

Weisfeiler-Leman on Graphs of Small Twin-Width

Irene Heinrich 

TU Darmstadt, Germany

Moritz Lichter 

RWTH Aachen University, Germany

Klara Pakhomenko 

Hasselt University, Diepenbeek, Belgium

Simon Raßmann 

TU Darmstadt, Germany

Abstract

Twin-width is a graph parameter introduced in the context of first-order model checking, and has since become a central parameter in algorithmic graph theory. While many algorithmic problems become easier on arbitrary classes of bounded twin-width, graph isomorphism on graphs of twin-width 4 and above is as hard as the general isomorphism problem. For each positive integer k , the k -dimensional Weisfeiler-Leman algorithm is an iterative color refinement algorithm that encodes structural similarities and serves as a fundamental tool for distinguishing non-isomorphic graphs. We show that the graph isomorphism problem for graphs of twin-width 1 can be solved by the 3-dimensional Weisfeiler-Leman algorithm, while there is no fixed k such that the k -dimensional Weisfeiler-Leman algorithm solves the graph isomorphism problem for graphs of twin-width 4. Moreover, we prove the conjecture of Bergougnoux, Gajarský, Guspiel, Hlinený, Pokrývka, and Sokolowski (ISAAC 2023) that stable graphs of twin-width 2 have bounded rank-width. This implies that isomorphism of these graphs is solved by a fixed dimension of the Weisfeiler-Leman algorithm.

2012 ACM Subject Classification Mathematics of computing → Graph algorithms; Theory of computation → Complexity theory and logic; Mathematics of computing → Paths and connectivity problems

Keywords and phrases twin-width, Weisfeiler-Leman algorithm, canonization, half-graph, rank-width

Digital Object Identifier 10.4230/LIPIcs.WG.2026.26

Related Version *Full Version*: <https://arxiv.org/abs/2601.05892> [28]

Funding *Irene Heinrich*: The research of this author has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (EngageS: grant agreement No. 820148)

Moritz Lichter: The research of this author has received funding by the European Union (ERC, SymSim, 101054974).

Klara Pakhomenko: This author is supported by the Flanders AI Research Program.

1 Introduction

Twin-width [11] is a graph parameter that has become a central parameter in algorithmic graph theory due to its significance in model checking: First-order model checking on graphs is fixed-parameter tractable when witnesses of small twin-width are provided. This generalizes similar results for classical graph classes like classes of bounded tree-width, rank-width, or genus, classes excluding a minor, or map graphs. Even stronger, bounded twin-width exactly describes the boundary of tractable first-order model checking on ordered graphs [8], tournaments [19], interval graphs [7], and permutation graphs [11].



© Irene Heinrich, Moritz Lichter, Klara Pakhomenko, and Simon Raßmann;
licensed under Creative Commons License CC-BY 4.0

52nd International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2026).

Editors: Jan Goedgebeur and Paweł Rzażewski; Article No. 26; pp. 26:1–26:15

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Twin-width is based on repeatedly merging pairs of vertices with similar neighborhoods, while keeping the total error at every merged set of vertices bounded (we postpone a formal definition to Section 2). For example, cographs are of twin-width 0 since every cograph can be contracted to a single vertex by repeatedly merging twins (which are vertices with identical neighborhoods). In fact, cographs are exactly the graphs of twin-width 0, see [11]. Hence, graphs of twin-width 0 can be recognized efficiently, and many algorithmic problems, including the graph isomorphism problem and even graph canonization, are polynomial-time solvable on this class [15].

The recognition results also hold for twin-width 1 graphs, where a polynomial-time recognition algorithm was given in [10], which was later improved to linear-time [2]. Moreover, graphs of twin-width 1 have bounded rank-width [10], which implies that isomorphism and canonization can be solved in polynomial time [24, 26].

In contrast to these positive results, recognition of graphs of twin-width 4 is NP-hard [5]. Since each graph can be transformed into a twin-width at most 4 graph by an $\Omega(\log(n))$ -subdivision [5] the global structure of twin-width 4 graphs is essentially unrestricted. In particular, graph isomorphism on the class of graphs of twin-width at most 4 is as hard as the isomorphism problem on all graphs. This leaves the cases of twin-width 2 and 3 open, where the structural complexity is not well-understood and where the complexity of both recognition and isomorphism is open. In contrast to hardness of isomorphism on graphs of twin-width at least 4, isomorphism of tournaments parameterized by their twin-width is fixed-parameter tractable [25], and thus polynomial-time solvable on every class of tournaments of bounded twin-width.

Through this paper we aim to provide a better understanding of the combinatorial complexity of graphs of small twin-width. One natural framework for this is provided by the Weisfeiler-Leman dimension which is a fundamental combinatorial approach to graph isomorphism that measures how much “distinguishing power” is needed to tell graphs apart. The Weisfeiler-Leman algorithm is a family of purely combinatorial polynomial-time algorithms which subsume most other combinatorial algorithms for the graph isomorphism problem. The 1-dimensional Weisfeiler-Leman algorithm, also known as color refinement or naive vertex classification, starts by coloring each vertex of the graph by its degree. The algorithm repeatedly refines the color of each vertex by the multiset of colors of its neighbors, until the partition induced by the coloring stabilizes. The multiset of colors is an isomorphism invariant: if they differ for two graphs, then they are not isomorphic. This simple heuristic identifies almost all graphs up to isomorphism [3]. Since almost all graphs have twin-width linear in their order [1], color refinement can identify graphs of unbounded twin-width.

Generalizing color refinement, the k -dimensional Weisfeiler-Leman algorithm colors k -tuples of vertices. The minimal dimension k , for which the k -dimensional Weisfeiler-Leman algorithm identifies every graph from some class $\mathcal{C}$ up to isomorphism, is the *Weisfeiler-Leman dimension* of $\mathcal{C}$. Many natural graph classes have bounded Weisfeiler-Leman dimension. This includes graph classes of bounded tree-width or bounded rank-width [24], graphs of bounded genus, and most generally classes excluding some minor [22]. However, the class of all graphs has unbounded Weisfeiler-Leman dimension, as exemplified by the *Cai-Fürer-Immerman (CFI) graphs* [14]. The k -dimensional Weisfeiler-Leman algorithm has a close correspondence to $(k+1)$ -variable first-order counting logic [14]. This connection is routinely exploited to translate bounds on the Weisfeiler-Leman dimension into definable canonizations in fixed-point logic with counting FP+C. This type of definability result implies that FP+C expresses exactly the polynomial-time decidable properties on these graphs [24]. FP+C-definable canonization is possible for all classes of bounded rank-width [24] and classes excluding a minor [22].

Our results. We investigate the Weisfeiler-Leman dimension of graphs of small twin-width. First, we show that the Weisfeiler-Leman dimension of a graph cannot be bounded in terms of its twin-width. Specifically, graphs of twin-width 4 have unbounded Weisfeiler-Leman dimension. Hence, no fixed-dimensional Weisfeiler-Leman algorithm distinguishes all pairs of non-isomorphic graphs in this class. Our result is based on subdivisions of CFI graphs.

In contrast to this, we show that the 3-dimensional Weisfeiler-Leman algorithm suffices to identify all graphs of twin-width at most 1, by showing that the algorithm can emulate the recognition algorithm of [10] in order to construct a canonical 1-contraction sequence for the graph, which can then be used to identify the graph. While it is known that graphs of twin-width 1 have bounded rank-width, and thus that some fixed-dimensional Weisfeiler-Leman algorithm correctly decides isomorphism, our argument shows that their Weisfeiler-Leman dimension is 2 or 3. Moreover, we show how our result translates to a more efficient $\text{FP}+\text{C}$ -definable canonization algorithm for graphs of twin-width 1.

Finally, we consider graphs of twin-width 2. It was shown in [6] that sparse graphs of twin-width 2 have bounded tree-width, and conjectured that stable graphs of twin-width 2 have bounded rank-width. We prove this conjecture by showing that graphs of twin-width 2 without a semi-induced half-graph H_t have rank-width $O(t)$. To achieve this, we first classify bipartite graphs of twin-width 1, and then leverage this classification to analyze contractions of red paths in contraction sequences of width 2. Using the same techniques, we also show that the tree-width of graphs of twin-width 2 without a $K_{t,t}$ -subgraph is bounded by $O(t)$, thus improving the bound given in [6]. In both of these cases, it follows that the $O(t)$ -dimensional Weisfeiler-Leman algorithm correctly decides isomorphism of these graphs.

Further related work. Twin-width has been extensively studied since its introduction [11], which among others established bounds in terms of boolean width (and thus rank-width and clique-width) and for minor-free classes. Further connections include tree-width [6, 32], and topological parameters as the genus [35]. Sharp bounds are known for planar graphs [30], and relations to other graph parameters have also been considered in restricted graph classes [12].

For small twin-widths, recognition has already been studied: While recognition is polynomial-time for twin-width at most 1 [10, 2] but NP-hard for twin-width 4 [5], the complexity of twin-width 2 and 3 remains open.

In this paper we study the relationship between the twin-width and the Weisfeiler-Leman dimension of a graph, which provides a complementary complexity measure: many algorithmically tractable classes have bounded Weisfeiler-Leman dimension, including graphs of bounded tree-width [23], rank-width [24], planar graphs [21, 34], and minor-free classes [22]. However, the relationship between twin-width and Weisfeiler-Leman dimension was previously explored only for tournaments [25], where bounded twin-width yields tractable isomorphism despite unbounded Weisfeiler-Leman dimension. Our study of this relationship for general graphs also resolves the conjecture from [6] on rank-width bounds for stable graphs of twin-width 2. Stable graphs of bounded twin-width have been studied independently in [17].

For full proofs of all statements, we refer to the full version of this paper [28].

2 Preliminaries

For a natural number k , set $[k] := \{1, 2, \dots, k\}$. The rank of a matrix M is denoted $\text{rk}(M)$.

Graphs. All graphs in this paper are finite and simple. A graph is coconnected if its complement is connected. For a graph G , we write $V(G)$ for its vertex set and $E(G)$ for its edge set. The *order* of G is $|V(G)|$. For a vertex set $S \subseteq V(G)$, we write $G[S]$ for the

subgraph induced by S in G . For two disjoint sets $L, R \subseteq V(G)$, we write $G[L, R]$ for the induced bipartite graph with parts L and R obtained from $G[L \cup R]$ by removing all internal edges of L and R , respectively. A bipartite graph $H[A, B]$ is a *semi-induced subgraph* of G if there is an embedding $\iota: V(H) \rightarrow V(G)$ such that the induced map $H[A, B] \rightarrow G[\iota(A), \iota(B)]$ is an isomorphism. If H is not a semi-induced subgraph of G , then G is *H -semi-free*.

For a graph G and a tuple $\mathbf{v} \in V(G)^k$, the *atomic type* $\text{atp}_G(\mathbf{v})$ is the isomorphism type of the ordered graph $G[\mathbf{v}]$, i.e., two tuples $\mathbf{v} \in V(G)^k$ and $\mathbf{w} \in V(H)^k$ are of the same atomic type exactly if $v_i \mapsto w_i$ defines an isomorphism between $G[\mathbf{v}]$ and $H[\mathbf{w}]$. A colored graph G^c is a tuple (G, c) of a graph G and a map c defined on $V(G)$, usually into the natural numbers.

Twin-width. A *trigraph* G is a graph with edges colored either red or black. Graphs are interpreted as trigraphs by coloring each edge black. For $v \in V(G)$ the *red degree* $\text{red-deg}_G(v)$ is the number of red edges incident to v . A *red component* of a trigraph G is a component of the graph obtained from G by removing the set of all black edges.

Two disjoint vertex subsets U and W of G are *fully connected* if every 2-set $\{u, w\}$ with $u \in U$ and $w \in W$ is a black edge. If no such pair is an edge of G , then U and W are *disconnected*. The sets U and W are *homogeneously connected* if they are either fully connected or disconnected. Given a partition $\mathcal{P}$ of $V(G)$, the *quotient graph* $G/\mathcal{P}$ is the trigraph with vertex set $V(G/\mathcal{P}) = \mathcal{P}$ such that two parts U and W of $\mathcal{P}$ are

joined via a black edge	if U and W are fully connected,
not adjacent	if U and W are disconnected, and
joined via a red edge	otherwise.

A *partition sequence* of an order- n trigraph G is a sequence $\mathcal{P}_n, \mathcal{P}_{n-1}, \dots, \mathcal{P}_1$ of partitions of $V(G)$ where $\mathcal{P}_n$ is the discrete partition and for each $i \in [n-1]$ the partition $\mathcal{P}_i$ is obtained by replacing two distinct parts P and Q of $\mathcal{P}_{i+1}$ by $P \cup Q$ (we call this a *merge* or a *contraction*). Equivalently, a partition sequence is given by the sequence of trigraphs $G/\mathcal{P}_i$ or the sequence of pairs of vertices to contract in these trigraphs, both are called *contraction sequences*. The *width* of a contraction sequence (or its associated partition sequence) is the maximum red degree over all trigraphs $G/\mathcal{P}_i$. If a partition or contraction sequence has width at most k , we call it a *k -partition sequence* or *k -contraction sequence*, respectively. The *twin-width* $\text{tw}_G(G)$ of G is the minimal width of a contraction sequence of G . The twin-width of a colored graph is the twin-width of the underlying uncolored graph. The *component twin-width* of G is the minimum over all contraction sequences of the maximal order of red components in the appearing trigraphs. Component twin-width is functionally equivalent to rank-width [9] and differs by at most a constant factor from clique-width [4].

Ranks, rank-width, and well-linked sets. Let G be a graph. Given two vertex sets $A, B \subseteq V(G)$, the *biadjacency matrix* of A and B , denoted by $\text{Adj}_G(A, B)$, is the $|A| \times |B|$ -matrix over $\mathbb{F}_2$ with rows indexed by vertices in A and columns indexed by vertices in B such that the (a, b) -th entry is 1 if and only if $ab \in E(G)$. We set $\text{rk}_G(A, B) := \text{rk}(\text{Adj}_G(A, B))$. The rank-function is subadditive: If $A, B, C \subseteq V(G)$ are pairwise disjoint, then $\text{rk}_G(A, B \cup C) \leq \text{rk}_G(A, B) + \text{rk}_G(A, C)$. For disjoint $A, B \subseteq V(G)$, the *rank-connectivity* of A and B in G is

$$\kappa_G^{\text{rk}}(A, B) := \min_{A \subseteq X \subseteq V(G) \setminus B} \text{rk}_G(X, V(G) \setminus X).$$

► **Lemma 1** (Monotonicity and subadditivity of the rank-connectivity function). *If G is a graph and A , B , and C are pairwise disjoint vertex subsets of G , then*

1. $\kappa_G^{\text{rk}}(A, B) \leq \kappa_G^{\text{rk}}(A, B \cup C)$, and
2. $\kappa_G^{\text{rk}}(A, B \cup C) \leq \kappa_{G-C}^{\text{rk}}(A, B) + \kappa_{G-B}^{\text{rk}}(A, C)$.

Proof. The first inequality is clear. For the second inequality we pick two cuts $V(G) = A'_B \dot{\cup} B' = A'_C \dot{\cup} C'$ with $A \subseteq A'_B \subseteq V(G) \setminus C$, $A \subseteq A'_C \subseteq V(G) \setminus B$, $B \subseteq B' \subseteq V(G) \setminus C$ and $C \subseteq C' \subseteq V(G) \setminus B$ such that $\text{rk}(A'_B, B') = \kappa_{G-C}^{\text{rk}}(A, B)$ and $\text{rk}(A'_C, C') = \kappa_{G-B}^{\text{rk}}(A, C)$. We obtain

$$\begin{aligned} \kappa_G^{\text{rk}}(A, B \cup C) &\leq \text{rk}(A'_B \cap A'_C, B' \cup C') \\ &\leq \text{rk}(A'_B, B') + \text{rk}(A'_C, C') \\ &= \kappa_{G-C}^{\text{rk}}(A, B) + \kappa_{G-B}^{\text{rk}}(A, C). \end{aligned} \quad \blacktriangleleft$$

A *rank-decomposition* of G is a rooted binary tree T together with a bijection between the leaves of T and the vertices of G . This way, every tree edge $e \in E(T)$ induces a cut of $V(G)$ into two parts A and B by partitioning the vertices according to the component of $T - e$ their leafs lie in. The *width* of the rank decomposition is the maximal rank $\text{rk}(A, B)$ across all these cuts, and the *rank-width* of G is the minimal width of a rank decomposition.

A *rk-well-linked* set in a graph is a subset $U \subseteq V(G)$ such that for any two disjoint sets $A, B \subseteq U$, we have $\kappa_G^{\text{rk}}(A, B) \geq \min\{|A|, |B|\}$.

► **Theorem 2** ([37, Theorem 5.2]). *Every graph of rank-width greater than k contains a rk-well-linked set of size k .*

There exist similar notions of well-linkedness for other connectivity parameters. For example, a vertex subset $U \subseteq V(G)$ is *mm-well-linked* if each cut $U = X \dot{\cup} (U \setminus X)$ contains a matching of size at least $\min\{|X|, |U \setminus X|\}$. This is equivalent to requiring that every two subsets $A, B \subseteq U$ are linked by $\min\{|A|, |B|\}$ vertex-disjoint paths.

► **Theorem 3** ([38, Lemma 3.4]). *Every graph of tree-width greater than k contains a mm-well-linked set of size k .*

The Weisfeiler-Leman algorithm. For a graph G , the k -dimensional Weisfeiler-Leman algorithm (short: k -WL) iteratively refines an isomorphism-invariant coloring of $V(G)^k$. The initial coloring is $\chi_{G,k}^0 : \mathbf{v} \mapsto \text{atp}_G(\mathbf{v})$. For $i \geq 0$ we set

$$\chi_{G,k}^{i+1}(\mathbf{v}) := \left(\chi_{G,k}^i(\mathbf{v}), \left\{ \left(\text{atp}_G(\mathbf{v}w), \chi_{G,k}^i(\mathbf{v}[w/1]), \dots, \chi_{G,k}^i(\mathbf{v}[w/k]) \right) \mid w \in V(G) \right\} \right),$$

where the notation $\mathbf{v}[w/i]$ denotes the tuple obtained from $\mathbf{v}$ by replacing its i -th entry with w . For some $i \in \mathbb{N}$, the colorings $\chi_{G,k}^i$ and $\chi_{G,k}^{i+1}$ are equivalent, that is, both colorings induce the same partition of $V(G)^k$ into color classes. If i is the minimal index where this happens, the *stable coloring* $\chi_{G,k}^{i+1}$ is denoted by $\chi_{G,k}$ and can be computed in $\mathcal{O}(k^2 n^{k+1} \log n)$ [31]. Two graphs G and H are *distinguished by k -WL* if there is a color c such that

$$|\{\mathbf{v} \in V(G)^k \mid \chi_{G,k}(\mathbf{v}) = c\}| \neq |\{\mathbf{w} \in V(H)^k \mid \chi_{H,k}(\mathbf{w}) = c\}|.$$

A graph G is *identified by k -WL* if the algorithm distinguishes G from every non-isomorphic graph H . The *WL dimension* of G is the smallest k such that k -WL identifies the graph. The WL dimension of a graph class $\mathcal{C}$ is the smallest k such that k -WL identifies all graphs in $\mathcal{C}$. For a more detailed discussion of the algorithm, see for example [14, 33].

The logic C_k is the k -variable fragment of first-order logic with the counting quantifier $\exists^{\geq i}x.\varphi$. For two graphs G and H that satisfy exactly the same formulas of C_k , we write $G \equiv_{C_k} H$. For a more detailed definition and relevant examples, see for example [14].

► **Theorem 4** (Theorem 5.2 of [14]). *Let $k \in \mathbb{N}$. Two graphs G and H are not distinguished by k -WL precisely if $G \equiv_{C_{k+1}} H$.*

The *bijective k -pebble game* is a round-based combinatorial game for two players, *Spoiler* and *Duplicator*. A *position* is a pair of tuples $(\mathbf{v}, \mathbf{w}) \in V(G)^\ell \times V(H)^\ell$ with $\ell \in [k] \cup \{0\}$. The initial position is $((), ())$. Let $((v_1, \dots, v_\ell), (w_1, \dots, w_\ell))$ be the position at the start of a round. If $\ell > 0$, then Spoiler can remove one pebble pair by picking some $i \in [\ell]$ and move to $((v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_\ell), (w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_\ell)) =: (\mathbf{v}', \mathbf{w}')$. If Spoiler does not remove a pebble pair, then $(\mathbf{v}', \mathbf{w}') := (\mathbf{v}, \mathbf{w})$. If Spoiler removed a pebble or $\ell < k$, then Duplicator picks a bijection $b : V(G) \rightarrow V(H)$ and Spoiler picks some $v \in V(G)$ and moves to $(\mathbf{v}'v, \mathbf{w}'b(v))$. Spoiler wins if at any point $\text{atp}_G(\mathbf{v}) \neq \text{atp}_H(\mathbf{w})$. Duplicator wins if Spoiler never wins. We say p is a *winning position* for Spoiler/Duplicator if Spoiler/Duplicator has a winning strategy for the game starting at p . We say Spoiler/Duplicator *wins* if $((), ())$ is a winning position for Spoiler/Duplicator.

► **Lemma 5** ([29]). *For all integers $k \geq 1$, two graphs G and H are distinguished by k -WL if and only if Spoiler wins the bijective $(k+1)$ -pebble game on G and H .*

Half-graphs and stability. For an integer k , the *half-graph* H_k is the bipartite graph with vertex set $V(H_k) = \{v_i : i \in [k]\} \cup \{w_i : i \in [k]\}$ such that $v_i w_j \in E(H_k)$ if and only if $i \leq j$.

A class of graphs is monadically stable if it does not transduce the class of all half-graphs. In particular, this implies that the class is H_k -semi-free for some $k \in \mathbb{N}$. Since we will only use this latter property of monadically stable classes, we will not formally define FO-transductions and instead refer to [17].

3 The Weisfeiler-Leman dimension of the class of twin-width 4 graphs is unbounded

► **Theorem 6.** *For all $k \in \mathbb{N}$, there are two non-isomorphic graphs of twin-width 4 and order $\mathcal{O}(k \log k)$ that are not distinguished by k -WL.*

Proof sketch. For each $k \in \mathbb{N}$, there is a pair of non-isomorphic cubic graphs of order $n = \mathcal{O}(k)$, known as *Cai-Fürer-Immerman* (CFI) graphs [14], which are not distinguished by $2k$ -WL. We consider the $2 \log n$ -subdivisions of these graphs, in which each edge is replaced by a path of length $2 \log n$. These subdivisions have twin-width at most 4 [5]. We show that subdividing the edges of a graph by any length increases its Weisfeiler-Leman dimension by at most a factor of 2 (full proof in the appendix). Therefore, subdivided CFI graphs provide the desired pair of twin-width 4 graphs not distinguished by k -WL. ◀

A full proof of this theorem and all other theorems in the paper can be found in [28].

4 Graphs of twin-width 1 have Weisfeiler-Leman dimension at most 3

Since the Weisfeiler-Leman dimension of twin-width 4 graphs is unbounded, we investigate whether it is bounded for smaller twin-width. The class of twin-width 0 graphs, that is, the class of cographs, has WL dimension 2, see [16]. The following is known about twin-width 1.

► **Lemma 7** ([10]). *Graphs of twin-width 1 have a 1-contraction sequence where there is a single red edge at every step.*

► **Lemma 8** ([24]). *For every $k \in \mathbb{N}$, the $(3k+4)$ -dimensional Weisfeiler-Leman algorithm identifies every graph of rank-width at most k .*

► **Corollary 9.** *Graphs of twin-width 1 have WL dimension at most 10.*

Proof. By Lemma 7, graphs of twin-width 1 have component twin-width 2 and, by Theorem 10 of [9], also bounded boolean width and bounded rank-width [13]. Inspecting the proof of Theorem 10 of [9] shows that graphs of component twin-width 2 have rank-width at most 2: The contraction tree of the red components along the contraction sequence of component twin-width 2 serves as rank decomposition. The claim then follows from Lemma 8. ◀

In the following, we investigate the construction in [10] and show a better bound explicitly: graphs of twin-width 1 have WL dimension at most 3. First, we show this for the case of prime graphs.

4.1 Prime graphs of twin-width 1

A vertex v of a graph G is *homogeneously connected* to a set of vertices $S \subseteq V(G)$ if v is adjacent to all or none of S in G . A set $M \subseteq G$ is a *module* if all vertices $v \in V(G) \setminus M$ are homogeneously connected to M . A module of a graph G is *trivial* if it consists of a single vertex or all vertices of G . A graph G is *prime* if it has only trivial modules. The proof of Lemma 7 shows that a prime graph of twin-width 1 has a 1-contraction sequence of the following form:

1. Guess the first contraction of a valid 1-contraction sequence. This creates one red edge.
2. Make every possible “safe” contraction, that is, contractions into the ends of the current red edge that create no additional red edges.
3. Contract the ends of the red edge. If there is only one vertex left, terminate. Otherwise, this will create exactly one more red edge. Repeat Step 2.

This contraction sequence is *canonical* in the sense that given the starting contraction, there is a description of it that is a complete invariant for colored prime graphs of twin-width 1: if the invariant is the same on two graphs, then the graphs are isomorphic (an isomorphism has to map vertices to vertices of the same color).

► **Theorem 10.** *There is a polynomial-time computable invariant I for colored prime graphs of twin-width 1: two colored twin-width 1 graphs G, H are isomorphic if and only if $I(G) = I(H)$.*

Proof sketch. We first define a function cs that, given the first vertices u and v of a graph G to contract, canonically describes the contraction sequence, that is, the number and atomic types of safe contractions in Step 2. It outputs $\mathbf{f}$ if guessing u and v did not produce a valid 1-contraction sequence. Hence, for all colored prime graphs $G^c = (G, c_G)$ and $H^c = (H, c_H)$ of twin-width 1, G^c and H^c are isomorphic if and only if for some $u, v \in V(G)$ and $u', v' \in V(H)$ it holds that $\text{cs}(G^c, u, v) = \text{cs}(H^c, u', v') \neq \mathbf{f}$. We turn this into an invariant I : For a colored prime graph G of twin-width 1, define

$$\text{CS}(G) := \{\text{cs}(G, u, v) \mid u, v \in V(G), \text{cs}(G, u, v) \neq \mathbf{f}\}.$$

Then $I(G) := \min_{\leq_{\text{lex}}} \text{CS}(G)$ is a complete invariant and computable in time $\mathcal{O}(|V|^5)$, since the canonical contraction sequence is computable in time $\mathcal{O}(|V|^3)$. ◀

By “unwinding” the canonical contraction sequence $I(G)$, we can *canonize* the graphs, that is, compute a canonical graph $C(G)$ isomorphic to G such that $C(G) = C(H)$ if and only if G is isomorphic to H (for all prime graphs G, H of twin-width 1).

Finally, we can leverage the canonical contraction sequence to show that the 3-dimensional Weisfeiler-Leman algorithm identifies colored prime graphs of twin-width 1.

► **Theorem 11.** *Let G and H be two colored prime graphs of twin-width 1 such that $G \not\cong H$. Then 3-WL distinguishes G and H .*

Proof sketch. We show via Lemma 5 that Spoiler wins with bijective 4-pebble game. Spoiler fixes the guessed contraction pair with two pebbles. Then Spoiler tracks the resulting sequence with two more pebbles and eventually will spot the difference. ◀

4.2 From prime graphs to all graphs

To lift the result to all graphs of twin-width 1, we use *modular decompositions*. A set of modules is *strong* if the modules in it are pairwise disjoint. For a connected and coconnected graph G , let M_G be the inclusion-wise maximal strong set of modules that are proper subsets of $V(G)$. The elements of M_G will also be referred to as *maximal modules* of G .

The set M_G is a uniquely determined partition of $V(G)$ [27]. We will call M_G the modular decomposition of G . The graph G can be canonically decomposed into maximal modules, always yielding prime quotient graphs.

► **Lemma 12** ([18]). *For every graph G , one of the following holds: G is disconnected, $\overline{G}$ is disconnected, or G/M_G is prime.*

► **Definition 13** (Modular decomposition tree [36]). *For a graph G , the modular decomposition tree $\text{ModTree}_G = (\mathcal{M}, E, \ell)$ consists of a set of modules $\mathcal{M} \subseteq 2^V$ of G , a relation $E \subseteq \mathcal{M}^2$ such that $(\mathcal{M}, E)$ is a directed tree, and a label function $\ell : \mathcal{M} \rightarrow \{\text{prime}, \text{series}, \text{parallel}, \text{single}\}$ labeling each module. It is defined recursively as follows:*

- *For a one-vertex graph G , label the single module $V(G)$ **single**.*
- *If G is connected and coconnected, G/M_G is prime. Label the root $V(G)$ **prime**, and for each $M \in M_G$, attach $\text{ModTree}_{G[M]}$ to it as a subtree.*
- *If G is disconnected, then label the root $V(G)$ **parallel**, and for each connected component C of G , attach $\text{ModTree}_{G[C]}$ to it as a subtree.*
- *If $\overline{G}$ is disconnected, then label the vertex $V(G)$ **series**, and for each connected component C of $\overline{G}$, attach $\text{ModTree}_{G[C]}$ to it as a subtree.*

Given a canonization of the prime graphs of a class, this decomposition lifts it to all graphs in the class: Since M_G is a set of modules of G , G/M_G has no red edges. By lexicographically ordering the subgraphs in each step, we can recursively define a canonization for the whole graph by using the canonization of the relevant subgraphs. To this end we use the canonization of *colored* prime graphs, coloring each vertex in the quotient prime graph with the isomorphism type of the module it represents.

► **Lemma 14.** *There is a polynomial-time canonization for colored graphs of twin-width 1. In particular, graph isomorphism for twin-width 1 graphs is polynomial-time decidable.*

Next, we turn to distinguishing graphs of twin-width 1 with Weisfeiler-Leman. First, we show that 2-WL distinguishes maximal modules in a graph (via the bijective 3-pebble game).

► **Lemma 15.** *Let G and H be connected and coconnected graphs and let $u, v \in V(G)$ and $u', v' \in V(H)$. If u and v are in the same maximal module of G and u' and v' are in different maximal modules of H , then $uv, u'v'$ is a winning position for Spoiler in the bijective 3-pebble game.*

Proof sketch. Spoiler follows inductively the modular decomposition tree bottom-up starting from the leaves. ◀

Lemma 15 allows us to establish the key result of this section, that derives the Weisfeiler-Leman dimension of a graph from its modules.

► **Definition 16.** *Let $<^*$ be a fixed order on the isomorphism types of colored graphs. Given a colored graph $G_c = (G, c)$, let $I := \{\text{itp}(G_c[M]) \mid M \in M_G\}$ be the set of isomorphism types of the colored maximal modules in G . Let $i_1 <^* \dots <^* i_k$ be an enumeration of I . We define the graph $G_c^* := (G/M_G, c^*)$ where $c^*(M) = k$ such that $\text{itp}(G_c[M]) = i_k$.*

► **Lemma 17.** *Let $G_c = (G, c)$ be a colored graph such that for its uncolored version G , the quotient graph G/M_G is prime, and let $k := |M_G|$ and $M_G = \{M_1, \dots, M_k\}$. Then*

$$\dim_{\text{WL}}(G_c) \leq \max \{\dim_{\text{WL}}(G_c^*), \dim_{\text{WL}}(G_c[M_1]), \dots, \dim_{\text{WL}}(G_c[M_k]), 2\}.$$

Proof sketch. According to Lemma 15, the Duplicator must map modules to modules, or lose. Now, if the Duplicator chooses a bijection such that modules of G_c are not mapped to corresponding isomorphic modules of H_c , Spoiler can place a pebble on a module in G_c that is not mapped to an isomorphic module in H_c , and play the winning strategy within that module. If this is not the case, we exploit the fact that $G_c^* \not\cong H_c^*$ because $G_c \not\cong H_c$. This translates to a winning strategy for Spoiler on G_c and H_c using the same number of pebbles: Pebbles in a module M of G_c or H_c correspond to a pebble placed on M in G_c^* or H_c^* . ◀

We are now ready to show the central result of this section.

► **Lemma 18.** *3-WL identifies colored graphs of twin-width 1.*

Proof. Let G be a colored graph of twin-width 1. We show by induction over $\text{ModTree}_G = (\mathcal{M}, E, \ell)$ that 3-WL identifies G . Since 3-WL identifies 1-vertex graphs it identifies all leaf modules labeled **single**. Now let $M \in \mathcal{M}$ be a module such that 3-WL identifies $G[M']$ for all children M' of M in ModTree_G . If M is labeled **parallel**, then $G[M]$ is the disjoint union of its children and, hence, can be identified by 3-WL. If M is labeled **series**, then $G[M]$ can be identified by 3-WL since 3-WL identifies a graph precisely if it identifies its complement. If M is labeled **prime**, then $G[M]^*$ is a prime colored graph of twin-width 1, and thus identified by 3-WL by Theorem 11. 3-WL also identifies all induced subgraphs of modules in $G[M]/M_{G[M]}$. Therefore, by Lemma 17, 3-WL also identifies $G[M]$. By induction, 3-WL identifies $G[M]$ for all $M \in \mathcal{M}$. Since the root module is $V(G)$, 3-WL identifies G . ◀

► **Theorem 19.** *The class of colored graphs of twin-width 1 has WL dimension 2 or 3.*

Finally, for all k , every polynomial-time decidable property of the class of all graphs of rank-width at most k is expressible in $\text{FP}+\text{C}$ [24] because it admits an $\text{FP}+\text{C}$ -definable canonization. This also applies to the class of twin-width 1 graphs because they have rank-width 2. However, the construction in [24] uses 2^{2k} -ary relations for rank-width k . Following the algorithm described in this section, the same can be achieved with 5-ary relations for twin-width 1 in $\text{FP}+\text{C}$ and hence can be evaluated in time $\mathcal{O}(n^5)$, see [28].

5 Stable classes of twin-width 2

As exploited in the previous section, graphs of twin-width 1 admit a contraction sequence which only ever contains a single red edge. In particular, they have bounded component twin-width and thus also bounded rank- and clique-width. This is not true for graphs of twin-width 2, since unit interval graphs have twin-width at most 2 [11] but unbounded rank-width [20]. Still, sparse graphs of twin-width 2 have bounded tree-width [6], with a polynomial bound on the tree-width in terms of the size of a forbidden semi-induced biclique. We strengthen and extend this result by showing that graph classes of twin-width 2 that are *monadically stable*, that is, forbid some half-graph H_t as a semi-induced subgraph, have their rank-width bounded by a linear function in the size of the forbidden semi-induced half-graph. This in turn also gives a linear upper bound on their Weisfeiler-Leman dimension.

5.1 Cuts induced by red edges in graphs of twin-width 1

In contraction sequences of twin-width 2, all red components are either red paths or cycles. When performing contractions involving inner vertices of such a red path or cycle, these contractions essentially behave like width-1 contractions, in the sense that they do not create additional red edges. This motivates our approach: We first analyze the structure that emerges from 1-contraction sequences, and then apply this understanding to the locally width-1 behaviour within general 2-contraction sequences. During this, it will often be beneficial to talk about partition sequences instead of contraction sequences.

We thus start our investigation by returning to graphs of twin-width 1 and consider the following question: Given a graph G with a 1-partition sequence $(\mathcal{P}_i)_{i \in [n]}$ and a red edge $PQ \in R(G/\mathcal{P}_i)$, what can we say about the graph $G[P, Q]$? Equivalently, which bipartite graphs $G[A, B]$ admit a 1-partition sequence whose final two parts are the two sides A and B ?

One example of such bipartite graphs are half-graphs:

► **Lemma 20.** *For every $t \in \mathbb{N}$, the half-graph H_t admits a 1-partition sequence whose final two parts are the two sides of the half-graph.*

Since twin-width is monotone with respect to induced subgraphs, it follows that every induced subgraph of a half-graph admits such a 1-partition sequence. We call such graphs *partial half-graphs*. We show that this is already the complete picture, that is, every bipartite graph induced by a red edge in a 1-partition sequence is a partial half-graph.

► **Lemma 21.** *For every graph G , every 1-partition sequence $(\mathcal{P}_i)_{i \in [n]}$ of G , and every two distinct parts $P, Q \in \mathcal{P}_i$, the graph $G[P, Q]$ is a partial half-graph.*

Proof sketch. We proceed by backward induction on the partition sequence. If $i = n$, all parts are singletons and the claim is clear. For the induction step, observe that when uncontracting two parts P and P' incident to a red edge $(P \cup P')Q$, at most one of the new parts can keep the red edge, while the other becomes homogeneously connected to Q . The claim then follows from the observation that the class of partial half-graphs is closed under adding isolated or universal vertices to either side of the bipartition. ◀

With this characterization of bipartite graphs induced by red edges in graphs of twin-width 1, we analyze the subgraphs forced by high connectivity between two parts.

► **Lemma 22.** *For every $t \in \mathbb{N}$, a partial half-graph $H[L, R]$ contains an induced half-graph H_t if and only if $\text{rk}_H(L, R) \geq t$.*

Proof sketch. Since deleting twins and isolated vertices neither changes the rank nor affects the existence of induced half-graphs, we may assume that H contains neither twins nor isolated vertices. In this case, ordering the vertices of both sides by inclusion of their neighborhoods reveals that H is a half-graph, and has the required rank. ◀

► **Lemma 23.** *For every $t \in \mathbb{N}$, if a partial half-graph $H[L, R]$ contains a matching of size $2t - 1$, then H contains a biclique $K_{t,t}$.*

Proof sketch. Either the smallest t elements of L and the largest t elements of R form a biclique, or the smallest $t - 1$ elements of L and the largest $t - 1$ elements of R form a vertex cover, which forbids a matching of size larger than $2t - 2$. ◀

5.2 Stable graphs of twin-width 2 have bounded rank-width

With these preparations, we show that H_k -semi-free graphs of twin-width 2 have rank-width $O(k)$. Our approach is similar to the one for sparse twin-width 2 [6]: Assuming G has large rank-width, then it contains a large rk-well linked set by Theorem 2. Investigating the 2-partition sequence on this well-linked set, we extract a single red edge inside a longer red path with good connectivity properties. Going back the partition sequence, we note that as long as the red path persists, the (un-)contraction sequence applied to the red edge of this path is actually a 1-partition sequence, since the extra red edges of the path forbid additional red edges between the two parts in question. Thus, we find a large bipartite graph of twin-width 1 with good connectivity properties, implying the existence of a large half-graph.

We start by showing that graphs of twin-width 2 that contain a large well-linked set must also contain a highly connected red path at some point along the partition sequence.

► **Lemma 24.** *Let $t \geq 0$ be an integer and $k := 2t + 9$. If G is a graph which contains a rk-well-linked set W of size $11k - 5$ and $(\mathcal{P}_i)_{i \in [n]}$ is a 2-partition sequence of G , then there exists some $i \in [n]$ and four parts $X_1, X_2, X_3, X_4 \in \mathcal{P}_i$ such that*

- $X_1 X_2 X_3 X_4$ forms a red 4-path in this order in $G/\mathcal{P}_i$,
- there is no red edge $X_1 X_4$,
- $\kappa_{G[X_1 \cup X_2 \cup X_3 \cup X_4]}^{\text{rk}}(X_1, X_4) \geq t$.

Proof sketch. There exists some part $Z \in \mathcal{P}_i$ for some $i \in [n]$ such that $k \leq |Z \cap W| < 2k$. Now, if the red component of Z in $G/\mathcal{P}_i$ contained at most 5 parts, then the cut between this component and its complement would have rank at most 5, which contradicts well-linkedness of W . Thus, we find at least one red 4-path starting at Z in $G/\mathcal{P}_i$. If no such red 4-path satisfies the rank connectivity condition in the claim, we again find a low-rank cut by splitting W according to the low-rank cuts in the red paths starting at Z . ◀

Next, we want to show that the red 4-path whose existence we proved in the previous lemma forces the existence of a large semi-induced half-graph.

► **Lemma 25.** *Let G be a graph with a 2-partition sequence $(\mathcal{P}_i)_{i \leq n}$ of G , and assume for some $m \in [n]$, there exists four parts $X_1, X_2, X_3, X_4 \in \mathcal{P}_m$ such that X_1, X_2, X_3, X_4 form a red path in this order in $G/\mathcal{P}_m$, there is no red edge $X_1 X_4$, and $\kappa_{G[X_1 \cup X_2 \cup X_3 \cup X_4]}^{\text{rk}}(X_1, X_4) \geq t$. Then G contains a semi-induced half-graph H_t .*

Proof sketch. Consider the last partition $\mathcal{P}_i$ before $\mathcal{P}_m$ in the 2-partition sequence in which there does not exist a red 4-path in $G[X_1 \cup X_2 \cup X_3 \cup X_4]$ whose central red edge is an X_2 - X_3 edge. Then, such a red 4-path whose central edge crosses the cut (X_2, X_3) exists in all trigraphs $G/\mathcal{P}_j$ for $i > j \leq m$ and furthermore, the central edge of this path is the unique red

edge crossing this cut in those trigraphs. Indeed, the red edge X_2X_3 is certainly the unique red edge connecting X_2 and X_3 in $G/\mathcal{P}_m$, and any further red edge cannot be contracted into the central edge of the red 4-path without creating red degree 3 in between. Thus, the induced partition sequence on the bipartite graph $G[X_2, X_3]$ is a 1-partition sequence between the partitions $\mathcal{P}_{i-1}$ and $\mathcal{P}_m$. Depending on where the existing red 4-path crossing the cut (X_2, X_3) broke when uncontracting from $\mathcal{P}_{i-1}$ to $\mathcal{P}_i$, we find a low-rank cut within $X_2 \cup X_3$ such that when removing one half of that cut, we no longer have any red edges crossing (X_2, X_3) in partitions $\mathcal{P}_j$ for $j \geq i$. This implies that on this smaller graph, we can extend the partial 1-partition sequence of $G[X_2, X_3]$ to a complete 1-partition sequence. But since the path $X_1X_2X_3X_4$ has large rank-connectivity, and we only removed vertices of a low-rank cut, we have $\text{rk}_G(X_2, X_3) \geq t$ (since otherwise the cut $(X_1 \cup X_2, X_3 \cup X_4)$ would have low rank). By Lemma 22, this implies that we find a half-graph H_t in $G[X_2, X_3]$. ◀

Combining the previous two lemmas yields our main theorem:

► **Theorem 26.** *Every H_t -semi-free graph G with twin-width at most 2 has rank-width at most $22t + 170$. In particular, every monadically stable graph class of twin-width at most 2 has bounded rank-width.*

This theorem generalizes the related result of [6] that every sparse graph class of twin-width at most 2 has bounded tree-width, because every sparse class of bounded rank-width also has bounded tree-width. But while this approach gives worse bounds on the tree-width than given in [6], we can actually improve their bounds by replacing their use of the excluded grid theorem for tree-width by the use of mm-well-linked sets as in our proof and using Lemma 23 instead of Lemma 22 to extract a large biclique from a highly connected partial half-graph. This then yields a linear upper bound on the tree-width in terms of a forbidden semi-induced biclique:

► **Theorem 27.** *Every graph of twin-width at most 2 which does not contain a $K_{t,t}$ -subgraph has tree-width at most $O(t)$.*

The authors of [6] also showed that their result does not generalize to graphs of twin-width 3: There exists a family of sparse graphs of twin-width 3 which has unbounded tree-width. Since sparse classes of bounded rank-width also have bounded tree-width, this same family also has unbounded rank-width. Thus, our theorem also does not generalize to twin-width 3. Finally, by Lemma 8, we get a bound on the Weisfeiler-Leman dimension.

► **Corollary 28.** *H_t -semi-free graphs of twin-width at most 2 are identified by the $O(t)$ -dimensional Weisfeiler-Leman algorithm.*

6 Conclusion

We considered the power of the Weisfeiler-Leman algorithm on graphs of small twin-width, showing that graphs of twin-width 1 have Weisfeiler-Leman dimension at most 3, while graphs of twin-width 4 have unbounded Weisfeiler-Leman dimension. Moreover, we showed that stable graphs of twin-width 2 have bounded rank-width, resolving a conjecture of [6], and also implying a bound on the Weisfeiler-Leman dimension. It remains open if the Weisfeiler-Leman dimension of graphs of twin-width 1 is 2 or 3. Likewise, the precise Weisfeiler-Leman dimension of stable graphs of twin-width 2 remains open. Finally, it is not clear whether general graphs of twin-width 2 and 3 have a bounded Weisfeiler-Leman dimension at all.

Every stable class of bounded twin-width is a transduction of a sparse class of bounded twin-width [17]. Since transductions of classes of bounded tree-width have bounded rank-width, a natural question is whether every stable class of twin-width 2 is a transduction of some sparse class of twin-width 2. If true, this would provide an alternative proof of our result that stable graphs of twin-width 2 have bounded rank-width, albeit with worse bounds.

Our techniques might extend to other width parameters where high connectivity in partial half-graphs implies the existence of certain substructures. An interesting candidate might be mim-width [39], which is similar to rank-width, with the rank function replaced by the size of a maximum semi-induced matching over a cut. Since partial half-graphs contain no non-trivial semi-induced matchings, one might expect that graphs of twin-width 2 have bounded mim-width. However, since the connectivity function underlying mim-width is not submodular, high mim-width does not imply the existence of large mim-well-linked sets, which were a main ingredient in our proof. Still, we conjecture that graphs of twin-width 2 have bounded mim-width. Since permutation graphs have mim-width 1 and include all graphs of twin-width 1 [2], this is true for graphs of twin-width 1. In general, twin-width and mim-width are incomparable, as exemplified by wall graphs and interval graphs [12].

References

- 1 Jungho Ahn, Kevin Hendrey, Donggyu Kim, and Sang-il Oum. Bounds for the Twin-Width of Graphs. *SIAM Journal on Discrete Mathematics*, 36(3):2352–2366, 2022. doi:10.1137/21M1452834.
- 2 Jungho Ahn, Hugo Jacob, Noleen Köhler, Christophe Paul, Amadeus Reinald, and Sebastian Wiederrecht. Twin-width one. In Olaf Beyersdorff, Michal Pilipczuk, Elaine Pimentel, and Kim Thang Nguyen, editors, *42nd International Symposium on Theoretical Aspects of Computer Science, STACS 2025, March 4-7, 2025, Jena, Germany*, volume 327 of *LIPIcs*, pages 6:1–6:19. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025. doi:10.4230/LIPIcs.STACS.2025.6.
- 3 László Babai, Paul Erdős, and Stanley M. Selkow. Random graph isomorphism. *SIAM J. Comput.*, 9(3):628–635, 1980. doi:10.1137/0209047.
- 4 Ambroise Baril, Miguel Couceiro, and Victor Lagerkvist. Improved bounds for twin-width parameter variants with algorithmic applications to counting graph colorings, 2025. doi:10.48550/arXiv.2509.05122.
- 5 Pierre Bergé, Édouard Bonnet, and Hugues Déprés. Deciding Twin-Width at Most 4 Is NP-Complete. In Mikołaj Bojańczyk, Emanuela Merelli, and David P. Woodruff, editors, *49th International Colloquium on Automata, Languages, and Programming (ICALP 2022)*, volume 229 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 18:1–18:20, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ICALP.2022.18.
- 6 Benjamin Bergougnoux, Jakub Gajarský, Grzegorz Guspiel, Petr Hliněný, Filip Pokrývka, and Marek Sokolowski. Sparse graphs of twin-width 2 have bounded tree-width. In Satoru Iwata and Naonori Kakimura, editors, *34th International Symposium on Algorithms and Computation, ISAAC 2023, December 3-6, 2023, Kyoto, Japan*, volume 283 of *LIPIcs*, pages 11:1–11:13. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023. doi:10.4230/LIPIcs.ISAAC.2023.11.
- 7 Édouard Bonnet, Dibyayan Chakraborty, Eun Jung Kim, Noleen Köhler, Raul Lopes, and Stéphan Thomassé. Twin-width VIII: delineation and win-wins. In Holger Dell and Jesper Nederlof, editors, *17th International Symposium on Parameterized and Exact Computation, IPEC 2022, Potsdam, Germany, September 7-9, 2022*, *LIPIcs*, pages 9:1–9:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.IPEC.2022.9.

- 8 Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Torunczyk. Twin-width IV: ordered graphs and matrices. *J. ACM*, 71(3):21, 2024. doi:10.1145/3651151.
- 9 Édouard Bonnet, Eun Jung Kim, Amadeus Reinald, and Stéphan Thomassé. Twin-width VI: the lens of contraction sequences. In Joseph (Seffi) Naor and Niv Buchbinder, editors, *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022*, pages 1036–1056. SIAM, 2022. doi:10.1137/1.9781611977073.45.
- 10 Édouard Bonnet, Eun Jung Kim, Amadeus Reinald, Stéphan Thomassé, and Rémi Watrigant. Twin-width and polynomial kernels. *Algorithmica*, 84(11):3300–3337, 2022. doi:10.1007/S00453-022-00965-5.
- 11 Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width I: tractable FO model checking. *J. ACM*, 69(1):3:1–3:46, 2022. doi:10.1145/3486655.
- 12 Nick Brettell, Andrea Munaro, Daniël Paulusma, and Shizhou Yang. Comparing width parameters on graph classes. *Eur. J. Comb.*, 127:104163, 2025. doi:10.1016/J.EJC.2025.104163.
- 13 Binh-Minh Bui-Xuan, Jan Arne Telle, and Martin Vatshelle. Boolean-width of graphs. *Theoretical Computer Science*, 412(39):5187–5204, 2011. doi:10.1016/j.tcs.2011.05.022.
- 14 Jin-Yi Cai, Martin Fürer, and Neil Immerman. An optimal lower bound on the number of variables for graph identification. *Combinatorica*, 12(4):389–410, December 1992. doi:10.1007/BF01305232.
- 15 Derek G. Corneil, H. Lerchs, and L. Stewart Burlingham. Complement reducible graphs. *Discret. Appl. Math.*, 3(3):163–174, 1981. doi:10.1016/0166-218X(81)90013-5.
- 16 Sergei Evdokimov, Ilia Ponomarenko, and Gottfried Tinhofer. Forestal algebras and algebraic forests (on a new class of weakly compact graphs). *Discrete Mathematics*, 225(1-3):149–172, 2000. doi:10.1016/S0012-365X(00)00152-7.
- 17 Jakub Gajarský, Michal Pilipczuk, and Szymon Torunczyk. Stable graphs of bounded twin-width. In Christel Baier and Dana Fisman, editors, *LICS '22: 37th Annual ACM/IEEE Symposium on Logic in Computer Science, Haifa, Israel, August 2 - 5, 2022*, pages 39:1–39:12. ACM, 2022. doi:10.1145/3531130.3533356.
- 18 T. Gallai. Transitiv orientierbare graphen. *Acta Mathematica Academiae Scientiarum Hungarica*, 18:25–66, 1967. doi:10.1007/BF02020961.
- 19 Colin Geniet and Stéphan Thomassé. First Order Logic and Twin-Width in Tournaments. In Inge Li Gørtz, Martin Farach-Colton, Simon J. Puglisi, and Grzegorz Herman, editors, *31st Annual European Symposium on Algorithms (ESA 2023)*, volume 274 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 53:1–53:14, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ESA.2023.53.
- 20 Martin Charles Golumbic and Udi Rotics. On the clique-width of some perfect graph classes. *Int. J. Found. Comput. Sci.*, 11(3):423–443, 2000. doi:10.1142/S0129054100000260.
- 21 Martin Grohe. Fixed-point logics on planar graphs. In *Thirteenth Annual IEEE Symposium on Logic in Computer Science, Indianapolis, Indiana, USA, June 21-24, 1998*, pages 6–15. IEEE Computer Society, 1998. doi:10.1109/LICS.1998.705639.
- 22 Martin Grohe. *Descriptive Complexity, Canonisation, and Definable Graph Structure Theory*, volume 47 of *Lecture Notes in Logic*. Cambridge University Press, 2017. doi:10.1017/9781139028868.
- 23 Martin Grohe and Julian Mariño. Definability and descriptive complexity on databases of bounded tree-width. In Catriel Beeri and Peter Buneman, editors, *Database Theory - ICDT '99, 7th International Conference, Jerusalem, Israel, January 10-12, 1999, Proceedings*, volume 1540 of *Lecture Notes in Computer Science*, pages 70–82. Springer, 1999. doi:10.1007/3-540-49257-7_6.
- 24 Martin Grohe and Daniel Neuen. Canonisation and definability for graphs of bounded rank width. *ACM Trans. Comput. Logic*, 24(1), January 2023. doi:10.1145/3568025.

- 25 Martin Grohe and Daniel Neuen. Isomorphism for Tournaments of Small Twin Width. In Karl Bringmann, Martin Grohe, Gabriele Puppis, and Ola Svensson, editors, *51st International Colloquium on Automata, Languages, and Programming (ICALP 2024)*, volume 297 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 78:1–78:20, Dagstuhl, Germany, 2024. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ICALP.2024.78.
- 26 Martin Grohe and Pascal Schweitzer. Isomorphism testing for graphs of bounded rank width. In Venkatesan Guruswami, editor, *IEEE 56th Annual Symposium on Foundations of Computer Science, FOCS 2015, Berkeley, CA, USA, 17-20 October, 2015*, pages 1010–1029. IEEE Computer Society, 2015. doi:10.1109/FOCS.2015.66.
- 27 Michel Habib and Christophe Paul. A survey of the algorithmic aspects of modular decomposition. *Computer Science Review*, 4(1):41–59, 2010. doi:10.1016/j.cosrev.2010.01.001.
- 28 Irene Heinrich, Moritz Lichter, Klara Pakhomenko, and Simon Raßmann. Weisfeiler-leman on graphs of small twin-width. *CoRR*, abs/2601.05892, 2026. doi:10.48550/arXiv.2601.05892.
- 29 Lauri Hella. Logical hierarchies in PTIME. *Inf. Comput.*, 129(1):1–19, 1996. doi:10.1006/INCO.1996.0070.
- 30 Petr Hliněný and Jan Jedelský. Twin-Width of Planar Graphs Is at Most 8, and at Most 6 When Bipartite Planar. In Kousha Etessami, Uriel Feige, and Gabriele Puppis, editors, *50th International Colloquium on Automata, Languages, and Programming (ICALP 2023)*, volume 261 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 75:1–75:18, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ICALP.2023.75.
- 31 Neil Immerman and Rik Sengupta. The k -dimensional weisfeiler-leman algorithm, 2019. arXiv:1907.09582.
- 32 Hugo Jacob and Marcin Pilipczuk. Bounding twin-width for bounded-treewidth graphs, planar graphs, and bipartite graphs. In Michael A. Bekos and Michael Kaufmann, editors, *Graph-Theoretic Concepts in Computer Science - 48th International Workshop, WG 2022, Tübingen, Germany, June 22-24, 2022, Revised Selected Papers*, volume 13453 of *Lecture Notes in Computer Science*, pages 287–299. Springer, 2022. doi:10.1007/978-3-031-15914-5_21.
- 33 Sandra Kiefer. *Power and Limits of the Weisfeiler-Leman Algorithm*. PhD thesis, RWTH Aachen University, 2020. doi:10.18154/RWTH-2020-03508.
- 34 Sandra Kiefer, Ilia Ponomarenko, and Pascal Schweitzer. The weisfeiler-leman dimension of planar graphs is at most 3. *J. ACM*, 66(6):44:1–44:31, 2019. doi:10.1145/3333003.
- 35 Daniel Král, Kristýna Pekárková, and Kenny Štorgel. Twin-Width of Graphs on Surfaces. In Rastislav Kráľovič and Antonín Kučera, editors, *49th International Symposium on Mathematical Foundations of Computer Science (MFCS 2024)*, volume 306 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 66:1–66:15, Dagstuhl, Germany, 2024. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.MFCS.2024.66.
- 36 John H. Muller and Jeremy Spinrad. Incremental modular decomposition. *J. ACM*, 36(1):1–19, January 1989. doi:10.1145/58562.59300.
- 37 Sang-il Oum and Paul D. Seymour. Approximating clique-width and branch-width. *J. Comb. Theory B*, 96(4):514–528, 2006. doi:10.1016/J.JCTB.2005.10.006.
- 38 B. A. Reed. *Tree Width and Tangles: A New Connectivity Measure and Some Applications*, pages 87–162. London Mathematical Society Lecture Note Series. Cambridge University Press, 1997. doi:10.1017/CB09780511662119.006.
- 39 Martin Vatshelle. *New Width Parameters of Graphs*. PhD thesis, University of Bergen, 2012. URL: <https://hdl.handle.net/1956/6166>.