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Brief paper

Non-overshooting output shaping for switched linear systems under arbitrary switching using eigenstructure assignment[☆]

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A B S T R A C T

We consider the analytical control design for switched linear multiple-input multiple-output (MIMO) systems subject to arbitrary switching signals. A state feedback controller design method is proposed to obtain an eigenstructure assignment ensuring that the closed-loop switched system is globally asymptotically stable, and the outputs achieve the non-overshooting tracking of a step reference. Our analysis indicates whether non-overshooting or even monotonic tracking is achievable for the given system and considered outputs, and provides a choice of possible eigenstructures to be assigned to the constituent subsystems. We derive a structural condition that verifies the feasibility of the chosen assignment. A constructive algorithm to obtain suitable feedback matrices is provided, and the method is illustrated with numerical examples.

1. Introduction

Switched systems find application in numerous engineering problems such as congestion control (Chen et al., 2023), automotive dynamics (Solmaz et al., 2008) regenerative power systems (Palejiya & Chen, 2016), aircraft control systems (Wang et al., 2016) precision motion systems (Heertjes & Nijmeijer, 2012). Accordingly switched systems have been studied for more than two decades resulting in several monographs on this subject (Liberzon, 2003; Sun & Ge, 2005, 2011). Presumably the majority of contributions is dedicated to the stability analysis, (Lin & Antsaklis, 2009; Shorten et al., 2007) and references therein.

In this paper we consider the control design problem for set-point tracking for switched systems under arbitrary switching. We utilise eigenstructure assignment methods that are familiar to control engineers. Such methods have been applied to switched systems to achieve stabilisation (Bajcinca & Flockerzi, 2015; Haimovich, 2016; Haimovich & Braslavsky, 2013; Honecker, Gernandt, et al., 2024; Wulff et al., 2021, 2009). The devised controller in Bajcinca and Flockerzi (2015) assigns the left eigenspace of the closed-loop subsystems to ensure stability for arbitrary switching. This can also be achieved by simultaneous triangularisation of the subsystems (Haimovich, 2016; Haimovich & Braslavsky, 2013), where generic sufficient conditions are investigated using Lie-algebra methods. Honecker, Gernandt, et al. (2024) provides a parameterisation of feasible eigenstructure assignments that guarantees stability for arbitrary switching.

This contribution extends these results to obtain a generalised control concept for shaping the outputs for a class of switched linear MIMO systems. Therefore we combine the concepts from Honecker, Gernandt, et al. (2024) and Schmid and Ntogramatzidis (2010) to achieve globally stabilising state-feedback for switched linear MIMO systems while ensuring a non-overshooting step reference tracking. In order to obtain stability we rectify the eigenstructure of the subsystems as proposed in Honecker, Gernandt, et al. (2024). This imposes restrictions on the assignable eigenstructure which are strongly dependent on the constituent subsystems. Employing the results of Schmid and Ntogramatzidis (2010) for non-overshooting step-reference tracking, we identify the feasible eigenstructure assignments for the given set of subsystems. For that matter, a partitioning is chosen that simultaneously assigns the modes of the subsystems to their mutual outputs. We present conditions to check the feasibility of such a partitioning for the eigenstructure assignment problem. Further, for a given choice of partitioning we obtain specific sets of eigenvalues as well as eigenvectors to be chosen from. It turns out that in most cases the choice of eigenvalues is only mildly restricted, whereas the choice of eigenvectors is limited, depending on the

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chosen partitioning and eigenvalues to be assigned. However in many cases these sets still provide enough freedom for design to shape the output dynamics as desired.

The manuscript is structured as follows. The following section gives the class of switched systems as well as the design problem considered. Section 3.1 summarises preliminary results for the eigenstructure assignment for LTI systems and adapts the formulations for the scheme considered in the paper. In Section 3.2 we recall the eigenstructure assignment for switched linear systems in Honecker, Gernandt, et al. (2024) and propose a sufficient condition for the simultaneous eigenvector rectification together with the simultaneous mode assignment to the outputs of the system. Our main results on shaping the step-tracking response of the switched system are presented in Section 4. We propose a method to design non-overshooting or monotonic step-tracking of the specified outputs for arbitrary switching. For both designs we present a constructive algorithm to achieve the desired performance. In Section 5 we illustrate both design approaches by an example and discuss their properties.

2. Problem definition

We introduce the system class considered and formulate the reference tracking problems to be addressed.

2.1. Switched linear system and control problem

Consider the MIMO switched linear system

$$\Sigma_{\mathrm{OL}}: \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

with $x(0) = x_0 \in \mathbb{R}^n$, where the piecewise constant switching signal $\sigma : \mathbb{R}^+ \to \mathcal{I} = \{1, 2\}$, indicates the active subsystem at time $t \geq 0$. We denote the LTI system (A_q, B_q, C) by Σ_q and refer to it as subsystem of (1), with $A_q \in \mathbb{R}^{n\times n}$, $B_q \in \mathbb{R}^{n\times m}$ has full column rank m, and $C \in \mathbb{R}^{p\times n}$ has full row rank p for each $q \in \mathcal{I}$. While the switching instants occurs autonomously, the current value $\sigma(t)$ of the switching signal is assumed to be known at any time instant $t \geq 0$ for the purpose of controller design, and the control input u is assumed to be able to adjust to these switches instantaneously.

The homogeneous part of (1) is given by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

Definition 1 (*Liberzon, 2003, p. 22*)**.** The switched system (2) is globally uniformly exponentially stable (GUES), if there exists $\gamma, \lambda > 0$ such that for all switching signals σ and all $x_0 \in \mathbb{R}^n$, the solution of (2) satisfies

$$\|x(t)\| \leq \gamma \|x_0\| e^{\lambda t}, \quad \text{for all } t \geq 0. \tag{3}$$

Note that (1) is Input-to-State-stable if (2) is GUES, Liberzon (2003, p. 45). Thus for the problems considered in the following we require GUES of the homogeneous part ensuring boundedness of the outputs.

Given the switched system (1) and a constant reference signal $r \in \mathbb{R}^p$, we consider the switched control law

$$u(t) = F_{\sigma(t)}x(t) + G_{\sigma(t)} \tag{4}$$

with feedback matrices $F_q \in \mathbb{R}^{m\times n}$ and feedforward matrices $G_q \in \mathbb{R}^m$, $q \in \mathcal{I}$, to be chosen, that yields the closed-loop control system

$$\Sigma_{\mathrm{CL}}: \begin{cases} \dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}F_{\sigma(t)})x(t) + B_{\sigma(t)}G_{\sigma(t)} \\ y(t) = Cx(t). \end{cases} \tag{5}$$

Our goal is to devise the control law (4) such that the state trajectories of Σ_{CL} converge to an equilibrium point $x_{\mathrm{ss}} \in \mathbb{R}^n$ with $y_{\mathrm{ss}} := Cx_{\mathrm{ss}} = r$. Following Ntogramatzidis et al. (2016), Schmid and Ntogramatzidis (2010) we shall additionally distinguish the transient characteristics of *monotonic* and *non-overshooting* outputs. Introducing the tracking error $e(t) := r - y(t)$, these notions are defined as follows.

Definition 2. The switched system Σ_{CL} achieves *non-overshooting tracking* of $r \in \mathbb{R}^p$ from the initial condition $x_0 \in \mathbb{R}^n$ under arbitrary switching, if the error converges and has no change of sign in any component under any switching signal, i.e. for each $k \in \{1, \ldots, p\}$, $\mathrm{sgn}(e_k(t))$ is constant for $t > 0$, where e_k is the kth component of e.

Definition 3. The switched system Σ_{CL} achieves *globally monotonic tracking* of $r \in \mathbb{R}^p$ under arbitrary switching if the error $e(t) \to 0$ converges monotonically in every output component, under any switching signal, from *any* initial condition $x_0 \in \mathbb{R}^n$.

We shall distinguish the following two design goals:

Problem 4 (*Non-Overshooting Set-Point Tracking*)**.** Given the switched system Σ_{OL} in (1) and the reference $r \in \mathbb{R}^p$, find matrices F_q and G_q and a set $\mathcal{X}_0 \subset \mathbb{R}^n$ such that Σ_{CL} in (5) is GUES and achieves non-overshooting tracking of r from all $x_0 \in \mathcal{X}_0$ under arbitrary switching.

Problem 5 (*Global Monotonic Set-Point Tracking*)**.** Given the switched system Σ_{OL} in (1) and the reference $r \in \mathbb{R}^p$, find matrices F_q and G_q such that Σ_{CL} in (5) is GUES and achieves globally monotonic tracking of r from all $x_0 \in \mathbb{R}^n$ under arbitrary switching.

2.2. Common steady state and design of the feedforward control

If the switched system is at an equilibrium state of the active subsystem immediately prior to switching, the system output may exhibit significant transients due to the switched dynamics, which renders constant set-point tracking infeasible. Such transients can be avoided if all subsystems share the same steady-state equilibrium point, (Wulff et al., 2005). In this spirit we require that the system of linear equations

$$\begin{bmatrix} A_1 & B_1 & 0 \\ A_2 & 0 & B_2 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{\mathrm{ss}} \\ u_{1,\mathrm{ss}} \\ u_{2,\mathrm{ss}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} \tag{6}$$

has the solution $(x_{\mathrm{ss}}, u_{1,\mathrm{ss}}, u_{2,\mathrm{ss}}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ for the given set-point $r \in \mathbb{R}^p$. Then the common steady-state x_{ss} for the subsystems Σ_q, $q \in \mathcal{I} = \{1, 2\}$ produces the output $y = r$ with constant inputs $u_{1,\mathrm{ss}}$ and $u_{2,\mathrm{ss}}$, respectively. The following is taken as a standing assumption throughout the paper.

Assumption 6. Each subsystem Σ_q is right invertible (Trentelman et al., 2002) and satisfies $n+p < 2m$. There exist matrices $F_q, q \in \mathcal{I}$ such that the homogeneous part of (5) is GUES, and a feasible set-point r is given such that $(x_{\mathrm{ss}}, u_{1,\mathrm{ss}}, u_{2,\mathrm{ss}})$ is a solution of (6).

For the design of the feedforward control G_q, $q \in \mathcal{I}$, in (4) we assume that the homogeneous part of (5) is GUES for F_q, $q \in \mathcal{I}$, and choose

$$G_q = -F_q x_{\mathrm{ss}} + u_{q,\mathrm{ss}}. \tag{7}$$

With the change of variable $\xi := x - x_{\mathrm{ss}}$ and the error term $e(t) = r - y(t)$, the control law (4) with feedforward (7) yields the closed-loop homogeneous switched system

$$\Sigma_{\xi}: \begin{cases} \dot{\xi}(t) = (A_{\sigma(t)} + B_{\sigma(t)}F_{\sigma(t)})\xi(t) \\ e(t) = C\,\xi(t), \end{cases} \tag{8}$$

equivalent to (5). If the state matrices F_q achieve GUES for (8), we conclude that ξ converges to the origin, the state vector x of Σ_{CL} converges to x_{ss}, $e(t) \to 0$, and the output y converges to r as $t \to \infty$.

The principal task will be the design of the feedback matrices F_q, $q \in \mathcal{I}$, and the identification of the set $\mathcal{X}_0$ of initial conditions under which the control law (4) is able to solve Problems 4 and 5. The main results of this paper provide conditions for the solvability of these problems for a given switched system Σ_{OL}. When the problems are solvable, we provide a constructive procedure to design controllers that solve the respective problems. It turns out that each of these dynamical properties may be assigned to specific outputs of the switched system.

3. Preliminary results

This section revisits and extends earlier results on eigenstructure assignment for LTI MIMO systems as well as eigenstructure rectification for switched systems, and presents a novel result on the simultaneous mode assignment to the outputs of the subsystems.

3.1. Eigenstructure assignment for LTI MIMO systems

In this section we revisit and generalise some results on eigenstructure assignment for right-invertible LTI systems Σ_q that achieve non-overshooting output tracking. For ease of exposition we drop the index q throughout this subsection and denote Σ_q by the triplet (A, B, C). Our analysis will require some tools from the geometry of linear systems (Basile & Marro, 1992).

The aim of our eigenstructure assignment is to render certain modes invisible from certain outputs. For this matter we introduce the matrices $C_{(k)} \in \mathbb{R}^{p_{(k)} \times n}$ such that $C_{(0)} := C$, and for each $k \in \{1, \dots, p\}$

$$C_{(k)} := C, \text{ with the } k\text{th row removed.} \tag{9}$$

Accordingly, we have $p_{(0)} = p$ and $p_{(k)} = p - 1$, $k \in \{1, \dots, p\}$. We will use $\Sigma_{(k)}$ to denote the LTI system obtained from the triple $(A, B, C_{(k)})$, with $\Sigma_{(0)}$ denoting the system (A, B, C).

Definition 7. For each $\Sigma_{(k)}$, with $k \in \{0, 1, \dots, p\}$, and for any $\lambda \in \mathbb{R}$, we define the *Rosenbrock system matrices*

$$R_{(k)}(\lambda) := \begin{bmatrix} \lambda I - A & B \\ C_{(k)} & 0 \end{bmatrix} \tag{10}$$

of dimension $(n+p_{(k)}) \times (n+m)$. Their kernels may be decomposed as

$$\mathrm{im} \begin{bmatrix} N_{(k)}(\lambda) \\ M_{(k)}(\lambda) \end{bmatrix} := \ker(R_{(k)}(\lambda)), \tag{11}$$

where $N_{(k)}(\lambda)$ and $M_{(k)}(\lambda)$ have the dimensions $n \times m_{(k)}(\lambda)$ and $m \times m_{(k)}(\lambda)$, respectively.

We denote by $\mathcal{V}^*_{(k)}$ the *largest output-nulling subspace* of $\Sigma_{(k)}$, i.e., the largest subspace $\mathcal{V}_{(k)}$ of $\mathbb{R}^n$ for which a matrix $F \in \mathbb{R}^{m \times n}$ exists such that $(A + BF)\,\mathcal{V}_{(k)} \subseteq \mathcal{V}_{(k)} \subseteq \ker(C)$. Any real matrix F satisfying this inclusion is called a *friend* of $\mathcal{V}_{(k)}$. The symbol $\mathcal{R}^*_{(k)}$ denotes the so-called *output-nulling reachability subspace* on $\mathcal{V}^*_{(k)}$. It is the maximum subspace reachable from the origin with state trajectories completely belonging to $\mathcal{V}^*_{(k)}$.

Remark 8. We denote the normal rank of $\Sigma_{(k)}$ by $r_{(k)} = \max_{\lambda \in \mathbb{C}} \mathrm{rank}(R_{(k)}(\lambda))$. Values $\lambda_0 \in \mathbb{C}$ for which $\mathrm{rank}\, R_{(k)}(\lambda_0) < r_{(k)}$ are called *invariant zeros* of $\Sigma_{(k)}$. As B and $C_{(k)}$ are of full rank, we have $r_{(k)} = n + p_{(k)}$. If λ is not an invariant zero of $\Sigma_{(k)}$, we have $m_{(k)}(\lambda) = n + m - r_{(k)} = m - p_{(k)}$, otherwise we have $m_{(k)}(\lambda_0) > m - p_{(k)}$.

To simplify notation when referring to the dimensions of the kernel of the Rosenbrock matrix as in (11), we shall assume that λ is not an invariant zero. Then we may drop the dependency of $m_{(k)}(\lambda)$ on λ, and assume that $N_{(k)}(\lambda)$ and $M_{(k)}(\lambda)$ have dimensions $n \times m_{(k)}$ and $m \times m_{(k)}$, respectively, where $m_{(k)} = m - p_{(k)}$.

To summarise several results of Schmid and Ntogramatzidis (2010) in a succinct manner, we introduce the following.

Definition 9.

(1) For any two positive integers $n_1 < n_2$, we let $2^{[n_1,n_2]}$ denote the set of all subsets of the integers in the interval $[n_1, n_2]$.
(2) For any pair of positive integers p, n with $p < n$, we say that a $(p+1)$-tuple of integers $(d_0, d_1, \dots, d_p)$ is a (p, n)-*partitioning* if $0 \le d_0$, $0 \le d_k \le 3$ for $k \ne 0$, and $\sum_{k=0}^{p} d_k = n$.
(3) Let $(d_0, d_1, \dots, d_p)$ be a (p, n)-partitioning.

   (a) Let $\mathcal{L}_k = \{\lambda_{k,1}, \dots, \lambda_{k,d_k}\} \subset \mathbb{R}$, for each $k \in \{0, 1, \dots, p\}$. We say that the sets of scalars $\mathcal{L}_k$ are compatible with $(d_0, d_1, \dots, d_p)$ if $card(\mathcal{L}_k) = d_k$ and $\mathcal{L}_{k_1} \cap \mathcal{L}_{k_2} = \emptyset$, for any distinct $k_1, k_2 \in \{0, 1, \dots, p\}$. We further assume that compatible sets are indexed such that

   $$\lambda_{k,1} < \lambda_{k,2} < \lambda_{k,3}, \quad k \in \{1, \dots, p\}. \tag{12}$$

   (b) Let $\mathcal{V}_k = \{v_{k,1}, \dots, v_{k,d_k}\} \subset \mathbb{R}^n$, for each $k \in \{0, 1, \dots, p\}$. We say that the sets $\mathcal{V}_k$ are *compatible with* $(d_0, d_1, \dots, d_p)$ if $card(\mathcal{V}_k) = d_k$.

Remark 10. A (p, n)-partitioning represents a desired allocation of modes per output component. The sets $\mathcal{L}_k$ represent choices of desired closed-loop eigenvalues, and the vectors in $\mathcal{V}_k$ represent the associated eigenvectors. Provided that $\cup_{k=0}^{p} \mathcal{V}_k$ is linearly independent, the method of Moore (1976) can be used to obtain a feedback matrix F such the control law $u = Fx$ for Σ_q ensures the closed-loop system has an eigenstructure in which the closed-loop eigenvalues in $\mathcal{L}_0$ will not contribute to any of the output components, and the modes in $\mathcal{L}_k$ will only contribute to the kth output. Constraining $d_k \le 3$ guarantees that at most three modes are associated with each output such that it consists of at most three exponential terms. This allows us to apply the Lemmas A.1 and A.2 in Schmid and Ntogramatzidis (2010) that give necessary and sufficient conditions for the absence of a sign change in the sum of two or three exponential functions.

Theorems 3.1, 3.2 and 3.3 from Schmid and Ntogramatzidis (2010) may be combined as follows.

Theorem 11. *Given the LTI system Σ_q, let the sets $\mathcal{L}_k = \{\lambda_{k,1}, \dots, \lambda_{k,d_k}\} \subset \mathbb{R}$, and $\mathcal{V}_k = \{v_{k,1}, \dots, v_{k,d_k}\} \subset \mathbb{R}^n$, $k \in \{0, 1, \dots, p\}$, both be compatible with the (p, n)-partitioning $(d_0, d_1, \dots, d_p)$. Assume $\mathcal{V} = \cup_{k=0}^{p} \mathcal{V}_k$ is linearly independent. Let $w_{k,i} \in \mathbb{R}^m$ be such that*

$$\left[v_{k,i}^\top \; w_{k,i}^\top \right]^\top \subset \ker(R_{(k)}(\lambda_{k,i})), \tag{13}$$

for each $\lambda_{k,i} \in \mathcal{L}_k$ and $v_{k,i} \in \mathcal{V}_k$. Let

$$F = -WV^{-1}, \tag{14}$$

where $V \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{m \times n}$ are given by

$$V = \left[v_{0,1} \dots v_{0,d_0} \;\; v_{1,1} \dots v_{p,d_p} \right], \tag{15a}$$

$$W = \left[w_{0,1} \dots w_{0,d_0} \;\; w_{1,1} \dots w_{p,d_p} \right]. \tag{15b}$$

Then

$$(A+BF)\,v_{k,i} = \lambda_{k,i}\,v_{k,i}, \tag{16a}$$

$$C_{(k)}\,v_{k,i} = 0. \tag{16b}$$

Defining

$$[\alpha_{0,1},\dots,\alpha_{0,d_0},\ \alpha_{1,1},\dots,\alpha_{p,d_p}]^\top := V^{-1}x_0, \tag{17}$$

the output y of Σ_q from the input $u = Fx$ is given by

$$y(t) = \begin{bmatrix} \alpha_{1,1}e^{\lambda_{1,1}t} + \cdots + \alpha_{1,d_1}e^{\lambda_{1,d_1}t} \\ \alpha_{2,1}e^{\lambda_{2,1}t} + \cdots + \alpha_{2,d_2}e^{\lambda_{2,d_2}t} \\ \vdots \\ \alpha_{p,1}e^{\lambda_{p,1}t} + \cdots + \alpha_{p,d_p}e^{\lambda_{p,d_p}t} \end{bmatrix}. \tag{18}$$

Proof. For each $k \in \{0, 1, \dots, p\}$ and $i \in \{1, \dots, d_k\}$, if $\lambda_{k,i}$, $v_{k,i}$ and $w_{k,i}$ satisfy (13), and if F is obtained from (14), then by Proposition 1 in Moore (1976), $\lambda_{k,i}$ and $v_{k,i}$ satisfy (16). Thus

$$C v_{k,i} = \begin{cases} 0 & \text{if } k = 0, \\ s_k & \text{if } k \in \{1, \dots, p\}, \end{cases} \tag{19}$$

for all $i \in \{1, \dots, d_k\}$, where s_k denotes some scalar multiple of the kth canonical basis vector of $\mathbb{R}^p$. Hence the closed-loop output is

$$y(t) = Ce^{(A+BF)t}x_0 = \sum_{k=0}^{p}\sum_{i=1}^{d_k} C v_{k,i}\,\alpha_{k,i}\,e^{\lambda_{k,i}t}, \tag{20}$$

and with (19) we obtain the form (18). □

Remark 12. The assigned closed-loop eigenstructure (16) ensures that d_0 of the closed-loop eigenvectors are in the nullspace of C. The remaining $n-d_0$ eigenvectors are chosen so that exactly d_k of them lie in the nullspace of $C_{(k)}$. Consequently, the modes in $\mathcal{L}_0$ are rendered invisible at all outputs, and the modes in $\mathcal{L}_k$ are visible only in the kth output component, for each $k \in \{1, \dots, p\}$.

Theorem 11 requires the existence of a (p, n)-partitioning, compatible sets $\mathcal{L}_k$, and linearly independent $\mathcal{V}_k$. For each $k \in \{0, 1, \dots, p\}$, (13) requires that for any closed loop eigenvalue-eigenvector pair $(\lambda_{k,i}, v_{k,i})$ with $\lambda_{k,i} \in \mathcal{L}_k$, we must have $v_{k,i} \in \mathcal{R}^*_{(k)}$. Thus a necessary condition for the existence of a friend F that assigns the desired eigenstructure (16) is that $\dim(\mathcal{R}^*_{(0)}) \geq d_0$, and $\dim(\mathcal{R}^*_{(k)}) \geq d_0 + d_k$, since $\mathcal{R}^*_{(0)} \subset \mathcal{R}^*_{(k)}$, and the vectors d_k selected from $\mathcal{R}^*_{(k)}$ must be independent from the d_0 vectors selected from $\mathcal{R}^*_{(0)}$. We define

Definition 13. The (p, n)-partitioning $(d_0, d_1, \dots, d_p)$ is *feasible for* Σ_q if

$$d_0 \leq \dim(\mathcal{R}^*), \tag{21}$$

$$d_k \leq \dim(\mathcal{R}^*_{(k)}) - d_0. \tag{22}$$

Conditions (21) and (22) are readily checked using the MATLAB GA toolbox (Basile & Marro, 1992). Note that the existence of a feasible partitioning is necessary but not sufficient for solving the problems addressed in this paper. Assuming at least one feasible partitioning for Σ_q exists, the task is to identify conditions ensuring the existence of suitable sets $\mathcal{L}_k$ and $\mathcal{V}_k$. We formally state this as:

Problem 14 (*Eigenstructure Assignment for LTI Systems*)**.** Let the (p, n)-partitioning $(d_0, d_1, \dots, d_p)$ be feasible for Σ_q and the sets $\mathcal{L}_k$ and $\mathcal{V}_k$, $k \in \{0, 1, \dots, p\}$, be compatible with $(d_0, d_1, \dots, d_p)$. Find a feedback matrix F that yields a closed-loop output given by (18).

For a given $\lambda \in \mathbb{R}$ and for each $k \in \{0, 1, \dots, p\}$ we introduce the subspaces

$$\mathcal{R}^*_{(k)}(\lambda) := \left\{ v \in \mathbb{R}^n : \exists w \in \mathbb{R}^m \,\middle|\, \begin{bmatrix} v^\top & w^\top \end{bmatrix}^\top \in \ker\big(R_{(k)}(\lambda)\big) \right\} \tag{23}$$

representing the closed-loop eigenvectors that can be assigned to satisfy (16). For a given set $\mathcal{L}_k \subset \mathbb{R}$ that is compatible with a given feasible (p, n)-partitioning,

$$\mathcal{R}^*_{(k)}(\mathcal{L}_k) := \sum_{\lambda \in \mathcal{L}_k} \mathcal{R}^*_{(k)}(\lambda) \tag{24}$$

denotes the subspace of $\mathbb{R}^n$ that is generated by these eigenvectors, for all $\lambda \in \mathcal{L}_k$. A sufficient condition for solving Problem 14 is given by

Theorem 15. *Let the (p, n)-partitioning $(d_0, d_1, \dots, d_p)$ be feasible for Σ_q, and let the sets $\mathcal{L}_k = \{\lambda_{k,1}, \dots, \lambda_{k,d_k}\} \subset \mathbb{R}$, $k \in \{0, 1, \dots, p\}$ be compatible with $(d_0, d_1, \dots, d_p)$. Then a feedback matrix F solving Problem 14 exists if, for every $S \in 2^{[1,p]}$,*

$$\dim\Big(\mathcal{R}^*_{(0)}(\mathcal{L}_0) + \sum_{k \in S} \mathcal{R}^*_{(k)}(\mathcal{L}_k) \Big) = d_0 + \sum_{k \in S} d_k. \tag{25}$$

Proof. We apply Proposition 40 with subspaces $\mathcal{P}_k = \mathcal{R}^*_{(k)}(\mathcal{L}_k)$. We obtain n linearly independent vectors $v_{k,i}$, $k \in \{0, 1, \dots, p\}$ and $i \in \{1, \dots, d_k\}$, such that each $v_{k,i} \in \mathcal{R}^*_{(k)}(\mathcal{L}_k)$. Solving (13) for each $v_{k,i}$, we obtain n vectors $w_{k,i}$. Since the $v_{k,i}$ are linearly independent, from Theorem 11, we obtain F in (14) such that (16)–(18) are satisfied. □

Remark 16. Ntogramatzidis et al. (2016) gives necessary and sufficient results for the existence of a feedback matrix F to solve Problem 14 for the case of monotonic outputs, in which a single mode contributes to each output component. This corresponds to the (p, n)-partitioning with $d_0 = n - p$ and $d_k = 1$, for all $k \in \{1, \dots, p\}$. Theorem 15 generalises this result to accommodate an arbitrary feasible (p, n)-partitioning for Σ_q.

3.2. Eigenstructure assignment for switched linear systems

In this section we revisit the feedback rectification results of Honecker, Gernandt, et al. (2024) yielding GUES of (1), and present a novel result on the simultaneous mode assignment to the outputs of the subsystems.

Definition 17. The subsystems Σ_q of (1) are called *simultaneously eigenvector rectifiable* if there exist linearly independent vectors $\mathcal{V} = \{v_i : i \in \{1, \dots, n\}\} \subset \mathbb{R}^n$, sets $\mathcal{L}_q = \{\lambda_{q,i} : i \in \{1, \dots, n\}\} \subset \mathbb{R}$, with $q \in \mathcal{I}$, and matrices $F_q \in \mathbb{R}^{m\times n}$ such that for each $i \in \{1, \dots, n\}$,

$$(A_q + B_qF_q)v_i = \lambda_{q,i}v_i\,. \tag{26}$$

Note that for simultaneous eigenvector rectification, we require the feedback matrices F_q to assign identical eigenvectors in $\mathcal{V}$ for all closed-loop subsystems, however the corresponding eigenvalues in $\mathcal{L}_q$ may be different. The following result from Honecker, Gernandt, et al. (2024) establishes that simultaneous eigenvector rectification with stable eigenvalues implies GUES.

Theorem 18. *Let the state feedback matrices $F_q \in \mathbb{R}^{m\times n}$ achieve simultaneous eigenvector rectification for Σ_q, for some $\mathcal{V} \subset \mathbb{R}^n$ and some stable sets of eigenvalues $\mathcal{L}_q \subset \mathbb{R}^-$, $q \in \mathcal{I}$. Then the homogeneous part of Σ_{CL} in (5) is GUES.*

Remark 19. An important contribution of Honecker, Gernandt, et al. (2024) is to obtain conditions on Σ_{OL} ensuring that simultaneous eigenvector rectification can be achieved. The system dimension requirement $n < 2m$ is shown to be necessary, which is accommodated in our Assumption 6.

To extend the eigenvector rectification methods of Honecker, Gernandt, et al. (2024) to accommodate systems with outputs, we extend the notation of Section 3.1 to the subsystems of (1).

Notation 20. *For each $q \in \mathcal{I}$ and $k \in \{0, 1, \ldots, p\}$, we define $\Sigma_{q,(k)} = (A_q, B_q, C_{(k)})$ with $C_{(k)}$ as in (9). For convenience we identify $\Sigma_{q,(0)} = \Sigma_q$ in this notation. Let $\mathcal{R}^*_{q,(k)}$ denote the largest output-nulling reachable subspace of $\Sigma_{q,(k)}$ and we identify $\mathcal{R}^*_q = \mathcal{R}^*_{q,(0)}$. For any $\lambda \in \mathbb{R}$, we define the Rosenbrock system matrices $R_{q,(k)}(\lambda)$ according to (10). Their kernels may be decomposed as $N_{q,(k)}(\lambda)$ and $M_{q,(k)}(\lambda)$ in (11). Let $\mathcal{L}_{q,k} = \{\lambda_{q,k,1}, \ldots, \lambda_{q,k,d_k}\} \subset \mathbb{R}$, $k \in \{0, 1, \ldots, p\}$ be sets compatible with $(d_0, d_1, \ldots, d_k)$ and indexed such that*

$$\lambda_{q,k,1} \le \lambda_{q,k,2} \le \lambda_{q,k,d_k}, \quad k \in \{1, \ldots, p\}. \tag{27}$$

The set of closed-loop eigenvalues in Theorem 18 is then obtained by $\mathcal{L}_q = \mathcal{L}_{q,0} \cup L_{q,1} \cup \cdots \cup L_{q,p}$.

Remark 21. As before (see Remark 8) we shall assume that λ is not an invariant zero of $\Sigma_{q,(k)}$, when referring to the dimensions of $N_{q,(k)}(\lambda)$. Then $N_{q,(k)}(\lambda) \in \mathbb{R}^{n \times m_{(k)}}$ for $q \in \{1, 2\}$. Note however, that this assumption is made to simplify notation only, and should not restrict the choice of λ in the design. In fact if λ is chosen to be an invariant zero of $\Sigma_{q,(k)}$, the dimension $m_{(k)}$ increases and thus may increase the intersection (28), providing additional freedom to choose the joint eigenvectors.

We can now formalise the eigenstructure assignment problem in order to achieve eigenvector rectification and simultaneous mode assignment for Σ_{CL} as follows:

Problem 22 (*Eigenstructure Assignment for Switched Linear Systems with Outputs*)**.** Given the switched system (1), a (p, n)-partitioning $(d_0, d_1, \ldots, d_p)$ simultaneously feasible for each subsystem Σ_q with compatible sets $\mathcal{L}_{q,k} = \{\lambda_{q,k,1}, \ldots, \lambda_{q,k,d_k}\} \subset \mathbb{R}$, $q \in \mathcal{I}$, and $\mathcal{V}_k \subset \mathbb{R}^n$, $k \in \{0, 1, \ldots, p\}$, find feedback matrices F_q for the feedback laws $u = F_q x$ to achieve simultaneous eigenvector rectification (26) and also yield closed-loop outputs in the form (18) for all Σ_q, $q \in \mathcal{I}$.

Note that the desired eigenvalue sets $\mathcal{L}_{q,k}$ may be different for the two subsystems, however the (p, n)-partitioning must be the same, in order to obtain outputs with the same number of modes in each component.

For each $k \in \{0, 1, \ldots, p\}$ all assignable eigenvectors $v_{k,i}$, $i \in \{1, \ldots, d_k\}$, are given by the following intersection

$$v_{k,i} \in \operatorname{im} N_{1,(k)}(\lambda_{1,k,i}) \cap \operatorname{im} N_{2,(k)}(\lambda_{2,k,i})\,. \tag{28}$$

In order to describe the intersection (28) as the span of rational matrix-valued functions we consider the scalars $\lambda_1, \lambda_2 \in \mathbb{R}$. Let $T_{(k)}(\lambda_1, \lambda_2)$ be a transformation matrix that converts $[N_{1,(k)}(\lambda_1)\ N_{2,(k)}(\lambda_2)]$ into reduced row-echelon form:

$$T_{(k)}(\lambda_1, \lambda_2)\begin{bmatrix} N_{1,(k)}(\lambda_1) & N_{2,(k)}(\lambda_2)\end{bmatrix} = \begin{bmatrix} E_{(k),11}(\lambda_1, \lambda_2) & E_{(k),12}(\lambda_1, \lambda_2) \\ 0 & E_{(k),22}(\lambda_1, \lambda_2) \\ 0 & 0 \end{bmatrix}, \tag{29}$$

where $E_{(k),11}, E_{(k),12} \in \mathbb{R}[s]^{m_{(k)} \times m_{(k)}}$, and the zero-row has the dimension $p_{(k)} \times 2m_{(k)}$, since $C_{(k)} N_{q,(k)}(\lambda_q) = 0$, $q \in \mathcal{I}$, i.e. the columns of all $N_{q,(k)}$ are orthogonal on $C_{(k)}$. Define the set

$$G_{(k)} := \big\{(\lambda_1, \lambda_2) : T_{(k)}(\lambda_1, \lambda_2) \text{ is invertible}\big\} \subset \mathbb{R} \times \mathbb{R}. \tag{30}$$

Adapting the result in Honecker, Gernandt, et al. (2024) we obtain

Proposition 23. *For any $(\lambda_1, \lambda_2) \in G_{(k)}$, let $Q_{(k)}(\lambda_1, \lambda_2)$ be the rational matrix-valued function given by*

$$Q_{(k)}(\lambda_1, \lambda_2) = T^{-1}_{(k)}(\lambda_1, \lambda_2)\begin{bmatrix} E_{(k),12}(\lambda_1, \lambda_2) \\ 0 \\ 0 \end{bmatrix}, \tag{31}$$

where $E_{(k),12}(\lambda_1, \lambda_2)$ is obtained by the reduced row-echelon form of $[N_{1,(k)}(\lambda_1)\ N_{2,(k)}(\lambda_2)]$ in (29). Then

$$\operatorname{im} N_{1,(k)}(\lambda_1) \cap \operatorname{im} N_{2,(k)}(\lambda_2) = \operatorname{im} Q_{(k)}(\lambda_1, \lambda_2)\,. \tag{32}$$

Remark 24. The sets $G_{(k)}$ are dense in $\mathbb{R}^2$ (Honecker, Gernandt, et al., 2024), and denote the subset of assignable eigenvalues for which $\operatorname{im} Q_{(k)}(\lambda_1, \lambda_2)$ is a valid representation of the intersection $\operatorname{im} N_{1,(k)}(\lambda_1) \cap \operatorname{im} N_{2,(k)}(\lambda_2)$. In case $T_{(k)}(\lambda_1, \lambda_2)$ is singular at points of interest $(\lambda_1^\star, \lambda_2^\star)$, further representations of the intersection can be obtained. At such points $(\lambda_1^\star, \lambda_2^\star)$ the reduced row-echelon form takes a different shape and is obtained by a different transformation $T^\star_{(k)}(\lambda_1^\star, \lambda_2^\star)$. Calculating $Q^\star_{(k)}(\lambda_1^\star, \lambda_2^\star)$ may increase the number of assignable linearly independent eigenvectors, see Honecker, Gernandt, et al. (2024) for an example.

The set $G_{(k)}$ provides pairs of suitable eigenvalues for solving Problem 22. In the remainder of this section we study structural properties of (1) to obtain a sufficient condition for the existence of a suitable partitioning leading to the solvability of Problem 22. To obtain a suitable (p, n)-partitioning and compatible sets of eigenvalues, we define for each $k \in \{0, 1, \ldots, p\}$,

$$d_{(k)} := \dim\Bigg(\sum_{(\lambda_1, \lambda_2) \in G_{(k)}} \operatorname{im} Q_{(k)}(\lambda_1, \lambda_2)\Bigg). \tag{33}$$

Thus $d_{(k)}$ represents the number of linearly independent eigenvectors that can be assigned in $\operatorname{im} Q_{(k)}(\lambda_1, \lambda_2)$ for distinct $\lambda_1 = \lambda_{1,k,i} \in \mathcal{L}_{1,k}$, $\lambda_2 = \lambda_{2,k,i} \in \mathcal{L}_{2,k}$, and $\mathcal{L}_{1,k} \times \mathcal{L}_{2,k} \subset G_{(k)}$. We obtain a polynomial representation of the image of $Q_{(k)}(\lambda_1, \lambda_2)$ by multiplying each column with the largest common denominator polynomial $q_j(\lambda_1, \lambda_2)$, $j \in \{1, \ldots, m_{(k)}\}$

$$P_{(k)}(\lambda_1, \lambda_2) = Q_{(k)}(\lambda_1, \lambda_2)\begin{bmatrix} q_1(\lambda_1, \lambda_2) & & 0 \\ & \ddots & \\ 0 & & q_{m_{(k)}}(\lambda_1, \lambda_2)\end{bmatrix}. \tag{34}$$

$P_{(k)}(\lambda_1, \lambda_2)$ can be written as a matrix polynomial with coefficient matrices $D_{(k),hl} \in \mathbb{R}^{n \times m_{(k)}}$

$$P_{(k)}(\lambda_1, \lambda_2) = \sum_{h=0}^{n}\sum_{l=0}^{n} D_{(k),hl}\lambda_1^h\lambda_2^l\,. \tag{35}$$

The following lemma adapted from Honecker, Gernandt, et al. (2024) permits the computation of $d_{(k)}$.

Lemma 25. *For each $k \in \{0, 1, \ldots, p\}$, let $D_{(k),hl}$ be the coefficient matrices in (35). Then*

$$d_{(k)} = \operatorname{rank}\big[D_{(k),00}\ \ldots\ D_{(k),hl}\ \ldots\ D_{(k),nn}\big]. \tag{36}$$

Proof. Let $\tilde{D}_k = [\tilde{D}_{(k)00}\ \ldots\ \tilde{D}_{(k)nn}]$ consist of all linearly independent rows of $\big[D_{(k),00}\ \ldots\ D_{(k),nn}\big]$ as in (36), and denote $\tilde{d}_k = \operatorname{rank} \tilde{D}_k$. Define

$$\tilde{P}_k(\lambda_1, \lambda_2) = \sum_{h=0}^{n}\sum_{l=0}^{n} \tilde{D}_{(k)hl}\lambda_1^h\lambda_2^l\,. \tag{37}$$

Assume there exists $\alpha \in \mathbb{R}^{\tilde{d}} \setminus \{0\}$ satisfying $\alpha^\top \tilde{P}_k(\lambda_1, \lambda_2) = 0$ for all $(\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}$. Then with (37) we get

$$\alpha^\top \tilde{P}_k(\lambda_1, \lambda_2) = \sum_{h=0}^{n}\sum_{l=0}^{n} \alpha^\top \tilde{D}_{(k)hl}\lambda_1^h\lambda_2^l = 0 \in \mathbb{R}^z. \tag{38}$$

In the ring $\mathbb{R}[\lambda_1, \lambda_2]$ over $\mathbb{R}$, the set of polynomials $\{\lambda_1^h \lambda_2^l\}_{h,l=0}^n$ is linearly independent. Hence (38) implies

$$\alpha^\top \tilde{D}_{(k)hl} = 0 \quad \text{for all } h, l \in \{1, \dots, n\}, \tag{39}$$

but this contradicts (36). Hence, there exists no $\alpha \in \mathbb{R}^{\tilde{d}} \setminus \{0\}$ satisfying $\alpha^\top \tilde{P}_k(\lambda_1, \lambda_2) = 0$ for all $(\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}$. For any dense set $G_{(k)} \subset \mathbb{R} \times \mathbb{R}$, this implies that

$$\dim\Big(\sum_{(\lambda_1,\lambda_2)\in G_{(k)}} \operatorname{im} \tilde{P}_k(\lambda_1, \lambda_2)\Big) = \tilde{d}_k\,. \tag{40}$$

With (34) we have $\tilde{d}_k = d_{(k)}$. □

Remark 26. As the considered intersection is valid for all pairs of eigenvalues $(\lambda_1, \lambda_2) \in G_{(k)}$, any selection of distinct eigenvalues within this set yields $d_{(k)}$ linearly independent eigenvectors.

Definition 27. A (p, n)-partitioning $(d_0, d_1, \dots, d_p)$ is feasible for the switched system Σ_{OL} in (1) if

$$d_k \le d_{(k)}, \quad k \in \{0, 1, \dots, p\}. \tag{41}$$

Remark 28. Note that (41) requires that the partitioning is feasible for both subsystems, but also requires that the intersections of each partition (28) provide at least the required number of linearly independent eigenvectors for each $k \in \{0, 1, \dots, p\}$.

The following lemma is straightforward:

Lemma 29. *A feasible (p, n)-partitioning $(d_0, d_1, \dots, d_p)$ exists for system Σ_{OL} if*

$$\sum_{k=1}^{p} \min(3, d_{(k)}) \ge n - d_{(0)}. \tag{42}$$

Notation 30. *Given a feasible (p, n)-partitioning $(d_0, d_1, \dots, d_p)$ for Σ_{OL} with compatible sets $\mathcal{L}_{q,k} = \{\lambda_{q,k,1}, \dots, \lambda_{q,k,d_k}\} \subset \mathbb{R}$, $q \in \mathcal{I}$, $k \in \{0, 1, \dots, p\}$, we define*

$$\mathcal{P}_{(k)}(\mathcal{L}_{1,k} \times \mathcal{L}_{2,k}) := \sum_{(\lambda_1,\lambda_2)\in \mathcal{L}_{1,k}\times\mathcal{L}_{2,k}} \operatorname{im} P_{(k)}(\lambda_1, \lambda_2). \tag{43}$$

The following result solves Problem 22 and provides a sufficient condition for the simultaneous eigenvector rectification together with the simultaneous mode assignment to the outputs of a switched linear system.

Theorem 31. *Let the (p, n)-partitioning $(d_0, d_1, \dots, d_p)$ be feasible for Σ_{OL} with compatible sets $\mathcal{L}_{q,k} = \{\lambda_{q,k,1}, \dots, \lambda_{q,k,d_k}\} \subset \mathbb{R}$, $q \in \mathcal{I}$, $k \in \{0, 1, \dots, p\}$, such that $\mathcal{L}_{1,k} \times \mathcal{L}_{2,k} \subset G_{(k)}$. Then feedback matrices F_q solving Problem 22 exist, if for every $S \in 2^{[1,p]}$ we have*

$$\dim\Big(\mathcal{P}_{(0)}(\mathcal{L}_{1,0} \times \mathcal{L}_{2,0}) + \sum_{k\in S} \mathcal{P}_{(k)}(\mathcal{L}_{1,k} \times \mathcal{L}_{2,k})\Big) = d_0 + \sum_{k\in S} d_k. \tag{44}$$

Proof. Identify the subspaces $\mathcal{P}_k$ in Proposition 40 with $\mathcal{P}_{(k)}(\mathcal{L}_{1,k}\times \mathcal{L}_{2,k})$ of the theorem. We obtain n linearly independent vectors $v_{k,i}$, $k \in \{0, 1, \dots, p\}$ and $i \in \{1, \dots, d_k\}$, such that each $v_{k,i} \in \mathcal{P}_{(k)}(\mathcal{L}_{1,k} \times \mathcal{L}_{2,k})$. Solving (13) for each $v_{k,i}$, we obtain n vectors $w_{q,k,i}$ for each subsystem Σ_q, $q \in \mathcal{I}$. Since the $v_{k,i}$ are linearly independent, we obtain $F_q = W_q V^{-1}$ with V in (15a) and $W_q = \big[w_{q,0,1} \dots w_{q,0,d_0}\ w_{q,1,1} \dots w_{q,p,d_p}\big]$. Then (26) is satisfied for each $q \in \mathcal{I}$ and with Theorem 11 the outputs of each subsystem Σ_q with feedback $u = F_q x$ take the form (18). □

4. Main results: shaping of the tracking response

In this section we present our main results on shaping the transient response of the switched system for constant reference tracking solving Problems 4 and 5, and provide a constructive procedure for the synthesis of the switched control law (4). The first result yields non-overshooting tracking of a step response from a set of initial states, under arbitrary switching. The second result achieves globally monotonic tracking of constant references under arbitrary switching, and requires stronger conditions on the properties of the subsystems. For both results we obtain feedback matrices that achieve simultaneous eigenvector rectification, while also assigning the closed-loop eigenvectors into suitable output-nulling subspaces. Asymptotic stability of the closed-loop switched system is guaranteed by assigning eigenvalues in $\mathbb{R}^-$ for each mode.

For the non-overshooting design, we identify a set of initial states from which the non-overshooting response will be achieved. This requires the following notation.

Notation 32. *Let $(d_0, d_1, \dots, d_p)$ be a feasible (p, n)-partitioning for Σ_{OL}, and let $\mathcal{V}_k = \{v_{k,1}, \dots, v_{k,d_k}\}$ be compatible sets of vectors for $k \in \{0, 1, \dots, p\}$. We define*

$$\mathcal{H}_0 := \operatorname{span}\big\{v_{0,1}, \dots, v_{0,d_0}\big\} \tag{45}$$

and for $k \in \{1, \dots, p\}$, we define the sets

$$\mathcal{H}_k := \Big\{\xi \in \mathbb{R}^n : \xi = \sum_{i=1}^{d_k} \alpha_{k,i}\, v_{k,i}\Big\}, \text{ with } \alpha_{k,i} \in \mathbb{R}. \tag{46}$$

If $d_k = 2$, the $\alpha_{k,i}$ satisfy $(\alpha_{k,1} + \alpha_{k,2})\alpha_{k,2} > 0$;
if $d_k = 3$, the $\alpha_{k,i}$ are such that none of the following hold:

I. $\alpha_{k,1}\alpha_{k,2} > 0$, $\alpha_{k,1}\alpha_{k,3} < 0$, and $|\alpha_{k,1} + \alpha_{k,2}| > |\alpha_{k,3}|$;
II. $\alpha_{k,2}\alpha_{k,3} > 0$, $\alpha_{k,1}\alpha_{k,2} < 0$, and $|\alpha_{k,1}| > |\alpha_{k,2} + \alpha_{k,3}|$;
III. $(\alpha_{k,2} + \alpha_{k,3})\alpha_{k,3} < 0$.

We also let $\pi_k(\xi)$, $k \in \{0, 1, \dots, p\}$ denote the orthogonal projection of $\xi \in \mathbb{R}^n$ onto $\operatorname{span}\{v_{k,1}, \dots, v_{k,d_k}\}$.

Remark 33. The conditions on the $\alpha_{k,i}$ above are derived from Lemmas A.1 and A.2 in Schmid and Ntogramatzidis (2010) and give necessary and sufficient conditions for the absence of a sign change in the sum of two, and sufficient conditions in the sum of three exponential functions. While the equilibrium $x_{ss} \in \mathcal{X}_0$, the set $\mathcal{X}_0$ may not contain a neighbourhood of x_{ss}.

The following theorem provides the controller synthesis to obtain non-overshooting set-point tracking.

Theorem 34 (*Non-Overshooting Outputs*)**.** *Let the switched system Σ_{OL} in (1) satisfy Assumption 6 and let $r \in \mathbb{R}^p$ be the desired set-point. Let the (p, n)-partitioning $(d_0, d_1, \dots, d_p)$ be feasible for Σ_{OL}, and let $\mathcal{L}_{q,k} \subset \mathbb{R}^-$ be compatible sets of desired closed-loop eigenvalues with $\mathcal{L}_{1,k} \times \mathcal{L}_{2,k} \subset G_{(k)}$, and satisfying (44). Let $\mathcal{V}_k$ be compatible sets of vectors with $v_{k,i} \in \operatorname{im} P_{(k)}(\lambda_{1,k,i}, \lambda_{2,k,i})$, $k \in \{0, 1, \dots, p\}$, $i \in \{1, \dots, d_k\}$ associated with $\mathcal{L}_{1,k} \times \mathcal{L}_{2,k} \subset G_{(k)}$. Let $\mathcal{H}_k$ in (46) be given in terms of these $v_{k,i}$. Then there exist feedback matrices F_q that solve Problem 4, with the set of initial states $\mathcal{X}_0$ given by*

$$\mathcal{X}_0 := \big\{x_0 \in \mathbb{R}^n : \pi_k(x_0 - x_{ss}) \in \mathcal{H}_k, k \in \{0, \dots, p\}\big\}. \tag{47}$$

Proof. For each $v_{k,i}$ and associated $\lambda_{q,k,i}$ we obtain $w_{q,k,i}$ from (13) and construct V, W_q as in (15). Then the control law (4) with F_q obtained from (14) simultaneously rectifies the subsystems Σ_q. As $\mathcal{L}_{q,k} \subset \mathbb{R}^-$, the closed-loop system Σ_{CL} is GUES, by Theorem 18.

Consider $\xi_0 = x_0 - x_{ss} \in \mathcal{X}_0$ and consider the closed-loop homogeneous switched system Σ_ξ in (8). Let $\{t_j : j \in \mathbb{N}\} \subset [0, \infty)$ be the sequence of switching instants, i.e. points of discontinuity of σ. Consider the switching instant t_j with $\sigma(t_j) = q$. The feedback matrices F_q together with subsystem Σ_q and partitioning $(d_0, d_1, \dots, d_p)$ satisfy the assumptions of Theorem 11. Thus the outputs of the closed-loop system Σ_ξ in (8) for $t \in [0, t_{j+1} - t_j)$ take the form

$$e(t_j + t) = \begin{bmatrix} \alpha_{1,1}e^{\lambda_{q,1,1}t} + \cdots + \alpha_{1,d_1}e^{\lambda_{q,1,d_1}t} \\ \vdots \\ \alpha_{p,1}e^{\lambda_{q,p,1}t} + \cdots + \alpha_{p,d_p}e^{\lambda_{q,p,d_p}t} \end{bmatrix}, \tag{48}$$

where $\alpha_{k,i}$, $k \in \{1, \dots, p\}$, $i \in \{1, \dots, d_k\}$, are the last $n - d_0$ entries of $V^{-1}\xi(t_j)$. By assumption, the $\alpha_{k,i}$, $i \in \{1, \dots, d_k\}$, satisfy the conditions that define $\mathcal{H}_k$, for $d_k = 2$ and $d_k = 3$, respectively, and we conclude that $\pi(\xi_0) \in \mathcal{H}_k$ for each k, and the output $e(t)$ is non-overshooting for $t \in [t_j, t_{j+1})$.

Note the eigenstructure of all Σ_q is rectified and the output matrix C is shared by all subsystems. Therefore we obtain the same $\alpha_{k,i}$, $k \in \{1 \dots, p\}$, $i \in \{1, \dots, d_k\}$, for each subsystem $q \in \mathcal{I}$. Furthermore also $\xi(t) \in \mathcal{X}_0$ for $t \in [t_j, t_{j+1})$ which ensures that the initial condition $\xi(t_{j+1}) \in \mathcal{X}_0$ guarantees non-overshooting of $e(t)$ for the subsequent switching interval. Since Σ_ξ is GUES $e(t) \to 0$. Accordingly, the set-point tracking y of Σ_{CL} in (5) is non-overshooting. □

Remark 35. If $x_0 \notin \mathcal{X}_0$ and $d_k = 2$, there exist switching sequences leading to an overshooting output, e.g. constant switching signals $\sigma(t) \equiv q$. However, we observed many examples of switching signals with no overshooting output even if $x_0 \notin \mathcal{X}_0$. In any case Theorem 34 guarantees GUES for every initial condition $x_0 \in \mathbb{R}^n$.

Remark 36. The convergence rate of the output at any time $t > 0$ is determined by the activated closed-loop poles from $\mathcal{L}_q$ assigned to the eigenvectors in $\mathcal{R}^*_{q,(k)}$, $k \neq 0$. Choosing identical eigenvalues for both closed-loop subsystems corresponding to the same output component results in a smooth output trajectory rendering the switching of the system dynamics invisible at the output in the sense that the outputs are continuously differentiable.

The following resembles the result in Honecker, Schmid, et al. (2024) establishing globally monotonic set-point tracking for arbitrary initial conditions.

Theorem 37 (*Monotonic Outputs*)**.** *Let the switched system Σ_{OL} satisfy Assumption 6. Let the (p, n)-partitioning with $d_0 = n-p$, $d_k = 1$, $k \in \{1, \dots, p\}$ be feasible for Σ_{OL}, with compatible sets of desired closed-loop eigenvalues $\mathcal{L}_{q,k} \subset \mathbb{R}^-$ such that $\mathcal{L}_{1,k} \times \mathcal{L}_{2,k} \subset G_{(k)}$, and satisfying (44). Then there exist feedback matrices F_q that solve Problem 5.*

Proof. With the definition of $d_{(0)}$ in (33) any choice $(\lambda_{1,0,i}, \lambda_{2,0,i}) \in G_{(0)}$ for distinct eigenvalues $\lambda_{1,0,i} \in \mathcal{L}_1, \lambda_{2,0,i} \in \mathcal{L}_2$, yields linearly independent eigenvectors $v_{0,i} \in \operatorname{im} P_{(0)}(\lambda_{1,0,i}, \lambda_{2,0,i})$, $i \in \{1, \dots, n-p\}$.

Since Condition (44) is satisfied we can choose the remaining p eigenvectors $v_{k,1} \in \operatorname{im} P_{(k)}(\lambda_{1,k,1}, \lambda_{2,k,1})$ with $\lambda_{q,k,1} \in \mathcal{L}_{q,k}$, for $k \in \{1, \dots, p\}$ such that $\{v_{k,i} : k \in \{0, 1, \dots, p\}, i \in \{1, \dots, d_k\}\}$ is a set of linearly independent vectors by Theorem 31. For each $v_{k,i}$ and associated $\lambda_{q,k,i}$ we obtain $w_{q,k,i}$ from (13). Construct V, W_q as in (15). Then the control law (4) with F_q obtained from (14) simultaneously rectifies the subsystems Σ_q. Since $\mathcal{L}_q \in \mathbb{R}^-$ we have GUES for Σ_{CL} with Theorem 18.

In each switching instant the outputs of Σ_ξ take the form (18), where only the first coefficient $\alpha_{k,1}$, $k \in \{1, \dots, p\}$ is non-zero. Hence each output-component for any subsystem is monotonically decreasing from any initial condition. Similar reasoning as in the proof of Theorem 34 yields globally monotonic step-reference tracking of Σ_{CL} for arbitrary switching. □

Theorem 37 can be considered as a special case of Theorem 34 in the sense that a fixed partitioning is applied for the eigenstructure assignment. Such partitioning together with the assignment of negative real eigenvalues ensures monotonic tracking of the output. Furthermore, the monotonic tracking is guaranteed for any initial condition $x_0 \in \mathbb{R}^n$. As no test on the initial states is necessary the d_0 eigenvalues in $\mathcal{L}_{q,0}$ do not need to be sorted as in (27). Therefore we may calculate the intersection (28) for arbitrary pairs of the sets $\mathcal{L}_{1,0}$ and $\mathcal{L}_{2,0}$ which may increase the choice of suitable eigenvectors and thus provide more freedom in the design.

For the monotonic design the requirements regarding the intersections (43) are more restrictive than for the non-overshooting design, e.g. we require $d_{(0)} \geq n - p$. Thus a given switched system (1) may be infeasible for the eigenstructure assignment for a monotonic response but a non-overshooting design may succeed. Obviously the non-overshooting design offers more flexibility in choosing the partitioning $(d_0, d_1, \dots, d_p)$ for which we only require that the d_k sum to n and $0 \leq d_k \leq 3$. Therefore monotonic and non-overshooting tracking can be combined in the same design to different outputs, e.g. choosing $d_k = 1$, monotonic tracking is assigned to the output y_k, regardless of the initial condition.

We propose the following algorithms to obtain the feedback matrices for this task. The first algorithm provides the analysis of the switched system and a feasible choice of partitioning for the design, whereas the second yields the feedback matrices for the control law and domain of initial states if needed.

Algorithm 1 (*Analysis*)**.**

(Input): Systems Σ_q, $q \in \mathcal{I}$ satisfying Assumption 6.

(Output): A set $\mathcal{D}$ of feasible (p, n)-partitionings for Σ_{OL}.

(1) For each $k \in \{0, 1, \dots, p\}$ and $q \in \mathcal{I}$, compute the Rosenbrock matrices $R_{q,(k)}(\lambda)$, matrix kernels $N_{q,(k)}$ and $M_{q,(k)}$, transformation matrices $T_{(k)}(\lambda_1, \lambda_2)$, dense sets $G_{(k)}$ and rational matrix functions $P_{(k)}(\lambda_1, \lambda_2)$, as in Section 3.2.

(2) Compute $d_{(k)}$ as in (36).
If $d_{(0)} + \sum_{k=1}^{p} \min\{3, d_{(k)}\} < n$: stop.

(3a) ***For non-overshooting design:*** Collect in $\mathcal{D}$ all feasible partitionings $(d_0, d_1, \dots, d_p)$ satisfying (41)

(3b) ***For monotonic design:*** If $(n-p, 1, \dots, 1)$ satisfies (41) then $\mathcal{D} = \{(n-p, 1, \dots, 1)\}$

Remark 38. If the condition in Step 2 fails for the transformation matrix $T_{(k)}(\lambda_1, \lambda_2)$ obtained in Step 1, then the algorithm stops since no feasible partitioning exists for this representation. However, it may be possible to find a different representation of the reduced row-echelon form at points $(\lambda_1^\star, \lambda_2^\star)$ with corresponding $T^\star_{(k)}(\lambda_1^\star, \lambda_2^\star)$, (see Remark 24), for which

$$d_{(0)} + \sum_{k=1}^{p} \min\{3, d_{(k)}\} \geq n \tag{49}$$

and then a feasible partitioning vector may be found.

Algorithm 2 (*Synthesis*)**.**

(Input): Systems Σ_q, $q \in \mathcal{I}$ satisfying Assumption 6. A feasible (p, n)-partitioning $(d_0, d_1, \dots, d_p)$ for Σ_{OL}. Sets of compatible desired closed-loop eigenvalues $\mathcal{L}_{q,k} \subset \mathbb{R}^-$ such that $\mathcal{L}_{1,k} \times \mathcal{L}_{2,k} \in G_{(k)}$.

(Output): Feedback matrices F_q and a domain of initial states $\mathcal{X}_0 \subseteq \mathbb{R}^n$ solving Problem 4 and Problem 5, respectively.

(4) Compute $\mathcal{P}_{(k)}(\mathcal{L}_{1,k} \times \mathcal{L}_{2,k})$ in (43).
If: Condition (44) in Theorem 31 holds continue,
Else: choose a different feasible partitioning and/or eigenvalues $\mathcal{L}_{1,k} \times \mathcal{L}_{2,k} \in G_{(k)}$.
Stop: if none of the partitionings in $\mathcal{D}$ satisfies (44).

(5) For $k \in \{0, 1, \ldots, p\}$ and $i \in \{1, \ldots, d_k\}$ select n linearly independent vectors

$$v_{k,i} \in \operatorname{im} P_{(k)}(\lambda_{1,k,i}, \lambda_{2,k,i}).$$

(6) For each $v_{k,i}$ and $\lambda_{q,k,i}$, $q \in \mathcal{I}$, $k \in \{0, 1, \ldots, p\}$, $i \in \{1, \ldots, d_k\}$, solve (13) for $w_{q,k,i}$ to obtain matrices V, W_q and F_q from (14)–(15), ensuring that the indexing of the columns of V and W_q correspond to the partitioning $(d_0, d_1, \ldots, d_p)$ and match those of the $\mathcal{L}_{q,k}$.

(7a) **For the non-overshoot design:** compute the set of initial states $\mathcal{X}_0$ in (47) using (46).

(7b) **For the monotonic design:** $\mathcal{X}_0 = \mathbb{R}^n$.

5. Numerical examples

The section illustrates the design procedure. The first example allows for non-overshooting design only, whereas for the second monotonic outputs are achieved.

5.1. Non-overshooting design

Consider the subsystems Σ_q of (1) with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & -4 & 1 & -1 & 0 \\ 2 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 4 & 0 & 0 & -5 & 0 & 3 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 3 & 0 & -3 & 0 & 0 \\ 0 & 1 & -1 & 4 & 0 \\ -1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 5 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & -4 & 3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 5 & 0 & -5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^\top,$$

and the set-point $r = \begin{bmatrix} 1 & -6 & 10 \end{bmatrix}^\top$. With (6) we obtain the joint steady state x_{ss} as well as steady-state control $u_{q,ss}$ for each subsystem

$$x_{ss} = \begin{bmatrix} -9 & 10 & 0 & 14.67 & 59.11 & 36.44 & -6 \end{bmatrix}^\top,$$
$$u_{1,ss} = \begin{bmatrix} 0 & 13.33 & 3.33 & 0 & 0 \end{bmatrix}^\top,$$
$$u_{2,ss} = \begin{bmatrix} -153.33 & -65.67 & 11 & -72.89 & 20 \end{bmatrix}^\top.$$

We have $n = 7$, $m = 5$ and $p = 3$, so Assumption 6 on the system dimensions is satisfied.

Evaluating Step 1 of Algorithm 2 we note that the intersection space for $k = 0$ is $\operatorname{im} P_{(0)}(\lambda_1, \lambda_2) = 0$. Therefore $d_{(0)} = 0$ and thus no mode can be hidden from all outputs.

Using (36) as in Step 2 it is readily verified that $d_{(k)} = 5$ for all $k \in \{1, \ldots, 3\}$ and therefore five linearly independent eigenvectors can be obtained for each intersection $\operatorname{im} N_{1,(k)}(\lambda_1) \cap \operatorname{im} N_{2,(k)}(\lambda_2)$. Therefore the (p, n)−partitioning $(d_0, d_1, d_2, d_3) = (0, 3, 3, 1)$ is a feasible choice for Σ_{OL} according to Definition 27. Further we choose the indexed eigenvalue sets

$$\mathcal{L}_{1,1} \times \mathcal{L}_{2,1} = \{-5, -4, -3\} \times \{-3, -2, -1\} \in G_{(1)}$$
$$\mathcal{L}_{1,2} \times \mathcal{L}_{2,2} = \{-8, -7, -6\} \times \{-7, -6, -4\} \in G_{(2)}$$
$$\mathcal{L}_{1,3} \times \mathcal{L}_{2,3} = \{-0.5\} \times \{-8\} \in G_{(3)}.$$

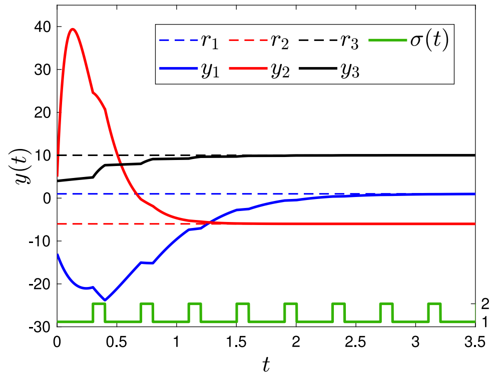

Fig. 1. Non-overshooting step reference tracking.

It is readily verified that (44) holds. Thus we can choose n linearly independent eigenvectors from the intersections obtained from the above sets of eigenvalues. We choose three eigenvectors of the output-nulling subspace $v_{1,i} \in \operatorname{im} P_{(1)}(\lambda_{1,1,i}, \lambda_{2,1,i})$, $i \in \{1, 2, 3\}$, three from the output-nulling subspace $v_{2,i} \in \operatorname{im} P_{(2)}(\lambda_{1,2,i}, \lambda_{2,2,i})$, $i \in \{1, 2, 3\}$, and $v_{3,1} \in \operatorname{im} P_{(3)}(\lambda_{1,3,1}, \lambda_{2,3,1})$.

Using Step 6 of Algorithm 2, we obtain the feedback matrices for each subsystem. By Theorem 34, the outputs of Σ_{CL} track the reference without overshooting if $x_0 - x_{ss} \in \mathcal{X}_0$. The set $\mathcal{X}_0$ is obtained as in (47) evaluating (46) as in Step 7a. Note for the evaluation of the conditions on the $\alpha_{k,i}$ in (46) the eigenvalues assigned for each output have to be sorted in ascending order (27) and paired accordingly, e.g. the eigenvalue $\lambda_{1,1,1} = -5$ has to be paired with the eigenvalue $\lambda_{2,1,1} = -3$, etc. To the third output only one mode is assigned hence this output tracks its reference monotonically from all initial states $x_0 \in \mathbb{R}^n$. Note, that also for $k = 0$ we have no constraints on the initial state. Therefore the sets $\mathcal{L}_{q,0}$, if they exist, are not indexed such that the eigenvalues may be paired arbitrarily, providing additional degrees of freedom for the design; see Section 5.2.

Of course we can also choose a different partitioning that satisfies $\sum d_k = n$ and $d_k \leq \min\{3, d_{(k)}\}$. This might be necessary, e.g. in case (41) fails for some initial choice of partitioning. For instance the partitioning $(d_0, d_1, d_2, d_3) = (0, 3, 2, 2)$ is also feasible and satisfies (41). Such design would lead to non-overshooting, but not necessarily monotonic, step-reference tracking in all three outputs. Since $d_{(0)} = 0$ and we have to allocate seven modes to three outputs with maximal three modes to each output. Thus we can at most choose one output for a monotonic design.

Fig. 1 shows a simulation of the closed-loop switched system for the partitioning $(d_0, d_1, d_2, d_3) = (0, 3, 3, 1)$ above, the initial state $x_0 = \begin{bmatrix} -17 & 4 & 0 & 2 & -7 & -3 & 5 \end{bmatrix}^\top$, and the periodic switching signal that activates Σ_1 for 0.3s followed by Σ_2 for the subsequent 0.1s. The parameters $\alpha_{k,i}$ obtained from $V^{-1}(x_0 - x_{ss})$ satisfy the conditions in (46). We observe that y_1 and y_2 are non-overshooting, whereas y_3 exhibits monotonic behaviour.

5.2. Monotonic design

For this example we adopt A_1, A_2, B_1, B_2 of Example 5.1 but remove one output by considering

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The reference $r = \begin{bmatrix} 1 & -6 \end{bmatrix}^\top$ satisfies (6) with joint steady state $x_{ss} = \begin{bmatrix} 7 & -6 & 0 & -8 & -26.67 & -22.67 & 0 \end{bmatrix}^\top$, and controls $u_{1,ss} = \begin{bmatrix} 0 & -8 & -2 & 0 & 0 \end{bmatrix}^\top$, $u_{2,ss} = \begin{bmatrix} 88 & 39 & -5 & 45.33 & -12 \end{bmatrix}^\top$.

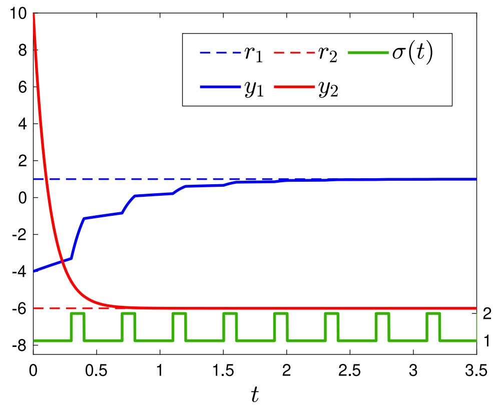

Fig. 2. Monotonic step reference tracking.

We obtain $d_{(0)} = 5$ and thus a monotonic design might succeed. We verify that the partitioning $(d_0, d_1, d_2) = (5, 1, 1)$ is feasible for Σ_{OL}. After calculating feasible sets of eigenvalues $G_{(k)}$, which are not displayed due to space restrictions, we choose the following sets of eigenvalues

$$\begin{aligned}\mathcal{L}_{1,0} \times \mathcal{L}_{2,0} &= \{-7, -3, -4, -5, -6\}\times \\ &\qquad \{-2, -1, -3, -4, -6\} \in G_{(0)}, \\ \mathcal{L}_{1,1} \times \mathcal{L}_{2,1} &= \{-0.5\} \times \{-7\} \in G_{(1)}, \\ \mathcal{L}_{1,2} \times \mathcal{L}_{2,2} &= \{-8\} \times \{-8\} \in G_{(2)}.\end{aligned}$$

Note that the closed-loop eigenvalues are the same as in the first example, but the sets need not to be indexed such that arbitrary pairing is possible. While this offers more flexibility for performance design of the switched system, it also gives more options in case the intersections (43) are not rich enough to span the whole state-space. A different pairing of the eigenvalues may lead to larger intersections such that Condition (44) is satisfied.

Algorithm 2 terminates with Step 6 as there is no restriction for the initial condition x_0. To highlight that we choose an initial condition that does not satisfy the conditions on $\alpha_{k,i}$ in (46): $x_0 = \begin{bmatrix}-14 & 10 & -1 & 2 & 52 & 35 & -9\end{bmatrix}^\top$. Fig. 2 shows the resulting response for the same switching signal as before. We observe that both outputs are monotonic. Moreover y_2 is even a smooth function, i.e. the switching instances are invisible. This is achieved by assigning identical eigenvalues for the modes assigned to the second output.

6. Conclusion

We investigate a state-feedback design approach for switched linear MIMO systems. By analysing eigenstructure properties of the constituent subsystems we obtain feasible partitionings that jointly assign modes to the respective outputs of the system. We provide a structural condition for the feasibility of the combined problem of mode-assignment to the outputs and eigenvector rectification of the subsystems. Our analysis provides a choice of partitionings, i.e. possibilities to assign certain properties to the respective outputs as well as sets of eigenvectors and eigenvalues to be assigned. The design procedure achieves eigenvector rectification for the constituent subsystems while assigning closed-loop poles in the open left complex half-plane. By that we obtain GUES as well as non-overshooting or monotonic output tracking for the specified outputs. While this contribution is focused on the design for two subsystems the methodology may be applicable to a larger number of subsystems $q \geq 3$. However the complexity of the expressions increase rapidly and conditions may turn out to be so strict that no feasible design may exist.

Appendix

Lemma 39 (*Radó's Theorem Radó, 1942*). *Let $\mathcal{P}_1, \ldots, \mathcal{P}_s$ be sets of an Euclidean space. There exist elements $p_i \in \mathcal{P}_i$ for all $i \in \{1, \ldots, s\}$ such that $\{p_1, \ldots, p_s\}$ is a set of linearly independent vectors if and only if given k numbers $\nu_1, \ldots, \nu_k$ such that $1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq s$ for all $k \in \{1, \ldots, s\}$, the union $\mathcal{P}_{\nu_1} \cup \mathcal{P}_{\nu_2} \cup \cdots \cup \mathcal{P}_{\nu_k}$ contains k linearly independent elements.*

Proposition 40. *Let the tuple $(d_0, d_1, \ldots, d_p)$ be such that $\sum_{k=0}^{p} d_k = n$. Let $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_p$ be subspaces of $\mathbb{R}^n$, such that $\dim \mathcal{P}_k = d_k$. There exists a set of linearly independent vectors $\{p_{k,i} : k \in \{0, 1 \ldots, p\}$ and $i \in \{1 \ldots, d_k\}\}$ such that each $p_{k,i} \in \mathcal{P}_k$ if and only if*

$$\dim\big(\mathcal{P}_0 + \sum_{k \in S} \mathcal{P}_k\big) \geq d_0 + \sum_{k \in S} d_k \quad \forall S \in 2^{[1,p]}. \tag{50}$$

Proof. Consider constants ν_i with $1 \leq \nu_1 < \cdots < \nu_k \leq s$. Since $\sum \mathcal{P}_i = \operatorname{span} \bigcup \mathcal{P}_i$, we have: $\mathcal{P}_{\nu_1} \cup \cdots \cup \mathcal{P}_{\nu_k}$ contains k linearly independent vectors if and only if $\mathcal{P}_{\nu_1} + \cdots + \mathcal{P}_{\nu_k}$ contains k linearly independent vectors. Thus $\dim(\mathcal{P}_{\nu_1} + \cdots + \mathcal{P}_{\nu_k}) = k$. With Lemma 39 it follows that there exist $p_i \in \mathcal{P}_i$ such that $\{p_1, \ldots, p_s\}$ is linearly independent if and only if

$$\forall S \in 2^{[1,s]} : \dim \textstyle\sum_{j \in S} \mathcal{P}_j \geq \operatorname{card}(S). \tag{51}$$

Consider the direct sum $\mathcal{P}_0 \oplus \mathcal{P}_k$, $k \in \{1 \ldots, p\}$. Let $[I_{d_0} \; 0]^\top$ be a basis of $\mathcal{P}_0$ and $[P_{k,1}^\top \; P_{k,2}^\top]^\top$ a basis of $\mathcal{P}_k$, $k \in \{1 \ldots, p\}$. We can find linearly independent vectors $\{p_{01}, \ldots, p_{0d_0}, p_1, \ldots, p_s\}$ such that $\operatorname{span}\{p_{01}, \ldots, p_{0d_0}\} \subset \mathcal{P}_0$ and $p_j \in \mathcal{P}_j$ for all $j \in \{1, \ldots, s\}$ if and only if there exist $\tilde{p}_j \in \mathcal{P}_{j,2}$ such that $\{\tilde{p}_1, \ldots, \tilde{p}_s\}$ is linearly independent. With (51) the assertion of the proposition follows. $\square$

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