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The connection of the acyclic disconnection and feedback arc sets – On an open problem of Figueroa et al.

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ABSTRACT

We examine the connection of two graph parameters, the size of a minimum feedback arcs set and the acyclic disconnection. A feedback arc set of a directed graph is a subset of arcs such that after deletion the graph becomes acyclic. The acyclic disconnection denotes the maximum number of colors that can be used in a vertex coloring such that after deletion of the monochromatic arcs the graph is acyclic. Figueroa et al. generalized the definitions of a feedback arc set and the acyclic disconnection to undirected graphs. They proposed a connection between the two parameters in the undirected setting. To be more precise, they showed that in wheel graphs and grid graphs the sum of the two parameters equals the number of vertices and claimed it to be true for outerplanar graphs. We extend their results by giving a characterization of graph classes for which the directed version of the proposed relation holds, answering an open question of Figueroa et al. We apply our characterization to almost trees (3) and wheel graphs. Moreover, we show that the proposed relation does not hold for outerplanar graphs neither in the directed nor in the undirected version. This proves Theorem 15 of Figueroa et al. to be wrong.

1. Introduction

We aim to study the connection of two graph parameters, the acyclic disconnection and the cardinality of a minimum feedback arc set. Figueroa et al. [8] propose a connection of these two measures and examine it in their paper. We address the two open problems stated in [8] and extend and correct the results.

Minimum feedback arc sets are a well-studied graph theoretical concept. Given a directed graph, a feedback arc set denotes a subset of arcs that leaves the graph acyclic after deletion. A minimum feedback arc set is such a set of minimum cardinality. The corresponding decision problem is one of Karp's original 21 $\mathcal{NP}$-complete problems [10]. Moreover, the problem remains $\mathcal{NP}$-hard when restricted to tournaments or bipartite tournaments [2,4]. Feedback arc sets are a broadly recognized solution concept when it comes to sorting inconsistent data. On tournament graphs, after multiple constant factor approximation algorithms [1,5,16] finally, Claire Kenyon-Mathieu and Warren Schu came up with a PTAS [11] for the minimum feedback arc set problem. On general graphs, there is an $O(\log n \log\log n)$-approximation algorithm [7,15], while the problem is known to be APX-hard [10,6], which means a large gap remains open.

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The less known acyclic disconnection was introduced in [13] as a generalization of the number of connected components of a graph. It denotes the maximal number of colors that can be used to color the vertices of a given directed graph such that every directed cycle contains a monochromatic arc, i.e., an arc where both end vertices are colored with the same color. In [12,14,9] the acyclic disconnection is studied for special families of digraphs like tournaments and multipartite tournaments and there are some attempts to connect it to other concepts in graph theory such as the dichromatic number [14] or the girth [3].

Since both - the feedback arc set and the acyclic disconnection - refer to the structure of cycles of a graph, Figueroa et al. [8] aimed to relate the feedback arc set and the acyclic disconnection. To state the proposed connection formally, we start with the definition of feedback arc sets and the acyclic disconnection.

1.1. Preliminaries and definitions

Given some undirected graph G, let $\mathcal{O}(G)$ denote the set of all possible orientations of G.

Definition 1 *(Feedback arc set).* Given a directed graph $D=(V,A)$. A *Feedback Arc Set* F of D is a subset of arcs, i.e., $F \subseteq A$, such that $(V, A\backslash F)$ is acyclic. We denote the minimum cardinality of such an arc set by

$$\tau_1(D) := \min_{F\subseteq A} |\{F \mid (V, A\backslash F) \text{ is acyclic}\}| .$$

Moreover, we define for an undirected graph G, $\tau_1(G)$ to be the maximum of $\tau_1(D)$ over all possible orientations D of G. More precisely, let

$$\tau_1(G) := \max_{D\in\mathcal{O}(G)} \tau_1(D).$$

Definition 2 *(Acyclic disconnection).* Given a directed graph $D=(V,A)$. An *externally acyclic coloring* of D is a surjective mapping $\varphi\colon V \to [\omega]$ for some $\omega \in \mathbb{Z}_{>0}$, such that $(V, A\setminus F)$ is acyclic, where $F := \{(v,w)\in A \mid \varphi(v)=\varphi(w)\}$ is the set of monochromatic arcs. The *acyclic disconnection* $\vec{\omega}(D)$ of a directed graph is the maximal ω, such that an externally acyclic coloring exists.

Moreover, the acyclic disconnection $\vec{\omega}(G)$ of an undirected graph G is defined as the minimum acyclic disconnection over all orientations of G. More precisely, let

$$\vec{\omega}(G) := \min_{D\in\mathcal{O}(G)} \vec{\omega}(D).$$

Figueroa et al. [8] examine the sum of the cardinality of the minimum feedback arc set and the acyclic disconnection for undirected graphs. They state that the sum of the two is equal to the number of vertices in the case of wheel graphs, grid graphs and outerplanar graphs, i.e. for G in one of these graph classes they claim

$$\tau_1(G) + \vec{\omega}(G) = |V(G)| . \tag{UC}$$

A graph class fulfills an even stronger condition if the following equation holds for any orientation D of a graph G in the graph class

$$\tau_1(D) + \vec{\omega}(D) = |V(G)| . \tag{DC}$$

Figueroa et al. propose two open problems to characterize graph classes fulfilling (UC) (Problem 1) and graph classes fulfilling (DC) (Problem 2), respectively. They suspect that both (UC) and (DC) hold for all planar graphs.

We will show that this is not the case, but that there exist some graph classes on which (UC) and (DC) do hold. One of them is almost trees (3), which we define in the following.

Definition 3 *(almost tree (k)).* A graph is an *almost tree* (k) for $k \in \mathbb{N}$, if for every 2-(vertex)-connected component C and every spanning tree T of C, there are at most k edges of C not contained in T, i.e. $|E(C)\setminus E(T)| \le k$.

1.2. Our contribution

We give a characterization of directed graphs that fulfill (DC), by linking it to the structure of minimal feedback arc sets. This gives an answer to Problem 2 posed in Figueroa et al. [8].

We apply our characterization to show that (DC) is satisfied by any orientation D of an almost tree (3) or a wheel graph. The result for almost trees is tight since there are almost trees (4) where (DC) is not fulfilled. Clearly, when (DC) is satisfied by every orientation of a graph G, (UC) holds as well. Thus, the results imply that (UC) holds for almost trees (3) and wheel graphs, where the latter was already shown in [8].

However, on the negative side it turns out that neither (DC) nor (UC) holds for all directed respectively undirected planar graphs. We give an example contradicting (UC) for outerplanar graphs, which proves Theorem 15 of Figueroa et al. [8] to be wrong. Moreover, the example implies that (DC) is not fulfilled for all outerplanar graphs, which contradicts the thought of Figueroa et al. that (DC) might hold for all planar graphs.

2. Characterization of directed graphs fulfilling (DC)

We start with some general statements. First, we show that the number of vertices is a lower bound on the sum of the acyclic disconnection and the size of a minimum feedback arc set. Then we give a characterization of graphs fulfilling (DC) which we apply in the subsequent section to several graph classes.

Let $D = (V, A)$ be a directed graph and $F \subseteq A$. We define $G[V, F]$ as the graph with vertices V and undirected edges corresponding to the arc set F. More explicit, for every arc $a = (v, w) \in F$, there is an edge $e = \{v, w\}$ in $G[V, F]$.

The *cycle rank* $r(G)$ of an undirected graph G denotes the minimal number of edges that needs to be deleted from G such that the graph becomes acyclic.

We restate the following observation which is already part of [8].

Observation 4 *([8], Remark 3). Let $D = (V, A)$ be a directed graph. A feedback arc set $F \subseteq A$ induces an externally acyclic coloring with ω colors, where ω is the number of connected components of $G[V, F]$.*

Lemma 5. *For any directed graph $D = (V, A)$,*

$$\tau_1(D) \geq |V| - \vec{\omega}(D).$$

Proof. Let F be a minimum feedback arc set, let ω be the number of connected components of $G[V, F]$, and let $\varphi\colon V \mapsto [\omega]$ be the coloring induced by F that exists by Observation 4. We get

$$\tau_1(D) = |F| \geq |V| - \omega \geq |V| - \vec{\omega}(D).$$

The first inequality holds by definition of ω as the number of connected components and since for any graph $G = (V, E)$ it holds that

$$0 \leq r(G) = |E| - |V| + \#\,\text{connected components of } G. \quad \square \tag{1}$$

Since the lemma can be applied to any orientation of an undirected graph, we obtain the following corollary.

Corollary 6. *For any graph $G = (V, E)$,*

$$\tau_1(G) \geq |V| - \vec{\omega}(G).$$

In our main theorem we connect (DC) to the structure of minimal feedback arc sets of a graph.

Theorem 7. *Let $D = (V, A)$ be a directed graph. Then, $\tau_1(D) + \vec{\omega}(D) = |V|$ if and only if for each minimal feedback arc set F of D it holds that $|F| \geq \tau_1(D) + r(G[V, F])$.*

Proof. First, assume there is a minimal feedback arc set F that violates the inequality from the theorem, i.e., $|F| < \tau_1(D) + r(G[V, F])$. Consider the coloring $\varphi\colon V \mapsto [\omega]$ induced by F that exists by Observation 4. Applying Equation (1), this yields

$$\vec{\omega}(D) \geq \omega = |V| - |F| + r(G[V, F]) > |V| - \tau_1(D).$$

This implies that $\tau_1(D) + \vec{\omega}(D) > |V|$ and thus the *only if* direction holds. Now, assume that $\tau_1(D) + \vec{\omega}(D) = |V|$ is violated. By Lemma 5 this means that $\tau_1(D) > |V| - \vec{\omega}(D)$. Let φ be an optimal externally acyclic coloring of D with $\vec{\omega}(D)$ colors. Let F be the set of all monochromatic arcs of D. Since φ is externally acyclic, F has to be a feedback arc set. Moreover the number of connected components of $G[V, F]$ is $\vec{\omega}(D)$ by Observation 4. Let $F' \subseteq F$ be a minimal feedback arc set contained in F. We claim that the number of connected components of $G[V, F']$ and $G[V, F]$ coincide. To see this, first note that the number of connected components in $G[V, F']$ is at least as large as in $G[V, F]$ since $F' \subseteq F$. Assume for contradiction that the number of connected components in $G[V, F']$ is strictly larger than in $G[V, F]$. By Observation 4 this implies the existence of an externally acyclic coloring with strictly more colors than $\vec{\omega}(D)$ which contradicts the maximality of $\vec{\omega}(D)$. Thus, the number of connected components of $G[V, F']$ is equal to $\vec{\omega}(D)$. Applying Equation (1), we conclude:

$$\tau_1(D) > |V| - \vec{\omega}(D) = \left|F'\right| - r(G[V, F']).$$

This yields that F' is a minimal feedback arc set but $\left|F'\right| < \tau_1(D) + r(G[V, F'])$. $\square$

Fig. 1. On the right an example of an externally acyclic coloring, where the red, square vertices form a color class, the blue, rhombe vertices form a second color class, and all white vertices form their own class. On the left a minimum feedback arc set for the same orientation is depicted with blue dashed edges.

From the theorem and the definition of $\tau_1(D)$, we obtain a necessary condition for a directed graph D to satisfy (DC).

Corollary 8. *Let D be a directed graph. If $\tau_1(D) + \vec{\omega}(D) = |V|$ then, the subgraph induced by each minimum feedback arc set of D is acyclic when disregarding arc directions.*

The reverse direction of the above corollary however does not hold, i.e. there are examples where each minimum feedback arc set of a directed graph D is acyclic when disregarding arc directions but still $\tau_1(D) > |V| - \vec{\omega}(D)$ as the following example shows.

Example 9. Let D be the graph on 12 vertices depicted in Fig. 1. Choosing a new color for each white vertex in the coloring of D in Fig. 1, we obtain an externally acyclic coloring with 8 colors, $\vec{\omega}(D) \geq 8$. The blue, dashed edges in Fig. 1 give a minimum feedback arc set, $\tau_1(D) = 5$. Moreover, every minimum feedback arc set of D is acyclic. However, we have $\tau_1(D) = 5 > 12 - 8 \geq 12 - \vec{\omega}(D)$.

3. Application to graph classes: almost trees, wheel graphs and outerplanar graphs

In this section we consider concrete graph classes and apply Theorem 7. We show that for any orientation of almost trees (3) and wheel graphs, (DC) holds. This directly implies that (UC) also holds for almost trees (3) and wheel graphs. The latter is already proven in Figueroa et al. [8] for wheel graphs. Concluding this section, we show that (UC) does not hold for outerplanar graphs.

Before we consider a concrete graph class, we state the following observation which allows to characterize possible cycles in a minimal feedback arc set. It follows directly from the minimality of F.

Observation 10. *Let F be a minimal feedback arc set of a digraph $D = (V, A)$ and $a \in F$. Then there exists a directed cycle C_a in D such that $F \cap C_a = \{a\}$.*

Proof. Assume that there exists an arc $a \in F$ such that for all cycles C in D, $(F \cap C) \setminus \{a\} \neq \emptyset$. Thus, $F \setminus \{a\}$ remains a feedback arc set. This is a contradiction to the minimality of F. □

3.1. Almost trees

In this subsection we show that (DC) holds for any orientation of almost trees (3), while there exists an almost tree (4) with an orientation violating (DC). The class of almost trees (3) includes graphs with a simple cyclic structure. Almost trees are essentially trees with a small number of additional edges. An almost tree (1) is also called a *cactus graph*. A formal definition of almost trees (k) was given in Definition 3.

Theorem 11. *Let D be an orientation of an almost tree (k). Then, we have $\tau_1(D) = |V| - \vec{\omega}(D)$ for $k \leq 3$. Moreover, there exists an oriented almost tree (4) D', such that $\tau_1(D') > |V| - \vec{\omega}(D')$.*

The following observation used in [8] allows us to focus on a 2-(vertex)-connected component instead of the whole graph and will be useful in the proof of Theorem 11.

Observation 12 *([8]). Let D be an orientation of a graph G. We can decompose every minimum (minimal) feedback arc set of D into minimum (minimal) feedback arc sets of the corresponding orientations of the 2-connected components of G. Similarly, every optimal externally acyclic coloring of D can be decomposed into optimal externally acylic colorings of the corresponding orientations of the 2-connected components of G.*

Also the interior dual of an outerplanar graph will be useful and is thus introduced here.

Definition 13 *(Interior dual).* An outerplanar 2-connected graph G has a unique outerplanar embedding. The interior dual T of G is a graph that has a vertex for each interior face of this embedding, i.e. for each face except the outer face. Two vertices of T are connected by an edge if the corresponding faces share an edge on their boundary. In particular, T is a tree.

Moreover, we will use the following lemma to characterize $\tau_1(G)$ for outerplanar graphs G by the *edge covering number* β'. Given a graph, the edge covering number denotes the minimal number of edges such that every vertex of the graph is adjacent to at least one of the edges.

Lemma 14 *(Lemma 13 [8]). Let G be an outerplanar 2-connected graph and let T be the interior dual of G. If $|V(T)| = 1$, then $\tau_1(G) = 1$. Otherwise,*

$$\tau_1(G) = \beta'(T).$$

Proof of Theorem 11. Let G be an almost tree (3). Because of Observation 12, we can analyze the 2-connected components of G separately. To be more precise, assume the graph is not 2-connected. Then there exists a vertex v such that deleting this vertex disconnects the graph and different connected components are attached to this vertex. We can consider each of these components separately and afterwards glue them back together at v. Gluing together means that the union of minimal feedback arc sets on the connected components is a minimal feedback arc set on the whole graph by Observation 12. Moreover, we get a maximum acyclic coloring by unifying maximum acyclic colorings of the connected components and identifying the color class at v. Thus, $\vec{\omega}(G)$ is the sum of the acyclic disconnection in the connected components minus the number of connected components minus one. Since the same is true for the vertices, it is enough to show the statement on 2-connected components. Thus, we can assume without loss of generality that G is 2-connected.

Consider an orientation D of G. If every minimal feedback arc set is acyclic, we can apply Theorem 7 to obtain the required equality. So assume F is a minimal feedback arc set of D and the subgraph induced by F contains an undirected cycle $C = \{v_1, .., v_\ell\}$ where for $i \in [\ell-1]$, $a_i \in F$ connects v_i and v_{i+1} and $a_\ell \in F$ connects v_ℓ and v_1. We use Observation 10 to understand the structure of G. It yields that there exist directed cycles $C_{a_i}, i \in [\ell]$ such that $C_{a_i} \cap F = \{a_i\}$. Let $P_i := C_{a_i} \setminus \{a_i\}$ be the path connecting the vertices v_i and v_{i+1} and in case of $i = \ell$ the vertices v_ℓ and v_1. Let H be the undirected subgraph of G consisting of C and the undirected equivalents of $C_{a_i}, i \in [\ell]$. Note that the subgraph of an almost tree (k) remains an almost tree (k). Thus, H is an almost tree (3). Moreover, observe that the undirected equivalent of $\bigcup_{i\in[\ell]} P_i$ spans the subgraph H and hence, contains a spanning tree T of H. We have for every $i \in [\ell]$ that $a_i \in H$ but $a_i \notin \bigcup_{i\in[\ell]} P_i$.[1] In particular, each a_i for $i \in [\ell]$ is not part of the spanning tree T which implies that $\ell \le 3$. Since C is a cycle, $\ell = 3$. From this, we can conclude that $\bigcup_{i\in[3]} P_i$ is actually equal to the spanning tree T. Otherwise, there would be more than three edges of H not contained in T, i.e. the edges a_1, a_2, a_3 and all edges that appear in $\bigcup_{i\in[3]} P_i$ but not in T. Furthermore, the undirected graph corresponding to H is equal to G. Assume that $H \neq G$. We can extend T to a spanning tree T' of G. Again, since $a_1, a_2, a_3 \notin T'$, there are no arcs beside a_1, a_2 and a_3 that are part of G but not T'. If G contains a vertex that is not in H this leads to a contradiction on the assumption that G is 2-connected. Thus, we observed that $G = H$.

In the following, we show that one of the paths P_1, P_2 or P_3 is equal to the union of the two other paths. Consider two of the three paths, say P_1 and P_2. We can assume without loss of generality that $P_1 \not\subset P_2$ and $P_2 \not\subset P_1$. A pair like this has to exist since otherwise there would be a hierarchy of the paths. Assume for contradiction, we have such a hierarchy, i.e., $P_1 \subset P_2 \subset P_3$. The inclusions are strict since any two paths differ in at least one endpoint. Both P_2 and P_3 have v_3 as an endpoint. Hence, $v_1 \notin P_2$. However, v_1 is one of the endpoints of P_1 and $P_1 \subset P_2$, which is a contradiction. Remember that $\bigcup_{i\in[3]} P_i$ is a spanning tree of $H = G$. Any two vertices of a tree are connected by a unique path. In $P_1 \cup P_2$, the endpoints v_1 and v_3 of P_3 are already connected. Thus, we can deduce that $P_3 \subseteq P_1 \cup P_2$. G has to be isomorphic to one of the two graphs on the left side in Fig. 2 when replacing arcs by undirected edges. Indeed, P_1 and P_2 share v_2 as an endpoint. Moreover, once the two paths split they won't share a vertex again. Otherwise, there would be a cycle in $T = P_1 \cup P_2$. This leaves us with two options for G, either P_1 and P_2 overlap in at least one edge (on the right in Fig. 2) or their edge sets are disjoint (on the left in Fig. 2). However, we can exclude the case where the edge sets of P_1 and P_2 intersect. Remember that the paths P_1, P_2 and P_3 are directed paths. Let u be the last vertex P_1 and P_2 have in common and let w_1 and w_2 be the next vertex of P_1 and P_2, respectively. In other words, w_1 lies on P_1 and not on P_2 and analogue for w_2. There are two options how to direct $\{u, w_1\}$ and $\{u, w_2\}$. Either as (u, w_1) and (u, w_2) or as (w_1, u) and (w_2, u). In both cases, we obtain a contradiction, as the segment (w_1, u, w_2) of P_3 cannot be directed consistently.

So we know that $P_3 = P_1 \cup P_2$ and that G is isomorphic to the graph depicted on the left in Fig. 2. Hence, G is an outerplanar graph. The interior dual of G has an edge covering number of 2 and by Lemma 14, this yields $\tau_1(G) = 2$. Furthermore, $F = \{a_1, a_2, a_3\}$ and thus $r(G[V, F]) = 1$. We can conclude that

$$|F| = 3 = \tau_1(D) + r(G[V, F]).$$

This allows to apply Theorem 7 which gives $\tau_1(D) = |V| - \vec{\omega}(D)$.

[1] Note that we abuse notation here and use a_i for the directed arc as well as for the undirected edges.

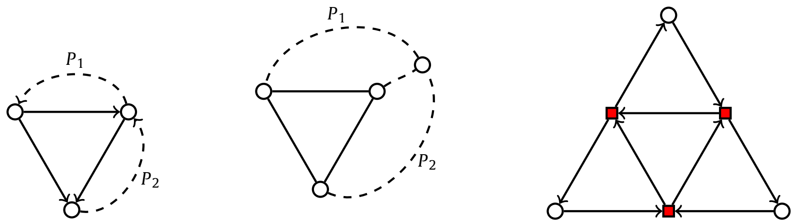

Fig. 2. On the left a directed almost tree (3) is depicted. Up to symmetry this is the only orientation such that a minimal feedback arc set can contain a cycle. On the right, an oriented almost tree (4) is depicted that does not fulfill (DC). The minimal feedback arc set consisting of the inner triangle violates the inequality from Theorem 7.

For $k = 4$ there exists an orientation such that (DC) is not fulfilled anymore. Consider the directed graph D' depicted on the right in Fig. 2. It consists of a triangle with an additional triangle per side, where each of those outer triangles forms a directed cycle. Again, we can determine the cardinality of a minimum feedback arc set using Lemma 14 since the graph is outerplanar and 2-connected: $\tau_1(D') = 3$. From coloring the inner triangle monochromatic and all remaining vertices with a different color, we get that $\vec{\omega}(D') \geq 4$ while $|V| = 6$. $\square$

For $k = 3$, we obtain the following result regarding (UC) as a direct consequence.

Corollary 15. *Let G be an almost tree (3). Then,* $\tau_1(G) = |V| - \vec{\omega}(G)$.

In Theorem 17 we present an almost tree (7) which does not fulfill (UC). For $k \in \{4, 5, 6\}$, we leave it open whether (UC) holds for almost trees (k).

3.2. Wheel graphs

Let W_n denote the wheel graph with n spokes. Figueroa et al. [8] proved that $\tau_1(W_n) = |V| - \vec{\omega}(W_n)$. We will apply Theorem 7 to improve upon this result and show that the equality actually holds for every orientation D of W_n.

Theorem 16. *For every orientation D of the wheel graph W_n with $n \geq 2$ spokes,*

$$\tau_1(D) = |V| - \vec{\omega}(D).$$

Proof. Consider an orientation D of W_n and a corresponding minimal feedback arc set F. We will show that $|F| \geq \tau_1(D) + r(G[V, F])$ and apply Theorem 7. If F contains no cycles, we can always apply Theorem 7 by definition of a minimal feedback arc set. Thus we may assume that F contains a cycle. We analyze the cycles that can be contained in F. First assume F contains the arcs of the outer cycle C_n as depicted in blue in Fig. 3 for the graph W_8. By Observation 10, F is actually equal to this set of arcs and moreover, every internal face of D has to be bounded by a directed cycle, see Fig. 3 for a possible realization of the orientation D. Note that C_n cannot be a directed cycle in D. Thus, by removing either the arcs pointing from the center vertex or the arcs pointing to the center vertex we obtain a feedback arc set of size at most $n/2$. Hence,

$$|F| = n \geq n/2 + 1 \geq \tau_1(D) + r(G[V, F]).$$

Next, assume that C_n is not contained in F. Thus, every cycle in F consists of a path P on the outer vertices and two arcs connecting the ends of the path to the center vertex. Let C be such a cycle with corresponding path P, see Fig. 3 for an example. Note that by the minimality of F, there cannot be an arc in F that connects the center vertex to an inner vertex of P. This implies, that the cycles in $G[V, F]$ are edge disjoint and furthermore, that the number of cycles in $G[V, F]$ is equal to the cycle rank $r(G[V, F])$. Let v be an endpoint of the path P. Let a be the arc incident to v on the path P and c the arc connecting v to the center vertex. By using Observation 10 again, there have to be two directed cycles C_a and C_c such that $F \cap C_a = \{a\}$ and $F \cap C_c = \{c\}$. Both directed cycles have to use the remaining arc b incident to v. Without loss of generality, we can assume that b is directed toward v. Thus, a and c are leaving v and b is the only arc entering v. This implies that every directed cycle containing arc a or c has to contain arc b as well. Hence, $F' = (F \setminus \{a, c\}) \cup \{b\}$ is a feedback arc set of D. We can modify F in this way for every cycle in $G[V, F]$ to obtain a feedback arc set F' of size $|F'| \leq |F| - r(G[V, F])$. Thus, again we can apply Theorem 7, since we have

$$|F| \geq |F'| + r(G[V, F]) \geq \tau_1(D) + r(G[V, F]). \quad \square$$

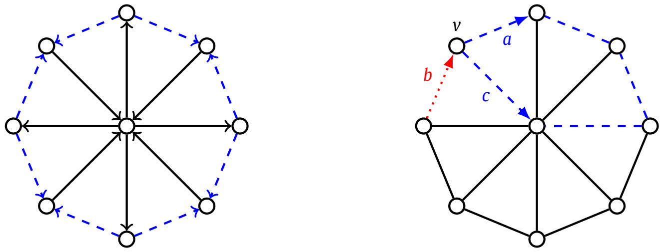

Fig. 3. On the left the outer cycle C_8 of the wheel graph W_8 is depicted and on the right a cycle inside a feedback arc set of the wheel graph W_8.

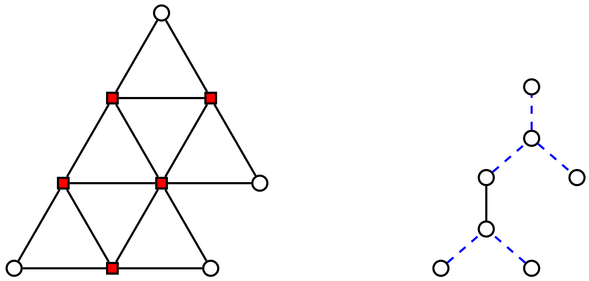

Fig. 4. On the left an externally acyclic coloring of the graph is depicted. Here the red, square vertices form a color class, while every white vertex corresponds to a different color class. On the right a minimum edge covering of the interior dual T is depicted with blue, dashed edges. In outerplanar graphs this was shown to correspond to a minimum feedback arc set.

3.3. Outerplanar graphs

In this section, we show that (UC) is not satisfied by outerplanar graphs. This contradicts Theorem 15 in [8] that states $\tau_1(G) = |V| - \vec{\omega}(G)$.

Let G be a 2-connected outerplanar graph with $|V(T)| > 1$, where T is the interior dual of G. By Lemma 14 and Corollary 6, we get that $\vec{\omega}(G) \geq |V| - \beta'(T)$. In the proof sketch for Lemma 14/Theorem 15 in [8], Figueroa et al. claim that a specific orientation (a so-called whirlpool orientation, where every interior face is bounded by a directed cycle) achieves the lower bound $\vec{\omega}(D) = |V| - \beta(T)$. However, this is not true as the following theorem shows.

Theorem 17. *There exists an outerplanar graph G on* 9 *vertices such that* $\tau_1(G) > 9 - \vec{\omega}(G)$.

Proof. Let G be the outerplanar graph and φ the corresponding coloring shown on the left in Fig. 4, where we choose a new color for each white vertex, while coloring all red vertices with the same color. After removing all monochromatic edges, the resulting undirected graph is acyclic. Thus, φ is externally acyclic for every orientation of G. Since φ uses 5 colors we can conclude that $\vec{\omega}(G) \geq 5$. On the other hand, the right side of Fig. 4 depicts the interior dual T of G with a minimum edge cover in blue. By Lemma 14, we have that $\tau_1(G) = \beta'(T) = 5 > 4 \geq |V| - \vec{\omega}(G)$. $\square$

Declaration of competing interest

The authors have no financial or personal relationships that could influence this work.

Data availability

No data was used for the research described in the article.

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