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On the non-vanishing of the D'Arcais polynomials

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Abstract
In this paper we invest in the non-vanishing of the Fourier coefficients of powers of the Dedekind eta function. This is reflected in non-vanishing properties of the D'Arcais polynomials. We generalize and improve results of Heim–Luca–Neuhauser and Żmija. We apply methods from algebraic number theory.

Keywords Algebraic number theory · Dedekind eta function · Generating functions · Recurrence relations

Mathematics Subject Classification Primary 05A17 · 11P82 · Secondary 05A20 · 11R04

1 Introduction

In this paper, we study the vanishing properties of the q-expansion of rth powers of the Dedekind η-function. Euler and Jacobi [10, 21] studied the odd cases $r = 1$ and $r = 3$ via explicit formulas involving pentagonal and triangular numbers. Serre [23] considered the even case and proved that the sequence of coefficients is lacunary if and only if $r \in S_{\text{even}} := \{0, 2, 4, 6, 8, 10, 14, 26\}$. Lehmer [13] conjectured that the coefficients for $r = 24$, which involves the discriminant function, called Ramanujan tau-function, never vanish. We also refer to Ono's speculation [20] for $r = 12$. In general not much is known for integer powers. In this paper we allow complex powers and consider all of them simultaneously, which leads to the study of D'Arcais polynomials $P_n^{\sigma}(x)$ [4, 8]. Polynomization has recently become a very active research field [15].

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The D'Arcais polynomials $P_n^{\sigma}(x)$ [1, 2, 6] dictate the properties of the coefficients of the powers of the Dedekind η-function [8, 19, 23]. The Dedekind η-function [21] is a modular form of weight 1/2 and defined on the complex upper half-plane $\mathbb{H}$. Let $\tau \in \mathbb{H}$ and $q := e^{2\pi i \tau}$. Then

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - q^n\right).$$

One of the crucial facts is that the nth coefficients of the q-expansion of zth powers of the infinite product $\prod_{n=1}^{\infty} (1 - q^n)$ are polynomial in z of degree n. These are the D'Arcais polynomials. In combinatorics they are called Nekrasov–Okounkov polynomials [3, 18, 24].

Let $\sigma(n) := \sum_{d|n} d$ and $z \in \mathbb{C}$. Then

$$\sum_{n=0}^{\infty} P_n^{\sigma}(z)\, q^n := \prod_{n=1}^{\infty} \left(1 - q^n\right)^{-z} = \exp\left(z \sum_{n=1}^{\infty} \sigma(n) \frac{q^n}{n} \right).$$

Nekrasov and Okounkov [3, 18, 24] discovered a new type of hook length formula derived from random partitions and the Seiberg–Witten theory.

Let λ be a partition of n denoted by $\lambda \vdash n$ with weight $|\lambda| = n$ and $\mathcal{H}(\lambda)$ the multiset of hook lengths associated with λ. Let $\mathcal{P}$ be the set of all partitions. Then the Nekrasov–Okounkov hook length formula is given by

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{m=1}^{\infty} \left(1 - q^m\right)^{z-1}. \tag{1}$$

Note that $A_n^{\sigma}(x) := n!\, P_n^{\sigma}(x) \in \mathbb{N}_0[x]$ are monic. Therefore, the zeros are algebraic integers. For example, this implies that the coefficients of $\prod_{n=1}^{\infty} (1 - q^n)^{\frac{1}{2}}$ are rational and non-vanishing.

By utilizing results from representation theory of simple complex Lie algebras, Kostant [11] proved that $P_n^{\sigma}\left(-(m^2 - 1)\right)$ does not vanish for $m \geq n$. Han deduced from (1) that for real x, we have $P_n^{\sigma}(x) \neq 0$ if $|x| \geq n^2 - 1$. Recently, these results have been significantly improved [7]. Let $c := 9.7226$ and $x \in \mathbb{C}$. Then $P_n^{\sigma}(x) \neq 0$ for all $|x| > c\,(n - 1)$. It is known that c can not be smaller than 9.72245. There is a conjecture [5] on the real part of the zeros of $P_n^{\sigma}(x)$. Recall that a polynomial is called Hurwitz polynomial or stable polynomial if the real part of its zeros is negative.

Conjecture (Heim, Neuhauser [5]) *Let $n \geq 1$. The D'Arcais polynomials $P_n^{\sigma}(x)$ divided by x are Hurwitz polynomials and the zeros are simple.*

In particular it would be interesting to know if there are no zeros on the imaginary axes. As a first result in this direction, the following is known.

Theorem 1 (Heim, Luca, Neuhauser [4]) *Let $m \geq 3$ and let ζ_m be a primitive root of unity. Then for all $n \in \mathbb{N}$:*

$$P_n^{\sigma}(\zeta_m) \neq 0.$$

In particular $P_n^{\sigma}(\pm i) \neq 0$.

Recently, Żmija [25] in his doctoral thesis developed a remarkable generalization of Theorem 1. Let $g : \mathbb{N} \longrightarrow \mathbb{Z}$ with $g(1) = 1$ be a normalized arithmetic function. Define

$$P_n^g(x) := \frac{x}{n} \sum_{k=1}^{n} g(k)\, P_{n-k}^g(x), \quad n \geq 1,$$

with initial value $P_0^g(x) = 1$ ([8, 25]).

Theorem 2 (Żmija [25]) *Let g be a normalized $\mathbb{Z}$-valued arithmetic function and $P_n^g(x)$ integer-valued polynomials for all $n \in \mathbb{N}$. Let $m \geq 3$ and ζ_m be a primitive root of unity. Let $A_n^g(x) := n!\, P_n^g(x)$. Assume*

(1) *modulo 5: none of the polynomials $A_3^g(x)$ and $A_4^g(x)$ is divisible by a monic irreducible polynomial of degree 2 over $\mathbb{F}_5$.*
(2) *modulo 7: none of the polynomials $A_r^g(x)$ for $2 \leq r \leq 6$ is divisible by a monic irreducible polynomial of degree 4 over $\mathbb{F}_7$.*
(3) *modulo 11: none of the polynomials $A_r^g(x)$ for $2 \leq r \leq 10$ is divisible by a monic irreducible polynomial over $\mathbb{F}_{11}$ that divides $x^{11^6-1} - 1$ and does not divide $x^{11^d-1} - 1$ for $1 \leq d \leq 10$, $d \neq 6$.*

Then $P_n^g(\zeta_m) \neq 0$ for all mth roots of unity ζ_m of order at least 3. In particular,

$$P_n^g(\pm i) \neq 0.$$

The proofs of Theorem 1 and Theorem 2 are based on a careful analysis of $A_n^g(x) \pmod{p}$ for all prime numbers p and an analytic argument based on properties of the Chebyshev function obtained by Rosser and Schoenfeld [22] and the utilization of the computer algebra system Mathematica.

In this paper we provide a new proof of the theorems and show that the non-vanishing of $P_n^g(x)$ at values related to roots of unity in essence depend on the decomposition $A_n^g(x) \pmod 2$ and $A_n^g(x) \pmod 3$ in $\mathbb{F}_2[x]$ and $\mathbb{F}_3[x]$. Another ingredient of our proofs will be the prime ideal decomposition of $p\mathcal{O}_K$ in the cyclotomic field $K = \mathbb{Q}(\zeta_m)$, where $\mathcal{O}_K$ is the ring of integers. Further, we introduce a certain algebraic integer α with $K = \mathbb{Q}(\alpha)$ and $p = 2$ or $p = 3$ does not divide the index $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$. Finally, we utilize the Dedekind–Kummer Theorem to obtain our results.

We begin by improving Theorem 1 and Theorem 2. The assumption that the polynomials $P_n^g(x)$ are integer-valued is no longer needed.

Theorem 3 *Let g be a normalized $\mathbb{Z}$-valued arithmetic function. Let* $g(3) \equiv 0, 1 \pmod 3$. *Let $m \geq 3$ and ζ_m be an mth primitive root of unity. Then for all $n \in \mathbb{N}$: $P_n^g(\zeta_m) \neq 0$. In particular, $P_n^g(\pm i) \neq 0$.*

Note that $\sigma(3) \equiv 1 \pmod 3$. Therefore $P_n^{\sigma}(\zeta_m) \neq 0$ for all $n \in \mathbb{N}$ and $m \geq 3$. Further, $P_n^{\sigma}(\zeta_2) = 0$ if and only if n is not a (generalized) pentagonal number.
To state our next result we introduce the following notation. Let $\mathcal{Z}_n^g$ be the set of zeros of $P_n^g(x)$. The set of all zeros is a subset of the set of algebraic integers and is denoted by

$$\mathcal{Z}^g = \bigcup_{n=1}^{\infty} \mathcal{Z}_n^g \subset \overline{\mathbb{Z}}.$$

Theorem 4 *Let g be a normalized integer-valued arithmetic function. Let $m \geq 3$ and ζ_m be an mth primitive root of unity. Let K be the mth cyclotomic field and $\mathcal{O}_K$ the ring of integers. Then we have the following.*

i) *Let a prime $p \neq 2$ exist such that $p \mid m$. Then*

$$\{\zeta_m + 2\beta \,:\, \beta \in \mathcal{O}_K\} \cap \mathcal{Z}^g = \emptyset.$$

ii) *Let $m \geq 1$ be not a power of* 3 *or twice a power of* 3. *Let* $g(3) \equiv 0, 1 \pmod 3$. *Then*

$$\{\zeta_m + 3\beta \,:\, \beta \in \mathcal{O}_K\} \cap \mathcal{Z}^g = \emptyset.$$

Corollary 1 *Let g be a normalized integer-valued arithmetic function. Let* $g(3) \equiv 0, 1 \pmod 3$. *Then we have for all $m \geq 3$ and $n \geq 1$ that $P_n^g(\zeta_m + 6\beta) \neq 0$ for all $\beta \in \mathbb{Z}[\zeta_m]$.*

2 Results from algebraic number theory

We recall basic notation and results. For further details, we refer to [12, 14, 16, 17].

2.1 Dedekind–Kummer

Let $K \supseteq \mathbb{Q}$ be a number field and $\mathcal{O}_K$ the ring of integers in K. There always exists an element α with $K = \mathbb{Q}(\alpha)$ called primitive element. Let us assume that $\alpha \in \mathcal{O}_K$. Let $\mathcal{O}_{K,\alpha} := \mathbb{Z}[\alpha]$, which is an order in K. The index

$$\kappa_\alpha := [\mathcal{O}_K : \mathcal{O}_{K,\alpha}].$$

is finite. Let $\mathrm{Min}_\alpha(x) \in \mathbb{Z}[x]$ be the minimal polynomial of α. Let Δ_K be the discriminant of K. The discriminant $\mathrm{disc}(\mathrm{Min}_\alpha)$ of the minimal polynomial satisfies

$$\mathrm{disc}(\mathrm{Min}_\alpha) = \kappa_\alpha^2 \, \Delta_K.$$

The norm $N(\mathfrak{a})$ of an ideal $\mathfrak{a} \neq \{0\}$ is defined as the finite index of $\mathfrak{a}$ in $\mathcal{O}_K$. Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_K$, different from $\{0\}$ and $\mathcal{O}_K$. Then $\mathfrak{a}$ decomposes uniquely into a finite product of prime ideals, up to the order, since $\mathcal{O}_K$ is a Dedekind domain. Let

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdot \mathfrak{p}_2^{e_2} \cdots \mathfrak{p}_g^{e_g}. \tag{2}$$

Then e_k denotes the ramification index of $\mathfrak{p}_k$. Let f_k be the residue class degree or inertial degree of $\mathfrak{p}_k$ determined by the norm $N(\mathfrak{p}_k) = p^{f_k}$. Here $\mathbb{Z} \cap \mathfrak{p}_k = p\mathbb{Z}$. A prime p is called ramified if at least one ramification index e_k in the decomposition of $\mathfrak{a} = p\mathcal{O}_K$ is larger than 1. It is known that a prime p is ramified if and only if $p \mid \Delta_K$.

Rather than first specifying the field, we start with an algebraic integer α. We define $K := \mathbb{Q}(\alpha)$ and have

$$K \supset \mathcal{O}_K \supseteq \mathcal{O}_{K,\alpha}.$$

We are interested in the prime ideal decomposition of the principal ideals $p\mathcal{O}_K$ in $\mathcal{O}_K$ and the associated ramification indices and inertia degrees. This is a consequence of the Dedekind–Kummer theorem.

Theorem 5 (Dedekind–Kummer) *Let α be an algebraic integer and the primitive element of an algebraic number field $K := \mathbb{Q}(\alpha)$. Let p be any prime with $p \nmid \kappa_\alpha$. Let* $\mathrm{Min}_\alpha(x)$ *be the minimal polynomial of α. Then*

$$p\mathcal{O}_K = \prod_k \mathfrak{p}_k^{e_k} \iff \mathrm{Min}_\alpha(x) \equiv \prod_k \left(\mathrm{Min}_{\alpha,k}(x)\right)^{e_k} \pmod{p},$$

where the polynomials $\mathrm{Min}_{\alpha,k}(x) \in \mathbb{F}_p[x]$ *are irreducible. Up to the order we have* $\mathrm{degree}_{\mathbb{F}_p} \mathrm{Min}_{\alpha,k}(x) = f_k$.

Next, we take a closer look at cyclotomic fields.

2.2 Cyclotomic fields

Let $m \geq 1$. We denote by ζ_m a primitive mth root of unity. Let $K_m := \mathbb{Q}(\zeta_m)$ be the mth cyclotomic field. Then $K = K_m$ has a power basis with $\mathcal{O}_K = \mathbb{Z}[\zeta_m]$. Moreover, $K_m/\mathbb{Q}$ is a Galois extension of degree $\varphi(m)$, where φ is the Euler totient function. Further, p is ramified in $\mathcal{O}_K$ if and only if $p \mid m$. Since we have a Galois extension, the prime ideal decomposition (2) simplifies:

$$p\mathcal{O}_K = \prod_{k=1}^{g} \mathfrak{p}_k^{e}, \text{ where } f_k = f \text{ and } \varphi(m) = e\, f\, g.$$

The inertial degree f of p in $\mathcal{O}_K$ can be explicitly calculated. Let m_p be the largest divisor of m, which is coprime to p. Then, it is well-known that f is the smallest positive integer such that

$$p^f \equiv 1 \pmod{m_p}.$$

Let R_p be the set of $m \geq 1$ such that the inertial degree f of $p\mathcal{O}_{K_m}$ is 1 (note that f is unique since cyclotomic fields are Galois extensions). This leads to

$$R_2 = \left\{2^\ell \,:\, \ell \in \mathbb{N}_0\right\} \text{ and } R_3 = \left\{2^a\,3^\ell \,:\, a \in \{0,1\} \text{ and } \ell \in \mathbb{N}_0\right\}.$$

2.3 Proof of Theorem 3

We first consider $A_n^g(x) \pmod{p}$.

Lemma 1 *Let g be a normalized $\mathbb{Z}$-valued arithmetic function. Let p be a prime number. Then we have*

$$A^g_{\ell\, p+r}(x) \equiv A_r^g(x)\, \left(A_p^g\,(x)\right)^\ell \pmod{p}, \quad \textit{where } 0 \leq r < p.$$

Further, let $\{P_n^g(x)\}_n$ be integer-valued polynomials. Then

$$A_p^g(x) \equiv x\,(x-1)\ldots(x-p+1) \pmod{p}.$$

Proof For $g=\sigma$ the proof is given in [4]. Let the polynomials be integer-valued. Then the proof is given by Żmija ([25], Lemma 5). The general case is proven in the same way. The basic ingredient is provided by

$$A_n^g(x) = x\left(\sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!}\; g(k)\; A^g_{n-k}(x)\right).$$

Then we reduce $\pmod{p}$ the following polynomials

$$A^g_{\ell\, p+1}(x), A^g_{\ell\, p+2}(x), \ldots, A^g_{(\ell+1)\, p}(x)$$

step by step to $A^g_{\ell\, p}(x)$. □

We have $A_0^g(x) = 1$, $A_1^g(x) = x$ and $A_2^g(x) = x\,(x+g(2))$. Therefore, $A_n^g(x) \pmod{2}$ decomposes into linear factors for all n.

Let $m = 2^\ell$, $\ell \geq 2$. Then we study $A_n^g(x) \pmod{3}$, which is essentially $A_3^g(x) \pmod{3}$.

Lemma 2 *Let g be a normalized $\mathbb{Z}$-valued arithmetic function. Then* $A_3^g(x) \equiv x^2\,\,(x+3g(2)) \pmod{2}$. *We have*

$$A_3^g(x) \equiv x\left(x^2 - g(3)\right) \pmod{3}.$$

Therefore, $A_3^g(x) \pmod{3}$ *decomposes into linear factors if and only if* $g(3) \equiv 0, 1 \pmod{3}$.

Proof We have

$$A_3^g(x) = x\left(x^2 + 3g(2)x + 2g(3)\right).$$

Then the solutions of $A_3^g(x) = 0$ are $x_1 = 0$ and

$$x_{2/3} = \frac{-3g(2) \pm \sqrt{9g(2)^2 - 8g(3)}}{2}.$$

Therefore, there $A_3^g(x)/x \pmod 3$ is irreducible if and only if $g(3)$ is not a quadratic residue $\pmod 3$. □

Suppose $P_n^g(\zeta_m) = 0$. Then Min_{ζ_m} divides $A_n^g(x)$. But since $2^\ell \notin R_3$, we have that $\mathrm{Min}_{\zeta_m}(x) \pmod 3$ does not decompose into linear factors, which is a contradiction to the assumption that $P_n^g(\zeta_m) = 0$.

Now, let $m \neq 2^\ell$ and $m \geq 3$. Then $A_n^g(x) \pmod 2$ decomposes into linear factors. Suppose $P_n^g(\zeta_m) = 0$. Then Min_{ζ_m} divides $A_n^g(x)$. But since $m \notin R_2$, the same argument as before leads to a contradiction. Therefore, Theorem 3 is proven.

2.4 Proof of Theorem 4

Let $m \geq 3$ and $K = \mathbb{Q}(\zeta_m) \supset \mathcal{O}_K$. The following result gives us control over the index $\kappa_\alpha = [\mathcal{O}_K : \mathbb{Z}[\alpha]]$ of some algebraic integer α with $K = \mathbb{Q}(\alpha)$.

Lemma 3 *Let ζ_m be a primitive root of unity for $m \geq 1$. Let p be a prime number and μ an integer, such that $p \mid \mu$. Let $K = \mathbb{Q}[\zeta_m]$ and $\mathcal{O}_K$ the ring of integers. Let*

$$\alpha := \alpha_\beta = \zeta_m + \mu\,\beta, \text{ where } \beta \in \mathcal{O}_K.$$

Let $\mathcal{O}_{K,\alpha} := \mathbb{Z}[\alpha]$ be the order associated to α. Then p is coprime to the index

$$\kappa_\alpha = \left[\mathcal{O}_K : \mathcal{O}_{K,\alpha}\right].$$

Proof We consider the exact sequence

$$0 \to \mathcal{O}_{K,\alpha} \hookrightarrow \mathcal{O}_K \twoheadrightarrow \mathcal{O}_K \,/\, \mathcal{O}_{K,\alpha} \to 0.$$

We apply the right exact functor $\otimes_\mathbb{Z} \mathbb{Z}/p\,\mathbb{Z}$ and obtain:

$$\mathcal{O}_{K,\alpha}/p\,\mathcal{O}_{K,\alpha} \to \mathcal{O}_K/p\,\mathcal{O}_K \twoheadrightarrow D \to 0.$$

Here we have

$$D = \left(\mathcal{O}_K \,/\, \mathcal{O}_{K,\alpha}\right) \Big/ \, p\left(\mathcal{O}_K \,/\, \mathcal{O}_{K,\alpha}\right).$$

We recall ([9]), let M be a $\mathbb{Z}$ module, then

$$M \otimes_{\mathbb{Z}} \mathbb{Z}/p\,\mathbb{Z} \simeq M/pM.$$

Since $\zeta_m \equiv \alpha \pmod{p\,\mathcal{O}_K}$, we obtain

$$\mathcal{O}_{K,\alpha}\big/p\,\mathcal{O}_{K,\alpha} \simeq \mathcal{O}_K\big/\,p\mathcal{O}_K.$$

Therefore, the module D has to be trivial. And we obtain:

$$p\,\mathcal{O}_K\big/\mathcal{O}_{K,\alpha} \simeq \mathcal{O}_K\big/\mathcal{O}_{K,\alpha}.$$

Since $\mathcal{O}_K\big/\mathcal{O}_{K,\alpha}$ is a finite abelian group the claim of the lemma follows ([14], Chapter 4). □

i) Let $\alpha = \zeta_m + 2\,\beta$ with $\beta \in \mathcal{O}_K$. Then $K = \mathbb{Q}[\alpha]$. We deduce that

$$2 \nmid \kappa_\alpha = [\mathcal{O}_K : \mathcal{O}_{K,\alpha}]$$

from Lemma 3. Therefore, we can apply the Dedekind–Kummer Theorem 5. We have that $\mathrm{Min}_\alpha(x) \pmod 2$ decomposes into linear factors $\Longleftrightarrow m = 2^\ell$, $\ell \geq 2$. On the other hand we have from Lemma 1 and $A_1^g(x) = x$, $A_2^g(x) = x(x + g(2))$ that

$$A_n^g(x) \pmod 2 \text{ decomposes into linear factors.} \tag{3}$$

Suppose $P_n^g(\zeta_m) = 0$ for $n \geq 1$ and $m \neq 2^\ell$, $\ell \geq 2$. Then $\mathrm{Min}_\alpha(x) \pmod 2$ has to divide $A_n^g(x) \pmod 2$. But this contradicts (3).

ii) Due to $g(3) \equiv 0, 1 \pmod 3$, we deduce from Lemma 1 and Lemma 2 that $A_n^g(x) \pmod 3$ decomposes into linear factors. With the same reasoning as in i) and the assumption on m we obtain the claim.

3 Open challenges and further study

The upcoming subsections contain several problems, discussions, and ideas for further study. In the following we consider the sequence of D'Arcais polynomials $\{P_n^\sigma(x)\}_n$. Some of the questions can also be asked for a more general type of polynomials $P_n^g(x)$, where g is a non-negative normalized integer-valued arithmetic function. For example, let $g(n) = n^d$, $(d \in \mathbb{N}_0)$ then $d = 1$ is related to Laguerre polynomials [8].

3.1 Imaginary axis

We believe that the D'Arcais polynomials are Hurwitz polynomials. The zero at $x = 0$ is irrelevant. It would be of interest to show that $x = 0$ is the only zero on the imaginary axis, in particular that

$$f_n(t) := P_n^\sigma(it)$$

is non-vanishing for all $t \in \mathbb{Z} \setminus \{0\}$. Theorem 4 with $m = 4$ and $p = 3$ implies that

$$f(3k \pm 1) \neq 0 \quad \text{for all } k \in \mathbb{Z}.$$

Moreover, an analysis of $A_n^{\sigma}(x) \pmod 7$ yields

$$P_n^{\sigma}(\pm 3i) \neq 0.$$

3.2 Results of Zmija

In this paper we generalized the results of Theorem 1 and Theorem 2 with the assumption that $g(3) \not\equiv 2 \pmod 3$. It would be interesting to clarify how this is compatible or related with the assumptions of Theorem 2. Note that we do not assume that $P_n^g(x)$ is integer-valued.

3.3 Vanishing properties on the unit circle

Let g be an integer-valued normalized arithmetic function and $g(3) \not\equiv 2 \pmod 3$. Let

$$\mathbb{U} := \{z \in \mathbb{C} \, : \, |z| = 1\}.$$

It follows from Theorem 3 that $P_n^g(x)$ is non-vanishing at all primitive roots of unity ζ_m for $m \geq 3$. Since $A_n^g(x) \in \mathbb{Z}[x]$ and normalized, zeros have to be algebraic integers. Let $g = \sigma$. Then $\zeta_1 = 1$ is not a zero and $\zeta_2 = -1$ is a zero if and only if n is not of the form $k(3k-1)/2$ for $k \in \mathbb{Z}$ (Euler's famous pentagonal theorem). We expect that if $P_n^{\sigma}(\alpha) = 0$ with $\alpha \in \mathbb{C}$ and $|\alpha| = 1$, then necessarily $\alpha = -1$.

3.4 Dedekind–Kummer approach

Find suitable algebraic integers α, as for example ζ_m $(m \geq 3)$ with $P_n^{\sigma}(\alpha) \neq 0$. And analyse the prime ideal decomposition of $p\mathcal{O}_K$ for $p = 2, 3$, and 5 and $K = \mathbb{Q}[\alpha]$ and the irreducible factors of the minimal polynomial $\pmod p$.

Vary α to β such that still the same number field is generated and that the index $[\mathcal{O}_K : \mathbb{Z}[\beta]]$ can be controlled.

3.5 Stretching the unit circle

Let $r \in \mathbb{N}$ or more generally let r be an algebraic integer. Let

$$\mathbb{U}_r := \{r\, z \in \mathbb{C} \, : \, |z| = 1\}.$$

Determine those $\alpha \in \mathbb{U}_r$ for which $P_n^{\sigma}(\alpha) \neq 0$ for all $n \in \mathbb{N}$ (or almost all n).

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Declarations

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