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Hecke eigenforms for meromorphic cusp forms

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Abstract

In this paper, we construct Hecke eigenforms for two families of quotient spaces of meromorphic cusp forms on $\mathrm{SL}_2(\mathbb{Z})$. We show that each quotient space in the first (resp. second family) is isomorphic as a Hecke module to the space S_{2k} (resp. M_{2k}) of cusp forms (resp. holomorphic modular forms) of the same weight on $\mathrm{SL}_2(\mathbb{Z})$.

Keywords Harmonic Maass forms · Hecke eigenforms · Lifting maps for meromorphic modular forms · Meromorphic cusp forms

Mathematics Subject Classification 11F11 · 11F25 · 11F37

1 Introduction and statement of results

Hecke theory for holomorphic modular forms is one of the most fruitful tools for (holomorphic) modular forms. It is therefore natural to seek some kind of Hecke structure also for more general spaces of meromorphic modular forms. For the space $M_{2-2k}^!$ of *weakly holomorphic modular forms*, i.e., such meromorphic modular forms whose poles, if any, are supported at cusps, this has been achieved by Guerzhoy in [12]. The main difference to the holomorphic theory is that Hecke operators increase the pole order at $i\infty$, so there are no truly weakly holomorphic eigenforms in the strict sense. Guerzhoy [12] instead factored out by a suitable Hecke-stable subspace and defined eigenforms for the quotient space. This subspace is distinguished in a number of ways. For a prime p and $n \in \mathbb{N}$ with $p \mid n$, the n-th Fourier coefficients of its elements are divisible by high powers of p, yielding congruences

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for coefficients of weakly holomorphic modular forms [13]. Moreover, elements of this space basically lead to vanishing period polynomials [5].

As an example, consider the weight 6 weakly holomorphic modular form

$$f_{6,i\infty}(z) := \frac{E_6(z)^3}{\Delta(z)} + 1488E_6(z) = q^{-1} - 73764q - 86241280q^2 + O\left(q^3\right).$$

Here and throughout, we let $q := e^{2\pi iz}$ and define the weight $2k$ *Eisenstein series*

$$E_{2k}(z) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n,$$

with $\sigma_r(n) := \sum_{d|n} d^r$ and B_k the k-th Bernoulli number. Moreover

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24} =: \sum_{n=1}^{\infty}\tau(n)q^n$$

is the weight 12 modular discriminant. Since the constant term in the Fourier expansion of $f_{6,i\infty}$ vanishes, $f_{6,i\infty}$ belongs to the subspace $S_6^! \subseteq M_6^!$ of *weakly holomorphic cusp forms*, i.e., those weakly holomorphic modular forms whose constant term vanishes. Setting $D := \frac{1}{2\pi i}\frac{\partial}{\partial z}$ and using the fact that D^{2k-1} sends $M_{2-2k}^!$ to $S_{2k}^!$ (see [9, Theorem 1.2]), one verifies that

$$D^5\left(\frac{E_8(z)}{\Delta(z)}\right) = -f_{6,i\infty}(z).$$

Since

$$\frac{E_8(z)}{\Delta(z)} = q^{-1} + 504 + 73764q + 2695040q^2 + O\left(q^3\right),$$

we see that $\frac{E_8}{\Delta} \notin S_{-4}^!$, so $f_{6,i\infty} \notin D^5(S_{-4}^!)$ because any solution to the differential equation $D^5(F) = -f_{6,i\infty}$ differs from $\frac{E_8}{\Delta}$ by a polynomial in z of degree at most 5, and thus $\frac{E_8}{\Delta}$ is the only modular solution. However, it turns out that for $m \in \mathbb{N}$ we have

$$f_{6,i\infty}\big|T_m - \sigma_5(m)f_{6,i\infty} \in D^5\left(S_{-4}^!\right), \tag{1.1}$$

where T_m denotes the m-th Hecke operator defined in (2.5). Denoting

$$\mathbb{J}_{2k}^{i\infty} := D^{2k-1}\left(M_{2-2k}^!\right) \qquad \text{and} \qquad \mathbb{J}_{2k,0}^{i\infty} := D^{2k-1}\left(S_{2-2k}^!\right), \tag{1.2}$$

we see that $f_{6,i\infty}$ has eigenvalue $\sigma_5(m)$ under the Hecke operator T_m in $S_6^!/\mathbb{J}_{6,0}^{i\infty}$. These eigenvalues match those of E_6, demonstrating the fact that

$$\left(S_{2k}^! \cap S_{2k}^\perp\right)\Big/\mathbb{J}_{2k}^{i\infty} \cong S_{2k} \qquad \text{and} \qquad \left(S_{2k}^! \cap S_{2k}^\perp\right)\Big/\mathbb{J}_{2k,0}^{i\infty} \cong M_{2k}$$

as Hecke modules (see [5, Theorem 1.2]). Here $S_{2k}^\perp$ denotes the subspace of forms which are orthogonal to S_{2k} under the (suitably regularized) Petersson inner product. In particular, if $k = 3$, the quotient space on the left-hand side of the second isomorphism is spanned by $f_{6,i\infty}$, while M_6 is spanned by E_6 (note that $S_6 = \{0\}$, so $S_6^! \cap S_6^\perp = S_6^!$).

Motivated by (1.1) and applications such as the Eichler–Shimura theory [5] and congruence properties [13], we construct appropriate subspaces generalizing $\mathbb{J}_{2k}^{i\infty}$ and $\mathbb{J}_{2k,0}^{i\infty}$ to meromorphic modular forms. *Meromorphic cusp forms*, originally studied by Petersson [20], have also been a flourishing subject in the recent past (see e.g. [1–3, 6]). These forms may have poles in the upper half-plane $\mathbb{H}$ but decay exponentially towards $i\infty$. One of the main problems in finding a notion of Hecke eigenform in this context is that Hecke operators move the pole, which makes it more complicated to find a good space to factor out by. We hence decompose the space of meromorphic cusp forms into certain natural Hecke-stable subspaces and then consider the action of the Hecke operators on these subspaces. Note that for a function $f:\mathbb{H}\to\mathbb{C}$, the space

$$\mathbb{S}_{2k}(f):=\langle f|_{2k}T_m\ :\ m\in\mathbb{N}\rangle$$

is the smallest Hecke-stable subspace containing f. We investigate such subspaces for certain natural choices of f. Specifically, for fixed $\mathfrak{z}\in\mathbb{H}$ and $k,\ell\in\mathbb{N}$ with $k\geq 2$, we choose for f the weight $2k$ elliptic Poincaré series $\Psi^{\mathfrak{z}}_{2k,-\ell}$, defined in (2.8). These Poincaré series are characterized by their growth at $\mathfrak{z}$ and span the subspace of meromorphic cusp forms orthogonal to holomorphic cusp forms by [21, Satz 7–8] (see also Lemma 2.2). We abbreviate

$$\mathbb{S}^{\mathfrak{z}}_{2k,-\ell}:=\mathbb{S}_{2k}\left(\Psi^{\mathfrak{z}}_{2k,-\ell}\right)=\left\langle\Psi^{\mathfrak{z}}_{2k,-\ell}|_{2k}T_m\ :\ m\in\mathbb{N}\right\rangle.$$

Our generalization of $\mathbb{J}^{i\infty}_{2k}$ and $\mathbb{J}^{i\infty}_{2k,0}$ to subspaces of $\mathbb{S}^{\mathfrak{z}}_{2k,-\ell}$ is motivated by an algebraicity conjecture of Gross and Zagier [11, Conjecture (4.4)], which has recently been proven by Li [16, Theorem 1.1] and Bruinier–Li–Yang [8]. To describe this, suppose that $(\lambda_m)_{m\geq 1}$ is a sequence for which $\lambda_m\neq 0$ for only finitely many m. Suppose λ_m are chosen such that for any weight $2k$ cusp form f with Fourier expansion $f(z)=\sum_{m=1}^{\infty}c_f(m)q^m$

$$\sum_{m=1}^{\infty}\lambda_m c_f(m)=0. \tag{1.3}$$

Gross and Zagier used such sequences to build linear combinations of the automorphic Green's function[1] $G_s(z,\mathfrak{z})$ evaluated at different points. They further conjectured that these combinations satisfy certain algebraic properties. For example, let $G_k^m(z,\mathfrak{z})$ be the extension of $G_k(z,\mathfrak{z})|_{0,\mathfrak{z}}T_m$ defined for all $z,\mathfrak{z}\in\mathbb{H}$ in [11, equation (5.7) on p. 251].[2] Then a special case of the algebraicity conjecture claims that if $\mathfrak{z}$ is a CM-point of fundamental discriminant D and $(\lambda_m)_{m\geq 1}$ satisfies (1.3), then

$$\sum_{m=1}^{\infty}\lambda_m m^{k-1}G_k^m(\mathfrak{z},\mathfrak{z})=u^2D^{1-k}\log|\beta|, \tag{1.4}$$

where β is an algebraic number lying in the class field of $\mathbb{Q}(\sqrt{D})$, and $u\in\mathbb{N}$ is half of the number of roots of unity in $\mathbb{Q}(\sqrt{D})$ (see [11, Conjecture 4.4 on p. 317]).

If g is modular of weight $2\ell-2k$ on $\mathrm{SL}_2(\mathbb{Z})$, then we define[3]

$$\mathcal{G}_g(z,\mathfrak{z})=\mathcal{G}_{2k,-\ell,g}(z,\mathfrak{z}):=\sum_{m=1}^{\infty}m^{2k-\ell}g(\mathfrak{z})\big|_{2\ell-2k}T_m q^m. \tag{1.5}$$

[1] The automorphic Green's function, defined in (2.10), is a non-holomorphic modular form of weight zero which is an eigenfunction under the Laplace operator (see (2.2)).

[2] Here and throughout, $f(z,\mathfrak{z})|_{k,z}\mathcal{O}$ (resp. $f(z,\mathfrak{z})|_{k,\mathfrak{z}}\mathcal{O}$) means that the operator $\mathcal{O}$ acts on the two-variable function f in the z-variable (resp. $\mathfrak{z}$-variable).

[3] Note that the sum does not converge everywhere; see Lemma 4.1 for details about the meromorphic continuation in z.

We omit the dependence on k and ℓ if it is clear from the context. Fix $k, \ell \in \mathbb{N}$ with $k \geq 2$ and $\mathfrak{z} \in \mathbb{H}$. Condition (1.3) naturally leads one to consider the spaces[4]

$$\mathbb{J}_{2k,-\ell}^{\mathfrak{z}} := \left\{ \mathcal{G}_g(z, \mathfrak{z}) : g \in R_{2-2k}^{\ell-1} \left(M_{2-2k}^! \right) \right\}, \tag{1.6}$$

$$\mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}} := \left\{ \mathcal{G}_g(z, \mathfrak{z}) : g \in R_{2-2k}^{\ell-1} \left(S_{2-2k}^! \right) \right\}. \tag{1.7}$$

Here, writing $z = x + iy$ throughout

$$R_{2-2k}^{\ell-1} := R_{2\ell-2k-2} \circ \cdots \circ R_{2-2k} \quad \text{with} \quad R_{2\kappa} := 2i \frac{\partial}{\partial z} + \frac{2\kappa}{y}$$

denotes the *Maass raising operator* repeated $\ell - 1$ times. It raises the weight of a modular object from $2 - 2k$ to $2\ell - 2k$. For $\ell = 2k$, Bol's identity (see [9, Lemma 2.1]) yields

$$R_{2-2k}^{2k-1} = -(4\pi)^{2k-1} D^{2k-1}, \tag{1.8}$$

so these spaces emulate the spaces defined in (1.2). We prove in Theorem 1.1 that $\mathbb{J}_{2k,-\ell}^{\mathfrak{z}}$ and $\mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}}$ are subspaces of $\mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$. One can thus view the spaces in (1.6) and (1.7) as images under the map $f \mapsto \mathcal{G}_{R^{\ell-1}(f)}(z, \mathfrak{z})$ from weakly holomorphic modular forms of weight $2-2k$ to meromorphic cusp forms of weight $2k$. The existence of a weakly holomorphic modular form with a prescribed growth towards $i\infty$ is governed by the Fourier coefficients of weight $2k$ cusp forms (see [20, Satz 7]), yielding a link with (1.3).

We next consider an example that emulates the behaviour from (1.1). Set

$$f_{6,i}(z) := \frac{\Delta(z)}{E_6(z)} = q + 480q^2 + 258804q^3 + 138542080q^4 + O\left(q^5\right), \tag{1.9}$$

$$\begin{aligned} g_5(z) &:= \frac{1}{\alpha} \frac{E_8(z)}{\Delta(z)} \left(j(z)^4 - 3480 j(z)^3 + 3838860 j(z)^2 - 1425282400 j(z) + 114237825024 \right), \\ &= \frac{1}{\alpha} \left(q^{-5} - 3126 q^{-1} + 26994415788736 q + 519615094283304960 q^2 \right) \\ &\quad + O\left(q^3\right) \in S_{-4}^!, \end{aligned} \tag{1.10}$$

$$\begin{aligned} g_7(z) &:= \frac{1}{\alpha} \frac{E_8(z)}{\Delta(z)} \Big(j(z)^6 - 4968 j(z)^5 + 9176868 j(z)^4 - 7736486240 j(z)^3 \\ &\quad + 2925506969154 j(z)^2 - 411526489432464 j(z) + 12317318339088384 \Big) \\ &= \frac{1}{\alpha} \left(q^{-7} - 16808 q^{-1} + 10625045828793993 q + 1689691172521357344768 q^2 \right) \\ &\quad + O\left(q^3\right) \in S_{-4}^!, \end{aligned}$$

where $\alpha := \frac{E_8(i)}{\Delta(i)} = 1187.006489\ldots$, $j(z) := \frac{E_4(z)^3}{\Delta(z)}$ is the *modular j-function*, and the Fourier expansions hold for $y > 1$. A straightforward calculation (using (4.5) below with $k = 3$, $\ell = 1$, and $\mathfrak{z} = i$) yields that

$$f_{6,i}(z)|_6 T_5 - \sigma_5(5) f_{6,i}(z) = -2^{-10} \mathcal{G}_{g_5}(z, i) \in \mathbb{J}_{6,-1,0}^{i},$$

$$f_{6,i}(z)|_6 T_7 - \sigma_5(7) f_{6,i}(z) = -2^{-10} \mathcal{G}_{g_7}(z, i) \in \mathbb{J}_{6,-1,0}^{i}.$$

This behaviour, which parallels (1.1), is not isolated. Much like $f_{6,i\infty}$, the function $f_{6,i} \in \mathbb{S}_{6,-1}^{i}$ generates the one-dimensional quotient space $\mathbb{S}_{6,-1}^{\mathfrak{z}} / \mathbb{J}_{6,-1,0}^{i}$, and the following theorem

[4] For a detailed explanation of the connection between condition (1.3) and (1.6) we refer to Remark 4.4.

gives an isomorphism as Hecke modules between the spaces S_{2k} and M_{2k} and the quotient spaces formed by factoring $\mathbb{S}^{\mathfrak{z}}_{2k,-\ell}$ out by the subspaces of $\mathbb{S}^{\mathfrak{z}}_{2k,-\ell}$ defined in (1.6) and (1.7).

Theorem 1.1 *For $k, \ell \in \mathbb{N}$ with $k \geq 2$, the spaces $\mathbb{J}^{\mathfrak{z}}_{2k,-\ell}$ and $\mathbb{J}^{\mathfrak{z}}_{2k,-\ell,0}$ are Hecke-stable subspaces of $\mathbb{S}^{\mathfrak{z}}_{2k,-\ell}$ and the following are isomorphisms of Hecke modules:*

$$\mathbb{S}^{\mathfrak{z}}_{2k,-\ell}\Big/\mathbb{J}^{\mathfrak{z}}_{2k,-\ell} \cong S_{2k}, \qquad \mathbb{S}^{\mathfrak{z}}_{2k,-\ell}\Big/\mathbb{J}^{\mathfrak{z}}_{2k,-\ell,0} \cong M_{2k}.$$

The example $f_{6,i}$ corresponds to E_6 under the second isomorphism $\mathbb{S}^{\mathfrak{z}}_{2k,-\ell}/\mathbb{J}^{\mathfrak{z}}_{2k,-\ell,0} \cong M_{2k}$ for $k=3$, $\ell=1$, and $\mathfrak{z}=i$. We further elaborate upon this in Section 5 and also show that

$$G(z) := \frac{\Delta(z)^2}{E_4(z)^3 + 15^3\Delta(z)} = q^2 - 4143q^3 + 16868385q^4 - 68686682635q^5 + O\left(q^6\right) \tag{1.11}$$

maps to Δ under the first isomorphism $\mathbb{S}^{\mathfrak{z}}_{2k,-\ell}/\mathbb{J}^{\mathfrak{z}}_{2k,-\ell} \cong S_{2k}$ for $k=6$, $\ell=1$, and $\mathfrak{z}=\frac{1+i\sqrt{7}}{2}$.

The paper is organized as follows. In Sect. 2, we introduce non-holomorphic and meromorphic modular forms such as harmonic Maass forms and meromorphic cusp forms, construct bases for these spaces, and recall certain operators. In Sect. 3, we review some properties of elliptic Poincaré series originally considered by Petersson [20, 21]. In Sect. 4, we investigate $\mathcal{G}_g(z,\mathfrak{z})$ and prove Theorem 1.1. Finally, in Sect. 5, we describe how to construct meromorphic Hecke eigenforms using the isomorphism that is used to prove Theorem 1.1 and demonstrate this construction by explaining how $f_{6,i}$ is chosen.

2 Preliminaries

2.1 Meromorphic modular forms and harmonic Maass forms

We let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ act on $\mathbb{H}$ via fractional linear transformations. As usual, for $\kappa \in \mathbb{Z}$ and a function $f : \mathbb{H} \to \mathbb{C}$, we define the *weight 2κ slash operator*

$$f(z)\big|_{2\kappa}\gamma := \det(\gamma)^{\kappa}(cz+d)^{-2\kappa} f(\gamma z). \tag{2.1}$$

We include the variable z in the notation for the slash operator $|_{2\kappa,z}$ and other operators if the variable is not clear from context. We say that f satisfies *weight 2κ modularity for* $\mathrm{SL}_2(\mathbb{Z})$ if $f|_{2\kappa}\gamma = f$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We call f a *meromorphic modular form* (for $\mathrm{SL}_2(\mathbb{Z})$) of weight 2κ if f is meromorphic on $\mathbb{H}$, satisfies weight 2κ modularity for $\mathrm{SL}_2(\mathbb{Z})$, and if there exists $n_0 \in \mathbb{Z}$ such that $f(z) \asymp e^{-2\pi n_0 y}$ as $z \to i\infty$. We say that f has a *pole at $i\infty$* if $n_0 < 0$. A meromorphic modular form is a meromorphic cusp form if $n_0 > 0$. On the other hand, if f has no poles in $\mathbb{H}$, then f is a *weakly holomorphic modular form*. *(Holomorphic) modular forms* are those weakly holomorphic modular forms with $n_0 \geq 0$ and *cusp forms* are those holomorphic modular forms with $n_0 > 0$.

We also require certain non-holomorphic modular forms. Following [7], a real-analytic function $f : \mathbb{H} \to \mathbb{C}$ is a *harmonic Maass form* of weight 2κ (for $\mathrm{SL}_2(\mathbb{Z})$) if it satisfies:

(1) The function f is modular of weight 2κ on $\mathrm{SL}_2(\mathbb{Z})$.
(2) The function f is annihilated by the *weight 2κ hyperbolic Laplace operator*

$$\Delta_{2\kappa} := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2i\kappa y\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2.2}$$

(3) There exists $c \in \mathbb{R}$ such that $f(z) = O(e^{cy})$ as $y \to \infty$.

For $\kappa < 0$ a harmonic Maass form of weight 2κ has a Fourier expansion (for some $n_0 \in \mathbb{N}_0$)

$$f(z) = \sum_{n \geq -n_0} c_f^+(n)q^n + c_f^-(0)y^{1-2\kappa} + \sum_{\substack{n \leq n_0 \\ n \neq 0}} c_f^-(n)\Gamma(1-2\kappa, -4\pi ny)q^n,$$

where for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ we let $\Gamma(s, y) := \int_y^\infty t^{s-1}e^{-t}dt$ denote the *incomplete gamma function*. The *principal part* of f at $i\infty$ is

$$\sum_{n<0} c_f^+(n)q^n + c_f^-(0)y^{1-2\kappa} + \sum_{n>0} c_f^-(n)\Gamma(1-2\kappa, -4\pi ny)q^n.$$

Defining $\xi_{2\kappa} := 2iy^{2\kappa}\overline{\frac{\partial}{\partial \overline{z}}}$, we have

$$\Delta_{2\kappa} = -\xi_{2-2\kappa} \circ \xi_{2\kappa},$$

and $\xi_{2\kappa}$ maps harmonic Maass forms of weight 2κ to weakly holomorphic modular forms of weight $2 - 2\kappa$ (see [7, Proposition 3.2]). We let $H_{2\kappa}^{\mathrm{cusp}}$ denote the space of harmonic Maass forms which map to cusp forms under $\xi_{2\kappa}$. Note that a harmonic Maass form f lies in $H_{2\kappa}^{\mathrm{cusp}}$ if and only if there is a polynomial $P \in \mathbb{C}[q]$ such that $f(z) - P(q)$ is bounded as $y \to \infty$.

2.2 Operators

For $m \in \mathbb{N}$, the *V-operator* is given by

$$f(z)\big|V_m := f(mz) = m^{-\kappa} f(z)\Big|_{2\kappa} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3}$$

We define the *U-operator* by

$$f(z)\big|U_m := m^{\kappa-1} \sum_{j=1}^{m} f(z)\Big|_{2\kappa} \begin{pmatrix} 1 & j \\ 0 & m \end{pmatrix}. \tag{2.4}$$

Note that if f is translation-invariant, then f has a (local) Fourier expansion of the type $f(z) = \sum_{n \in \mathbb{Z}} c\,(n; y)\, q^n$. We have

$$f(z)\big|V_m = \sum_{n \in \mathbb{Z}} c\,(n; my)\, q^{mn} \quad \text{and} \quad f(z)\big|U_m = \sum_{n \in \mathbb{Z}} c\left(mn; \frac{y}{m}\right) q^n.$$

Combining these operators, one defines the *m-th Hecke operator*[5] ($m \in \mathbb{N}$)

$$f(z)\big|_{2\kappa} T_m := \sum_{n \in \mathbb{Z}} \sum_{r | \gcd(m,n)} r^{2\kappa-1} c\left(\frac{mn}{r^2}; \frac{r^2 y}{m}\right) q^n = \sum_{r|m} r^{2\kappa-1} f(z)\big|U_{\frac{m}{r}} \circ V_r. \tag{2.5}$$

If f satisfies weight 2κ modularity for $\mathrm{SL}_2(\mathbb{Z})$, then so does $f|_{2\kappa}T_m$ (see e.g. [19, Proposition 2.3]). For $m \in \mathbb{N}$, we let

$$\mathcal{M}_m := \{M \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \ : \ \det(M) = m\} \tag{2.6}$$

[5] Note that there exist multiple normalizations of T_m (differing by a power of m) in the literature. The above definition (2.5) is chosen so that it preserves integrality of Fourier coefficients for $\kappa \in \mathbb{N}$.

be the set of 2×2 integer matrices of determinant m and choose a set of representatives

$$\mathcal{R}_m := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = m,\ d > 0,\ b \pmod{d} \right\}$$

for the right cosets in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{M}_m$ (see e.g. [22, p. 38]). By (2.5), (2.4), and (2.3), we have

$$f(z)\big|_{2\kappa} T_m = m^{\kappa-1} \sum_{M \in \mathcal{R}_m} f(z)\big|_{2\kappa} M. \tag{2.7}$$

We also require the well-known commutator relations.

Lemma 2.1 *Let $f : \mathbb{H} \to \mathbb{C}$ be a real-analytic modular form of weight $2\kappa \in 2\mathbb{Z}$, i.e., a real-analytic function on $\mathbb{H}$ which transforms like a modular form of weight 2κ. Then*

$$R_{2\kappa}\left(f\big|_{2\kappa} T_m \right) = \frac{1}{m} R_{2\kappa}(f)\big|_{2\kappa+2} T_m.$$

Proof Plugging the last equalities in (2.3) and (2.4) into (2.5) and using that the raising operator commutes with the slash operator on $\mathrm{GL}_2^+(\mathbb{R})$ yields the claim. $\square$

2.3 Elliptic Poincaré series

For $z, \mathfrak{z} \in \mathbb{H}$, let $X_{\mathfrak{z}}(z) := \frac{z-\mathfrak{z}}{z-\overline{\mathfrak{z}}}$, $r_{\mathfrak{z}}(z) := |X_{\mathfrak{z}}(z)|$. For $\kappa \in \mathbb{Z}$, the weight 2κ *elliptic expansion around* $\mathfrak{z}$ of $f : \mathbb{H} \to \mathbb{C}$ is defined as

$$f(z) = (z - \overline{\mathfrak{z}})^{-2\kappa} \sum_{n \in \mathbb{Z}} c_{\mathfrak{z}}\left(r_{\mathfrak{z}}(z), n\right) X_{\mathfrak{z}}(z)^n,$$

for $r = r_{\mathfrak{z}}(z)$ sufficiently small.[6] If f is meromorphic on $\mathbb{H}$, then there exists $n_{\mathfrak{z},0}$ such that $c_{\mathfrak{z}}(r, n) = 0$ for $n < n_{\mathfrak{z},0}$ and $c_{\mathfrak{z}}(r, n)$ is independent of r; in this case, we simply write $c_{\mathfrak{z}}(n)$. The *principal part* of f around $\mathfrak{z}$ is given by

$$(z - \overline{\mathfrak{z}})^{-2\kappa} \sum_{n=n_{\mathfrak{z},0}}^{-1} c_{\mathfrak{z}}(n) X_{\mathfrak{z}}(z)^n.$$

For $k \in \mathbb{N}$ with $k \geq 2$ and $\ell \in \mathbb{Z}$, we define the *elliptic Poincaré series* of weight $2k$ as

$$\Psi_{2k,\ell}^{\mathfrak{z}}(z) := \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \psi_{2k,\ell}^{\mathfrak{z}}(z)\Big|_{2k,z} \gamma, \tag{2.8}$$

with seed

$$\psi_{2k,\ell}^{\mathfrak{z}}(z) := (z - \overline{\mathfrak{z}})^{-2k} X_{\mathfrak{z}}(z)^{\ell}. \tag{2.9}$$

Denote by $\omega_{\mathfrak{z}}$ half of the size of the stabilizer of $\mathfrak{z}$ in $\mathrm{SL}_2(\mathbb{Z})$. Petersson [21, Satz 7] showed that if $\ell \not\equiv -k \pmod{\omega_{\mathfrak{z}}}$, then $\Psi_{2k,\ell}^{\mathfrak{z}}$ vanishes identically. Moreover, for $\ell < 0$ with $\ell \equiv -k \pmod{\omega_{\mathfrak{z}}}$, [21, Satz 7] implies that $\Psi_{2k,\ell}^{\mathfrak{z}}$ has principal part $2\omega_{\mathfrak{z}}(z - \overline{\mathfrak{z}})^{-2k} X_{\mathfrak{z}}(z)^{\ell}$ in its elliptic expansion around the point $\mathfrak{z}$ and is holomorphic at points in $\mathbb{H} \cup \{i\infty\}$ which are not $\mathrm{SL}_2(\mathbb{Z})$-equivalent to $\mathfrak{z}$. Moreover, Petersson showed in [21, Sätze 7–9] that these Poincaré series generate $\mathbb{S}_{2k}$ (see also [15, Section 4.3]).

Lemma 2.2 *The space $\mathbb{S}_{2k}$ is generated by $\{\Psi_{2k,\ell}^{\mathfrak{z}} : \ell \in \mathbb{Z},\ \mathfrak{z} \in \mathbb{H}\}$.*

[6] Considered by Petersson in [20, (2a.16)] for holomorphic functions.

We also require a two-variable Poincaré series that is $\mathrm{SL}_2(\mathbb{Z})$-invariant, has singularities in $\mathbb{H}$, and is an eigenfunction under the Laplace operator Δ_0 in both variables. For $z, \mathfrak{z} \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the *automorphic Green's function* is defined by

$$G_s(z, \mathfrak{z}) := \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} g_s^{\mathbb{H}}(z, \gamma\mathfrak{z}). \tag{2.10}$$

Here $g_s^{\mathbb{H}}(z, \mathfrak{z}) := -Q_{s-1}(1 + \frac{|z-\mathfrak{z}|^2}{2y\mathfrak{y}}) = -Q_{s-1}(\cosh(d(z, \mathfrak{z})))$ with Q_s the Legendre function of the second kind (see [17, 14.2.1]), $d(z, \mathfrak{z}) := \log(\frac{|z-\overline{\mathfrak{z}}|+|z-\mathfrak{z}|}{|z-\overline{\mathfrak{z}}|-|z-\mathfrak{z}|})$ denotes the hyperbolic distance between z and $\mathfrak{z}$, and $\mathfrak{y} := \mathrm{Im}(\mathfrak{z})$ throughout. For further properties of $G_s(z, \mathfrak{z})$, see [10, 14].

2.4 Harmonic Poincaré series

We also require a special family of harmonic Maass forms. To describe it, for $w > 0$ set

$$\mathcal{M}_{\kappa,s}(-w) := w^{-\frac{\kappa}{2}} M_{-\frac{\kappa}{2}, s-\frac{1}{2}}(w),$$

where $M_{a,b}(c)$ denotes the M-Whittaker function (see [17, 13.14.2]). For $k, m \in \mathbb{N}$ define

$$\varphi_{2-2k,-m}(z) := -\mathcal{M}_{2-2k,k}(-4\pi m y)e^{-2\pi i m x}.$$

For $k \geq 2$ we let $\mathcal{F}_{2-2k,-m}$ be the weight $2 - 2k$ *harmonic Maass–Poincaré series*

$$\mathcal{F}_{2-2k,-m}(z) := \frac{1}{(2k-1)!} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \varphi_{2-2k,-m}(z)\big|_{2-2k}\gamma,$$

where $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$. The function $\mathcal{F}_{2-2k,-m}$ has principal part $e^{-2\pi i m z}$ (see the remark before [4, Corollary 6.12]) and $\{\mathcal{F}_{2-2k,-m} : m \in \mathbb{N}\}$ is a basis for the space H_{2-2k}^{cusp}.

3 Properties of elliptic Poincaré series

To prove Theorem 1.1, we require some properties of the elliptic Poincaré series. We first investigate the relation between the Hecke operators on $\Psi_{2k,\ell}^{\mathfrak{z}}(z)$ in z and $\mathfrak{z}$.

Proposition 3.1 *For $k \in \mathbb{N}$ with $k \geq 2$, $\ell \in \mathbb{Z}$, and $n \in \mathbb{N}$, we have,*

$$\Psi_{2k,\ell}^{\mathfrak{z}}(z)\Big|_{2k,z} T_n = \left(\frac{n}{\mathfrak{y}}\right)^{2k+\ell} \left(\mathfrak{y}^{2k+\ell}\Psi_{2k,\ell}^{\mathfrak{z}}(z)\right)\Big|_{-2k-2\ell,\mathfrak{z}} T_n.$$

In order to prove Proposition 3.1, we require another Poincaré series. For $n \in \mathbb{N}$, we choose a set of representatives of the left cosets in $\mathcal{M}_n / \mathrm{SL}_2(\mathbb{Z})$ (see (2.6)),

$$\mathcal{L}_n := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = n,\ d > 0,\ b \pmod{a} \right\}$$

and define

$$\Psi_{2k,\ell,n}^{\mathfrak{z}}(z) := n^{k-1} \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \sum_{M \in \mathcal{L}_n} \psi_{2k,\ell}^{\mathfrak{z}}(z)\Big|_{2k,z} M\gamma \tag{3.1}$$

with $\psi_{2k,\ell}^{\mathfrak{z}}$ as in (2.9). It is not hard to see that $\Psi_{2k,\ell,n}^{\mathfrak{z}}$ are well-defined. They can be constructed from the functions $\Psi_{2k,\ell}^{\mathfrak{z}}$ as follows.

Lemma 3.2 *We have*

$$\Psi^{\mathfrak{z}}_{2k,\ell,n}(z) = \Psi^{\mathfrak{z}}_{2k,\ell}(z)\big|_{2k,z} T_n.$$

Proof Swapping the left coset representatives $\mathcal{L}_n$ appearing in (3.1) with right coset representatives $\mathcal{R}_n$ and combining with (2.7) and (2.8) gives the result. □

We have the following relation between $\Psi^{\mathfrak{z}}_{2k,\ell}$ and $\Psi^{\mathfrak{z}}_{2k,\ell,n}$.

Lemma 3.3 *For $k \in \mathbb{N}$ with $k \geq 2$, $\ell \in \mathbb{Z}$, and $n \in \mathbb{N}$, we have*

$$n^{2k+\ell}\left(\mathrm{y}^{2k+\ell}\Psi^{\mathfrak{z}}_{2k,\ell}(z)\right)\Big|_{-2k-2\ell,\mathfrak{z}} T_n = \mathrm{y}^{2k+\ell}\Psi^{\mathfrak{z}}_{2k,\ell,n}(z).$$

Proof By (2.5), plugging in (2.4) and (2.3) to evaluate the U- and V-operators, we have

$$\left(\mathrm{y}^{2k+\ell}\Psi^{\mathfrak{z}}_{2k,\ell}(z)\right)\big|_{-2k-2\ell,\mathfrak{z}} T_n = n^{-2k-\ell-1}\mathrm{y}^{2k+\ell}\sum_{r|n} r^{2k}\sum_{j=0}^{\frac{n}{r}-1}\Psi^{\frac{r}{n}(r\mathfrak{z}+j)}_{2k,\ell}(z). \tag{3.2}$$

On the other hand, we may rewrite $\mathrm{y}^{2k+\ell}$ times the seed from (3.1) as

$$\mathrm{y}^{2k+\ell}n^{k-1}\sum_{M\in\mathcal{L}_n}\psi^{\mathfrak{z}}_{2k,\ell}(z)\Big|_{2k,z} M = \mathrm{y}^{2k+\ell}\frac{1}{n}\sum_{r|n} r^{2k}\sum_{j=0}^{r-1}\psi^{\mathfrak{z}}_{2k,\ell}\left(\frac{r^2 z + rj}{n}\right).$$

We make the change of variables $r \mapsto \frac{n}{r}$ and plug in the definition of $\psi^{\mathfrak{z}}_{2k,\ell}$ to obtain

$$\begin{aligned}
&\mathrm{y}^{2k+\ell}\frac{1}{n}\sum_{r|n}\left(\frac{n}{r}\right)^{2k}\sum_{j=0}^{\frac{n}{r}-1}\psi^{\mathfrak{z}}_{2k,\ell}\left(\frac{\left(\frac{n}{r}\right)^2 z - \frac{n}{r}j}{n}\right)\\
&= \mathrm{y}^{2k+\ell}n^{2k-1}\sum_{r|n} r^{-2k}\sum_{j=0}^{\frac{n}{r}-1}\left(\frac{nz-rj}{r^2}-\overline{\mathfrak{z}}\right)^{-2k} X_{\mathfrak{z}}\left(\frac{nz-rj}{r^2}\right)^{\ell}\\
&= \mathrm{y}^{2k+\ell}\frac{1}{n}\sum_{r|n} r^{2k}\sum_{j=0}^{\frac{n}{r}-1}\left(z-\frac{r^2\overline{\mathfrak{z}}+rj}{n}\right)^{-2k} X_{\frac{r^2\mathfrak{z}+rj}{n}}(z)^{\ell} = \mathrm{y}^{2k+\ell}\frac{1}{n}\sum_{r|n} r^{2k}\sum_{j=0}^{\frac{n}{r}-1}\psi^{\frac{r^2\mathfrak{z}+rj}{n}}_{2k,\ell}(z).
\end{aligned}$$

Therefore, plugging this into (3.1), using (2.8), and comparing with (3.2), we obtain

$$\begin{aligned}
\mathrm{y}^{2k+\ell}\Psi^{\mathfrak{z}}_{2k,\ell,n}(z) &= \mathrm{y}^{2k+\ell}\frac{1}{n}\sum_{r|n} r^{2k}\sum_{j=0}^{\frac{n}{r}-1}\sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})}\psi^{\frac{r^2\mathfrak{z}+rj}{n}}_{2k,\ell}(z)\Big|_{2k,z}\gamma\\
&= \mathrm{y}^{2k+\ell}\frac{1}{n}\sum_{r|n} r^{2k}\sum_{j=0}^{\frac{n}{r}-1}\Psi^{\frac{r^2\mathfrak{z}+rj}{n}}_{2k,\ell}(z) = n^{2k+\ell}\left(\mathrm{y}^{2k+\ell}\Psi^{\mathfrak{z}}_{2k,\ell}(z)\right)\Big|_{-2k-2\ell,\mathfrak{z}} T_n.
\end{aligned}$$

□

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1 Combining Lemma 3.2 and Lemma 3.3, we conclude that

$$\Psi^{\mathfrak{z}}_{2k,\ell}(z)\Big|_{2k,z} T_n = \Psi^{\mathfrak{z}}_{2k,\ell,n}(z) = \left(\frac{n}{\mathrm{y}}\right)^{2k+\ell}\left(\mathrm{y}^{2k+\ell}\Psi^{\mathfrak{z}}_{2k,\ell}(z)\right)\Big|_{-2k-2\ell,\mathfrak{z}} T_n.$$

□

We also need the following relation between $\Psi^{\mathfrak{z}}_{2k,-\ell}$ and $\mathcal{F}_{2-2k,-m}$.

Lemma 3.4 *For $\ell \in \mathbb{N}$ and $y > \max\{\mathrm{Im}(\gamma\mathfrak{z}) \ :\ \gamma \in \Gamma\}$, we have the Fourier expansion*

$$\Psi^{\mathfrak{z}}_{2k,-\ell}(z) = \frac{(-1)^k 2^{3-2k}\pi \mathrm{y}^{\ell-2k}}{(\ell-1)!} \sum_{m=1}^{\infty} R^{\ell-1}_{2-2k}\left(\mathcal{F}_{2-2k,-m}(\mathfrak{z})\right) q^m.$$

Proof We proceed by induction. The case $\ell = 1$ is [6, (3.10) and (3.11)].

Now assume that the lemma holds for some $\ell \in \mathbb{N}$. We multiply both sides of the lemma by $\mathrm{y}^{2k-\ell}$ and apply raising in weight $2\ell - 2k$ in $\mathfrak{z}$ (for $y > \mathrm{Im}(\gamma\mathfrak{z})$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$), yielding

$$\sum_{\gamma\in \mathrm{SL}_2(\mathbb{Z})} \left((z-\overline{\mathfrak{z}})^{-2k} R_{2\ell-2k,\mathfrak{z}}\left(\mathrm{y}^{2k-\ell} X_{\mathfrak{z}}(z)^{-\ell}\right)\right)\Big|_{2k,z}\gamma$$
$$= \frac{(-1)^k 2^{3-2k}\pi}{(\ell-1)!} \sum_{m=1}^{\infty} R^{\ell}_{2-2k}(\mathcal{F}_{2-2k,-m}(\mathfrak{z}))e^{2\pi i m z}.$$

Here we note that the contribution to the sum over m from the principal parts $e^{-2\pi i m\gamma\mathfrak{z}}$ of $\mathcal{F}_{2-2k,-m}(\mathfrak{z})$ at each cusp $\gamma\mathfrak{z}$ converges absolutely and locally uniformly because of the restriction $y > \mathrm{Im}(\gamma\mathfrak{z})$. We then compute

$$R_{2\ell-2k,\mathfrak{z}}\left(\mathrm{y}^{2k-\ell} X_{\mathfrak{z}}(z)^{-\ell}\right) = \frac{\mathrm{y}^{2k-\ell-1}\,(z-\overline{\mathfrak{z}})^{\ell}}{(z-\mathfrak{z})^{\ell+1}}((2k-\ell)(z-\mathfrak{z}) + 2i\ell\mathrm{y} + (2\ell-2k)(z-\mathfrak{z})),$$

where the first term comes from differentiating $\mathrm{y}^{2k-\ell}$, the second comes from differentiating $(z-\mathfrak{z})^{-\ell}$, and the third comes from the term $\frac{2\ell-2k}{\mathrm{y}}$. We then rewrite the right-hand side as

$$\ell \frac{\mathrm{y}^{2k-\ell-1}(z-\overline{\mathfrak{z}})^{\ell+1}}{(z-\mathfrak{z})^{\ell+1}} = \ell \mathrm{y}^{2k-\ell-1} X_{\mathfrak{z}}(z)^{-\ell-1}.$$

Hence

$$\ell \mathrm{y}^{2k-\ell-1} \Psi^{\mathfrak{z}}_{2k,-\ell-1}(z) = \frac{(-1)^k 2^{3-2k}\pi}{(\ell-1)!} \sum_{m=1}^{\infty} R^{\ell}_{2-2k}\left(\mathcal{F}_{2-2k,-m}(\mathfrak{z})\right) q^m.$$

Dividing both sides by $\ell\mathrm{y}^{2k-\ell-1}$ then yields the claim for $\ell + 1$, completing the proof. □

4 Proof of Theorem 1.1

Before proving Theorem 1.1, we use Lemma 3.4 to extend the definition of $\mathcal{G}_g(z, \mathfrak{z})$.

Lemma 4.1 *Let $k \in \mathbb{N}_{\geq 2}$. For $g \in R^{\ell-1}_{2-2k}(H^{\mathrm{cusp}}_{2-2k})$ and $\mathfrak{z} \in \mathbb{H}$, $\mathcal{G}_g(z, \mathfrak{z})$ converges for y sufficiently large (depending on $\mathfrak{z}$) and has a meromorphic continuation to $\mathbb{H}$.*

Proof Choose $F \in H^{\mathrm{cusp}}_{2-2k}$ such that $g = R^{\ell-1}_{2-2k}(F)$. Since $\{\mathcal{F}_{2-2k,-n} : n \in \mathbb{N}\}$ is a basis (see Section 2.4) for the space H^{cusp}_{2-2k}, there exist $c(n) \in \mathbb{C}$ and $n_0 \in \mathbb{N}$ such that

$$F = \sum_{n=1}^{n_0} c(n)\mathcal{F}_{2-2k,-n}.$$

By comparing principal parts (recall (2.5) and the principal part q^{-n} of $\mathcal{F}_{2-2k,-n}$), we have

$$n^{1-2k}\mathcal{F}_{2-2k,-n} = \mathcal{F}_{2-2k,-1}\big|_{2-2k}T_n. \tag{4.1}$$

Thus we obtain

$$F = \sum_{n=1}^{n_0} c(n)n^{2k-1}\mathcal{F}_{2-2k,-1}\big|_{2-2k}T_n.$$

Therefore we obtain

$$g = \sum_{n=1}^{n_0} c(n)n^{2k-1}R_{2-2k}^{\ell-1}\left(\mathcal{F}_{2-2k,-1}\big|_{2-2k}T_n\right).$$

Plugging this into (1.5), for y sufficiently large (depending on $\mathfrak{z}$, where we use the convergence by Lemma 3.4) we have

$$\mathcal{G}_g(z,\mathfrak{z}) = \sum_{m=1}^{\infty} m^{2k-\ell}\sum_{n=1}^{n_0} c(n)n^{2k-1}R_{2-2k}^{\ell-1}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\big|_{2-2k}T_n\right)\big|_{2\ell-2k}T_m q^m. \tag{4.2}$$

We then use Lemma 2.1 $\ell-1$ times to rewrite the right-hand side of (4.2) as

$$\sum_{m=1}^{\infty} m^{2k-\ell}\sum_{n=1}^{n_0} c(n)n^{2k-\ell}R_{2-2k}^{\ell-1}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\right)\big|_{2\ell-2k}T_n\circ T_m q^m.$$

Since Hecke operators commute (see for instance [22, equation (42)]), we can rewrite this as

$$\sum_{m=1}^{\infty} m^{2k-\ell}\sum_{n=1}^{n_0} c(n)n^{2k-\ell}R_{2-2k}^{\ell-1}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\right)\big|_{2\ell-2k}T_m q^m\big|_{2\ell-2k,\mathfrak{z}}T_n.$$

We next interchange the sums. This is valid due to the growth condition

$$\left|m^{2k-\ell}n^{2k-\ell}R_{2-2k}^{\ell-1}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\right)\big|_{2\ell-2k}T_m\big|_{2\ell-2k}T_n\right| \ll_{\mathfrak{z}} e^{cnm\left(\mathrm{Im}(\mathfrak{z})+\frac{1}{\mathrm{Im}(\mathfrak{z})}\right)}$$

implied by the convergence of the series in [10, Theorem 3.1]. Therefore (4.2) becomes

$$\mathcal{G}_g(z,\mathfrak{z}) = \sum_{n=1}^{n_0} c(n)n^{2k-\ell}\left(\sum_{m=1}^{\infty} m^{2k-\ell}R_{2-2k}^{\ell-1}(\mathcal{F}_{2-2k,-1}(\mathfrak{z}))\big|_{2\ell-2k}T_m q^m\right)\Bigg|_{2\ell-2k,\mathfrak{z}}T_n. \tag{4.3}$$

We use Lemma 2.1 again $\ell-1$ times to rewrite the sum inside the parentheses on the right-hand side of (4.3) as

$$\sum_{m=1}^{\infty} m^{2k-1}R_{2-2k}^{\ell-1}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\big|_{2-2k}T_m\right)q^m.$$

We next employ (4.1) again followed by Lemma 3.4 to rewrite this as

$$\sum_{m=1}^{\infty} R_{2-2k}^{\ell-1}\left(\mathcal{F}_{2-2k,-m}(\mathfrak{z})\right)q^m = \frac{(-1)^k 2^{2k-3}(\ell-1)!}{\pi}\mathfrak{y}^{2k-\ell}\Psi_{2k,-\ell}^{\mathfrak{z}}(z).$$

Plugging this into (4.3), we obtain

$$\mathcal{G}_g(z,\mathfrak{z}) = \frac{(-1)^k 2^{2k-3}(\ell-1)!}{\pi}\sum_{n=1}^{n_0} c(n)n^{2k-\ell}\left(\mathfrak{y}^{2k-\ell}\Psi_{2k,-\ell}^{\mathfrak{z}}(z)\right)\big|_{2\ell-2k,\mathfrak{z}}T_n. \tag{4.4}$$

Since $\Psi_{2k,-\ell}^{\mathfrak{z}}$ is meromorphic and well-defined for all $z \in \mathbb{H}$ and the sum is finite, we obtain a meromorphic continuation for $\mathcal{G}_g(z,\mathfrak{z})$. □

To prove Theorem 1.1,we next show that we have Hecke-stable subspaces of $\mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$.

Lemma 4.2 *The spaces $\mathbb{J}_{2k,-\ell}^{\mathfrak{z}}$ and $\mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}}$ are Hecke-stable subspaces of $\mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$.*

Proof We first show that $\mathbb{J}_{2k,-\ell}^{\mathfrak{z}}$ and $\mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}}$ are subspaces of $\mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$. Recalling (1.6) and (1.7), both are vector spaces due to the linearity in g in the definition (1.5) and the fact that $R_{2-2k}^{\ell-1}(M_{2-2k}^!)$ and $R_{2-2k}^{\ell-1}(S_{2-2k}^!)$ are vector spaces. Noting that $\mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}} \subseteq \mathbb{J}_{2k,-\ell}^{\mathfrak{z}}$ because $R_{2-2k}^{\ell-1}(S_{2-2k}^!) \subseteq R_{2-2k}^{\ell-1}(M_{2-2k}^!)$, it remains to show $\mathbb{J}_{2k,-\ell}^{\mathfrak{z}} \subseteq \mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$. For this, let $\mathcal{G}_g(z,\mathfrak{z})$ with $g \in R_{2-2k}^{\ell-1}(M_{2-2k}^!)$. We use Lemma 3.3 followed by Lemma 3.2 to rewrite (4.4) as

$$\mathcal{G}_g(z,\mathfrak{z}) = \frac{(-1)^k 2^{2k-3}(\ell-1)!}{\pi} \mathrm{y}^{2k-\ell} \sum_{n=1}^{n_0} c(n) \Psi_{2k,-\ell}^{\mathfrak{z}}(z)\big|_{2k,z} T_n. \tag{4.5}$$

Thus $\mathcal{G}_g(z,\mathfrak{z}) \in \mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$, so $\mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}} \subseteq \mathbb{J}_{2k,-\ell}^{\mathfrak{z}} \subseteq \mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$ are subspaces.

Finally, to show that the subspaces are Hecke-stable, we apply $|_{2k,z}T_r$ to both sides of (4.5) to obtain

$$\mathcal{G}_g(z,\mathfrak{z})\big|_{2k,z} T_r = \frac{(-1)^k 2^{2k-3}(\ell-1)!}{\pi} \sum_{n=1}^{n_0} c(n) \mathrm{y}^{2k-\ell} \Psi_{2k,-\ell}^{\mathfrak{z}}(z)\big|_{2k,z} T_n \circ T_r. \tag{4.6}$$

Since the Hecke operators commute, we may interchange them to obtain

$$\mathrm{y}^{2k-\ell} \Psi_{2k,-\ell}^{\mathfrak{z}}(z)\big|_{2k,z} T_n \circ T_r = \mathrm{y}^{2k-\ell} \Psi_{2k,-\ell}^{\mathfrak{z}}(z)\big|_{2k,z} T_r \circ T_n. \tag{4.7}$$

We then use Lemma 3.2 followed by Lemma 3.3 (with $\ell \mapsto -\ell$) to rewrite

$$\mathrm{y}^{2k-\ell} \Psi_{2k,-\ell}^{\mathfrak{z}}(z)\big|_{2k,z} T_r = r^{2k-\ell} \left(\mathrm{y}^{2k-\ell} \Psi_{2k,-\ell}^{\mathfrak{z}}(z)\right)\big|_{2\ell-2k,\mathfrak{z}} T_r. \tag{4.8}$$

Plugging (4.7) and (4.8) into the right-hand side of (4.6) yields

$$\mathcal{G}_g(z,\mathfrak{z})\big|_{2k,z} T_r = \frac{(-1)^k 2^{2k-3}(\ell-1)!}{\pi} r^{2k-\ell} \sum_{n=1}^{n_0} c(n) \left(\mathrm{y}^{2k-\ell} \Psi_{2k,-\ell}^{\mathfrak{z}}(z)\right)\big|_{2\ell-2k,\mathfrak{z}} T_r \big|_{2k,z} T_n.$$

We next interchange the Hecke operator T_n in z inside with T_r in $\mathfrak{z}$ to obtain

$$\begin{aligned}\mathcal{G}_g(z,\mathfrak{z})\big|_{2k,z} T_r &= r^{2k-\ell} \left(\frac{(-1)^k 2^{2k-3}(\ell-1)!}{\pi} \sum_{n=1}^{n_0} c(n) \left(\mathrm{y}^{2k-\ell} \Psi_{2k,-\ell}^{\mathfrak{z}}(z)\right)\big|_{2k,z} T_n \right)\bigg|_{2\ell-2k,\mathfrak{z}} T_r \\ &= r^{2k-\ell} \mathcal{G}_g(z,\mathfrak{z})\big|_{2\ell-2k,\mathfrak{z}} T_r,\end{aligned} \tag{4.9}$$

using (4.5) in the last step. Plugging (4.9) into the definition (1.5) and then interchanging Hecke operators in $\mathfrak{z}$, we obtain that

$$\mathcal{G}_g(z,\mathfrak{z})\big|_{2k,z} T_r = \sum_{m=1}^{\infty} (mr)^{2k-\ell} g(\mathfrak{z})\big|_{2\ell-2k} T_r \circ T_m q^m = r^{2k-\ell} \mathcal{G}_{g|_{2\ell-2k}T_r}(z,\mathfrak{z}) \tag{4.10}$$

Since $M_{2-2k}^!$ and $S_{2-2k}^!$ are both Hecke-stable, Lemma 2.1 implies that $R_{2-2k}^{\ell-1}(M_{2-2k}^!)$ and $R_{2-2k}^{\ell-1}(S_{2-2k}^!)$ are both Hecke-stable as well. □

For the proof of Theorem 1.1 we require certain isomorphisms.

Lemma 4.3 *The following spaces are isomorphic as Hecke modules,*

$$H_{2-2k}^{\mathrm{cusp}}/M_{2-2k}^{!} \cong S_{2k}, \qquad H_{2-2k}^{\mathrm{cusp}}/S_{2-2k}^{!} \cong M_{2k}.$$

Proof By [9, Theorems 1.1 and 1.2 and Lemma 2.1] (also see Lemma 2.1 for the normalization), D^{2k-1} is an isomorphism from H_{2-2k}^{cusp} to $S_{2k}^{!} \cap S_{2k}^{\perp}$ which, by (1.8) and Lemma 2.1, essentially commutes with the Hecke operators in the sense that

$$D^{2k-1}\left(f\big|_{2-2k}T_m\right) = m^{1-2k}D^{2k-1}(f)\big|_{2k}T_m.$$

We then consider the reduction of this map modulo $D^{2k-1}(M_{2-2k}^{!})$ (resp. $D^{2k-1}(S_{2-2k}^{!})$) and use the homomorphism theorem. Since D^{2k-1} is a bijection from H_{2-2k}^{cusp} to $S_{2k}^{!} \cap S_{2k}^{\perp}$, the kernel of the composition of D^{2k-1} followed by the reduction map to $D^{2k-1}(M_{2-2k}^{!})$ (resp. $D^{2k-1}(S_{2-2k}!)$) is $M_{2-2k}^{!}$ (resp. $S_{2-2k}^{!}$). Therefore

$$H_{2-2k}^{\mathrm{cusp}}/M_{2-2k}^{!} \cong \left(S_{2k}^{!} \cap S_{2k}^{\perp}\right)/D^{2k-1}\left(M_{2-2k}^{!}\right),$$
$$H_{2-2k}^{\mathrm{cusp}}/S_{2-2k}^{!} \cong \left(S_{2k}^{!} \cap S_{2k}^{\perp}\right)/D^{2k-1}\left(S_{2-2k}^{!}\right).$$

Now using the isomorphism in [5, Theorem 1.2] completes the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 In order to prove the isomorphisms, for $k, \ell \in \mathbb{N}$ with $k \geq 2$, we consider the map $\varrho_{2-2k,\ell}^{\mathfrak{z}} : H_{2-2k}^{\mathrm{cusp}} \mapsto \mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$ given by

$$\varrho_{2-2k,\ell}^{\mathfrak{z}}(\mathcal{F})(z) := \mathcal{G}_{R_{2-2k}^{\ell-1}(\mathcal{F})}(z,\mathfrak{z}). \tag{4.11}$$

We claim that $\varrho_{2-2k,\ell}^{\mathfrak{z}}$ is a bijection between H_{2-2k}^{cusp} and $\mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$ and essentially commutes with the Hecke operators in the sense that

$$\varrho_{2-2k,\ell}^{\mathfrak{z}}(\mathcal{F})\big|_{2k,z}T_m = m^{2k-1}\varrho_{2-2k,\ell}^{\mathfrak{z}}\left(\mathcal{F}\big|_{2-2k}T_m\right). \tag{4.12}$$

We first show how the claim follows from this: Note that by definition

$$\varrho_{2-2k,\ell}^{\mathfrak{z}}\left(M_{2-2k}^{!}\right) = \mathbb{J}_{2k,-\ell}^{\mathfrak{z}} \qquad \text{and} \qquad \varrho_{2-2k,\ell}^{\mathfrak{z}}\left(S_{2-2k}^{!}\right) = \mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}}.$$

Hence if $\varrho_{2-2k,\ell}^{\mathfrak{z}}$ is a Hecke-equivariant isomorphism from H_{2-2k}^{cusp} to $\mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$, then the quotient map $\varrho_{2-2k,\ell}^{\mathfrak{z}} \pmod{\mathbb{J}_{2k,-\ell}^{\mathfrak{z}}}$ (resp. $\varrho_{2-2k,\ell}^{\mathfrak{z}} \pmod{\mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}}}$) has kernel $M_{2-2k}^{!}$ (resp. $S_{2-2k}^{!}$), yielding the Hecke-equivariant isomorphisms

$$\mathbb{S}_{2k,-\ell}^{\mathfrak{z}}/\mathbb{J}_{2k,-\ell}^{\mathfrak{z}} \cong H_{2-2k}^{\mathrm{cusp}}/M_{2-2k}^{!}, \qquad \mathbb{S}_{2k,-\ell}^{\mathfrak{z}}\Big/\mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}} \cong H_{2-2k}^{\mathrm{cusp}}/S_{2-2k}^{!}. \tag{4.13}$$

After establishing (4.13), the theorem immediately follows by Lemma 4.3.

It therefore remains to show that $\varrho_{2-2k,\ell}^{\mathfrak{z}}$ is an isomorphism and satisfies (4.12). Since $g \mapsto \mathcal{G}_g$ is linear, it suffices to prove that it sends a basis of H_{2-2k}^{cusp} to a basis of $\mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$. From Section 2.4 we know that $\{\mathcal{F}_{2-2k,-r} : r \in \mathbb{N}\}$ is a basis for H_{2-2k}^{cusp}, while $\{\Psi_{2k,-\ell}^{\mathfrak{z}}|_{2k}T_r : r \in \mathbb{N}\}$ is a basis for $\mathbb{S}_{2k,-\ell}^{\mathfrak{z}}$ by definition of the space. We claim that

$$\varrho^{\mathfrak{z}}_{2-2k,\ell}\left(\mathcal{F}_{2-2k,-r}\right) = \frac{(-1)^k 2^{2k-3}(\ell-1)!}{\pi} \mathrm{y}^{2k-\ell} \Psi^{\mathfrak{z}}_{2k,-\ell}(z)\big|_{2k,z} T_r, \tag{4.14}$$

which then immediately implies that $\varrho^{\mathfrak{z}}_{2-2k,\ell}$ is an isomorphism.

To show (4.14), we plug (4.1) into Lemma 3.4 and then apply Lemma 2.1, yielding that for y sufficiently large, depending on $\mathfrak{z}$, we have

$$\Psi^{\mathfrak{z}}_{2k,-\ell}(z) = \frac{(-1)^k 2^{3-2k} \pi \mathrm{y}^{\ell-2k}}{(\ell-1)!} \sum_{m=1}^{\infty} m^{2k-\ell} R^{\ell-1}_{2-2k}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\right)\big|_{2\ell-2k} T_m q^m. \tag{4.15}$$

Combining (4.8) with (4.15) and then interchanging the Hecke operators yields

$$\begin{aligned} &\mathrm{y}^{2k-\ell} \Psi^{\mathfrak{z}}_{2k,-\ell}(z)\big|_{2k,z} T_r \\ &\quad = \frac{(-1)^k 2^{3-2k}\pi}{(\ell-1)!} r^{2k-\ell} \sum_{m=1}^{\infty} m^{2k-\ell} R^{\ell-1}_{2-2k}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\right)\big|_{2\ell-2k} T_r \big|_{2\ell-2k} T_m q^m. \end{aligned} \tag{4.16}$$

We then use Lemma 2.1 $\ell-1$ times to rewrite

$$R^{\ell-1}_{2-2k}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\right)\big|_{2\ell-2k} T_r = r^{\ell-1} R^{\ell-1}_{2-2k}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\big|_{2-2k} T_r\right).$$

The right-hand side of (4.16) then becomes

$$\frac{(-1)^k 2^{3-2k}\pi r^{2k-1}}{(\ell-1)!} \sum_{m=1}^{\infty} m^{2k-\ell} R^{\ell-1}_{2-2k}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\big|_{2-2k} T_r\right)\big|_{2\ell-2k} T_m q^m.$$

We next use (4.1) to rewrite

$$r^{2k-1} R^{\ell-1}_{2-2k}\left(\mathcal{F}_{2-2k,-1}(\mathfrak{z})\big|_{2-2k} T_r\right) = R^{\ell-1}_{2-2k}\left(\mathcal{F}_{2-2k,-r}(\mathfrak{z})\right).$$

Plugging back into (4.16) gives

$$\mathrm{y}^{2k-\ell} \Psi^{\mathfrak{z}}_{2k,-\ell}(z)\big|_{2k,z} T_r = \frac{(-1)^k 2^{3-2k}\pi}{(\ell-1)!} \sum_{m=1}^{\infty} m^{2k-\ell} R^{\ell-1}_{2-2k}\left(\mathcal{F}_{2-2k,-r}(\mathfrak{z})\right)\big|_{2\ell-2k} T_m q^m.$$

Comparing with the definitions (1.5) and (4.11), we obtain (4.14). It remains to show that (4.12) holds. Applying (4.10) followed by Lemma 2.1, we have

$$\varrho^{\mathfrak{z}}_{2-2k,\ell}(\mathcal{F})(z)\big|_{2k,z} T_m = \mathcal{G}_{R^{\ell-1}_{2-2k}(\mathcal{F})|_{2\ell-2k} T_m}(z,\mathfrak{z}) = m^{2k-1} \varrho^{\mathfrak{z}}_{2-2k,\ell}\left(\mathcal{F}\big|_{2-2k} T_m\right)(z),$$

yielding (4.12). □

We end this section with a remark about the motivation for our definition and its relation with Gross and Zagier's conjecture [11, Conjecture 4.4, p. 317].

Remark 4.4 Let $(\lambda_r)_{r\geq 1}$ be a sequence of complex numbers, only finitely many of which are $\neq 0$. Note that, using (4.1),

$$\sum_{r=1}^{\infty} \lambda_r r^{2k-1} \mathcal{F}_{2-2k,-1}|_{2-2k} T_r = \sum_{r=1}^{\infty} \lambda_r \mathcal{F}_{2-2k,-r} \in M^!_{2-2k}$$

if and only if there exists a weakly holomorphic modular form with principal part $\sum_{r=1}^{\infty} \lambda_r q^{-r}$ at $i\infty$. Applying Riemann–Roch (for example, see [20, Satz 1]), such a weakly holomorphic modular form exists if and only if for every $f \in S_{2k}$

$$\sum_{r=1}^{\infty} \lambda_r c_f(r) = 0.$$

Hence (1.3) is equivalent to

$$g = g_\lambda := \sum_{r=1}^{\infty} \lambda_r R_{2-2k}^{\ell-1}\left(\mathcal{F}_{2-2k,-r}\right) \in R_{2-2k}^{\ell-1}\left(M_{2-2k}^{!}\right),$$

and (4.5) constructs from such g a sum $\mathcal{G}_g$ of the $\Psi_{2k,-\ell}^{\mathfrak{z}}(z)|_{2k,z}T_n$ analogous to the sum appearing in the formula (1.4) conjectured by Gross and Zagier, but in higher weight. Since $\mathcal{G}_g \in \mathbb{J}_{2k,-\ell}^{\mathfrak{z}}$ if and only if $g \in R_{2-2k}^{\ell-1}(M_{2-2k}^{!})$, this formed the motivation for the definitions of the subspaces $\mathbb{J}_{2k,-\ell}^{\mathfrak{z}}$ and $\mathbb{J}_{2k,-\ell,0}^{\mathfrak{z}}$.

5 Examples of meromorphic Hecke eigenforms

In this section, we consider two examples of meromorphic Hecke eigenforms which are counterparts of holomorphic modular forms under the isomorphism in Theorem 1.1. The first, $f_{6,i}$ as defined in (1.9), has the same eigenvalues as the Eisenstein series E_6. The second example, G defined in (1.11), has the same eigenvalues as the cusp form Δ.

5.1 A meromorphic Hecke eigenform associated to an Eisenstein series

Here, we investigate $f_{6,i}$ given in the introduction and explain how it is defined. Noting that $S_6 = \{0\}$ and $M_6 = \mathbb{C}E_6$ Theorem 1.1 and (4.13) yields that for every $\ell \in \mathbb{N}$

$$\{0\} = S_6 \cong \mathbb{S}_{6,-\ell}^{\mathfrak{z}}/\mathbb{J}_{6,-\ell}^{\mathfrak{z}} \cong H_{-4}^{\mathrm{cusp}}/M_{-4}^{!}, \qquad \mathbb{C}E_6 = M_6 \cong \mathbb{S}_{6,-\ell}^{\mathfrak{z}}/\mathbb{J}_{6,-\ell,0}^{\mathfrak{z}} \cong H_{-4}^{\mathrm{cusp}}/S_{-4}^{!}.$$

Choosing the generator $\mathcal{F}_{-4,-1}$ of $H_{-4}^{\mathrm{cusp}}/S_{-4}^{!}$, by (4.12) we see that $\varrho_{6,\ell}^{\mathfrak{z}}(\mathcal{F}_{-4,-1})$ is a Hecke eigenform in $\mathbb{S}_{6,-\ell}^{\mathfrak{z}}/\mathbb{J}_{6,-\ell,0}^{\mathfrak{z}}$ with eigenvalues $n^{2k-1}\sigma_5(n)$ under T_n. We choose $\ell = 1$ and $\mathfrak{z} = i$ and explicitly compute $\varrho_{6,1}^{i}(\mathcal{F}_{-4,-1})$. By (4.4) we have

$$\varrho_{6,1}^{i}\left(\mathcal{F}_{-4,-1}\right) = \mathcal{G}_{\mathcal{F}_{-4,-1}}(z,i) = -\frac{8}{\pi}\Psi_{6,-1}^{i}(z).$$

Normalizing yields the eigenform

$$\frac{1}{2^7\pi\alpha}\Psi_{6,-1}^{i}(z) = \frac{\Delta(z)}{E_6(z)} = f_{6,i}(z).$$

The calculation given in the introduction illustrates that $f_{6,i}$ is a meromorphic Hecke eigenform corresponding to (a constant multiple of) E_6 under the isomorphism in Theorem 1.1. Specifically, the eigenvalues under T_m for $m \in \{5, 7\}$ are directly computed and shown to match the eigenvalues $\sigma_5(m)$ of E_6 under T_m.

5.2 A meromorphic Hecke eigenform associated to a cusp form

In this example we establish a meromorphic counterpart of Ramanujan's Δ-function under the isomorphism in Theorem 1.1. Let $\mathfrak{z} = \frac{1+\sqrt{7}i}{2}$ to be the unique CM point of discriminant

-7 in the standard fundamental domain. Furthermore let $\Omega := \Omega_{-7}$ the Chowla–Selberg period of discriminant -7 (for an explicit formula in terms of the Γ-function see e.g. [22, equation (97)]). With this we can explicitly evaluate (see [22, p. 87]),

$$E_4(\mathfrak{z}) = 15\Omega^4, \quad E_6(\mathfrak{z}) = 27\sqrt{7}\Omega^6, \quad \Delta(\mathfrak{z}) = -\Omega^{12}, \quad j(\mathfrak{z}) = -15^3. \tag{5.1}$$

Together with the valence formula this implies that the weight 12 modular form

$$F(z) := E_4(z)^3 + 15^3\Delta(z) = 1 + 4095q + 98280q^2 + 17805060q^3 + O\left(q^4\right) \in M_{12}$$

has a simple zero in $\mathfrak{z}$ and vanishes nowhere else in the fundamental domain.

We claim that $G = \frac{\Delta^2}{F}$, defined in (1.11), is a meromorphic Hecke eigenform whose eigenvalues agree with those of Δ. For this, we consider the Maass–Poincaré series $\mathcal{F}_{-10,-1}$ as a non-trivial representative of the space $H_{-10}^{\text{cusp}}/M_{-10}^! \cong S_{12} = \mathbb{C}\Delta$, where we use the first isomorphism in Lemma 4.3. As in the previous example, the image of this Poincaré series under the map $\varrho_{12,1}^{\mathfrak{z}}$ yields a constant multiple of the elliptic Poincaré series $\Psi_{12,-1}^{\mathfrak{z}}$. By [18, Theorem 1.1], for any prime p

$$\mathcal{F}_{-10,-1}|T_p - p^{-11}\tau(p)\mathcal{F}_{-10,-1}$$

is a weakly holomorphic modular form. Set

$$g(z) := \frac{E_4(z)^2E_6(z)}{\Delta(z)^2} = q^{-2} + 24q^{-1} - 196560 - 47709536q - 3688365156q^2 + O\left(q^3\right) \in M_{-10}^!.$$

We find that

$$\mathcal{F}_{-10,-1}|T_2 - 2^{-11}\tau(2)\mathcal{F}_{-10,-1} = 2^{-11}g,$$
$$\mathcal{F}_{-10,-1}|T_3 - 3^{-11}\tau(3)\mathcal{F}_{-10,-1} = 3^{-11}(j - 768)g.$$

To see this, note that the principal parts are equal in in each case, wherefore the difference of both sides of the equality must be a holomorphic modular form of weight -10 and hence 0.

We may use these to compute the first few Fourier coefficients of $\Psi_{12,-1}^{\mathfrak{z}}$ in terms of $\alpha := \mathcal{F}_{-10,-1}(\mathfrak{z})$ and $\beta := g(\mathfrak{z}) = \frac{6075\sqrt{7}}{\Omega^{10}}$, where we employ (5.1) for the last equality. Namely, by Lemma 3.4, up to a multiplicative constant, we have, for y sufficiently large,

$$\Psi_{12,-1}^{\mathfrak{z}}(z) = \alpha q + (\beta - 24\alpha)q^2 + (252\alpha - 4143\beta)q^3 + O\left(q^4\right). \tag{5.2}$$

In order to show that G is indeed a meromorphic Hecke eigenform corresponding to Δ, we use Theorem 1.1 and show that G equals (up to a constant multiple of Δ itself) the elliptic Poincaré series $\Psi_{12,-1}^{\mathfrak{z}}$. Since $\Psi_{12,-1}^{\mathfrak{z}}$ and G in (1.11) both have weight 12 and a simple pole in $\mathfrak{z}$ and nowhere else in the fundamental domain, it follows that $\frac{\Psi_{12,-1}^{\mathfrak{z}}}{G}$ is a modular function, holomorphic in $\mathbb{H}$ with at most a simple pole at $i\infty$. Comparing Fourier coefficients in (1.11) and (5.2) we find that

$$\frac{\Psi_{12,-1}^{\mathfrak{z}}(z)}{G(z)} = \alpha q^{-1} + (4119\alpha + \beta) + 196884\alpha q + O\left(q^2\right) = \alpha(j(z) + 3375) + \beta = \alpha\frac{F(z)}{\Delta(z)} + \beta.$$

It follows therefore that, again up to a constant factor,

$$\Psi_{12,-1}^{\mathfrak{z}} = \beta G + \alpha\Delta.$$

Since we can ignore the term $\alpha\Delta$, Theorem 1.1 yields our claim that G is a meromorphic Hecke eigenform corresponding to Δ. To illustrate this we can compute

$$G(z)|T_2 - (-24)G(z) = q + 16868409q^2 + 279687514914333q^3 + O\left(q^4\right), \tag{5.3}$$

$$\begin{aligned} 2^{11}g(z)|T_2 &= q^{-4} + 24q^{-2} + 2048q^{-1} - 402751440 - 7553771839488q + O\left(q^2\right) \\ &= g(z)\left(j(z)^2 - 1512j(z) + 374784\right) \end{aligned} \tag{5.4}$$

$$\begin{aligned} 3^{11}g(z)|T_3 &= q^{-6} + 24q^{-3} - 34820210880 - 27948629556463536q + O\left(q^2\right) \\ &= g(z)\left(j(z)^4 - 3000j(z)^3 + 2784384j(z)^2 - 842201064j(z) + 52796307708\right). \end{aligned} \tag{5.5}$$

As predicted by Theorem 1.1, evaluating the polynomials in j obtained in (5.4) and (5.5) in $\mathfrak{z}$ (i.e., $j(\mathfrak{z}) = -15^3$), yields exactly the coefficients of q^2 and q^3 in (5.3).

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References

1. Bengoechea, P.: Corps quadratiques et formes modulaires (2013). (Ph.D. thesis)
2. Berndt, B., Bialek, P., Yee, A.: Formulas of Ramanujan for the power series coefficients of certain quotients of Eisenstein series. Int. Math. Res. Not. **2002**, 1077–1109 (2002)
3. Bialek, P.: Ramanujan's formulas for the coefficients in the power series expansions of certain modular forms, University of Illinois at Urbana-Champaign (1995). (Ph. D. thesis)
4. Bringmann, K., Folsom, A., Ono, K., Rolen, L.: Harmonic Maass forms and mock modular forms: theory and applications, Amer. Math. Soc. Colloq. Publ. **64**, Amer. Math. Soc., Providence, RI (2017)
5. Bringmann, K., Guerzhoy, P., Kent, Z., Ono, K.: Eichler-shimura theory for mock modular forms. Math. Ann. **355**, 1085–1121 (2013)
6. Bringmann, K., Kane, B.: Ramanujan-like formulas for Fourier coefficients of all meromorphic cusp forms. Adv. Math. **373**, 107308 (2020)
7. Bruinier, J., Funke, J.: On two geometric theta lifts. Duke Math. J. **125**, 45–90 (2004)
8. Bruinier, J., Li, Y., Yang, T.: Deformations of theta integrals and a conjecture of Gross-Zagier
9. Bruinier, J., Ono, K., Rhoades, R.: Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues. Math. Ann. **342**, 673–693 (2008)
10. Fay, J.: Fourier coefficients of the resolvent kernel for a Fuchsian group. J. Reine Angew. Math. **293–294**, 143–203 (1977)
11. Gross, B., Zagier, D.: Heegner points and derivatives of L-series. Invent. Math. **84**, 225–320 (1986)
12. Guerzhoy, P.: Hecke operators for weakly holomorphic modular forms and supersingular congruences. Proc. Am. Math. Soc. **136**, 3051–3059 (2008)
13. Guerzhoy, P., Kent, Z., Ono, K.: p-adic coupling of mock modular forms and shadows. Proc. Natl. Acad. Sci. U.S.A. **107**, 6169–6174 (2010)
14. Hejhal, D.: The Selberg trace formula for $\mathrm{PSL}_2(\mathbb{R})$, Lect. Notes Math. **1001**, Springer, Berlin (1983)

15. Imamoḡlu, Ö., O'Sullivan, C.: Parabolic, hyperbolic and elliptic Poincaré series. Acta Arith. **139**, 199–228 (2009)
16. Li, Y.: Algebraicity of higher Green functions at a CM point, preprint arXiv:2106.13653 (2021)
17. NIST Digital Library of Mathematatal functions, http://dlmf.nist.gov/, Release 1.1.3 of 2021-09-15. Olver, F., Olde Daalhuis, A., Lozier, D., Schneider, B., Boisvert, R., Clark, C., Miller, B., Saunders, B., Cohl, H., McClain, M. eds
18. Ono, K.: A mock theta function for the Delta-function, Combinatorial number theory: Proceedings of the 2007 Integers Conference, de Gruyter, Berlin, pp. 141–156 (2009)
19. Ono, K.: The web of modularity: arithmetic of the coefficients of modular forms and q-series. CBMS 102, Amer. Math. Soc (2004)
20. Petersson, H.: Konstruktion der Modulformen und der zu gewissen Grenzkreisgruppen gehörigen automorphen Formen von positiver reeller Dimension und die vollständige Bestimmung ihrer Fourierkoeffizienten, S.-B. Heidelberger Akad. Wiss. Math. Nat. Kl , 415–474 (1950)
21. Petersson, H.: Über automorphe Orthogonalfunktionen und die Konstruktion der automorphen Formen von positiver reeller Dimension. Math. Ann. **127**, 33–81 (1954)
22. Zagier, D.: Elliptic modular forms and their applications, in "The 1-2-3 of modular forms". Springer, New York (2008)