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Solving multi-leader-follower games via parametric optimization

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HIGHLIGHTS

- Analysis and solution of Multi-Leader–Follower Games via Multi-Parametric Optimization.
- Converting complex game structures into single optimization problems via potential functions.
- Numerical solution methods based on reformulations using Multi-Parametric and Nonlinear Optimization as well as Mixed Integer and Disjunctive Programming.

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ABSTRACT

Hierarchical Nash game models such as Multi-Leader–Follower games have recently attracted the interest of researchers as they are an important modeling tool in various applications to study a strategic decision-making process of individuals, where the individuals are split into a hierarchy of different groups. In this paper, we address these game models using a parametric optimization approach in order to reduce the number of levels to a single-level game. In particular, we further consider the potential followers' game to reduce the complexity of the problem. On the one hand, we study various mathematical structures and properties of such nonlinear hierarchical games. On the other hand, we provide suitable numerical solution approaches for the resulting single-level, but nonconvex, non-smooth or disjunctive optimization problems based on the specific theoretical properties. Finally, we finish the paper with computational results for some numerical experiments that support our approaches.

1. Introduction

Multi-Leader–Follower Games (MLFGs) are hierarchical games where multiple leaders make decisions first, anticipating optimal reactions from followers, who in turn respond to the leaders' strategies. This framework, often modeled through a Stackelberg–Nash equilibrium, combines Stackelberg leadership with Nash equilibrium to capture both sequential leader–follower interactions and simultaneous competition among leaders. MLFGs are widely applied in competitive markets, resource allocation, and strategic decision-making in economics, engineering, and environmental management, with each leader optimizing while taking into account both the responses of the followers and the strategies of rival leaders.

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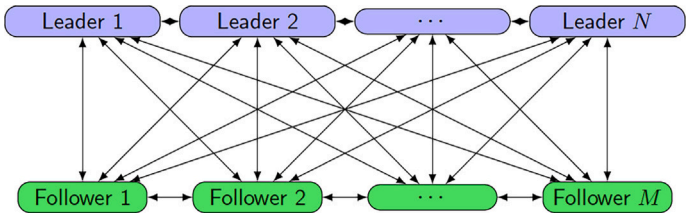


Fig. 1. Sketch of Multi-Leader–Follower Game.

A bilevel multi-leader–follower game involving N leaders in the upper-level and M followers in the lower-level, as illustrated in Fig. 1, is mathematically formulated as:

$$\left.\begin{aligned} &\min_{x_\nu \in \mathbb{R}^{n_\nu}} \ \theta_\nu(x_\nu, x_{-\nu}, y) \\ &\text{s.t.} \quad x_\nu \in X_\nu, \\ &\qquad y_j \in \underset{z \in \mathbb{R}^{m_j}}{\operatorname{argmin}} \{\, \tau_j(z, y_{-j}, x_\nu, x_{-\nu}) \mid z \in Y_j(x) \} \text{ for } j = 1, \ldots, M \end{aligned}\right\} \text{ for } \nu = 1, \ldots, N. \tag{1}$$

Lei et al. [1] modeled strategic bidding in day-ahead energy and reserve markets as a multi-leader–follower game. Distributed energy resource aggregators act as leaders, setting bidding prices and quantities to maximize profits, while an independent system operator and a distribution system operator, act as followers, clearing the markets and ensuring distribution security.

Early multi-leader–follower formulations date back to Sherali [2], who established equilibrium existence results in a Cournot setting. Later works expanded the framework to weaker existence conditions [3,4], electricity market applications [5,6], stochastic extensions [7], and generalized Nash equilibrium reformulations [8]. Further developments include robust equilibrium results for quadratic models [9] and reformulations via MPECs, nonlinear programs, and complementarity problems [10].

Despite the growing popularity of multi-leader–follower game (MLFG) models, numerical methods for their solution remain underdeveloped. Early approaches relied on diagonalization schemes such as Gauss-Jacobi and Gauss-Seidel [5], while others addressed a single nonlinear complementarity problem derived from strong stationarity conditions for MPECs [10], offering improved efficiency. Smoothing and relaxation strategies have also been proposed [3,11].

Recently, attention has shifted to reformulating MLFGs as single-level problems to leverage established optimization theory [12, 13], although such approaches often target specific structural cases. On the theoretical front, Wang et al. [14] studied convex bilevel problems with separable follower objectives, while Kulkarni [15] identified MLFG subclasses that guarantee the existence of an equilibrium and characterized solutions via potential structures—results later extended in [16–18]. Since Monderer and Shapley's foundational work on potential games [19], numerous variants, including ordinal, exact, weighted and quasi-potential games, have been adapted to MLFG settings.

Applications of these ideas include the work of Gugat, Schuster, and Steffensen [20], who modeled the European gas market as a dynamic multilevel partial differential equation-constrained optimal control problem and, by recasting participant Nash games as potential games, reduced the original four-level structure to a bilevel problem. Zewde and Kassa [21] proposed an equivalent reformulation of hierarchical MLFGs as multilevel problems with single decision makers at each level, allowing a multi-parametric solution to the resulting case of single-leader single-follower problem. Their framework handles both separable and non-separable objectives, with the latter factored into opponent-dependent and common terms—generalizing the quasi-potential model of Kulkarni and Shanbhag [17]. Later, this formulation was extended to MLFGs with convex nonlinear constraints [22]. Several formulations and solution methods for MLFGs are also discussed in [23].

Franci et al. [51] develop a mixed-integer Gauss–Seidel algorithm with global convergence for potential games, while Hori et al. [50] reformulated MLFGs as single-level variational inequalities using a smoothed followers' response function. Theoretical and numerical studies on multi-leader–follower games (MLFGs) with potential structures have also been proposed in [7,43,45].

This paper investigates a bilevel MLFG involving a potential followers' game and develops solution methodologies therefor. The remainder of the paper is organized as follows. Section 2 presents some background on MLFGs and potential games including the introduction of the followers' best response mapping and the Stackelberg-Nash equilibrium for MLFGs. In Section 3 we then establish a reformulation of the followers' problem as a generalized potential game under suitable assumptions. Next, we recast the MLFG as a Nash game using an explicit followers' solution and furthermore develop a multi-parametric approach that leads to a mixed-integer GNEP formulation. Section 4 provides some numerical illustrations of the previously mentioned approaches. Finally Section 5 finalizes the paper with some concluding remarks and future directions.

2. Multi-leader–follower game

2.1. Preliminaries

John Nash introduced the well-known concept of Nash equilibrium [24,25] in non-cooperative games involving two or more players. In such a game, called the Nash game or Nash equilibrium problem (NEP), all players are assumed to know the objective functions of other players, and make decisions to choose their own strategies at the same time by taking into account the strategies

of other players. When no player has an incentive to unilaterally deviate from their current strategy, the resulting strategy profile constitutes a Nash equilibrium.

Consider a standard Nash game denoted by a tuple $(\mathcal{N}, (X_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ where $\mathcal{N} = \{1, 2, \dots, N\}$ is the set of players, $X_i \subseteq \mathbb{R}^{n_i}$ is the fixed set of strategies for player i, $X = \prod_{i=1}^{N} X_i$ is the combined strategy set and $u_i : X \to \mathbb{R}$ is the individual utility (or pay-off) function of player i. The strategies of all players are collectively denoted by the vector $x = (x_1, \dots, x_N) = (x_i, x_{-i}) \in X \subseteq \mathbb{R}^n$ where $n = \sum_{i=1}^{N} n_i$ and $n_{-i} = n - n_i$. The player i selects a (pure) strategy $x_i \in X_i \subseteq \mathbb{R}^{n_i}$ independently of the strategies of the other players, collectively denoted by $x_{-i} \in X_{-i} \subset \mathbb{R}^{n_{-i}}$, i.e., $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$.

Given x_{-i}, player i hence solves the following optimization problem with respect to his own decision variable x_i:

$$\begin{aligned} &\min_{x_i} u_i(x_i, x_{-i}) \\ &s.t.\ \ x_i \in X_i. \end{aligned} \tag{2}$$

Definition 2.1 (Nash equilibrium). A point $x^* = (x_i^*, x_{-i}^*) \in X = \prod_{i=1}^{N} X_i$ is called a Nash Equilibrium of the strategic game $(\mathcal{N}, (X_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ if

$$u_i(x_i^*, x_{-i}^*) \le u_i(x_i, x_{-i}^*) \quad \forall x_i \in X_i, \forall i \in \mathcal{N}. \tag{3}$$

Theorem 2.2 ([26]). *Let $X_i \subseteq \mathbb{R}^{n_i}$, $(i \in \mathcal{N})$ be nonempty, convex, and compact. If the function $u_i : X \to \mathbb{R}$ is continuous in x and quasi-convex in x_i for all $i \in \mathcal{N}$, then there exists a pure strategy Nash equilibrium.*

In standard Nash games the players influence each other only through their utility functions u_i while their feasible sets remain independent. However, many practical settings involve shared resources, such as budgets or infrastructure, which restrict the strategies of all players and lead to coupled feasible sets. This shift in modeling was introduced by Debreu [26] and later extended by Arrow and Debreu [27] and Rosen [28], resulting in game frameworks in which the admissible strategies of each player also depend on the strategies of others.

A generalized game (also called an abstract economy due to Debreu's work in [26,27]) is described as a tuple $(\mathcal{N}, (K_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ where $K_i : X_{-i} \rightrightarrows X_i$ is a set-valued mapping from X_{-i} to a subset of X_i and $u_i : X = \prod_{i=1}^{N} X_i \to \mathbb{R}$ is the utility function of player i. The set $K_i(x_{-i}) \subseteq \mathbb{R}^{n_i}$ defines the ith player's feasible set given the other players' strategy vector x_{-i}. Typically, it is described explicitly as

$$K_i(x_{-i}) = \left\{ x_i \in X_i \mid g_i(x_i, x_{-i}) \le 0 \right\}, \tag{4}$$

where $g_i : X \to \mathbb{R}^{q_i}$ is a vector-valued constraint function. If g_i does not depend on x_{-i} for any $i \in \mathcal{N}$, we obtain the standard Nash game. In other words if the feasible sets $K_i(x_{-i})$ do not depend on the rival players' strategies, such that we have $K_i(x_{-i}) = X_i$ for some fixed set $X_i \subseteq \mathbb{R}^{n_i}$ and all $i \in \mathcal{N}$ then the GNEP reduces to a standard NEP.

Definition 2.3. [Generalized Nash equilibrium] A strategy vector $x^* = (x_i^*, x_{-i}^*)$ is called a generalized Nash equilibrium for the game $(\mathcal{N}, (K_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ if for all $i \in \mathcal{N}$

$$x_i^* \in \underset{x_i \in K_i(x_{-i}^*)}{\operatorname{argmin}} u_i(x_i, x_{-i}^*). \tag{5}$$

Alternatively, the generalized Nash equilibrium can be characterized by defining the best-response of the player i in a strategic form game as a set-valued mapping $\gamma_i : X_{-i} \rightrightarrows X_i$

$$\gamma_i(x_{-i}) = \underset{x_i \in K_i(x_{-i})}{\operatorname{argmin}} u_i(x_i, x_{-i}).$$

Note that for each feasible $x_{-i} \in X_{-i}$, the set $\gamma_i(x_{-i})$ may or may not be a singleton. Let the joint best-response set be defined as

$$\gamma(x) = \{ x \in X \mid \forall i \in \mathcal{N} : \ x_i \in \gamma_i(x_{-i}) \},$$

then, a joint strategy vector x^* is a generalized Nash equilibrium if and only if it satisfies the fixed point condition

$$x^* \in \gamma(x^*).$$

The next result by Dutang specifies some conditions on the GNEP that guarantee the existence of a generalized Nash equilibrium.

Theorem 2.4 (Dutang [46]). *Let N players be characterized by an action space X_i, a set-valued mapping K_i from X_{-i} to a subset of X_i, and an objective function $u_i : X \to \mathbb{R}$. Assume that for all players $i \in \mathcal{N}$ we have*

(i) the set X_i is a nonempty, convex and compact subset of a Euclidean space,
(ii) the mapping K_i is both upper and lower semi-continuous in X_{-i},
(iii) for all $x_{-i} \in X_{-i}$ the set $K_i(x_{-i})$ is nonempty, closed and convex,
(iv) u_i is continuous on the graph $Gr(K_i)$,[1]

[1] *$Gr(K_i)$ denotes the graph of a set-valued mapping K_i, defined by $Gr(K_i) = \{(x_{-i}, x_i) \in X_{-i} \times X_i : x_i \in K_i(x_{-i})\}$.*

(v) $\forall x \in X,\ x_i \to u_i(x_i, x_{-i})$ *is quasi-convex on* $K_i(x_{-i})$.

Then there exists a generalized Nash equilibrium for the associated generalized Nash game.

A multitude of approaches have been developed and analyzed to solve GNEPs. For detailed reviews of generalized Nash equilibrium problems and related methods, please see the surveys, e.g., in [29].

2.2. Potential game

A strategic game is called a potential game if it admits a potential function that captures the impact of a player's unilateral deviation. The potential functions provide a unified measure of player incentives, capturing disagreement or drift towards a Nash equilibrium. They can replace the individual utility functions while preserving essential game properties, including Nash equilibria and best responses. In particular, if a finite game (*i.e.*, with a finite number of players and strategy sets) admits a potential function, then under suitable conditions this function attains a global optimum, guaranteeing the existence of at least one pure-strategy Nash equilibrium. Since being introduced by Monderer and Shapley [19] in 1996, potential games have received increasing attention and various notions of potential games including exact potential, weighted potential, ordinal potential, generalized ordinal potential, best-response potential and pseudo-potential games have been introduced and studied in the literature (see *e.g.* [19,47,48]).

Definition 2.5 (Potential game). A standard Nash game $(\mathcal{N}, (X_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ is called a *potential game* if there exists a (potential) function $\Phi : X \to \mathbb{R}$ such that for each $i \in \mathcal{N}$ and $x_i, x_i' \in X_i$ and $x_{-i} \in X_{-i}$

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = \Phi(x_i, x_{-i}) - \Phi(x_i', x_{-i}). \tag{6}$$

A finite potential game always has a Nash equilibrium. Also, if each X_i is compact and u_i is a continuous function, such an exact potential game also possesses a Nash equilibrium.

Note that any minimizer of $\Phi(x)$ is a Nash equilibrium; however, the converse does not generally hold (see [19,33,43,44] for counterexamples and further discussion on the relationship between Nash equilibria of potential games and minimizers of a potential function).

Definition 2.6 (Weighted potential game). A game $(\mathcal{N}, (X_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ is a *weighted potential game* if there exists a (weighted potential) function $\Phi : X \to \mathbb{R}$ and a positive weight vector $(w_i)_{i\in\mathcal{N}}$ such that for each $i \in \mathcal{N}$ and $x_i, x_i' \in X_i$ and $x_{-i} \in X_{-i}$

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = w_i(\Phi(x_i, x_{-i}) - \Phi(x_i', x_{-i})).$$

The corresponding potential game for the GNEP was introduced in Definition 2.1 of [30] and is stated as follows.

Definition 2.7 (Generalized Potential Game). A GNEP $(\mathcal{N}, (K_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ is a generalized potential game if:

(a) There exists a nonempty, closed set $D \subseteq \mathbb{R}^n$ such that, for all $i = 1, \dots, N$ it holds

$$K_i(x_{-i}) = \left\{x_i \in X_i : (x_i, x_{-i}) \in D\right\}, \tag{7}$$

where $X_i \subseteq \mathbb{R}^{n_i}$ are nonempty, closed sets such that $\prod_{i=1}^N X_i \cap D \neq \emptyset$ (*i.e.*, the "feasible set" of the game is non-empty).

(b) There exists a continuous function $\Phi : \mathbb{R}^n \to \mathbb{R}$ such that for all i, for all x_{-i} (such that $K_i(x_{-i})$ is non-empty) and for all $y_i, z_i \in K_i(x_{-i})$

$$u_i(y_i, x_{-i}) - u_i(z_i, x_{-i}) > 0$$

implies

$$\Phi(y_i, x_{-i}) - \Phi(z_i, x_{-i}) \geq \sigma(u_i(y_i, x_{-i}) - u_i(z_i, x_{-i})), \tag{8}$$

where $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a forcing function:

$$\lim_{k\to\infty} \sigma(t_k) = 0 \implies \lim_{k\to\infty} t_k = 0.$$

Note that in the above definition if (i) in condition (b) a function Φ exists such that (8) holds with equality and with the forcing function σ equal to the identity and (ii) the set D is given by the Cartesian product of lower dimensional sets X_i, i.e., $D = X = \prod_{i=1}^N X_i$, with $X_i \subseteq \mathbb{R}^{n_i}$, then the generalized potential game GPG is just an exact potential NEP as defined by Monderer and Shapley [19].

If the set D in condition (a) of Definition 2.7 is defined as an inequality constraint $D = \{x \in \mathbb{R}^n : g(x) \leq 0\}$ where $g : \mathbb{R}^n \to \mathbb{R}^q$ is continuous, then the feasible set in (7) can be described as

$$K_i(x_{-i}) = \left\{x_i \in X_i : g(x_i, x_{-i}) \leq 0\right\}. \tag{9}$$

In the cases where the X_i are convex and the g_ℓ are jointly convex with respect to all variables x, then the aggregated set $K(x)$ will become convex.

Condition (b) of Definition 2.7 states that there is a single function Φ that, in some sense, reflects the changes in the players' objective functions. It easily follows that a global solution to the following optimization problem

$$\min_x \Phi(x) \quad \text{s.t.} \quad x \in D, \quad x_i \in X_i,\ i = 1, \dots, N \tag{10}$$

yields a Nash equilibrium [30].

If (i) the objective functions either do not depend on the other players' variables, $u_i(x) = u_i(x_i)$ or (ii) the objective functions have a common term β which is the same for all players plus an additional individual cost related only to x_i, i.e., $u_i(x) = \beta(x) + \alpha_i(x_i)$, then condition (b) of Definition 2.7 is satisfied with the potential functions $\Phi(x) = \sum_{i=1}^{N} u_i(x_i)$ or $\Phi(x) = \beta(x) + \sum_{i=1}^{N} \alpha_i(x_i)$, respectively.

Some practical applications whose formulation give rise to a GPG are discussed in [30]. Another example of a potential GNEP structure arises in the economic multilevel modeling of the European gas market as, e.g., discussed in [20,31].

2.3. Problem description of MLFG

Let the leaders' game be a NEP that consists of N players and is given by

$$\min_{x_\nu \in \mathbb{R}^{n_\nu}} \theta_\nu(x_\nu, x_{-\nu}, y) \quad \text{s.t.} \quad x_\nu \in X_\nu \subseteq \mathbb{R}^{n_\nu}, \qquad \text{for} \qquad \nu = 1, \dots, N \tag{11}$$

where $\theta_\nu : X \times Y \to \mathbb{R}$ (with $X = \prod_{\nu=1}^{N} X_\nu$) is the objective function of leader ν which depends on both his own strategy variables x_ν as well as on the vector $x_{-\nu}$ of all other leaders' strategies and furthermore the followers' joint strategy vector y. In case of an upper-level GNEP we replace the feasible set X_ν of player ν by $S_\nu(x_{-\nu}) \subseteq X_\nu \subseteq \mathbb{R}^{n_\nu}$ such that it might also depend on the rival players' strategies. However, we assume it to be defined by inequality constraints such as

$$S_\nu(x_{-\nu}) = \left\{ x_\nu \in X_\nu \mid g_\nu(x_\nu, x_{-\nu}) \leq 0 \right\}, \tag{12}$$

where $g_\nu(\cdot, x_{-\nu}) : \mathbb{R}^{n_\nu} \to \mathbb{R}^{p_\nu}$.

Furthermore, the joint strategy vector of the followers $y = (y_j, y_{-j}) \in \mathbb{R}^m$, where $m = \sum m_j$, is determined through a Nash game at the lower-level modeled by the following set of optimization problems:

$$\min_{y_j \in \mathbb{R}^{m_j}} \quad \tau_j(y_j, y_{-j}, x) \quad \text{s.t.} \quad y_j \in Y_j(x), \qquad \text{for } j = 1, \dots, M, \tag{13}$$

where each set of strategies $Y_j(x) \subseteq \mathbb{R}^{m_j}$ is supposed to be nonempty, closed and convex and generally described by inequality constraints, that is $Y_j(x) = \left\{ y_j \in \mathbb{R}^{m_j} \mid g_j(y_j, x) \leq 0 \right\}$. Moreover, let $Y(x)$ be the aggregated feasible set of the followers' joint strategy vector y.

The corresponding multi-leader–follower game is then described by (1). Note that here we restrict ourselves to the case of a standard NEP that is parameterized by the upper-level (or leaders') strategy vector x.

Now, fix $\bar{x} \in X$ and consider the followers' game defined in (13). According to Definition 2.1, a joint strategy vector $y^* \in Y(\bar{x})$ is said to be a Nash equilibrium of the lower-level problem (13) if for all $j = 1, \dots, M$ and all $y_j \in Y_j(\bar{x})$ the following condition is satisfied:

$$\tau_j(y_j^*, y_{-j}^*, \bar{x}) \leq \tau_j(y_j, y_{-j}^*, \bar{x}). \tag{14}$$

This observation will now be further discussed and used to describe the equilibria of the MLFG (1).

2.4. Stackelberg–Nash equilibrium

Consider the followers' problem (13). Given any fixed leaders' strategy vector $\bar{x} \in X$ (or an element of the aggregation of (12)), we can define the follower j's optimal decision mapping for all strategy vectors y_{-j} (i.e., for all followers except j) by $\mathcal{R}_j^F(\cdot, \bar{x}) : Y_{-j}(\bar{x}) \rightrightarrows Y_j(\bar{x})$ (with $Y_{-j}(\bar{x})$ being the aggregation of all sets $Y_k(\bar{x})$ except for $k = j$)

$$\mathcal{R}_j^F(y_{-j}, \bar{x}) = \operatorname*{argmin}_{z \in \mathbb{R}^{m_j}} \left\{ \tau_j(z, y_{-j}, \bar{x}) \mid z \in Y_j(\bar{x}) \right\}. \tag{15}$$

Thus given $\bar{x} \in X$ a condition to characterize a Nash equilibrium of the lower-level NEP is given by

$$y_j \in \mathcal{R}_j^F(y_{-j}, \bar{x}) \qquad \forall\, j \in \{1, .., M\}.$$

Now, this condition can be used to define the so-called followers' best response mapping which determines the (set of) best or equilibrium replies of the followers for any feasible leaders' strategy choice x.

Definition 2.8 (Followers' Best Response Mapping). Given the leaders $\nu = 1, .., N$ and furthermore the followers $j = 1, .., M$. Let the feasible sets of the leaders be given by $X = \prod_{\nu=1}^{N} X_\nu \subseteq \mathbb{R}^n$. Moreover, let for any $x \in X$ the feasible set of the followers be given by $Y(x) = \prod_{j=1}^{M} Y_j(x) \subseteq \mathbb{R}^m$. Let for any $x \in X$ the follower's individual utility function be given by $\tau_j(\cdot, x) : \mathbb{R}^m \to \mathbb{R}$. Then we define the Followers' Best Response Mapping $\mathcal{BR} : X \rightrightarrows Y(x)$ by

$$\mathcal{BR}(x) := \{\, y \in Y(x) \mid y_j \in \mathcal{R}_j^F(y_{-j}, x) \quad \forall\, j \in \{1, .., M\} \,\}.$$

Note that $\mathcal{BR}$ denotes a set-valued mapping. Hence, in general for any $x \in X$ the value $\mathcal{BR}(x)$ might be a subset of the parameterized feasible set $Y(x)$ which is also known as the followers' optimal reaction set [49]. Furthermore, if for some $\bar{x}$ for all $j = 1, .., M$

the set $\mathcal{R}_j^F(y_{-j}, \bar{x})$ is single-valued, so is $\mathcal{BR}(\bar{x})$. Moreover, if this assumption holds true for all feasible $x \in X$, then we replace the set-valued mapping $\mathcal{BR}$ with the function $\vartheta : X \to Y(x)$ that assigns each leaders' strategy choice $\bar{x}$ the unique followers' optimal response $\bar{y}$ so that $\vartheta(\bar{x}) = \bar{y}$, where $\mathcal{BR}(\bar{x}) = \{\bar{y}\}$.

Next, we can also define the optimal decision mapping on the upper-level of the game. The optimal decision of the leader ν is defined by $\mathcal{R}_\nu^L : X_{-\nu} \rightrightarrows X_\nu$ with

$$\mathcal{R}_\nu^L(x_{-\nu}) = \underset{z_\nu \in X_\nu}{\operatorname{argmin}} \left\{ \theta_\nu(z_\nu, x_{-\nu}, y) \mid y \in \mathcal{BR}(z_\nu, x_{-\nu}) \right\} .$$

Note, that in case where the followers' best response mapping is not single-valued this definition for the optimal reaction of the leaders corresponds to what is known as optimistic case (as the leaders can choose freely among the followers' responses $y \in \mathcal{BR}(x)$). Moreover, it is not assumed that each leader chooses the same followers' response $y \in \mathcal{BR}(x)$. In the context of Stackelberg games (i.e., games that incorporate a single leader with multiple followers on the lower level) various regularization methods have been proposed in the literature to overcome such drawbacks (see *e.g.* in [49]).

Definition 2.9 (Stackelberg-Nash Equilibrium). Given the conditions of Definition 2.8, a strategy profile $(x^*, y^*) \in X \times \mathbb{R}^m$ is said to be a Stackelberg-Nash Equilibrium (SNE) of the MLFG (1) if the following conditions hold:

(i) For all $\nu = 1, \dots, N$:

$$x_\nu^* \in \mathcal{R}_\nu^L(x_{-\nu}^*)$$

(ii) $y^* \in \mathcal{BR}(x^*)$.

Notice, that in the single-valued case the flexibility for the leaders and the followers to choose an element of $\mathcal{BR}(x^*)$ vanishes and therefore the complexity of the bilevel game reduces significantly, as the additional variable y^* can be replaced by $\vartheta(x^*)$.

Remark 2.10. The bilevel MLFG (1) displays two partial games, namely the lower-level game (13) and the upper-level game (11). The equilibrium of the entire game (1) is a pure strategy subgame perfect Nash equilibrium, while the equilibria in each partial game are Nash equilibria [32].

3. Solution strategies for bilevel MLFGs with a potential lower-level game

In this section we now discuss different solution approaches for bilevel MLFGs with potential followers' games. In case the followers' NEP in MLFG (1) has an explicit solution (1) can be reformulated as a GNEP of the upper-level variables only. The presentation is divided into three parts: the first part introduces the structural assumptions on the objective functions defining the followers' Nash game in order to obtain a potential problem on the lower-level. The second part then examines explicit solutions for quadratic potential followers' games, while the third part introduces a multi-parametric method applicable to general potential followers' games.

3.1. Followers' generalized potential game

Assumption 1. Given $x \in X$, assume that for all $j = 1, \dots, M$ the followers' objective functions in (13) can be described as

$$\tau_j(y, x) = \alpha_j(y_j, x) + \bar{\alpha}_j(y_{-j}, x) + \lambda_j(x)\beta(y, x) \quad \text{for all } y \in Y(x), \tag{16}$$

where $\tau_j(\cdot, x)$, $\beta_j(\cdot, x) : Y(x) \to \mathbb{R}$, $\alpha_j(\cdot, x) : Y_j(x) \to \mathbb{R}$, $\bar{\alpha}_j(\cdot, x) : Y_{-j}(x) \to \mathbb{R}$ and $\lambda_j : X \to \mathbb{R}_{>0}$.

Theorem 3.1. *If Assumption 1 holds for all $x \in X$, then there exists a weighted potential function $\pi(y, x)$ such that for all $y_j, y_j^* \in Y_j(x)$ and $y_{-j} \in Y_{-j}(x)$ it holds*

$$\tau_j(y_j^*, y_{-j}, x) - \tau_j(y_j, y_{-j}, x) = \lambda_j(x)[\pi(y_j^*, y_{-j}, x) - \pi(y_j, y_{-j}, x)] . \tag{17}$$

Proof. The proof follows the same ideas as the proof of Lemma 3.4 in [22]. □

Remark 3.2. Note that when $\lambda_j(x)$ is a positive constant function for all $j = 1, \dots, M$, then problem (13) coincides with the quasi-potential game considered in [17,18,33].

The follower's potential problem will be

$$P(x) : \quad \min_{y \in \mathbb{R}^m} \quad \pi(y, x) \quad \text{s.t.} \quad y \in Y(x), \tag{18}$$

where $Y(x) = \{y \in \mathbb{R}^m \mid g(y, x) \leq 0\}$, $\pi(\cdot, x) : \mathbb{R}^m \to \mathbb{R}$ and $g(\cdot, x) : \mathbb{R}^m \to \mathbb{R}^p$.

Theorem 3.3. *Let $\bar{x} \in X$ be fixed. If $y^* \in \mathbb{R}^m$ is an optimal solution of (18), then y^* is a Nash equilibrium of (13).*

Proof. Fix $\bar{x}$ and apply Lemma 2.1 in [19]. □

Assumption 2. Given $x \in X$, assume the following conditions hold for the followers' problem:

(i) $\tau_j(\cdot, x)$ is continuous in y for all $j = 1, .., M$.

(ii) $Y_j(x)$ is nonempty and compact for all $j = 1, \dots, M$.

Remark 3.4. Under the Assumptions 2, the potential problem $P(x)$ admits a global minimizer and hence there exists a Nash equilibrium for (13). Moreover, if the feasible sets $Y_j(x)$ are convex for all $j = 1, .., M$ and in addition the associated potential function $\pi(\cdot, x)$ is strictly convex, then $P(x)$ admits a unique solution and thus $\mathcal{BR}(x)$ is single-valued. Hence, if such stronger conditions hold true for all $x \in X$, we can replace the set-valued best response mapping $\mathcal{BR}$ with the followers' best response function $\vartheta : X \to \mathbb{R}^m$.

Furthermore, if we have additional differentiability properties for π, we can characterize the global minimizer of (13) in terms of stationarity properties such as, e.g., variational inequalities as is discussed in [13].

3.2. Explicit solution for a quadratic follower's problem

Consider the followers' NEP with quadratic objective functions and polyhedral constraints

$$\left.\begin{array}{ll} \min\limits_{y_j} & [\frac{1}{2} y_j^T Q_F^j y_j - b_F^j(x)^T y_j] + [\frac{1}{2} y^T Q y - b(x)^T y] \\ \text{s.t.} & y_j - c_j(x) \geq 0 \end{array}\right\} \quad \text{for} \quad j = 1, \dots, M \tag{19}$$

where $Q_F^j \in \mathbb{R}^{m_j \times m_j}$, $b_F^j(x) \in \mathbb{R}^{1 \times m_j}$, $b(x) \in \mathbb{R}^{1 \times m}$, $c(x) \in \mathbb{R}^{1 \times m}$, $Q \in \mathbb{R}^{m \times m}$ for $m = \sum_{j=1}^M m_j$ and $Y_j(x) = \{y_j \in \mathbb{R}^{m_j} \mid y_j - c_j(x) \geq 0\}$. The problem (19) satisfies the Assumption 1 and hence admits a potential function which is given by

$$\pi(y, x) = \sum_{j=1}^{M} [\frac{1}{2} y_j^T Q_F^j y_j - b_F^j(x)^T y_j] + [\frac{1}{2} y^T Q y - b(x)^T y].$$

Therefore, the potential problem $P(x)$ for (19) will be

$$\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \pi(y, x) \\ \text{s.t.} & y - c(x) \geq 0\,. \end{array} \tag{20}$$

Remark 3.5. The problem structure in (19) generalizes the formulations in [12,34] in such a way that the analytical discussion in [12,34] has been conducted assuming just a single follower in the lower-level problem with a remark on possible extensions to a multi-follower case with separable objective functions. In contrast to this, the expression in (19) also contains $\frac{1}{2} y^T Q y$ which is a non-separable common term across all the objective functions with respect to the followers' decision variable. Moreover, in [12] each of the involved matrices Q_y^j is assumed to be positive definite and diagonal, whereas here we assume that the joint matrices Q and Q_F are block diagonal (with respect to the followers' dimensions), where each block is a symmetric and positive matrix and the matrix $(Q + Q_F)$ is regular and contains only nonnegative elements.

Cesarone et al. [35] developed a bilevel reformulation of a single-leader multi-follower (SLMF) game for multi-portfolio selection with Environmental, Social, and Governance criteria. They demonstrated that when the objectives of the followers are differentiable, quadratic, and convex, the equilibria align with stationary points of a convex potential problem. Likewise, Lampariello et al. [36] showed that a quadratic, convex, potential-based lower-level NEP in portfolio selection reduces to a hierarchical SLMF game and into a standard bilevel optimization problem.

Lemma 3.6. *Assume that problem* (13) *is given by* (19) *and for all* $j = 1, \dots, M$, Q_F^j *and* Q^j *are symmetric positive definite matrices and* $(Q + Q_F)$ *is a regular matrix with nonnegative elements. Then the potential problem of the follower* (20) *has a unique solution* $\vartheta(x)$ *for any given leader's strategy vector* $x \in \mathbb{R}^n$. *In particular,* $\vartheta(x)$ *is given by the function*

$$\vartheta(x) = \max\left\{(Q_F + Q)^{-1}(b_F(x) + b(x)),\ c(x)\right\} \tag{21}$$

where Q *and* Q_F *are block diagonal matrices given by* $Q = diag(Q^1, \dots, Q^M)$ *and* $Q_F = diag(Q_F^1, \dots, Q_F^M) \in \mathbb{R}^{m \times m}$, *and* $b : \mathbb{R}^n \to \mathbb{R}^m$, $b_F : \mathbb{R}^n \to \mathbb{R}^m$ *and* $c : \mathbb{R}^n \to \mathbb{R}^m$ *are vector-valued functions with* $b(x)$ *and* $b_F(x)$ *being block column matrices defined by* $b^T(x) = (b^1(x), \dots, b^m(x)) \in \mathbb{R}^{1 \times m}$, $b_F^T(x) = (b_F^{1,T}(x), \dots, b_F^{M,T}(x)) \in \mathbb{R}^{1 \times m}$ *and* $c^T(x) \in \mathbb{R}^{1 \times m}$ *for any* $x \in \mathbb{R}^n$.

Proof. Given the assumptions, the objective is a strictly convex function, the KKT conditions are necessary and sufficient for a global minimizer and the minimizer is unique for all leader strategies x.

Define the Lagrangian of (20) by

$$\mathcal{L}(y, \lambda, x) = \pi(y, x) - \lambda^T (y - c(x))\,.$$

where $\lambda \in \mathbb{R}^m$ is the Lagrangian. The KKT conditions (which are also necessary and sufficient for a global minimizer) for (20) are

(i) $\nabla_y \mathcal{L} = (Q_F + Q)y - (b_F(x) + b(x)) - \lambda = 0$
(ii) $0 \leq \quad \lambda \ \perp \ y - c(x) \quad \geq 0$

Plugging the expression $\lambda = (Q_F + Q)y - (b_F(x) + b(x))$ of the Lagrangian from (i) into condition (ii) gives

$$0 \leq \quad (Q_F + Q)y - (b_F(x) + b(x)) \;\perp\; y - c(x) \quad \geq 0\,.$$

Next, we can replace the inequality $(Q_F + Q)y - (b_F(x) + b(x)) \geq 0$ as follows:

$$(Q_F + Q)y - (Q_F + Q)(Q_F + Q)^{-1}(b_F(x) + b(x)) \geq 0\,.$$

Taking out the common term, we get

$$(Q_F + Q)[y - (Q_F + Q)^{-1}(b_F(x) + b(x))] \geq 0\,.$$

Since by our assumptions $(Q_F + Q)^{-1}$ is a matrix of monotone kind[2] we can multiply the above inequality with $(Q_F + Q)^{-1}$ and hence replace it by

$$y - (Q_F + Q)^{-1}(b_F(x) + b(x)) \geq 0.$$

Next, we use this to further simplify condition (ii) to the complementarity problem

$$0 \leq \quad y - (Q_F + Q)^{-1}(b_F(x) + b(x)) \;\perp\; y - c(x) \quad \geq 0$$

which can be rewritten equivalently as (using an NCP function as the *minimum function*)

$$0 = \min\{\, y - (Q_F + Q)^{-1}(b_F(x) + b(x)),\; y - c(x)\}\,.$$

Extracting y, we obtain

$$0 = y + \min\{-(Q_F + Q)^{-1}(b_F(x) + b(x)), -c(x)\}\,.$$

Therefore, we obtain

$$\vartheta(x) = \max\left\{(Q_F + Q)^{-1}(b_F(x) + b(x)),\; c(x)\right\}\,.$$

□

Note that the inequalities and complementarity conditions as well as both the min and max operators are applied component-wise to a vector and furthermore, that Q_F and Q are positive definite and symmetric.

Remark 3.7. Hence, if the followers' problems are quadratic problems defined as in (20), then the followers' optimal reaction sets $\mathcal{BR}(x)$ are single-valued for each x and we can replace the optimal response mapping $\mathcal{BR}$ with the optimal response function ϑ for which in this case by (21) we even have an explicit expression. This can furthermore be used to reduce the bilevel problem (1) to a single-level problem which is either a NEP or a GNEP (depending on the feasible sets $S_\nu(x_{-\nu})$ or X_ν, respectively):

$$\left.\begin{array}{l}\min\limits_{x_\nu} \tilde{\theta}_\nu(x_\nu, x_{-\nu}) \\ \text{s.t.} \quad x_\nu \in S_\nu(x_{-\nu})\end{array}\right\} \text{ for } \quad \nu = 1, \ldots, N, \tag{22}$$

where $\tilde{\theta}_\nu(x) = \theta_\nu(x, \vartheta(x))$ for $\vartheta(x) = \max\left\{(Q_F + Q)^{-1}(b_F(x) + b(x)),\; c(x)\right\}$.

Problem (22) can then be solved by standard solution methods for an NEP such as a Gauss–Seidel type method as, e.g., proposed by Facchinei et al. [30] or GNEPs, respectively.

Moreover, in case the leaders' problem (11) also admits a potential function $\Phi(x, y)$, such that (22) becomes an optimization problem with variable $x \in X$ we obtain a nonlinear (possibly nonsmooth) optimization problem given by

$$\begin{array}{l}\min\limits_{x \in \mathbb{R}^n} \bar{\Phi}(x) \\ \text{s.t.} \quad x \in X,\end{array} \tag{23}$$

where $\bar{\Phi}(x) = \Phi(x, \vartheta(x))$ and $\vartheta(x) = \max\left\{(Q_F + Q)^{-1}(b_F(x) + b(x)),\; c(x)\right\}$. This problem can then be solved using any existing (nonsmooth) nonlinear optimization solver.

Remark 3.8. If the components $b_{F,j}(\cdot)$, $b_j(\cdot)$ and $c_j(\cdot)$ of the functions b, b_F and c are individually convex in x, then the components ϑ_j of best response function $\vartheta(x)$ are also convex in x as a component-wise maximum of a positive sum of convex functions (under the assumption that $(Q_F + Q)$ has only nonnegative entries). This property can be used to develop conditions such that the reduced NEP or GNEP admits an equilibrium.

Definition 3.9. Let $\theta : \mathbb{R}^m \to \mathbb{R}$ be a continuous function. We call θ unilaterally monotone increasing, if for any unilateral change of variables, i.e., the evaluation of θ in $y_z = (y_1, .., y_{j-1}, z, y_{j+1}, .., y_m)$ with $z \in \mathbb{R}$ instead of y_j where $z \leq y_j$ for some $j \in \{1, \ldots, m\}$ it holds that $\theta(y_z) \leq \theta(y)$.

[2] For a real matrix $A = (a_{ij})$ of monotone kind there follows $x \geq 0$ from $Ax \geq 0$, that is, $\det(A)$ is different from zero (the symbol $\geq$ with vectors is understood to apply to each component). Collatz [37] has shown that for a matrix A of monotone kind it is necessary and sufficient that all elements of the inverse matrix A^{-1} are nonnegative.

Lemma 3.10. *Let the assumptions of Lemma 3.6 hold and assume further that $(Q_F + Q)$ has only nonnegative entries. Moreover, assume that $\theta_\nu(x, y)$ is (quasi-) convex in (x, y) and unilaterally monotone increasing in y. Moreover, assume that the components of b_F, b, and c are convex in x. Then $\tilde{\theta}_\nu$ is (quasi-)convex in x.*

Proof. By assumption the components of the function $(Q_F + Q)^{-1}(b_F(x) + b(x))$ and of $c(x)$ are convex in x. Hence, the components of the best response function $\vartheta(x)$ are also convex in x as a component-wise maximum of convex functions. To show the (quasi-)convexity of $\tilde{\theta}_\nu(x)$, let $x, \bar{x}$ be feasible and assume that any convex combination of them is also feasible. Let $\lambda \in (0, 1)$ and define

$$\bar{y} = \vartheta(\lambda x + (1-\lambda)\bar{x})$$
$$\tilde{y} = \lambda\,\vartheta(x) + (1-\lambda)\,\vartheta(\bar{x}).$$

By the convexity of the components of ϑ, it follows that $\bar{y} \le \tilde{y}$ (component-wise). Next define the vectors

$$y_{[j]} = (\bar{y}_1, \dots, \bar{y}_j, \tilde{y}_{j+1}, \dots, \tilde{y}_m) \qquad \text{for} \quad j = 1, \dots, m-1$$

and moreover $y_{[0]} = \tilde{y}$ and $y_{[m]} = \bar{y}$. Due to the monotonicity assumptions on θ_ν we have

$$\theta_\nu(\lambda x + (1-\lambda)\bar{x}, y_{[j]}) \le \theta_\nu(\lambda x + (1-\lambda)\bar{x}, y_{[j-1]}) \qquad \text{for} \quad j = 1, \dots, m.$$

Therefore, $\theta_\nu(\lambda x + (1-\lambda)\bar{x}, y_{[m]}) \le \theta_\nu(\lambda x + (1-\lambda)\bar{x}, y_{[0]})$, so that by the (quasi-)convexity we obtain

$$\begin{aligned}
\tilde{\theta}_\nu(\lambda x + (1-\lambda)\bar{x}) &= \theta_\nu(\lambda x + (1-\lambda)\bar{x}, \vartheta(\lambda x + (1-\lambda)\bar{x})) \\
&= \theta_\nu(\lambda x + (1-\lambda)\bar{x}, \bar{y}) \\
&\le \theta_\nu(\lambda x + (1-\lambda)\bar{x}, \tilde{y}) \\
&= \theta_\nu(\lambda x + (1-\lambda)\bar{x}, \lambda\,\vartheta(x) + (1-\lambda)\,\vartheta(\bar{x})) \\
&\le \max(\theta_\nu(x, \vartheta(x)), \theta_\nu(\bar{x}, \vartheta(\bar{x}))) \\
&= \max(\tilde{\theta}_\nu(x), \tilde{\theta}_\nu(\bar{x})).
\end{aligned} \tag{24}$$

If the quasi-convexity of θ_ν in (x, y) is replaced by (strict) convexity in (x, y) then the last two equations in (24) have to be replaced accordingly and will yield the result. □

Theorem 3.11. *Let the assumptions of Lemma 3.10 and Theorem 2.4 hold. Then the MLFG (1) with lower-level given by (19) has a Stackelberg-Nash equilibrium $(x^*, \vartheta(x^*))$ where x^* is a (generalized) Nash equilibrium of (22) and $\vartheta(x^*)$ is the unique Nash equilibrium of (19) given by (21).*

Proof. By the assumptions of Lemma 3.10 and Theorem 2.4, problem (22) admits a Nash equilibrium x^* such that for any fixed $\bar{y}$ and for all $x_\nu \in S_\nu(x_{-\nu})$, $\nu = 1, \dots, N$, $\theta_\nu(x^*, \bar{y}) \le \theta_\nu(x_\nu, x^*_{-\nu}, \bar{y})$. In particular, for $\bar{y} = \vartheta(x^*)$ where $\vartheta(x)$ is an optimal solution of (20) given by (21), we will have

$$\theta_\nu(x^*, y^*) \le \theta_\nu(x_\nu, x^*_{-\nu}, y^*) \quad \text{for all} \quad x_\nu \in S_\nu(x_{-\nu}), \quad \nu = 1, \dots, N.$$

Moreover, for a fixed feasible x^* the optimal solution of (20) is given by $\vartheta(x^*)$ and by Theorem 3.3 the vector $\vartheta(x^*)$ is an optimal solution of (19). Therefore, we have

$$\tau_j(x^*, y^*) \le \tau_j(x^*, y_j, y^*_{-j}) \quad \text{for all} \quad y_j \in Y_j(x^*), \quad j = 1, \dots, M$$

Hence, $(x^*, \vartheta(x^*))$ is a Stackelberg-Nash equilibrium of (1). □

Now, instead of assuming a particular structure for the lower-level problem we make the following assumption to state the existence result of an explicit solution for the follower's potential problem (18).

Assumption 3. Assume that the lower-level NEP (13) for each feasible $x \in X$ admits an explicit optimal best response function $\vartheta(x)$, which is lower semi-continuous and convex.

Remark 3.12. Suppose Assumption 3 is satisfied and $\theta_\nu(x, y)$ is quasi-convex in (x, y) and unilaterally monotonically increasing in y. Then, given Assumptions (i)-(iv) of Theorem 2.4, problem (1) admits a Stackelberg-Nash equilibrium.

3.3. Multi-parametric solution for a potential lower-level problem

A multi-parametric solution approach for multilevel optimization reformulates the innermost problem as a parametric program, treating upper-level variables as parameters. Solving this problem globally yields parametric solutions that can be substituted into higher levels, providing a complete map of optimal solutions over parameter spaces divided into critical regions where each solution remains valid [38].

Problem (18) is an optimization problem parameterized by x, so it is possible to implement a parametric optimization approach to solve the potential game of the follower.

The Lagrangian of (18) is defined as

$$\mathcal{L}(y, \lambda, \mu, x) = \pi(y, x) + \lambda^T g(y, x) \tag{25}$$

where $\lambda \in \mathbb{R}^l$ is the Lagrangian multiplier vector associated with the inequality constraints.

If we fix $x = x_0$ in (18), we will have a classical nonlinear problem,

$$P(x_0): \quad \min_y \quad \pi(y, x_0) \quad \text{s.t.} \quad y \in Y(x_0). \tag{26}$$

To analyze the behavior of a local solution $y(x)$ of (18) when x is subject to perturbation, we made the following assumptions.

Assumption 4.

(i) The functions defining problem (18) are twice continuously differentiable in (y, x) in a neighborhood of (y^*, x_0).
(ii) The second order sufficient conditions for a local minimum of problem (26) hold at y^* with associated Lagrange multiplier λ^*.
(iii) The gradients $\nabla_y g_k(y^*, x_0)$, for all k such that $g_i(y^*, x_0) = 0$ are linearly independent (LI).
(iv) Strict complementary slackness (SCS) holds at (y^*, x_0), *i.e.*, $\lambda_k^* > 0$ for all k such that $g_k(y^*, x_0) = 0$.

Corollary 3.13 (Fiacco, 1976). *Under the Assumption 4, a first-order estimation of the optimal solution* $[y(x), \lambda(x)]$ *in a neighborhood of* x_0 *is*

$$\begin{bmatrix} y(x) \\ \lambda(x) \end{bmatrix} = \begin{bmatrix} y^* \\ \lambda^* \end{bmatrix} - M_0^{-1} \cdot N_0 \cdot (x - x_0) + o(\|x\|) \tag{27}$$

where $[y^*, \lambda^*] = [y(x_0), \lambda(x_0)]$, $M_0 = M(x_0)$, $N_0 = N(x_0)$, *and* $M(x)$ *and* $N(x)$ *are defined as*

$$M(x) = \left[\begin{array}{c|c} \nabla^2_{yy}\mathcal{L} & \nabla_y g \\ \hline -\lambda \nabla_y^T g & diag(-g_i) \end{array}\right]$$

$$N(x) = [\nabla^2_{yx}\mathcal{L}, -\lambda_1 \nabla_x^T g_1, \ldots, -\lambda_p \nabla_x^T g_p]^T$$

and $\phi(x) = o(\|x\|)$ *means that* $\phi(x)/\|x\| \to 0$ *as* $x \to x_0$.

When (18) involves only linear constraints, we apply the algorithm of Dua et al. [39] to solve it over the entire parameter space for convex problems. The space over which each piecewise linear approximation (27) remains optimal, known as *critical region* ($\mathcal{CR}$), is defined by,

$$\mathcal{CR} = \begin{cases} \check{g}(y(x), x) \le 0, & \text{Feasibility conditions,} \\ \check{\lambda}(x) \ge 0, & \text{Optimality conditions,} \end{cases}$$

where $\check{g}$ corresponds to the inactive polyhedral constraints and $\check{\lambda}$ to the Lagrangian multipliers of the active constraints [39]. The optimal solution $y(x)$ can be approximated explicitly as a piecewise linear function,

$$y(x) = \begin{cases} C_1 + K_1 x, & \text{for} \quad x \in \mathcal{CR}_1 \\ C_2 + K_2 x, & \text{for} \quad x \in \mathcal{CR}_2 \\ \qquad \vdots & \\ C_p + K_p x, & \text{for} \quad x \in \mathcal{CR}_p, \end{cases} \tag{28}$$

where C_i are column vectors, K_i are real matrices, and $\mathcal{CR}_i \subseteq \mathbb{R}^m$ is the i^{th} critical region.

Remark 3.14. The procedure above is valid for convex objectives with linear constraints. Non-convex cases can be handled via the branch-and-bound approach in [40], while nonlinear constraints can be treated using the method in [41].

Plugging the follower's multi-parametric solution (28) into the leader's game (11), we have a Nash game with a disjunctive programming structure

$$\left.\begin{array}{l} \min\limits_{x_\nu \in X_\nu} \theta_\nu(x_\nu, x_{-\nu}, y(x)) \\ \text{s.t.} \quad x_\nu \in X_\nu, \\ \qquad \bigvee\limits_{t=1,\ldots,p} \left[\; y(x) = y^t(x) = C_t + K_t x \text{ for } x \in \mathcal{CR}_t \; \right] \end{array}\right\} \text{for } \nu = 1, \ldots, N. \tag{29}$$

The critical regions are determined as $\mathcal{CR}_t = \{x : \check{g}(y^t(x), x) \le 0, \ \check{\lambda}^t(x) \ge 0\}$ where $\check{g}$ corresponds to the inactive constraints and $\check{\lambda}^t$ to the associated Lagrangian multipliers. For simplicity, let us denote the critical regions by

$$\mathcal{CR}_t = \{x : D_{t_k} x \le e_{t_k}, \ k = 1, \ldots, q_t\} \quad \text{for all} \quad t = 1, \ldots, p.$$

By introducing the binary variable and using the Big-M method, problem (29) can be reformulated as a mixed integer Nash game in which for each $\nu = 1, \dots, N$ leader ν solves a MINLP

$$\begin{aligned}
&\min_{x_\nu \in X_\nu} \ \theta_\nu(x_\nu, x_{-\nu}, y(x)) \\
&\text{s.t.} \quad x_\nu \in X_\nu, \\
&\qquad y(x) = \textstyle\sum_{t=1}^{p} \delta^t y^t(x) = \sum_{t=1}^{p} \delta^t (C_t + K_t x), \\
&\qquad D_{t_k} x \le e_{t_k} + M(1-\delta^t) \quad \text{for all} \quad k = 1, \dots, q_t \ \& \ t = 1, \dots, p, \\
&\qquad \textstyle\sum_{t=1}^{p} \delta^t = 1, \\
&\qquad \delta^t \in \{0,1\} \quad \text{for all} \quad t = 1, \dots, p,
\end{aligned} \tag{30}$$

where M is a sufficiently large scalar that makes the particular constraint redundant when its indicator variable is not selected (*i.e.*, $\delta^t = 0$).

Problem (30) can be solved using solution methods for mixed integer NEP such as a method proposed by Sagratella in [42].

Remark 3.15. If the leaders' problem (11) admits a potential function $\Phi(x, y)$, then (22) will simply become an optimization problem with variable x given by,

$$\begin{aligned}
&\min_{x} \Phi(x, y(x)) \\
&\text{s.t.} \quad x \in X, \\
&\qquad y(x) = \textstyle\sum_{t=1}^{p} \delta^t y^t(x) = \sum_{t=1}^{p} \delta^t (C_t + K_t x), \\
&\qquad D_{t_k} x \le e_{t_k} + M(1-\delta^t) \quad \text{for all} \quad k = 1, \dots, q_t \quad \text{and} \quad t = 1, \dots, p, \\
&\qquad \textstyle\sum_{t=1}^{p} \delta^t = 1, \\
&\qquad \delta^t \in \{0,1\} \quad \text{for all} \quad t = 1, \dots, p\,.
\end{aligned} \tag{31}$$

Problem (31) is an MINLP and can be solved using standard solution methods for MINLP such as Gurobi Optimizer.

4. Numerical examples

In this section, we will present numerical examples to show how the proposed solution methods are implemented. The first example is a Nash game consisting of three leaders constrained by a two-follower Nash game with quadratic objectives and linear constraints. Following the discussion in Section 3.2, the follower's decision variables will be explicitly computed as a function of the leaders' decision variables and used in the upper-level Nash game. The second and the third examples are a two-leader two-follower games, where the lower-level problem is solved by implementing multi-parametric algorithms, and then the obtained linear piecewise solution is plugged into the upper-level problem to create a disjunctive programming problem which is then transformed into a mixed-integer Nash game.

Example 4.1. Consider a three-leader two-follower problem where the leaders' GNEP is given by

$$\left.\begin{aligned} &\min_{x_\nu} \theta_\nu(x_\nu, x_{-\nu}, y) \\ &\text{s.t.} \quad x_\nu \in X_\nu \subseteq \mathbb{R}^3 \end{aligned}\right\} \ \text{for} \quad \nu = 1, 2, 3, \tag{32}$$

where three different variants of leaders' objective functions are considered. In the first two variants, we used simple convex, quadratic functions $\theta_\nu(x, y) = \alpha_\nu x_\nu^2 + \phi(x, y)$, where $\phi(x, y) = e^{\|x\|^2}(\gamma_1^T y_1 + \gamma_2^T y_2)$ tested for different values of α, γ_1 and γ_2, whereas for the third variant setup we applied the following objectives: $\theta_1(x, y) = 5e^{x_1} - 8y_{22}^4 + \phi(x, y)$, $\theta_2(x, y) = y_{11}y_{21}e^{2x_2} + 10x_2^2 + \phi(x, y)$, and $\theta_3(x, y) = y_{11}^2 e^{3x_3} + \phi(x, y)$ where $\phi(x, y) = y_{12}(y_{11}^2 - 4)\sqrt{x_1^2 + x_2^2 + x_3^2}$, and $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -2 \le x_1 \le 2, -1 \le x_2 \le 1, 0 \le x_3 \le 5, \|x\|_2 \le 2\}$.

Moreover, the joint strategy vector of the followers $y = (y_j, y_{-j}) \in \mathbb{R}^4$ is obtained through a Nash game with quadratic objective function and polyhedral constraints

$$\left.\begin{aligned} &\min_{y_j} \varphi_j(y, x) = [\tfrac{1}{2} y_j^T Q_F^j y_j - b_F^j(x)^T y_j] + [\tfrac{1}{2} y^T Q y - b(x)^T y] \\ &\text{s.t.} \quad y_j - c_j(x) \ge 0 \end{aligned}\right\} \ \text{for} \quad j = 1, 2, \tag{33}$$

where for $j = 1, 2$, $y_j = \begin{bmatrix} y_{j1} \\ y_{j2} \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $b_F^1(x)^T = \begin{pmatrix} \max(x_1^2, x_3^2) \\ x_2^2 e^{x_2} + x_3 e^{x_2} \end{pmatrix}$,

$$b_F^2(x)^T = \begin{pmatrix} \ln(1 + e^{x_1} + e^{x_2}) \\ 5x_1^2 + 7x_2^2 + 2x_1x_2 + x_3 - 4 \end{pmatrix}, \ b(x)^T = \begin{pmatrix} \sqrt{x_1^2 + x_2^2 + x_3^2} \\ 4x_1 - x_2 + 5x_3 + 6 \\ 2x_1^2 + 3x_3^2 + 4x_1x_3 \\ x_2 + 2 \end{pmatrix},$$

$$Q = \begin{pmatrix} 4 & 1 & 2 & 3 \\ 1 & 5 & 1 & 0 \\ 2 & 1 & 6 & 2 \\ 3 & 0 & 2 & 7 \end{pmatrix}, Q_F^1 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, Q_F^2 = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix},$$

$c_1(x) = \begin{pmatrix} x_1 + 2x_2 - 3x_3 - 5 \\ x_1 - 2x_2 + 5x_3 - 8 \end{pmatrix}$, and $c_2(x) = \begin{pmatrix} 2x_1 + 3x_2 + x_3 - 4 \\ 5x_1 - 3x_2 + 4x_3 - 10 \end{pmatrix}$.

The corresponding potential game for (33) will be

$$\begin{aligned} &\min_y \pi(y, x) = \textstyle\sum_{j=1}^{2} [\tfrac{1}{2} y_j^T Q_F^j y_j - b_F^j(x)^T y_j] + [\tfrac{1}{2} y^T Q y - b(x)^T y] \\ &\text{s.t.} \quad y - c(x) \geq 0\,. \end{aligned} \tag{34}$$

Then, the parametric solution for (34) will be explicitly expressed as

$$\vartheta(x) = \max\left\{(Q_F + Q)^{-1}(b_F(x) + b(x)),\ c(x)\right\}\,.$$

where $c(x) = \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}$, $b_F(x)^T = \begin{pmatrix} b_F^1(x)^T \\ b_F^2(x)^T \end{pmatrix}$, and $Q_F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix}$.

Problem (32) admits a potential function $\Phi(x, y) = 5e^{x_1} + y_{11}y_{21}e^{2x_2} + y_{11}^2 e^{3x_3} + 10x_2^2 - 8y_{22}^4 + y_{12}(y_{11}^2 - 4)\sqrt{x_1^2 + x_2^2 + x_3^2}$. Then, the leader's potential problem will be

$$\begin{aligned} &\min_x \bar{\Phi}(x) \\ &\text{s.t.} \quad x \in X \subseteq \mathbb{R}^9, \end{aligned} \tag{35}$$

where $\bar{\Phi}(x) = \Phi(x, \vartheta(x))$. Solving (35), we get an optimal solution $(x_1, x_2, x_3) = (-1.6144, 1.0000, 0.6274)$ with the corresponding follower's decision values $(y_{11}, y_{12}, y_{21}, y_{22}) = (-0.1286, 0.7234, -0.2728, 1.4923)$. Fig. 2 displays the evolution of the leaders' decision variables for the third variant setup and the initial vector $x_0 = (0, 0, 0)$.

Neglecting the fact that the leaders' game has a potential structure, we can solve the resulting NEP using Gauss-Seidel type method as, e.g., proposed by Facchinei et al. [30]. If we use the explicit solution $\vartheta(x)$ in (32), we obtain

$$\left.\begin{aligned} &\min_{x_\nu} \tilde{\theta}_\nu(x_\nu, x_{-\nu}) \\ &\text{s.t.} \quad x_\nu \in X_\nu \subseteq \mathbb{R}^3, \end{aligned}\right\} \text{ for } \quad \nu = 1, 2, 3\,. \tag{36}$$

where $\vartheta(x) = \max\left\{(Q_F + Q)^{-1}(b_F(x) + b(x)),\ c(x)\right\}$ and $\tilde{\theta}_\nu(x) = \theta_\nu(x, y^*(x))$ for $\nu = 1, 2, 3$. Here, Fig. 3 shows the evolution of the leaders' decision variables for the first variant setup and the initial vector $x_0 = (0, 0, 0)$. Furthermore, in Fig. 4 we display the values of the difference between the new iterate (from the last outer, i.e., Gauss-Seidel iteration) and the previous one, that is $\|x^{k+1} - x^k\|_2$. If the difference becomes smaller than the tolerance 10^{-8}, we stop the outer iteration procedure.

Finally, in Table 1 we display the iteration numbers for both methods and a solution code (SC) for the three problem variants of Example 1 and four different initial data ($x_0 - 0 = (0, 0, 0)$ and three randomly generated $x_0 - r \in [0, 1]^3$), respectively. Since we

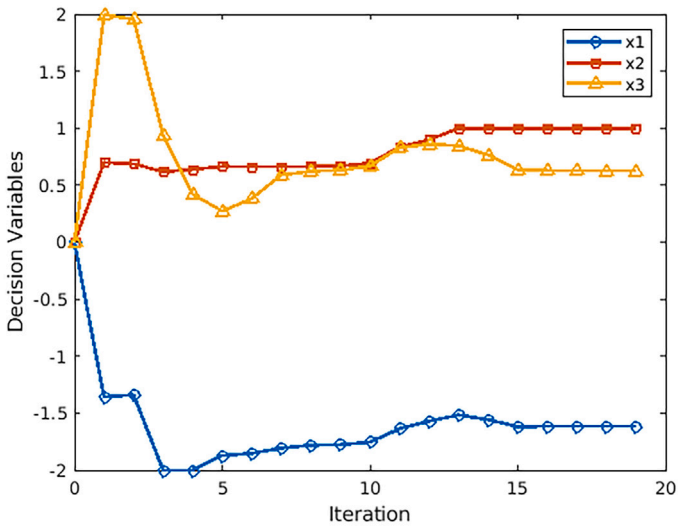


Fig. 2. Decision variables vs iterations.

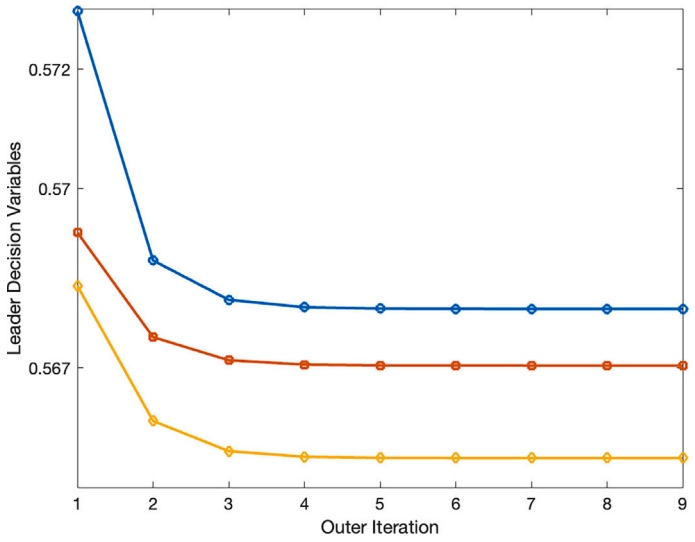


Fig. 3. Decision variables vs outer iterations.

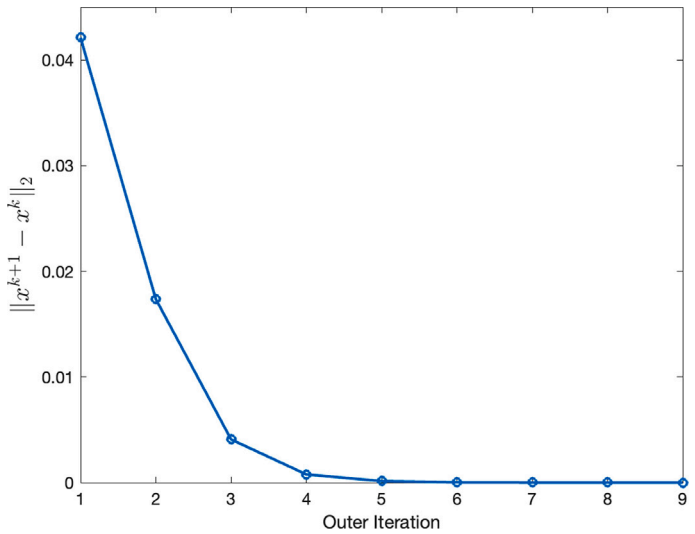


Fig. 4. Difference in the outer decision variables vs outer iterations $k = 1, 2, .., 9$.

Table 1
Number of inner (i.e., NLP solver) iterations (for both NLP and GS methods with outer iterations in brackets).

	Example 1 - Variant 1			Example 1 - Variant 2			Example 1 - Variant 3		
Initial x_0	# (NLP)	#(GS)	SC	# (NLP)	#(GS)	SC	# (NLP)	# (GS)	SC
$x_0 - 0$	6	14(3)	1	8	58 (9)	1	19	213(20)	0
$x_0 - 1$	8	25(4)	1	20	59(9)	2	13	17 (2)	2
$x_0 - 2$	9	22(4)	1	10	61(9)	1	8	28(4)	2
$x_0 - 3$	8	25(4)	1	9	62(9)	1	13	24(4)	1

are interested in comparing the solution approaches themselves, both methods have been implemented as a MATLAB code using the nonlinear optimization solver `fmincon` as a black box solver with the option `sqp` and default tolerances. Moreover, we have set the maximum outer (GS) iteration number to `maxiter = 20`. The solution code is as follows: SC = 0 - the GS method is unable to find a solution within the prescribed tolerances, SC = 1 - both methods yield identical solutions, SC = 2 - the solution provided by the GS method is worse in terms of the objective function value. It becomes clear that using the potential structure of the GNEP not only significantly reduces the number of (solver) iterations but might also lead to better solutions than applying an NEP solver such as the GS method.

Example 4.2. In this example we consider a two-leader two-follower game with a potential structure at both levels to show how the multi-parametric procedure is implemented to solve such problems.

$$\begin{cases} \min\limits_{x_1} \theta_1(x,y) = 2x_1^2 - x_1 - (y_1^2 + y_2^2) - 3(y_1 + y_2) \\ \min\limits_{x_2} \theta_2(x,y) = 4x_2^2 + 2x_2 - (y_1^2 + y_2^2) - 3(y_1 + y_2) \end{cases}$$

$$s.t. \begin{cases} \begin{cases} \min\limits_{y_1} \varphi_1(x,y) = y_1^2 + (1 - x_1)y_2 - 2\ln(y_1 + y_2 + 4) \\ \min\limits_{y_2} \varphi_2(x,y) = -y_2^2 + (1 - x_2)y_1 - 3\ln(y_1 + y_2 + 4) \end{cases} \\ s.t. \begin{cases} 3x_1 + x_2 - 2y_1 - y_2 \le 2.5 \\ x_1 - 2x_2 - 3y_1 + 4y_2 \le -3 \\ 0 \le y_1, y_2 \le 2 \\ 0 \le x_1, x_2 \le 2 \end{cases} \end{cases} \tag{37}$$

Since we have a potential structure in both levels, problem (37) can be equivalently reformulated as

$$\min_x \theta(x,y) = 2x_1^2 + 4x_2^2 - x_1 + 2x_2 - (y_1^2 + y_2^2) - 3(y_1 + y_2)$$

$$s.t. \begin{cases} \min\limits_y \varphi(x,y) = \frac{1}{2}y_1^2 - \frac{1}{3}y_2^2 - \ln(y_1 + y_2 + 4) \\ s.t. \begin{cases} 3x_1 + x_2 - 2y_1 - y_2 \le 2.5 \\ x_1 - 2x_2 - 3y_1 + 4y_2 \le -3 \\ 0 \le y_1, y_2 \le 2 \\ 0 \le x_1, x_2 \le 2 \end{cases} \end{cases} \tag{38}$$

The inner problem in (38) is a multi-parametric problem with parameter $x = (x_1, x_2)$:

$$\min_y \varphi(x,y) = \frac{1}{2}y_1^2 - \frac{1}{3}y_2^2 - \ln(y_1 + y_2 + 4)$$

$$s.t. \begin{cases} 3x_1 + x_2 - 2y_1 - y_2 \le 2.5 \\ x_1 - 2x_2 - 3y_1 + 4y_2 \le -3 \\ 0 \le y_1, y_2 \le 2 \\ 0 \le x_1, x_2 \le 2 \end{cases} \tag{39}$$

By implementing a multi-parametric approach, we obtain an optimal parametric solution to the problem (39) with their corresponding critical regions (see Fig. 5):

$$y^1(x) = \begin{bmatrix} 1.1818x_1 + 0.1818x_2 - 0.6364 \\ 0.6364x_1 + 0.6364x_2 - 1.2273 \end{bmatrix}$$

in the critical region $CR^1 = \{(x_1, x_2) : -0.9941x_1 + 0.1085x_2 \le -0.5005, 0.8913x_1 + 0.4534x_2 \ge 0.9040,\ x_1 \le 2,\ 0 \le x_2 \le 2\}$. Here, $y_1^1(x) = 1.1818x_1 + 0.1818x_2 - 0.6364$, $y_2^1(x) = 0.6364x_1 + 0.6364x_2 - 1.2273$, $D_{11}(x) = -0.9941x_1 + 0.1085x_2 + 0.5005$, $D_{12}(x) = -0.8913x_1 - 0.4534x_2 + 0.9040$.

In the region $CR^2 = \{(x_1, x_2) : 0.9941x_1 - 0.1085x_2 \le 0.5005,\ x_1 \ge 0,\ 0 \le x_2 \le 2\}$

$$y^2(x) = \begin{bmatrix} -0.1406x_1 + 0.2811x_2 + 0.7631 \\ -0.3554x_1 + 0.7108x_2 - 0.1777 \end{bmatrix}$$

which is $y_1^2(x) = -0.1406x_1 + 0.2811x_2 + 0.7631$, $y_2^2(x) = -0.3554x_1 + 0.7108x_2 - 0.1777$, $D_{21}(x) = 0.9941x_1 - 0.1085x_2 - 0.5005$ and $D_{22}(x) = 0$.

In the region $CR^3 = \{(x_1, x_2) : -0.9941x_1 + 0.1085x_2 \le -0.5005,\ 0.8913x_1 + 0.4534x_2 \le 0.9040,\ x_2 \ge 0\}$

$$y^3(x) = \begin{bmatrix} -0.1432x_1 + 0.2864x_2 + 0.9523 \\ -0.3574x_1 + 0.7148x_2 - 0.0357 \end{bmatrix}$$

That means, $y_1^3(x) = -0.1432x_1 + 0.2864x_2 + 0.9523$, $y_2^3(x) = -0.3574x_1 + 0.7148x_2 - 0.035$, $D_{31}(x) = -0.9941x_1 + 0.1085x_2 + 0.5005$, and $D_{32}(x) = 0.8913x_1 + 0.4534x_2 - 0.9040$.

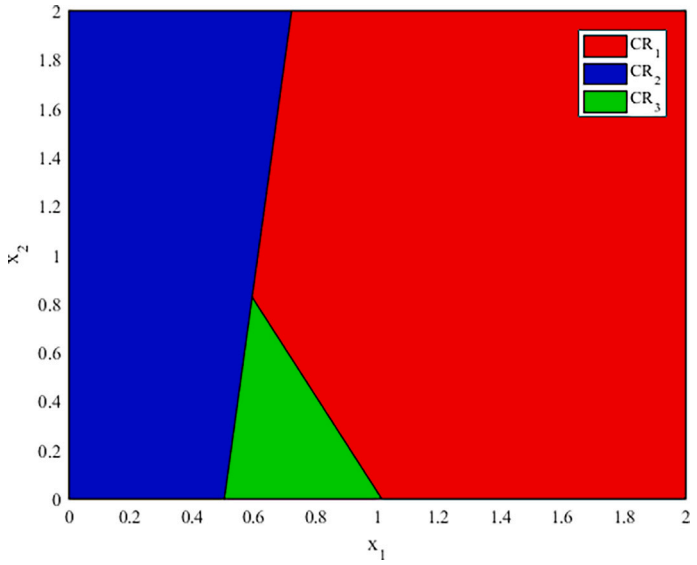


Fig. 5. Critical regions for parametric problem (39).

Using the follower's parametric solution in the leader's problem of the potential game (38) will be a disjunctive programming problem

$$\begin{aligned}
&\min_{x} \theta(x, y(x)) = 2x_1^2 + 4x_2^2 - x_1 + 2x_2 - (y_1^2 + y_2^2) - 3(y_1 + y_2) \\
&\text{s.t.} \quad 0 \le x_1, x_2 \le 2, \\
&\qquad y(x) = \textstyle\sum_{t=1}^{3} \delta^t y^t(x), \\
&\qquad D_{ts}(x) \le M(1 - \delta^t) \quad \text{for all} \quad t = 1, 2, 3, \quad s = 1, 2, \\
&\qquad \textstyle\sum_{t=1}^{5} \delta^t = 1, \\
&\qquad \delta^t \in \{0, 1\} \quad \text{for all} \quad t = 1, 2, 3,
\end{aligned} \tag{40}$$

In order to globally solve (40) we used the Gurobi Optimizer version 12.0.0 (build v12.0.0rc1) within Matlab. We set the following options: `MIPfocus = 1` and `TimeLimit = 600`. Starting with the initial guess $(x_1, x_2) = (0, 0)$ the problem was solved in 113 MIP iterations and 0.08s running time. The objective function value $\theta(x^*) = 0.3115$ was reported with a MIP gap of $1.08e-6$. The solution given by Gurobi is $x* = (0.2858, 1.643)$ and $y^*(x^*) = (1.185, 0.8884) \in CR^2$ and we have $\delta = (0, 1, 0)$. Standard approaches for example apply the KKT conditions to (39) and solve the resulting GNEP using Gauss-Seidel type methods. Applied to Example 4.2, this yields a final objective value of $\theta(x^*) = -5.232$ with $x* = (0.7059, 0)$ and associated $y^* = (1.235, 0)$ within 3 outer iterations, 15 solver iterations and 0.16s running time. Starting both methods with the initial guess $(x_1, x_2) = (2, 2)$ the solutions remain the same (for both methods). However, the GS method then requires 0.33s running time, whereas Gurobi still requires 0.09s.

For more advanced (highly nonlinear) MIP models, one could apply the same methodology; however, this requires a MIP solver capable of handling such types of advanced MINLPs.

Example 4.3. Consider the following bilevel two-leader two-follower game adapted from [22]:

$$\begin{cases} \min\limits_{x_1} \theta_1(x, y) = Q_1 x_1^2 + b_1 x_1 - x^T Q_c x - y^T Q_F y - c^T y \\ \min\limits_{x_2} \theta_2(x, y) = Q_2 x_2^2 + b_2 x_1 - x^T Q_c x - y^T Q_F y - c^T y \end{cases}$$

$$s.t. \begin{cases} \begin{cases} \min\limits_{y_1} \varphi_1(x, y) = y_1^2 + (1 - x_1) y_2 - (x_2^2 + 2) \ln(y_1 + y_2 + 4) \\ \min\limits_{y_2} \varphi_2(x, y) = -y_2^2 + (1 - x_2) y_1 - (x_1^2 + 3) \ln(y_1 + y_2 + 4) \end{cases} \\ s.t. \begin{cases} x_1 + 2y_1 - y_2 \le 2, \; x_1 - x_2 - y_1 + y_2 \le 1, \\ 0 \le x_1, \; x_2, \; y_1, \; y_2 \le 2. \end{cases} \end{cases} \tag{41}$$

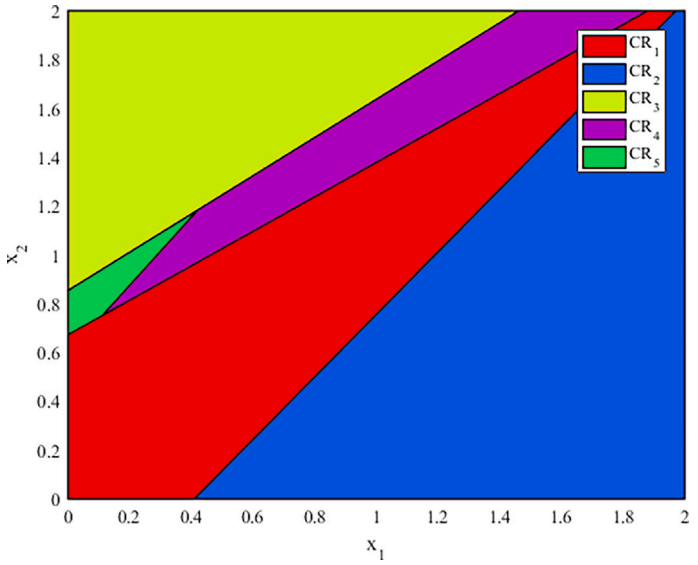


Fig. 6. Critical regions for the inner problem in (42).

The objective functions at both levels satisfy Assumption 1, so an equivalent potential problem for (41) is given by

$$\begin{aligned} &\min_x \theta(x,y) = x^T Q x + b^T x - y^T Q_F y - c^T y \\ &s.t. \begin{cases} \min_y \varphi(x,y) = \dfrac{y_1^2}{x_2^2+2} - \dfrac{y_2^2}{x_1^2+3} - \ln(y_1+y_2+4) \\ s.t. \begin{cases} x_1 + 2y_1 - y_2 \le 2,\ x_1 - x_2 - y_1 + y_2 \le 1, \\ 0 \le x_1,\ x_2,\ y_1,\ y_2 \le 2, \end{cases} \end{cases} \end{aligned} \tag{42}$$

where $Q = diag(Q_1, Q_2) - Q_c$ and $b = (b_1, b_2)^T$. The inner-level problem in (42) is solved by a multi-parametric approach to obtain an optimal solution with corresponding critical regions (see Fig. 6). In the critical region CR^1,

$$y^1(x) = \begin{bmatrix} -3.8630x_1 + 3.6393x_2 + 1.2237 \\ -4.8630x_1 + 4.6393x_2 + 2.2237 \end{bmatrix}$$

with $D_{11}(x) = 0.7881x_1 - 0.6156x_2 - 0.3224$, $D_{12}(x) = -0.5767x_1 + 0.8170x_2 - 0.5498$, and $D_{13}(x) = 0$. In the region CR^2,

$$y^2(x) = \begin{bmatrix} -6.5148x_1 + 4.5813x_2 + 3.3640 \\ -7.5148x_1 + 5.5813x_2 + 4.3640 \end{bmatrix}$$

with $D_{21}(x) = -0.7881x_1 + 0.6156x_2 + 0.3224$, $D_{22}(x) = 0$, and $D_{23}(x) = 0$. In the region CR^3,

$$y^3(x) = \begin{bmatrix} 0.2870x_2 - 0.1114 \\ 0.0107x_2 + 1.9786 \end{bmatrix}$$

with $D_{31}(x) = 0.6167x_1 - 0.7872x_2 + 0.6723$, $D_{32}(x) = 0$, and $D_{33}(x) = 0$.

In the region CR^4,

$$y^4(x) = \begin{bmatrix} -14.7758x_1 + 10.1658x_2 - 2.4725 \\ -15.7758x_1 + 11.1658x_2 - 1.4725 \end{bmatrix}$$

with $D_{41}(x) = -0.8125x_1 + 0.5829x_2 - 0.3480$, $D_{42} = -0.6167x_1 + 0.7872x_2 - 0.6723$, and $D_{43}(x) = 0.5767x_1 - 0.8170x_2 + 0.55498$.

In the region CR^5,

$$y^5(x) = \begin{bmatrix} -1.6379x_1 + 1.0111x_2 + 2.2657 \\ -2.6379x_1 + 2.0111x_2 + 3.2657 \end{bmatrix}$$

with $D_{51}(x) = 0.8125x_1 - 0.5829x_2 + 0.3480$, $D_{52} = -0.6167x_1 + 0.7872x_2 - 0.6723$, and $D_{53}(x) = 0.5767x_1 - 0.8170x_2 + 0.55498$.

Plugging these parametric solutions into the outer problem of (42), we get a disjunctive programming problem

$$\begin{aligned}
&\min_x \theta(x, y(x)) = x^T Q x + b^T x - y^T Q_F y - c^T y \\
&\text{s.t.} \quad 0 \le x_1, x_2,\ y_1, y_2 \le 2, \\
&\qquad y(x) = \textstyle\sum_{t=1}^{5} \delta^t y^t(x), \\
&\qquad D_{ts}(x) \le M(1-\delta^t) \quad \text{for all} \quad t = 1, \dots, 5, \quad s = 1, \dots, 3, \\
&\qquad \textstyle\sum_{t=1}^{5} \delta^t = 1, \\
&\qquad \delta^t \in \{0, 1\} \quad \text{for all} \quad t = 1, \dots, 5,
\end{aligned} \tag{43}$$

For this problem, we build a random set of 20 test problems for various objectives $\theta(x, y(x))$. We chose $Q = diag(q_L^j)$ with $q_L^j \in [1, 10]$, $b_i \in [-10, 10]$, $Q_F = diag(q_F^j)$ with $q_F^j \in [0, 1]$ and $c_i \in (0, 1)$. To solve the test problems we used the same setting as for Example 4.2. The number of iterations varies between 12 and 148 iterations with 46 iterations and 0.074s running time on average. Moreover, 80% of the solutions belong to CR^3 and the remaining 20% belong to CR^4. Here, the MIP gap varies between $7.8E-5$ and $1E-16$.

5. Conclusions

This paper proposed a parametric optimization framework for nonlinear multi-leader–follower games with potential follower structures. By embedding the parametric response of the follower, the problem is reduced to a mixed integer GNEP that preserves equilibrium properties while allowing for specialized numerical methods. An explicit solution was derived for quadratic follower problems, and numerical experiments demonstrated some potential of this approach compared to other approaches. Hence, it defines a suitable alternative solution approach in the case of an inner problem that admits a potential structure. Future work will address large-scale and more complex settings such as uncertainties in the data.

CRediT authorship contribution statement

Sonja Steffensen: Writing – review & editing, Writing – original draft, Software, Project administration, Funding acquisition, Conceptualization. **Addis Belete Zewde:** Writing – review & editing, Writing – original draft, Software, Funding acquisition, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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