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Asymptotic behavior of renewal processes with random time

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ABSTRACT

For renewal processes with random time, extensions of well-known asymptotic relations such as the elementary renewal theorem, Blackwell's renewal and Smith's key renewal theorem are given. Whereas, in these results, time tends to infinity, the limits here are taken with respect to a sequence of parameters of respective random-time distributions satisfying some condition.

1. Introduction

In renewal theory, the elementary renewal theorem, Blackwell's renewal theorem and Smith's key renewal theorem are well known as fundamental asymptotic results. For introductions to renewal theory and properties of renewal processes, we refer to, e.g., Feller (1971), Kulkarni (2017), Mitov and Omey (2014) and Pinsky and Karlin (2011). We consider extensions of such asymptotic results to renewal processes with random time in this work, where the time is independent of the respective renewal process and the limits are taken with respect to sequences of parameters of random-time distributions.

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of non-negative, independent and identically distributed (iid) random variables on a probability space $(\Omega, \mathfrak{A}, P)$ with cumulative distribution function (cdf) F, where F is non-degenerate at 0, and let $\mu = E(X_1)$, where $1/\mu$ is interpreted as zero if $\mu = \infty$, throughout. Then, the sequence of occurrence times $(S_n)_{n\in\mathbb{N}_0}$, where $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$, $n \in \mathbb{N}$, defines a renewal process, and the corresponding renewal counting process $(N(t))_{t\geq 0}$ is defined by $N(t) = \sum_{n=1}^\infty \mathbb{1}_{[0,t]}(S_n)$, $t \geq 0$. The random variables $X_1, X_2, \ldots$ are called interoccurrence times. The occurrence times and counting variables are linked via the equivalence $N(t) \geq n \iff S_n \leq t$ for all $n \in \mathbb{N}_0$ and $t \geq 0$. The renewal function $E(N(\cdot))$ is given by $E(N(t)) = \sum_{n=1}^\infty F^{*n}(t)$, $t \geq 0$, where F^{*n} denotes the n-fold convolution of F. $E(N(t))$ is known to be finite for every $t \geq 0$. Asymptotic results where the time t tends to infinity are of major interest in renewal theory; such as the basic ones — the elementary renewal theorem, Blackwell's renewal theorem and Smith's key renewal theorem with a wide range of applications.

All asymptotic relations throughout will be stated for non-delayed renewal processes as defined above. Nonetheless, the results in this article can also be expressed in terms of delayed renewal processes for which we refer to a detailed discussion in Rauwolf (2023).

In the following, let $(T_\vartheta)_{\vartheta\in\Theta}$ be a collection of non-negative random variables on the same probability space as the renewal process, where T_ϑ, which is referred to as a random (inspection) time, has a left-continuous cdf G_ϑ with $G_\vartheta(x) = P(T_\vartheta < x)$, $x \geq 0$, and $P(T_\vartheta < \infty) = 1$. The parametric family of distributions $\mathcal{P} := \{G_\vartheta | \vartheta \in \Theta\}$, $\Theta \subseteq \mathbb{R}^k$ for some $k \in \mathbb{N}$, is assumed to be identifiable.

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Throughout, $(T_\vartheta)_{\vartheta\in\Theta}$ are assumed to be independent of the underlying renewal process. Such random inspection times have been previously discussed in, e.g., Cox (1962) who presents properties of the number of occurrences up to a random time and in Herff et al. (1997); see also Liu and Peña (2016). Rauwolf and Kamps (2023) study the inspection paradox in a random-time version. Moreover, in Badía and Cha (2013), Badía and Sangüesa (2015), Salehi et al. (2012) and the references therein, results can be found on the preservation of aging properties in connection with a random time. Kamps and Rauwolf (2023) point out a connection with record values. Asmussen et al. (1999) study the asymptotic behavior of $P(N(T) > k)$ as $k \to \infty$, where the random variable T satisfies certain additional assumptions.

We consider sequences of random times $(T_{\vartheta_i})_{i\in\mathbb{N}}$, $T_{\vartheta_i} \sim G_{\vartheta_i}$, with $\vartheta_i \in \Theta$, $i \in \mathbb{N}$. Via the choice $\mathcal{P} = \{\mathbb{1}_{(t,\infty)}(\cdot) | t \in [0,\infty)\}$, i.e., the family of degenerate distributions, all results for a deterministic time t (tending to infinity) are included in this setup.

Section 2 contains some motivation and preparatory considerations as well as the main assumption imposed on the parameter sequences of the random-time distributions along with two lemmas of transfer. Extensions of Blackwell's renewal theorem and Smith's key renewal theorem are subject matter of Section 3. Random-time versions of the elementary renewal theorem and asymptotic results regarding the variance of the number of renewals up to a random time are given in Section 4.

2. Preparatory considerations

We start with two motivating examples in the situation of common renewal processes and then present a helpful tool needed for the transfer of specific types of asymptotic relations to the setting with random time. As a first "random-time limit" analogue of Blackwell's renewal theorem, an asymptotic expression in terms of a sequence of parameters for which the corresponding expected values $E(T_{\vartheta_i})$ tend to infinity as $i \to \infty$ is derived.

Example 2.1. Let $\mathcal{P} = \{G_\eta | \eta \in (0,\infty)\}$ with $G_\eta(t) = 1 - e^{-\eta t}$, $t \geq 0$, be the family of exponential distributions. For $\alpha \geq 0$ and for $T_\eta \sim G_\eta$, we obtain

$$E(N(T_\eta+\alpha)) - E(N(T_\eta)) = \int_\alpha^\infty E(N(z))\eta e^{-\eta(z-\alpha)}dz - E(N(T_\eta)) = (e^{\eta\alpha}-1)E(N(T_\eta)) - e^{\eta\alpha}\int_0^\alpha E(N(z))\eta e^{-\eta z}dz.$$

Using the Laplace–Stieltjes transform $\hat{F}(\eta) = \int_0^\infty e^{-\eta x}dF(x)$ of F, the renewal function at random time can be expressed as $E(N(T_\eta)) = \frac{\hat{F}(\eta)}{1-\hat{F}(\eta)}$ (see, e.g., Cox, 1962, pp. 42–44). An asymptotic result is easily obtained by taking a sequence of parameters $(\eta_i)_{i\in\mathbb{N}} \subseteq (0,\infty)$ for which $E(T_{\eta_i}) \to \infty$ as $i \to \infty$ which means requiring $\eta_i \to 0$ as $i \to \infty$. In analogy to Blackwell's renewal theorem, we have

$$\lim_{i\to\infty}\left[E(N(T_{\eta_i}+\alpha)) - E(N(T_{\eta_i}))\right] = \lim_{i\to\infty}(e^{\eta_i\alpha}-1)\frac{\hat{F}(\eta_i)}{1-\hat{F}(\eta_i)} = \frac{\alpha}{\mu}$$

for every $\alpha \geq 0$ by applying L'Hôspital's rule. It should be noted that there is no restriction on the cdf F of the interoccurrence times.

For distributions other than the exponential distribution, a direct calculation of this limit will not be possible, in general. Furthermore, the following example shows that a condition stronger than expected random times tending to infinity is needed in order to obtain a general (w.r.t. F) analogue in random time to Blackwell's renewal theorem.

Example 2.2. Let F be non-arithmetic. We consider random inspection times T_b, $b > 0$, from the family of twopoint distributions $\mathcal{P} = \{G_b | b \in (0,\infty)\}$ with $G_b(t) = (1-p)\mathbb{1}_{(0,b]}(t) + \mathbb{1}_{(b,\infty)}(t)$, $t \geq 0$, where $p \in (0,1)$ is fixed. Then, $E(N(T_b)) = pE(N(b))$ and $E(N(T_b+\alpha)) = (1-p)EN(\alpha) + pEN(b+\alpha)$ holds for all $b > 0$ and $\alpha \geq 0$. Thus, considering the sequence of parameters $(b_i)_{i\in\mathbb{N}} \subseteq (0,\infty)$ with $b_i \to \infty$ as $i \to \infty$ (for which $E(T_{b_i}) \to \infty$ as $i \to \infty$), we find

$$\lim_{i\to\infty}\left[E(N(T_{b_i}+\alpha)) - E(N(T_{b_i}))\right] = p\frac{\alpha}{\mu} + (1-p)E(N(\alpha)),$$

which, however, is not equal to α/μ in general as Blackwell's theorem might suggest. Note that equality holds if X_1 has an exponential distribution.

In order to establish a random-time analogue of Blackwell's renewal theorem valid for arbitrary interoccurrence-time distributions, a suitable condition to be imposed on a sequence $(T_{\vartheta_i})_{i\in\mathbb{N}}$ of random times is its convergence in probability to infinity.

Lemma 2.3 (*Lemma of Transfer*). *Let $(T_{\vartheta_i})_{i\in\mathbb{N}}$ be a sequence of random inspection times with $(\vartheta_i)_{i\in\mathbb{N}} \subseteq \Theta$ such that*

$$\lim_{i\to\infty} P(T_{\vartheta_i} > x) = 1 \quad \text{for all } x > 0. \tag{T}$$

Then, for a measurable and locally bounded function $\phi : [0,\infty) \to \mathbb{R}$, i.e., a function bounded on every compact interval $[a,b] \subset [0,\infty)$,

$$\lim_{t\to\infty}\phi(t) = c \in (-\infty,\infty) \quad \Longrightarrow \quad \lim_{i\to\infty} E(\phi(T_{\vartheta_i})) = c.$$

Proof. Due to the assumptions, $\phi(T_{\vartheta_i})$ converges to c in probability as $i \to \infty$, and thus the assertion follows from the dominated convergence theorem. □

Lemma 2.3 thus states: Whenever an expression containing a random inspection time may be written in the form $E(\phi(T_\vartheta)) = \int_0^\infty \phi(t) dG_\vartheta(t)$ and whenever a sequence of parameters exists in Θ for which (T) is satisfied, then a limit result in renewal theory which is of the form $\phi(t) \xrightarrow{t\to\infty} c$ extends to the random-time setting with the same limit. Condition (T) is satisfied in Example 2.1 but not in Example 2.2. In the following example with Weibull distributions, we see that (T) may or may not be satisfied depending on the choice of parameter sequences.

Example 2.4. Consider the family $\mathcal{P}$ of Weibull distributions with distribution functions $G_{(\alpha,\lambda)}(t) = 1 - e^{-\lambda t^\alpha}$, $t \geq 0$, $\alpha, \lambda > 0$. Then, for α fixed and a sequence $(\lambda_i)_{i\in\mathbb{N}}$ tending to zero, $E(T_{\lambda_i}) \to \infty$, $i \to \infty$ and (T) is valid. However, for some $\lambda < 1$ fixed and a sequence $(\alpha_i)_{i\in\mathbb{N}}$ tending to zero, the expected values $E(T_{\alpha_i})$ tend to infinity as $i \to \infty$, although (T) is not fulfilled.

The following simple lemma turns out to be useful in Section 4 to derive a random-time version of the elementary renewal theorem as well as the limit behavior of $Var(N(T_{\vartheta_i}))$ normalized by $E(T_{\vartheta_i})$ or $Var(T_{\vartheta_i})$ as $i \to \infty$.

Lemma 2.5. *Let $(T_{\vartheta_i})_{i\in\mathbb{N}}$ be a sequence of random inspection times with $(\vartheta_i)_{i\in\mathbb{N}} \subseteq \Theta$ such that $E(T_{\vartheta_i}) < \infty$, $i \in \mathbb{N}$, and $\lim_{i\to\infty} E(T_{\vartheta_i}) = \infty$. Then, for a measurable and locally bounded function $\gamma : [0,\infty) \to \mathbb{R}$, we find*

$$\lim_{t\to\infty} \frac{\gamma(t)}{t} = c \in (-\infty, \infty) \quad \Longrightarrow \quad \lim_{i\to\infty} \frac{E(\gamma(T_{\vartheta_i}))}{E(T_{\vartheta_i})} = c.$$

3. Blackwell's renewal theorem and Smith's key renewal theorem with random time

Based on the considerations in Section 2, "random-time analogues" of Blackwell's renewal theorem and Smith's key renewal theorem are given in Theorems 3.1 and 3.2, respectively, both of which can be derived directly by an application of Lemma 2.3. It can be shown that, for random inspection times and non-arithmetic renewal processes, Blackwell's renewal theorem also implies Smith's key renewal theorem and vice versa (cf. Rauwolf, 2023), as is well known for the classical theorems.

Theorem 3.1. *Let $(T_{\vartheta_i})_{i\in\mathbb{N}}$ be a sequence of random inspection times with $(\vartheta_i)_{i\in\mathbb{N}} \subseteq \Theta$ such that $E(T_{\vartheta_i}) < \infty$, $i \in \mathbb{N}$, and Condition (T) is satisfied.*

(i) If F is non-arithmetic, then $\lim_{i\to\infty} \left[E(N(T_{\vartheta_i} + \alpha)) - E(N(T_{\vartheta_i}))\right] = \frac{\alpha}{\mu}$ for all $\alpha \geq 0$.

(ii) If F is d-arithmetic, then $\lim_{i\to\infty} \left[E(N(T_{\vartheta_i} + nd)) - E(N(T_{\vartheta_i}))\right] = \frac{nd}{\mu}$ for all $n \in \mathbb{N}_0$.

Proof. (i) The function $\phi : t \mapsto E(N(t+\alpha) - N(t))$, $t \geq 0$, is locally bounded. Furthermore, due to Blackwell's renewal theorem in the non-arithmetic case, $\lim_{t\to\infty} \phi(t) = \frac{\alpha}{\mu}$. Applying Lemma 2.3 then leads to

$$\lim_{i\to\infty} \left[E(N(T_{\vartheta_i} + \alpha)) - E(N(T_{\vartheta_i}))\right] = \lim_{i\to\infty} E(\phi(T_{\vartheta_i})) = \lim_{i\to\infty} \int_0^\infty \phi(t) dG_{\vartheta_i}(t) = \frac{\alpha}{\mu}.$$

(ii) Follows analogously. □

Obviously, the classical Blackwell theorem is contained by choosing degenerate distributions in $t_i > 0$, $i \in \mathbb{N}$, for the random times $(T_{t_i})_{i\in\mathbb{N}}$ and letting $\lim_{i\to\infty} t_i = \infty$. A version of Smith's key renewal theorem can be stated for random time in a similar way by utilizing Condition (T).

Theorem 3.2. *Let $(T_{\vartheta_i})_{i\in\mathbb{N}}$ be a sequence of random inspection times with $(\vartheta_i)_{i\in\mathbb{N}} \subseteq \Theta$ such that $E(T_{\vartheta_i}) < \infty$, $i \in \mathbb{N}$, and Condition (T) is satisfied.*

(i) If F is non-arithmetic and $h : \mathbb{R} \to \mathbb{R}$ is a function which is zero for negative arguments and is directly Riemann integrable on $[0,\infty)$, then

$$\lim_{i\to\infty} \int_0^\infty E(h(T_{\vartheta_i} - x)) dE(N(x)) = \frac{1}{\mu} \int_0^\infty h(y) dy.$$

(ii) If F is d-arithmetic, T_{ϑ_i} have distributions on $\mathbb{N}_0$ and if $h : \mathbb{R} \to [0,\infty)$ is a function which is zero for negative arguments and satisfies $\sum_{k=0}^\infty h(kd) < \infty$, then

$$\lim_{i\to\infty} \sum_{n=1}^\infty E(h(T_{\vartheta_i} d - S_n)) = \frac{d}{\mu} \sum_{k=0}^\infty h(kd).$$

Proof. (i) We apply Lemma 2.3 to the function $\phi(t) = \int_0^t h(t-x)dE(N(x))$, $t \geq 0$, which converges to $\frac{1}{\mu}\int_0^\infty h(y)dy$ for $t \to \infty$ by Smith's key renewal theorem. Then,

$$\int_0^\infty E(h(T_{\vartheta_i} - x))dE(N(x)) = \int_0^\infty \int_x^\infty h(t-x)dG_{\vartheta_i}(t)dE(N(x)) = \int_0^\infty \phi(t)dG_{\vartheta_i}(t)$$

and hence the proof. (ii) follows by conditioning on S_n and on T_{ϑ_i} and applying Smith's key renewal theorem as well as Lemma 2.3. □

In the particular case of choosing the sequence of degenerate distributions at $i \in \mathbb{N}$, the assertion of Theorem 3.2 (ii) that $\lim_{i\to\infty}\sum_{n=1}^\infty E(h(id - S_n)) = \frac{d}{\mu}\sum_{k=0}^\infty h(kd)$ coincides with the classical form of Smith's key renewal theorem in the arithmetic case. As is the case for the classical result with deterministic time t, the random-time analogue to Smith's key renewal theorem 3.2 has a wide range of applications. Within renewal theory, some applications include asymptotic results for distributions and expected values of quantities such as the age $U(T) = T - S_{N(T)}$ and the residual lifetime $V(T) = S_{N(T)+1} - T$ for some random time T. We refer to Rauwolf (2023) for details.

4. The elementary renewal theorem and the variance of $N(T)$ with random time

By applying Lemma 2.3, a random-time analogue to the elementary renewal theorem is given by $E\left(N(T_{\vartheta_i})/T_{\vartheta_i}\right) \overset{i\to\infty}{\longrightarrow} 1/\mu$, assuming $(\vartheta_i)_{i\in\mathbb{N}}$ is a sequence of parameters satisfying (T). However, it suffices to consider a sequence of random inspection times for which only the expectations tend to infinity in order to derive another version of the elementary renewal theorem. This version is given in Corollary 4.1, the proof of which is obvious by means of Lemma 2.5.

Corollary 4.1. *Let $(T_{\vartheta_i})_{i\in\mathbb{N}}$ be a sequence of random inspection times with $(\vartheta_i)_{i\in\mathbb{N}} \subseteq \Theta$ such that $E(T_{\vartheta_i}) < \infty$, $i \in \mathbb{N}$, and $\lim_{i\to\infty} E(T_{\vartheta_i}) = \infty$. Then,*

$$\lim_{i\to\infty}\frac{E(N(T_{\vartheta_i}))}{E(T_{\vartheta_i})} = \frac{1}{\mu}.$$

The above random-time version of the elementary renewal theorem was mentioned in Herff et al. (1997) under the stronger condition $E(X_1^2) < \infty$. Both versions of the elementary renewal theorem given above coincide with the classical version when choosing the family $\mathcal{P} = \{\mathbb{1}_{(t,\infty)}(\cdot) \mid t > 0\}$ of degenerate distributions.

The limit behavior of the variance of $N(T_{\vartheta_i})$ as $i \to \infty$ is of interest as well. Considering the variance of $N(t)$ for some $t > 0$ in a common renewal process, it is well known that it behaves asymptotically linear as $t \to \infty$ in the sense

$$\lim_{t\to\infty}\frac{Var(N(t))}{t} = \frac{Var(X_1)}{\mu^3}, \tag{1}$$

assuming $E(X_1^2) < \infty$ (see, e.g., Pinsky and Karlin, 2011). For a random inspection time T_ϑ, we note that the second moment $E(N(T_\vartheta)^2)$ is finite if $E(T_\vartheta^2) < \infty$ due to

$$\begin{aligned} E(N(T_\vartheta)^2) &= \int_0^\infty E(N(t)^2)dG_\vartheta(t) = \int_0^\infty \left[2\int_0^t E(N(t-y))dE(N(y)) + E(N(t))\right]dG_\vartheta(t) \\ &\leq 2\int_0^\infty [E(N(t))]^2 dG_\vartheta(t) + E(N(T_\vartheta)), \end{aligned}$$

which, by the elementary renewal theorem, is seen to be finite if $E(T_\vartheta^2) < \infty$.

The behavior of the variance of $N(T_\vartheta)$ is influenced primarily by the way the random inspection time behaves. In particular, limits of $Var(N(T_{\vartheta_i}))/E(T_{\vartheta_i})$ and $Var(N(T_{\vartheta_i}))/Var(T_{\vartheta_i})$ as $i \to \infty$ can be derived with respect to the asymptotic behavior of $Var(T_{\vartheta_i})/E(T_{\vartheta_i})$ as $i \to \infty$ by applying Lemmas 2.3 and 2.5.

Theorem 4.2. *Assume F to be non-arithmetic and $E(X_1^2) < \infty$. Let $(T_{\vartheta_i})_{i\in\mathbb{N}}$ be a sequence of random inspection times with $(\vartheta_i)_{i\in\mathbb{N}} \subseteq \Theta$ such that $E(T_{\vartheta_i}^2) < \infty$, $i \in \mathbb{N}$, and Condition (T) is satisfied. Moreover, let $\lim_{i\to\infty} Var(T_{\vartheta_i})/E(T_{\vartheta_i}) = c$, say.*

(i) If $c = 0$, then

$$\lim_{i\to\infty}\frac{Var(N(T_{\vartheta_i}))}{E(T_{\vartheta_i})} = \frac{Var(X_1)}{\mu^3}.$$

(ii) If $c \in (0, \infty]$, then

$$\lim_{i\to\infty}\frac{Var(N(T_{\vartheta_i}))}{Var(T_{\vartheta_i})} = \frac{1}{\mu^2} + \frac{Var(X_1)}{\mu^3}\frac{1}{c} \quad \left(\text{with } \frac{1}{c} = 0 \text{ if } c = \infty\right).$$

Proof. By the law of total variance,

$$Var(N(T_{\vartheta_i})) = E(Var(N(T_{\vartheta_i})|T_{\vartheta_i})) + Var(E(N(T_{\vartheta_i})|T_{\vartheta_i})),$$

where, using (1) and Lemma 2.5, the first summand satisfies

$$\lim_{i\to\infty} \frac{E(Var(N(T_{\vartheta_i})|T_{\vartheta_i}))}{E(T_{\vartheta_i})} = \frac{Var(X_1)}{\mu^3}.$$

To derive the limit behavior of the second summand, we use the representation

$$\begin{aligned} & Var(E(N(T_{\vartheta_i})|T_{\vartheta_i})) - \frac{1}{\mu^2}Var(T_{\vartheta_i}) \\ = & \int_0^\infty E^2(N(t))dG_{\vartheta_i}(t) - E^2(N(T_{\vartheta_i})) - \frac{1}{\mu^2}E(T_{\vartheta_i}^2) + \frac{1}{\mu^2}E^2(T_{\vartheta_i}) \\ = & \int_0^\infty \left(E(N(t)) - \frac{t}{\mu}\right)^2 dG_{\vartheta_i}(t) - \left(E(N(T_{\vartheta_i})) - \frac{E(T_{\vartheta_i})}{\mu}\right)^2 + 2\int_0^\infty \left(E(N(t)) - \frac{t}{\mu}\right)\frac{t}{\mu}dG_{\vartheta_i}(t) - 2\left(E(N(T_{\vartheta_i})) - \frac{E(T_{\vartheta_i})}{\mu}\right)\frac{E(T_{\vartheta_i})}{\mu}. \end{aligned}$$

The second order property of $E(N(\cdot))$, i.e., $E(N(t)) - t/\mu \to E(X_1^2)/(2\mu^2) - 1$ as $t \to \infty$, together with Lemma 2.3 implies

$$\lim_{i\to\infty} \int_0^\infty \left(E(N(t)) - \frac{t}{\mu}\right)^2 dG_{\vartheta_i}(t) = \lim_{i\to\infty} \left(E(N(T_{\vartheta_i})) - \frac{E(T_{\vartheta_i})}{\mu}\right)^2 = \left(\frac{E(X_1^2)}{2\mu^2} - 1\right)^2.$$

Therefore, we obtain

$$\lim_{i\to\infty} \frac{1}{E(T_{\vartheta_i})}\left(Var(E(N(T_{\vartheta_i})|T_{\vartheta_i})) - \frac{1}{\mu^2}Var(T_{\vartheta_i})\right) = \lim_{i\to\infty}\left[\frac{2\int_0^\infty \left(E(N(t)) - \frac{t}{\mu}\right)\frac{t}{\mu}dG_{\vartheta_i}(t)}{E(T_{\vartheta_i})} - \frac{2}{\mu}\left(E(N(T_{\vartheta_i})) - \frac{E(T_{\vartheta_i})}{\mu}\right)\right] = 0$$

by applying Lemma 2.5 and from this, (i) follows directly. In case (ii), we obtain $E(Var(N(T_{\vartheta_i})|T_{\vartheta_i}))/Var(T_{\vartheta_i}) \to Var(X_1)/(c\mu^3)$ and $Var(E(N(T_{\vartheta_i})|T_{\vartheta_i}))/Var(T_{\vartheta_i}) \to 1/\mu^2$ as $i \to \infty$, and thus the assertion. □

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No data was used for the research described in the article.

References

Asmussen, S., Klüppelberg, C., Sigman, K., 1999. Sampling at subexponential times, with queueing applications. Stochastic Process. Appl. 265–286.
Badía, F.G., Cha, J.H., 2013. Preservation properties of a renewal process stopped at a random dependent time. Probab. Engrg. Inform. Sci. 27, 163–175.
Badía, F.G., Sangüesa, C., 2015. The DFR property for counting processes stopped at an independent random time. J. Appl. Probab. 52, 574–585.
Cox, D.R., 1962. Renewal Theory. Methuen, London.
Feller, W., 1971. second ed. An Introduction to Probability Theory and Its Applications, vol. II, John Wiley, New York.
Herff, W., Jochems, B., Kamps, U., 1997. The inspection paradox with random time. Stat. Pap. 38, 103–110.
Kamps, U., Rauwolf, D., 2023. A record-values property of a renewal process with random inspection time. Statist. Probab. Lett. 195, 109785.
Kulkarni, V.G., 2017. Modeling and Analysis of Stochastic Systems, third ed. CRC Press, Boca Raton.
Liu, P., Peña, E.A., 2016. Sojourning with the homogeneous Poisson process. Am. Stat. 70, 413–423.
Mitov, K.V., Omey, E., 2014. Renewal Processes. Springer, Cham.
Pinsky, M.A., Karlin, S., 2011. An Introduction to Stochastic Modeling, fourth ed. Academic Press, Burlington.
Rauwolf, D., 2023. Renewal Processes with Random Time (Doctoral Thesis). RWTH Aachen University.
Rauwolf, D., Kamps, U., 2023. Quantifying the inspection paradox with random time. Am. Stat. 77, 274–282.
Salehi, E.T., Badía, F.G., Asadi, M., 2012. Preservation properties of a homogeneous Poisson process stopped at an independent random time. Statist. Probab. Lett. 82, 574–585.