

Subleading effects in soft-gluon emission at one-loop in massive QCD

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ABSTRACT: We provide the last missing ingredient necessary to approximate one-loop amplitudes in QCD with massive quarks in the limit of vanishing energy of a single gluon up to terms suppressed by this energy. Our main result is a soft operator acting in color and spin space that manipulates the momenta of the hard partons while keeping them on-shell and respecting momentum conservation. Additionally, we provide a complete expression for the subleading term of the expansion of an arbitrary tree-level amplitude in the limit where the momenta of a massless quark and a massless anti-quark of the same flavor become collinear. This limit is necessary to obtain the one-loop soft approximation whenever the process involves such a quark-anti-quark pair. Interestingly, the result involves a high-energy limit.

KEYWORDS: Factorization, Renormalization Group, Higher-Order Perturbative Calculations, Resummation

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1 Introduction

In a previous publication [1], we have studied the subleading single-soft-gluon asymptotics of one-loop amplitudes in QCD without massive quarks. There are, nevertheless, interesting settings where the masses of quarks play a role and should not be neglected. It is the purpose of the present work to provide a formula for this case as well. Whenever soft emissions from massive fermions are discussed, it is worth consulting the literature on QED. The most relevant works for us are refs. [2–4] (see also the review [5]). Certainly, the gluon self-interaction makes the soft approximation substantially more complex in QCD, but gluon emissions from a single massive-quark line manifest some “abelian” properties which are given by the same expressions as in QED. It turns out that this is also the place where some interesting subtleties of factorization at subleading order are hidden. In particular, there are contributions due to hard virtual momenta, which are not contained in the one-loop amplitude for the process without gluon.

The soft asymptotics of amplitudes for the scattering of massive quarks only accompanied by a single gluon is entirely contained in a soft operator. The structure of the approximation is constrained by the Ward identity satisfied by the original amplitude and gauge-fixing independence of the amplitude with the gluon removed on which the operator is applied. This latter property requires the soft operator to at most change the color and spin of the quarks, while keeping their momenta on-shell and satisfying momentum conservation. We exploit these constraints to organize our result in a compact and transparent form.

The result is only complete if we also allow for massless quarks and gluons and not just massive quarks. Just as was the case in massless QCD, the formula for the soft operator, given as a sum over pairs of partons, is identical for every parton combination. At leading order of the soft expansion, this is a consequence of the well-known spin independence of

the eikonal approximation. At subleading order, spin effects do play a role but amount to little-group transformations, which are the same for a massless quark and a massless gluon up to a normalization factor due to the different total spin.

Although we build upon ref. [1], we do not forget to mention some of the preceding studies of the topic most relevant to us in QED [6, 7], QCD [8, 9], SCET [10–12] and gravity [13–15]. All of them derive from the classic works of Low [16], Burnett and Kroll [17], and Del Duca [18]. Interestingly, soft emissions at next-to-leading power still offer avenues for exploration even at tree-level as demonstrated by the very recent refs. [19–23].

A side result of the study presented in ref. [1] was the determination of the subleading behaviour of tree-level amplitudes in the two-parton collinear limit. In the case of splittings $\bar{q} \rightarrow \bar{q}g$ and $g \rightarrow gg$ the results were expressed in terms of universal factors and process-dependent amplitudes. A key element to obtain the expressions was the use of the subleading soft approximation for the final state gluons. As far as the splitting $g \rightarrow q\bar{q}$ is concerned, the structure of the result was determined, but one term was left unexpressed through universal factors and process-dependent amplitudes unlike what would be expected from a factorization-based result. The reason for this shortcoming was the lack of an expression for the subleading soft asymptotics due to a (anti-)quark. Since subleading tree-level collinear asymptotics are a key ingredient for the one-loop subleading-soft approximation, it is desirable to have all results provided in the most general and intuitively appealing form possible. We present the last missing piece needed for the splitting $g \rightarrow q\bar{q}$ in the present publication.

The paper is organized as follows. In the next section, we define the problem and quote a minimum of formulae to keep the publication reasonably self-contained. Afterwards, we discuss the master integrals present in the result. Although these are known for a long time, we reproduce the most relevant expressions for completeness. Our main result is presented in section 4. There, we not only provide the formula for the soft operator, but also discuss pole cancellation, and, most importantly, the intricacies of the derivation. Appendix A on the subleading collinear limit for $g \rightarrow q\bar{q}$ closes the publication.

2 Definitions

We consider the process

$$0 \rightarrow a_1(p_1, \sigma_1, c_1) + \cdots + a_n(p_n, \sigma_n, c_n) + g(q, \sigma_{n+1}, c_{n+1}) , \quad a_i \in \{q, \bar{q}, g\} , \quad (2.1)$$

where a_i , p_i , σ_i , c_i are the respective flavors, momenta, polarizations, and colors of the partons. We do not make a distinction between (anti-)quarks of different flavor as QCD interactions are flavor independent. The singled-out gluon is assumed to be soft, i.e. $q^0 \ll |p_i^0|$. We allow for massive quarks, $p_i^2 \neq 0$, $a_i \in \{q, \bar{q}\}$ for at least one¹ i . The process may include any number of color singlets, which we omit in the discussion for the sake of simplicity of notation. The reduced process without the soft gluon is

$$0 \rightarrow a_1(p'_1, \sigma'_1, c'_1) + \cdots + a_n(p'_n, \sigma'_n, c'_n) . \quad (2.2)$$

¹The result is valid beyond QCD and flavor-changing interactions might allow for amplitudes with a single massive quark.

Amplitudes for a process with m partons, with $m = n + 1$ or $m = n$, are expanded in the bare strong coupling constant g_s^B as follows.

$$M_m \equiv (g_s^B)^{m-2} \left[M_m^{(0)} + \frac{(\mu^2 e^{\gamma_E})^{-\epsilon} \alpha_s^B}{(4\pi)^{1-\epsilon}} M_m^{(1)} + \mathcal{O}((\alpha_s^B)^2) \right], \quad \alpha_s^B \equiv \frac{(g_s^B)^2}{4\pi}. \quad (2.3)$$

The results are presented in dimensional regularization with spacetime dimension $d \equiv 4 - 2\epsilon$. The strong coupling constant is left unrenormalized, but mass counterterms are included as they are needed for gauge-fixing independence of the amplitudes. Of course, LSZ normalization factors are also included. The massive quarks are considered as decoupled, i.e. there are no heavy quark loops.

The structure of the soft approximation at one-loop,

$$\begin{aligned} & |M_g^{(1)}(\{p_i\}, q)\rangle \approx \\ & \mathbf{S}^{(0)}(\{p_i\}, \{p'_i\}, q) |M^{(1)}(\{p'_i\})\rangle + \mathbf{S}^{(1)}(\{p_i\}, \{p'_i\}, q) |M^{(0)}(\{p'_i\})\rangle \\ & + \int_0^1 dx \left(\sum_{i, m_i=0} \mathbf{J}_i^{(1)}(x, p'_i, q) |H_{g,i}^{(0)}(x, \{p'_i\}, q)\rangle + \sum_{i, a_i=g} \tilde{\mathbf{J}}_i^{(1)}(x, p'_i, q) |H_{\tilde{q},i}^{(0)}(x, \{p'_i\}, q)\rangle \right) \\ & + \sum_{\substack{i \neq j \\ m_i=0 \\ m_j=0}} \sum_{\substack{\tilde{a}_i \neq a_i \\ \tilde{a}_j \neq a_j}} \tilde{\mathbf{S}}_{a_i a_j \leftarrow \tilde{a}_i \tilde{a}_j, ij}^{(1)}(p'_i, p'_j, q) |M^{(0)}(\{p'_i\})\rangle \big|_{a_j \rightarrow \tilde{a}_j}^{a_i \rightarrow \tilde{a}_i}, \end{aligned} \quad (2.4)$$

has been determined in ref. [1]. $|M_g^{(1)}\rangle$ is the one-loop amplitude for process (2.1) considered as a vector in color-and-spin space, while $|M^{(0)}\rangle$, $|M^{(1)}\rangle$ are the tree-level and one-loop amplitudes for process (2.2). The definitions of the modified $(n+1)$ -parton tree-level amplitudes in the collinear limit, $|H_{g,i}^{(0)}\rangle$ and $|H_{\tilde{q},i}^{(0)}\rangle$, as well as the definitions of the operators $\mathbf{J}_i^{(1)}$, $\tilde{\mathbf{J}}_i^{(1)}$ and $\tilde{\mathbf{S}}^{(1)}$ can be found in ref. [1]. The extension of the one-loop soft operator $\mathbf{S}^{(1)}$ to the massive case is the subject of the present publication. The tree-level soft operator $\mathbf{S}^{(0)}$ is given by²

$$\mathbf{P}_g(\sigma, c) \mathbf{S}^{(0)} = - \sum_i \mathbf{T}_i^c \otimes \mathbf{S}_i^{(0)}, \quad \mathbf{S}_i^{(0)} = \frac{p_i \cdot \epsilon^*}{p_i \cdot q} \left(1 + \sum_j (p_i - p'_i) \cdot \partial_j \right) + \frac{1}{2p_i \cdot q} F_{\mu\nu} (J_i^{\mu\nu} - \mathbf{K}_i^{\mu\nu}), \quad (2.5)$$

where $\mathbf{P}_g(\sigma, c)$ is a surjection operator from $(n+1)$ -parton to n -parton color-and-spin space that removes the gluon state by projecting out the polarization σ and color c . The respective gluon polarization vector is $\epsilon^* \equiv \epsilon^*(\mathbf{q}, \sigma)$. The tensor-product notation separates operators acting in color space from operators acting in spin space. $\mathbf{T}_i^c$ is a standard color operator acting on the color state of parton i , and $F_{\mu\nu}$ is the abelian field-strength tensor in the momentum representation,

$$F_{\mu\nu} = i(q_\mu \epsilon_\nu^* - q_\nu \epsilon_\mu^*). \quad (2.6)$$

²At the subleading level in the soft expansion, it is not necessary to make a distinction between p_i and p'_i . On the one hand, this results in a lack of strict power-counting in the soft expansion parameter. On the other hand, however, the expressions are more compact. The latter property is particularly welcome in the massive case.

Finally, $J_i^{\mu\nu}$ is the four-dimensional angular-momentum tensor operator,

$$J_i^{\mu\nu} \equiv i(p_i^\mu \partial_i^\nu - p_i^\nu \partial_i^\mu), \quad \partial_i^\mu \equiv \partial/\partial p_{i\mu}, \quad (2.7)$$

and $\mathbf{K}_i^{\mu\nu}$ is a spin-space operator acting on the polarization state of parton i

$$\mathbf{K}_i^{\mu\nu} |\dots, \sigma'_i, \dots\rangle \equiv \sum_{\sigma_i} K_{a_i, \sigma_i \sigma'_i}^{\mu\nu} |\dots, \sigma_i, \dots\rangle, \quad (2.8)$$

defined through its matrix elements $K_{a, \sigma \sigma'}^{\mu\nu}$,

$$\begin{aligned} \sum_{\sigma'} K_{q, \sigma \sigma'}^{\mu\nu} \bar{u}(\mathbf{p}_i, \sigma') &\equiv J_i^{\mu\nu} \bar{u}(\mathbf{p}_i, \sigma) - \frac{1}{2} \bar{u}(\mathbf{p}_i, \sigma) \sigma^{\mu\nu}, \\ \sum_{\sigma'} K_{g, \sigma \sigma'}^{\mu\nu} \epsilon_\alpha^*(\mathbf{p}_i, \sigma') &\equiv \left(J_i^{\mu\nu} g_{\alpha\beta} + i(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu) \right) \epsilon^{*\beta}(\mathbf{p}_i, \sigma) + \text{terms proportional to } p_{i\alpha}. \end{aligned} \quad (2.9)$$

The matrix elements of $\mathbf{K}_i$ can be obtained with the help of Lorentz transformations of spinors and polarization vectors. For example

$$S(\Lambda) u(\mathbf{p}, \sigma) = \sum_{\sigma'} \mathcal{D}_{\sigma' \sigma}^{(1/2)}(R) u(\mathbf{p}', \sigma'), \quad \Lambda = e^\omega, \quad S = e^{-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}}, \quad \mathbf{p}' = \Lambda \mathbf{p}, \quad (2.10)$$

where $\mathcal{D}_{\sigma' \sigma}^{(1/2)}(R)$ is the Wigner rotation matrix for the little-group rotation R , which depends on the conventions used in the definition of the spinor. For more details, see ref. [1].

3 Master integrals

The one-loop soft operator $\mathbf{S}^{(1)}$ is given in section 4 in terms of six master integrals. One of those is the standard one-loop integral $A_0(m^2)$ which vanishes for $m = 0$ and is otherwise given by

$$A_0(m^2) = \mu^{2\epsilon} e^{\epsilon\gamma_E} \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{[l^2 - m^2]} = \frac{1}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{m^2}\right) + \mathcal{O}(\epsilon). \quad (3.1)$$

For convenience, we also define

$$\left(-\frac{\mu^2 s_{ij}}{s_{iq} s_{jq}} \right)^\epsilon M_{0i} \equiv \frac{A_0(m_i^2)}{m_i^2}, \quad (3.2)$$

with

$$s_{ij} \equiv 2p_i \cdot p_j + i0^+, \quad s_{iq} \equiv 2p_i \cdot q + i0^+, \quad s_{jq} \equiv 2p_j \cdot q + i0^+. \quad (3.3)$$

The remaining five master integrals,

$$\begin{aligned} M_{1i} &\equiv (p_i \cdot q) I_{1110}, & M_{1j} &\equiv (p_j \cdot q) I_{1101}, \\ M_{2j} &\equiv \frac{1}{2} (p_i \cdot p_j) I_{1011}, & M_{2i} &\equiv \frac{1}{2} (p_i \cdot p_j) I_{0111}, \\ M_3 &\equiv (p_i \cdot q)(p_j \cdot q) I_{1111}, \end{aligned} \quad (3.4)$$

are defined in terms of the generic integral $I_{\alpha_1\alpha_2\alpha_3\alpha_4}$,

$$\left(-\frac{\mu^2 s_{ij}}{s_{iq}s_{jq}}\right)^\epsilon I_{\alpha_1\alpha_2\alpha_3\alpha_4} \equiv (\mu^2 e^{\gamma_E})^\epsilon \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{[l^2]^{\alpha_1} [(l+q)^2]^{\alpha_2} [p_i \cdot (l+q)]^{\alpha_3} [-p_j \cdot l]^{\alpha_4}}. \quad (3.5)$$

Each expression in the square brackets on the r.h.s. of eq. (3.5) contains an implicit $+i0^+$. The integrals have been calculated in ref. [24] with exact dependence on the spacetime dimension d . For most applications at subleading order in the soft expansion, only a Laurent expansion to $\mathcal{O}(\epsilon^0)$ is needed. Assuming $m_i \neq 0$ and $m_j \neq 0$, there is

$$\begin{aligned} M_{1k} &= \frac{1}{2\epsilon^2} + \frac{\ln(-\alpha_k)}{2\epsilon} + \frac{1}{4} \ln^2(-\alpha_k) + \frac{5\pi^2}{24} + \mathcal{O}(\epsilon), \\ M_{2k} &= \frac{1}{v} \left(-\frac{l_x}{2\epsilon} + \frac{1}{2} \ln(-\alpha_k) l_x - \ln(v) l_x - \frac{1}{4} l_x^2 - \text{Li}_2(x) + \frac{\pi^2}{6} \right) + \mathcal{O}(\epsilon), \\ M_3 &= \frac{1}{\epsilon^2} + \frac{\ln(-\alpha_i) + \ln(-\alpha_j)}{2\epsilon} + \frac{1}{2} \ln(-\alpha_i) \ln(-\alpha_j) - \frac{1}{4} l_x^2 - \frac{7\pi^2}{12} + \mathcal{O}(\epsilon), \end{aligned} \quad (3.6)$$

with

$$\begin{aligned} \alpha_i &\equiv \frac{m_i^2 s_{jq}}{s_{ij} s_{iq}}, & \alpha_j &\equiv \frac{m_j^2 s_{iq}}{s_{ij} s_{jq}}, & v &\equiv \sqrt{1 - 4\alpha_i \alpha_j} = \sqrt{1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}}, \\ x &\equiv \frac{1-v}{1+v}, & l_x &\equiv \ln(x) + 2\pi i \theta(s_{ij}). \end{aligned} \quad (3.7)$$

The analytic continuation of $\ln(x)$ to timelike kinematics, $s_{ij} > 0$, follows from the relation

$$x = \alpha_i \alpha_j (1+x)^2 = \left(\frac{m_i m_j (1+x)}{-s_{ij}} \right)^2, \quad (3.8)$$

which implies

$$l_x = \ln(-\alpha_i) + \ln(-\alpha_j) + 2\ln(1+x) = 2\ln(m_i m_j (1+x)) - 2\ln(-s_{ij}). \quad (3.9)$$

Since $|x| \leq 1$, $1+x$ does not cross the branch cut of the logarithm. On the one hand, expressing l_x in terms of $\ln(-\alpha_i)$, $\ln(-\alpha_j)$ and $\ln(1+x)$ allows for easy analytic continuation and provides direct access to mass singularities. On the other hand, however, expressions for the integrals are more compact in terms of l_x .

In case $m_j = 0$ and $m_i \neq 0$, M_{1i} remains unchanged, but there is

$$\begin{aligned} M_{1j} &= 0, & M_{2j} &= -M_{1i}, \\ M_{2i} &= \frac{1}{2\epsilon^2} - \frac{\ln(-\alpha_i)}{2\epsilon} + \frac{1}{4} \ln(-\alpha_i)^2 + \frac{3\pi^2}{8} + \mathcal{O}(\epsilon), \\ M_3 &= \frac{3}{2\epsilon^2} + \frac{\ln(-\alpha_i)}{2\epsilon} - \frac{1}{4} \ln(-\alpha_i)^2 - \frac{5\pi^2}{24} + \mathcal{O}(\epsilon). \end{aligned} \quad (3.10)$$

In case $m_i = 0$ and $m_j \neq 0$, the master integrals are obtained by swapping i and j . Finally, if $m_i = m_j = 0$ then

$$M_{1k} = M_{2k} = 0, \quad M_3 = \frac{2}{\epsilon^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon). \quad (3.11)$$

It has been proved in ref. [24] that the leading term of the soft expansion is regular at vanishing masses. It follows from eq. (4.3) that $M_3 + M_{2j}$ and $M_{2i} + M_{1j}$ are regular at $m_j = 0$. Similarly, $M_3 + M_{2i}$ and $M_{2j} + M_{1i}$ are regular at $m_i = 0$. This implies that the sum of the five master integrals

$$M_3 + M_{2i} + M_{2j} + M_{1i} + M_{1j} \quad (3.12)$$

is regular at both $m_i = 0$ and $m_j = 0$.

4 The one-loop soft operator

The one-loop soft operator can be decomposed into non-abelian (NA) and abelian (A) contributions. It is a sum over partons i and j with a summand that contains the subleading soft behavior due to i .

$$\mathbf{P}_g(\sigma, c) \mathbf{S}^{(1)} = \sum_{i \neq j} \left(-\frac{\mu^2 s_{ij}}{s_{iq} s_{jq}} \right)^\epsilon \left[2i f^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \otimes \mathbf{S}_{ij}^{(\text{NA})} + \mathbf{T}_i^c \mathbf{T}_i \cdot \mathbf{T}_j S_{ij}^{(\text{A})} \right] \quad (4.1)$$

$$+ C_F \sum_i \mathbf{T}_i^c \frac{2\epsilon(1-\epsilon)(1+2\epsilon)}{1-2\epsilon} \frac{A_0(m_i^2)}{m_i^2} \left(g^{\mu\nu} - \frac{2p_i^\mu p_i^\nu}{m_i^2} \right) \frac{F_{\mu\rho}}{s_{iq}} \mathbf{K}_{i\nu}{}^\rho,$$

$$\mathbf{S}_{ij}^{(\text{NA})} = -g_{ij}^{(1)} \mathbf{S}_i^{(0)} + \epsilon S_{ij}^{(\text{A})} + a_i \left(g^{\mu\nu} - \frac{2p_i^\mu p_i^\nu}{m_i^2} \right) \frac{F_{\mu\rho}}{s_{iq}} \mathbf{K}_{i\nu}{}^\rho$$

$$+ \left(b_{ij} \alpha_j g^{\mu\nu} + c_{ij} \alpha_j \frac{2p_j^\mu p_j^\nu}{m_j^2} + b_{ji} \frac{2p_j^\mu p_i^\nu}{s_{ij}} + c_{ji} \frac{2p_i^\mu p_j^\nu}{s_{ij}} \right) \frac{F_{\mu\rho}}{s_{iq}} (J_i - \mathbf{K}_i)_\nu{}^\rho \quad (4.2)$$

$$+ \frac{b_{ji} + c_{ji} + 2\alpha_j(b_{ij} + c_{ij})}{v^2} \frac{2p_i^\mu p_j^\nu F_{\mu\nu}}{s_{ij} s_{iq}} \frac{2p_i^\rho p_j^\sigma}{s_{ij}} (J_i - \mathbf{K}_i)_{\rho\sigma},$$

$$-g_{ij}^{(1)} = \frac{M_{1i} + M_{1j} + (1-2\alpha_j) M_{2i} + (1-2\alpha_i) M_{2j} + (1-2\alpha_i - 2\alpha_j) M_3}{2g_{ijq}}, \quad (4.3)$$

$$S_{ij}^{(\text{A})} = -\frac{8}{v^2} \left(\frac{2\epsilon}{1-2\epsilon} M_{1i} + (v^2 - 1) M_{2j} \right) \frac{p_i^\mu p_j^\nu i F_{\mu\nu}}{s_{ij} s_{iq}},$$

$$a_i = \frac{(1-\epsilon)(1-2\epsilon^2) M_{0i} - 2\epsilon M_{1i}}{1-2\epsilon}, \quad (4.4)$$

$$b_{ij} = \frac{-M_{1i} - M_{1j} + (1-2\alpha_i)(M_{2i} - M_{2j}) + M_3}{g_{ijq}}, \quad (4.5)$$

$$c_{ij} = \left[\left((1-2\alpha_j - (1-2\epsilon)(1-2\alpha_i)) M_{1i} - (1-2\alpha_i + 2g_{ijq}) \right. \right. \quad (4.6)$$

$$\left. - (1-2\epsilon)(1-2\alpha_j + 2g_{ijq}) \right) M_{1j} + \left(2(1-2\epsilon)(v^2 - g_{ijq}) + 2\epsilon(1-2\alpha_i)^2 \right) M_{2i} + \left(2g_{ijq} - 2(1-\epsilon)v^2 \right) M_{2j}$$

$$\left. + \left((1-2\alpha_i - (1-2\epsilon)(1-2\alpha_j)) M_3 \right] \frac{1}{2(1-2\epsilon)g_{ijq}^2},$$

$$g_{ijq} = \frac{4 \det(G)}{s_{ij} s_{iq} s_{jq}} = 1 - \alpha_i - \alpha_j, \quad (4.7)$$

where G is the Gram matrix of p_i, p_j and q . Several comments are in order.

1. The soft operator is not UV renormalized as implied by the definition of the matrix elements (2.3). The UV renormalized version is given by

$$\mathbf{S}^{(1)} - \frac{\beta_0}{2\epsilon} \mathbf{S}^{(0)}, \quad \beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_l, \quad (4.8)$$

with n_l the number of massless quark flavors as we have decoupled heavy quarks.

2. The coefficient $g_{ij}^{(1)}$ determines the leading behavior of the soft operator. It has been originally obtained in ref. [24].³
3. The subleading behavior of the soft operator when i is a massless quark is identical to that when i is a gluon, up to the difference in the value of the spin operators $\mathbf{K}_i$, which are then diagonal in the helicity basis.
4. The abelian contribution is a new feature of the massive case not present in massless QCD. Our result for $S_{ij}^{(A)}$ agrees with the corresponding expression in QED provided in ref. [2]. The term in the second line of eq. (4.1) does not affect unpolarized matrix elements, but is necessary at the amplitude level. It was first determined for QED in refs. [25, 26]. This term can in principle be combined with $S_{ij}^{(A)}$ using the following relation valid by virtue of color conservation for summands independent of j .

$$\sum_{i \neq j} \mathbf{T}_i^c \mathbf{T}_i \cdot \mathbf{T}_j = -C_F \sum_i \mathbf{T}_i^c. \quad (4.9)$$

5. Contributions proportional to the master integral A_0 originate in the hard-region of the soft expansion, see section 4.3. We keep them in the soft operator $\mathbf{S}^{(1)}$ in order to maintain the structural minimality of eq. (2.4). In an analysis based on effective field theory, these contributions would be separated as they correspond to the propagation of hard modes rather than soft modes.⁴
6. The term in the second line of eq. (4.1) as well as the third term in eq. (4.2) could as well be expressed through $J_i - \mathbf{K}_i$ because

$$\left(g^{\mu\nu} - \frac{2p_i^\mu p_i^\nu}{m_i^2} \right) \frac{F_{\mu\rho}}{s_{iq}} J_{i\nu}{}^\rho = 0. \quad (4.10)$$

7. If $m_i = 0$ then the last term in eq. (4.2) can be combined with the term proportional to b_{ji} . Indeed, equations of motion imply in this case

$$\frac{2p_i^\mu p_j^\nu F_{\mu\nu}}{s_{ij}s_{iq}} \frac{2p_i^\rho p_j^\sigma}{s_{ij}} (J_i - \mathbf{K}_i)_{\rho\sigma} = -\frac{2p_j^\mu p_i^\nu}{s_{ij}} \frac{F_{\mu\rho}}{s_{iq}} (J_i - \mathbf{K}_i)_\nu{}^\rho. \quad (4.11)$$

If i is a gluon, then transversality of the process-dependent amplitude $|M^{(0)}\rangle$ implies furthermore

$$p_j^\mu p_i^\nu F_{\mu\rho} (J_i - \mathbf{K}_i)_\nu{}^\rho = 0. \quad (4.12)$$

³The minus sign in front of $g_{ij}^{(1)}$ in eq. (4.2) is necessary to compensate for the opposite sign of the strong coupling constant assumed in ref. [24].

⁴See for example ref. [3] for a discussion within QED.

8. If $m_i \neq 0$ then the coefficient of the last term in eq. (4.2) is related to b_{ij} and c_{ij} due to momentum-conservation constraints. Indeed, the soft operator must satisfy

$$\mathbf{S}^{(1)}|_{\text{momentum derivatives}}|f(P')\rangle = 0, \quad P' \equiv \sum_i p'_i, \quad (4.13)$$

where $|f\rangle$ is an arbitrary function of the sum of the momenta p'_i of the particles in the reduced process. This translates to

$$[\mathbf{S}_{ij}^{(\text{NA})} - \mathbf{S}_{ji}^{(\text{NA})}]|_{\text{momentum derivatives}}|f(P')\rangle = 0. \quad (4.14)$$

The first term in eq. (4.2) satisfies this constraint because

$$\mathbf{S}_i^{(0)}|_{\text{momentum derivatives}}|f(P')\rangle = -\epsilon^* \cdot \frac{\partial}{\partial P'}|f(P')\rangle, \quad (4.15)$$

which is independent of i , and because the coefficient, $-g_{ij}^{(1)}$, of $\mathbf{S}_i^{(0)}$ is symmetric in i and j , while the color factor is anti-symmetric. For the remaining terms containing J_i one obtains the relation specified in the last term of eq. (4.2).

4.1 Spurious-pole cancellation

The pole terms of the soft operator (4.1) depend on the mass configuration.

1. $m_i \neq 0$ and $m_j \neq 0$

$$\begin{aligned} \mathbf{S}_{ij}^{(\text{NA})} &= \frac{1}{\epsilon} \left[\frac{1}{\epsilon} + \frac{1}{2} \left(\ln(-\alpha_i) + \ln(-\alpha_j) - \frac{l_x}{v} \right) \right] \mathbf{S}_i^{(0)} + \mathcal{O}(\epsilon^0), \\ S_{ij}^{(\text{A})} &= \frac{1}{\epsilon} \left[-\frac{8}{v^2} \left(1 + \frac{2\alpha_i\alpha_j l_x}{v} \right) \frac{p_i^\mu p_j^\nu}{s_{ij}s_{iq}} iF_{\mu\nu} \right] + \mathcal{O}(\epsilon^0). \end{aligned} \quad (4.16)$$

2. $m_i \neq 0$ and $m_j = 0$

$$\begin{aligned} \mathbf{S}_{ij}^{(\text{NA})} &= \frac{1}{\epsilon} \left[\frac{1}{\epsilon} \mathbf{S}_i^{(0)} + 4 \frac{p_j^\mu p_j^\nu}{s_{ij}s_{jq}} F_{\mu\rho} (J_i - \mathbf{K}_i)_\nu{}^\rho \right] + \mathcal{O}(\epsilon^0), \\ S_{ij}^{(\text{A})} &= \frac{1}{\epsilon} \left[-8 \frac{p_i^\mu p_j^\nu}{s_{ij}s_{iq}} iF_{\mu\nu} \right] + \mathcal{O}(\epsilon^0). \end{aligned} \quad (4.17)$$

3. $m_i = 0$ and $m_j \neq 0$

$$\begin{aligned} \mathbf{S}_{ij}^{(\text{NA})} &= \frac{1}{\epsilon} \left[\frac{1}{\epsilon} \mathbf{S}_i^{(0)} + 4 \frac{p_i^\mu p_j^\nu - p_j^\mu p_i^\nu}{s_{ij}s_{iq}} F_{\mu\rho} (J_i - \mathbf{K}_i)_\nu{}^\rho \right] + \mathcal{O}(\epsilon^0), \\ S_{ij}^{(\text{A})} &= 0, \end{aligned} \quad (4.18)$$

4. $m_i = 0$ and $m_j = 0$

$$\begin{aligned} \mathbf{S}_{ij}^{(\text{NA})} &= \frac{1}{\epsilon} \left[\frac{1}{\epsilon} \mathbf{S}_i^{(0)} + 4 \frac{p_i^\mu p_j^\nu - p_j^\mu p_i^\nu}{s_{ij}s_{iq}} F_{\mu\rho} (J_i - \mathbf{K}_i)_\nu{}^\rho + 4 \frac{p_j^\mu p_j^\nu}{s_{ij}s_{jq}} F_{\mu\rho} (J_i - \mathbf{K}_i)_\nu{}^\rho \right] + \mathcal{O}(\epsilon^0), \\ S_{ij}^{(\text{A})} &= 0, \end{aligned} \quad (4.19)$$

Using the known singularities of massive one-loop gauge-theory amplitudes [27], one may obtain the pole terms of the soft approximation,⁵

$$\begin{aligned}
 & \mathbf{P}_g(\sigma, c) |M_g^{(0)}\rangle \approx \\
 & \mathbf{P}_g(\sigma, c) \mathbf{S}^{(0)} |M^{(1)}\rangle \\
 & + \sum_{i \neq j} 2if^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \otimes \left(-\frac{\mu^2 s_{ij}}{s_{iq} s_{jq}} \right)^\epsilon \frac{1}{\epsilon} \left[\frac{1}{\epsilon} + \frac{1}{2} \left(\ln(-\alpha_i) + \ln(-\alpha_j) - \frac{l_x}{v} \right) \right] \mathbf{S}_i^{(0)} |M^{(0)}\rangle \quad (4.20) \\
 & + \sum_{i \neq j} \mathbf{T}_i^c \mathbf{T}_i \cdot \mathbf{T}_j \frac{1}{\epsilon} \left[-\frac{8}{v^2} \left(1 + \frac{2\alpha_i \alpha_j l_x}{v} \right) \frac{p_i^\mu p_j^\nu}{s_{ij} s_{iq}} iF_{\mu\nu} \right] |M^{(0)}\rangle + \mathcal{O}(\epsilon^0) .
 \end{aligned}$$

The pole terms of the soft operator match this formula for $m_i \neq 0$ and $m_j \neq 0$ as confirmed by eq. (4.16). The r.h.s. of eq. (4.20) is regular if either $m_i = 0$ or $m_j = 0$,

$$\begin{aligned}
 \mathbf{P}_g(\sigma, c) |M_g^{(0)}\rangle & \approx \mathbf{P}_g(\sigma, c) \mathbf{S}^{(0)} |M^{(1)}\rangle \\
 & + \sum_{i \neq j} 2if^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \otimes \left(-\frac{\mu^2 s_{ij}}{s_{iq} s_{jq}} \right)^\epsilon \frac{1}{\epsilon} \left[\frac{1}{\epsilon} \right] \mathbf{S}_i^{(0)} |M^{(0)}\rangle \quad (4.21) \\
 & + \sum_{i \neq j} \mathbf{T}_i^c \mathbf{T}_i \cdot \mathbf{T}_j \frac{1}{\epsilon} \left[-8 \frac{p_i^\mu p_j^\nu}{s_{ij} s_{iq}} iF_{\mu\nu} \right] |M^{(0)}\rangle + \mathcal{O}(\epsilon^0) .
 \end{aligned}$$

Eqs. (4.17), (4.18) and (4.19) clearly differ from this result. However, if either i or j is massless then it is necessary to include jet-operator and flavor-off-diagonal soft contributions in eq. (2.4). In particular, if j is massless then⁶

$$\begin{aligned}
 \mathbf{P}_g(\sigma, c) \int_0^1 dx \mathbf{J}_j^{(1)} |H_{g,j}^{(0)}\rangle & = \sum_{i \neq j} 2if^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \otimes \frac{1}{\epsilon} \left[-4 \frac{p_j^\mu p_j^\nu}{s_{ij} s_{jq}} F_{\mu\rho} (J_i - \mathbf{K}_i)_\nu{}^\rho \right] |M^{(0)}\rangle \\
 & + \sum_{i \neq j} 2if^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \otimes \frac{1}{\epsilon} \left[-4 \frac{p_i^\mu p_j^\nu - p_j^\mu p_i^\nu}{s_{ij} s_{jq}} F_{\mu\rho} (J_j - \mathbf{K}_j)_\nu{}^\rho \right] |M^{(0)}\rangle \\
 & + \sum_{i \neq j} \mathbf{T}_j^c \mathbf{T}_j \cdot \mathbf{T}_i \frac{1}{\epsilon} \left[-8 \frac{p_j^\mu p_j^\nu}{s_{ij} s_{jq}} iF_{\mu\nu} \right] |M^{(0)}\rangle + \mathcal{O}(\epsilon^0) \\
 & + \text{terms canceled by an } \tilde{\mathbf{S}}^{(1)} \text{ contribution if } j \text{ is a (anti-)quark} . \quad (4.22)
 \end{aligned}$$

The sum of each of eqs. (4.17), (4.18), (4.19) and eq. (4.22) for either j or i or both, depending on the mass configuration, matches the pole terms of eq. (4.21).

4.2 Finite contributions

As in the case of pole terms in section 4.1, the finite part of the soft operator depends on the mass configuration. Below we list the values of the finite parts of the coefficients defined in eqs. (4.1) and (4.2).

⁵This is a generalization of eq. (4.78) in ref. [1].

⁶This result is obtained by summing eqs. (4.80) and (4.82) from ref. [1] and exchanging i and j . The term in the first line in eq. (4.22) contains $(J_i - \mathbf{K}_i)$ instead of $(J_j - \mathbf{K}_j)$ which it should have contained according to eq. (4.80) of ref. [1]. The reason for the difference is that symmetry w.r.t. the exchange of the summation indices i and j has been used in ref. [1]. In the present case, i and j cannot be exchanged, since only one of the partons is massless, and there is no jet-operator contribution for massive quarks.

1. $m_i \neq 0$ and $m_j \neq 0$

$$\begin{aligned}
 -g_{ij}^{(1)} = & \frac{1}{v} \left[\left(\frac{(1-2\alpha_j)\ln(-\alpha_i) + (1-2\alpha_i)\ln(-\alpha_j)}{4g_{ijq}} - \ln(v) - \frac{1}{4}l_x \right) l_x - \text{Li}_2(x) + \frac{\pi^2}{6} \right] \\
 & + \frac{(\ln(-\alpha_i) - \ln(-\alpha_j))^2 + l_x^2 + 4\pi^2}{8g_{ijq}} + \frac{1}{2}\ln(-\alpha_i)\ln(-\alpha_j) - \frac{1}{4}l_x^2 - \frac{7\pi^2}{12},
 \end{aligned} \tag{4.23}$$

$$\epsilon S_{ij}^{(A)} = -\frac{8}{v^2} \left[1 + \frac{2\alpha_i\alpha_j l_x}{v} \right] \frac{p_i^\mu p_j^\nu iF_{\mu\nu}}{s_{ij}s_{iq}}, \tag{4.24}$$

$$\begin{aligned}
 S_{ij}^{(A)} = & -\frac{8}{v^2} \left[2 + \ln(-\alpha_i) + \frac{\alpha_i\alpha_j}{v} \left((4\ln(v) - 2\ln(-\alpha_j) + l_x) l_x + 4\text{Li}_2(x) - \frac{2\pi^2}{3} \right) \right] \\
 & \times \frac{p_i^\mu p_j^\nu iF_{\mu\nu}}{s_{ij}s_{iq}},
 \end{aligned} \tag{4.25}$$

$$a_i = -2\ln(-\alpha_i) - 2\ln(-s_{ij}) + 2\ln(-s_{jq}), \tag{4.26}$$

$$b_{ij} = \frac{1}{4g_{ijq}} \left[\left(\ln(-\alpha_j) - \ln(-\alpha_i) + \frac{1-2\alpha_i}{v} 2l_x \right) (\ln(-\alpha_i) - \ln(-\alpha_j)) - l_x^2 - 4\pi^2 \right], \tag{4.27}$$

$$\begin{aligned}
 c_{ij} = & \frac{1}{2g_{ijq}} \left[\frac{(\alpha_i - \alpha_j) \left((\ln(-\alpha_i) - \ln(-\alpha_j))^2 + l_x^2 + 4\pi^2 \right) + 2v(\ln(-\alpha_i) - \ln(-\alpha_j)) l_x}{2g_{ijq}} \right. \\
 & \left. + (\ln(-\alpha_j) - \ln(-\alpha_i) - 2(1-2\alpha_i)) \frac{l_x}{v} + 2(\ln(-\alpha_i) - \ln(-\alpha_j)) \right].
 \end{aligned} \tag{4.28}$$

2. $m_i \neq 0$ and $m_j = 0$

$$\begin{aligned}
 -g_{ij}^{(1)} = & \frac{\alpha_i \ln^2(-\alpha_i) + (1+5\alpha_i)\frac{\pi^2}{6}}{2(1-\alpha_i)}, \\
 \epsilon S_{ij}^{(A)} = & -8 \frac{p_i^\mu p_j^\nu iF_{\mu\nu}}{s_{ij}s_{iq}}, & S_{ij}^{(A)} = & (-16 - 8\ln(-\alpha_i)) \frac{p_i^\mu p_j^\nu iF_{\mu\nu}}{s_{ij}s_{iq}}, \\
 a_i = & -2\ln(-\alpha_i) - 2\ln(-s_{ij}) + 2\ln(-s_{jq}), & b_{ij} = & \frac{-\alpha_i \ln^2(-\alpha_i) + (1-7\alpha_i)\frac{\pi^2}{6}}{1-\alpha_i}, \\
 c_{ij} = & 4 + \frac{2\alpha_i \ln(-\alpha_i)}{1-\alpha_i} + \frac{\alpha_i(\ln^2(-\alpha_i) + \pi^2)}{(1-\alpha_i)^2}.
 \end{aligned} \tag{4.29}$$

3. $m_i = 0$ and $m_j \neq 0$

$$\begin{aligned}
 -g_{ij}^{(1)} = & \frac{\alpha_j \ln^2(-\alpha_j) + (1+5\alpha_j)\frac{\pi^2}{6}}{2(1-\alpha_j)}, & \epsilon S_{ij}^{(A)} = & 0, & S_{ij}^{(A)} = & 0, & a_i = & 0, \\
 b_{ij} = & -\frac{\ln^2(-\alpha_j) + \pi^2}{1-\alpha_j}, & c_{ij} = & -\frac{2\ln(-\alpha_j)}{1-\alpha_j} - \frac{\alpha_j(\ln^2(-\alpha_j) + \pi^2)}{(1-\alpha_j)^2}.
 \end{aligned} \tag{4.30}$$

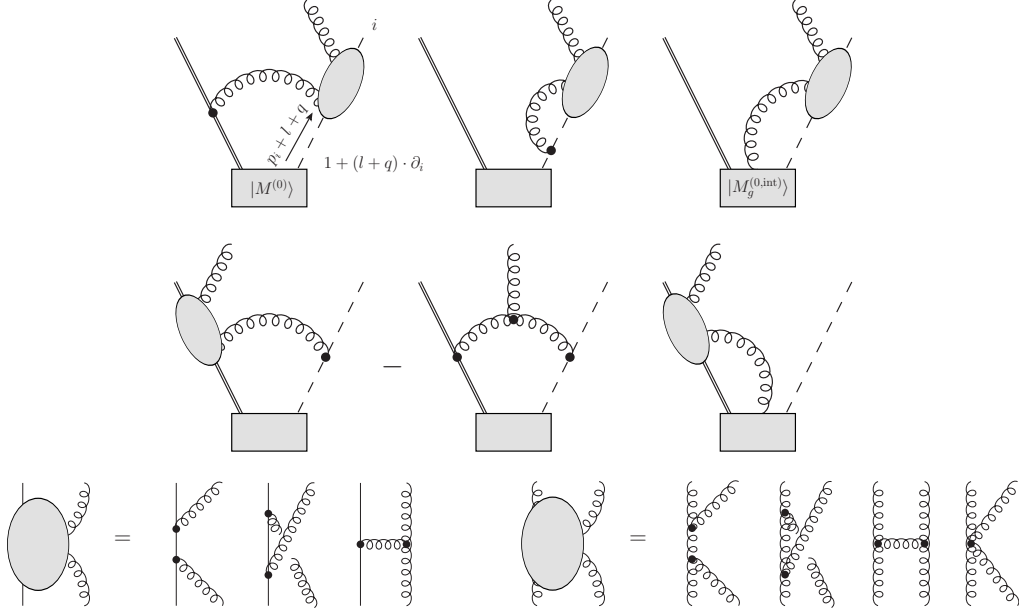


Figure 1. Diagrams included in the calculation of the soft operator. A double line is eikonal, while a dashed line corresponds to a (anti-)quark or a gluon. The shaded rectangle is a process-dependent sum of tree-level diagrams. The eikonal approximation is identical for (anti-)quarks and gluons if expressed through color operators. In this approximation, the quartic gluon vertex does not contribute. It does contribute, however, at subleading order of the soft expansion.

4. $m_i = 0$ and $m_j = 0$

$$-g_{ij}^{(1)} = \frac{\pi^2}{12}, \quad \epsilon S_{ij}^{(A)} = 0, \quad S_{ij}^{(A)} = 0, \quad a_i = 0, \quad b_{ij} = \frac{\pi^2}{6}, \quad c_{ij} = 4. \quad (4.31)$$

4.3 Derivation

The result (4.1) for $\mathbf{S}^{(1)}$ is a sum of two contributions of different kind. The first of those is obtained by evaluating the diagrams in figure 1 after expanding their integrands in the loop momentum and the momentum of the soft gluon. These two momenta are both taken to be formally of the same order, $l^\mu \sim q^\mu \sim \lambda$ with λ the soft-expansion parameter. This defines the soft-region contribution within the expansion-by-regions method [28] (see also refs. [29, 30]). Notice that it is necessary to expand the process-dependent part of the amplitude in the soft momenta as demonstrated in the first diagram of figure 1. This expansion is responsible for a subset of momentum derivatives present in the soft operator. In case parton i is a gluon, it is necessary to account for the transversality of the process-dependent amplitude. This may generate subleading contributions, because the momentum leaving the process-dependent amplitude on line i is either $p_i + q$, if there are no interactions on line j , or $p_i + l + q$. It is this momentum, and not just p_i , that contracted with the process-dependent amplitude yields a vanishing contribution.

While constructing the integrand for the soft-region contribution to $\mathbf{S}^{(1)}$ care must be taken of gluon emissions from internal lines of the process-dependent part of the amplitude, as depicted for example in the third diagram of figure 1. Following refs. [16, 17], we replace

the amplitude with internal-line emissions only, $|M_g^{(0,\text{int})}\rangle$, by derivatives of the non-radiative amplitude $|M^{(0)}\rangle$, both defined in figure 1.⁷ To this end, we first insert a gluon in the diagrams of $|M^{(0)}\rangle$ in all possible ways. We then approximate the resulting radiative amplitude $|M_g^{(0)}\rangle$ in the soft-gluon limit,

$$\mathbf{P}(\sigma, c) |M_g^{(0)}(\{p_i\}, q)\rangle \approx - \sum_i \mathbf{T}_i^c \frac{p_i \cdot \epsilon^*}{p_i \cdot q} (1 + q \cdot \tilde{\partial}_i) |M^{(0)}(\{p_i\})\rangle + \mathbf{P}(\sigma, c) |M_g^{(0,\text{int})}(\{p_i\}, q=0)\rangle, \quad (4.32)$$

where $\tilde{\partial}_i$ is a differential operator which vanishes when applied to the spinor or polarization vector on line i , while $\tilde{\partial}_i = \partial_i$ otherwise. We have ignored spin effects in (4.32), because they are represented by an anti-symmetric tensor, which would cancel in the next step anyway. By the Ward identity, the left-hand side of (4.32) vanishes for $\epsilon^* = q$,

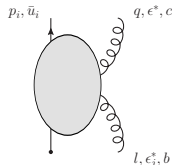
$$0 = - \sum_i \mathbf{T}_i^c (q \cdot \tilde{\partial}_i) |M^{(0)}(\{p_i\})\rangle + \mathbf{P}(q, c) |M_g^{(0,\text{int})}(\{p_i\}, q=0)\rangle, \quad (4.33)$$

where $\mathbf{P}(q, c)$ projects out the scalar polarization by replacing ϵ^* with q . A derivative in q^μ yields

$$\mathbf{P}_g(\mu, c) |M_g^{(0,\text{int})}\rangle = \sum_i \mathbf{T}_i^c \tilde{\partial}_i^\mu |M^{(0)}\rangle, \quad (4.34)$$

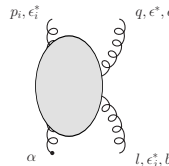
where $\mathbf{P}(\mu, c)$ strips the amplitude of the polarization vector and leaves the four-vector index μ instead. The derivatives on the right-hand side of eq. (4.34) complete the set of derivatives occurring in the soft operator.

Remarkably, the soft-operator for a massless (anti-)quark on line i is exactly the same as that for a gluon. This is a consequence of a relationship between double emissions from quark and gluon lines, which is valid at leading and subleading order. In particular, the relevant contributions have the following structure in the Feynman gauge used in the present calculation.⁸



$$= [\mathbf{T}_i^c, \mathbf{T}_i^b] \left(\mathcal{C}_{\text{NA}} \bar{u}_i + \mathcal{C}_{\text{NA}}^{\mu\nu} \bar{u}_i \frac{1}{2} \sigma_{\mu\nu} \right) + \{ \mathbf{T}_i^c, \mathbf{T}_i^b \} \left(\mathcal{C}_A \bar{u}_i + \mathcal{C}_A^{\mu\nu} \bar{u}_i \frac{1}{2} \sigma_{\mu\nu} \right) + \mathcal{O}(\lambda^0), \quad (4.35)$$

and



$$= [\mathbf{T}_i^c, \mathbf{T}_i^b] \left(\mathcal{C}_{\text{NA}} \epsilon_{i\alpha}^* + \mathcal{C}_{\text{NA}}^{\mu\nu} i(\epsilon_{i\mu}^* g_{\nu\alpha} - \epsilon_{i\nu}^* g_{\mu\alpha}) \right) + \{ \mathbf{T}_i^c, \mathbf{T}_i^b \} \left(\mathcal{C}_A \epsilon_{i\alpha}^* + \mathcal{C}_A^{\mu\nu} i(\epsilon_{i\mu}^* g_{\nu\alpha} - \epsilon_{i\nu}^* g_{\mu\alpha}) \right) + \mathbf{T}_i^c \mathbf{T}_i^b \frac{(\epsilon_i^* \cdot \epsilon^*) \epsilon_{j\alpha}^*}{2p_i \cdot q} + \mathbf{T}_i^b \mathbf{T}_i^c \frac{(\epsilon_i^* \cdot \epsilon_j^*) \epsilon_\alpha^*}{2p_i \cdot l} + \mathcal{O}(\lambda^0), \quad (4.36)$$

⁷It may seem that the process-dependent part of the amplitude is not an amplitude itself, because an external spinor or polarization vector is missing on the dashed line. However, if a subleading contribution is considered, as is the case of internal-line emissions, the dashed line is also eikonal.

⁸The result for double-gluon emission from a quark line can be obtained with the help of next-to-eikonal Feynman rules presented in ref. [31].

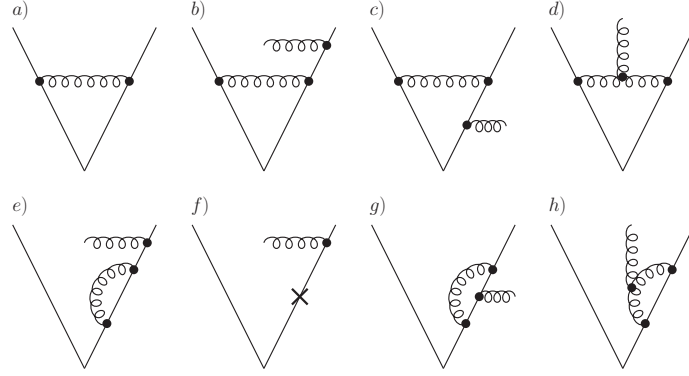


Figure 2. Diagrams for $\gamma^* \rightarrow q\bar{q}$ (a)) and $\gamma^* \rightarrow q\bar{q}g$ (b-h)). The cross in diagram f) represents a mass-renormalization counterterm.

where we have used transversality, $p_i^\alpha = -l^\alpha - q^\alpha$. The common coefficients $\mathcal{C}_{\text{NA},A}$ and $\mathcal{C}_{\text{NA},A}^{\mu\nu}$ are linear in ϵ^* and ϵ_j^* and depend on the momenta p_i , l and q . We do not reproduce their values as they are lengthy and not particularly illuminating. Eq. (4.35) is also valid for a massive quark with exactly the same coefficients. The vector ϵ_j^* is not assumed transverse to l as it represents the remaining part of the diagram with attachment to line j . A propagator is only included on the line with momentum $p_i + l + q$. The terms on the last line in eq. (4.36) yield scaleless integrals. Integrals over l corresponding to diagrams containing the left-hand sides of eqs. (4.35) and (4.36) are thus identical if expressed through $J_i - \mathbf{K}_i$ according to eqs. (2.9).

For the actual calculation, we have found it practical to perform a Passarino-Veltman reduction followed by an integration-by-parts reduction using KIRA [32, 33]. As a final comment, we point out that the result of integration contains the leading behavior due to j and leading and subleading behavior due to i . We have removed the leading behavior due to j in eq. (4.1). It is restored by the summation over $i \neq j$.

The second kind of contributions to $\mathbf{S}^{(1)}$ has a less obvious origin. Consider an amplitude for a process with no massless quarks and only one gluon. Application of the expansion-by-regions method in the soft-gluon limit yields

$$|M_g^{(1)}\rangle \Big|_{\text{hard region}} + |M_g^{(1)}\rangle \Big|_{\text{soft region}} = \mathbf{S}^{(0)} |M^{(1)}\rangle + \mathbf{S}^{(1)} |M^{(0)}\rangle + \text{higher-order soft-expansion terms} , \quad (4.37)$$

where the hard-region contribution is obtained by expanding diagrams in the soft-gluon momentum while keeping the dependence on the loop momentum exact. This expression suggests a one-to-one correspondence between the two terms on each side after appropriate truncation of the expansion. It turns out, however, that there is mismatch

$$|M_g^{(1)}\rangle \Big|_{\text{hard region}} \neq \mathbf{S}^{(0)} |M^{(1)}\rangle + \text{higher-order soft-expansion terms} . \quad (4.38)$$

Let us consider the process $\gamma^* \rightarrow q\bar{q}g$ with a massive quark in order to elucidate the origin of the mismatch. The relevant one-loop diagrams are depicted in figure 2. We omit the LSZ normalization factors as they are the same on both sides of (4.38). $|M_g^{(1)}\rangle \Big|_{\text{hard region}}$ is obtained by expanding the integrands of diagrams b)-h) in $q^\mu \sim \lambda$. Diagrams c) and d)

contribute at $\mathcal{O}(\lambda^0)$, b), g) and h) at $\mathcal{O}(1/\lambda)$, and e) and f) at $\mathcal{O}(1/\lambda^2)$. The validity of the leading-power soft factorization requires the sum of e), f), g) and h) to only contribute at $\mathcal{O}(\lambda^0)$. Indeed, the leading, $\mathcal{O}(1/\lambda)$, behavior of the amplitude for $\gamma^* \rightarrow q\bar{q}g$ is obtained by multiplying a) with eikonal factors which results in the leading behavior of b) without any further contributions. Let F_a denote the value of diagram a). Similarly, let F_b , F_{cd} and $F_{e\dots h}$ denote the values of diagram b), the sum of c) and d), and the sum of e)-h), each expanded to $\mathcal{O}(\lambda^0)$. With this notation

$$|M^{(0)}\rangle = F_a, \quad |M_g^{(1)}\rangle \Big|_{\text{hard region}} = F_b + F_{cd} + F_{e\dots h} + \mathcal{O}(\lambda). \quad (4.39)$$

It follows from the definition (2.5) of the tree-level soft operator that

$$F_b = \left(\mathbf{S}^{(0)} - \mathbf{S}^{(0, \text{Ward})} \right) F_a, \quad \mathbf{S}^{(0, \text{Ward})} \equiv \sum_i \mathbf{T}_i^c \epsilon^* \cdot \tilde{\partial}_i. \quad (4.40)$$

In order to obtain the rest of the contributions, we exploit the Ward identity

$$\left[F_b + F_{cd} + F_{e\dots h} \right]_{\epsilon^*=q} = 0. \quad (4.41)$$

Substitution of eq. (4.40) yields

$$\left[\mathbf{S}^{(0, \text{Ward})} \right]_{\epsilon^*=q} F_a = \left[F_{cd} + F_{e\dots h} \right]_{\epsilon^*=q}. \quad (4.42)$$

Since both $\mathbf{S}^{(0, \text{Ward})}$ and F_{cd} are independent of q , there is

$$\mathbf{S}^{(0, \text{Ward})} F_a = F_{cd} + \epsilon^* \cdot \frac{\partial}{\partial q} \left[F_{e\dots h} \right]_{\epsilon^*=q}. \quad (4.43)$$

In consequence

$$|M_g^{(1)}\rangle \Big|_{\text{hard region}} = \mathbf{S}^{(0)} |M^{(1)}\rangle + \left(F_{e\dots h} - \epsilon^* \cdot \frac{\partial}{\partial q} \left[F_{e\dots h} \right]_{\epsilon^*=q} \right) + \mathcal{O}(\lambda). \quad (4.44)$$

Although we have obtained an expression for the mismatch (4.38) using a specific example, it should be clear at this point that the result only depends on external line diagrams and is thus general. An explicit calculation yields the second contribution to $\mathbf{S}^{(1)}$

$$\begin{aligned} & \sum_i \mathbf{T}_i^c \frac{1-\epsilon}{1-2\epsilon} \left((1-2\epsilon^2)C_A + 2\epsilon(1+2\epsilon)C_F \right) \frac{A_0(m_i^2)}{m_i^2} \left(g^{\mu\nu} - \frac{2p_i^\mu p_i^\nu}{m_i^2} \right) \frac{F_{\mu\rho}}{s_{iq}} \mathbf{K}_{i\nu}{}^\rho \\ &= \sum_i \mathbf{T}_i^c \left[\left(\frac{1}{\epsilon} + \ln \left(\frac{\mu^2}{m_i^2} \right) + 2 \right) C_A + 2C_F + \mathcal{O}(\epsilon) \right] \left(g^{\mu\nu} - \frac{2p_i^\mu p_i^\nu}{m_i^2} \right) \frac{F_{\mu\rho}}{s_{iq}} \mathbf{K}_{i\nu}{}^\rho. \end{aligned} \quad (4.45)$$

The coefficient of C_F in $F_{e\dots h}$ satisfies the Ward identity on its own and the derivative contribution in eq. (4.44) vanishes. On the other hand, the correct value of the coefficient of C_A requires a non-vanishing derivative contribution. The term proportional to C_A has been included in the coefficient a_i of $\mathbf{S}_{ij}^{(\text{NA})}$ (see eq. (4.4)), because it is required to cancel a spurious divergence that would otherwise be present there. The term proportional to C_F has been kept separate in eq. (4.1).

The fact that there are non-trivial hard-region contributions in subleading-soft factorization in QED starting already at one-loop has been pointed out in refs. [25, 26].⁹ Later, in ref. [3], these effects have been related to matching coefficients in the abelian version of HQET (Heavy Quark Effective Theory). In HQET, our result (4.45) corresponds to corrections to the chromomagnetic operator which are known to three-loop order [34].¹⁰

4.4 Numerical example

In order to illustrate the usage of eq. (4.1), we perform a numerical study for two processes: 1) $e^+e^- \rightarrow t\bar{t}g$, which is the process discussed at the end of section 4.3; 2) $e^+e^- \rightarrow b\bar{b}t\bar{t}g$, which involves quarks with different masses. As in ref. [1], the one-loop n -particle amplitudes $|M^{(1)}\rangle$ as well as their derivatives are calculated with RECOLA [35, 36] linked to COLLIER [37–40] for the evaluation of tensor and scalar one-loop integrals. For the evaluation of the one-loop $(n+1)$ -particle amplitudes, $|M_g^{(1)}\rangle$, we instead link RECOLA to CUTTOOLS [41] for tensor reduction and ONELOOP [42, 43] for the evaluation of scalar integrals at quadruple precision. The quality of the approximations is demonstrated with the help of

$$\Delta_{\text{LP/NLP}} \equiv \frac{1}{N} \sum_{\substack{\text{singular} \\ \text{colour flows } \{c\} \\ \text{helicities } \{\sigma\}}} \left| \frac{[\langle \{c, \sigma\} | M_g^{(1)} \rangle - \langle \{c, \sigma\} | M_g^{(1)} \rangle_{\text{LP/NLP}}]_{\mathcal{O}(\epsilon^0)}}{[\langle \{c, \sigma\} | M_g^{(1)} \rangle]_{\mathcal{O}(\epsilon^0)}} \right|, \quad (4.46)$$

where LP (leading power) stands for soft expansion up to $\mathcal{O}(1/\lambda)$, while NLP (next-to-leading power) up to $\mathcal{O}(\lambda^0)$. The sum runs over all colour-flow and helicity configurations for which the amplitude has a soft singularity. The number of such configurations is denoted by N . The quantity $\Delta_{\text{LP/NLP}}$ is sensitive not only to color effects, but also to spin effects. It is thus better suited than a test on matrix elements summed over polarizations, which are not sensitive to the majority of spin effects. The results of numerical evaluations are presented in figure 3 and clearly confirm the correctness of the formulae at least for massive quarks and to $\mathcal{O}(\epsilon^0)$.

5 Future directions

The formulae that we have provided in this publication and in [1] are sufficient to approximate one-loop QCD amplitudes up to the first non-singular term of the soft expansion. Although we have only considered massive quarks, the result would be very similar with massive scalars. In that case, one would need to modify the single-line contributions described in section 4.3 and given for massive (anti-)quarks in eq. (4.45). In practice, our results can be used to replace the exact amplitude as long as the energy of the gluon is substantially smaller than any other momentum or mass. Clearly, future work should be directed at the emission of two or more soft gluons at one-loop, as well as at the emission of a single gluon beyond the one-loop level. Nevertheless, there remains another problem much closer to the results we have presented: what happens when the momentum of the gluon is much smaller than any other three-momenta, but comparable to the mass of a quark? Surely, this must be

⁹The C_F term at $\mathcal{O}(\epsilon^0)$ in eq. (4.45) matches the right-hand side of eq. (28) in ref. [26] with appropriate charges.

¹⁰After renormalization in the $\overline{\text{MS}}$ scheme, (4.45) is equal to the one-loop contribution in eq. (14) of ref. [34].

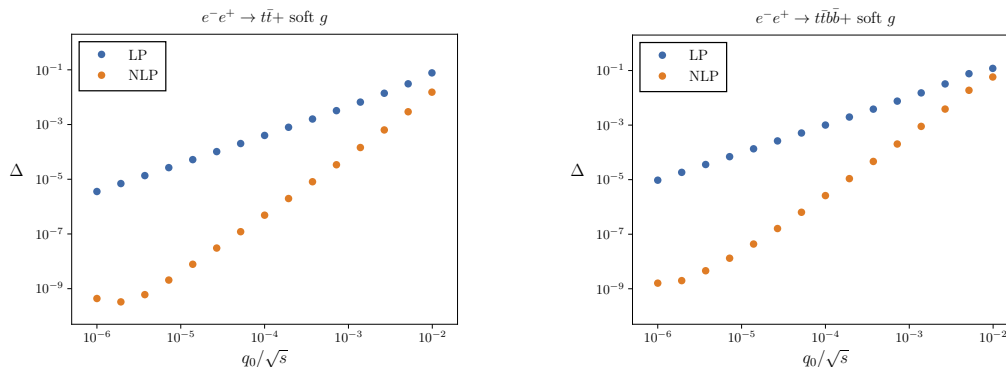


Figure 3. Relative error $\Delta_{\text{LP/NLP}}$ of the one-loop soft approximation to leading power (LP) and subleading power (NLP). The energy, q_0 , of the soft gluon is normalised to the centre-of-mass energy, $\sqrt{s}$, of the process. The apparent breakdown of the approximation at low soft-gluon energies is due to the limited numerical precision of the one-loop integrals in ONELOOP which impacts the result for the $(n + 1)$ -particle amplitudes.

described by a formula like eq. (4.1), but the jet operators should now be non-trivial functions of the energy-to-mass ratio. This case has been studied in QED very recently [44]. The determination of the relevant functions will be our next objective.

Finally, let us point out that our factorization formula (2.4) is written in a form which is not easily comparable to the operator formalism of SCET or direct diagrammatic next-to-eikonal exponentiation. This might make it more difficult to apply in an existing resummation framework. On the other hand, our general results are simpler and more compact than partial results derived previously with other methods. It seems that formalisms aiming at resummation make some simplifications less obvious. For this reason, we believe that our presentation can be a valuable starting point of further studies and might lead to novel insights, perhaps even improvements of existing resummation frameworks.

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A Subleading collinear limit of the tree-level $g \rightarrow q\bar{q}$ splitting

Let us consider a process with $n + 1$ external particles with particle i a quark and particle $n + 1$ an anti-quark. The kinematics of the process is defined by momenta $\{k_j\}_{j=1}^{n+1}$. The collinear limit under study corresponds to $l_\perp \rightarrow 0$ in the Sudakov parameterization

$$k_{n+1} \equiv xp_i + l_\perp - \frac{l_\perp^2}{2x} \frac{q}{p_i \cdot q}, \quad \text{with } l_\perp \cdot p_i = l_\perp \cdot q = q^2 = 0, \quad (\text{A.1})$$

$$k_i \equiv (1-x)p_i - l_\perp - \frac{l_\perp^2}{2(1-x)} \frac{q}{p_i \cdot q}, \quad \text{and } k_j \equiv p_j + \mathcal{O}(l_\perp^2), \quad j \neq i, j \leq n. \quad (\text{A.2})$$

The tree-level amplitude of the $(n+1)$ -parton process exhibits poles at kinematic configurations that allow for on-shell intermediate states. Hence

$$|R_I^{(0)}\rangle = \frac{1}{2p_i \cdot p_I} \frac{1}{\sqrt{x_I(1-x_I)}} \quad \text{Diagram} \quad , \quad x_I = \frac{m_I^2 - p_I^2}{2p_i \cdot p_I} . \quad (\text{A.8})$$

Here, I is a subset of at most $n-3$ of the n particles excluding i , so that the result contains two legitimate amplitudes (not splitting functions). The mass of the intermediate state has been denoted with m_I , and p_I is the sum of the momenta of the particles in I . The dashed lines represent (anti-)quarks, gluons and possibly other particles, if the theory is more general than just QCD.

We do not employ the subleading soft asymptotics to obtain the last contribution, $|C^{(0)}\rangle$, as was done in ref. [1] for the other splittings. Instead, we notice that this term corresponds to the limit $|x| \rightarrow \infty$. This limit “eikonalises” the collinear quarks, because it effectively makes them infinitely “harder” than any other particle in the process. Hence, the result is simply given by a straight eikonal line with multiple gluons attached as is typical of high energy scattering

$$|C^{(0)}\rangle = -2\delta_{-\sigma_{n+1}\sigma_i} \sum \quad \text{Diagram} \quad . \quad (\text{A.9})$$

The sum is taken over all possible tree-level diagrams (disconnected shaded circles) with at least two gluons attaching to the eikonal line. The single-gluon case is already taken care of by the leading collinear asymptotics. There is always one eikonal vertex more than there are eikonal propagators. The result for an outgoing quark-anti-quark pair follows from¹²

$$\bar{u}(p_i, \sigma_i) \gamma^\mu v(p_i, \sigma_{n+1}) = -2p_i^\mu \delta_{-\sigma_{n+1}\sigma_i} . \quad (\text{A.10})$$

The factor of i in the last eikonal vertex due to the spinor contraction comes from the fact that $|C^{(0)}\rangle$ is an amplitude.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

¹²The sign depends on spinor conventions used.

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