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Boundary conditions for hyperbolic relaxation systems with characteristic boundaries

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Abstract

This paper is concerned with initial-boundary-value problems of general multi-dimensional hyperbolic relaxation systems with characteristic boundaries. For the characteristic case, we redefine a Generalized Kreiss condition (GKC) which is essentially necessary to have a well-behaved relaxation limit. Under this characteristic GKC and a Shizuta-Kawashima-like condition, we derive reduced boundary conditions for the relaxation limit solving the corresponding equilibrium systems and justify the validity thereof. The key of the derivation is to select an elaborate version of the characteristic GKC by invoking the Shizuta-Kawashima-like condition. In contrast to the existing results, the present one does not assume that the boundary is non-characteristic for either the relaxation or equilibrium systems.

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1. Introduction

Consider first-order partial differential equations (PDEs) with a small parameter:

$$U_t + \sum_{j=1}^{d} F_j(U)_{x_j} = \frac{1}{\epsilon} Q(U) \tag{1.1}$$

defined in $t > 0$ and $x = (x_1, x_2, ..., x_d) \in \Omega \subset \mathbb{R}^n$. Here $U = U(x,t) \in \mathbb{R}^n$ is the unknown vector-valued function, $F_j(U)$ $(j = 1, 2, ..., d)$ and $Q(U)$ are smooth functions of U, and ϵ is a small positive parameter called relaxation time. Such equations are referred to as *relaxation systems* and describe many important non-equilibrium processes including chemically reactive flows [10], compressible viscoelastic flows [6,36], traffic flows [2,13]. They also arise in non-equilibrium thermo-dynamics [15,25,41], the kinetic theory [3,9,17,24], nonlinear optics [5], neuroscience [7] and so on.

For such a small-parameter problem, the main interest is to study the limit as ϵ tends to zero, the so-called zero relaxation limit. To see the limit, we notice that the source term in the aforementioned examples usually has (or can be easily transformed into) the form

$$Q(U) = \begin{pmatrix} 0 \\ q(U) \end{pmatrix}$$

with $q(U) \in \mathbb{R}^r$ consisting of r linearly independent functions of U. Accordingly, we write

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F_j(U) = \begin{pmatrix} f_j(u,v) \\ g_j(u,v) \end{pmatrix}, \quad Q(U) = \begin{pmatrix} 0 \\ q(u,v) \end{pmatrix}.$$

With this partition, it is easy to see that the (formal) limit satisfies $q(u, v) = 0$. From this equation one can usually get $v = h(u)$. Thus the limit can be determined by solving the so-called *equilibrium system*

$$u_t + \sum_{j=1}^{d} f_i(u, h(u))_{x_j} = 0.$$

For initial-value problems, this formal limit was justified in [32,34] under the structural stability condition proposed therein. Moreover, it was shown in [32,34,35] that the stability condition holds for almost all examples mentioned above, implies the hyperbolicity of both the relaxation and equilibrium systems, and is the same as the sub-characteristic condition [20] for relaxation systems with $n = 2$ and $r = 1$. In what follows, we only consider the systems satisfying the structural stability condition.

The present work focuses on initial-boundary-value problems (IBVPs) with prescribed boundary conditions. In this case, a fundamental task is to seek boundary conditions (called *reduced boundary conditions, rBCs*) for the equilibrium system. Without such boundary conditions, the equilibrium system alone is generally inadequate to determine the relaxation limit even if it exists, which is in contrast to the initial-value problems. To illustrate this task, we consider a simple example

$$\begin{aligned} u_t^\epsilon + 3u_x^\epsilon + v_x^\epsilon &= 0, \\ v_t^\epsilon + u_x^\epsilon + v_x^\epsilon &= -v^\epsilon/\epsilon \end{aligned} \tag{1.2}$$

defined for $x \geq 0$. Since the coefficient matrix has two positive eigenvalues, the classical theory [1,11] requires two boundary conditions

$$u^\epsilon(0,t) = g(t), \quad v^\epsilon(0,t) = h(t)$$

and proper initial data for (1.2) to solve (u^ϵ, v^ϵ) with each fixed $\epsilon > 0$. It is easy to see that the formal limit (u^0, v^0) satisfies $v^0 = 0$ and the equilibrium system

$$u_t^0 + 3u_x^0 = 0, \qquad x > 0. \tag{1.3}$$

To determine the limit u^0 on $x > 0$, a boundary condition is indispensable for (1.3). Remark that this boundary condition should be determined merely by the system (1.2) and its prescribed initial and boundary conditions, since so is the solution sequence (u^ϵ, v^ϵ). Because $u^\epsilon(0,t) = g(t)$ for each fixed ϵ, it is natural to guess that $u^0(0,t) = g(t)$. However, this is wrong if $h(t) \neq 0$ and the correct boundary condition is

$$u^0(0,t) = g(t) + \frac{h(t)}{3}.$$

This mismatch is caused by the so-called boundary-layer due to the small relaxation parameter, and the boundary-layer phenomenon is well-known in fluid dynamics since L. Prandtl. More details about this can be founded in Section 4. This example indicates that deriving the rBC is not trivial.

The above fundamental task was proposed in [32] over three decades ago and partially resolved under a non-characteristic assumption [32,33]. In this pioneered work it was observed that the structural stability condition, together with the Uniform Kreiss condition (UKC) [12,16], is not enough for the existence of relaxation limit for IBVPs. Note that the UKC is an essentially necessary criterion for the well-posedness of (multi-dimensional) hyperbolic systems [12,23]. To remedy this, a so-called Generalized Kreiss condition (GKC) was proposed in [32,33] for linear version of (1.1) with constant coefficients:

$$F_j(U) = A_j U, \quad Q(U) = QU. \tag{1.4}$$

Here A_j $(j = 1, 2, ..., d)$ and Q are constant matrices. Like the UKC for the well-posedness, the GKC is essentially necessary for the existence of relaxation limit. Under the GKC, the reduced boundary condition was obtained.

The goal of the present work is to remove the non-characteristic assumption in [33]. To proceed, we follow [16,23,33] and assume $\Omega = \{x_1 > 0\}$. For such a domain, the boundary $\partial\Omega = \{x_1 = 0\}$ is called characteristic for (1.1) with (1.4) if the coefficient matrix A_1 has zero eigenvalues. Because the boundary could be characteristic for either the relaxation system or the equilibrium system, there are the following four different cases:

Relaxation system	N	C	N	C
Equilibrium system	N	N	C	C
studied in	[33]	[39]	[40]	Present work

N: non-characteristic, C: characteristic

The non-characteristic assumption in [33] means that the boundary is non-characteristic for both the relaxation and equilibrium systems, corresponding to the first column of the table. In [39, 40], the boundary is assumed to be non-characteristic only for one of the two systems. The present work allows the boundary to be characteristic for both the systems. In this sense, the non-characteristic assumptions in [33,39,40] are completely removed.

At this point, let us make the following remarks. (1) Characteristic IBVPs are classical [23] and the double characteristic case occurs in many situations, such as the Grad's moment system defined in the half-space $x_1 > 0$ [18]. (2) Compared to the non-characteristic cases, characteristic ones possess new mathematical challenges. At least, the GKC or UKC needs to be redefined because they involve the inverse of A_1. (3) It is a common practice that the theory of hyperbolic IBVPs starts with linear problems with constant coefficients defined in the half space $\{x_1 > 0\}$ [1,16,23]. In fact, the IBVPs in a general smooth domain can be converted to finitely many similar problems in the half-space and one in the whole space by means of a local coordinate change and a partition of unity [23]. Therefore, the half-space is representative. On the other hand, the linear theory has its own interest and turns out to be essential for studying nonlinear problems [21,22]. (4) Last but not least, deriving the rBCs is a fundamental task also for studying singular limiting problems of other PDEs.

In this work, we redefine the crucial GKC by combining the ideas in [23,33]. Under this characteristic-GKC (c-GKC) and a Shizuta-Kawashima-like condition used in [8], we derive rBCs for the relaxation limit solving the corresponding equilibrium system. To justify the derivation, we show that the rBCs satisfy the (characteristic) UKC [23] for the equilibrium system. Particularly, our rBCs do not involve the characteristic modes corresponding to the zero eigenvalue for the equilibrium system. Details can be founded in Theorem 4.3. The key step of the derivation is to select an elaborate version of the c-GKC by invoking the Shizuta-Kawashima-like condition. Having this version, we can adapt the subtle analytical expansions of matrices in [40] to the present problem. At last, the validity of the rBC is established by using the energy method in [11] under the assumptions that the initial data are given in equilibrium and consistent with boundary conditions. Non-equilibrium initial data may cause interactions of initial and boundary layers, which are beyond the scope of this paper. It would be interesting to study the interactions of the initial and boundary layers. Moreover, the stiff well-posedness is another important topic to be studied.

Now we briefly review the existing literature on IBVPs of hyperbolic relaxation systems. Unlike ours, all of those works are about specific relaxation systems. For the Jin-Xin model [14], there are a series of papers [27–30] devoted to studying the relaxation limit and boundary-layer behaviors. Particularly, the stiff well-posedness of the IBVPs for the Jin–Xin model and the Katsoulakis–Tzavaras model was proved, even for initial data not at equilibrium. About the one-dimensional Kerr-Debye model in nonlinear optics, the authors in [5] studied the relaxation limit of a specific IBVP. In [2], the authors derived the rBC for a nonlinear discrete-velocity model of traffic flows by solving a boundary Riemann problem. In [19], the one-dimensional Broadwell model of the nonlinear Boltzmann equation has been studied in the presence of boundary-layers. For the Grad's moment closure systems with Maxwell boundary conditions, the GKC is verified in [37] for a simplified moment system and the solvability of boundary-layer equations for some flow problems is investigated in [18]. Particularly, the Shizuta-Kawashima-like condition was tacitly verified in [18]. The interested reader is referred to [4,19,31,38] for further related works.

This paper is organized as follows. Basic assumptions and notations are listed in Section 2. In Section 3, we present the characteristic Generalized Kreiss condition for IBVPs of relaxation systems. Section 4 is devoted to the derivation of the reduced boundary conditions. Finally, in Section 5 we prove the validity of the reduced boundary condition.

2. Basic assumptions

In this section, we list all the key assumptions used in this paper.

2.1. Assumptions for the relaxation system

For the linear system (1.1) with constant coefficients (1.4):

$$U_t + \sum_{j=1}^{d} A_j U_{x_j} = \frac{1}{\epsilon} QU, \tag{2.1}$$

we firstly recall the structural stability condition.

Definition 2.1 *(Structural stability condition).* This condition consists of the following three items:
(i) There is an invertible $n \times n$ matrix P and an invertible $r \times r$ matrix S such that

$$PQ = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} P.$$

(ii) There exists a symmetric positive definite matrix A_0, called the symmetrizer, such that

$$A_0 A_j = A_j^T A_0, \qquad j = 1, 2, ..., d.$$

Here and below the superscript T means the transpose of matrices or vectors.
(iii) The hyperbolic part and the source term are coupled in the following sense

$$A_0 Q + Q^T A_0 \leq -P^T \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} P.$$

Here I_r is the unit matrix of order r.

Remark 2.1. It should be noted that the stability condition was proposed in [32,34] originally for nonlinear problems. For the zero-relaxation limit, the sufficiency of the structural stability condition has been justified there for general multi-D nonlinear systems. On the other hand, it was shown in [35] that the structural stability condition has its root in non-equilibrium thermodynamics and is satisfied by many classical physical models.

Additionally, we also assume that the symmetrizer A_0 and the source term Q satisfy

$$A_0 Q = Q^T A_0. \tag{2.2}$$

This additional condition corresponds to the celebrated Onsager reciprocal relation and is satisfied by a large class of physically motivated relaxation models as shown in [35].

Under the structural stability condition and (2.2), we may as well assume that the matrices A_j $(j = 1, 2, ..., d)$ are all symmetric, the symmetrizer A_0 equals to I_n, and $Q = \mathrm{diag}(0, S)$ with S a symmetric negative definite matrix (see the discussion in Section 2 of [40]). Corresponding to the partition of Q, we write

$$A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}, \quad A_j = \begin{pmatrix} A_{j11} & A_{j12} \\ A_{j12}^T & A_{j22} \end{pmatrix}, \quad j = 2, ..., d. \tag{2.3}$$

Unlike previous works [33,39,40], we allow both A_1 and A_{11} to be singular. Correspondingly, we introduce the following notations. Let $R_0 \in \mathbb{R}^{n\times n_0}$ be a full-rank matrix satisfying $A_1 R_0 = 0$ with n_0 the multiplicity of the zero eigenvalue of A_1, and $R_1 \in \mathbb{R}^{n\times(n-n_0)}$ be a matrix such that (R_1, R_0) is invertible. Notice that the choice of R_0 and R_1 is not unique. Denote the inverse matrix by

$$\begin{pmatrix} L_1 \\ L_0 \end{pmatrix} = (R_1, R_0)^{-1}.$$

Then we have the decomposition

$$I_n = R_1 L_1 + R_0 L_0. \tag{2.4}$$

Furthermore, we denote $\widehat{A}_1 = R_1^T A_1 R_1$. From the relation

$$\begin{pmatrix} R_1^T \\ R_0^T \end{pmatrix} A_1 \begin{pmatrix} R_1 & R_0 \end{pmatrix} = \begin{pmatrix} \widehat{A}_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.5}$$

we see that $\widehat{A}_1 \in \mathbb{R}^{(n-n_0)\times(n-n_0)}$ is an invertible matrix which has the same number of positive and negative eigenvalues as A_1.

Moreover, we introduce the characteristic decomposition for the matrix A_{11}. Denote by n_{10} the multiplicity of the zero eigenvalue of A_{11}. Since A_{11} is symmetric, we introduce an orthonormal matrix (P_1, P_0) satisfying

$$\begin{pmatrix} P_1^T \\ P_0^T \end{pmatrix} \begin{pmatrix} P_1 & P_0 \end{pmatrix} = I_{n-r}, \qquad \begin{pmatrix} P_1^T \\ P_0^T \end{pmatrix} A_{11} \begin{pmatrix} P_1 & P_0 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.6}$$

Here Λ_1 is an $(n - r - n_{10}) \times (n - r - n_{10})$ invertible diagonal matrix, P_1 and P_0 are $(n - r) \times (n - r - n_{10})$ and $(n - r) \times n_{10}$ matrices respectively.

Furthermore, we need

Definition 2.2 *(Shizuta-Kawashima-like condition).* The kernel of matrix Q does not contain the eigenvectors of A_1 associated with the zero eigenvalue:

$$\ker(A_1) \cap \ker(Q) = \{0\}. \tag{2.7}$$

Remark that (2.7) is much weaker than the usual Shizuta-Kawashima condition in [26]:

$$\ker\left(\sum_{j=1}^{d}\omega_j A_j - \lambda I_n\right) \cap \ker(Q) = \{0\}$$

for any $\lambda \in \mathbb{R}$ and any $(\omega_1, \omega_2, ..., \omega_d) \in \mathbb{R}^d \setminus \{0\}$. This weaker version was also used in [8] for the existence of shock profiles. It is satisfied by all the three cases studied in [33,39,40]. Indeed, when A_1 is invertible as assumed in [33,40], then (2.7) is obviously true because $\ker(A_1) = \{0\}$; when A_{11} is invertible as in [39], it can be seen from the partition (2.3) that $\ker(A_1)$ does not contain any nonzero vectors in $\ker(Q) = \mathbb{R}^{n-r} \times \{0\}$.

We end this subsection with the following equivalent version of the condition (2.7).

Proposition 2.1. *The Shizuta-Kawashima-like condition is equivalent to the invertibility of the matrix $R_0^T Q R_0$. Here $R_0 \in \mathbb{R}^{n\times n_0}$ consists of n_0 eigenvectors of A_1 associated with the zero eigenvalue of multiplicity n_0.*

Proof. If $R_0^T Q R_0$ is not invertible, there exists a nonzero $x \in \mathbb{R}^{n_0}$ such that $x^T R_0^T Q R_0 x = 0$. By the expression $Q = \mathrm{diag}(0, S)$ with $S < 0$, we deduce that $R_0 x \in \ker(Q)$. This contradicts to (2.7) since $A_1 R_0 x = 0$. Conversely, if there is a nonzero $x \in \mathbb{R}^n$ such that $A_1 x = Qx = 0$, then there exists a nonzero $y \in \mathbb{R}^{n_0}$ such that $x = R_0 y$. Thus $R_0^T Q R_0 y = R_0^T Q x = 0$. This completes the proof. □

2.2. *Assumptions for the boundary condition*

For the relaxation system (2.1) defined in the half-space $\{x_1 > 0\}$, we allow the boundary $x_1 = 0$ to be characteristic. Namely, the symmetric matrix A_1 may have zero as its eigenvalue. Denote by

$$n_+ = \text{the number of positive eigenvalues of } A_1.$$

According to the classical theory [1,11] for IBVPs of first-order hyperbolic systems, n_+ boundary conditions

$$BU(0, \hat{x}, t) = b(\hat{x}, t) \tag{2.8}$$

should be prescribed at the boundary $x_1 = 0$ for (2.1). Here $\hat{x} = (x_2, x_3, ..., x_d)$ and B is a constant $n_+ \times n$-matrix of full-rank. Moreover, this boundary condition should satisfy the Uniform Kreiss Condition (UKC) [16,23] and

$$BR_0 = 0, \tag{2.9}$$

for the well-posedness. The relation (2.9) means that the characteristic mode corresponding to the zero eigenvalue is not involved in the boundary condition. Its necessity was illustrated in [23] for the characteristic IBVPs.

In studying the relaxation limit as $\epsilon \to 0$, it was observed in [33] that the UKC and the structural stability condition are not enough to guarantee the existence of relaxation limit for

IBVPs. To remedy this, a so-called Generalized Kreiss condition (GKC) was proposed in [33]. To state the GKC, we recall the following terminology introduced in [33].

Definition 2.3. Let $n \times n$-matrix M have precisely k $(0 \le k \le n)$ stable eigenvalues. A full-rank $n \times k$-matrix R_M^S is called a right-stable matrix of M if

$$MR_M^S = R_M^S S_M,$$

where S_M is a $k \times k$ stable matrix. Similarly, we can define the right-unstable matrix R_M^U.

For the non-characteristic case where A_1 is invertible, we can define

$$M = M(\xi, \omega, \eta) = A_1^{-1}\left(\eta Q - \xi I_n - i\sum_{j=2}^{d}\omega_j A_j\right) \tag{2.10}$$

for $\eta \ge 0$, ξ a complex number satisfying $Re\xi > 0$, and $\omega = (\omega_2, ..., \omega_d) \in \mathbb{R}^{d-1}$. Denote by $R_M^S(\xi, \omega, \eta)$ the right-stable matrix for $M = M(\xi, \omega, \eta)$. By Lemma 2.3 in [33], the matrix $M(\xi, \omega, \eta)$ has n_+ stable eigenvalues and thereby $R_M^S(\xi, \omega, \eta)$ is an $n \times n_+$-matrix. With this notation, the GKC can be stated as

Definition 2.4 *(Generalized Kreiss condition [33]).* There exists a constant $c_K > 0$ such that

$$|\det\{BR_M^S(\xi, \omega, \eta)\}| \ge c_K\sqrt{\det\{R_M^{S*}(\xi, \omega, \eta)R_M^S(\xi, \omega, \eta)\}}$$

for all $\eta \ge 0$, $\omega \in \mathbb{R}^{d-1}$ and ξ with $Re\xi > 0$. Here the superscript $*$ means the conjugate transpose of matrices.

Remark 2.2. Let us mention that the original UKC [16] is exactly the GKC with $\eta \equiv 0$. As well-known, the UKC is difficult to be verified, although it is conceptually essential. Compared with the UKC, the GKC was also proposed for general problems and additionally accounts for the source terms. Therefore, it is naturally more difficult to be verified than the UKC. On the other hand, the GKC can be verified if a given boundary condition satisfies the easily-checked strictly dissipative condition [33]. For specific problems with specific boundary conditions, the GKC can be concretized as in [28] for the Jin-Xin model and in [37] for a linearized moment closure system.

For the characteristic case where A_1 is not invertible, the matrix $M = M(\xi, \omega, \eta)$ and thereby the GKC are not defined. The main goal of the present paper is to study such a case. Particularly, in Section 3 we introduce a characteristic GKC.

To sum up, our basic assumptions for general linear relaxation systems (2.1) defined in the half-space are the structural stability condition, the Shizuta-Kawashima-like condition, the two assumptions in (2.2) and (2.9), and the characteristic Generalized Kreiss condition (to be defined in Definition 3.1).

3. Characteristic Generalized Kreiss condition

Following [33], we consider the eigenvalue problem corresponding to the linear system (2.1):

$$\begin{aligned} &\xi \widehat{U} + A_1 \widehat{U}_{x_1} + i \sum_{j=2}^{d} \omega_j A_j \widehat{U} = \eta Q \widehat{U}, \\ &B \widehat{U}|_{x_1=0} = 0. \end{aligned} \tag{3.1}$$

Here $\eta \geq 0$, $\omega = (\omega_2, \omega_3, ..., \omega_d) \in \mathbb{R}^{d-1}$, and ξ is a complex number with $Re\xi > 0$. If (3.1) has a bounded solution $\widehat{U} = \widehat{U}(x_1)$ for certain parameters satisfying $\eta > 0$ and $Re\xi > 0$, then

$$U^{\epsilon}(x_1, x_2, ..., x_d, t) = \exp\left(\frac{\xi t}{\eta \epsilon} + \sum_{j=2}^{d} \frac{i \omega_j x_j}{\eta \epsilon}\right) \widehat{U}\left(\frac{x_1}{\eta \epsilon}\right)$$

is the solution to the linear system in (2.1) with homogeneous boundary conditions:

$$\begin{aligned} &U_t + \sum_{j=1}^{d} A_j U_{x_j} = \frac{1}{\epsilon} Q U, \\ &B U|_{x_1=0} = 0. \end{aligned}$$

Since $Re\xi > 0$ and $\eta > 0$, this solution U^{ϵ} exponentially increases for $t > 0$ as ϵ goes to zero. In order to have a well-behaved zero relaxation limit, such exponentially increasing solutions should be excluded by proposing proper constraints on the boundary conditions (2.8).

Moreover, we need the following fact.

Lemma 3.1 *(Proposition A.1 in [39]). Let D be a symmetric positive definite matrix and E be a symmetric matrix. Then the complex matrix $D + iE$ is invertible.*

With this, we rewrite the linear system of ordinary differential equations (ODEs) in (3.1). Notice that $R_0^T A_1 = 0$. We multiply R_0^T on the left of (3.1) to obtain

$$R_0^T G(\xi, \omega, \eta) \widehat{U} = 0 \quad \text{with} \quad G(\xi, \omega, \eta) = \eta Q - \xi I_n - i \sum_{j=2}^{d} \omega_j A_j.$$

Thanks to (2.4), we have

$$R_0^T G(\xi, \omega, \eta) R_0 (L_0 \widehat{U}) + R_0^T G(\xi, \omega, \eta) R_1 (L_1 \widehat{U}) = 0.$$

Set

$$G_{kl}(\xi, \omega, \eta) = R_k^T G(\xi, \omega, \eta) R_l, \qquad k, l = 0, 1. \tag{3.2}$$

By Lemma 3.1, it is easy to see that $G_{00}(\xi, \omega, \eta)$ is invertible. Then we have

$$L_0\widehat{U} = -G_{00}^{-1}(\xi,\omega,\eta)G_{01}(\xi,\omega,\eta)(L_1\widehat{U}). \tag{3.3}$$

On the other hand, multiplying R_1^T on the left of (3.1) yields

$$(R_1^T A_1 R_1)(L_1\widehat{U})_{x_1} = G_{11}(\xi,\omega,\eta)(L_1\widehat{U}) + G_{10}(\xi,\omega,\eta)(L_0\widehat{U}). \tag{3.4}$$

Recall $\widehat{A}_1 = R_1^T A_1 R_1$ in (2.5). We deduce from (3.3) and (3.4) that

$$(L_1\widehat{U})_{x_1} = M(\xi,\omega,\eta)(L_1\widehat{U}), \tag{3.5}$$

where

$$M(\xi,\omega,\eta) = \widehat{A}_1^{-1}\Big[G_{11}(\xi,\omega,\eta) - G_{10}(\xi,\omega,\eta)G_{00}^{-1}(\xi,\omega,\eta)G_{01}(\xi,\omega,\eta)\Big] \tag{3.6}$$

is an $(n-n_0)\times(n-n_0)$ matrix. Consequently, the equation in (3.1) has been rewritten as (3.3) and (3.5). Moreover, the initial condition $B\widehat{U}|_{x_1=0} = 0$ in (3.1) can be rewritten as

$$B\widehat{U}|_{x_1=0} = BR_1(L_1\widehat{U})|_{x_1=0} = 0 \tag{3.7}$$

due to the decomposition (2.4) and the assumption (2.9).

To solve (3.5) with the initial condition (3.7), we need the following important fact about the matrix $M(\xi,\omega,\eta)$ defined in (3.6).

Lemma 3.2. *For any $\eta \ge 0$ and any ξ with $Re\xi > 0$, the matrix $M(\xi,\omega,\eta)$ has precisely n_+ stable eigenvalues and $(n-n_0-n_+)$ unstable eigenvalues.*

Proof. Firstly, we show that $M(\xi,\omega,\eta)$ has no purely imaginary eigenvalues. Otherwise, if $M(\xi,\omega,\eta)$ has a purely imaginary eigenvalue $i\kappa$ with $\kappa\in\mathbb{R}$, then we have

$$\det\{i\kappa I_{n-n_0} - M(\xi,\omega,\eta)\} = 0.$$

Since $\widehat{A}_1$ is invertible, it follows that

$$\det\Big\{G_{11}(\xi,\omega,\eta) - G_{10}(\xi,\omega,\eta)G_{00}^{-1}(\xi,\omega,\eta)G_{01}(\xi,\omega,\eta) - i\kappa\widehat{A}_1\Big\} = 0.$$

From the relation

$$\begin{aligned}&\begin{pmatrix} G_{11} - G_{10}G_{00}^{-1}G_{01} & 0 \\ 0 & G_{00}\end{pmatrix} - i\kappa\begin{pmatrix}\widehat{A}_1 & 0 \\ 0 & 0\end{pmatrix} \\ &= \begin{pmatrix} I & -G_{10}G_{00}^{-1} \\ 0 & I\end{pmatrix}\left[\begin{pmatrix} G_{11} & G_{10} \\ G_{01} & G_{00}\end{pmatrix} - i\kappa\begin{pmatrix}\widehat{A}_1 & 0 \\ 0 & 0\end{pmatrix}\right]\begin{pmatrix} I & 0 \\ -G_{00}^{-1}G_{01} & I\end{pmatrix},\end{aligned}$$

we see that

$$\det\left\{\begin{pmatrix} G_{11} & G_{10} \\ G_{01} & G_{00}\end{pmatrix} - i\kappa\begin{pmatrix}\widehat{A}_1 & 0 \\ 0 & 0\end{pmatrix}\right\} = \det\left\{G_{11} - G_{10}G_{00}^{-1}G_{01} - i\kappa\widehat{A}_1\right\}\det\{G_{00}\} = 0.$$

By (3.2) and (2.5), we have

$$\begin{pmatrix} G_{11} & G_{10} \\ G_{01} & G_{00}\end{pmatrix} - i\kappa\begin{pmatrix}\widehat{A}_1 & 0 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} R_1^T \\ R_0^T\end{pmatrix}\left(\eta Q - \xi I - i\sum_{j=2}^{d}\omega_j A_j - i\kappa A_1\right)\begin{pmatrix} R_1 & R_0\end{pmatrix}.$$

Since (R_1, R_0) is invertible, this relation implies that

$$\det\left\{\eta Q - \xi I - i\sum_{j=2}^{d}\omega_j A_j - i\kappa A_1\right\} = 0.$$

On the other hand, we know from Lemma 3.1 that $\eta Q - \xi I - i\sum_{j=2}^{d}\omega_j A_j - i\kappa A_1$ is invertible, which leads to a contradiction. Thus we conclude that $M(\xi,\omega,\eta)$ has no purely imaginary eigenvalue.

Since the eigenvalues of $M(\xi,\omega,\eta)$ are continuous with respect to the parameters, the numbers of stable and unstable eigenvalues for $M(\xi,\omega,\eta)$ are invariant. Thus it suffices to show that $M(\xi,0,0) = -\xi\widehat{A}_1^{-1}$ has n_+ stable eigenvalues and $(n-n_0-n_+)$ unstable eigenvalues. Since $Re\xi > 0$, the result follows from the relation (2.5) and the fact that A_1 has n_+ positive eigenvalues, n_0 zero eigenvalues and $(n-n_0-n_+)$ negative eigenvalues. This completes the proof. □

By this lemma, the right-stable matrix $R_M^S(\xi,\omega,\eta)$ of $M(\xi,\omega,\eta)$ is an $(n-n_0)\times n_+$ matrix. Thus, we can easily see from (3.3), (3.5) and (3.7) that the linear system of ODEs (3.1) has no bounded solutions if

$$\det\{BR_1R_M^S(\xi,\omega,\eta)\} \neq 0.$$

Following the ideas in [16,23,33], we introduce

Definition 3.1 *(Characteristic Generalized Kreiss condition).* There exists a constant $c_K > 0$ such that

$$|\det\{BR_1R_M^S(\xi,\omega,\eta)\}| \geq c_K\sqrt{\det\{R_M^{S*}(\xi,\omega,\eta)R_M^S(\xi,\omega,\eta)\}}$$

for all $\eta \geq 0$, $\omega \in \mathbb{R}^{d-1}$ and ξ with $Re\xi > 0$.

Remark 3.1. When $\eta = 0$, this characteristic GKC is just the Uniform Kreiss Condition (UKC) for characteristic IBVPs [23], which implies the well-posedness of the IBVPs. On the other hand, when $n_0 = 0$, the matrix R_0 is void and R_1 can be the identity matrix I_n. In this case, we have $\widehat{A}_1 = A_1$ and

$$M(\xi,\omega,\eta)=A_1^{-1}\left(\eta Q-\xi I_n-i\sum_{j=2}^{d}\omega_j A_j\right),$$

which is just the matrix in (2.10). Thus this characteristic GKC is an extension of the original GKC in Definition 2.4.

Remark 3.2. Notice that the choice of R_0 and R_1 is not unique. However, the characteristic GKC does not rely on the special choice. Indeed, if we have another $(\widetilde{R}_0,\widetilde{R}_1)$ such that $A_1\widetilde{R}_0=0$ and $(\widetilde{R}_1,\widetilde{R}_0)$ is invertible, then it can be expressed in terms of (R_1,R_0) as

$$\widetilde{R}_0=R_0D_0,\quad \widetilde{R}_1=R_1C_1+R_0C_0, \tag{3.8}$$

where $D_0\in\mathbb{R}^{n_0\times n_0}$ and $C_1\in\mathbb{R}^{(n-n_0)\times(n-n_0)}$ are invertible, and $C_0\in\mathbb{R}^{n_0\times(n-n_0)}$. Since $A_1R_0=0$ and A_1 is symmetric, we see from (3.8) that

$$\widetilde{R}_1^TA_1\widetilde{R}_1=C_1^T(R_1^TA_1R_1)C_1.$$

Moreover, it follows from (3.8) and the definition in (3.2) that

$$\begin{aligned}\widetilde{G}_{11}-\widetilde{G}_{10}\widetilde{G}_{00}^{-1}\widetilde{G}_{01}&=\widetilde{R}_1^T\left(G-G\widetilde{R}_0(\widetilde{R}_0^TG\widetilde{R}_0)^{-1}\widetilde{R}_0^TG\right)\widetilde{R}_1\\&=(R_1C_1+R_0C_0)^T\left(G-GR_0(R_0^TGR_0)^{-1}R_0^TG\right)(R_1C_1+R_0C_0).\end{aligned}$$

Notice that $\left(G-GR_0(R_0^TGR_0)^{-1}R_0^TG\right)R_0=0$. We have

$$\begin{aligned}\widetilde{G}_{11}-\widetilde{G}_{10}\widetilde{G}_{00}^{-1}\widetilde{G}_{01}&=C_1^TR_1^T\left(G-GR_0(R_0^TGR_0)^{-1}R_0^TG\right)R_1C_1\\&=C_1^T(G_{11}-G_{10}G_{00}^{-1}G_{01})C_1\end{aligned}$$

and thereby

$$\widetilde{M}(\xi,\omega,\eta)=(\widetilde{R}_1^TA_1\widetilde{R}_1)^{-1}(\widetilde{G}_{11}-\widetilde{G}_{10}\widetilde{G}_{00}^{-1}\widetilde{G}_{01})=C_1^{-1}M(\xi,\omega,\eta)C_1.$$

Thus the right-stable matrix for $\widetilde{M}(\xi,\omega,\eta)$ is $\widetilde{R}_M^S(\xi,\omega,\eta)=C_1^{-1}R_M^S(\xi,\omega,\eta)$. Since $BR_0=0$, we deduce from (3.8) that

$$|\det\{B\widetilde{R}_1\widetilde{R}_M^S(\xi,\omega,\eta)\}|=|\det\{BR_1R_M^S(\xi,\omega,\eta)\}|.$$

On the other hand, by using Lemma 3.3 in [33] we can see

$$\begin{aligned}\sqrt{\det\{R_M^{S*}(\xi,\omega,\eta)R_M^S(\xi,\omega,\eta)\}}&=\sqrt{\det\{\widetilde{R}_M^{S*}(\xi,\omega,\eta)C_1^*C_1\widetilde{R}_M^S(\xi,\omega,\eta)\}}\\&\geq c_0\sqrt{\det\{\widetilde{R}_M^{S*}(\xi,\omega,\eta)\widetilde{R}_M^S(\xi,\omega,\eta)\}}\end{aligned}$$

with a constant $c_0 > 0$. In conclusion, the characteristic GKC holds with $(\widetilde{R}_1, \widetilde{R}_0)$ if so does it with (R_1, R_0). This means that the characteristic GKC is an intrinsic property of the system (2.1) together with the boundary condition (2.8).

4. Reduced boundary conditions

In this section, we derive the reduced boundary condition for the relaxation limit. Unlike those in our previous works [33,39,40], both the coefficient matrices A_1 and A_{11} may have zero eigenvalues. Denote

$$n_{1+} = \text{the number of positive eigenvalues of } A_{11}.$$

Recall Section 2.2 that n_0 and n_+ represent the numbers of zero and positive eigenvalues for A_1, respectively. Moreover, n_{10} represents the multiplicity of the zero eigenvalue of A_{11}. This section has four subsections and we start with

4.1. Asymptotic expansions

In this subsection, we review the asymptotic expansions in [40]. Set

$$y = \frac{x_1}{\epsilon}, \quad z = \frac{x_1}{\sqrt{\epsilon}}$$

and consider an approximate solution of the form

$$U_\epsilon(x_1, \hat{x}, t) = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}(x_1, \hat{x}, t) + \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix}(y, \hat{x}, t) + \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix}(z, \hat{x}, t) + \sqrt{\epsilon}\begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix}(z, \hat{x}, t). \quad (4.1)$$

Here the partition is the same as that in (2.3), the first term $(\bar{u}, \bar{v})$ is the outer solution, and the other terms are boundary-layer corrections satisfying

$$(\mu_0, \nu_0)(\infty, \hat{x}, t) = (\mu_1, \nu_1)(\infty, \hat{x}, t) = (\mu_2, \nu_2)(\infty, \hat{x}, t) = 0. \quad (4.2)$$

It is not difficult to see that the outer solution solves the equilibrium system

$$\bar{u}_t + A_{11}\bar{u}_{x_1} + \sum_{j=2}^{d} A_{j11}\bar{u}_{x_j} = 0, \quad (4.3)$$

$$\bar{v} = 0. \quad (4.4)$$

To derive equations for the boundary-layer correction terms, we denote by $\mathcal{L}$ the differential operator

$$\mathcal{L}(U) := \partial_t U + \sum_{j=1}^{d} A_j \partial_{x_j} U - QU/\epsilon$$

and write

$$\mathcal{L}(U_\epsilon) = \mathcal{L}(U_{outer}) + \mathcal{L}(U_{layer}), \tag{4.5}$$

where

$$U_{outer} = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}, \qquad U_{layer} = \begin{pmatrix} \mu_0 + \mu_1 + \sqrt{\epsilon}\mu_2 \\ \nu_0 + \nu_1 + \sqrt{\epsilon}\nu_2 \end{pmatrix}.$$

By (4.3) and (4.4), we have

$$\begin{aligned}
\mathcal{L}(U_{outer}) &= \partial_t \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} + \sum_{j=1}^{d} \begin{pmatrix} A_{j11} & A_{j12} \\ A_{j12}^T & A_{j22} \end{pmatrix} \partial_{x_j} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} - \frac{1}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \sum_{j=1}^{d} A_{j12}^T \partial_{x_j} \bar{u} \end{pmatrix}.
\end{aligned} \tag{4.6}$$

On the other hand, we compute:

$$\begin{aligned}
\mathcal{L}(U_{layer}) = {} & \partial_t \begin{pmatrix} \mu_0 + \mu_1 + \sqrt{\epsilon}\mu_2 \\ \nu_0 + \nu_1 + \sqrt{\epsilon}\nu_2 \end{pmatrix} + \sum_{j=2}^{d} \begin{pmatrix} A_{j11} & A_{j12} \\ A_{j12}^T & A_{j22} \end{pmatrix} \partial_{x_j} \begin{pmatrix} \mu_0 + \mu_1 + \sqrt{\epsilon}\mu_2 \\ \nu_0 + \nu_1 + \sqrt{\epsilon}\nu_2 \end{pmatrix} \\
& + \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \left[\frac{1}{\epsilon} \partial_y \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} + \frac{1}{\sqrt{\epsilon}} \partial_z \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix} + \partial_z \begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix} \right] \\
& - \frac{1}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \mu_0 + \mu_1 + \sqrt{\epsilon}\mu_2 \\ \nu_0 + \nu_1 + \sqrt{\epsilon}\nu_2 \end{pmatrix}.
\end{aligned} \tag{4.7}$$

The asymptotic solution U_ϵ should make $\mathcal{L}(U_\epsilon)$ as small as possible. Thus we let the coefficients of ϵ^{-1}, $\epsilon^{-1/2}$, ϵ^0 in (4.7) be zero and obtain

$$\begin{aligned}
O\left(\epsilon^{-1}\right): \quad & \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \partial_y \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \mu_0 + \mu_1 \\ \nu_0 + \nu_1 \end{pmatrix}, \\
O\left(\epsilon^{-1/2}\right): \quad & \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \partial_z \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix}, \\
O\left(\epsilon^{0}\right): \quad & \partial_t \mu_1 + A_{11}\partial_z \mu_2 + A_{12}\partial_z \nu_2 + \sum_{j=2}^{d} \left[A_{j11}\partial_{x_j}\mu_1 + A_{j12}\partial_{x_j}\nu_1 \right] = 0.
\end{aligned} \tag{4.8}$$

Here, for the coefficient of ϵ^0, only the first $(n-r)$ components and the z-dependent terms are considered. In the first equation of (4.8), the unknowns (μ_0, ν_0) and (μ_1, ν_1) should be considered separately since y and z are independent variables, which gives

$$\nu_1 = 0, \tag{4.9}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \partial_y \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix}. \tag{4.10}$$

By (4.9), the second equation in (4.8) becomes

$$A_{11}\partial_z \mu_1 = 0, \tag{4.11}$$

$$A_{12}^T \partial_z \mu_1 = S\nu_2. \tag{4.12}$$

If the matrix A_{11} is invertible, we obtain $\mu_1 = 0$ from (4.11) and (4.2). This corresponds to the non-characteristic case in [33] and there is no boundary-layer of scale $O(\sqrt{\epsilon})$.

Recall the decomposition of A_{11} in (2.6). We multiply P_1^T on the left of (4.11) and use (4.2) to obtain

$$P_1^T \mu_1 = 0, \tag{4.13}$$

which means

$$\mu_1 = (P_1 P_1^T + P_0 P_0^T)\mu_1 = P_0(P_0^T \mu_1).$$

By using (4.9) and (4.12), we multiply P_0^T on the last equation of (4.8) to obtain

$$\partial_t (P_0^T \mu_1) + \left[P_0^T A_{12} S^{-1} A_{12}^T P_0 \right] \partial_{zz} (P_0^T \mu_1) + \sum_{j=2}^{d} (P_0^T A_{j11} P_0) \partial_{x_j} (P_0^T \mu_1) = 0. \tag{4.14}$$

On the other hand, multiplying P_1^T on the last equation of (4.8) yields

$$P_1^T A_{11} \partial_z \mu_2 + P_1^T A_{12} \partial_z \nu_2 + P_1^T \sum_{j=2}^{d} A_{j11} \partial_{x_j} \mu_1 = 0.$$

Moreover, by (2.6) and (4.12) we get

$$P_1^T \mu_2 = -\Lambda_1^{-1} \left[P_1^T A_{12} S^{-1} A_{12}^T \partial_z \mu_1 - \sum_{j=2}^{d} P_1^T A_{j11} \int_z^\infty \partial_{x_j} \mu_1 \right]. \tag{4.15}$$

Note that it is not necessary for our purpose to determine μ_2 itself.

In summary, we have

$$\begin{cases} (4.4) \ \Rightarrow \ \bar{v} = 0, \\ (4.9) \ \Rightarrow \ \nu_1 = 0, \\ (4.13) \ \Rightarrow \ P_1^T \mu_1 = 0, \end{cases} \qquad \begin{cases} (4.3) \ \Rightarrow \ \bar{u}, \\ (4.10) \ \Rightarrow \ (\mu_0, \nu_0), \\ (4.14) \ \Rightarrow \ P_0^T \mu_1, \end{cases} \qquad \begin{cases} (4.15) \Rightarrow \ P_1^T \mu_2, \\ (4.12) \Rightarrow \ \nu_2. \end{cases}$$

Note that the second set of equations (4.3), (4.10) and (4.14) are differential equations while others are algebraic relations. In particular, (4.3) is the equilibrium system governing the relaxation limit. By our basic assumptions, the equilibrium system is symmetrizable hyperbolic and the coefficient matrix A_{11} has n_{1+} positive eigenvalues. From the classical theory of hyperbolic PDEs [11,1], n_{1+} boundary conditions should be given to solve the equilibrium system (4.3).

4.2. Boundary-layer equations

In order to derive the n_{1+} boundary conditions for the equilibrium system, we turn to the boundary-layer equations (4.10) and (4.14). Note that the coefficient matrix A_1 in (4.10) may not be invertible which differs from our previous work [40].

Observe that the boundary-layer equation (4.10) for (μ_0, ν_0) is just the equation in (3.1) with $\xi = 0$, $\omega = 0$, and $\eta = 1$. Thus we use the results in Section 3 to obtain

$$L_0 \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} = -G_{00}^{-1}(0,0,1)G_{01}(0,0,1)L_1 \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} \tag{4.16}$$

and

$$L_1 \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix}_y = M(0,0,1)L_1 \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix}, \tag{4.17}$$

which correspond to those in (3.3) and (3.5). Notice that

$$-G_{00}^{-1}(0,0,1)G_{01}(0,0,1) = -(R_0^T Q R_0)^{-1} R_0^T Q R_1$$

and $M(0,0,1) = \widehat{A}_1^{-1}\widehat{Q}$ with

$$\widehat{A}_1 = R_1^T A_1 R_1, \qquad \widehat{Q} = R_1^T Q R_1 - R_1^T Q R_0 (R_0^T Q R_0)^{-1} R_0^T Q R_1. \tag{4.18}$$

As in Section 3, once the ODE system (4.17) is solved, the relation (4.16) can be used to determine $L_0 \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix}$. Consequently, we can obtain the correction term (μ_0, ν_0) as

$$\begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} = R_0 L_0 \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} + R_1 L_1 \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix}. \tag{4.19}$$

To solve (4.17), we analyze the matrices in (4.18). Firstly, we select an elaborate matrix R_1 as follows. Recall Remark 3.2 that the choice of R_1 is not unique.

Proposition 4.1. *Partition the matrix R_0 as*

$$R_0 = \begin{pmatrix} R_{01} \\ R_{02} \end{pmatrix}, \qquad R_{01} \in \mathbb{R}^{(n-r)\times n_0}, \qquad R_{02} \in \mathbb{R}^{r\times n_0}.$$

Under the Shizuta-Kawashima-like condition, it follows that $r \geq n_0$ and the matrix R_{02} is of full-rank. Let $R_{02}^{\perp} \in \mathbb{R}^{r\times(r-n_0)}$ be a matrix such that $(R_{02}, R_{02}^{\perp})$ is invertible. Then (R_0, R_1) is invertible with

$$R_1 = \begin{pmatrix} I_{n-r} & 0 \\ 0 & R_{02}^{\perp} \end{pmatrix}. \tag{4.20}$$

Proof. With the above partition of R_0, we compute

$$R_0^T Q R_0 = \begin{pmatrix} R_{01}^T & R_{02}^T \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} R_{01} \\ R_{02} \end{pmatrix} = R_{02}^T S R_{02}.$$

Thanks to Proposition 2.1, $R_{02}^T S R_{02}$ is invertible. Since S is symmetric negative definite, it follows that $r \geq n_0$ and R_{02} is of full-rank. The invertibility of

$$(R_1, R_0) = \begin{pmatrix} I_{n-r} & 0 & R_{01} \\ 0 & R_{02}^{\perp} & R_{02} \end{pmatrix}$$

is obvious. This completes the proof. □

Remark 4.1. In what follows, we focus on the case $r > n_0$ unless otherwise specified. For $r = n_0$, the matrix $R_{02}^{\perp}$ is void and similar conclusions can be obtained easily.

Having R_1 selected above, we compute $\widehat{A}_1$ and $\widehat{Q}$ defined in (4.18):

$$\widehat{A}_1 = R_1^T A_1 R_1 = \begin{pmatrix} I_{n-r} & 0 \\ 0 & (R_{02}^{\perp})^T \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \begin{pmatrix} I_{n-r} & 0 \\ 0 & R_{02}^{\perp} \end{pmatrix} = \begin{pmatrix} A_{11} & \widehat{A}_{12} \\ \widehat{A}_{12}^T & \widehat{A}_{22} \end{pmatrix}$$

with

$$\widehat{A}_{12} = A_{12} R_{02}^{\perp}, \qquad \widehat{A}_{22} = (R_{02}^{\perp})^T A_{22} R_{02}^{\perp} \tag{4.21}$$

and

$$\begin{aligned} \widehat{Q} &= R_1^T [Q - Q R_0 (R_0^T Q R_0)^{-1} R_0^T Q] R_1 \\ &= R_1^T \left[\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} - \begin{pmatrix} 0 \\ S R_{02} \end{pmatrix} \left(R_{02}^T S R_{02} \right)^{-1} \begin{pmatrix} 0 & R_{02}^T S \end{pmatrix} \right] R_1 \\ &= \begin{pmatrix} I_{n-r} & 0 \\ 0 & (R_{02}^{\perp})^T \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & S - S R_{02} \left(R_{02}^T S R_{02} \right)^{-1} R_{02}^T S \end{pmatrix} \begin{pmatrix} I_{n-r} & 0 \\ 0 & R_{02}^{\perp} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \widehat{S} \end{pmatrix} \end{aligned} \tag{4.22}$$

with

$$\widehat{S} = (R_{02}^{\perp})^T \Big(S - S R_{02} (R_{02}^T S R_{02})^{-1} R_{02}^T S \Big) R_{02}^{\perp}. \tag{4.23}$$

Since $(R_{02}, R_{02}^{\perp})$ is invertible and S is symmetric negative definite, it is not difficult to show that $\widehat{S}$ is also symmetric negative definite for $r > n_0$.

Having the block-diagonal form of $\widehat{Q}$ in (4.22), the solution of the boundary-layer equation (4.17) can be attributed to the case studied in [40].

Proposition 4.2 *(Propositions 3.2 and 3.3 in [40]). Assume that the relaxation system* (2.1) *satisfies the structural stability condition, together with the assumption in* (2.2)*, and the Kawashima-like condition* (2.7)*. Then we have the following conclusions with notations given in* (2.6)*,* (4.21)*, and* (4.23)*.*

(i) $r - n_0 \geq n_{10}$ *and the matrix*

$$K := \widehat{A}_{12}^T P_0 \in \mathbb{R}^{(r-n_0)\times n_{10}}$$

is of full-rank.

(ii) *The solution to* (4.17) *can be expressed as*

$$L_1 \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} = \begin{pmatrix} N \\ \widetilde{K} \end{pmatrix} w, \tag{4.24}$$

where $w \in \mathbb{R}^{r-n_0-n_{10}}$ *satisfies the ordinary differential equation*

$$\partial_y w = \big(\widetilde{K}^T X \widetilde{K}\big)^{-1} \big(\widetilde{K}^T \widehat{S} \widetilde{K}\big) w, \qquad y \geq 0. \tag{4.25}$$

Here $\widetilde{K} \in \mathbb{R}^{(r-n_0)\times(r-n_0-n_{10})}$ *is the orthogonal complement of* K*,* $X \in \mathbb{R}^{(r-n_0)\times(r-n_0)}$ *and* $N \in \mathbb{R}^{(n-r)\times(r-n_0-n_{10})}$ *are defined by*

$$X = \widehat{A}_{22} - \widehat{A}_{12}^T P_1 \Lambda_1^{-1} P_1^T \widehat{A}_{12}$$
$$N = -P_1 \Lambda_1^{-1} P_1^T \widehat{A}_{12} \widetilde{K} + P_0 (K^T K)^{-1} \Big(\big(K^T \widehat{S} \widetilde{K}\big) \big(\widetilde{K}^T \widehat{S} \widetilde{K}\big)^{-1} \big(\widetilde{K}^T X \widetilde{K}\big) - K^T X \widetilde{K} \Big).$$

Moreover, the coefficient matrix $\big(\widetilde{K}^T X \widetilde{K}\big)^{-1} \big(\widetilde{K}^T \widehat{S} \widetilde{K}\big)$ *in* (4.25) *is invertible with* $(n_+ - n_{1+} - n_{10})$ *stable eigenvalues.*

This proposition is just the Propositions 3.2 and 3.3 in [40] with n, r, A_1 and Q replaced by $(n - n_0)$, $(r - n_0)$, $\widehat{A}_1$ and $\widehat{Q}$ respectively. Thanks to this proposition, $L_1 \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix}$ can be uniquely determined by solving the ODE system (4.25), defined in the half-space $y \geq 0$, if proper initial conditions are prescribed. Thus the boundary-layer terms (μ_0, ν_0) can be determined by using (4.16) and (4.19).

Next we turn to the boundary-layer equation (4.14) for the other correction term (μ_1, ν_1). This time-dependent equation is a parabolic system of second-order partial differential equations, which can be shown as

Proposition 4.3. *The coefficient matrix $P_0^T A_{12} S^{-1} A_{12}^T P_0$ in (4.14) is symmetric negative definite.*

Proof. Since S is symmetric negative definite, it suffices to show that $A_{12}^T P_0$ has full-rank. From Proposition 4.2 (i) and (4.21), we know that

$$n_{10} = \mathrm{rank}(K) = \mathrm{rank}\big((R_{02}^{\perp})^T A_{12}^T P_0\big) \le \mathrm{rank}(A_{12}^T P_0).$$

This completes the proof. □

Due to this proposition, we know that n_{10} boundary conditions should be given for (4.14).

4.3. Boundary conditions for the asymptotic expansion

In the previous subsections, we have analyzed the equations for the outer solution and boundary-layer correction terms. It is pointed out that the hyperbolic system (4.3) and the parabolic system (4.14) need proper boundary conditions while the ODE system (4.25) requires initial conditions. Here we turn to these initial and boundary conditions.

Substituting the asymptotic expansion (4.1) into the boundary condition in (2.8) and matching the coefficient of order $O(1)$, we obtain

$$B\begin{pmatrix} \bar{u} + \mu_1 + \mu_0 \\ \bar{v} + \nu_1 + \nu_0 \end{pmatrix}(0, \hat{x}, t) = b(\hat{x}, t). \tag{4.26}$$

Due to $BR_0 = 0$ assumed in (2.9), we have $B = B(R_1L_1 + R_0L_0) = BR_1L_1$ and thereby

$$BR_1L_1\begin{pmatrix} \bar{u} + \mu_1 + \mu_0 \\ \bar{v} + \nu_1 + \nu_0 \end{pmatrix}(0, \hat{x}, t) = b(\hat{x}, t).$$

By using (4.4), (4.9), (4.13) and Proposition 4.2, the relation (4.26) can be reduced to

$$BR_1L_1\begin{pmatrix} \bar{u} + P_0(P_0^T\mu_1) \\ 0 \end{pmatrix}(0, \hat{x}, t) + BR_1\begin{pmatrix} N \\ \widetilde{K} \end{pmatrix} w(0, \hat{x}, t) = b(\hat{x}, t).$$

On the other hand, we deduce from $L_1R_1 = I_{n-n_0}$ and the expression of R_1 in (4.20) that

$$L_1\begin{pmatrix} I_{n-r} & 0 \\ 0 & R_{02}^{\perp} \end{pmatrix} = I_{n-n_0} = \begin{pmatrix} I_{n-r} & 0 \\ 0 & I_{r-n_0} \end{pmatrix} \quad\Rightarrow\quad L_1\begin{pmatrix} I_{n-r} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{n-r} \\ 0 \end{pmatrix}.$$

Moreover, we partition $B = (B_u, B_v)$ as in (2.3) and compute $BR_1 = (B_u, B_vR_{02}^{\perp})$. Thus the relation (4.26) becomes

$$B_u\bar{u}(0,\hat{x},t)+B_uP_0(P_0^T\mu_1)(0,\hat{x},t)+\left(B_uN+B_vR_{02}^{\perp}\widetilde{K}\right)w(0,\hat{x},t)=b(\hat{x},t).$$

For the ODE system (4.25) to have a bounded solution in $y\geq 0$, the initial data $w(0,\hat{x},t)$ should be given on the stable subspace span$\{R_2^S\}$ with R_2^S the right-stable matrix of the coefficient matrix $\left(\widetilde{K}^TX\widetilde{K}\right)^{-1}\left(\widetilde{K}^TS\widetilde{K}\right)$. Proposition 4.2 indicates that R_2^S should be an $(r-n_0-n_{10})\times(n_+-n_{1+}-n_{10})$ matrix. Thus we express $w(0,\hat{x},t)=R_2^Sw^S(\hat{x},t)$ with $w^S\in\mathbb{R}^{n_+-n_{1+}-n_{10}}$. Using this, we finally obtain

$$B_u\bar{u}(0,\hat{x},t)+B_uP_0(P_0^T\mu_1)(0,\hat{x},t)+\left(B_uNR_2^S+B_vR_{02}^{\perp}\widetilde{K}R_2^S\right)w^S(\hat{x},t)=b(\hat{x},t). \quad (4.27)$$

From this relation, we will derive the reduced boundary condition for $\bar{u}$ and determine $P_0^T\mu_1(0,\hat{x},t)$ and $w^S(\hat{x},t)$.

4.4. Reduced boundary conditions

To derive the reduced boundary condition from (4.27), we follow [33,40] and compute the right-stable matrix $R_M^S(\xi,\omega,\eta)$ in the characteristic GKC for sufficiently large η. Recall that both the characteristic GKC and the matrix defined in (3.6):

$$M(\xi,\omega,\eta)=\widehat{A}_1^{-1}\Big[G_{11}(\xi,\omega,\eta)-G_{10}(\xi,\omega,\eta)G_{00}^{-1}(\xi,\omega,\eta)G_{01}(\xi,\omega,\eta)\Big]$$

involve the matrix R_1. Firstly, we have

Lemma 4.1. *With the elaborately selected matrix R_1 in (4.20), the matrix $M(\xi,\omega,\eta)$ has the following form*

$$M(\xi,\omega,\eta)=\begin{pmatrix}A_{11} & \widehat{A}_{12}\\ \widehat{A}_{12}^T & \widehat{A}_{22}\end{pmatrix}^{-1}\left[\eta\begin{pmatrix}0&0\\0&\widehat{S}\end{pmatrix}-\begin{pmatrix}\xi I_{n-r}+C(\omega) & \star\\ \star & \star\end{pmatrix}\right]+O\Big(\frac{1}{\eta}\Big) \quad (4.28)$$

for sufficiently large η. Here

$$C(\omega)=i\sum_{j=2}^{d}\omega_jA_{j11},$$

the notation $\star$ means a matrix independent of the parameter η and the details are omitted.

Proof. Recall the definition in (3.2) that

$$G_{kl}(\xi,\omega,\eta)=\eta(R_k^TQR_l)+R_k^THR_l \quad \text{with} \quad H=-\xi I-i\sum_{j=2}^{d}\omega_jA_j. \quad (4.29)$$

For sufficiently large η, it is not difficult to compute the inverse matrix

$$G_{00}^{-1}(\xi,\omega,\eta)=\frac{1}{\eta}(R_0^TQR_0)^{-1}-\frac{1}{\eta^2}(R_0^TQR_0)^{-1}(R_0^THR_0)(R_0^TQR_0)^{-1}+O\Big(\frac{1}{\eta^3}\Big).$$

Thus we have

$$
\begin{aligned}
& G_{11}(\xi,\omega,\eta) - G_{10}(\xi,\omega,\eta)G_{00}^{-1}(\xi,\omega,\eta)G_{01}(\xi,\omega,\eta) \\
&= \eta\Big(R_1^T Q R_1 + \frac{1}{\eta} R_1^T H R_1\Big) - \eta\Big(R_1^T Q R_0 + \frac{1}{\eta} R_1^T H R_0\Big) \\
&\quad \Big((R_0^T Q R_0)^{-1} - \frac{1}{\eta}(R_0^T Q R_0)^{-1}(R_0^T H R_0)(R_0^T Q R_0)^{-1}\Big)\Big(R_0^T Q R_1 + \frac{1}{\eta} R_0^T H R_1\Big) + O\Big(\frac{1}{\eta}\Big) \\
&= \eta \widehat{Q} + \widehat{H} + O\Big(\frac{1}{\eta}\Big)
\end{aligned}
$$

with coefficients

$$
\begin{aligned}
\widehat{Q} &= R_1^T Q R_1 - R_1^T Q R_0 (R_0^T Q R_0)^{-1} R_0^T Q R_1, \\
\widehat{H} &= R_1^T H R_1 - (R_1^T Q R_0)(R_0^T Q R_0)^{-1}(R_0^T H R_1) - (R_1^T H R_0)(R_0^T Q R_0)^{-1}(R_0^T Q R_1) \\
&\quad + (R_1^T Q R_0)(R_0^T Q R_0)^{-1}(R_0^T H R_0)(R_0^T Q R_0)^{-1}(R_0^T Q R_1).
\end{aligned}
$$

Notice that the matrix $\widehat{Q}$ has been computed in (4.22) and $\widehat{H}$ can be reorganized as

$$
\widehat{H} = \Big[R_1^T - (R_1^T Q R_0)(R_0^T Q R_0)^{-1} R_0^T\Big] H \Big[R_1 - R_0 (R_0^T Q R_0)^{-1}(R_0^T Q R_1)\Big].
$$

By the definition of H in (4.29), we can write

$$
H = \begin{pmatrix} -\xi I_{n-r} - C(\omega) & \star \\ \star & \star \end{pmatrix} \quad \text{with} \quad C(\omega) = i \sum_{j=2}^{d} \omega_j A_{j11}.
$$

Moreover, we use the expression of R_1 in (4.20) to compute

$$
\begin{aligned}
& R_1 - R_0 (R_0^T Q R_0)^{-1} R_0^T Q R_1 \\
&= \begin{pmatrix} I_{n-r} & 0 \\ 0 & R_{02}^{\perp} \end{pmatrix} - \begin{pmatrix} R_{01} \\ R_{02} \end{pmatrix} \Big(R_{02}^T S R_{02}\Big)^{-1} \begin{pmatrix} R_{01}^T & R_{02}^T \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & S R_{02}^{\perp} \end{pmatrix} \\
&= \begin{pmatrix} I_{n-r} & \star \\ 0 & \star \end{pmatrix}.
\end{aligned}
$$

Thus it follows that

$$
\widehat{H} = \begin{pmatrix} I_{n-r} & 0 \\ \star & \star \end{pmatrix} \begin{pmatrix} -\xi I_{n-r} - C(\omega) & \star \\ \star & \star \end{pmatrix} \begin{pmatrix} I_{n-r} & \star \\ 0 & \star \end{pmatrix} = - \begin{pmatrix} \xi I_{n-r} + C(\omega) & \star \\ \star & \star \end{pmatrix}.
$$

This completes the proof. $\square$

Observe that, without the high-order term $O(1/\eta)$, the expression of $M(\xi,\omega,\eta)$ in (4.28) is the same as that in [40] by dropping out the symbol "$\widehat{\ }$". Thus we can refer to the argument in Section 4 of [40], with a series of subtle matrix transformations, to have the following lemma. Note that the high-order term does not affect the argument in [40].

Lemma 4.2 *([40]). As η goes to infinity, the right-stable matrix for $M(\xi,\omega,\eta)$ can be expressed by*

$$R_M^S(\xi,\omega,\infty)=\begin{pmatrix}\Big[P_1-P_0\big[\xi I_{n_{10}}+P_0^TC(\omega)P_0\big]^{-1}(P_0^TC(\omega)P_1)\Big]R_1^S & P_0 & NR_2^S\\ 0 & 0 & \widetilde{K}R_2^S\end{pmatrix}.$$

Here the matrices $\widetilde{K}$ and N are given in Proposition 4.2. Moreover, $R_1^S\in\mathbb{R}^{(n-r)\times n_{1+}}$ is the right-stable matrix for

$$M_1(\xi,\omega)=-\Lambda_1^{-1}\Big(\big[\xi I+P_1^TC(\omega)P_1\big]-P_1^TC(\omega)P_0\big[\xi I+P_0^TC(\omega)P_0\big]^{-1}P_0^TC(\omega)P_1\Big)$$

and $R_2^S\in\mathbb{R}^{(r-n_0-n_{10})\times(n_+-n_{1+}-n_{10})}$ is the right-stable matrix for

$$M_2=(\widetilde{K}^TX\widetilde{K})^{-1}(\widetilde{K}^T\widehat{S}\widetilde{K}).$$

Remark 4.2. M_2 is just the coefficient matrix for the ODE system (4.25). The matrix $M_1(\xi,\omega)$ is related to the Uniform Kreiss Condition (UKC) for the equilibrium system (4.3). To see this, we consider the corresponding eigenvalue problem of (4.3):

$$\xi\widehat{u}+A_{11}\widehat{u}_{x_1}+C(\omega)\widehat{u}=0\quad\text{with}\quad C(\omega)=i\sum_{j=2}^{d}\omega_jA_{j11}.$$

As in (3.3) and (3.5), we can easily deduce that

$$\begin{aligned}P_0^T\widehat{u}=&-[\xi I+P_0^TC(\omega)P_0]^{-1}P_0^TC(\omega)P_1(P_1^T\widehat{u}),\\(P_1^T\widehat{u})_{x_1}=&-\Lambda_1^{-1}\Big(\big[\xi I+P_1^TC(\omega)P_1\big]-P_1^TC(\omega)P_0\big[\xi I+P_0^TC(\omega)P_0\big]^{-1}P_0^TC(\omega)P_1\Big)(P_1^T\widehat{u})\\\equiv&\,M_1(\xi,\omega)(P_1^T\widehat{u}).\end{aligned}$$

By using the same argument as that in Section 3, the UKC for the boundary condition $\bar{B}\bar{u}(0,\hat{x},t)=\bar{b}(\hat{x},t)$ of the equilibrium system reads as

$$|\det\{\bar{B}P_1R_1^S\}|\geq c_K\sqrt{\det\{R_1^{S*}R_1^S\}}$$

with R_1^S the right-stable matrix for $M_1(\xi,\omega)$.

Now we state our main result with the partition $B=(B_u,B_v)$ as in Subsection 4.3.

Theorem 4.3. *Assume that the relaxation system (2.1) satisfies the structural stability condition, the assumption in (2.2), and the Kawashima-like condition (2.7). Moreover, we assume that the boundary condition (2.8) satisfies the assumption (2.9) and the characteristic generalized Kreiss condition. Then there exists a full-rank matrix $B_o \in \mathbb{R}^{n_{1+}\times n_+}$ such that the relation*

$$B_o B_u \bar{u}(0, \hat{x}, t) = B_o b(\hat{x}, t), \tag{4.30}$$

as a boundary condition for the equilibrium system (4.3), satisfies $B_o B_u P_0 = 0$ and the Uniform Kreiss Condition [23] for the characteristic IBVPs:

$$|det\{B_0 B_u P_1 R_1^S\}| \geq \bar{c}_K \sqrt{det\{R_1^{S*} R_1^S\}}$$

with $\bar{c}_K$ a positive constant and R_1^S the right-stable matrix of $M_1(\xi, \omega)$.

Proof. The proof is quite similar to that in [40]. For completeness, we present the full details here. Taking $\eta \to \infty$ in the characteristic GKC, we have

$$|\det\{B R_1 R_M^S(\xi, \omega, \infty)\}| \geq c_K \sqrt{\det\{R_M^{S*}(\xi, \omega, \infty) R_M^S(\xi, \omega, \infty)\}}.$$

This indicates the invertibility of $B R_1 R_M^S(\xi, \omega, \infty)$. By using the expression of $R_M^S(\xi, \omega, \infty)$ in Lemma 4.2, we can write

$$B R_1 R_M^S(\xi, \omega, \infty) = (Y_1, Y_2, Y_3)$$

with

$$\begin{aligned} Y_1 &= B_u\Big[P_1 - P_0\Big[\xi I_{n_{10}} + P_0^T C(\omega) P_0\Big]^{-1}(P_0^T C(\omega) P_1)\Big]R_1^S, \\ Y_2 &= B_u P_0, \qquad Y_3 = B_u N R_2^S + B_v R_{02}^{\perp} \widetilde{K} R_2^S. \end{aligned}$$

Notice that $Y_1 \in \mathbb{R}^{n_+\times n_{1+}}$, $Y_2 \in \mathbb{R}^{n_+\times n_{10}}$ and $Y_3 \in \mathbb{R}^{n_+\times(n_+-n_{1+}-n_{10})}$. Then there is a full-rank matrix $B_o \in \mathbb{R}^{n_{1+}\times n_+}$ such that

$$B_o \begin{pmatrix} Y_2 & Y_3 \end{pmatrix} = 0.$$

Taking $\widetilde{B}_o \in \mathbb{R}^{(n_+-n_{1+})\times n_+}$ such that $\begin{pmatrix} B_o \\ \widetilde{B}_o \end{pmatrix}$ is invertible, it follows that

$$\begin{pmatrix} B_o \\ \widetilde{B}_o \end{pmatrix} B R_1 R_M^S(\xi, \omega, \infty) = \begin{pmatrix} B_o \\ \widetilde{B}_o \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 & Y_3 \end{pmatrix} = \begin{pmatrix} B_o Y_1 & 0 & 0 \\ \widetilde{B}_o Y_1 & \widetilde{B}_o Y_2 & \widetilde{B}_o Y_3 \end{pmatrix}.$$

By Lemma 4.2, R_2^S (in Y_3) is a right-stable matrix of $M_2 = (\widetilde{K}^T X \widetilde{K})^{-1} \widetilde{K}^T \widehat{S} \widetilde{K}$, which is independent of parameters ξ and ω. Thus B_o and $\widetilde{B}_o$ are independent of the parameters and we have

$$
\begin{aligned}
|\det\{B_o Y_1\}| = &\left|\det\{(\widetilde{B}_o Y_2,\ \widetilde{B}_o Y_3)\}^{-1}\right| \left|\det\left\{\begin{pmatrix} B_o \\ \widetilde{B}_o \end{pmatrix}\right\}\right| \left|\det\{B R_1 R_M^S(\xi,\omega,\infty)\}\right| \\
&\geq c_1 \left|\det\{B R_1 R_M^S(\xi,\omega,\infty)\}\right|
\end{aligned}
$$

with $c_1 > 0$ a constant independent of the parameters. Since $B_o Y_2 = B_o B_u P_0 = 0$, it follows from the expression of Y_1 that $B_o Y_1 = B_o B_u P_1 R_1^S$. Thus the last inequality becomes

$$
|\det\{B_o B_u P_1 R_1^S\}| \geq c_1 \left|\det\{B R_1 R_M^S(\xi,\omega,\infty)\}\right|. \tag{4.31}
$$

Next we use Lemma 4.2 and rewrite

$$
R_M^S(\xi,\omega,\infty) = \begin{pmatrix} P_1 R_1^S & P_0 & N R_2^S \\ 0 & 0 & \widetilde{K} R_2^S \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ X_1 & I & 0 \\ 0 & 0 & I \end{pmatrix},
$$

where $X_1 = -[\xi I_{n_{10}} + P_0^T C(\omega) P_0]^{-1}(P_0^T C(\omega) P_1) R_1^S$. Moreover, it follows from the expression of N in Proposition 4.2:

$$
N = X_2 \widetilde{K} + P_0 X_3,
$$

with X_2 and X_3 clearly defined and independent of (ξ,ω), that

$$
\begin{pmatrix} P_1 R_1^S & P_0 & N R_2^S \\ 0 & 0 & \widetilde{K} R_2^S \end{pmatrix} = \begin{pmatrix} I & X_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} P_1 R_1^S & P_0 & 0 \\ 0 & 0 & \widetilde{K} R_2^S \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & X_3 R_2^S \\ 0 & 0 & I \end{pmatrix}.
$$

Thus we deduce from Lemma 3.3 in [33] that

$$
\begin{aligned}
\sqrt{\det\{R_M^{S*}(\xi,\omega,\infty) R_M^S(\xi,\omega,\infty)\}} &\geq c_2 \sqrt{\det\left\{\begin{pmatrix} R_1^{S*} P_1^T \\ P_0^T \end{pmatrix} \begin{pmatrix} P_1 R_1^S & P_0 \end{pmatrix}\right\} \det\left\{R_2^{S*} \widetilde{K}^T \widetilde{K} R_2^S\right\}} \\
&= c_2 \sqrt{\det\{R_1^{S*} R_1^S\} \det\left\{R_2^{S*} \widetilde{K}^T \widetilde{K} R_2^S\right\}}
\end{aligned}
$$

with $c_2 > 0$ depending only on X_2. Here we have used the orthogonality of P_0 and P_1. Note that $R_2^{S*} \widetilde{K}^T \widetilde{K} R_2^S$ is positive definite and independent of the parameters (ξ,ω). We conclude from the last inequality, together with (4.31) and the characteristic GKC, that there is a constant $\bar{c}_K$ such that

$$
|\det\{B_0 B_u P_1 R_1^S\}| \geq \bar{c}_K \sqrt{\det\{R_1^{S*} R_1^S\}}.
$$

This completes the proof. □

Thanks to Theorem 4.3, the outer solution $\bar{u}$ can be uniquely determined by solving the equilibrium system (4.3) with the reduced boundary condition (4.30) and proper initial data. Furthermore, we can use (4.27) to obtain the desired boundary conditions for the parabolic system (4.14) and initial conditions for the ODE system (4.25). Indeed, multiplying $\widetilde{B}_o$ on the left of (4.27) yields

$$\widetilde{B}_o \left(B_u P_0, \quad B_u N R_2^S + B_v R_{02}^{\perp} \widetilde{K} R_2^S \right) \begin{pmatrix} P_0^T \mu_1 \\ w^S \end{pmatrix} = \widetilde{B}_o b(\hat{x}, t) - \widetilde{B}_o B_u \bar{u}(0, \hat{x}, t). \tag{4.32}$$

Once $\bar{u}$ is solved, this is a system of $(n_+ - n_{1+})$ linear algebraic equations for the n_{10} variables $P_0^T \mu_1$ and $(n_+ - n_{1+} - n_{10})$ variables w^S. Following the proof of Theorem 4.3, we notice that the coefficient matrix is just $\widetilde{B}_o(Y_2, Y_3)$, which is invertible. Thus $P_0^T \mu_1$ and w^S can be uniquely determined from (4.32), giving the desired boundary and initial conditions.

5. Validity

In this section, we show the validity of the reduced boundary condition (4.30) by examining the discrepancy between the exact solution U^ϵ to the IBVP (2.1) with (2.8) and the solution to the equilibrium system with (4.30). For this purpose, we will estimate the L^2-error between U^ϵ and the asymptotic solution U_ϵ constructed in Section 3 when ϵ is small.

5.1. Asymptotic solutions

In order to focus on the boundary-layer behaviors, we choose the initial data such that the initial-layer can be neglected. To this end, the initial value $U_0(x_1, \hat{x})$ for the relaxation system (2.1) is taken to be in equilibrium:

$$U_0(x_1, \hat{x}) = \begin{pmatrix} u_0(x_1, \hat{x}) \\ 0 \end{pmatrix}, \tag{5.1}$$

where $u_0(x_1, \hat{x})$ represents the first $(n-r)$ components of $U_0(x_1, \hat{x})$. Moreover, we assume that the initial value $U_0(x_1, \hat{x})$ and the boundary condition in (2.8) are compatible:

$$BU_0(0, \hat{x}) = b(\hat{x}, 0), \qquad \text{for } \hat{x} \in \mathbb{R}^{d-1}. \tag{5.2}$$

From (5.1) and (5.2), we see that $B_o B_u u_0(0, \hat{x}) = B_o b(\hat{x}, 0)$, meaning that the reduced boundary condition (4.30) for the equilibrium system is compatible with the initial value $u_0(x_1, \hat{x})$.

About the asymptotic solution U_ϵ in (4.1), we recall from (4.4), (4.9), (4.13) that the coefficients satisfy

$$\bar{v} = 0, \quad \nu_1 = 0, \quad P_1^T \mu_1 = 0.$$

The terms $\bar{u}$, $P_0^T \mu_1$ and (μ_0, ν_0) solve the differential equations (4.3), (4.14) and (4.10) respectively. According to Theorem 4.3 and the discussion at the end of Section 4, proper boundary and initial conditions have been determined for these equations. The initial data for the PDEs (4.3) and (4.14) are the same as those in our previous work [40]:

$$\bar{u}(x_1, \hat{x}, 0) = u_0(x_1, \hat{x}), \qquad P_1^T \mu_1(z, \hat{x}, 0) \equiv 0,$$

which imply

$$U_\epsilon(x_1, \hat{x}, 0) = U_0(x_1, \hat{x}) = U^\epsilon(x_1, \hat{x}, 0). \tag{5.3}$$

At last, we determine μ_2 and ν_2 from the relations (4.15) and (4.12). Note that μ_2 is not fully determined by the equation (4.15). The interested reader is referred to Section 5.1 in [40] for further details. For the coefficients thus determined, we make the following regularity assumption.

Assumption 5.1. The expansion coefficients in (4.1) satisfy

$$\begin{cases} \bar{u},\ P_0^T \mu_1 \in L^2([0, T] \times H^1(\mathbb{R}^+ \times \mathbb{R}^{d-1})), \\ (\mu_0, \nu_0),\ (\mu_2, \nu_2) \in H^1([0, T] \times \mathbb{R}^+ \times \mathbb{R}^{d-1}). \end{cases}$$

Note that this regularity assumption implies the initial data $U_0 \in H^1(\mathbb{R}^+ \times \mathbb{R}^{d-1})$ according to the classical existence result [1].

5.2. *Error estimate*

Denote the difference between the exact solution U^ϵ and the asymptotic solution U_ϵ by

$$W := U^\epsilon - U_\epsilon.$$

We firstly derive the equation for W. To this end, we recall the linear operator

$$\mathcal{L}(U) := \partial_t U + \sum_{j=1}^{d} A_j \partial_{x_j} U - QU/\epsilon$$

and compute $\mathcal{L}(W) = \mathcal{L}(U^\epsilon) - \mathcal{L}(U_\epsilon)$. Since U^ϵ satisfies the equation (2.1), we know that $\mathcal{L}(U^\epsilon) = 0$. On the other hand, from (4.5) and (4.6) we know that

$$\mathcal{L}(U_\epsilon) = \mathcal{L}(U_{outer}) + \mathcal{L}(U_{layer}), \qquad \mathcal{L}(U_{outer}) = \begin{pmatrix} 0 \\ \sum_{j=1}^{d} A_{j12}^T \partial_{x_j} \bar{u} \end{pmatrix}.$$

Furthermore, we use (4.8) to compute $\mathcal{L}(U_{layer})$ in (4.7) as

$$\mathcal{L}(U_{layer}) = \begin{pmatrix} \partial_t \mu_0 + \sum_{j=2}^{d} \Big[A_{j11} \partial_{x_j} \mu_0 + A_{j12} \partial_{x_j} \nu_0 \Big] \\ \partial_t \nu_0 + \sum_{j=2}^{d} \Big[A_{j12}^T \partial_{x_j} (\mu_0 + \mu_1) + A_{j22} \partial_{x_j} \nu_0 \Big] + A_{12}^T \partial_z \mu_2 + A_{22} \partial_z \nu_2 \end{pmatrix}$$
$$+ \sqrt{\epsilon} \left[\partial_t \begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix} + \sum_{j=2}^{d} \begin{pmatrix} A_{j11} & A_{j12} \\ A_{j12}^T & A_{j22} \end{pmatrix} \partial_{x_j} \begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix} \right].$$

In summary, we can write

$$\mathcal{L}(U_\epsilon) = \mathcal{L}(U_{outer}) + \mathcal{L}(U_{layer}) = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

with E_1 and E_2 clearly defined. By the regularity Assumption 5.1, we see that $E_2 \in L^2([0,T] \times \mathbb{R}^+ \times \mathbb{R}^{d-1})$. Moreover, since

$$\int_{[0,T]\times R^+ \times R^{d-1}} \left|(\mu_0, \nu_0)\left(\frac{x_1}{\epsilon}, \hat{x}, t\right)\right|^2 dt dx_1 d\hat{x} = \epsilon \int_{[0,T]\times R^+ \times R^{d-1}} \left|(\mu_0, \nu_0)(y, \hat{x}, t)\right|^2 dt dy d\hat{x},$$

we have $\|E_1\|_{L^2([0,T]\times R^+ \times R^{d-1})} \le C\sqrt{\epsilon}$. Consequently, the equation for W is

$$\partial_t W + \sum_{j=1}^d A_j \partial_{x_j} W = \frac{1}{\epsilon}\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} W - \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \tag{5.4}$$

Next we turn to the initial and boundary conditions for W. Clearly, it follows from (5.3) that

$$W(x_1, \hat{x}, 0) \equiv 0. \tag{5.5}$$

From the boundary conditions in (2.8) and (4.26), it is easy to see that the boundary value of W satisfies

$$BW(0, \hat{x}, t) = \sqrt{\epsilon}\begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix}(0, \hat{x}, t) \equiv g(\hat{x}, t). \tag{5.6}$$

By Assumption 5.1, we have $\mu_2(0, \hat{x}, t), \nu_2(0, \hat{x}, t) \in L^2([0,T] \times \mathbb{R}^{d-1})$ and thereby

$$\|g\|_{L^2([0,T]\times\mathbb{R}^{d-1})} \le C\sqrt{\epsilon}.$$

Combining (5.4)-(5.6), we have the following IBVP for W:

$$\begin{cases} \partial_t W + \sum_{j=1}^d A_j \partial_{x_j} W = \frac{1}{\epsilon}\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} W - \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \\ BW(0, \hat{x}, t) = g(\hat{x}, t), \\ W(x_1, \hat{x}, 0) = 0. \end{cases}$$

To estimate W, we follow [11] and decompose $W = W_1 + W_2$. Here W_1 satisfies

$$\begin{cases} \partial_t W_1 + \sum_{j=1}^d A_j \partial_{x_j} W_1 = \frac{1}{\epsilon}\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} W_1 - \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \\ R_+^T W_1(0, \hat{x}, t) = 0, \\ W_1(x_1, \hat{x}, 0) = 0, \end{cases} \tag{5.7}$$

and W_2 satisfies

$$\begin{cases} \partial_t W_2 + \sum_{j=1}^{d} A_j \partial_{x_j} W_2 = \frac{1}{\epsilon}\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} W_2, \\ BW_2(0, \hat{x}, t) = g(\hat{x}, t) - BW_1(0, \hat{x}, t), \\ W_2(x_1, \hat{x}, 0) = 0. \end{cases} \tag{5.8}$$

Note that the boundary condition for W_1 is constructed artificially and the matrix R_+ consists of eigenvectors of A_1 associated with the positive eigenvalues. Since A_1 is symmetric, we can take R_+ such that (R_+, R_-, R_0) is an orthonormal matrix satisfying

$$\begin{pmatrix} R_+^T \\ R_-^T \\ R_0^T \end{pmatrix} A_1 \begin{pmatrix} R_+ & R_- & R_0 \end{pmatrix} = \begin{pmatrix} \Lambda_+ & & \\ & \Lambda_- & \\ & & 0 \end{pmatrix}. \tag{5.9}$$

Here Λ_+ and Λ_- are diagonal matrices whose entries are n_+ positive eigenvalues and $(n - n_0 - n_+)$ negative eigenvalues of A_1. Based on this decomposition, we take $R_1 = (R_+, R_-)$. Recall Remark 3.2 that the characteristic GKC does not rely on the choice of R_0 and R_1.

For IBVPs (5.7) and (5.8), we have the following conclusions.

Lemma 5.1. *The IBVP (5.7) has an unique solution $W_1 = W_1(x_1, \hat{x}, t)$ satisfying*

$$\begin{aligned} &\max_{t\in[0,T]} \|W_1(\cdot,\cdot,t)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})} + \|R_1^T W_1|_{x_1=0}\|^2_{L^2([0,T]\times\mathbb{R}^{d-1})} \\ &\le C(T)\Big(\|E_1\|^2_{L^2([0,T]\times\mathbb{R}^+\times\mathbb{R}^{d-1})} + \epsilon\|E_2\|^2_{L^2([0,T]\times\mathbb{R}^+\times\mathbb{R}^{d-1})}\Big). \end{aligned}$$

Here $C(T)$ is a generic constant depending only on T.

Lemma 5.2. *The IBVP (5.8) has an unique solution $W_2 = W_2(x_1, \hat{x}, t)$ satisfying*

$$\begin{aligned} &\max_{t\in[0,T]} \|W_2(\cdot,\cdot,t)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})} + \|R_1^T W_2|_{x_1=0}\|^2_{L^2([0,T]\times\mathbb{R}^{d-1})} \\ &\le C(T)\Big(\|R_1^T W_1|_{x_1=0}\|^2_{L^2([0,T]\times\mathbb{R}^{d-1})} + \|g\|^2_{L^2([0,T]\times\mathbb{R}^{d-1})}\Big). \end{aligned}$$

Recall that $\|E_2\|_{L^2([0,T]\times\mathbb{R}^+\times\mathbb{R}^{d-1})} \le C$, $\|E_1\|_{L^2([0,T]\times\mathbb{R}^+\times\mathbb{R}^{d-1})} \le C\sqrt{\epsilon}$, and $\|g\|_{L^2([0,T]\times\mathbb{R}^{d-1})} \le C\sqrt{\epsilon}$. The above two lemmas immediately give

Theorem 5.3. *Assume that the relaxation system (2.1) satisfies the structural stability condition, the assumption in (2.2), and the Kawashima-like condition (2.7). Assume that the boundary condition (2.8) satisfies the assumption (2.9) and the characteristic generalized Kreiss condition. The initial data $U_0 \in H^1(\mathbb{R}^+ \times \mathbb{R}^{d-1})$ given in (5.1) is assumed to be in equilibrium and consistent with the boundary condition. Namely, the relation (5.2) holds. In addition, we assume* *the regularity Assumption 5.1 and the constant hyperbolicity assumption [1]. Then there exists a constant $C(T) > 0$ such that the error estimate*

$$\|(U^\epsilon - U_\epsilon)(\cdot,\cdot,t)\|_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})} \le C(T)\epsilon^{1/2}$$

holds for all time $t \in [0,T]$. Here U^ϵ and U_ϵ respectively represent the exact solution and the asymptotic solution to the relaxation system (1.4) with boundary condition (2.8) and initial data (5.1).

From this theorem and the expansion of U_ϵ in (4.1), we can immediately see the L^2-convergence of U^ϵ to the solution $(\bar{u}, \bar{v})$ of the equilibrium system with the reduced boundary condition (4.30) as ϵ goes to zero.

It remains to prove Lemma 5.1 and Lemma 5.2, which is similar to those in [40]. For completeness, we present the details here.

Proof of Lemma 5.1. Clearly, the boundary condition $R_+^T W_1(0,\hat{x},t) = 0$ satisfies the Uniform Kreiss Condition. Thus it follows from the existence theory in [1] that there exists a unique solution $W_1 \in C([0,T]; L^2(\mathbb{R}^+\times\mathbb{R}^{d-1}))$.

For the estimate, we multiply (5.7) with W_1^T from the left to obtain

$$\frac{d}{dt}(W_1^T W_1) + \sum_{j=1}^{d}(W_1^T A_j W_1)_{x_j} = \frac{2}{\epsilon} W_1^T \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} W_1 - 2(W_1^I)^T E_1 - 2(W_1^{II})^T E_2.$$

Here W_1^I represents the first $(n-r)$ components of W_1 and W_1^{II} represents the other r components. From the last equation, we use the negative-definiteness of S and derive

$$\begin{aligned}\frac{d}{dt}(W_1^T W_1) + \sum_{j=1}^{d}(W_1^T A_j W_1)_{x_j} &\le -\frac{c_0}{\epsilon}|W_1^{II}|^2 + 2|(W_1^{II})^T E_2| + 2|(W_1^I)^T E_1| \\ &\le -\frac{c_0}{2\epsilon}|W_1^{II}|^2 + \frac{2\epsilon}{c_0}|E_2|^2 + |E_1|^2 + |W_1|^2\end{aligned}$$

with $c_0 > 0$ a constant. Integrating the last inequality over $(x_1,\hat{x}) \in [0,+\infty)\times\mathbb{R}^{d-1}$ yields

$$\begin{aligned}&\frac{d}{dt}\|W_1(\cdot,\cdot,t)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})} - \int_{\mathbb{R}^{d-1}} W_1^T(0,\hat{x},t)A_1 W_1(0,\hat{x},t)d\hat{x} \\ &\le \frac{2\epsilon}{c_0}\|E_2\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})} + \|E_1\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})} + \|W_1\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})}. \qquad (5.10)\end{aligned}$$

Since $A_1 = R_+\Lambda_+R_+^T + R_-\Lambda_-R_-^T$ and $R_+^T W_1(0,\hat{x},t) = 0$, it follows that

$$\begin{aligned}-W_1^T(0,\hat{x},t)A_1W_1(0,\hat{x},t) &= -W_1^T(0,\hat{x},t)(R_+\Lambda_+R_+^T + R_-\Lambda_-R_-^T)W_1(0,\hat{x},t) \\ &= -W_1^T(0,\hat{x},t)R_-\Lambda_-R_-^T W_1(0,\hat{x},t) \\ &\ge c_1 W_1^T(0,\hat{x},t)R_-R_-^T W_1(0,\hat{x},t)\end{aligned}$$

$$= c_1 |R_1^T W_1(0, \hat{x}, t)|^2 \tag{5.11}$$

with c_1 a positive constant. Applying Gronwall's inequality to (5.10), we have

$$\max_{t\in[0,T]} \|W_1(\cdot,\cdot,t)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})} \le Ce^{CT}\left(\epsilon\|E_2\|^2_{L^2([0,T]\times\mathbb{R}^+\times\mathbb{R}^{d-1})} + \|E_1\|^2_{L^2([0,T]\times\mathbb{R}^+\times\mathbb{R}^{d-1})}\right).$$

At last, we use (5.11) and integrate (5.10) over $t \in [0, T]$ to obtain

$$\|R_1^T W_1|_{x_1=0}\|^2_{L^2([0,T]\times\mathbb{R}^{d-1})} \le C(T)\left(\epsilon\|E_2\|^2_{L^2([0,T]\times\mathbb{R}^+\times\mathbb{R}^{d-1})} + \|E_1\|^2_{L^2([0,T]\times\mathbb{R}^+\times\mathbb{R}^{d-1})}\right).$$

This completes the proof of Lemma 5.1. □

Proof of Lemma 5.2. By Remark 3.1, the Uniform Kreiss Condition is implied by the characteristic GKC. Thus the existence theory in [1] indicates that there exists a unique solution $W_2 \in C([0,T]; L^2(\mathbb{R}^+ \times \mathbb{R}^{d-1}))$.

Next we adapt the method in [11] to obtain the estimate. Define the Fourier transform of the solution to (5.8) with respect to $\hat{x}$:

$$\widetilde{W}_2(x_1,\omega,t) = \int\limits_{\mathbb{R}^{d-1}} e^{i\omega\hat{x}} W_2(x_1,\hat{x},t)d\hat{x}, \qquad \omega = (\omega_2,\omega_3,\ldots,\omega_d) \in \mathbb{R}^{d-1}$$

and the Laplace transform with respect to t:

$$\widehat{W}_2(x_1,\omega,\xi) = \int_0^\infty e^{-\xi t}\widetilde{W}_2(x_1,\omega,t)dt, \qquad Re\xi > 0.$$

Then we deduce from (5.8) that

$$\begin{cases} A_1\partial_{x_1}\widehat{W}_2 = \left(\eta Q - \xi I_n - i\sum_{j=2}^d \omega_j A_j\right)\widehat{W}_2, \\ B\widehat{W}_2(0,\omega,\xi) = \widehat{g}(\omega,\xi) - B\widehat{W}_1(0,\omega,\xi), \\ \|\widehat{W}_2(\cdot,\omega,\xi)\|_{L^2(\mathbb{R}^+)} < \infty \quad \text{for a.e. } \xi,\ \omega. \end{cases} \tag{5.12}$$

Here $\eta = 1/\epsilon$. From Section 3, the differential equations in (5.12) can be reduced to

$$\partial_{x_1}(R_1^T\widehat{W}_2) = M(\xi,\omega,\eta)(R_1^T\widehat{W}_2), \tag{5.13}$$

$$R_0^T\widehat{W}_2 = -G_{00}^{-1}(\xi,\omega,\eta)G_{01}(\xi,\omega,\eta)(R_1^T\widehat{W}_2). \tag{5.14}$$

Since $BR_0 = 0$ and $I_n = R_0^T R_0 + R_1^T R_1$, the boundary condition can be written as

$$(BR_1)(R_1^T\widehat{W}_2)(0,\omega,\xi) = \widehat{g}(\omega,\xi) - (BR_1)(R_1^T\widehat{W}_1)(0,\omega,\xi). \tag{5.15}$$

Let $R_M^S = R_M^S(\xi, \omega, \eta)$ and $R_M^U = R_M^U(\xi, \omega, \eta)$ be the respective right-stable and right-unstable matrices of $M = M(\xi, \omega, \eta)$:

$$M R_M^S = R_M^S M^S, \qquad M R_M^U = R_M^U M^U,$$

where M^S is a stable-matrix and M^U is an unstable-matrix. In view of the Schur decomposition, we may choose R_M^S and R_M^U such that

$$\begin{pmatrix} R_M^{S*} \\ R_M^{U*} \end{pmatrix} \begin{pmatrix} R_M^S & R_M^U \end{pmatrix} = I_n.$$

Recall that the superscript $*$ denotes the conjugate transpose. Then from (5.13) we obtain

$$\begin{pmatrix} R_M^{S*} \\ R_M^{U*} \end{pmatrix} \partial_{x_1}(R_1^T \widehat{W}_2) = \begin{pmatrix} M^S & \\ & M^U \end{pmatrix} \begin{pmatrix} R_M^{S*} \\ R_M^{U*} \end{pmatrix} (R_1^T \widehat{W}_2).$$

Since $\|\widehat{W}_2(\cdot, \omega, \xi)\|_{L^2(\mathbb{R}^+)} < \infty$ for a.e. (ξ, ω) and M^U is an unstable-matrix, it must be

$$R_M^{U*}(R_1^T \widehat{W}_2) = 0.$$

Thus the boundary condition in (5.15) becomes

$$BR_1(R_1^T \widehat{W}_2)(0, \omega, \xi) = BR_1 R_M^S R_M^{S*}(R_1^T \widehat{W}_2)(0, \omega, \xi) = \widehat{g}(\omega, \xi) - BR_1(R_1^T \widehat{W}_1)(0, \omega, \xi).$$

Since the matrix $(BR_1 R_M^S)^{-1}$ is uniformly bounded due to the characteristic GKC, we conclude that

$$\begin{aligned} \left|(R_1^T \widehat{W}_2)(0, \omega, \xi)\right|^2 &= \left|R_M^S R_M^{S*}(R_1^T \widehat{W}_2)(0, \omega, \xi)\right|^2 \\ &= \left|R_M^S (BR_1 R_M^S)^{-1} \left[\widehat{g}(\omega, \xi) - BR_1(R_1^T \widehat{W}_1)(0, \omega, \xi)\right]\right|^2 \\ &\le C\left(|\widehat{g}(\omega, \xi)|^2 + |(R_1^T \widehat{W}_1)(0, \omega, \xi)|^2\right). \end{aligned}$$

By Parseval's identity, the last inequality leads to

$$\begin{aligned} &\int_{\mathbb{R}^{d-1}} \int_0^\infty e^{-2tRe\xi} |(R_1^T W_2)(0, \hat{x}, t)|^2 dt d\hat{x} \\ &\le C\Big(\int_{\mathbb{R}^{d-1}} \int_0^\infty e^{-2tRe\xi} \left|g(\hat{x}, t)\right|^2 dt d\hat{x} + \int_{\mathbb{R}^{d-1}} \int_0^\infty e^{-2tRe\xi} \left|(R_1^T W_1)(0, \hat{x}, t)\right|^2 dt d\hat{x}\Big) \\ &\le C\Big(\int_{\mathbb{R}^{d-1}} \int_0^\infty \left|g(\hat{x}, t)\right|^2 dt d\hat{x} + \int_{\mathbb{R}^{d-1}} \int_0^\infty \left|(R_1^T W_1)(0, \hat{x}, t)\right|^2 dt d\hat{x}\Big) \quad \text{for } Re\xi > 0. \end{aligned}$$

Because the right-hand side is independent of $Re\xi$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \int_0^\infty |(R_1^T W_2)(0,\hat{x},t)|^2 dt d\hat{x} \\
&\leq C\Big(\int_{\mathbb{R}^{d-1}} \int_0^\infty \big|g(\hat{x},t)\big|^2 dt d\hat{x} + \int_{\mathbb{R}^{d-1}} \int_0^\infty \big|(R_1^T W_1)(0,\hat{x},t)\big|^2 dt d\hat{x} \Big).
\end{aligned}$$

By using the trick from [11], the integral interval $[0,\infty)$ in the last inequality can be changed to $[0,T]$:

$$\|R_1^T W_2|_{x_1=0}\|^2_{L^2([0,T]\times\mathbb{R}^{d-1})} \leq C\Big(\|R_1^T W_1|_{x_1=0}\|^2_{L^2([0,T]\times\mathbb{R}^{d-1})} + \|g\|^2_{L^2([0,T]\times\mathbb{R}^{d-1})} \Big). \tag{5.16}$$

At last, we multiply the equation in (5.8) with W_2^T from the left to get

$$\frac{d}{dt}(W_2^T W_2) + \partial_{x_1}(W_2^T A_1 W_2) + \sum_{j=2}^d \partial_{x_j}(W_2^T A_j W_2) \leq 0.$$

Integrating the last inequality over $(x_1,\hat{x}) \in [0,+\infty)\times\mathbb{R}^{d-1}$ and $t\in[0,T]$ yields

$$\begin{aligned}
\max_{t\in[0,T]} \|W_2(\cdot,\cdot,t)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})} &\leq C \int_{\mathbb{R}^{d-1}} \int_0^T \big|(W_2^T A_1 W_2)(0,\hat{x},t)\big|^2 dt d\hat{x} \\
&= C \int_{\mathbb{R}^{d-1}} \int_0^T \big|(W_2^T R_1)(R_1^T A_1 R_1)(R_1^T W_2)(0,\hat{x},t)\big|^2 dt d\hat{x} \\
&\leq C \int_{\mathbb{R}^{d-1}} \int_0^T |(R_1^T W_2)(0,\hat{x},t)|^2 dt d\hat{x}.
\end{aligned}$$

This together with (5.16) completes the proof. □

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