On canonical basis and path basis

Von der Fakultät für Mathematik und Naturwissenschaften der
RWTH Aachen University zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften genehmigte Dissertation

vorgelegt von

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Tag der mündlichen Prüfung: 2. Juni 2026

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   - Felix Röhrich and Yuhuai Zhou. *Low-Rank Tensor Recovery via Theta Nuclear p-Norm.* 2025. DOI: 10.48550/arXiv.2504.08911

Aachen, den 9. Juni 2026

Abstract

We show for quantum groups over classical Lie algebras that there is a unitriangular base change between the canonical basis constructed by Lusztig and Kashiwara and the path monomial basis constructed by Littelmann for every simple highest weight module. After specialization over an algebraically closed field, we prove that, in these cases, there is also a unitriangular base change between the dual canonical basis and the path vector basis constructed by Littelmann.

It was conjectured by Littelmann that for any path model the associated path vector basis has a unitriangular base change to the dual canonical basis for any symmetrizable Kac-Moody Lie algebra. In addition to the above result, we provide further evidence for the conjecture by proving that an element of the dual canonical basis agrees with the corresponding path vector up to terms neglegible by a partial order if it is compatible with the dual of the Frobenius splitting. This suggests that the conjecture holds for many elements of the dual canonical basis, since Qin proved that the cluster monomials belong to the dual canonical basis and Song proved that the cluster monomials are compatible with the dual of the Frobenius splitting.

Based on the observed compatibility between the dual canonical basis and the path vectors, the author was posed the question, whether the path vectors form a perfect basis in the sense of Berenstein and Kazhdan, or Baumann, Kamnitzer and Knutson. We show that this is not the case already in rank 2.

As part of our investigation, we define a modified version of Littelmann's root operators in the case of Lakshmibai-Seshadri paths for symmetrizable Kac-Moody Lie algebras. Although the operators are incompatible with the crystal structure, they still share interesting properties with the root operators. In particular, we conjecturally obtain a formula for the action (left multiplication) of a Chevalley generator on an element of the canonical basis similar to the one by Kashiwara. Here the leading term is with respect to a partial order defined by Littelmann; in general, the order is incompatible with the crystal structure and the leading term differs from the one in Kashiwara's formula. We prove this formula in the classical types for certain paths.

Finally, we use Lusztig's tight monomial cone to study the duals of Leclerc's imaginary vectors. This allows us to show for all finite types that the canonical basis is not fully compatible with the Frobenius morphism and its splitting. Here compatible means that an element of the canonical basis gets mapped to either zero or an element of the canonical basis, in a way that is compatible with the crystal structure. Furthermore, we give a examples for semi-tight monomials arising from Lusztig's tight monomial cone for all finite types. In both cases, examples were only known in the simply-laced case before.

Acknowledgments

First and foremost, I would like to express my deepest gratitude to Xin Fang. Thank you for your interest in my work, encouragement, insightful discussions, suggestions, proofreading, and continuous support throughout my doctoral studies.

I would also like to thank my advisor Ghislain Fourier for granting me the opportunity to pursue my PhD in Aachen and for pushing me to participate in conferences and summer schools. Especially the summer school in Nordfjordeid will remain a fond memory.

I am grateful to everyone at the ART and ANT chairs for the friendly atmosphere and nice discussions. Special thanks go to Wolfgang Krass for all the help with organizational and bureaucratic matters, and to Lars Göttgens for his help with OSCAR.

Finally, I want to thank my family, particularly my mother, for their support and encouragement throughout this time. Their unwavering faith in me has been a constant source of motivation.

I acknowledge funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - project number 442047500 - through the Collaborative Research Center "Sparsity and Singular Structures" (SFB 1481).

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1. Introduction

1.1. **Background.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and $U_q(\mathfrak{g})$ be its quantized enveloping algebra. In 1990 Lusztig [37] constructed the *canonical basis* for the negative part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ in the types A, D and E. Lusztig's starting point for the construction were bases of PBW type, that is, bases which give a Poincaré-Birkhoff-Witt basis of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, when specializing q to 1. The bases of PBW type are constructed via Lusztig's braid automorphisms, and thus depend on the choice of a reduced decomposition of the longest element of the Weyl group of $\mathfrak{g}$. However, they span a $\mathbb{Z}[q^{-1}]$-submodule L which is independent of the choice of a reduced decomposition, and the images of the bases of PBW type under the projection $\pi\colon L \to L/q^{-1}L$ form a $\mathbb{Z}$-basis B of $L/q^{-1}L$. Finally, there is a unique basis $\mathcal{B}(\infty)$ of L such that $\pi(\mathcal{B}(\infty)) = B$ and that is invariant under the bar involution, the $\mathbb{Q}$-algebra homomorphism fixing the generators of $U_q^-(\mathfrak{g})$ and sending q to q^{-1}. The fact that $\mathcal{B}(\infty)$ is unique and independent of the choice of a reduced decomposition motivates the name *canonical basis*.

One year later, in 1991, Lusztig proved the existence of canonical basis in full generality using topological methods [39]. Meanwhile Kashiwara [22] constructed the *global crystal basis*, whose definition shared similarities to the canonical basis, via purely algebraic methods in the same year. It was later shown by Grojnowski and Lusztig [17] that both bases coincide.

Canonical bases have been intensively studied since their inception due to their remarkable properties. For example: positivtiy of the structure constants of multiplication and comultiplication; for every dominant weight λ, the action of the canonical basis on the highest weight vector u_λ produces the canonical basis of the highest weight module $V(\lambda)$; the combinatorial machinery of crystals, which plays an important role in representation theory.

Since $U_q^-(\mathfrak{g})$ can be endowed with an inner product in a natural way, this gives rise to the *dual canonical basis* (or *upper global crystal basis*) $\mathcal{B}(\infty)^\vee$. Around 2000 Fomin and Zelevensky [13] introduced *cluster algebras* to study the dual canonical basis and total positivity. Since then, cluster algebras have developed into a completely independent area of mathematical research with connections to, for example, Poisson geometry, integrable systems and higher Teichmüller spaces [26].

The *path vector basis* was introduced by Littelmann [35] to complete the program of standard monomial theory which was initiated by Lakshmibai, Seshadri and Musili [29]. The term *standard monomial theory* stems from the work of Hodge [18] who constructed a monomial basis of the homogeneous coordinate ring of the Grassmann variety $\mathrm{Gr}(d, n)$ which is indexed by standard Young tableaux. The goal of the program was to extend his results to Kac-Moody Lie algebras by constructing bases of the graded dual of a simple highest weight module which have nice geometric and combinatorial properties. We explain how this is achieved by the path vectors; for simplicity assume that $\mathfrak{g}$ is simply-laced.

In the setting of Kac-Moody Lie algebras, the role of the Young tableaux is played by Littelmann's path model [36]. A particularly nice path model is the one generated by the straight-line path $\pi_\lambda\colon [0,1], t \mapsto t\lambda$. Here each path in the path model $B(\pi_\lambda)$ corresponds to a so-called *Lakshmibai-Seshadri path (LS path) of shape* λ. Recall that an LS path π is a tuple $(\tau_1, \ldots, \tau_r; t_1, \ldots, t_r)$ consisting of elements $\tau_1 > \tau_2 > \cdots > \tau_r$ of the Weyl group which are linearly ordered with respect to

the Bruhat order, and rational numbers $0 = t_0 < t_1 < t_2 < \cdots < t_r = 1$ satisfying certain integrality constraints. Let l be the smallest positive even integer such that $lt_k \in \mathbb{Z}$ for all $k \in [1, r]$ and fix a primitive $2l$-th root of unity $\zeta \in R$, where R is the extension of $\mathbb{Z}$ by adjoining all roots of unity. We define the R-modules

$$V_1(\lambda) := V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} R, \quad V_\zeta(\lambda) := V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} R.$$

For the first tensor product, we use the specialization $q \mapsto 1$, and for the second tensor product we use the specialization $q \mapsto \zeta$.

We define the vectors

$$x_\pi = u_{\tau_1\lambda}^{lt_1} \otimes u_{\tau_2\lambda}^{l(t_2-t_1)} \otimes \cdots \otimes u_{\tau_r\lambda}^{(t_r-t_{r-1})} \in V_\zeta(\lambda)^{\otimes l},$$

and

$$x^\pi = (u_{\tau_1\lambda}^\vee)^{lt_1} \otimes (u_{\tau_2\lambda}^\vee)^{l(t_2-t_1)} \otimes \cdots \otimes (u_{\tau_r\lambda}^\vee)^{(t_r-t_{r-1})} \in (V_\zeta(\lambda)^\vee)^{\otimes l}.$$

Here $u_{\tau_i\lambda}$ is the extremal weight vector of weight $\tau_i\lambda$ and $u_{\tau_i\lambda}^\vee$ is its dual vector.

Littelmann [35] showed that there is a $U_1(\mathfrak{g})$-module structure on $V_\zeta(\lambda)^{\otimes l}$, and with respect to this structure we have an embedding of $U_1(\mathfrak{g})$-modules

$$\phi\colon V_1(\lambda) \to V_\zeta(\lambda)^{\otimes l}.$$

The *path vector* associated to the LS path π is defined as

$$p_\pi := \phi^\vee(x^\pi) \in V_1(\lambda)^\vee.$$

The path vectors have various nice properties: they are compatible with Demazure modules, that is, for each $w \in W$ the set

$$\{p_\pi \mid \pi = (\tau_1, \ldots; t_1, \ldots) \in B(\pi_\lambda), \tau_1 \leq w\}$$

is a basis of the Demazure module $V_{1,w}(\lambda)^\vee := U_1(\mathfrak{g})^+ u_{w\lambda}^\vee$; and there is a simple combinatorial way to choose a basis, the *standard monomials*, of $V_1(\lambda+\mu)^\vee$ from the monomials $p_\pi p_{\pi'}$, where $p_\pi \in V_1(\lambda)^\vee$ and $p_{\pi'} \in V_1(\mu)^\vee$. Based on their properties, Littelmann conjectured the following.

Conjecture 1 ([35]). *For every dominant weight λ, there is a unitriangular base change from the dual canonical basis of $V_1(\lambda)$ to the path vector basis.*

The base change is supposed to have the following properties:

First, the base change is compatible with the underyling combinatorial structure of the crystal base. This makes sense, since it was shown by Kashiwara [23] that the crystal base $B(\lambda)$ is (as a crystal [32]) isomorphic to the path model $B(\pi_\lambda)$ introduced by Littelmann [36]. Since all path models for λ are equivalent [36], we have $B(\lambda) \cong B(\eta)$ for any dominant path η ending in λ. For a path $\pi \in B(\eta)$, we denote by $b_\pi \in B(\lambda)$ the corresponding element under this isomorphism.

Second, the base change is supposed to be with respect to a linearisation of a partial order $\triangleright$ defined on the path model $B(\pi_\lambda)$ (cf. Section 2.5, LS path order). Note that the partial order naturally extends to concatenations of LS paths.

The *path monomial basis* was also introduced by Littelmann [35]. This basis, which we denote by $\mathcal{P}(\pi_\lambda)$, was constructed from the combinatorics of LS paths. It seems to have received little attention and only served as a tool to prove that the path vectors form a basis of $V_1(\lambda)^\vee$. To this end, it was shown by Littelmann that there is a unitriangular base change from the dual basis of $\mathcal{P}(\pi_\lambda)$ to the path vector basis. Thus, instead of trying to prove Conjecture 1 directly, one may prove that there is a unitriangular base change from the path monomial basis $\mathcal{P}(\pi_\lambda)$ to the

canonical basis of $V(\lambda)$. This has approach has the advantage that the indirection via the Frobenius splitting and specialization at a root of unity can be avoided.

1.2. **Outline of the thesis.** In this thesis, we study the four aforementioned bases and, in particular, investigate the base change between the path monomial basis and the canonical basis of $V(\lambda)$ (Section 3), and the base change between the path vector basis and the dual canonical basis of $V_1(\lambda)$ (Section 6). We also take a closer look at the duals of Leclerc's imaginary vectors [30] from the perspective of Lusztig's tight monomial cone [40] (Section 4). Since the definition of path vectors relies on the quantum Frobenius splitting, we also study the amount of compatibility of the canonical basis with the quantum Frobenius morphism and its splitting (Section 5). Here compatible means that the Frobenius morphism and its splitting map an element of the canonical basis to either zero or an element of the canonical basis, in a way that is compatible with the crystal structure.

1.3. **Main results.** Let us give a brief overview of the structure and main results of this thesis.

We begin by recalling the mathematical background underlying this thesis and introducing the corresponding notation in Section 2. The overarching topics are quantized enveloping algebras, canonical bases and Littelmann's path model.

In Section 3 we introduce a modified version of the root operators introduced by Littelmann [36]. These operators $\tilde{e}_{i;n}$, $\tilde{f}_{i;n}$ act on the n-th local minimum of the function $\langle h_i, \pi(t)\rangle$ in a similar way as the root operators would act on the global minimum. Of primary interest is the operator $\tilde{f}_{i;1}$ which acts on the first local minimum. The idea behind the operator is to mimic the action of the Chevalley generator f_i on an element of $V_1(\lambda) \hookrightarrow V_\zeta(\lambda)^{\otimes l}$ on the left-most tensor factor where it can act non-trivially. Based on the observation that relations in $V_\zeta(\lambda)^{\otimes l}$ do not depend on the root of unity, we formulate the following conjecture, which is central to Section 3.

Conjecture 2 (Conjecture 3.9)**.** *Let λ be a dominant weight and η be a dominant path of weight λ. Then, for any path $\pi \in B(\eta)$ and $i \in I$,*

$$f_i^{(n)} G_\lambda(b_\pi) = \sum F^{i,n}_{\pi,\pi'} G_\lambda(b_{\pi'}),$$

where $F^{i,n}_{\pi,\pi'} \in \mathbb{Z}[q, q^{-1}]$ and the sum ranges over $\pi' \in B(\eta)$ with the following properties:

(i) $\pi'(t) \succeq \pi(t) - n\alpha_i$ for all $t \in [0,1]$.

(ii) Let $x \in [0,1]$ be maximal such that $\pi'(t) - \pi(t)$ is a multiple of α_i for all $t \in [0,x]$. Assume that $x < 1$ and let $\epsilon > 0$ be such that on $[x, x+\epsilon]$ the path π is supported on $\tau \in W$, the path π' is supported on $\sigma \in W$. Then, $\sigma < \max\{\tau, s_i\tau\}$, and if $\pi'(x) - \pi(x) = -n\alpha_i$, then already $\sigma < \tau$.

(iii) Let $t_0 \in [0,1]$ be maximal such that $\langle h_i, \pi(t)\rangle$ is decreasing on $[0,t_0]$, and let $t_1 \in [t_0, 1]$ be maximal such that $\langle h_i, \pi(t)\rangle$ is increasing on $[t_0, t_1]$ and $M := \langle h_i, \pi(t_1)\rangle - \langle h_i, \pi(t_0)\rangle \in \mathbb{Z}$. If $n \leq M$, then $\tilde{f}^n_{i;1}\pi \neq 0$ and

$$F^{i,n}_{\pi, \tilde{f}^n_{i;1}\pi} = \begin{bmatrix} \epsilon_{i;1}(\pi) + n \\ n \end{bmatrix}_i.$$

Let us briefly explain the notation used in the conjecture for the case $\eta = \pi_\lambda$:

In (i) the partial order $\succeq$ is defined on $P \otimes_{\mathbb{Z}} \mathbb{R}$, where P is the weight lattice, by $\lambda \succeq \mu$ if $\lambda - \mu$ is an $\mathbb{R}_{\geq 0}$-linear combination of simple roots.

In (ii), when we say that π is supported on $\tau \in W$, we mean the following: If $\pi = (\tau_1, \ldots, \tau_r; t_1, \ldots, t_r)$ and $x < 1$, then ϵ can be chosen such that $[x, x+\epsilon] \subseteq [t_{k-1}, t_k]$ for some k and τ_k is called the *support* of π on this interval.

In (iii), note that the modified root operator $\tilde{f}_{i;1}$ is not compatible with the crystal structure on $B(\lambda)$, in general, and $\epsilon_{i;1}(\pi)$ is an analogue to the function ϵ_i on the crystal $B(\lambda)$.

We also obtain a conjecture on the Chevalley generator e_i.

Conjecture 3 (Conjecture 3.15)**.** *Let λ be a dominant weight and η be a dominant path of weight λ. Then, for any path $\pi \in B(\eta)$ and $i \in I$,*

$$e_i G_\lambda(b_\pi) = \sum E^i_{\pi,\pi'} G_\lambda(b_{\pi'})$$

where $E^i_{\pi,\pi'} \in \mathbb{Z}[q, q^{-1}]$ and the sum ranges over all $\pi' \in B(\eta)$ with the following properties:

(i) $\pi' \succ_+ \pi$.

(ii) Let $x \in [0,1]$ be maximal such that $\pi'(t) - \pi(t)$ is a multiple of α_j for some $j \in I$ and all $t \in [0, x]$. Assume that $x < 1$ and let $\epsilon > 0$ be such that on $[x, x+\epsilon]$ the path π is supported on $\tau \in W$ and the path π' is supported on $\sigma \in W$. Then, $\sigma < \max\{\tau, s_j\tau\}$, and if $\pi'(x) - \pi(x) = 0$, then already $\sigma < \tau$.

Here we write $\pi' \succeq_+ \pi$ if $\pi'(t) \succeq \pi(t)$ for all $t \in [0,1]$ and $\pi' \trianglelefteq \pi$. One interesting consequence of Conjecture 3 is that the $\mathbb{Z}[q, q^{-1}]$-submodules

$$V_{\succ_+\pi}(\lambda) := \bigoplus_{\pi' \succ_+ \pi} \mathbb{Z}[q, q^{-1}] G_\lambda(b_{\pi'}) \quad \text{and} \quad V_{\succeq_+\pi}(\lambda) := \bigoplus_{\pi' \succeq_+ \pi} \mathbb{Z}[q, q^{-1}] G_\lambda(b_{\pi'}).$$

are, in fact, $U^+_{\mathbb{Z}}(\mathfrak{g})$-submodules of $V(\lambda)$. They may be viewed as generalizations of Demazure modules.

As a first step we verify Conjectures 2 and 3 for fundamental weights of classical type (Propositions 3.19 and 3.20) and for both fundamental weights of type G_2 (Lemma 3.23). In fact, in these cases the path monomial basis coincides with the canonical basis (Proposition 3.18 and Example 3.22). Using the compatibility of the canonical basis with tensor products, allows us to prove Conjectures 2 and 3 inductively for dominant weights and certain paths. In particular, this yields the main result of Section 3.

Theorem 4 (Theorem 3.31)**.** *Let $\mathfrak{g}$ be a classical Lie algebra or of type G_2 and let $\lambda = m_1\varpi_1 + \cdots + m_N\varpi_N$ be a dominant weight. Fix the dominant path $\eta := \pi_{m_1\varpi_1} * \cdots * \pi_{m_N\varpi_N}$. Then, there is a unitriangular base change from path monomial basis $\mathcal{P}(\eta)$ to the canonical basis of $V(\lambda)$ for any linearisation of $\triangleright$ on $B(\eta)$.*

In the dual picture, this translates to the main result of Section 6.

Theorem 5 (Theorem 6.16)**.** *Let $\mathfrak{g}$ be a classical Lie algebra or of type G_2 and let $\lambda = m_1\varpi_1 + \cdots + m_N\varpi_N$ be a dominant weight. Fix the dominant path $\eta := \pi_{m_1\varpi_1} * \cdots * \pi_{m_N\varpi_N}$. Then, there is a unitriangular base change from the path vectors basis to the dual canonical basis of $V_1(\lambda)$ for any linearisation of $\triangleright$ on $B(\eta)$.*

The theorem shows that for a specific choice of the dominant path, Conjecture 1 holds in classical types and in type G_2. This gives hope that the conjecture holds for any dominant path which is a concatenation of LS paths. The conjecture is further supported our other results of Section 6.

Let $\mathrm{Fr}'^{\vee}\colon U_\zeta^-(\mathfrak{g})^\vee \to U_1^-(\mathfrak{g})^\vee$ be the dual Frobenius splitting and $S_\infty\colon B(\infty)\to B(\infty)$ be Kashiwara's similarity map [23]. In the simply-laced case, S_∞ operates by sending $\tilde{f}_{i_1}^{a_1}\cdots\tilde{f}_{i_r}^{a_r}b_\infty$ to $\tilde{f}_{i_1}^{la_1}\cdots\tilde{f}_{i_r}^{la_r}b_\infty$.

Proposition 6 (Proposition 6.14)**.** *Let λ be a dominant weight, π be an LS path of shape λ. If $\mathrm{Fr}'^{\vee}(G(S_\infty(b_\pi))^\vee) = G(b_\pi)^\vee$, then*

$$G_\lambda(b_\pi)^\vee = p_\pi + \sum_{\pi'\in B(\pi_\lambda)} z_{\pi,\pi'}p_{\pi'},$$

where $z_{\pi,\pi'}\in R$ is zero unless $\pi'\rhd\pi$.

Qin [45] recently proved that the cluster monomials belong to dual canonical basis and Song [49] showed that the dual of Frobenius splitting is compatible with the cluster monomials. Although Song's assumptions are not satisfied in our setting, his result suggests, in view of the above proposition, that many elements of the dual canonical basis agree with path vectors up to terms which are greater with respect to $\rhd$.

Using the full compatibility of the canonical basis with the quantum Frobenius splitting in small rank, allows us to verify Conjecture 1 in these cases.

Proposition 7 (Proposition 6.12)**.** *Assume that $\mathfrak{g}$ is of type A_1, A_2, A_3 or B_2. For any dominant weight λ, there is a unitriangular base change from the path vector basis to the dual canonical basis of $V_1(\lambda)$ for any linearisation of $\rhd$.*

We also study the path vector basis for the type A_2 in more detail. The central piece is the following proposition, which explicitly describes the evaluation of path vectors. Here we make use of the fact that an LS path of shape λ can be intepreted as an element of *LS fan of monoids* LS_λ^+ of degree 1 [11]. The LS path $\pi = (\tau_1,\dots,\tau_r;t_1,\dots,t_r)$ is identified with the element

$$t_1e_{\tau_1} + (t_2-t_1)e_{\tau_2} + \cdots + (t_r - t_{r-1})e_{\tau_r}.$$

Proposition 8 (Proposition 6.5)**.** *Assume that $\mathfrak{g}$ is of type A_2. Let λ be an dominant weight and let π be an LS path of shape λ. Write*

$$\pi = a_{121}e_{s_1s_2s_1} + a_{21}e_{s_2s_1} + a_{12}e_{s_1s_2} + a_2e_{s_2} + a_1e_{s_1} + a_0e_{\mathrm{id}}.$$

Then, the evaluation of the path vector p_π is given by

$$p_\pi(f_1^{(n_1)}f_2^{(n_2)}f_1^{(n_3)}u_\lambda) = \begin{cases} \binom{\lfloor\langle h_1,\lambda\rangle(a_{12}+a_1)\rfloor}{\lfloor n_3-\langle h_1,\lambda\rangle(a_{121}+a_{21})\rfloor}, & \text{if } \mu = \mathrm{wt}\,\pi,\\ 0, & \text{otherwise,}\end{cases}$$

where $\mu = \lambda - (n_1+n_3)\alpha_1 + n_2\alpha_2$; and

$$p_\pi(f_2^{(n_1)}f_1^{(n_2)}f_2^{(n_3)}u_\lambda) = \begin{cases} \binom{\lfloor\langle h_2,\lambda\rangle(a_{21}+a_2)\rfloor}{\lfloor n_3-\langle h_2,\lambda\rangle(a_{121}+a_{12})\rfloor}, & \text{if } \mu' = \mathrm{wt}\,\pi,\\ 0, & \text{otherwise,}\end{cases}$$

where $\mu' = \lambda - n_2\alpha_1 + (n_1+n_3)\alpha_2$.

Using the proposition, we are able to show that the path vector basis is not perfect in the sense of Baumann, Kamnitzer and Knutson [2]. In addition, we can express the dual canonical basis elements in terms of path vectors and provide an explicit description of the base change matrix (Theorem 6.10).

In Section 4, we begin by deriving the defining inequalities of Lusztig's tight monomial cone from a generalization of the higher order Serre relations. The inequalities defining the *Lusztig cone* $L_{\underline{i}}$ were obtained by Lusztig [40] from the

positivity properties of a quadratic form. The purpose of the quadratic form was to decide whether a given monomial of $U_q^-(\mathfrak{g})$ belongs to the canonical basis (called *tight* in this case), or is an $\mathbb{N}$-linear combination of canonical basis elements (called *semi-tight* in this case). Although the conditions of the Lusztig cone are sufficient for a monomial to be tight in the types A_1, A_2, A_3 [40], A_4 [42] and B_2 [54], it was shown by Reineke [47] that this is no longer true for A_6.

We will give examples of semi-tight monomials arising from the Lusztig cone for A_5, B_3, C_3, D_4 and G_2 in Section 4.6. In view of the above, for each type this is smallest rank that such examples can occur. All examples have one thing in common: they are closely related to the duals of Leclerc's imaginary vectors [30]. We will make this precsise shortly; first, we recall the background of the imaginary vectors.

Berenstein and Zelevinsky [4] conjectured that for any two elements b_1 and b_2 of the dual canonical basis, their product b_1b_2 is an element of $q^{\mathbb{Z}}\mathcal{B}(\infty)^\vee$ if and only if b_1 and b_2 q-commute. If the conjecture was true, then any $b \in \mathcal{B}(\infty)^\vee$ would be a *real vector*, that is, $b^2 \in q^{\mathbb{Z}}\mathcal{B}(\infty)^\vee$; otherwise, there would exist vectors $b \in \mathcal{B}(\infty)^\vee$ which are *imaginary*, that is, $b^2 \notin q^{\mathbb{Z}}\mathcal{B}(\infty)^\vee$. Leclerc [30] provided examples of imaginary vectors in A_5, B_3, C_3, D_4 and G_2; thereby disproving the conjecture of Berenstein and Zelevinsky.

The duals of these imaginary vectors occur in various (counter)examples: Kashiwara and Saito [25] show that in A_5 there are elements of the canonical basis (when understood as perverse sheaves) whose singular support is not irreducible; and Baumann [1] used them to show that in A_5 and D_4 there exist elements in the canonical basis which are incompatible with the quantum Frobenius morphism and its splitting.

Let $b \in \mathcal{B}(\infty)^\vee$ be one of Leclerc's imaginary vectors and denote by $\theta \in U_q^-(\mathfrak{g})$ the unique element such that $(\theta, b) = 1$. We write ξ for the element θ^2 after normalization by q-binomial coefficients. The vector ξ is exactly our example for a semi-tight monomial in A_5, B_3, C_3, D_4 and G_2. To see that they are semi-tight, we use the fact that an element $u \in U_{\mathbb{Z}}^-(\mathfrak{g})$ is semi-tight only if $(u, u) \in \mathbb{N} + qA$. Here A is the subalgebra of $\mathbb{Q}(q)$ consisting of all rational functions regular at $q = 0$. For this purpose, we compute the evaluation of the inner product in regards to θ and ξ.

Theorem 9 (Theorem 4.19)**.** *We have modulo qA: $(\theta^{[l]}, \theta^{[l]}) = 1$, $(\theta^{[2l]}, \xi^{[l]}) = 1$ and $(\xi^{[l]}, \xi^{[l]}) = l + 1$.*

Here $\theta^{[l]}$ and $\xi^{[l]}$ are generalizations of θ and ξ (see Theorem 4.17); we have $\theta^{[1]} = \theta$ and $\xi^{[1]} = \xi$. We also obtain the following criterion.

Proposition 10 (Proposition 4.11)**.** *Let $\underline{i} = (i_1, \ldots, i_r)$ be a sequence in I such that $s_{\underline{i}}$ is a reduced decomposition. For $(a_1, \ldots, a_r) \in L_{\underline{i}}$ assume that $(f_{\underline{i}}^{(\underline{a})}, f_{\underline{i}}^{(\underline{a})}) \in 1 + qA$. Then, $f_{\underline{i}}^{(\underline{a})}$ is an element of the canonical basis of $U_q^-(\mathfrak{g})$.*

For the simply-laced case, this was already observed by Reineke [47], where he used the positivity of the structure constantants of multplication [38] to prove the statement. However, since the positivity does not hold in general, his argument does not work in the non-simply-laced case. The key to our proof is the following observation.

Proposition 11 (Proposition 4.10)**.** *Let $\underline{i} = (i_1, \dots, i_r)$ be a sequence in I such that $s_{\underline{i}}$ is a reduced decomposition. Then, the Lusztig cone $L_{\underline{i}}$ is contained in the adapted string cone $S_{\underline{i}}(\infty)$.*

The adapted string cone $S_{\underline{i}}(\infty)$ [34] consists of the tuples $(a_1, \dots, a_r) \in \mathbb{Z}_{\geq 0}$ such that $b := \tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\infty \neq 0$ and for all $k \in [1, r]$

$$\tilde{e}_{i_k} \tilde{f}_{i_{k+1}}^{a_{k+1}} \cdots \tilde{f}_{i_r}^{a_r} b_\lambda = 0.$$

Motivated by this result, we investigate how the Lusztig cone $L_{\underline{i}}$ behaves under braid moves. Alas, $L_{\underline{i}}$ is in general not preserved under the braid moves (Lemma 4.14).

In Section 5, we study the compatibility of the canonical basis with the quantum Frobenius morphism and its splitting. The motivation behind this is that x_π is the leading term of element $G(b_{l\pi})$ with respect to a preorder $\blacktriangleright$ on $V_\zeta(\lambda)^{\otimes l}$ (cf. (2.21)). The preorder has the property that $\pi \triangleright \pi'$ if and only if $x_\pi \blacktriangleright x_{\pi'}$. Using this fact, one can see that Conjecture 1 holds if

$$\mathrm{Fr}'(G(b_\pi)) = G(b_{l\pi}) + \sum_{\substack{\pi' \in B(l\lambda) \\ \pi' \lhd l\pi}} z_{\pi,\pi'} G(b_{\pi'}).$$

In particular, Conjecture 1 holds if the canonical basis is fully compatible with the quantum Frobenius splitting, that is, $\mathrm{Fr}'(G(b_\pi)) = G(b_{l\pi})$ for all $\pi \in B(\pi_\lambda)$.

We first show that there indeed types where we have full compatibility.

Proposition 12 (Propositions 5.4 and 5.5)**.** *The canonical basis is fully compatible with the quantum Frobenius morphism and its splitting if $\mathfrak{g}$ is of type A_1, A_2, A_3 or B_2.*

However, using the elements θ and ξ from Section 4, we see that the canonical basis is not fully compatible with the quantum Frobenius morphism and its splitting, in general.

Proposition 13 (Section 5.2)**.** *For $\mathfrak{g}$ of type A_n ($n \geq 5$), B_n ($n \geq 3$), C_n ($n \geq 3$), D_n ($n \geq 4$), or G_2, the canonical basis is not fully compatible with the quantum Frobenius morphism and its splitting.*

Thus, in the final part of Section 5, we consider different partial orders which have some degree of compatibility with the quantum Frobenius splitting. By this we mean that they would imply the desired equation above if the partial order is weaker than $\triangleright$. However, all the partial orders we discuss turn out to be incomparable to $\triangleright$.

1.4. **Outlook.** Conjecture 1 remains open for the path π_λ except for the small rank cases covered in Proposition 7. If Conjecture 2 could be verified for π_λ, even if only in the classical types, this would be an important step, since, from a geometric standpoint, the path π_λ has nice properties (for example for Seshadri stratifications [11]). For this purpose, it would interesting to study how the partial order induced by π_λ compares to the one by induced $\pi_{m_1\varpi_1} * \cdots * \pi_{m_N\varpi_N}$.

It was recently shown by Qin [45] that the cluster monomials are contained in the dual canonical basis for all symmetrizable Kac-Moody Lie algebras. Thus, it is interesting to determine the relation between the path vector basis and the cluster monomials. Possible candidates for path vectors to be cluster monomials are those

where the LS path has a support which is compatible with a reduced decomposition of w_0 in the sense of [8].

For the elements $\xi^{[l]}$ an explicit description in terms of the canonical basis was given by Baumann [1] in A_5 and D_4. Our results suggest that a similar structure holds in B_3, C_3 and G_2 as well; it would be nice to have a uniform description for all these types.

The operators $\tilde{f}_{i;1}$ and $\tilde{e}_{i;1}$ introduced in Section 3 are only defined for LS paths (and their concatenations). It would be interesting to see whether they can be extended to more general paths.

Although we have verified Conjecture 2 for fundamental weights of classical type and for both fundamental weights of type G_2, the underlying reason why it holds in these cases remains unclear. A deeper understanding may reveal a new perspective on the canonical basis.

2. Preliminaries

In this thesis we mainly follow the notation and conventions of Kashiwara [22] and Kimura [27]. In examples we use the conventions of Bourbaki [7], Plates I - IX for the labeling of Dynkin diagrams and simple roots.

The notation for the path model is based on [36]. In this thesis a path will always mean a concatenation of LS paths unless stated otherwise.

For path monomials and path vectors the notation is similar to [35].

2.1. **q-integers.** Let $\mathbb{Q}(q)$ the field of rational functions in the indeterminate q over $\mathbb{Q}$. We denote by A the $\mathbb{Q}$-subalgebra of $\mathbb{Q}(q)$ consisting of functions which are regular at $q = 0$.

We define the q-integer

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{for } n \in \mathbb{Z},$$

the q-factorial

$$[n]! := [n]_q[n-1]_q \cdots [1]_q \quad \text{for } n \in \mathbb{Z}_{\geq 0},$$

and q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n][n-1]_q \cdots [n-k+1]}{[k][k-1] \cdots [1]} \quad \text{for } n \in \mathbb{Z} \text{ and } k \in \mathbb{N}.$$

It is well-known $[n]$, $[n]!$ and $\begin{bmatrix} n \\ k \end{bmatrix}$ are elements of $\mathbb{Z}[q, q^{-1}]$.

From the definition it is easy to see that

$$q^{n(n-1)/2}[n]! \in 1 + qA \quad \text{and} \quad q^{n(m-n)} \begin{bmatrix} m \\ n \end{bmatrix} \in 1 + qA. \tag{2.1}$$

Binomial identities. We list some identities for binomial coefficients from Sections 1.3 and 34.1 of [38].

Lemma 2.1. *Let $a \in \mathbb{N}$. We have*

$$\sum_{t=0}^{a} (-1)^t q^{t(1-a)} \begin{bmatrix} a \\ t \end{bmatrix} = 0.$$

Lemma 2.2. *Let $a \in \mathbb{Z}$ and $t \in \mathbb{N}$. We have*

$$\begin{bmatrix} a \\ t \end{bmatrix} = (-1)^t \begin{bmatrix} -a+t-1 \\ t \end{bmatrix}.$$

Lemma 2.3. *Let $a', a'' \in \mathbb{Z}$ and $t \in \mathbb{N}$. We have*

$$\begin{bmatrix} a'+a'' \\ t \end{bmatrix} = \sum_{t'+t''=t} q^{a't''-a''t'} \begin{bmatrix} a' \\ t' \end{bmatrix} \begin{bmatrix} a'' \\ t'' \end{bmatrix}.$$

Lemma 2.4. *Let $l \in \mathbb{N}$ and ζ be a primitive l-th or $2l$-th root of unity over $\mathbb{Z}$ if l is odd, and a primitive $2l$-th root of unity over $\mathbb{Z}$ if l is even. Let $a \in \mathbb{Z}$ and $t \in \mathbb{N}$. Write $a = a_0 + la_1$ with $a_0, a_1 \in \mathbb{Z}$ and $t = t_0 + lt_1$ with $t_0, t_1 \in \mathbb{Z}_{\geq 0}$ such that $0 \leq a_0, t_0 \leq l-1$. Then,*

$$\begin{bmatrix} a \\ t \end{bmatrix}_\zeta = \zeta^{(a_0t_1 - a_1t_0)l + (a_1+1)t_1l^2} \begin{bmatrix} a_0 \\ t_0 \end{bmatrix}_\zeta \binom{a_1}{t_1}.$$

Here $\begin{bmatrix} a \\ t \end{bmatrix}_\zeta$ is the evaluation of $\begin{bmatrix} a \\ t \end{bmatrix}$ at $q = \zeta$.

2.2. **Quantized enveloping algebras.**

Root Datum. Throughout this thesis we fix a *root datum* consisting of

(i) A finite dimensional $\mathbb{Q}$-vector space $\mathfrak{h}$,
(ii) a finite index set I,
(iii) a linearly independent set $\{\alpha_i \mid i \in I\}$ of $\mathfrak{h}^*$ (the set of *simple roots*),
(iv) a subset $\{h_i \mid i \in I\}$ of $\mathfrak{h}$ (the set of *simple coroots*),
(v) a $\mathbb{Q}$-valued symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$, and
(vi) a lattice $P \subseteq \mathfrak{h}^*$ (the *weight lattice*) together with the natural pairing $\langle\cdot,\cdot\rangle\colon P^\vee \otimes P \to \mathbb{Z}$, where $P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P,\mathbb{Z})$ is the dual lattice,

such that for $i \in I$ the following properties are fulfilled:

(i) $(\langle h_i, \alpha_j\rangle)_{ij}$ is a generalized Cartan matrix,
(ii) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$,
(iii) $\langle h_i, \lambda\rangle = 2(\alpha_i,\lambda)/(\alpha_i,\alpha_i)$ for $\lambda \in P$,
(iv) $\alpha_i \in P$ and $h_i \in P^\vee$.

We assume that the root datum is *simply-connected*, that is, there exist $\varpi_i \in P$ such that $\langle h_i, \varpi_j\rangle = \delta_{ij}$ for any $i, j \in I$.

Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subseteq P$ be the *root lattice*; we set $Q_+ := \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $Q_- := -Q_+$. For $\xi = \sum_{i\in I} \xi_i\alpha_i \in Q$, define $\mathrm{tr}(\xi) = \sum_{i\in I}\xi_i$. We call a weight $\lambda \in P$ *dominant* if $\langle h_i, \lambda\rangle \geq 0$ for all $i \in I$, and denote set of dominant weights by P_+. A weight λ is *regular* if $\langle h_i, \lambda\rangle \neq 0$ for all $i \in I$. We define the *Weyl vector* $\rho := \sum_{i\in I}\varpi_i$.

The weight lattice P has a partial order given by $\lambda \succeq \mu$ if $\lambda - \mu \in Q_+$, that is, $\lambda - \mu$ is a $\mathbb{Z}_{\geq 0}$-linear combination of simple roots.

Kac-Moody Lie algebra. Let $\mathfrak{g}$ be the *symmetrizable Kac-Moody Lie algebra* associated with the Cartan matrix $(\langle h_i, \alpha_j\rangle)_{ij}$. That is to say, $\mathfrak{g}$ is the $\mathbb{Q}$-vector space generated by e_i, f_i $(i \in I)$ and $\mathfrak{h}$ with the Lie bracket

$$[h, h'] = 0, \quad [h, e_i] = \langle h, \alpha_i\rangle e_i, \quad [h, f_i] = -\langle h, \alpha_i\rangle f_i, \quad [e_i, f_j] = \delta_{ij} h_i,$$

for $i, j \in I$ and $h, h' \in \mathfrak{h}$, and the relations for $i \neq j$

$$\mathrm{ad}(e_i)^{1-\langle h_i,\alpha_j\rangle} e_j = 0, \quad \mathrm{ad}(f_i)^{1-\langle h_i,\alpha_j\rangle} f_j = 0.$$

We write $\mathfrak{g}_i$ for Kac-Moody Lie subalgebra of $\mathfrak{g}$ generated by e_i, f_i; it is isomorphic to $\mathfrak{sl}_2$.

An element $\alpha \in Q$ is called a *root* of $\mathfrak{g}$ if the *root space*

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \langle h, \alpha\rangle x \text{ for all } h \in \mathfrak{h}\}$$

is non-trivial. A root α is called *positive* (*negative*) if $\alpha \in Q_+$ ($\alpha \in Q_-$).

Weyl Group. For $i \in I$ define the *simple reflection* $s_i\colon \mathfrak{h}^* \to \mathfrak{h}^*$,

$$s_i\lambda = \lambda - \langle h_i, \lambda\rangle\alpha_i. \tag{2.2}$$

The group W generated by the s_i $(i \in I)$ is called the *Weyl group*. The Weyl group also acts on $\mathfrak{h}$ via

$$s_i h = h - \langle h, \alpha_i\rangle h_i. \tag{2.3}$$

We have $\langle h, s_i\lambda\rangle = \langle s_i h, \lambda\rangle$ for $h \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$.

A root α is called *real* if there exists $w \in W$ and $i \in I$ such that $\alpha = w\alpha_i$; otherwise it is called *imaginary*. For a real root α, we write $h_\alpha := wh_i$ for its coroot; this definition does not depend on the choice of w and i.

For a finite sequence $\underline{i} = (i_1, \dots, i_r)$ in I we write

$$s_{\underline{i}} = s_{i_1} \cdots s_{i_r}. \tag{2.4}$$

Given $w \in W$, its *length* $\ell(w)$ is the minimal integer r such that $w = s_{\underline{i}}$ for some sequence $\underline{i} = (i_1, \dots, i_r)$ in I; such a word $s_{\underline{i}} = s_{i_1} \cdots s_{i_r}$ is called a *reduced decomposition* of w. The set of all sequences $\underline{i}$ in I such that $s_{\underline{i}}$ is a reduced decomposition of w is denoted by $\mathcal{H}(w)$.

For $w, w' \in W$ we write $w > w'$ if w is greater than w' in the Bruhat order.

For a weight $\lambda \in P$, we denote its stabilizer subgroup in W by W_λ; we have

$$W_\lambda = \langle s_i \mid \langle h_i, \lambda \rangle = 0, i \in I \rangle.$$

Each coset in W/W_λ has a unique representative of minimal length; we write W^λ for the set of these representatives.

If the root datum is of finite type, then W is a finite group and has unique element w_0 of greatest length.

Quantized enveloping algebra. The *quantized enveloping algebra* $U_q(\mathfrak{g})$ is the algebra over the rational function field $\mathbb{Q}(q)$ generated by e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ subject to the following relations for $i, j \in I$ and $h, h' \in P^\vee$:

(i) $q^0 = 1$, $q^h q^h = q^{h+h'}$,
(ii) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i$, $q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$
(iii) $[e_i, f_j] = \delta_{ij}(t_i - t_i^{-1})/(q_i - q_i^{-1})$, and
(iv) $\sum_{k=0}^{a} (-1)^k e_i^{(k)} e_j e_i^{(a-k)} = \sum_{k=0}^{a} (-1)^k f_i^{(k)} f_j f_i^{(a-k)} = 0$ for $i \neq j$, $a = 1 - \langle h_i, \alpha_j \rangle$.

Here we set $q_i = q^{(\alpha_i, \alpha_i)/2}$, $t_i = q_i^{h_i}$; $e_i^{(n)} = e_i^n/[n]_i!$ and $f_i^{(n)} = f_i^n/[n]_i!$ are the *divided powers* as usual, where $[n]_i$, $[n]_i!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_i$ are obtained by replacing q with q_i.

We will also use the notations

$$\begin{bmatrix} t_i; n \\ k \end{bmatrix} := \prod_{j=1}^{k} \frac{q_i^{n+1-j} t_i - q_i^{-(n+1-j)} t_i^{-1}}{q_i^j - q_i^{-j}} \tag{2.5}$$

for $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, and

$$\begin{bmatrix} t_i \\ k \end{bmatrix} := \begin{bmatrix} t_i; 0 \\ k \end{bmatrix}.$$

For $n \in \mathbb{Z}$ we use the convention that $f_i^{(n)} = e_i^{(n)} = 0$ if $n < 0$.

For a sequence $\underline{i} = (i_1, \dots, i_r)$ in I and $\underline{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$, we set

$$f_{\underline{i}}^{(\underline{a})} := f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} \in U_q^-(\mathfrak{g}). \tag{2.6}$$

Weight grading. The algebra $U_q(\mathfrak{g})$ has a natural Q-grading; for $\xi \in Q$ the graded component is

$$U_q(\mathfrak{g})_\xi = \{u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\langle h, \xi \rangle} u \text{ for all } h \in P^\vee\}.$$

If $u \in U_q(\mathfrak{g})_\xi$, we say that ξ is the *weight* of u and denote it by $\operatorname{wt} u$.

Automorphisms. The *bar involution* is the $\mathbb{Q}$-algebra automorphism $-\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ given by

$$\overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q} = q^{-1}, \quad \overline{q^h} = q^{-h}. \tag{2.7}$$

The $*$-*involution* is the $\mathbb{Q}(q)$-algebra antiautomorphism $*\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ given by

$$*e_i = e_i, \quad *f_i = f_i, \quad *q^h = q^{-h}. \tag{2.8}$$

Hopf algebra structure. There are two common choices for the coproduct on $U_q(\mathfrak{g})$:

$$\begin{aligned} \Delta_+(q^h) &= q^h \otimes q^h, \\ \Delta_+(e_i) &= e_i \otimes 1 + t_i \otimes e_i, \\ \Delta_+(f_i) &= f_i \otimes t_i^{-1} + 1 \otimes f_i; \end{aligned}$$

and

$$\begin{aligned} \Delta_-(q^h) &= q^h \otimes q^h, \\ \Delta_-(e_i) &= e_i \otimes t_i^{-1} + 1 \otimes e_i, \\ \Delta_-(f_i) &= f_i \otimes 1 + t_i \otimes f_i, \end{aligned} \tag{2.9}$$

In this thesis, we will use Δ_- as coproduct and simply write Δ henceforth.

With respect to Δ, the counit $\epsilon\colon U_q(\mathfrak{g}) \to \mathbb{Q}(q)$ is

$$\epsilon(e_i) = 0, \quad \epsilon(f_i) = 0, \quad \epsilon(q^h) = 1, \tag{2.10}$$

and the antipode $S\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is given by

$$S(e_i) = -e_i t_i, \quad S(f_i) = -t_i^{-1} f_i, \quad S(q^h) = q^{-h}. \tag{2.11}$$

Triangular decomposition. Let $U_q^-(\mathfrak{g})$ be the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by the f_i $(i \in I)$ and $U_q^+(\mathfrak{g})$ the subalgebra generated by the e_i $(i \in I)$. Then, we have the *triangular decomposition* [38]

$$U_q(\mathfrak{g}) \cong U_q^-(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q)[P^\vee] \otimes_{\mathbb{Q}(q)} U_q^+(\mathfrak{g}).$$

Here, $\mathbb{Q}(q)[P^\vee]$ is the group algebra of $P^\vee$ over $\mathbb{Q}(q)$.

PBW-type bases. For $i \in I$ there is a $\mathbb{Q}(q)$-algebra automorphism $T_i\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ given by [38]

$$\begin{aligned} T_i(e_i) &= -f_i t_i, \\ T_i(f_i) &= -t_i^{-1} e_i, \\ T_i(f_j) &= \sum_{r+s=-\langle h_i, \alpha_j \rangle} (-1)^r q_i^r f_i^{(r)} f_j f_i^{(s)} \\ T_i(e_j) &= \sum_{r+s=-\langle h_i, \alpha_j \rangle} (-1)^r q_i^{-r} e_i^{(s)} e_j e_i^{(r)} \\ T_i(q^h) &= q^{h - \langle h, \alpha_i \rangle \alpha_i} \end{aligned} \tag{2.12}$$

Remark 2.5. T_i is the automorphism $T''_{i,-1}$ in [38].

Let $w \in W$ and fix a reduced decomposition $\underline{i} = (i_1, \dots, i_r)$ of w. Then, we have a sequence of positive roots

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad k \in [1, r],$$

and to each β_k we associate for $n \geq 0$ the element

$$f_{\beta_k}^{(n)} := T_{i_1} \cdots T_{i_{k-1}}(f_{i_k}^{(n)}) \in U_q^-(\mathfrak{g}). \tag{2.13}$$

We denote the $\mathbb{Q}(q)$-subalgebra of $U_q^-(\mathfrak{g})$ generated by $f_{\beta_1}, \dots, f_{\beta_r}$ by $U_q^-(\mathfrak{g}, w)$. As implied by the notation $U_q^-(\mathfrak{g}, w)$ only depends on w and not on the choice of $\underline{i}$ [38, Proposition 40.2.1]. The elements of the shape

$$f_{\beta_1}^{(a_1)} \cdots f_{\beta_r}^{(a_r)} \quad \text{for } a_1, \dots, a_r \in \mathbb{Z}_{\geq 0}$$

form a basis (called *PBW-type* basis) of $U_q^-(\mathfrak{g}, w)$ [38, Proposition 40.2.1].

If $\mathfrak{g}$ is of finite type, then $U_q^-(\mathfrak{g}) = U_q^-(\mathfrak{g}, w_0)$.

Integrable modules. Let M be a $U_q(\mathfrak{g})$-module. For any $\lambda \in P$, we define its *weight space*

$$M_\lambda = \{m \in M \mid q^h m = q^{\langle h, \lambda \rangle} m \text{ for all } h \in P^\vee\}.$$

If $M_\lambda \neq 0$, we say that λ is a *weight* of M.

A $U_q(\mathfrak{g})$-module M is said to be *integrable* if

(i) $M = \bigoplus_{\lambda \in P} M_\lambda$,
(ii) M_λ is finite-dimensional for any $\lambda \in P$, and
(iii) e_i and f_i act locally nilpotent on M for any $i \in I$.

In this thesis we will only consider integrable modules.

For a dominant weight $\lambda \in P_+$, let $V(\lambda)$ be the *integrable highest weight module* of highest weight λ; we denote its highest weight vector by u_λ. The elements in the orbit $W\lambda$ are called *extremal weights* of $V(\lambda)$. It is well-known [38] that $V(\lambda)$ is simple and that it is isomorphic to

$$U_q(\mathfrak{g}) \Big/ \Big(\sum\nolimits_{i \in I} U_q(\mathfrak{g}) e_i + \sum\nolimits_{h \in P^\vee} U_q(\mathfrak{g})(q^h - q^{\langle h, \lambda \rangle}) + \sum\nolimits_{i \in I} U_q(\mathfrak{g}) f_i^{\langle h_i, \lambda \rangle + 1} \Big).$$

There is a unique non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on $V(\lambda)$ with values in $\mathbb{Q}$ [22] such that

(i) $(u_\lambda, u_\lambda) = 1$,
(ii) $(q^h u, v) = (u, q^h v)$ for $u, v \in V(\lambda)$ and $h \in P^*$,
(iii) $(e_i u, v) = (u, q_i t_i^{-1} f_i v)$ for $u, v \in V(\lambda)$ and $i \in I$,
(iv) $(f_i u, v) = (u, q_i^{-1} t_i e_i v)$ for $u, v \in V(\lambda)$ and $i \in I$.

The *dual* $V(\lambda)^\vee$ of $V(\lambda)$ is, as vector space, defined as the graded dual with respect to the weight space decomposition; the $U_q(\mathfrak{g})$-module structure is defined via the vector space isomorphism $V(\lambda) \to V(\lambda)^\vee, u \mapsto (u, \cdot)$. In particular, $V(\lambda)^\vee$ is also an integrable highest weight module of highest weight λ, and $V(\lambda)^\vee \cong V(\lambda)$. Note that the structure differs from the usual dual module structure for Hopf algebras, which is defined via the antipode. Such a module would have highest weight $w_0\lambda$ if $\mathfrak{g}$ is of finite type, which is not isomorphic to $V(\lambda)$ in general; if $\mathfrak{g}$ is not of finite type, it would not be a highest weight module.

Let $w \in W$ and fix reduced decomposition $w = s_{i_1} \cdots s_{i_r}$. We define the vector

$$u_{w\lambda} := f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} u_\lambda,$$

where $a_k = \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_r} \lambda \rangle$ for $k \in [1, r]$. This definition is independent of the choice of the reduced decomposition of w due to the quantum Verma identities [38, Proposition 39.3.7].

The *Demazure module* $V_w(\lambda)$ is the $U_q^+(\mathfrak{g})$-submodule of $V(\lambda)$ generated by $u_{w\lambda}$:

$$V_w(\lambda) = U_q^+(\mathfrak{g}) u_{w\lambda}.$$

2.3. **Canonical basis.** We recall the definition of the canonical basis of $U_q^-(\mathfrak{g})$ and $V(\lambda)$ following Kashiwara [22].

$\mathbb{Z}$-forms. Let $U_{\mathbb{Z}}(\mathfrak{g})$ be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$, t_i, and $\begin{bmatrix} t_i; a \\ n \end{bmatrix}$ $(i \in I,\ a \in \mathbb{Z},\ n \geq 0)$. Let $U_{\mathbb{Z}}^-(\mathfrak{g})$ be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q^-(\mathfrak{g})$ generated by $f_i^{(n)}$.

For a dominant weight λ, we define

$$V_{\mathbb{Z}}(\lambda) := U_{\mathbb{Z}}^-(\mathfrak{g}) u_\lambda. \tag{2.14}$$

q-Boson algebra. For $i \in I$, let $e_i', e_i'' \colon U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ be the skew-derivations given by

$$e_i' f_j = q_i^{-\langle h_i, \alpha_j \rangle} f_j e_i' + \delta_{ij} \text{ and } e_i'' f_j = q_i^{\langle h_i, \alpha_j \rangle} f_j e_i'' + \delta_{ij}.$$

Here f_j acts on $U_q^-(\mathfrak{g})$ by left multiplication.

Kashiwara [22] showed that e_i' and e_i'' that are uniquely determined by the equation

$$[e_i, u] = \frac{t_i e_i'' u - t_i^{-1} e_i' u}{q_i - q_i^{-1}} \quad (u \in U_q^-(\mathfrak{g})). \tag{2.15}$$

The *q-Boson algebra* $B_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$-algebra generated by e_i', f_i $(i \in I)$ subject to the relations

(i) $e_i' f_j = q_i^{-\langle h_i, \alpha_j \rangle} f_j e_i' + \delta_{ij}$,

(ii) $\sum_{k=0}^{a} (-1)^k e_i'^{(k)} e_j' e_i'^{(a-k)} = \sum_{k=0}^{a} (-1)^k f_i^{(k)} f_j f_i^{(a-k)} = 0$ for $i \neq j$, $a = 1 - \langle h_i, \alpha_j \rangle$.

$U_q^-(\mathfrak{g})$ is a left $B_q(\mathfrak{g})$-module [22, Lemma 3.4.2].

Lemma 2.6 ([22])**.** *For $n, m \geq 0$, we have*

$$e_i'^n f_i^{(m)} = \sum_{k=0}^{n} q_i^{-2nm + k(n+m) - k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_i f_i^{(m-k)} e_i'^{n-k}$$

and for $j \neq i$

$$e_i'^n f_j^{(m)} = q_i^{-\langle h_i, \alpha_j \rangle nm} f_j^{(m)} e_i'^n.$$

The formulas were stated without proof; we provide one for completeness.

Proof. The second formula is clear; we prove the first one. For $n = 1$, induction on m gives

$$\begin{aligned} e_i' f_i^{(m)} &= [m]_i^{-1} (q_i^{-2} f_i (q_i^{-2(m-2)} f_i^{(m-1)} e_i' + q_i^{-m+2} f_i^{(m-2)}) + f_i^{(n-1)}) \\ &= q_i^{-2m} f_i^{(m)} e_i' + q_i^{-m+1} [m]_i^{-1} (q_i^{-1} [m-1]_i + q_i^{m-1}) f_i^{(m-1)} \\ &= q_i^{-2m} f_i^{(m)} e_i' + q_i^{-m+1} f_i^{(m-1)}. \end{aligned}$$

Now, induction on n gives

$$\begin{aligned}
e_i'^n f_i^{(m)} &= \sum_{k=0}^{n-1} q_i^{-2(n-1)m+k(n-1+m)-k(k-1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_i e_i' f_i^{(m-k)} e_i'^{n-1-k} \\
&= \sum_{k=0}^{n-1} q_i^{-2(m-k)} q_i^{-2(n-1)m+k(n-1+m)-k(k-1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_i f_i^{(m-k)} e_i'^{n-k} \\
&+ \sum_{k=0}^{n-1} q_i^{-m+k-1} q_i^{-2(n-1)m+k(n-1+m)-k(k-1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_i f_i^{(m-1-k)} e_i'^{n-1-k} \\
&= \sum_{k=0}^{n} q_i^{-2nm+k(n+m)-k(k-1)/2} \left(q_i^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_i + q_i^{-n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_i \right) f_i^{(m-k)} e_i'^{n-k} \\
&= \sum_{k=0}^{n} q_i^{-2nm+k(m+n)-k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_i f_i^{(m-k)} e_i'^{n-k}.
\end{aligned}$$

□

Inner product. By [22] there is a unique non-degenerate symmetric bilinear form $(\cdot,\cdot)$ on $U_q^-(\mathfrak{g})$ with values in $\mathbb{Q}$ such that

$$(1,1) = 1 \text{ and } (f_i u, v) = (u, e_i' v) \text{ for } u, v \in U_q^-(\mathfrak{g}).$$

Canonical basis of $U_q^-(\mathfrak{g})$. Kashiwara [22] showed that for any $i \in I$

$$U_q^-(\mathfrak{g}) = \bigoplus_{n \geq 0} f_i^{(n)} \ker e_i'.$$

Let $\tilde{e}_i, \tilde{f}_i \colon U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ be the operators defined by

$$\tilde{e}_i(f_i^{(n)} u) = f_i^{(n-1)} u \text{ and } \tilde{f}_i(f_i^{(n)} u) = f_i^{(n+1)} u \text{ for } u \in \ker e_i'.$$

They are commonly referred to as *Kashiwara operators*.

Let $L(\infty)$ be the A-submodule of $U_q^-(\mathfrak{g})$ generated by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot 1$ for $r \geq 0$ and $i_1, \ldots, i_r \in I$. Denote by $B(\infty) \subseteq L(\infty)/qL(\infty)$ the images of vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot 1$ and write $b_\infty \in B(\infty)$ for the image of 1 under the projection $L(\infty) \to L(\infty)/qL(\infty)$.

There is an isomorphism [22, Theorem 6]

$$G \colon L(\infty)/qL(\infty) \to U_{\mathbb{Q}}^-(\mathfrak{g}) \cap L(\infty) \cap \overline{L(\infty)}$$

of $\mathbb{Q}$-vector spaces.

The image of $B(\infty)$ under G is called the *(lower) global crystal basis* or *canonical basis* of $U_q^-(\mathfrak{g})$; we denote it by $\mathcal{B}(\infty)$. We refer to its dual basis $\mathcal{B}(\infty)^\vee$ in the graded dual $U_q^-(\mathfrak{g})^\vee$ as the *dual canonical basis*.

Theorem 2.7 ([24, Theorem 2.1.1])**.** *The crystal base $B(\infty)$ is invariant under the $*$-involution, $*B(\infty) = B(\infty)$.*

Corollary 2.8. *The canonical basis $\mathcal{B}(\infty)$ is invariant under the $*$-involution; more precisely, $*G(b) = G(*b)$ for any $b \in B(\infty)$.*

From Propositions 5.1.2 and 5.1.3 in [22] and the definition of the canonical basis, we obtain the following criterion.

Theorem 2.9. *Let $u \in U_q^-(\mathfrak{g})$. If $u \in U_{\mathbb{Z}}^-(\mathfrak{g})$, $\bar{u} = u$ and $(u,u) \in 1 + qA$, then either u or $-u$ is in $\mathcal{B}(\infty)$.*

Theorem 2.10 ([22])**.** *Let $b \in B(\infty)$ and $i \in I$. We have*

$$e_i G(b) = [\varphi_i(b) + 1]_i G(\tilde{e}_i b) + \sum E^i_{b,b'} G(b'), \tag{2.16}$$

where $E^i_{b,b'} \in \mathbb{Z}[q, q^{-1}]$ is zero unless $\varphi_i(b') > \varphi_i(b) + 1$; and

$$f_i G(b) = [\epsilon_i(b) + 1]_i G(\tilde{f}_i b) + \sum F^i_{b,b'} G(b') \tag{2.17}$$

where $F^i_{b,b'} \in \mathbb{Z}[q, q^{-1}]$ is zero unless $\epsilon_i(b') > \epsilon_i(b) + 1$.

Theorem 2.11 ([21, Proposition 5.3.1])**.** *Let $b \in B(\infty)$ and $i \in I$. We have*

$$e_i G(b)^\vee = [\epsilon_i(b)]_i G(\tilde{e}_i b)^\vee + \sum_{b' \in B(\infty)} E^i_{b,b'} G(b')^\vee,$$

where $E^i_{b,b'} \in \mathbb{Z}[q, q^{-1}]$ is zero unless $\varphi_j(b') \le \varphi_j(b)$ for all $j \in I$; and

$$f_i G(b)^\vee = [\varphi_i(b)]_i G(\tilde{f}_i b)^\vee + \sum_{b' \in B(\infty)} F^i_{b,b'} G(b')^\vee,$$

where $F^i_{b,b'} \in \mathbb{Z}[q, q^{-1}]$ is zero unless $\epsilon_j(b') \le \epsilon_j(b)$ for all $j \in I$.

We present examples of the canonical basis where an explicit description is known.

Example 2.12. The canonical basis of $U_q^-(\mathfrak{g})$ in type A_1 is $\mathcal{B}(\infty) = \{f^{(n)} \mid n \ge 0\}$.

Example 2.13 ([37])**.** The canonical basis of $U_q^-(\mathfrak{g})$ in type A_2 is

$$\mathcal{B}(\infty) = \{f_1^{(a)} f_2^{(b)} f_1^{(c)} \mid b \ge a + c\} \cup \{f_2^{(a)} f_1^{(b)} f_2^{(c)} \mid b \ge a + c\}.$$

Example 2.14 ([53, Theorem 2.2])**.** The canonical basis of $U_q^-(\mathfrak{g})$ in type A_3 consists of the following elements:

(i) if $a \ge d \ge f$ and $b \ge e$:
$$f_1^{(c)} f_2^{(b+c)} f_3^{(a+b+c)} f_1^{(e)} f_2^{(d+e)} f_1^{(f)},$$

(ii) if $a \ge d \ge f$ and $d + e \ge a + b$:
$$f_2^{(b)} f_3^{(a+b)} f_1^{(c+e)} f_2^{(c+d+e)} f_3^{(c)} f_1^{(f)},$$

(iii) if $d \ge f$, $e \ge b$ and $a + b \ge d + e$:
$$f_2^{(c)} f_1^{(c+e)} f_2^{(c)} f_3^{(a+b+c)} f_2^{(d+e)} f_1^{(f)},$$

(iv) if $f \ge d \ge a$ and $e \ge b$:
$$f_3^{(a)} f_2^{(b+d)} f_3^{(b)} f_1^{(c+e+f)} f_2^{(c+e)} f_3^{(c)},$$

(v) if $f \ge d \ge a$ and $b + d \ge e + f$:
$$f_1^{(c)} f_3^{(a)} f_2^{(b+c+d)} f_3^{(b+c)} f_1^{(e+f)} f_2^{(e)},$$

(vi) if $d \ge a$, $b \ge e$ and $e + f \ge b + d$:
$$f_3^{(a)} f_2^{(b+d)} f_1^{(c+e+f)} f_2^{(c)} f_3^{(b+c)} f_2^{(e)},$$

(vii) if $a \ge d$, $f \ge d$ and $d + e \ge a + b$:
$$f_2^{(b)} f_3^{(a+b)} f_2^{(d)} f_1^{(c+e+f)} f_2^{(c+e)} f_3^{(c)},$$

(viii) if $a \geq d$, $f \geq d$ and $b + d \geq e + f$:

$$f_1^{(c)} f_2^{(b+c)} f_3^{(a+b+c)} f_2^{(d)} f_1^{(e+f)} f_2^{(e)},$$

(ix) if $f \geq d$, $e \geq b$ and $a + b \geq d + e$:

$$\sum_{0 \leq k \leq c} \begin{bmatrix} d+e-a-b \\ k \end{bmatrix} f_2^{(b)} f_3^{(a+b+k)} f_2^{(d)} f_1^{(c+e+f)} f_2^{(c+e)} f_3^{(c-k)},$$

(x) if $a \geq d$, $b \geq e$ and $e + f \geq b + d$:

$$\sum_{0 \leq k \leq c} \begin{bmatrix} b+d-e-f \\ k \end{bmatrix} f_1^{(c-k)} f_2^{(b+c)} f_3^{(a+b+c)} f_2^{(d)} f_1^{(e+f+k)} f_2^{(e)},$$

(xi) if $d \geq f \geq a$, $b \geq e$:

$$\sum_{0 \leq k \leq e} \begin{bmatrix} f-d \\ k \end{bmatrix} f_1^{(c)} f_3^{(a)} f_2^{(b+c+d+k)} f_3^{(b+c)} f_1^{(e+f)} f_2^{(e-k)},$$

(xii) if $f \geq d$, $e \geq b$ and $a + b \geq d + e$:

$$\sum_{0 \leq k \leq c+e} \begin{bmatrix} f-d \\ k \end{bmatrix} f_3^{(a)} f_2^{(b+d+k)} f_3^{(b)} f_1^{(c+e+f)} f_2^{(c+e-k)} f_3^{(c)},$$

(xiii) if $d \geq a \geq f$ and $e \geq b$:

$$\sum_{0 \leq k \leq b} \begin{bmatrix} a-d \\ k \end{bmatrix} f_2^{(b-k)} f_3^{(a+b)} f_1^{(c+e)} f_2^{(c+d+e+k)} f_3^{(c)} f_1^{(f)},$$

(xiv) if $d \geq a \geq f$ and $b \geq e$:

$$\sum_{0 \leq k \leq b+c} \begin{bmatrix} a-d \\ k \end{bmatrix} f_1^{(c)} f_2^{(b+c-k)} f_3^{(a+b+c)} f_1^{(e)} f_2^{(d+e+k)} f_1^{(f)}.$$

Example 2.15 ([54, Theorem 2.2])**.** The canonical basis of $U_q^-(\mathfrak{g})$ in type B_2 consists of the following elements:

(i) if $c \geq a$ and $b \geq d$:

$$f_2^{(a)} f_1^{(b+c)} f_2^{(2b+c)} f_1^{(d)},$$

(ii) if $c + 2d \geq a + 2b$ and $a + b \geq c + d$:

$$f_1^{(b)} f_2^{(a+2b)} f_1^{(c+d)} f_2^{(c)},$$

(iii) if $c \leq a$ and $b \geq d$:

$$\sum_{0 \leq k \leq 2b+c} \begin{bmatrix} c-a \\ k \end{bmatrix}_2 f_2^{(a+k)} f_1^{(b+c)} f_2^{(2b+c-k)} f_1^{(d)},$$

(iv) if $c \geq a$ and $b \leq d$:

$$\sum_{0 \leq k \leq b+c} \begin{bmatrix} b-d \\ k \end{bmatrix}_1 f_2^{(a)} f_1^{(b+c-k)} f_2^{(2b+c)} f_1^{(d+k)},$$

(v) if $c \leq a$, $b \leq d$ and $c + d \geq a + b$:

$$\sum_{\substack{0 \leq k \leq 2b+c \\ 0 \leq l \leq b+c}} \begin{bmatrix} c-a \\ k \end{bmatrix}_2 \begin{bmatrix} b-d-k \\ l \end{bmatrix}_1 f_2^{(a+k)} f_1^{(b+c-l)} f_2^{(2b+c-k)} f_1^{(d+l)},$$

(vi) if $c \le a$, $b \le d$ and $c + 2d \le a + 2b$:

$$\sum_{\substack{0\le k\le 2b+c\\ 0\le l\le b+c}} \begin{bmatrix} c-a-2l \\ k \end{bmatrix}_2 \begin{bmatrix} b-d \\ l \end{bmatrix}_1 f_2^{(a+k)} f_1^{(b+c-l)} f_2^{(2b+c-k)} f_1^{(d+l)}.$$

Lusztig datum. If $\mathfrak{g}$ is of finite type, then there is a unitriangular base change from the canonical basis to any PBW-type basis [37]. Fix a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_N}$ and put $\underline{i} := (i_1, \dots, i_N)$. Then, any $G(b)$ for $b \in B(\infty)$ can be uniquely written as

$$G(b) = f_{\beta_1}^{(a_1)} \cdots f_{\beta_N}^{(a_N)} + \sum z_{\underline{a},\underline{a}'} f_{\beta_1}^{(a_1')} \cdots f_{\beta_N}^{(a_N')},$$

where $\underline{a} = (a_1, \dots, a_N)$ and $\underline{a}' = (a_1', \dots, a_N')$ are in $\mathbb{Z}_{\ge 0}^N$ and $z_{\underline{a},\underline{a}'} \in q\mathbb{Z}[q]$ is zero unless $\underline{a} < \underline{a}'$ in the lexicographic order.

The tuple $\underline{a}$ is called the *Lusztig datum* or *PBW datum* of b with respect to $\underline{i}$.

Canonical basis of $V(\lambda)$. Let λ be a dominant weight. From the theory of $U_q(\mathfrak{sl}_2)$ it is known that for any $i \in I$ there is a direct sum decomposition

$$V(\lambda) = \bigoplus_{n\ge 0} f_i^{(n)} \ker e_i.$$

Let $\tilde{e}_i, \tilde{f}_i \colon V(\lambda) \to V(\lambda)$ be the operators defined by

$$\tilde{e}_i(f_i^{(n)} u) = f_i^{(n-1)} u \text{ and } \tilde{f}_i(f_i^{(n)} u) = f_i^{(n+1)} u \text{ for } u \in \ker e_i.$$

Let $- \colon V(\lambda) \to V(\lambda)$, be the involution given by $\overline{Fu_\lambda} = \bar{F} u_\lambda$.

Let $L(\lambda)$ be the A-submodule of $V(\lambda)$ generated by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot 1$ for $r \ge 0$ and $i_1, \dots, i_r \in I$. Denote by $B(\lambda) \subseteq L(\lambda)/qL(\lambda)$ the images of vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot 1$ and write $b_\lambda \in B(\lambda)$ for the image of u_λ under the projection $L(\lambda) \to L(\lambda)/qL(\lambda)$.

There is an isomorphism [22, Theorem 6]

$$G_\lambda \colon L(\lambda)/qL(\lambda) \to V_{\mathbb{Q}}(\lambda) \cap L(\lambda) \cap \overline{L(\lambda)}$$

of $\mathbb{Q}$-vector spaces.

The image of $B(\lambda)$ under G_λ is called the *global crystal basis* or *canonical basis* of $V(\lambda)$; we denote it by $\mathcal{B}(\lambda)$. We refer to its dual basis $\mathcal{B}(\lambda)^\vee$ in the graded dual $V(\lambda)^\vee$ as the *dual canonical basis* of $V(\lambda)$.

The $U_q^-(\mathfrak{g})$-linear homomorphism $U_q^-(\mathfrak{g}) \to V(\lambda)$ sending 1 to u_λ induces a surjective homomorphism [22, Theorem 5]

$$\mathrm{pr}_\lambda \colon L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda). \tag{2.18}$$

As a consequence of Theorems 5 and 6 of [22] we have the following theorem.

Theorem 2.16. *Let λ be a dominant weight. Then, for any $b \in B(\infty)$,*

$$G(b)u_\lambda = G_\lambda(\mathrm{pr}_\lambda(b)).$$

In particular,

$$\mathcal{B}(\lambda) = \{G(b)u_\lambda \mid b \in B(\infty), \mathrm{pr}_\lambda(b) \ne 0\}.$$

From Propositions 5.1.1 and 5.1.3 in [22] and the definition of the canonical basis of $V(\lambda)$, we obtain the following criterion.

Theorem 2.17. *Let $u \in V(\lambda)$. If $u \in V_{\mathbb{Z}}(\lambda)$, $\bar{u} = u$ and $(u, u) \in 1 + qA$, then either u or $-u$ is in $\mathcal{B}(\infty)$.*

For $w \in W$ we define the set

$$B_w(\lambda) := \{\tilde{f}_{i_1}^{a_1} \dots \tilde{f}_{i_r}^{a_r} b_\lambda \mid s_{i_1} \cdots s_{i_r} \text{ is a reduced decomposition of } w\}. \tag{2.19}$$

2.4. **Crystals.** The definition of a crystal was first given by Kashiwara [24] as abstraction of the crystal bases $B(\lambda)$ and $B(\infty)$. We recall the definition given in *loc.cit.*.

A *crystal* is a set B together with maps

$$\tilde{e}_i \colon B \to B \sqcup \{0\}, \quad \tilde{f}_i \colon B \to B \sqcup \{0\}$$
$$\epsilon_i \colon B \to \mathbb{Z} \cup \{-\infty\}, \varphi_i \colon B \to \mathbb{Z} \cup \{-\infty\}, \mathrm{wt} \colon B \to P$$

such that for any $i \in I$ and $b \in B$:

(i) $\varphi_i(b) = \epsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$;
(ii) if $\tilde{e}_i b \in B$, then $\tilde{f}_i \tilde{e}_i b = b$, $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\epsilon_i(\tilde{e}_i b) = \epsilon_i(b) - 1$, and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$;
(iii) if $\tilde{f}_i b \in B$, then $\tilde{e}_i \tilde{f}_i b = b$, $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\epsilon_i(\tilde{f}_i b) = \epsilon_i(b) + 1$, and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$;
(iv) if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

Here 0 is a ghost element, which is not contained in B; the addition on $\mathbb{Z} \cup \{\infty\}$ is extended from $\mathbb{Z}$ by

$$-\infty + x = -\infty \text{ for any } x \in \mathbb{Z} \cup \{-\infty\}.$$

A crystal B is called *upper normal* if $\epsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n b \neq 0\}$ for any $b \in B$ and $i \in I$; similarly, it is called *lower normal* if $\varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n b \neq 0\}$ for any $b \in B$ and $i \in I$. It is called *normal* if it is both upper and lower normal. Note that if B is upper normal (lower normal), then for any $b \in B$ and $i \in I$ there exists $n \in \mathbb{N}$ such that $\tilde{e}_i^n b = 0$ ($\tilde{f}_i^n b = 0$).

A *morphism* between two crystals B_1 and B_2 is a map $\psi \colon B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$, such that for $b \in B_1$:

(i) $\psi(0) = 0$;
(ii) if $\psi(b) \in B_2$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\epsilon_i(\psi(b)) = \epsilon_i(b)$, $\varphi_i(\psi_i(b)) = \varphi_i(b)$;
(iii) if $\psi(b), \psi(\tilde{e}_i b) \in B_2$, then $\tilde{e}_i \psi(b) = \psi(\tilde{e}_i b)$;
(iv) if $\psi(b), \psi(\tilde{f}_i b) \in B_2$, then $\tilde{f}_i \psi(b) = \psi(\tilde{f}_i b)$.

A morphism is called *strict* if it commutes with all $\tilde{f}_i$ and $\tilde{e}_i$.

For two crystals B_1 and B_2 their tensor product $B_1 \otimes B_2$ is defined as follows:

(i) $B_1 \otimes B_2 := B_1 \times B_2$ as a set and elements $(b_1, b_2) \in B_1 \otimes B_2$ are written as $b_1 \otimes b_2$;
(ii) $\epsilon_i(b_1 \otimes b_2) = \max(\epsilon_i(b_1), \epsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle)$;
(iii) $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2)\rangle)$;
(iv) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$.

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are given by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \epsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{else;} \end{cases}$$

and

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \epsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{else;} \end{cases}$$

with the convention $b_1 \otimes 0 = 0 \otimes b_2 = 0$.

Canonical bases of tensor products. Let λ and μ be dominant weights. We want to define the canonical basis of the tensor product $V(\lambda) \otimes V(\mu)$. For that purpose, we need to introduce a suitable bar involution on $V(\lambda) \otimes V(\mu)$. Since the action of $U_q(\mathfrak{g})$ on $V(\lambda) \otimes V(\mu)$ is given via the coproduct Δ, we first need to extend the bar involution on $U_q(\mathfrak{g})$ to $\Delta(U_q(\mathfrak{g}))$.

There are two natural ways to do this. Let $u \in U_q(\mathfrak{g})$; the first choice is $\overline{\Delta(u)} := \sum \overline{u_{(1)}} \otimes \overline{u_{(2)}}$ and the second choice is $\Delta(\bar{u})$. Both involutions are linked via the *quasi-$\mathcal{R}$-matrix* Θ, more precisely, $\Delta(\bar{u})\Theta = \Theta\overline{\Delta(u)}$.

To define the quasi-$\mathcal{R}$-matrix we need to consider the completion

$$U_q^+(\mathfrak{g}) \mathbin{\widehat{\otimes}} U_q^-(\mathfrak{g}) := \bigoplus_{\xi \in Q} \prod_{\xi = \xi' + \xi''} U_q^+(\mathfrak{g})_{\xi'} \otimes U_q^-(\mathfrak{g})_{\xi''}.$$

Theorem 2.18 ([38, Theorem 4.1.2])**.** *The quasi-$\mathcal{R}$-matrix Θ is the unique element in $U_q^+(\mathfrak{g}) \mathbin{\widehat{\otimes}} U_q^-(\mathfrak{g})$ such that $\epsilon(\Theta) = 1$ and $\Delta(\bar{u})\Theta = \Theta\overline{\Delta(u)}$.*

Let B be a basis of $U_q^-(\mathfrak{g})$ such that $B_\xi := B \cap U_q^-(\mathfrak{g})_\xi$ is a basis of $U_q^-(\mathfrak{g})_\xi$ for any $\xi \in Q_-$. Let $B^\vee = \{b^\vee \mid b \in B\}$ be the dual basis of B with respect to $(\cdot, \cdot)$. Then, $\Theta = \sum_{\xi \in Q_-} \Theta_\xi$, where

$$\Theta_\xi = \prod_{i \in I} (q_i - q_i^{-1})^{\xi_i} \sum_{b \in B_\xi} (b^\vee)^+ \otimes b \in U_q^+(\mathfrak{g})_{-\xi} \otimes U_q^-(\mathfrak{g})_\xi,$$

for $\xi = -\sum_{i \in I} \xi_i \alpha_i$ and $+ \colon U_q^-(\mathfrak{g}) \to U_q^+(\mathfrak{g})$ is $\mathbb{Q}(q)$-algebra isomorphism defined by $f_i \mapsto e_i$ for $i \in I$.

Remark 2.19. The element Θ_ξ looks different from the one in [38] due to the different choice of coproduct and bilinear form.

Corollary 2.20 ([38, Corollary 4.1.3])**.** *We have $\Theta\bar{\Theta} = \bar{\Theta}\Theta = 1$.*

Example 2.21. If $\mathfrak{g}$ is of type A_2, the quasi-$\mathcal{R}$-matrix is

$$\Theta = \sum_{n \geq 0} q^{n(n-1)/2} \gamma_n e^{(n)} \otimes f^{(n)},$$

where $\gamma_n = \prod_{k=1}^n (q^k - q^{-k})$.

Example 2.22. Assume that $\mathfrak{g}$ is of finite type and fix a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_N}$. For $k \in [1, N]$ put

$$\Theta^{[k]} = \sum_{n \geq 0} q^{n(n-1)/2} \gamma_n T_{i_1} \cdots T_{i_{k-1}}(e_{i_k}^{(n)}) \otimes T_{i_1} \cdots T_{i_{k-1}}(f_{i_k}^{(n)}),$$

where $\gamma_n = \prod_{k=1}^n (q^k - q^{-k})$. Then, by [19, Section 8.30], $\Theta = \Theta^{[1]} \cdots \Theta^{[N]}$.

Let λ and μ be dominant weights. We define the involution $\Psi \colon V(\lambda) \otimes V(\mu) \to V(\lambda) \otimes V(\mu)$ by

$$\Psi(u \otimes v) = \Theta(\bar{u} \otimes \bar{v}). \tag{2.20}$$

Let λ and μ be dominant weights. We define a preorder $\blacktriangleright$ on $V(\lambda) \otimes V(\mu)$ by $u \otimes v \blacktriangleright u' \otimes v'$ if and only if

$$\begin{gathered} \operatorname{wt} u' \succ \operatorname{wt} u \text{ and } u' \in U_q^+(\mathfrak{g})u, \text{ or} \\ u' \in \mathbb{Q}(q)u, \operatorname{wt} v' \succ \operatorname{wt} v \text{ and } v' \in U_q^+(\mathfrak{g})v. \end{gathered} \tag{2.21}$$

Theorem 2.23 ([38, Theorem 27.3.2])**.** *Let $b_1 \in B(\lambda)$ and $b_2 \in B(\mu)$. We have*

$$G(b_1 \otimes b_2) = G_\lambda(b_1) \otimes G_\mu(b_2) + \sum_{b_1' \otimes b_2' \in B(\lambda) \otimes B(\mu)} z_{b_1,b_2;b_1',b_2'} G_\lambda(b_1') \otimes G_\mu(b_2'),$$

where $z_{b_1,b_2;b_1',b_2'} \in q\mathbb{Z}[q]$ is zero unless $G_\lambda(b_1') \otimes G_\mu(b_2') \blacktriangleleft G_\lambda(b_1) \otimes G_\mu(b_2)$.

Let $\phi\colon V(\lambda+\mu) \to V(\lambda) \otimes V(\mu)$ be the unique $U_q(\mathfrak{g})$-module homomorphism sending $u_{\lambda+\mu}$ to $u_\lambda \otimes u_\mu$. The following lemma relates the involution on $V(\mu+\lambda)$ to the involution defined on the image of ϕ.

Lemma 2.24. *For $u \in V(\lambda) \otimes V(\mu)$ we have $\phi(\bar{u}) = \Psi(\phi(u))$.*

Proof. Let $F \in U_q^-(\mathfrak{g})$, then

$$\bar{F} \cdot (u_\lambda \otimes u_\mu) = \Delta(\bar{F})\Theta(u_\lambda \otimes u_\mu) = \Theta\overline{\Delta(F)}(u_\lambda \otimes u_\mu) = \Psi(F \cdot (u_\lambda \otimes u_\mu)).$$

□

The most important consequence of the above lemma is that the image of the canonical basis of $V(\lambda+\mu)$ under ϕ is the canonical basis of the Cartan component of $V(\lambda) \otimes V(\mu)$. Thus, we can discuss the canonical basis of $V(\lambda+\mu)$ using the explicit description of the canonical basis of $V(\lambda) \otimes V(\mu)$.

2.5. **The path model.** We recall the definition of the Littelmann path model following [36].

A *rational piecewise linear path* is a piecewise linear, continuous map $\pi\colon [0,1] \to P_\mathbb{R} := P \otimes_\mathbb{Z} \mathbb{R}$ such that all turning points are rational. We identify the paths π and η if there exists a piecewise linear, non-decreasing, surjective map $\phi\colon [0,1] \to [0,1]$ such that $\pi = \phi \circ \eta$. Denote by Π the set of all rational piecewise linear paths π with $\pi(0) = 0$ and $\pi(1) \in P$. We say a path of Π is *dominant* if its image is contained in the dominant Weyl chamber. We write π_λ for the straight-line path $t \mapsto t\lambda$.

For a path $\pi \in \Pi$ and $i \in I$ we write $H_i\colon [0,1] \to \mathbb{R}, t \mapsto \langle h_i, \pi(t)\rangle$. We say that $H_i(t)$ *attains a local minimum at* $t = t_0$ if there exists an $\epsilon > 0$ such that $H_i(t) \geq H_i(t_0)$ for $|t - t_0| < \epsilon$ and $H_i(t) > H_i(t_0)$ for $t_0 - \epsilon < t < t_0$ or $t_0 < t < t_0 + \epsilon$.

For $i \in I$ let m_i be the global minimum attained by $H_i(t)$ and let $l_i, r_i\colon [0,1] \to [0,1]$ be defined as

$$l_i(t) := \min\{1, H_i(s) - m_i \mid t \leq s \leq 1\}, \text{ and}$$
$$r_i(t) := \max\{0, 1 + m_i - H_i(s) \mid 0 \leq s \leq t\}.$$

The *root operators* $\tilde{e}_i, \tilde{f}_i\colon \Pi \to \Pi \cup \{0\}$ are given by

$$(\tilde{e}_i\pi)(t) = \begin{cases} \pi(t) + r_i(t)\alpha_i, & \text{if } r_i(0) = 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$(\tilde{f}_i\pi)(t) = \begin{cases} \pi(t) - l_i(t)\alpha_i, & \text{if } l_i(1) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We define

$$\operatorname{wt}\pi := \pi(1), \tag{2.22}$$

$$\epsilon_i(\pi) := -\min\{H_i(t) \mid t \in [0,1]\} \cap \mathbb{Z}, \tag{2.23}$$

$$\varphi_i(\pi) := \epsilon_i(\pi) + \langle h_i, \operatorname{wt}\pi\rangle. \tag{2.24}$$

For a dominant path $\eta \in \Pi$ ending in $\lambda \in P_+$ the set

$$B(\eta) := \{\tilde{f}_{i_1} \dots \tilde{f}_{i_r}\eta \mid r \geq 0, i_k \in I \text{ for } k \in [1,r]\} \setminus \{0\} \tag{2.25}$$

is a crystal [32]. It was shown by Kashiwara [23] that $B(\eta)$ is isomorphic to $B(\lambda)$. For $\pi \in B(\eta)$ we denote the corresponding element in $B(\lambda)$ by b_π.

The action of the simple reflections s_i on $B(\eta)$ given by

$$s_i\pi = \begin{cases} \tilde{f}_i^{\langle h_i, \operatorname{wt}\pi\rangle}\pi, & \text{if } \langle h_i, \operatorname{wt}\pi\rangle \geq 0, \\ \tilde{e}_i^{-\langle h_i, \operatorname{wt}\pi\rangle}\pi, & \text{if } \langle h_i, \operatorname{wt}\pi\rangle < 0, \end{cases}$$

extends to an action of the Weyl group W on $B(\eta)$ [36].

Lakshmibai-Seshadri paths. Let λ be a dominant weight. A *Lakshmibai-Seshadri path (LS path) of shape* λ is a pair $\pi := (\tau_1, \dots \tau_r; t_1, \dots t_r)$ consisting of a sequence of Weyl group elements $\tau_1 > \dots > \tau_r$ in W^λ and sequence of rational numbers $t_0 := 0 < t_1 < \dots < t_r = 1$, fulfilling the following condition: for each $k \in [1, r-1]$ there exists a maximal chain

$$\tau_k = \sigma_1 \xleftarrow{\beta_1} \sigma_2 \xleftarrow{\beta_2} \dots \xleftarrow{\beta_m} \sigma_{m+1} = \tau_{k+1}$$

such that

$$t_k\langle h_{\beta_j}, \sigma_j\lambda\rangle \in \mathbb{Z} \text{ for all } j \in [1,m].$$

For notational convenience, we allow $\tau_k = \tau_{k+1}$ which is understood as

$$(\tau_1, \dots, \tau_k, \tau_{k+2}, \dots, \tau_r; t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_r).$$

The path π is identified with the piecewise linear path $\pi(t)$ defined by

$$\pi(t) = (t - t_{n-1})\tau_n\lambda + \sum_{k=1}^{n-1}(t_k - t_{k-1})\tau_k\lambda,$$

for $t \in [t_{n-1}, t_n]$ and $n \in [1,r]$.

The set $\{\tau_1, \dots, \tau_r\}$ is called the *support* of π; we denote it by $\operatorname{supp}\pi$.

The initial direction of π is defined as $\iota(\pi) := \tau_1$. Let η be a dominant path ending in $\lambda \in P_+$ such that η is a concatenation of LS paths. If $\pi' \in B(\eta)$ and $\pi \cong \pi'$ via the isomorphism $B(\pi_\lambda) \cong B(\eta)$, we put $\iota(\pi') := \iota(\pi)$.

In this thesis we will only consider LS paths and concatenations thereof.

LS path order. Let λ be a dominant weight. Littelmann [35] introduced a partial order $\triangleright$ on the crystal $B(\pi_\lambda)$. For two LS paths $\pi = (\tau_1, \dots, \tau_r; t_1, \dots, t_r)$ and $\pi' = (\sigma_1, \dots, \sigma_s; d_1, \dots, d_s)$ of shape λ, we write $\pi \triangleright \pi'$, if $\tau_1 > \sigma_1$, or $\tau_1 = \sigma_1$ and $t_1 > d_1$, or $\tau_1 = \sigma_1$, $t_1 = d_1$ and $\tau_2 > \sigma_2$, etc.

Example 2.25. Assume that $\mathfrak{g}$ is of type A_2 and consider the module $V(2\rho)$. We have the paths $\pi_1 = (s_1s_2s_1, \mathrm{id}; \frac{1}{2}, 1)$, $\pi_2 = (s_2s_1, s_1; \frac{1}{2}, 1)$, $\pi_3 = (s_2s_1; 1)$, $\pi_4 = (s_1s_2; 1)$. We have $\pi_1 \triangleright \pi_2, \pi_3, \pi_4$; $\pi_3 \triangleright \pi_2$; and the path π_4 is incomparable to π_2 and π_3.

Let η be a dominant path ending in λ. By assumption η is a concatenation of LS paths and thus $\triangleright$ extends to a partial order on $B(\eta)$.

Using the isomorphism $B(\lambda) \cong B(\eta)$ we can transfer the partial order from the path model $B(\eta)$ to the crystal base $B(\lambda)$, which we denote by $\triangleright_\eta$ to make the dependence on the dominant path clear.

Adapted strings. Let $w \in W$ and let $\underline{i} = (i_1, \ldots, i_r)$ be a sequence in I such that $s_{\underline{i}}$ a reduced decomposition of w. A vector $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r_{\geq 0}$ is called an *adapted string* of $b \in B(\lambda)$ if

$$b = \tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\lambda$$

and for all $k \in [1, r]$

$$\tilde{e}_{i_k} \tilde{f}_{i_{k+1}}^{a_{k+1}} \cdots \tilde{f}_{i_r}^{a_r} b_\lambda = 0.$$

We denote by $S_{\underline{i}}(\lambda)$ the set of all $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r_{\geq 0}$ such that $\underline{a}$ is an adapted string of some $b \in B(\lambda)$; and we put $S_{\underline{i}}(\infty) = \bigcup S_{\underline{i}}(\lambda)$, where the union ranges over all dominant weights λ.

Let $C_{\underline{i}}(\infty)$ be the real cone spanned by $S_{\underline{i}}(\infty)$ and let $C_{\underline{i}}(\lambda)$ be the real polytope defined by the inequalities

$$a_p \leq \langle h_{i_p}, \lambda \rangle - \sum_{p < k \leq r} \langle h_{i_p}, \alpha_{i_k} \rangle a_k \quad \text{for } p \in [1, r]. \tag{2.26}$$

Proposition 2.26 ([34, Proposition 1.5])**.** *The following holds:*

(i) $C_{\underline{i}}(\infty)$ is a rational cone and $S_{\underline{i}}(\infty)$ is the set of its integral points.

(ii) $C_{\underline{i}}(\lambda)$ is a rational polytope and $S_{\underline{i}}(\lambda)$ is the set of its integral points.

Let $>$ denote the lexicographic order on $S_{\underline{i}}(\infty)$ and $S_{\underline{i}}(\lambda)$.

Proposition 2.27 ([34, Proposition 10.3])**.** *Let $w \in W$ and let $\underline{i} = (i_1, \ldots, i_r)$ be a sequence in I such that $s_{\underline{i}}$ a reduced decomposition of w. Let $b \in B_w(\infty)$ and let $\underline{a} = (a_1, \ldots, a_r)$ be its adapted string. Then,*

$$f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} = G(b) + \sum z_{b,b'} G(b'),$$

where $z_{b,b'} \in qA$ is zero unless $\underline{a}' > \underline{a}$ for the adapted string $\underline{a}'$ of b'.

In particular, the set

$$\{f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} \mid (a_1, \ldots, a_r) \in S_{\underline{i}}(\infty)\}$$

is a basis of $U_q^-(\mathfrak{g}, w)$.

3. Canonical basis and path bases

Let λ be a dominant weight and η a dominant path ending in λ (remember that η is a concatenation of LS paths). Littlemann [35] associated to each path $\pi \in B(\eta)$ a monomial $F(\pi) \in U_{\mathbb{Z}}^-(\mathfrak{g})$ and showed that the set $\{F(\pi)u_\lambda \mid \pi \in B(\eta)\}$ (the *path monomial basis*) is a basis of $V_{\mathbb{Z}}(\lambda)$. In this section we study the relation between this basis and the canonical basis of $V(\lambda)$ from the perspective of Conjecture 2.

For this purpose, we introduce the *modified root operators* $\tilde{e}_{i;n}$ and $\tilde{f}_{i;n}$ which act (roughly speaking) on the n-th local minimum of the function $\langle h_i, \pi(t)\rangle$.

After recalling the definition of $F(\pi)$ we investigate the relation between the path monomial basis and the canonical basis for fundamental weights. It turns out that for fundamental weights of classical type both bases coincide. This is also the case for both fundamental weights of type G_2. Using the compatibility of the canonical basis with tensor products, we use this result to prove Conjecture 2 in A_n, B_n, C_n, D_n and G_2 for the dominant path $\eta = \pi_{m_1\varpi_1} * \cdots * \pi_{m_n\varpi_N}$. As an immediate consequence we obtain a unitriangular base change between the path monomial basis and the canonical basis with respect to the partial order $\triangleright$ on $B(\eta)$.

Finally, we discuss how these results for simple hightest weight modules can be extended to $U_q^-(\mathfrak{g})$.

3.1. Modified root operators.

The root operators $\tilde{e}_i$ and $\tilde{f}_i$ operate on the first and last global minimum of $H_i(t)$, respectively. However, the following proposition guarantees that under certain conditions we may act on an arbitrary local minimum of $H_i(t)$ and still obtain an LS path. This motivates the definition of an analogue to the root operators which act on a local minimum of $H_i(t)$.

Proposition 3.1 ([36, Proposition 4.7])**.** *Let $\pi = (\tau_1, \ldots, \tau_r; t_1, \ldots, t_r)$ be an LS path of shape λ. Suppose that $H_i(t)$ attains a local minimum at $t = t_k$.*

(i) Suppose there exists $x < t_k$ such that $H_i(x) = H_i(t_k) + 1$ and $H_i(t) \geq H_i(t_k)$ for $t \in [x, t_k]$. Then there exist $x \leq y < t_{k'} \leq t_k$, such that

$$H_i(x) = H_i(y) > H_i(t) > H_i(t_{k'}) = H_i(t_k)$$

for all $y < t < t_{k'}$ and the function is strictly decreasing on $[y, t_{k'}]$. In addition,

$$(\tau_1, \ldots, \tau_l, s_i\tau_l, \ldots, s_i\tau_{k'}, \tau_{k'+1}, \ldots, \tau_r; t_1, \ldots, t_l, y, t_{l+1}, \ldots, t_r)$$

is an LS path of shape λ. Here, l is uniquely determined by the condition that $y \in [t_l, t_{l+1}]$.

(ii) Suppose there exists $y > t_k$ such that $H_i(y) = H_i(t_k) + 1$ and $H_i(t) \geq H_i(t_k)$ for $t \in [t_k, y]$. Then there exist $t_k \leq t_{k'} < x \leq y$, such that

$$H_i(t_k) = H_i(t_{k'}) < H_i(t) < H_i(x) = H_i(y)$$

for all $t_{k'} < t < x$ and the function is strictly increasing on $[t_{k'}, x]$. In addition,

$$(\tau_1, \ldots, \tau_{k'}, s_i\tau_{k'+1}, \ldots, s_i\tau_l, \tau_l, \ldots, \tau_r; t_1, \ldots, t_{l-1}, x, t_l, \ldots, t_r)$$

is an LS path of shape λ. Here, l is uniquely determined by the condition that $x \in [t_{l-1}, t_l]$.

For $i \in I$ and $n \in \mathbb{N}$, we define the *modified root operators* $\tilde{e}_{i;n}, \tilde{f}_{i;n} \colon B(\pi_\lambda) \to B(\pi_\lambda) \cup \{0\}$ on an LS path $\pi = (\tau_1, \ldots, \tau_r; t_1, \ldots, t_r)$ of shape λ. The definition

of the modified root operators extends to tensor products (concatenations) of LS paths in the natural way.

Let $x_1 \in [0,1]$ be maximal such that $H_i(t)$ is decreasing on $[0,x_1]$, let x_1^- be minimal with $H_i(x_1^-) = H_i(x_1)$, and let x_1^+ be maximal such that $H_i(t)$ is increasing on $[x_1, x_1^+]$. Let $y^- \in [0, x_1^-]$ be minimal such that $H_i(y^-) - H_i(x_1) \leq 1$ and $H_i(t)$ is strictly decreasing on $[y^-, x_1]$. Assume that $x_1^- = t_k$ and denote by π^- the path

$$(\tau_1, \ldots, \tau_l, s_i\tau_l, \ldots, s_i\tau_k, \tau_{k+1}, \ldots, \tau_r; t_1, \ldots, t_l, y^-, t_{l+1}, \ldots, t_r).$$

Note that π^- is not necessarily an LS path.

We define

$$\tilde{e}_{i;1}\pi = \begin{cases} 0, & \text{if } H_i(y^-) - H_i(x_1) < 1; \\ \pi^-, & \text{if } H_i(y^-) - H_i(x_1) = 1. \end{cases}$$

Let $y^+ \in [x_1, x_1^+]$ be maximal such that $H_i(y^+) - H_i(x_1) \leq 1$ and $H_i(t)$ is strictly increasing on $[x_1, y^+]$. Assume that $x_1 = t_k$ and denote by π^+ the path

$$(\tau_1, \ldots, \tau_k, s_i\tau_{k+1}, \ldots, s_i\tau_l, \tau_l, \ldots, \tau_r; t_1, \ldots, t_{l-1}, y^+, t_l, \ldots, t_r).$$

Note that π^+ is not necessarily an LS path.

We define

$$\tilde{f}_{i;1}\pi = \begin{cases} 0, & \text{if } H_i(y^+) - H_i(x_1) < 1; \\ \pi^+, & \text{if } H_i(y^+) - H_i(x_1) = 1. \end{cases}$$

Lemma 3.2. *Let η be a dominant path. The operators $\tilde{e}_{i;1}, \tilde{f}_{i;1}$ are well-defined defined maps from $B(\eta)$ to $B(\eta) \cup \{0\}$.*

Proof. First, assume that $\eta = \pi_\lambda$ for a dominant weight λ. In this case, the lemma follows directly from the definitions of $\tilde{e}_{i;1}$ and $\tilde{f}_{i;1}$; we explain this for $\tilde{e}_{i;1}$. The case for $H_i(y^-) - H_i(x_1) < 1$ is clear and if $H_i(y^-) - H_i(x_1) = 1$, then the conditions of Proposition 3.1 are satisfied and thus π^- is an LS path.

In the general case, η is a concatenation of LS paths; here the arguments from the proof of Theorem 7 in [33] can used to see that the modified root operators are well-defined on $B(\eta)$. □

Furthermore, we define

$$\epsilon_{i;1}(\pi) := \max\{k \mid \tilde{e}_{i;1}^k \pi \neq 0\}, \quad \varphi_{i;1}(\pi) := \max\{k \mid \tilde{f}_{i;1}^k \pi \neq 0\}.$$

If $x_1^+ < 1$, we may repeat the above procedure on $[x_1^+, 1]$ to obtain x_2, x_2^-, x_2^+ and define $\tilde{e}_{i;2}, \tilde{f}_{i;2}$, etc. However, we will focus on the operators $\tilde{e}_{i;1}$ and $\tilde{f}_{i;1}$ in this thesis.

The following figures illustrate the difference between the action of the root operators $\tilde{e}_i, \tilde{f}_i$ and the modified root operators $\tilde{e}_{i;1}, \tilde{f}_{i;1}$ on $H_i(t)$.

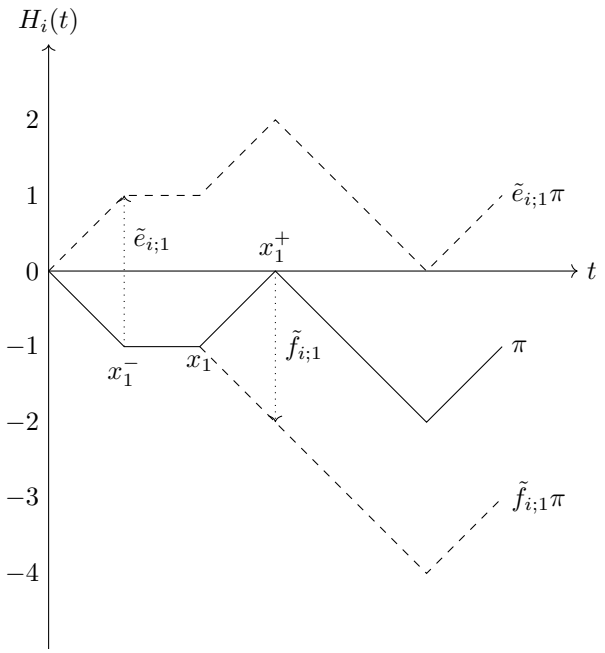


FIGURE 1. Action of the modified root operators $\tilde{e}_{i;1}$ and $\tilde{f}_{i;1}$ on $H_i(t)$.

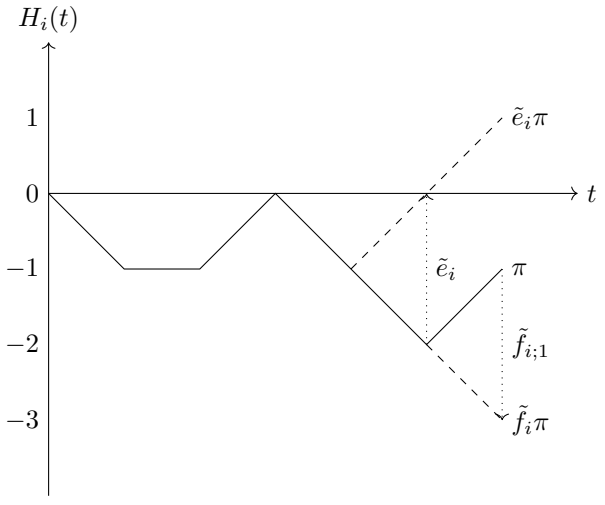


FIGURE 2. Action of the root operators $\tilde{e}_i$ and $\tilde{f}_i$ on $H_i(t)$.

Example 3.3. Let $\mathfrak{g}$ be of type A_2 and $\lambda = 4\rho$. We consider the LS path $\pi = (s_2s_1, s_1, \mathrm{id}; \frac{3}{8}, \frac{3}{4}, 1)$. Then,

$$\tilde{f}_{1;1}\pi = (s_1s_2s_1, s_2s_1, s_1, \mathrm{id}; \tfrac{1}{4}, \tfrac{3}{8}, \tfrac{3}{4}, 1) \text{ and } \tilde{f}_{1;2}\pi = \tilde{f}_1\pi = (s_2s_1, s_1; \tfrac{3}{8}, 1).$$

The modified root operators satisfy properties similar to the root operators.

Lemma 3.4. *We have* $\mathrm{wt}(\tilde{e}_{i;1}\pi) = \mathrm{wt}(\pi) + \alpha_i$ *if* $\tilde{e}_{i;1}\pi \neq 0$, *and* $\mathrm{wt}(\tilde{f}_{i;1}\pi) = \mathrm{wt}(\pi) - \alpha_i$ *if* $\tilde{f}_{i;1}\pi \neq 0$.

Proof. This follows directly from the definition of $\tilde{e}_{i;1}$ and $\tilde{f}_{i;1}$. □

Lemma 3.5. *We have* $\tilde{f}_{i;1}\tilde{e}_{i;1}\pi = \pi$, *if* $\tilde{e}_{i;1}\pi \neq 0$.

Proof. Let x_1^-, x_1, x_1^+, y^- and y^+ be as in the definition of $\tilde{e}_{i;1}$ above; and similarly let $x_1'^-, x_1', x_1'^+$, y^- and y'^+ be the corresponding times for $\pi' := \tilde{e}_{i;1}\pi$. It follows from the definition that $x_1' = y^-$ and $y'^+ = x_1^-$. Thus, applying $\tilde{f}_{i;1}$ to π' reverts the operation of $\tilde{e}_{i;1}$ on π. □

Contrary to the root operators, $\tilde{f}_{i;1}\pi \neq 0$ does not imply $\tilde{e}_{i;1}\tilde{f}_{i;1}\pi = \pi$, as the following example shows.

Example 3.6. Let $\mathfrak{g}$ be of type A_2 and $\lambda = 2\rho$. For the path $\pi = (s_2s_1, s_1; \frac{1}{2}, 1)$ we do not have $\tilde{e}_{i;1}\tilde{f}_{i;1}\pi = \pi$, since

$$\tilde{e}_{1;1}\tilde{f}_{1;1}\pi = \tilde{e}_{1;1}(s_1s_2s_1, s_1; \tfrac{1}{2}, 1) = (s_1s_2s_1, \mathrm{id}; \tfrac{1}{2}, 1).$$

3.2. **Bases associated to paths.** Littelmann [35] constructed a monomial basis of $V_{\mathbb{Z}}(\lambda)$ from the combinatorics of LS paths. The basis served as a tool to prove that the path vectors form a basis and received little attention otherwise. We are going to study this basis in more detail by comparing it to the canonical basis.

We recall the construction from [35]: Let η be a dominant path ending in λ (remember that η is assumed to be a concatenation of LS paths). Let $\pi \in B(\eta)$ and write

$$\pi = (\tau_1, \dots; t_1, \dots) \otimes \cdots \otimes (\tau_1', \dots; t_1', \dots).$$

Fix a reduced decomposition $s_{i_1} \dots s_{i_q}$ of τ_1, put $\pi_0 := \pi$ and define

$$a_k := \epsilon_{i_k;1}(\pi_{k-1}) \text{ and } \pi_k := \tilde{e}_{i_k;1}^{a_k}\pi_{k-1} \quad \text{for } k \in [1, q].$$

Now, π_q has the shape

$$(\mathrm{id}; 1) \otimes (\sigma_1, \dots; d_1, \dots) \otimes \cdots \otimes (\sigma_1', \dots; d_1', \dots).$$

Repeating the procedure for the remaining factors, we eventually reach $\pi_r = \eta$.

By [33], the sequence $\underline{i} = (i_1, \dots, i_r)$ defines a reduced decomposition of $\iota(\pi)$. In addition, we have $a_k > 0$ for all $k \in [1, r]$.

We define the *path monomial*

$$F(\pi, \underline{i}) := f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)}. \tag{3.1}$$

Generally, statements will not depend on specific choice of $\underline{i}$; in such cases we simply write $F(\pi)$.

Example 3.7. Assume that $\mathfrak{g}$ is of type A_2 and let $\lambda = 4\rho$ and $\eta = \pi_\lambda$. For the LS path $\pi := (s_1s_2s_1, s_2s_1, s_1, \mathrm{id}; \frac{1}{4}, \frac{3}{8}, \frac{3}{4}, 1)$ we have $F(\pi, (1,2,1)) = f_1 f_2^{(3)} f_1^{(3)}$ and $F(\pi, (2,1,2)) = f_2^{(2)} f_1^{(4)} f_2$.

For each $\pi \in B(\eta)$ choose a sequence $\underline{i}_\pi$ in I as above. Then, it was shown by Littelmann [35] that the set $\{F(\pi, \underline{i}_\pi)u_\lambda \mid \pi \in B(\eta)\}$ is a basis of $V_{\mathbb{Z}}(\lambda)$.

Since the result is independent of the reduced decomposition chosen for each $\pi \in B(\eta)$, we write

$$\mathcal{P}(\eta) := \{F(\pi)u_\lambda \mid \pi \in B(\eta)\}, \tag{3.2}$$

where for each π an arbitrary reduced decomposition of $\iota(\pi)$ is fixed. We call $\mathcal{P}(\eta)$ the *path monomial basis* associated to η.

3.3. **Partial orders on the path model.** We introduce a new partial order $\succeq$ on the path model $B(\eta)$ for a dominant path η. Here η is not necessarily a concatenation of LS paths. The partial order $\succeq$ defined on the weight lattice P extends to $P_\mathbb{R} = P \otimes_\mathbb{Z} \mathbb{R}$ in a natural way: For $\lambda, \mu \in P_\mathbb{R}$, we say $\lambda \succeq \mu$ if $\lambda - \mu$ is a $\mathbb{R}_{\geq 0}$-linear combination of simple roots.

Let λ be a dominant weight and let η be a (arbitrary) dominant path ending in λ. For $\pi, \pi' \in B(\eta)$ we write $\pi \succeq \pi'$ if

$$\pi(t) - \pi'(t) \succeq 0 \quad \text{for all } t \in [0,1]. \tag{3.3}$$

Lemma 3.8. *Let η be a dominant path of weight λ. The relation $\succeq$ is a partial order on $B(\eta)$.*

Proof. Reflexivity is clear, for antisymmetry note that $\pi \succeq \pi'$ and $\pi' \succeq \pi$ implies that $\pi(t) - \pi'(t) = 0$ for all $t \in [0,1]$, and thus $\pi = \pi'$. For transitivity, assume that $\pi \succeq \pi'$ and $\pi' \succeq \pi''$. Then,

$$\pi(t) - \pi''(t) = (\pi(t) - \pi'(t)) + (\pi'(t) - \pi''(t)) \succeq 0 \quad \text{for all } t \in [0,1].$$

□

If $\eta = \pi_\lambda$, that is, π and π' are LS paths of shape λ, we further define the partial order $\succeq_+$ as

$$\pi \succeq_+ \pi' \text{ if and only if } \pi \succeq \pi' \text{ and } \pi \trianglelefteq \pi'. \tag{3.4}$$

3.4. **Conjectures on the canonical basis.** Motivated by the conjecture that there is unitriangular base change between path vectors and the dual canonical basis, we investigated the relation between the path monomial basis and the canonical basis. This revealed an interesting structure on the canonical basis, which is incompatible with the crystal structure. We conjecture the following formula, which should not be confused with the similar looking one from Kashiwara in (2.17).

Conjecture 3.9. *Let λ be a dominant weight and η be a dominant path of weight λ. Then, for any path $\pi \in B(\eta)$ and $i \in I$,*

$$f_i^{(n)} G_\lambda(b_\pi) = \sum F^{i,n}_{\pi,\pi'} G_\lambda(b_{\pi'}),$$

where $F^{i,n}_{\pi,\pi'} \in \mathbb{Z}[q, q^{-1}]$ and the sum ranges over $\pi' \in B(\eta)$ with the following properties:

- *(i) $\pi'(t) \succeq \pi(t) - n\alpha_i$ for all $t \in [0,1]$.*
- *(ii) Let $x \in [0,1]$ be maximal such that $\pi'(t) - \pi(t)$ is a multiple of α_i for all $t \in [0,x]$. Assume that $x < 1$ and let $\epsilon > 0$ be such that on $[x, x+\epsilon]$ the path π is supported on $\tau \in W$, the path π' is supported on $\sigma \in W$. Then, $\sigma < \max\{\tau, s_i\tau\}$, and if $\pi'(x) - \pi(x) = -n\alpha_i$, then already $\sigma < \tau$.*

(iii) Let $t_0 \in [0,1]$ be maximal such that $\langle h_i, \pi(t)\rangle$ is decreasing on $[0,t_0]$, and let $t_1 \in [t_0,1]$ be maximal such that $\langle h_i, \pi(t)\rangle$ is increasing on $[t_0,t_1]$ and $M := \langle h_i, \pi(t_1)\rangle - \langle h_i, \pi(t_0)\rangle \in \mathbb{Z}$. If $n \leq M$, then $\tilde{f}_{i;1}^n \pi \neq 0$ and

$$F^{i,n}_{\pi,\tilde{f}_{i;1}^n\pi} = \begin{bmatrix} \epsilon_{i;1}(\pi)+n \\ n \end{bmatrix}_i.$$

Remark 3.10. Let us make more precise what we mean, when we say that the path π is supported on $\tau \in W$ in Conjecture 3.9 (ii). By assumption η is a concatenation of LS paths, that is, $\eta = \pi_{\lambda_1} * \cdots * \pi_{\lambda_m}$ for a decomposition of λ into dominant weights $\lambda_1, \ldots, \lambda_m$. Thus, π can be written as $\pi = \pi_1 * \cdots * \pi_m$ with $\pi_k \in B(\pi_{\lambda_k})$ for $k \in [1,m]$. Assume that π is parametrized such that $\pi(t) = \pi_1(1) + \cdots + \pi_{k-1}(1) + \pi_k(mt-(k-1))$ for $t \in [(k-1)/m, k/m]$. Then, if $x < 1$, an $\epsilon > 0$ can be chosen such that $[x, x+\epsilon] \subseteq [(k-1)/m, k/m]$ for some $k \in [1,m]$. Assume that $\pi_k = (\tau_1, \ldots, \tau_r; t_1 \ldots, t_r)$. Then, ϵ can be further refined such that $[mx-(k-1), m(x+\epsilon)-(k-1)] \subseteq [t_{j-1}, t_j]$ for some $j \in [1,r]$. We say that on this interval π_k is *supported* on $\tau_j \in W^{\lambda_k}$. Hence, the support of π on $[x, x+\epsilon]$ is τ_j.

Corollary 3.11. *In Conjecture 3.9, if $\tilde{f}_{i;1}^n \pi \neq 0$, then $\tilde{f}_{i;1}^n \pi \trianglerighteq \pi'$ for any π' with $F^{i,n}_{\pi,\pi'} \neq 0$.*

Proof. Let $x \in [0,1]$ be maximal such that $\pi'(t) = \tilde{f}_{i;1}^n \pi(t)$ for all $t \in [0,x]$ and let $y \in [0,1]$ be minimal such that $\tilde{f}_{i;1}^n \pi(y) - \pi(y) = -n\alpha_i$. Assume that $\pi' \neq \tilde{f}_{i;1}^n \pi$. Thus, $x < 1$ and there exists $\epsilon > 0$ such that on $[x, x+\epsilon]$ the path $\tilde{f}_{i;1}^n \pi$ is supported on $\tau \in W$ and the path π' is supported on $\sigma \in W$.

First, consider $x \geq y$. Note that $\tilde{f}_{i;1}^n \pi(t) - \pi(t) = -n\alpha_i$ on $[y,1]$ and hence $\pi'(x) - \pi(x) = -n\alpha_i$. Since π is also supported on τ for $t \in [x, x+\epsilon]$, we have by Conjecture 3.9 (ii) that $\sigma < \tau$. Hence, $\pi' \triangleleft \tilde{f}_{i;1}^n \pi$.

Second, consider $x < y$. Note that $\langle h_i, \tilde{f}_{i;1}^n \pi(t)\rangle$ is decreasing on $[x, x+\epsilon]$, hence $s_i\tau \leq \tau$. In addition, observe that π is supported on $s_i\tau$ or τ. Now, Conjecture 3.9 (i) implies that we cannot have $\sigma > \tau$. Thus, either $\sigma = s_i\tau < \tau$ or it follows from Conjecture 3.9 (ii) that $\sigma < \max\{s_i\tau, \tau\} = \tau$. Hence, $\pi' \triangleleft \tilde{f}_{i;1}^n \pi$. □

Example 3.12. Assume that $\mathfrak{g}$ is of type A_2 and let $\lambda = 4\rho$ and $\eta = \pi_\lambda$.

(i) For $\pi = (s_2s_1, s_1, \mathrm{id}; \frac{1}{8}, \frac{1}{4}, 1)$ the corresponding element of canonical basis of $V(\lambda)$ is $f_2f_1u_\lambda$. We have

$$f_1(f_2f_1u_\lambda) = f_1^{(2)}f_2u_\lambda + f_2f_1^{(2)}u_\lambda,$$

where the respective paths are

$$\pi_1 := (s_1s_2, \mathrm{id}; \tfrac{1}{4}, 1) \text{ and } \pi_2 := (s_2s_1, s_1, \mathrm{id}; \tfrac{1}{8}, \tfrac{1}{2}, 1).$$

For π_1 a simple computation gives $\pi_1(\frac{1}{8}) - \pi(\frac{1}{8}) = -\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$, $\pi_1(\frac{1}{4}) - \pi(\frac{1}{4}) = \pi_1(1) - \pi(1) = -\alpha_1$. Similarly, we obtain for π_2 that $\pi_2(\frac{1}{8}) - \pi(\frac{1}{8}) = \pi_2(\frac{1}{4}) - \pi(\frac{1}{4}) = 0$, and $\pi_2(\frac{1}{2}) - \pi(\frac{1}{2}) = \pi_2(1) - \pi(1) = -\alpha_1$. Thus, $\pi_1, \pi_2 \trianglerighteq \pi - \alpha_1$ and the conjecture holds.

(ii) For $\pi = (s_2s_1, s_1, \mathrm{id}; \frac{3}{8}, \frac{3}{4}, 1)$ the corresponding element of canonical basis of $V(\lambda)$ is $f_2^{(3)}f_1^{(3)}u_\lambda$. We have

$$f_1(f_2^{(3)}f_1^{(3)}u_\lambda) = f_2^{(2)}f_1^{(4)}f_2u_\lambda + f_2^{(3)}f_1^{(4)}u_\lambda.$$

The respective paths are $\pi_1 := (s_1s_2s_1, s_2s_1, s_1, \mathrm{id}; \frac{1}{4}, \frac{3}{8}, \frac{3}{4}, 1)$ and $\pi_2 := (s_1s_2, s_2, \mathrm{id}; \frac{1}{2}, \frac{3}{4}, 1)$. Here, the conjecture holds, since $\epsilon_{1;1}(\pi) = 0$, $\tilde{f}_{1;1}\pi = \pi_1$ and $\pi_1 \triangleright \pi_2$.

The following example shows that, in general, the coefficient of the leading term of Conjecture 3.9 is not well-behaved.

Example 3.13. Assume that $\mathfrak{g}$ is of type A_2 and let $\lambda = 4\rho$ and $\eta = \pi_\lambda$. For the path $\pi = (s_1s_2s_1, s_2s_1, s_1, \mathrm{id}; \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ the corresponding element of the canonical basis is $f_1 f_2^{(4)} f_1^{(3)} u_\lambda$. A computation gives

$$f_1^{(2)}(f_1 f_2^{(4)} f_1^{(3)} u_\lambda) = ([4] + [2]) f_2^{(2)} f_1^{(6)} f_2^{(2)} + [3] f_2 f_1^{(6)} f_2^{(3)} u_\lambda + [3] f_2^{(3)} f_1^{(6)} f_2 u_\lambda,$$

where $f_2^{(2)} f_1^{(6)} f_2^{(2)}$ is the element of the canonical basis corresponding to the path $\tilde{f}_{i;1}^2 \pi = (s_1s_2s_1, s_1; \frac{1}{2}, 1)$.

Proposition 3.14. *Assume that Conjecture 3.9 holds for the dominant path η ending in the dominant weight λ. Then, there is a unitriangular base change between the path monomial basis $\mathcal{P}(\eta)$ and the canonical basis of $V(\lambda)$ for any linearisation of the partial order $\trianglerighteq$ on $B(\eta)$.*

Proof. The proof is by induction on the partial order $\trianglerighteq$. The case $\pi = \eta$ is clear, thus we can assume that $\pi \triangleright \eta$ and that the statement holds for all $\pi' \in B(\eta)$ with $\pi' \triangleleft \pi$. Fix a sequence $\underline{i} = (i_1, \ldots, i_r)$ in I as in the construction of $F(\pi, \underline{i})$ and put $\pi_k := \tilde{f}_{i_k;1}^{a_k} \cdots \tilde{f}_{i_r;1}^{a_r} \eta$ and $\underline{i}_k = (i_k, \ldots, i_r)$ for $k \in [1, r]$. We have $\pi = \pi_1 \triangleright \pi_2$, hence using the induction hypothesis we can write

$$F(\pi_2, \underline{i}_2) u_\lambda = G_\lambda(b_{\pi_2}) + \sum_{\pi' \in B(\eta)} z_{\pi_2, \pi'} G_\lambda(b_{\pi'}),$$

where $z_{\pi_2,\pi'} \in \mathbb{Z}[q, q^{-1}]$ is zero unless $\pi' \triangleleft \pi_2$. We claim that applying $f_{i_1}^{(a_1)}$ to both sides gives

$$F(\pi, \underline{i}) u_\lambda = G_\lambda(b_\pi) + \sum F_{\pi_2, \pi''}^{i_1, a_1} G_\lambda(b_{\pi''}) + \sum z_{\pi_2, \pi'} F_{\pi', \pi'''}^{i_1, a_1} G_\lambda(b_{\pi'''}),$$

where the first sum ranges over $\pi'' \in B(\eta)$ with $\pi'' \triangleleft \pi$ and second sum ranges over $\pi''' \in B(\eta)$ with $\pi''' \triangleleft \pi$.

Indeed, the coefficient of $G_\lambda(b_\pi)$ is 1 by Conjecture 3.9 (iii), since π_2 fulfills the conditions by construction, and $\pi'' \triangleleft \pi$ by Corollary 3.11.

To see $\pi''' \triangleleft \pi$, let $x \in [0, 1]$ be maximal with $\pi_2(t) = \pi'(t)$ for all $t \in [0, x]$ and let $y \in [0, 1]$ be maximal such that $\langle h_i, \pi(t) \rangle$ is decreasing on $[0, y]$. Note that $\langle h_i, \pi_2(t) \rangle$ is increasing on $[0, y]$.

If $x \geq y$, then the claim also follows from Corollary 3.11, since $\pi = \tilde{f}_{i;1}^n \pi_2 \triangleright \tilde{f}_{i;1}^n \pi'$. For $x < y$, let $x' \in [0, 1]$ be maximal such that $\pi'''(t) - \pi'(t)$ is a multiple of α_i for all $t \in [0, x']$. Observe that for $t \in [0, \min\{x, x'\}]$ we have $\pi'''(t) - \tilde{f}_{i;1}^n \pi(t) \succeq 0$, and if for some $t \in [0, \min\{x, x'\}]$ we have $\pi'''(t) - \tilde{f}_{i;1}^n \pi(t) \succ 0$, then $\pi''' \triangleleft \tilde{f}_{i;1}^n \pi$. Hence, we may assume that $\pi'''(t) = \tilde{f}_{i;1}^n \pi(t)$ for all $t \in [0, \min\{x, x'\}]$.

If $x' < x$, let $\epsilon > 0$ be such that on $[x', x' + \epsilon]$ the path π_2 is supported on $\tau \in W$, the path π' is supported on $\sigma \in W$ and the path π''' is supported on $\nu \in W$. It follows that $\nu < \max\{\sigma, s_i\sigma\} = s_i\tau$, where the inequality is due to Conjecture 3.9 (ii). Since $\tilde{f}_{i;1}^n \pi$ is supported on $s_i\tau$ for $t \in [x', x' + \epsilon]$, we have $\pi''' \triangleleft \tilde{f}_{i;1}^n \pi$.

If $x' \geq x$, let $\epsilon > 0$ be such that on $[x, x+\epsilon]$ the path π_2 is supported on $\tau \in W$, the path π' is supported on $\sigma \in W$ and the path π''' is supported on $\nu \in W$. Since $\pi' \lhd \pi$, we have $\sigma < \tau$ and thus $\nu \leq \max\{\sigma, s_i\sigma\} < s_i\tau$, where the first inequality is due to Conjecture 3.9 (ii). Since $\tilde{f}_{i;1}^n \pi$ is supported on $s_i\tau$ for $t \in [x, x+\epsilon]$, we have $\pi''' \lhd \tilde{f}_{i;1}^n \pi$. □

Conjecture 3.15. *Let λ be a dominant weight and η be a dominant path of weight λ. Then, for any path $\pi \in B(\eta)$ and $i \in I$,*

$$e_i G_\lambda(b_\pi) = \sum E^i_{\pi,\pi'} G_\lambda(b_{\pi'})$$

where $E^i_{\pi,\pi'} \in \mathbb{Z}[q, q^{-1}]$ and the sum ranges over all $\pi' \in B(\eta)$ with the following properties:

(i) $\pi' \succ_+ \pi$.

(ii) Let $x \in [0,1]$ be maximal such that $\pi'(t) - \pi(t)$ is a multiple of α_j for some $j \in I$ and all $t \in [0,x]$. Assume that $x < 1$ and let $\epsilon > 0$ be such that on $[x, x+\epsilon]$ the path π is supported on $\tau \in W$ and the path π' is supported on $\sigma \in W$. Then, $\sigma < \max\{\tau, s_j\tau\}$, and if $\pi'(x) - \pi(x) = 0$, then already $\sigma < \tau$.

Corollary 3.16. *Assume that Conjecture 3.15 holds for the dominant path η ending in the dominant weight λ. Then, the sets*

$$V_{\succ_+\pi}(\lambda) := \bigoplus_{\pi' \in B(\eta), \pi' \succ_+ \pi} \mathbb{Z}[q, q^{-1}] G_\lambda(b_{\pi'}) \quad \text{and}$$

$$V_{\succeq_+\pi}(\lambda) := \bigoplus_{\pi' \in B(\eta), \pi' \succeq_+ \pi} \mathbb{Z}[q, q^{-1}] G_\lambda(b_{\pi'}).$$

are $U_q^+(\mathfrak{g})$-submodules of $V(\lambda)$.

Combining Conjectures 3.9 and 3.15, we can refine Theorem 2.23 as follows:

Theorem 3.17. *Assume that Conjecture 3.15 holds for the dominant path η_λ ending in λ and Conjecture 3.9 holds for the dominant path η_μ ending in μ. Then,*

$$G(b_{\pi_1} \otimes b_{\pi_2}) = G_\lambda(b_{\pi_1}) \otimes G_\mu(b_{\pi_2}) + \sum z_{\pi_1,\pi_2;\pi_1',\pi_2'} G_\lambda(b_{\pi_1'}) \otimes G_\mu(b_{\pi_2'}),$$

where $z_{\pi_1,\pi_2;\pi_1',\pi_2'} \in q\mathbb{Z}[q]$ is zero unless $\pi_1' \succ_+ \pi_1$ and $\pi_2'(t) \succeq \pi_2(t) - (\mathrm{wt}(\pi_1') - \mathrm{wt}(\pi_1))$ for all $t \in [0,1]$. In particular, $\pi_1' * \pi_2' \succ_+ \pi_1 * \pi_2$.

Proof. Let $\pi_1' \in B(\eta_\lambda)$ and $\pi_2' \in B(\eta_\mu)$ such that $z_{\pi_1,\pi_2;\pi_1',\pi_2'} \neq 0$. The fact that $z_{\pi_1,\pi_2;\pi_1',\pi_2'} = 0$, unless $\pi_1' \succ_+ \pi_1$ and $\pi_2'(t) \succeq \pi_2(t) - (\mathrm{wt}(\pi_1') - \mathrm{wt}(\pi_1))$ for all $t \in [0,1]$, is an immediate consequence of Conjecture 3.15 and Conjecture 3.9, respectively. Since $\pi_1' \succ_+ \pi_1$, we have for $t \in [0, \frac{1}{2}]$

$$(\pi_1' * \pi_2')(t) - (\pi_1 * \pi_2)(t) = \pi_1'(2t) - \pi_1(2t) \succeq 0;$$

and since $\pi_2' \succ \pi_2 - \xi$, with $\xi := \mathrm{wt}(\pi_1') - \mathrm{wt}(\pi_1)$, we have for $t \in [\frac{1}{2}, 1]$

$$(\pi_1' * \pi_2')(t) - (\pi_1 * \pi_2)(t) = \xi + \pi_2'(2t-1) - \pi_2(2t-1) \succeq \xi + (-\xi) = 0.$$

We conclude that $\pi_1' * \pi_2' \succ_+ \pi_1 * \pi_2$. □

3.5. **Comparsion of bases for fundamental weights.** We investigate the veracity of Conjecture 3.9 by comparing the path monomial basis to the canonical basis for fundamental weights. In this section we assume that $\mathfrak{g}$ is of finite type.

We first consider fundamental weights of classical type. A fundamental weight ϖ_i is said to be *of classical type* if $\langle h_\beta, \varpi_i\rangle \leq 2$ for all positive roots β.

Proposition 3.18. *Let λ be a fundamental weight of classical type and let π be an LS path of shape λ. For any reduced decomposition $\underline{i} = (i_1, \ldots, i_r)$ of $\iota(\pi)$ we have*

$$G_\lambda(b_\pi) = F(\pi, \underline{i})u_\lambda.$$

Proof. We prove the statement by induction on the length of $\iota(\pi)$. If $\ell(\iota(\pi)) = 0$, then $\pi = \pi_\lambda$ and $G_\lambda(b_\pi) = F(\pi, \underline{i})u_\lambda = u_\lambda$. Now, assume that the length of $\iota(\pi)$ is greater than 0 and write $F(\pi, \underline{i}) = f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)}$. Since λ is of classical type, the length of any $\mathfrak{sl}_2$-chain is at most 2. Therefore, every LS path has the shape $(\tau, \sigma; \frac{1}{2}, 1)$ with $\tau, \sigma \in W$ (we allow $\tau = \sigma$ and this is understood as the path $(\tau; 1)$). From this, one sees that for any $i \in I$ the function $H_i(t) = \langle h_i, \pi(t)\rangle$ has a unique minimum if $\epsilon_i(\pi) > 0$; in particular, we have $\epsilon_{i_1}(\pi) = \epsilon_{i_1;1}(\pi) = a_1$. Using the induction hypothesis on $\pi' := \tilde{e}_i^{a_1}\pi = \tilde{e}_{i_1;1}^{a_1}\pi$ (note that $\pi' \neq 0$), we obtain

$$G_\lambda(b_{\pi'}) = f_{i_2}^{(a_2)} \cdots f_{i_r}^{(a_r)} u_\lambda.$$

Now, using (2.17), we see that

$$f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} u_\lambda = f_{i_1}^{(a_1)} G_\lambda(b_{\pi'}) = G_\lambda(b_\pi) + \sum F_{\pi,\pi''}^{i_1,a_1} G_\lambda(b_{\pi''}),$$

where $F_{\pi,\pi''}^{i_1,a_1} = 0$ unless $\epsilon_{i_1}(\pi'') > a_1$. However, by construction $\mathrm{wt}(\pi') + \alpha_i$ is not a weight of $V(\lambda)$ (the length of any $\mathfrak{sl}_2$-chain is at most 2). Thus, the only term on the right hand side of the above equation is $G_\lambda(b_\pi)$. The proposition follows. □

Proposition 3.19. *Conjecture 3.9 holds for fundamental weights of classical type.*

Proof. Let λ be a fundamental weight of classical type and let $\pi = (\tau, \sigma; \frac{1}{2}, 1)$ be an LS path of shape λ (we allow $\tau = \sigma$ for convenience). For any $i \in I$ and $n \geq 0$ we have $f_i^{(n)} G_\lambda(b_\pi) = \sum F_{\pi,\pi'}^{i,n} G_\lambda(b_{\pi'}) \in V_w(\lambda)$ by Proposition 3.2.3 [24], where $w = \max\{\tau, s_i\tau\}$. Thus, if for some π' in the above sum $F_{\pi,\pi'}^{i} \neq 0$, then $\pi' = (\tau', \sigma'; \frac{1}{2}, 1)$ for some $\tau' \leq w$. In particular, $\pi'(\frac{1}{2}) - \pi(\frac{1}{2}) = \frac{1}{2}(\tau'\lambda - \tau\lambda) =: \xi \succeq -\alpha_i$. Furthermore, $\pi'(1) - \pi(1) = \xi + \frac{1}{2}(\sigma'\lambda - \sigma\lambda) = -\alpha_i$. This proves (i) and (ii); (iv) follows from Proposition 3.18. The claim that $F_{\pi,\pi'}^{i,n} \in \mathbb{Z}[q, q^{-1}]$ follows directly from Theorem 2.10. □

Proposition 3.20. *Conjecture 3.15 holds for fundamental weights of classical type.*

Proof. Let λ be a fundamental weight of classical type and let $\pi = (\tau, \sigma; \frac{1}{2}, 1)$ be an LS path of shape λ (we allow $\tau = \sigma$ for convenience). For any $i \in I$ we have $e_i G_\lambda(b_\pi) = \sum E_{\pi,\pi'}^{i} G_\lambda(b_{\pi'}) \in V_\tau(\lambda)$ by Proposition 3.2.3 [24]. Thus, if for some π' in the above sum $E_{\pi,\pi'}^{i} \neq 0$, then $\pi' = (\tau', \sigma'; \frac{1}{2}, 1)$ for some $\tau' \leq \tau$. In particular, $\pi'(\frac{1}{2}) - \pi(\frac{1}{2}) = \frac{1}{2}(\tau'\lambda - \tau\lambda) =: \xi \in Q_+$. Furthermore, $\pi'(1) - \pi(1) = \xi + \frac{1}{2}(\sigma'\lambda - \sigma\lambda) = \alpha_i$. This proves (i) and (ii). The claim that $E_{\pi,\pi'}^{i} \in \mathbb{Z}[q, q^{-1}]$ follows directly from Theorem 2.10. □

Marsh [41] investigated the canonical basis of fundamental weights from the perspective of standard monomials.

Remark 3.21. Marsh [41] used the orientation of the Dynkin diagram of F_4 opposite to the one in this thesis. Also, Marsh's computations were made following Lusztig's conventions [38], whereas this thesis follows Kashiwara's conventions [22]. We have adapted Marsh's results accordingly.

We recall the definition of standard monomials for a fundamental weight of classical type. Let $\tau, \sigma \in W^\lambda$. The tuple (τ, σ) is called an *admissible pair* if there is a sequence

$$\tau = \tau_1 > \cdots > \tau_{r+1} = \sigma$$

such that for each $k \in [1, r]$ there exists $i_k \in I$ with $\tau_k = s_{i_k}\tau_{k+1}$ and $\langle h_{i_k}, \tau_{k+1}\lambda\rangle = 2$.

The *standard monomial* $Q(\tau, \sigma)$ associated to an admissible pair (τ, σ) is defined as follows:

(i) If $\tau = \sigma$, let $s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of τ and put $a_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_r}\lambda\rangle$. We define

$$Q(\tau, \tau) := f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} u_\lambda.$$

The definition is independent of the choice of the reduced decomposition due to the quantum Verma identity.

(ii) If $\tau > \sigma$ and $\tau = \tau_1 > \cdots > \tau_r = \sigma$ is a sequence as above. We define

$$Q(\tau, \sigma) := f_{i_1} \cdots f_{i_r} Q(\sigma, \sigma).$$

The definition is independent of the choice of the sequence.

There is an alternative description of standard monomials [41, Definition 4.3], which is equivalent to the one given above [41, Theorem 4.4]. Using this definition it is easy to see that, for the LS path $\pi = (\tau, \sigma; \frac{1}{2}, 1)$, the standard monomial $Q(\tau, \sigma)$ agrees with the definition of $F(\pi, \underline{i})$. In particular, we recover Theorem 5.1 of [41], that is, the standard monomial basis and the canonical basis agree for fundamental weights of classical type.

Let us now consider fundamental weights which are not of classical type.

Example 3.22. Assume that $\mathfrak{g}$ is of type G_2. For both fundament weights the canonical basis agrees with the path basis, as the following explicit description shows. The canonical basis of $V(\varpi_1)$ is

$$\{u_{\varpi_1}, f_1u_{\varpi_1}, f_2f_1u_{\varpi_1}, f_1f_2f_1u_{\varpi_1}, f_1^{(2)}f_2f_1u_{\varpi_1}, f_2f_1^{(2)}f_2f_1u_{\varpi_1}, f_1f_2f_1^{(2)}f_2f_1u_{\varpi_1}\}.$$

The corresponding LS paths of shape ϖ_1 are

$$\begin{aligned}\{&(\mathrm{id}; 1), (s_1; 1), (s_2s_1; 1), (s_1s_2s_1, s_2s_1; \tfrac{1}{2}, 1)\\ &(s_1s_2s_1; 1), (s_2s_1s_2s_1; 1), (s_1s_2s_1s_2s_1; 1)\}.\end{aligned}$$

The canonical basis of $V(\varpi_2)$ is

$$\begin{aligned}\{&u_{\varpi_2}, f_2u_{\varpi_2}, f_1f_2u_{\varpi_2}, f_1^{(2)}f_2u_{\varpi_2}, f_1^{(3)}f_2u_{\varpi_2}, f_2f_1^{(2)}f_2u_{\varpi_2}, f_2f_1^{(3)}f_2u_{\varpi_2},\\ &f_1f_2f_1^{(2)}f_2u_{\varpi_2}, f_2^{(2)}f_1^{(3)}f_2u_\lambda, f_1f_2f_1^{(3)}f_2u_{\varpi_2}, f_1f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2},\\ &f_1^{(2)}f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}, f_1^{(3)}f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}, f_2f_1^{(3)}f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}\}.\end{aligned}$$

The corresponding LS paths of shape ϖ_2 are

$$\{(\mathrm{id};1),(s_2;1),(s_1s_2,s_2;\tfrac{1}{3},1),(s_1s_2,s_2;\tfrac{2}{3},1),(s_1s_2;1),(s_2s_1s_2,s_1s_2,s_2;\tfrac{1}{2},\tfrac{2}{3},1),$$
$$(s_2s_1s_2,s_1s_2;\tfrac{1}{2},1),(s_1s_2s_1s_2,s_2s_1s_2,s_1s_2,s_2;\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3},1),(s_2s_1s_2;1),$$
$$(s_1s_2s_1s_2,s_2s_1s_2,s_1s_2;\tfrac{1}{3},\tfrac{1}{2},1),(s_1s_2s_1s_2,s_2s_1s_2;\tfrac{1}{3},1),$$
$$(s_1s_2s_1s_2,s_2s_1s_2;\tfrac{2}{3},1),(s_1s_2s_1s_2;1),(s_2s_1s_2s_1s_2;1)\}.$$

Lemma 3.23. *Conjectures 3.9 and 3.15 hold for both fundamental weights in type G_2.*

Proof. For ϖ_1 this follows from Propositions 3.19 and 3.20, since ϖ_1 is of classical type. For ϖ_2, we present a table for the action of f_1 and f_2 on the canonical basis of $V(\varpi_2)$. The result can then be verified by comparing with Example 3.22.

	f_1	f_2
u_{ϖ_2}	0	$f_2u_{\varpi_2}$
$f_2u_{\varpi_2}$	$f_1f_2u_{\varpi_2}$	0
$f_1f_2u_{\varpi_2}$	$[2]_1f_1^{(2)}f_2u_{\varpi_2}$	0
$f_1^{(2)}f_2u_{\varpi_2}$	$[3]_1f_1^{(3)}f_2u_{\varpi_2}$	$f_2f_1^{(2)}f_2u_{\varpi_2}$
$f_1^{(3)}f_2u_{\varpi_2}$	0	$f_2f_1^{(3)}f_2u_{\varpi_2}$
$f_2f_1^{(2)}f_2u_{\varpi_2}$	$f_1f_2f_1^{(2)}f_2u_{\varpi_2}$	0
$f_2f_1^{(3)}f_2u_{\varpi_2}$	$f_1f_2f_1^{(3)}f_2u_{\varpi_2}$	$[2]_2f_2^{(2)}f_1^{(3)}f_2u_\lambda$
$f_1f_2f_1^{(2)}f_2u_{\varpi_2}$	$[2]_1f_1f_2f_1^{(3)}f_2u_{\varpi_2}$	$[3]_1f_2^{(2)}f_1^{(3)}f_2u_\lambda$
$f_2^{(2)}f_1^{(3)}f_2u_\lambda$	$f_1f_2^{(2)}f_1^{(3)}f_2u_\lambda$	0
$f_1f_2f_1^{(3)}f_2u_{\varpi_2}$	0	$f_1f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}$
$f_1f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}$	$[2]_1f_1^{(2)}f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}$	0
$f_1^{(2)}f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}$	$[3]_1f_1^{(3)}f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}$	0
$f_1^{(3)}f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}$	0	$f_2f_1^{(3)}f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}$
$f_2f_1^{(3)}f_2^{(2)}f_1^{(3)}f_2u_{\varpi_2}$	0	0

□

Marsh [41] showed, using a generalized definition for standard monomials, that in F_4 for the fundamental weight ϖ_3 there exist standard monomials which no longer belong to the canonical basis. We compare her example to the corresponding element in the path basis.

She considered the standard monomial

$$(f_4f_3-qf_3f_4)f_3f_2f_3f_1f_2f_3u_\lambda,$$

which corresponds to the LS path

$$\pi=(s_3s_4s_2s_3s_1s_2s_3,s_3s_2s_3s_1s_2s_3;\tfrac{1}{2},1).$$

Marsh showed that

$$((f_4f_3-qf_3f_4)f_3f_2f_3f_1f_2f_3u_\lambda,(f_4f_3-qf_3f_4)f_3f_2f_3f_1f_2f_3u_\lambda)=q^2+q^4.$$

Hence, by Theorem 2.17 is not an element of the canonical basis.

Here, the element of the path monomial basis is $F(\pi,\underline{i})u_\lambda = f_3^{(2)}f_4f_2f_3f_1f_2f_3u_\lambda$, with $\underline{i}=(3,4,2,3,1,2,3)$. We compute

$$\begin{aligned}
&(f_3^{(2)}f_4f_2f_3f_1f_2f_3u_\lambda, f_3^{(2)}f_4f_2f_3f_1f_2f_3u_\lambda)\\
&=(f_4f_2f_3f_1f_2f_3u_\lambda, f_4f_2f_3f_1f_2f_3u_\lambda)\\
&=(1+q^2)(f_2f_3f_1f_2f_3u_\lambda, f_2f_3f_1f_2f_3u_\lambda)\\
&=(1+q^2)(f_3f_1f_2f_3u_\lambda, f_3f_1f_2f_3u_\lambda)\\
&=(1+q^2)(f_1f_2f_3u_\lambda, f_1f_2f_3u_\lambda)\\
&=(1+q^2)(f_2f_3u_\lambda, f_2f_3u_\lambda)\\
&=(1+q^2)(f_3u_\lambda, f_3u_\lambda)\\
&=1+q^2.
\end{aligned}$$

Noting that $(2,1,1,1,1,1)$ is also the adapted string of π with respect to $\underline{i}$, we conclude that $F(\pi,\underline{i})=G_{\varpi_3}(b_\pi)$ using Theorem 2.17 and Proposition 2.27.

In fact, using OSCAR [44] we were able to verify that in $V(\varpi_3)$, we indeed have $F(\pi,\underline{i})=G_{\varpi_3}(b_\pi)$ for all $\pi\in B(\pi_{\varpi_3})$ and any choice of $\underline{i}$ such that $s_{\underline{i}}=\iota(\pi)$.

However, we found an example for ϖ_2, where the path basis no longer coincides with the canonical basis. Consider the LS path

$$\pi=(s_3s_1s_2s_3s_4s_3s_1s_2s_3s_1s_2, s_1s_2s_3s_4s_3s_1s_2s_3s_1s_2, s_2s_3s_1s_2s_3s_1s_2; \tfrac{1}{4},\tfrac{1}{2},1)$$

of weight $-\varpi_1+2\varpi_4$ and fix $\underline{i}=(3,1,2,3,4,3,1,2,3,1,2)$. Then, associated monomial is

$$F(\pi,\underline{i})=f_3f_1f_2^{(2)}f_3^{(2)}f_4^{(2)}f_3f_1f_2^{(2)}f_3^{(2)}f_1f_2.$$

Before we are able to compute the value of $(F(\pi,\underline{i})u_\lambda, F(\pi,\underline{i})u_\lambda)$, we need some intermediate computations. For this we put

$$\begin{aligned}
v_1&=f_1f_2^{(2)}f_3^{(2)}f_4^{(2)}f_3f_1f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda,\\
v_2&=f_3f_1f_2^{(3)}f_3^{(2)}f_4^{(2)}f_1^{(2)}f_2f_3^{(2)}f_2u_\lambda.
\end{aligned}$$

First, we compute

$$\begin{aligned}
&(f_4^{(2)}f_3f_1f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda, f_4^{(2)}f_3f_1f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda)\\
&=(1+q^2+q^4)(f_3f_1f_2^{(2)}f_3^{(2)}f_1f_2, f_3f_1f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda)\\
&=(1+2q^2+2q^4+q^6)(f_1f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda, f_4^{(2)}f_1f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda)\\
&=(1+2q^2+2q^4+q^6)(f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda, f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda)\\
&=(1+2q^2+2q^4+q^6)(f_3^{(2)}f_1f_2u_\lambda, f_3^{(2)}f_1f_2u_\lambda)\\
&=(1+2q^2+2q^4+q^6)(f_1f_2u_\lambda, f_1f_2u_\lambda)\\
&=(1+2q^2+2q^4+q^6)(f_2u_\lambda, f_2u_\lambda)\\
&=1+2q^2+2q^4+q^6
\end{aligned}$$

and

$$\begin{aligned}
&(f_4^{(2)}f_3f_1f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda, f_3f_4^{(2)}f_1f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda)\\
&=q^2(f_3f_1f_2^{(2)}f_3^{(2)}f_1f_2, f_3f_1f_2^{(2)}f_3^{(2)}f_1f_2u_\lambda)\\
&=q^2+q^4.
\end{aligned}$$

Next, we compute

$$\begin{aligned}
&(v_1, v_1)\\
&= (f_2^{(2)} f_3^{(2)} f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda, f_2^{(2)} f_3^{(2)} f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda)\\
&= (f_3^{(2)} f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda, f_3^{(2)} f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda)\\
&= (f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda, f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda)\\
&+ (q^2 + 2 + q^{-2})(f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda, f_3 f_4^{(2)} f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda)\\
&= (1 + 2q^2 + 2q^4 + q^6) + (1 + 3q^2 + 3q^4 + q^6)\\
&= 2 + 5q^2 + 5q^4 + 2q^6
\end{aligned}$$

and

$$\begin{aligned}
&(v_1, v_2)\\
&= (f_2^{(2)} f_3^{(2)} f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda, f_3 f_2^{(3)} f_3^{(2)} f_4^{(2)} f_1^{(2)} f_2 f_3^{(2)} f_2 u_\lambda)\\
&= (f_3^{(2)} f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda, f_3 f_2 f_3^{(2)} f_4^{(2)} f_1^{(2)} f_2 f_3^{(2)} f_2 u_\lambda)\\
&= (f_4^{(2)} f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda, f_2 f_3 f_4^{(2)} f_1^{(2)} f_2 f_3^{(2)} f_2 u_\lambda + f_3 f_4^{(2)} f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda)\\
&= (f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda, 0 + f_3 f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda)\\
&= (q^2 + q^4)(f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda, f_1 f_2^{(2)} f_3^{(2)} f_1 f_2 u_\lambda)\\
&= q^2 + q^4.
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
&(F(\pi, \underline{i})u_\lambda, F(\pi, \underline{i})u_\lambda)\\
&= (1 + q^2)(v_1, v_1 + v_2)\\
&= (1 + q^2)((2 + 5q^2 + 5q^4 + 2q^6) + (q^2 + q^4))\\
&= (1 + q^2)(2 + 6q^2 + 6q^4 + 2q^6)\\
&= 2 + 8q^2 + 12q^4 + 8q^6 + 2q^8.
\end{aligned}$$

Hence, by Theorem 2.17, the vector $F(\pi, \underline{i})u_\lambda$ is not an element of the canonical basis.

For the path π we further verified using OSCAR [44] that for any choice of $\underline{i}$ such that $s_{\underline{i}} = \iota(\pi)$, we have $F(\pi, \underline{i}) \neq G_\lambda(b_\pi)$.

3.6. **Proof of the conjecture for special cases.** In the previous section we verified Conjecture 3.9 for fundamental weights of classical type and for both fundamental weights in G_2. We now aim to use tensor products to lift these results to general dominant weights $\lambda = \lambda_1\varpi_1 + \cdots + \lambda_N\varpi_N$, where the dominant path is $\pi_{\lambda_1\varpi_1} * \cdots * \pi_{\lambda_N\varpi_N}$.

First, let us state some results on the canonical basis of tensor products, which we will need in the proofs of Propositions 3.27 and 3.28.

Lemma 3.24. *For integers $m, n \geq 0$ we the following identity:*

$$q^{n(m-n)} = \sum_{k=0}^{n} q^{-k(m+k)} q^{(n-k)(n-k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=k+1}^{n} (q^{m+j} - q^{-m-j}).$$

Proof. We prove the lemma by induction on n. For $n = 0$ the claim is trivial; thus, we may assume that $n > 0$ and that the claim holds for $n-1$. Using the q-binomial identity

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{-n+k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

we may split the sum into two parts. We will use the notation $\{t\} := q^t - q^{-t}$ for better readability.

For the first part we compute

$$\begin{aligned}
&\sum_{k=0}^{n-1} q^{-k(m+k)} q^{(n-k)(n-k-1)/2} q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} \prod_{j=k+1}^{n} \{m+j\} \\
&= q^{n-1}\{m+n\} \sum_{k=0}^{n-1} q^{-k(m+k)} q^{(n-k-1)(n-k-2)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} \prod_{j=k+1}^{n-1} \{m+j\} \\
&= q^{n-1}\{m+n\} q^{(n-1)(m+n-1)} \\
&= q^{n(m+n)} - q^{(n-2)(m+n)}.
\end{aligned}$$

For the second part we compute

$$\begin{aligned}
&\sum_{k=1}^{n} q^{-k(m+k)} q^{(n-k)(n-k-1)/2} q^{-n+k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \prod_{j=k+1}^{n} \{m+j\} \\
&= \sum_{k=0}^{n-1} q^{-(k+1)(m+k+1)} q^{(n-k-1)(n-k-2)/2} q^{-n+k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix} \prod_{j=k+2}^{n} \{m+j\} \\
&= q^{-(m+n)} \sum_{k=0}^{n-1} q^{-k(m+k+1)} q^{(n-k-1)(n-k-2)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} \prod_{j=k+1}^{n-1} \{m+j+1\} \\
&= q^{-(m+n)} q^{(n-1)(m+n)} \\
&= q^{(n-2)(m+n)}.
\end{aligned}$$

Adding both parts together gives the desired identity. □

Proposition 3.25. *Let λ and μ be dominant weights. Let $\psi\colon V(\lambda+\mu) \to V(\lambda) \otimes V(\mu)$ be the embedding given by $u_{\lambda+\mu} \mapsto u_\lambda \otimes u_\mu$, and $\psi_0\colon B(\lambda+\mu) \to B(\lambda) \otimes B(\mu)$ be the embedding given by $b_{\lambda+\mu} \mapsto b_\lambda \otimes b_\mu$. Let $b \in B(\lambda+\mu)$. Assume that $\psi_0(b) = b_1 \otimes b_2$ and write*

$$\psi(G_{\lambda+\mu}(b)) = \sum z_{b_1,b_2;b_1',b_2'} G_\lambda(b_1') \otimes G_\mu(b_2').$$

If $\varphi_i(b_1) > \epsilon_i(b_2)$, then

$$z_{\pi_1,\pi_2;\tilde{e}_i^n b_1, \tilde{f}_i^n b_2} = q_i^{n(\varphi_i(b_1)+n)} \begin{bmatrix} \epsilon_i(b_2)+n \\ n \end{bmatrix}_i,$$

and if $\varphi_i(b_1) \le \epsilon_i(b_2)$, then

$$z_{b_1,b_2;\tilde{e}_i^n b_1, \tilde{f}_i^n b_2} = q_i^{n(\epsilon_i(b_2)+n)} \begin{bmatrix} \varphi_i(b_1)+n \\ n \end{bmatrix}_i.$$

Proof. Let $\Psi\colon V(\lambda) \otimes V(\mu) \to V(\lambda) \otimes V(\mu)$ be the involution defined in (2.20). The coefficients $z_{b_1,b_2;b_1',b_2'}$ are uniquely determined by the requirements that $\psi(G_{\lambda+\mu}(b))$ is Ψ-invariant, $z_{b_1,b_2;b_1,b_2} = 1$, $z_{b_1,b_2;b_1',b_2'} \in q\mathbb{Z}[q]$ for $b_1' \otimes b_2' \neq b_1 \otimes b_2$, and

$z_{b_1,b_2;b_1',b_2'} = 0$ unless $G_\lambda(b_1') \otimes G_\mu(b_2') \blacktriangleleft G_\lambda(b_1) \otimes G_\mu(b_2)$. Therefore, we only need to consider the summands $\Theta_{-n\alpha_i}$ $(n \geq 0)$ of the quasi-$\mathcal{R}$-matrix (cf. Theorem 2.18).

We prove the case $\varphi_i(\pi_1) > \epsilon_i(\pi_2)$ by induction on n; the case $\varphi_i(\pi_1) \leq \epsilon_i(\pi_2)$ is similar. For $n = 0$ there is nothing to show; thus, let $n > 0$ and assume that the lemma holds for all $k \in [0, n-1]$. Note that due to the assumption $\varphi_i(\pi_1) > \epsilon_i(\pi_2)$, we have $q_i^{n(\varphi_i(b_1)+n)} \begin{bmatrix} \epsilon_i(b_2)+n \\ n \end{bmatrix}_i \in q\mathbb{Z}[q]$.

We claim that $z_{\pi_1,\pi_2;\tilde{e}_i^n b_1, \tilde{f}_i^n b_2} = q_i^{n(\varphi_i(b_1)+n)} \begin{bmatrix} \epsilon_i(b_2)+n \\ n \end{bmatrix}_i$. Indeed, under this choice the application of Ψ to $\psi(G_{\lambda+\mu}(b))$ gives the summand

$$\sum_{k=0}^{n} q_i^{-k(\varphi_i(b_1)+k)} q_i^{(n-k)(n-k-1)/2} \begin{bmatrix} \epsilon_i(b_2)+n \\ n-k \end{bmatrix}_i \begin{bmatrix} \epsilon_i(b_2)+k \\ k \end{bmatrix}_i$$
$$\prod_{j=k+1}^{n} (q_i^{\varphi_i(b_1)+j} - q_i^{-\varphi_i(b_1)-j}) G_\lambda(b_{\tilde{e}_i^n b_1}) \otimes G_\mu(b_{\tilde{f}_i^n b_2}).$$

Using the identitiy

$$\begin{bmatrix} \epsilon_i(b_2)+n \\ n-k \end{bmatrix}_i \begin{bmatrix} \epsilon_i(b_2)+k \\ k \end{bmatrix}_i = \begin{bmatrix} \epsilon_i(b_2)+n \\ n \end{bmatrix}_i \begin{bmatrix} n \\ k \end{bmatrix}_i.$$

and Lemma 3.24 the above sum becomes

$$q_i^{n(\varphi_i(b_1)+n)} \begin{bmatrix} \epsilon_i(b_2)+n \\ n \end{bmatrix}_i G_\lambda(b_{\tilde{e}_i^n b_1}) \otimes G_\mu(b_{\tilde{f}_i^n b_2}).$$

Now, the uniqueness of $z_{\pi_1,\pi_2;\tilde{e}_i^n b_1, \tilde{f}_i^n b_2}$ implies the claim. □

Lemma 3.26. *Let λ be a dominant weight and η be a dominant path of weight λ. Assume that Conjecture 3.15 holds for $V(\lambda)$ and η. Let $\pi \in B(\eta)$ and let $N \subseteq V(\lambda)$ be the minimal $\mathbb{Z}[q, q^{-1}]$-submodule such that $U_{\mathbb{Z}}^+(\mathfrak{g}) G_\lambda(b_\pi) \subseteq N$ and there exists a subset $B' \subseteq B(\eta)$ with $N = \bigoplus_{\pi' \in B'} \mathbb{Z}[q, q^{-1}] G_\lambda(b_{\pi'})$. Then, any $\pi' \in B'$ has the following property:*

Let $x \in [0, 1]$ be maximal such that $\pi'(t) - \pi(t)$ is a multiple of α_j for some $j \in I$ and all $t \in [0, x]$. Assume that $x < 1$ and let $\epsilon > 0$ be such that on $[x, x+\epsilon]$ the path π is supported on $\tau \in W$ and the path π' is supported on $\sigma \in W$. Then, $\sigma < \max\{\tau, s_j\tau\}$, and if $\pi'(x) - \pi(x) = 0$, then already $\sigma < \tau$.

Proof. Let $\pi'' \in B'$ and put $\xi := \mathrm{wt}(\pi'') - \mathrm{wt}(\pi)$. We prove the lemma by induction on $\mathrm{tr}(\xi)$. If $\mathrm{tr}(\xi) = 0$, then $\pi'' = \pi$ and $x = 1$, so there is nothing to show. Thus, we may assume that $\mathrm{tr}(\xi) > 0$ and that the claim holds for all $\pi' \in B'$ with $\mathrm{tr}(\mathrm{wt}(\pi') - \mathrm{wt}(\pi)) < \mathrm{tr}(\xi)$. By the definition of N, there must exist $i \in I$ and $\pi' \in B(\eta)_{\mathrm{wt}(\pi)+\xi-\alpha_i} \cap B'$ such that $\xi - \alpha_i \in Q_+$ and $E^i_{\pi',\pi''} \neq 0$.

Fix $j \in I$. Let $x \in [0, 1]$ be maximal such that $\pi''(t) - \pi(t)$ is a multiple of α_j for all $t \in [0, x]$. Let $x' \in [0, 1]$ be maximal such that $\pi'(t) - \pi(t)$ is a multiple of α_j for all $t \in [0, x']$. Assume that $x < 1$.

We claim that $x' \geq x$. Indeed, if this was not the case, then we could choose $\epsilon > 0$ such that $x' + \epsilon < x$ and on $[x', x' + \epsilon]$ the path π is supported on $\tau \in W$, the path π' is supported on $\sigma \in W$, the path π'' is supported on $\nu \in W$. Now, Conjecture 3.15 (ii) implies that $\nu < \max\{\sigma, s_j\sigma\}$ and the induction hypothesis that $\sigma < \max\{\tau, s_j\tau\}$. Hence, we have

$$\max\{\nu, s_j\nu\} < \max\{\sigma, s_j\sigma\} < \max\{\tau, s_j\tau\} = \max\{\nu, s_j\nu\},$$

where the inequalities are strict, since $\sigma \notin \{\nu, s_j\nu\} = \{\tau, s_j\tau\}$; a contradiction.

Thus, we have that $\pi'(t) - \pi(t)$ is a multiple of α_j for all $t \in [0, x]$. Let $\epsilon > 0$ be such that on $[x, x+\epsilon]$ the path π is supported on $\tau \in W$ and the path π' is supported on $\sigma \in W$, and the path π'' is supported on $\nu \in W$. If $x' > x$, we have that $\sigma \in \{\tau, s_j\tau\}$ and Conjecture 3.15 (ii) implies that $\nu < \max\{\sigma, s_j\sigma\} = \max\{\tau, s_j\tau\}$, since $\pi''(t) - \pi'(t)$ is a multiple of α_j for all $t \in [0, x]$. If $x' = x$, then either $\nu \in \{\sigma, s_j\sigma\}$, or we have $\nu < \max\{\sigma, s_j\sigma\}$ by Conjecture 3.15 (ii), since $\pi''(t) - \pi'(t)$ is a multiple of α_j for all $t \in [0, x]$. In addition, the induction hypothesis implies that $\sigma < \max\{\tau, s_j\tau\}$. Hence, $\nu \leq \max\{\sigma, s_j\sigma\} < \max\{\tau, s_j\tau\}$, where the second inequality is strict, since $\sigma \notin \{\tau, s_j\tau\}$.

If $\pi''(x) - \pi(x) = 0$, then we see from

$$0 = \pi''(x) - \pi(x) = (\pi''(x) - \pi'(x)) + (\pi'(x) - \pi(x))$$

that $\pi''(x) - \pi'(x) = 0$ and $\pi'(x) - \pi(x) = 0$ due to Conjecture 3.15 (i). If $x' > x$, we have either $\sigma = \tau$ or $\sigma = s_j\tau < \tau$ by Conjecture 3.15 (i) and thus $\nu < \sigma \leq \tau$ by Conjecture 3.15 (ii). If $x' = x$, we have either $\nu = \sigma$ or $\nu = s_j\sigma < \sigma$, by Conjecture 3.15 (i), or $\nu < \sigma$, by Conjecture 3.15 (ii). Hence, we have $\nu \leq \sigma < \tau$ by the induction hypothesis. □

Proposition 3.27. *Let λ and μ be dominant weights. Assume that Conjectures 3.9 and 3.15 hold for both $V(\lambda)$ and $V(\mu)$ with the respective dominant paths η_λ and η_μ. Then, Conjecture 3.9 also holds for $V(\lambda + \mu)$ and the path $\eta_\lambda * \eta_\mu$.*

Proof. Let $\pi_1 \in B(\eta_\lambda)$ and $\pi_2 \in B(\eta_\mu)$ such that $\pi_1 * \pi_2 \in B(\eta_\lambda * \eta_\mu)$. Let $\psi\colon V(\lambda+\mu) \to V(\lambda) \otimes V(\mu)$ be the embedding given by $u_{\lambda+\mu} \mapsto u_\lambda \otimes u_\mu$. We write

$$\psi(G_{\lambda+\mu}(b_{\pi_1*\pi_2})) = \sum z_{\pi_1,\pi_2;\pi_1',\pi_2'} G_\lambda(b_{\pi_1'}) \otimes G_\mu(b_{\pi_2'}).$$

Fix $\pi_1' \in B(\eta_\lambda)$ and $\pi_2' \in B(\eta_\mu)$ with $z_{\pi_1,\pi_2;\pi_1',\pi_2'} \neq 0$. Since Conjectures 3.9 and 3.15 hold for both $V(\lambda)$ and $V(\mu)$ with the respective dominant paths η_λ and η_μ, we have by Theorem 3.17 that $\pi_1' \succeq_+ \pi_1$ and $\pi_2'(t) \succeq \pi_2(t) - (\pi_1'(1) - \pi_1(1))$ for all $t \in [0, 1]$.

Proof of (i). Consider a fixed k in the sum on the right-hand side of

$$f_i^{(n)}(G_\lambda(b_{\pi_1'}) \otimes G_\mu(b_{\pi_2'})) = \sum_{k=0}^{n} q^{k(n-k)} t_i^k f_i^{(n-k)} G_\lambda(b_{\pi_1'}) \otimes f_i^{(k)} G_\mu(b_{\pi_2'}).$$

We have by Conjecture 3.9 for $V(\lambda)$ and η_λ that any $\pi_1'' \in B(\eta_\lambda)$ with $F^{i,n-k}_{\pi_1',\pi_1''} \neq 0$ fulfills $\pi_1''(t) - \pi_1'(t) \succeq -(n-k)\alpha_i$ for all $t \in [0, 1]$. Similarly, we have by Conjecture 3.9 for $V(\mu)$ and η_μ that any $\pi_2'' \in B(\eta_\mu)$ with $F^{i,k}_{\pi_2',\pi_2''} \neq 0$ fulfills $\pi_2''(t) - \pi_2'(t) \succeq -k\alpha_i$ for all $t \in [0, 1]$. Thus, we have for $t \in [0, 1/2]$ that

$$\begin{aligned}
&(\pi_1'' * \pi_2'')(t) - (\pi_1 * \pi_2)(t) \\
&= \pi_1''(2t) - \pi_1(2t) \\
&= (\pi_1''(2t) - \pi_1'(2t)) + (\pi_1'(2t) - \pi_1(2t)) \\
&\succeq -(n-k)\alpha_i,
\end{aligned}$$

and we have for $t \in [1/2, 1]$ that

$$\begin{aligned}
&(\pi_1'' * \pi_2'')(t) - (\pi_1 * \pi_2)(t) \\
&= \pi_1''(1) - \pi_1(1) + \pi_2''(2t-1) - \pi_2(2t-1) \\
&= (\pi_1''(1) - \pi_1'(1)) + (\pi_1'(1) - \pi_1(1)) \\
&\quad + (\pi_2''(2t-1) - \pi_2'(2t-1)) + (\pi_2'(2t-1) - \pi_2(2t-1)) \\
&\succeq -(n-k)\alpha_i + (\pi_1'(1) - \pi_1(1)) - k\alpha_i - (\pi_1'(1) - \pi_1(1)) \\
&= -n\alpha_i.
\end{aligned}$$

Proof of (ii). We assume $n = 1$ and prove the case $n > 1$ later by induction. Thus, we can write

$$f_i(G_\lambda(b_{\pi_1'}) \otimes G_\mu(b_{\pi_2'})) = \sum F^i_{\pi_1',\pi_2';\pi_1'',\pi_2''} G_\lambda(b_{\pi_1''}) \otimes G_\mu(b_{\pi_2''}).$$

Fix $\pi_1'' \in B(\eta_\lambda)$ and $\pi_2'' \in B(\eta_\mu)$ with $F^i_{\pi_1',\pi_2';\pi_1'',\pi_2''} \neq 0$.

Let $x' \in [0,1]$ be maximal such that $(\pi_1' * \pi_2')(t) - (\pi_1 * \pi_2)(t)$ is a multiple of α_i for all $t \in [0, x']$. Let $x'' \in [0,1]$ be maximal such that $(\pi_1'' * \pi_2'')(t) - (\pi_1 * \pi_2)(t)$ is a multiple of α_i for all $t \in [0, x'']$. A similar argument as in the proof of Lemma 3.26 shows that $x' \geq x''$.

Assume that $x'' < 1$ and let $\epsilon > 0$ be such that on $[x'', x'' + \epsilon]$ the path $\pi_1 * \pi_2$ is supported on $\tau \in W$ and the path $\pi_1' * \pi_2'$ is supported on $\sigma \in W$, and the path $\pi_1'' * \pi_2''$ is supported on $\nu \in W$.

First, consider $x'' < 1/2$. Let $\epsilon > 0$ be such that on $[2x'', 2x'' + 2\epsilon]$ the path π_1 is supported on $\tau \in W$ and the path π_2' is supported on $\sigma \in W$, and the path π_2'' is supported on $\nu \in W$. If $x' > x''$, we have that $\sigma \in \{\tau, s_i\tau\}$ and Conjecture 3.9 (ii) implies that $\nu < \max\{\sigma, s_i\sigma\} = \max\{\tau, s_i\tau\}$, since $\pi_1''(t) - \pi_1'(t)$ is a multiple of α_i for all $t \in [0, x'']$. If $x' = x''$, we have that $\sigma < \max\{\tau, s_i\tau\}$ by Lemma 3.26 and Conjecture 3.9 (ii) implies that $\nu \leq \max\{\sigma, s_i\sigma\} < \max\{\tau, s_i\tau\}$, where the second inequality comes from the fact that $\sigma \notin \{\tau, s_i\tau\}$.

If $(\pi_1'' * \pi_2'')(x'') - (\pi_1 * \pi_2)(x'') = -\alpha_i$, then we see from

$$\begin{aligned}
-\alpha_i &= (\pi_1'' * \pi_2'')(x'') - (\pi_1 * \pi_2)(x'') \\
&= (\pi_1''(2x'') + \pi_1'(2x'')) + (\pi_1'(2x'') - \pi_1(2x'')).
\end{aligned}$$

that $\pi_1''(x'') - \pi_1'(x'') = -\alpha_i$, by Conjecture 3.9 (i), and $\pi_1'(x'') - \pi_1(x'') = 0$, by Conjecture 3.15 (i). Now, similar arguments for $x' > x''$ and $x' = x''$ show $\nu < \tau$.

Second, consider $x'' \geq 1/2$. Then, $\pi_1''(1) - \pi_1(1)$ is a multiple of α_i and hence $\pi_1'(1) - \pi_1(1)$ must be as well. In particular, $G_\lambda(b_{\pi_1'}) \otimes G_\mu(b_{\pi_2'})$ is obtained from $G_\lambda(b_{\pi_1}) \otimes G_\mu(b_{\pi_2})$ via the action of

$$\Theta_{-k\alpha_i} = q_i^{k(k-1)/2} \prod_{j=1}^{k} (q_i^j - q_i^{-j}) e_i^{(k)} \otimes f_i^{(k)}$$

for some $k \geq 0$. This follows from the definitions of the quasi-$\mathcal{R}$-matrix (cf. Theorem 2.18), of the involution on $V(\lambda) \otimes V(\mu)$ (cf. (2.20)), and of the canonical basis of $V(\lambda) \otimes V(\mu)$ (cf. Theorem 2.23).

If $x' > x''$, we have that $\sigma \in \{\tau, s_i\tau\}$ and Conjecture 3.9 (ii) for $V(\mu)$ and η_μ implies that $\nu < \max\{\sigma, s_i\sigma\} = \max\{\tau, s_i\tau\}$, since $\pi_2''(t) - \pi_2'(t)$ is a multiple of α_i for all $t \in [0, 2x'' - 1]$. If $x' = x''$, we have that $\nu \leq \max\{\sigma, s_i\sigma\} < \max\{\tau, s_i\tau\}$,

where both inequalities follow from by Conjecture 3.9 (ii) for $V(\mu)$ and η_μ. For the second inequality note that $\sigma \notin \{\tau, s_i\tau\}$.

If $(\pi_1'' * \pi_2'')(x'') - (\pi_1 * \pi_2)(x'') = -\alpha_i$, then we see from

$$\begin{aligned}-\alpha_i &= (\pi_1'' * \pi_2'')(x'') - (\pi_1 * \pi_2)(x'') \\ &= ((\pi_1'' * \pi_2'')(x'') - (\pi_1' * \pi_2')(x'')) + ((\pi_1' * \pi_2')(x'') - (\pi_1 * \pi_2)(x''))\end{aligned}$$

that $(\pi_1'' * \pi_2'')(x'') - (\pi_1' * \pi_2')(x'') = -\alpha_i$, by the proof of (i) above, and $(\pi_1' * \pi_2')(x'') - (\pi_1 * \pi_2)(x'') = 0$, by Theorem 3.17. Hence, by the definition of $\pi_1'' * \pi_2''$, either $\pi_1'' = \pi_1'$ and $\pi_2''(2x''-1) - \pi_2'(2x''-1) = -\alpha_i$, or $\pi_2'' = \pi_2'$ and $\pi_1''(1) - \pi_1'(1) = -\alpha_i$. Furthermore, $\pi_2'(2x''-1) - \pi_2(2x''-1) = \pi_1(1) - \pi_1'(1) = -k\alpha_i$ (recall that $G_\mu(b_{\pi_2'})$ is obtained from the action of $\Theta_{-k\alpha_i}$). In both cases, similar arguments for $x' > x''$ and $x' = x''$ show $\nu < \tau$.

Finally, we prove $n > 1$ by induction. Let $\pi \in B(\eta_\lambda * \eta_\mu)$ and write

$$f_i^{(n-1)} G_{\lambda+\mu}(b_\pi) = \sum F_{\pi,\pi'}^{i,n-1} G_{\lambda+\mu}(b_{\pi'}).$$

Fix $\pi' \in B(\eta_\lambda * \eta_\mu)$ with $F_{\pi,\pi'}^{i,n-1} \neq 0$ and write

$$f_i G_{\lambda+\mu}(b_{\pi'}) = \sum F_{\pi',\pi''}^{i,1} G_{\lambda+\mu}(b_{\pi''}).$$

Fix $\pi'' \in B(\eta_\lambda * \eta_\mu)$ with $F_{\pi',\pi''}^{i,1} \neq 0$.

Let $y' \in [0,1]$ be maximal such that $\pi'(t) - \pi(t)$ is a multiple of α_i for all $t \in [0, y']$. Let $y'' \in [0,1]$ be maximal such that $\pi''(t) - \pi(t)$ is a multiple of α_i for all $t \in [0, y'']$. A similar argument as in the proof of Lemma 3.26 shows that $y' \geq y''$.

Assume that $y'' < 1$ and let $\epsilon > 0$ be such that on $[y'', y'' + \epsilon]$ the path π is supported on $\tau \in W$, the path π' is supported on $\sigma \in W$, and the path π'' is supported on $\nu \in W$. If $y'' < y'$, we may assume $y'' + \epsilon \leq y'$ and thus $\nu < \max\{\sigma, s_i\sigma\} = \max\{\tau, s_i\tau\}$ by Conjecture 3.9 (ii). If $y'' = y'$, then either $\nu \in \{\sigma, s_i\sigma\}$, or we have $\nu < \max\{\sigma, s_i\sigma\}$ by Conjecture 3.9 (ii), since $\pi''(t) - \pi'(t)$ is a multiple of α_i for all $t \in [0, y'']$. Hence, we obtain from the induction hypothesis that $\nu \leq \max\{\sigma, s_i\sigma\} < \max\{\tau, s_i\tau\}$.

If $\pi''(y'') - \pi(y'') = -n\alpha_i$, then we see from

$$-n\alpha_i = \pi''(y'') - \pi(y'') = (\pi''(y'') - \pi'(y'')) + (\pi'(y'') - \pi(y''))$$

that $\pi''(y'') - \pi'(y'') = -\alpha_i$ and $\pi'(y'') - \pi(y'') = -(n-1)\alpha_i$ due to Conjecture 3.9 (i). Now, similar arguments for $y'' < y'$ and $y'' = y'$ show $\nu < \tau$.

Proof of (iii). Let $t_0 \in [0,1]$ be maximal such that $\langle h_i, (\pi_1 * \pi_2)(t)\rangle$ is decreasing on $[0, t_0]$ and let $t_1 \in [t_0, 1]$ be maximal such that $\langle h_i, (\pi_1 * \pi_2)(t)\rangle$ is increasing on $[t_0, t_1]$ and $M := \langle h_i, (\pi_1 * \pi_2)(t_1)\rangle - \langle h_i, (\pi_1 * \pi_2)(t_0)\rangle \in \mathbb{Z}$. Let $n \leq M$, then $\tilde{f}_{i;1}^n \pi \neq 0$.

First, consider $t_1 \leq 1/2$. In this case $\tilde{f}_{i;1}^n(\pi_1 * \pi_2) = \tilde{f}_{i;1}^n \pi_1 * \pi_2$ and $\epsilon_{i;1}(\pi_1 * \pi_2) = \epsilon_{i;1}(\pi_1)$. Observe that in $\psi(G_{\lambda+\mu}(b_{\pi_1 * \pi_2}))$ the only summand from which the term $G_\lambda(b_{\tilde{f}_{i;1}^n \pi_1}) \otimes G_\mu(b_{\pi_2})$ can be obtained by the action of $f_i^{(n)}$ is $G_\lambda(b_{\pi_1}) \otimes G_\mu(b_{\pi_2})$. We obtain by Conjecture 3.9 and Corollary 3.11 that

$$\begin{aligned}&f_i^{(n)} G_\lambda(b_{\pi_1}) \otimes G_\mu(b_{\pi_2}) \\ &= \begin{bmatrix} \epsilon_{i;1}(\pi_1) + n \\ n \end{bmatrix}_i G_\lambda(b_{\tilde{f}_{i;1}^n \pi_1}) \otimes G_\mu(b_{\pi_2}) + \sum F_{\pi_1;\pi_1'}^{i,n} G_\lambda(b_{\pi_1'}) \otimes G_\mu(b_{\pi_2}),\end{aligned}$$

where the sum ranges over $\pi_1' \in B(\eta_\lambda)$ with $\pi_1' \lhd \tilde{f}_{i;1}^n \pi_1$.

Second, consider $t_1 > 1/2$. In this case $\tilde{f}_{i;1}^n(\pi_1 * \pi_2) = \tilde{f}_{i;1}^{n_1}\pi_1 * \tilde{f}_{i;1}^{n_2}\pi_2$ for some $n_1, n_2 \geq 0$ with $n_1 + n_2 = n$. Observe that $\langle h_i, \pi_1(t)\rangle$ has a unique local minimum; hence, $\tilde{e}_{i;1}^k \pi_1 = \tilde{e}_i^k \pi_1$ and $\tilde{f}_{i;1}^k \pi_1 = \tilde{f}_i^k \pi_1$ for all $k \geq 0$. Furthermore, note that in $\psi(G_{\lambda+\mu}(b_{\pi_1*\pi_2}))$ the only summands from which the term $G_\lambda(b_{\tilde{f}_{i;1}^{n_1}\pi_1}) \otimes G_\mu(b_{\tilde{f}_{i;1}^{n_2}\pi_2})$ can be obtained by the action of $f_i^{(n)}$ are $G_\lambda(b_{\tilde{e}_i^{n_2-k}\pi_1}) \otimes G_\mu(b_{\tilde{f}_{i;1}^{n_2-k}\pi_2})$ for $k \in [0, n_2]$. This follows from Theorem 2.10 for $G_\lambda(b_{\tilde{f}_{i;1}^{n_1}\pi_1})$ and from Conjecture 3.9 for $G_\mu(b_{\tilde{f}_{i;1}^{n_2}\pi_2})$. For the coefficient of $G_\lambda(b_{\tilde{e}_i^{n_2-k}\pi_1}) \otimes G_\mu(b_{\tilde{f}_{i;1}^{n_2-k}\pi_2})$ in $\psi(G_{\lambda+\mu}(b_{\pi_1*\pi_2}))$ we may use an analogue of Proposition 3.25, since

$$n_2 \leq \langle h_i, \pi_2(2t_1 - 1)\rangle - \langle h_i, \pi_2(\max\{0, 2t_0 - 1\})\rangle$$

and thus Conjecture 3.9 (iii) is applicable for $f_i^{(n_2-k)} G_\mu(b_{\pi_2})$.

If $t_0 < 1/2$, then $n_1 = \varphi_i(\pi_1)$, $n_2 = n - n_1$ and $\epsilon_{i;1}(\pi_1 * \pi_2) = \epsilon_{i;1}(\pi_1)$. Based on our previous considerations, we only need to consider the sum

$$\sum_{k=0}^{n} q_i^{(n_2-k)(\varphi_i(\pi_1)+n_2-k)} q_i^{k(n-k)} t_i^k f_i^{(n-k)} G_\lambda(b_{\tilde{e}_i^{n_2-k}\pi_1}) \otimes f_i^{(k)} G_\mu(b_{\tilde{f}_{i;1}^{n_2-k}\pi_2})$$

in $f_i^{(n)}\psi(G_{\lambda+\mu}(b_{\pi_1*\pi_2}))$. Hence, the coefficient of $G_\lambda(b_{\tilde{f}_{i;1}^{n_1}\pi_1}) \otimes G_\mu(b_{\tilde{f}_{i;1}^{n_2}\pi_2})$ is

$$\begin{aligned}
&\sum_{k=0}^{n} q_i^{k(n-k)} q_i^{(n_2-k)(n-k)} q_i^{-k(\epsilon_i(\pi_1)+n_1)} \begin{bmatrix} \epsilon_{i;1}(\pi_1) + n_1 \\ n-k \end{bmatrix}_i \begin{bmatrix} n_2 \\ k \end{bmatrix}_i \\
&= \sum_{k=0}^{n} q_i^{n_2(n-k)-k(\epsilon_i(\pi_1)+n_1)} \begin{bmatrix} \epsilon_{i;1}(\pi_1) + n_1 \\ n-k \end{bmatrix}_i \begin{bmatrix} n_2 \\ k \end{bmatrix}_i \\
&= \begin{bmatrix} \epsilon_{i;1}(\pi_1) + n \\ n \end{bmatrix}_i.
\end{aligned}$$

The last equality follows from Lemma 2.3.

If $t_0 \geq 1/2$, then $n_1 = 0$, $n_2 = n$ and $\epsilon_{i;1}(\pi_1 * \pi_2) = \epsilon_{i;1}(\pi_1) + \epsilon_{i;1}(\pi_2)$. Based on our previous considerations, we only need to consider the sum

$$\sum_{k=0}^{n} q_i^{(n-k)(\epsilon_{i;1}(\pi_2)+n-k)} q_i^{k(n-k)} t_i^k f_i^{(n-k)} G_\lambda(b_{\tilde{e}_i^{n-k}\pi_1}) \otimes f_i^{(k)} G_\mu(b_{\tilde{f}_{i;1}^{n-k}\pi_2})$$

in $f_i^{(n)}\psi(G_{\lambda+\mu}(b_{\pi_1*\pi_2}))$. Hence, the coefficient of $G_\lambda(b_{\pi_1}) \otimes G_\mu(b_{\tilde{f}_{i;1}^{n}\pi_2})$ is

$$\begin{aligned}
&\sum_{k=0}^{n} q_i^{(n-k)(\epsilon_{i;1}(\pi_2)+n-k)} q_i^{k(n-k)} q_i^{-k\epsilon_i(\pi_1)} \begin{bmatrix} \epsilon_{i;1}(\pi_1) \\ n-k \end{bmatrix}_i \begin{bmatrix} \epsilon_{i;1}(\pi_2) + n \\ k \end{bmatrix}_i \\
&= \sum_{k=0}^{n} q_i^{(n-k)(\epsilon_{i;1}(\pi_2)+n)-k\epsilon_i(\pi_1)} \begin{bmatrix} \epsilon_{i;1}(\pi_1) \\ n-k \end{bmatrix}_i \begin{bmatrix} \epsilon_{i;1}(\pi_2) + n \\ k \end{bmatrix}_i \\
&= \begin{bmatrix} \epsilon_{i;1}(\pi_1) + \epsilon_{i;1}(\pi_2) + n \\ n \end{bmatrix}_i.
\end{aligned}$$

The last equality follows from Lemma 2.3. □

Proposition 3.28. *Let λ and μ be dominant weights. Assume that Conjectures 3.9 and 3.15 hold for both $V(\lambda)$ and $V(\mu)$ with the respective dominant paths η_λ and η_μ. Then, Conjecture 3.15 also holds for $V(\lambda+\mu)$ and the path $\eta_\lambda * \eta_\mu$.*

Proof. Let $\pi_1 \in B(\eta_\lambda)$ and $\pi_2 \in B(\eta_\mu)$ such that $\pi_1 * \pi_2 \in B(\eta_\lambda * \eta_\mu)$. Let $\psi\colon V(\lambda+\mu) \to V(\lambda)\otimes V(\mu)$ be the embedding given by $u_{\lambda+\mu} \mapsto u_\lambda \otimes u_\mu$. We write

$$\psi(G_{\lambda+\mu}(b_{\pi_1*\pi_2})) = \sum z_{\pi_1,\pi_2;\pi_1',\pi_2'} G_\lambda(b_{\pi_1'}) \otimes G_\mu(b_{\pi_2'}).$$

Fix $\pi_1' \in B(\eta_\lambda)$ and $\pi_2' \in B(\eta_\mu)$ with $z_{\pi_1,\pi_2;\pi_1',\pi_2'} \neq 0$. Since Conjectures 3.9 and 3.15 hold for both $V(\lambda)$ and $V(\mu)$ with the respective dominant paths η_λ and η_μ, we have by Theorem 3.17 that $\pi_1' * \pi_2' \succeq_+ \pi_1 * \pi_2$.

Proof of (i). We have by Conjecture 3.15 (i) that

$$\begin{aligned} e_i(G_\lambda(b_{\pi_1'}) \otimes G_\mu(b_{\pi_2'})) &= e_i G_\lambda(b_{\pi_1'}) \otimes t_i^{-1} G_\mu(b_{\pi_2'}) + G_\lambda(b_{\pi_1'}) \otimes e_i G_\mu(b_{\pi_1'}) \\ &= \sum E^i_{\pi_1',\pi_2';\pi_1'',\pi_2''} G_\lambda(b_{\pi_1''}) \otimes G_\mu(b_{\pi_2''}), \end{aligned}$$

where $E^i_{\pi_1,\pi_2;\pi_1',\pi_2'} \in \mathbb{Z}[q,q^{-1}]$ and the sum ranges over all $\pi_1'' \in B(\eta_\lambda)$ and $\pi_2'' \in B(\eta_\mu)$ with $\pi_1'' * \pi_2'' \succ_+ \pi_1' * \pi_2'$. Hence, $\pi_1'' * \pi_2'' \succ_+ \pi_1 * \pi_2$ for such π_1'' and π_2''.

Proof of (ii). Fix $j \in I$, and $\pi_1'' \in B(\eta_\lambda)$ and $\pi_2'' \in B(\eta_\mu)$ with $E^i_{\pi_1',\pi_2';\pi_1'',\pi_2''} \neq 0$.

Let $x' \in [0,1]$ be maximal such that $(\pi_1' * \pi_2')(t) - (\pi_1 * \pi_2)(t)$ is a multiple of α_j for all $t \in [0,x']$. Let $x'' \in [0,1]$ be maximal such that $(\pi_1'' * \pi_2'')(t) - (\pi_1 * \pi_2)(t)$ is a multiple of α_j for all $t \in [0,x'']$. A similar argument as in the proof of Lemma 3.26 shows that $x' \geq x''$.

Assume that $x'' < 1$ and let $\epsilon > 0$ be such that on $[x'', x''+\epsilon]$ the path $\pi_1 * \pi_2$ is supported on $\tau \in W$ and the path $\pi_1' * \pi_2'$ is supported on $\sigma \in W$, and the path $\pi_1'' * \pi_2''$ is supported on $\nu \in W$.

First, consider $x'' < 1/2$. Here $\nu < \max\{\tau, s_j\tau\}$ is an immediate consequence of Lemma 3.26 for $V(\lambda)$ and η_λ. If $(\pi_1'' * \pi_2'')(x'') - (\pi_1 * \pi_2)(x'') = 0$, then

$$\pi_1''(2x'') - \pi_1(2x'') = (\pi_1'' * \pi_2'')(x'') - (\pi_1 * \pi_2)(x'') = 0.$$

Hence $\nu < \tau$ by Lemma 3.26 for $V(\lambda)$ and η_λ.

Second, consider $x'' \geq 1/2$. Then, $\pi_1''(1) - \pi_1(1)$ is a multiple of α_j and hence $\pi_1'(1) - \pi_1(1)$ must be as well. In particular, $G_\lambda(b_{\pi_1'}) \otimes G_\mu(b_{\pi_2'})$ is obtained from $G_\lambda(b_{\pi_1}) \otimes G_\mu(b_{\pi_2})$ via the action of $\Theta_{-k\alpha_j}$ for some $k \geq 0$. If $x' > x''$, we have that $\sigma \in \{\tau, s_j\tau\}$ and Conjecture 3.15 (ii) for $V(\mu)$ and η_μ implies that $\nu < \max\{\sigma, s_j\sigma\} = \max\{\tau, s_j\tau\}$, since $\pi_2''(t) - \pi_2'(t)$ is a multiple of α_j for all $t \in [0, 2x''-1]$. If $x' = x''$, we have that $\sigma < \max\{\tau, s_j\tau\}$ by Conjecture 3.9 (ii) for $V(\mu)$ and η_μ and Conjecture 3.15 (ii) for $V(\mu)$ and η_μ implies that $\nu \leq \max\{\sigma, s_j\sigma\} < \max\{\tau, s_j\tau\}$, where the second inequality comes from the fact that $\sigma \notin \{\tau, s_j\tau\}$.

If $(\pi_1'' * \pi_2'')(x'') - (\pi_1 * \pi_2)(x'') = 0$, then we see from

$$\begin{aligned} 0 &= (\pi_1'' * \pi_2'')(x'') - (\pi_1 * \pi_2)(x'') \\ &= ((\pi_1'' * \pi_2'')(x'') - (\pi_1' * \pi_2')(x'')) + ((\pi_1' * \pi_2')(x'') - (\pi_1 * \pi_2)(x'')) \end{aligned}$$

that $(\pi_1'' * \pi_2'')(x'') - (\pi_1' * \pi_2')(x'') = 0$, by the proof of (i) above, and $(\pi_1' * \pi_2')(x'') - (\pi_1 * \pi_2)(x'') = 0$, by Theorem 3.17. By the definition of $\pi_1'' * \pi_2''$, either $\pi_1'' = \pi_1'$ or $\pi_2'' = \pi_2'$. In the latter case, we would have $\pi_1''(1) - \pi_1'(1) = \alpha_j$, which is not possible under our current assumptions. Hence, $\pi_1'' = \pi_1'$ and $\pi_2''(2x''-1) - \pi_2'(2x''-1) = 0$. Furthermore, $\pi_2'(2x''-1) - \pi_2(2x''-1) = \pi_1(1) - \pi_1'(1) = -k\alpha_j$ (recall that $G_\mu(b_{\pi_2'})$

is obtained from the action of $\Theta_{-k\alpha_j}$). Now, similar arguments for $x' > x''$ and $x' = x''$ show $\nu < \tau$. $\square$

A special case of Propositions 3.27 and 3.28 is for the tensor product $V(m\lambda) \otimes V(n\lambda)$. In this case, it is possible to give an explicit description of the embedding $V((m+n)\lambda) \hookrightarrow V(m\lambda) \otimes V(n\lambda)$ in terms of LS paths.

Lemma 3.29. *Let λ be a dominant weight and $m, n \geq 1$. The (unique) embedding of crystals*

$$B((m+n)\lambda) \hookrightarrow B(m\lambda) \otimes B(n\lambda)$$

sending $b_{(m+n)\lambda}$ to $b_{m\lambda} \otimes b_{n\lambda}$ is in the language of LS paths given by

$$(\tau_1, \ldots, \tau_r; t_1, \ldots, t_r) \mapsto (\tau_1, \ldots, \tau_r; t'_1, \ldots, t'_r) \otimes (\tau_1, \ldots, \tau_r; t''_1, \ldots, t''_r)$$

where $t'_k = \min\{(m+n)t_k/m, 1\}$ and $t''_k = ((m+n)t_k - mt'_k)/n$.

Proof. We verify that $\pi' := (\tau_1, \ldots, \tau_r; t'_1, \ldots, t'_r)$ and $\pi'' := (\tau_1, \ldots, \tau_r; t''_1, \ldots, t''_r)$ are LS paths of shape $m\lambda$ and $n\lambda$, respectively.

Let $\tau_1 = w_1 > \cdots > w_{p+1} = \mathrm{id}$ be a maximal chain on which π is supported, and let $\beta_1, \ldots, \beta_p$ be positive roots such that $w_{k+1} = \beta_k w_k$ for $k \in [1, p]$. Assume $t_1 < m/(m+n)$, then we have for π':

$$t'_1 \langle h_{\beta_1}, \tau_1 m\lambda \rangle = \frac{(m+n)t_1}{m} \langle h_{\beta_1}, \tau_1 m\lambda \rangle = t_1 \langle h_{\beta_1}, \tau_1 (m+n)\lambda \rangle \in \mathbb{Z}.$$

The remaining integrality constraints for π' follow similarly. Assume $t_1 \geq m/(m+n)$, then we have π'':

$$\begin{aligned} t''_1 \langle h_{\beta_1}, \tau_1 n\lambda \rangle &= \frac{(m+n)t_1 - mt'_1}{n} \langle h_{\beta_1}, \tau_1 n\lambda \rangle \\ &= t_1 \langle h_{\beta_1}, \tau_1 (m+n)\lambda \rangle - t'_1 \langle h_{\beta_1}, \tau_1 m\lambda \rangle \in \mathbb{Z}. \end{aligned}$$

The remaining integrality constraints for π'' follow similarly. Thus, both π' and π'' are LS paths of shape $m\lambda$ and $n\lambda$, respectively.

Let $\psi \colon B(\pi_{(m+n)\lambda}) \to B(\pi_{m\lambda}) \otimes B(\pi_{n\lambda})$ denote the map $\pi \mapsto \pi' \otimes \pi''$. Since the embedding is unique and clearly $\pi_{(m+n)\lambda} \mapsto \pi_{m\lambda} \otimes \pi_{n\lambda}$, it is sufficient to show that ψ commutes with $\tilde{e}_i$ and $\tilde{f}_i$ for all $i \in I$.

Fix $i \in I$. Let t_{k^-} be the last time the global minimum of $\langle h_i, \pi(t) \rangle|_{[0, m/(m+n)]}$ is attained and let t_{k^+} be the last time the global minimum of $\langle h_i, \pi(t) \rangle|_{[m/(m+n), 1]}$ is attained. Now,

$$\epsilon_i(\pi') = -\langle h_i, \pi(t_{k^-}) \rangle \quad \text{and} \quad \varphi_i(\pi') = \langle h_i, \pi(m/(m+n)) - \pi(t_{k^-}) \rangle$$

and

$$\epsilon_i(\pi'') = \langle h_i, \pi(m/(m+n)) - \pi(t_{k^+}) \rangle \quad \text{and} \quad \varphi_i(\pi'') = \langle h_i, \pi(1) - \pi(t_{k^+}) \rangle.$$

From this we can clearly see that $\varphi_i(\pi') > \epsilon_i(\pi'')$ is equivalent to $\pi(t_{k^-}) < \pi(\tau_{k^+})$, and hence $\psi(\tilde{f}_i \pi) = \tilde{f}_i(\pi' \otimes \pi'')$ and $\psi(\tilde{e}_i \pi) = \tilde{e}_i(\pi' \otimes \pi'')$. $\square$

From the explicit description of the embedding in Lemma 3.29 we see that the partial order $\triangleright$ on $B(\pi_{(m+n)\lambda})$ is equivalent to the partial order $\triangleright$ on $B(\pi_{m\lambda} * \pi_{n\lambda})$. Thus, we have the following result.

Corollary 3.30. *Let λ be a dominant weight. Assume that Conjectures 3.9 and 3.15 hold for $V(\lambda)$ and the dominant path π_λ. Then, both conjectures also hold for $V(m\lambda)$ and the dominant path $\pi_{m\lambda}$ for all $m \geq 1$.*

Theorem 3.31. *Let $\mathfrak{g}$ be a classical Lie algebra or of type G_2 and let $\lambda = m_1\varpi_1 + \cdots + m_N\varpi_N$ be a dominant weight. Fix the dominant path $\eta := \pi_{m_1\varpi_1} * \cdots * \pi_{m_N\varpi_N}$. Then, there is a unitriangular base change from path monomial basis $\mathcal{P}(\eta)$ to the canonical basis of $V_{\mathbb{Z}}(\lambda)$ for any linearisation of $\triangleright$ on $B(\eta)$.*

Proof. If λ is a fundamental weight, Lemmas 3.19 and 3.20 imply that Conjectures 3.9 and 3.15 hold for π_λ. Applying Corollary 3.30 we see that both conjectures hold for $V(m\varpi_i)$ and the path $\pi_{m\varpi_i}$ for any $i \in I$ and $m \geq 0$. Using Proposition 3.27, we conclude that, in particular, Conjecture 3.9 holds for $V(\lambda)$ and η. Now, this theorem is a consequence of Proposition 3.14. □

3.7. **Paths at infinity.** In the previous sections we discussed how to construct a basis of an integrable highest weight module $V(\lambda)$ using the combinatorics of LS paths. Now, we want to briefly show how to extend this construction to obtain a basis of $U_q^-(\mathfrak{g})$.

Let $\underline{\eta} = (\eta_1, \eta_2, \dots)$ be an infinite sequence of dominant paths, such that

$$\sum_{t=1}^{\infty} \langle h_i, \operatorname{wt} \eta_t \rangle = \infty \quad \text{for all } i \in I.$$

For $\underline{\eta}$ we construct the crystal $B(\underline{\eta})$ in the following. For $k \leq l$ we have an injective map

$$\phi_{k,l} \colon B(\eta_1 * \cdots * \eta_k) \to B(\eta_1 * \cdots * \eta_l), \quad \pi \mapsto \pi * \eta_{k+1} * \cdots * \eta_l.$$

As a set, the crystal $B(\underline{\eta})$ is defined as

$$\varinjlim B(\eta_1 * \cdots * \eta_k).$$

We consider the elements of $B(\underline{\eta})$ as infinite sequences $\underline{\pi} = (\pi_1, \pi_2, \dots)$ of paths with $\pi_1 \in B(\eta_1)$, $\pi_2 \in B(\eta_2)$, etc. For $\underline{\pi}$ let k be minimal such that $\pi_t = \eta_t$ for all $t \geq k$ and put $k_i := \min\{t \geq k \mid \langle h_i, \operatorname{wt}(\eta_t) \rangle > 0\}$. The crystal structure on $B(\underline{\eta})$ is then given by

$$\epsilon_i(\underline{\pi}) = \epsilon_i(\pi_1 * \cdots * \pi_k), \quad \varphi_i(\underline{\pi}) = \epsilon_i(\underline{\pi}) + \operatorname{wt}(\underline{\pi}),$$

$$\operatorname{wt}(\underline{\pi}) = \sum_{t=1}^{\infty} \operatorname{wt}(\pi_t) - \operatorname{wt}(\eta_t),$$

and

$$\tilde{e}_i\underline{\pi} = \begin{cases} 0, & \text{if } \epsilon_i(\underline{\pi}) = 0 \\ (\pi_1^-, \dots, \pi_k^-, \eta_{k+1}, \dots), & \text{if } \epsilon_i(\underline{\pi}) > 0, \end{cases}$$

where $\pi_1^-, \dots, \pi_k^-$ are defined by $\pi_1^- * \cdots * \pi_k^- = \tilde{e}_i(\pi_1 * \cdots * \pi_k)$; and

$$\tilde{f}_i\underline{\pi} = (\pi_1^+, \dots, \pi_{k_i}^+, \eta_{k_i+1}, \dots),$$

where $\pi_1^+, \dots, \pi_{k_i}^+$ are defined by $\pi_1^+ * \cdots * \pi_{k_i}^+ = \tilde{f}_i(\pi_1 * \cdots * \pi_{k_i})$.

Theorem 3.32. *The crystal $B(\underline{\eta})$ is isomorphic to $B(\infty)$.*

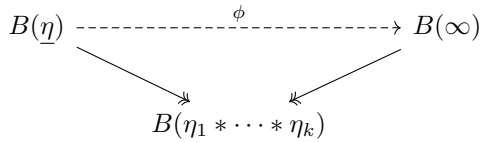
Proof. By definition, any element of $B(\underline{\eta})$ and $B(\infty)$ can be written as $\tilde{f}_{i_1} \dots \tilde{f}_{i_r}\underline{\eta}$ and $\tilde{f}_{i_1} \dots \tilde{f}_{i_r} b_\infty$, respectively. We claim that $\phi \colon B(\underline{\eta}) \to B(\infty)$ defined by

$$\tilde{f}_{i_1} \dots \tilde{f}_{i_r}\underline{\eta} \mapsto \tilde{f}_{i_1} \dots \tilde{f}_{i_r} b_\infty.$$

is a (well-defined) crystal isomorphism. To prove this it is sufficient to show that

$$\tilde{f}_{i_1} \dots \tilde{f}_{i_r} b_\infty = \tilde{f}_{j_1} \dots \tilde{f}_{j_s} b_\infty,$$

if and only if $\tilde{f}_{i_1} \dots \tilde{f}_{i_r} \underline{\eta} = \tilde{f}_{j_1} \dots \tilde{f}_{j_s} \underline{\eta}$. However, this is clear, since ϕ is the unique map such that the diagram



commutes for all $k \geq 1$. Here, the projection $B(\underline{\eta}) \to B(\eta_1 * \cdots * \eta_k)$, is given by

$$\underline{\pi} \mapsto \begin{cases} \pi_1 * \cdots * \pi_k, & \text{if } \pi_t = \eta_t \text{ for all } t > k, \\ 0, & \text{otherwise.} \end{cases}$$

□

Remark 3.33. A similar approach to construct a "path model" for $U_q^-(\mathfrak{g})$ is due to Li and Zhang [31]. They considered piecewise linear paths $\pi\colon [0,\infty) \to P_{\mathbb{R}}$ and showed that for a dominant path such that $\lim_{t\to\infty} \langle h_i, \pi(t)\rangle = \infty$ for all $i \in I$ one obtains a crystal isomorphic to $B(\infty)$.

The construction of path monomials and the partial order $\triangleright$ extend to $B(\underline{\eta})$ in a straightforward way. For the partial order on $B(\infty)$ induced by the isomorphism $B(\underline{\eta}) \cong B(\infty)$ we use the notation $\triangleright_{\underline{\eta}}$.

The following result is then a simple consequence of Proposition 3.27.

Theorem 3.34. *Let $\underline{\eta} = (\eta_1, \eta_2, \dots)$ be an infinite sequence of dominant paths. Assume that Conjectures 3.9 and 3.15 hold for $V(\operatorname{wt} \eta_k)$ and the dominant path η_k for all $k \geq 1$ and $\sum_{t=1}^{\infty} \langle h_i, \operatorname{wt} \eta_t\rangle = \infty$ for all $i \in I$. Then, the set $\mathcal{P}(\underline{\eta}) := \{F(\underline{\pi}) \mid \underline{\pi} \in B(\underline{\eta})\}$ is a basis of $U_q^-(\mathfrak{g})$ and there is a unitriangular base change from this basis to the canonical basis of $U_q^-(\mathfrak{g})$ for any linearisation of $\triangleright$ on $B(\underline{\eta})$.*

Corollary 3.35. *Assume that $\mathfrak{g}$ is a classical Lie algebra or of type G_2. Let $\underline{\eta} = (\varpi_1, \dots, \varpi_N, \varpi_1, \dots, \varpi_N, \dots)$, where N is the rank of $\mathfrak{g}$. Then, there is a unitriangular base change from $\mathcal{P}(\eta)$ to the canonical basis of $U_q^-(\mathfrak{g})$ for any linearisation of $\triangleright$ on $B(\underline{\eta})$.*

4. Lusztig's tight monomial cone revisited

Lusztig [40] investigated (in the simply-laced case) under which conditions a monomial of $U_q^-(\mathfrak{g})$ belongs to the canonical basis. He called a monomial *tight* if it is an element of the canonical basis, and *semi-tight* if it is an $\mathbb{N}$-linear combination of elements of the canonical basis. His examples showed that in A_2 and A_3 a monomial is tight if its exponent vector belongs to the cone $L_{\underline{i}}$ for some sequence $\underline{i}$ in I such that $s_{\underline{i}}$ is a reduced decomposition of w_0. He thus posed the question under which circumstances the monomial $f_{\underline{i}}^{(\underline{a})}$ for $\underline{a} \in L_{\underline{i}}$ belongs to the canonical basis. First evidence that not all $\underline{a} \in L_{\underline{i}}$ have this property was provided by Marsh in [42]. Later, Reineke [47] showed for A_6 that there is, in fact, a 6-dimensional region in which the property does not hold.

Independent of the original question, Lusztig cones were studied in other contexts. For one, Caldero, Marsh and Morier-Genoud [9] showed that the cone is simplicial and parametrizes a basis for adapted algebras, that is, it is closely related to cluster algebras. In another case, Fang, Fourier and Reineke [12] discovered that the quantum degree cone embeds into the negative Lusztig cone.

Let $\underline{i} = (i_1, \ldots, i_r)$ be a sequence in I. The *Lusztig cone* $L_{\underline{i}} \subseteq \mathbb{Z}^r_{\geq 0}$ is the cone consisting of tuples $(a_1, \ldots, a_r) \in \mathbb{Z}^r_{\geq 0}$ such that for every pair $1 \leq p < q \leq r$ with $i_p = i_q := i$ we have

$$a_p + a_q + \sum_{p<k<q} \langle h_i, \alpha_{i_k} \rangle a_k \leq 0. \tag{4.1}$$

We refer to this as the *Lusztig cone condition*.

First, we present an alternative motivation for the cone condition by deriving it from a quantum analogue of the adjoint action.

Second, we investigate under which conditions on the Lusztig cone $L_{\underline{i}}$ the sequence $\underline{i}$ gives rise to a reduced decomposition $s_{\underline{i}}$. Furthermore, we show that if $s_{\underline{i}}$ is a reduced decomposition, then $L_{\underline{i}}$ is contained in the adapted string cone $S_{\underline{i}}(\infty)$. This leads to the following observation: If $\underline{a} \in L_{\underline{i}}$ is such that $(f_{\underline{i}}^{(\underline{a})}, f_{\underline{i}}^{(\underline{a})}) \in 1 + qA$, then $f_{\underline{i}}^{(\underline{a})}$ is an element of the canonical basis.

Third, we study the duals of Leclerc's imaginary vectors from Section 2 of [30]. This gives rise to an interesting family of elements in the canonical basis, which serve as counterexamples to various conjectures. As an application, we show in Section 5.2 that in general the quantum Frobenius morphism and its splitting do not preserve the canonical basis.

Fourth and finally, we give examples of semi-tight monomials arising from Lusztig cones for A_5, B_3, C_3, D_4 and G_2. These examples complement the results of Reineke [47].

4.1. Alternative motivation for Lusztig cones.

In this section we fix a sequence $\underline{i} = (i_1 \ldots, i_r)$ in I and an element $i \in I$. For $\varepsilon \in \{\pm 1\}$ and integers n and $a_1, \ldots, a_r$ we define

$$F_{n;a_1,\ldots,a_r;\varepsilon} = \sum_{k+k'=n} (-1)^k q_i^{\varepsilon k \eta_{n;a_1,\ldots,a_r}} f_i^{(k)} f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} f_i^{(k')}. \tag{4.2}$$

where $\eta_{n;a_1,\ldots,a_r} = n + \sum_{j=1}^r \langle h_i, \alpha_{i_j} \rangle a_j - 1$.

Recall from [38, Chapter 7] that for $r = 1$, $a_1 \geq 1$ and $n > -\langle h_i, \alpha_{i_1} \rangle a_1$ we have the so called *higher order quantum Serre relations* $F_{n;a_1;\epsilon} = 0$. We show that the

results and arguments of *loc.cit.* can be generalized to $F_{n;a_1,\dots,a_r;\varepsilon}$. This gives rise to (4.1) in a natural way.

Lemma 4.1. *Suppose that $\eta_{n;a_1,\dots,a_r}=0$, then for any $j\in I$ we have*

$$\begin{aligned}&e_jF_{n;a_1,\dots,a_r;\varepsilon}-F_{n;a_1,\dots,a_r;\varepsilon}e_j\\&=\sum_{i_t=j}\frac{q_j^{a_t-1+\eta[t]}}{q_j-q_j^{-1}}t_jF_{n;a_1,\dots,a_t-1,\dots,a_r;-1}-\frac{q_j^{-a_t+1-\eta[t]}}{q_j-q_j^{-1}}t_j^{-1}F_{n;a_1,\dots,a_t-1,\dots,a_r;+1}\end{aligned}$$

where $\eta[t]:=n+\sum_{j=1}^{t-1}\langle h_i,\alpha_{i_j}\rangle a_j-1$.

Proof. We use the commutation relation [38, Corollary 3.1.9]

$$e_jf_j^{(n)}=f_j^{(n)}e_j+\frac{q_j^{n-1}t_j-q_j^{-n+1}t_j^{-1}}{q_j-q_j^{-1}}f_j^{(n-1)}$$

on $e_jF_{n;a_1,\dots,a_r;\varepsilon}-F_{n;a_1,\dots,a_r;\varepsilon}e_j$. If $j=i$, we obtain from the commutation with $f_i^{(k)}$ and $f_i^{(k')}$ the term

$$\begin{aligned}&\sum_{k+k'=n}(-1)^k\frac{q_i^{k-1}t_i-q_i^{-k+1}t_i^{-1}}{q_i-q_i^{-1}}f_i^{(k-1)}f_{i_1}^{(a_1)}\cdots f_{i_r}^{(a_r)}f_i^{(k')}\\&+\sum_{k+k'=n}(-1)^k\frac{q_i^{k}t_i-q_i^{-k}t_i^{-1}}{q_i-q_i^{-1}}f_i^{(k)}f_{i_1}^{(a_1)}\cdots f_{i_r}^{(a_r)}f_i^{(k'-1)}.\end{aligned}$$

Performing the index shift $k\mapsto k+1$ in the first sum and $k'\mapsto k'+1$ in the second sum, we see obtain

$$\begin{aligned}&\sum_{k+k'=n-1}(-1)^{k+1}\frac{q_i^{k}t_i-q_i^{-k}t_i^{-1}}{q_i-q_i^{-1}}f_i^{(k)}f_{i_1}^{(a_1)}\cdots f_{i_r}^{(a_r)}f_i^{(k')}\\&+\sum_{k+k'=n-1}(-1)^{k}\frac{q_i^{k}t_i-q_i^{-k}t_i^{-1}}{q_i-q_i^{-1}}f_i^{(k)}f_{i_1}^{(a_1)}\cdots f_{i_r}^{(a_r)}f_i^{(k')}.\end{aligned}$$

Noting the opposite signs, we see that the sums above cancel each other.

Further, for each $t\in[1,r]$ with $i_t=j$ we get

$$\begin{aligned}&\sum_{k+k'=n}(-1)^k\frac{q_j^{a_t-1+\langle h_j,\alpha_i\rangle k+\eta[t]}t_j-q_j^{-a_t+1-\langle h_j,\alpha_i\rangle k-\eta[t]}t_j^{-1}}{q_j-q_j^{-1}}\\&\qquad\times f_i^{(k)}f_{i_1}^{(a_1)}\cdots f_{i_t}^{(a_t-1)}\cdots f_{i_r}^{(a_r)}f_i^{(k')}.\end{aligned}$$

By assumption on $\eta_{n;a_1,\dots,a_r}$, we have

$$0=\eta_{n;a_1,\dots,a_t,\dots,a_r}=\eta_{n;a_1,\dots,a_t-1,\dots,a_r}+\langle h_i,\alpha_j\rangle.$$

Therefore, observing that

$$q_i^{k\eta_{n;a_1,\dots,a_t-1,\dots,a_r}}=q_i^{-k\langle h_i,\alpha_j\rangle}=q_j^{-k\langle h_j,\alpha_i\rangle}$$

yields the desired identity. □

Lemma 4.2. *We have the following relation:*

$$-q_i^{\varepsilon(\eta_{n;a_1,\dots,a_r}+n+1)}f_iF_{n;a_1,\dots,a_r;\varepsilon}+F_{n;a_1,\dots,a_r;\varepsilon}f_i=[n+1]_iF_{n+1;a_1,\dots,a_r;\varepsilon}.$$

Proof. We will write $\eta := \eta_{n;a_1,\dots,a_r}$ for simplicity. The left-hand side is equal to

$$\sum_{k+k'=n} (-1)^k q_i^{\varepsilon k\eta}(-q_i^{\varepsilon(\eta+n+1)}[k+1]_i f_i^{(k+1)} f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} f_i^{(k')} \\ + [k'+1]_i f_i^{(k)} f_{i_1}^{(a_1)} \dots f_{i_r}^{(a_r)} f_i^{(k'+1)})$$

Performing an index shift $k \mapsto k-1$, one observes for the powers of q_i that

$$q_i^{\varepsilon(k-1)\eta} q_i^{\varepsilon(\eta+(n-1)+1)} = q_i^{\varepsilon k\eta} q_i^{\varepsilon(k+k')} = q_i^{\varepsilon k(\eta+1)} q_i^{\varepsilon k'}.$$

Note that $n = k+k'$, thus n shifts with k. After a similar computation for the shift $k' \mapsto k'-1$, we obtain

$$\sum_{k+k'=n+1} (-1)^k q_i^{\varepsilon k(\eta+1)}(q_i^{\varepsilon k'}[k]_i + q_i^{-\varepsilon k}[k']) f_i^{(k)} f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} f_i^{(k')} \\ = [n+1]_i F_{n+1;a_1,\dots,a_r;\varepsilon}.$$

Here we used that $\eta + 1 = \eta_{n;a_1,\dots,a_r} + 1 = \eta_{n+1;a_1,\dots,a_r}$. □

Proposition 4.3 ([38, Proposition 3.2.7])**.** *Assume that $u \in U_q^-(\mathfrak{g})$ commutes with e_i for all $i \in I$. Then u is a multiple of 1.*

Proposition 4.4. *We have $F_{n;a_1,\dots,a_r;\varepsilon} = 0$, if $\eta_{n;a_1,\dots,a_r} > -1$.*

Proof. First, consider the case $\eta_{n;a_1,\dots,a_r} = 0$. Clearly, the statement is true if any $a_t < 0$ for $t \in [1,r]$; hence, we may assume $a_t \geq 0$. Suppose that the proposition holds when replacing any a_t with $a_t - 1$. Then, it follows from Lemma 4.1 that $F_{n;a_1,\dots,a_r;\epsilon}$ commutes with e_j for any $j \in I$, since

$$\eta_{n;a_1,\dots,a_t-1,\dots,a_r} \geq \eta_{n;a_1,\dots,a_r} > -1.$$

Using Proposition 4.3, we see that $F_{n;a_1,\dots,a_r;\varepsilon}$ must be scalar multiple of 1. However, $F_{n;a_1,\dots,a_r;\varepsilon} \notin U_q^-(\mathfrak{g})_0$ and hence $F_{n;a_1,\dots,a_r;\varepsilon} = 0$.

Now, consider the case $\eta_{n;a_1,\dots,a_r} > 0$; we argue by induction n. For fixed $a_1,\dots,a_r$ we have just proved the base case where n is such that $\eta_{n;a_1,\dots,a_r} = 0$. Noting $\eta_{n-1;a_1,\dots,a_r} \geq 0$, we see that the induction hypothesis is applicable to $\eta_{n-1;a_1,\dots,a_r}$. Hence, $F_{n-1;a_1,\dots,a_r;\varepsilon} = 0$. Using Lemma 4.2, we see that $[n+1]_i F_{n;a_1,\dots,a_r;\varepsilon} = 0$. Clearly, $[n+1]_i \neq 0$ and thus $F_{n;a_1,\dots,a_r;\varepsilon} = 0$. □

Let $\xi \in Q_-$. The monomials $f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)}$ with $a_1\alpha_{i_1} + \dots + a_r\alpha_{i_r} = \xi$ span the weight space $U_q^-(\mathfrak{g})_\xi$. Thus, the condition $\eta_{n;a_1,\dots,a_r} > -1$ expresses which power n of the adjoint action annihilates the weight space $U_q^-(\mathfrak{g})_\xi$. Since n and $a_1,\dots,a_r$ are integers, $\eta_{n;a_1,\dots,a_r} > -1$ is equivalent to

$$n + \sum_{j=1}^{r} \langle h_i, \alpha_{i_j}\rangle a_j > 0.$$

That is to say, this is exactly the case, when the Lusztig cone condition (4.1) is violated.

Corollary 4.5. *For any n and $a_1,\dots,a_r \geq 0$ with $\eta_{n;a_1,\dots,a_r} > -1$, we have*

$$f_i^{(n)} f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} = \sum_{l+l'=n;\eta<l'\leq n} \gamma_{l'} f_i^{(l)} f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} f_i^{(l')},$$

where $\gamma_{l'} = (-1)^{(l'+1)} q_i^{l'\eta} \sum_{k=0}^{\eta} (-1)^k q_i^{k(1-n)} \begin{bmatrix} l' \\ k \end{bmatrix}_i$.

Proof. For $\eta := \eta_{n;a_1,\dots,a_r}$, we obtain from Proposition 4.4 that

$$\sum_{k=0}^{\eta}(-1)^k q_i^{k(1-n)} F_{n-k;a_1,\dots,a_r;-1} f_i^{(k)} = 0.$$

On the other hand, we can write the sum above as

$$\begin{aligned}&\sum_{k=0}^{\eta}\sum_{l+l'=n-k}(-1)^{k+l}q_i^{k(1-n)-l(\eta-k)} f_i^{(l)} f_{i_1}^{(a_1)}\cdots f_{i_r}^{(a_r)} f_i^{(l')} f_i^{(k)}\\ &= \sum_{l+l'=n}\tilde{\gamma}_{l,l'} f_i^{(l)} f_{i_1}^{(a_1)}\cdots f_{i_r}^{(a_r)} f_i^{(l')},\end{aligned}$$

where

$$\tilde{\gamma}_{l,l'} := \sum_{k=0}^{\eta}(-1)^{k+l}q_i^{k(1-n)-l(\eta-k)}\begin{bmatrix}l'\\k\end{bmatrix}_i = (-1)^l q_i^{-l\eta}\sum_{k=0}^{\eta}(-1)^k q_i^{k(1-l')}\begin{bmatrix}l'\\k\end{bmatrix}_i.$$

If $0 \le l' \le \eta$, we may replace the range of summation of $\tilde{\gamma}_{l,l'}$ to $0 \le k \le l'$, since binomial coefficient vanishes otherwise. Hence, for such l' we have by Lemma 2.1 that the sum vanishes, unless $l' = 0$. In this case, we get the coefficient $(-1)^n q_i^{-n\eta}$.

Back in the general case, we put

$$\gamma_{l'} := (-1)^{-n+1} q_i^{n\eta}\gamma_{l,l'},$$

which makes sense, since the right-hand only depends on l'. Now, applying the above considerations to

$$\sum_{l+l'=n}\gamma_{l'} f_i^{(l)} f_{i_1}^{(a_1)}\cdots f_{i_r}^{(a_r)} f_i^{(l')} = 0$$

yields the desired identity. □

4.2. **Relation to adapted strings.** We show that the Lusztig cone $L_{\underline{i}}$ is contained in the adapted string cone $S_{\underline{i}}(\infty)$ if $s_{\underline{i}}$ is a reduced decomposition. For this reason first investigate under which conditions on $L_{\underline{i}}$ the sequence $\underline{i}$ gives rise to a reduced decomposition $s_{\underline{i}}$.

Lemma 4.6. *Let $\underline{i} = (i_1, \dots, i_r)$ be a sequence in I. There exists a weight λ in the Tits cone such that*

$$\langle h_{i_k}, s_{i_{k+1}}\cdots s_{i_r}\lambda\rangle > 0$$

for all $1 \le k \le r$ if and only if $s_{\underline{i}}$ is a reduced decomposition.

Proof. For an element $w \in W$ it is a well-known fact that $\ell(s_i w) > \ell(w)$ if and only if $w^{-1}\alpha_i$ is a positive root. Thus, if $s_{\underline{i}}$ is a reduced decomposition, we can choose any dominant regular weight λ. Indeed, $\ell(s_{i_k}\cdots s_{i_r}) > \ell(s_{i_{k+1}}\cdots s_{i_r})$ is equivalent to $\langle s_{i_r}\cdots s_{i_{k+1}} h_{i_k}, \lambda\rangle > 0$ (since λ is dominant and regular), that is $\langle h_{i_k}, s_{i_{k+1}}\cdots s_{i_r}\lambda\rangle > 0$.

Conversely, assume that there exists a weight λ in the Tits cone such that $\langle h_{i_k}, s_{i_{k+1}}\cdots s_{i_r}\lambda\rangle > 0$ for all $1 \le k \le r$. Then, λ is conjugate to a dominant weight λ^+ by some $w \in W$. Fix a reduced decomposition of $w = s_{j_1}\cdots s_{j_q}$. Using the same argument as above we see that $s_{\underline{i}}(s_{j_1}\cdots s_{j_q})$ is a reduced decomposition of $s_{\underline{i}} w$. In particular, $s_{\underline{i}}$ is a reduced decomposition. □

Lemma 4.7. *Let λ be a dominant weight and $w \in W$. Choose a sequence $\underline{i} = (i_1, \dots, i_r)$ in I such that $s_{\underline{i}}$ is a reduced decomposition of w and put*

$$a_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_r} \lambda \rangle.$$

Then for any pair $p < q$ with $i_p = i_q =: i$, we have $a_p + a_q + \sum_{p<k<q} \langle h_i, \alpha_{i_k} \rangle a_k = 0$. In particular, $(a_1, \dots, a_r) \in L_{\underline{i}}$.

Proof. It follows from

$$\langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_q} \lambda \rangle + \langle h_{i_k}, \alpha_{i_{k+1}} \rangle \langle h_{i_{k+1}}, s_{i_{k+2}} \cdots s_{i_r} \lambda \rangle = \langle h_{i_k}, s_{i_{k+2}} \cdots s_{i_r} \lambda \rangle$$

that

$$(a_p + a_q) + \sum_{p<k<q} \langle h_i, \alpha_{i_k} \rangle a_k = \langle h_i, s_{i_q} \cdots s_{i_r} \lambda \rangle + \langle h_i, s_{i_{q+1}} \cdots s_{i_r} \lambda \rangle = 0.$$

□

Lemma 4.8. *Assume that $\mathfrak{g}$ is of finite type. Let $\underline{i} = (i_1, \dots, i_r)$ be a sequence in I. Assume that there exists $(a_1, \dots, a_r) \in L_{\underline{i}}$ such that $a_k > 0$ for all $k \in [1, r]$ and such that for every pair $p < q$ with $i_p = i_q =: i$ we have*

$$a_p + a_q + \sum_{p<k<q} \langle h_i, \alpha_{i_k} \rangle a_k = 0.$$

Then, $s_{\underline{i}}$ is a reduced decomposition.

Proof. For $i \in I$, put $\lambda_i := 0$ if $i_k \neq i$ for all $k \in [1, r]$, otherwise put $\lambda_i := a_p + \sum_{p<k\leq r} \langle h_{i_p}, \alpha_{i_k} \rangle a_k$, where p is chosen maximally with $i_p = i$. Then, we have

$$\langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_r} \lambda \rangle = a_k > 0 \quad \text{for all } k \in [1, r].$$

Now, the claim follows from Lemma 4.6.

□

Remark 4.9. It is unclear to the author whether the assumption

$$a_p + a_q + \sum_{p<k<q} \langle h_i, \alpha_{i_k} \rangle a_k = 0$$

in the above lemma is necessary.

Proposition 4.10. *Let $\underline{i} = (i_1, \dots, i_r)$ be a sequence in I such that $s_{\underline{i}}$ is a reduced decomposition. Then, $L_{\underline{i}} \subseteq S_{\underline{i}}(\infty)$.*

Proof. Let $\underline{a} = (a_1, \dots, a_r) \in L_{\underline{i}}$. We construct a dominant weight λ such that $\underline{a}$ is an element of the string polytope $S_{\underline{i}}(\lambda)$; then the proposition follows. For $i \in I$, define $\lambda_i := \max_{i_k = i} a_k$ (with the convention that $\lambda_i = 0$ if i does not occur in $\underline{i}$) and put $\lambda := \sum_{i \in I} \lambda_i \varpi_i$. Let $p \in [1, r]$ and let q be maximal such that $i_q = i_p$. If $p = q$, then

$$a_p \leq \langle h_{i_p}, \lambda \rangle \leq \langle h_{i_p}, \lambda \rangle - \sum_{p<k\leq r} \langle h_{i_p}, \alpha_{i_k} \rangle a_k;$$

and if $p < q$, we have

$$\begin{aligned} a_p &\leq -a_q - \sum_{p<k<q} \langle h_{i_p}, \alpha_{i_k} \rangle a_k \\ &\leq \langle h_{i_p}, \lambda \rangle - 2a_q - \sum_{p<k<q} \langle h_{i_p}, \alpha_{i_k} \rangle a_k \\ &\leq \langle h_{i_p}, \lambda \rangle - \sum_{p<k\leq r} \langle h_{i_p}, \alpha_{i_k} \rangle a_k. \end{aligned}$$

This shows that the all inequalities (2.26) for the string polytope are satisfied. □

Proposition 4.11. *Let $\underline{i} = (i_1, \ldots, i_r)$ be a sequence in I such that $s_{\underline{i}}$ is a reduced decomposition. For $(a_1, \ldots, a_r) \in L_{\underline{i}}$ assume that $(f_{\underline{i}}^{(\underline{a})}, f_{\underline{i}}^{(\underline{a})}) \in 1 + qA$. Then, $f_{\underline{i}}^{(\underline{a})}$ is an element of the canonical basis of $U_q^-(\mathfrak{g})$.*

Proof. By definition we have $f_{\underline{i}}^{(\underline{a})} \in U_{\mathbb{Z}}^-(\mathfrak{g})$ and $\overline{f_{\underline{i}}^{(\underline{a})}} = f_{\underline{i}}^{(\underline{a})}$. Together with the assumption that $(f_{\underline{i}}^{(\underline{a})}, f_{\underline{i}}^{(\underline{a})}) \in 1 + qA$, this means all conditions of Theorem 2.9 are fulfilled. Thus, $f_{\underline{i}}^{(\underline{a})}$ is an element of the canonical basis up to a sign. Since $s_{\underline{i}}$ is a reduced decomposition, Proposition 4.10 implies that $(a_1, \ldots, a_r) \in S_{\underline{i}}(\infty)$. Finally, Proposition 2.27 allows us to conclude the proposition. □

Remark 4.12. Let λ be a dominant weight and $\pi = (\tau_1, \ldots, \tau_r; t_1, \ldots, t_r)$ be an LS path of shape λ. Fix a sequence $\underline{i} = (i_1, \ldots, i_p)$ in I such that $s_{\underline{i}}$ is a reduced decomposition of τ_1. Let $(a_1, \ldots, a_p)$ be the adapted string of π with respect to $\underline{i}$ and assume that $(a_1, \ldots, a_p) \in L_{\underline{i}}$. We put $\pi_k := \tilde{e}_{i_k}^{a_k} \cdots \tilde{e}_{i_1}^{a_1} \pi$ for $k \in [0, p]$.

Examples suggest that for each $k \in [1, p]$ the first local minimum attained by the function $\langle h_{i_k}, \pi_{k-1}(t) \rangle$ is a global minimum. If this is the case, then we would have that $F(\pi, \underline{i}) = f_{i_1}^{(a_1)} \cdots f_{i_p}^{(a_p)}$ is also an element of the adapted string basis.

4.3. **Compatibility with braid moves.** Let $\underline{i}$ be a finite sequence in I such that $s_{\underline{i}}$ is a reduced decomposition. It is known how $S_{\underline{i}}(\infty)$ behaves under braid moves [34], thus it is natural to ask under which conditions the cone $L_{\underline{i}}$ is preserved by the braid moves.

Proposition 4.13 ([34, Proposition 2.3])**.** *In A_2 the bijection between $S_{(1,2,1)}(\infty)$ and $S_{(2,1,2)}(\infty)$ induced by the 3-move is given by*

$$(a_1, a_2, a_3) \mapsto (\max\{a_3, a_2 - a_1\}, a_1 + a_3, \min\{a_2 - a_3, a_1\}).$$

Lemma 4.14. *Let $\underline{i} = (i_1, \ldots, i_r)$ be a sequence in I such that $s_{\underline{i}}$ is a reduced decomposition. Assume that a 3-move can be applied to (i_{p-1}, i_p, i_{p+1}) for $p \in [2, r-1]$. Let $(a_1, \ldots, a_r) \in S_{\underline{i}}(\infty)$ and denote by $(a'_1, \ldots, a'_r)$ its image under the 3-move.*

If $(a_1, \ldots, a_r) \in L_{\underline{i}}$, then $(a'_1, \ldots, a'_r) \in L_{\underline{i}}$ if and only if $a_p = a_{p-1} + a_{p+1}$.

Proof. Let $x, y \in [1, r]$ with $x < y$ and $i_x = i_y$. If $x, y < p-1$, $x, y > p+1$, or $x < p-1$ and $y > p+1$, then the Lusztig cone condition for x and y is unchanged by the 3-move.

Note that $a_p \geq a_{p-1} + a_{p+1}$, since $(a_1, \ldots, a_r) \in L_{\underline{i}}$. If we had $a_p > a_{p-1} + a_{p+1}$, then $a'_{p-1} + a'_{p+1} - a'_p = a_p - (a_{p-1} + a_{p+1}) > 0$, that is $(a'_1, \ldots, a'_r) \notin L_{\underline{i}}$. For the

converse, we will verify the Lusztig cone condition (4.1) for two cases, the others being similar. First, consider $x \in [1, p-2]$ such that $i_x = i_{p-1}$. We have

$$\begin{aligned} & a'_x + a'_p + \sum_{x<k<p} \langle h_{i_x}, \alpha_{i_k} \rangle a'_k \\ &= a_x + (a_{p-1} + a_{p+1}) + \sum_{x<k<p-1} \langle h_{i_x}, \alpha_{i_k} \rangle a_k + \langle h_{i_x}, \alpha_{i_p} \rangle a_{p+1} \\ &= a_x + a_{p-1} + \sum_{x<k<p-1} \langle h_{i_x}, \alpha_{i_k} \rangle a_k \leq 0. \end{aligned}$$

Second, consider $x \in [1, p-2]$ such that $i_x = i_p$. We have

$$\begin{aligned} & a'_x + a'_{p-1} + \sum_{x<k<p-1} \langle h_{i_x}, \alpha_{i_k} \rangle a'_k \\ &= a_x + a_{p+1} + \sum_{x<k<p-1} \langle h_{i_x}, \alpha_{i_k} \rangle a_k \\ &= a_x + a_p + \sum_{x<k<p} \langle h_{i_x}, \alpha_{i_k} \rangle a_k \leq 0. \end{aligned}$$

□

The lemma shows that the Lusztig cone is in general not compatible with braid moves.

4.4. **Imaginary vectors and Lusztig cones.** An element of the dual canonical basis $b \in \mathcal{B}(\infty)^\vee$ is called *real* if $b^2 \in q^{\mathbb{Z}} \mathcal{B}(\infty)^\vee$, otherwise *imaginary*. Leclerc gave an example of an imaginary vector for each of the types A_5, B_3, C_3, D_4, and G_2 in section 2 of [30]. Assume that $\mathfrak{g}$ is one of these types and that $b \in \mathcal{B}(\infty)^\vee$ is the corresponding example. Let $\theta \in \mathcal{B}(\infty)$ be the unique element with $(b, \theta) = 1$. We claim that it is a monomial and has the following shape:

$$\theta = (f_{i_1}^{(a_1)} \cdots f_{i_p}^{(a_p)})(f_{j_1}^{(b_1)} \cdots f_{j_q}^{(b_q)})(f_{i_1}^{(a_1)} \cdots f_{i_p}^{(a_p)}). \tag{4.3}$$

Indeed, the element θ can be obtained from b in the following way: Using the fact that $B(\infty)$ is an orthonormal basis with respect to the bilinear form on $L(\infty)/qL(\infty)$ (which is induced by that of $U_q^-(\mathfrak{g})$), one sees that it is sufficient to know the Lusztig datum (PBW datum) of b to find θ. This information was already given by Leclerc in the same paper [30]. From here, one uses the crystal structure on Lusztig data to compute the adapted string of b with respect to the same reduced decomposition as the one used by Leclerc. Finally, one can use Proposition 4.25 to see that the monomial corresponding to the adpated string is indeed in the canonical basis.

Remark 4.15. For the crystal structure on Lusztig data we refer the reader to [5].

Similarly to the concept of real and imaginary vectors, we consider the "square" of θ:

$$\begin{aligned} \xi = & (f_{i_1}^{(a_1)} \cdots f_{i_p}^{(a_p)})(f_{j_1}^{(b_1)} \cdots f_{j_q}^{(b_q)})(f_{i_1}^{(2a_1)} \cdots f_{i_p}^{(2a_p)}) \\ & \cdot (f_{j_1}^{(b_1)} \cdots f_{j_q}^{(b_q)})(f_{i_1}^{(a_1)} \cdots f_{i_p}^{(a_p)}). \end{aligned} \tag{4.4}$$

A close observation of θ and ξ reveals the following properties:

(P1) The sets $\{i_1 \ldots, i_p\}$ and $\{j_1, \ldots, j_q\}$ form a partition of I,

(P2) $(\alpha_{i_r}, \alpha_{i_s}) = 0$ for all $1 \leq r < s \leq p$ and $(\alpha_{j_r}, \alpha_{j_s}) = 0$ for all $1 \leq r < s \leq q$,
(P3) $3a_k + \sum_{t=1}^{q} \langle h_{i_k}, \alpha_{j_t} \rangle b_{j_t} = 0$ for $1 \leq k \leq p$, and
(P4) $b_k + \sum_{t=1}^{p} \langle h_{j_k}, \alpha_{i_t} \rangle a_{i_t} = 0$ for $1 \leq k \leq q$.

Remark 4.16. (i) Property (P2) implies that ξ is indeed a square of θ up to q-binomials.
(ii) Properties (P1) and (P2) imply that

$$|\{i_1 \ldots, i_p\}| = p \quad \text{and} \quad |\{j_1 \ldots, j_q\}| = q.$$

In particular, each $i \in I$ occurs exactly once in either $i_1, \ldots, i_p$ or $j_1, \ldots, j_q$.
(iii) Properties (P3) and (P4) are the Lusztig cone condition in (4.1) for ξ.

This leads to the following classification problem: Determine all finite-dimensional simple complex Lie algebras $\mathfrak{g}$, together with pairs of monomials (θ, ξ), as above, such that properties (P1) - (P4) hold.

Theorem 4.17. *Every triple $(\mathfrak{g}, \theta, \xi)$ fulfilling the properties (P1) - (P4) belongs to one of the following 10 families:*

(i) Type A_5

$$\theta_1^{[l]} = (f_2^{(l)} f_4^{(l)})(f_1^{(l)} f_3^{(2l)} f_5^{(l)})(f_2^{(l)} f_4^{(l)})$$
$$\xi_1^{[l]} = (f_2^{(l)} f_4^{(l)})(f_1^{(l)} f_3^{(2l)} f_5^{(l)})(f_2^{(2l)} f_4^{(2l)})(f_1^{(l)} f_3^{(2l)} f_5^{(l)})(f_2^{(l)} f_4^{(l)})$$

(ii) Type A_5

$$\theta_2^{[l]} = (f_1^{(l)} f_3^{(2l)} f_5^{(l)})(f_2^{(3l)} f_4^{(3l)})(f_1^{(l)} f_3^{(2l)} f_5^{(l)})$$
$$\xi_2^{[l]} = (f_1^{(l)} f_3^{(2l)} f_5^{(l)})(f_2^{(3l)} f_4^{(3l)})(f_1^{(2l)} f_3^{(4l)} f_5^{(2l)})(f_2^{(3l)} f_4^{(3l)})(f_1^{(l)} f_3^{(2l)} f_5^{(l)})$$

(iii) Type B_3

$$\theta_3^{[l]} = f_2^{(l)}(f_1^{(l)} f_3^{(2l)}) f_2^{(l)}$$
$$\xi_3^{[l]} = f_2^{(l)}(f_1^{(l)} f_3^{(2l)}) f_2^{(2l)}(f_1^{(l)} f_3^{(2l)}) f_2^{(l)}$$

(iv) Type B_3

$$\theta_4^{[l]} = (f_1^{(l)} f_3^{(2l)}) f_2^{(3l)}(f_1^{(l)} f_3^{(2l)})$$
$$\xi_4^{[l]} = (f_1^{(l)} f_3^{(2l)}) f_2^{(3l)}(f_1^{(2l)} f_3^{(4l)}) f_2^{(3l)}(f_1^{(l)} f_3^{(2l)})$$

(v) Type C_3

$$\theta_5^{[l]} = f_2^{(l)}(f_1^{(l)} f_3^{(l)}) f_2^{(l)}$$
$$\xi_5^{[l]} = f_2^{(l)}(f_1^{(l)} f_3^{(l)}) f_2^{(2l)}(f_1^{(l)} f_3^{(l)}) f_2^{(l)}$$

(vi) Type C_3

$$\theta_6^{[l]} = (f_1^{(l)} f_3^{(l)}) f_2^{(3l)}(f_1^{(l)} f_3^{(l)})$$
$$\xi_6^{[l]} = (f_1^{(l)} f_3^{(l)}) f_2^{(3l)}(f_1^{(2l)} f_3^{(2l)}) f_2^{(3l)}(f_1^{(l)} f_3^{(l)})$$

(vii) Type D_4

$$\theta_7^{[l]} = f_2^{(l)}(f_1^{(l)} f_3^{(l)} f_4^{(l)}) f_2^{(l)}$$
$$\xi_7^{[l]} = f_2^{(l)}(f_1^{(l)} f_3^{(l)} f_4^{(l)}) f_2^{(2l)}(f_1^{(l)} f_3^{(l)} f_4^{(l)}) f_2^{(l)}$$

(viii) Type D_4

$$\theta_8^{[l]} = (f_1^{(l)} f_3^{(l)} f_4^{(l)}) f_2^{(3l)} (f_1^{(l)} f_3^{(l)} f_4^{(l)})$$
$$\xi_8^{[l]} = (f_1^{(l)} f_3^{(l)} f_4^{(l)}) f_2^{(3l)} (f_1^{(2l)} f_3^{(2l)} f_4^{(2l)}) f_2^{(3l)} (f_1^{(l)} f_3^{(l)} f_4^{(l)})$$

(ix) Type G_2

$$\theta_9^{[l]} = f_1^{(l)} f_2^{(l)} f_1^{(l)}$$
$$\xi_9^{[l]} = f_1^{(l)} f_2^{(l)} f_1^{(2l)} f_2^{(l)} f_1^{(l)}$$

(x) Type G_2

$$\theta_{10}^{[l]} = f_2^{(l)} f_1^{(3l)} f_2^{(l)}$$
$$\xi_{10}^{[l]} = f_2^{(l)} f_1^{(3l)} f_2^{(2l)} f_1^{(3l)} f_2^{(l)}$$

Proof. First, observe that (P1) and (P2) together imply that there exactly two ways to label the Dynkin diagram of $\mathfrak{g}$ with $\{i_1, \ldots, i_p\}$ and $\{j_1, \ldots, j_q\}$ and second, observe that (P3) and (P4) define a linear system over $\mathbb{R}^{p+q}$ with variables $a_1, \ldots, a_p, b_1, \ldots, b_q$. Determining a positive integral solution to linear system is thus equivalent to finding a monomial fulfilling (P1) - (P4). Once a labeling is chosen, it is straightforward to solve the linear system. Since we only care about positive integral solutions, we can abort our computations as soon as one coordinate is ≤ 0 and a solution will always mean positive integral solution. We now proceed with a case-by-case analysis.

Type A_n. First, consider the labeling

$$j_1 \text{ — } i_1 \text{ — } j_2 \text{ — } i_2 \text{ — } \cdots$$

(P3) and (P4) give the following system of linear equations:

$$\left.\begin{aligned} b_1 - a_1 &= 0 \\ 3a_1 - b_1 - b_2 &= 0 \\ b_2 - a_1 - a_2 &= 0 \\ 3a_2 - b_2 - b_3 &= 0 \\ b_3 - a_2 - a_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} b_1 &= a_1 \\ b_2 &= 3a_1 - b_1 = 2a_1 \\ a_2 &= b_2 - a_1 = a_1 \\ b_3 &= 3a_2 - b_2 = a_1 \\ a_3 &= b_3 - a_2 = 0 \end{aligned}$$

Observe that taking $p < 2$ or $q < 3$ only admits the trivial solution for the linear system; we also cannot take $p \geq 3$, since in this case $a_3 = 0$ would not be positive, and from (P1) it follows that q cannot be greater than 3. Thus, the linear system only has a solution if $p = 2$, $q = 3$, $\{i_1, i_2\} = \{2, 4\}$ and $\{j_1, j_2, j_3\} = \{1, 3, 5\}$. Fixing the order $(i_1, i_2, j_1, j_2, j_3) = (2, 4, 1, 3, 5)$ gives the solutions $(l, l, l, 2l, l)$ for $l \in \mathbb{N}$.

The second possible labeling is

$$i_1 \text{ — } j_1 \text{ — } i_2 \text{ — } j_2 \text{ — } \cdots$$

(P3) and (P4) give the following system of linear equations:

$$\left.\begin{aligned} 3a_1 - b_1 &= 0 \\ b_1 - a_1 - a_2 &= 0 \\ 3a_2 - b_1 - b_2 &= 0 \\ b_2 - a_2 - a_3 &= 0 \\ 3a_3 - b_2 - b_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} b_1 &= 3a_1 \\ a_2 &= b_1 - a_2 = 2a_1 \\ b_2 &= 3a_2 - b_1 = 3a_1 \\ a_3 &= b_2 - a_2 = a_1 \\ b_3 &= 3a_3 - b_2 = 0 \end{aligned}$$

It only has a solution if $p = 3$, $q = 2$, $\{i_1, i_2, i_3\} = \{1, 3, 5\}$ and $\{j_1, j_2\} = \{2, 4\}$. Fixing $(i_1, i_2, i_3, j_1, j_2) = (1, 3, 5, 2, 4)$ gives the solutions $(l, 2l, l, 3l, 3l)$ for $l \in \mathbb{N}$.

Type B_n. First, consider the labeling

$$\cdots \text{—} \; i_2 \text{—} \; j_2 \text{—} \; i_1 \Rightarrow\!\!\!\!\!\!\times \; j_1$$

(P3) and (P4) give the following system of linear equations:

$$\left.\begin{aligned} b_1 - 2a_1 &= 0 \\ 3a_1 - b_1 - b_2 &= 0 \\ b_2 - a_1 - a_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} b_1 &= 2a_1 \\ b_2 &= 3a_1 - b_1 = a_1 \\ a_2 &= b_2 - a_1 = 0 \end{aligned}$$

It only has a solution if $p = 1$, $q = 2$, $i_1 = 2$, $\{j_1, j_2\} = \{1, 3\}$. Fixing $(i_1, j_1, j_2) = (2, 1, 3)$ gives the solutions $(l, l, 2l)$ for $l \in \mathbb{N}$.

The second possible labeling is

$$\cdots \text{—} \; j_2 \text{—} \; i_2 \text{—} \; j_1 \Rightarrow\!\!\!\!\!\!\times \; i_1$$

(P3) and (P4) give the following system of linear equations:

$$\left.\begin{aligned} 3a_1 - 2b_1 &= 0 \\ b_1 - a_1 - a_2 &= 0 \\ 3a_2 - b_1 - b_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} b_1 &= \tfrac{3}{2}a_1 \\ a_2 &= b_1 - a_1 = \tfrac{1}{2}a_1 \\ b_2 &= 3a_2 - b_1 = 0 \end{aligned}$$

It only has a solution if $p = 2$, $q = 1$ and $\{i_1, i_2\} = \{1, 3\}$, and $j_1 = 2$. Fixing $(i_1, i_2, j_1) = (1, 3, 2)$ gives the solutions $(l, 2l, 3l)$ for $l \in \mathbb{N}$.

Type C_n. First, consider the labeling

$$\cdots \text{—} \; i_2 \text{—} \; j_2 \text{—} \; i_1 \Leftarrow\!\!\!\!\!\!\times \; j_1$$

(P3) and (P4) give the following system of linear equations:

$$\left.\begin{aligned} b_1 - a_1 &= 0 \\ 3a_1 - 2b_1 - b_2 &= 0 \\ b_2 - a_1 - a_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} b_1 &= a_1 \\ b_2 &= 3a_1 - 2b_1 = a_1 \\ a_2 &= b_2 - a_1 = 0 \end{aligned}$$

It only has a solution if $p = 1$, $q = 2$, $i_1 = 2$ and $\{j_1, j_2\} = \{1, 3\}$. Fixing $(i_1, j_1, j_2) = (2, 1, 3)$ gives the solutions (l, l, l) for $l \in \mathbb{N}$.

The second possible labeling is

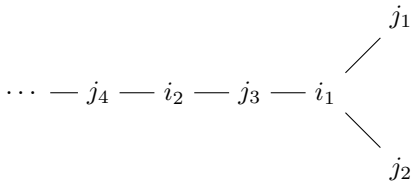
$$\cdots \text{—} \; j_2 \text{—} \; i_2 \text{—} \; j_1 \Leftarrow\!\!\!\!\!\!\times \; i_1$$

(P3) and (P4) give the following system of linear equations:

$$\left.\begin{aligned} 3a_1 - 2b_1 &= 0 \\ b_1 - a_1 - a_2 &= 0 \\ 3a_2 - b_1 - b_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} b_1 &= \tfrac{3}{2}a_1 \\ a_2 &= b_1 - a_1 = \tfrac{1}{2}a_1 \\ b_2 &= 3a_2 - b_1 = 0 \end{aligned}$$

It only has a solution if $p = 2$, $q = 1$, $\{i_1, i_2\} = \{1, 3\}$ and $j_1 = 2$. Fixing $(i_1, i_2, j_1) = (1, 3, 2)$ gives the solutions $(l, l, 3l)$ for $l \in \mathbb{N}$.

Type $D_n, n \geq 4$. First, consider the labeling

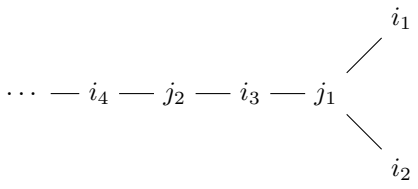


(P3) and (P4) give the following system of linear equations:

$$\left.\begin{aligned} b_1 - a_1 &= 0 \\ b_2 - a_1 &= 0 \\ 3a_1 - b_1 - b_2 - b_3 &= 0 \\ b_3 - a_1 - a_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} b_1 &= a_1 \\ b_2 &= a_1 \\ b_3 &= 3a_1 - b_1 - b_2 = a_1 \\ a_2 &= a_1 - b_3 = 0 \end{aligned}$$

It only has a solution if $p = 1$, $q = 3$, $i_1 = 2$ and $\{j_1, j_2, j_3\} = \{1, 3, 4\}$. Fixing $(i_1, j_1, j_2, j_3) = (2, 1, 3, 4)$ gives the solutions (l, l, l, l) for $l \in \mathbb{N}$.

The second possible labeling is

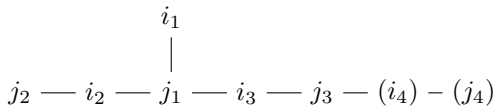


(P3) and (P4) give the following system of linear equations:

$$\left.\begin{aligned} 3a_1 - b_1 &= 0 \\ 3a_2 - b_1 &= 0 \\ b_1 - a_1 - a_2 - a_3 &= 0 \\ 3a_3 - b_1 - b_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} b_1 &= 3a_1 \\ a_2 &= \tfrac{1}{3}b_1 = a_1 \\ a_3 &= b_1 - a_1 - a_2 = a_1 \\ b_2 &= 3a_3 - b_1 = 3a_1 - 3a_1 = 0 \end{aligned}$$

It only has a solution if $p = 3$, $q = 1$, $\{i_1, i_3, i_4\} = \{1, 3, 4\}$ and $j_1 = 2$. Fixing $(i_1, i_2, i_3, j_1) = (1, 3, 4, 2)$ gives the solutions $(l, l, l, 3l)$ for $l \in \mathbb{N}$.

Type E_6, E_7, E_8. By looking at the Dynkin diagram we see that



and

$$\begin{array}{ccccccccccc} & & & & j_1 & & & & & & \\ & & & & | & & & & & & \\ i_2 & — & j_2 & — & i_1 & — & j_3 & — & i_3 & — & (j_4) - (i_4) \end{array}$$

are the possible labelings. However, starting at i_1 for the first and j_1 for the second, we see that the computations are the same as in D_4. Therefore (P1) - (P4) cannot be satisfied simultaneously.

Type F_4. By looking at the Dynkin diagram we see that

$$i_1 — j_1 \Rightarrow\!\!\!\!\!\!= i_2 — j_2$$

and

$$j_1 — i_1 \Rightarrow\!\!\!\!\!\!= j_2 — i_2$$

are the possible labelings. However starting at i_1 for the first and at j_1 for the second gives the second B_3 and first B_3 case, respectively. Therefore (P1) - (P4) cannot be satisfied simultaneously.

Type G_2. The two possible labelings are

$$i_1 \Lleftarrow\!\!\!\!\!\!= j_1 \quad \text{and} \quad j_1 \Lleftarrow\!\!\!\!\!\!= i_1$$

In the first case fixing $(i_1, j_1) = (1, 2)$ gives the solutions (l, l) for $l \in \mathbb{N}$ and in the second fixing $(i_1, j_1) = (2, 1)$ gives the solutions $(l, 3l)$ for $l \in \mathbb{N}$. □

Remark 4.18. We just considered the case where $\mathfrak{g}$ is a finite-dimensional Lie algebra. However, it is a natural question to ask if there are monomials arising from (infinite-dimensional) symmetrizable Kac-Moody Lie algebras. Fix $k \in \{1, \cdots, p\}$ and $l \in \{1, \ldots, q\}$ such that $\langle h_{i_k}, \alpha_{j_l}\rangle\langle h_{j_l}, \alpha_{i_k}\rangle > 3$ (such a pair must exist by (P1) and the assumption that $\mathfrak{g}$ is infinite-dimensional). However, combining properties (P3) and (P4) we obtain

$$\begin{aligned} 0 = 3a_k + \sum_{t=1}^{q} \langle h_{i_k}, \alpha_{j_t}\rangle b_t &\leq 3a_k + \langle h_{i_k}, \alpha_{j_l}\rangle b_l \\ &\leq 3a_k - \sum_{s=1}^{p} \langle h_{i_k}, \alpha_{j_l}\rangle\langle h_{j_l}, \alpha_{i_s}\rangle a_s \\ &\leq 3a_k - \langle h_{i_k}, \alpha_{j_l}\rangle\langle h_{j_l}, \alpha_{i_k}\rangle a_k. \end{aligned}$$

This is a contradiction to our choice of k and l. In particular, the above inequality only has a positive solution if $\mathfrak{g}$ is finite-dimensional.

In the following fix $(\mathfrak{g}, \theta, \xi)$ to be one of the ten cases in Theorem 4.17.

Theorem 4.19. *We have*

(i) $(\theta^{[l]}, \theta^{[l]}) \in 1 + qA$,
(ii) $(\theta^{[2l]}, \xi^{[l]}) \in 1 + qA$, *and*
(iii) $(\xi^{[l]}, \xi^{[l]}) \in (l + 1) + qA$.

Remark 4.20. In particular Theorem 4.19 states that $\xi^{[l]}$ is not an element of the canonical basis for $l \geq 1$. This shows Theorem 2.1 of both [52] and [51] does not hold in the generality stated there.

Proof. (1) will be a consequence of Corollary 4.27. Let θ and ξ be as in (4.3) and (4.4); we begin with the proof of (2). For this, rewrite the bilinear form into

$$(\theta^{[2l]},\xi^{[l]}) = [2a_1]_{i_1}!^{-2}\cdots[2a_p]_{i_p}!^{-2}[2b_1]_{j_1}!^{-1}\cdots[2b_q]_{j_q}!^{-1} \times (1, e_{i_1}'^{2a_1}\cdots e_{i_p}'^{2a_p}e_{j_1}'^{2b_1}\cdots e_{j_q}'^{2b_q}e_{i_1}'^{2a_1}\cdots e_{i_p}'^{2a_p}\xi^{[l]})$$

and by expanding the action of the e_i' this becomes

$$\text{(I)}\qquad \sum_{k_1=0}^{a_1}\sum_{l_1=a_1-k_1}^{2a_1-k_1}\cdots\sum_{k_p=0}^{a_p}\sum_{l_p=a_p-k_p}^{2a_p-k_p} A(\underline{k},\underline{l})B(\underline{k},\underline{l}),$$

where $\underline{k}=(k_1,\dots,k_p)$, $\underline{l}=(l_1,\dots,l_p)$,

$$\begin{aligned}
&A(\underline{k},\underline{l}) = \\
&\prod_{s=1}^{p}[2a_s]_{i_s}!^{-2}q_{i_s}^{-4a_s^2+3k_sa_s-k_s(k_s-1)/2}\begin{bmatrix}2a_s\\k_s\end{bmatrix}_{i_s}q_{i_s}^{-(2a_s-k_s)\sum_{t=1}^{q}\langle h_{i_s},\alpha_{j_t}\rangle b_t}\\
&\times q_{i_s}^{-2(2a_s-k_s)2a_s+l_s(4a_s-k_s)-l_s(l_s-1)/2}\begin{bmatrix}2a_s-k_s\\l_s\end{bmatrix}_{i_s}q_{i_s}^{-(2a_s-k_s-l_s)\sum_{t=1}^{q}\langle h_{i_s},\alpha_{j_t}\rangle b_t}\\
&\times q_{i_s}^{-2(2a_s-k_s-l_s)a_s+(2a_s-k_s-l_s)(3a_s-k_s-l_s)-(2a_s-k_s-l_s)(2a_s-k_s-l_s-1)/2}\\
&\times q_{i_s}^{-4a_s(a_s-k_s)+(a_s-k_s)(3a_s-k_s)-(a_s-k_s)(a_s-k_s-1)/2}\begin{bmatrix}2a_s\\a_s-k_s\end{bmatrix}_{i_s}\\
&\times q_{i_s}^{-2(a_s+k_s)(2a_s-l_s)+(2a_s-l_s)(3a_s+k_s-l_s)-(2a_s-l_s)(2a_s-l_s-1)/2}\begin{bmatrix}a_s+k_s\\2a_s-l_s\end{bmatrix}_{i_s}\\
&\times q_{i_s}^{-2(-a_s+k_s+l_s)^2+2(-a_s+k_s+l_s)^2-(k_s+l_s-a_s)(k_s+l_s-a_s-1)/2},
\end{aligned}$$

and

$$\begin{aligned}
&B(\underline{k},\underline{l}) = \\
&\prod_{t=1}^{q}[2b_t]_{j_t}!^{-1}q_{j_t}^{-2b_t\sum_{s=1}^{p}\langle h_{j_t},\alpha_{i_s}\rangle(a_s-k_s)}q_{j_t}^{-4b_t^2+3b_t^2-b_t(b_t-1)/2}\begin{bmatrix}2b_t\\b_t\end{bmatrix}_{j_t}\\
&\times q_{j_t}^{-b_t\sum_{s=1}^{p}\langle h_{j_t},\alpha_{i_s}\rangle(2a_s-l_s)}q_{j_t}^{-2b_t^2+2b_t^2-b_t(b_t-1)/2}.
\end{aligned}$$

Let us explain the summation ranges in (I) for $s\in[1,p]$. The upper bound on k_s is due to the fact the first f_{i_s} in ξ has exponent a_s, and the lower bound of l_s comes from the requirement that for the last f_{i_s} we have

$$e_{i_s}'^{2a_s-k_s-l_s}f_{i_s}^{(a_s)}\neq 0,$$

that is $2a_s-k_s-l_s\leq a_s$.

For every factor of the product $\prod_{s=1}^{p}$ in $A(\underline{k},\underline{l})$ we have

$$\begin{aligned}
&q_{i_s}^{-k_s(k_s-1)/2-l_s(l_s-1)/2-(2a_s-k_s-l_s)(2a_s-k_s-l_s-1)/2-(a_s-k_s)(a_s-k_s-1)/2}\\
&\times q_{i_s}^{-(2a_s-l_s)(2a_s-l_s-1)/2-(k_s+l_s-a_s)(k_s+l_s-a_s-1)/2}\\
&\times [2a_s]_{i_s}!^{-2}\begin{bmatrix}2a_s\\k_s\end{bmatrix}_{i_s}\begin{bmatrix}2a_s-k_s\\l_s\end{bmatrix}_{i_s}\begin{bmatrix}2a_s\\a_s-k_s\end{bmatrix}_{i_s}\begin{bmatrix}a_s+k_s\\2a_s-l_s\end{bmatrix}_{i_s}\in 1+qA,
\end{aligned}$$

and for every factor of $\prod_{t=1}^{q}$ in $B(\underline{k},\underline{l})$ we have

$$q_{j_t}^{-b_t(b_t-1)}[2b_t]_{j_t}!^{-1}\begin{bmatrix}2b_t\\ b_t\end{bmatrix}_{j_t}\in 1+qA.$$

Now, from $B(\underline{k},\underline{l})$ we extract

$$\prod_{t=1}^{q} q_{j_t}^{-b_t\sum_{s=1}^{p}\langle h_{j_t},\alpha_{i_s}\rangle(4a_s-2k_s-l_s)}=\prod_{s=1}^{p} q^{-(4a_s-2k_s-l_s)\sum_{t=1}^{q}(\alpha_{i_s},\alpha_{i_t})b_t}.$$

using the identity $q_{j_t}^{\langle h_{j_t},\alpha_{i_s}\rangle}=q^{(\alpha_{i_s},\alpha_{i_t})}$. Combining this with the remaining terms in $A(\underline{k},\underline{l})$ we get

$$\prod_{s=1}^{p} q^{\frac{1}{2}(\alpha_{i_s},\alpha_{i_s})(-9a_s^2+2a_sk_s-2a_sl_s+2k_s^2+2k_sl_s+2l_s^2)-(8a_s-4k_s-2l_s)\sum_{t=1}^{q}(\alpha_{i_s},\alpha_{i_t})b_t}.$$

Using the inner product version of (P3), $3(\alpha_{i_s},\alpha_{i_s})a_s=-2\sum_{t=1}^{q}(\alpha_{i_s},\alpha_{j_t})b_t$, the above expression becomes

$$q^{\frac{1}{2}(\alpha_{i_s},\alpha_{i_s})(15a_s^2-10a_sk_s-8a_sl_s+2k_s^2+2k_sl_s+2l_s^2)}.$$

Accounting for the remaining $\prod_{t=1}^{q} q_{j_t}^{-b_t^2}$ from $B(\underline{k},\underline{l})$, one sees that (I) is equal to

$$q^{\frac{3}{2}\sum_{s=1}^{p}(\alpha_{i_s},\alpha_{i_s})a_s^2-\frac{1}{2}\sum_{t=1}^{q}(\alpha_{j_t},\alpha_{j_t})b_t^2}\sum_{k_1=0}^{a_1}\sum_{l_1=a_1-k_1}^{2a_1-k_1}\cdots\sum_{k_p=0}^{a_p}\sum_{l_p=a_p-k_p}^{2a_p-k_p} q^{Q_1(\underline{k},\underline{l})},$$

where $Q_1(\underline{k},\underline{l})$ is given by

$$\sum_{s=1}^{p}\tfrac{1}{2}(\alpha_{i_s},\alpha_{i_s})((k_s-a_s)^2+(l_s-a_s)^2+(k_s+l_s-2a_s)^2+2a_s(3a_s-2k_s-l_s)).$$

Using the inner product version of (P3) again and the one of (P4), $(\alpha_{j_t},\alpha_{j_t})b_t=-2\sum_{s=1}^{q}(\alpha_{j_t},\alpha_{i_s})a_s$, we see that

$$q^{\frac{3}{2}\sum_{s=1}^{p}(\alpha_{i_s},\alpha_{i_s})a_s^2-\frac{1}{2}\sum_{t=1}^{q}(\alpha_{j_t},\alpha_{j_t})b_t^2}=1.$$

It remains to consider $Q_1(\underline{k},\underline{l})$. Looking at the summation ranges of (I) we see that $Q_1(\underline{k},\underline{l})$ is nonnegative, and it is zero if and only if $k_s=l_s=a_s$. This shows that every summand in (I) is in qA except for the case where $k_s=l_s=a_s$ for all $1\le s\le p$; it follows that $(\theta^{[2l]},\xi^{[l]})\in 1+qA$.

The proof of (3) is similar to the one of (2). We begin again by expanding

$$\begin{aligned}(\xi^{[l]},\xi^{[l]})=&[a_1]_{i_1}!^{-2}\cdots[a_p]_{i_p}!^{-2}[b_1]_{j_1}!^{-2}\cdots[b_q]_{j_q}!^{-2}[2a_1]_{i_1}!^{-1}\cdots[2a_p]_{i_p}!^{-1}\\ &\times(1,e_{i_1}'^{a_1}\cdots e_{i_p}'^{a_p}e_{j_1}'^{b_1}\cdots e_{j_q}'^{b_q}e_{i_1}'^{2a_1}\cdots e_{i_p}'^{2a_p}e_{j_1}'^{b_1}\cdots e_{j_q}'^{b_q}e_{i_1}'^{a_1}\cdots e_{i_p}'^{a_p}\xi)\end{aligned}$$

into

$$\text{(II)}\qquad \sum_{k_1=0}^{a_1}\sum_{l_1=0}^{a_1-k_1}\sum_{k_1'=0}^{a_1-k_1}\sum_{l_1'=2a_1'-k_1-l_1-k_1'}^{2a_1-k_1'}\cdots\sum_{k_p=0}^{a_p}\sum_{l_p=0}^{a_p-k_p}\sum_{k_p'=0}^{a_p-k_p}\sum_{l_p'=2a_p'-k_p-l_p-k_p'}^{2a_p-k_p'}$$
$$\sum_{m_1=0}^{b_1}\cdots\sum_{m_q=0}^{b_q}A(\underline{k},\underline{l},\underline{k}',\underline{l}',\underline{m})B(\underline{k},\underline{l},\underline{k}',\underline{l}',\underline{m}),$$

where $\underline{k} = (k_1, \dots, k_p)$, $\underline{l} = (l_1, \dots, l_p)$, etc.,

$$
\begin{aligned}
&A(\underline{k}, \underline{l}, \underline{k}', \underline{l}', \underline{m}) = \\
&\prod_{s=1}^{p} [a_s]_{i_s}!^{-2} [2a_s]_{i_s}!^{-1} q_{i_s}^{-2a_s^2+2k_s a_s - k_s(k_s-1)/2} \begin{bmatrix} a_s \\ k_s \end{bmatrix}_{i_s} q_{i_s}^{-(a_s-k_s)\sum_{t=1}^{q} \langle h_{i_s}, \alpha_{j_t} \rangle b_t} \\
&\times q_{i_s}^{-2(a_s-k_s)2a_s + l_s(3a_s-k_s) - l_s(l_s-1)/2} \begin{bmatrix} a_s - k_s \\ l_s \end{bmatrix}_{i_s} q_{i_s}^{-(a_s-k_s-l_s)\sum_{t=1}^{q} \langle h_{i_s}, \alpha_{j_t} \rangle b_t} \\
&\times q_{i_s}^{-2(a_s-k_s-l_s)a_s + (a_s-k_s-l_s)(2a_s-k_s-l_s) - (a_s-k_s-l_s)(a_s-k_s-l_s-1)/2} \\
&\times q_{i_s}^{-2(a_s-k_s)2a_s + k_s'(3a_s-k_s) + k_s'(k_s'-1)/2} \begin{bmatrix} 2a_s \\ k_s' \end{bmatrix}_{i_s} q_{i_s}^{-(2a_s-k_s')\sum_{t=1}^{q} \langle h_{i_s}, \alpha_{j_t} \rangle (b_t - m_t)} \\
&\times q_{i_s}^{-2(2a_s-k_s')(2a_s-l_s) + l_s'(4a_s-l_s-k_s') - l_s'(l_s'-1)/2} \begin{bmatrix} 2a_s \\ l_s' \end{bmatrix}_{i_s} q_{i_s}^{-(2a_s-k_s'-l_s')\sum_{t=1}^{q} \langle h_{i_s}, \alpha_{j_t} \rangle m_t} \\
&\times q_{i_s}^{-2(2a_s-k_s'-l_s')(k_s+l_s) + (2a_s-k_s'-l_s')(2a_s+k_s+l_s-k_s'-l_s') - (2a_s-k_s'-l_s')(2a_s-k_s'-l_s'-1)/2} \\
&\times q_{i_s}^{-2a_s(a_s-k_s-k_s') + (a_s-k_s-k_s')(2a_s-k_s-k_s') - (a_s-k_s-k_s')(a_s-k_s-k_s'-1)/2} \begin{bmatrix} a_s \\ a_s-k_s-k_s' \end{bmatrix}_{i_s} \\
&\times q_{i_s}^{-2(k_s+k_s')(2a_s-l_s-l_s') + (2a_s-l_s-l_s')(2a_s+k_s-l_s+k_s'-l_s') - (2a_s-l_s-l_s')(2a_s-l_s-l_s'-1)/2} \\
&\quad \times \begin{bmatrix} k_s + k_s' \\ -2a_s + k_s + l_s + k_s' + l_s' \end{bmatrix}_{i_s} \\
&\times q_{i_s}^{-2(-2a_s+k_s+l_s+k_s'+l_s')^2 + 2(-2a_s+k_s+l_s+k_s'+l_s')^2} \\
&\times q_{i_s}^{-(-2a_s+k_s+l_s+k_s'+l_s')(-2a_s+k_s+l_s+k_s'+l_s'-1)/2},
\end{aligned}
$$

and

$$
\begin{aligned}
&B(\underline{k}, \underline{l}, \underline{k}', \underline{l}', \underline{m}) = \\
&\prod_{t=1}^{q} [b_t]_{j_t}!^{-2} q_{j_t}^{-b_t \sum_{s=1}^{p} \langle h_{j_t}, \alpha_{i_s} \rangle (a_s - k_s)} q_{j_t}^{-2b_t^2 + 2m_t b_t - m_t(m_t-1)/2} \begin{bmatrix} b_t \\ m_t \end{bmatrix}_{j_t} \\
&\times q_{j_t}^{-(b_t-m_t)\sum_{s=1}^{p} \langle h_{j_t}, \alpha_{i_s} \rangle (2a_s - l_s)} q_{j_t}^{-2(b_t-m_t)b_t + (b_t-m_t)(2b_t-m_t) - (b_t-m_t)(b_t-m_t-1)/2} \\
&\times q_{j_t}^{-b_t \sum_{s=1}^{p} \langle h_{j_t}, \alpha_{i_s} \rangle (a_s - k_s - k_s')} q_{j_t}^{-2(b_t-m_t)b_t + (b_t-m_t)(2b_t-m_t) + (b_t-m_t)(b_t-m_t-1)/2} \\
&\quad \times \begin{bmatrix} b_t \\ b_t - m_t \end{bmatrix}_{j_t} \\
&\times q_{j_t}^{-m_t \sum_{s=1}^{p} \langle h_{j_t}, \alpha_{i_s} \rangle (2a_s - l_s - l_s')} q_{j_t}^{-2m_t^2 + 2m_t^2 - m_t(m_t-1)/2}.
\end{aligned}
$$

Taking care of the q-integers and q-binomials as above and rearranging the remaining terms in $A(\underline{k}, \underline{l}, \underline{k}', \underline{l}', \underline{m})$ and $B(\underline{k}, \underline{l}, \underline{k}', \underline{l}', \underline{m})$, we can write (II) as

$$
\begin{aligned}
&q^{3\sum_{s=1}^{p} (\alpha_{i_s}, \alpha_{i_s}) a_t^2 - \sum_{t=1}^{q} (\alpha_{i_t}, \alpha_{i_t}) b_t^2} \sum_{k_1=0}^{a_1} \sum_{l_1=0}^{a_1-k_1} \sum_{k_1'=0}^{a_1-k_1} \sum_{l_1'=2a_1'-k_1-l_1-k_1'}^{2a_1-k_1'} \cdots \\
&\sum_{k_p=0}^{a_p} \sum_{l_p=0}^{a_p-k_p} \sum_{k_p'=0}^{a_p-k_p} \sum_{l_p'=2a_p'-k_p-l_p-k_p'}^{2a_p-k_p'} \sum_{m_1=0}^{b_1} \cdots \sum_{m_q=0}^{b_q} q^{Q_2(\underline{k}, \underline{l}, \underline{k}', \underline{l}', \underline{m})},
\end{aligned}
$$

where $Q_2(\underline{k},\underline{l},\underline{k}',\underline{l}',\underline{m})$ is given by

$$\begin{aligned}&\sum_{s=1}^{p}\tfrac{1}{2}(\alpha_{i_s},\alpha_{i_s})((k_s+l_s-a_s)^2+(k_s'+\tfrac{1}{2}l_s'-a_s)^2\\&+(k_s+k_s'-a_s)^2+(l_s+\tfrac{1}{2}l_s'-a_s)^2+(2a_s-l_s')(2a_s-2k_s-k_s'-l_s))\\&+\sum_{1\le t\le q,c_{s,t}\neq 0}c_{s,t}(m_t+c_{s,t}^{-1}(\alpha_{i_s},\alpha_{j_t})l_s')^2,\end{aligned}$$

and the coefficients $c_{s,t}\in\mathbb{Q}_{\ge 0}$ are uniquely determined by the equations

$$\begin{aligned}c_{s,t}&=0 && \text{if } (\alpha_{i_s},\alpha_{j_t})=0,\\ \tfrac{3}{4}(\alpha_{i_s},\alpha_{i_s})&=\sum_{1\le t\le q,c_{s,t}\neq 0}c_{s,t}^{-1}(\alpha_{i_s},\alpha_{j_t})^2 && \text{for } 1\le s\le p,\\ (\alpha_{j_t},\alpha_{j_t})&=\sum_{s=1}^{p}c_{s,t} && \text{for } 1\le t\le q.\end{aligned}$$

Note that $c_{s,t}=0$ if and only if $(\alpha_{i_s},\alpha_{j_t})=0$. By our previous arguments, it is clear that

$$q^{3\sum_{s=1}^{p}(\alpha_{i_s},\alpha_{i_s})a_t^2-\sum_{t=1}^{q}(\alpha_{i_t},\alpha_{i_t})b_t^2}=1.$$

Thus, we only have to consider the zeros of $Q_2(\underline{k},\underline{l},\underline{k}',\underline{l}',\underline{m})$, which occur for $l_s=a_s-k_s$, $k_s'=a_s-k_s$, $l_s'=2k_s$, $m_t=-2k_sc_{s,t}^{-1}(\alpha_{i_s},\alpha_{i_t})$. By (P1) the sets $\{i_1,\dots,i_p\}$ and $\{j_1,\dots,j_q\}$ form a partition of I, especially since I is connected, the solution is already uniquely determined by fixing k_1. Finally, a case-by-case analysis shows that $(\xi^{[l]},\xi^{[l]})\in(l+1)+qA$. □

Example 4.21. Let $\mathfrak{g}$ be of type G_2. We consider (9) of Theorem 4.17:

$$\theta^{[a]}=f_1^{(a)}f_2^{(a)}f_1^{(a)}\text{ and }\xi^{[a]}=f_1^{(a)}f_2^{(a)}f_1^{(2a)}f_2^{(a)}f_1^{(a)}.$$

We want to explicitly determine the set of zeros S_a of

$$Q_2(\underline{k},\underline{l},\underline{k},\underline{l}',\underline{m})=Q_2(k,l,k',l',m)$$

from the previous proof on the domain defined by $0\le k\le a$, $0\le k'\le a-k$, $0\le l\le a-k$, $2a-k-l-k'\le l'\le 2a-k'$ and $0\le m\le a$. For this we understand $Q_2(k,l,k',l',m)$ as an element of the polynomial ring $\mathbb{Q}[k,l,k',l',m]$ with the lexicographic order $k>l>k'>l'>m$. In this case the only coefficient is $c_{1,2}=6$ and thus

$$\begin{aligned}Q_2(k,l,k',l',m)&=(k+k'-a)^2+(k'+\tfrac{1}{2}l'-a)^2+(k+l-a)^2\\&\quad+(l+\tfrac{1}{2}l'-a)^2+(2a-l')(2a-2k-k'-l)+6(m-\tfrac{1}{2}l')^2.\end{aligned}$$

We now consider the ideal J generated by

$$k+k'-a,\quad k'+\tfrac{1}{2}l'-a,\quad k+l-a,\quad l+\tfrac{1}{2}l'-a,\quad m-\tfrac{1}{2}l'.$$

Computing a Groebner basis of J with respect to the above lexicographic order and applying the division algorithm to $Q_2(k,l,k',l',m)$ yields

$$\begin{aligned} Q_2(k,l,k',l',m) = {} & (k-m)(2k+2k'+2l+2l'+2m-8a) \\ & + (l+m-a)(2l+2l'-4a) \\ & + (k'+m-a)(2k'+2l'-4a) \\ & + (l'-2m)(2l'-4m). \end{aligned}$$

Therefore, $S_1 = \{x, y\}$, where $x = (0,1,1,0,0)$ and $y = (1,0,0,2,1)$, and for $a > 1$ we have $S_a = (S_{a-1} + x) \cup (S_{a-1} + y)$. In particular, all zeros lie on the lattice spanned by x and y.

Remark 4.22. In the cases (1), (2), (7) and (8) of Theorem 4.17, Baumann [1] found an explicit description of $\xi^{[l]}$ in terms of the canonical basis. More precisely, he showed that

$$\xi^{[l]} = G(b_{l,1}) + \ldots + G(b_{l,l+1})$$

for certain $b_{l,1}, \ldots, b_{l,l+1} \in B(\infty)$. This agrees with Theorem 4.19 (3), thus we expect to find a similar description in the remaining cases as well. The following corollary is a special case of this for $l = 1$.

Corollary 4.23. *We have $\xi^{[1]} - \theta^{[2]} \in \mathcal{B}(\infty)$.*

Proof. It is obvious that $\xi^{[1]} - \theta^{[2]} \in U_{\mathbb{Z}}^-(\mathfrak{g})$ and $\overline{\xi^{[1]} - \theta^{[2]}} = \xi^{[1]} - \theta^{[2]}$. Furthermore, we have

$$(\xi^{[1]} - \theta^{[2]}, \xi^{[1]} - \theta^{[2]}) = (\xi^{[1]}, \xi^{[1]}) - 2(\xi^{[1]}, \theta^{[2]}) + (\theta^{[2]}, \theta^{[2]}) = 1 \mod qA.$$

Thus, Theorem 2.9 implies that $\xi^{[1]} - \theta^{[2]} \in \pm\mathcal{B}(\infty)$.

Write $\theta^{[1]} = f_{\underline{i}}^{(\underline{a})}$ and $\xi^{[1]} = f_{\underline{j}}^{(\underline{b})}$. Since $\underline{a} \in L_{\underline{i}}$ and $\underline{b} \in L_{\underline{j}}$ fulfill the conditions of Lemma 4.8, we know that $s_{\underline{i}}$ and $s_{\underline{j}}$ are reduced decompositions. By Proposition 4.10, $\underline{a} \in S_{\underline{i}}(\infty)$ and $\underline{b} \in S_{\underline{j}}(\infty)$. We may consider $\underline{a}$ as an element of $S_{\underline{j}}(\infty)$ by extending it to the right with zeros. Thus, we have $\underline{b} < \underline{a}$ with respect to the lexicographic order on $S_{\underline{j}}(\infty)$. Hence, Proposition 2.27 allows us to conclude that $\xi^{[1]} - \theta^{[2]} \in \mathcal{B}(\infty)$. □

Corollary 4.24. *We have $\theta^{[l]}, \xi^{[l]} \in U_{\mathbb{Z}}^-(\mathfrak{g}) \cap L(\infty) \cap \overline{L(\infty)}$ for all $l \in \mathbb{N}$.*

Proof. By definition $\theta^{[l]}, \xi^{[l]} \in U_{\mathbb{Z}}^-(\mathfrak{g})$, $\theta^{[l]} = \overline{\theta^{[l]}}$ and $\xi^{[l]} = \overline{\xi^{[l]}}$. Theorem 4.19 together with Proposition 5.1.3 of [22] imply that $\theta^{[l]}, \xi^{[l]} \in L(\infty)$. □

4.5. **A family of monomials in the canonical basis.** Let $\mathcal{M}$ the subset of $U_q^-(\mathfrak{g})$ consisting of the monomials $f_{i_1}^{(a_1)} \cdots f_{i_n}^{(a_n)}$ satisfying the following properties:

(i) $\underline{i} := (i_1, \ldots, i_n)$ is such that $s_{\underline{i}}$ is a reduced decomposition,
(ii) $(a_1, \ldots, a_n) \in L_{\underline{i}}$,
(iii) every $i \in I$ occurs in $\underline{i}$ at most twice, and
(iv) if $i_1 = i_r$ for some $r > 1$, then

$$f_{i_2}^{(a_2)} \cdots f_{i_r}^{(a_r - a_1)} \cdots f_{i_n}^{(a_n)} \in \mathcal{M}.$$

Let $\nu \in Q_-$, we define $\mathcal{M}_\nu := \mathcal{M} \cap U_q^-(\mathfrak{g})_\nu$.

Proposition 4.25. *Every element of $\mathcal{M}$ belongs to the canonical basis.*

Proof. Let $\nu \in Q_-$ and $u \in \mathcal{M}_\nu$. We prove the proposition by induction on $\mathrm{tr}(\nu)$. If $\mathrm{tr}(\nu) = 0$, clearly $u = 1$ and the statement is true. Now, let $\mathrm{tr}(\nu) < 0$ and assume that the claim is true for all $u' \in \mathcal{M}_{\nu'}$ with $\mathrm{tr}(\nu') > \mathrm{tr}(\nu)$. Since $s_{\underline{i}}$ is a reduced decomposition, by Proposition 4.11 it is sufficient to check that $(u, u) \in 1 + qA$. For this write $u = f_{i_1}^{(a_1)} \cdots f_{i_n}^{(a_n)}$ as in the definition of $\mathcal{M}$.

By property (iii) i_1 occurs at most twice in $\underline{i}$. First, assume that i_1 only occurs once in $\underline{i}$. Then,

$$\begin{aligned}
&(f_{i_1}^{(a_1)} \cdots f_{i_n}^{(a_n)}, f_{i_1}^{(a_1)} \cdots f_{i_n}^{(a_n)}) \\
&= [a_1]_{i_1}!^{-1} (f_{i_2}^{(a_2)} \cdots f_{i_n}^{(a_n)}, e_{i_1}'^{a_1} f_{i_1}^{(a_1)} \cdots f_{i_n}^{(a_n)}) \\
&= [a_1]_{i_1}!^{-1} q_{i_1}^{-2a_1^2 + 2a_1^2 - a_1(a_1-1)/2} (f_{i_2}^{(a_2)} \cdots f_{i_n}^{(a_n)}, f_{i_2}^{(a_2)} \cdots f_{i_n}^{(a_n)})
\end{aligned}$$

Observe that $u' := f_{i_2}^{(a_2)} \cdots f_{i_n}^{(a_n)} \in \mathcal{M}$ and $\mathrm{tr}(\mathrm{wt}\, u') > \mathrm{tr}(\mathrm{wt}\, u)$. Using the induction hypothesis, we have $(u', u') \in 1 + qA$ and since $[a_1]_{i_1}!^{-1} q_{i_1}^{-a_1(a_1-1)/2} \in 1 + qA$, the induction proceeds in this case.

Now, consider the case where i_1 occurs twice in $\underline{i}$ and let r denote the second index with $i_r = i_1$. Put $\gamma = \sum_{t=2}^{r-1} \langle h_{i_1}, \alpha_{i_t} \rangle a_t$. We compute

$$\begin{aligned}
&(f_{i_1}^{(a_1)} \cdots f_{i_n}^{(a_n)}, f_{i_1}^{(a_1)} \cdots f_{i_n}^{(a_n)}) \\
&= [a_1]_i!^{-1} (f_{i_2}^{(a_2)} \cdots f_{i_n}^{(a_n)}, e_{i_1}'^{a_1} f_{i_1}^{(a_1)} \cdots f_{i_n}^{(a_n)}) \\
&= \sum_{k=0}^{a_1} [a_1]_{i_1}!^{-1} q_{i_1}^{-2a_1^2 + 2ka_1 - k(k-1)/2} \begin{bmatrix} a_1 \\ k \end{bmatrix}_{i_1} \\
&\quad \times (f_{i_2}^{(a_2)} \cdots f_{i_n}^{(a_n)}, f_{i_1}^{(a_1-k)} e_{i_1}'^{a_1-k} f_{i_2}^{(a_2)} \cdots f_{i_n}^{(a_n)}) \\
&= \sum_{k=0}^{a_1} [a_1]_{i_1}!^{-1} [a_1 - k]_{i_1}!^{-1} q_{i_1}^{-2a_1^2 + 2ka_1 - k(k-1)/2} \begin{bmatrix} a_1 \\ k \end{bmatrix}_{i_1} \\
&\quad \times (e_{i_1}'^{a_1-k} f_{i_2}^{(a_2)} \cdots f_{i_n}^{(a_n)}, e_{i_1}'^{a_1-k} f_{i_2}^{(a_2)} \cdots f_{i_n}^{(a_n)}) \\
&= \sum_{k=0}^{a_1} [a_1]_{i_1}!^{-1} [a_1 - k]_{i_1}!^{-1} q_{i_1}^{-2a_1^2 + 2ka_1 - k(k-1)/2} \begin{bmatrix} a_1 \\ k \end{bmatrix}_{i_1} \\
&\quad \times q_{i_1}^{-2(a_1-k)\gamma} q_{i_1}^{-4(a_1-k)a_r + 2(a_1-k)(a_1-k+a_r) - (a_1-k)(a_1-k-1)} \\
&\quad \times (f_{i_2}^{(a_2)} \cdots f_{i_r}^{(a_r - a_1 + k)} \cdots f_{i_n}^{(a_n)}, f_{i_2}^{(a_2)} \cdots f_{i_r}^{(a_r - a_1 + k)} \cdots f_{i_n}^{(a_n)}).
\end{aligned}$$

We have

$$q_{i_1}^{-k(k-1)/2 - (a_1-k)(a_1-k-1)} [a_1]_{i_1}!^{-1} [a_1 - k]_{i_1}!^{-1} \begin{bmatrix} a_1 \\ k \end{bmatrix}_{i_1} \in 1 + qA$$

and by property (iv) we know that $u_k' := f_{i_2}^{(a_2)} \cdots f_{i_r}^{(a_r - a_1 + k)} \cdots f_{i_n}^{(a_n)} \in \mathcal{M}$ for any $0 \le k \le a_1$ and $\mathrm{tr}(\mathrm{wt}\, u_k') > \mathrm{tr}(\mathrm{wt}\, u)$. Thus, the induction hypothesis can be applied to u_k' and one obtains $(u_k', u_k') \in 1 + qA$. The remaining powers of q_{i_1} can be written as

$$\eta = q_{i_1}^{2(a_1-k)^2 - 2(a_1-k)(a_1 + a_r - \gamma)}.$$

From property (ii), we obtain $a_1 + a_r - \gamma \le 0$; it follows that $\eta = 1$ for $k = a_1$ and $\eta \in qA$ for $k < a_1$. Thus, only the summand for $k = a_1$ is in $1 + qA$ and the remaining ones are in qA. Hence, $(u, u) \in 1 + qA$ and the induction proceeds. $\square$

Example 4.26. Let $\{i_1, \dots, i_p\}$ and $\{j_1, \dots, j_q\}$ be disjoint subsets of I. Let $\underline{i}$ be a shuffle of $(i_1, \dots, i_p, i_p, \dots, i_1)$ and $(j_1, \dots, j_q)$, and let $\underline{a} = (a_1, \dots, a_{2p+q}) \in L_{\underline{i}}$. Then $f_{\underline{i}}^{(\underline{a})}$ is an element of $\mathcal{M}$ and therefore in the canonical basis.

Note that $\underline{i}$ is not necessarily reduced, however the subsequence consisting of the i_k with $a_k \neq 0$ is. This follows from Lemma 4.8 after modifying $\underline{a}$ appropriately.

The following corollary is now an immediate consequence of the above example.

Corollary 4.27. *We have $\theta^{[l]} \in \mathcal{B}(\infty)$ for all $l \in \mathbb{N}$.*

In particular, this proves Theorem 4.19 (i). We also recover the canonical basis of $U_q^-(\mathfrak{sl}_3)$ as presented in [37]. The result also follows from the proof of Proposition 42.1.5 of [38].

Corollary 4.28. *If $\mathfrak{g}$ is of type A_2, the canonical basis of $U_q^-(\mathfrak{g})$ is*

$$\mathcal{B}(\infty) = \{f_1^{(a)} f_2^{(b)} f_1^{(c)} \mid b \geq a + c\} \cup \{f_2^{(a)} f_1^{(b)} f_2^{(c)} \mid b \geq a + c\}.$$

Proof. The inclusion "$\supseteq$" follows directly from Proposition 4.25. Denote the right-hand side by B. To see the equality, it is sufficient to check that the cardinality of $B \cap U_q^-(\mathfrak{g})_\xi$ equals $\dim U_q^-(\mathfrak{g})_\xi$ for all $\xi \in Q_-$. Write $\xi = -n_1\alpha_1 - n_2\alpha_2$. Using Kostant's partition function, it is easy to see that $\dim U_q^-(\mathfrak{g})_\xi = \min\{n_1, n_2\}$. Now, if $n_2 \geq n_1$, all elements of $B \cap U_q^-(\mathfrak{g})_\xi$ are of the form $f_1^{(a)} f_2^{(n_2)} f_1^{(c)}$ with $a + c = n_1$. Thus, B contains exactly n_1 elements of weight ξ, matching the dimension. The case $n_1 \geq n_2$ is similar. □

4.6. **Semi-tight monomials arising from Lusztig cones.** We show that in A_5, B_3, C_3, D_4 and G_2 a reduced decomposition $\underline{i}$ and vector $\underline{a}$ can be chosen such that $\underline{a} \in L_{\underline{i}}$ but $f_{\underline{i}}^{(\underline{a})}$ is not in the canonical basis. Due to the results of Xi [54] in B_2 and Marsh [42] in A_4 it is known that in these cases every monomial associated with a point in the Lusztig cone is in the canonical basis. Thus, the examples we present are minimal in the sense that for each type, it is the smallest rank that such examples exist.

(i) Type A_5. For the reduced decomposition

$$\underline{i} = (2, 4, 1, 3, 5, 2, 4, 1, 3, 5, 2, 4, 1, 3, 5)$$

we have

$$z_1 = (1, 1, 1, 2, 1, 2, 2, 1, 2, 1, 1, 1, 0, 0, 0) \text{ and}$$
$$z_2 = (0, 0, 1, 2, 1, 3, 3, 2, 4, 2, 3, 3, 1, 2, 1).$$

Since $f_{\underline{i}}^{(z_{1/2})} = \xi_{1/2}^{[1]}$ and $(\xi_{1/2}^{[1]}, \xi_{1/2}^{[1]}) \in 2 + qA$ by Theorem 4.19, Theorem 2.9 implies that $f_{\underline{i}}^{(z_{1/2})}$ is not in the canonical basis. The argument for the remaining cases is similar.

(ii) Type B_3. For the reduced decomposition

$$\underline{i} = (2, 1, 3, 2, 1, 3, 2, 1, 3)$$

we have

$$z_3 = (1, 1, 2, 2, 1, 2, 1, 0, 0) \text{ and } z_4 = (0, 1, 2, 3, 2, 4, 3, 1, 2).$$

(iii) Type C_3. For the reduced decomposition

$$\underline{i} = (2,1,3,2,1,3,2,1,3)$$

we have

$$z_5 = (1,1,1,2,1,1,1,0,0) \text{ and } z_6 = (0,1,1,3,2,2,3,1,1).$$

(iv) Type D_4. For the reduced decomposition

$$\underline{i} = (2,1,3,4,2,1,3,4,2,1,3,4)$$

we have

$$z_7 = (1,1,1,1,2,1,1,1,1,0,0,0) \text{ and}$$
$$z_8 = (0,1,1,1,3,2,2,2,3,1,1,1).$$

(v) Type G_2. For the reduced decomposition

$$\underline{i} = (1,2,1,2,1,2)$$

we have

$$z_9 = (1,1,2,1,1,0) \text{ and } z_{10} = (0,1,3,2,3,1).$$

We now take a closer look at the semi-tight monomials in G_2. A computation using OSCAR [44] shows that

$$f_{\underline{i}}^{(z_9+z_{10})} = f_1 f_2^{(2)} f_1^{(5)} f_2^{(3)} f_1^{(4)} f_2 = G(b_1) + G(b_2) + G(b_3) + G(b_4) + G(b_5)$$

where

$$b_1 = \tilde{f}_1 \tilde{f}_2^2 \tilde{f}_1^5 \tilde{f}_2^3 \tilde{f}_1^4 \tilde{f}_2 b_\infty,$$
$$b_2 = \tilde{f}_1 \tilde{f}_2^3 \tilde{f}_1^8 \tilde{f}_2^3 \tilde{f}_1 b_\infty,$$
$$b_3 = \tilde{f}_1^2 \tilde{f}_2^2 \tilde{f}_1^6 \tilde{f}_2^4 \tilde{f}_1^2 b_\infty,$$
$$b_4 = \tilde{f}_1^2 \tilde{f}_2^3 \tilde{f}_1^5 \tilde{f}_2^2 \tilde{f}_1^3 \tilde{f}_2 b_\infty,$$
$$b_5 = \tilde{f}_1^2 \tilde{f}_2^4 \tilde{f}_1^8 \tilde{f}_2^2 b_\infty.$$

This suggests that there is a two-dimensional region which is not in the canonical basis. This region seems to behave nicely, and hence an explicit description of the corresponding monomials might be possible. Having an explicit description of the monomials in the canonical basis was one of the key ingredients for Xi to write down the canonical basis in A_3 [53] and B_2 [54]. Thus, this could be a starting point for obtaining a characterization in G_2.

5. Quantum Frobenius and the canonical basis

We will first show that both the Frobenius morphism and its splitting are fully compatible (defined below) with the canonical basis if $\mathfrak{g}$ is of type A_1, A_2, A_3, or B_2. Then, we will present counterexamples to full compatibility in types A_5, B_3, C_3, D_4 and G_2. Finally, we discuss which partial orders on $B(\infty)$ yield compatibility up to a filtration and compare them to the partial order $\triangleright_{\pi_\lambda}$ (cf. Section 2.5, LS path order).

Let l be a positive integer and for $i \in I$ let l_i be the smallest positive integer such that $l_i(\alpha_i, \alpha_i)/2 \in l\mathbb{Z}$. We define a new root datum where the simple roots are $\{\alpha_i^* := l_i\alpha_i\}$, the simple coroots are $\{h_i^* := l_i^{-1}h_i\}$, and the weight lattice is

$$P^* = \{\lambda \in P \mid \langle h_i, \lambda\rangle \in l_i\mathbb{Z} \text{ for all } i \in I\}.$$

Remark 5.1. The lattice P^* should not be confused with dual lattice $P^\vee$ of P.

We write $\mathfrak{g}^*$ for the associated Kac-Moody Lie algebra, and correspondingly $U_q(\mathfrak{g}^*)$ for the associated quantized enveloping algebra. By abuse of notation we denote the generators of $\mathfrak{g}^*$ and $U_q(\mathfrak{g}^*)$ also by e_i, f_i ($i \in I$) and h or q^h ($h \in \mathfrak{h}$), respectively. Note that $\operatorname{wt} f_i = \alpha_i$ for $f_i \in U_q(\mathfrak{g})$, while $\operatorname{wt} f_i = \alpha_i^* = l_i\alpha_i$ for $f_i \in U_q(\mathfrak{g}^*)$. Furthermore, we write $q_i^* = q^{(\alpha_i^*, \alpha_i^*)/2} = q_i^{l_i^2}$, $[n]_i^* = (q_i^{*n} - q_i^{*-n})/(q_i^* - q_i^{*-1})$, etc.

Let R be an integral domain. We assume that R contains a primitive l-th or $2l$-th root of unity if l is odd, and a primitive $2l$-th root of unity if l is even. Denote this (primitive) root of unity by ζ. The ring R becomes a $\mathbb{Z}[q, q^{-1}]$-algebra via $q \mapsto \zeta$. We now have the change of rings

$$U_\zeta^-(\mathfrak{g}) = U_\mathbb{Z}^-(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} R \text{ and } U_\zeta^-(\mathfrak{g}^*) = U_\mathbb{Z}^-(\mathfrak{g}^*) \otimes_{\mathbb{Z}[q,q^{-1}]} R.$$

Lusztig [38] constructed two R-algebra homomorphisms: the *Frobenius morphism*

$$\mathrm{Fr}\colon U_\zeta^-(\mathfrak{g}) \to U_\zeta^-(\mathfrak{g}^*), \quad \mathrm{Fr}(f_i^{(n)}) = f_i^{(n/l_i)},$$

here $f_i^{(n/l_i)}$ is understood to be 0 if l_i does not divide n; and the *Frobenius splitting*

$$\mathrm{Fr}'\colon U_\zeta^-(\mathfrak{g}^*) \to U_\zeta^-(\mathfrak{g}), \quad \mathrm{Fr}'(f_i^{(n)}) = f_i^{(l_i n)}.$$

under the following assumptions on the root datum:

(i) for any $i \neq j$ in I such that $l_j \geq 2$, we have $l_i \geq -\langle h_i, \alpha_j\rangle + 1$;
(ii) there is no sequence $i_1, \ldots, i_p, i_{p+1} = i_1$ in I such that $p \geq 3$ is odd and $(\alpha_{i_s}, \alpha_{i_{s+1}}) < 0$ for $s \in [1, p]$, or equivalently, the associated Dynkin diagram has no odd cycles.

Remark 5.2. Observe that both maps agree with the weight grading, that is, for $f_i \in U_\zeta^-(\mathfrak{g})$

$$\operatorname{wt} f_i^{(n)} = n\alpha_i = nl_i^{-1}\alpha_i^* = \operatorname{wt}\mathrm{Fr}(f_i^{(n)}),$$

and for $f_i \in U_\zeta^-(\mathfrak{g}^*)$

$$\operatorname{wt} f_i^{(n)} = n\alpha_i^* = nl_i\alpha_i = \operatorname{wt}\mathrm{Fr}'(f_i^{(n)}).$$

It has already been noted by Lusztig [38] that (i) is not necessary for the existence of Fr', that (ii) may replaced with the assumption that l is odd, and that (ii) is always fulfilled if $\mathfrak{g}$ is of finite type. The map Fr was later also constructed by McGerty [43] using Hall algebras under the assumption that l is coprime to

$(\alpha_i, \alpha_i)/2$ for all $i \in I$. In this case, he was able to prove the existence of Fr without any further restrictions on the root datum. It is conjectured that both maps exist without any restrictions.

Song [49] recently investigated both maps in the context of cluster algebras. He proved, under the assumption that l is odd and coprime to $(\alpha_i, \alpha_i)/2$ for all $i \in I$, that duals of the Frobenius morphism and its splitting are compatible with the cluster monomials.

Both maps play an important role in representation theory and geometry; for example: Fr is the obvious quantum analogue of the Frobenius morphism for reductive groups over a field of finite characteristic, which plays an important role in their representation theory. Kumar and Littelmann [28] showed that the dual of Fr can be viewed as characteristic zero analogue of the geometric Frobenius splitting. The path vectors, which we will discuss in detail in Section 6, were constructed by Littelmann [35] using the Frobenius splitting.

On the level of crystals Kashiwara [23] constructed an embedding $S_\infty \colon B(\infty) \to B(\infty)$, which has very similar properties to the Frobenius splitting. For any $b \in B(\infty)$ and $i \in I$:

$$S_\infty(\tilde{f}_i b) = \tilde{f}_i^{l_i} S_\infty(b), \quad S_\infty(\tilde{e}_i b) = \tilde{e}_i^{l_i} S_\infty(b),$$
$$\epsilon_i(S_\infty(b)) = l_i \epsilon_i(b), \quad \varphi_i(S_\infty(b)) = l_i \varphi_i(b), \quad \mathrm{wt}(S_\infty(b)) = \mathrm{wt}(b).$$

It is thus interesting to study the compatibility of the canonical basis with the Frobenius morphism and its splitting. This was first done by Baumann [1].

Following *loc.cit.*, for $b \in B(\infty)$ we say that $G(b)$ is *compatible* with the Frobenius morphism if

$$\mathrm{Fr}(G(b)) = \begin{cases} G(b'), & \text{if there is } b' \in B(\infty) \text{ with } b = S_\infty(b'), \\ 0, & \text{else.} \end{cases}$$

Note for the first case that S_∞ is an embedding, as such there is at most one preimage and the condition makes sense. If the above property holds for all $b \in B(\infty)$, we say that the canonical basis is *fully compatible* with the Frobenius morphism.

Similarly, we say that $G(b)$ is *compatible* with the Frobenius splitting if

$$\mathrm{Fr}'(G(b)) = G(S_\infty(b));$$

and if this property holds for all $b \in B(\infty)$, we say that the canonical basis is *fully compatible* with the Frobenius splitting.

5.1. **Compatibility.**

Lemma 5.3 ([38, Lemma 35.1.5])**.** *For $a \in \mathbb{Z}$ and $t \in \mathbb{N}$, we have the following equality in R:*

$$\begin{bmatrix} a \\ t \end{bmatrix}_i^* = \begin{bmatrix} l_i a \\ l_i t \end{bmatrix}_i.$$

Proposition 5.4. *Assume that $\mathfrak{g}$ is of type A_1, A_2, A_3, or B_2. Then, the Frobenius splitting is fully compatible with the canonical basis.*

Proof. Let $b \in B(\infty)$. Then, we know (cf. Proposition 5.8) that $G(S_\infty(b))$ occurs with coefficient 1 in the expansion of $\mathrm{Fr}'(G(b))$ in the canonical basis of $U_\zeta^-(\mathfrak{g})$. Thus, it is sufficient to show that $\mathrm{Fr}'(G(b))$ is an element of the canonical basis to prove compatibility.

Note that A_1 and A_2 are simply-laced, that is $l_i = l_j$ for any $i, j \in I$. Hence, it is clear from the definition of the canonical basis in A_1 (Example 2.12) and A_2 (Example 2.13) that it is compatible the Frobenius splitting.

We prove the A_3-case using the explicit description of the canonical basis from Example 2.14. A_3 is simply-laced, thus $l_i = l_j$ for any $i, j \in I$; we write t for this integer. It follows that the Frobenius splitting is compatible with the monomials (i) - (viii). The computation for the elements (ix) - (xiv) is similar; we restrict ourselves to verifying (ix). In this case the element in the canonical basis of $U_\zeta^-(\mathfrak{g}^*)$ is

$$u = \sum_{0 \le k \le c} \begin{bmatrix} d+e-a-b \\ k \end{bmatrix}^* f_2^{(b)} f_3^{(a+b+k)} f_2^{(d)} f_1^{(c+e+f)} f_2^{(c+e)} f_3^{(c-k)},$$

where $f \ge d$, $e \ge b$ and $a + b \ge d + e$. The image under the Frobenius splitting is

$$\mathrm{Fr}'(u) = \sum_{0 \le k \le c} \begin{bmatrix} td+te-ta-tb \\ tk \end{bmatrix} f_2^{(tb)} f_3^{(ta+tb+tk)} f_2^{(td)} f_1^{(tc+te+tf)} f_2^{(tc+te)} f_3^{(tc-tk)}.$$

Using Lemma 2.4, we see that

$$\mathrm{Fr}'(u) = \sum_{0 \le k \le tc} \begin{bmatrix} td+te-ta-tb \\ k \end{bmatrix} f_2^{(tb)} f_3^{(ta+tb+k)} f_2^{(td)} f_1^{(tc+te+tf)} f_2^{(tc+te)} f_3^{(tc-k)},$$

since the q-binomial coefficient vanishes unless k is a multiple of t. Therefore, $\mathrm{Fr}'(u)$ is an element of the canonical basis of $U_\zeta^-(\mathfrak{g}^*)$.

We prove the B_2-case using the explicit description of the canonical basis from Example 2.15. If l is odd, then $l_1 = l_2$ and the same arguments as for A_3 may be used. Now, assume that l is even; so in $\mathfrak{g}$ the root α_1 is long, while in $\mathfrak{g}^*$ the root α_1^* is short. We only consider the cases (i), (iii) and (v); the other cases are similar. Note that $2l_1 = l_2$ and since the in equalities in (i) - (vi) define a cone we may restrict ourselves to $l_1 = 1$ and $l_2 = 2$ without loss of generality.

For (i) the element in the canonical basis of $U_\zeta^-(\mathfrak{g}^*)$ is

$$u = f_1^{(a)} f_2^{(b+c)} f_1^{(2b+c)} f_2^{(d)},$$

where $c \ge a$ and $b \ge d$. The image under the Frobenius splitting is

$$\mathrm{Fr}'(u) = f_1^{(a)} f_2^{(2b+2c)} f_1^{(2b+c)} f_2^{(2d)}.$$

We need that $2b+2c \ge a+2b+c$ and $4b+2c \ge 2b+2c+2d$ for $f_1^{(a)} f_2^{(2b+2c)} f_1^{(2b+c)} f_2^{(2d)}$ to be in the canonical basis of $U_\zeta^-(\mathfrak{g})$. This is case, since the inequalities are equivalent to $c \ge a$ and $b \ge d$, respectively.

For (iii) the element in the canonical basis of $U_\zeta^-(\mathfrak{g})$ is

$$u = \sum_{0 \le k \le 2b+c} \begin{bmatrix} c-a \\ k \end{bmatrix}_1^* f_1^{(a+k)} f_2^{(b+c)} f_1^{(2b+c-k)} f_2^{(d)},$$

where $c \le a$ and $b \ge d$. The image under the Frobenius splitting is

$$\mathrm{Fr}'(u) = \sum_{0 \le k \le 2b+c} \begin{bmatrix} c-a \\ k \end{bmatrix}_1 f_1^{(a+k)} f_2^{(2b+2c)} f_1^{(2b+c-k)} f_2^{(2d)}.$$

Applying the $*$-invariance of the canonical basis (Corollary 2.8) to (iv), we see that

$$\sum_{0 \le k \le b'+c'} \begin{bmatrix} b'-d' \\ k \end{bmatrix}_1 f_1^{(d'+k)} f_2^{(2b'+c')} f_1^{(b'+c'-k)} f_2^{(a')}$$

is an element of the canonical basis of $U_\zeta^-(\mathfrak{g})$ for $c' \geq a'$ and $b' \leq d'$. Choosing $a' = 2d$, $b' = c$, $c' = 2b$, and $d' = a$, we see that $\mathrm{Fr}'(u)$ is an element of the canonical basis of $U_\zeta^-(\mathfrak{g})$.

For (v) the element in the canonical basis of $U_\zeta^-(\mathfrak{g}^*)$ is

$$u = \sum_{\substack{0\leq j\leq 2b+c\\ 0\leq k\leq b+c}} \begin{bmatrix} c-a \\ j \end{bmatrix}_1^* \begin{bmatrix} b-d-j \\ k \end{bmatrix}_2^* f_1^{(a+j)} f_2^{(b+c-k)} f_1^{(2b+c-j)} f_2^{(d+k)}.$$

where $c \leq a$, $b \leq d$ and $c + d \geq a + b$. The image under the Frobenius splitting is

$$\mathrm{Fr}'(u) = \sum_{\substack{0\leq j\leq 2b+c\\ 0\leq k\leq b+c}} \begin{bmatrix} c-a \\ j \end{bmatrix}_1 \begin{bmatrix} 2b-2d-2j \\ 2k \end{bmatrix}_2 f_1^{(a+j)} f_2^{(2b+2c-2k)} f_1^{(2b+c-j)} f_2^{(2d+2k)}.$$

Using the $*$-invariance on (vi), we see that

$$\sum_{\substack{0\leq j\leq 2b'+c'\\ 0\leq k\leq b'+c'}} \begin{bmatrix} c'-a'-2k \\ j \end{bmatrix}_2 \begin{bmatrix} b'-d' \\ k \end{bmatrix}_1 f_1^{(d'+k)} f_2^{(2b'+c'-j)} f_1^{(b'+c'-k)} f_2^{(a'+j)}$$

is an element of the canonical basis of $U_\zeta^-(\mathfrak{g})$ for $c' \leq a'$, $b' \leq d'$ and $c'+2d' \leq a'+2b'$. Choose $a' = 2d$, $b' = c$, $c' = 2b$, and $d' = a$. Observing that the first binomial coefficient vanishes unless j is even, we see that $\mathrm{Fr}'(u)$ is an element of the canonical basis of $U_\zeta^-(\mathfrak{g})$. □

Proposition 5.5. *Assume that $\mathfrak{g}$ is of type A_1, A_2, A_3, or B_2. Then, the Frobenius morphism is fully compatible with the canonical basis.*

Proof. For an element u of the canonical basis of $U_\zeta^-(\mathfrak{g})$, it is sufficient to show that $\mathrm{Fr}(u) = 0$ if u is not in the image of the Frobenius splitting, since $\mathrm{Fr} \circ \mathrm{Fr}' = \mathrm{id}$.

The statement for A_1 and A_2 is clear from the definition of the Frobenius morphism; for A_3 we will again consider only the case (ix) from Example 2.14:

$$u = \sum_{0\leq k\leq c} \begin{bmatrix} d+e-a-b \\ k \end{bmatrix} f_2^{(b)} f_3^{(a+b+k)} f_2^{(d)} f_1^{(c+e+f)} f_2^{(c+e)} f_3^{(c-k)},$$

where $f \geq d$, $e \geq b$ and $a+b \geq d+e$. From the definition of the Frobenius morphism we see that $\mathrm{Fr}(u) = 0$, unless l divides all exponents. In particular, l must divide $d + (c + e) - (a + b + k) - (c - k) = d + e - a - b$. Hence, for a summand not to vanish, k must be a multiple of l, as otherwise the q-binomial would be zero in R by Lemma 2.4. It follows that every summand in the definition of u vanishes under Fr unless a, b, c, d, e and f are multiples of l. In this case, the arguments in the proof for the Frobenius splitting imply that u is indeed in the image of the Frobenius splitting.

For B_2, we only consider the case (v) from Example 2.15:

$$u = \sum_{\substack{0\leq j\leq 2b+c\\ 0\leq k\leq b+c}} \begin{bmatrix} c-a \\ j \end{bmatrix}_2 \begin{bmatrix} b-d-j \\ k \end{bmatrix}_1 f_2^{(a+j)} f_1^{(b+c-k)} f_2^{(2b+c-j)} f_1^{(d+k)},$$

where $c \leq a$, $b \leq d$ and $c + d \geq a + b$. Any summand vanishes under the Frobenius morphism unless l_1 divides $b + c - k$ and $d + k$, and l_2 divides $a + j$ and $2b + c - j$.

Since $l_1 \in \{l_2/2, l_2\}$, we must have that l_1 divides

$$(2b + c - j) - (b + c - k) - (d + k) = b - d - j.$$

Thus, any summand vanishes unless k is divisible by l_1. Furthermore, we must have that $l_2 \in \{l_1, 2l_1\}$ divides

$$2(b + c - k) - (2b + c - j) - (a + j) = c - a - 2k.$$

Since, $2k$ is divisble by l_2, in particular, $c - a$ must be divisible by l_2. Hence, any summand vanishes unless j is divisble by l_2. We conclude that every summand in the definition of u vanishes under Fr unless a and c are multiples of l_2, and b and d are multiples of l_1. Now, the arguments in the proof for the Frobenius splitting imply that u is indeed in the image of the Frobenius splitting. □

Remark 5.6. The result of Proposition 5.4 and 5.5 was already mentioned in [1] as an observation by Littelmann, however no proof was provided.

5.2. **Counterexamples to full compatibility.** The first counterexamples to full compatibility of the Frobenius morphism and splitting with the canonical basis were provided by Baumann [1] for the types A_5 and D_4. The monomials he used in the counterexamples are our monomials $\xi_1^{[l]}$, $\xi_2^{[l]}$ for A_5 and $\xi_7^{[l]}$, $\xi_8^{[l]}$ for D_4 (see Theorem 4.17). We present a different argument for A_5 and D_4; the results for B_3, C_3 and G_2 are new.

First, we present the counterexamples for the Frobenius splitting. For simplicity assume that l is coprime to $(\alpha_i, \alpha_i)/2$ for all $i \in I$. Let $\mathfrak{g}$ be of type A_5, B_3, C_3, D_4 or G_2 and let θ and ξ be chosen from the list in Theorem 4.17 correspondingly. Then, $\xi^{[1]} - \theta^{[2]} \in \mathcal{B}(\infty)$ by Corollary 4.23. We have

$$\mathrm{Fr}'(\xi^{[1]} - \theta^{[2]}) = \xi^{[l]} - \theta^{[2l]}$$

and computing the bilinear form gives

$$(\xi^{[l]} - \theta^{[2l]}, \xi^{[l]} - \theta^{[2l]}) = (\xi^{[l]}, \xi^{[l]}) - 2(\xi^{[l]}, \theta^{[2l]}) + (\theta^{[2l]}, \theta^{[2l]}) = l.$$

Hence, $\xi^{[l]} - \theta^{[2l]} \notin \mathcal{B}(\infty)$, by Theorem 2.9.

For the counterexamples for the Frobenius morphism, we need to distinguish between types.

If $\mathfrak{g}$ is of type A_5, D_4 or G_2, we consider $l = 2$ (assuming that l is coprime to $(\alpha_i, \alpha_i)/2$ for all $i \in I$). Then,

$$\mathrm{Fr}(\xi^{[1]} - \theta^{[2]}) = -\theta^{[1]} \notin \mathcal{B}(\infty).$$

This follows directly from the fact that $\theta^{[1]} \in \mathcal{B}(\infty)$ by Corollary 4.27.

If $\mathfrak{g}$ is of type C_3, we consider $l = 2$ and the bilinear form determined by $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$ and $(\alpha_3, \alpha_3) = 4$. Thus, $l_1 = l_2 = 2$ and $l_3 = 1$. We have

$$f_2(f_1 f_3) f_2^{(2)} (f_3 f_1) f_2 - f_2^{(2)} (f_1^{(2)} f_3^{(2)}) f_2^{(2)} \in \mathcal{B}(\infty),$$

by Corollary 4.23, and

$$\mathrm{Fr}(f_2(f_1 f_3) f_2^{(2)} (f_3 f_1) f_2 - f_2^{(2)} (f_1^{(2)} f_3^{(2)}) f_2^{(2)}) = -f_2(f_1 f_3^{(2)}) f_2.$$

Since $f_2(f_1 f_3^{(2)}) f_2$ is an element of the canonical basis of $U_\zeta^-(\mathfrak{g}^*)$ (type B_3) by Corollary 4.27, $-f_2(f_1 f_3^{(2)}) f_2$ is not.

If $\mathfrak{g}$ is of type B_3, we consider $l = 4$ and the bilinear form determined by $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 4$ and $(\alpha_3, \alpha_3) = 2$. Thus, $l_1 = l_2 = 2$ and $l_3 = 4$. We have

$$f_2(f_1 f_3^{(2)}) f_2^{(2)} (f_3^{(2)} f_1) f_2 - f_2^{(2)} (f_1^{(2)} f_3^{(4)}) f_2^{(2)} \in \mathcal{B}(\infty),$$

by Corollary 4.23, and

$$\mathrm{Fr}(f_2(f_1 f_3^{(2)}) f_2^{(2)} (f_3^{(2)} f_1) f_2 - f_2^{(2)} (f_1^{(2)} f_3^{(4)}) f_2^{(2)}) = -f_2(f_1 f_3) f_2.$$

Since $f_2(f_1 f_3) f_2$ is an element of the canonical basis of $U_\zeta^-(\mathfrak{g}^*)$ (type C_3) by Corollary 4.27, $-f_2(f_1 f_3) f_2$ is not.

Remark 5.7. In view of Remark 4.22, we expect that the counterexamples for the Frobenius morphism can be generalized to all $l \geq 2$ that are coprime to $(\alpha_i, \alpha_i)/2$ for all $i \in I$.

5.3. **Compatibility up to filtration.** We have seen in the previous section that, in general, the Frobenius morphism and its splitting are not fully compatible with the canonical basis. However, this is not necessary to prove the unitriangular base change between the canonical basis and the path vectors (Conjecture 1). For a dominant weight λ, it is sufficient to find a partial order $\geq$ on $B(\infty)$ such that $b \geq b'$ implies $\mathrm{pr}_\lambda(b) \triangleright_{\pi_\lambda} \mathrm{pr}_\lambda(b')$ for all $b, b' \in B(\infty)$ and we have compatibility up to filtration with respect to this partial order. That is to say, considering an appropriate partial order allows us to neglect undesired terms introduced by the Frobenius splitting. We present several partial orders that yield compatibility with the Frobenius splitting at least up to a filtration and compare them to the partial order $\triangleright_{\pi_\lambda}$. Here we assume that $\mathfrak{g}$ is of finite type and that $\underline{i}$ is a sequence in I such that $s_{\underline{i}}$ is a reduced decomposition of w_0.

Lexicographic order on adapated strings. Let $\geq$ be the total order on the crystal $B(\infty)$ given by the lexicographic order on the adapted string cone $S_{\underline{i}}(\infty)$. Littelmann showed that there is a unitriangular base change between the canonical basis and the adapted string basis (cf. Theorem 2.27). The following result is an easy consequence of this fact.

Proposition 5.8. *The Frobenius splitting is compatible with the canonical basis up to a filtration with respect to the lexicographic order on the string cone.*

However, the following example shows that the total order $\geq$ is not compatible with $\triangleright_{\pi_\lambda}$.

Example 5.9. Assume that $\mathfrak{g}$ is of type A_2 and let $\lambda = 2\rho$. We consider

$$B(\lambda)_0 = \{\pi_0 = (s_1 s_2 s_1, \mathrm{id}; \tfrac{1}{2}, 1), \pi_1 = (s_2 s_1, s_1; \tfrac{1}{2}, 1), \pi_2 = (s_1 s_2, s_2; \tfrac{1}{2}, 1)\}.$$

Then, $\pi_0 \triangleright \pi_1, \pi_2$ and the paths π_1 and π_2 are incomparable. If we now fix the reduced decomposition $s_1 s_2 s_1$, $\pi_2 > \pi_0 > \pi_1$, and if we fix $s_2 s_1 s_2$, $\pi_1 > \pi_0 > \pi_2$. In both cases π_0 is neither the greatest or smallest element, hence the partial order $\triangleright_{\pi_\lambda}$ cannot linearized in a way compatible with $\geq$.

Lexicographic order on Lusztig data. Let $\geq$ be the total order on the crystal $B(\infty)$ given by the lexicographic order on the Lusztig data with respect to $\underline{i}$. Baumann [1, Lemma 6.2] showed that the Frobenius splitting is compatible with this order up to a filtration. However, the following example shows that this order is not compatible with $\triangleright_{\pi_\lambda}$.

Example 5.10. Assume that $\mathfrak{g}$ is of type A_2 and let $\lambda = 2\rho$. We consider

$$B(\lambda)_0 = \{\pi_0 = (s_1s_2s_1, \mathrm{id}; \tfrac{1}{2}, 1), \pi_1 = (s_2s_1, s_1; \tfrac{1}{2}, 1), \pi_2 = (s_1s_2, s_2; \tfrac{1}{2}, 1)\}.$$

Then, $\pi_0 \triangleright \pi_1, \pi_2$ and the paths π_1 and π_2 are incomparable. If we fix the reduced decomposition $s_1s_2s_1$, the Lusztig data are $(1,1,1)$ for π_0, $(0,2,0)$ for π_1 and $(2,0,2)$ for π_2. Hence, $\pi_2 > \pi_0 > \pi_1$. Since π_0 is neither the greatest nor the smallest element, the partial order $\triangleright_{\pi_\lambda}$ cannot be linearized in a way compatible with $\geq$. For the same reasons choosing the reduced decomposition $s_2s_1s_2$ does not yield compatibility either.

String order. For $(b_1, b_2), (b_1', b_2') \in B(\infty) \times B(\infty)$, we write $(b_1, b_2) \approx (b_1', b_2')$ if

(i) there exists $i \in I$ such that $\epsilon_i(b_1) = \epsilon_i(b_2)$ and $(b_1', b_2') = (\tilde{f}_i b_1, \tilde{f}_i b_2)$, or
(ii) there exists $i \in I$ such that $\epsilon_i(b_1) = \epsilon_i(b_2) > 0$ and $(b_1', b_2') = (\tilde{e}_i b_1, \tilde{e}_i b_2)$, or
(iii) $(b_1', b_2') = (*b_1, *b_2)$.

For $b_1, b_2 \in B(\infty)$, we write $b_1 \leq b_2$ if b_1 and b_2 have the same weight and if for all sequences

$$(b_1, b_2) \approx (b_1', b_2') \approx \ldots \approx (b_1'', b_2''),$$

one has $\epsilon_i(b_1'') \leq \epsilon_i(b_2'')$ for all $i \in I$.

The relation $\geq$ is called *string order*; it is a partial order on $B(\infty)$ by Proposition 2.6 of [1]. Baumann showed in *loc.cit.* that there is the following compatibility up to filtration.

Proposition 5.11 ([1, Proposition 6.1])**.** *Let $b, b' \in B(\infty)$.*

(i) If $G(b')$ occurs in the expansion of $\mathrm{Fr}(G(S_\infty(b)))$ in the canonical basis, then $b \leq b'$. Moreover, $G(b)$ occurs with coefficient 1.
(ii) If $G(b')$ occurs in the expansion of $\mathrm{Fr}'(G(b))$ in the canonical basis, then $b \leq b'$. Moreover, $G(S_\infty(b))$ occurs with coefficient 1.

This amount of compatibility would be sufficient to prove the unitriangular base change between the canonical basis and the path vectors if the string order was compatible with $\triangleright_{\pi_\lambda}$. However, the next two examples, when combined together, show that this is not the case.

Example 5.12. The string order in A_3 is trivial by Example 2.7 of [1]. However, if a dominant weight λ is large enough, Theorem 6.10 shows that the dual canonical basis of $V(\lambda)$ and the path vector basis do not coincide. Thus, the string order is too coarse for a meaningful comparison of both bases.

Example 5.13. In type A_4 consider the elements $b = \tilde{f}_3^4\tilde{f}_2^4\tilde{f}_1^4\tilde{f}_4^4\tilde{f}_3^4\tilde{f}_2^4 b_\infty$ and $b' = \tilde{f}_2\tilde{f}_1^3\tilde{f}_4\tilde{f}_3^7\tilde{f}_2^7\tilde{f}_1\tilde{f}_4^3\tilde{f}_3 b_\infty$ of $B(\infty)$. We know from Example 2.7 of [1] that $b < b'$. Consider $\lambda = 5\rho$. Then, b is realized through the LS path

$$\pi = (s_3s_4s_2s_3s_1s_2, s_4s_2s_3s_1s_2, s_3s_1s_2, s_2, \mathrm{id}; \tfrac{1}{5}, \tfrac{4}{15}, \tfrac{2}{5}, \tfrac{4}{5}, 1)$$

and b' is realized through the LS path

$$\pi' = (s_2s_3s_4s_1s_2s_3s_2s_1, s_3s_4s_1s_2s_3s_2s_1, s_3s_4s_3s_1s_2, s_3s_4s_3s_2, s_2; \tfrac{1}{15}, \tfrac{1}{5}, \tfrac{3}{10}, \tfrac{2}{5}, 1).$$

Since $\iota(\pi)$ and $\iota(\pi')$ are incomparable in the Bruhat order, π and π' are incomparable with respect to $\triangleright_{\pi_\lambda}$. In fact, this statement is true for every dominant weight λ such that $\mathrm{pr}_\lambda(b), \mathrm{pr}_\lambda(b') \neq 0$.

Mirkovic-Vilonen polytope order. Kamnitzer [20, Theorem 7.1] showed that there is a bijection $b \mapsto \mathrm{Pol}(b)$ between $B(\infty)$ and the set of stable Mirković-Vilonen polytopes (MV polytopes). Thus, there is a partial order on $B(\infty)$ given by $b \leq_{\mathrm{pol}} b'$ if $\mathrm{Pol}(b) \subseteq \mathrm{Pol}(b')$. Baumann [1, Proposition 3.1] showed that this order is equivalent to the following order $\geq$.

Let $\{\varpi_i^\vee \mid i \in I\} \subseteq P^\vee$ such that $\langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}$ for any $i, j \in I$. Fix $\underline{i} = (i_1, \ldots, i_N) \in \mathcal{H}(w_0)$ and put

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \text{ and } \gamma_k = -s_{i_1} \cdots s_{i_k} \varpi_{i_k}^\vee.$$

For $b, b' \in B(\infty)$, we write $b \leq b'$ if for all $\underline{i} \in \mathcal{H}(w_0)$ and $k \in [1, N]$ we have

$$\sum_{t=1}^{k} \langle \gamma_k, \beta_t \rangle n_t \leq \sum_{t=1}^{k} \langle \gamma_k, \beta_t \rangle n'_t,$$

where $(n_1, \ldots, n_N)$ and $(n'_1, \ldots, n'_N)$ are the Lusztig data of b and b' with respect to $\underline{i}$, respectively.

We have the following compatibility up to filtration.

Proposition 5.14 ([1, Proposition 6.3])**.** *Let $b, b' \in B(\infty)$.*

(i) If $G(b')$ occurs in the expansion of $\mathrm{Fr}(G(b))$ in the canonical basis, then $b \leq S_\infty(b')$.

(ii) If $G(b')$ occurs in the expansion of $\mathrm{Fr}'(G(b))$ in the canonical basis, then $b' \leq b$.

The following example shows that the MV polytope order is not compatible with $\rhd_{\pi_\lambda}$.

Example 5.15. In type A_3 consider $b = (\tilde{e}_1\tilde{e}_3)\tilde{e}_2^2(\tilde{e}_1\tilde{e}_3)b_\infty$ and $b' = \tilde{e}_2(\tilde{e}_1\tilde{e}_3)^2\tilde{e}_2 b_\infty$. Then, $b < b'$ by Example 3.2 of [1]. Note that $w_1 = s_1s_3s_2s_1s_3$ and $w' = s_2s_1s_3s_2$ are both reduced decompositions; and that w and w' are incomparable in the Bruhat order. It follows that for every dominant weight λ, such that $\mathrm{pr}_\lambda(b), \mathrm{pr}_\lambda(b') \neq 0$, the associated LS paths are incomparable.

LS path order. Fix a sequence of dominant paths $\underline{\eta} = (\eta_1, \eta_2, \ldots)$ such that

$$\sum_{k=1}^{\infty} \langle h_i, \mathrm{wt}\, \eta_k \rangle = \infty \text{ for any } i \in I,$$

and Conjectures 3.9 and 3.15 hold for all $V(\mathrm{wt}\, \eta_k)$ with respect to η_k. The following result is an immediate consequence of the unitriangular base change between the canonical basis and the path monomial basis for $\underline{\eta}$ (Theorem 3.34).

Proposition 5.16. *Let $b, b' \in B(\infty)$.*

(i) If $G(b')$ occurs in the expansion of $\mathrm{Fr}(G(b))$ in the canonical basis, then $b \trianglerighteq_{\underline{\eta}} S_\infty(b')$. Moreover, if $b = S_\infty(b')$, then $G(b')$ occurs with coefficient 1.

(ii) If $G(b')$ occurs in the expansion of $\mathrm{Fr}'(G(b))$ in the canonical basis, then $b' \trianglerighteq_{\underline{\eta}} S_\infty(b)$. Moreover, $G(S_\infty(b))$ occurs with coefficient 1.

6. Path Vectors

We first study the path vectors in A_2. Through an elementary computation, we obtain a closed formula to compute the evaluation of path vectors on monomials. This allows us to give an explicit description of the base change from the path vector basis to the dual canonical basis. Furthermore, we show that the path vectors are not a perfect basis in the sense of Baumann, Kamnitzer and Knutson [2].

In the second part of this section, we compare the path vectors with the dual canonical basis in the general case. We show that Conjecture 1 holds true if $\mathfrak{g}$ is of type A_1, A_2, A_3 or B_2, that is there is a unitriangular base change from the path vectors to the dual canonical basis in these cases. Additionally, we prove that if element of the dual canonical basis is compatible with dual Frobenius splitting, then it agrees with the corresponding path vector up to terms which are greater with respect to $\triangleright$. Finally, we show that there is a unitriangular base change from the path vector basis to the dual canonical basis for the dominant path $\eta = \pi_{m_1\varpi_1} * \cdots * \pi_{m_N\varpi_N}$ if $\mathfrak{g}$ is a classical Lie algebra or of type G_2.

Let $U_{\mathbb{Z}}(\mathfrak{g})$ be the Kostant $\mathbb{Z}$-form of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Recall that $U_{\mathbb{Z}}(\mathfrak{g})$ is the $\mathbb{Z}$-subalgebra of $U(\mathfrak{g})$ generated by the divided powers $e_i^{(n)} := e_i^n/n!$, $f_i^{(n)} := f_i^n/n!$ and

$$\binom{h_i;k}{n} := \frac{(h_i+k)(h_i+k-1)\cdots(h_i+k-n+1)}{n!}$$

for $i \in I$, $k \in \mathbb{Z}$, and $n \in \mathbb{Z}_{\geq 0}$.

For a dominant weight λ, we denote the simple $U(\mathfrak{g})$-module of highest weight λ by $M(\lambda)$; we write v_λ for the highest weight vector of $V(\lambda)$.

Let R the ring obtained from $\mathbb{Z}$ by adjoining all roots of unity. We denote the *Weyl module* over R by

$$M_R(\lambda) := M_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} R,$$

where $M_{\mathbb{Z}}(\lambda) := U_{\mathbb{Z}}^-(\lambda)v_\lambda$ is the $\mathbb{Z}$-form of $M(\lambda)$.

The path vectors defined by Littelmann [35] are a basis of $M_R(\lambda)^\vee$; we recall their construction.

We define $d_i := (\alpha_i, \alpha_i)/2$ and $d := \operatorname{lcm} d_i$. The diagonal matrix $(d_i)_{i\in I}$ is the symmetrizer of the Cartan matrix associated to $\mathfrak{g}$.

Let $\pi = (\tau_1, \ldots, \tau_r; t_1, \ldots, t_r)$ be an LS path of shape λ and let l be the smallest positive integer such that $lt_k \in \mathbb{N}$ for all $k \in [1, r]$ and l is divisble by $2d$. Put $l' := ld$, $l_i := ld_i/d$ and let $\zeta \in R$ be a primitive $2l'$-th root of unity.

We construct a new root datum where the simple roots are $\{\alpha_i' := (d/d_i)\alpha_i \mid i \in I\}$, the simple coroots are $\{h_i' := (d_i/d)h_i \mid i \in I\}$ and the weight lattice is

$$P' := \{\lambda \in P \mid \langle h_i, \lambda\rangle \in (d/d_i)\mathbb{Z} \text{ for all } i \in I\}.$$

Littelmann [35] showed that there is an R-algebra homomorphism $\Phi\colon U_R(\mathfrak{g}) \to U_\zeta(\mathfrak{g}')$,

$$e_i^{(n)} \mapsto e_i^{(l_i n)}, \qquad f_i^{(n)} \mapsto f_i^{(l_i n)}, \quad \binom{h_i;a}{n} \mapsto \begin{bmatrix} t_i; l_i a \\ l_i n \end{bmatrix}.$$

Remark 6.1. For the restriction of Φ to the homomorphism $U_R^-(\mathfrak{g}) \to U_\zeta^-(\mathfrak{g}')$ this follows from the results of Lusztig on the Frobenius splitting [38]; let us explain how: Starting from the root datum of $\mathfrak{g}'$ and the integer l', following Lusztig we

define a new root datum with simple roots $\{\alpha_i^* := l_i\alpha_i' \mid i \in I\}$, simple coroots $\{h_i^* := l_i^{-1}h_i' \mid i \in I\}$ and weight lattice

$$P^* := \{\lambda \in P' \mid \langle h_i', \lambda\rangle \in l_i\mathbb{Z} \text{ for all } i \in I\}.$$

Let $\mathfrak{g}^*$ be the corresponding Kac-Moody Lie algebra and $U_q(\mathfrak{g}^*)$ the associated quantum group.

By construction, we have for any $i \in I$,

$$\frac{l_i(\alpha_i', \alpha_i')}{2} = \frac{ld_i}{d}\left(\frac{d}{d_i}\right)^2 \frac{(\alpha_i, \alpha_i)}{2} = ld = l'.$$

Thus, we are in indeed in the situation of [38]. Since l_i is even, we have

$$\zeta^{(\alpha_i^*, \alpha_i^*)/2} = \zeta^{l^2(\alpha_i, \alpha_i)/2} = \zeta^{l'(ld_i/d)} = \zeta^{l'l_i} = 1.$$

As $\mathfrak{g}$ and $\mathfrak{g}^*$ are of the same type, the above equation implies that we have an ismorphism of R-algebras $U_1(\mathfrak{g}) \cong U_\zeta(\mathfrak{g}^*)$. Since $U_R(\mathfrak{g})$ embeds into $U_1(\mathfrak{g})$ via

$$e_i^{(n)} \mapsto e_i^{(n)}, \quad f_i^{(n)} \mapsto f_i^{(n)}, \quad \binom{h_i; a}{n} \mapsto \begin{bmatrix} h_i^*; a \\ n \end{bmatrix},$$

we obtain the desired homomorphism $U_R^-(\mathfrak{g}) \to U_\zeta^-(\mathfrak{g}')$. Note however, that compared to [38] this homomorphism does not respect the weight grading. More precisely, we have $\alpha_i^* = l\alpha_i$, $h_i^* = l^{-1}\alpha_i$ and thus $M_R(\lambda) \cong V_1^*(l\lambda)$.

We endow $V_\zeta'(\lambda)$ with a $U_R(\mathfrak{g})$-module structure $\circ$ via Φ. With respect to this structure we have an embedding of $U_R(\mathfrak{g})$-modules

$$M_R(\lambda) \xrightarrow{\phi} V_\zeta'(l\lambda) \xrightarrow{\psi} V_\zeta'(\lambda)^{\otimes l}.$$

Here, ϕ is the homomorphism induced by Φ sending v_λ to $u_{l\lambda}$ and ψ is the unique homomorphism sending $u_{l\lambda}$ to $u_\lambda^{\otimes l}$.

We define the weight vectors

$$x_\pi := u_{\tau_1\lambda}^{\otimes lt_1} \otimes \cdots \otimes u_{\tau_2\lambda}^{\otimes l(t_2-t_1)} \otimes \cdots \otimes u_{\tau_r\lambda}^{\otimes l(t_r-t_{r-1})} \tag{6.1}$$

and

$$x^\pi := (u_{\tau_1\lambda}^\vee)^{\otimes lt_1} \otimes \cdots \otimes (u_{\tau_2\lambda}^\vee)^{\otimes l(t_2-t_1)} \otimes \cdots \otimes (u_{\tau_r\lambda}^\vee)^{\otimes l(t_r-t_{r-1})}, \tag{6.2}$$

where $u_{\tau_k\lambda}^\vee$ is dual of $u_{\tau_k\lambda}$. We have

$$x_\pi \in V_\zeta'(\lambda)^{\otimes l} \text{ and } x^\pi \in (V_\zeta'(\lambda)^\vee)^{\otimes l} \cong (V_\zeta'(\lambda)^{\otimes l})^\vee.$$

For the following, recall the preorder $\blacktriangleright$ defined in (2.21) and the monomial $F(\pi)$ defined in Section 3.2.

Remark 6.2. Due to the choice of l and the crystal structure on $B(\lambda)^{\otimes l}$, we have by Theorem 2.23 that

$$\psi(G_{l\lambda}(b_{l\pi})) = x_\pi + \sum \text{smaller terms w.r.t. } \blacktriangleright .$$

Lemma 6.3 ([35, Lemma 3])**.** *Let λ be a dominant weight and let π be an LS path of shape λ. Then,*

$$F(\pi) \circ u_\lambda^{\otimes l} = x_\pi + \sum \textit{smaller terms w.r.t. } \blacktriangleright .$$

The *path vector* $p_\pi \in M_R(\lambda)^\vee$ associated π is

$$p_\pi := (\psi\phi)^\vee(x^\pi); \tag{6.3}$$

it is a weight vector of weight $\operatorname{wt}\pi$.

Theorem 6.4 ([35, Theorem 3])**.** *Let λ be a dominant weight. The set $\{p_\pi \mid \pi \in B(\pi_\lambda)\}$ is a basis of $M_R(\lambda)^\vee$.*

6.1. **The A_2-case.** Throughout this section, we assume that $\mathfrak{g}$ is of type A_2.

In order to state the results in a concise way, we use the representation of LS paths as elements of a fan of monoids as in [11].

Let $\mathfrak{C}$ be a maximal chain in W^λ and denote by $\mathbb{Q}^{\mathfrak{C}}$ the free vector space over $\mathbb{Q}$ with basis $\{e_\tau \mid \tau \in \mathfrak{C}\}$. Assume that $\mathfrak{C}$ is given by $\tau_1 > \tau_2 > \ldots > \tau_r > \tau_{r+1} = \mathrm{id}$ and let $\beta_1, \ldots, \beta_r$ be positive roots such that $\tau_k = s_{\beta_k}\tau_{k+1}$ for $k \in [1, r]$. The *Lakshmibai–Seshadri lattice* (*LS lattice*) $\mathrm{LS}_{\lambda,\mathfrak{C}}$ consists of all vectors $\underline{a} = \sum_{k=1}^r a_k e_{\tau_k}$ such that

$$a_1\langle h_{\beta_1}, \tau_1\lambda\rangle \in \mathbb{Z}, (a_1+a_2)\langle h_{\beta_2}, \tau_2\lambda\rangle \in \mathbb{Z}, \ldots, (a_1+\cdots+a_r)\langle h_{\beta_r}, \tau_r\lambda\rangle \in \mathbb{Z},$$
$$\text{and } a_1+\cdots+a_{r+1} \in \mathbb{Z}.$$

The intersection of the LS lattice with the positive quadrant $\mathrm{LS}_{\lambda,\mathfrak{C}} \cap \mathbb{Q}^{\mathfrak{C}}_{\geq 0}$ is a monoid, we denote it by $\mathrm{LS}^+_{\lambda,\mathfrak{C}}$.

The sum $a_1 + \cdots + a_{r+1}$ is called the *degree* of $\underline{a}$. For $m \geq 0$ we write $\mathrm{LS}^+_{\lambda,\mathfrak{C}}(m)$ for the subset of elements of degree m in $\mathrm{LS}^+_{\lambda,\mathfrak{C}}$. The *Lakshmibai-Seshadri fan of monoids* LS^+_λ is defined as the union of the monoids $\mathrm{LS}^+_{\lambda,\mathfrak{C}}$ over all maximal chains $\mathfrak{C}$ in W^λ. The definition of the degree extends naturally to LS^+_λ and we write $\mathrm{LS}^+_\lambda(m)$ for the subset of elements of degree m in LS^+_λ.

The LS paths of shape λ may be identified with the set $\mathrm{LS}^+_\lambda(1)$: For $\pi = (\tau_1, \ldots, \tau_r; t_1, \ldots, t_r)$ we may interpret the LS path π as the vector

$$t_1 e_{\tau_1} + (t_2 - t_1)e_{\tau_2} + \cdots + (t_r - t_{r-1})e_{\tau_r}.$$

Proposition 6.5. *Let λ be an dominant weight and let π be an LS path of shape λ. Write*

$$\pi = a_{121}e_{s_1s_2s_1} + a_{21}e_{s_2s_1} + a_{12}e_{s_1s_2} + a_2e_{s_2} + a_1e_{s_1} + a_0e_{\mathrm{id}}.$$

Then, the evaluation of the path vector p_π is given by

$$p_\pi(f_1^{(n_1)}f_2^{(n_2)}f_1^{(n_3)}u_\lambda) = \begin{cases} \binom{\lfloor\langle h_1,\lambda\rangle(a_{12}+a_1)\rfloor}{\lfloor n_3 - \langle h_1,\lambda\rangle(a_{121}+a_{21})\rfloor}, & \text{if } \mu = \operatorname{wt}\pi, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mu = \lambda - (n_1+n_3)\alpha_1 + n_2\alpha_2$; and

$$p_\pi(f_2^{(n_1)}f_1^{(n_2)}f_2^{(n_3)}u_\lambda) = \begin{cases} \binom{\lfloor\langle h_2,\lambda\rangle(a_{21}+a_2)\rfloor}{\lfloor n_3 - \langle h_2,\lambda\rangle(a_{121}+a_{12})\rfloor}, & \text{if } \mu' = \operatorname{wt}\pi, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mu' = \lambda - n_2\alpha_1 + (n_1+n_3)\alpha_2$.

Proof. We prove the first formula; the proof for the second one is similar. It is sufficient to consider the case $\mu = \operatorname{wt}\pi$, since p_π is a weight vector of weight $\operatorname{wt}\pi$.

Let l be as in the definition of p_π; then, we have

$$\begin{aligned} p_\pi(f_1^{(n_1)} f_2^{(n_2)} f_1^{(n_3)} u_\lambda) &= x^\pi (f_1^{(ln_1)} f_2^{(ln_2)} f_1^{(ln_3)}) \cdot u_\lambda^{\otimes l}) \\ &= x^\pi \left(\sum \zeta^\bullet f_1^{(p_1)} f_2^{(q_1)} f_1^{(r_1)} u_\lambda \otimes \cdots \otimes f_1^{(p_l)} f_2^{(q_l)} f_1^{(r_l)} u_\lambda \right) \end{aligned}$$

where the sum ranges over all possible combinations such that $ln_1 = p_1 + \ldots + p_l$, $ln_2 = q_1 + \ldots + q_l$, $ln_3 = r_1 + \ldots + r_l$ and $\zeta^\bullet$ stands for some power of ζ.

We write $x_\pi = u_1 \otimes u_2 \otimes \cdots \otimes u_l$. Note that to determine evaluation of x^π we only need to consider the summands where $f_1^{(p_1)} f_2^{(q_1)} f_1^{(r_1)} \in Ru_1$, $f_1^{(p_2)} f_2^{(q_2)} f_1^{(r_2)} \in Ru_2$, $\ldots$, and $f_1^{(p_l)} f_2^{(q_l)} f_1^{(r_l)} \in Ru_l$. Thus, $q_1, \ldots, q_l$ are fixed and $p_1, \ldots, p_l$ are already determined once $r_1, \ldots, r_l$ are known. As a special case, p_j and r_j are fixed if $u_j \in \{u_{s_1 s_2 s_1 \lambda}, u_{s_2 s_1 \lambda}, u_{s_2 \lambda}\}$.

We write the coproduct in the following way

$$\Delta^{[l]}(f_i^{(n)}) = \sum q^{k_1 m_1} t_i^{m_1} f_i^{(k_1)} \otimes q^{k_2 m_2} t_i^{m_2} f_i^{(k_2)} \otimes \cdots \otimes f_i^{(k_l)}$$

where $m_j = \sum_{t=j+1}^{l} k_j$. Thus, the coefficient $\zeta^\bullet$ is explicitly given by $\zeta^{c(\underline{r})}$, where $c(\underline{r}) = \sum_{j=1}^{l} x_{\underline{r},j} + x'_{\underline{r},j} + x''_{\underline{r},j}$ and

$$\begin{aligned} x_{\underline{r},j} &= (\langle h_1, \lambda \rangle - r_j) \sum_{t=j+1}^{l} r_t, \\ x'_{\underline{r},j} &= (\langle h_2, \lambda \rangle + r_j - q_j) \sum_{t=j+1}^{l} q_t, \\ x''_{\underline{r},j} &= (\langle h_1, \lambda \rangle - 2r_j + q_j - p_j) \sum_{t=j+1}^{l} p_t. \end{aligned}$$

Introducing the function

$$z_{\underline{r},j,t} = r_t(\langle h_1, \lambda \rangle - r_j) + q_t(\langle h_2, \lambda \rangle + r_j - q_j) + p_t(\langle h_1, \lambda \rangle - 2r_j + q_j - p_j)$$

allows us to write

$$c(\underline{r}) = \sum_{j=1}^{l} \sum_{t=j+1}^{l} z_{\underline{r},j,t}.$$

We have the following possible values for $z_{\underline{r},j,t}$:

(i) if $u_j = u_{s_1 s_2 s_1}$ and u_t is arbitrary, then $z_{\underline{r},j,t} = 0$;
(ii) if $u_j = u_t = u_{s_1 s_2 \lambda}$, then $z_{\underline{r},j,t} = \langle h_1, \lambda \rangle (r_t - r_j)$;
(iii) if $u_j = u_{s_1 s_2 \lambda}$ and $u_t = u_{s_1 \lambda}$, then $z_{\underline{r},j,t} = \langle h_1, \lambda \rangle (r_t - r_j)$;
(iv) if $u_j = u_{s_1 s_2 \lambda}$ and $u_t = u_{s_2 \lambda}$, then $z_{\underline{r},j,t} = \langle h_2, \lambda \rangle r_j$;
(v) if $u_j = u_t = u_{s_2 s_1 \lambda}$, then $z_{\underline{r},j,t} = 0$;
(vi) if $u_j = u_{s_2 s_1 \lambda}$ and $u_t = u_{s_1 \lambda}$, then $z_{\underline{r},j,t} = \langle h_2, \lambda \rangle p_t$;
(vii) if $u_j = u_{s_2 s_1 \lambda}$ and $u_t = u_{s_2 \lambda}$, then $z_{\underline{r},j,t} = 0$;
(viii) if $u_j = u_t = u_{s_1 \lambda}$, then $z_{\underline{r},j,t} = \langle h_1, \lambda \rangle (r_t - r_j)$;
(ix) if $u_j = u_t = u_{s_2 \lambda}$, then $z_{\underline{r},j,t} = 0$;
(x) if u_j is arbitrary and $u_t = u_\lambda$, then $z_{\underline{r},j,t} = 0$.

Now, using Lemma 42.1.2 of [38], we obtain

$$f_1^{(p_j)} f_2^{(q_j)} f_1^{(r_j)} u_\lambda = \begin{bmatrix} \langle h_1, \lambda \rangle \\ r_1 \end{bmatrix} u_j.$$

Hence, the $p_\pi(f_1^{(n_1)} f_2^{(n_2)} f_1^{(n_3)} u_\lambda)$ evaluates to

$$\sum \zeta^{c(\underline{r})} \begin{bmatrix} \langle h_1, \lambda \rangle \\ r_1 \end{bmatrix} \cdots \begin{bmatrix} \langle h_1, \lambda \rangle \\ r_l \end{bmatrix},$$

where the sum ranges over all admissible combinations of $r_1, \ldots, r_l$ such that $r_1 + \ldots + r_l = ln_1$. In other words, for the sum the only r_j that contribute are the ones for $u_j \in \{u_{s_1 s_2 \lambda}, u_{s_1 \lambda}\}$.

Comparing (ii), (iii) and (viii) with Lemma 2.3, we obtain

$$p_\pi(f_1^{(n_1)} f_2^{(n_2)} f_1^{(n_3)} u_\lambda) = \zeta^d \begin{bmatrix} l\langle h_1, \lambda \rangle (a_{12} + a_1) \\ ln_3 - l\langle h_1, \lambda \rangle (a_{121} + a_{21}) \end{bmatrix};$$

the upper part of binomial is the total number of how often $u_{s_1 s_2 \lambda}$ and $u_{s_1 \lambda}$ occur in x_π, while the lower part is the "number of actions by f_1" which can be distribued onto the corresponding factors in the tensor product.

We claim that

$$\zeta^d = \begin{cases} 1, & \text{if } \operatorname{supp} \pi \subseteq \{s_1 s_2 s_1, s_2 s_1, s_2, \mathrm{id}\}, \\ 1, & \text{if } \operatorname{supp} \pi \subseteq \{s_1 s_2 s_1, s_1 s_2, s_1, \mathrm{id}\}, \\ q^{la_2 \langle h_2, \lambda \rangle (ln_3 - la_{121} \langle h_1, \lambda \rangle)}, & \text{if } \operatorname{supp} \pi \subseteq \{s_1 s_2 s_1, s_1 s_2, s_2, \mathrm{id}\}, \\ q^{la_{21} \langle h_2, \lambda \rangle (l(a_{121} + a_{21} + a_1) \langle h_1, \lambda \rangle - ln_3)}, & \text{if } \operatorname{supp} \pi \subseteq \{s_1 s_2 s_1, s_2 s_1, s_1, \mathrm{id}\}. \end{cases}$$

To see this, we proceed case by case. For the purposes of computing the value of $\sum_{j=1}^{l} \sum_{t=j+1}^{l} z_{\underline{r},j,t}$, we may assume that $a_{121} = a_0 = 0$ without loss of generality due to (i) and (x).

Suppose that π is supported on the chain $s_1 s_2 s_1 > s_2 s_1 > s_2 > \mathrm{id}$. In this case, all $z_{\underline{r},j,t}$ are zero; hence,

$$\sum_{j=1}^{l} \sum_{t=j+1}^{l} z_{\underline{r},j,t} = 0.$$

Suppose that π is supported on the chain $s_1 s_2 s_1 > s_1 s_2 > s_1 > \mathrm{id}$. In this case, the contribution of the summands for $u_j = u_{s_1 s_2 \lambda}$ and $u_t = u_{s_1 \lambda}$ was already absorbed by Lemma 2.3; hence

$$\sum_{j=1}^{l} \sum_{t=j+1}^{l} z_{\underline{r},j,t} = 0.$$

Suppose that π is supported on the chain $s_1 s_2 s_1 > s_1 s_2 > s_2 > \mathrm{id}$. In this case, we only need to consider the summands for $u_j = u_{s_1 s_2 \lambda}$ and $u_t = u_{s_2 \lambda}$. Since $u_{s_2 \lambda}$ occurs la_2-times in x_π, $u_{s_1 s_2 s_1 \lambda}$ occurs la_{121}-times in x_π and $\sum_{k=1}^{l} r_k = ln_3$, this gives

$$\sum_{j=1}^{l} \sum_{t=j+1}^{l} z_{\underline{r},j,t} = la_2 \langle h_2, \lambda \rangle (ln_3 - la_{121} \langle h_1, \lambda \rangle).$$

Suppose that π is supported on the chain $s_1 s_2 s_1 > s_2 s_1 > s_1 > \mathrm{id}$. In this case, we only need to consider the summands for $u_j = u_{s_2 s_1 \lambda}$ and $u_t = u_{s_1 \lambda}$; the summands for $u_j = u_t = u_{s_1 \lambda}$ were already absorbed by Lemma 2.3. Since $u_{s_2 s_1 \lambda}$

occurs la_{21}-times in x_π, this gives

$$\begin{aligned}\sum_{j=1}^{l}\sum_{t=j+1}^{l} z_{\underline{r},j,t} &= la_{21}\langle h_2,\lambda\rangle(ln_1 - la_{121}\langle h_1, s_2s_1\lambda\rangle)\\ &= la_{21}\langle h_2,\lambda\rangle(l(a_{121}+a_{21}+a_1)\langle h_1,\lambda\rangle - ln_3),\end{aligned}$$

where last equality follows from

$$ln_1 + ln_3 = la_{121}\langle h_1, s_2s_1\lambda\rangle + l(a_{121}+a_{21}+a_1)\langle h_1,\lambda\rangle.$$

Thus, the claim is proven in all cases.

We compute the explicit value of the binomial for each chain using Lemma 2.4.

Suppose that π is supported on $s_1s_2s_1 > s_2s_1 > s_2 > \mathrm{id}$. Then, $a_{12} = a_1 = 0$, and $\langle h_1,\lambda\rangle(a_{121}+a_{21}) = n_3$; as such

$$\begin{bmatrix} l\langle h_1,\lambda\rangle(a_{12}+a_1)\\ ln_3 - l\langle h_1,\lambda\rangle(a_{121}+a_{21})\end{bmatrix} = \binom{\langle h_1,\lambda\rangle(a_{12}+a_1)}{n_3 - \langle h_1,\lambda\rangle(a_{121}+a_{21})} = 1.$$

Suppose that π is supported on $s_1s_2s_1 > s_1s_2 > s_1 > \mathrm{id}$. By the integrality condition,

$$a_{121}\langle h_2, s_1s_2\lambda\rangle = a_{121}\langle h_1,\lambda\rangle \in \mathbb{Z} \text{ and } \langle h_1,\lambda\rangle(a_{121}+a_{12}+a_1)\in\mathbb{Z},$$

and therefore $\langle h_1,\lambda\rangle(a_{12}+a_1)\in\mathbb{Z}$, as well. Note that $a_{21}=0$ and ζ is an even root of unity, that is $\zeta^{l^2}=1$; hence

$$\begin{bmatrix} l\langle h_1,\lambda\rangle(a_{12}+a_1)\\ ln_3 - l\langle h_1,\lambda\rangle(a_{121}+a_{21})\end{bmatrix} = \binom{\langle h_1,\lambda\rangle(a_{12}+a_1)}{n_3 - \langle h_1,\lambda\rangle(a_{121}+a_{21})}.$$

Suppose that π is supported on $s_1s_2s_1 > s_1s_2 > s_2 > \mathrm{id}$. By the intregality constraints,

$$a_{121}\langle h_1,\lambda\rangle\in\mathbb{Z}, \langle h_1, s_2\lambda\rangle(a_{121}+a_{12})\in\mathbb{Z}, \langle h_2,\lambda\rangle(a_{121}+a_{12}+a_2)\in\mathbb{Z}.$$

We write $lk_1+k_0 = la_{12}\langle h_1,\lambda\rangle$, where $k_1\in\mathbb{Z}_{\geq 0}$ and $k_0\in[0,l-1]$. Then, we have for the binomial (using that $a_1=0$)

$$\begin{aligned}&\zeta^d\begin{bmatrix} l\langle h_1,\lambda\rangle(a_{12}+a_1)\\ ln_3 - l\langle h_1,\lambda\rangle(a_{121}+a_{21})\end{bmatrix}\\ &= \zeta^{l(n_3-\langle h_1,\lambda\rangle a_{121})(k_0+la_2\langle h_2,\lambda\rangle)}\binom{\lfloor a_{12}\langle h_1,\lambda\rangle\rfloor}{n_3 - a_{121}\langle h_1,\lambda\rangle}.\end{aligned}$$

Using the integrality constraints, we see that

$$\begin{aligned}&la_{12}\langle h_1,\lambda\rangle - la_2\langle h_2,\lambda\rangle\\ &= l(a_{121}+a_{12})\langle h_1, s_2\lambda\rangle - la_{121}\langle h_1,\lambda\rangle - l(a_{121}+a_{12}+a_2)\langle h_2,\lambda\rangle\in l\mathbb{Z}.\end{aligned}$$

Hence, $k_0 + la_2\langle h_2,\lambda\rangle$ is even and coefficient in front of the binomial is 1.

Suppose π is supported on $s_1s_2s_1 > s_2s_1 > s_1 > \mathrm{id}$. Using that $a_{12}=0$, we have

$$\begin{aligned}&\zeta^d\begin{bmatrix} la_1\langle h_1,\lambda\rangle\\ ln_3 - l(a_{121}+a_{21})\langle h_1,\lambda\rangle\end{bmatrix}\\ &= \zeta^d\begin{bmatrix} la_1\langle h_1,\lambda\rangle\\ l(a_{121}+a_{21}+a_1)\langle h_1,\lambda\rangle - ln_3\end{bmatrix}\\ &= \zeta^{l((a_{121}+a_{21}+a_1)\langle h_1,\lambda\rangle - n_3)(k_0+la_{21}\langle h_2,\lambda\rangle)}\binom{\lfloor a_1\langle h_1,\lambda\rangle\rfloor}{(a_{121}+a_{21}+a_1)\langle h_1,\lambda\rangle - n_3},\end{aligned}$$

where $k_1\in\mathbb{Z}_{\geq 0}$ and $k_0\in[0,l-1]$ are such that $lk_1+k_0 = la_1\langle h_1,\lambda\rangle$.

Using the integrality constraints, we see that

$$\begin{aligned} &la_{21}\langle h_2, \lambda\rangle - la_1\langle h_1, \lambda\rangle \\ &= l(a_{121} + a_{21})\langle h_2, s_1\lambda\rangle - la_{121}\langle h_2, \lambda\rangle - l(a_{121} + a_{21} + a_1)\langle h_1, \lambda\rangle \in l\mathbb{Z}. \end{aligned}$$

Hence, $k_0 + la_{21}\langle h_2, \lambda\rangle$ is even and the coefficient in front of the binomial is 1. $\square$

As an application of the proposition, we can answer whether the path vectors form a perfect basis. Let λ be a dominant weight. A linear basis B of $M_R(\lambda)$ is called *perfect* [2] if its has an upper normal crystal structure such that

(i) $v_\lambda \in B$,
(ii) each $b \in B$ is a weight vector,
(iii) for each $i \in I$ and each $b \in B$, we have

$$e_i b = \epsilon_i(b)\tilde{e}_i b + \sum_{b' \in B} E^i_{b,b'} b',$$

where $E^i_{b,b'} \in R$ is zero unless $\epsilon_i(b') < \epsilon_i(b) - 1$.

This is the same definition as the one given by Berenstein and Kazhdan [3] up to the normalization in (iii). Comparing with Theorem 2.11, we see that perfect bases are close to the dual canonical basis. As a special case, the dual canonical basis is a perfect basis.

We define $e_i^{\max} b = e_i^{(\epsilon_i(b))} b$; the notation makes sense, since $e_i^{(\epsilon_i(b)+1)} b = 0$.

It follows from the crystal structure on B and (iii) that $e_i^{\max}$ must be injective on the set $\{b \in B \mid \epsilon_i(b) = n\}$ for all $n \in \mathbb{N}$.

Remark 6.6. Recall that the action of $U_R(\mathfrak{g})$ on $M_R(\lambda)^\vee$ is given by

$$e_i p(v) = p(f_i v), \quad f_i p(v) = p(e_i v), \quad \binom{h_i; a}{n} p(v) = p\left(\binom{h_i; a}{n} v\right),$$

for $p \in M_R(\lambda)^\vee$ and $v \in M_R(\lambda)$.

Example 6.7. Let $\lambda = 2\alpha_1 + 2\alpha_2$ and consider the paths

$$\pi_1 = (s_1 s_2 s_1, \mathrm{id}; \tfrac{1}{2}, 1) \quad \text{and} \quad \pi_2 = (s_2 s_1, s_1; \tfrac{1}{2}, 1).$$

Both paths have weight 0. Using Proposition 6.5, we compute

$$p_{\pi_1}(f_1^{(n_1)} f_2^{(2)} f_3^{(n_3)}) = \binom{0}{n_3 - 1}, \quad p_{\pi_2}(f_1^{(n_1)} f_2^{(2)} f_3^{(n_3)}) = \binom{1}{n_3 - 1},$$

for $n_1 + n_3 = 2$. Thus, for both path vectors n_1 may be at most 1 for the evaluation to be non-zero; hence, $\epsilon_1(p_{\pi_1}) = \epsilon_1(p_{\pi_2}) = 1$. The weight space $M_R(\lambda)_{\alpha_1}$ is two-dimensional with basis $\{f_1 f_2^{(2)}, f_2^{(2)} f_1\}$. Using Proposition 6.5 again shows that

$$e_1 p_{\pi_1}(f_2^{(2)} f_1) = e_1 p_{\pi_2}(f_2^{(2)} f_1) = 1 = p_\pi(f_2^{(2)} f_1),$$

and

$$e_1 p_{\pi_1}(f_1 f_2^{(2)}) = e_1 p_{\pi_2}(f_1 f_2^{(2)}) = 0 = p_\pi(f_1 f_2^{(2)}),$$

where $\pi = (s_2 s_1, \mathrm{id}; \tfrac{1}{2}, 1)$. Thus, $e_1^{\max}$ is not injective on the set $\{p_\pi \mid \pi \in B(\pi_\lambda), \epsilon_1(\pi) = 1\}$ and the path vectors are not a perfect basis.

Let λ be a dominant weight and let

$$\pi = (\tau_1, \dots, \tau_r; t_1, \dots, t_r) \text{ and } \pi' = (\sigma_1, \dots, \sigma_s; d_1, \dots, d_s)$$

be LS paths of shape λ. We want to determine the evaluation of p_π on $G_\lambda(b_{\pi'})$. For this note that Theorem 2.23 gives

$$G_\lambda(b_{\pi'}) \circ u_\lambda^{\otimes l} = x_{\pi'} + \sum \text{terms smaller w.r.t. } \blacktriangleright .$$

In particular, x_π occurs in the sum above only if $\pi \lhd \pi'$, that is, $p_\pi(G_\lambda(b_{\pi'})) \neq 0$ only if $\pi \lhd \pi'$. To decide whether x_π occurs, we take a closer look at the action of the quasi-$\mathcal{R}$-matrix on $x_{\pi'}$.

We fix the reduced decomposition $w_0 = s_1 s_2 s_1$. From description of the quasi-$\mathcal{R}$-matrix in Example 2.22, we see that it suffices to understand the action of $e_{\beta_1}^{(n)} \otimes f_{\beta_1}^{(n)}$, $e_{\beta_2}^{(n)} \otimes f_{\beta_2}^{(n)}$, and $e_{\beta_3}^{(n)} \otimes f_{\beta_3}^{(n)}$ on $x_{\pi'}$, where $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + \alpha_2$, and $\beta_3 = \alpha_2$.

For $\beta \in \{\beta_1, \beta_2, \beta_3\}$ assume that $u_1 \otimes \cdots \otimes u_l$ occurs in the sum from the action of $e_\beta^{(n)} \otimes f_\beta^{(n)}$ on $x_{\pi'}$. Note that for $u_1 \otimes \cdots \otimes u_l$ to be equal to x_π, all u_k ($k \in [1, l]$) must be extremal weight vectors and $u_{k+1} \succ u_k$ for all $k \in [1, l-1]$. Due to the fact that f_β has to act "to the right of" e_β the action of $e_{\beta_2}^{(n)} \otimes f_{\beta_2}^{(n)}$ cannot produce x_π. Thus, we only need to consider the actions for $e_1^{(n)} \otimes f_1^{(n)}$ and $e_2^{(n)} \otimes f_2^{(n)}$.

To mimic the action of $e_i^{(n)} \otimes f_i^{(n)}$ on the level of LS paths, we define the operators

$$\flat_i \pi = \begin{cases} \tilde{e}_{i;1} \tilde{f}_{i;-1} \pi, & \text{if } t_0 < t_1, \\ 0, & \text{else,} \end{cases} \quad \text{and} \quad \sharp_i \pi = \begin{cases} \tilde{f}_{i;1} \tilde{e}_{i;-1}, \pi & \text{if } t_0 < t_1, \\ 0, & \text{else.} \end{cases}$$

where t_0 (t_1) is the first (last) time $H_i = \langle h_i, \pi(t) \rangle$ attains a local minimum and $\tilde{e}_{i;-1}$ ($\tilde{f}_{i;-1}$) is the operator $\tilde{e}_{i;n}$ ($\tilde{f}_{i;n}$) for the last local minimum (cf. Section 3.1 for the definition of $\tilde{e}_{i;n}$ and $\tilde{f}_{i;n}$). Clearly, we have:

Lemma 6.8. *The operators $\flat_i$ and $\sharp_i$ have the following properties:*

(i) If $\flat_i \pi \neq 0$, then $\operatorname{wt}(\flat_i \pi) = \operatorname{wt} \pi$ and $\flat_i \pi \lhd \pi$.
(ii) If $\sharp_i \pi \neq 0$, then $\operatorname{wt}(\sharp_i \pi) = \operatorname{wt} \pi$ and $\sharp_i \pi \rhd \pi$.

Lemma 6.9. *Let $m, n \in \mathbb{Z}_{\geq 0}$. Then,*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{m-k}{n-k} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases}$$

Proof. We have

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{m-k}{n-k} = \sum_{k=0}^{m} (-1)^k \binom{m}{n} \binom{n}{k} = \binom{m}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \delta_{n,0}.$$

□

Theorem 6.10. *Let λ be a dominant weight and let*

$$\pi = a_{121} e_{s_1 s_2 s_1} + a_{21} e_{s_2 s_1} + a_{12} e_{s_1 s_2} + a_2 e_{s_2} + a_1 e_{s_1} + a_0 e_{\mathrm{id}}$$

be an LS path of shape λ.

If π is supported on the chain $s_1 s_2 s_1 > s_2 s_1 > s_1 > \mathrm{id}$, then

$$G(b_\pi)^\vee = \sum_{k=0}^{M} (-1)^k \binom{M}{k} p_{\sharp_1^k \pi},$$

where $M = \lfloor \min\{a_{21}\langle h_2, \lambda\rangle, a_1\langle h_1, \lambda\rangle\}\rfloor$.

If π *is supported on the chain* $s_1s_2s_1 > s_1s_2 > s_2 > \mathrm{id}$*, then*

$$G(b_\pi)^\vee = \sum_{k=0}^{M} (-1)^k \binom{M}{k} p_{\sharp_2^k \pi},$$

where $M = \lfloor \min\{a_{12}\langle h_1, \lambda\rangle, a_2\langle h_2, \lambda\rangle\}\rfloor$.

If π *is supported on chain* $s_1s_2s_1 > s_1s_2 > s_1 > \mathrm{id}$ *or the chain* $s_1s_2s_1 > s_2s_1 > s_2 > \mathrm{id}$*, then*

$$G(b_\pi)^\vee = p_\pi.$$

Proof. We use the explicit description of the canonical basis from Example 2.13 to compute the evaluation of p_π.

Assume that π is supported on the chain $s_1s_2s_1 > s_1s_2 > s_1 > \mathrm{id}$. We compute the adapted string (m_1, m_2, m_3) of π with respect to the reduced decomposition $s_2s_1s_2$:

$$\begin{aligned} m_1 &= a_{121}\langle h_2, s_1s_2\lambda\rangle, \\ m_2 &= (a_{121} + a_{12})\langle h_1, s_2\lambda\rangle + a_1\langle h_1, \lambda\rangle, \\ m_3 &= (a_{121} + a_{12})\langle h_2, \lambda\rangle. \end{aligned}$$

We have $m_2 \geq m_1 + m_3$. Thus, $G_\lambda(b_\pi) = f_2^{(m_1)} f_1^{(m_2)} f_2^{(m_3)} u_\lambda$. Furthermore, any element of the canonical basis in $\mathcal{B}(\lambda)_{\mathrm{wt}\,\pi}$ can be written as $f_2^{(n_1)} f_1^{(n_2)} f_2^{(n_3)} u_\lambda$. By Proposition 6.5, we have

$$p_\pi(f_2^{(n_1)} f_1^{(n_2)} f_2^{(n_3)} u_\lambda) = \begin{pmatrix} 0 \\ n_3 - (a_{121} + a_{12})\langle h_2, \lambda\rangle \end{pmatrix}.$$

Thus, $p_\pi(G_\lambda(b_\pi)) = 1$ and $G_\lambda(b_\pi)$ is the unique element of the canonical basis such that the evaluation of p_π is non-zero.

The argument for the case that π is supported on the chain $s_1s_2s_1 > s_2s_1 > s_2 > \mathrm{id}$ is almost identical; we omit the details.

Assume that π is supported on the chain $s_1s_2s_1 > s_2s_1 > s_1 > \mathrm{id}$. We only consider the case $a_1\langle h_1, \lambda\rangle \geq a_{21}\langle h_2, \lambda\rangle$; the other case is similar. We compute the adapted string (m_1, m_2, m_3) of π with respect to the reduced decomposition $s_2s_1s_2$:

$$\begin{aligned} m_1 &= a_{121}\langle h_2, s_1s_2\lambda\rangle + a_{21}\langle h_2, s_1\lambda\rangle, \\ m_2 &= a_{121}\langle h_1, s_2\lambda\rangle + (a_{21} + a_1)\langle h_1, \lambda\rangle, \\ m_3 &= a_{121}\langle h_2, \lambda\rangle. \end{aligned}$$

By assumption, we have $m_2 \geq m_1 + m_3$ and $M = \lfloor a_{21}\langle h_2, \lambda\rangle\rfloor$. Using Lemma 6.9, we see that

$$\begin{aligned} \sum_{k=0}^{M} (-1)^k \binom{M}{k} p_{\sharp_1^k \pi}(f_2^{(n_1)} f_1^{(n_2)} f_2^{(n_3)} u_\lambda) &= \sum_{k=0}^{M} (-1)^k \binom{M}{k} \binom{M-k}{n_3 - a_{121}\langle h_2, \lambda\rangle - k} \\ &= \begin{cases} 1, & \text{if } n_3 = a_{121}\langle h_2, \lambda\rangle, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

We conclude that $p_\pi(G_\lambda(b_\pi)) = 1$ and that $G_\lambda(b_\pi)$ is the unique element of the canonical basis such that the evaluation of $\sum_{k=0}^{M} (-1)^k \binom{M}{k} p_{\sharp_1^k \pi}$ is non-zero.

The argument for the case that π is supported on the chain $s_1s_2s_1 > s_1s_2 > s_2 > \mathrm{id}$ is almost identical; we omit the details. □

Remark 6.11. The base change in the opposite direction, from the dual canonical basis to the path vectors, was already proved by Caldero and Littelmann [8]; albeit only up to a root of unity in the coefficients.

6.2. **Comparison with the dual canonical basis.** We present all the evidence we have that there is a unitriangular base change from the path vector basis to the dual canonical basis.

Proposition 6.12. *Assume that $\mathfrak{g}$ is of type A_1, A_2, A_3 or B_2. For any dominant weight λ, there is a unitriangular base change from the path vector basis to the dual canonical basis of $M_R(\lambda)$ for any linearisation of $\triangleright$.*

Proof. Let π be an LS path of shape λ and $b \in B(\infty)$ such that $\mathrm{pr}_\lambda(b) = b_\pi$. We use the same notation as in the the definition of p_π.

By Proposition 5.4 the Frobenius splitting is fully compatible with the canonical basis. Thus, the homomorphism $M_R(\lambda) \to V'_\zeta(l\lambda)$ sends $G_\lambda(b_\pi)$ to $G_{l\lambda}(b_{l\pi})$. Let $\psi\colon V'_\zeta(l\lambda) \to V'_\zeta(\lambda)^{\otimes l}$ be the unique homomorphism sending $u_{l\lambda}$ to $u_\lambda^{\otimes l}$. Comparing with Remark 6.2, we see that

$$\begin{aligned} p_{\pi'}(G_\lambda(b_\pi)) &= x^{\pi'}(\psi\phi(G_\lambda(b_\lambda))) \\ &= x^{\pi'}(\psi(G_{l\lambda}(b_{l\lambda}))) \\ &= x^{\pi'}(x_\pi + \sum \text{smaller terms w.r.t. } \blacktriangleright). \end{aligned}$$

Since $\pi \triangleright \pi'$ if and only if $x_\pi \blacktriangleright x_{\pi'}$, it follows that $p_\pi(G_\lambda(b_\pi)) = 1$ and $p_{\pi'}(G_\lambda(b_\pi)) = 0$ unless $\pi \trianglerighteq \pi'$. Thus, we may choose any linearisation of $\triangleright$ to obtain a unitriangular base change from the path vector basis to the dual canonical basis. □

Proposition 6.13. *Assume that $\mathfrak{g}$ is a classical Lie algebra or of type G_2. For any $i \in I$ and $m \geq 1$, there is a unitriangular base change from the path vector basis to the dual canonical basis of $M_R(m\varpi_i)$.*

Proof. Put $\lambda := m\varpi_i$. We have seen in Corollary 3.30 that for $V_\mathbb{Z}(\lambda)$ there is a unitriangular base change from the basis $\{F(\pi) \mid \pi \in B(\pi_\lambda)\}$ to the canonical basis for any linearisation of $\triangleright$. Thus, for $\pi, \pi' \in B(\pi_\lambda)$ we have

$$p_{\pi'}(G_\lambda(b_\pi)) = p_{\pi'}(F(\pi)) + \sum_{\substack{\pi'' \in B(\pi_\lambda) \\ \pi'' \lhd \pi}} z_{\pi,\pi''} p_{\pi'}(F(\pi'')).$$

Applying Lemma 6.3 shows that $p_\pi(G(b_\pi)) = 1$ and $p_{\pi'}(G(b_\pi)) = 0$ unless $\pi \trianglerighteq \pi'$. □

Proposition 6.14. *Let λ be a dominant weight, π be an LS path of shape λ and let $b \in B(\infty)$ be such that $\mathrm{pr}_\lambda(b) = b_\pi$. Let Fr' be the Frobenius splitting as in the definition of p_π. If $\mathrm{Fr}'^\vee(G(S_\infty(b))^\vee) = G(b)^\vee$, then*

$$G_\lambda(b_\pi)^\vee = p_\pi + \sum_{\pi' \in B(\pi_\lambda)} z_{\pi,\pi'} p_{\pi'},$$

where $z_{\pi,\pi'} \in R$ is zero unless $\pi' \triangleright \pi$.

Proof. We use the same notation as in the the definition of p_π. To prove the proposition we compare the evaluation of $G_\lambda(b_\pi)^\vee$ and p_π. We have the commutating

diagram:

$$\begin{array}{ccc} V'_\zeta(l\lambda)^\vee & \hookrightarrow & U^-_\zeta(\mathfrak{g}')^\vee \\ \downarrow{\scriptstyle \phi^\vee} & & \downarrow{\scriptstyle \Phi^\vee} \\ M_R(\lambda)^\vee & \hookrightarrow & U^-_R(\mathfrak{g}^*)^\vee \end{array}$$

It follows from Theorem 2.16 and the commutating diagram that

$$\phi^\vee(G_{l\lambda}(b_{l\pi})^\vee) = G_\lambda(b_\pi)^\vee.$$

Now, let $\pi' = (\sigma_1, \ldots, \sigma_p; d_1, \ldots, d_p)$ be an LS path of shape λ. We have

$$\begin{aligned} &G_\lambda(b_\pi)^\vee(F(\pi')v_\lambda) \\ &= \phi^\vee(G_{l\lambda}(b_{l\pi})^\vee)(F(\pi')v_\lambda) \\ &= G_{l\lambda}(b_{l\pi})^\vee \left(G_{l\lambda}(b_{l\pi'}) + \sum\nolimits_{\pi'' \in B(\pi_\lambda); \pi'' \lhd \pi} z_{b',b''} G(b_{\pi''}) \right). \end{aligned}$$

The last equality is a consequence of Remark 6.2 in view of Lemma 6.3.

It follows that $G_\lambda(b_\pi)^\vee(F(\pi')v_\lambda) = 0$ unless $\pi' \trianglerighteq \pi$ and $G_\lambda(b_\pi)^\vee(F(\pi)v_\lambda) = 1$. Thus, the proposition follows, since the path vectors have the same properties by definition. □

6.3. **Standard monomials.** Let $\lambda_1, \ldots, \lambda_N$ be dominant weights and put $\lambda := \lambda_1 + \cdots + \lambda_N$. The inclusion $M_R(\lambda) \to M_R(\lambda_1) \otimes \cdots \otimes M_R(\lambda_N)$ induces a surjection $M_R(\lambda_1)^\vee \otimes \cdots \otimes M_R(\lambda_N)^\vee \to M_R(\lambda)^\vee$. Thus, we can intrepret the elements of $M_R(\lambda_1)^\vee \otimes \cdots \otimes M_R(\lambda_N)^\vee$ as functions on $M_R(\lambda)$ via multiplication. Let $\pi_1, \ldots, \pi_N$ be LS paths of shape $\lambda_1, \ldots, \lambda_N$, respectively. A monomial of path vectors $p_{\pi_1} \cdots p_{\pi_N}$ is called *standard* if $\pi_1 * \cdots * \pi_N \in B(\pi_{\lambda_1} * \cdots * \pi_{\lambda_N})$.

Theorem 6.15 ([35, Theorem 4])**.** *Let $\lambda_1, \ldots, \lambda_N$ be dominant weights and put $\lambda := \lambda_1 + \cdots + \lambda_N$. The set of standard monomials form a basis of $M_R(\lambda)^\vee$.*

The standard monomials can also be considered as a type of path vectors. The following results shows that for certain standard monomials there is a unitriangular base change to the dual canonical basis.

Theorem 6.16. *Let $\mathfrak{g}$ be a classical Lie algebra or of type G_2 and let $\lambda = m_1\varpi_1 + \cdots + m_N\varpi_N$ be a dominant weight. Fix the dominant path $\eta := \pi_{m_1\varpi_1} * \cdots * \pi_{m_N\varpi_N}$. Then, there is a unitriangular base change from the path vectors basis to the dual canonical basis of $M_R(\lambda)$ for any linearisation of $\triangleright$ on $B(\eta)$.*

Proof. This is a consequence of Theorem 3.31. The proof is similar to the one of Proposition 6.13. □

Appendix A. Low-Rank Tensor Recovery via Theta Nuclear p-Norm

Given a vector space $X = \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d}$, a *linear measurement map* is a linear map $A\colon X \to \mathbb{R}$. Suppose we have maps $A_1, \ldots, A_m$, with $m \ll n_1 \ldots n_d$, and *linear measurements* $b_i \in \mathbb{R}$. The goal of *row-rank tensor recovery* is to find an element $x \in X$, such that $A_i(x) = b_i$ for all $i \in [1, m]$. In other words, we are trying to solve an underdetermined system of linear equations. To make the solution unique, we further require the rank of x to minimal. However, rank minimization is NP-hard in general. Other methods which are tractable in matrix case ($d = 2$) are also NP-hard for general tensors. We refer to [10], [14] and [50], [15] for details.

To overcome this problem, Rauhut and Stojanac [46] used relaxations of theta bodies, which were introduced by Gouveia et al. [16]. Motivated by their results, the goal of [48] is to investigate algebraic and geometric properties of the relaxations in the more general context of nuclear p-norms. The first step for this is to find ideals $I_p \subseteq \mathbb{R}[X]$, such that the unit ball B_p of the nuclear p-norm is the convex hull of the real algebraic variety $\mathcal{V}_{\mathbb{R}}(I_p)$, and to determine their reduced Groebner bases. This was done by this author in [48]; and we will discuss the results in the following. The results make it possible to formulate the relaxations as a semidefinite program with which the recovery problem can be solved efficiently [6].

A.1. **Notation.** Let $d \in \mathbb{N}$ and $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. We denote by $[\mathbf{n}]$ the product set $[n_1] \times \cdots \times [n_d]$, where $[n_i] = \{1, \ldots, n_i\}$. For elements $a, b \in [\mathbf{n}]$ we write $a < b$, if a is smaller than b with respect to the lexicographic order. Further, define

$$(a \wedge b)_i = \min\{a_i, b_i\}, \quad (a \vee b)_i = \max\{a_i, b_i\}$$

and

$$(a \bar{\wedge} b)_i = \begin{cases} \min\{a_i, b_i\} & \text{if } a_i \neq b_i, \\ n_i & \text{else.} \end{cases}, \quad (a \bar{\vee} b)_i = \begin{cases} \max\{a_i, b_i\} & \text{if } a_i \neq b_i, \\ n_i & \text{else.} \end{cases}$$

The tensor product $X = \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d}$ comes with a canonical basis $\{e_a \mid a \in [\mathbf{n}]\}$ given by the tensor product of the standard bases of $\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_d}$. Hence, we can write $x \in X$ uniquely as $x = \sum_{a \in [\mathbf{n}]} x_a e_a$ and obtain an isomorphism $X \cong \mathbb{R}^{n_1 \cdots n_d}$.

We consider $\mathbb{R}[X]$, the polynomial ring over $\mathbb{R}$ with variables x_a where a ranges over $[\mathbf{n}]$. For $\alpha \in \mathbb{N}_0^{[\mathbf{n}]}$, we write

$$x^\alpha = \prod_{a \in [\mathbf{n}]} x_a^{\alpha_a} = x_{1\ldots1}^{\alpha_{1\ldots1}} \cdots x_{n_1\ldots n_d}^{\alpha_{n_1\ldots n_d}}$$

Note that, a priori, the polynomial ring $\mathbb{R}[X]$ no longer sees the multilinear structure of X and only the linear structure of $\mathbb{R}^{n_1 \cdots n_d}$, but it is still possible to describe certain properties, such as being rank 1, by polynomial equations.

A.2. **Algebraic Description.** On the space X, we define the norm $\|\cdot\|_p$ via the isomorphism $X \cong \mathbb{R}^{n_1 \cdots n_d}$. That is, for $x \in X$ it is given by

$$\|x\|_p := \left(\sum\nolimits_{a \in [\mathbf{n}]} |x_a|^p \right)^{1/p}.$$

The *nuclear p-norm* is the defined through its unit ball B_p, where

$$B_p := \operatorname{conv}\{x \in X \mid \|x\|_p = 1, \operatorname{rk} x = 1\}.$$

We will now determine ideals I_p such that B_p is the convex hull of the real algebraic variety $\mathcal{V}_{\mathbb{R}}(I_p)$.

Lemma A.1. *A nonzero tensor $x \in X$ is rank 1, if and only if $x_a x_b - x_{a\wedge b} x_{a\vee b} = 0$ for all $a, b \in [\mathbf{n}]$. In other words, the set of rank-1 tensors is the variety of the ideal*

$$I_0 := \langle x_a x_b - x_{a\wedge b} x_{a\vee b} \mid a, b \in [\mathbf{n}] \rangle$$

Proof. First, assume that x is rank 1, i.e. $x = x^{(1)} \otimes \cdots \otimes x^{(d)}$. Then by definition

$$x_{a\wedge b} = x^{(1)}_{\min\{a_1,b_1\}} \cdots x^{(d)}_{\min\{a_d,b_d\}} \text{ and } x_{a\vee b} = x^{(1)}_{\max\{a_1,b_1\}} \cdots x^{(d)}_{\max\{a_d,b_d\}},$$

thus $x_a x_b - x_{a\wedge b} x_{a\vee b} = 0$.

To show that x is rank 1 if $x_a x_b - x_{a\wedge b} x_{a\vee b} = 0$, it is sufficient to find $x^{(1)} \in \mathbb{R}^{n_1}, \ldots, x^{(d)} \in \mathbb{R}^{n_d}$, such that

$$x_a = (x^{(1)} \otimes \cdots \otimes x^{(d)})_a$$

for all $a \in [n]$. Since $x \neq 0$, there exists some $s \in [\mathbf{n}]$ with $x_s \neq 0$. Put $x^{(1)} = (x_{1s_2\ldots s_d}, \ldots, x_{n_1 s_2 \ldots s_d})$ and $x^{(i)} = (x_{s_1\ldots 1\ldots s_d}/x_s, \ldots, x_{s_1\ldots n_i \ldots s_d}/x_s)$ for $i > 1$. Now

$$(x^{(1)} \otimes \cdots \otimes x^{(d)})_a = x^{(1)}_{a_1} \cdots x^{(d)}_{a_d} = (x_s)^{-d+1} x_{a_1 s_2 \ldots s_d} \cdots x_{s_1 \ldots s_{d-1} a_d}.$$

By assumption, we have the identity

$$x_{a_1 s_2 \ldots s_d} x_{s_1 a_2 \ldots s_d} = x_{s_1 s_2 \ldots s_d} x_{a_1 a_2 \ldots s_d} = x_s x_{a_1 a_2 \ldots s_d}.$$

Thus, the above expression reduces to $(x_s)^{-d+2} x_{a_1 a_2 \ldots s_d} \ldots x_{s_1 \ldots s_{d-1} a_d}$. Iterating the argument, we obtain the desired identity. □

For $p = 1$, the extremal points are the standard basis vectors e_a. It follows that every coordinate is in $\{-1, 0, 1\}$, giving the equation $x_a^3 - x_a = 0$ for all $a \in [\mathbf{n}]$. Additionally, only one coordinate should not vanish, so $x_a x_b = 0$ for $a \neq b$, and we require $\sum_{a\in[\mathbf{n}]} x_a^2 - 1$ so that the non-zero entry is ± 1. Altogether we find

$$I_1 = \langle x_a^3 - x_a, x_a x_b, \sum\nolimits_{a\in[\mathbf{n}]} x_a^2 - 1 \rangle \tag{A.1}$$

If $2 \leq p < \infty$ is an even number, the defining polynomial is clear since $\|X\|_p = 1$ is equivalent to $\sum_{a\in[\mathbf{n}]} x_a^p - 1 = 0$. We obtain

$$I_p = \langle \sum\nolimits_{a\in[\mathbf{n}]} x_a^p - 1, x_a x_b - x_{a\wedge b} x_{a\vee b} \rangle. \tag{A.2}$$

If p was odd, I_p would not be the desired ideal since $\mathcal{V}_{\mathbb{R}}(I_p)$ in this case is unbounded.

For $p = \infty$, every coordinate of the extreme points is either -1 or $+1$ giving us

$$I_\infty = \langle x_a^2 - 1, x_a x_b - x_{a\wedge b} x_{a\vee b} \rangle. \tag{A.3}$$

A.3. **Essentials of Groebner Bases.** We first need to fix a total order on the monomials. Here we use the *graded reverese lexicographic order*, *grevlex* for short (sometimes also *degrevlex*), where $x^\alpha > x^\beta$ if the first nonzero entry from the right of $\alpha - \beta$ is negative. In our case, this means $x_{1\ldots 11} > x_{1\ldots 12} > \cdots > x_{21\ldots 1} > \cdots > x_{n_1\ldots n_d}$. With respect to such an ordering, for a nonzero polynomial $f = \sum a_\alpha x^\alpha$ we have its

- multidegree $\operatorname{multideg}(f) = \max\{\alpha \mid a_\alpha \neq 0\}$,
- leading coefficient $\operatorname{LC}(f) = a_{\operatorname{multideg}(f)}$,
- leading monomial $\operatorname{LM}(f) = x^{\operatorname{multideg}(f)}$ and
- leading term $\operatorname{LT}(f) = \operatorname{LC}(f) \operatorname{LM}(f)$.

Given an ordered set $\mathcal{G} = \{g_1, \dots, g_n\} \subseteq \mathbb{R}[X]$, every $f \in \mathbb{R}$ can be written as

$$f = c_1 g_1 + \ldots + c_n g_n + r$$

with $c_1, \dots, c_n, r \in \mathbb{R}[X]$ and $\operatorname{multideg}(r) < \operatorname{multideg}(g_i)$ using a division algorithm. We say f reduces to r modulo $\mathcal{G}$, $f \to_{\mathcal{G}} r$.

In general, the remainder r depends on the ordering of $g_1, \dots, g_n$, in particular if $\mathcal{G}$ is an arbitrary generating set of some ideal $I \subseteq \mathbb{R}[X]$, it cannot be used in conjunction with the division algorithm to determine ideal membership.

The set $\mathcal{G}$ is called Gröbner basis of I (with respect to the monomial ordering) if $(\operatorname{LM}(\mathcal{G})) = (\operatorname{LM}(I))$. In this case, it has the important property that r is unique, in the sense that it no longer depends on the order of $\mathcal{G}$. Hence, for a polynomial $f \in \mathbb{R}[X]$ we have $f \in I$ if and only if r is zero, i.e. $f \to_{\mathcal{G}} 0$.

To determine whether $\mathcal{G}$ is a Gröbner basis can be done via S-polynomials. The S-polynomial of $f_1, f_2 \in \mathbb{R}[X]$ is defined as

$$S(f_1, f_2) = \frac{\operatorname{lcm}(\operatorname{LT}(f_1), \operatorname{LT}(f_2))}{\operatorname{LT}(f_1)} f_1 - \frac{\operatorname{lcm}(\operatorname{LT}(f_1), \operatorname{LT}(f_2))}{\operatorname{LT}(f_2)} f_2$$

Theorem A.2 (Buchberger's Criterion)**.** *A subset $\mathcal{G} = \{g_1, \dots, g_n\}$ of an ideal $I \subseteq \mathbb{R}[X]$ is a Gröbner basis of I if and only if $\mathcal{G}$ generates I and*

$$S(g_i, g_j) \to_{\mathcal{G}} 0$$

for all $g_i, g_j \in \mathcal{G}$.

$\mathcal{G}$ is called *reduced* if for all $g \in \mathcal{G}$ we have $\operatorname{LC}(g) = 1$ and no monomial of g is contained in $\operatorname{LT}(\mathcal{G} \setminus \{g\})$.

A.4. **Groebner bases for I_p.** The goal of this section is to prove the following theorem case by case.

Theorem A.3. *We have the following reduced Groebner bases*

- $\mathcal{G}_0 := \{x_a x_b - x_{a \wedge b} x_{a \vee b} \mid a, b \in [\mathbf{n}], a < b \text{ and } a_i > b_i \text{ for some } i\}$ *for* I_0,
- $\mathcal{G}_1 := \{x_a x_b \mid a, b \in [\mathbf{n}], a < b\} \cup \{\sum_{a \in [\mathbf{n}]} x_a^2 - 1\} \cup \{x_a^3 - x_a \mid a \in [\mathbf{n}], a > (1, \dots, 1)\}$ *for* I_1,
- $\mathcal{G}_p := \{\sum_{a \in [\mathbf{n}]} x_a^p - 1\} \cup \mathcal{G}_0$ *for* I_p *with* $2 \leq p < \infty$, *and*
- $\mathcal{G}_\infty := \{x_a^2 - 1 \mid a \in [\mathbf{n}]\} \cup \{x_a x_b - x_{a \bar{\wedge} b} x_{a \bar{\vee} b} \mid a, b \in [\mathbf{n}], a < b \text{ and } a_i > b_i \text{ or } a_i = b_i < n_i \text{ for some } i\}$ *for* I_∞.

The case for $p = 2$ was already proven in [46]. The case $2 \leq p < \infty$ is based on their work, except for the fact that they relied on matricization to prove their results, while we work directly with tensors. The main benefit is an easier description of $\mathcal{G}_p$.

For the other cases, we used the computer algebra system OSCAR [44] to compute examples for the Gröbner bases of the ideals I_p.

Proposition A.4. *Let $x_{a^1} \cdots x_{a^k} \in \mathbb{R}[X]$. Then $x_{a^1} \cdots x_{a^k}$ reduces to $x_{b^1} \cdots x_{b^k}$ modulo $\mathcal{G}_0$, where $b_i^j = \min(\sqcup_{s=1}^k \{a_i^s\} \setminus \sqcup_{s=1}^{j-1} \{b_i^s\})$.*

Proof. Assume that $x_{a^1} \cdots x_{a^k} \in \mathbb{R}[X]$ is divisible by $x_{a^1} x_{a^2} - x_{a^1 \wedge a^2} x_{a^1 \vee a^2}$. Polynomial division by $x_{a^1} x_{a^2} - x_{a^1 \wedge a^2} x_{a^1 \vee a^2}$ can now be purely expressed by an operation on the tuples $(a_i^1, \dots, a_i^k)$ for $1 \leq i \leq d$. In particular, the operations for different indices are independent of each other. Thus, we can reduce to the case $d = 1$, i.e. $X = \mathbb{R}^n$. Now, it is clear that the operation is simply sorting $a^1, \dots, a^k$ ascending. □

Corollary A.5. *$\mathcal{G}_0$ is the reduced Gröbner basis of I_0.*

Proof. $\mathcal{G}_0$ is a generating set of I_0, since the condition on $a, b \in [\mathbf{n}]$ in the definition of $\mathcal{G}_0$ are minimal such that $x_a x_b - x_{a\wedge b} x_{a\vee b} \neq 0$. We now consider the S-polynomial of two elements $x_a x_b - x_{a\wedge b} x_{a\vee b}$ and $x_c x_d - x_{c\wedge d} x_{c\vee d}$ of $\mathcal{G}_0$. Note that the two are coprime if $\{a, b\}$ and $\{c, d\}$ are disjoint. We will compute the case $b = d$, the other cases are similar. We have

$$S(x_a x_b - x_{a\wedge b} x_{a\vee b}, x_c x_b - x_{c\wedge b} x_{c\vee b}) = x_a x_{c\wedge b} x_{c\vee b} - x_c x_{a\wedge b} x_{a\vee b}$$

and comparing the index sets for both monomials shows that

$$\{a_i, (c \wedge b)_i, (c \vee b)_i\} = \{a_i, b_i, c_i\} = \{c_i, (a \wedge b)_i, (a \vee b)_i\}.$$

By Proposition A.4 we have $S(x_a x_b - x_{a\wedge b} x_{a\vee b}, x_c x_b - x_{c\wedge b} x_{c\vee b}) \to_{\mathcal{G}_0} 0$, hence $\mathcal{G}_0$ is a Gröbner basis of I_0. The reducedness is clear. □

Corollary A.6. *$\mathcal{G}_p$ is the reduced Gröbner basis of I_p for $2 \leq p < \infty$.*

Proof. The polynomials $\sum_{a\in[\mathbf{n}]} x_a^p - 1$ and $x_a x_b - x_{a\wedge b} x_{a\vee b}$ are always coprime, since their respective leading terms are $x_{1\dots1}^p$ and $x_a x_b$ with neither a nor b being $x_{1\dots1}$. Hence, the statement follows from Corollary A.5. □

Proposition A.7. *$\mathcal{G}_1$ is the reduced Gröbner basis of I_1.*

Proof. We begin with the computation of the S-polynomials. Looking at the definition of $\mathcal{G}_1$ we see: All polynomials in the first set are monomials; thus, their S-polynomials are necessarily 0. Furthermore, all leading monomials of the second and third set are coprime and consequently, their S-polynomials are 0. It remains to check

$$S(x_a x_b, x_a^3 - x_a) = x_a x_b \to_{\mathcal{G}_1} 0, \quad S(x_a x_b, x_b^3 - x_b) = x_a x_b \to_{\mathcal{G}_1} 0,$$

and

$$S(x_{1\dots1} x_b, \sum\nolimits_{a\in[\mathbf{n}]} x_a^2 - 1) = \left(\sum\nolimits_{a>(1,\dots,1), a\neq b} x_a^2 x_b\right) + x_b^3 - x_b \to_{\mathcal{G}_1} 0.$$

One easily sees that $\mathcal{G}_1$ is a generating set of I_1 and thus $\mathcal{G}_1$ is a Gröbner basis of I_1; the reducedness is clear. □

Lemma A.8. *For $a, b \in [\mathbf{n}]$, we have $x_a x_b - x_{a\bar{\wedge}b} x_{a\bar{\vee}b} \in I_\infty$.*

Proof. Without restriction, we can assume $a < b$. Note that $x_a x_b - x_{a\bar{\wedge}b} x_{a\bar{\vee}b}$ is non-zero if and only if $a_i > b_i$ or $a_i = b_i < n_i$ for some $i \in [d]$. We assume that $a_i = b_i < n_i$ for some i, as otherwise $a\bar{\wedge}b = a \wedge b$ and $a\bar{\vee}b = a \vee b$, hence the claim follows from the definition of I_∞. Since $a < b$, there exists $j < i$ such that $a_j < b_j$. Together, this gives $(a\bar{\wedge}b)_i = n_i > b_i = a_i \vee b_i$ and $(a\bar{\wedge}b)_j = a_j < b_j = a_j \vee b_j$, so $x_{a\bar{\wedge}b} x_{a\vee b} - x_{a\wedge b} x_{a\bar{\vee}b} \in I_\infty$, where we used the fact that $a \wedge b = (a\bar{\wedge}b) \wedge (a \vee b)$ and $a\bar{\vee}b = (a\bar{\wedge}b) \vee (a \vee b)$. Now,

$$\begin{aligned} x_a x_b - x_{a\bar{\wedge}b} x_{a\bar{\vee}b} = {} & x_a x_b - x_{a\wedge b} x_{a\vee b} + x_{a\wedge b} x_{a\bar{\wedge}b} (x_{a\bar{\wedge}b} x_{a\vee b} - x_{a\wedge b} x_{a\bar{\vee}b}) \\ & + x_{a\bar{\wedge}b} x_{a\bar{\vee}b} (x_{a\wedge b}^2 - 1) - x_{a\wedge b} x_{a\vee b} (x_{a\bar{\wedge}b}^2 - 1). \end{aligned}$$

□

Lemma A.9. *$\mathcal{G}_\infty$ generates I_∞.*

Proof. It is sufficient to show that for any $a, b \in [\mathbf{n}]$ we can write $x_a x_b - x_{a\wedge b} x_{a\vee b}$ in terms of elements of $\mathcal{G}_\infty$, but this is simply

$$(x_a x_b - x_{a\barwedge b} x_{a\barvee b}) - (x_{a\wedge b} x_{a\vee b} - x_{(a\wedge b)\barwedge(a\vee b)} x_{(a\wedge b)\barvee(a\vee b)}).$$

□

Proposition A.10. *Let $x_{a^1} \dots x_{a^k} \in \mathbb{R}[X]$. For $j \in [d]$ and $i \in [n_j]$ put $s_{i,j} := \#\{1 \le l \le k \mid a^l_j = i\}$ and write $s_{i,j} = l_{i,j} + \delta_{i,j}$ with $l_{i,j}$ even and $\delta_{i,j} \in \{0,1\}$. Let $l = k - \min_j \sum_i l_{i,j}$ and $S_j = \{i \mid \delta_{i,j} = 1\}$. For $1 \le r \le l$ we put $b^r_j = \min((S_j \setminus \cup_{s=1}^{r-1}\{b^s_j\}) \cup \{n_j\})$. Then, $x_{a^1} \dots x_{a^k}$ reduces to $x_{b^1} \dots x_{b^l}$ modulo $\mathcal{G}_\infty$.*

Proof. Similarly to Proposition A.4, one can show that $x_{a^1} \dots x_{a^k}$ modulo

$$R = \{x_a x_b - x_{a\barwedge b} x_{a\barvee b} \mid a, b \in [\mathbf{n}], a < b \text{ and } a_i > b_i \text{ or } a_i = b_i < n_i \text{ for some } i\}$$

reduces to $x_{b^1} \dots x_{b^k}$ where $b^r_j = \min((S_j \setminus \cup_{s=1}^{r-1}\{b^s_j\}) \cup \{n_j\})$ for $1 \le r \le k$.

Now assume that $x_{a^1} \cdots x_{a^k}$ is already reduced modulo R. Clearly, it is only divisible by $x^2_{\mathbf{n}} - 1$ and exactly $\frac{1}{2}(\min_j \sum_i l_{i,j})$-times. □

Corollary A.11. *$\mathcal{G}_\infty$ is the reduced Gröbner basis of I_∞.*

Proof. Lemma A.9 asserts that $\mathcal{G}_\infty$ generates I_∞ and by Proposition A.10 computation of the S-polynomials gives

$$S(x^2_a - 1, x_a x_b - x_{a\barwedge b} x_{a\barvee b}) = x_a x_{a\barwedge b} x_{a\barvee b} - x_b \to_{\mathcal{G}_\infty} 0,$$

since we have (in the notation of Proposition A.10) $S_j = \{b_j\}$ and $l = 2$ for $x_a x_{a\barwedge b} x_{a\barvee b}$, and similarly

$$S(x_a x_b - x_{a\barwedge b} x_{a\barvee b}, x_a x_c - x_{a\barwedge c} x_{a\barvee c}) = x_b x_{a\barwedge c} x_{a\barvee c} - x_c x_{a\barwedge b} x_{a\barvee b} \to_{\mathcal{G}_\infty} 0$$

Hence, $\mathcal{G}_\infty$ is a Gröbner basis for I_∞. One easily checks that G_∞ is also reduced. □

Index of notation

$[m,n]$	$\{m,\dots,n\}$ for $m,n\in\mathbb{Z}_{\geq 0}$ with $m\leq n$
$\succeq$	partial order on the weight lattice/path model, 20/38
$\succeq_+$	partial order on LS paths, 38
$\blacktriangleright$	preorder defined on $V(\lambda)\otimes V(\mu)$, 30
$\triangleright$	partial order on $B(\eta)$ for a dominant path η, 32
$\triangleright_\eta$	partial order induced on $B(\lambda)$ by $\triangleright$ on $B(\eta)$, 32
A	subalgebra of $\mathbb{Q}(q)$ of functions regular at $q=0$
$B(\infty)$	crystal basis of $U_q^-(\mathfrak{g})$, 25
$\mathcal{B}(\infty)$	canonical basis of $U_q^-(\mathfrak{g})$, 25
$B(\lambda)$	crystal basis of $V(\lambda)$, 28
$\mathcal{B}(\lambda)$	canonical basis of $V(\lambda)$, 28
b_∞	image of 1 under the projection $L(\infty)\to L(\infty)/qL(\infty)$, 25
b_λ	image of u_λ under the projection $L(\lambda)\to L(\lambda)/qL(\lambda)$, 28
b_π	element of $B(\lambda)$ corresponding to the path π, 31
Δ	coproduct of $U_q(\mathfrak{g})$, 22
$\tilde{e}_{i;1}$	modified root operator, 34
$\epsilon_{i;1}(\pi)$	$\max\{n\geq 0 \mid \tilde{e}_{i;1}^n\pi\neq 0\}$, 34
$F(\pi)$	$F(\pi,\underline{i})$ for an arbitrary choice of $\underline{i}$, 37
$F(\pi,\underline{i})$	path monomial associated to π and $\underline{i}$, 37
Fr	Frobenius morphism, 77
Fr'	Frobenius splitting, 77
$f_{\underline{i}}^{(\underline{a})}$	$f_{i_1}^{(a_1)}\cdots f_{i_r}^{(a_r)}$ for $\underline{i}=(i_1,\dots,i_r)$ and $\underline{a}=(a_1,\dots,a_r)$
$\tilde{f}_{i;1}$	modified root operator, 34
$\phi_{i;1}(\pi)$	$\max\{n\geq 0 \mid \tilde{f}_{i;1}^n\pi\neq 0\}$, 34
$\mathfrak{g}$	symmetrizable Kac-Moody Lie algebra, 20
$\mathcal{H}(w)$	set of sequences $\underline{i}$ in I such that $s_{\underline{i}}$ is a red. dec. of $w\in W$, 20
$\iota(\pi)$	intial direction of π, 31
$M(\lambda)$	simple $U(\mathfrak{g})$-module of highest weight λ
$M(\lambda)^\vee$	graded dual of $M(\lambda)$
P	weight lattice, 20
p_π	path vector associated to an LS path π, 87
π_λ	straight-line path $t\mapsto t\lambda$, 31
Q	root lattice, 20
S_∞	similarity map, 78
$S_{\underline{i}}(\infty)$	adapted string cone, 33
$S_{\underline{i}}(\lambda)$	adapted string polytope, 33
s_i	simple reflection for $i\in I$, 20
$U(\mathfrak{g})$	universal enveloping algebra of $\mathfrak{g}$
$U_q(\mathfrak{g})$	quantized enveloping algebra of $\mathfrak{g}$, 21
u_λ	highest weight vector of $V(\lambda)$
$V(\lambda)$	simple $U_q(\mathfrak{g})$-module of highest weight λ, 23
$V(\lambda)^\vee$	graded dual of $V(\lambda)$, 23
v_λ	highest weight vector of $M(\lambda)$
W	Weyl group of $\mathfrak{g}$, 20
w_0	longest element of the Weyl group W, 20

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