shifts in time, we can assume without loss of generality that $t_0 = 0$ and write $P_r := P_r((x_0, 0))$. For now we assume $0 < r < R_s$ fixed and decrease the upper bound if needed. There exists $\rho \in [0, r)$ such that

$$(r-\rho)^2 \sup_{P_\rho} |\nabla \boldsymbol{m}|^2 = \max_{\sigma\in[0,r]} (r-\sigma)^2 \sup_{P_\sigma} |\nabla \boldsymbol{m}|^2\,.$$

This is due to the fact that P_σ is a compact set on which $|\nabla \boldsymbol{m}|$ is continuous and thus the function $\sigma \mapsto \sup_{P_\sigma} |\nabla \boldsymbol{m}|^2$ is continuous on $[0, r]$. The value $\sigma = r$ can be excluded since the non-negative function vanishes there. As said, we therefore find a point $(\tilde{x}, \tilde{t}) \in P_\rho$ such that

$$|\nabla \boldsymbol{m}(\tilde{x}, \tilde{t})| = \sup_{P_\rho} |\nabla \boldsymbol{m}| =: s_0\,.$$

Now two cases can occur: either

$$(r-\rho)^2 s_0^2 \leq 1 \;\text{ or }\; (r-\rho)^2 s_0^2 > 1\,.$$

In the first case we easily conclude by the definitions of ρ and s_0

$$\sup_{P_{r/2}} |\nabla \boldsymbol{m}|^2 \leq \frac{4}{r^2}(r-\rho)^2 s_0^2 \leq \frac{4}{r^2}\,,$$

from which the claim follows. In the second case we first see that we then have

$$s_0^{-2} < (r-\rho)^2\,. \tag{3.6}$$

We now use stereographic coordinates centered at $\tilde{x}$ and define the rescaled function $\tilde{\boldsymbol{m}}$

$$\tilde{\boldsymbol{m}}(x,t) = \boldsymbol{m}\Big(s_0^{-1}x, \tilde{t} + s_0^{-2}t\Big)\,.$$

By the triangle inequality and (3.6) we find

$$P_{1/s_0}((\tilde{x},\tilde{t})) \subset P_{r-\rho}((\tilde{x},\tilde{t})) \subset P_r((x_0,0))\,. \tag{3.7}$$

We want to use this observation to find a parabolic cylinder on which $\tilde{\boldsymbol{m}}$ is defined. To this end, let $(x,t) \in P_{1/2}((\tilde{x},0))$. Then, for the norm of the coordinate representation in $\mathbb{R}^2$ we find

$$|\boldsymbol{\Phi}_{\tilde{x}}^{-1}(x)| = \tan\left(\frac{d(x,\tilde{x})}{2}\right) \leq \tan\left(\frac{1}{4}\right) < \frac{1}{2}\,.$$

And hence

$$|s_0^{-1}\boldsymbol{\Phi}_{\tilde{x}}^{-1}(x)| < \frac{s_0^{-1}}{2} \leq \tan\left(\frac{s_0^{-1}}{2}\right).$$

We conclude that $(\boldsymbol{\Phi}_{\tilde{x}}(s_0^{-1}\boldsymbol{\Phi}_{\tilde{x}}^{-1}(x)), \tilde{t} + s_0^{-2}t) \in P_{1/s_0}((\tilde{x},\tilde{t}))$ and hence with (3.7) that $\tilde{\boldsymbol{m}}$ is well-defined and smooth on $P_{1/2}((\tilde{x},0))$.
Thus, as in Definition 3.10, we find that $\tilde{\boldsymbol{m}}$ solves the equation

$$\tilde{\boldsymbol{m}}_t = \mu(\Delta\tilde{\boldsymbol{m}} + |\nabla\tilde{\boldsymbol{m}}|^2\tilde{\boldsymbol{m}}) + \kappa s_0^{-2}\langle\tilde{\boldsymbol{m}},\tilde{\boldsymbol{v}}\rangle\Pi_{\tilde{\boldsymbol{m}}}\tilde{\boldsymbol{v}}\,,$$

on $P_{1/2}(\tilde{x},0)$, with $\mu(x) = \lambda^2(x)/\lambda^2(s_0^{-1}x)$ in stereographic coordinates centered at $\tilde{x}$. By the above inclusion we also have

$$\sup_{P_{1/2}((\tilde{x},0))} |\nabla\tilde{\boldsymbol{m}}|^2 \leq s_0^{-2}\sup_{P_r}|\nabla\boldsymbol{m}|^2 \leq s_0^{-2}\frac{(r-\rho)^2 s_0^2}{r^2} = \frac{(r-\rho)^2}{r^2} \leq 1 \tag{3.8}$$

and

$$|\nabla\tilde{\boldsymbol{m}}|(0,0) = s_0^{-1}|\nabla\boldsymbol{m}|(\tilde{x},\tilde{t}) = 1\,. \tag{3.9}$$

Hence, by setting $\tilde{\kappa} = \kappa s_0^{-2}$ we can invoke Definition 3.9 and apply it to $\tilde{\boldsymbol{m}}$ on $P_\tau((\tilde{x},0))$ with $0 < \tau \leq 1/2$. Then by virtue of (3.8) the inequality has the form

$$\begin{aligned}\|\nabla\tilde{\boldsymbol{m}}(0)\|^2_{H^{k-1}(B_{\tau/2})} &+ \int_{-(\tau/2)^2}^0 \|\nabla\tilde{\boldsymbol{m}}(t)\|^2_{H^k(B_{\tau/2})}\mathrm{d}t \\ &\leq C\left(1+\frac{1}{\tau^2}+\tilde{\kappa}\right)\int_{-\tau^2}^0 \|\nabla\tilde{\boldsymbol{m}}(t)\|^2_{H^{k-1}(B_\tau)}\mathrm{d}t + C\tilde{\kappa}\tau^4\,,\end{aligned}$$

where $B_s = B_s(\tilde{x})$. Note that whenever T appeared in the prefactors in the original inequality it is to be interpreted as length of the time interval on which the inequality is applied. With the help of the Sobolev embedding $H^2(B_{1/16}) \hookrightarrow L^\infty(B_{1/16})$ we can start now at (3.9) and use the above inequality three times to obtain

$$\begin{aligned}1 &= |\nabla\tilde{\boldsymbol{m}}(\tilde{x},0)|^2 \\ &\leq C\|\nabla\tilde{\boldsymbol{m}}(0)\|^2_{H^2(B_{1/16})} && (k=3, \tau=1/8) \\ &\leq C(1+\tilde{\kappa})\int_{-(1/8)^2}^0 \|\nabla\tilde{\boldsymbol{m}}(t)\|^2_{H^2(B_{1/8})}\mathrm{d}t + C\tilde{\kappa} && (k=2, \tau=1/4)\end{aligned}$$

$$\leq C(1+\tilde{\kappa})^2 \int_{-(1/4)^2}^{0} \|\nabla \tilde{\boldsymbol{m}}(t)\|^2_{H^1(B_{1/4})} \mathrm{d}t + C\tilde{\kappa}(1+\tilde{\kappa}) \qquad (k=1, \tau = 1/2)$$

$$\leq C(1+\tilde{\kappa})^3 \int_{-(1/2)^2}^{0} \|\nabla \tilde{\boldsymbol{m}}(t)\|^2_{L^2(B_{1/2})} \mathrm{d}t + C\tilde{\kappa}(1+\tilde{\kappa})^2 \,, \tag{3.10}$$

where the coefficients at the right denote the combination used in the inequality to arrive at the subsequent row. Also note that the constant C changes from line to line. Now, we compute for $t \in [-\frac{1}{4}, 0]$

$$\begin{aligned}
\|\nabla \tilde{\boldsymbol{m}}(t)\|^2_{L^2(B_{1/2}(\tilde{x}))} &= \int_{B_{1/2}(\tilde{x})} |\nabla \tilde{\boldsymbol{m}}(t)|^2 \, \mathrm{d}\sigma \\
&= \int_{B_{\tan(1/4)}(0)} |D\boldsymbol{m}|^2 (s_0^{-1} x, s_0^{-2} t + \tilde{t}) \frac{\mathrm{d}x}{s_0^2} \\
&= \int_{B_{s_0^{-1}\tan(1/4)}(0)} |D\boldsymbol{m}|^2 (x, s_0^{-2} t + \tilde{t}) \mathrm{d}x \\
&\leq \int_{B_{\tan(s_0^{-1}/2)}(0)} |D\boldsymbol{m}|^2 (x, s_0^{-2} t + \tilde{t}) \mathrm{d}x \\
&\leq \sup_{s \in [\tilde{t} - s_0^{-2}, \tilde{t}]} \int_{B_{s_0^{-1}}(\tilde{x})} |\nabla \boldsymbol{m}(s)|^2 \, \mathrm{d}\sigma \\
&\leq \sup_{s \in [t_0 - r^2, t_0]} \int_{B_r(x_0)} |\nabla \boldsymbol{m}(s)|^2 \, \mathrm{d}\sigma \leq E_{\mathrm{s}} \,,
\end{aligned}$$

where we once again used

$$s_0^{-1} \tan(1/4) \leq \tan(s_0^{-1}/2) \quad \text{and} \quad P_{s_0^{-1}}((\tilde{x}, 0)) \subset P_r((x_0, t_0)) \,.$$

Therefore, inserting this estimate in (3.10) and resubstituting $\tilde{\kappa}$ we arrive at

$$1 \leq C E_{\mathrm{s}} (1 + \kappa s_0^{-2})^3 + C\kappa s_0^{-2} (1 + \kappa s_0^{-2})^3 \leq C E_{\mathrm{s}} (1 + \kappa R_{\mathrm{s}}^2)^3 + C\kappa R_{\mathrm{s}}^2 (1 + \kappa R_{\mathrm{s}}^2)^3 \,,$$

as $s_0^{-1} \leq (r - \rho) \leq r \leq R_{\mathrm{s}}$. The constants appearing here depend on the Sobolev embedding constants and the constants in Definition 3.9 and hence are independent of $\boldsymbol{m}$, E_{s}, and R_{s}. Consequently, we can choose R_{s} and E_{s}, independently of $\boldsymbol{m}$, such that the right hand side is less than one, a contradiction. This implies that the first case must hold and the proof is complete. □

As said, we can use this result to show that solutions with no energy concentration at a point can be smoothly extended.

Corollary 3.13. *Let* $\boldsymbol{m} \in C^\infty(\mathbb{S}^2 \times (0, T_0))$ *solve* (SMHF). *Then if there exists* $0 < R < R_\mathrm{s}$ *such that for all* $0 < T < T_0$

$$\sup_{(x,t)\in\mathbb{S}^2\times[0,T]} \mathcal{E}(\boldsymbol{m}(t), B_R(x)) < E_\mathrm{s}\,, \tag{3.11}$$

then $\boldsymbol{m}$ *can be smoothly extended to* $\mathbb{S}^2 \times (0, T_0 + \delta)$ *for some* $\delta > 0$.

Proof. Assume T_0 is the maximal existence time of $\boldsymbol{m}$. Then by Definition 3.3 we find that

$$\lim_{T\nearrow T_0} \|\nabla \boldsymbol{m}(T)\|_{L^\infty(\mathbb{S}^2)} = \infty\,.$$

However, by Definition 3.12 and the assumption on the supremum we know that for all $x \in \mathbb{S}^2$ we have

$$|\nabla \boldsymbol{m}(x, T)| \leq \frac{2}{R}\,,$$

a contradiction. Thus, T_0 is not the maximal existence time and the solution can be smoothly extended. □

Now we finally have all the preparations done to prove Theorem 3. Again, we follow and adapt the proof of the theorem for the harmonic map heat flow in [63] and [62]. In fact, our proof holds for all values of κ, in particular $\kappa = 0$. Therefore, it can also be used to reproduce the result for the harmonic map heat flow. In the following we try to be as precise as possible and give all the necessary details that in the existing literature are often omitted or imprecise. Some arguments used can found in [29].

Proof of Theorem 3. The proof is divided into five steps.
Step 1: Local existence. For $\boldsymbol{m}_0 \in H^1(\mathbb{S}^2, \mathbb{S}^2)$, by density of $C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ we find a sequence $\{\boldsymbol{m}_{0,k}\} \subset C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ such that $\boldsymbol{m}_{0,k} \to \boldsymbol{m}_0$ as $k \to \infty$ in $H^1(\mathbb{S}^2, \mathbb{S}^2)$. Let $\boldsymbol{m}_k$ solve (SMHF) with initial data $\boldsymbol{m}_{0,k}$ as provided by Definition 3.3. By the H^1-convergence we conclude that $e(\boldsymbol{m}_{0,k})$ converges in L^1 to $e(\boldsymbol{m}_0)$. Hence by Vitali's convergence theorem [3, cf. Theorem 1.23] we find $0 < R_0 < R_\mathrm{s}/2$ such that

$$\sup_{x\in\mathbb{S}^2} \mathcal{E}(\boldsymbol{m}_{0,k}, B_{2R_0}(x)) \leq \frac{E_\mathrm{b}}{2}\,,$$

for all $k \in \mathbb{N}$, where $E_\mathrm{b} = \min\{E_V, E_\mathrm{s}\}$. Therefore, as described before, setting

$$T_0 = \inf_k \frac{E_\mathrm{b} R_0^2}{C\mathcal{E}(\boldsymbol{m}_{0,k})}\,,$$

we get as in (3.3) and (3.4)

$$\sup_{(x,t)\in\mathbb{S}^2\times[0,T_0]} \mathcal{E}(\boldsymbol{m}_k, B_{R_0}(x)) < E_{\mathrm{b}}\,, \tag{3.12}$$

and

$$\|\boldsymbol{m}_k\|^2_{V^{T_0}} \le C\mathcal{E}(\boldsymbol{m}_{0,k})\left(1+\frac{T_0}{R_0^2}+C_\kappa R_0^2 T_0\right) \le C\left(\mathcal{E}(\boldsymbol{m}_0)+\frac{C_\kappa}{\mathcal{E}(\boldsymbol{m}_0)}\right),$$

as $\mathcal{E}(\boldsymbol{m}_{0,k}) \to \mathcal{E}(\boldsymbol{m}_0)$ for $k\to\infty$ by the H^1-continuity of $\mathcal{E}$. Note that $0 < T_0 \le T_k$ for all k, where T_k is the maximal existence time of $\boldsymbol{m}_k$. This is due to the fact that by virtue of (3.12) and Definition 3.13 we find $T_k > T_0$ and therefore we can consider $(\boldsymbol{m}_k)$ to be a sequence of maps defined on $[0,T_0]$ with $\|\boldsymbol{m}_k\|^2_{V^{T_0}}$ uniformly bounded.

This implies that $\boldsymbol{m}_k \in L^2_t H^2_x \cap H^1_t L^2_x(\mathbb{S}^2\times[0,T_0])$ with uniform bounds. As a result, after extracting a subsequence, we can assume weak convergence of $\boldsymbol{m}_k \rightharpoonup \boldsymbol{m}$ in $L^2_t H^2_x$ and $\boldsymbol{m}_{k,t} \rightharpoonup \boldsymbol{m}_t$ in $L^2_t L^2_x$. Also we note that the $\boldsymbol{m}_k$ are uniformly bounded in $H^1_{(t,x)}$ which compactly embeds into $L^2_{(t,x)}$ by the Rellich–Kondrachov theorem. Hence, after extracting another subsequence, we assume that $\boldsymbol{m}_k \to \boldsymbol{m}$ strongly in $L^2_{(t,x)}$. We want to improve this to strong convergence $\boldsymbol{m}_k \to \boldsymbol{m}$ in $L^2_t H^1_x$. Indeed, using (2.7) we can rewrite the difference of the gradient norms as

$$|\nabla \boldsymbol{m}_k|^2 - |\nabla \boldsymbol{m}|^2 = -\langle \boldsymbol{m}_k, \Delta \boldsymbol{m}_k\rangle + \langle \boldsymbol{m}, \Delta\boldsymbol{m}\rangle = -\langle \boldsymbol{m}_k - \boldsymbol{m}, \Delta\boldsymbol{m}_k\rangle + \langle \boldsymbol{m}, \Delta\boldsymbol{m} - \Delta\boldsymbol{m}_k\rangle.$$

Therefore, we conclude

$$\begin{aligned}
&\Big|\int_0^{T_0}\int_{S^2} |\nabla\boldsymbol{m}_k|^2 - |\nabla\boldsymbol{m}|^2 \,\mathrm{d}\sigma\mathrm{d}t\Big| \\
&\le \int_0^{T_0}\int_{S^2} |\Delta\boldsymbol{m}_k|\,|\boldsymbol{m}_k-\boldsymbol{m}|\,\mathrm{d}\sigma\mathrm{d}t + \Big|\int_0^{T_0}\int_{S^2}\langle\boldsymbol{m},\Delta\boldsymbol{m}-\Delta\boldsymbol{m}_k\rangle\,\mathrm{d}\sigma\mathrm{d}t\Big| \\
&\le \underbrace{\|\Delta\boldsymbol{m}_k\|_{L^2_{(x,t)}}}_{\le\|\boldsymbol{m}_k\|^2_{V^{T_0}}\le M} \|\boldsymbol{m}_k-\boldsymbol{m}\|_{L^2_{(x,t)}} + \Big|\int_0^{T_0}\int_{S^2}\langle\boldsymbol{m},\Delta\boldsymbol{m}-\Delta\boldsymbol{m}_k\rangle\,\mathrm{d}\sigma\mathrm{d}t\Big| \quad \longrightarrow 0\,,
\end{aligned}$$

as $k\to\infty$ by the $L^2_{(x,t)}$-convergence and weak $L^2_t H^2_x$-convergence. Thus, $\boldsymbol{m}_k \to \boldsymbol{m}$ in $L^2_t H^1_x$ since it converges weakly and in norm. By extracting another subsequence, we can assume that $\boldsymbol{m}_k \to \boldsymbol{m}$ and $\nabla\boldsymbol{m}_k \to \nabla\boldsymbol{m}$ almost everywhere in $\mathbb{S}^2\times[0,T_0]$. Since (3.12) holds, we conclude that we can apply Definition 3.12 to find that for any $t_* \in (0, T_0)$ we get

$$|\nabla \boldsymbol{m}_k(x,t)| \leq C \max\left\{\frac{1}{R_0}, \frac{1}{\sqrt{t_*}}\right\} \tag{3.13}$$

for all $(x,t) \in \mathbb{S}^2 \times (t_*, T_0]$ and by pointwise convergence almost everywhere we conclude that

$$\|\nabla \boldsymbol{m}\|_{L^\infty(\mathbb{S}^2 \times (t_*, T_0])} \leq C \max\left\{\frac{1}{R_0}, \frac{1}{\sqrt{t_*}}\right\}$$

by definition of the essential supremum. Moreover, convergence almost everywhere of $\boldsymbol{m}_k$ also implies that $|\boldsymbol{m}(x,t)| = 1$ for almost all $(x,t) \in \mathbb{S}^2 \times (0, T_0]$.
We want to show now that $\boldsymbol{m}$ is a weak solution of (SMHF). Multiplying (SMHF), in which we exchange $\boldsymbol{m}$ with $\boldsymbol{m}_k$, by the product $\varphi \boldsymbol{v}$ with test functions $\varphi \in C_0^\infty([0, T_0], \mathbb{R})$ and $\boldsymbol{v} \in C^\infty(\mathbb{S}^2, \mathbb{R}^3)$ and integrating in space and time we find

$$\begin{aligned} 0 = &\int_0^{T_0} \int_{\mathbb{S}^2} \langle \boldsymbol{m}_{k,t}, \varphi \boldsymbol{v}\rangle - \langle \nabla \boldsymbol{m}_k, \varphi \nabla \boldsymbol{v}\rangle + \kappa \langle \boldsymbol{m}_k, \varphi \langle \boldsymbol{\nu}, \boldsymbol{v}\rangle \boldsymbol{\nu}\rangle \, \mathrm{d}\sigma \mathrm{d}t \\ &+ \int_0^{T_0} \int_{\mathbb{S}^2} |\nabla \boldsymbol{m}_k|^2 \langle \boldsymbol{m}_k, \varphi \boldsymbol{v}\rangle - \kappa \langle \boldsymbol{m}_k, \boldsymbol{\nu}\rangle^2 \langle \boldsymbol{m}_k, \varphi \boldsymbol{v}\rangle \mathrm{d}\sigma \mathrm{d}t \,. \end{aligned} \tag{3.14}$$

In the first integral we can use the weak convergence of $\boldsymbol{m}_k$ in $L_t^2 H_x^1$ and of $\boldsymbol{m}_{k,t}$ in $L_t^2 L_x^2$ in order to replace them in the limit with $\boldsymbol{m}$ and $\boldsymbol{m}_t$, respectively. For the second integral we need to work a little more. We want to check convergence and subtract the limit from the first term to get

$$\begin{aligned} I_1 :&= \int_0^{T_0} \int_{\mathbb{S}^2} |\nabla \boldsymbol{m}_k|^2 \langle \boldsymbol{m}_k, \varphi \boldsymbol{v}\rangle - |\nabla \boldsymbol{m}|^2 \langle \boldsymbol{m}, \varphi \boldsymbol{v}\rangle \mathrm{d}s \mathrm{d}t \\ &= \int_0^{T_0} \int_{\mathbb{S}^2} |\nabla \boldsymbol{m}_k|^2 \langle \boldsymbol{m}_k - \boldsymbol{m}, \varphi \boldsymbol{v}\rangle + \left(|\nabla \boldsymbol{m}_k|^2 - |\nabla \boldsymbol{m}|^2\right) \langle \boldsymbol{m}, \varphi \boldsymbol{v}\rangle \, \mathrm{d}\sigma \mathrm{d}t \,. \end{aligned}$$

Both summands converge to zero as $k \to \infty$, the first one by $L^2_{(x,t)}$-convergence and the fact that $L_t^2 W_x^{1,4} \subset L_t^2 H_x^2$ by spatial Sobolev embedding, the second one by the $L_t^2 H_x^1$-convergence. Hence, we find that $I_1 \to 0$ as $k \to \infty$.
For the second term of the second integral in (3.14) we deduce from the fact that $\boldsymbol{m}_k \to \boldsymbol{m}$ in $L^2_{(x,t)}$ and $|\boldsymbol{m}_k| = |\boldsymbol{m}| = 1$ a.e. that $\boldsymbol{m}_k \to \boldsymbol{m}$ in $L^3_{(x,t)}$ and hence

$$\int_0^{T_0} \int_{\mathbb{S}^2} \langle \boldsymbol{m}_k, \boldsymbol{\nu}\rangle^2 \langle \boldsymbol{m}_k, \varphi \boldsymbol{v}\rangle \mathrm{d}\sigma \mathrm{d}t \longrightarrow \int_0^{T_0} \int_{\mathbb{S}^2} \langle \boldsymbol{m}, \boldsymbol{\nu}\rangle^2 \langle \boldsymbol{m}, \varphi \boldsymbol{v}\rangle \mathrm{d}\sigma \mathrm{d}t \,, \text{ as } k \to \infty \,.$$

Consequently, performing the limit $k \to \infty$ in (3.14) we find

$$0 = \int_0^{T_0} \left(\int_{\mathbb{S}^2} \langle \boldsymbol{m}_t, \boldsymbol{v}\rangle - \langle \nabla \boldsymbol{m}, \nabla \boldsymbol{v}\rangle + \kappa \langle \boldsymbol{m}, \langle \boldsymbol{\nu}, \boldsymbol{v}\rangle \boldsymbol{\nu}\rangle \right.$$

$$+|\nabla \boldsymbol{m}|^2\langle \boldsymbol{m}, \boldsymbol{v}\rangle - \kappa\langle \boldsymbol{m}, \boldsymbol{\nu}\rangle^2\langle \boldsymbol{m}, \boldsymbol{v}\rangle \mathrm{d}\sigma\Big)\varphi \mathrm{d}t\,.$$

Since φ was arbitrary, we conclude by the fundamental lemma of the calculus of variations that for almost all $t \in (0, T)$ it holds

$$0 = \int_{\mathbb{S}^2} \langle \boldsymbol{m}_t, \boldsymbol{v}\rangle - \langle \nabla \boldsymbol{m}, \nabla \boldsymbol{v}\rangle + \kappa\langle \boldsymbol{m}, \langle \boldsymbol{\nu}, \boldsymbol{v}\rangle \boldsymbol{\nu}\rangle + |\nabla \boldsymbol{m}|^2\langle \boldsymbol{m}, \boldsymbol{v}\rangle - \kappa\langle \boldsymbol{m}, \boldsymbol{\nu}\rangle^2\langle \boldsymbol{m}, \boldsymbol{v}\rangle \mathrm{d}\sigma\,,$$

i.e., $\boldsymbol{m}$ is a weak solution of (SMHF).
For attainment of the initial data we want to improve $L^2_{(x,t)}$-convergence to L^2_x-convergence uniformly in time, i.e., $C^0_t L^2_x$-convergence. To this end, we invoke the Arzelà–Ascoli theorem. We compute for $t, s \in [0, T_0]$

$$\begin{aligned}
\|\boldsymbol{m}_k(t) - \boldsymbol{m}_k(s)\|_{L^2} &= \| \int_s^t \boldsymbol{m}_{k,\tau}(\tau)\mathrm{d}\tau\|_{L^2} \\
&\le \Big(\int_{\mathbb{S}^2} | \int_s^t \boldsymbol{m}_{k,\tau}(\tau)\mathrm{d}\tau|^2 \,\mathrm{d}\sigma\Big)^{1/2} \\
&\le |s-t|^{1/2}\Big(\int_{\mathbb{S}^2}\int_0^{T_0} |\boldsymbol{m}_{k,\tau}(\tau)|^2 \mathrm{d}\tau\,\mathrm{d}\sigma\Big)^{1/2} < M|s-t|^{1/2}\,.
\end{aligned}$$

Consequently, $(\boldsymbol{m}_k)$ is equicontinuous seen as sequence of maps from $[0, T_0]$ to $L^2(\mathbb{S}^2, \mathbb{R}^3)$. Next, we need to show pointwise in time relative compactness of the sets $\{\boldsymbol{m}_k(t)\} \subset L^2(\mathbb{S}^2, \mathbb{S}^2)$ for $t \in [0, T_0]$. As $\|\boldsymbol{m}_k(t)\|_{H^1} \le \|\boldsymbol{m}_k\|_{V^{T_0}} \le \sqrt{M}$, we see that $\{\boldsymbol{m}_k(t)\}$ is a bounded set in $H^1(\mathbb{S}^2, \mathbb{R}^3)$ and by the Rellich–Kondrachov theorem relatively compact in $L^2(\mathbb{S}^2, \mathbb{S}^2)$. Consequently, by the Arzelà–Ascoli theorem, extracting a further subsequence we establish $\boldsymbol{m}_k \to \boldsymbol{m}$ in $C^0([0, T_0], L^2(\mathbb{S}^2, \mathbb{S}^2))$.
Thus, we have shown that firstly $\boldsymbol{m}_k(t) \to \boldsymbol{m}(t)$ for $k \to \infty$, uniformly in t, secondly $\boldsymbol{m}_k(t) \to \boldsymbol{m}_k(0) = \boldsymbol{m}_{0,k}$ for $t \to 0$, equicontinuously, and thirdly $\boldsymbol{m}_{0,k} \to \boldsymbol{m}_0$ by the choice of the $\boldsymbol{m}_{0,k}$. Hence for $\varepsilon > 0$ given, we can choose $\delta > 0$ such that for all $0 < t < \delta$ it holds $\|\boldsymbol{m}_k(t) - \boldsymbol{m}_{0,k}\|_{L^2} < \varepsilon/3$ for all $k \in \mathbb{N}$. Then we pick $k_0 \in N$ such that for all $k > k_0$ we have $\|\boldsymbol{m}_k(t) - \boldsymbol{m}(t)\|_{L^2} < \varepsilon/3$ for all $0 < t < \delta$ and $\|\boldsymbol{m}_{0,k} - \boldsymbol{m}_0\|_{L^2} < \varepsilon/3$. Then it follows that for all $0 < t < \delta$ and some $k > k_0$

$$\|\boldsymbol{m}(t) - \boldsymbol{m}_0\|_{L^2} \le \|\boldsymbol{m}(t) - \boldsymbol{m}_k(t)\|_{L^2} + \|\boldsymbol{m}_k(t) - \boldsymbol{m}_{0,k}\|_{L^2} + \|\boldsymbol{m}_{0,k} - \boldsymbol{m}_0\|_{L^2} < \varepsilon\,.$$

Thus, $\lim_{t\to 0} \boldsymbol{m}(t) = \boldsymbol{m}_0$ in L^2. By [22, cf. 5.9.2 Theorem 4] we know that $L^2_t H^2_x \cap H^1_t L^2_x$ embeds into $C^0_t H^1_x$, hence we proved that $\boldsymbol{m}$ attains $\boldsymbol{m}_0$ in $t = 0$ also in $H^1(\mathbb{S}^2, \mathbb{S}^2)$.

Finally, we want to show that $\boldsymbol{m}$ is smooth on $\mathbb{S}^2 \times (0, T_0)$. To this end, let $0 < t^* < T_0$ arbitrary but such that $\boldsymbol{m}_k(t^*) \to \boldsymbol{m}(t^*)$ strongly in $H^1(\mathbb{S}^2, \mathbb{R}^3)$ by convergence almost everywhere in time. Recall that by (3.13) we have that $\nabla \boldsymbol{m}_k$ is uniformly bounded in $L^\infty(\mathbb{S}^2 \times (t^*/2, T_0))$. Therefore, applying Definition 3.8 twice, first for $k = 2$ and then $k = 3$, with radius and time chosen suitably, we find that $\boldsymbol{m}_k(t^*)$ is uniformly bounded in $H^3(\mathbb{S}^2, \mathbb{R}^3)$ since we can cover $\mathbb{S}^2$ by finitely many balls. Consequently, by extracting an additional subsequence, we can assume that $\boldsymbol{m}_k(t^*) \rightharpoonup \boldsymbol{m}^*$ in $H^3(\mathbb{S}^2, \mathbb{R}^3)$. Invoking Definition 3.3 we find that (SMHF) admits a smooth, unique solution with initial data $\boldsymbol{m}^*$. By uniqueness of the limit we conclude that $\boldsymbol{m}(t^*) = \boldsymbol{m}^*$ and thus $\boldsymbol{m}$ is smooth on $\mathbb{S}^2 \times (t^*, T_0)$ by the uniqueness of the solution. Since t^* was arbitrarily small, we find that $\boldsymbol{m}$ is smooth on $\mathbb{S}^2 \times (0, T_0)$.

By Definition 3.13 we obtain $T_1 > T_0$ such that $\boldsymbol{m}$ can be smoothly extended to $\mathbb{S}^2 \times (0, T_1)$. Moreover, by Definition 3.5 we know that $\mathcal{E}(\boldsymbol{m}(t)) \leq \mathcal{E}(\boldsymbol{m}(s)) \leq \mathcal{E}(\boldsymbol{m}_0)$ for $0 \leq s \leq t < T_1$.

In the case $T_1 < \infty$, the time T_1 is characterized by

$$\limsup_{t \nearrow T_1} \mathcal{E}(\boldsymbol{m}, B_R(x_k)) \geq E_{\mathrm{b}}$$

for all $0 < R$ and for points $x_1, x_2, \ldots$ where the energy concentration occurs. We call these points *blowup points in space-time*. Note that there has to exist at least one of these points. In the case $T_1 = \infty$ we refer to the asymptotic case following since then $\boldsymbol{m}$ is already a global solution in time.

We proceed with $T_1 < \infty$. We want to show that there can only exist finitely many blowup points. We choose a finite collection $x_1, \ldots, x_{K_1}$ of these points and choose $R > 0$ small enough such that the balls $B_{2R}(x_k)$ are disjoint and such that

$$t_0 := T_1 - \frac{E_{\mathrm{b}} R_1^2}{4 C_1 \mathcal{E}(\boldsymbol{m}_0)} > 0$$

holds, where the constant $C_1 > 0$ is the one from Definition 3.5. Furthermore, we pick $t_0 < t_k < T_1$ such that

$$\mathcal{E}(\boldsymbol{m}(t_k), B_R(x_k)) \geq \frac{E_{\mathrm{b}}}{2}.$$

Then by virtue of Definition 3.6 this yields

$$\mathcal{E}(\boldsymbol{m}(t_0), B_{2R}(x_k)) \geq \mathcal{E}(\boldsymbol{m}(t_k), B_R(x_k)) - C_1 \frac{t_k - t_0}{R^2} \mathcal{E}(\boldsymbol{m}_0)$$

$$\geq \frac{E_{\mathrm{b}}}{2} - C_1 \frac{T_1 - t_0}{R^2} \mathcal{E}(\boldsymbol{m}_0) = \frac{E_{\mathrm{b}}}{4}\,. \tag{3.15}$$

Accordingly, invoking Definition 3.4, we arrive at the estimate

$$\mathcal{E}(\boldsymbol{m}_0) \geq \mathcal{E}(\boldsymbol{m}(t_0)) \geq \sum_{k=1}^{K_1} \mathcal{E}(\boldsymbol{m}(t_0), B_{2R}(x_k)) \geq K_1 \frac{E_{\mathrm{b}}}{4}\,.$$

Thus, $K_1 \leq 4\mathcal{E}(\boldsymbol{m}_0)/E_{\mathrm{b}}$ and hence there can only be finitely many points $\{x_k : 1 \leq k \leq K_1\}$ where the energy concentration occurs.

Finally, applying the same argument as in Definition 3.13 locally on sets $Q \subset\subset \mathbb{S}^2 \times [0, T_1] \setminus \{(x_1, T_1), \ldots, (x_{K_1}, T_1)\}$, we find that $\boldsymbol{m}$ is smooth on Q as we can find $R > 0$ such that

$$\sup_{(x,t)\in Q} \mathcal{E}(\boldsymbol{m}, B_R(x)) < E_{\mathrm{b}}\,.$$

Therefore, $\boldsymbol{m}$ extends to a smooth solution on $\mathbb{S}^2 \times [0, T_1] \setminus \{(x_1, T_1), \ldots, (x_{K_1}, T_1)\}$.

Step 2: Global solution. Again, we assume $T_1 < \infty$ since otherwise $\boldsymbol{m}$ is the desired global solution. By Definition 3.4 we know that $\mathcal{E}(\boldsymbol{m}(t)) \leq \mathcal{E}(\boldsymbol{m}_0)$ for all $t \in [0, T_1)$. Consequently, $\boldsymbol{m}(t)$ is uniformly bounded in $H^1(\mathbb{S}^2, \mathbb{R}^3)$ for all $t \in [0, T_1)$. As a result, we find $\boldsymbol{m}_0^{(1)} \in H^1(\mathbb{S}^2, \mathbb{R}^3)$ such that $\boldsymbol{m}(t) \rightharpoonup \boldsymbol{m}_0^{(1)}$ in $H^1(\mathbb{S}^2, \mathbb{R}^3)$ as $t \nearrow T_1$. Weak lower semicontinuity of the H^1-norm implies

$$\mathcal{E}(\boldsymbol{m}(T_1)) \leq \liminf_{s \nearrow T_1} \mathcal{E}(\boldsymbol{m}(s)) \leq \mathcal{E}(\boldsymbol{m}(t))$$

for all $0 < t < T_1$.

By the Rellich–Kondrachov theorem we can assume that $\boldsymbol{m}(t) \to \boldsymbol{m}_0^{(1)}$ strongly in $L^2(\mathbb{S}^2, \mathbb{R}^3)$ as $t \nearrow T_1$ and therefore $\boldsymbol{m}_0^{(1)}$ is $\mathbb{S}^2$-valued almost everywhere. In fact, on $\mathbb{S}^2 \setminus \{x_1, \ldots, x_{K_1}\}$ we have that $\boldsymbol{m}_0^{(1)} = \boldsymbol{m}(T_1)$ by the smooth extension explained above and hence $\boldsymbol{m}_0^{(1)}$ is the unique weak limit of $\boldsymbol{m}(t)$ in $H^1(\mathbb{S}^2, \mathbb{S}^2)$ as $t \nearrow T_1$.

Similarly, as in (3.15) we find a $t_1 < T_1$ such that for all $t_1 < t < T_1$ we have

$$\mathcal{E}(\boldsymbol{m}(t), B_{2R}(x_k)) \geq \frac{E_{\mathrm{b}}}{4}$$

for all $1 \leq k \leq K_1$. Thus, we gain the energy estimate

$$\mathcal{E}(\boldsymbol{m}_0^{(1)}) = \lim_{R\to 0} \mathcal{E}\left(\boldsymbol{m}_0^{(1)}, \mathbb{S}^2 \setminus \bigcup_{k=1}^{K_1} B_{2R}(x_k)\right)$$

$$
\begin{aligned}
&= \lim_{R\to 0}\lim_{t\nearrow T_1}\left(\mathcal{E}(\boldsymbol{m}(t),\mathbb{S}^2) - \sum_{k=1}^{K_1}\mathcal{E}(\boldsymbol{m}(t),B_{2R}(x_k))\right)\\
&\leq \mathcal{E}(\boldsymbol{m}_0) - K_1\frac{E_{\mathrm{b}}}{4}\,.
\end{aligned}
$$

Now, setting $T_0 = 0$, $\boldsymbol{m}_0^{(0)} = \boldsymbol{m}_0$, and $\boldsymbol{m}^{(0)} = \boldsymbol{m}$, we can construct iteratively by reproducing the previous steps a sequence of solutions $\boldsymbol{m}^{(n)}$ to (SMHF) on $C^0((T_n,T_{n+1}),H^1(\mathbb{S}^2,\mathbb{S}^2))$ with initial data $\boldsymbol{m}^{(n)}(T_n) = \boldsymbol{m}_0^{(n-1)}$ for $n \in \mathbb{N}$ such that $\boldsymbol{m}^{(n)}(t) \rightharpoonup \boldsymbol{m}_0^{(n+1)}$ in $H^1(\mathbb{S}^2,\mathbb{R}^3)$ as $t \nearrow T_{n+1}$. Additionally, there exist finitely many singular points $x_1^{(n)},\ldots,x_{K_{n+1}}^{(n)} \in \mathbb{S}^2$ such that $\boldsymbol{m}^{(n)}$ is smooth on $\mathbb{S}^2\times(T_n,T_{n+1})\setminus\{(x_1^{(n)},T_{n+1}),\ldots,(x_{K_{n+1}}^{(n)},T_{n+1})\}$. The points are characterized by energy concentration

$$
\limsup_{t\nearrow T_{n+1}}\mathcal{E}(\boldsymbol{m}^{(n)}(t),B_R(x_k)) \geq E_{\mathrm{b}}\,.
$$

Moreover, we have the energy estimate

$$
\mathcal{E}(\boldsymbol{m}_0^{(n+1)}) \leq \mathcal{E}(\boldsymbol{m}_0^{(n)}) - K_{n+1}\frac{E_{\mathrm{b}}}{4}
$$

which implies that since the energy is nonnegative that

$$
\sum_{l=1}^{n} K_l \leq \mathcal{E}(\boldsymbol{m}_0)\frac{4}{E_{\mathrm{b}}}\,,
$$

for all $n \in \mathbb{N}$. As a result, the total number of singular points is finite and hence we only need to consider a finite number of time intervals, i.e., there exists an $N \in \mathbb{N}$ such that $\boldsymbol{m}^{(N)}$ is defined and smooth on $\mathbb{S}^2\times(T_N,\infty)$. For the behavior at infinity we refer to the asymptotic case, *Step 4*.
Defining $\boldsymbol{m}$ now by $\boldsymbol{m}(t) = \boldsymbol{m}^{(n)}(t)$ for $t \in (T_n,T_{n+1})$ and $\boldsymbol{m}(T_n) = \boldsymbol{m}_0^{(n)}$ we have found a function in $\mathbb{S}^2\times(0,\infty)\to\mathbb{S}^2$ which is smooth on $\mathbb{S}^2\times(0,\infty)$ away from at most finitely many points $(\bar{x}_k,\bar{t}_k), 0<\bar{t}_k<\infty, 1\leq k\leq\bar{K}$ and is a weak solution of (SMHF) with initial data $\boldsymbol{m}_0$, since the $\boldsymbol{m}^{(n)}$ are weak solutions. Moreover, as shown in *Step 1*, $\boldsymbol{m}$ assumes the initial data $\boldsymbol{m}_0$ continuously in $H^1(\mathbb{S}^2,\mathbb{R}^3)$ at $t=0$.
Step 3: Uniqueness. We proceed as in step (5°) of the proof of Theorem 6.6 in [63]. For the sake of completeness we write down the arguments in detail.
We assume that $\boldsymbol{m}_1,\boldsymbol{m}_2 \in L_t^2H_x^2\cap H_t^1L_x^2$ are two weak solutions of (SMHF) to the same initial data $\boldsymbol{m}_0 \in H^1(\mathbb{S}^2,\mathbb{S}^2)$ on some interval $(0,T_0)$. Let $\boldsymbol{\phi} = \boldsymbol{m}_1 - \boldsymbol{m}_2$.

Certainly, $\boldsymbol{\phi}(t) \in H^1 \cap L^\infty(\mathbb{S}^2, \mathbb{R}^3)$ for every $0 < t < T_0$, so we can use it as a test function against $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ in (3.1). Subtracting both equations from each other yields

$$0 = \int_{\mathbb{S}^2} \langle \boldsymbol{\phi}_t, \boldsymbol{\phi} \rangle + |\nabla \boldsymbol{\phi}|^2 - |\boldsymbol{\phi}|^2 |\nabla \boldsymbol{m}_1|^2 - \langle \boldsymbol{m}_2, \boldsymbol{\phi} \rangle \langle \nabla \boldsymbol{\phi}, \nabla \boldsymbol{m}_1 + \nabla \boldsymbol{m}_2 \rangle$$
$$- \kappa \langle \boldsymbol{\phi}, \boldsymbol{\nu} \rangle (\langle \boldsymbol{\nu}, \boldsymbol{\phi} \rangle - \langle \boldsymbol{m}_1 + \boldsymbol{m}_2, \boldsymbol{\nu} \rangle \langle \boldsymbol{m}_2, \boldsymbol{\phi} \rangle) + \kappa \langle \boldsymbol{m}_1, \boldsymbol{\nu} \rangle^2 \langle \boldsymbol{\phi}, \boldsymbol{\phi} \rangle \, \mathrm{d}\sigma.$$

Hence, recalling the already established fact $|\boldsymbol{m}_1| = |\boldsymbol{m}_2| = 1$, from which also follows $|\boldsymbol{\phi}| \leq 2$, we can now integrate in time from 0 to $t \in (0, T)$ for a $t_0 < T < T_0$ still to be chosen to get

$$\begin{aligned}
&\frac{1}{2} \|\boldsymbol{\phi}(t)\|_{L^2}^2 \\
&\quad \leq \int_0^t \int_{\mathbb{S}^2} -|\nabla \boldsymbol{\phi}|^2 + |\nabla \boldsymbol{m}_1|^2 |\boldsymbol{\phi}|^2 + |\nabla \boldsymbol{m}_1 + \nabla \boldsymbol{m}_2| |\boldsymbol{\phi}| |\nabla \boldsymbol{\phi}| + C\kappa |\boldsymbol{\phi}|^2 \, \mathrm{d}\sigma \mathrm{d}t \\
&\quad \leq \int_0^t \int_{\mathbb{S}^2} -\frac{1}{2} |\nabla \boldsymbol{\phi}|^2 + C(|\nabla \boldsymbol{m}_1|^2 + |\nabla \boldsymbol{m}_2|^2) |\boldsymbol{\phi}|^2 + C\kappa |\boldsymbol{\phi}|^2 \, \mathrm{d}\sigma \mathrm{d}t \\
&\quad \leq \int_0^t -\frac{1}{2} \|\nabla \boldsymbol{\phi}\|_{L^2}^2 + C(\|\nabla \boldsymbol{m}_1\|_{L^4}^2 + \|\nabla \boldsymbol{m}_2\|_{L^4}^2) \|\boldsymbol{\phi}\|_{L^4}^2 + C\kappa \|\boldsymbol{\phi}\|_{L^2}^2 \, \mathrm{d}t \\
&\quad \leq \int_0^t -\frac{1}{2} \|\nabla \boldsymbol{\phi}\|_{L^2}^2 \, \mathrm{d}t + C\kappa T \sup_{s \in [0,T]} \|\boldsymbol{\phi}(s)\|_{L^2}^2 \\
&\qquad + C \left(\left(\int_0^t \|\nabla \boldsymbol{m}_1\|_{L^4}^4 \, \mathrm{d}t \right)^{1/2} + \left(\int_0^t \|\nabla \boldsymbol{m}_2\|_{L^4}^4 \, \mathrm{d}t \right)^{1/2} \right) \left(\int_0^t \|\boldsymbol{\phi}\|_{L^4}^4 \mathrm{d}t \right)^{1/2}
\end{aligned} \tag{3.16}$$

where we first applied Young's inequality to absorb a positive term of $\frac{1}{2} \|\nabla \boldsymbol{\phi}\|_{L^2}$ and then the Hölder inequality. The term stemming from the anisotropy we crudely bounded by pulling the L^2-norm of $\boldsymbol{\phi}$ out of the integral as a supremum and then estimating $t \leq T$.

We proceed by estimating the final line. Let $\delta > 0$. By (2.7), which implies that $|\nabla \boldsymbol{m}_i|^2 \leq |\Delta \boldsymbol{m}_i| \leq |\nabla^2 \boldsymbol{m}_i|$ we get for $i = 1, 2$

$$\int_0^t \|\nabla \boldsymbol{m}_i\|_{L^4}^4 \mathrm{d}t \leq \int_0^T \|\nabla^2 \boldsymbol{m}_i\|_{L^2}^2 \, \mathrm{d}t < \delta^2,$$

if we choose $T = T(\delta, \boldsymbol{m}_1, \boldsymbol{m}_2)$ sufficiently small by the absolute continuity of the Lebesgue integral. Integrability of $\|\nabla^2 \boldsymbol{m}_i\|_{L^2}$ follows from the fact that $\boldsymbol{m}_i \in L^2([0, T_0], H^2)$. For the integral involving the L^4-norm of $\boldsymbol{\phi}$ we invoke the Sobolev imbedding $W^{1,1}(\mathbb{S}^2, \mathbb{R}^3) \hookrightarrow L^2(\mathbb{S}^2, \mathbb{R}^3)$ [5, cf. Theorem 2.21], which gives us

$$\|\boldsymbol{\phi}\|_{L^4}^4 = \||\boldsymbol{\phi}|^2\|_{L^2}^2 \le C\Big(\||\boldsymbol{\phi}|^2\|_{L^1} + \|\nabla|\boldsymbol{\phi}|^2\|_{L^1}\Big)^2 \le C\Big(\|\boldsymbol{\phi}\|_{L^2}^2 + \|\boldsymbol{\phi}\|_{L^2}\|\nabla\boldsymbol{\phi}\|_{L^2}\Big)^2,$$

Consequently, we obtain

$$\begin{aligned}\int_0^t \|\boldsymbol{\phi}\|_{L^4}^4 \mathrm{d}t &\le C\int_0^t \|\boldsymbol{\phi}\|_{L^2}^4 + \|\boldsymbol{\phi}\|_{L^2}^2\|\nabla\boldsymbol{\phi}\|_{L^2}^2 \mathrm{d}t \\ &\le CT\left(\sup_{s\in[0,T]}\|\boldsymbol{\phi}(s)\|_{L^2}^2\right)^2 + \sup_{s\in[0,T]}\|\boldsymbol{\phi}(s)\|_{L^2}^2\int_0^t \|\nabla\boldsymbol{\phi}\|_{L^2}^2 \mathrm{d}t \\ &\le \left(C\delta \sup_{s\in[0,T]}\|\boldsymbol{\phi}(s)\|_{L^2}^2 + \frac{1}{4\delta}\int_0^t \|\nabla\boldsymbol{\phi}\|_{L^2}^2\mathrm{d}t\right)^2,\end{aligned}$$

where in the last step we used Young's inequality and that $A^2 + B^2 \le (A+B)^2$ for nonnegative A, B. Furthermore, we also chose that $T < 1$ holds. Inserting these estimates into (3.16), we arrive at

$$\frac{1}{2}\|\boldsymbol{\phi}(t)\|_{L^2}^2 \le \int_0^t -\frac{1}{4}\|\nabla\boldsymbol{\phi}\|_{L^2}^2 \,\mathrm{d}t + C(\kappa T + \delta)\sup_{s\in[0,T]}\|\boldsymbol{\phi}(s)\|_{L^2}^2 \le C\delta \sup_{s\in[0,T]}\|\boldsymbol{\phi}(s)\|_{L^2}^2, \tag{3.17}$$

decreasing T such that $\kappa T < \delta$, if necessary. Note that the constant C here only depends on the geometry of $\mathbb{S}^2$ (through the Sobolev constant) and otherwise ordinary numbers. Hence, we can set δ such that $C\delta < 1/4$, then pick $T = T(\delta, \boldsymbol{m}_1, \boldsymbol{m}_2, \kappa)$ accordingly such that the above assumptions hold. For this T we can then fix $t \in [0, T]$ such that

$$\|\boldsymbol{\phi}(t)\|_{L^2}^2 = \sup_{s\in[0,T]}\|\boldsymbol{\phi}(s)\|_{L^2}^2 .$$

As a result, (3.17) holds for this t and we conclude $\|\boldsymbol{\phi}(t)\|_{L^2} = 0$ and thus $\boldsymbol{m}_1 = \boldsymbol{m}_2$ a.e. on $[0, T]$ by the definition of t as maximal point. Iterating this process now with the end point of the interval as new starting point, we see that we can extend the interval of coincidence maximally $I := [0, \tilde{T})$ with $0 < \tilde{T} \le T_0$. We claim $\tilde{T} = T_0$. Assume not. Let $(t_k) \subset I$ be a sequence such that $t_k \to \tilde{T}$. Since $\boldsymbol{m}_1, \boldsymbol{m}_2 \in C^0([0, T_0), L^2)$, the same holds for $\boldsymbol{\phi}$ and we deduce

$$0 = \lim_{t_k\to\tilde{T}}\|\boldsymbol{\phi}(t_k)\|_{L^2} = \|\boldsymbol{\phi}(\tilde{T})\|_{L^2},$$

But this contradicts the maximality of $\tilde{T}$. Hence, $\boldsymbol{\phi} = 0$ on $[0, T_0)$, which means that uniqueness of the solution with initial data $\boldsymbol{m}_0 \in H^1(\mathbb{S}^2, \mathbb{S}^2)$ is proved.

Step 4: Asymptotics. We want to study the behavior of $\boldsymbol{m}$ as $t \to \infty$. Two cases can occur: either $\boldsymbol{m}$ has bounded local energy beyond some time $T > 0$ or there is a blowup at infinity, i.e., energy concentration. We start with the first case. Assume that there exist $T > 1$ and $R > 0$ such that

$$\sup_{(x,t)\in\mathbb{S}^2\times(T,\infty)} \mathcal{E}(\boldsymbol{m}(t), B_R(x)) < \frac{E_\mathrm{b}}{2}.$$

By Definition 3.12 this implies that $\nabla\boldsymbol{m}$ is uniformly bounded on $\mathbb{S}^2 \times (T, \infty)$. In addition, let $(t_j) \subset \mathbb{R}$ be an arbitrary sequence with $t_j \to \infty$. Then, by Definition 3.7 we find a uniform constant $C > 0$ such that

$$\int_{t_j-1}^{t_j} \int_{\mathbb{S}^2} |\nabla^2\boldsymbol{m}|^2 \,\mathrm{d}\sigma\mathrm{d}t \le C\mathcal{E}(\boldsymbol{m}_0)(1 + R^{-1} + R^2)$$

Therefore, applying Definition 3.8—similarly as in *Step 1*—we find that $\boldsymbol{m}(t_j)$ is uniformly bounded in $H^2(\mathbb{S}^2, \mathbb{R}^3)$.
Hence, that after extracting a subsequence we get $\boldsymbol{m}_t(t_j) \to 0$ in $L^2(\mathbb{S}^2, \mathbb{R}^3)$ and $\boldsymbol{m}(t_j) \rightharpoonup \boldsymbol{m}_\infty$ weakly in $H^2(\mathbb{S}^2, \mathbb{R}^3)$ and strongly in $W^{1,4}(\mathbb{S}^2, \mathbb{S}^2)$ by the Rellich–Kondrachov theorem. Hence, we can pass into the limit $j \to \infty$ in (3.1) and conclude that $\boldsymbol{m}_\infty$ is a weak solution of (EL) and thus a smooth function on $\mathbb{S}^2$ by Definition 2.6. Moreover, we do not have to attach any harmonic maps at infinity, i.e., $K_\infty = 0$.
In the other case, there exist sequences $t_j \to \infty$ and $(x_j^0) \subset \mathbb{S}^2$ such that

$$\liminf_{j\to\infty} \mathcal{E}(\boldsymbol{m}(t_j), B_R(x_j^0)) \ge \frac{E_\mathrm{b}}{2}$$

for all $R > 0$. By compactness of $\mathbb{S}^2$ we can extract a subsequence such that $x_j^0 \to x_0$ and we can assume

$$\liminf_{j\to\infty} \mathcal{E}(\boldsymbol{m}(t_j), B_{2R}(x_0)) \ge \frac{E_\mathrm{b}}{2},$$

for all $R > 0$. Let $x_1, \dots, x_{K_\infty}$ be a finite amount of points such that the above holds. Note here that we always have the same fixed sequence in time. We call these points *blowup points at infinity* corresponding to (t_j). Choosing $R > 0$ such that the balls $B_{2R}(x_k)$ are mutually disjoint, we find

$$\mathcal{E}(\boldsymbol{m}_0) \ge \mathcal{E}(\boldsymbol{m}(t_j)) \ge \sum_{k=1}^{K_\infty} \mathcal{E}(\boldsymbol{m}(t_j), B_{2R}(x_k)) \ge K_\infty \frac{E_\mathrm{b}}{4},$$

for j sufficiently large. Accordingly, the number of blowup points at infinity K_∞ is bounded. Here we also note that the bound on K_∞ is independent of the sequence (t_j).

Thus, we can achieve that K_∞ is maximal and such that the points $x_1, \dots, x_{K_\infty}$ satisfy

$$\liminf_{j\to\infty} \mathcal{E}(\boldsymbol{m}(t_j), B_{2R}(x_k)) \geq \frac{E_\mathrm{b}}{2},$$

for all $R > 0$ and $1 \leq k \leq K_\infty$.

This means in particular that we cannot add another point $y \in \mathbb{S}^2$ distinct of the x_k to the set $\{x_1, \dots, x_{K_\infty}\}$ such that the above inequality holds, i.e., for all $y \in \mathbb{S}^2 \setminus \{x_1, \dots, x_{K_\infty}\}$ there exists an $R(y) > 0$ such that

$$\limsup_{j\to\infty} \mathcal{E}(\boldsymbol{m}(t_j), B_{2R(y)}(y)) < \frac{E_\mathrm{b}}{2}.$$

Note that this implies that $x_k \notin B_{R(y)}(y)$ for all $1 \leq k \leq K_\infty$ as otherwise we could find a ball around x_k inside $B_{2R(y)}(y)$ accumulating energy larger than $E_\mathrm{b}/2$, contradicting the definition of $R(y)$.

For $Q \subset\subset \mathbb{S}^2 \setminus \{x_1, \dots, x_{K_\infty}\}$ we assert that by the smoothness of $\boldsymbol{m}$ on $\overline{Q}$ the function $y \mapsto R(y)$ is continuous on $\overline{Q}$ and hence find by the compactness an $R = R(Q) > 0$ such that

$$\limsup_{j\to\infty} \mathcal{E}(\boldsymbol{m}(t_j), B_{2R}(y)) < \frac{E_\mathrm{b}}{2}$$

for all $y \in Q$. Then we set

$$\tau = \tau(Q) = \frac{E_\mathrm{b} R^2}{4C_1 \mathcal{E}(\boldsymbol{m}_0)},$$

where C_1 is again the constant from Definition 3.5. Since we can always decrease R without breaking the inequality, we can assume that always $\tau \leq 1$ and $R < R_\mathrm{s}$ hold, where R_s is the constant from Definition 3.12. Thus, we conclude by the statement of said lemma that there exists $j_0(Q) \in \mathbb{N}$ such that for all $j > j_0(Q)$ and $y \in Q$ it holds

$$\sup_{t_j \leq t \leq t_j+\tau} \mathcal{E}(\boldsymbol{m}(t), B_R) \leq \mathcal{E}(\boldsymbol{m}(t_j), B_{2R}(y)) + C_1 \frac{\tau}{R^2} \mathcal{E}(\boldsymbol{m}_0) < E_\mathrm{b}. \tag{3.18}$$

We now assume that $j > j_0(Q)$ holds. By compactness of $\overline{Q}$, we find a finite set of points $y_1, \dots, y_M$ such that $\overline{Q} \subset \cup_{i=1}^M B_R(y_i)$. Then, by means of Definition 3.7, we assert that

$$\int_{t_k}^{t_k+\tau}\int_{B_R(y_i)}|\nabla^2\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t\leq C\mathcal{E}(\boldsymbol{m}_0)\Big(1+\frac{\tau}{R^2}+C_\kappa R^2\tau^2\Big)=C(\boldsymbol{m}_0,Q,\kappa)$$

for all $1\leq i\leq M$. Using the fact that the $B_R(y_i)$ cover Q, we hence conclude

$$\int_{t_k}^{t_k+\tau}\int_{\Omega}|\nabla^2\boldsymbol{m}|^2\,\mathrm{d}\sigma\mathrm{d}t\leq C(\boldsymbol{m}_0,Q,\kappa)\,.$$

We want to stress that the constant and τ directly depend on Q.
Now let $(Q_l)_{l\in\mathbb{N}}\subset\mathbb{S}^2\setminus\{x_1,\ldots,x_{K_\infty}\}$ be an exhaustion of relatively compact sets such that $\overline{Q}_l\subset Q_{l+1}$ for all $l\in\mathbb{N}$. By the above bound we find for $l\in\mathbb{N}$ arbitrary but fixed a sequence of times $(t_{j,l})_{j\in\mathbb{N}}$ such that

$$t_j\leq t_{j,l}\leq t_j+\tau(Q_l)$$

and

$$\|\boldsymbol{m}(t_{j,l})\|_{H^2(Q_l)}\leq C_l\,.$$

Moreover, in view of Definition 3.5 which yields that

$$\int_{t_j}^{t_j+\tau}\int_{\mathbb{S}^2}|\boldsymbol{m}_t|^2\,\mathrm{d}\sigma\mathrm{d}t\longrightarrow 0\ ,\ \text{ as }\ j\to\infty\,,$$

we can also assume that $\|\boldsymbol{m}_t(t_{j,l})\|_{L^2(\mathbb{S}^2)}\to 0$ for $j\to\infty$. Extracting a subsequence we can even suppose that $\boldsymbol{m}(t_{j,l})\to\boldsymbol{m}_{\infty,l}$ in $W^{1,q}(Q_l,\mathbb{R}^3)$ for any $q\geq 2$ as $j\to\infty$. This implies $|\boldsymbol{m}_{\infty,l}(x)|=1$ for almost all $x\in Q_l$. Hence, going into the limit in (3.1) with the integral restricted to Q_l and a test function in $H^1_0\cap L^\infty(Q_l,\mathbb{R}^3)$, we conclude that $\boldsymbol{m}_{\infty,l}$ is a weak solution of (EL) on Q_l.
After doing this procedure for all $l\in\mathbb{N}$, we return to the original sequence (t_j). By Definition 3.4 we find that $\boldsymbol{m}(t_j)\rightharpoonup\boldsymbol{m}_\infty$ as $j\to\infty$ weakly in $H^1(\mathbb{S}^2,\mathbb{S}^2)$ and strongly in $L^2(\mathbb{S}^2,\mathbb{S}^2)$. We want to show that on Q_l it holds

$$\boldsymbol{m}_\infty\big|_{Q_l}=\boldsymbol{m}_{\infty,l}$$

for all $l\in\mathbb{N}$. To this end, we fix $l\in\mathbb{N}$ and estimate

$$\begin{aligned}&\|\boldsymbol{m}_\infty-\boldsymbol{m}_{\infty,l}\|_{L^2(Q_l)}\\ &\leq\|\boldsymbol{m}_\infty-\boldsymbol{m}(t_j)\|_{L^2(Q_l)}+\|\boldsymbol{m}(t_j)-\boldsymbol{m}(t_{j,l})\|_{L^2(Q_l)}+\|\boldsymbol{m}_{\infty,l}-\boldsymbol{m}(t_{j,l})\|_{L^2(Q_l)}\,.\end{aligned}$$

The first and third term on the right hand side can be made arbitrarily small by the strong L^2-convergence. The second term can be estimated by means of the fundamental theorem of calculus for space-time Sobolev functions [22, 5.9.2 Theorem 2] and Jensen's inequality as

$$\begin{aligned}\|\boldsymbol{m}(t_j)-\boldsymbol{m}(t_{j,l})\|^2_{L^2(Q_l)} &= \int_{Q_l}\Big|\int_{t_j}^{t_{j,l}} \boldsymbol{m}_t \,\mathrm{d}t\Big|^2 \mathrm{d}\sigma \\ &\le (t_{j,l}-t_j)\int_{Q_l}\int_{t_j}^{t_{j,l}} |\boldsymbol{m}_t|^2 \,\mathrm{d}t\,\mathrm{d}\sigma \\ &\le \underbrace{\tau(Q_l)}_{\le 1}\int_{t_j}^{\infty}\int_{Q_l} |\boldsymbol{m}_t|^2\,\mathrm{d}\sigma\,\mathrm{d}t \longrightarrow 0\,,\ \text{as } j\to\infty\,,\end{aligned}$$

by the integrability of $\|\boldsymbol{m}_t\|^2_{L^2(\mathbb{S}^2)}$ over $[0,\infty)$ according to Definition 3.4. Hence, we have shown that $\boldsymbol{m}_\infty = \boldsymbol{m}_{\infty,l}$ on Q_l for all $l \in \mathbb{N}$. Since the Q_l exhaust $\mathbb{S}^2 \setminus \{x_1,\dots,x_{K_\infty}\}$, and since $\boldsymbol{m}_{\infty,l} \in H^2(Q_l,\mathbb{S}^2)$ and since it is a weak solution of (EL) on Q_l, we conclude that $\boldsymbol{m}_\infty$ is in $H^2_{\mathrm{loc}}(\mathbb{S}^2,\mathbb{S}^2)$ and it is a weak solution of (EL) on $\mathbb{S}^2 \setminus \{x_1,\dots,x_{K_\infty}\}$. By Definition 2.7 we can extend $\boldsymbol{m}_\infty$ to a smooth solution of (EL) on $\mathbb{S}^2$.

Step 5: Blowup analysis. As we have seen, there are two types of singular points: blowup points in space-time as identified in *Step 2* and blowup points at infinity in time as dealt with in *Step 4*. We want to give a precise description for the limit behavior in the vicinity of the first type of points, the second type of points are analyzed further after that. We call the set of blowup points in space-time $\mathcal{S}$ such that $|\mathcal{S}| = \overline{K}$.

Let $(x_0,T)\in\mathcal{S}$. For any $r>0$ we set $B_r := B_r(x_0)$. Then

$$\limsup_{t\nearrow T}\mathcal{E}(\boldsymbol{m}(t),B_R)\ge E_{\mathrm{b}}$$

for all $R>0$. By finiteness of $\mathcal{S}$ we get that x_0 is an isolated point and we can choose $R_0>0$ such that $B_{2R_0}\times\{T\}\cap\mathcal{S}\setminus\{(x_0,T)\}=\emptyset$. We want to apply a result like Theorem 1 in [55]. We can mainly repeat the proof there and adjust it to our setting. We begin with the equivalent of Proposition 2.1 in [55], namely asserting that $\lim_{t\nearrow T}\int_{B_R}|\nabla\boldsymbol{m}(t)|^2\,\mathrm{d}\sigma$ exists and is finite for any $0<R<2R_0$. To this end, let $-\infty\le m\le M\le\infty$ and $t_i,s_i\nearrow T$ be such that

$$\limsup_{t\to T}\int_{B_R} e(\boldsymbol{m}(t))\,\mathrm{d}\sigma=\lim_{i\to\infty}\int_{B_R} e(\boldsymbol{m}(t_i))\,\mathrm{d}\sigma = M+\int_{B_R} e(\boldsymbol{m}(T))\,\mathrm{d}\sigma\,, \qquad (3.19)$$

and

$$\liminf_{t\to T}\int_{B_R} e(\boldsymbol{m}(t))\,\mathrm{d}\sigma = \lim_{i\to\infty}\int_{B_R} e(\boldsymbol{m}(s_i))\,\mathrm{d}\sigma = m + \int_{B_R} e(\boldsymbol{m}(T))\,\mathrm{d}\sigma\,. \tag{3.20}$$

Then we need to show that $m = M$. We define $r_j = R/j$ for $j \in \mathbb{N}$. As shown in *Step 1*, we know that $\boldsymbol{m}(t)$ converges strongly in H^1 to $\boldsymbol{m}(T)$ on any compact subset of $B_R \setminus \{x_0\}$. Thus, we can choose a subsequence $t_{i(j)}$ such that holds

$$\int_{B_R\setminus B_{r_j}} |e(\boldsymbol{m}(t_{i(j)})) - e(\boldsymbol{m}(T))|\,\mathrm{d}\sigma \le \frac{1}{2}\int_{B_{r_j}} e(\boldsymbol{m}(T))\,\mathrm{d}\sigma\,,$$

and such that in view of (3.19) we have

$$\int_{B_R} e(\boldsymbol{m}(t_{i(j)}))\,\mathrm{d}\sigma - \int_{B_R} e(\boldsymbol{m}(T))\,\mathrm{d}\sigma \ge M - \frac{1}{2}\int_{B_{r_j}} e(\boldsymbol{m}(T))\,\mathrm{d}\sigma\,.$$

Moreover, by picking elements later in the subsequence $i(j)$ if needed, we can also assure that $s_j \le t_{i(j)}$. Combining the two estimates and redefining $t_j = t_{i(j)}$ we find

$$\int_{B_{r_j}} e(\boldsymbol{m}(t_j))\,\mathrm{d}\sigma \ge M \tag{3.21}$$

with $s_j \le t_j$ for all $j \in \mathbb{N}$. Let $\delta > 0$ and j_0 such that $r_j < \delta$ for all $j \ge j_0$. Then, invoking Definition 3.6 we gain for all $j \ge j_0$ the estimate

$$\int_{B_\delta} e(\boldsymbol{m}(s_j))\,\mathrm{d}\sigma \ge \int_{B_{r_j}} e(\boldsymbol{m}(t_j))\,\mathrm{d}\sigma - C\frac{t_j - s_j}{(\delta - r_j)^2}\mathcal{E}(\boldsymbol{m}_0)\,.$$

By (3.21) and again under usage of strong H^1-convergence of $\boldsymbol{m}(s_j)$ to $\boldsymbol{m}(T)$ on $B_R \setminus B_\delta$, this gives us

$$\begin{aligned}
&\lim_{j\to\infty}\int_{B_R} e(\boldsymbol{m}(s_j))\,\mathrm{d}\sigma \\
&= \lim_{j\to\infty}\left(\int_{B_R\setminus B_\delta} e(\boldsymbol{m}(s_j))\,\mathrm{d}\sigma + \int_{B_\delta} e(\boldsymbol{m}(s_j))\,\mathrm{d}\sigma\right) \\
&\ge \lim_{j\to\infty}\left(\int_{B_R\setminus B_\delta} e(\boldsymbol{m}(s_j))\,\mathrm{d}\sigma + \int_{B_{r_j}} e(\boldsymbol{m}(t_j))\,\mathrm{d}\sigma - C\frac{t_j - s_j}{(\delta - r_j)^2}\mathcal{E}(\boldsymbol{m}_0)\right) \\
&\ge \int_{B_R\setminus B_\delta} e(\boldsymbol{m}(T))\,\mathrm{d}\sigma + M\,,
\end{aligned}$$

since we have

$$C\frac{t_j - s_j}{(\delta - r_j)^2}\mathcal{E}(\boldsymbol{m}_0) \longrightarrow C\frac{T - T}{\delta^2}\mathcal{E}(\boldsymbol{m}_0) = 0\,,\ \text{as}\ j \to \infty\,.$$

As $\delta > 0$ was arbitrary, we conclude that $m \geq M$ and thus $m = M$.
Therefore, we gain

$$\lim_{t \nearrow T} \int_{B_R} e(\boldsymbol{m}(t))\,\mathrm{d}\sigma = L + \int_{B_R} e(\boldsymbol{m}(T))\,\mathrm{d}\sigma\,,$$

for some $L \in \mathbb{R}$. In view of strong L^2-convergence of $\boldsymbol{m}(t)$ to $\boldsymbol{m}(T)$ we deduce that

$$\lim_{t \nearrow T} \int_{B_R} |\nabla \boldsymbol{m}(t)|^2\,\mathrm{d}\sigma = L + \int_{B_R} |\nabla \boldsymbol{m}(T)|^2\,\mathrm{d}\sigma\,. \tag{3.22}$$

We note here that a priori L depends on R. We show independence with the following argument. Let $L(R)$ and $L(R')$ be the limits for R and R', respectively. Without loss of generality we can assume that $R < R'$. Then, since $\boldsymbol{m}(t)$ converges strongly in H^1 to $\boldsymbol{m}(T)$ on $B_{R'} \setminus B_R$, we get

$$\begin{aligned}
L(R') + \int_{B_{R'}} |\nabla \boldsymbol{m}(T)|^2\,\mathrm{d}\sigma &= \lim_{t \nearrow T} \int_{B_{R'}} |\nabla \boldsymbol{m}(t)|^2\,\mathrm{d}\sigma \\
&= \lim_{t \nearrow T} \left(\int_{B_R} |\nabla \boldsymbol{m}(t)|^2\,\mathrm{d}\sigma + \int_{B_{R'} \setminus B_R} |\nabla \boldsymbol{m}(t)|^2\,\mathrm{d}\sigma \right) \\
&= L(R) + \int_{B_R} |\nabla \boldsymbol{m}(T)|^2\,\mathrm{d}\sigma + \int_{B_{R'} \setminus B_R} |\nabla \boldsymbol{m}(T)|^2\,\mathrm{d}\sigma \\
&= L(R) + \int_{B_{R'}} |\nabla \boldsymbol{m}(T)|^2\,\mathrm{d}\sigma\,,
\end{aligned}$$

whence $L(R) = L(R') =: L$.
We fix now $R = R_0$. We take the sequences $t_j \nearrow T$ and $r_j \to 0$ such that (3.21) holds for R_0 with $M = L$. As (t_j) is bounded, we can assume that $1 < t_j / r_j^2$ and $r_j < 1$. Using stereographic coordinates centered at x_0, we define the rescaled functions $\boldsymbol{v}_j$ by

$$\boldsymbol{v}_j(x) = \boldsymbol{m}(r_j x, -r_j^2 t + t_j)\,.$$

These maps are certainly well-defined on $\mathbb{R}^2 \times [0,1]$ and solve the equation (cf. Definition 3.10)

$$\Delta_{\mathbb{R}^2} \boldsymbol{v}_j + |\nabla \boldsymbol{v}_j|^2 = \lambda^2(r_j x)\Big(-\boldsymbol{v}_{j,t} - \kappa r_j^2 \boldsymbol{G}_j\Big) =: \boldsymbol{w}_j,$$

where the $\boldsymbol{G}_j$ are smooth and uniformly bounded functions. Then, asserting that $\lambda^2(r_j x) \leq 2$ for all $j \in \mathbb{N}$, for any bounded $\Omega \subset \mathbb{R}^2$ we obtain by Definition 3.4 that

$$\int_0^1 \int_\Omega \lambda^4(r_j x) |\partial_t \boldsymbol{v}|^2\,\mathrm{d}x\mathrm{d}t = \int_0^1 \int_\Omega r_j^4 \lambda^4(r_j x) |\boldsymbol{m}_t|^2 (r_j x, -r_j^2 t + t_j)\,\mathrm{d}x\mathrm{d}t$$

$$\leq 2 \int_{t_j - r_j^2}^{t_j} \int_{r_j \Omega} \lambda^2(x) |\boldsymbol{m}_t|^2 \, \mathrm{d}x\mathrm{d}t$$
$$\leq 2 \int_{t_j - r_j^2}^{t_j} \int_{\mathbb{S}^2} |\boldsymbol{m}_t|^2 \, \mathrm{d}\sigma\mathrm{d}t \longrightarrow 0 \,, \text{ as } j \to \infty \,,$$

by the absolute continuity of the Lebesgue integral. Additionally, we gain

$$\int_0^1 \int_\Omega \lambda^4(r_j x) r_j^4 |\boldsymbol{G}_j|^2 \, \mathrm{d}x\mathrm{d}t \leq C|\Omega| r_j^4 \longrightarrow 0 \,, \text{ as } j \to \infty \,.$$

Thus, we can conclude that there exists a sequence $(s_j) \subset [0,1]$ such that $\tau_j := \boldsymbol{w}_j(\cdot, s_j) \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ as $j \to \infty$. Moreover, by the conformal invariance of the Dirichlet energy we have

$$\int_{\mathbb{R}^2} |D\boldsymbol{v}_j(s_j)|^2 \, \mathrm{d}x = \int_{\mathbb{S}^2} |\nabla \boldsymbol{m}(t_j - r_j^2 s_j)|^2 \, \mathrm{d}\sigma \leq \mathcal{E}(\boldsymbol{m}_0) \,.$$

As a result, we can apply Proposition 1.2 in [55] and which provides us with harmonic maps $(\omega_i)_{i=0}^p$, mapping $\mathbb{S}^2 \to \mathbb{S}^2$, and sequences of scales $(\mu_j^i)_{i=1}^p \subset \mathbb{R}_+$ with $\lim_{j\to\infty} \mu_j^i = 0$ and points $(b_j^i)_{i=1}^p \subset \mathbb{R}^2$ with $\lim_{j\to\infty} b_j^i = b^i \in \mathbb{R}^2$ satisfying

$$\frac{\mu_j^\ell}{\mu_j^i} + \frac{\mu_j^i}{\mu_j^\ell} + \frac{|b_j^\ell - b_j^i|^2}{\mu_j^\ell \mu_j^i} \longrightarrow \infty \,, \text{ as } j \to \infty \,, \tag{3.23}$$

for $i \neq \ell$, such that if we define

$$\boldsymbol{f}_j(x) = \omega_0(\boldsymbol{\Phi}(x)) + \sum_{i=1}^p \left(\omega_i \left(\boldsymbol{\Phi} \left(\frac{x - b_j^i}{\mu_j^i} \right) \right) - \omega_i(-\hat{e}_3) \right),$$

then $\boldsymbol{v}_j(s_j) - \boldsymbol{f}_j$ converges to zero strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$ as $j \to \infty$. These are not yet the sequences of scales and points used in (ii) and (iii) of the theorem because we still need to rescale by r_j. Hence, we define $\boldsymbol{F}_j \colon B_{R_0} \to \mathbb{R}^3$ in stereographic coordinates centered at x_0 by

$$\boldsymbol{F}_j(x) = \boldsymbol{f}_j\left(\frac{x}{r_j}\right) - \omega_0(-\hat{e}_3) = \sum_{i=0}^p \left(\omega_i \left(\boldsymbol{\Phi} \left(\frac{x - a_j^i}{\lambda_j^i} \right) \right) - \omega_i(-\hat{e}_3) \right),$$

where $a_j^0 = 0$, $\lambda_j^0 = r_j$, and $a_j^i = r_j b_j^i$ and $\lambda_j^i = r_j \mu_j^i$ for $1 \leq i \leq p$. Then, $\lim_{j\to\infty} a_j^i = 0$ and $\lim_{j\to\infty} \lambda_j^i = 0$ for all $0 \leq i \leq p$. We readily check that the new parameters still satisfy

$$\frac{\lambda_j^\ell}{\lambda_j^i} + \frac{\lambda_j^i}{\lambda_j^\ell} + \frac{|a_j^\ell - a_j^i|^2}{\lambda_j^\ell \lambda_j^i} \longrightarrow \infty\,, \text{ as } j \to \infty\,, \tag{3.24}$$

i.e., (1.3).

It remains to show the strong convergence of $\boldsymbol{m}(T_j) - \boldsymbol{F}_j$ to $\boldsymbol{m}(T)$ in $H^1(B_{R_0})$ with $T_j = t_j - s_j r_j^2$. To this end, we want to show weak convergence and convergence in norm as $H^1(B_{R_0})$ is a Hilbert space. We already know that $\boldsymbol{m}(T_j)$ converges weakly to $\boldsymbol{m}(T)$ in $H^1(B_{R_0})$ since $T_j \nearrow T$ as $j \to \infty$. Thus, for weak convergence it suffices to check that $\boldsymbol{F}_j \rightharpoonup 0$ in $H^1(B_{R_0})$. To this end, we first we note that $\boldsymbol{F}_j$ is uniformly bounded in $H^1(B_{R_0})$ since it is a finite sum of the functions $\omega_j^i \colon B_{R_0} \to \mathbb{R}^3$ defined by

$$\omega_j^i(x) = \hat{\omega}_i\left(\frac{x - a_j^i}{\lambda_j^i}\right) - \hat{\omega}_i(\infty)\,.$$

in stereographic coordinates centered at x_0, where $\hat{\omega}_k = \omega_k \circ \boldsymbol{\Phi} \colon \mathbb{R}^2 \to \mathbb{S}^2$ is the representation of ω in stereographic coordinates centered at $\hat{e}_3$.

These functions are pointwise bounded and satisfy

$$\begin{aligned}\left\|\nabla \omega_j^i\right\|_{L^2(B_{R_0})}^2 &= \int_{B_{\tilde{R}_0}(0)} |D\hat{\omega}_i|^2\left(\tfrac{x-a_j^i}{\lambda_j^i}\right)\frac{\mathrm{d}x}{(\lambda_j^i)^2} \\ &\le \int_{\mathbb{R}^2} |D\hat{\omega}_i|^2\,\mathrm{d}x = \|\nabla\omega_i\|_{L^2(\mathbb{S}^2)}^2 < \infty\end{aligned}$$

for all $0 \le i \le p$. Here and later we denote by $\tilde{R}_0$ the corresponding radius of R_0 in stereographic coordinates, such that $B_{R_0} = \boldsymbol{\Phi}_{\overline{x}}(B_{\tilde{R}_0}(0))$. Therefore, we can extract a subsequence such that $\boldsymbol{F}_j \rightharpoonup \boldsymbol{F}$ in $H^1(B_{R_0}, \mathbb{R}^3)$ for some $\boldsymbol{F} \in H^1(B_{R_0}, \mathbb{R}^3)$. Now, for $x \in B_{\tilde{R}_0}(0) \setminus \{0\}$ we have

$$\frac{x - a_j^i}{\lambda_j^i} \longrightarrow \infty\,, \text{ as } j \to \infty\,,$$

since $a_j^i \to 0$, such that pointwise we deduce

$$\hat{\omega}_i\left(\frac{x - a_j^i}{\lambda_j^i}\right) - \hat{\omega}_i(\infty) \longrightarrow 0\,, \text{ as } j \to \infty$$

for all $0 \le i \le p$. Thus, since pointwise limit almost everywhere and weak limit have to coincide, we conclude that $F = 0$ almost everywhere in B_{R_0}, i.e., $\boldsymbol{F}_j \rightharpoonup 0$ in $H^1(B_{R_0})$. By the Rellich–Kondrachov theorem we also conclude that $\boldsymbol{F}_j \to 0$ in $L^2(B_{R_0})$.

In order to prove convergence in norm we only need to consider the gradient, since strong L^2-convergence of $\boldsymbol{m}(T_j) - \boldsymbol{F}_j$ to $\boldsymbol{m}(T)$ is already established. For readability purposes we now set $B_r =: B_r(0) \subset \mathbb{R}^2$. However, we still keep $B_{R_0} = B_{R_0}(\overline{x}) \subset \mathbb{S}^2$. We also recall that a ball of radius r on $\mathbb{R}^2$ in stereographic coordinates corresponds to a ball of geodesic radius $2\arctan(r)$ on $\mathbb{S}^2$. See also Definition 2.3 for the notational conventions of the following computations.

Let $\varepsilon > 0$ be arbitrary. We want to find $j_0 = j_0(\varepsilon) \in \mathbb{N}$ such that for all $j > j_0$ we get

$$\left| \|\nabla \boldsymbol{m}(T_j) - \nabla \boldsymbol{F}_j\|^2_{L^2(B_{R_0})} - \|\nabla \boldsymbol{m}(T)\|^2_{L^2(B_{R_0})} \right| \le \varepsilon .$$

To this end, let $K_0 > 0$ be a number to be chosen later and assume that j is large enough such that $\tilde{R}_0 / r_j > K_0$. Then we can compute by using the definition of $\boldsymbol{v}_j$ and $\boldsymbol{f}_j$, and the conformal invariance of the Dirichlet energy

$$\begin{aligned}
&\left| \|\nabla \boldsymbol{m}(T_j) - \nabla \boldsymbol{F}_j\|^2_{L^2(B_{R_0})} - \|\nabla \boldsymbol{m}(T)\|^2_{L^2(B_{R_0})} \right| \\
&= \left| \int_{B_{K_0}} |\nabla \boldsymbol{m}(T_j) - \nabla \boldsymbol{F}_j|^2 \,\mathrm{d}\sigma - \int_{B_{K_0}} |\nabla \boldsymbol{m}(T)|^2 \,\mathrm{d}\sigma \right| \\
&= \left| \int_{B_{\tilde{R}_0/r_j}} |D\boldsymbol{v}_j(s_j) - D\boldsymbol{f}_j|^2 \,\mathrm{d}x - \int_{B_{K_0}} |\nabla \boldsymbol{m}(T)|^2 \,\mathrm{d}\sigma \right| \\
&= \left| \int_{B_{K_0}} |D\boldsymbol{v}_j(s_j) - D\boldsymbol{f}_j|^2 \,\mathrm{d}x + \int_{B_{\tilde{R}_0/r_j} \setminus B_{K_0}} |D\boldsymbol{v}_j(s_j) - D\boldsymbol{f}_j|^2 \,\mathrm{d}x - \int_{B_{K_0}} |\nabla \boldsymbol{m}(T)|^2 \,\mathrm{d}\sigma \right| \\
&\le \underbrace{\int_{B_{K_0}} |D\boldsymbol{v}_j(s_j) - D\boldsymbol{f}_j|^2 \,\mathrm{d}x}_{=:I_1(K_0)} + \underbrace{\left| \int_{B_{\tilde{R}_0/r_j} \setminus B_{K_0}} |D\boldsymbol{v}_j(s_j)|^2 \,\mathrm{d}x - \int_{B_{K_0}} |\nabla \boldsymbol{m}(T)|^2 \,\mathrm{d}\sigma \right|}_{=:I_2(K_0)} \\
&\quad + \underbrace{\int_{B_{\tilde{R}_0/r_j} \setminus B_{K_0}} |D\boldsymbol{f}_j|^2 \,\mathrm{d}x + 2 \left| \int_{B_{\tilde{R}_0/r_j} \setminus B_{K_0}} \langle D\boldsymbol{v}_j(s_j), D\boldsymbol{f}_j \rangle \,\mathrm{d}x \right|}_{=:I_3(K_0)} ,
\end{aligned} \tag{3.25}$$

where in order to arrive to the last line we expanded the square in the second integral before applying the triangle inequality. We will consider each of the terms I_i separately.

First, for $K > 0$ we find $j_1 = j_1(K) \in \mathbb{N}$ such that for all $j \geq j_1$ it holds

$$I_1(K) = \int_{B_K} |D\boldsymbol{v}_j(s_j) - D\boldsymbol{f}_j|^2 \, \mathrm{d}x < \frac{\varepsilon}{4} \tag{3.26}$$

by the $H^1_{\mathrm{loc}}(\mathbb{R}^2)$-convergence provided by Proposition 1.2 in [55].
For I_2 we first estimate

$$\begin{aligned} I_2(K) &= \Big| \int_{B_{\tilde{R}_0/r_j} \setminus B_K} |D\boldsymbol{v}_j(s_j)|^2 \, \mathrm{d}x - \int_{B_{R_0}} |\nabla \boldsymbol{m}(T)|^2 \, \mathrm{d}\sigma \Big| \\ &= \Big| \int_{B_{\tilde{R}_0/r_j}} |D\boldsymbol{v}_j(s_j)|^2 \, \mathrm{d}x - \int_{B_{R_0}} |\nabla \boldsymbol{m}(T)|^2 \, \mathrm{d}\sigma - L + L - \int_{B_K} |D\boldsymbol{v}_j(s_j)|^2 \, \mathrm{d}x \Big| \\ &\leq \Big| \int_{B_{R_0}} |\nabla \boldsymbol{m}(T_j)|^2 \, \mathrm{d}\sigma - \int_{B_{R_0}} |\nabla \boldsymbol{m}(T)|^2 \, \mathrm{d}\sigma - L \Big| + \Big| L - \int_{B_K} |D\boldsymbol{v}_j(s_j)|^2 \, \mathrm{d}x \Big| . \end{aligned} \tag{3.27}$$

Now, we compute for $K > 1$ to be chosen subsequently

$$\begin{aligned} &\int_{B_K} |D\boldsymbol{v}_j(s_j)|^2 \, \mathrm{d}x = \int_{B_{r_j K}} |D\boldsymbol{m}(T_j)|^2 \, \mathrm{d}x = \int_{B_{2\arctan(r_j K)}(\overline{x})} |\nabla \boldsymbol{m}(T_j)|^2 \, \mathrm{d}\sigma \\ &= \mathcal{E}(\boldsymbol{m}(T_j), B_{2\arctan(r_j K)}(\overline{x})) - \kappa \int_{B_{2\arctan(r_j K)}(\overline{x})} 1 - \langle \boldsymbol{m}(T_j), \boldsymbol{\nu} \rangle^2 \, \mathrm{d}\sigma . \end{aligned} \tag{3.28}$$

For $r_j \leq 2$ we assert that

$$2\arctan(r_j K) > 2\arctan(r_j) \geq r_j .$$

Hence, we can invoke Definition 3.6, and get in combination with the definition of r_j in (3.21) that

$$\begin{aligned} \mathcal{E}(\boldsymbol{m}(T_j), B_{2\arctan(r_j K)}(\overline{x})) &\geq \mathcal{E}(\boldsymbol{m}(t_j), B_{r_j}) - C_1 \frac{t_j - T_j}{(2\arctan(r_j K) - r_j)^2} \mathcal{E}(\boldsymbol{m}_0) \\ &\geq L - C_1 \frac{r_j^2}{(2\arctan(r_j K) - r_j)^2} \mathcal{E}(\boldsymbol{m}_0) , \end{aligned}$$

where we used $t_j - T_j = s_j r_j^2 \leq r_j^2$.
Now, we choose $K_1 > 1$ such that for all $K \geq K_1$ it holds

$$\frac{C_1}{4} \mathcal{E}(\boldsymbol{m}_0) \frac{1}{(K-1)^2} < \frac{\varepsilon}{16} .$$

By Taylor's theorem there exists a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and

$$2\arctan(r_jK) - r_j = r_j\Big(2K - 1 + g(r_j^2K^3)\Big)\,.$$

Hence, there exists $j_{2,1}(K) \in \mathbb{N}$ such that for all $j \geq j_{2,1}(K)$ it holds

$$2\arctan(r_jK) - r_j \geq 2r_j(K-1)\,,$$

and additionally,

$$\Big|\kappa \int_{B_{2\arctan(r_jK)}(\overline{x})} 1 - \langle \boldsymbol{m}(T_j), \boldsymbol{\nu}\rangle^2 \,\mathrm{d}\sigma\Big| < \frac{\varepsilon}{16}\,,$$

by the absolute continuity of the Lebesgue integral and the fact that the integrand is uniformly bounded by the number 2. Putting these estimates into (3.28), we therefore conclude that for all $K \geq K_1$ and $j \geq j_1(K)$ it holds

$$\int_{B_K} |D\boldsymbol{v}_j(s_j)|^2 \,\mathrm{d}x \geq L - \frac{\varepsilon}{8}\,.$$

Now, we fix $0 < \delta < 1$ such that

$$\int_{B_\delta} |\nabla \boldsymbol{m}(T)|^2 \,\mathrm{d}\sigma < \frac{\varepsilon}{16}\,.$$

Then, in view of (3.22) we can find $j_{2,2} = j_{2,2}(K,\delta) \in \mathbb{N}$ such that for all $j \geq j_{2,2}$ it holds $\tan(\delta/2)/r_j > K$ and

$$\int_{B_\delta} |\nabla \boldsymbol{m}(T_j)|^2 \,\mathrm{d}\sigma < L + \int_{B_\delta} |\nabla \boldsymbol{m}(T)|^2 \,\mathrm{d}\sigma + \frac{\varepsilon}{16} < L + \frac{\varepsilon}{8}\,.$$

This yields then for these j that

$$\begin{aligned}\int_{B_K} |D\boldsymbol{v}_j(s_j)|^2 \,\mathrm{d}x &\leq \int_{B_{\tan(\delta/2)/r_j}} |D\boldsymbol{v}_j(s_j)|^2 \,\mathrm{d}x \\ &= \int_{B_{\tan(\delta/2)}} |D\boldsymbol{m}(T_j)|^2 \,\mathrm{d}x \\ &= \int_{B_\delta} |\nabla \boldsymbol{m}(T_j)|^2 \,\mathrm{d}\sigma < L + \frac{\varepsilon}{8}\end{aligned}$$

Finally, there exists $j_{2,3}$, such that according to (3.22) we have for all $j \geq j_{2,3}$ that

$$\Big|\int_{B_{R_0}} |\nabla \boldsymbol{m}(T_j)|^2 \,\mathrm{d}\sigma - \int_{B_{R_0}} |\nabla \boldsymbol{m}(T)|^2 \,\mathrm{d}\sigma - L\Big| < \frac{\varepsilon}{8}\,. \tag{3.29}$$

Setting $j_2(K) = \max\{j_{2,1}(K), j_{2,2}(K,\delta), j_{2,3}\}$ we therefore have shown that for all $K \geq K_1$ and $j \geq j_2(K)$ it holds

$$\Big| \int_{B_K} |D\boldsymbol{v}_j(s_j)|^2 \, \mathrm{d}x - L \Big| < \frac{\varepsilon}{8}, \tag{3.30}$$

and thus, in combination with (3.29) and (3.27) we obtain

$$I_2(K) \leq \frac{\varepsilon}{4}. \tag{3.31}$$

For I_3 we set $\tilde{\varepsilon} = \min\{\varepsilon, \varepsilon^2/(16\mathcal{E}(\boldsymbol{m}_0))\}$ and fix a value

$$K_2 > \max\{b^i : 1 \leq i \leq p\} + 2,$$

such that for all $K \geq K_2$ it holds

$$\int_{\mathbb{R}^2 \setminus B_K} |D\omega_0|^2 \, \mathrm{d}x < \frac{\tilde{\varepsilon}}{8(p+1)}. \tag{3.32}$$

Such a K_2 exists, since $\omega_0 \in H^1(\mathbb{S}^2, \mathbb{R}^3)$. Then we choose $j_3 = j_3(K) \in \mathbb{N}$ such that for all $j \geq j_2$ it holds, $|b_j^i - b^i| < 1$ and

$$\int_{\mathbb{R}^2 \setminus B_{1/\mu_j^i}} |D\omega_i|^2 \, \mathrm{d}x < \frac{\tilde{\varepsilon}}{8(p+1)}.$$

for all $1 \leq i \leq p$. Again, by the integrability of $|\nabla\omega_i|^2$ and the fact that $\mu_j^i \to 0$ we can find such a j_2.
With these definitions we obtain therefore for all $K \geq K_2$ and $j \geq j_2$ and $1 \leq i \leq p$ that

$$B_1(b_j^i) \subset B_2(b^i) \subset B_K.$$

And thus, by applying the change of coordinates $y = (x - b_j^i)/\mu_j^i$, we find

$$\begin{aligned}
\int_{B_{\tilde{R}_0/r_j} \setminus B_K} \left| D\left(\omega_i\left(\tfrac{x - b_j^i}{\mu_j^i}\right)\right)\right|^2 \mathrm{d}x &\leq \int_{\mathbb{R}^2 \setminus B_K} |D\omega_i|^2 \left(\tfrac{x - b_j^i}{\mu_j^i}\right) \frac{\mathrm{d}x}{(\mu_j^i)^2} \\
&\leq \int_{\mathbb{R}^2 \setminus B_1(b_j^i)} |D\omega_i|^2 \left(\tfrac{x - b_j^i}{\mu_j^i}\right) \frac{\mathrm{d}x}{(\mu_j^i)^2} \\
&= \int_{\mathbb{R}^2 \setminus B_{1/\mu_j^i}} |D\omega_i|^2 \, \mathrm{d}x < \frac{\tilde{\varepsilon}}{8(p+1)}.
\end{aligned}$$

Hence, we gain for all $K \geq K_2$ and $j \geq j_2$ that

$$\begin{aligned}\int_{B_{\tilde{R}_0/r_j}\setminus B_K} |D\boldsymbol{f}_j|^2 \,\mathrm{d}x &\leq 2\int_{B_{\tilde{R}_0/r_j}\setminus B_K} |D\omega_0|^2 \,\mathrm{d}x + 2\sum_{i=1}^{p}\int_{B_{\tilde{R}_0/r_j}\setminus B_K} |D\omega_i|^2 \,\mathrm{d}x \\ &< 2\frac{\tilde{\varepsilon}}{8(p+1)} + 2\sum_{k=1}^{p}\frac{\tilde{\varepsilon}}{8(p+1)} \leq \frac{\tilde{\varepsilon}}{4} \leq \frac{\varepsilon}{4}\,. \end{aligned} \tag{3.33}$$

Likewise, we estimate by means of the Cauchy–Schwarz inequality

$$\begin{aligned}&\Big|\int_{B_{\tilde{R}_0/r_j}\setminus B_K} \langle D\boldsymbol{v}_j(s_j), D\boldsymbol{f}_j\rangle \,\mathrm{d}x\Big| \\ &\leq \left(\int_{B_{\tilde{R}_0/r_j}\setminus B_K} |D\boldsymbol{v}_j(s_j)|^2 \,\mathrm{d}x\right)^{1/2}\left(\int_{B_{\tilde{R}_0/r_j}\setminus B_K} |D\boldsymbol{f}_j|^2 \,\mathrm{d}x\right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^2} |D\boldsymbol{v}_j(s_j)|^2 \,\mathrm{d}x\right)^{1/2}\left(\frac{\tilde{\varepsilon}}{4}\right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^2} |D\boldsymbol{m}(T_j)|^2 \,\mathrm{d}x\right)^{1/2}\frac{\varepsilon}{8\sqrt{\mathcal{E}(\boldsymbol{m}_0)}} \leq \frac{\varepsilon}{8}\,, \end{aligned} \tag{3.34}$$

using Definition 3.4 and the definition of $\tilde{\varepsilon}$. Consequently, we have shown that for all $K \geq K_2$ and $j \geq j_2(K)$ it holds

$$I_3(K) \leq \frac{\varepsilon}{2}\,. \tag{3.35}$$

We have now all parts together to prove the desired convergence in norm. We set $K_0 = \max\{K_1, K_2\}$ and $j_0 = \max\{j_1(K_0), j_2(K_0), j_3(K_0)\}$. Increasing j_0 if necessary, we can ensure that $\tilde{R}_0/r_j > K_0$ for all $j \geq j_0$. Then for all $j \geq j_0$ we obtain plugging the estimates (3.26), (3.31), and (3.35) into (3.25) the estimate

$$\left|\|\nabla\boldsymbol{m}(T_j) - \nabla\boldsymbol{F}_j\|^2_{L^2(B_{R_0})} - \|\nabla\boldsymbol{m}(T)\|^2_{L^2(B_{R_0})}\right| \leq \varepsilon\,.$$

Thus, convergence in norm is shown and hence $\boldsymbol{m}(T_j) - \boldsymbol{F}_j \to \boldsymbol{m}(T)$ strongly in $H^1(\mathbb{S}^2, \mathbb{R}^3)$.

For the blowup behavior of $\boldsymbol{m}$ at infinity we can apply the theory of approximate harmonic maps directly. By Definition 3.4 we know that $\|\boldsymbol{m}_t(t)\|^2_{L^2}$ is integrable over $[0, \infty)$ and thus we can pick a sequence $t_j \to \infty$ such that $\boldsymbol{m}_t(t_j) \to 0$ in $L^2(\mathbb{S}^2, \mathbb{R}^3)$. Furthermore by the very same lemma we conclude that $\boldsymbol{m}(t_j)$ is a bounded sequence in $H^1(\mathbb{S}^2, \mathbb{R}^3)$. Consequently, we can extract a subsequence such that for $\boldsymbol{m}_j := \boldsymbol{m}(t_j)$ we find $\boldsymbol{m}_j \rightharpoonup \boldsymbol{m}_\infty$ weakly in $H^1(\mathbb{S}^2, \mathbb{R}^3)$ and $\boldsymbol{m}_j \to \boldsymbol{m}_\infty$ strongly in $L^p(\mathbb{S}^2, \mathbb{R}^3)$ by virtue of the Rellich–Kondrachov theorem and Hölder

inequality for any $p \geq 2$. This implies that $|\boldsymbol{m}_\infty(x)| = 1$ for almost every $x \in \mathbb{S}^2$ and thus $\boldsymbol{m}_\infty \in H^1(\mathbb{S}^2, \mathbb{S}^2)$.
Consequently, in the language of [69], $(\boldsymbol{m}_j) \subset H^1(\mathbb{S}^2, \mathbb{S}^2)$ solves the equation

$$\Delta \boldsymbol{m}_j + |\nabla \boldsymbol{m}_j|^2 \boldsymbol{m}_j = \tau_j \,,$$

with tension fields

$$\tau_j = \boldsymbol{m}_t(t_j) - \kappa \langle \boldsymbol{m}_j, \boldsymbol{\nu} \rangle (\boldsymbol{\nu} - \langle \boldsymbol{m}_j, \boldsymbol{\nu} \rangle \boldsymbol{m}_j)$$

such that

$$\tau_j \to -\kappa \langle \boldsymbol{m}_\infty, \boldsymbol{\nu} \rangle (\boldsymbol{\nu} - \langle \boldsymbol{m}_\infty, \boldsymbol{\nu} \rangle \boldsymbol{m}_\infty) \,, \text{ as } j \to \infty$$

in $L^2(\mathbb{S}^2, \mathbb{R}^3)$. Applying Theorem 1.2 in [69] we conclude the existence of the harmonic maps $\omega_1, \dots, \omega_p$, sequences of points $(a_j^1) \dots, (a_j^p)$, and scales $(\lambda_j^1) \dots, (\lambda_j^p)$ for some $p \in \mathbb{N}$ satisfying (iv) – (vi) of the theorem such that (1.3) and (1.4) hold. □

Remark 3.14. In both cases of blowup at a finite time or at infinity we have only stated the H^1-convergence of the respective blowup trees. However, it is a result from the theory of approximate harmonic maps that the convergence is also true in L^∞-norm [69, 46]. This is the so-called no-neck-property of the blowup. In particular, we have

$$\Big\| \boldsymbol{m}(t_j) - \boldsymbol{m}(\bar{t}) - \sum_{i=1}^{p} \omega_j^i \Big\|_{L^\infty} \longrightarrow 0 \,, \text{ as } j \to \infty \,,$$

where the norm is taken over the appropriate set for finite time blowup or blowup at infinity. Moreover, in both cases, in the respective stereographic coordinates (centered at x_0 or at the north pole) the sequences (λ_j^i) and (a_j^i) satisfy

$$\boldsymbol{m}(\lambda_j^i x + a_j^i, t_j) \longrightarrow \omega_i(x) \,, \text{ as } j \to \infty \,,$$

in $H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^3)$ and in $L^\infty_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^3)$ for all $1 \leq i \leq p$ [69, cf. p. 20]. ⊡

3.2. Axisymmetric maps

In this section we inspect the skyrmionic map heat flow in the case of axisymmetric maps which we introduced in Section 2.3. We show that for axisymmetric initial data the solution to the SMHF is also axisymmetric. This is a consequence of the fact that the energy functional is invariant under the actions induced by $O(3)_{\hat{e}_3}$. This observation allows us to reduce the problem to a one-dimensional parabolic equation on the interval $[0,\pi]$.

Proposition 3.15. *Let $\boldsymbol{m}_0 \in H^1(\mathbb{S}^2,\mathbb{S}^2)$ be axisymmetric and let $h_0\colon [0,\pi] \to \mathbb{R}$ be its profile with $h_0(0) = m\pi$, $h_0(\pi) = n\pi$ for some $m,n \in \mathbb{Z}$.*
Then the solution $\boldsymbol{m}$ to (SMHF) *with initial data $\boldsymbol{m}_0$ provided by Theorem 3 is also axisymmetric for all $t \in (0,T)$, where $T > 0$ is the maximal existence time until the first blowup, if present. So there exists a map $h : C^\infty([0,\pi] \times (0,T))$ such that*

$$\boldsymbol{m}(\theta,\varphi,t) = \boldsymbol{\Psi}(h(\theta,t),\varphi)\,. \tag{3.36}$$

This map satisfies

$$\begin{aligned} h_t &= h_{\theta\theta} + \frac{\cos\theta}{\sin\theta} h_\theta - \frac{\sin 2h}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2h-2\theta), \\ h(\theta,0) &= h_0(\theta) \quad \textit{for all } \theta \in [0,\pi], \\ h(0,t) &= m\pi, \quad h(\pi,t) = n\pi\,. \end{aligned} \tag{3.37}$$

Proof. Let $R \in O(3)_{\hat{e}_3}$ be arbitrary. We want to show that $\boldsymbol{m}_R$ also solves (SMHF). Generally, we can write the matrix R as

$$R = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

for some $A \in O(2)$. Then, writing $x \in \mathbb{S}^2$ in stereographic coordinates centered at the north pole, i.e., $x = (x_1,x_2)^T \in \mathbb{R}^2$, we have

$$R.x = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Consequently, we can write for the representation of the map $\boldsymbol{m}_R$ in stereographic coordinates

$$\boldsymbol{m}_R = R^{-1}\boldsymbol{m} \circ A\,.$$

Using the identity

$$\Delta(u \circ B) = (\Delta u) \circ B$$

for any $u \in C^2(\mathbb{R}^n,\mathbb{R})$ and $B \in O(n)$ we get by using stereographic coordinates and the above notation

$$\Delta_{\mathbb{S}^2}(\boldsymbol{m} \circ R) = \lambda^{-2}(x)\Delta_{\mathbb{R}^2}(\boldsymbol{m} \circ A)$$

$$
\begin{aligned}
&= \lambda^{-2}(x)(\Delta_{\mathbb{R}^2}\boldsymbol{m}) \circ A \\
&= (\Delta_{\mathbb{S}^2}\boldsymbol{m}) \circ R\,.
\end{aligned}
$$

In (2.11) and (2.12) we have already seen that it holds

$$
|\nabla \boldsymbol{m}_R(x)|^2 = |\nabla \boldsymbol{m}|^2(R.x)\,, \quad \boldsymbol{\nu}(R.x) = R\boldsymbol{\nu}(x) \quad \text{and} \quad \langle \boldsymbol{m}_R, \boldsymbol{\nu}\rangle(x) = \langle \boldsymbol{m}, \boldsymbol{\nu}\rangle(R.x)\,.
$$

Combining the pieces, we conclude

$$
\begin{aligned}
&\Big(\Delta \boldsymbol{m}_R + \kappa\langle \boldsymbol{m}_R, \boldsymbol{\nu}\rangle\boldsymbol{\nu} + \boldsymbol{m}_R\Big(|\nabla \boldsymbol{m}_R|^2 - \kappa\langle \boldsymbol{m}_R, \boldsymbol{\nu}\rangle^2\Big)\Big)(x,t) \\
&= R^{-1}\Big(\Delta \boldsymbol{m} + \kappa\langle \boldsymbol{m}, \boldsymbol{\nu}\rangle\boldsymbol{\nu} + \boldsymbol{m}\Big(|\nabla \boldsymbol{m}|^2 - \kappa\langle \boldsymbol{m}, \boldsymbol{\nu}\rangle^2\Big)\Big)(R.x,t) \\
&= R^{-1}\,\partial_t \boldsymbol{m}(R.x,t) \\
&= \partial_t \boldsymbol{m}_R(x,t)\,.
\end{aligned}
$$

Hence, $\boldsymbol{m}_R$ solves (SMHF) with initial data

$$
\boldsymbol{m}_R(x,0) = R^{-1}\boldsymbol{m}(R.x,0) = R^{-1}\boldsymbol{m}_0(R.x) = \boldsymbol{m}_0(x)\,,
$$

since $\boldsymbol{m}_0$ is axisymmetric by assumption. As a result, by the uniqueness of the solution it follows that $\boldsymbol{m}_R(t) = \boldsymbol{m}(t)$ for all $t \in (0,T)$. As $R \in O(3)_{\hat{e}_3}$ was arbitrary, we deduce that $\boldsymbol{m}(t)$ is axisymmetric. Therefore, for every $t \in (0,T)$ we get a map $h(\theta,t)$ such that (3.36) holds. Smoothness in θ of h follows from Definition 2.13 and the smoothness of $\boldsymbol{m}$. In the same way we can show smoothness in t. Inserting the representation (3.36) of $\boldsymbol{m}$ into (SMHF) gives us (3.37) as in the proof of Definition 2.17. □

Of course, we can also go the other way round and construct solutions of the skyrmionic map heat flow equation by means of solutions of the profile equation (3.37).

Corollary 3.16. *Let $h_0 \in \mathcal{PF}^1$ with $h_0(0) = m\pi$, $h_0(\pi) = n\pi$ for some $m,n \in \mathbb{Z}$. Then there exists $T > 0$ and a solution $h \in C^\infty([0,\pi] \times (0,T))$ for the parabolic problem*

$$
\begin{aligned}
h_t &= h_{\theta\theta} + \frac{\cos\theta}{\sin\theta}h_\theta - \frac{\sin 2h}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2h-2\theta), \\
h(\theta,0) &= h_0(\theta) \qquad \text{for all } \theta \in [0,\pi], \\
h(0,t) &= m\pi, \quad h(\pi,t) = n\pi\,.
\end{aligned}
$$

Moreover, the function $\boldsymbol{m}(\theta,\varphi,t) = \boldsymbol{\Psi}(h(\theta,t),\varphi)$ solves (SMHF) *with initial value $\boldsymbol{m}_0(\theta,\varphi) = \boldsymbol{\Psi}(h_0(\theta),\varphi)$.*

Proof. By Definition 2.15 we deduce that $\boldsymbol{m}_0 \in H^1(\mathbb{S}^2, \mathbb{S}^2)$. Therefore, by the previous proposition we conclude the statement. $\square$

We have established that the flow preserves the axisymmetry up until the point of a possible blowup. Furthermore, the flow also preserves the boundary value of the profile which splits the function space into finer topological classes than the mapping degree.

Definition 3.17. For $m, n \in \mathbb{Z}$ we define

$$E_{m,n} = \{h \in \mathcal{PF} : h(0) = m\pi, h(\pi) = n\pi\}. \qquad \boxdot$$

Accordingly, if $h_0 \in E_{m,n}$ then for the solution h of (3.37) it holds that $h(., t) \in E_{m,n}$ for all $t \in (0, T)$.
The symmetry also constrains the only possible location of a blowup to be at the poles and the separating harmonic function is an axisymmetric map. In fact, the only axisymmetric harmonic maps are of the form

$$\boldsymbol{u}(\theta, \varphi) = \boldsymbol{\Psi}(\beta(\theta), \varphi),$$

where

$$\beta_{a,b}(\theta) = 2\arctan\left(a \tan\left(\frac{\theta}{2}\right)\right) + b\pi$$

with $a \in \mathbb{R}$ and $b \in \{0, 1\}$. This restricts these maps to degree 0 and ± 1. The following lemma is an adaption of the results in [6]. A similar result has also been stated in [4], but to our understanding the proof provided is not correct.

Lemma 3.18. *Let $\boldsymbol{m}_0 \in H^1(\mathbb{S}^2, \mathbb{S}^2)$ be axisymmetric and let $\boldsymbol{m}\colon \mathbb{S}^2 \times (0, T) \to \mathbb{S}^2$ be the solution to* (SMHF) *with initial data $\boldsymbol{m}_0$ provided by Theorem 3. If $\boldsymbol{m}$ blows up at $(x_0, T) \in \mathbb{S}^2 \times (0, \infty]$ in the sense that for all $R > 0$ and some sequence $T_j \nearrow T$ it holds*

$$\limsup_{j \to \infty} \int_{B_R(x_0)} |\nabla \boldsymbol{m}(T_j)|^2 \, d\sigma > \frac{E_b}{2},$$

then $x_0 = \pm \hat{e}_3$.
Furthermore, we can choose the sequences $(\lambda_j^i) \subset \mathbb{R}$ and $(a_j^i) \subset \mathbb{R}^2$ such that for the harmonic maps ω_i separating at x_0 along the sequence $T_j \nearrow T$ as in (1.2) *and* (1.4) *it holds that ω_i is axisymmetric for all $1 \leq i \leq p$ with nonconstant profile β_i. In addition, for the function $h \in E_{m,n}$ satisfying* (3.36) *it holds for every $0 < \theta_0 < \pi$ that*

$$h\left(2\arctan\left(\lambda_j^i \tan\left(\frac{\theta}{2}\right)\right), T_j\right) \longrightarrow \beta_i(\theta) + k\pi \quad , \text{ as } j \to \infty,$$

for some $k \in \mathbb{Z}$ and uniformly for all $\theta \in [0, \theta_0]$. For blowup at the south pole, a similar result holds by a change of coordinates.

Remark 3.19. Defining for $\theta \in (0, \pi)$ fixed the sequence

$$\theta_j^i = 2 \arctan\left(\lambda_j^i \tan\left(\frac{\theta}{2}\right)\right),$$

we therefore conclude that $\theta_j^i \to 0$ and $h(\theta_j^i, T_j) \to \beta_i(\theta) + k\pi \notin \pi\mathbb{Z}$ as $j \to \infty$. ⊡

Proof. By the axisymmetry it follows that $R.x_0$ is also a singular point for all $R \in O(3)_{\hat{e}_3}$ and because $O(3)_{\hat{e}_3}$ is a Lie-group, by the finiteness of the singular set, only $x_0 = \pm\hat{e}_3$ is possible, since for these points we have $R.x_0 = x_0$. Without loss of generality we assume now $x_0 = \hat{e}_3$ is the only blowup occurring at $0 < T \leq \infty$. We now need to do a finer analysis in the blowup analysis *Step 5* of the proof of Theorem 3. We treat the cases of finite time blowup and blowup at infinity simultaneously. In both cases, the sequences λ_j^i and a_j^i are chosen such that we have the energy estimate

$$\int_{B_{\lambda_j^i}(a_j^i)} |D\boldsymbol{m}(T_j)|^2 \, \mathrm{d}x > \varepsilon_i > 0 \tag{3.38}$$

for all $j \in \mathbb{N}$ and $1 \leq i \leq p$ (cf. the proofs of Proposition 1.2 in [55] and of Theorem 1.2 in [69]). The ε_i are fixed positive numbers. Moreover, as explained in Definition 3.14, we have the convergence

$$\boldsymbol{m}(\lambda_j^i x + a_j^i, T_j) \longrightarrow \omega_i(x)\,. \tag{3.39}$$

in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$, where both maps are their representations in stereographic coordinates centered at the north pole, since it holds $x_0 = \hat{e}_3$.
For every $1 \leq i \leq p$ fixed we want to construct a new harmonic map $\tilde{\omega}_i$ from ω_i which is axisymmetric. To lighten the notation, we will drop the index i for now. We use stereographic coordinates centered at the north pole. We claim that the sequence a_j/λ_j is bounded in $\mathbb{R}^2$. Assume not. Then we find a subsequence such that $a_j/\lambda_j \to \infty$. In particular, we can assume that $a_j > \lambda_j/2\sin(\pi/j)$ for all $j \in \mathbb{N}$. As the euclidean distance between two points on a circle with radius r and angle φ between them is given by $2\sin(\varphi/2)$, we can therefore find j balls with radius λ_j and center points with distance $|a_j|$ to the origin which are pairwise disjoint. By the axisymmetry of $\boldsymbol{m}$ we then get the estimate (3.38) on each of these

balls. This implies then by Definition 3.4

$$\mathcal{E}(\boldsymbol{m}_0) \geq \frac{1}{2}\int_{\mathbb{R}^2} |D\boldsymbol{m}|^2 \,\mathrm{d}x \geq \frac{j\varepsilon_i}{2} \longrightarrow \infty\,, \text{ as } j \to \infty\,,$$

a contradiction. Consequently, the sequence a_j/λ_j is bounded and by the Bolzano-Weierstraß theorem we can extract a subsequence such that $a_j/\lambda_j \to \xi$ for some $\xi \in \mathbb{R}^2$. Now, we define the new harmonic map $\tilde{\omega}\colon \mathbb{S}^2 \to \mathbb{S}^2$ through its representation in stereographic coordinates centered at the origin by

$$\tilde{\omega}(x) = \omega(x - \xi)\,.$$

For this we first assert that translations preserve harmonicity for maps $\mathbb{R}^2 \to \mathbb{S}^2$ and thus, we can conclude that $\tilde{\omega}$ is harmonic with

$$\tilde{\omega}(-\hat{e}_3) = \tilde{\omega}(\infty) = \omega(\infty) = \omega(-\hat{e}_3)\,.$$

We now claim

$$\boldsymbol{m}(\lambda_j x) \longrightarrow \tilde{\omega}(x)\,, \text{ as } j \to \infty\,, \tag{3.40}$$

in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$ and also $H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$. In order to prove this, we first define $y_j = \xi - a_j/\lambda_j$ and find that $y_j \to 0$ as $j \to \infty$. Let $U \subset \mathbb{R}^2$ be a compact set. Then there is an open set $V \supset U$ such that $\overline{V}$ is compact and for j large enough we have $U + y_j \subset V$. Then we gain for $x \in U$

$$\begin{aligned}
&|\boldsymbol{m}(\lambda_j x) - \tilde{\omega}(x)| \\
&= |\boldsymbol{m}(\lambda_j(x - \xi + y_j) + a_j) - \omega(x - \xi)| \\
&\leq |\boldsymbol{m}(\lambda_j(x - \xi + y_j) + a_j) - \omega(x - \xi + y_j)| + |\omega(x - \xi + y_j) - \omega(x - \xi)| \\
&\leq \|\boldsymbol{m}(\lambda_j \cdot + a_j) - \omega\|_{L^\infty(V-\xi)} + \|\nabla\omega\|_{L^\infty(V-\xi)}|y_j| \longrightarrow 0\,, \text{ as } j \to \infty\,,
\end{aligned}$$

independently of x, by the convergence of $\boldsymbol{m}$ to ω in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$ and the smoothness of ω. Thus, we have shown L^∞_{loc}-convergence. A very similar argument can be used to show H^1_{loc}-convergence by using a substitution $x \mapsto x - y_j$ in the integrals.

We want to show that $\tilde{\omega}$ is axisymmetric, i.e., that for all $R \in O(3)_{\hat{e}_3}$ and all $x \in \mathbb{S}^2$ it holds $\tilde{\omega}_R(x) = \tilde{\omega}(x)$. For $x \neq -\hat{e}_3$ we can use the stereographic coordinates centered at the north pole. Then, as explained in Section 2.3, there is $A \in O(2)$ representing the action $R.x = Ax$ in these coordinates. Using this we conclude from the axisymmetry of $\boldsymbol{m}$ that

$$\begin{aligned}\boldsymbol{m}(\lambda_j x) = \boldsymbol{m}_R(\lambda_j x) = R^{-1}\boldsymbol{m}(A\lambda_j x) = R^{-1}\boldsymbol{m}(\lambda_j A x) \\ \longrightarrow R^{-1}\tilde{\omega}(Ax) = \tilde{\omega}_R(x)\,, \text{ as } j \to \infty\,,\end{aligned}$$

in $L^\infty_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^3)$. By the uniqueness of the limit we conclude that $\tilde{\omega}_R = \tilde{\omega}$ on $\mathbb{S}^2 \setminus \{-\hat{e}_3\}$ and by continuity on the whole sphere. Since R was arbitrary, we have shown that $\tilde{\omega}$ is axisymmetric and we can express it in polar stereographic coordinates as

$$\tilde{\omega}(r, \varphi) = \boldsymbol{\Psi}(\beta(\theta(r)), \varphi)$$

for some β with parameter $a > 0$.

Doing the preceding procedure for all $1 \leq i \leq p$, we find axisymmetric maps $\tilde{\omega}_i$ with nonconstant profile β_i. From the condition (1.3) and the established fact that a^i_j/λ^i_j is bounded for all i it follows that already

$$\frac{\lambda^i_j}{\lambda^k_j} + \frac{\lambda^k_j}{\lambda^i_j} \longrightarrow \infty\,, \text{ as } j \to \infty$$

has to hold. Hence we can set $a^i_j = 0$ and $\omega_i = \tilde{\omega}_i$ for all $1 \leq i \leq p$. The blowup tree result, i.e., convergence as in (1.4) including energy identity and no neck property follow now from (3.40) as in [69]. Consequently, we have thus proven the first assertion of the lemma.

It remains to show the convergence of the profiles. To this end, we again fix $1 \leq i \leq p$ and drop the index. Let $0 < \theta_0 < \pi$ be arbitrary. We recall that the distance of a point $x(\theta, \varphi) \in \mathbb{S}^2$ parametrized by $(\theta, \varphi) \in [0, (\theta_0/2)] \times (0, 2\pi]$ to the north pole in stereographic coordinates is given by $r(\theta) = \tan(\theta/2)$. Thus, the set expressed by $[0, \theta_0] \times [0, 2\pi)$ in spherical coordinates (which is a spherical cap around the north pole) corresponds to the compact set $\Omega = \overline{B_{\tan(\theta_0/2)}}(0) \subset \mathbb{R}^2$ in stereographic coordinates. Hence, taking the points $x(\theta, \varphi) \in \mathbb{S}^2$ parametrized by $\theta \in [0, \theta_0/2]$ and $\varphi = 0$ we find in view of (3.40) the uniform convergence

$$|\omega(\tan(\theta/2), 0) - \boldsymbol{m}(\lambda_j \tan(\theta/2), 0)| \longrightarrow 0\,, \text{ as } j \to \infty$$

on $[0, \theta_0]$, where both maps are now parametrized in polar stereographic coordinates. For $\theta \in [0, \theta_0]$ we define

$$\theta_j = 2\arctan\left(\lambda_j \tan\left(\frac{\theta}{2}\right)\right).$$

Then for the respective profiles β and h we conclude from the components

$$
\begin{aligned}
|\sin(h(\theta_j, T_j)) - \sin(\beta(\theta))| &\longrightarrow 0 \\
|\cos(h(\theta_j, T_j)) - \cos(\beta(\theta))| &\longrightarrow 0\,, \text{ as } j \to \infty\,,
\end{aligned}
$$

again independently of θ. Using the trigonometric identities

$$
\begin{aligned}
\sin(x) - \sin(y) &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \\
\cos(x) - \cos(y) &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right),
\end{aligned}
$$

this then implies that $h(\theta_j, T_j) \to \beta(\theta) + m\pi$ for some $m \in \mathbb{N}$, again uniformly. By the continuity of the functions involved this m is independent of θ. This concludes the proof. □

For the harmonic map heat flow there exist stronger results, e.g., [36, 6]. There, the authors prove a blowup tree result for the profiles themselves, in particular independent of the sequence (T_j) in the first source. The way they do it, is by dealing with the parabolic equation for the profile and do similar steps as for the general case. We presume that one could reproduce these results for the skyrmionic map heat flow as well. However, for our purposes, Definition 3.18 is sufficient.

4. Existence of a Saddle Point

In this chapter, we want to prove the existence of a saddle point of $\mathcal{E}$ in the class of admissible functions in $H^1(\mathbb{S}^2, \mathbb{S}^2)$ with mapping degree 0. To this end, we will take the skyrmionic map heat flow and show that for a special class of initial data of degree 0 no blowup can occur, meaning that the limit solution $\boldsymbol{m}_\infty$ indeed is a critical point of $\mathcal{E}$ that has the same degree as the initial data. Then, we will show that in any neighborhood of such a limit function there is another function with lower energy which is sufficient for $\boldsymbol{m}_\infty$ being a saddle point.
We return to the axisymmetric setting dealt with in Section 2.3.

4.1. Comparison principle for profiles

Since the evolution equation for the profiles (3.37) is a semi-linear parabolic equation in one space dimension, we find that we can show an important tool – a so-called comparison principle. This principle states that solutions without crossing points initially will not develop them at later times. Moreover, if one of the functions is only a super- or subsolution of (3.37) then this property holds 'in one direction'. We begin with the first case.

Lemma 4.1. *Let $h_1 \in E_{m_1,n_1}$, and $h_2 \in E_{m_2,n_2}$ be two solutions of* (3.37) *with $h_1(\theta, 0) \leq h_2(\theta, 0)$ for all $\theta \in [0, \pi]$. Then $h_1(\theta, t) \leq h_2(\theta, t)$ for all $(\theta, t) \in [0, \pi] \times [0, T)$, where $T > 0$ is the maximal existence time of the solutions.*

Proof. First, $h_1(\theta, 0) \leq h_2(\theta, 0)$ implies $m_1 \leq m_2$ and $n_1 \leq n_2$. We define $\Delta = h_2 - h_1$ and obtain $\Delta_0 = h_2(0) - h_1(0) \geq 0$ and $\Delta(0, t) = (m_2 - m_1)\pi =: m_0\pi \geq 0$ and $\Delta(\pi, t) = (n_2 - n_1)\pi =: n_0\pi \geq 0$ for all $t \in [0, T)$. Since h_1 and h_2 solve (3.37), we check that Δ solves

$$
\begin{aligned}
0 &= \Delta_t - \Delta_{\theta\theta} - \frac{\cos\theta}{\sin\theta}\Delta_\theta + \frac{\sin 2h_2 - \sin 2h_1}{2\sin^2\theta} + \frac{\kappa}{2}(\sin(2h_2 - 2\theta) - \sin(2h_1 - 2\theta)) \\
&= \Delta_t - \Delta_{\theta\theta} - \frac{\cos\theta}{\sin\theta}\Delta_\theta + \Bigg(\underbrace{\kappa\frac{\sin\Delta}{\Delta}\cos(h_2 + h_1 - 2\theta)}_{=:c_1(\theta,t)} + \underbrace{\frac{\sin\Delta}{\Delta}\frac{\cos(h_2 + h_1)}{\sin^2\theta}}_{=:c_2(\theta,t)}\Bigg)\Delta\,,
\end{aligned}
$$

where in the last step we used the trigonometric identity

$$\sin(2a) - \sin(2b) = 2\sin(a-b)\cos(a+b)\,.$$

We want to apply Definition A.5. For this, let $\tilde{T} \in (0, T)$ be arbitrary. Then Δ certainly solves the equation on $I_{\tilde{T}} = (0, \pi) \times (0, \tilde{T}]$ with initial data $\Delta_0 \geq 0$ and boundary values $\Delta(0, t) = m_0\pi$ and $\Delta(\pi, t) = n_0\pi$. We have to show that $c_1 + c_2$ is bounded from below.
First of all, as a product of bounded functions the function c_1 is bounded from below by $-\delta_1$ for some $\delta_1 > 0$. Hence, we only need to consider c_2. We see that c_2 is a product of bounded functions and $\sin^{-2}$ which is positive on $(0, \pi)$ and diverges to $+\infty$ for $\theta \to 0$ and $\theta \to \pi$. Thus, since we want a lower bound for c_2, it suffices to inspect the sign of

$$f(\theta) := \frac{\sin\Delta}{\Delta}\cos(h_2 + h_1)$$

close to 0 and π.
Now, four cases can occur: $m_0 = 0$ or $m_0 \geq 1$, and $n_0 = 0$ or $n_0 \geq 1$. In the equality case (value 0) we can show that f is indeed positive in a neighborhood of the respective point. In the other case, this does not hold in general. However, it quickly follows that Δ itself must be positive already in a neighborhood of the respective point such that we can apply the maximum principle on a smaller interval where all functions involved are bounded.
In order to demonstrate the arguments that then can be used also in the other cases, we consider the case $m_0 = 0$ and $n_0 \geq 1$.
To this end, we recall that h_1 and h_2 are continuous on $[0, \pi] \times [0, T)$ and thus uniformly continuous on $\overline{I}_{\tilde{T}}$. Therefore, on the one hand there exists $0 < \theta_1 < \frac{\pi}{8}$ such that for $i = 1, 2$ it holds

$$|h_i(\theta, t) - m_i\pi| < \frac{\pi}{8} \quad \text{for all } (\theta, t) \in [0, \theta_1] \times [0, \tilde{T}]\,.$$

This then implies that

$$|\Delta(\theta, t)| = |\Delta(\theta, t) - m_0\pi| < \frac{\pi}{4} \quad \text{for all } \theta \in [0, \theta_1] \times [0, \tilde{T}]\,.$$

Moreover, it therefore holds that

$$\begin{aligned} &|h_2(\theta, t) + h_1(\theta, t) - 2\theta - (m_1 + m_2)\pi| \\ &\leq |h_2(\theta, t) - m_2\pi| + |h_1(\theta, t) - m_1\pi| + 2|\theta| \leq \frac{\pi}{2} \end{aligned}$$

for all $(\theta, t) \in [0, \theta_1] \times [0, \tilde{T}]$. Consequently, since by the assumption $m_0 = 0$ it follows $m_1 + m_2 = 2m_1$, we obtain by the 2π-periodicity of cos that

$$f(\theta) = \frac{\sin \Delta}{\Delta} \cos(h_2 + h_1 - 2\theta - (m_1 + m_2)\pi) \geq 0 \quad \text{for all } (\theta, t) \in [0, \theta_1] \times [0, \tilde{T}] .$$

On the other hand, there exists $0 < \theta_2 < \pi$ such that for $i = 1, 2$ it holds

$$|h_i(\theta, t) - n_i \pi| < \frac{\pi}{4} \quad \text{for all } (\theta, t) \in [\theta_2, \pi] \times [0, \tilde{T}] .$$

This yields

$$\Delta(\theta, t) \geq n_0 \pi - |\Delta(\theta, t) - n_0 \pi| \geq \pi - |h_2(\theta, t) - n_2 \pi| - |h_1(\theta, t) - n_1 \pi| > \frac{\pi}{2} ,$$

for all $(\theta, t) \in [\theta_2, \pi] \times [0, \tilde{T}]$. In particular, we have $\Delta(\theta_2, t) > 0$ for all $t \in [0, \tilde{T}]$. The first estimate and the fact that c_2 is continuous on $[\theta_1, \theta_2] \times [0, \tilde{T}]$ then implies that there exists a global constant $\delta_2 > 0$ such that $c_2(\theta, t) > -\delta_2$ for all $(\theta, t) \in [0, \theta_2] \times (0, \tilde{T}]$. Thus, $c := c_1 + c_2$ is bounded from below by $-\delta_1 - \delta_2$ on $[0, \theta_2] \times (0, \tilde{T})$. As a result, invoking Definition A.5 we deduce $\Delta \geq 0$ on $[0, \theta_2] \times (0, \tilde{T})$ and since $\Delta > 0$ on $[\theta_2, \pi] \times (0, \tilde{T})$ we obtain $\Delta \geq 0$ on $I_{\tilde{T}}$. As $\tilde{T}$ was arbitrary, we conclude that $\Delta \geq 0$ on $I \times (0, T)$. □

Since Definition A.5 also works for super- and subsolutions, we can prove in the same way the following two statements.

Lemma 4.2. *Let $h_1 \in E_{m_1, n_1}$ be a solution of* (3.37)*, and let $h_2 \in E_{m_2, n_2}$ satisfy*

$$h_t \geq h_{\theta\theta} + \frac{\cos \theta}{\sin \theta} h_\theta - \frac{\sin 2h}{2 \sin^2 \theta} - \frac{\kappa}{2} \sin(2h - 2\theta) .$$

Alternatively, let $h_1 \in E_{m_1, n_1}$ satisfy

$$h_t \leq h_{\theta\theta} + \frac{\cos \theta}{\sin \theta} h_\theta - \frac{\sin 2h}{2 \sin^2 \theta} - \frac{\kappa}{2} \sin(2h - 2\theta) .$$

and let $h_2 \in E_{m_2, n_2}$ be a solution of (3.37)*.*
In both cases, if $h_1(\theta, 0) \leq h_2(\theta, 0)$ for all $\theta \in [0, \pi]$, then $h_1(\theta, t) \leq h_2(\theta, t)$ for all $(\theta, t) \in [0, \pi] \times (0, T)$, where $T > 0$ is the maximal existence time of the solutions.

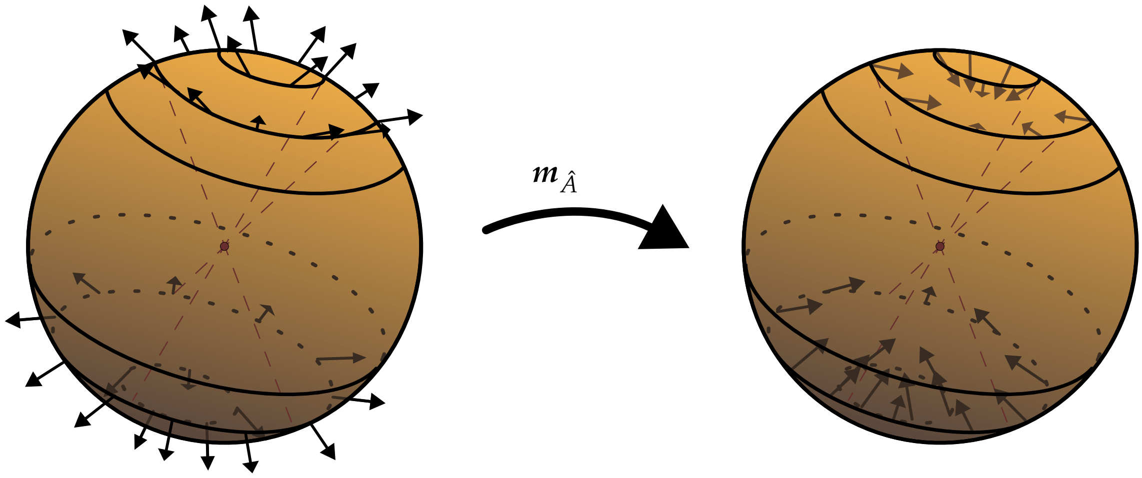

Figure 4.1.: The action of $\boldsymbol{m}_{\hat{A}}$ is precisely the inversion of all base points of the vector field $\boldsymbol{m}$ at the origin.

4.2. Hemispheric maps

We notice that the energy is invariant under the additional transformation $\boldsymbol{m}_{\hat{A}} \mapsto \boldsymbol{m} \circ \hat{A}$, where $\hat{A}\colon \mathbb{S}^2 \to \mathbb{S}^2$ is the antipodal map. This follows from the fact that $\nabla \boldsymbol{m}_{\hat{A}} = -(\nabla \boldsymbol{m}) \circ \hat{A}$ and $\boldsymbol{\nu} \circ \hat{A} = -\boldsymbol{\nu}$.
In spherical coordinates the action has the form

$$\hat{A}(\theta, \varphi) = \begin{cases} (\pi - \theta, \varphi + \pi) & \text{for } \varphi \in [0, \pi)\,, \\ (\pi - \theta, \varphi - \pi) & \text{for } \varphi \in [\pi, 2\pi)\,. \end{cases}$$

For axisymmetric $\boldsymbol{m}$, this yields

$$\boldsymbol{m}_{\hat{A}}(\theta, \varphi) = \begin{pmatrix} \cos(\varphi + \pi) \sin h(\pi - \theta) \\ \sin(\varphi + \pi) \sin h(\pi - \theta) \\ \cos h(\pi - \theta) \end{pmatrix} = \begin{pmatrix} \cos(\varphi) \sin(-h(\pi - \theta)) \\ \sin(\varphi) \sin(-h(\pi - \theta)) \\ \cos(-h(\pi - \theta)) \end{pmatrix}.$$

Note that this computation is valid for all values of $\varphi \in [0, 2\pi)$ by the 2π-periodicity of sin and cos.
Thus, defining $h_{\hat{A}}(\theta) := 2\pi k - h(\pi - \theta)$ for some $k \in \mathbb{Z}$, we see that the transformation $\boldsymbol{m} \mapsto \boldsymbol{m}_{\hat{A}}$ induces a transformation on the level of profiles of $h \mapsto h_{\hat{A}}$ and the axisymmetry of $\boldsymbol{m}$ is preserved under this transformation. As a result, if we want to examine functions which are axisymmetric *and* invariant under the

antipodal map, we need to determine all maps $h\colon [0,\pi] \to \mathbb{R}$ such that

$$h(\theta) = 2\pi k - h(\pi - \theta) \text{ for all } \theta \in [0,\pi]\,. \tag{4.1}$$

In this case, the value of k is determined by the end points

$$k = \frac{m+n}{2}\,,$$

if $h \in E_{m,n}$. The point $\theta = \frac{\pi}{2}$ is of special interest of this class of functions. First, we have $h(\frac{\pi}{2}) = k\pi$. Setting $x = \theta - \frac{\pi}{2}$ and defining the function

$$f(x) = h\Big(x + \frac{\pi}{2}\Big) - k\pi\,,$$

we see that f is an odd function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Subsequently, we use the expression "h is odd around the point $\frac{\pi}{2}$" for this behavior. Consequently, any function h for which (4.1) holds is determined by its values on $[0, \frac{\pi}{2}]$ already. Moreover, all even derivatives of h vanish at $\frac{\pi}{2}$.

Definition 4.3. We call a map $\boldsymbol{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ *hemispheric* if and only if it is axisymmetric and satisfies $\boldsymbol{m} = \boldsymbol{m}_{\hat{A}}$. Similarly, we call a profile function $h\colon [0,\pi] \to \mathbb{R}$ a *hemispheric profile* if and only if $h = h_{\hat{A}}$.
We define the set of hemispheric profiles with boundary values $h(0) = m\pi$ and $h(\pi) = n\pi$ for $m, n \in \mathbb{Z}$ as

$$H_{m,n} = \big\{h \in E_{m,n} : h_{\hat{A}} = h\big\}\,. \qquad \boxdot$$

As with axisymmetry, the hemispheric symmetry is also preserved under the SMHF. This is again due to the fact that the equation is invariant under the transformation $\boldsymbol{m} \mapsto \boldsymbol{m}_{\hat{A}}$, which we will show on the level of the profile.

Proposition 4.4. *Let $\boldsymbol{m}_0$ be hemispheric with hemispheric profile $h_0 \in H_{m,n}$ for some $m, n \in \mathbb{Z}$.*
Then the solution $\boldsymbol{m}$ to (SMHF) *provided by Theorem 3 is also hemispheric for all $t \in (0,T)$, where $T > 0$ is the maximal existence time until the first blowup, if present. In particular, $h(\cdot,t) \in H_{m,n}$ for all $t \in (0,T)$.*

Proof. Since hemispheric implies axisymmetric, by Definition 3.15 we obtain the profile $h(\theta,t)$ which solves (3.37). Now we compute

$$\partial_t h_{\hat{A}}(\theta) = -\,\partial_t h(\pi - \theta)$$

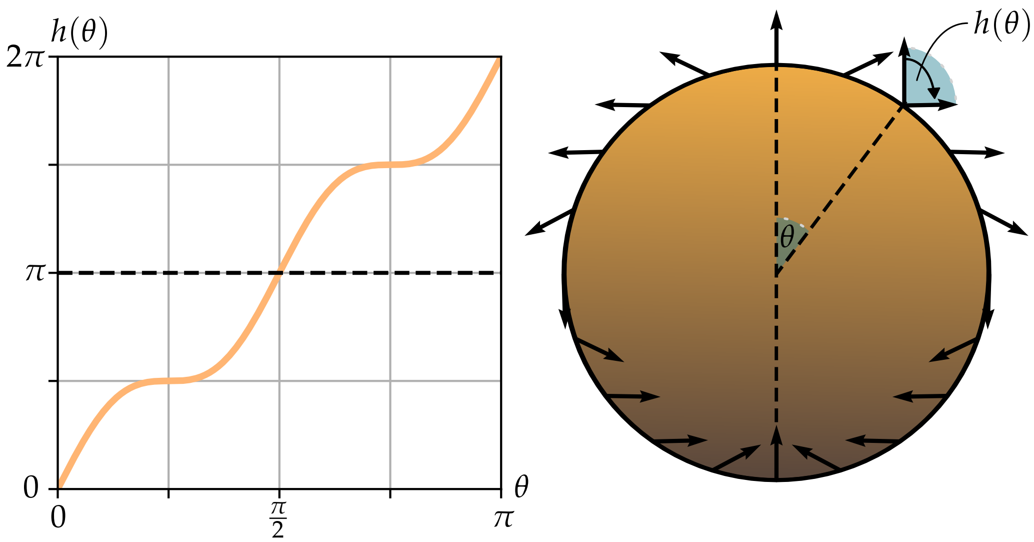

Figure 4.2.: Example of a hemispheric profile $h \in H_{0,2}$.

$$\begin{aligned} &= -\partial_{\theta\theta}h(\pi-\theta) + \frac{\cos(\pi-\theta)}{\sin(\pi-\theta)}\partial_\theta h(\pi-\theta) + \sin 2h(\pi-\theta) \\ &\quad + \frac{\kappa}{2}\frac{\sin(2h(\pi-\theta)-2(\pi-\theta))}{\sin(\pi-\theta)} \\ &= \partial_{\theta\theta}h_{\hat{A}}(\theta) - \frac{\cos\theta}{\sin\theta}\partial_\theta h_{\hat{A}}(\theta) - \frac{\sin 2h_{\hat{A}}(\theta)}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2h_{\hat{A}}(\theta)-2\theta)\,, \end{aligned}$$

where we induced a sign change in the last step via $\cos(\pi-\theta) = -\cos\theta$, whereas $\sin(\pi-\theta) = \sin\theta$, and

$$\sin(2h(\pi-\theta)-2(\pi-\theta)) = -\sin(2(\pi-h(\pi-\theta))-2\theta) = -\sin(2h_{\hat{A}}(\theta)-2\theta)\,.$$

Therefore, $h_{\hat{A}}$ solves (3.37) as well with initial data $(h_0)_{\hat{A}} = h_0$. As in the proof of Definition 3.15, we deduce that $\boldsymbol{m}_{\hat{A}} = \boldsymbol{m}$ for all $t \in (0,T)$ by the uniqueness of the solution, from which follows that $\boldsymbol{m}$ is hemispheric. □

Remark 4.5. As stated before, for hemispheric profiles the point $\theta = \frac{\pi}{2}$ is an inflection point with $h(\frac{\pi}{2}) = \pi k = \pi\frac{m+n}{2}$. The point is therefore stable under the flow which of course is also expected as h is continuous in t and has integer value at $\theta = \frac{\pi}{2}$. Indeed, plugging in the point into the differential equation yields

$$h_t(\tfrac{\pi}{2}) = h_{\theta\theta}(\tfrac{\pi}{2}) + \frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})}h_\theta(\tfrac{\pi}{2}) - \frac{\sin 2h(\frac{\pi}{2})}{2\sin^2(\frac{\pi}{2})} - \frac{\kappa}{2}\sin(2h(\tfrac{\pi}{2})-\pi) = 0\,.$$

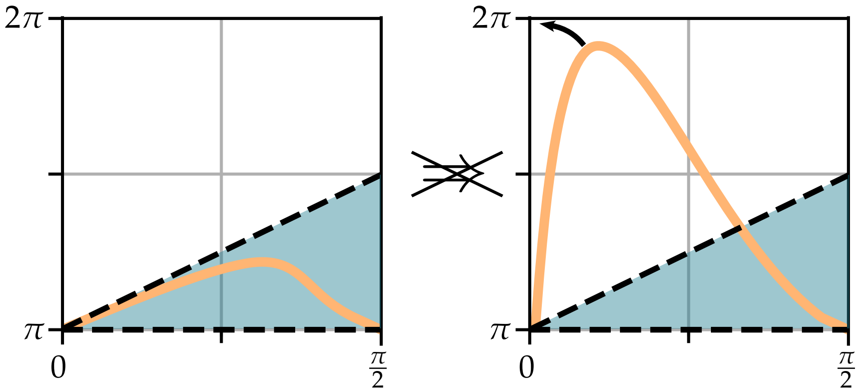

Figure 4.3.: Since the profile h is confined in the wedge for all times, a blowup is prohibited as this requires the profile making a jump of at least $\pm\pi$ at 0.

Hence, for hemispheric initial data, it suffices to consider the initial value problem of (3.37) on the half interval $[0, \frac{\pi}{2}]$ with boundary values $h(0) = m\pi$ and $h(\frac{\pi}{2}) = k\pi$. $\boxdot$

We now consider initial hemispheric profiles $h_0 \in H_{1,1}$, then clearly also $k = 1$. Additionally, by Definition 2.16, we obtain $Q(\boldsymbol{m}_0) = 0$ for the corresponding field. Hence, if a solution of the SMHF flows into a stationary solution (i.e., no blowup occurs), then we have found a skyrmion solution. This is the result of the following proposition.

Proposition 4.6. *Let $\boldsymbol{m}_0$ be hemispheric with profile $h_0 \in H_{1,1}$ which satisfies*

$$\pi \leq h_0(\theta) \leq \pi + \theta \quad \text{for all } \theta \in \left[0, \frac{\pi}{2}\right]. \tag{4.2}$$

Then the solution $\boldsymbol{m}$ to (SMHF) *provided by Theorem 3 has existence time $T = \infty$ and no blowup occurs at infinity. Therefore, $\boldsymbol{m}_\infty$ is a hemispheric solution to* (EL) *with $Q(\boldsymbol{m}_\infty) = 0$. Moreover, h_∞ satisfies* (4.2)*.*

Remark 4.7. An admissible profile that satisfies the condition is clearly $h_0 \equiv \pi$. It is unclear whether all initial data eventually flows into the same stationary solution. For $\kappa > 24$, a solution $\boldsymbol{m}_\infty$ obtained this way is however distinct from the minimizer $\boldsymbol{m}_{\min}$ of the energy $\mathcal{E}$ in the class of axisymmetric maps with degree 0. This follows from the fact that for the profile of the minimizer it holds $h < \pi$ on $(0, \pi)$ [59, Prop. 3.1]. $\boxdot$

Proof. By Definition 4.4 we know that $\boldsymbol{m}$ is hemispheric for all times and completely determined by its hemispheric profile h on $(0, \frac{\pi}{2})$. We split the proof up

by first showing that the profile is confined in the wedge as in (4.2) for all times. Then we prove that this already implies that no blowup can occur.
We claim that

$$\pi \leq h(\theta,t) \leq \pi + \theta \text{ for all } (\theta,t) \in \left[0,\frac{\pi}{2}\right] \times (0,T)\,.$$

Since $\theta \mapsto \pi + \theta$ is a solution to (3.37), the upper bound follows immediately from Definition 4.1. Hence, we only need to show the lower bound. To this end, we can adapt the statement of Definition 4.2 to functions on the half interval $[0,\frac{\pi}{2}]$ as $h(\frac{\pi}{2},t) = \pi$ is fixed for all t. Hence, it suffices to show that the constant map $g(\theta,t) = \pi$ satisfies

$$g_t \leq g_{\theta\theta} + \frac{\cos\theta}{\sin\theta} g_\theta - \frac{\sin 2g}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2g - 2\theta)$$

for all $(\theta,t) \in [0,\frac{\pi}{2}] \times (0,T)$. Since g is constant, this reduces to

$$0 \leq -\frac{\kappa}{2}\sin(-2\theta) = \frac{\kappa}{2}\sin(2\theta)\,,$$

which is certainly true for $\theta \in [0,\frac{\pi}{2}]$. Therefore, the lower bound is also established.
Now, assume that $\boldsymbol{m}$ blows up at T, regardless of whether T is finite or infinite. Then by Definition 3.18, there are $\lambda_k \searrow 0$ and $t_k \nearrow T$ such that for any $0 < \theta < \pi$, if we define

$$\theta_k = 2\arctan\left(\lambda_k \tan\left(\frac{\theta}{2}\right)\right),$$

then $h(\theta_k,t_k) \to \beta(\theta) + \pi$. If we fix $0 < \theta < \pi$, then $\theta_k \searrow 0$ and $\beta(\theta) + \pi \neq \pi$ because β is nonconstant. However, by the already stated bounds we obtain

$$\pi \leq h(\theta_k,t_k) \leq \pi + \theta_k \longrightarrow \pi\,, \text{ as } k \to \infty\,,$$

and hence $h(\theta_k,t_k) \to \pi \neq \beta(\theta) + \pi$, a contradiction. Therefore, $\boldsymbol{m}$ cannot blowup at T and consequently $T = \infty$ and $\boldsymbol{m}_0$ is homotopic to $\boldsymbol{m}_\infty$ which implies $Q(\boldsymbol{m}_\infty) = Q(\boldsymbol{m}_0) = 0$ and also that $\boldsymbol{m}_\infty$ is hemispheric. □

In the same way we can prove the non-blowup of another class of hemispheric solutions.

Proposition 4.8. *Let $\kappa \geq 4$ and let $\boldsymbol{m}_0$ be hemispheric with profile $h_0 \in H_{0,2}$ which satisfies*

$$0 \leq h_0(\theta) \leq 2\theta \quad \textit{for all } \theta \in \left[0,\frac{\pi}{2}\right]. \tag{4.3}$$

Then the solution $\boldsymbol{m}$ *to* (SMHF) *provided by Theorem 3 has existence time* $T = \infty$ *and no blowup occurs at infinity. Therefore,* $\boldsymbol{m}_\infty$ *is a hemispheric solution to* (EL) *with* $Q(\boldsymbol{m}_\infty) = 0$*. Moreover,* h_∞ *satisfies* (4.3)*.*

Proof. First, we assert that $Q(\boldsymbol{m}_0) = 0$. This follows from Definition 2.16 and the fact that $h_0(0) = 0$ and $h_0(\pi) = 2\pi$.
As in the previous proof, we know that $\boldsymbol{m}$ is hemispheric for all times and completely determined by its hemispheric profile h on $(0, \frac{\pi}{2})$. The statement then follows if we assert that (4.3) is preserved under the flow. The lower bound has been shown in the previous proof (note that the equation is invariant under $h \mapsto h - \pi$). The upper bound follows from Definition 4.2, again adapted to the half interval. As computed in Definition 2.19, the function $g(\theta, t) = 2\theta$ satisfies

$$g'' + \frac{\cos\theta}{\sin\theta}g' - \frac{\sin 2g}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2g - 2\theta) = \left(2 - \frac{\kappa}{2}\right)\sin(2\theta) \leq 0 = g_t\,,$$

for all $(\theta, t) \in [0, \frac{\pi}{2}] \times (0, T)$ since $\kappa \geq 4$. Therefore, the upper bound is also established and thus no blowup can occur. □

Remark 4.9. Since $\theta \mapsto \theta$ is a solution to (3.37), we can replace the lower bound in (4.3) by θ and reduce the set of possible initial profiles for which then the solution h is guaranteed to stay in the narrower wedge

$$\theta \leq h(\theta, t) \leq 2\theta \quad \text{for all } \theta \in \left[0, \tfrac{\pi}{2}\right], \tag{4.4}$$

for all times $t \in (0, \infty)$. ⊡

4.3. Proof of existence of a saddle point: Theorem 1

Finally, we can prove the existence of a saddle point of $\mathcal{E}$ for certain κ. We begin with the proof of Theorem 1 and will proceed with the proof of Theorem 2 in the following section. The idea is to take the hemispheric solution $\boldsymbol{m}_\infty$ obtained in Definition 4.6 from an initial data $\boldsymbol{m}_0$ with a special profile h_0. By constructing a perturbation of the resulting profile h_∞, we can find a map $\boldsymbol{m}_-$ close to $\boldsymbol{m}_\infty$ with $\mathcal{E}(\boldsymbol{m}_-) < \mathcal{E}(\boldsymbol{m}_\infty)$, showing that $\boldsymbol{m}_\infty$ is not a local minimum. Since $\boldsymbol{m}_\infty$ stems from a flow along which energy decreases, we conclude that $\boldsymbol{m}_\infty$ indeed is a saddle point.
Before we begin, we first define the wedge $W \subset \mathbb{R}^2$ in which the graph of profiles satisfying condition (4.2) lies for all times since this set will appear in a lot of

arguments.

$$W = \left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le \frac{\pi}{2}, \pi \le y \le \pi + x\right\}.$$

For the remainder of this section, we will use a shorter notation for the denotation of subsets of $\mathbb{R}^2$: we suppress the range of the first variable x which is always assumed to be $[0, \frac{\pi}{2}]$ if not stated differently and by only write the bounds for the second variable y. For instance, we write $W = \{\pi \le y \le \pi + x\}$.

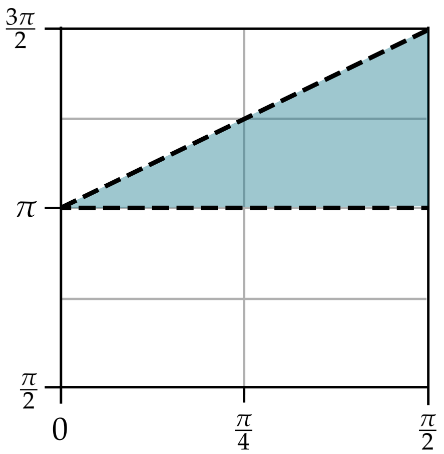

Figure 4.4.: The set W.

We begin by defining the initial profile, for which we then gain an energy bound. For $\kappa > 0$ we let $0 < \theta_0(\kappa) < \frac{\pi}{2}$, chosen later, and define the initial profile as

$$h_{0,\kappa}(\theta) = \begin{cases} \pi + \theta & \text{for } \theta \in [0, \theta_0(\kappa)]\,, \\ \pi - \frac{\theta_0(\kappa)}{\frac{\pi}{2} - \theta_0(\kappa)}\left(\theta - \frac{\pi}{2}\right) & \text{for } \theta \in (\theta_0(\kappa), \pi - \theta_0(\kappa)]\,, \\ \theta & \text{for } \theta \in (\pi - \theta_0(\kappa), \pi]\,. \end{cases} \tag{4.5}$$

This profile is continuous and piecewise linear. Consequently, we find that it belongs to $\mathcal{PF}^1$. Moreover, it is hemispheric and satisfies the wedge condition (4.2). Thus, $h_{0,\kappa}$ qualifies as a valid initial profile for the flow.

Lemma 4.10. *For $\kappa > 4$, we can choose $\theta_0(\kappa)$ such that $h_{0,\kappa}$ satisfies the energy bound*

$$E(h_{0,\kappa}) \le \pi\sqrt{\frac{\kappa}{2} - 1}\,.$$

Proof. Since $h_{0,\kappa}$ is a hemispheric profile, it suffices to compute the energy on the interval $[0, \frac{\pi}{2}]$ and multiply by 2. For readability, we set $h = h_{0,\kappa}$ and define

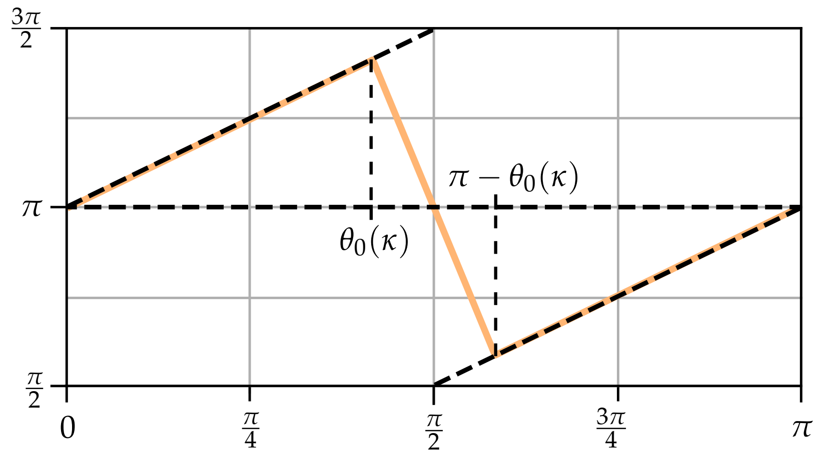

Figure 4.5.: The function $h_{0,\kappa}$ in dependence of $\theta_0(\kappa)$.

$\delta = \frac{\pi}{2} - \theta_0(\kappa)$. Moreover, we set

$$a = \frac{\theta_0(\kappa)}{\frac{\pi}{2} - \theta_0(\kappa)} = \frac{\frac{\pi}{2} - \delta}{\delta}.$$

We gain

$$\begin{aligned}
E(h) &= \int_0^{\frac{\pi}{2}} h'^2 \sin\theta + \frac{\sin^2 h}{\sin\theta} + \kappa \sin^2(h - \theta)\sin\theta \,\mathrm{d}\theta \\
&= \int_0^{\theta_0(\kappa)} \sin\theta + \frac{\sin^2(\pi + \theta)}{\sin\theta} + \kappa\sin^2(\pi + \theta - \theta)\sin\theta\,\mathrm{d}\theta \\
&\quad + \int_{\theta_0(\kappa)}^{\frac{\pi}{2}} a^2 \sin\theta + \frac{\sin^2\big(\pi - a(\theta - \frac{\pi}{2})\big)}{\sin\theta} + \kappa\sin^2\big(\pi - a(\theta - \tfrac{\pi}{2}) - \theta\big)\sin\theta\,\mathrm{d}\theta \\
&= 2(1 - \cos\theta_0(\kappa)) + a^2\cos\theta_0(\kappa) + \underbrace{\int_{\theta_0(\kappa)}^{\frac{\pi}{2}} \frac{\sin^2\big(a(\theta - \frac{\pi}{2})\big)}{\sin\theta}\,\mathrm{d}\theta}_{=:I_1} \\
&\quad + \kappa\underbrace{\int_{\theta_0(\kappa)}^{\frac{\pi}{2}} \cos^2\big((1+a)(\theta - \tfrac{\pi}{2})\big)\sin\theta\,\mathrm{d}\theta}_{=:I_2}.
\end{aligned}$$

For the integrals we first estimate $\sin\theta_0(\kappa) \leq \sin\theta \leq 1$, since $\theta \in [\theta_0(\kappa), \frac{\pi}{2}] \subset [0, \frac{\pi}{2}]$. By definition of δ we also have $\cos\theta_0(\kappa) = \sin\delta$ and $\sin\theta_0(\kappa) = \cos\delta$. Consequently, we can estimate I_1 using the well known antiderivative of $\sin^2$ as

$$I_1 \leq \frac{1}{\cos\delta}\int_{\theta_0(\kappa)}^{\frac{\pi}{2}} \sin^2\big(a(\theta - \tfrac{\pi}{2})\big)\,\mathrm{d}\theta$$

$$
\begin{aligned}
&= \frac{1}{\cos\delta}\frac{2a(\frac{\pi}{2}-\theta_0(\kappa)) - \sin(2a(\frac{\pi}{2}-\theta_0(\kappa)))}{4a} \\
&= \frac{\delta}{2\cos\delta}\left(1 - \frac{\sin(2\delta)}{2\theta_0(\kappa)}\right) \\
&= \frac{\delta}{2}\left(\frac{1}{\cos\delta} - \frac{\sin(\delta)}{\frac{\pi}{2}-\delta}\right),
\end{aligned}
$$

where we used that $a(\frac{\pi}{2}-\theta_0(\kappa)) = \theta_0(\kappa)$, and $2\theta_0(\kappa) \le \pi$. For I_2 we proceed similarly, using the antiderivative of $\cos^2$. We find

$$
\begin{aligned}
I_2 &\le \int_{\theta_0(\kappa)}^{\frac{\pi}{2}} \cos^2\big((1+a)(\theta-\tfrac{\pi}{2})\big)\,\mathrm{d}\theta \\
&= \frac{2(1+a)(\frac{\pi}{2}-\theta_0(\kappa)) + \sin(2(1+a)(\frac{\pi}{2}-\theta_0(\kappa)))}{4(1+a)}.
\end{aligned}
$$

Using $1+a = \frac{\pi}{2}/\delta$ we see that the term involving the sine reduces to $\sin\pi$ such that it vanishes. As a result, we obtain

$$
I_2 \le \frac{\delta}{2}.
$$

We can combine the estimates to retrieve

$$
\begin{aligned}
E(h) &\le 2(1-\sin\delta) + \frac{1}{\delta^2}\left(\frac{\pi}{2}-\delta\right)^2\sin\delta + \frac{\delta}{2}\left(\frac{1}{\cos\delta} - \frac{\sin\delta}{\frac{\pi}{2}-\delta}\right) + \frac{\kappa\delta}{2} \\
&\le 2 + \left(-2+1-\frac{\pi}{\delta}+\frac{\pi^2}{4\delta^2}\right)\sin\delta + \frac{\delta}{2}\left(\frac{1}{\cos\delta} - \frac{\sin\delta}{\frac{\pi}{2}-\delta}\right) + \frac{\kappa\delta}{2} \\
&\le \frac{\pi^2}{4\delta} + \frac{\delta\kappa}{2} + 2 - \pi\frac{\sin\delta}{\delta} - \sin\delta + \frac{\delta}{2}\left(\frac{1}{\cos\delta} - \frac{\sin\delta}{\frac{\pi}{2}-\delta}\right).
\end{aligned}
$$

Now, we inspect the function

$$
f(\theta) = 2 - \pi\frac{\sin\delta}{\delta} - \sin\delta + \frac{\delta}{2}\left(\frac{1}{\cos\delta} - \frac{\sin\delta}{\frac{\pi}{2}-\delta}\right).
$$

We can see that $f(0) = 2-\pi < 0$, hence, for $0 < \delta$ small, we see that f is negative. In fact, it holds that for $\delta < \frac{\pi}{2}$ we have $f < -1$, which can easily be verified. Consequently, we arrive at the estimate

$$
E(h) \le -1 + \frac{\pi^2}{4\delta} + \frac{\delta\kappa}{2}.
$$

Setting $\delta = \frac{\pi}{\sqrt{2\kappa}}$—and hence fixing $\theta_0(\kappa)$—we assert that $\delta < \frac{\pi}{2}$ holds for $\kappa > 4$. The desired energy bound then follows directly upon summing up both terms involving δ. □

For the remainder of this section, we fix $\kappa > 4$ and let $h := h_\kappa$ be the smooth profile of the hemispheric solution $\boldsymbol{m}_\infty$ obtained from the initial data $\boldsymbol{m}_{0,\kappa}$ with profile $h_{0,\kappa}$ as in Definition 4.6. This implies that h satisfies the wedge condition (4.2), solves (2.18), and that $Q(\boldsymbol{m}_\infty) = 0$. Additionally, as the energy decreases along the flow according to Definition 3.4, we immediately obtain with the above lemma the energy bound

$$E(h) \leq \pi\sqrt{\frac{\kappa}{2}} - 1\,. \tag{4.6}$$

We want to analyze h and its behavior for increasing κ in the following lemmata by exhausting the information we have of it. One observation that follows from this is a global bound on its derivative.

Lemma 4.11. *Let $\kappa \geq 4$ and let $h \in H_{1,1}$ be a solution of* (2.18) *that satisfies the wedge condition* (4.2). *Then, for all $\theta \in [0, \pi]$ it holds*

$$h'(\theta) \leq 1\,.$$

Proof. Since h is a hemispheric profile, we deduce that h' is an even function at $\frac{\pi}{2}$. Therefore, we can restrict ourselves to the interval $[0, \frac{\pi}{2}]$. We argue by contradiction. To this end, we assume that there exists $\theta \in [0, \frac{\pi}{2}]$ such that $h'(\theta) > 1$. From the wedge condition we conclude that $h'(0) \leq 1$ and $h'(\frac{\pi}{2}) \leq 0$. Thus, $\theta \in (0, \frac{\pi}{2})$ and by Rolle's theorem and the smoothness of h we can assume without loss of generality that $h'(\theta) > 1$ and $h''(\theta) = 0$, i.e., we choose the maximum of h'. This corresponds to an inflection point of h. As we have fixed θ now, we suppress it as argument of h and its derivatives. Then, since h is a solution to (2.18) and $\sin^2\theta > 0$, we have

$$\begin{aligned}
0 &= 2\sin^2\theta\left(h'' + \frac{\cos\theta}{\sin\theta}h' - \frac{\sin 2h}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2h - 2\theta)\right) \\
&> 2\cos\theta\sin\theta - \sin 2h - \kappa\sin(2h - 2\theta)\sin^2\theta \\
&= \sin 2\theta - \sin 2h - \kappa\sin(2h - 2\theta)\sin^2\theta =: f(\theta)\,.
\end{aligned}$$

Since $(\theta, h) \in W$, we get $h - \theta \in [\frac{\pi}{2}, \pi]$ which implies that $\sin(h - \theta) \geq 0$ and $\sin(2h - 2\theta) \leq 0$. Now, either $(\theta, h) \in \{y \leq \frac{3\pi}{2} - x\} \cap W$ or $(\theta, h) \in \{y > \frac{3\pi}{2} - x\} \cap W$. In the first case, we rewrite the function f using a basic trigonometric

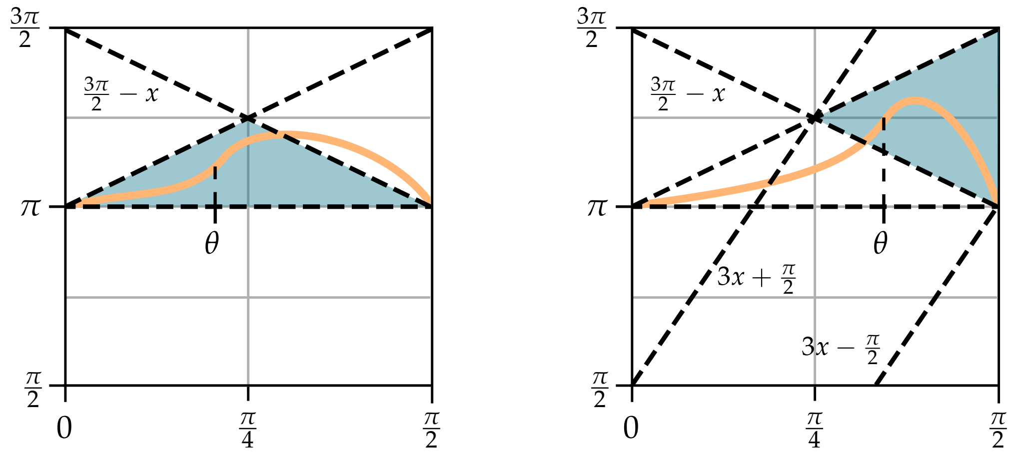

Figure 4.6.: In the first case (left), the inflection point is in the marked set $\{y \leq \frac{3\pi}{2} - x\} \cap W$. In the second case (right), the inflection point is in the marked set $\{y \geq \frac{3\pi}{2} - x\} \cap \{3x - \frac{\pi}{2} < y < 3x + \frac{\pi}{2}\} \cap W$.

identity as

$$f(\theta) = -2\cos(h+\theta)\sin(h-\theta) - \kappa \sin(2h - 2\theta)\sin^2\theta\,.$$

Since $(\theta, h) \in \{y \leq \frac{3\pi}{2} - x\} \cap W$, we have $h + \theta \in [\pi, \frac{3\pi}{2}]$ and hence $\cos(h+\theta) \leq 0$. Together with the aforementioned bounds this leads to $f(\theta) \geq 0$, a contradiction to $f < 0$.

In the second case, we write f using (A.T1) and the above form as

$$f(\theta) = \frac{1}{2}\sin(h-\theta)\big(\kappa\cos(h-3\theta) - 2\kappa\cos(h-\theta) + (\kappa-4)\cos(h+\theta)\big)\,.$$

Since we are now in the other case, we find $\cos(h+\theta) > 0$ and by the assumption on κ we get that the last term in the parentheses is nonnegative. Moreover, as $h - \theta \in [\frac{\pi}{2}, \pi]$ we conclude that $\cos(h-\theta) < 0$. Finally, we have $(\theta, h(\theta)) \in \{3x - \frac{\pi}{2} < y < 3x + \frac{\pi}{2}\} \cap W$ which implies that $\cos(h - 3\theta) \geq 0$. Therefore, we deduce that all three terms in the parentheses are nonnegative and as a result $f(\theta) \geq 0$, again a contradiction. Hence, we arrive at the conclusion that both cases lead to a contradiction and thus we have finished the proof. □

Corollary 4.12. *The function $\theta \mapsto \pi + \theta - h(\theta)$ is monotonically increasing. The function $\theta \mapsto \sin^2(h-\theta)$ is monotonically increasing on the interval $[0, \frac{\pi}{2}]$ and monotonically decreasing on the interval $[\frac{\pi}{2}, \pi]$.*

Proof. The first statement immediately follows from the previous lemma. For the second one we compute

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\sin^2(h-\theta) = 2\sin(h-\theta)\cos(h-\theta)(h'-1) = \sin(2h-2\theta)(h'-1)\,.$$

Now, since $(\theta, h(\theta)) \in W$ for $\theta \le \frac{\pi}{2}$, we conclude $\sin(2h-2\theta) \le 0$ and as by the previous lemma $h'-1 \le 0$ we find that the product is nonnegative. The behavior on the interval $[\frac{\pi}{2}, \pi]$ follows from the fact that then $\sin(2h-2\theta) \ge 0$, by the hemispheric symmetry. □

Remark 4.13. Since sin is a positive monotonically increasing function on $[0, \frac{\pi}{2}]$, we can conclude that the function $\theta \mapsto \sin^2(h-\theta)\sin\theta$ is also monotonically increasing on the interval $[0, \frac{\pi}{2}]$ and monotonically decreasing on the interval $[\frac{\pi}{2}, \pi]$. ⊡

In combination with the energy bound we can obtain the following pointwise result from the previous corollary.

Lemma 4.14. *For any* $0 < \theta_* < \frac{\pi}{2}$*, we obtain that*

$$h_\kappa(\theta) \longrightarrow \pi + \theta\ ,\ \text{as}\ \kappa \to \infty\,,$$

uniformly on $[0, \theta_*]$*. Likewise, we have that*

$$h_\kappa(\theta) \longrightarrow \theta\ ,\ \text{as}\ \kappa \to \infty\,,$$

uniformly on $[\pi - \theta_*, \pi]$.

Proof. We only show the first statement as the other one follows by the hemispheric symmetry. Assume that the statement is not true. Then there exists a $0 < \theta_* < \frac{\pi}{2}$ such that for a sequence $\kappa \to \infty$ there exist $\theta_\kappa \in [0, \theta_*]$ such that $\pi + \theta_\kappa - h_\kappa(\theta_\kappa) \ge \varepsilon$ for some $0 < \varepsilon < \frac{\pi}{2}$. By the previous corollary this implies that $\pi + \theta_* - h_\kappa(\theta_*) \ge \varepsilon$ for all κ. Which again with the remark implies for all $\theta \in [\theta_*, \frac{\pi}{2}]$ that

$$\sin^2(h_\kappa(\theta)-\theta)\sin\theta \ge \sin^2(h_\kappa(\theta_*)-\theta_*)\sin\theta_* \ge \sin^2\varepsilon\sin\theta_* > 0\,.$$

Invoking now the energy bound (4.6) we find that

$$\pi\sqrt{2\kappa} \ge E(h_\kappa) \ge \frac{\kappa}{2}\int_{\theta_*}^{\frac{\pi}{2}} \sin^2(h_\kappa(\theta)-\theta)\sin(\theta)\,\mathrm{d}\theta$$

$$\geq \frac{\kappa}{2}\sin^2(\varepsilon)\sin(\theta_*)\int_{\theta_*}^{\frac{\pi}{2}} \mathrm{d}\theta$$
$$\geq C\kappa\,.$$

Now, since κ increases faster than $\sqrt{\kappa}$, we arrive at a contradiction for $\kappa \to \infty$. □

We see that in the limit $\kappa \to \infty$ the profile converges into a sawtooth function with a very steep drop at $\frac{\pi}{2}$. In the following lemma we provide an upper bound on the rate with which the modulus of the derivative at that point increases. Recall that since h satisfies the wedge condition, it must hold $h'(\frac{\pi}{2}) \leq 0$.

Lemma 4.15. *Let $\kappa > 4$. Then it holds*

$$-h'(\tfrac{\pi}{2}) \leq 2\sqrt{E(h)\sqrt{\kappa}} \leq C\sqrt{\kappa}\,.$$

Proof. We let $0 < \delta := \delta(\kappa)$ to be chosen later. Then, let $\theta_\delta = \frac{\pi}{2} - \delta$ and we compute by adding a zero and using the fundamental theorem of calculus

$$\begin{aligned}
-h'(\tfrac{\pi}{2})\delta &= -\int_{\theta_\delta}^{\frac{\pi}{2}} h'(\tfrac{\pi}{2})\sin(\tfrac{\pi}{2}) - h'(\theta)\sin(\theta)\,\mathrm{d}\theta - \int_{\theta_\delta}^{\frac{\pi}{2}} h'(\theta)\sin(\theta)\,\mathrm{d}\theta \\
&= -\int_{\theta_\delta}^{\frac{\pi}{2}}\int_{\theta}^{\frac{\pi}{2}} \big(h'\sin\big)'(\tau)\,\mathrm{d}\tau\mathrm{d}\theta - \int_{\theta_\delta}^{\frac{\pi}{2}} h'(\theta)\sin(\theta)\,\mathrm{d}\theta \\
&= \underbrace{-\int_{\theta_\delta}^{\frac{\pi}{2}}\int_{\theta}^{\frac{\pi}{2}} \frac{\sin 2h}{2\sin\tau} + \frac{\kappa}{2}\sin(2h-2\tau)\sin\tau\,\mathrm{d}\tau\mathrm{d}\theta}_{=:I_1} \underbrace{- \int_{\theta_\delta}^{\frac{\pi}{2}} h'(\theta)\sin(\theta)\,\mathrm{d}\theta}_{=:I_2}\,,
\end{aligned}$$

where we used the fact that

$$\big(h'\sin\big)'(\theta) = h''\sin\theta + h'\cos\theta = \frac{\sin 2h}{2\sin\theta} + \frac{\kappa}{2}\sin(2h-2\theta)\sin\theta$$

as h is a solution to (2.18). Now, for I_1 we recall that $\sin(2h) \geq 0$ and $\sin(2h - 2\theta) \leq 0$ because $(\theta, h(\theta)) \in W$. Therefore, taking into account the sign, we can estimate the first term by means of the Hölder inequality as

$$\begin{aligned}
I_1 &\leq -\frac{\kappa}{2}\int_{\theta_\delta}^{\frac{\pi}{2}}\int_{\theta}^{\frac{\pi}{2}} \sin(2h-2\tau)\sin\tau\,\mathrm{d}\tau\mathrm{d}\theta \\
&= -\kappa\int_{\theta_\delta}^{\frac{\pi}{2}}\int_{\theta}^{\frac{\pi}{2}} \sin(h-\tau)\cos(h-\tau)\sin\tau\,\mathrm{d}\tau\mathrm{d}\theta \\
&\leq \sqrt{2\kappa}\int_{\theta_\delta}^{\frac{\pi}{2}}\left(\frac{\kappa}{2}\int_{\theta}^{\frac{\pi}{2}} \sin^2(h-\tau)\sin\tau\,\mathrm{d}\tau\right)^{1/2}\left(\int_{\theta}^{\frac{\pi}{2}}\sin\tau\,\mathrm{d}\tau\right)^{1/2}\mathrm{d}\theta \\
&\leq \sqrt{2\kappa\delta E_2(h)}\int_{\theta_\delta}^{\frac{\pi}{2}}\mathrm{d}\theta \leq \sqrt{2E_2(h)\delta}\sqrt{\kappa}\delta = \sqrt{2E_2(h)\delta}\,,
\end{aligned}$$

if we set $\delta = 1/\sqrt{\kappa}$, and

$$E_2(h) = \frac{\kappa}{2}\int_0^{\frac{\pi}{2}} \sin^2(h-\theta)\sin\theta\,\mathrm{d}\theta\,,$$

for the anisotropy part of the energy. For the second term I_2 we define the Dirichlet part of the energy as

$$E_1(h) = \frac{1}{2}\int_0^{\pi} h'^2\sin\theta + \frac{\sin^2 h}{\sin\theta}\,\mathrm{d}\theta$$

and obtain by means of the Hölder inequality

$$\begin{aligned} I_2 &\le \left(\int_{\theta_\delta}^{\frac{\pi}{2}} h'(\theta)^2\sin\theta\,\mathrm{d}\theta\right)^{1/2}\left(\int_{\theta_\delta}^{\frac{\pi}{2}}\sin\theta\,\mathrm{d}\theta\right)^{1/2} \\ &\le \sqrt{2E_1(h)\delta} \end{aligned}$$

Now, since $E(h) = E_1(h) + E_2(h)$, we can combine the estimates to obtain

$$-h'(\tfrac{\pi}{2}) \le 2\sqrt{\frac{E(h)}{\delta}} = 2\sqrt{E(h)\sqrt{\kappa}}\,.$$

Finally, using the (worse) estimate $E(h) \le \pi\sqrt{\kappa/2}$ from (4.6), we arrive at the conclusion. □

In order to prove Theorem 1, we want to construct a perturbation of the profile h such that the resulting map has lower energy. To this end, we first compute the energy difference.

Lemma 4.16. *Let $g \in C^\infty([0,\pi])$ with $g(0) = g(\pi) = 0$ and define the map $h_g = h + g$. Then, we have*

$$\begin{aligned} E(h_g) - E(h) = \frac{1}{2}\int_0^{\pi} g'^2\sin\theta + \left(\frac{\cos 2h}{\sin^2\theta} + \kappa\cos(2h-2\theta)\right)\sin^2 g\sin\theta\,\mathrm{d}\theta \\ + \frac{1}{2}\int_0^{\pi}\left(\frac{\sin 2h}{\sin^2\theta} + \kappa\sin(2h-2\theta)\right)\left(\frac{\sin 2g}{2} - g\right)\mathrm{d}\theta\,. \qquad (4.7) \end{aligned}$$

Proof. We want to compute $E(h_g)$. To this end, we first compute the individual terms appearing. We gain

$$\begin{aligned} {h_g'}^2 &= (h' + g')^2 = h'^2 + g'^2 + 2h'g'\,, \\ \sin^2(h_g) &= \sin^2(h+g) = (\sin h\cos g + \sin g\cos h)^2 \\ &= \sin^2 h\cos^2 g + \sin^2 g\cos^2 h + 2\sin h\cos g\sin g\cos h \end{aligned}$$

$$
\begin{aligned}
&= \sin^2 h(1-\sin^2 g) + \sin^2 g \cos^2 h + \frac{\sin 2h \sin 2g}{2} \\
&= \sin^2 h + \cos 2h \sin^2 g + \sin 2h \frac{\sin 2g}{2}\,,
\end{aligned}
$$

using basic trigonometric identities. Likewise,

$$
\sin^2\big(h_g-\theta\big) = \sin^2(h-\theta) + \cos(2h-2\theta)\sin^2 g + \sin(2h-2\theta)\frac{\sin 2g}{2}\,,
$$

from which follows

$$
\begin{aligned}
E(h_g) = E(h) &+ \frac{1}{2}\int_0^\pi g'^2 \sin\theta + \left(\frac{\cos 2h}{\sin^2\theta} + \kappa\cos(2h-2\theta)\right)\sin^2 g \sin\theta\,\mathrm{d}\theta \\
&+ \frac{1}{2}\int_0^\pi 2h'g'\sin\theta + \left(\frac{\sin 2h}{\sin^2\theta} + \kappa\sin(2h-2\theta)\right)\frac{\sin 2g}{2}\sin\theta\,\mathrm{d}\theta\,.
\end{aligned}
$$

Now we apply integration by parts on the term involving $h'g'\sin\theta$ and use the fact that h is a solution of (2.18) to obtain

$$
\begin{aligned}
2\int_0^\pi h'g'\sin\theta\,\mathrm{d}\theta &= -2\int_0^\pi \left(h'' + \frac{\cos\theta}{\sin\theta}h'\right) g\sin\theta\,\mathrm{d}\theta \\
&= -\int_0^\pi \left(\frac{\sin 2h}{\sin^2\theta} + \kappa\sin(2h-2\theta)\right) g\sin\theta\,\mathrm{d}\theta\,.
\end{aligned}
$$

Inserting this into the previous equation yields the desired result. □

Due to the fact that the Euler–Lagrange equation of the profile is nonlinear, we do not simply get the second variation of E for the energy difference. However, if we expand the equation now for small g, i.e., let $\varepsilon > 0$ and set $g_\varepsilon = \varepsilon g$, we see that the first line of the right-hand-side in (4.7) is of second order in ε while the second line is of third order. Thus, we can write

$$
\begin{aligned}
E(h_{g_\varepsilon}) - E(h) &= \frac{\varepsilon^2}{2}\int_0^\pi g'^2\sin\theta + \left(\frac{\cos 2h}{\sin^2\theta} + \kappa\cos(2h-2\theta)\right) g^2\sin\theta\,\mathrm{d}\theta + \mathcal{O}(\varepsilon^3) \\
&= \frac{\varepsilon^2}{2}\,\delta^2 E[h](g,g) + \mathcal{O}(\varepsilon^3)\,. \qquad (4.8)
\end{aligned}
$$

Now, the term of second order is indeed twice the second variation of E at h evaluated twice in g, which follows immediately by deriving twice in ε and then setting $\varepsilon = 0$. In order to show the existence of a map with energy lower than $E(h)$, we need to show that the second variation is negative for some g since we can control the higher order terms for ε small. The remainder of this section is devoted to the proof of this claim.

We define the map $g\colon [0,\pi] \to \mathbb{R}$ as follows.

$$g(\theta) = (h'(\theta) - 1)\sin\theta\,. \tag{4.9}$$

Then, we have $g(0) = g(\pi) = 0$, which implies that g is a valid perturbation. We compute the first derivative of g as

$$g'(\theta) = h''\sin\theta + h'\cos\theta - \cos\theta = \left(\frac{\sin 2h}{2\sin^2\theta} + \frac{\kappa}{2}\sin(2h - 2\theta)\right)\sin\theta - \cos\theta\,, \tag{4.10}$$

where we again used that h is a solution to (2.18).

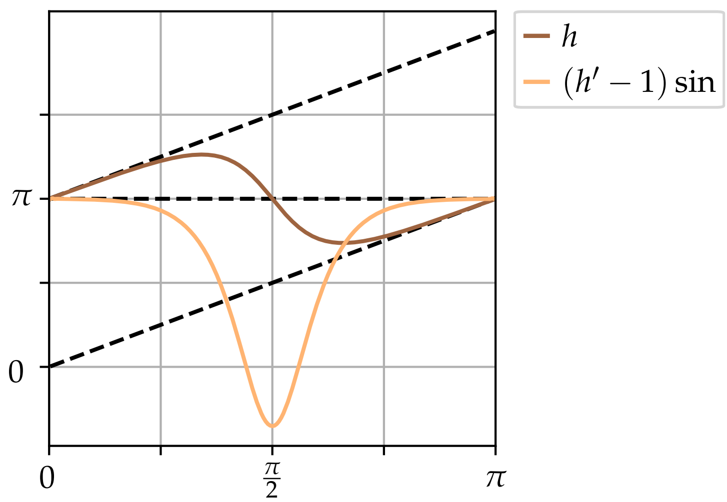

Figure 4.7.: Plot of h and the function g for $\kappa = 20$.

As a consequence of Definition 4.11, we find that $g(\theta) \leq 0$ for all $\theta \in [0,\pi]$. Moreover, we observe that $2g'\sin$ is exactly equal to $-f$ in the proof of said lemma. And as we have shown there that $f \geq 0$ for all $(\theta, h(\theta)) \in W$, we conclude that $g' \leq 0$ on $[0, \frac{\pi}{2}]$ and by symmetry $g' \geq 0$ on $[\frac{\pi}{2}, \pi]$, as g is an even function at $\frac{\pi}{2}$ and sin is nonnegative on the whole interval. Consequently, g is nonpositive on $[0,\pi]$ and attains its minimum at $\frac{\pi}{2}$.
We want to argue heuristically, why our choice of g is a good perturbation to lower the energy of h. As $h \in H_{1,1}$, this implies that $\boldsymbol{m}_\infty$ is equal to $-\hat{e}_3$ at the equator. At those points, the normal $\boldsymbol{\nu}$ lies in the x_1x_2-plane. Hence, the energy contribution of the anisotropy, which stems from the integral over $1 - \langle \boldsymbol{m}_\infty, \boldsymbol{\nu}\rangle^2$, is maximal at the equator. The idea now is that if we wiggle the vector field $\boldsymbol{m}_\infty$ at the equator slightly in any direction, we will lower the anisotropy energy more than we would get through an increase in the Dirichlet energy. This especially

implies that we break the symmetry of having hemispheric maps. If we restrict ourselves to wiggles in an axisymmetric manner, on the level of the profile this means that we have to perturb the profile h around $\frac{\pi}{2}$ such that it no longer is hemispheric. The perturbation g achieves exactly this. As we have seen, g is nonpositive and an even function at $\frac{\pi}{2}$ which breaks the symmetry in the most effective way. Additionally, it has its minimum at $\frac{\pi}{2}$, resulting in its largest effect being in fact at the equator. Moreover, as will see, it also balances the Dirichlet energy contribution in a precise manner. We quantify the above argument now in the proof of the following proposition.

Proposition 4.17. *There exists $\kappa_0 \geq 4$ such that for all $\kappa \geq \kappa_0$ we have*

$$\delta^2 E[h](g,g) < 0\,.$$

Proof. We compute the second variation at h of g and gain

$$\delta^2 E[h](g,g) = \int_0^\pi g'^2 \sin\theta + \left(\frac{\cos 2h}{\sin^2\theta} + \kappa\cos(2h-2\theta)\right) g^2 \sin\theta \,\mathrm{d}\theta\,.$$

We want to rewrite the terms not involving g' by inserting the definition of g once and then integrating by parts, note that $g = 0$ at the boundary. For the first one we obtain

$$\begin{aligned}\int_0^\pi \frac{\cos 2h}{\sin^2\theta} g^2 \sin\theta\,\mathrm{d}\theta &= \int_0^\pi \cos 2h(h'-1) g\,\mathrm{d}\theta \\ &= \int_0^\pi -\frac{\sin 2h}{2\sin\theta} g' \sin\theta - \cos(2h) g\,\mathrm{d}\theta\,.\end{aligned}$$

Likewise, we can compute the second term as

$$\begin{aligned}\int_0^\pi \kappa\cos(2h-2\theta) g^2 \sin\theta\,\mathrm{d}\theta &= \int_0^\pi \kappa\cos(2h-2\theta)(h'-1) g \sin^2\theta\,\mathrm{d}\theta \\ &= \int_0^\pi -\frac{\kappa}{2}\sin(2h-2\theta)(g'\sin^2\theta + 2g\sin\theta\cos\theta)\,\mathrm{d}\theta\,.\end{aligned}$$

Consequently, we can rewrite the second variation using (4.10) as

$$\begin{aligned}\delta^2 E[h](g,g) = \int_0^\pi g'^2 \sin\theta - \underbrace{\left(\frac{\sin 2h}{2\sin\theta} + \frac{\kappa}{2}\sin(2h-2\theta)\sin\theta\right)}_{=g'+\cos\theta} g' \sin\theta\,\mathrm{d}\theta \\ + \int_0^\pi -\cos 2hg - \kappa\sin(2h-2\theta) g \sin\theta\cos\theta\,\mathrm{d}\theta \\ = \int_0^\pi -g'\frac{1}{2}\sin(2\theta) - \cos 2hg - \kappa\sin(2h-2\theta) g\sin\theta\cos\theta\,\mathrm{d}\theta\end{aligned}$$

$$= 2\int_0^{\frac{\pi}{2}} (\cos 2\theta - \cos 2h) g \,\mathrm{d}\theta - 2\kappa \int_0^{\frac{\pi}{2}} \sin(2h - 2\theta) g \sin\theta \cos\theta \,\mathrm{d}\theta\,, \tag{4.11}$$

where we have used that the integrands of both integrals are even at $\frac{\pi}{2}$, such that we can restrict the domain of integration to $[0, \frac{\pi}{2}]$ and multiply the result by 2. Since $\cos 2\theta - \cos 2h = 2\sin(h-\theta)\sin(h+\theta) \le 0$ for $(\theta, h(\theta)) \in W$, we find in combination with the nonpositivity of g that the first integral is positive. For the second integral we assert that $\sin(2h - 2\theta) \le 0$ for $(\theta, h(\theta)) \in W$ such that including the signs of g and in front of the integral we find that this integral is negative. Hence, depending on which of the two integrals dominates, we can conclude that the second variation is negative or positive. It turns out that the first integral can be bounded from above by a constant with respect to κ while the second integral including the factor κ can be bounded from above by a constant times $-\sqrt{\kappa}$. As a result, for κ large enough, we will be able to establish negativity of the second variation.

Before we start with the estimates of the individual integrals, we first show that we can in fact narrow down the domain of integration even further. To this end, by the trigonometric identity (A.T4) we can rewrite all the terms in the integrals that get multiplied by g as follows

$$\begin{aligned}&\cos 2\theta - \cos 2h - \kappa \sin(2h - 2\theta)\sin\theta\cos\theta \\ &= \frac{1}{2}\sin(h-\theta)(\kappa \sin(h - 3\theta) + (4-\kappa)\sin(h+\theta))\,.\end{aligned}$$

Now since $(\theta, h(\theta)) \in W$, we find that for $\theta \le \frac{\pi}{4}$ we have $(\theta, h(\theta)) \in W \cap \{3x \le y \le \pi + 3x\} \cap \{\pi - x \le y \le \frac{3\pi}{2} - x\}$, from which follows

$$\frac{1}{2}\underbrace{\sin(h-\theta)}_{\ge 0}(\kappa \underbrace{\sin(h-3\theta)}_{\ge 0} + \underbrace{(4-\kappa)}_{<0}\underbrace{\sin(h+\theta)}_{\le 0}) \ge 0\,.$$

Hence, as $g \le 0$ on the whole interval of integration, we deduce that the total integral from 0 to $\frac{\pi}{4}$ is negative (since the integrand does not vanish everywhere). Therefore, we obtain

$$\delta^2 E[h](g,g) < \underbrace{2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\theta \, g \,\mathrm{d}\theta}_{=:I_1} + \underbrace{2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\cos 2h \, g \,\mathrm{d}\theta}_{=:I_2}$$

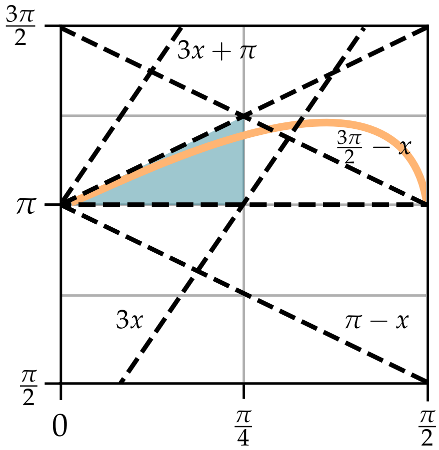

Figure 4.8.: On the interval $[0, \frac{\pi}{4}]$ the graph of h is in the set $W \cap \{x \le \frac{\pi}{4}\}$ and hence the integrand is negative.

$$- 2\kappa \underbrace{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2h - 2\theta) g \sin\theta \cos\theta \, d\theta}_{=:I_3}, \tag{4.12}$$

We begin with the bound for I_1. We insert the definition of g and integrate by parts, which yields

$$\begin{aligned} I_1 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\theta (h' - 1) \sin\theta \, d\theta \\ &= (h - \theta - \pi) \cos 2\theta \sin\theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (h - \theta - \pi)(\cos\theta \cos 2\theta - 2 \sin\theta \sin 2\theta) \, d\theta \\ &= \frac{\pi}{2} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \underbrace{(\pi + \theta - h)}_{\ge 0} \underbrace{(\cos\theta \cos 2\theta - 2 \sin\theta \sin 2\theta)}_{\le 0} \, d\theta \le \frac{\pi}{2}, \end{aligned} \tag{4.13}$$

where we used the fact that the function $\theta \mapsto \cos\theta \cos 2\theta - 2 \sin\theta \sin 2\theta$ is negative on the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$.

For I_2 we insert again the definition of g and integrate by parts. We obtain

$$\begin{aligned} I_2 &= - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2h (h' - 1) \sin\theta \, d\theta \\ &= - \frac{\sin 2h}{2} \sin\theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 2h}{2} \cos\theta + \cos 2h \sin\theta \, d\theta \\ &= \frac{\sin 2h(\frac{\pi}{4}) - 1}{2\sqrt{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} (\sin 2h - \sin 2\theta) \cos\theta + (\cos 2h - \cos 2\theta) \sin\theta \, d\theta, \end{aligned}$$

where in the last step we inserted a zero and used the fact that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\sin 2\theta\cos\theta + \cos 2\theta \sin\theta \,\mathrm{d}\theta = -\frac{1}{2\sqrt{2}}\,.$$

Now, the first summand is always negative. For the integral, we rewrite the integrand again by virtue of the trigonometric identity (A.T5) to gain

$$\begin{aligned}
&\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin 2h - \sin 2\theta)\cos\theta + (\cos 2h - \cos 2\theta)\sin\theta \,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(h-\theta)(3\cos(h+2\theta) - \cos h)\,\mathrm{d}\theta \\
&\le \frac{1}{\sqrt{2\kappa}}\Bigg(\underbrace{\frac{\kappa}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2(h-\theta)\sin\theta\,\mathrm{d}\theta}_{\le E(h)}\Bigg)^{1/2}\Bigg(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \underbrace{\frac{(3\cos(h+2\theta)-\cos h)^2}{\sin\theta}}_{\le 16}\,\mathrm{d}\theta\Bigg)^{1/2} \\
&\le \sqrt{\frac{\pi E(h)}{2\kappa}}\,,
\end{aligned}$$

where we used the Hölder inequality in the second to last step, and the bound of the integrand of the second integral in the last step follows from studying the function $(x,y) \mapsto (3\cos(x+y) - \cos x)^2 \sin^{-1} x$ on the domain $W \cap \{\frac{\pi}{4} \le x \le \frac{\pi}{2}\}$. All in all, we obtain the estimate

$$I_2 \le \sqrt{\frac{\pi E(h)}{2\kappa}}\,. \tag{4.14}$$

As we have seen in (4.6), $E(h)$ increases like $\sqrt{\kappa}$ such that we can conclude that I_2 can be bounded from above by an arbitrarily small number for κ large enough. Finally, we want to find a bound for I_3, but this time *from below* because we have not included the minus sign in front of the integral. First, we let $\theta(\kappa) \in [\frac{3\pi}{8}, \frac{\pi}{2}]$, to be chosen later. Then we note that $\sin(2h - 2\theta)(h' - 1) \ge 0$ on the domain of integration by Definition 4.11 and the fact that $(\theta, h(\theta)) \in W$. Moreover, the function $\theta \mapsto \sin^2\theta\cos\theta$ is also nonnegative on the interval of integration. Hence, decreasing the domain of integration to $[\frac{\pi}{4}, \theta(\kappa)]$ only decreases the value of the integral. Additionally, it holds that

$$\sin^2\theta\cos\theta \ge \sin^2\theta(\kappa)\cos\theta(\kappa)\,, \tag{4.15}$$

for all $\theta \in [\frac{\pi}{4}, \theta(\kappa)]$. This follows from the fact that the function has a maximum at $2\arctan(\sqrt{2-\sqrt{3}}) \le \frac{3\pi}{8}$ and then monotonically decreases to 0 at $\frac{\pi}{2}$. Moreover,

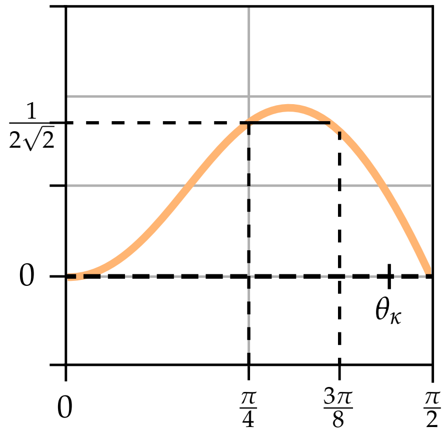

Figure 4.9.: Plot of the function $\theta \mapsto \sin^2\theta\cos\theta$. By the choice of $\theta(\kappa)$, we can guarantee that (4.15) holds.

the function attains the value $(2\sqrt{2})^{-1} = \sin^2(\frac{\pi}{4})\cos(\frac{\pi}{4})$ at an argument $\theta < \frac{3\pi}{4}$. Putting this together, we get the bound

$$\begin{aligned} I_3 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2h - 2\theta)(h' - 1)\sin^2\theta\cos\theta\,\mathrm{d}\theta \\ &\geq \sin^2\theta(\kappa)\cos\theta(\kappa)\int_{\frac{\pi}{4}}^{\theta(\kappa)} \sin(2h-2\theta)(h'-1)\,\mathrm{d}\theta \\ &= \sin^2\theta(\kappa)\cos\theta(\kappa)\ \sin^2(h-\theta)\Big|_{\frac{\pi}{4}}^{\theta(\kappa)} \\ &= \sin^2\theta(\kappa)\cos\theta(\kappa)\Big(\sin^2(h(\theta(\kappa)) - \theta(\kappa)) - \sin^2(h(\tfrac{\pi}{4}) - \tfrac{\pi}{4})\Big)\,, \end{aligned}$$

where we used that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\sin^2(h-\theta) = 2\sin(h-\theta)\cos(h-\theta)(h'-1) = \sin(2h-2\theta)(h'-1)\,.$$

Invoking Definition 4.14, we find that $\sin^2(h(\frac{\pi}{4}) - \frac{\pi}{4}) \to 0$ as $\kappa \to \infty$. The idea now is to choose $\theta(\kappa)$ such that $\sin^2(h(\theta(\kappa)) - \theta(\kappa))$ is bounded away from 0 which means that $\theta(\kappa)$ has to converge to $\frac{\pi}{2}$. However, this implies that $\sin^2\theta(\kappa)\cos\theta(\kappa)$ converges to 0 as $\kappa \to \infty$. Since we multiply the whole integral by κ, if suffices to choose $\theta(\kappa)$ such that $\sin^2\theta(\kappa)\cos\theta(\kappa)$ converges to 0 slower than $1/\kappa$.

We define $\theta(\kappa) \in (\frac{\pi}{4}, \frac{\pi}{2})$ to be the unique point such that

$$h(\theta(\kappa)) = \frac{3\pi}{4} + \theta(\kappa)\,.$$

Existence of such a point follows from the intermediate value theorem since h is continuous and $h(0) = \pi > \frac{3\pi}{4}$ and $h(\frac{\pi}{2}) = \pi < \frac{5\pi}{4} = \frac{3\pi}{4} + \frac{\pi}{2}$. Uniqueness follows from Definition 4.11 – once h has crossed the line $\frac{3\pi}{4} + \theta$, it can never surpass it again since that would require $h' > 1$. We also define $\delta = \frac{\pi}{2} - \theta(\kappa)$. Then we get the identities

$$\sin\theta(\kappa) = \cos\delta\,, \text{ and } \cos\theta(\kappa) = \sin\delta\,.$$

In view of Definition 4.14, we observe that $\theta(\kappa) \to \frac{\pi}{2}$ as $\kappa \to \infty$ from which follows that $\delta \to 0$ as $\kappa \to \infty$. Coming back to the expression for I_3, we see that by definition of $\theta(\kappa)$ we have $\sin^2(h(\theta(\kappa)) - \theta(\kappa)) = \sin^2(\frac{3\pi}{4}) = \frac{1}{2}$ for all κ. Moreover, with the above identities we conclude that

$$I_3 \geq \sin\delta\cos^2\delta\left(\frac{1}{2} - o(1)\right). \tag{4.16}$$

Where $o(1)$ is with respect to $\kappa \to \infty$. Consequently, we assert that our estimate depends linearly of lowest order in δ. Therefore, the question at hand now is only the rate of convergence of δ to 0.
We claim that there exist constants $C > 0$ and $\kappa_0 \geq 4$ such that

$$\frac{1}{C\sqrt{\kappa}} \leq \delta \leq \frac{C}{\sqrt{\kappa}} \tag{4.17}$$

for all $\kappa \geq \kappa_0$. For the upper bound we argue as in the proof of Definition 4.14. We recall that as stated in Definition 4.13 we have that $\sin^2(h-\theta)\sin\theta$ is monotonically increasing on $[\theta(\kappa), \frac{\pi}{2}]$ such that with the energy bound (4.6) we find that

$$\begin{aligned} C\sqrt{\kappa} \geq E(h) &\geq \frac{\kappa}{2}\int_{\theta(\kappa)}^{\frac{\pi}{2}} \sin^2(h-\theta)\sin\theta\,\mathrm{d}\theta \\ &\geq \frac{\kappa}{2}\underbrace{\sin^2(h(\theta(\kappa)) - \theta(\kappa))}_{=\frac{1}{2}}\sin\theta(\kappa)\delta \geq \frac{\kappa}{4\sqrt{2}}\delta\,, \end{aligned}$$

where we used that $\theta(\kappa) > \frac{\pi}{4}$ such that $\sin\theta(\kappa) > 1/\sqrt{2}$. Hence, for any $\kappa \geq 4$

$$\delta \leq \frac{C}{\sqrt{\kappa}}\,.$$

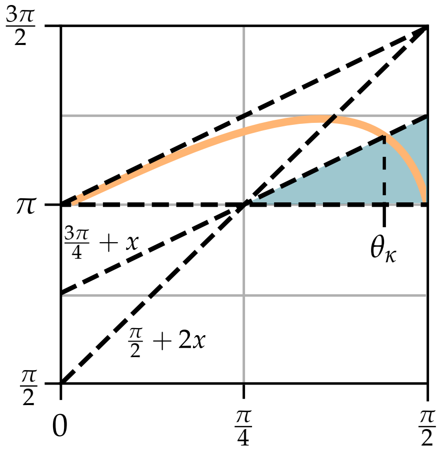

Figure 4.10.: Definition of $\theta(\kappa)$ as the crossing point of h and $\theta \mapsto \frac{3\pi}{4} + \theta$. Moreover, the graph lies in the set $W \cap \{y \leq \frac{3\pi}{4} + x\} \subset W \cap \{y \leq \frac{\pi}{2} + 2x\}$ for $\theta \in [\theta(\kappa), \frac{\pi}{2}]$.

For the lower bound we use the definition of $\theta(\kappa)$ and apply Taylor's theorem with the remainder term in integral form to obtain

$$\frac{\pi}{4} - \delta = h(\theta(\kappa)) - h(\tfrac{\pi}{2}) = -h'(\tfrac{\pi}{2})\delta + \int_{\theta(\kappa)}^{\frac{\pi}{2}} h''(\theta)(\theta - \theta(\kappa))\,\mathrm{d}\theta\,.$$

Now, as already stated above, once we have passed $\theta(\kappa)$, the value of $h(\theta)$ stays below $\frac{3\pi}{4} + \theta$, i.e., for all $\theta \in [\theta(\kappa), \frac{\pi}{2}]$ we have $(\theta, h(\theta)) \in W \cap \{y \leq \frac{3\pi}{4} + x\} \subset W \cap \{y \leq \frac{\pi}{2} + 2x\}$. This implies using the trigonometric identity (A.T3) that

$$\begin{aligned} &\frac{\sin 2h}{2\sin^2\theta} + \frac{\kappa}{2}\sin(2h - 2\theta) \\ &= \frac{1}{8\sin^2\theta}\Big(2\kappa \underbrace{\sin(2h-2\theta)}_{\leq 0} - \kappa \underbrace{\sin(2h-4\theta)}_{\geq 0} + \underbrace{(4-\kappa)}_{\leq 0}\underbrace{\sin 2h}_{\geq 0}\Big) \leq 0\,. \end{aligned}$$

Therefore, using the fact that h is a solution to the Euler–Lagrange equation, we gain the inequality

$$h''(\theta) \leq -\frac{\cos\theta}{\sin\theta} h'(\theta)$$

for all $\theta \in [\theta(\kappa), \frac{\pi}{2}]$. Putting this into the remainder term we obtain by partial integration

$$\frac{\pi}{4} - \delta \leq -h'(\tfrac{\pi}{2})\delta - \int_{\theta(\kappa)}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta} h'(\theta)(\theta - \theta(\kappa))\,\mathrm{d}\theta$$

$$= -h'(\tfrac{\pi}{2})\delta - \underbrace{h\frac{\cos\theta}{\sin\theta}(\theta-\theta(\kappa))\Big|_{\theta(\kappa)}^{\frac{\pi}{2}}}_{=0} + \int_{\theta(\kappa)}^{\frac{\pi}{2}} h\Big(-\frac{\theta-\theta(\kappa)}{\sin^2\theta} + \frac{\cos\theta}{\sin\theta}\Big)\mathrm{d}\theta\,.$$

In the integral the first term is negative. Using the fact that $h\sin^{-1}$ is bounded from above on the interval we estimate further

$$\begin{aligned}\int_{\theta(\kappa)}^{\frac{\pi}{2}} h\Big(-\frac{\theta-\theta(\kappa)}{\sin^2\theta} + \frac{\cos\theta}{\sin\theta}\Big)\mathrm{d}\theta &\leq C\int_{\theta(\kappa)}^{\frac{\pi}{2}} \cos\theta\,\mathrm{d}\theta\\ = C\int_{\theta(\kappa)}^{\frac{\pi}{2}} \sin(\tfrac{\pi}{2}-\theta)\,\mathrm{d}\theta &\leq C\int_{\theta(\kappa)}^{\frac{\pi}{2}} (\tfrac{\pi}{2}-\theta)\,\mathrm{d}\theta \leq C\delta^2\,.\end{aligned}$$

Taking the already established upper bound for δ, we find κ_0 such that $C\delta < 1$ for all $\kappa \geq \kappa_0$. Applying this result in the above inequality and rearranging yields

$$\frac{\pi}{4} \leq \big(-h'(\tfrac{\pi}{2}) + 2\big)\delta \leq C\sqrt{\kappa}\delta\,,$$

invoking Definition 4.15 and the fact that $\kappa \geq 4$. Dividing by $C\sqrt{\kappa}$ we arrive at the lower bound.

Having established (4.17), we come back to (4.16). We write

$$\kappa I_3 \geq \kappa\delta\bigg(\frac{\sin(\delta)}{\delta}\cos^2(\delta)\Big(\frac{1}{2} - o(1)\Big)\bigg)\,.$$

The term in parentheses converges to $1/2$ as $\kappa \to \infty$. Hence, increasing κ_0 if necessary, we conclude that for $\kappa \geq \kappa_0$ we have

$$\kappa I_3 \geq \frac{\kappa}{4}\delta \geq C\sqrt{\kappa} \tag{4.18}$$

for some constant $C > 0$.

Inserting all estimates (4.13), (4.14), and (4.18) into (4.12) we find that for $\kappa \geq \kappa_0$ it holds

$$\delta^2 E[h](g,g) \leq \frac{\pi}{2} + \sqrt{\frac{\pi E(h)}{\kappa}} - C\sqrt{\kappa}\,.$$

Increasing κ_0 if necessary, we can ensure that the right-hand side is indeed negative since the positive terms are bounded from above. □

With this result at hand, it is now a simple task to show that the hemispheric solution $\boldsymbol{m}_\infty$ is a saddle point of the energy functional.

Proof of Theorem 1. We set $\boldsymbol{m} = \boldsymbol{m}_\infty$ and let $\kappa_0 \geq 4$ be the constant from the previous proposition. We set $\kappa \geq \kappa_0$ to be arbitrary but fixed. As $\boldsymbol{m}$ is a solution

to the Euler–Lagrange equation (EL), it is a critical point of $\mathcal{E}$. We need to show that in any H^1-neighborhood of $\boldsymbol{m}_\infty$ there exist two elements where one has lower and the other has higher energy than $\boldsymbol{m}_\infty$.
As the energy is nonincreasing along the flow as shown in Definition 3.4, we find for any $\tilde{\varepsilon} > 0$ an element $\boldsymbol{m}_+$ with $\|\boldsymbol{m} - \boldsymbol{m}_+\|_{H^1} < \tilde{\varepsilon}$ and $\mathcal{E}(\boldsymbol{m}_+) > \mathcal{E}(\boldsymbol{m})$. Therefore, we only need to assert the existence of an element $\boldsymbol{m}_-$ with $\|\boldsymbol{m} - \boldsymbol{m}_-\|_{H^1} < \tilde{\varepsilon}$ and $\mathcal{E}(\boldsymbol{m}_-) < \mathcal{E}(\boldsymbol{m})$ for $\boldsymbol{m}$ to be a saddle point. To this end, we set $\boldsymbol{m}_-$ to be the axisymmetric map with the profile $h_{g_\varepsilon} = h + \varepsilon g$, where g is the function defined in (4.9). Then, according to Definition 4.17, there exists a constant $M > 0$ such that due to (4.8) we have

$$E(h_{g_\varepsilon}) - E(h) \leq -M\varepsilon^2 + \mathcal{O}(\varepsilon^3) < 0$$

for ε small enough. This of course implies $\mathcal{E}(\boldsymbol{m}_-) < \mathcal{E}(\boldsymbol{m})$. Consequently, it remains to show that $\|\boldsymbol{m} - \boldsymbol{m}_-\|_{H^1} < \tilde{\varepsilon}$ for ε small. To this end, we invoke (2.16). First,

$$\|h - h_{g_\varepsilon}\|_{H^1}^2 = \varepsilon^2 \|g\|_{H^1}^2 \leq C\varepsilon^2\,,$$

since g is a smooth function on $[0, \pi]$. Secondly, we have

$$\int_0^\pi \left|h - h_{g_\varepsilon}\right|^2 |h' h'_{g_\varepsilon}| \sin\theta \mathrm{d}\theta \leq \varepsilon^2 \|h' - 1\|_{L^\infty}^2 \int_0^\pi \Big((h')^2 + \varepsilon |h' g'| \Big) \sin\theta \mathrm{d}\theta \leq C\varepsilon^2\,,$$

where again we use that h and g are smooth and thus are bounded and have bounded derivatives. As a result, by virtue of (2.16) we ultimately arrive at

$$\|\boldsymbol{m} - \boldsymbol{m}_{g_\varepsilon}\|_{H^1} \leq C\varepsilon < \tilde{\varepsilon}$$

for ε small enough. Hence, we have found an element $\boldsymbol{m}_-$ in the H^1-neighborhood of $\boldsymbol{m}_\infty$ with lower energy than $\boldsymbol{m}_\infty$. Consequently, we conclude that $\boldsymbol{m}_\infty$ is a saddle point of the energy functional $\mathcal{E}$. □

4.4. Proof of existence of a saddle point: Theorem 2

In this section we want to show that there is a $0 < \kappa_1 < 4$ for which there exists a saddle point of $\mathcal{E}$ in the space of axisymmetric solutions for every $\kappa > \kappa_1$, i.e., we prove Theorem 2. The idea of the proof is twofold.
In the first step, we repeat the arguments of the previous section and show that the critical points obtained in Definition 4.9 are saddle points of the energy

functional $\mathcal{E}$ for every $\kappa \geq 4$. In this case, the estimates simplify because we can in fact show negativity of the integrand of the second variation of the energy functional on the whole interval.
Then in a second step, we show that we can go beyond $\kappa = 4$ and find saddle points for also smaller values of κ. For this step we do not argue by means of the skyrmionic map heat flow but rather use functional analytic arguments. We depart from the critical point obtained in Definition 2.19 for $\kappa = 4$ and show that it is a saddle point of $\mathcal{E}$ by means of spectral analysis of the second variation operator of the reduced axisymmetric energy. Then we argue by perturbation in κ that for values close to 4 we obtain a critical point which is still a saddle point. In the end it is unclear whether the critical points obtained for $\kappa > 4$ by these two methods are the same. However, we conjecture that they are.

4.4.1. Saddle points for $\kappa > 4$

For the course of this subsection we fix $\kappa > 4$ and we set $h = h_\kappa$ to be the smooth profile of the hemispheric solution $\boldsymbol{m}_\kappa$ of (EL) obtained from the initial data $\boldsymbol{m}_0$ with profile $\theta \mapsto 2\theta$ as in Definition 4.8. By Definition 4.9 and since $\theta \mapsto 2\theta$ clearly satisfies the wedge condition stated there, we know that h satisfies that condition too, i.e., its graph fulfills

$$(\theta, h(\theta)) \in W := \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{\pi}{2}, x \leq y \leq 2x\right\}.$$

Note that W is a different set than the one used in the previous section.

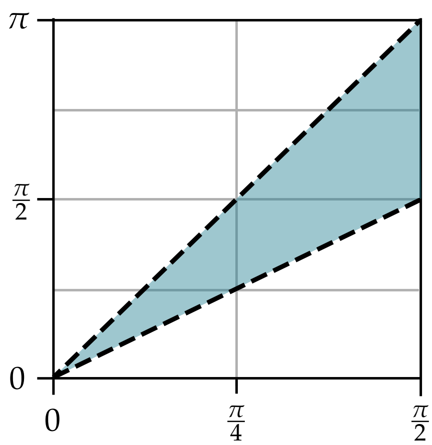

Figure 4.11.: The set W.

We begin with the first observation complementary to that of Definition 4.11.

Lemma 4.18. *Let $\kappa \geq 4$ and let $h \in H_{0,2}$ be a solution of* (2.18) *that satisfies the wedge condition* (4.4)*. Then, for all $\theta \in [0, \pi]$ it holds*

$$h'(\theta) \geq 1 \,.$$

Proof. We can almost repeat the proof of Definition 4.11 verbatim, only changing the sets in which the graph lies and interchanging the signs in the inequalities. By the hemispheric symmetry, it suffices to show the claim for $\theta \in [0, \frac{\pi}{2}]$. We argue by contradiction and assume that there exists $\theta \in [0, \frac{\pi}{2}]$ such that $h'(\theta) < 1$. From the wedge condition we conclude that $h'(0) \geq 1$ and $h'(\frac{\pi}{2}) \geq 2$. Thus, $\theta \in (0, \frac{\pi}{2})$ and by Rolle's theorem and the smoothness of h we can assume without loss of generality that $h'(\theta) < 1$ and $h''(\theta) = 0$, i.e., we choose the minimum of h'. This corresponds to an inflection point of h. As we have fixed θ now, we suppress it as argument of h and its derivatives. Then, since h is a solution to (2.18) and $\sin^2\theta > 0$, we have

$$\begin{aligned} 0 &= 2\sin^2\theta\left(h'' + \frac{\cos\theta}{\sin\theta}h' - \frac{\sin 2h}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2h - 2\theta)\right) \\ &< 2\cos\theta\sin\theta - \sin 2h - \kappa\sin(2h-2\theta)\sin^2\theta \\ &= \sin 2\theta - \sin 2h - \kappa\sin(2h-2\theta)\sin^2\theta =: f(\theta)\,. \end{aligned}$$

Since $(\theta, h) \in W$, we get $h - \theta \in [0, \frac{\pi}{2}]$ which implies that $\sin(h - \theta) \geq 0$ and $\sin(2h - 2\theta) \geq 0$. Now, either $(\theta, h) \in \{y \leq \frac{\pi}{2} - x\} \cap W$ or $(\theta, h) \in \{y > \frac{\pi}{2} - x\} \cap W$. In the first case, we rewrite the function f using a basic trigonometric identity as

$$f(\theta) = -2\cos(h+\theta)\sin(h-\theta) - \kappa\sin(2h-2\theta)\sin^2\theta\,.$$

Since $(\theta, h) \in \{y \leq \frac{\pi}{2} - x\} \cap W$, we have $h + \theta \in [0, \frac{\pi}{2}]$ and hence $\cos(h + \theta) \geq 0$. Together with the aforementioned bounds this leads to $f(\theta) \leq 0$, a contradiction to $f > 0$.

In the second case, we write f using (A.T2) and the above form as

$$f(\theta) = \frac{1}{2}\sin(h-\theta)(2\kappa\sin(h-2\theta)\sin\theta - \kappa\cos(h-\theta) + (\kappa-4)\cos(h+\theta))\,.$$

Since we are now in the other case, we find $\cos(h + \theta) < 0$ and by the assumption on κ we get that the last term in the parentheses is nonpositive. Moreover, as $h - \theta \in [0, \frac{\pi}{2}]$ we conclude that $\cos(h - \theta) > 0$. Finally, $(\theta, h(\theta)) \in W$ implies $h - 2\theta \in [-\frac{\pi}{2}, 0]$ such that $\sin(h - 2\theta) \leq 0$. Therefore, we deduce that all

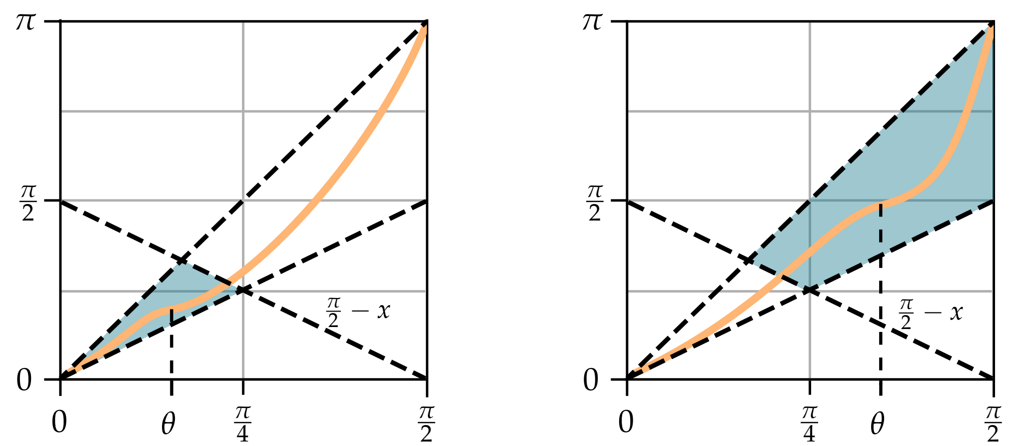

Figure 4.12.: In the first case (left), the inflection point is in the set $\{y \leq \frac{\pi}{2} - x\} \cap W$. In the second case (right), the inflection point is in the set $\{y > \frac{\pi}{2} - x\} \cap W$.

three terms in the parentheses are nonpositive and as a result $f(\theta) \leq 0$, again a contradiction. Hence, we arrive at the conclusion that both cases lead to a contradiction and thus we have finished the proof. □

We define the same map $g\colon [0, \pi] \to \mathbb{R}$ as in the previous section for the perturbation of the profile.

$$g(\theta) = (h'(\theta) - 1)\sin\theta\,.$$

Then, by the previous lemma, we have $g(\theta) \geq 0$ for all $\theta \in [0, \pi]$. This is already enough to show that the second variation of the energy functional evaluated at g is negative.

Proposition 4.19. *For all $\kappa \geq 4$ we have*

$$\delta^2 E[h](g, g) < 0\,.$$

Proof. Since the computations leading to (4.11) do not depend on the specifics of the profile h, we obtain the same expression

$$\delta^2 E[h](g, g) = 2\int_0^{\frac{\pi}{2}} (\cos 2\theta - \cos 2h - \kappa \sin(2h - 2\theta)\sin\theta\cos\theta) g \,\mathrm{d}\theta\,.$$

We define the term in parentheses as the function $f(\theta)$ and we want to show that it is nonpositive on the interval $[0, \frac{\pi}{2}]$. To this end, let $\theta \in [0, \frac{\pi}{2}]$ be arbitrary. Since $(\theta, h(\theta)) \in W$, we have $h - \theta \in [0, \frac{\pi}{2}]$ and hence $\sin(h - \theta) \geq 0$ and also $\sin(2h - 2\theta) \geq 0$.

Moreover, two cases can occur. Either $(\theta, h(\theta)) \in \{y \leq \pi - x\} \cap W$ or $(\theta, h(\theta)) \in \{y > \pi - x\} \cap W$. In the first case, this implies that $h + \theta \in [0, \pi]$ and hence $\sin(h + \theta) \geq 0$. We can use the trigonometric identity (A.T4) to rewrite f as

$$f(\theta) = \frac{1}{2}\sin(h - \theta)(\kappa \sin(h - 3\theta) + (4 - \kappa)\sin(h + \theta)).$$

By the assumption on κ we conclude that the second term in the parentheses is nonpositive. Moreover, since for all $\theta \in [0, \frac{\pi}{2}]$ we have $(\theta, h(\theta)) \in W \subset \{3x - \pi \leq y \leq 3x\}$, we assert that $h - 3\theta \in [-\pi, 0]$ and hence $\sin(h - 3\theta) \leq 0$, making the first term in the parentheses nonpositive as well. Therefore, we conclude that $f(\theta) \leq 0$ in this case.

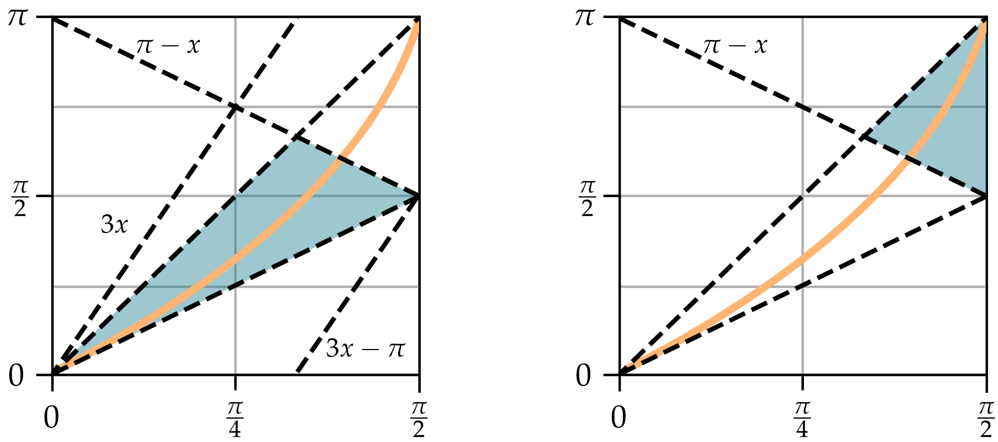

Figure 4.13.: The two regimes in which the graph of h lies. The first case is depicted on the left, the second case on the right.

In the second case, we get that $\sin(h + \theta) < 0$. We rewrite f using the standard addition formula for the cosine as

$$f(\theta) = 2\sin(h - \theta)\sin(h + \theta) - \kappa \sin(2h - 2\theta)\sin^2\theta,$$

from which we immediately obtain in combination with the already stated bounds which hold in all cases that $f(\theta) \leq 0$.

As a result, we deduce that $f(\theta) \leq 0$ for all $\theta \in [0, \frac{\pi}{2}]$. Since $g(\theta) \geq 0$ for all $\theta \in [0, \pi]$, we find that the integrand of the second variation is nonpositive on the whole interval of integration. Moreover, the integrand is also not constantly 0 since h is not constant. Thus, we conclude

$$\delta^2 E[h](g, g) = 2\int_0^{\frac{\pi}{2}} f(\theta)g(\theta)\,\mathrm{d}\theta < 0.$$

□

Corollary 4.20. *For all $\kappa > 4$ the map $\boldsymbol{m}_\kappa$ defined by its profile h_κ is a saddle point of the energy functional $\mathcal{E}$ in the sense that for any neighborhood of $\boldsymbol{m}_\kappa$ in $H^1(\mathbb{S}^2, \mathbb{S}^2)$ there exist maps $\boldsymbol{m}_-$ and $\boldsymbol{m}_+$ inside that neighborhood such that $\mathcal{E}(\boldsymbol{m}_-) < \mathcal{E}(\boldsymbol{m}_\kappa) < \mathcal{E}(\boldsymbol{m}_+)$.*

Proof. The proof is analogous to the one of Theorem 1. With the perturbation g defined above, we can find a map $\boldsymbol{m}_-$ with lower energy than $\boldsymbol{m}_\kappa$. A map with higher energy can be obtained by the skyrmionic map heat flow. □

Note that the heat flow argument to find a map with higher energy than the limit map does not work for the case $\kappa = 4$ since the initial data $\theta \mapsto 2\theta$ is already a critical point of the energy functional and thus already a stationary solution to the flow. With the results from the following subsection we see how we can overcome this problem.

4.4.2. Saddle points in neighborhood of $\kappa = 4$

In this subsection we analyze the critical point $\boldsymbol{m}$ obtained in Definition 2.19 for $\kappa = 4$. We show that it is a saddle point of the energy functional $\mathcal{E}$. Then in a second step, we show that by means of the implicit function theorem we can find a saddle point for all κ in a neighborhood of $\kappa = 4$. This allows us to go beyond the value $\kappa = 4$.

Proposition 4.21. *For $\kappa = 4$, the axisymmetric map $\boldsymbol{m}$ defined by its profile $h(\theta) = 2\theta$ is a saddle point of $\mathcal{E}$ in the sense that for any neighborhood of $\boldsymbol{m}$ in $H^1(\mathbb{S}^2, \mathbb{S}^2)$ there exist maps $\boldsymbol{m}_-$ and $\boldsymbol{m}_+$ inside that neighborhood such that $\mathcal{E}(\boldsymbol{m}_-) < \mathcal{E}(\boldsymbol{m}) < \mathcal{E}(\boldsymbol{m}_+)$.*

Proof. In Definition 2.19 we have already shown that $\boldsymbol{m}$ is a critical point of the energy $\mathcal{E}$ for $\kappa = 4$. We want to show that it is a saddle point.
As in the proof of Theorem 1, it is sufficient to find test functions $g_1, g_2 \in E_{0,0}$ such that $\delta^2 E[h](g_1, g_1) < 0$ and $\delta^2 E[h](g_2, g_2) > 0$.
We compute for a map $g \in E_{0,0}$ the second variation of the energy for the profile h and the value $\kappa = 4$ as stated in (4.8) using a trigonometric identity to simplify the terms. We find

$$\begin{aligned}\delta^2 E[h](g,g) &= \int_0^\pi g'^2 \sin\theta + \left(\frac{\cos 2h}{\sin^2\theta} + \kappa\cos(2h - 2\theta)\right) g^2 \sin\theta \,\mathrm{d}\theta \\ &= \int_0^\pi g'^2 \sin\theta + \left(\frac{\cos 4\theta}{\sin^2\theta} + 4\cos 2\theta\right) g^2 \sin\theta \,\mathrm{d}\theta\end{aligned}$$

$$= \int_0^\pi \left(-g'' - \frac{\cos\theta}{\sin\theta} g' + \left(\frac{1}{\sin^2\theta} - 4 \right) g \right) g \sin\theta \, d\theta$$
$$= \int_0^\pi \langle (-\hat{\mathcal{L}} - 4) g, g \rangle \sin\theta \, d\theta \, ,$$

where we have defined the operator $\hat{\mathcal{L}}$ as

$$\hat{\mathcal{L}} = \frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) - \frac{1}{\sin^2\theta} \, .$$

This operator is the Sturm–Liouville operator of the general Legendre equation with $m = 1$ in the variable $\cos\theta$. This corresponds to the Laplace–Beltrami operator on the 2-sphere $\mathbb{S}^2$ for functions with polar dependence $e^{i\varphi}$. For further information consult Section A.5. $\hat{\mathcal{L}}$ has a discrete spectrum with eigenvalues $-l(l+1)$ for $l \geq 1$ and the associated Legendre functions $P_l^1(\cos\theta)$ as eigenfunctions. Since we are actually interested in the eigenvalues of the shifted operator $\mathcal{L} = -\hat{\mathcal{L}} - 4$, we find that the eigenvalues of $\mathcal{L}$ are given by $\lambda_l = l(l+1) - 4$ with the same eigenfunctions. Then the first eigenvalue is $\lambda_1 = -2$ with eigenfunction $P_1^1(\cos\theta) = \sin\theta$ and the second eigenvalue is $\lambda_2 = 2$ with eigenfunction $P_2^1(\cos\theta) = \sin 2\theta$. All further eigenvalues are positive, too. Consequently, if we set $g_1 = \sin\theta$ and $g_2 = \sin 2\theta$, we find

$$\delta^2 E[h](g_1, g_1) = -2 \int_0^\pi \sin^2\theta \sin\theta \, d\theta < 0$$

and

$$\delta^2 E[h](g_2, g_2) = 2 \int_0^\pi \sin^2 2\theta \sin\theta \, d\theta > 0 \, .$$

Hence, $\boldsymbol{m}$ is a saddle point of $\mathcal{E}$. □

The operator $\hat{\mathcal{L}}$ that appeared in the previous proof will play in important role in the following. We recall the subsequent definition from Section A.5. The maximal domain of $\hat{\mathcal{L}}$, $D(\hat{\mathcal{L}})$, is defined by

$$D(\hat{\mathcal{L}}) := \left\{ f \in L^2_{\sin}((0,\pi)) : f, f' \sin \in \mathrm{AC_{loc}}((0,\pi)), \quad \hat{\mathcal{L}} f \in L^2_{\sin}((0,\pi)) \right\} ,$$

where the space $L^2_{\sin}((0,\pi))$ is the Hilbert space of square integrable functions with respect to the measure $\sin\theta d\theta$ on $(0,\pi)$ and $\mathrm{AC_{loc}}((0,\pi))$ is the space of locally absolutely continuous functions [71, cf. Part 4]. Moreover, by Definition A.34 we know that $D(\hat{\mathcal{L}})$ endowed with the graph norm

$$\|f\|_D = \|f\|_{L^2_{\sin}} + \|\hat{\mathcal{L}} f\|_{L^2_{\sin}} \, ,$$

is a Banach space.

Lemma 4.22. *There exist* $0 < \kappa_1 < 4 < \kappa_2$ *such that for all* $\kappa \in (\kappa_1, \kappa_2)$ *there exists a map* $h_\kappa \in E_{0,2}$ *which is a solution to* (2.18) *for that respective* κ*. The map* $\kappa \mapsto h_\kappa$ *is differentiable as map* $(\mathbb{R}, |\cdot|) \to (E_{0,2}, \|\cdot\|_D)$.

Proof. The idea is to use the implicit function theorem. To this end, we first rewrite (2.18) as an equation for the function $\eta \in C^2([0,\pi])$ using $h_\eta = 2\theta + \eta$. For this, we use the addition formulas for the sine and cosine functions to obtain

$$\begin{aligned}
&h_\eta'' + \frac{\cos\theta}{\sin\theta} h_\eta' - \frac{\sin 2h_\eta}{2\sin^2\theta} - \frac{\kappa}{2}\sin(2h_\eta - 2\theta) \\
&= \eta'' + \frac{\cos\theta}{\sin\theta}\eta' + 2\frac{\cos\theta}{\sin\theta} - \left(4\frac{\cos\theta}{\sin\theta} + (\kappa-4)\sin 2\theta\right)\frac{\cos 2\eta}{2} \\
&\quad + \left(4 - \frac{1}{\sin^2\theta} - (\kappa-4)\cos 4\theta\right)\frac{\sin 2\eta}{2} \\
&=: T(\eta,\kappa)\,,
\end{aligned} \tag{4.19}$$

where T is the operator to which we want to apply the implicit function theorem. By construction and Definition 2.19 we know that $T(0,4) = 0$. We formally compute the Gateaux derivative of T with respect to η in direction $g \in C^2([0,\pi])$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} T(\eta + tg, \kappa)\Big|_{t=0} &= g'' + \frac{\cos\theta}{\sin\theta} g' + \left(4\frac{\cos\theta}{\sin\theta} + (\kappa-4)\sin 2\theta\right)\sin 2\eta\; g \\
&\quad + \left(4 - \frac{1}{\sin^2\theta} - (\kappa-4)\cos 4\theta\right)\cos 2\eta\; g \\
&=: \mathcal{L}(\eta,\kappa) g\,.
\end{aligned}$$

We see that this derivative is a linear operator and a candidate for the Fréchet derivative of T. We assert that $\mathcal{L}(0,4)$ has the form

$$\mathcal{L}(0,4) = \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) - \frac{1}{\sin^2\theta} + 4 = \hat{\mathcal{L}} + 4\,,$$

where $\hat{\mathcal{L}}$ is the operator introduced above. Since we know that -4 is inside the resolvent set of $\hat{\mathcal{L}}$, we conclude that $(\hat{\mathcal{L}} + 4)^{-1}$ exists as an operator from the range of $\hat{\mathcal{L}}$ to its maximal domain $D(\hat{\mathcal{L}})$. As explained in Section A.5 we know that the eigenfunctions span $L^2_{\sin}([0,\pi])$, hence the codomain of T is in fact $L^2_{\sin}([0,\pi])$. Consequently, if we fix now T to be an operator $T\colon D(\hat{\mathcal{L}}) \times \mathbb{R} \to L^2_{\sin}([0,\pi])$, we assert that the operator $\mathcal{L}(0,4)$ is a bounded linear operator from $D(\hat{\mathcal{L}})$ to $L^2_{\sin}([0,\pi])$. Thus, we deduce that $\mathcal{L}(0,4)$ is a Banach space isomorphism. It remains to show that T is in fact continuously Fréchet differentiable. As we have seen, the Gateaux derivative in the first component is the linear operator $\mathcal{L}(\eta,\kappa)$. Hence, it suffices if we show that for all $(\eta,\kappa) \in D(\hat{\mathcal{L}}) \times \mathbb{R}$ the operator $\mathcal{L}(\eta,\kappa)$ is a bounded operator in the operator norm induced by $\|\cdot\|_D$ and that it is continuous as a map $(\eta,\kappa) \mapsto \mathcal{L}(\eta,\kappa)$. Starting from the definition of $\mathcal{L}(\eta,\kappa)$, we see that we can express it as

$$\begin{aligned}\mathcal{L}(\eta,\kappa) = \hat{\mathcal{L}} + \left(4\frac{\cos\theta}{\sin\theta} + (\kappa-4)\sin 2\theta\right)\sin 2\eta + \big(4-(\kappa-4)\cos 4\theta\big)\cos 2\eta \\ + \frac{1}{\sin^2\theta}(1-\cos 2\eta) =: \hat{\mathcal{L}} + f(\eta,\kappa)\,,\end{aligned} \tag{4.20}$$

where $f(\eta,\kappa)$ is in $L^2_{\sin}$ as according to Definition A.35 it holds that $\eta\sin^{-1} \in L^4_{\sin}([0,\pi])$ and hence $\sin(2\eta)\sin^{-1}$ and $(1-\cos 2\eta)\sin^{-2}$ are in $L^2_{\sin}([0,\pi])$. For the other, non-singular parts of f that estimate holds by the boundedness of all functions involved. Consequently, for $g \in D(\mathcal{L})$ we can compute

$$\|\mathcal{L}(\eta,\kappa)g\|_{L^2_{\sin}} \le \|\hat{\mathcal{L}}g\|_{L^2_{\sin}} + \|f(\eta,\kappa)\|_{L^2_{\sin}}\|g\|_{L^\infty} \le (1+\|f(\eta,\kappa)\|_{L^2_{\sin}})\|g\|_D\,,$$

where we also used Definition A.36. Thus, we find that $\mathcal{L}(\eta,\kappa)$ is a bounded linear operator from $D(\hat{\mathcal{L}})$ to $L^2_{\sin}([0,\pi])$ and the Fréchet derivative in the first component is given by $D_\eta T(\eta,\kappa) = \mathcal{L}(\eta,\kappa)$. To show continuity of that component, we infer from the above identity that for (η_1,κ_1) and (η_2,κ_2) in $D(\hat{\mathcal{L}}) \times \mathbb{R}$ we have

$$\|\mathcal{L}(\eta_1,\kappa_1) - \mathcal{L}(\eta_2,\kappa_2)\| \le \|f(\eta_1,\kappa_1) - f(\eta_2,\kappa_2)\|_{L^2_{\sin}}\,.$$

Using the uniform continuity of the trigonometric functions and invoking the lemmata A.35 and A.36 again, we find that we obtain a bound of the form

$$\begin{aligned}&\|\mathcal{L}(\eta_1,\kappa_1) - \mathcal{L}(\eta_2,\kappa_2)\| \\ &\le C\left(\left\|\frac{\eta_1-\eta_2}{\sin}\right\|_{L^2_{\sin}} + \left\|\frac{\eta_1-\eta_2}{\sin}\right\|^2_{L^4_{\sin}} + (1+\kappa_2)\|\eta_1-\eta_2\|_{L^\infty} + |\kappa_1-\kappa_2|\right) \\ &\le C\Big((1+\kappa_2)\|\eta_1-\eta_2\|_D + \|\eta_1-\eta_2\|_D^2 + |\kappa_1-\kappa_2|\Big)\,.\end{aligned} \tag{4.21}$$

This suffices to show that the map $(\eta,\kappa) \mapsto \mathcal{L}(\eta,\kappa)$ is continuous.
Continuous Fréchet differentiability in the second component is straightforward as the operator T is linear in κ, i.e.,

$$D_\kappa T(\eta,\kappa) = \frac{1}{2}(\cos 2\eta - \sin 2\eta)\,.$$

This map is multiplication by a bounded function. Continuity in (η, κ) follows again by the uniform continuity of the trigonometric functions and Definition A.36.

As a result, we can apply the implicit function theorem and find neighborhoods $U \subset D(\mathcal{L})$ of 0 and $(\kappa_1, \kappa_2) \subset \mathbb{R}_{\geq 0}$ of 4, and a function $\eta \colon (\kappa_1, \kappa_2) \to U$ which is continuous with respect to the norm on $D(\hat{\mathcal{L}})$ such that $\eta_4 = 0$ and $T(\eta_\kappa, \kappa) = 0$ for all $\kappa \in (\kappa_1, \kappa_2)$. Setting $h_\kappa = 2\theta + \eta_\kappa$, we find that h_κ is a solution to (2.18) for all $\kappa \in (\kappa_1, \kappa_2)$. Moreover, by Definition A.32, η_κ vanishes at the boundary, such that we assert that h_κ is in $E_{0,2}$. The continuity of the map $\kappa \mapsto h_\kappa$ follows from the continuity of the map $\kappa \mapsto \eta_\kappa$.

Finally, since for $\kappa = 0$ there are no solutions to (2.18) in $E_{0,2}$ as the only harmonic map with mapping degree 0 is the constant map, we can assert that $\kappa_1 > 0$. □

Remark 4.23. Since the Euler–Lagrange equation stems from the first variation of the energy E, it is no surprise that the derivative of the operator T in the proof is the integrand of the second variation of the energy. This implies that for the function $h_\eta = 2\theta + \eta$ we can, in fact, write the second variation of the energy as

$$\delta^2 E[h_\eta](g, g) = \int_0^\pi \langle -\mathcal{L}(\eta, \kappa) g, g \rangle \sin\theta \, \mathrm{d}\theta \,,$$

where $\mathcal{L}(\eta, \kappa)$ is the operator defined in the proof of the previous proposition. ⊡

Using the fact that the negative eigenvalue of the operator $\mathcal{L}(0, 4)$ is bounded away from zero, we can use the closeness of η_κ to 0 to show that for those η_κ the second variation of the energy is still negative for the first eigenfunction of the unperturbed operator.

Proposition 4.24. *Let $h_\kappa = 2\theta + \eta_\kappa$ be the solution to* (2.18) *for $\kappa \in (\kappa_1, \kappa_2)$ as in Definition 4.22 and set $\boldsymbol{m}_\kappa$ to be the axisymmetric map with profile h_κ. Then $\boldsymbol{m}_\kappa$ is a saddle point of the energy functional $\mathcal{E}$ in the sense that for any neighborhood of $\boldsymbol{m}_\kappa$ in $H^1(\mathbb{S}^2, \mathbb{S}^2)$ there exist maps $\boldsymbol{m}_-$ and $\boldsymbol{m}_+$ inside that neighborhood such that $\mathcal{E}(\boldsymbol{m}_-) < \mathcal{E}(\boldsymbol{m}_\kappa) < \mathcal{E}(\boldsymbol{m}_+)$.*

Proof. As in the previous proofs on saddle points, it suffices to find test functions g_1 and g_2 such that $\delta^2 E[h_\kappa](g_1, g_1) < 0$ and $\delta^2 E[h_\kappa](g_2, g_2) > 0$. We use the same test functions as in the proof of Definition 4.21 and set $g_1 = \sin\theta$ and $g_2 = \sin 2\theta$. As explained in the previous remark, this yields for $l = 1, 2$

$$\delta^2 E[h_\eta](g_l, g_l) = \int_0^\pi \langle -\mathcal{L}(\eta_\kappa, \kappa) g_l, g_l \rangle \sin\theta \, \mathrm{d}\theta$$

$$
\begin{aligned}
&= \int_0^\pi \langle -\mathcal{L}(0,4)g_l, g_l\rangle \sin\theta \,\mathrm{d}\theta + \int_0^\pi \langle (\mathcal{L}(0,4) - \mathcal{L}(\eta_\kappa,\kappa))g_l, g_l\rangle \sin\theta\,\mathrm{d}\theta \\
&= \lambda_l \|g_l\|^2_{L^2_{\sin}} + \langle (\mathcal{L}(0,4) - \mathcal{L}(\eta_\kappa,\kappa))g_l, g_l\rangle_{L^2_{\sin}}\,,
\end{aligned}
$$

where λ_l is the eigenvalue of the operator $-\mathcal{L}(0,4)$ corresponding to the eigenfunction g_l. We know that $\lambda_1 = -2$ and $\lambda_2 = 2$.
As shown in (4.21), the map $(\eta,\kappa) \mapsto \mathcal{L}(\eta,\kappa)$ is continuous with respect to the operator norm induced by the norm on $D(\hat{\mathcal{L}}) \times \mathbb{R}$ such that

$$
\begin{aligned}
|\langle (\mathcal{L}(0,4) - \mathcal{L}(\eta_\kappa,\kappa))g_l, g_l\rangle_{L^2_{\sin}}| &\le \|\mathcal{L}(0,4) - \mathcal{L}(\eta_\kappa,\kappa)\| \|g_l\|_D \|g_l\|_{L^2_{\sin}} \\
&\le C\sqrt{1+\lambda_l^2}\Big(\|\eta_\kappa\|_D + \|\eta_\kappa\|_D^2 + |\kappa - 4|\Big)\|g_l\|^2_{L^2_{\sin}} \\
&\le C\sqrt{1+\lambda_l^2}\Big(|\kappa-4| + |\kappa-4|^2\Big)\|g_l\|^2_{L^2_{\sin}}\,,
\end{aligned}
$$

where we used that $\|g_l\|_D^2 = (1+\lambda_l^2)\|g_l\|^2_{L^2_{\sin}}$ and in the last step we applied the differentiability of the function $\kappa \mapsto \eta_\kappa$ as map $\mathbb{R} \to D(\hat{\mathcal{L}})$ provided by Definition 4.22 in combination with the mean value theorem. We can choose κ_1 and κ_2 close enough to 4 such that this term is smaller than $|\lambda_l|/2$ for $l = 1,2$. Consequently, we find for these values of κ that

$$
\begin{aligned}
\delta^2 E[h_\kappa](g_1,g_1) &\le -\|g_1\|^2_{L^2_{\sin}} < 0\,, \\
\delta^2 E[h_\kappa](g_2,g_2) &\ge \|g_2\|^2_{L^2_{\sin}} > 0\,.
\end{aligned}
$$

Therefore, $\boldsymbol{m}_\kappa$ is a saddle point of $\mathcal{E}$ for $\kappa \in (\kappa_1,\kappa_2)$. □

Remark 4.25. For $\kappa \in (4,\kappa_2)$ we have now two ways to obtain a critical point of the energy functional $\mathcal{E}$. The first one is the one via the skyrmionic map heat flow as in Definition 4.9. The second one is the one via the implicit function theorem as presented in this subsection. It is not clear to us if these two methods yield the same critical points. Moreover, it is not guaranteed that the critical points obtained via the implicit function theorem still obey the hemispheric symmetry. Our conjecture is that both methods yield the same critical points, since we conjecture that for every $\kappa \ge 4$ there only exists one critical point of the energy functional $\mathcal{E}$ with profiles in $E_{0,2}$. ⊡

4.4.3. Stitching together both regimes

With the results from the previous two subsections, we can now prove Theorem 2.

Proof of Theorem 2. We let κ_1 be the one from Definition 4.22. Then for $\kappa \in (\kappa_1, 4)$ we set h_κ to be the profile from Definition 4.24. For $\kappa = 4$ we set $h_4 = 2\theta$ and for $\kappa > 4$ we set h_κ to be the one from Definition 4.20. We denote the corresponding maps by $\boldsymbol{m}_\kappa$. We assert that $Q(\boldsymbol{m}_\kappa) = 0$ according to Definition 2.16 as $h_\kappa \in E_{0,2}$. Moreover, by the results of Definition 4.24, Definition 4.21, and Definition 4.20 we find that $\boldsymbol{m}_\kappa$ is a saddle point of the energy functional $\mathcal{E}$ in the claimed sense. The saddle point is different from the one obtained in Theorem 1 since the profiles in the latter are in $H_{0,0}$ and the ones in this theorem are in $E_{0,2}$. □

5. Numerical Experiments

In this chapter, we present some basic numerical experiments for the skyrmionic map heat flow in the axisymmetric setting to gain some insights about its behavior and dependence on the parameter κ. That is, we solve (3.37) numerically for different initial data and κ. After explaining the numerical setting, we start with some mild initial settings which converge to the minimizer found in [59]. Then, we study the profile of the saddle points obtained in the previous chapter. Moreover, we find evidence that the value of κ_0 in Theorem 1 is in fact its lower bound $\kappa_0 = 4$, a result we were not able to prove. We also check the quality of the bounds found in (4.6) and Definition 4.15. Finally, we present an experiment in which we observe a behavior that could be explained by a finite time blowup.

5.1. Numerical setting

The experiments are conducted using the Python library `py-pde` [72] in version `0.33.2`. This package solves partial differential equations numerically "using the methods of lines with a finite-difference approximation of the differential operators." Since we are only interested in the qualitative behavior of the heat flow, we deem this package adequate for our purposes. In all subsequent cases, we run the following code to obtain the data that we then plot using the `matplotlib` package [34]. The input values, initial data function `h_0`, as well as the interval length and boundary conditions are set according to the experiment and stated in the respective section. The value of `t_range` corresponds to the end time of the simulation and is always set after some experimentation to a value such that the solution visually converges to a stationary solution. The same applies to `n_grid`, the number of grid points, where we try to find a balance between accuracy and computational time. We always pick an odd number of grid points to ensure that one points lies exactly at the center point $\frac{\pi}{2}$. The tracker option is used to track the data through time to be then able to analyze it and plot it.

```
import pde
from math import pi

kappa = 2
n_grid = 301
grid = pde.CartesianGrid([[0, pi]], [n_grid], periodic=[False])
field = pde.ScalarField.from_expression(grid, "h_0(x)")
bc = [{'value': pi},{'value': pi}]
p = pde.PDE({"h": f"laplace(h) + cos(x)/sin(x)*d_dx(h) - sin(2*h)/(
                                  2*sin(x)**2) - {kappa}/2*sin(2*h-
                                  2*x)"},bc=bc)
storage = pde.MemoryStorage()
result = p.solve(field, t_range=5, method="scipy", tracker=['
                                  progress', storage.tracker(0.1)])
```

The results of the simulations should only be viewed as a first step toward a better understanding of the flow and as an attempt to gain better intuition about its behavior. One problem, for example, is that the equation is of a singular type with coefficients diverging at the boundary. From the documentation of the package it is not clear how this is handled. Inspecting the results, there are indeed some numerical artifacts at the boundary in the form of oscillations. For this reason, in the upcoming plots, one should not consider the results at the boundaries as accurate. Moreover, we observe that the results can exhibit jumps of multiples of π at the boundaries—a behavior that could be explained by finite time blowup of the solution but is surprising since we solve the equation with fixed boundary conditions.

Nonetheless, since this thesis is not supposed to be a numerical study, we do not go further into details of the numerics but rather focus on the qualitative behavior of the results.

5.2. Minimizer

For the first experiment, we start with the initial data

$$h_0(\theta) = \pi - \sin\theta$$

and boundary conditions $h(0) = h(\pi) = \pi$. Since this profile is not hemispheric, we expect the solution to converge to the global minimizer among the axisymmetric maps, which is the map found in [59]. In fact, we verify this loosely by

taking different initial data which all converge to the same minimizer (as long as they are not hemispheric since that symmetry is preserved). This observation lets us assume that there is a unique minimizer in the class of axisymmetric maps (modulo antipodal symmetry, see [59, Sec. 3.1.3]).
We want to study the behavior for varying κ. We identify two regimes: For $\kappa < 4$, the solution converges to a hemispheric map which can be seen as a sine-like perturbation of the constant map $\theta \mapsto \pi$. For larger κ, the minimizer transforms into a map which consists of a drop from π at $\theta = 0$ to the profile of the normal field ν, i.e., $\theta \mapsto \theta$. This is the expected behavior as shown in [59].
Some further general observations regarding the flow are: in the small κ regime convergence is slower than in the large κ regime. Moreover, for large κ we need a finer grid size to guarantee adequate resolution of the solution. To illustrate the necessity for this heuristically, recall that for large κ the minimizer is close to the map $\theta \mapsto \theta$ with a steep drop at $\theta = 0$. This implies that there is an interval $[\delta_1, \delta_2]$ such that $\sin^2(h - \theta) > 1/2$ for all θ in that interval. Since the energy of the minimizer is bounded from above by 8 [50, Lemma 7], we obtain the following estimate

$$8 \geq \kappa \int_{\delta_1}^{\delta_2} \sin^2(h - \theta) \sin\theta \, \mathrm{d}\theta \geq \frac{\kappa}{2} \sin\delta_1 (\delta_2 - \delta_1) \, .$$

Therefore, in order to resolve this interval, the grid size should be chosen of at least order $O(1/\sqrt{\kappa})$.

5.2.1. Small κ regime

In the small κ regime, we analyze the behavior of the solution in the range $0 < \kappa < 6$ in order to also observe the transition from the hemispheric to the non-hemispheric minimizer. For all values of κ we set `t_range` to 15 and `n_grid` to 301.
In Figure 5.1 we show exemplarily the flow for $\kappa = 4$. Here and in all following plots of the flow for fixed κ it holds that lighter colors imply later snapshots in time. We see that the solution converges to a sine-like hemispheric profile. It should be noted that the convergence is very slow at the end, which points to the fact that a non-hemispheric profile cannot flow into a hemispheric one in finite time.
In Figure 5.2 we show the minimizers for $\kappa \in \{0, \ldots, 6\}$. We see that for $\kappa \leq 4$ the minimizer is hemispheric while for $\kappa > 4$ the symmetry is broken. Additionally, it should be noted that if we choose as initial profile the map $\theta \mapsto \pi + \sin\theta$, the

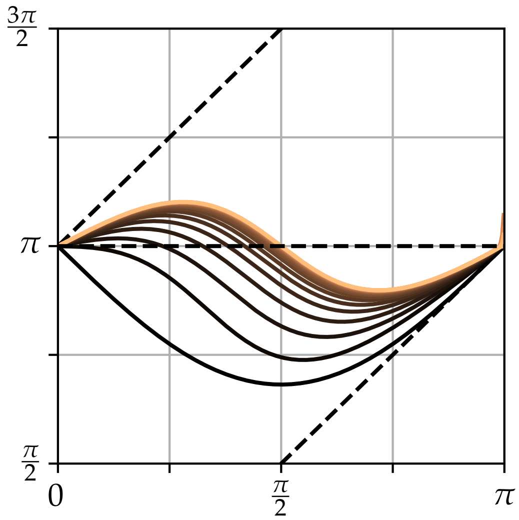

Figure 5.1.: Flow for $\kappa = 4$.

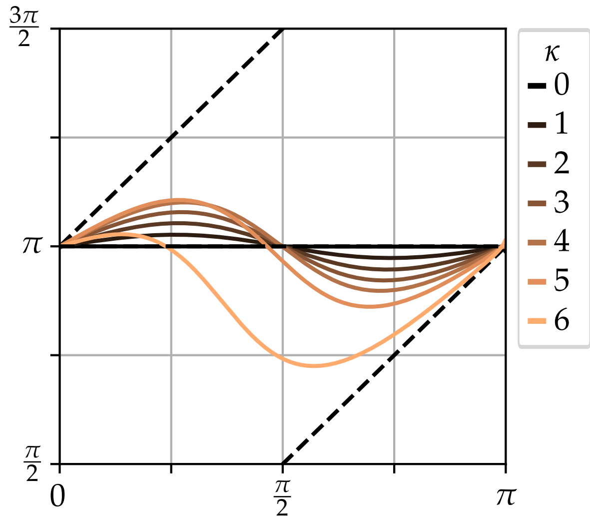

Figure 5.2.: Minimizers for the small κ regime.

minimizers for $\kappa \geq 5$ are different from the ones shown, and are in fact their antipodal transforms. In [59] these are called "upper" and "lower" minimizing profiles. Consequently, for $\kappa = 4$ we can identify a bifurcation point where the hemispheric minimizer splits into two distinct minimizers. It should be noted that the value $\kappa = 4$ is not rigorously proven but just inferred from the numerical results. This analysis is also done with a finer resolution of the κ-range. That there has to be a transition at some point is clear since the hemispheric profile becomes a saddle point for some value $\kappa_0 \geq 4$.

We can also give another argument for why the minimizer for small κ should be hemispheric. To this end, we take the Euler–Lagrange equation of the profile and take a perturbative ansatz $h = \pi + \epsilon g$ with $g \in E_{0,0}$ and $\kappa = \varepsilon$ small. We recall that $\theta \mapsto \pi$ is the (only) solution for $\kappa = 0$. Then we expand the equation in ε and get

$$0 = \varepsilon\left(g'' + \frac{\cos\theta}{\sin\theta}g' - \frac{1}{\sin^2\theta}g + \frac{1}{2}\sin 2\theta\right) + O(\varepsilon^2)\,.$$

In the first order we identify the operator $\hat{\mathcal{L}}$ dealt with in Section A.5 such that we obtain the equation

$$\hat{\mathcal{L}}g = -\frac{1}{2}\sin 2\theta\,.$$

On the right-hand side we detect exactly the second eigenfunction of the operator $\hat{\mathcal{L}}$ with eigenvalue -6 such that we conclude that the solution to the first order

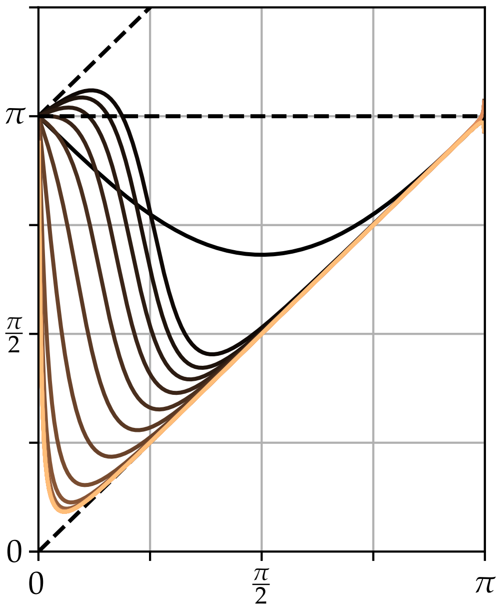

Figure 5.3.: Flow for $\kappa = 36$.

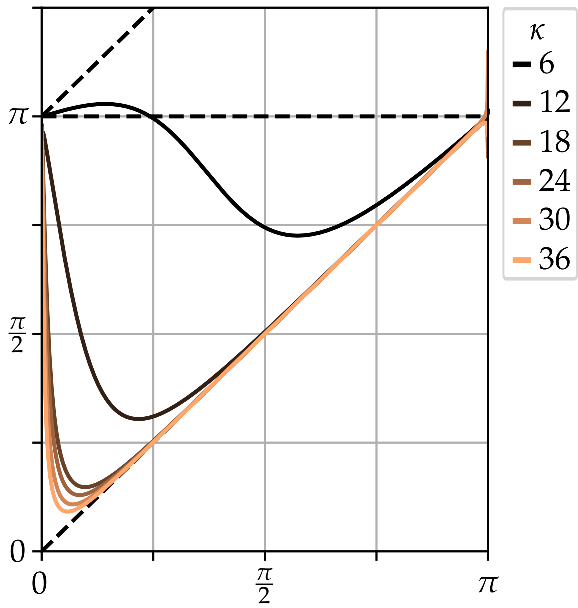

Figure 5.4.: Minimizers for the large κ regime.

equation is given by

$$g(\theta) = \frac{1}{12} \sin 2\theta \,.$$

This is indeed a hemispheric perturbation of the map $\theta \mapsto \pi$ and also the kind of profile we observe in the simulations for small κ.

5.2.2. Large κ regime

In the large κ regime, we analyze the behavior of the solution for $\kappa \geq 6$. We set `t_range` to 2 since the convergence is much faster and `n_grid` to 1001 to guarantee stability of the simulation.

In Figure 5.3, the flow for $\kappa = 36$ is displayed. The snapshots are taken at time intervals of length 0.1. We observe between the initial profile and the first snapshot that the profile quickly transforms into a step-like profile. Afterward, the step moves to the left until the profile converges to the minimizer.

In Figure 5.4, we show the minimizers for $\kappa \in \{6, 12, 18, 24, 30, 36\}$. We observe clearly the pointwise convergence behavior of the minimizers to the identity map as κ grows. This is in line with the results of [59, Sec. 3.2.4.]. Consequently, from a physical point of view, for the field $\boldsymbol{m}$, with growing κ we observe a concentration of structure at the north pole which corresponds to the formation of a skyrmion.

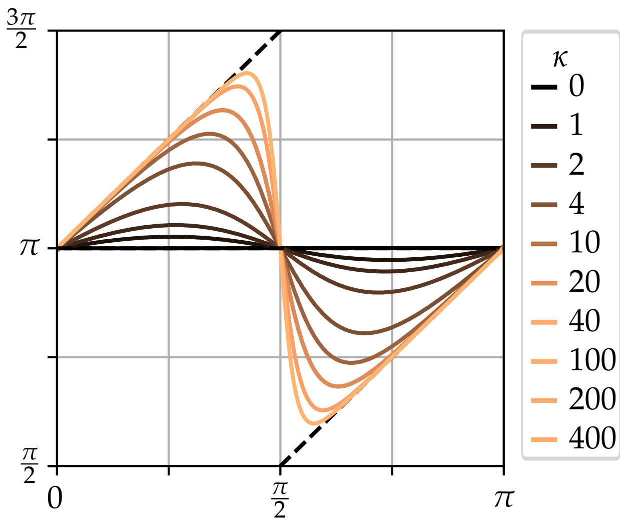

Figure 5.5.: Hemispheric solutions in $H_{1,1}$ for different κ.

5.3. Saddle points

5.3.1. Saddle points of first type

For the analysis of the saddle points of first type dealt with in Theorem 1, we start with the initial data

$$h_0(\theta) = \pi$$

and boundary conditions $h(0) = h(\pi) = \pi$. This map is clearly hemispheric, however it is not the profile $h_{0,\kappa}$ defined in (4.5). Nonetheless, as was the case for the minimizer, we observe that all hemispheric initial data tested, which satisfy the wedge condition (4.2), converge to the same profile. This lets us assume that this limit for these initial data is unique.

In Figure 5.5 the solutions for different values of κ in a range between 0 and 400 are displayed. For $\kappa \leq 4$, we observe that the flow converges to the same functions as in the previous section, i.e., the conjectured hemispheric minimizer. Then, as κ increases, and since the flow stays in the class of hemispheric maps, we observe that the solution is distinct from the minimizer. We can clearly observe the pointwise convergence to $\theta \mapsto \pi + \theta$ on the interval $[0, \frac{\pi}{2})$ and to $\theta \mapsto \theta$ on the interval $(\frac{\pi}{2}, \pi]$ as proven in Definition 4.14.

Now we want to compute numerically an upper bound on the value of κ_0 in Theorem 1. We recall that it is the constant from Definition 4.17, such that the second variation of the energy is negative for the function $g = (h' - 1)\sin$. With

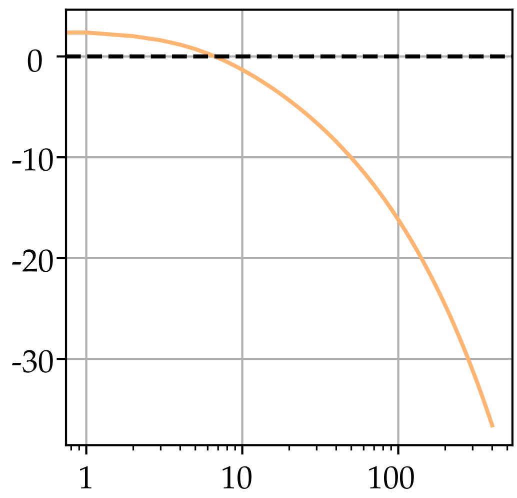

Figure 5.6.: Semi-logarithmic plot of $\delta^2 E[h](g,g)$ in dependence of κ.

the profiles h_κ at hand, we can indeed compute the second variation of the energy and check for which values of κ it is negative. Note that this only gives us an upper bound for κ_0 since we only check the second variation for a specific function g. We do this by computing the integral in (4.11) numerically. We make use of the methods `gradient` and `integral` of the `py-pde` package to compute the gradient and the integral of the functions involved.
In Figure 5.6 we show the result of the computation. We observe that the graph crosses the x-axis at $\kappa \approx 6.7$, thus giving us a numerical upper bound for κ_0 of approximately 6.7. This supports our conjecture that $\kappa_0 = 4$.
Lastly, we aim to evaluate the quality of the bounds found in (4.6) and Definition 4.15. To this end, we compute the energy for the profiles h_κ and the value of $h'(\frac{\pi}{2})$ by means of a central difference method. Then we compare those values with the bounds.
In Figure 5.7 we show the energy of the profiles h_κ in dependence of κ and given bounds. The light orange line corresponds to the energy computed from the profiles h_κ numerically. The brown line shows the bound from Equation (4.6). As we can see, the upper bounds do not quite match the actual energy. This mainly stems from the factor $\pi/\sqrt{2}$ being too large. In fact, the bound $2\sqrt{\kappa} - 1$ is much sharper and almost traces the energy perfectly as seen with the black line.
In Figure 5.8 the values of $-h'_\kappa(\frac{\pi}{2})$ in dependence of κ and given bounds are displayed. The light orange line corresponds to the derivative computed from the profiles h_κ numerically via the central difference method. The light brown line shows the bound from Definition 4.15 where for $E(h)$ we use the numerical

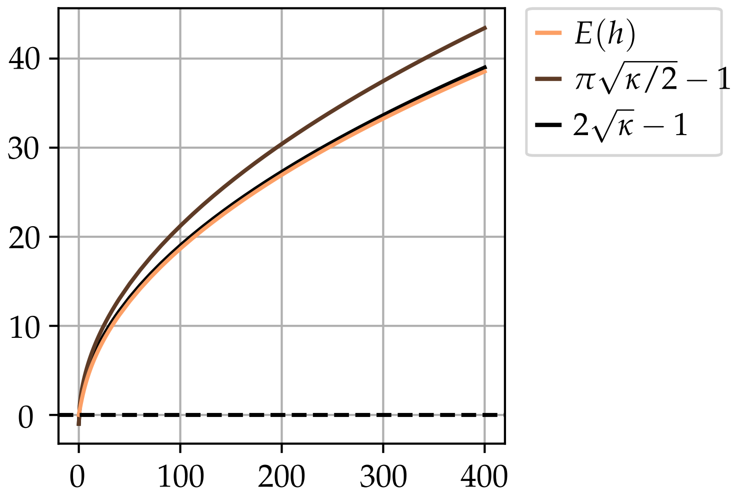

Figure 5.7.: Bounds for $E(h)$ in dependence of κ.

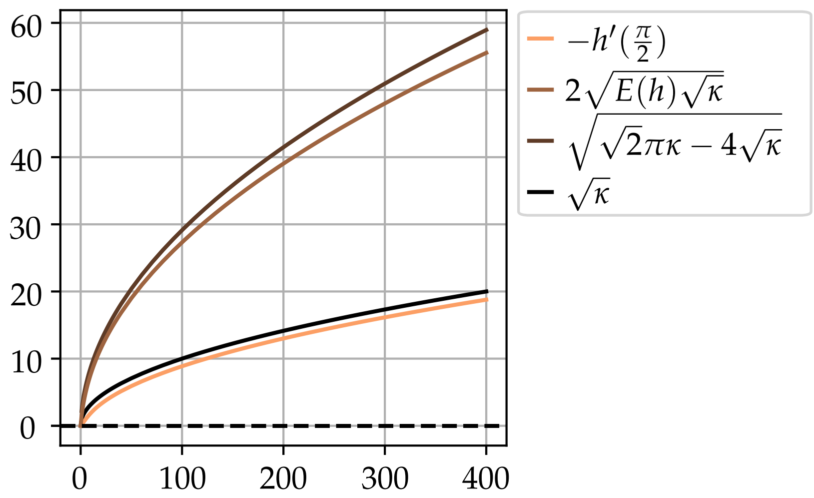

Figure 5.8.: Bounds for $-h'(\frac{\pi}{2})$ in dependence of κ.

values as computed for Figure 5.7. The dark brown line shows the bound where we used for $E(h)$ the bound from Equation (4.6). It is evident that the bounds we found leave room for improvement. The only minor improvement when taking the "real" energy indicates that the formula of the bound itself is deficient. We get the correct functional dependence as we can infer from the black line which shows a bound of just $\sqrt{\kappa}$, which actually is a very good upper bound. Consequently, the constants we have rigorously obtained are too large. This suggests that some of estimates used in the proofs were too crude.

5.3.2. Saddle points of second type

For the analysis of the saddle points dealt with in Theorem 2, we choose the initial data

$$h_0(\theta) = 2\theta$$

and boundary conditions $h(0) = 0$ and $h(\pi) = 2\pi$. For $\kappa > 4$ we thus obtain the critical points as constructed in Definition 4.20. This initial data is hemispheric and therefore the flow will stay in that class. However, for the saddle points in the class of maps $E_{0,2}$ obtained via the implicit function theorem in Definition 4.24 we have not proven that property.

We want to give an argument why we conjecture that they are indeed hemispheric. Taking the ansatz $h = 2\theta + \varepsilon g$ with $g \in E_{0,0}$ and $\kappa = 4 + \varepsilon$ for small ε, we can expand the Euler–Lagrange equation (or rather expand $T(\varepsilon g, 4+\varepsilon)$ from (4.19)) in ε to obtain the equation

$$0 = \varepsilon\left(g'' + \frac{\cos\theta}{\sin\theta}g' - \frac{1}{\sin^2\theta}g + 4g - \frac{1}{2}\sin 2\theta\right) + O(\varepsilon^2)\,.$$

In the first order we identify again the Legendre operator $\hat{\mathcal{L}}$ and gain

$$(\hat{\mathcal{L}} + 4)g = \frac{1}{2}\sin 2\theta\,.$$

This is readily solved by

$$g(\theta) = -\frac{1}{4}\sin 2\theta\,,$$

which is a function in $H_{0,0}$. Consequently, the function h is hemispheric. Therefore, we expect the saddle points from Theorem 2 to be hemispheric. This argument also supports the conjecture that the saddle points from Definition 4.20 and Definition 4.24 are the same for $\kappa \in (4, \kappa_2)$.

We return to the numerical analysis. The first finding that we make is that the flow does not seem to be numerically stable. This becomes manifest in the observation that the hemispheric symmetry gets broken after the flow has ostensibly converged to a stationary solution. Increasing the grid size does not prevent this behavior. In Figures 5.9 and 5.10 and we show exemplarily the flow for $\kappa = 6$ for a grid size `n_grid` of 1001 and `t_range` of 15. As before, the lighter the lines, the more time has passed. We observe that the solution quickly becomes almost stationary and then after some phase of "metastability" the hemispheric symmetry is broken with the profile moving to the right—first slowly and then faster.

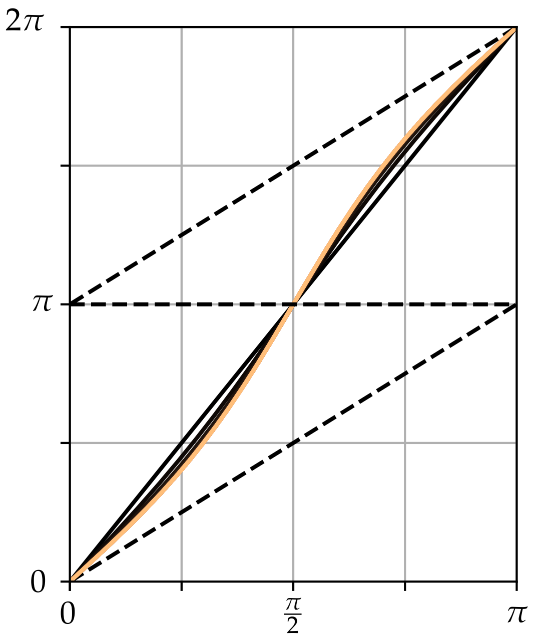

Figure 5.9.: Plot for the flow for $\kappa = 6$ for $t \in [0, 4]$.

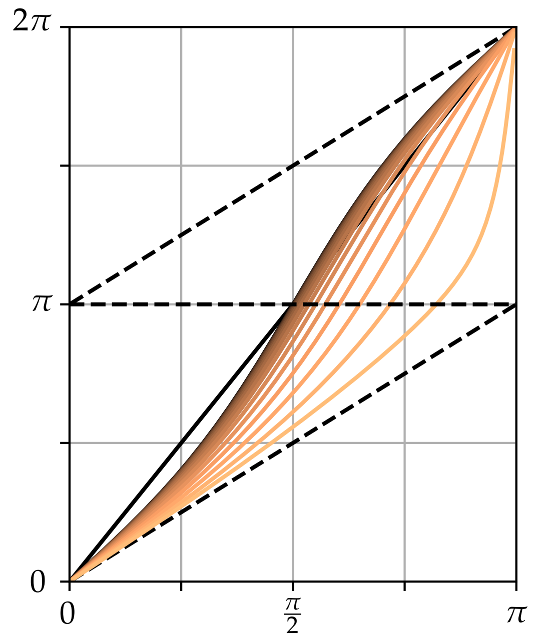

Figure 5.10.: Plot for the flow for $\kappa = 6$ for $t \in [10, 15]$.

Our assumption is that this happens because of tiny numerical inaccuracies at the inflection point that are enough to not let the solution fall back. We have not observed this behavior for the saddle points in $H_{1,1}$. All in all, this is of course not a surprising behavior since we are dealing with saddle points that by definition are unstable. Nonetheless, it seems like that the saddle points in $E_{0,2}$ are much more unstable than the ones in $H_{1,1}$.

To address the numerical instability, we change the problem so that we only solve it on the half interval $[0, \frac{\pi}{2}]$ with the boundary conditions $h(0) = 0$ and $h(\frac{\pi}{2}) = \pi$. Since the flow preserves the hemispheric symmetry, this is equivalent to solving the problem on the whole interval with the boundary conditions $h(0) = 0$ and $h(\pi) = 2\pi$ such that this does not pose a restriction. Indeed, doing so, we do not observe the symmetry breaking anymore because the solver enforces the symmetry due to the boundary conditions.

We divide the analysis into two regimes: $\kappa \geq 4$ and $\kappa < 4$. In the first regime the convergence of the flow is guaranteed by Definition 4.8. In the second regime, we observe a blowup behavior for small κ which we will discuss in detail.

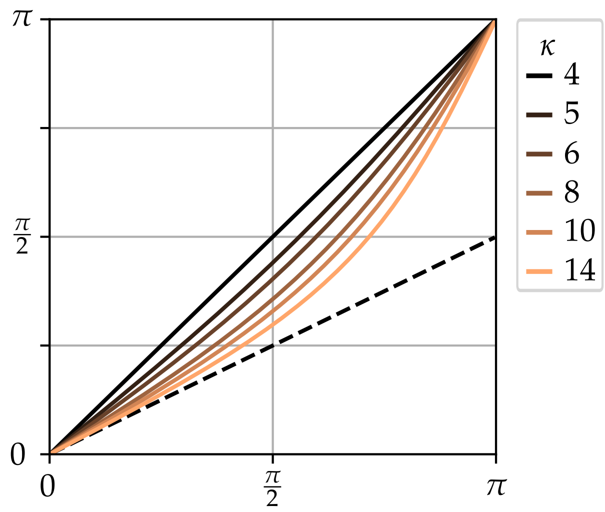

Figure 5.11.: Hemispheric solutions in $E_{0,2}$ for $\kappa \geq 4$.

Analysis of $\kappa \geq 4$ regime

When we solve the flow for $\kappa \geq 4$, we recall that we are in the situation of Definition 4.9 where we know that the flow is guaranteed to converge to a stationary solution. We set `n_grid` to 301 and `t_range` to 5.
In Figure 5.11 we show different solutions to (EL) for $\kappa \in \{4, 5, 6, 8, 10, 14\}$ only on the interval used in the simulation. As expected, the graphs of the solutions are inside the wedge $\{x \leq y \leq 2x\}$. Moreover, for increasing κ we also see again that the solutions converge pointwise to the identity map on the interval $[\frac{\pi}{2}, \pi]$. In some sense, we can view these solutions complementary to the $H_{1,1}$ saddle points: on the interval $[0, \frac{\pi}{2}]$ the $E_{0,2}$ saddle points are close to the identity map while the $H_{1,1}$ saddle points are close to the map $\theta \mapsto \pi + \theta$. Then at $\theta = \frac{\pi}{2}$ both maps do a jump in opposite directions such that on the interval $[\frac{\pi}{2}, \pi]$ they have inverted their roles.

Analysis of $\kappa < 4$ regime: Possible finite time blowup

In the case $0 < \kappa < 4$, we set `n_grid` to 301 and `t_range` to 15. We make an interesting observation: the flow converges to a stationary solution only for $\kappa \gtrsim 3.3$. For smaller values of κ, the flow seems to diverge in finite time at $\theta = 0$. In Figure 5.12, such a blowup is displayed for $\kappa = 3$. We observe that the profile first bulges very slowly to the left, followed by an abrupt blowup at $\theta = 0$. Then,

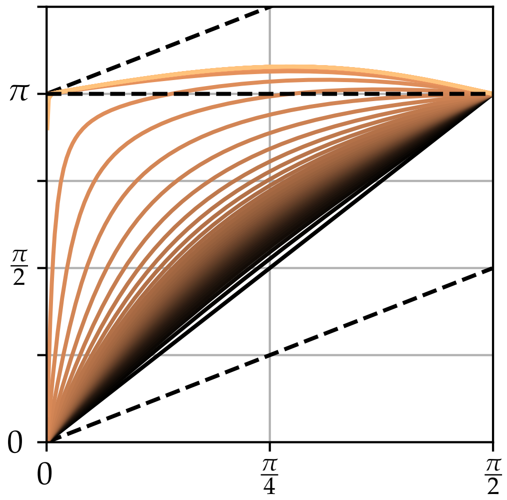

Figure 5.12.: Blowup behavior for $\kappa = 3$.

after the blowup, the solution converges into the stationary solution in the class $H_{1,1}$, which is the presumed minimizer, i.e., stable.
We recall that these blowups could be numerical artifacts due to the occasional ill behavior of the solver at the boundaries. However, also decreasing the grid size does not prevent this behavior. Moreover, a finite time blowup for this kind of initial data in case of the harmonic map heat flow has been proven in [11]. Therefore, since we are in the small κ regime where the effect of the Laplacian dominates, we conjecture that this is indeed a finite time blowup and not a numerical artifact. Consequently, this would imply that κ_1 from Theorem 2 is bounded from below approximately by 3.3. Moreover, the same blowup behavior can be observed if we choose as initial data a map in $E_{0,2}$ which is not hemispheric. This holds for any κ. For instance, if we let the flow in Figure 5.10 evolve further, we would observe a blowup at $\theta = \pi$ (not pictured). This suggests that in the class of maps $E_{0,2}$ the only solutions of (2.17) are the hemispheric ones.

6. Outlook and Open Questions

In this final chapter, we address open questions and possible further directions of research which can be studied with the theory developed in the present work. Most of the conjectures or loose threads originate from observations of the numerical simulations and have already been partly mentioned there. Herein, we outline two possible directions: on the one hand, one could further study the critical points obtained in this work. Especially the dependence on the parameter κ seems to hold a rich structure. On the other hand, a direction of a more academic nature concerns the further study of the skyrmionic map heat flow.

Regarding the first direction, the main result of this work is the existence of two distinct types of saddle points for the energy functional $\mathcal{E}$ for a specific set of κ. We did not address the question of uniqueness of these critical points in their given class, nor did we characterize them further. For instance, one could compute the Morse index of a saddle point. The spectral analysis in Section 4.4.2 indicates that the Morse index of the saddle point dealt with in this section is 1 since the second variation only had one negative eigenvalue for the map $\theta \mapsto 2\theta$ and $\kappa = 4$. However, we only deal with the second variation of the reduced energy functional, i.e., the Morse index in the class of axisymmetric maps is likely 1. Whether this carries over to the full energy is a question to be answered. To this end, one could use the approach used in [59] and expand the functions in the polar variable φ via Fourier expansion. This gives rise to a sequence of Fourier modes for which some results have already been established, such as positivity of the higher order modes of the second variation of the energy functional [59]. This could lead to an upper bound for the Morse index or even a precise value.

Furthermore, the numerical simulations strongly suggest that the critical points found in their class are unique. Nonetheless, a rigorous proof of this statement would be desirable. We also conjecture that the saddle point of Theorem 1 is the minimizer of the energy functional $\mathcal{E}$ in the class of hemispheric maps. We recall that for all values of $\kappa > 0$ the result of Definition 4.6 guarantees the existence of a critical map in the class of hemispheric maps. The result of Theorem 1 then characterizes these as saddle points for values of κ above κ_0, where $\kappa_0 \geq 4$. We have found a numerical upper bound for κ_0 of approximately 6.7 but we conjecture that actually $\kappa_0 = 4$. In addition, we hypothesize that for $\kappa < 4$ the hemispheric map is actually the minimizer of the energy functional $\mathcal{E}$ in the class of axisymmetric maps and furthermore, the global minimizer. This is backed up by the observation in the numerical simulations of a bifurcation of the hemispheric critical point into three critical points at this value of $\kappa = 4$. We assume that, from this value on, the stable upper and lower minimizing profiles separate, and the hemispheric profile becomes unstable. A rigorous study of this phenomenon could uncover interesting results.

Regarding the saddle points of second type found in Theorem 2, it is of interest what happens when $\kappa \to 0$. Evidently, a transition occurs since the critical point seems to "disappear" in this process.

The second line of possible research concerns the skyrmionic map heat flow itself. As stated in the introduction, a lot of inspiration for the methods used in this thesis was taken from the theory of the harmonic map heat flow. It is therefore natural to ask which of the existing results can be reproduced for the skyrmionic map heat flow.

One result by Chang, Ding and Ye about the HMHF is the existence of initial data for which the solution blows up in finite time [11]. The simulations for nonhemispheric profiles in the class $E_{0,2}$ give rise to the assumption that this is indeed also possible for the SMHF. A rigorous proof of the existence of finite time blowups would be desirable. In their proof, the authors also make use of a comparison principle similar to the one proved in Definition 4.1. From a physics point of view, further understanding blowups could also be connected to the question of how to create or destroy skyrmions since by a blowup the mapping degree (skyrmion number) of the map can be changed.

We examined the flow in the axisymmetric setting. The HMHF has also been studied in the equivariant setting, i.e., when the maps are of the form $\boldsymbol{m}(\theta, \varphi) = (\cos(k\varphi)\sin h(\theta), \sin(k\varphi)\sin h(\theta), \cos h(\theta))^T$ for some $k \in \mathbb{N}$. For instance, in a recent result by Jendrej and Lawrie, the authors establish *continuous in time bubbling* at blowup points for such maps [36]. It might be of interest to study these maps in our setting, too. A problem here might be that the anisotropy term will not simplify to a term that only depends on θ but will also have a φ-dependence.

As mentioned in the introduction, the models of Skyrmions in the plane were the starting point of the study of these micromagnetic structures in the literature.

There might be a potential use of applying an analogous heat flow in this case. One of the difficulties here is that we lose the compactness of the domain. One approach to overcome this problem could be to use a concentration-compactness principle. In [28] this was used to show long-time existence of the HMHF of maps from $\mathbb{R}^2$ to $\mathbb{S}^2$ for certain initial conditions. The authors used an equivariant setting whose special case of axisymmetric maps has already been studied in the Skyrmion setting [45, 27].

Another interesting approach would be to study a rescaled energy functional

$$\tilde{\mathcal{E}}(\boldsymbol{m}) = \frac{1}{\sqrt{\kappa}}\mathcal{E}(\boldsymbol{m}) .$$

As we have seen in the exposition in Section 4.3, for the saddle point in the class $H_{1,1}$, the energy scales like $\mathcal{E}(\boldsymbol{m}) \sim \sqrt{\kappa}$, where both terms, the Dirichlet energy and the anisotropy, contribute with the same order of magnitude. Also for the saddle points in the class $E_{0,2}$ we observe numerically that the energy scales like $\sqrt{\kappa}$. By rescaling the energy functional, we obtain something that is reminiscent of the study of the Cahn–Hilliard model by Modica and Mortola: a gradient term with a vanishing prefactor in the limit $\kappa \to \infty$ and a potential term with two separated zeros $\pm\boldsymbol{\nu}$. By transferring their results to our setting, we would expect that in the limit $\kappa \to \infty$ the functional $\tilde{\mathcal{E}}$ will Γ-converge into something that measures the perimeter of the sets $\{\boldsymbol{m} = \boldsymbol{\nu}\}$ and $\{\boldsymbol{m} = -\boldsymbol{\nu}\}$.

Finally, in our model of the thin ferromagnet we assumed that the anisotropy is of *easy-normal* type, i.e., we considered $\kappa > 0$. In the literature, the case of *easy-tangential* anisotropy has also been studied [15, 60], i.e., when $\kappa < 0$. This implies that the normal field $\boldsymbol{\nu}$ is an unstable solution and it is unclear how the stable configurations look like in this case. There is no reason why this model would not be suited to be studied with the methods developed in this work for $\kappa < 0$, especially with the skyrmionic map heat flow in the axisymmetric setting. On the level of profiles this means that the solution lines $\theta \mapsto \theta + \pi\mathbb{Z}$ are not attracting but repelling for the skyrmionic map heat flow. Instead, the lines $\theta \mapsto \theta + \frac{\pi}{2}\mathbb{Z}$ are of attracting type. It would be insightful to determine the stable solutions in this case and whether one could use these observations to rigorously prove the findings.

A. Appendix

A.1. Trigonometric identities

In this section we collect some more advanced trigonometric identities which are used in several formulas. The proofs basically always follow from the following well-known addition and product formulas for the sine and cosine functions:

$$\begin{aligned}
\sin(x+y) &= \sin x \cos y + \cos x \sin y\,, \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y\,, \\
\sin x \sin y &= \frac{1}{2}(\cos(x-y) - \cos(x+y))\,, \\
\cos x \cos y &= \frac{1}{2}(\cos(x+y) + \cos(x-y))\,, \\
\sin x \cos y &= \frac{1}{2}(\sin(x+y) + \sin(x-y))\,, \\
\sin x + \sin y &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\
\cos x - \cos y &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).
\end{aligned}$$

Lemma A.1. *For all $x, y \in \mathbb{R}$ it holds*

$$\begin{aligned}
&\sin(2y-2x)\sin^2 x \\
&= \frac{1}{2}\sin(y-x)(2\cos(y-x) - \cos(y-3x) - \cos(y+x)) && \text{(A.T1)} \\
&= \frac{1}{2}\sin(y-x)(\cos(y-x) - 2\sin x \sin(y-2x) - \cos(y+x)) && \text{(A.T2)} \\
&= \frac{1}{4}(-\sin 2y - \sin(2y-4x) + 2\sin(2y-2x))\,, && \text{(A.T3)}
\end{aligned}$$

and

$$\sin(2y-2x)\sin x \cos x = \frac{1}{2}\sin(y-x)(\sin(y-3x) - \sin(y+x))\,, \qquad \text{(A.T4)}$$

and

$$\frac{1}{2}(\sin 2y - \sin 2x)\cos x + (\cos 2y - \cos 2x)\sin x$$
$$= \frac{1}{2}\sin(y-x)(3\cos(y+2x) - \cos y)\,. \qquad \text{(A.T5)}$$

Proof. Ad A.T1: We have

$$\begin{aligned}\sin(2y-2x)\sin^2 x &= 2\sin(y-x)\cos(y-x)\sin^2 x\\ &= \sin(y-x)\cos(y-x)(1-\cos 2x)\\ &= \sin(y-x)(\cos(y-x) - \cos(y-x)\cos(2x))\\ &= \frac{1}{2}\sin(y-x)(2\cos(y-x) - \cos(y-3x) - \cos(y+x))\,.\end{aligned}$$

Ad A.T2: Applying the addition formula

$$\begin{aligned}&\cos(y-x) - \cos(y-3x)\\ &= -2\sin\left(\frac{(y-x)+(y-3x)}{2}\right)\sin\left(\frac{(y-x)-(y-3x)}{2}\right)\\ &= -2\sin(y-2x)\sin x\end{aligned}$$

to (A.T1) we arrive at (A.T2).
Ad A.T3: We use the triple product formula

$$\begin{aligned}&\sin a\sin b\sin c\\ &= \frac{1}{4}(-\sin(a+b+c) + \sin(-a+b+c) + \sin(a-b+c) + \sin(a+b-c))\end{aligned}$$

where we now set $a = 2y - 2x$, $b = x$ and $c = x$. We find

$$\sin(2y-2x)\sin^2 x = \frac{1}{4}(-\sin(2y) - \sin(2y-4x) + 2\sin(2y-2x))\,.$$

Ad A.T4: We compute

$$\begin{aligned}\sin(2y-2x)\sin x\cos x &= \sin(y-x)\cos(y-x)\sin 2x\\ &= \frac{1}{2}\sin(y-x)(\sin(y-3x) - \sin(y+x))\,,\end{aligned}$$

where we used $y - 3x = (y-x) - 2x$ and $y + x = (y-x) + 2x$.
Ad A.T5: We compute

$$\frac{1}{2}(\sin 2y - \sin 2x)\cos x + (\cos 2y - \cos 2x)\sin x$$

$$
\begin{aligned}
&= \sin(y-x)(\cos(y+x)\cos x - 2\sin(y+x)\sin x) \\
&= \sin(y-x)\bigg(\frac{1}{2}(\cos(y+x+x)+\cos(y+x-x)) \\
&\qquad\qquad - (\cos(y+x+x)-\cos(y+x-x))\bigg) \\
&= \frac{1}{2}\sin(y-x)(3\cos(y+2x)-\cos y)\,.
\end{aligned}
$$

□

A.2. Some results from analysis

Lemma A.2. *Let $K \subset \mathbb{R}$ be compact. Moreover, let $f \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ and $g \in C^{k,\alpha}(K, \mathbb{R})$ for some $k \in \mathbb{N}$ and $0 < \alpha < 1$. Then $f \circ g \in C^{k,\alpha}(K, \mathbb{R}^n)$.*

Proof. By just considering individual components we can assume without loss of generality that $f\colon \mathbb{R} \to \mathbb{R}$. First, we note that by consecutive application of the chain and product rule $f \circ g$ is indeed k times differentiable. We need to show Hölder continuity. An explicit formula for $(f \circ g)^{(l)}$ for $1 \le l \le k$ is given by Faà di Bruno's formula which we do not consult here but we rather state a handier, but sufficient version. For all $x \in K$ we have

$$
(f \circ g)^{(l)}(x) = f'(g(x))g^{(l)}(x) + F_l(x)\,,
$$

where $F_l \in C^{k-l+1}(K, \mathbb{R})$. We prove this formula by induction. For $l = 1$ we indeed have by the chain rule

$$
(f \circ g)^{(1)}(x) = f'(g(x))g'(x)\,,
$$

such that $F_1 \equiv 0 \in C^{k+1}(K, \mathbb{R})$. Now, let the statement be true for $1 \le l < k$ fixed. Then

$$
\begin{aligned}
(f \circ g)^{(l+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big((f \circ g)^{(l)}(x)\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\Big(f'(g(x))g^{(l)}(x) + F_l(x)\Big) \\
&= f'(g(x))g^{(l+1)}(x) + f''(g(x))g'(x)g^{(l)}(x) + F_l'(x)\,.
\end{aligned}
$$

Hence, defining $F_{l+1}(x) = f''(g(x))g'(x)g^{(l)}(x) + F_l'(x)$, we see that $F_{l+1} \in C^{k-l}(K, \mathbb{R})$, since f is smooth, $g'g^{(l)} \in C^{k-l}(K, \mathbb{R})$ by the product rule, and

$F_l' \in C^{k-l}(K,\mathbb{R})$ by the induction hypothesis. Thus, the formula holds. For $l = k$ we consequently get

$$(f \circ g)^{(k)}(x) = f'(g(x))g^{(k)}(x) + F(x)$$

with $F \in C^1(K,\mathbb{R})$. Using this representation we can now compute for $x, y \in K$

$$\begin{aligned}
&|(f \circ g)^{(k)}(x) - (f \circ g)^{(k)}(y)| \\
&= |f'(g(x))g^{(k)}(x) + F(x) - f'(g(y))g^{(k)}(y) - F(y)| \\
&\leq \max_{z\in K} |f'(g(z))||g^{(k)}(x) - g^{(k)}(y)| + \max_{z\in K} |g^{(k)}(z)||f'(g(x)) - f'(g(y))| \\
&\quad + |F(x) - F(y)| \\
&\leq C|x-y|^\alpha,
\end{aligned}$$

where in the last step we have used the Hölder continuity of $g^{(k)}$ and the fact that $f' \circ g$ and F are continuously differentiable, hence Hölder continuous on the compact set K, where they also attain their maximum, accordingly. As a result, $f \circ g \in C^{k,\alpha}(K,\mathbb{R})$. □

In the same manner we can show the opposite composition:

Lemma A.3. *Let $K \subset \mathbb{R}$ be compact. Moreover, let $f \in C^\infty(K,\mathbb{R})$ and $g \in C^{k,\alpha}(\mathbb{R},\mathbb{R})$ for some $k \in \mathbb{N}$ and $0 < \alpha < 1$. Then $g \circ f \in C^{k,\alpha}(K,\mathbb{R}^n)$.*

Lemma A.4. *Let for $U \subset \mathbb{R}$ open and convex be $f \in C^{k,\alpha}(U,\mathbb{R})$ for $k \in \mathbb{N}$ and $0 < \alpha < 1$. If we write for $x_0 \in U$ by Taylor's theorem*

$$f(x) = \sum_{n=1}^{k} \frac{1}{n!} f^{(n)}(y)(x-x_0)^n + (x-x_0)^k g(x)\,, \tag{A.1}$$

with $g(x_0) = 0$, then $|g(x)| \leq C|x-x_0|^\alpha$.

Proof. We write the Taylor formula of order $k-1$ using the Lagrange form of the remainder

$$f(x) = \sum_{n=1}^{k-1} \frac{1}{n!} f^{(n)}(y)(x-x_0)^n + \frac{1}{k!} f^{(k)}(\xi)(x-x_0)^k$$

for some ξ between x and x_0. Subtracting this from (A.1) we find

$$(x-x_0)^k g(x) = \frac{1}{k!}(x-x_0)^k \Big(f^{(k)}(\xi) - f^{(k)}(x_0)\Big)\,.$$

For $x \neq x_0$ then

$$|g(x)| \leq C\left|f^{(k)}(\xi) - f^{(k)}(x_0)\right| \leq C|x - x_0|^{\alpha}.$$

follows by the Hölder continuity of $f^{(k)}$ and the fact that $|\xi - x_0| \leq |x - x_0|$. □

Recall the definition of the parabolic cylinder for $U \subset \mathbb{R}^n$ and $T > 0$ to be $U_T = U \times (0, T]$. We denote with $C^2_1(U_T)$ the space of functions $U_T \to \mathbb{R}$ that are twice continuously differentiable in the space variable and once continuously differentiable in the time variable.

Lemma A.5 (Variant of a maximum principle)**.** *Let $I := (a, b) \subset \mathbb{R}$ be an open bounded interval and assume $u \in C^2(I_T) \cup C(\overline{I}_T)$ satisfies*

$$\begin{aligned} &0 \leq u_t - u'' + d(u, x)u' + c(u, x)u \\ &u|_{t=0} = u_0 \\ &u(a, t) = g(t),\ u(b, t) = h(t), \end{aligned}$$

where d and c are sufficiently smooth functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g, h\colon [0, T] \to [0, \infty)$ are continuous functions satisfying the compatibility conditions $g(0) = u_0(a)$ and $h(0) = u_0(b)$. Additionally, assume there exists $\delta > 0$ such that $c(u(x, t), x) > -\delta$ holds for all $(x, t) \in I_T$. Then it holds that $u_0 \geq 0$ on $\overline{I}$ implies $u \geq 0$ on I_T.
Similarly, if u instead satisfies

$$0 \geq u_t - u'' + d(u, x)u' + c(u, x)u,$$

with the same initial and boundary conditions, just that g and h are nonpositive, then it holds that $u_0 \leq 0$ implies $u \leq 0$ on I_T.

Proof. We define

$$v(x, t) = u(x, t)e^{-\delta t}.$$

Then v satisfies

$$\begin{aligned} &0 \leq v_t - v'' + d(u, x)v' + (c(u, x) + \delta)v \\ &v|_{t=0} = u_0 \\ &v(a, t) \geq 0,\ v(b, t) \geq 0 \end{aligned}$$

and the statement follows from the usual maximum principle for parabolic equations. For sake of completeness in this simple one-dimensional case we give the proof. Assume that $\min_{\overline{I}_T} v < 0$ which is attained at $(x_0, t_0) \in \overline{I}_T$, i.e.,

$v(x_0, t_0) < 0$. As on $\bar{I}_T \setminus I_T$ we have $v \geq 0$, we conclude $(x_0, v_0) \in I_T$. In light of $v \in C^2(I_T) \cup C(\bar{I}_T)$, this therefore yields

$$v(x_0, t_0) < 0\,, \quad v'(x_0, t_0) = 0\,, \quad v''(x_0, t_0) \geq 0\,, \text{ and } v_t(x_0, t_0) \leq 0\,.$$

Consequently, inserting this point into the equation we obtain

$$0 \leq \underbrace{v_t(x_0, t)}_{\leq 0} - \underbrace{v''(x_0, t)}_{\geq 0} + d(u, x_0)\underbrace{v'(x_0, t)}_{=0} + \underbrace{(c(u, x_0) + \delta)}_{>0}\underbrace{v(x_0, t)}_{<0} < 0\,,$$

a contradiction. Therefore, $v \geq 0$ on I_T, which then implies the statement since $e^{-\delta t} > 0$ for all t. The second case is proven by using the first case with $-u$ and $-u_0$. □

A.3. Riemannian geometry

In this section we give a brief summary of the theory of derivatives on Riemannian manifolds. We will merely collect the main definitions and basic results. To this end, we mainly follow [43, Chapters 4 and 5].
We start with the definition of a vector bundle on a smooth manifold [43, pp. 382]. Let M be a smooth manifold of dimension n. We call (E, π, M) a smooth vector bundle over M of rank m, if E is a smooth manifold with a surjective smooth map $\pi\colon E \to M$ such that the following conditions are satisfied:

1. For each $p \in M$ the fiber $E_p = \pi^{-1}(p)$ is a vector space of dimension m.
2. For each $p \in M$ there exists an open neighborhood $U \subset M$ of p and a diffeomorphism $\phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^m$ such that
   - $\pi = \pi_1 \circ \phi$ where $\pi_1\colon U \times \mathbb{R}^m \to U$ is the projection onto the first factor.
   - For each $q \in U$ the restriction $\phi|_{E_q}\colon E_q \to \{q\} \times \mathbb{R}^m$ is a linear isomorphism.

The map ϕ is called a local trivialization of E over U. The vector space E_p is called the fiber of E over p. A section of E is a map $s\colon M \to E$ such that $\pi \circ s = \mathrm{Id}_M$. We denote by $\Gamma(E)$ the space of smooth sections of E, i.e., when s is a smooth map. When we do not require any regularity apart from measurability, we call s a rough section and denote the space of rough sections by $\Gamma_{\mathrm{r}}(E)$.

The classic example of a vector bundle is the tangent bundle TM of a smooth manifold M. Then the set of smooth sections $\Gamma(TM)$ is the set of smooth vector fields on M, also denoted by $\mathfrak{X}(M)$.
We call a map $g\colon E \times_M E \to \mathbb{R}$ a bundle metric on E if for each $p \in M$ the restriction $g_p\colon E_p \times E_p \to \mathbb{R}$ is an inner product on E_p. If a bundle metric exists, we define the induced norm $|\cdot|$ on E such that on E_p we have $|v|_p = \sqrt{g_p(v,v)}$.
For the rest of the section let (M, g) be a compact smooth Riemannian manifold of dimension n without boundary, i.e., M is a smooth manifold and g is a Riemannian metric. By $\mathrm{d}v(g)$ we denote the volume form on M given in local coordinates by

$$\mathrm{d}v(g) = \sqrt{|\det g|}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \,.$$

Recall that elements in the tangent space at a point $p \in M$, denoted by T_pM, can be represented as derivations of C^∞ functions at p. In local coordinates $(x_1, \ldots, x_n)$ we get a basis for T_pM via $\partial_1, \ldots, \partial_n$, where with ∂_μ we denote the partial derivative in direction x_μ evaluated at p. Consequently, we can write for $v \in T_pM$

$$v = v^\mu \partial_\mu \,.$$

Note that we apply Einstein summation convention. Likewise, we can express elements in the cotangent space $\omega \in T_p^*M$ as

$$\omega = \omega_\mu \mathrm{d}x^\mu \,,$$

where $\mathrm{d}x^\mu$ is the dual basis defined via $\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$, the Kronecker delta.
Recall that the Riemannian metric is a map $TM \times_M TM \to \mathbb{R}$ such that it forms an inner product on T_pM for every $p \in M$. Therefore, g is a bundle metric on the tangent bundle TM. For $v, w \in T_pM$ with $p \in M$ we write

$$g_p(v, m) = \langle v, w \rangle_g \,.$$

In local coordinates $(x_1, \ldots, x_n)$ this can be expressed as

$$\langle v, w \rangle_g = g_{\mu\nu} v^\mu w^\nu \,.$$

The entries $g_{\mu\nu}$ are the components of the Gram matrix, $g_{\mu\nu} = g_p(\partial_\mu, \partial_\nu)$.
The metric induces a bundle isomorphism between the tangent bundle TM and the cotangent bundle T^*M via $\imath_g\colon TM \to T^*M$, $\imath_g(v) = g(v, \cdot)$. Expressed in local

coordinates this gives us

$$\imath_g(v) = g_{\mu\nu} v^\mu \mathrm{d}x^\nu \,.$$

Then g also induces an inner product on T^*M via

$$\hat{g}(\omega, \sigma) = \langle \omega, \sigma \rangle_g = \left\langle \imath_g^{-1}\omega, \imath_g^{-1}\sigma \right\rangle_g ,$$

which in coordinates reads

$$\langle \omega, \sigma \rangle_g = g^{\mu\nu} \omega_\mu \sigma_\nu \,,$$

where $g^{\mu\nu}$ are the entries of the inverse matrix of $g_{\mu\nu}$, such that $g^{\mu\nu} g_{\nu\gamma} = \delta^\mu_\gamma$.
Let $\phi\colon M \to N$ be a diffeomorphism between two smooth manifolds. Then the differential of ϕ is a bundle map $\mathrm{d}\phi\colon TM \to TN$ defined for $p \in M$ and $v \in T_pM$ as

$$\mathrm{d}\phi\, v(f) = v(f \circ \phi) \,,$$

for all $f \in C^\infty(N)$. This implies $\mathrm{d}\phi\, v \in T_{\phi(p)}N$. For any multilinear map $\alpha\colon TN \times_N \ldots \times_N TN \to \mathbb{R}$ we define the pullback of α via ϕ as

$$\begin{aligned} \phi^*\alpha\colon TM \times_M \ldots \times_M TM \to \mathbb{R} \\ (\phi^*\alpha)_p(v_1, \ldots, v_k) = \alpha_{\phi(p)}(\mathrm{d}\phi\, v_1, \ldots, \mathrm{d}\phi\, v_k) \,. \end{aligned}$$

The pullback of a map $N \to \mathbb{R}$ is defined as the composition with ϕ.
Likewise, we can define a pushforward of maps on the cotangent bundle T^*M via ϕ. For a multilinear map $\beta\colon T^*M \times_M \ldots \times_M T^*M \to \mathbb{R}$ we define the pushforward of β via ϕ as

$$\begin{aligned} \phi_*\beta\colon T^*N \times_N \ldots \times_N T^*N \to \mathbb{R} \\ (\phi_*\beta)_p(\omega_1, \ldots, \omega_k) = \beta_{\phi^{-1}(p)}(\phi^*\omega_1, \ldots, \phi^*\omega_k) \,. \end{aligned}$$

An isometry between two Riemannian manifolds (M, g) and (N, h) is a diffeomorphism $\phi\colon M \to N$ such that we have $\phi^*h = g$.

Lemma A.6. *Let $\phi\colon M \to N$ and $\psi\colon N \to P$ be isometries between Riemannian manifolds (M, g), (N, h), and (P, k). Then $\psi \circ \phi$ is an isometry.*

Proof. This follows from the fact that by virtue of the chain rule we have $(\psi \circ \phi)^*k = \phi^*(\psi^*k) = \phi^*h = g$. □

Lemma A.7. *Let $\phi\colon M \to N$ be an isometry between two Riemannian manifolds (M, g) and (N, h). Then $\phi_*\hat{g} = \hat{h}$.*

Proof. We fix $p \in M$. Then for $\omega, \sigma \in T^*_{\phi(p)}N$ we get by the definitions of the pushforward and the inner product on the cotangent bundles

$$\begin{aligned}(\phi_*\hat{g})_{\phi(p)}(\omega,\sigma) &= \hat{g}_p(\phi^*\omega, \phi^*\sigma)\\ &= g_p(\imath_g^{-1}\circ\phi^*\omega, \imath_g^{-1}\circ\phi^*\sigma)\\ &= (\phi^*h)_{\phi^{-1}(p)}(\imath_g^{-1}\circ\phi^*\omega, \imath_g^{-1}\circ\phi^*\sigma)\\ &= h_{\phi(p)}(\mathrm{d}\phi\circ\imath_g^{-1}\circ\phi^*\omega, \mathrm{d}\phi\circ\imath_g^{-1}\circ\phi^*\sigma)\\ &= \hat{h}_{\phi(p)}(\imath_h\circ\mathrm{d}\phi\circ\imath_g^{-1}\circ\phi^*\omega, \imath_h\circ\mathrm{d}\phi\circ\imath_g^{-1}\circ\phi^*\sigma)\,.\end{aligned}$$

Consequently, the statement follows if we show that the following diagram commutes

$$\begin{array}{ccc} TM & \xrightarrow{\mathrm{d}\phi} & TN \\ \imath_g\big\downarrow & & \big\downarrow\imath_h \\ T^*M & \xleftarrow[\phi^*]{} & T^*N \end{array}\,.$$

To this end, we compute for $v \in T_pM$ the map $\phi^*\circ\imath_h\circ\mathrm{d}\phi(v) \in T_p^*M$. Let $w \in T_pM$ be arbitrary. Then

$$\begin{aligned}(\phi^*\circ\imath_h\circ\mathrm{d}\phi)(v)[w] &= \phi^*(\underbrace{\imath_h(\mathrm{d}\phi\,v)}_{=h_{\phi(p)}(\mathrm{d}\phi\,v,\cdot)})[w] = h_{\phi(p)}(\mathrm{d}\phi\,v, \mathrm{d}\phi\,w)\\ &= (\phi^*h)_p(v,w) = g_p(v,w)\\ &= \imath_g(v)[w]\,.\end{aligned}$$

Hence $\phi^*\circ\imath_h\circ\mathrm{d}\phi = \imath_g$, which concludes the proof. □

Lemma A.8. *Let $\phi\colon M\to N$ be an isometry between two oriented Riemannian manifolds (M,g) and (N,h) and let $f\colon N\to\mathbb{R}$ be smooth. Then*

$$\int_M f\circ\phi \mathrm{d}v(g) = \int_N f\mathrm{d}v(h)\,.$$

Proof. By [43, Ex. 2.42], we find that $\phi^*\mathrm{d}v(h) = \pm\mathrm{d}v(g)$, where the sign is determined by whether ϕ is orientation preserving or reversing, respectively. Then by the rule $(f\circ\phi)\,\phi^*\mathrm{d}v(h) = \phi^*(f\mathrm{d}v(h))$ we get the statement by the diffeomorphism invariance of the integral [44, Prop. 16.6] which states

$$\int_M f\circ\phi\,\mathrm{d}v(g) = \pm\int_M \phi^*(f\,\mathrm{d}v(h)) = \int_N f\,\mathrm{d}v(h)\,,$$

where the sign cancels out in the last step since we gain the same sign from the invariance of the integral formula due to the orientation preservation/reversal of ϕ. □

We define the (r,s)-tensor bundle on M as $T^{(r,s)}TM$ for $(r,s) \in \mathbb{N}_0^2$ as

$$T^{(r,s)}TM = \underbrace{TM \otimes \cdots \otimes TM}_{r \text{ copies}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s \text{ copies}},$$

where by $\otimes$ we denote the tensor product of vector bundles defined as the standard tensor product on the fibers.
Given a tensor $F \in T^{(r,s)}TM$, we can express it in a given local frame $(\partial_1, \ldots, \partial_n)$ of TM and corresponding dual basis $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$ as

$$F = F^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_s}.$$

We call the components and indices corresponding to TM factors in the tensor product contravariant while those corresponding to T^*M factors we call covariant. By applying the isomorphism $\imath_g$ on the i-th contravariant component of the tensor product, which we call lowering of indices, we can construct a new tensor $\check{F} \in T^{(r-1,s+1)}TM$. Vice versa, applying the inverse $\imath_g^{-1}$ on the j-th covariant component, we gain a tensor $\hat{F} \in T^{(r+1,s-1)}TM$. This corresponds to raising an index. Expressed in coordinates we can write the respective components as

$$\check{F}^{\mu_1 \ldots \mu_{i-1} \mu_{i+1} \ldots \mu_r}_{\nu_1 \ldots \nu_r \nu_{s+1}} = g_{\nu_{s+1} \mu_i} F^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} \text{ and } \hat{F}^{\mu_1 \ldots \mu_r \mu_{r+1}}_{\nu_1 \ldots \nu_{j-1} \nu_{j+1} \nu_r} = g^{\mu_{r+1} \nu_j} F^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}.$$

It is common to use the same symbol of the tensor after having raised or lowered an index.
We can define an inner product on the tensor bundles in the natural way by using the metric g of the tangent and cotangent bundle in each component of the product. For tensors $F, G \in T^{(r,s)}TM$ with

$$F = \underbrace{v_1 \otimes \cdots \otimes v_r}_{\in TM} \otimes \omega_1 \otimes \cdots \otimes \omega_s$$
$$G = \underbrace{w_1 \otimes \cdots \otimes w_r}_{\in TM} \otimes \sigma_1 \otimes \cdots \otimes \sigma_s$$

with $v_i, w_i \in TM$ and $\omega_i, \sigma_i \in T^*M$ we define thus

$$\langle F, G \rangle_g = \langle v_1, w_1 \rangle_g \cdots \langle v_r, w_r \rangle_g \cdot \langle \omega_1, \sigma_1 \rangle_g \cdots \langle \omega_s, \sigma_s \rangle_g . \tag{A.2}$$

In local coordinates this means

$$\langle F, G \rangle_g = g^{\nu_1\delta_1} \dots g^{\nu_s\delta_s} g_{\mu_1\gamma_1} \dots g_{\mu_r\gamma_r} F^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s} G^{\gamma_1\dots\gamma_r}_{\delta_1\dots\delta_s} .$$

This indeed induces a norm on the tensors via $|F|_g = \sqrt{\langle F, F \rangle_g}$. In the following we will omit the subscript g in the scalar product and norm if the metric is clear from the context. Since $T^{(r,s)}TM$ is a vector bundle over M, we have defined a bundle metric by $\langle \cdot, \cdot \rangle_g$.

Finally, by $\mathrm{tr}^i_j \colon T^{(r,s)}TM \to T^{(r-1,s-1)}TM$ we denote the trace, or also called contraction, of the i-th covariant component with the j-th contravariant component of the tensor as

$$\mathrm{tr}^i_j F = \delta^{\nu_i}_{\mu_j} F^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s} \partial_{\mu_1} \otimes \dots \otimes \widehat{\partial_{\mu_j}} \otimes \dots \otimes \partial_{\mu_r} \otimes \mathrm{d}x^{\nu_1} \otimes \widehat{\mathrm{d}x^{\nu_i}} \otimes \dots \otimes \mathrm{d}x^{\nu_s} ,$$

where $\widehat{\partial_{\mu_j}}$ and $\widehat{\mathrm{d}x^{\nu_i}}$ indicate that the respective component is omitted. This means, we sum over all entries with $\mu_i = \nu_j$. Likewise, we define the trace over two components of the same type (covariant or contravariant) by first raising or respectively lowering one of the two indices first and then applying the trace. Technically, for the trace of two covariant components this translates in coordinates to

$$\mathrm{tr}^{ij} F = g^{\nu_i\nu_j} F^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_r} \otimes \mathrm{d}x^{\nu_1} \otimes \dots \otimes \widehat{\mathrm{d}x^{\nu_i}} \otimes \dots \otimes \widehat{\mathrm{d}x^{\nu_j}} \otimes \dots \otimes \mathrm{d}x^{\nu_s} ,$$

so we just contract using the inverse metric. We get the likewise representation for tr_{ij}, i.e., when contracting two contravariant components, by using g itself. Whenever in coordinate notation of a tensor two times the same index appears with one raised and the other lowered that implies that the trace of these components is taken. This of course is also implied by the Einstein summation convention. For instance,

$$F^{\mu}_{\mu\nu} \mathrm{d}x^{\nu} := \mathrm{tr}^1_1 (F^{\mu}_{\delta\nu} \partial_\mu \otimes \mathrm{d}x^{\delta} \otimes \mathrm{d}x^{\nu}) .$$

Recall that by $\mathfrak{X}(M)$ we denote the vector fields on M, i.e., the smooth sections of the tangent bundle TM, $\mathfrak{X}(M) = \Gamma(TM)$. Analogously, we define (r,s)-tensor fields as $\mathfrak{T}^{(r,s)}(M) = \Gamma(T^{(r,s)}TM)$. Then

$$\begin{aligned} \mathfrak{T}^{(0,0)}(M) &= C^\infty(M) \\ \mathfrak{T}^{(1,0)}(M) &= \mathfrak{X}(M) \\ \mathfrak{T}^{(0,1)}(M) &= \Omega_1(M) , \end{aligned}$$

where by $\Omega_1(M)$ we denote the space of 1-forms on M.
Given a set of local coordinates $(x_1, \dots, x_n)$ on an open set $U \subset M$ we can express $F \in \mathfrak{T}^{(r,s)}(M)$ as

$$F = F^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s},$$

where each coefficient $F^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}$ is a smooth real-valued function on U.
We adapt a very permissive notation from [66], which is the $*$-notation (not to be confused with the Hodge star). As we can already see, a precise notation of tensors is very extensive and the $*$-notation can help to alleviate that, especially when we are only interested in finding bounds or leading order terms. Citing from [66]: "We denote by $A * B$ any tensor field which is a (real) linear combination of tensor fields, each formed by starting with the tensor field $A \otimes B$, using the metric to switch the type of any number of T^*M components to TM components, or vice versa (that is, raising or lowering some indices) taking any number of contractions, and switching any number of components in the product. Here, the algorithm for arriving at a certain expression $A * B$ must be independent of the particular choice of tensors A and B of their respective types, and hence we are free to estimate $|A * B| \leq C|A||B|$ for some constant C which will depend neither on A nor B." It should be added that the constant C does depend on the specifics of the linear combination, though, and is not global. As an example, let $A \in T^{(2,3)}TM$ and $B \in T^{(2,1)}TM$ then any of the following expressions is included in $A * B$

$$\begin{aligned} A^{\mu\nu}_{\gamma\delta\lambda} B^{\gamma\sigma}_{\mu}\, \partial_\nu \otimes \partial_\sigma \otimes \mathrm{d}x^\delta \otimes \mathrm{d}x^\lambda &= A * B \\ 3A^{\mu\nu}_{\gamma\delta\lambda} B^{\gamma\lambda}_{\mu}\, \partial_\nu \otimes \mathrm{d}x^\delta + g_{\mu\nu} A^{\mu\sigma}_{\gamma\delta\lambda} B^{\nu\lambda}_{\mu}\, \partial_\nu \otimes \mathrm{d}x^\delta \otimes \mathrm{d}x^\gamma \otimes \mathrm{d}x^\lambda &= A * B \\ \langle A, B \rangle &= A * B \end{aligned}$$

Building on this, we can also now extend the notion of $*$-products of arbitrary length $A_1 * A_2 * \dots * A_n$ and powers $A^{*n} := A * \dots * A$, n times. By convention $A^{*0} = 1$, the real number. Then we define $*(A_1, A_2, \dots, A_n)$ by which we mean any $*$-product with the given tensors in any $*$-power, including 0. For instance

$$A_1^{*3} * A_2 * A_4 + 62 A_2^{*3} * A_3^{*2} = *(A_1, A_2, A_3, A_4, A_5)\,.$$

If we have indexed tensors A_i for $1 \leq i \leq n$, then we write

$$\mathop{\ast}_{1\leq i\leq n} (A_i) := \ast(A_1, \ldots, A_n)\,.$$

Now we want to review the concept of covariant derivation and definition of the Levi-Civita connection on M. By [43] we have the following definition.

Definition A.9. Let $\pi\colon E \to M$ be a smooth vector bundle over a smooth manifold M with or without boundary, and let $\Gamma(E)$ denote the space of smooth sections of E. A connection in E is a map

$$\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

written $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

1. $\nabla_X Y$ is linear over $C^\infty(M)$ in X : for $f_1, f_2 \in C^\infty(M)$ and $X_1, X_2 \in \mathfrak{X}(M)$,

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y\,.$$

2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y : for $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \Gamma(E)$,

$$\nabla_X(a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2.$$

3. ∇ satisfies the following product rule: for $f \in C^\infty(M)$,

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y. \qquad \boxdot$$

A connection $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ on the tangent bundle TM is called covariant derivative. It can be uniquely extended to a connection in each tangent bundle $T^{(r,s)}TM$ which we will also denote by ∇ such that the following holds [43, Prop. 4.15]

1. In $T^{(1,0)}TM = TM, \nabla$ agrees with the given connection.

2. In $T^{(0,0)}TM = M \times \mathbb{R}, \nabla$ is given by ordinary differentiation of functions:

$$\nabla_X f = Xf$$

3. ∇ obeys the following product rule with respect to tensor products:

$$\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G).$$

4. ∇ commutes with all contractions: if "tr" denotes a trace on any pair of indices, one covariant and one contravariant, then

$$\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F).$$

This connection also satisfies the following additional property:

5a. ∇ obeys the following product rule with respect to the natural pairing between a covector field ω and a vector field Y :

$$\nabla_X \langle \omega, Y \rangle = \langle \nabla_X \omega, Y \rangle + \langle \omega, \nabla_X Y \rangle.$$

By the Fundamental Theorem of Riemannian Geometry [43, Thm. 5.10] there exists a unique connection ∇ on the tangent space of (M, g) that satisfies the additional properties:

5b. ∇ is compatible with g, i.e., for $X, Y, Z \in \mathfrak{X}(M)$ follows

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \,.$$

6. ∇ is symmetric, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y] \,,$$

where $[X, Y]$ denotes the Lie bracket.

We call this connection the Levi-Civita connection on M. For a given local frame $(\partial_1, \ldots, \partial_n)$ of TM and dual basis $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$ we define the so-called Christoffel symbols Γ as

$$\nabla_{\partial_\mu} \partial_\nu = \Gamma^\delta_{\mu\nu} \partial_\delta \,.$$

Note that even though we write it like the component of a (1,2)-tensor, the Christoffel symbol is no tensor as it is only defined locally and does not transform as tensors do under change of coordinates.
We can interpret the Levi-Civita connection as a linear operator $\nabla \colon \mathfrak{T}^{(r,s)}(M) \to \mathfrak{T}^{(r,s+1)}(M)$ as it adds an additional slot for an element in the tangent space, the direction in which we derive. For 1-forms $\omega_1, \ldots, \omega_r$ and vector fields $Y_1, \ldots, Y_s$ we thus write

$$\begin{aligned}\nabla F(\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s, X) &:= (\nabla_X F)(\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s) \\ &= X(F(\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s)) - \sum_{k=1}^{r} F(\omega_1, \ldots, \nabla_X \omega_k, \ldots, \omega_r, Y_1, \ldots, Y_s)\end{aligned}$$

$$- \sum_{l=1}^{s} F(\omega_1, \dots, \omega_r, Y_1, \dots, \nabla_X Y_l, \dots, Y_s) \,.$$

How to compute $\nabla_X \omega$ for a 1-form ω follows from 5a. In local coordinates this reads

$$\nabla F = \left(\partial_\delta F^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} + \sum_{k=1}^{r} \Gamma^{\mu_k}_{\delta\sigma} F^{\mu_1 \dots \sigma \dots \mu_r}_{\nu_1 \dots \nu_s} - \sum_{l=1}^{s} \Gamma^{\sigma}_{\delta \nu_l} F^{\mu_1 \dots \mu_r}_{\nu_1 \dots \sigma \dots \nu_s} \right)$$
$$\partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s} \otimes \mathrm{d}x^\delta \,. \tag{A.3}$$

We call ∇F the total covariant derivative of F. If $f \in C^\infty(M, \mathbb{R})$ then we derive by the above formula that

$$\nabla (fF) = F \otimes \nabla f + f \nabla F \,. \tag{A.4}$$

Moreover, by the product rule with respect to tensor products we assert that with the $*$-notation it holds

$$\nabla (A * B) = \nabla A * B + A * \nabla B \tag{A.5}$$

for tensors A, B.
Since for $F \in \mathfrak{T}^{(r,s)}(M)$ the total covariant derivative is again a tensor $\nabla F \in \mathfrak{T}^{(r,s+1)}(M)$ we can subsequently apply the total covariant derivative to F

$$\nabla^k F = \underbrace{\nabla \dots \nabla}_{k \text{ times}} F \in \mathfrak{T}^{(r,s+k)}(M) \,.$$

for $k \in \mathbb{N}_0$ with the convention $\nabla^0 F = F$. Applying the product rule (A.5) recursively, we obtain by induction

$$\nabla^k (A * B) = \sum_{j=0}^{k} \nabla^j A * \nabla^{k-j} B \,. \tag{A.6}$$

By taking out the term with $j = k$ and rewriting $\nabla^{k-j} B = \nabla^{k-1-j} \nabla B$ we find the useful formula

$$\nabla^k (A * B) = \nabla^k A * B + \nabla^{k-1} (A * \nabla B) \,. \tag{A.7}$$

We want to introduce a way to express the (higher) covariant derivative locally using the $*$-notation. For this, we introduce for a fixed set of local coordinates on an open set U of M the symbol $\boldsymbol{\partial}^k F$ for $k \in \mathbb{N}_0$ as

$$(\boldsymbol{\partial}^k F)_{i_1 \dots i_k} = \partial_{i_1} \dots \partial_{i_k} F \,,$$

with the convention $\partial^0 F = F$. On the right hand side we have the m-th partial derivative of F in the directions given by the indices $i_1, ..., i_k$ in the given coordinates. Hence, $\partial^k F$ can be interpreted as a k-th order derivative $D^k F$ on U. Note that F here can be a tensor field of any rank and its indices were just suppressed. Like the Christoffel symbols, $\partial^k F$ is not a tensor. Now we can write (A.3) in a short manner as

$$\nabla F = \partial F + \Gamma * F\,.$$

Likewise, we can write the k-th covariant derivative on the set U as

$$\nabla^k F = \partial^k F + \sum_{j=0}^{k-1} \left(\mathop{\Huge *}_{i=0}^{k-1} (\partial^i \Gamma) \right) * \nabla^j F\,. \tag{A.8}$$

We prove this statement by induction. For $k = 1$ this is the identity given above. Assume the equation holds for $k \in \mathbb{N}$, then we compute

$$\begin{aligned}
\nabla^{k+1} F &= \nabla(\nabla^k F) = \partial(\nabla^k F) + \Gamma * \nabla^k F \\
&= \partial \left(\partial^k F + \sum_{j=0}^{k-1} \left(\mathop{\Huge *}_{i=0}^{k-1} (\partial^i \Gamma) \right) * \nabla^j F \right) + \Gamma * \nabla^k F \\
&= \partial^{k+1} F + \sum_{j=0}^{k-1} \partial \left(\mathop{\Huge *}_{i=0}^{k-1} (\partial^i \Gamma) \right) * \nabla^j F + \left(\mathop{\Huge *}_{i=0}^{k-1} (\partial^i \Gamma) \right) * \partial \nabla^j F + \Gamma * \nabla^k F \\
&= \partial^{k+1} F + \sum_{j=0}^{k-1} \left(\mathop{\Huge *}_{i=0}^{k} (\partial^i \Gamma) \right) * \nabla^j F + \left(\mathop{\Huge *}_{i=0}^{k-1} (\partial^i \Gamma) \right) * (\nabla^{j+1} F + \Gamma * \nabla^j F) + \Gamma * \nabla^k F \\
&= \partial^{k+1} F + \sum_{j=0}^{k} \left(\mathop{\Huge *}_{i=0}^{k} (\partial^i \Gamma) \right) * \nabla^j F\,,
\end{aligned}$$

which is the statement for $k + 1$.

We define the divergence of tensor fields $F \in \mathfrak{T}^{(r,s)}(M)$ with $s \geq 1$ with respect to the metric g as the trace of the covariant derivative of F with respect to its last two covariant components, i.e.

$$\begin{aligned}
\operatorname{div}\colon \mathfrak{T}^{(r,s)}(M) &\to \mathfrak{T}^{(r,s-1)}(M) \\
\operatorname{div}(F) &= \operatorname{tr}^{s,s+1}(\nabla F)\,.
\end{aligned}$$

Note that this definition is different from the usual definition of the divergence which applies to vector fields, i.e., $X \in \mathfrak{T}^{(1,0)}(M)$. However, we can retrieve the notion of the divergence of a vector field by first applying the isomorphism $\imath_g$ to X and then applying the divergence operator defined here (note that the covariant derivative and $\imath_g$ commute). The reason we chose to define the divergence in this way is that it becomes the formal adjoint of the covariant derivative with respect to the L^2-inner product on tensor fields (see below). With the other definition the divergence is the formal adjoint of a gradient operator which we do not use here. Note that in [43, p. 149 5-16] the definition of the divergence fits to our definition here. Using (A.4), we find the following product rule for the divergence

$$\operatorname{div}(fF) = \operatorname{tr}^{s,s+1}(F \otimes \nabla f) + f \operatorname{div} F\,.$$

In particular, if $F \in \mathfrak{T}^{(0,1)}(M)$ is a 1-form, then we have

$$\operatorname{div}(fF) = \langle F, \nabla f \rangle + f \operatorname{div} F\,. \tag{A.9}$$

We define the Laplace-Beltrami operator Δ on M to be the trace of the second total covariant derivative with respect to the covariant slots introduced by the derivative, or in other words, the divergence of the covariant derivative of the tensor field.

$$\begin{aligned} \Delta \colon \mathfrak{T}^{(r,s)}(M) &\to \mathfrak{T}^{(r,s)}(M) \\ \Delta F = \operatorname{div}(\nabla F) &= \operatorname{tr}^{s+1,s+2}(\nabla^2 F) \end{aligned}$$

In local coordinates a convenient way to write Δf for $f \in C^\infty(M)$ is

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu f \right),$$

where by $|g|$ we denote the determinant of the matrix representing g in the coordinates. The commutator of the Laplacian with the covariant derivative can be expressed using the Riemann curvature tensor Rm by [66, cf. (2.1.6)]

$$\nabla(\Delta F) - \Delta(\nabla F) = \nabla \mathrm{Rm} * F + \mathrm{Rm} * \nabla F = \nabla(\mathrm{Rm} * F)\,,$$

where in the last step we used (A.5). We can use this to also find a higher order commutator:

$$\nabla^k(\Delta F) = \Delta(\nabla^k F) + \nabla^k(\mathrm{Rm} * F)\,. \tag{A.10}$$

We prove this statement by induction. For $k = 1$ this is the identity given above. Assume the equation holds for $k \in \mathbb{N}$, then we compute

$$\nabla^{k+1}(\Delta F) = \nabla(\nabla^k(\Delta F))$$

$$
\begin{aligned}
&= \nabla\Big(\Delta(\nabla^k F) + \nabla^k(\mathrm{Rm} * F)\Big) \\
&= \nabla(\Delta(\nabla^k F)) + \nabla^{k+1}(\mathrm{Rm} * F) \\
&= \Big(\Delta(\nabla(\nabla^k F)) + \nabla(\mathrm{Rm} * \nabla^k F)\Big) + \nabla^{k+1}(\mathrm{Rm} * F) \\
&= \Delta(\nabla^{k+1} F) + \nabla \mathrm{Rm} * \nabla^k F + \mathrm{Rm} * \nabla^{k+1} F + \nabla^{k+1}(\mathrm{Rm} * F) \\
&= \Delta(\nabla^{k+1} F) + \nabla^{k+1}(\mathrm{Rm} * F)\,,
\end{aligned}
$$

where in the last step we used (A.6) in order to absorb the additional terms into $\nabla^{k+1}(\mathrm{Rm} * F)$.

Finally, we can introduce integration by parts on M for tensor fields. We recall that M is a manifold without boundary. As shown in [43, p. 149 5-16], for tensor fields $F \in \mathfrak{T}^{(r,s)}(M)$ and $G \in \mathfrak{T}^{(r,s+1)}(M)$ we have

$$\int_M \langle \nabla F, G \rangle \mathrm{d}v(g) = -\int_M \langle F, \operatorname{div} G \rangle \mathrm{d}v(g)\,.$$

This shows us how the divergence is the formal adjoint of $-\nabla$ with respect to this inner product. In particular, from the definitions we deduce that for tensor fields $F, G \in \mathfrak{T}^{(r,s)}(M)$ we get

$$\int_M \langle \Delta F, G \rangle \mathrm{d}v(g) = -\int_M \langle \nabla F, \nabla G \rangle \mathrm{d}v(g) = \int_M \langle F, \Delta G \rangle \mathrm{d}v(g)\,.$$

Moreover, with this rule we immediately get the divergence theorem in the case of a manifold without boundary. Let $X \in \mathfrak{X}(M)$ be a vector field then

$$\int_M \operatorname{div} X \,\mathrm{d}x = 0\,,$$

where the divergence of X is defined as noted above.

A.4. Sobolev spaces on Riemannian manifolds

In this section we want to present a brief collection on results of Sobolev spaces on (compact) Riemannian manifolds. We want to do this in terms of the covariant derivative. The standard literature for this point of view with extensive results are [5] and [30]. However, the main definition of the Sobolev spaces in both of these sources is imprecise and unsatisfactory as it lacks the notion of how the derivatives of single elements in the spaces can be interpreted. Therefore, we try to give a more precise definition of the Sobolev spaces by first defining L^p-spaces for sections of vector bundles on manifolds, from which then the Sobolev spaces

can be derived in a natural way. This approach has been used for instance in [21] (even though in the setting of open manifolds). For the notation and definitions used from Riemannian geometry we refer to the previous section.
For the remainder of this section let M be a closed, oriented, smooth manifold of dimension n. We endow M with its Borel-algebra.

A.4.1. L^p-spaces on manifolds

Let (E, π, M) be a smooth vector bundle over M of rank m with trivializations $(U_i, \phi_i)_{i\in I}$ and fibers $E_x := \pi^{-1}(x)$. Moreover, assume there exists a bundle metric on E, i.e., a family of scalar products $\langle\cdot,\cdot\rangle_x$ on E_x, depending smoothly on $x \in M$. These scalar products induce a norm on each E_x, denoted by $|\cdot|_x$. Consequently, for a smooth section $u \in \Gamma(E)$ the function $x \mapsto |u(x)|_x$ on M is measurable and can be integrated. As a result, we can define the L^p-norm of a rough section $u \in \Gamma_{\mathrm{r}}(E)$ as

Definition A.10 (L^p-norm and L^p-spaces of sections on manifolds)**.** Let $1 \le p \le \infty$ and let μ be a measure on M. For $u \in \Gamma_{\mathrm{r}}(E)$ we define the L^p-norm of u for $1 < p < \infty$ as

$$\|u\|_{L^p(\Gamma(E),\mu)} = \left(\int_M |u(x)|_x^p \,\mathrm{d}\mu\right)^{1/p}.$$

and for $p = \infty$ we define

$$\|u\|_{L^\infty(\Gamma(E),\mu)} = \operatorname*{ess\,sup}_{x\in M} |u(x)|_x := \inf\Big\{a > 0 : \mu(\{x \in M : |u(x)|_x > a\}) = 0\Big\}.$$

We define the L^p-space of sections of E, $L^p(\Gamma(E), \mu)$, as the completion of $\Gamma(E)$ under $\|\cdot\|_{L^p(\Gamma(E),\mu)}$. ⊡

Care has to be taken when one considers now an element $u \in L^p(\Gamma(E), \mu)$. Surely, u need not be a smooth section anymore. As is the case with the standard L^p-spaces, we would like to consider $L^p(\Gamma(E))$ as the space of equivalence classes of rough sections, i.e., sections of E without any smoothness (nor continuity) assumption, with finite L^p-norm, where two rough sections are equivalent if they coincide almost everywhere. We state this in the following proposition.

Proposition A.11. *Let $\mathcal{L}^p(\Gamma_{\mathrm{r}}(E), \mu)$ be the space of rough sections of E with finite L^p-norm. Then*

$$L^p(\Gamma(E), \mu) \cong \mathcal{L}^p(\Gamma_{\mathrm{r}}(E), \mu)/\sim,$$

where $\sim$ denotes the equivalence relation of being equal almost everywhere.

Proof. To this end, we present a procedure for picking a representative for $u \in L^p(\Gamma(E))$. By the definition of the completion, let $(u_k) \subset \Gamma(E)$ be a Cauchy sequence representing u. We define a map $T\colon \Gamma_r(E) \to \{M \to \mathbb{R}^m\}$ in the following way. According to [37, Thm. 2.1.3] we can choose trivializations $(U_i, \phi_i)_{i\in\mathcal{I}}$ belonging to the vector bundle (E, π, M) such that they are linear isometries on the second component, meaning that for all $x \in U_i$ and $v \in E_x$ we have

$$|\mathrm{Pr}_2 \circ \phi_i(v)| = |v|_x$$

for all $i \in I$, where Pr_2 denotes the projection onto the second component. The norm on the left-hand side denotes the Euclidean norm. Since M is compact, we can choose a finite cover $(U_i)_{1\le i\le I}$ of M. Set $A_i = U_i \setminus \bigcup_{j=1}^{i-1} U_j$. Then $(A_i)_{1\le i\le I}$ is a measurable partition of M. Let $\phi\colon E \to M \times \mathbb{R}^m$ be the map such that $\phi|_{\pi^{-1}(A_i)} = \phi_i$. It is a measurable map with measurable inverse ϕ^{-1} (by taking ϕ_i^{-1} on each A_i) but not a diffeomorphism in general. However, $\mathrm{Pr}_2 \circ \phi$ restricts to an isometry on E_x for all $x \in M$. Therefore, we define the map T as

$$Tu = \mathrm{Pr}_2 \circ \phi \circ u\,.$$

Then by construction $|Tu(x)| = |u(x)|_x$ for all $x \in M$ and T is a linear map. Hence, we find that $\|u\|_{L^p(\Gamma(E),\mu)} = \|Tu\|_{L^p(M,\mathbb{R}^m,\mu)}$ for all $u \in \Gamma_r(E)$ which implies

$$\|Tu_k - Tu_l\|_{L^p(M,\mathbb{R}^m,\mu)} = \|u_k - u_l\|_{L^p(\Gamma(E))}\,,$$

such that we deduce that (Tu_k) is a Cauchy sequence in the Banach space $L^p(M, \mathbb{R}^m, \mu)$. Since for the latter space we know of the identification of elements by their representatives almost everywhere, we can choose a representative $v\colon \mathbb{S}^2 \to \mathbb{R}^m$ for the limit of (Tu_k). Now we define the map $S\colon \{M \to \mathbb{R}^m\} \to \Gamma_r(E)$ by $S(f)(x) = \phi^{-1}(x, f(x))$ and set Sv to be the representative of u. We need to verify that this procedure is well defined, i.e. independent of the choice of the Cauchy sequence $(u_k) \subset \Gamma(E)$ representing u, and then of the representative v chosen. Let v and v' be the respective limits of the Cauchy sequences (Tu_k) and (Tu'_k) representing u. Then $v = v'$ almost everywhere. This follows from

$$\begin{aligned}
&\|v - v'\|_{L^p(M,\mathbb{R}^m)}\\
&\le \|v - Tu_k\|_{L^p(M,\mathbb{R}^m)} + \|Tu_k - Tu'_k\|_{L^p(M,\mathbb{R}^m)} + \|Tu'_k - v'\|_{L^p(M,\mathbb{R}^m)}\\
&= \|v - Tu_k\|_{L^p(M,\mathbb{R}^m)} + \|u_k - u'_k\|_{L^p(\Gamma(E))} + \|Tu'_k - v'\|_{L^p(M,\mathbb{R}^m)}\\
&\quad \longrightarrow 0\,, \text{ as } k \to \infty\,,
\end{aligned}$$

where the first and last term vanish as $k \to \infty$ by the definition of v and v', respectively, and the middle term vanishes as $k \to \infty$ since (u_k) and (u'_k) represent the same element in $L^p(\Gamma(E))$. Hence by definition of $L^p(M, \mathbb{R}^m, \mu)$ we have $v = v'$ almost everywhere, so they represent the same element. Now we show that if we choose two functions $v, v' \colon M \to \mathbb{R}^m$, which coincide almost everywhere, then Sv and Sv' coincide also almost everywhere. But this follows immediately again from the isometry property of T and the fact that S is the inverse of T. This yields

$$\|Sv - Sv'\|_{L^p(\Gamma)} = \|TS(v - v')\|_{L^p(M,\mathbb{R}^m)} = \|v - v'\|_{L^p(M,\mathbb{R}^m)} = 0\,,$$

which implies that $Sv = Sv'$ almost everywhere.
Conversely, if $u \in \mathcal{L}^p(\Gamma_{\mathrm{r}}(E), \mu)/\sim$, then we can pick a representative still denoted by u. Then $Tu \in L^p(M, \mathbb{R}^m, \mu)$ and by density we can choose a sequence $(v_k) \subset C^\infty(M, \mathbb{R}^m)$ such that $\lim_{k\to\infty} v_k = Tu$ in $L^p(M, \mathbb{R}^m, \mu)$. Then we define $u_k = Sv_k$ and find that

$$\|u - u_k\|_{L^p(\Gamma(E))} = \|Tu - TSv_k\|_{L^p(M,\mathbb{R}^m)} = \|Tu - v_k\|_{L^p(M,\mathbb{R}^m)} \longrightarrow 0\,, \text{ as } k \to \infty\,.$$

Hence u_k is a Cauchy sequence in $L^p(\Gamma(E))$ representing u. Since T and S are linear, this procedure of selecting representatives is linear. Thus, we have shown the desired isomorphism. □

With this new point of view as a space of rough sections, we see that the map T defined in the proof in fact extends to a map $T \colon L^p(\Gamma(E), \mu) \to L^p(M, \mathbb{R}^m, \mu)$, which is a linear isometry with inverse S, i.e., an isometric isomorphism. This gives immediate rise to the following corollary.

Corollary A.12. *The space $L^p(\Gamma(E), \mu)$ is isometrically isomorphic to $L^p(M, \mathbb{R}^m, \mu)$, where $L^p(M, \mathbb{R}^m, \mu)$ is the usual L^p-space of functions on M with values in $\mathbb{R}^m$ and measure μ.*

In particular, the space $L^p(\Gamma(E), \mu)$ features all the properties of $L^p(M, \mathbb{R}^m, \mu)$, of which the most important ones are:

1. $L^p(\Gamma(E), \mu)$ is a separable Banach space.

2. If $1 < p < \infty$, then $L^p(\Gamma(E), \mu)$ is a reflexive space, with its dual space identified with $L^q(\Gamma(E), \mu)$, where q is the Hölder conjugate of p.

3. For $p = 2$, $L^2(\Gamma(E), \mu)$ is a Hilbert space with inner product given by

$$\langle u, v \rangle_{L^2} = \int_M \langle u, v \rangle_x \mathrm{d}\mu \,.$$

This whole proof is due to [2], as we were not able to find a proof in the literature. If (M, g) is a Riemannian manifold and if we take as the vector bundle any of the tensor bundles $T^{(r,s)}TM$ then a bundle metric is induced by the Riemannian metric g as $\langle \cdot, \cdot \rangle_g$, as defined in (A.2). Using as the measure μ_g on M the one induced by the volume form $\mathrm{d}v(g)$, we can henceforth define the L^p-spaces of tensor fields on M as above and denote them with $L^p_{(r,s)}(M) := L^p(\mathfrak{T}^{(r,s)}(M), \mu_g)$. We want to extend the notion of L^p-spaces also to the $*$-notation introduced in Section A.3. For two tensor fields $F \in L^p_{(r,s)}(M), G \in L^q_{(r',s')}(M)$, the $*$-product $F * G$ is a linear combination of rough tensor fields of not necessarily the same rank. We define the L^p-norm of $F * G$ as the L^p-norm of the scalar function $|F * G|$, which of course is also only abstract. Nonetheless, when one is only interested in computing bounds, this notation serves useful and, by the condition $|F * G| \leq C|F||G|$, we can always make use of the Hölder inequality to find

$$\|F * G\|_{L^\sigma} \leq C \|F\|_{L^p_{(r,s)}} \|G\|_{L^q_{(r',s')}} \,,$$

as long as $1/p + 1/q = 1/\sigma$. The constant C depends neither on F and G nor on (M, g).

A.4.2. Sobolev spaces on Riemannian manifolds

Let (M, g) now be a Riemannian manifold. For $F \in \mathfrak{T}^{(r,s)}(M)$ we note that for every $k \in \mathbb{N}_0$ the tensor field $\nabla^k F$ is an element of $\mathfrak{T}^{(r,s+k)}(M)$. With the definition of the L^p-spaces for tensor fields we can now define the Sobolev spaces of tensor fields on M.

Definition A.13 (Sobolev norm and Sobolev spaces on manifolds)**.** Let $k \in \mathbb{N}_0$. For $F \in \mathfrak{T}^{(r,s)}(M)$ and $1 \leq p < \infty$ we define the Sobolev norm

$$\|F\|_{W^{k,p}_{(r,s)}(M)} = \left(\sum_{j=1}^{k} \|\nabla^j F\|^p_{L^p_{(r,s+j)}(M)} \right)^{1/p}$$

and for $p = \infty$ we define

$$\|u\|_{W^{k,\infty}_{(r,s)}(M)} = \max_{0 \leq j \leq k} \|\nabla^j u\|_{L^\infty_{(r,s)}(M)} \,.$$

We define the Sobolev space $W^{k,p}_{(r,s)}(M)$ to be the completion of $\mathfrak{T}^{(r,s)}(M)$ under $\|\cdot\|_{W^{k,p}_{(r,s)}(M)}$. ⊡

Remark A.14 (Notation and shorthands)**.** Usually it is clear from which space $\mathfrak{T}^{(r,s)}(M)$ the tensor fields originate, so we drop the subscript denoting the tensor type in the notations of the norms for simplification and readability, and write $W^{k,p}_{(r,s)}(M) = W^{k,p}(M) = W^{k,p}$, when also the manifold is clear from the context. Obviously, $W^{0,p}_{(r,s)}(M) = L^p_{(r,s)}(M)$. Moreover, if $p = 2$, we write $H^k_{(r,s)}(M) = W^{k,2}_{(r,s)}(M)$. Lastly, for the space of functions $M \to \mathbb{R}$, we write $W^{k,p}_{(0,0)}(M) = W^{k,p}(M, \mathbb{R})$. ⊡

For the space $W^{k,p}(M, \mathbb{R})$, our definition coincides with the definition used in [5] and [30]. However, by having defined the L^p-spaces for tensor fields, we can now make sense of what $\nabla^j u$ is for $u \in W^{k,p}(M, \mathbb{R})$ and $1 \leq j \leq k$, as in these textbooks it is not handled with satisfactory care. To explain, let $F \in W^{k,p}_{(r,s)}$. Then there exists a Cauchy sequence $(F_k) \subset \mathfrak{T}^{(r,s)}(M)$ representing F. By the definition of the norm we conclude that $\nabla^j F_k$ is a Cauchy sequence in $L^p_{(r,s+j)}(M)$ for all $j \in \{0, \dots, k\}$. Hence, we can define $\nabla^j F$ as the equivalence class of $\nabla^j F_k$ in $L^p_{(r,s+j)}(M)$, which is well defined. Moreover, by the characterization of the L^p-spaces by their representatives almost everywhere, we can represent $\nabla^j F$ by a rough tensor field in $\Gamma_{\mathrm{r}}(T^{(r,s+j)}TM)$. This notation can be confusing, as F itself is not necessarily a smooth section of $T^{(r,s)}TM$, but rather a rough section, and the covariant derivative is only defined for these smooth sections. Therefore, the question arises how to interpret the covariant derivative of a rough tensor field. To this end, we define an extension of the covariant derivative as

$$\begin{aligned} \nabla \colon W^{1,p}_{(r,s)}(M) &\to L^p_{(r,s+1)}(M) \\ F &\mapsto \nabla F \,, \end{aligned}$$

where ∇F is precisely the element obtained as explained above as the equivalence class of the elements ∇F_k where $(F_k) \subset \mathfrak{T}^{(r,s)}(M)$ is representing F. By this definition ∇ is a bounded linear operator between Banach spaces. Moreover, ∇ extends also to a bounded linear operator $W^{k,p}_{(r,s)}(M) \to W^{k-1,p}_{(r,s+1)}(M)$, and also the repeated application

$$\nabla^j \colon W^{k,p}_{(r,s)}(M) \to W^{k-j,p}_{(r,s+j)}(M)$$

for $j \leq k$ is still a bounded linear operator. Due to the density of smooth sections in $W^{k,p}_{(r,s)}(M)$, these extensions are therefore well-defined and all the properties of the covariant derivative collected in Section A.3 hold.

We briefly summarize some facts about Sobolev spaces on manifolds, which mostly hold due to the fact that M is compact.

1. We recall that the definition of the tensor norm $|\cdot|$ depends on the choice of the Riemannian metric g. However, the spaces $W^{k,p}(M,\mathbb{R})$ are independent of the choice of g and the respective norms are all equivalent [30, Prop. 2.2].

2. If we use a finite atlas $(U_i, \phi_i)_{i\in I}$ of M with a smooth partition of unity $(\rho_i)_{i\in I}$ subordinate to the $(U_i)_{i\in I}$, then $u \in W^{k,p}(M,\mathbb{R})$ if and only if $\rho_i\, \varphi_i^* u \in W^{k,p}(U_i,\mathbb{R})$ for all $i \in I$, where by the latter we denote the usual Sobolev space on subsets of $\mathbb{R}^n$. This is especially independent of the choice of atlas and partition of unity [64, Sec. 4.3.].

3. $H^k(M,\mathbb{R})$ is a Hilbert space with inner product given by

$$\langle u, v\rangle_{H^k} = \sum_{j=1}^{k} \int_M \left\langle \nabla^j u, \nabla^j v \right\rangle \mathrm{d}v(g)$$

   for $u, v \in H^k(M,\mathbb{R})$.

4. All embeddings known from the theory of Sobolev spaces on compact domains of $\mathbb{R}^n$ also hold. In particular,

   Theorem A.15 (Rellich–Kondrachov theorem [30, Thm. 2.9])**.** *Let (M, g) be a smooth, compact Riemannian manifold of dimension n.*

   (i) For any integers $j \geq 0$ and $m \geq 1$, any real number $q \geq 1$, and any real number p such that $1 \leq p < nq/(n-mq)$, the space $W^{j+m,q}(M,\mathbb{R})$ embeds into $W^{j,p}(M,\mathbb{R})$ compactly. In particular, for any $q \in [1, n)$ real and any $p \geq 1$ such that $1/p > 1/q - 1/n$, the embedding of $H_1^q(M)$ in $L^p(M)$ is compact.

   (ii) For any integer $k \geq 0$ and real number $q \geq 1$, $W^{k,q}(M)$ embeds into $C^\alpha(M,\mathbb{R})$ compactly for any $\alpha \in [0, 1)$ such that $k - \alpha > n/q$. In particular, the embedding of $W^{2,q}(M,\mathbb{R})$ in $C^0(M,\mathbb{R})$ is compact for any $q > n$.

We can define the $W^{k,p}$-norm also for measurable subsets $V \subset M$ by restricting the integrals to the respective subset:

$$\|F\|_{W^{k,p}(V)} = \left(\sum_{j=1}^{k} \|\nabla^j F\|^p_{L^p(V)}\right)^{1/p} = \left(\sum_{j=1}^{k} \int_V |\nabla^j F|^p \mathrm{d}v(g)\right)^{1/p}.$$

Let (U,ϕ) be a chart of M and let $K \subset U$ be compact and connected. Then all derivatives of the Christoffel symbols are bounded on K and we find by virtue of (A.8) that

$$|\nabla^k F|_g \le C \sum_{j=0}^{k} |\partial^j F|_g \le C \sum_{j=0}^{k} |\nabla^j F|_g\,, \tag{A.11}$$

on K, where we define the norm of the symbol $\partial^i u$ introduced in Section A.3 as if it was a tensor. Note that this still implies contraction via the metric.
Therefore, we can express a norm equivalent to $\|F\|_{W^{k,p}(K)}$ using only the chart (U,ϕ) and the partial derivatives. We denote it by

$$\|F\|_{\hat{W}^{k,p}(K)} = \left(\sum_{j=0}^{k} \int_K |\partial^j F|_g^p \mathrm{d}v(g)\right)^{1/p}. \tag{A.12}$$

This does not coincide with the standard $W^{k,p}$-norm of tensor fields $\mathbb{R}^n \supset \phi(K) \to \mathbb{R}^m$ because we still have the components of the metric contributing, through $|\cdot|_g$ and $\mathrm{d}v(g)$. Moreover, from (A.11) it becomes clear that we can only make this equivalence when considering the total $W^{k,p}$-norm and cannot generalize it to individual parts of the derivatives. In particular, in general we have

$$\|\nabla^j F\|_{L^p(K)} \neq \|\partial^j F\|_{L^p(K)}\,.$$

Lemma A.16. *Let M be a compact Riemannian manifold of dimension n and let (U,ϕ) be a chart on which M is conformally flat, i.e., there exists a smooth positive function $\lambda\colon U \to \mathbb{R}$ such that $g = \lambda^2 \overline{g}$ on U, where $\overline{g}$ is the Euclidean metric on $\mathbb{R}^n$. Let $K \subset U$ be compact and connected. Then there exist constants $0 < c < C$ such that for all $u \in C^\infty(M,\mathbb{R})$, $1 \le p \le \infty$, and $j \in \mathbb{N}_0$ we have*

$$c\|\partial^j u\|_{L^p(K)} \le \|D^j \hat{u}\|_{L^p(\phi(K))} \le C\|\partial^j u\|_{L^p(K)}\,,$$

where $\hat{u} = u \circ \phi$ denotes the coordinate representation of u on U and D^j is the j-th derivative on $\mathbb{R}^n$.

Proof. Since K is compact, there are constants $a, A > 0$ such that $a \le \lambda^{-1} \le A$ on K. The volume form in the coordinates is given by $\mathrm{d}v(g) = \lambda^2\, \mathrm{d}x$. Additionally, the tensor norm of a tensor field $F \in \mathfrak{T}^{(r,s)}(M)$ is given by $|F|_g = \lambda^{r-s}|F|_{\overline{g}}$, where $|F|_{\overline{g}}$ is precisely the square sum of all components of F. We find a relation between the p-norms of the partial derivative symbol $\partial^k u$ on K and the derivatives of $\hat{u}$ on

$\phi(K)$ by

$$\begin{aligned}\|\partial^j u\|^p_{L^p(K)} &= \int_K |\partial^j u|^p_g \mathrm{d}v(g) \\ &= \int_K \lambda^{-jp} |\partial^j u|^p_g \mathrm{d}v(g) \\ &= \int_{\phi(K)} \lambda^{-(jp-2)} |D^j \hat{u}|^p_g \,\mathrm{d}x \\ &\leq A^{jp-2} \|D^j \hat{u}\|^p_{L^p(\phi(K))} \\ &\leq \left(\frac{A}{a}\right)^{jp-2} \|\partial^j u\|^p_{L^p(K)}\,,\end{aligned}$$

in the case $jp \geq 2$. We used the equality $\partial^j u = D^j \hat{u}$ in the coordinates which holds exactly by the definition of the partial derivative on manifolds. In the last step we repeated the procedure from the first inequality by inserting $1 < a^{-(jp-2)}\lambda^{-(jp-2)}$ into the integral and identifying the original norm. If $jp < 2$, we get the same result just with a and A interchanged. This proves the lemma. □

We conclude by (A.12) that

Corollary A.17. *In the same setting as in the previous lemma, there exist constants* $0 < c_1 < c_2 < c_3$ *such that*

$$c_1 \|u\|_{W^{k,p}(K)} \leq c_2 \|u\|_{\hat{W}^{k,p}(K)} \leq \|\hat{u}\|_{W^{k,p}(\phi(K))} \leq c_3 \|u\|_{W^{k,p}(K)}\,.$$

With this observation we are able to prove the following version of the Gagliardo–Nirenberg interpolation inequality for compact manifolds. To our surprise there exist no proofs of the general Gagliardo–Nirenberg interpolation inequality for compact manifolds in the literature. The inequality for domains in $\mathbb{R}^n$ has the following form.

Theorem A.18 (Gagliardo–Nirenberg in $\mathbb{R}^n$ [53, pp. 125])**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a measurable, open and connected domain with* C^1*-boundary. Let* $1 \leq q \leq \infty$*, and let* l *and* m *be non-negative integers such that* $l < m$*. Furthermore, let* $1 \leq r \leq \infty$*,* $p \geq 1$*, and* $\theta \in [0,1]$ *be such that the relations*

$$\frac{1}{p} = \frac{l}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\theta}{q}, \qquad \frac{l}{m} \leq \theta \leq 1$$

hold. Then for $u \in L^q(\Omega) \cap W^{m,r}(\Omega)$

$$\|D^l u\|_{L^p(\Omega)} \leq C \|u\|^{\theta}_{W^{m,r}(\Omega)} \|u\|^{1-\theta}_{L^q(\Omega)}\,,$$

with one exceptional case: if $r > 1$ and $m - l - \frac{n}{r}$ is a non-negative integer, then the additional assumption $\frac{l}{m} \leq \theta < 1$ (notice the strict inequality) is needed. In any case, the constant $C > 0$ depends on the parameters l, m, n, q, r, θ, on the domain Ω, but not on u.

Remark A.19. We allow Ω to be bounded or unbounded. In the latter case, the inequality also holds if we replace $\|u\|_{W^{m,r}(\Omega)}$ by $\|D^m u\|_{L^r(\Omega)}$. However, for our purposes the inequality with the $W^{m,r}$-norm is sufficient. ⊡

If $1 \leq l < k$ then one Gagliardo–Nirenberg inequality reads as follows (by checking that the given parameters satisfy the conditions of the theorem above)

$$\|D^l u\|_{L^{2k/l}(\Omega)} \leq C\|u\|^{1-l/k}_{L^\infty(\Omega)} \|u\|^{l/k}_{H^k(\Omega)} \,. \tag{A.13}$$

We prove this inequality now for the case of compact Riemannian manifolds of dimension 2 with the covariant derivative replacing the standard derivative. Before we do this, we need a technical lemma about the topology of compact manifolds.

Lemma A.20. *Let M be a compact manifold of dimension n and let $(U_i, \phi_i)_{i\in\mathcal{I}}$ be a finite atlas of M. Then there exists a covering $(K_j)_{j\in\mathcal{J}}$ of M by compact sets K_j such that for each $j \in \mathcal{J}$ there exists an $i \in \mathcal{I}$ such that $K_j \subset U_i$.*

Proof. For each $x \in M$ there exists a chart $(U_{i(x)}, \phi_{i(x)})$ with $i(x) \in \mathcal{I}$ such that $x \in U_{i(x)}$. As a manifold, M is locally compact Hausdorff. Therefore, for each $x \in M$ there exists a relatively compact neighborhood V_x with $\overline{V}_x \subset U_{i(x)}$. Since M is compact, we can find a finite subcover $(V_{x_j})_{j\in\mathcal{J}}$ of M. We define $K_j = \overline{V}_{x_j}$ for $j \in \mathcal{J}$. Then $(K_j)_{j\in\mathcal{J}}$ is the desired covering. □

Remark A.21. If the atlas is of no importance, by adjusting it, we can always assume without loss of generality that $\mathcal{J} = \mathcal{I}$ and $K_i \subset U_i$ for all $i \in \mathcal{I}$. ⊡

Proposition A.22 (Special case of Gagliardo–Nirenberg inequality)**.** *Let M be a compact Riemannian manifold of dimension 2. Then for $1 \leq l < k$ and $u \in H^k(M, \mathbb{R})$ we have*

$$\|\nabla^l u\|_{L^{2k/l}} \leq C\|u\|^{1-l/k}_{L^\infty} \|u\|^{l/k}_{H^k} \,,$$

where the constant $C > 0$ depends on l, k, and the manifold M, but not on u. Note that since M is of dimension 2 and $k \geq 2$, we have $H^k(M, \mathbb{R}) \hookrightarrow L^\infty(M, \mathbb{R})$.

Proof. Since M is of dimension 2, it is locally conformally flat [43, cf. p. 222], and since it is compact, we can find a finite atlas $(U_i, \phi_i)_{i\in\mathcal{I}}$ such that the metric is conformally flat on each chart and such that by the above lemma we hence find a covering $(K_i)_{i\in\mathcal{I}}$ of M by compact sets $K_i \subset U_i$. Assume the inequality holds on the sets K_i. Then we obtain by simple arguments of integration over unions of sets that

$$\begin{aligned}\|\nabla^l u\|_{L^{2k/l}(M)} &\leq \sum_{i\in\mathcal{I}} \|\nabla^l u\|_{L^{2k/l}(K_i)} \\ &\leq C\sum_{i\in\mathcal{I}} \|u\|^{1-l/k}_{L^\infty(K_i)}\|u\|^{l/k}_{H^k(K_i)} \leq C\|u\|^{1-l/k}_{L^\infty(M)}\|u\|^{l/k}_{H^k(M)}\,.\end{aligned}$$

Therefore, it is sufficient to show the inequality on a single compact set on which the metric is conformally flat. Let $K \subset M$ be such a set with metric $g = \lambda^2\overline{g}$, where $\overline{g}$ is the Euclidean metric.
Invoking (A.11), we can write

$$\|\nabla^l u\|_{L^{2k/j}(K)} \leq C\sum_{j=0}^{l} \|\partial^j u\|_{L^{2k/l}(K)} \leq C\sum_{j=0}^{l} \|\partial^j u\|_{L^{2k/j}(K)}\,, \tag{A.14}$$

where in the last step we used the Hölder inequality on bounded domains since $j \leq l$. Note that for the $j = 0$ term we identify the norm of u with the L^∞-norm. For $j > 0$ we inspect each summand separately. We find by Definition A.16, Definition A.17, and (A.13) that

$$\begin{aligned}\|\partial^j u\|_{L^{2k/j}(K)} &\leq C\|D^j\hat{u}\|_{L^{2k/j}(\phi(K))} \\ &\leq C\|\hat{u}\|^{1-j/k}_{L^\infty(\phi(K))}\|\hat{u}\|^{j/k}_{H^k(\phi(K))} \\ &\leq C\|u\|^{1-j/k}_{L^\infty(K)}\|u\|^{j/k}_{H^k(K)}\,,\end{aligned}$$

where we identified $\|\hat{u}\|_{L^\infty(\phi(K))}$ with $\|u\|_{L^\infty(K)}$. Inserting this into the sum above we arrive at

$$\begin{aligned}\|\nabla^l u\|_{L^{2k/l}(K)} &\leq C\sum_{j=0}^{l} \|u\|^{1-j/k}_{L^\infty(K)}\|u\|^{j/k}_{H^k(K)} \\ &= C\|u\|_{L^\infty(K)}\sum_{j=0}^{l}\left(\frac{\|u\|_{H^k(K)}}{\|u\|_{L^\infty(K)}}\right)^{j/k} \\ &\leq C\|u\|_{L^\infty(K)}\left(1+\left(\frac{\|u\|_{H^k(K)}}{\|u\|_{L^\infty(K)}}\right)^{l/k}\right) \\ &= C\Big(\|u\|^{1-l/k}_{L^\infty(K)}\|u\|^{l/k}_{H^k(K)} + \|u\|_{L^\infty(K)}\Big)\,,\end{aligned}$$

where we used the the standard estimate for polynomials $\sum_{j=0}^{l} x^j \leq C(1+x^l)$ for $x \geq 0$. Now, since $k \geq 2$ and M is of dimension 2 we have that by Sobolev embedding $H^k(K,\mathbb{R}) \hookrightarrow L^\infty(K,\mathbb{R})$, such that we can absorb the last term into the first one, which proves the proposition. □

Remark A.23. For any compact subset $K \subset M$, we can repeat the proof for these sets (by using the covering $(K_i \cap K)_{i\in\mathcal{I}}$) and find the same inequality for the norms restricted to K. ⊡

Corollary A.24. *Let M be a compact Riemannian manifold of dimension 2. Then for $1 \leq l < k$ and $F \in H^k_{(r,s)}(M)$ we have*

$$\|\nabla^l F\|_{L^{2k/l}} \leq C\|F\|_{L^\infty}^{1-l/k}\|F\|_{H^k}^{l/k}.$$

Proof. As in the proof of the previous preposition we can restrict ourselves to a compact set with a single chart (U,ϕ) with a conformally flat metric. Applying again (A.11) we find a similar bound as in (A.14). Then the norm of $\partial^j F$ splits into the norms of the components of F in these coordinates and we can proceed as in the proof of the previous proposition on each component separately. Each right hand side is then dominated by the whole tensor norm again. □

With this inequality at hand, we can now prove similar results for manifolds as in the flat case as presented in [65, Prop. 3.6, Prop. 3.7].

Proposition A.25. *Let M be a compact Riemannian manifold of dimension 2. Then for all $k \in \mathbb{N}$ and $u,v \in H^k(M,\mathbb{R}) \cap L^\infty(M,\mathbb{R})$ we have*

$$\|\nabla^k(u\,v)\|_{L^2} \leq C(\|u\|_{L^\infty}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{L^\infty}).$$

Proof. Using (A.6) we find by virtue of the Hölder inequality with $1/2 = m/2k + l/2k$ that

$$\begin{aligned}
\|\nabla^k(u\,v)\|_{L^2} &= \sum_{l+m=k} \|\nabla^l u * \nabla^m v\|_{L^2} \\
&\leq C \sum_{l+m=k} \||\nabla^l u||\nabla^m v|\|_{L^2} \\
&\leq C \sum_{l+m=k} \|\nabla^l u\|_{L^{2k/l}}\|\nabla^m v\|_{L^{2k/m}} \\
&\leq C \sum_{l+m=k} \|u\|_{L^\infty}^{1-l/k}\|u\|_{H^k}^{l/k}\|v\|_{L^\infty}^{1-m/k}\|v\|_{H^k}^{m/k},
\end{aligned}$$

where we used the Gagliardo–Nirenberg inequality from the previous proposition in the last step. For each summand we apply now Young's inequality using the fact that $1 - l/k = m/k$

$$\|u\|_{L^\infty}^{1-l/k}\|u\|_{H^k}^{l/k}\|v\|_{L^\infty}^{1-m/k}\|v\|_{H^k}^{m/k} \leq C\|u\|_{L^\infty}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{L^\infty} ,$$

which gives us the desired inequality. □

Proposition A.26. *Let M be a compact Riemannian manifold of dimension 2. Then for all $k \in \mathbb{N}$ and $u, v \in H^k(M,\mathbb{R}) \cap L^\infty(M,\mathbb{R})$ we have*

$$\|uv\|_{H^k} \leq C(\|u\|_{L^\infty}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{L^\infty}) .$$

Proof. This basically follows from the previous proposition. We bound the H^k-norm of the product of u and v first by using the equivalence of the 2-norm and 1-norm on finite dimensional vector spaces to estimate the sum of the L^2-norms of the derivatives, and then applying the previous proposition on each summand. Based on that, we obtain

$$\begin{aligned} \|uv\|_{H^k} &\leq C \sum_{l=0}^{k} \|\nabla^l(uv)\|_{L^2} \\ &\leq C \sum_{l=0}^{k} \|u\|_{L^\infty}\|v\|_{H^l} + \|u\|_{H^l}\|v\|_{L^\infty} \\ &\leq C\|u\|_{L^\infty}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{L^\infty} , \end{aligned}$$

since by definition of the H^k-norm we have $\|u\|_{H^l} \leq \|u\|_{H^k}$ for $l \leq k$. □

Remark A.27. This shows us that for $k \geq 2$ the space $H^k(M,\mathbb{R})$ is an algebra with respect to the pointwise multiplication of functions, as again $H^k(M,\mathbb{R}) \hookrightarrow L^\infty(M,\mathbb{R})$ by Sobolev embedding. ⊡

Proposition A.28. *Let M be a compact Riemannian manifold of dimension 2. Then for all $k, j, l \in \mathbb{N}$ and $F \in H^k_{(r,s)} \cap L^\infty_{(r,s)}(M)$, $G \in H^k_{(r',s')} \cap L^\infty_{(r',s')}(M)$ we have*

$$\begin{aligned} \|\nabla^k(F * G)\|_{L^2} &\leq C(\|F\|_{L^\infty}\|G\|_{H^k} + \|F\|_{H^k}\|G\|_{L^\infty}) , \\ \|F * G\|_{H^k} &\leq C(\|F\|_{L^\infty}\|G\|_{H^k} + \|F\|_{H^k}\|G\|_{L^\infty}) \\ \|F^{*j}\|_{H^k} &\leq C\|F\|_{L^\infty}^{j-1}\|F\|_{H^k} \\ \|F^{*j} * G^{*l}\|_{H^k} &\leq C\|F\|_{L^\infty}^{j-1}\|G\|_{L^\infty}^{l-1}(\|F\|_{H^k}\|G\|_{L^\infty} + \|F\|_{L^\infty}\|G\|_{H^k}) . \end{aligned}$$

Proof. In the same way as the previous propositions, by applying (A.6) and Definition A.24, we can prove the first and second inequalities. The third inequality follows from the second one by induction and the last one follows from the second one and the third one. □

A.4.3. Vector valued functions

When we consider vector valued functions from the Riemannian manifold M, that is $\boldsymbol{u}\colon M \to \mathbb{R}^m$ for some $m \in \mathbb{N}$, we can define the Sobolev space $W^{k,p}(M, \mathbb{R}^m)$ by the same means as for scalar functions. As before, we first take a more general approach. To this end, for a given vector bundle (E, π, M) of rank k we define the vector bundle (E^m, π^m, M) for $k \in \mathbb{N}$ to be the Whitney sum of k-times the vector bundle E and call it the m-th power of E. Then the fiber $(\pi^m)^{-1}(x)$ is the direct sum of m copies of E_x and we can identify an element $\boldsymbol{\xi} \in E^m$ by $(\xi_1, \dots, \xi_m)$ with $\xi_i \in E$. If there exists a bundle metric $\langle \cdot, \cdot \rangle_E$ on E, we can define a bundle metric on E^m by

$$\langle \xi, \eta \rangle_{E^m} = \sum_{i=1}^{k} \langle \xi_i, \eta_i \rangle_E$$

for $\xi, \eta \in E^m$. Then, by Section A.4.1, the space $L^p(\Gamma(E^m))$ is well-defined and isometrically isomorphic to $L^p(M, \mathbb{R}^{km})$. Letting now $E = T^{(r,s)}TM$, then we can extend the covariant derivative to tensor fields $\boldsymbol{F} \in \Gamma((T^{(r,s)}TM)^m) := \mathfrak{T}^{(r,s)}(M, m)$ by defining $\nabla \boldsymbol{F} = (\nabla F_1, \dots, \nabla F_m)$. Then we can define the Sobolev space $W^{k,p}_{(r,s)}(M, m)$ as the completion of $\mathfrak{T}^{(r,s)}(M, m)$ with respect to the norm

$$\|\boldsymbol{F}\|_{W^{k,p}} = \left(\sum_{j=1}^{m} \int_M |\nabla^j \boldsymbol{F}|^p \mathrm{d}v(g) \right)^{1/p} \tag{A.15}$$

for $1 \leq p < \infty$, and

$$\|\boldsymbol{F}\|_{W^{k,\infty}} = \max_{0 \leq j \leq k} \operatorname*{ess\,sup}_{x \in M} |\nabla^j \boldsymbol{F}(x)|$$

for $p = \infty$.

Remark A.29. We use boldface letters to denote elements in m-th power vector bundles and simply call them *vector valued*, where the term vector refers to the fact that each element decomposes into its m components which we will denote in normal font and with index.

We identify $\mathfrak{T}^{(0,0)}(M,m)$ with $C^\infty(M,\mathbb{R}^m)$ and write $W^{k,p}(M,\mathbb{R}^m)$ for the space $W^{k,p}_{(0,0)}(M,m)$. Hence, by these conventions $\boldsymbol{u} = (u_1,\dots,u_m) \in W^{k,p}(M,\mathbb{R}^m)$ is a vector valued function with components $u_i \in W^{k,p}(M,\mathbb{R})$ for all $1 \le i \le m$, and $\nabla\boldsymbol{u}$ is its vector valued covariant derivative. ⊡

Remark A.30. Sometimes we use the notation of the scalar product of two vector valued tensors of different rank to express only the scalar product of the vector components. When we do so, we add a subscript to the scalar product. For instance, for a function $\boldsymbol{u} \in W^{1,2}(M,\mathbb{R}^m)$ we write

$$\langle \boldsymbol{u}, \nabla\boldsymbol{u}\rangle_{\mathbb{R}^m} = \sum_{i=1}^m \boldsymbol{u}_i \,\nabla\boldsymbol{u}_i\,,$$

which is now a (rough) 1-form on M. With this notation and the product rule, we obtain the following identity for the covariant derivative of the scalar product of two vector valued functions $\boldsymbol{u},\boldsymbol{v} \in W^{1,2}(M,\mathbb{R}^m)$:

$$\nabla\langle \boldsymbol{u},\boldsymbol{v}\rangle = \langle \nabla\boldsymbol{u},\boldsymbol{v}\rangle_{\mathbb{R}^m} + \langle \boldsymbol{u},\nabla\boldsymbol{v}\rangle_{\mathbb{R}^m}\,.$$ ⊡

By the equivalence of p-norms on $\mathbb{R}^m$, we can write an equivalent $W^{k,p}$-norm for vector valued functions $\boldsymbol{u} = (u_1,\dots,u_m)^T$ as

$$\|\boldsymbol{u}\|_{W^{k,p}(M,\mathbb{R}^m)} = \left(\sum_{i=1}^m \|u_i\|^p_{W^{k,p}(M,\mathbb{R})}\right)^{1/p},$$

and for $p=\infty$

$$\|\boldsymbol{u}\|_{W^{k,\infty}(M,\mathbb{R}^m)} = \max_{\substack{0\le j\le k\\ 1\le i\le m}} \operatorname*{ess\,sup}_{x\in M} |\nabla^j u_i(x)|\,.$$

We conclude that $W^{k,p}(M,\mathbb{R}^m)$ is isomorphic to $W^{k,p}(M,\mathbb{R}) \times \ldots \times W^{k,p}(M,\mathbb{R})$, m times. Consequently, the Sobolev spaces of vector valued functions on compact manifolds exhibit the same properties as the scalar spaces. In particular, $H^k(M,\mathbb{R}^m)$ is still a Hilbert space where the inner product is given by the integral over the bundle metric.

Moreover, by the equivalence of norms and the isometric isomorphism of the spaces, we can conclude that the Sobolev embeddings also hold for vector valued functions. Also, the Gagliardo–Nirenberg inequality carries over.

We extend the $*$-notation to elements $\boldsymbol{F} \in W^{k,p}_{(r,s)}(M,m)$ and $\boldsymbol{G} \in W^{k,p}_{(r',s')}(M,m)$ by setting

$$\boldsymbol{F} * \boldsymbol{G} = \sum_{1 \le i,j \le m} F_i * G_j + \sum_{j=1}^{m} \boldsymbol{F} * G_j + F_j * \boldsymbol{G}\,.$$

Note that this notation includes both the cases that whether $\boldsymbol{F} * \boldsymbol{G}$ is vector valued or not. However, if we are just interested on bounds on the respective norms, this notation is convenient to cover both cases. Then, by definition of the bundle metric on the m-th power of the tensor bundles, we conclude that $|\boldsymbol{F} * \boldsymbol{G}| \le C|\boldsymbol{F}||\boldsymbol{G}|$ still holds. Also the product rule $\nabla(\boldsymbol{F} * \boldsymbol{G}) = \nabla \boldsymbol{F} * \boldsymbol{G} + \boldsymbol{F} * \nabla \boldsymbol{G}$ can be easily verified. Hence, the vector version of Definition A.28 holds as well.

Proposition A.31. *Let M be a compact Riemannian manifold of dimension 2. Then for all $k, j, l \in \mathbb{N}$ and $\boldsymbol{F} \in H^k_{(r,s)} \cap L^\infty_{(r,s)}(M,m)$, $G \in H^k_{(r',s')} \cap L^\infty_{(r',s')}(M,m)$ we have*

$$\begin{aligned}
\|\nabla^k(\boldsymbol{F} * \boldsymbol{G})\|_{L^2} &\le C(\|\boldsymbol{F}\|_{L^\infty}\|\boldsymbol{G}\|_{H^k} + \|\boldsymbol{F}\|_{H^k}\|\boldsymbol{G}\|_{L^\infty})\,,\\
\|\boldsymbol{F} * \boldsymbol{G}\|_{H^k} &\le C(\|\boldsymbol{F}\|_{L^\infty}\|\boldsymbol{G}\|_{H^k} + \|\boldsymbol{F}\|_{H^k}\|\boldsymbol{G}\|_{L^\infty})\,,\\
\|\boldsymbol{F}^{*j}\|_{H^k} &\le C\|\boldsymbol{F}\|^{j-1}_{L^\infty}\|\boldsymbol{F}\|_{H^k}\\
\|\boldsymbol{F}^{*j} * \boldsymbol{G}^{*l}\|_{H^k} &\le C\|\boldsymbol{F}\|^{j-1}_{L^\infty}\|\boldsymbol{G}\|^{l-1}_{L^\infty}(\|\boldsymbol{F}\|_{H^k}\|\boldsymbol{G}\|_{L^\infty} + \|\boldsymbol{F}\|_{L^\infty}\|\boldsymbol{G}\|_{H^k})\,.
\end{aligned}$$

We also extend the divergence operator to vector valued tensor fields. Let $\boldsymbol{F} = (F_1, \ldots, F_m) \in \mathfrak{T}^{(r,s)}(M,m)$ be such a field, then its divergence is defined as

$$\operatorname{div} \boldsymbol{F} = (\operatorname{div} F_1, \ldots, \operatorname{div} F_m) \in \mathfrak{T}^{(r,s-1)}(M,m)\,.$$

This definition should not be confused with the divergence of a vector field in $\mathbb{R}^m$.
With this definition, the integration by parts rule holds also for vector valued fields. Let $\boldsymbol{F} \in \mathfrak{T}^{(r,s)}(M,m)$ and $\boldsymbol{G} \in \mathfrak{T}^{(r,s+1)}(M,m)$, the integration by parts formula reads

$$\int_M \langle \nabla \boldsymbol{F}, \boldsymbol{G}\rangle \mathrm{d}v(g) = -\int_M \langle \boldsymbol{F}, \operatorname{div} \boldsymbol{G}\rangle \mathrm{d}v(g)\,.$$

This follows from the definition of the tensor scalar product of vector valued functions and the integration by parts formula for scalar functions.

A.5. Sturm–Liouville theory: The Legendre operator

We consider the Sturm–Liouville problem of the associated Legendre equation for the value $m = 1$ in the variable $x = \cos\theta$ on the interval $(0, \pi)$. That is, we study the equation

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}h\right) - \frac{m^2}{\sin^2\theta}h = 0$$

for $m = 1$. This equation is well-known as the the azimuthal angle part of the Laplace equation after using separation of variables in spherical coordinates. We define the Legendre operator $\hat{\mathcal{L}}$ as

$$\hat{\mathcal{L}} = \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta} - \frac{1}{\sin^2\theta}\,.$$

In this section we want to briefly collect some properties of this operator and its eigenfunctions resulting from the Sturm–Liouville theory. Further details can be found in [71]. The Sturm–Liouville problem is to find the eigenvalues and eigenfunctions of this operator. It can be identified as a singular Sturm–Liouville problem since the weight function $\sin\theta$ vanishes at the endpoints of the interval. For general Sturm–Liouville problems this can pose problems such as a non-discrete spectrum of the operator. In our case however, it is well know that a complete set of eigenfunctions is given by the associated Legendre functions

$$\Psi_l(\theta) = P_l^1(\cos\theta) = \frac{1}{2^l l!}\sin\theta\frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}}(x^2-1)^l\big|_{x=\cos\theta}$$

for $l \in \mathbb{N}$. The corresponding discrete eigenvalues are given by

$$\hat{\mathcal{L}}\Psi_l = -l(l+1)\Psi_l\,.$$

Moreover, the eigenfunctions are orthogonal with respect to the inner product on the weighted L^2-space $L^2_{\sin}((0,\pi))$, which is defined as the space of square integrable functions with respect to the measure $\sin\theta\mathrm{d}\theta$ on the interval $(0,\pi)$. This space is a Hilbert space with inner product given by

$$\langle f, g\rangle_{L^2_{\sin}} = \int_0^\pi fg\sin\theta\mathrm{d}\theta\,.$$

The eigenfunctions are orthogonal in the sense that

$$\langle \Psi_l, \Psi_k\rangle_{L^2_{\sin}} = c_{lk}\delta_{lk}\,,$$

for some normalization constant c_{lk}. Moreover, they form a complete set of functions in the Hilbert space $L^2_{\sin}((0,\pi))$.

We want to show that the Legendre operator $\hat{\mathcal{L}}$ is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_{L^2_{\sin}}$. To this end, we first need to define which is the domain of the operator, since it is an unbounded operator. The maximal domain $D(\hat{\mathcal{L}})$ is defined by

$$D(\hat{\mathcal{L}}) := \left\{ f \in L^2_{\sin}((0,\pi)) : f, f'\sin \in \mathrm{AC}_{\mathrm{loc}}((0,\pi)), \quad \hat{\mathcal{L}} f \in L^2_{\sin}((0,\pi)) \right\},$$

where $\mathrm{AC}_{\mathrm{loc}}((0,\pi))$ is the space of locally absolutely continuous functions [71, cf. Part 4]. We want to show that being in this space already implies stronger properties of the functions than just being absolutely continuous. For example, note that we do not require any boundary conditions on the functions. First of all though, note that the space of smooth, compactly supported functions, $C_0^\infty((0,\pi))$, is a subset of $D(\hat{\mathcal{L}})$. Moreover, $C_0^\infty((0,\pi))$ is dense in $L^2_{\sin}((0,\pi))$. Hence, $D(\hat{\mathcal{L}})$ is dense in $L^2_{\sin}((0,\pi))$ as well. Hereinafter, we follow the exposition in [1] adapted to our case.

Proposition A.32. *The Legendre operator $\hat{\mathcal{L}}$ is self-adjoint on its domain $D(\hat{\mathcal{L}})$ with respect to the inner product $\langle\cdot,\cdot\rangle_{L^2_{\sin}}$.*

Proof. Let $\phi(\theta) = \sin(\theta)$. This is the first eigenfunction of the Legendre operator $\hat{\mathcal{L}}$ with eigenvalue -2. Also clearly, ϕ is in $D(\hat{\mathcal{L}})$.
Let $f, g \in D(\hat{\mathcal{L}})$. Then the Lagrange identity for the operator $\hat{\mathcal{L}}$ reads

$$g\hat{\mathcal{L}} f - f\hat{\mathcal{L}} g = \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\big(\sin\theta(f'g - fg')\big). \tag{A.16}$$

Setting $g = \phi$ and integrating over $(0,\pi)$ after multiplying by $\sin\theta$, we find

$$\int_0^\pi (\hat{\mathcal{L}} - 2) f\,\phi \sin\theta\,\mathrm{d}\theta = f'\sin^2\theta - f\sin\theta\cos\theta\Big|_0^\pi.$$

Now, since $f, \phi \in D(\hat{\mathcal{L}})$, we see that by the Cauchy–Schwarz inequality the left-hand side exists and thus, the limits on the right-hand side exist as well. We denote the limits by $\Phi_0(f)$ and $\Phi_\pi(f)$, respectively, i.e.,

$$\Phi_0(f) = \lim_{\theta\to 0}\Big(f'\sin^2\theta - f\sin\theta\cos\theta\Big)$$

and for $\Phi_\pi(f)$ likewise. We integrate the Lagrange identity again, but now from 0 to $\theta \in (0,\pi)$, which yields

$$\int_0^\theta (\hat{\mathcal{L}} - 2) f\ \sin^2\tau\,\mathrm{d}\tau = f'\sin^2\theta - f\sin\theta\cos\theta - \Phi_0(f). \tag{A.17}$$

Dividing by $\sin^3\theta$ and rearranging gives us

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\sin\theta}f\right) = \frac{\Phi_0(f)}{\sin^3\theta} + \frac{B(\theta)}{\sin\theta}\,, \tag{A.18}$$

where

$$B(\theta) = \frac{1}{\sin^2\theta}\int_0^\theta (\hat{\mathcal{L}}-2)f\sin^2\tau\,\mathrm{d}\tau\,.$$

We find the following bound for $B(\theta)$:

$$|B(\theta)| \leq \frac{1}{\sin^2\theta}\Big(\|\hat{\mathcal{L}}f\|_{L^2_{\sin}} + 2\|f\|_{L^2_{\sin}}\Big)\Big(\int_0^\theta \sin^3\tau\,\mathrm{d}\tau\Big)^{1/2} \leq C\Big(\|\hat{\mathcal{L}}f\|_{L^2_{\sin}} + 2\|f\|_{L^2_{\sin}}\Big)\,,$$

where $C > 0$ is global. Integrating (A.18), now from $t \in (0, \frac{\pi}{2})$ to $\frac{\pi}{2}$, gives us

$$f(t) = \left(f(\tfrac{\pi}{2}) - \int_t^{\frac{\pi}{2}} \frac{\Phi_0(f)}{\sin^3\theta}\,\mathrm{d}\theta - \int_t^{\frac{\pi}{2}} \frac{B(\theta)}{\sin\theta}\,\mathrm{d}\theta\right)\sin t\,.$$

The first integral can be evaluated to

$$\int_t^{\frac{\pi}{2}} \frac{\Phi_0(f)}{\sin^3\theta}\,\mathrm{d}\theta = \frac{\Phi_0(f)}{2}\left(\ln\tan\left(\frac{t}{2}\right) - \frac{\cos t}{\sin^2 t}\right),$$

whereas for the second integral we find using the antiderivative of $\sin^{-1}$ the bound

$$\Big|\int_t^{\frac{\pi}{2}} \frac{B(\theta)}{\sin\theta}\,\mathrm{d}\theta\Big| \leq C\Big(\|\hat{\mathcal{L}}f\|_{L^2_{\sin}} + 2\|f\|_{L^2_{\sin}}\Big)|\ln\tan\left(\frac{t}{2}\right)|\,. \tag{A.19}$$

Since

$$\ln\tan\left(\frac{t}{2}\right)\sin t \longrightarrow 0\,, \text{ as } t \to 0\,,$$

we see that we can write $f(t)$ as

$$f(t) = G(t) - \frac{\Phi_0}{2}\frac{\cos t}{\sin t}\,,$$

where G is a bounded function on $[0, \frac{\pi}{2}]$. Now, since $f \in L^2_{\sin}((0,\pi))$, we deduce that $\Phi_0(f) = 0$ must hold as $x \mapsto \cos x \sin^{-1} x$ is not square integrable on $(0,\pi)$. Thus, we can write

$$f(t) = f(\tfrac{\pi}{2})\sin t + \hat{G}(t)\,, \tag{A.20}$$

with

$$\hat{G}(t) = -\sin t\int_t^{\frac{\pi}{2}} \frac{B(\theta)}{\sin\theta}\,\mathrm{d}\theta\,,$$

which satisfies the following estimate as seen above

$$|\hat{G}(t)| \leq C\Big(\|\hat{\mathcal{L}}f\|_{L^2_{\sin}} + 2\|f\|_{L^2_{\sin}}\Big)\,|\ln\tan\Big(\frac{t}{2}\Big)|\,\sin(t)\,,$$

from which follows $\hat{G}(t) \to 0$ as $t \to 0$ and ultimately,

$$f(t) \longrightarrow 0\,, \text{ as } t \to 0\,.$$

Going back to (A.17), we assert that

$$f'(t)\sin t = f(t)\cos t + B(t)\sin t\,.$$

Recalling that B is bounded, we find that

$$f'(t)\sin t \longrightarrow 0\,, \text{ as } t \to 0\,.$$

We can repeat the same arguments for $t \to \pi$ by integrating (A.18) from $\frac{\pi}{2}$ to t to conclude that both

$$f(t) \longrightarrow 0 \text{ and } f'(t)\sin t \longrightarrow 0\,, \text{ as } t \to \pi\,.$$

Since $f \in D(\hat{\mathcal{L}})$ was arbitrary, we conclude that when integrating the Lagrange identity (A.16) the boundary terms vanish, i.e., for all $f, g \in D(\hat{\mathcal{L}})$ it holds

$$\langle \hat{\mathcal{L}}f, g\rangle_{L^2_{\sin}} - \langle f, \hat{\mathcal{L}}g\rangle_{L^2_{\sin}} = 0\,.$$

This implies that $\hat{\mathcal{L}}$ is symmetric. Let $\hat{\mathcal{L}}_0$ be the restriction of $\hat{\mathcal{L}}$ to $C_0^\infty((0,\pi))$. Then the adjoint operator $\hat{\mathcal{L}}_0^*$ is given by $\hat{\mathcal{L}}$. This implies due to the symmetry of $\hat{\mathcal{L}}$ the following inclusion in the sense of operators

$$\hat{\mathcal{L}} \subset \hat{\mathcal{L}}^{**} \subset \hat{\mathcal{L}}^* \subset \hat{\mathcal{L}}_0^* = \hat{\mathcal{L}}\,,$$

which shows that $\hat{\mathcal{L}}$ is self-adjoint. □

We collect the fact that the functions in $D(\hat{\mathcal{L}})$ vanish at the endpoints in the following statement.

Corollary A.33. *Let $f \in D(\hat{\mathcal{L}})$. Then it holds*

$$f(0) = f(\pi) = 0 \quad \textit{and} \quad \lim_{\theta\to 0} f'(\theta)\sin(\theta) = \lim_{\theta\to\pi} f'(\theta)\sin(\theta) = 0\,.$$

We define the graph norm on $D(\hat{\mathcal{L}})$ as

$$\|f\|_D^2 = \|f\|_{L^2_{\sin}}^2 + \|\hat{\mathcal{L}}f\|_{L^2_{\sin}}^2 .$$

Then we can show

Lemma A.34. *$D(\hat{\mathcal{L}})$ endowed with $\|\cdot\|_D$ is a Banach space.*

Proof. $D(\hat{\mathcal{L}})$ is dense in the Hilbert space $L^2_{\sin}((0,\pi))$. Then, the adjoint of $\hat{\mathcal{L}}$ is a closed operator but since $\hat{\mathcal{L}}$ is self-adjoint, we see that $\hat{\mathcal{L}}$ is a closed operator. The statement then follows from the definition of a closed operator. □

We conclude with two useful estimates for the functions in $D(\hat{\mathcal{L}})$.

Lemma A.35. *For any $p \geq 2$ and $f \in D(\hat{\mathcal{L}})$ we have*

$$\Big\|\frac{f}{\sin}\Big\|_{L^p_{\sin}} \leq C\|f\|_D ,$$

where the constant $C > 0$ depends only on p.

Proof. In the proof of Definition A.32 we have seen in (A.20) that for $\theta \in (0, \frac{\pi}{2})$ we can write

$$\frac{f(\theta)}{\sin(\theta)} = f(\tfrac{\pi}{2}) - \sin t \int_t^{\frac{\pi}{2}} \frac{B(\theta)}{\sin\theta}\,\mathrm{d}\theta .$$

For $\theta \in (\frac{\pi}{2}, \pi)$ we can get a similar expression. In any case, we find by the estimate in (A.19) that

$$\Big|\frac{f(\theta)}{\sin(\theta)}\Big|^p \leq C\|f\|_D^p \,|\ln\tan\Big(\frac{\theta}{2}\Big)|^p . \tag{A.21}$$

Now, for $\theta \to 0$ we have that $\ln\tan(\frac{\theta}{2})$ behaves like $\ln\theta$. Since the logarithm diverges slower than $\sin\theta$ vanishes, we assert that

$$|\ln\tan\Big(\frac{\theta}{2}\Big)|^p \sin\theta \longrightarrow 0\,, \text{ as } \theta \to 0\,,$$

which can be verified by L'Hospital's rule. For $\theta \to \pi$ we get the same result. Therefore, we conclude that the function

$$\theta \mapsto \Big|\frac{f(\theta)}{\sin(\theta)}\Big|^p \sin\theta$$

is bounded on $[0,\pi]$ and thus integrable. With the estimate (A.21) we also obtain the bound on the norm. □

Lemma A.36. *Let $f \in D(\hat{\mathcal{L}})$. Then for some global constant $C > 0$ it holds*

$$\|f\|_{L^\infty} \leq C\|f\|_D \,.$$

Proof. By Definition A.33, the function $ff' \sin$ vanishes at the endpoints of the interval $(0, \pi)$. Hence, by partial integration we find that

$$\langle -\hat{\mathcal{L}}f, f \rangle = \int_0^\pi f(-\hat{\mathcal{L}}f) \, \sin\theta \, \mathrm{d}\theta \, = \int_0^\pi f'^2 \sin\theta \, \mathrm{d}\theta + \int_0^\pi \frac{f^2}{\sin\theta} \, \mathrm{d}\theta \,,$$

from which we conclude by Young's inequality that

$$\|f'\|^2_{L^2_{\sin}} + \|\frac{f}{\sin}\|^2_{L^2_{\sin}} \leq \|f\|^2_{L^2_{\sin}} + \|\hat{\mathcal{L}}f\|^2_{L^2_{\sin}} = \|f\|^2_D \,.$$

This implies with $f(0) = 0$ that for $\theta \in (0, \pi)$ we have

$$\frac{1}{2} f(\theta)^2 = \int_0^\theta ff' \, \mathrm{d}\tau \leq \|f'\|^2_{L^2_{\sin}} + \|\frac{f}{\sin}\|^2_{L^2_{\sin}} \leq \|f\|^2_D \,.$$

The statement then follows immediately. □

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