

SPECIAL ISSUE 95th Annual Meeting of the International Association
 of Applied Mathematics and Mechanics (GAMM)

RESEARCH ARTICLE **OPEN ACCESS**

A Comparative Study of Polygonal Element Formulations for Linear Elasticity

Ajay Kumar Pasupuleti¹  | Mathias Reichle² | Anna Hellers² | Njomza Pacolli³  | Stefanie Reese^{3,4} |
 Hagen Holthusen⁵ | Sven Klinkel²  | Carolin Birk¹ 

¹Institute of Structural Analysis, University of Duisburg-Essen, Essen, Germany | ²Chair and Institute of Structural Analysis and Dynamics, RWTH Aachen University, Aachen, Germany | ³Institute of Applied Mechanics, RWTH Aachen University, Aachen, Germany | ⁴University of Siegen, Siegen, Germany | ⁵Institute of Applied Mechanics, University of Erlangen-Nuremberg, Erlangen, Germany

Correspondence: Ajay Kumar Pasupuleti (ajay.pasupuleti@uni-due.de)

Received: 28 July 2025 | **Revised:** 29 October 2025 | **Accepted:** 11 November 2025

ABSTRACT

With the rising complexity of architectural design, a growing interest in topology optimization, and the increasing need to model crack propagation, there is a higher demand for highly flexible geometry descriptions that go hand in hand with robust element formulations. Methods relying on polygonal meshes, such as the virtual element method (VEM), element formulations based on Wachspress shape functions, and the scaled boundary finite element method (SBFEM) have proven to be versatile in solving physical problems. Arbitrarily shaped elements are beneficial when it comes to highly localized meshes, treatment of hanging nodes, and flexibility of the meshing of complex domains. This paper provides a comparative study of polygonal element formulations, focusing specifically on the VEM, the semi-analytical SBFEM, and the fully discretized SBFEM. The study evaluates these methods in terms of stability, convergence behavior, and flexibility in the context of linear elasticity. The potential of the formulations is evaluated by applying them to several benchmark problems and by comparing the results. In doing so, the flexibility of the considered formulations is also proven. To this end, several indicators are considered, such as error norms, convergence rates, and displacements.

1 | Introduction

Often, real-world solid mechanics problems involve domains that are complex and thus hard to represent precisely using conventional triangular and quadrilateral meshes. Additionally, there may be local phenomena in specific regions, such as stress singularities, that demand refined meshes. For such problems, using uniform meshes can be computationally costly; hence, non-uniform meshes are generally used. As a result, rapid transition in element sizes on quadrilateral meshes would result in elements that are distorted, leading to numerical challenges. In this regard, polygonal meshes, with an arbitrary number of edges, can provide advantages.

Various numerical techniques are available for solving physical problems on polygonal meshes. The conventional finite element

method (FEM) cannot be applied to a polygonal mesh. However, versions of FEM exist that can be used on polygonal meshes, such as Wachspress basis functions [1], metric coordinate method [2], natural neighbor-based coordinates [3], maximum entropy shape functions [4, 5], and mean value coordinate shape functions [6]. The variants of polygonal shape functions in the FEM are well summarized in [7]. Other numerical techniques for solving partial differential equations, such as the scaled boundary finite element method (SBFEM) [8, 9] and virtual element method (VEM) [10] can be applied straightforwardly to a mesh containing elements with an arbitrary number of edges. These methods can also handle the elements with hanging nodes inherently. Hence, in this work, the generality of SBFEM and VEM to solve the problems on n – sided polygonal meshes is reiterated. In addition to this, in the framework of linear elasticity, the accuracy of these methods is compared by solving two example problems.

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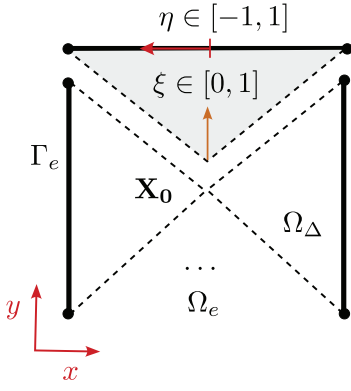


FIGURE 1 | Representation of polygonal element in SBFEM framework.

The outline of this article is as follows: Section 2 briefly discusses the different numerical techniques that are to be compared, Section 3 presents two numerical examples, and in Section 4 conclusions are made.

2 | Numerical Techniques

The governing equation in linear elasticity, for a body described by the open set $\Omega \in \mathbb{R}^2$ and closed by the boundary Γ , is given by

$$\begin{aligned} \mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} &= \mathbf{0}, \quad \boldsymbol{\sigma} = \mathbf{C} \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \mathbf{L} \mathbf{u}, \\ \mathbf{L} &= \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}^T \text{ in } \Omega, \end{aligned} \quad (1)$$

where $\boldsymbol{\sigma} = [\sigma_x \quad \sigma_y \quad \tau_{xy}]^T$ and $\boldsymbol{\varepsilon} = [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}]^T$ are stress and strain in Voigt notation, $\mathbf{u} = [u_x \quad u_y]^T$ is the displacement vector with displacements in x -, y - directions, $\mathbf{b}$ is the body load vector and $\mathbf{C}$ is the elasticity matrix. The solution to the governing equation is unique by prescribing admissible boundary conditions on Γ . In this work, the body loads ($\mathbf{b}$) are neglected.

In the remainder of this section, the formulations of SBFEM and VEM for polygonal elements are discussed.

2.1 | Scaled Boundary Finite Element Method (SBFEM)

The SBFEM is a semi-analytical solution technique developed by Song and Wolf [8] in the late 1990s. In this technique, the solution of the problem is obtained numerically along the boundary of the element or the circumferential direction (η) and analytically in the radial direction (ξ), hence the name semi-analytical approach (SA). However, solving a general nonlinear problem with SBFEM is quite challenging. This has been overcome in the works [9, 11], wherein shape functions are introduced in both boundary (η) and scaling (ξ) directions, thus replacing the analytical solution in radial direction, hence the name fully discretized approach (FD).

In SBFEM, an n -sided polygon is made up of multiple triangular sectors, as shown in Figure 1, with a corresponding scaling center

($\mathbf{X}_0$). This method can be applied to any domain that satisfies the star convexity.

2.1.1 | Parametrization of the Geometry

Let Ω be discretized into n_{poly} number of polygonal elements and let Ω_e define each of these elements such that $\Omega = \bigcup_{i=1}^{n_{\text{poly}}} \Omega_{e_i}$. There exists also a scaling center ($\mathbf{X}_0$) for each polygonal element, such that complete boundary of the element is visible from this point. In each polygonal element with the origin shifted to scaling center ($\mathbf{X}_0$), each triangular sector (Ω_Δ) in Ω_e is parametrized by circumferential coordinate ($\eta \in [-1, 1]$) and radial coordinate ($\xi \in [0, 1]$). With such a parametrization, any point (x, y) inside the Ω_Δ can be represented by

$$x(\xi, \eta) = \xi \mathbf{N}(\eta) \mathbf{x}^b, \quad y(\xi, \eta) = \xi \mathbf{N}(\eta) \mathbf{y}^b, \quad (2)$$

where $\mathbf{N}(\eta)$ is a row vector of shape functions corresponding to a line element, and $\mathbf{x}^b$ and $\mathbf{y}^b$ are the vectors with nodal x - and y -coordinates of the line element. For linear shape functions along the circumferential direction, $\mathbf{N}(\eta)$ is given by

$$\mathbf{N}(\eta) = [N_1(\eta) \quad N_2(\eta)], \quad (3)$$

with $N_1(\eta) = \frac{1}{2}(1 - \eta)$ and $N_2(\eta) = \frac{1}{2}(1 + \eta)$.

With the parametrization defined in Equation (2) within Ω_Δ , the gradient of any field in the parametric space can be mapped to the gradient in the actual space with the help of Jacobian matrix given by

$$\begin{bmatrix} \partial(\cdot)/\partial \xi \\ \partial(\cdot)/\partial \eta \end{bmatrix} = \mathbf{J} \begin{bmatrix} \partial(\cdot)/\partial x \\ \partial(\cdot)/\partial y \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix} \begin{bmatrix} \mathbf{N}(\eta) \mathbf{x}^b & \mathbf{N}(\eta) \mathbf{y}^b \\ \mathbf{N}(\eta)_{,\eta} \mathbf{x}^b & \mathbf{N}(\eta)_{,\eta} \mathbf{y}^b \end{bmatrix}, \quad (4)$$

where $\mathbf{N}(\eta)_{,\eta}$ is the derivative of shape function vector with respect to η . The inverse mapping can be obtained by taking the inverse of $\mathbf{J}$.

2.1.2 | Semi-Analytical Solution Approach

Using the isoparametric concept, the shape functions ($\mathbf{N}(\eta)$) used for parametrization of geometry in Equation (2) are utilized to approximate the displacement along the boundary of the triangular sector. The displacement in Ω_Δ is then represented by

$$\mathbf{u}_h(\xi, \eta) = \tilde{\mathbf{N}}(\eta) \hat{\mathbf{u}}(\xi), \quad \text{with } \tilde{\mathbf{N}}(\eta) = [N_1(\eta) \mathbf{I}_{2 \times 2} \quad N_2(\eta) \mathbf{I}_{2 \times 2}], \quad (5)$$

where $\hat{\mathbf{u}}(\xi)$ is the function representing the nodal displacement in radial direction of the triangular sector and $\mathbf{I}_{2 \times 2}$ is the identity matrix. In order to obtain the closed form solution for $\hat{\mathbf{u}}(\xi)$, as outlined in [12], the differential operator ($\mathbf{L}$) in Equation (1), with no body loads, is first replaced using the definition of gradient in Equation (4). Then, by applying the principle of virtual work and

integrating in the circumferential coordinate η results in a second-order ordinary differential equation in terms of radial coordinate ξ which is given by

$$\xi^2 \mathbf{E}_0^e \hat{\mathbf{u}}^e(\xi)_{,\xi\xi} + \xi (\mathbf{E}_0^e + (\mathbf{E}_1^e)^T - \mathbf{E}_1^e) \hat{\mathbf{u}}^e(\xi)_{,\xi} - \mathbf{E}_2^e \hat{\mathbf{u}}^e(\xi) = \mathbf{0}, \quad (6)$$

where $\hat{\mathbf{u}}^e(\xi)$ is the vector of nodal displacement functions related to all nodes of Ω_e , $\mathbf{E}_i^e, i = 0, 1, 2$; are the coefficient matrices corresponding to complete polygonal element (Ω_e) obtained by assembling the coefficient matrices ($\mathbf{E}_i^{\Omega_{\Delta s}}$) corresponding to each of the triangular sectors in finite-element sense, that is

$$\mathbf{E}_i^e = \sum_{s=1}^{n_s} \mathbf{E}_i^{\Omega_{\Delta s}}, \quad (7)$$

where n_s is the number of triangular sectors in a polygonal element. For detailed definitions of $\mathbf{E}_i^{\Omega_{\Delta s}}, i = 0, 1, 2$, refer to [13].

The second-order ordinary differential equation in Equation (6) is converted into first-order and solved analytically using matrix function solution as described in [12]. Following this approach leads to the stiffness matrix, which is given by

$$\mathbf{K}_e = \Phi^{(q)} (\Phi^{(u)})^{-1}, \quad (8)$$

where $\Phi^{(q)}$ and $\Phi^{(u)}$ are real transformation matrices obtained by performing the Schur decomposition of the Hamiltonian matrix $\mathbf{Z}$ associated with the first-order differential equation and by subsequent block-diagonalization.

2.1.3 | Fully Discretized Solution Approach

In the fully discretized version of SBFEM, the shape functions in the circumferential direction, as used in Equation (5), are retained. However, additional shape functions are also introduced in the radial direction. This yields

$$\mathbf{u}_h = \tilde{\mathbf{N}}(\eta) \hat{\mathbf{N}}(\xi) \hat{\mathbf{u}}, \quad \text{with} \quad \hat{\mathbf{N}}(\xi) = [\hat{N}_1(\xi) \mathbf{I}_{4 \times 4} \quad \hat{N}_2(\xi) \hat{\mathbf{I}}]. \quad (9)$$

$\hat{\mathbf{N}}(\xi)$ is a matrix containing the shape functions in scaling direction defined as $\hat{N}_1(\xi) = \xi$ and $\hat{N}_2(\xi) = 1 - \xi$ in the linear case and the modified identity matrix $\hat{\mathbf{I}}^T = [\mathbf{I}_{2 \times 2} \quad \mathbf{I}_{2 \times 2}]$. The vector $\hat{\mathbf{u}}$ contains the displacements of the scaling center and of the nodes such that $\hat{\mathbf{u}}^T = [\hat{\mathbf{u}}_1 \quad \hat{\mathbf{u}}_2 \quad \hat{\mathbf{u}}_0]$. In analogy to the displacements, the virtual displacements are approximated by $\delta \mathbf{u}_h = \tilde{\mathbf{N}}(\eta) \hat{\mathbf{N}}(\xi) \delta \hat{\mathbf{u}}$.

In the framework of SBFEM, the strains are expressed in terms of the displacements as

$$\boldsymbol{\varepsilon} = \mathbf{b}_1(\eta) \frac{\partial \mathbf{u}}{\partial \xi} + \frac{1}{\xi} \mathbf{b}_2(\eta) \frac{\partial \mathbf{u}}{\partial \eta}, \quad (10)$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are given by

$$\mathbf{b}_1 = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \mathbf{N}(\eta)_{,\eta} \mathbf{y}^b & 0 \\ 0 & -\mathbf{N}(\eta)_{,\eta} \mathbf{x}^b \\ -\mathbf{N}(\eta)_{,\eta} \mathbf{x}^b & \mathbf{N}(\eta)_{,\eta} \mathbf{y}^b \end{bmatrix},$$

$$\mathbf{b}_2 = \frac{1}{|\mathbf{J}|} \begin{bmatrix} -\mathbf{N}(\eta) \mathbf{y}^b & 0 \\ 0 & \mathbf{N}(\eta) \mathbf{x}^b \\ \mathbf{N}(\eta) \mathbf{x}^b & -\mathbf{N}(\eta) \mathbf{y}^b \end{bmatrix}. \quad (11)$$

Introducing now the approximation of the displacement field yields the approximated strains, defined as

$$\boldsymbol{\varepsilon}_h = \mathbf{B} \hat{\mathbf{u}} \quad \text{and} \quad \delta \boldsymbol{\varepsilon}_h = \mathbf{B} \delta \hat{\mathbf{u}}, \quad (12)$$

where the operator $\mathbf{B}$ is given as

$$\mathbf{B} = \mathbf{b}_1(\eta) \tilde{\mathbf{N}}(\eta) \hat{\mathbf{N}}(\xi)_{,\xi} + \frac{1}{\xi} \mathbf{b}_2(\eta) \tilde{\mathbf{N}}(\eta)_{,\eta} \hat{\mathbf{N}}(\xi). \quad (13)$$

By introducing the approximation of the strains and displacements into the weak form of equilibrium, the stiffness matrix of each triangular sector can be derived as

$$\mathbf{K}_s = \int_{\Omega_{\Delta}} \mathbf{B}^T \mathbf{C} \mathbf{B} d\mathbf{A}. \quad (14)$$

The element stiffness matrix is then derived from the assembly of the sectional stiffness matrices and is defined as

$$\mathbf{K}_e = \sum_{s=1}^{n_s} \mathbf{K}_s. \quad (15)$$

2.2 | Virtual Element Formulation With Stabilization

In comparison to the SBFEM or the classical FEM, VEM does not require an explicit computation of shape functions. Moreover, a projection onto a polynomial subspace is introduced. In general, this projection yields a rank deficient stiffness matrix. Thus, an additional stabilization term is needed. Recently, stabilization-free approaches have also been developed to circumvent the need for a stabilization term, see, for example, [14, 15]. In this contribution, a virtual element formulation with stabilization based on reduced integration is presented.

2.2.1 | Consistency Term

The geometry Ω with its boundary Γ is divided into several elements with the domain Ω_e and the boundary Γ_e . In VEM, the displacement $\mathbf{u}_h$ is approximated through a projection onto a polynomial subspace $\mathbf{u}_h \mapsto \Pi(\mathbf{u}_h) = \mathbf{u}_\pi$, see [10, 16], since the displacement is not known within the element. This projection leads to a split of the displacement into a so-called projection part and a remainder

$$\mathbf{u}_h = \mathbf{u}_\pi + (\mathbf{u}_h - \mathbf{u}_\pi). \quad (16)$$

For a linear polynomial ansatz, the gradient $\nabla \mathbf{u}_\pi$ is constant over the element. Thus, taking the orthogonality relation of the gradients of Equation (16) into account [17], an expression for $\nabla \mathbf{u}_\pi$ is obtained

$$\nabla \mathbf{u}_\pi = \frac{1}{\Omega_e} \int_{\Omega_e} \nabla \mathbf{u}_h d\Omega = \frac{1}{\Omega_e} \int_{\Gamma_e} \mathbf{u}_h \otimes \mathbf{N}_i d\Gamma. \quad (17)$$

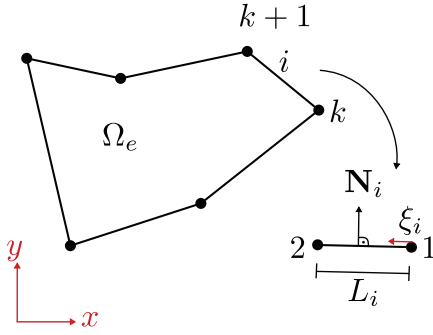


FIGURE 2 | Representation of polygonal element in VEM framework.

Here, $\mathbf{N}_i$ denotes the outward normal vector along the boundary Γ_e for every i -th edge. Since the displacement field $\mathbf{u}_h$ is known along the boundary of the element, it can be computed by introducing a linear ansatz for each edge, see Figure 2 and [18]. The residual vector $\mathbf{R}_0$ and the corresponding stiffness matrix $\mathbf{K}_0$ of the consistency part are obtained by using the automatic differentiation tool JAX [19].

2.2.2 | Stability Term

The choice of stabilization is based on the remainder in Equation (16). Here, the idea of reduced integration with hourglass stabilization from finite element technologies is considered, where a Taylor series expansion of the constitutive quantities is carried out at the center of the element, see, for example, [20, 21]. Polygonal reference elements are introduced, where barycentric coordinates (Wachspress basis functions or mean value coordinates) are defined, [3]. Altogether, this yields the VEM-STc+ formulation. In this study, mean value coordinates are used. To this end, the isoparametric concept is employed, where the (initial) geometry $\mathbf{x}_h$ and displacement field of the remainder $\mathbf{u}_h^r$ are discretized as follows

$$\mathbf{x}_h = \mathbf{N}(\tilde{\xi}) \mathbf{X}, \quad \mathbf{u}_h^r = \mathbf{N}(\tilde{\xi}) \mathbf{U}^r, \quad (18)$$

where $\mathbf{X}$ denotes the nodal position vector and $\mathbf{U}^r$ denotes the element nodal displacements, which are based on the remainder. Here, the mean value shape functions defined on the polygonal reference element are stored in the shape function matrix $\mathbf{N} = (\mathbf{N}_1 \mathbf{I}_2, \dots, \mathbf{N}_i \mathbf{I}_2)$, while i denotes the number of nodes per reference element and $\mathbf{I}_2$ the 2×2 identity matrix. The shape functions depend on the natural coordinate vector $\tilde{\xi} = (\tilde{\xi}, \tilde{\eta})^T$. With this information, a Taylor series approximation of the stress in Nye's notation $\hat{\sigma}$ is carried out at the center of the element. Depending on the number of nodes per element, the Taylor series of $\hat{\sigma}$ is expressed as follows

$$\hat{\sigma}(\hat{\varepsilon}) \approx \hat{\sigma} \Big|_{\tilde{\xi}=0} + \mathbf{c}^{hg} \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!j!} \frac{\partial^{i+j} \hat{\varepsilon}}{\partial \tilde{\xi}^i \partial \tilde{\eta}^j} \Big|_{\tilde{\xi}=0} \tilde{\xi}^i \tilde{\eta}^j \right). \quad (19)$$

To circumvent a linearization, the real material tangent is replaced by its deviatoric part

$$\mathbf{c}^{hg} = \frac{\mu^{\text{eff}}}{3} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mu^{\text{eff}} = \frac{1}{2} \sqrt{\frac{\text{tr}(\text{dev}(\boldsymbol{\sigma}^0)^2)}{\text{tr}(\text{dev}(\boldsymbol{\varepsilon}^0)^2)}} \quad (20)$$

with μ^{eff} denoting the effective shear modulus. The quantities $\boldsymbol{\sigma}^0$ and $\boldsymbol{\varepsilon}^0$ denote the stress and strain evaluated for $i = j = 0$ in Equation (19). By employing a Taylor series, analytical integration of the weak form is enabled, where the residual vector $\mathbf{R}_{\text{stab}}$ is obtained. Based on this, the stiffness matrix of the stabilization term yields

$$\mathbf{K}_{\text{stab}} = \frac{\partial \mathbf{R}_{\text{stab}}}{\partial \mathbf{U}}, \quad (21)$$

with $\mathbf{U}$ denoting the element displacement vector. The total residual vector and stiffness matrix at element level are

$$\mathbf{R}_e = \mathbf{R}_0 + \mathbf{R}_{\text{stab}}, \quad \mathbf{K}_e = \mathbf{K}_0 + \mathbf{K}_{\text{stab}}. \quad (22)$$

3 | Numerical Examples

The numerical methods discussed in Section 2 are applied to two numerical examples. These examples demonstrate the flexibility of applying the methods to solve a conventional linear elasticity problem, wherein the domain is discretized by using elements with an arbitrary number of edges. As stated in Section 2, in this section the results related to the semi-analytical approach of SBFEM are denoted as SBFEM-SA and the fully discretized SBFEM as SBFEM-FD. The VEM method with the stabilization based on Taylor series expansion is denoted as VEM-STc+.

Having established the applicability of the method on a mesh with arbitrary element shapes using the patch test, the condition number of the reduced stiffness matrix in each methodology is compared. In the second example, the conventional plate with a hole problem is solved. For the patch test, the Lamé constants are chosen as $\lambda = \mu = 0.4 \text{ N/m}^2$ and for the plate with a hole problem, Young's modulus is chosen as 100 N/m^2 and $\nu = 0.3$. For both examples, the plane strain assumption is considered. A discretized relative error measure as defined in Equation (23) is used in comparing the accuracy of the numerical solution ($\mathbf{u}_h$) obtained using different numerical techniques. Note that $\mathbf{u}_a$ in Equation (23) refers to the analytical solution,

$$e_u = \frac{\|\mathbf{u}_a - \mathbf{u}_h\|_{L^2}}{\|\mathbf{u}_a\|_{L^2}}. \quad (23)$$

3.1 | Patch Test

The domain discretized with elements containing different numbers of edges starting from 3 to 7 is shown in Figure 3, as presented in [22], and corresponding coordinates of the nodes are given in Table 1. For this patch test, a manufactured solution that is linear in x - and y - directions is chosen as given in Equation (24). To solve the problem, along all the boundary edges, the analytical

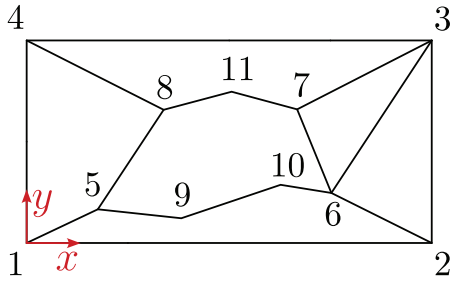


FIGURE 3 | Mesh for the patch test.

TABLE 1 | Node numbers and their coordinates corresponding to mesh in Figure 3.

Node number	Coordinates (x [m], y [m])
1	(0,0)
2	(2,0)
3	(2,1)
4	(0,1)
5	(0.333, 0.1667)
6	(1.5, 0.25)
7	(1.333, 0.6667)
8	(0.6667, 0.6667)
9	(0.75, 0.125)
10	(1.25, 0.2917)
11	(1, 0.75)

TABLE 2 | Condition number in different numerical techniques for patch test.

	VEM-STc+	SBFEM-SA	SBFEM-FD
$\text{cond}(\mathbf{K}_{\text{red}})$	12.8249	5.8258	14.2451

solution is applied as a Dirichlet boundary condition in both x - and y - directions

$$\mathbf{u}_a = \begin{bmatrix} x + 0.5y \\ 0.5x + y \end{bmatrix} \times 10^{-3} \text{ m}. \quad (24)$$

The linear elasticity problem is solved on this polygonal mesh using all the numerical techniques discussed in Section 2. As expected, all the numerical techniques are capable of solving the problem without any additional changes to the underlying framework. As given in Equation (24), the assumed analytical solution is linear. Linear ansatz functions have been adopted in all the numerical techniques. So, the assumed solution can be represented exactly. Hence, the error norm in each of the numerical techniques is obtained as a numerical zero, and the patch test is passed.

Table 2 shows the condition number of the reduced stiffness matrix in each of the numerical techniques. It can be learned that the condition number in SBFEM-SA is the least compared

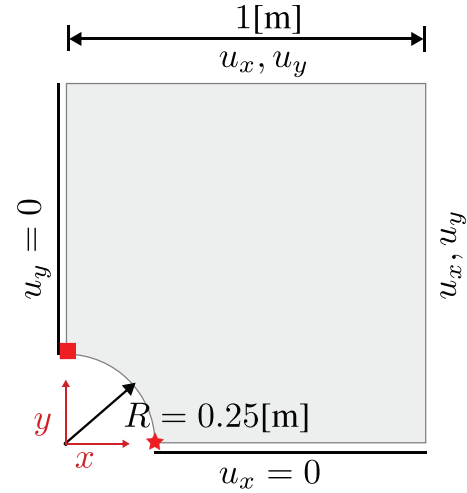


FIGURE 4 | A quarter of infinite plate with a hole analyzed in the current example.

to other techniques. However, the condition number in all the numerical techniques is of similar order.

3.2 | Plate With a Hole

In this example, the classical example of an infinite plate with a hole in the center is studied. Again, the main goal here is to show the flexibility of applying the discussed numerical methods to a polygonal mesh and compare the accuracy and the condition number. This is a well studied example in the literature, for example, in [23]. The corresponding analytical solution is

$$\begin{bmatrix} u_x(r, \theta) \\ u_y(r, \theta) \end{bmatrix} = q_x \frac{R}{8\mu} \begin{bmatrix} \frac{r}{R}(\kappa + 1) \cos(\theta) + \frac{2R}{r}((1 + \kappa) \cos(\theta) + \cos(3\theta) - \frac{2R^3}{r^3} \cos(3\theta)) \\ \frac{r}{R}(\kappa - 3) \sin(\theta) + \frac{2R}{r}((1 - \kappa) \sin(\theta) + \sin(3\theta) - \frac{2R^3}{r^3} \sin(3\theta)) \end{bmatrix}, \quad (25)$$

where (r, θ) refers to the polar coordinates of the 2D problem and u_x, u_y are the displacements in x - and y - directions. $R = 0.25$ m is the radius of the hole in the plate, μ is the shear modulus, and the parameter $\kappa = 3 - 4\nu$ for plane strain assumption with ν being Poisson's ratio. The stress applied in x - direction (q_x) is chosen as 10 N/m^2 for all the simulations performed on uniform and locally refined meshes. A side note to mention here is, all the meshes for this example are generated using the PolyMesher package of Matlab discussed in [24]. To generate the locally refined meshes, an exponential density function $d(\mathbf{x}) = \exp(10(x-1)^2(y-1)^2)$ is used.

The boundary conditions are shown in Figure 4 and they are as follows: the analytical solution is applied as a Dirichlet boundary condition on the top and right edges, and on the left and bottom edges, symmetry boundary conditions are applied so that just a quarter of the infinite plate is simulated for the study.

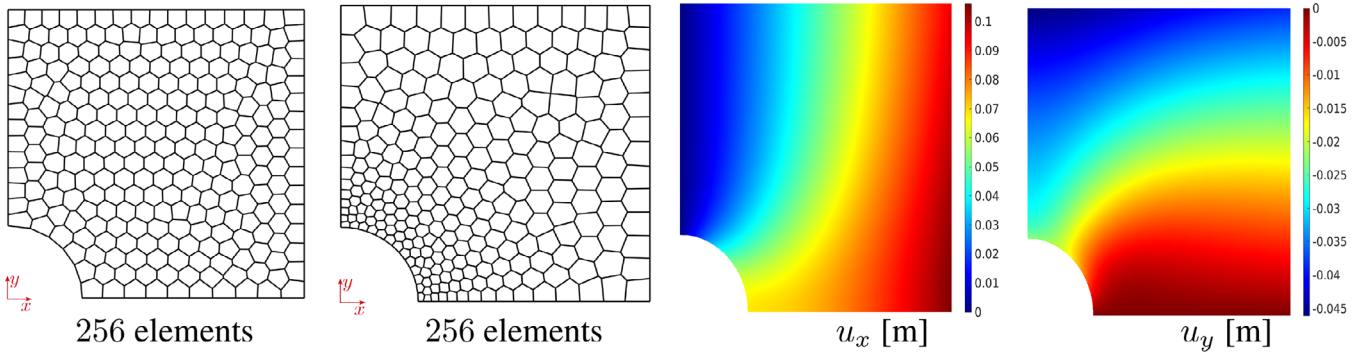


FIGURE 5 | Qualitative displacement solution obtained using uniform and locally refined meshes: The deformation plots are for the locally refined mesh. They are obtained using SBFEM-SA approach. As the qualitative plots look similar in all the numerical techniques and meshes, the others are omitted.

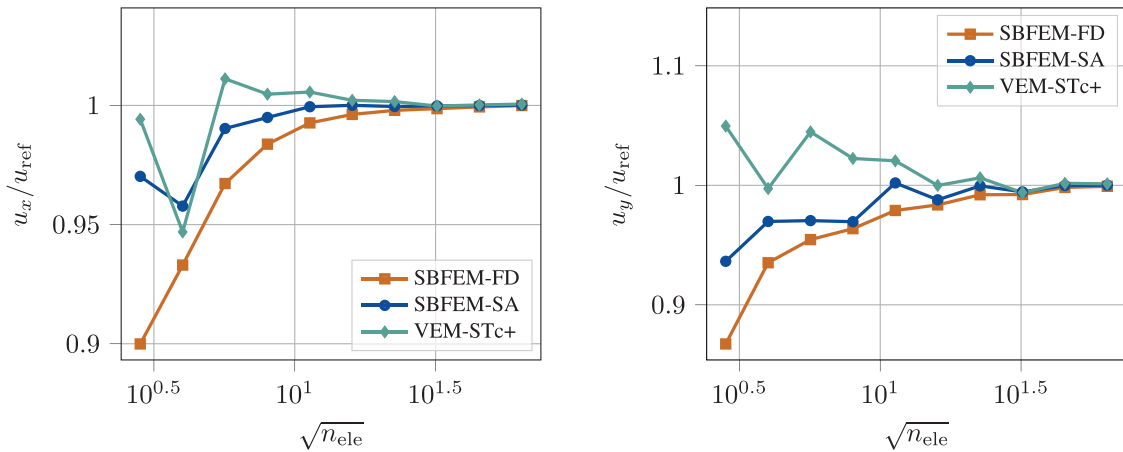


FIGURE 6 | Convergence study of displacements in x - and y - directions at the highlighted points in Figure 4 of infinite plate with a hole example on a uniformly refined meshes with different number of elements using SBFEM-SA, SBFEM-FD, and VEM.

In this example, in addition to the error norm defined in Equation (23) and the condition number, the displacement in x - and y - directions are compared at star and square points labeled in Figure 4.

A set of simulations on meshes with varying number of elements as powers of 2, that is, 8, 16 $\dots$ 4096, 8192, are performed. Figure 5 shows samples of uniform and locally refined meshes with 256 elements and corresponding color-maps of displacement in x - and y - directions obtained using SBFEM-SA. As qualitatively all the numerical techniques deliver same results, which are comparable to the solution in the literature, only the displacement plots obtained using SBFEM-SA are presented.

Figure 6 shows the results of the simulation performed on a uniform mesh using different numerical techniques. On comparing the displacement in Figure 6 at the highlighted points in Figure 3, it can be seen that as the number of elements increases, the displacements obtained using numerical results match the analytical solution. However, in VEM it is observed that for coarse meshes the solution oscillates before converging to the actual solution.

Figure 7 shows similar results of the simulations performed on a locally refined mesh. It can be observed that the obtained

displacement for coarse mesh with local refinement is better when compared to the uniform mesh. This is because the local refinement is in the region close to the points where the displacements are measured, and hence, as the mesh is fine in this region, the results are improved.

In Figure 8, the error norm and condition number are compared for the simulations performed on a uniform mesh and locally refined mesh using different numerical techniques. The error norm defined in Equation (23) is plotted against the square root of number of elements ($\sqrt{n_{ele}}$), and it can be seen that, as $\sqrt{n_{ele}}$ increases (that is, the element size becomes smaller), the error norm in all the numerical techniques becomes smaller. It can also be observed that the rate of convergence is close to the optimal rate, which is 2. Another interesting finding is that for VEM and SBFEM-SA, the condition numbers of the reduced stiffness matrices are of similar order. However, for SBFEM-FD, as the mesh becomes finer, it is observed that the condition number is one order higher as compared to other approaches.

4 | Conclusion

In this work, semi-analytical and fully discretized versions of the SBFEM and VEM have been successfully applied to solve linear

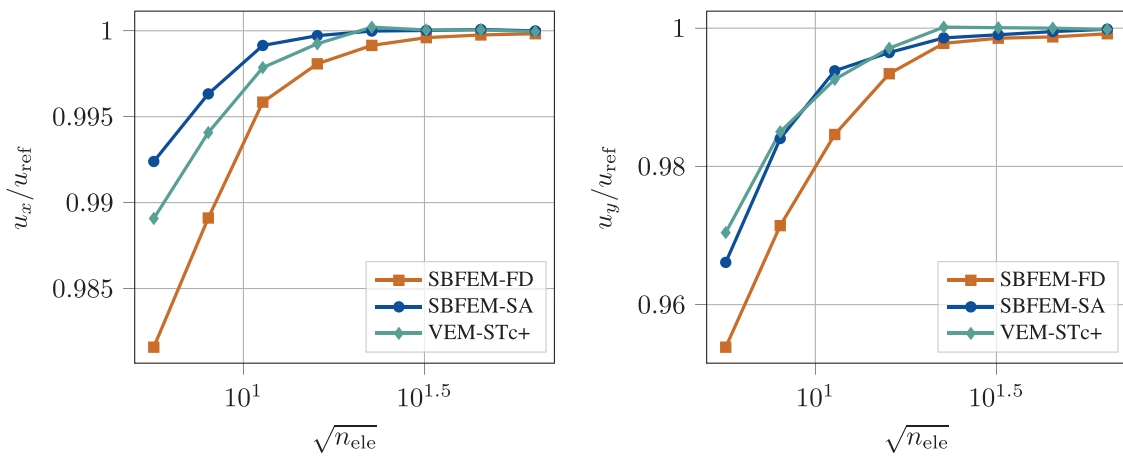


FIGURE 7 | Convergence study of displacements in x - and y - directions at the highlighted points in Figure 4 of infinite plate with a hole example on a locally refined meshes with different number of elements using SBFEM-SA, SBFEM-FD, and VEM.

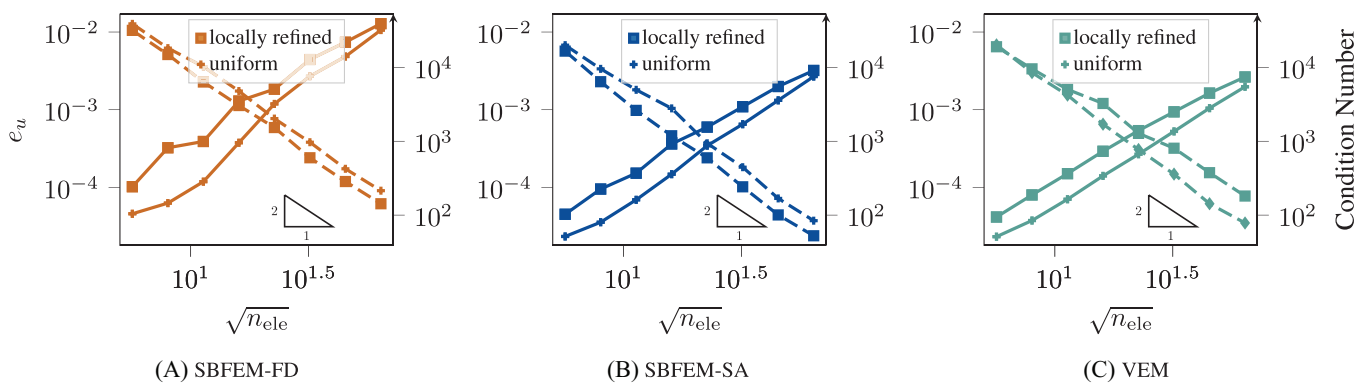


FIGURE 8 | Error norm (e_u) and condition number of reduced stiffness matrix obtained using SBFEM-FD, SBFEM-SA, and VEM formulations for plate with a hole example. Dashed line corresponds to error norm and solid line to the condition number.

elasticity problems on polygonal meshes with an arbitrary number of edges. All the numerical techniques demonstrate optimal convergence rates when performing an h – refinement study. As the mesh becomes finer, the condition numbers of SBFEM-SA and VEM are in the same order. However, for SBFEM-FD it is one order higher.

Acknowledgments

The authors would like to thank the German Research Foundation (Deutsche Forschungsgemeinschaft – DFG) for financial support of the research unit FOR5492 “Polytope Mesh Generation and Finite Element Analysis Methods for Problems in Solid Mechanics” under project number 495926269.

Open access funding enabled and organized by Projekt DEAL.

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