

Hamiltonian Cycles in Certain Graphs and Out-arc Pancyclic Vertices in Tournaments

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Preface

Graph theory is a delightful research area in discrete mathematics, and its results have many applications in computer science, social and natural science.

The Hamiltonian problem is concerned with conditions under which a graph contains a spanning cycle. Named after Sir William Rowan Hamilton, this problem traces its origins to the 1850s. Although this problem is **NP**-complete, in the last 100 years many sufficient conditions for a graph to be Hamiltonian have been found and many related topics have also been discussed, for example, pancyclicity, proper colored Hamiltonian paths or cycles, etc.

In the first part of this thesis, we introduce new sufficient conditions for a graph to be Hamiltonian and some other results on related topics.

Chapter 1 will be devoted to an introduction of the terminology and some well-known results about Hamiltonian graphs.

Generally speaking, there are two important types of sufficiency conditions with respect to the cyclic properties of graphs. The first is based on the so-called numerical conditions, of which probably degree conditions are the most well-known; structural conditions form the second important type. A typical example of the latter is forbidden subgraph conditions. In Chapter 2, we give some new sufficient conditions for a graph to be Hamiltonian by combining those two types.

Let $G = (V, E)$ be a connected graph. The distance between two vertices x and y in G , denoted by $d_G(x, y)$, is the length of a shortest path between x and y . A graph G is called almost distance-hereditary if each connected induced subgraph H of G has the property that $d_H(u, v) \leq d_G(u, v) + 1$ for every pair of vertices u and v in H . In Section 2.1, we prove that every 2-connected, 2-heavy and almost distance-hereditary graph is Hamiltonian, where a graph G is 2-heavy if the degrees of at least two end vertices of each claw in G are greater than or equal to $\frac{|V(G)|}{2}$.

In Section 2.2, we assume that G is a connected claw-free graph satisfying the following condition: For every maximal induced path P of length $p \geq 2$ with end vertices u and v holds $d(u) + d(v) \geq |V(G)| - p + 2$. Under this assumption, we prove that G has a 2-dominating induced cycle, and then G is Hamiltonian.

In Section 2.3, we consider the graph G satisfying this condition: For each vertex u in G it holds: $\max\{d(x) \mid d(x, u) = 2 \text{ and } x \neq u\} \geq \frac{|V(G)|}{2}$. It is proved that G is Hamiltonian if G is 3-connected, claw-free and hourglass-free.

In 1997, J. Bang-Jensen and G. Gutin conjectured that an edge-colored complete graph G has a properly colored Hamiltonian path if and only if G has a spanning subgraph consisting of a properly colored path C_0 and a (possibly empty) collection of properly colored cycles $C_1, C_2, \dots, C_d$ such that $V(C_i) \cap V(C_j) = \emptyset$ provided $0 \leq i < j \leq d$. In Chapter 3, we prove this conjecture.

Another interesting topic in graph theory is the research on the existence of a 2-factor in a graph. In Chapter 4, we consider a special 2-factor, the complementary cycles in jump graphs and we give a characterization for jump graphs containing complementary cycles.

In the second part of this thesis, we consider tournaments. T. Yao, Y. Guo and K. Zhang (*Discrete Appl. Math.* **99** (2000), 245 – 249) proved that every strong tournament contains a vertex x such that each arc going out of x is pancyclic. Moreover, they conjectured that each k -strong tournament contains k vertices whose out-arcs are pancyclic. A. Yeo (*J. Graph Theory* **50** (2005), 212 – 219) proved that each 3-strong tournament contains two vertices x, y such that all out-arcs of x and y are pancyclic. In addition, A. Yeo gave an infinite class of k -strong tournaments for all $k \in \mathbb{N}$, each of which contains at most 3 vertices whose out-arcs are pancyclic, and gave a conjecture that each 2-strong tournament contains 3 vertices whose out-arcs are pancyclic.

In Section 6.1, we extend the result of Yeo mentioned above and prove that each 2-strong tournament contains two vertices v_1 and v_2 such that every out-arc of v_i is pancyclic for $i = 1, 2$. Our proof yields a polynomial algorithm to find such two vertices.

In Section 6.2, we prove that each 3-strong tournament contains at least 3 vertices whose out-arcs are pancyclic. That implies that the conjecture of Yao, Guo and Zhang is true for the 3-strong tournament but the conjecture of Yeo is still open only for exactly 2-strong tournaments.

Because there are k -strong tournaments for all $k \in \mathbb{N}$ containing at most 3 vertices whose out-arcs are pancyclic, it gives rise to an interesting problem: How many vertices does a tournament contain such that all out-arcs of those vertices are 4-pancyclic? In Section 6.3, we show that each s -strong tournament with $s \geq 2$ contains at least $s + 1$ vertices whose out-arcs are 4-pancyclic.

The completion of this dissertation was made possible through the support and cooperation of many individuals. I am grateful for my advisor, Prof. Dr. Yubao Guo, who introduced me to graph theory and provided thoughtful guidance and encouragement. I would like to thank Prof. Dr. Eberhard Triesch for his support as co-referee. I also extend special thanks to Prof. Dr. h.c. Hubertus Th. Jongen for the generous support during the last few years. Thanks also to Dipl.-Math. Hans-Erik Giesen, PD Dr. Harald Günzel, Prof. Dr. Gregory Gutin, Dr. Guido Helden, M.Sc. Wei Meng, M.Sc. Ruijuan Li, Prof. Dr. Shengjia Li and Mrs. Roswitha Gilleszen for the excellent cooperation and helpful advices. Moreover, I would like to thank Prof. Dr. Gerhard Hiß for admitting me as an associate member of “Graduiertenkolleg: Hierarchie und Symmetrie in mathematischen Modellen (DFG), RWTH Aachen University, Germany”.

Finally, I owe my greatest debts to my family. Particularly, I would like to thank my wife, Ping Zhang, and our parents for their sacrifices and support. Special thanks to my daughter, Xiaoyue, who reminds me daily that miracles exist everywhere around us.

Contents

Preface	i
I Some Hamiltonian Graphs and Related Topics	1
1 Introduction	3
1.1 Terminology and Notation	3
1.2 Some Well-known Results on the Hamiltonian Problem	5
1.2.1 Four Classical Results on Hamiltonian Graphs	5
1.2.2 Some Generalizations	6
1.2.3 Forbidden Subgraphs and Closure Concept	7
2 Hamiltonian Graphs	11
2.1 Almost Distance-hereditary Graphs	11
2.2 Graphs with Ore-type Condition	17
2.3 Graphs with Fan-type Condition	24
3 Properly Colored Hamiltonian Paths	29
3.1 PCHP Conjecture	29
3.2 The Proof for PCHP Conjecture	30
4 Complementary Cycles in Jump Graphs	41
4.1 Some Results on Jump Graphs	41
4.2 Complementary Cycles in Jump Graphs	43
II Out-arc Pancyclic Vertices in Tournaments	55
5 Introduction	57
5.1 Terminology and Notation	57
5.2 Some Results on Tournaments	58
6 Out-arc Pancyclicity of Vertices in Tournaments	63
6.1 Out-arc Pancyclic Vertices in 2-strong Tournaments	63
6.2 Out-arc Pancyclic Vertices in 3-strong Tournaments	66
6.3 Out-arc 4-pancyclic Vertices	71
Bibliography	107
Index	113

Part I

**Some Hamiltonian Graphs
and
Related Topics**

Chapter 1

Introduction

In this thesis, we only consider finite graphs without loops and multiple edges. At this point, we introduce some definitions and notations as well as some well-known results of Hamiltonian graphs. For those not defined here we refer to [21]. The definitions and notations of digraphs will be introduced in Chapter 5.

1.1 Terminology and Notation

Let $G = (V(G), E(G))$ be a graph. The notations $|V(G)|$ and $|E(G)|$ are called the *order* and the *size* of G , respectively. For a vertex u of G and a subgraph H of G , the subset $\{v \in V(H) \mid uv \in E(G)\}$ is defined as the *neighborhood* of u in H , denoted by $N_H(u)$; $|N_H(u)|$ is the *degree* of u with respect to H , denoted by $d_H(u)$. If there is no confusion, the degree $d_G(u)$ and the neighborhood $N_G(u)$ of u are simply denoted by $d(u)$ and $N(u)$, respectively. Sometimes we also use $N[u]$ to denote the set $N(u) \cup \{u\}$. The set of all edges incident with u is denoted by $I_G(u)$ or $I(u)$. Note that $|I_G(u)| = d_G(u)$. The symbols $\Delta(G)$, $\delta(G)$, $\omega(G)$, $\kappa(G)$ and $\alpha(G)$ denote the *maximum degree*, the *minimum degree*, the *number of components*, the *connectivity* and the *independence number* of G , respectively.

For a subset V' of $V(G)$, the subgraph of G *induced* by V' is denoted by $G[V']$ and $G - V'$ denotes the induced subgraph $G[V \setminus V']$. Let E' be a subset of $E(G)$. Similarly, $G[E']$ ($G - E'$, respectively) denotes the subgraph of G *induced* by E' ($E \setminus E'$, respectively).

We say that a vertex $x \in V(G)$ is *locally connected* if $G[N_G(x)]$ is a connected graph; otherwise x is said to be *locally disconnected*. A locally connected vertex x is said to be *eligible* if $G[N_G(x)]$ is not a clique; otherwise we say that x is *simplicial*. The set of locally connected (eligible, simplicial, locally disconnected) vertices of G will be denoted by $V_{LC}(G)$ ($V_{EL}(G)$, $V_{SI}(G)$, $V_{LD}(G)$, respectively). Thus, the sets $V_{EL}(G)$, $V_{SI}(G)$ and $V_{LD}(G)$ are pairwise disjoint, $V_{EL}(G) \cup V_{SI}(G) = V_{LC}(G)$ and $V_{LC}(G) \cup V_{LD}(G) = V$. If $V_{LC}(G) = V(G)$, we say that the graph G is *locally connected*.

The *distance* between two vertices x and y in G , denoted by $d(x, y)$, is the length of a shortest path between x and y . Moreover, the distance between a vertex $u \in V(G)$ and a subgraph H of G is defined as $\min\{d(u, x) \mid x \in V(H)\}$, denoted by $d(u, H)$. Especially, if G is connected, then $0 \leq d(u, H) < \infty$ for all $u \in V(G)$. Let $t = \max\{d(u, H) \mid u \in V(G)\}$. Denote $N_k(H) = \{u \in V(G) \mid d(u, H) = k\}$ for all k , $0 \leq k \leq t$. We also use the symbols $N(H) = N_1(H)$ and $N[H] = N_1(H) \cup V(H)$. For $V(H) = \{v\}$, we write $N_k(H)$

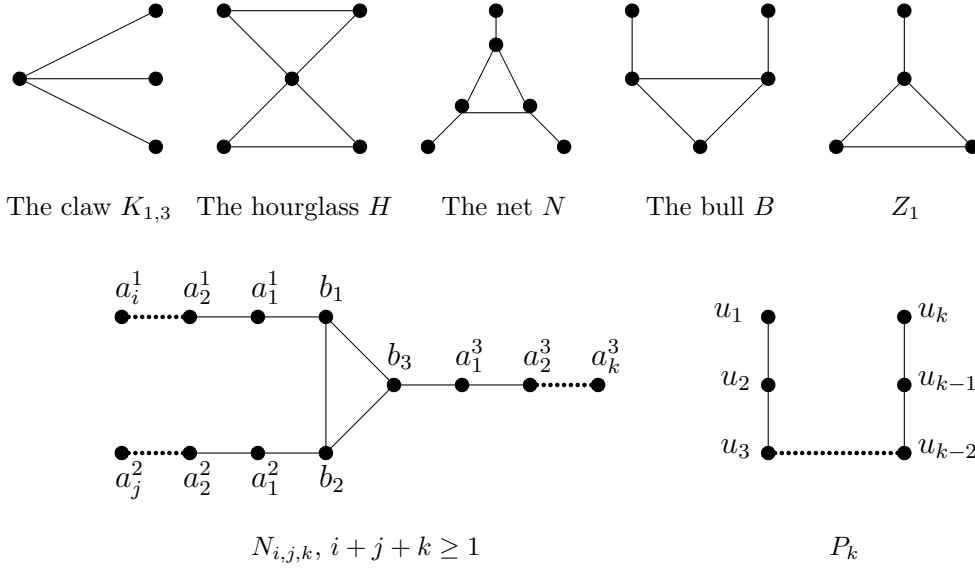


Figure 1.1: Some graphs used as forbidden subgraphs

also with $N_k(v)$. In addition, the collection $\{N_k(v) \mid 0 \leq k \leq t\}$ is called the *hanging* of G at v , denoted also by $h_v = (L_0, L_1, L_2, \dots, L_t)$.

As usual, P_n , C_n and K_n are the *path*, *cycle* and *complete graph* of order n , respectively. $K_{r,s}$ is the *complete bipartite graph* with two partite sets containing r and s vertices. In particular, $K_{1,s}$ is called a *star*. K_n^- denotes the graph obtained from K_n by deleting an edge, while K_n^+ denotes the graph obtained from K_n by adding a new vertex and joining it to exactly one vertex of K_n . The graph $K_3 \circ K_1$ is obtained from K_3 by adding a new vertex y_x only adjacent to x for every vertex x of K_3 , which is also called a *net* and denoted by N . We say two graphs G and H are disjoint if they have no vertex in common, and denote their union by $G + H$; it is called the *disjoint union* of G and H . The disjoint union of k copies of G is denoted by kG .

For given graphs $F_1, F_2, \dots, F_k$, we say that G is $\{F_1, F_2, \dots, F_k\}$ -free if G contains no induced subgraph isomorphic to any F_i ($1 \leq i \leq k$). The complete bipartite graph $K_{1,3}$ (sometimes called a *claw*) or graphs very closely related to claw are central to most forbidden subgraphs results to date (see Figure 1.1).

G is said to be *chordal* if every cycle of length greater than 3 possesses a chord, which is an edge joining two nonconsecutive vertices of the cycle.

A cycle C (a path P , respectively) of G is called an *induced cycle* (*path*, respectively) if $G[V(C)] = C$ ($G[V(P)] = P$, respectively). An induced path P of G is said to be *maximal* if G contains no path P' satisfying $V(P) \subset V(P')$ and $V(P') \setminus V(P) \neq \emptyset$. An induced cycle C of G is called *maximum* if G contains no induced cycle longer than C . Let $k \in \mathbb{N}$. A cycle C is called *k-dominating* if $d(x, C) \leq k$ for all $x \in V(G)$. A cycle C *doubly dominates* G if every vertex of G has at least two neighbors in $V(C)$.

A cycle (path, respectively) in a graph is called a *Hamiltonian cycle* (*Hamiltonian path*, respectively) if it contains all vertices of G . A graph is said to be *Hamiltonian* if it contains a Hamiltonian cycle.

In Section 2.1, we consider almost distance-hereditary graphs. A graph G is called *distance-hereditary* if each connected induced subgraph H of G has the property that $d_H(x, y) = d_G(x, y)$ for every pair of vertices x and y in H . A graph G is called *almost distance-hereditary* if each connected induced subgraph H of G has the property that $d_H(x, y) \leq d_G(x, y) + 1$ for every pair of vertices x and y in H .

It is clear that a connected graph G is almost distance-hereditary if and only if for every pair of non-adjacent vertices x and y the length of each induced path from x to y is equal to either $d_G(x, y)$ or $d_G(x, y) + 1$. Some characterizations and properties of distance-hereditary graphs and almost distance-hereditary graphs will be introduced in Section 2.1

1.2 Some Well-known Results on the Hamiltonian Problem

It is well-known that the Hamiltonian cycle problem is **NP**-complete, and many sufficient conditions with respect to various parameters have been found. The Hamiltonian problem is generally considered to determine conditions under which a graph contains a spanning cycle. Named after Sir William Rowan Hamilton, this problem traces its origins to the 1850s. Today, however, the constant stream of results in this area continues to supply us with new and interesting theorems and still further questions.

1.2.1 Four Classical Results on Hamiltonian Graphs

Beginning with Dirac's theorem [37] from 1952, the approach taken to find sufficient conditions for a graph to be Hamiltonian usually involved some sort of edge density condition: providing enough edges to overcome any obstructions to the existence of a Hamiltonian cycle.

Theorem 1.1 (Dirac [37]) *If G is a graph of order n such that $\delta(G) \geq n/2$, then G is Hamiltonian.*

Later, Ore [71] improved Dirac's theorem. Ore's result relaxes Dirac's condition and only requires the degrees for pairs of non-adjacent vertices in graphs.

Theorem 1.2 (Ore [71]) *If a graph G of order n such that $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices u and v , then G is Hamiltonian.*

In [20], Bondy and Chvátal extended Ore's theorem in a very useful way. The k -(degree) closure of G , denoted by $C_k(G)$, is defined as the graph obtained by recursively joining pairs of non-adjacent vertices whose degree sum is at least k , until no such pair remains. Their fundamental result is the following:

Theorem 1.3 (Bondy and Chvátal [20]) *A graph G of order n is Hamiltonian if and only if $C_n(G)$ is Hamiltonian.*

The fourth fundamental result was introduced by Chvátal and Erdős [35].

Theorem 1.4 (Chvátal and Erdős [35]) *If G is a graph with $\alpha(G) \leq \kappa(G)$, then G is Hamiltonian.*

1.2.2 Some Generalizations

Many generalizations of Theorems 1.1-1.4 have been found. A very interesting approach was introduced by Fan [39]. By Fan's result, we need not consider "all pairs of non-adjacent vertices", but only a particular subset of these pairs.

Theorem 1.5 (Fan [39]) *Let G be a 2-connected graph with $n \geq 3$ vertices. If for all vertices u and v with $d(u, v) = 2$, $\max\{d(u), d(v)\} \geq \frac{n}{2}$ holds, then G is Hamiltonian.*

Fan's theorem is significant for several reasons. First, it is a direct generalization of Dirac's theorem. But more importantly, Fan's theorem opened an entirely new avenue for investigation – one that incorporates some of the local structure, along with a density condition. When attempting to find new adjacent results, one must not only consider the "degree bounds", but also the set of vertices for which this bounds applies.

Theorem 1.5 was strengthened in [14], where the same conditions were shown to imply that the graph is pancyclic, with a few minor exceptions.

Using a generalized degree approach based upon neighborhood union, new sufficient conditions were found. Let G be a graph and let S be a subset of vertices of G . The *degree of S* is defined to be

$$d_G(S) = |\cup_{v \in S} N(v)|.$$

The first use of the generalized degree condition was to provide another generalization of Dirac's theorem.

Theorem 1.6 (Faudree et al. [42]) *If G is a 2-connected graph of order n such that $d_G(S) \geq (2n - 1)/3$ for each $S = \{x, y\}$ where x, y are independent vertices of G , then G is Hamiltonian.*

In 1989, Lindquister [67] was able to show that Fan's restriction to vertices at distance two could also be used with generalized degrees, providing an improvement to Theorem 1.6.

Theorem 1.7 (Lindquister [67]) *If G is a 2-connected graph of order n satisfying $d_G(S) \geq (2n - 1)/3$ for every set $S = \{x, y\}$ of vertices at distance 2 in G , then G is Hamiltonian.*

Asratyan and Khachatryan [3] introduced an Ore-type adjacency condition that is reminiscent of Fan's use of vertices at distance two. Let $G_2(x)$ denote the subgraph of G induced by those vertices at distance at most 2 from x .

Theorem 1.8 (Asratyan and Khachatryan [3]) *Let G be a graph of order n . Suppose that whenever $d(x) \leq (n - 1)/2$ and y is a vertex at distance 2 from x , $d(x) + d_{G_2(x)}(y) \geq |V(G_2(x))|$, then G is Hamiltonian.*

By combining with toughness ($t(G)$), new results were obtained.

Theorem 1.9 (Jung [64]) *Let G be a 1-tough graph of order $n \geq 11$ such that $d(x) + d(y) \geq n - 4$ for all non-adjacent vertices x, y . Then G is Hamiltonian.*

Theorem 1.10 (Bauer et al. [13]) *Let G be a 2-tough graph of order n such that $d(x) + d(y) + d(z) \geq n$ for all non-adjacent vertices x, y, z . Then G is Hamiltonian.*

Theorem 1.11 (Bauer et al. [12]) *Let G be a 3-connected and 1-tough graph on $n \geq 35$ vertices. If for all vertices x, y with $d(x, y) = 2$, $\max\{d(x), d(y)\} \geq \frac{1}{2}(n - 4)$ holds, then G is Hamiltonian.*

1.2.3 Forbidden Subgraphs and Closure Concept

The first result on the Hamiltonian graph with forbidden subgraphs is due to Goodman and Hedetniemi [55].

Theorem 1.12 (Goodman and Hedetniemi [55]) *If G is a 2-connected $\{K_{1,3}, Z_1\}$ -free graph, then G is Hamiltonian.*

Since 1980, a number of fundamental results were proved, which are of the type: If particular pairs of induced subgraphs are forbidden in a 2-connected graph, then the graph is Hamiltonian. The following results are notable among them:

Theorem 1.13 (Oberly and Sumner [70]) *A connected, locally connected and $K_{1,3}$ -free graph on $n > 3$ is Hamiltonian.*

Theorem 1.14 (Duffus et al. [38]) *If G is a 2-connected $\{K_{1,3}, N\}$ -free graph, then*

(a) *if G is 2-connected, then G is Hamiltonian,*

(b) *if G is connected, then G is traceable.*

Theorem 1.15 (Shi [77]) *Let G be a 2-connected graph on $n > 3$ vertices. If G is claw-free and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v with $d(u, v) = 2$, then G is Hamiltonian.*

By combining two results of [17] and [40], one can see the following:

Theorem 1.16 (Bedrossian [17], Faudree et al. [40]) *Let R and S be two connected graphs ($R, S \neq P_3$) and G a 2-connected graph of order n . Then G is $\{R, S\}$ -free implies G is Hamiltonian if and only if $R = K_{1,3}$ and S is one of the graphs N , P_6 , $N_{0,1,2}$, $N_{0,0,2}$, or $N_{0,0,3}$ (when $n \geq 10$), or a connected induced subgraph of one of these graphs.*

A new and powerful tool for dealing with Hamiltonian problems in claw-free graphs was developed by Ryjáček [73]. This tool has not only allowed people to attack new questions, but also provided ways to prove a number of the previously established results in much simpler ways. In the following, we denote the *circumference* of a graph G by $c(G)$.

Let $x \in V_{EL}(G)$ and let $B_x = \{uv \mid u, v \in N_G(x), uv \notin E(G)\}$. Denote $G'_x = (V(G), E(G) \cup B_x)$ (i.e. G'_x is obtained from G by adding to $G[N_G(x)]$ all missing edges). The graph G'_x is called the local completion of G at x . The following proposition shows that the local completion operation preserves the claw-freeness and the value of the circumference of G . Note that this graph is different from the well-known degree sum closure due to Bondy and Chvátal [20] or any of several other closures that have been developed. For more information on closures, see [26], [28] and [74].

Proposition 1.17 (Ryjáček [73]) *Let G be a claw-free graph and let $x \in V_{EL}(G)$ be an eligible vertex of G . Then*

- (i) *the graph G'_x is claw-free,*
- (ii) *$c(G'_x) = c(G)$.*

Apparently, if $x \in V_{EL}(G)$, then x becomes simplicial in G'_x and, if $V_{EL}(G'_x) \neq \emptyset$, the local completion operation can be applied repeatedly to another vertex. Ryjáček [73] introduced the following concept:

Let G be a claw-free graph. A graph H is a closure of G , denoted $H = cl(G)$, if

- (i) *there is a sequence of graphs $G_1, \dots, G_t$ and vertices $x_1, \dots, x_{t-1}$ such that $G_1 = G$, $G_t = H$, $x_i \in V_{EL}(G_i)$ and $G_{i+1} = (G_i)_{x_i}'$, $i = 1, \dots, t-1$,*
- (ii) *$V_{EL}(H) = \emptyset$.*

The following result summarizes basic properties of the closure operation.

Theorem 1.18 (Ryjáček [73]) *Let G be a claw-free graph. Then*

- (i) *the closure $cl(G)$ is well-defined,*
- (ii) *there is a triangle-free graph H such that $cl(G)$ is the line graph of H ,*
- (iii) *$c(G) = c(cl(G))$.*

Ryjáček [73] also applied his closure to show the following:

Theorem 1.19 (Ryjáček [73]) *Every 7-connected claw-free graph is Hamiltonian.*

This still leaves open the fundamental conjecture of Matthews and Sumner [68].

Conjecture 1.20 (Matthews and Sumner [68]) *Every 4-connected claw-free graph is Hamiltonian.*

In [25], it was shown that this conjecture holds in the class of hourglass-free graphs. From the following theorem, we see that the closure of a claw-free and hourglass-free graph is also claw-free and hourglass-free.

Theorem 1.21 (Brousek et al. [32]) *Let G be a claw-free and hourglass-free graph. Then $cl(G)$ is also claw-free and hourglass-free.*

In [31] and [32], Brousek et al. introduced some sufficient conditions for a graph to be Hamiltonian.

Theorem 1.22 (Brousek et al. [31])

- (i) *Every 3-connected $\{K_{1,3}, P_7\}$ -free graph is Hamiltonian.*
- (ii) *Every 3-connected $\{K_{1,3}, N_{0,0,4}\}$ -free graph is Hamiltonian.*

(iii) *Every 3-connected $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}$ -free graph is Hamiltonian.*

(iv) *Every 3-connected $\{K_{1,3}, N_{1,1,2}\}$ -free graph is Hamiltonian.*

Theorem 1.23 (Brousek et al. [32])

(i) *Every 3-connected $\{K_{1,3}, H, P_8\}$ -free graph is Hamiltonian.*

(ii) *Every 3-connected $\{K_{1,3}, H, N_{0,0,5}\}$ -free graph is Hamiltonian.*

(iii) *Every 3-connected $\{K_{1,3}, H, N_{1,1,4}\}$ -free graph is Hamiltonian.*

Chapter 2

Hamiltonian Graphs

In this chapter, we shall give some new sufficient conditions for a graph to be Hamiltonian by combining two important types, namely, numerical conditions and structural conditions.

2.1 Almost Distance-hereditary Graphs

The characterizations and properties of distance-hereditary graphs have been studied in several papers, e.g. Howorka [62], Bandelt and Mulder [5], D'Atri and Moscarini [36], Hammer and Maffray [60], etc. We summarize those as follows:

Theorem 2.1 *For any connected graph $G = (V, E)$ the following statements are equivalent:*

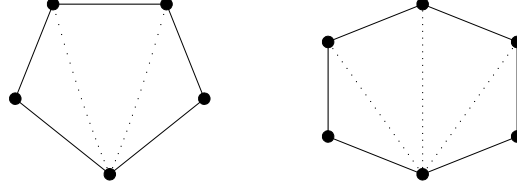
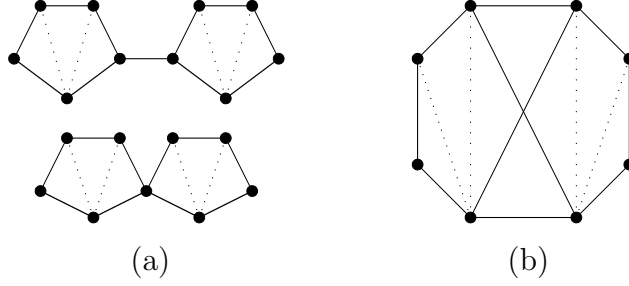
- (1) *G is a distance-hereditary graph.*
- (2) *Every cycle of length at least five in G has two crossing chords.*
- (3) *Every induced path in G is a shortest path.*
- (4) *For every hanging $h_u = (L_0, L_1, \dots, L_t)$ of G and every pair of vertices $x, y \in L_i$ ($1 \leq i \leq t$) that are in the same component of $G - L_{i-1}$, $N_{L_{i-1}}(x) = N_{L_{i-1}}(y)$.*

Theorem 2.2 *Suppose $h_u = (L_0, L_1, \dots, L_t)$ is the hanging of a connected distance-hereditary graph G at u . Then every L_i induces a graph which contains no induced path of 4 vertices. Moreover, if G is bipartite, then every L_i is an independent set of G .*

A characterization and some properties of almost distance-hereditary graphs have been given by Aider [1].

Theorem 2.3 (Aider [1]) *A graph G is almost distance-hereditary if and only if G contains neither a $2C_5$ -configuration nor a C_n -configuration for $n \geq 6$ as induced subgraphs.*

A C_n -configuration is a cycle on n vertices without chords or with chords, all of which are incident with a common vertex (see Figure 2.1). $2C_5$ -configurations have two types which are obtained by combining two C_5 -configurations (see Figure 2.2), where dashed lines represent edges, which may belong to the graphs.

Figure 2.1: C_5, C_6 -configurationsFigure 2.2: $2C_5$ -configurations

Corollary 2.4 (Aider [1]) *Let $G = (V, E)$ be an almost distance-hereditary graph and let $h_u = (L_0, L_1, \dots, L_t)$ be the hanging of G at $u \in V$. Then the following are true:*

- (1) *If x, y are two nonadjacent vertices in $G[L_i]$, which are connected by a path whose vertices have levels greater than i , then $N_{L_{i-1}}(x) = N_{L_{i-1}}(y)$.*
- (2) *There is no induced path of length 4 in $G[L_1]$ and there is no induced path of length 5 in $G[L_i]$ for any $i \geq 2$.*

In 2002, Hsieh, Ho, Hsu and Ko [63] presented an $O(|V| + |E|)$ -time algorithm to solve the Hamiltonian problem on distance-hereditary graphs.

In the following, we consider the Hamiltonian problem on almost distance-hereditary graphs.

Let $G = (V, E)$ be a graph with $|V| = n$. A vertex v of G is called *heavy* if $d(v) \geq \frac{n}{2}$. If v is not heavy, then we call it *light*. A claw of G is called *1-heavy* (*2-heavy*, respectively) if at least one (two, respectively) of its end vertices is (are, respectively) heavy. The graph G is *1-heavy* (*2-heavy*, respectively) if all claws of G are 1-heavy (2-heavy, respectively). Note that every claw-free graph is 2-heavy.

A cycle C is called *heavy* if it contains all heavy vertices of G ; it is called *extendable* if there exists a longer cycle in G containing all vertices of C .

Lemma 2.5 (Bollobás and Brightwell [18], Shi [77]) *Every 2-connected graph contains a heavy cycle.*

The following lemma is a variation of the closure lemma by Bondy and Chvátal [20].

Lemma 2.6 *Let G be a graph and $u, v \in V(G)$ be two nonadjacent heavy vertices. If $G + uv$ has a cycle C containing all heavy vertices of G , then G has a cycle containing all vertices of C .*

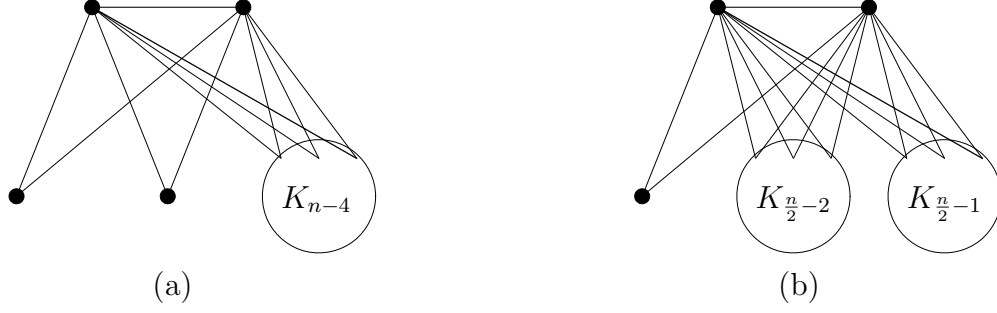


Figure 2.3: (a) A graph $K_2 \vee (2K_1 + K_{n-4})$; (b) A graph $K_2 \vee (K_1 + K_{\frac{n}{2}-2} + K_{\frac{n}{2}-1})$

The following sufficient conditions are for the existence of a Hamiltonian cycle in claw-free graphs.

Theorem 2.7 (Broersma and Veldman [30]) *Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, N_{0,2,2}\}$ -free, then G is Hamiltonian.*

Theorem 2.8 (Faudree et al. [44]) *Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, H\}$ -free, then G is Hamiltonian.*

Theorems 1.15, 2.7 and 2.8 are generalized as follows:

Theorem 2.9 (Broersma et al. [28]) *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is 2-heavy and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v with $d_G(u, v) = 2$ and $\max\{d(u), d(v)\} < \frac{n}{2}$, then G is Hamiltonian.*

Theorem 2.10 (Broersma et al. [28]) *Let G be a 2-connected graph. If G is 2-heavy, and moreover $\{P_7, N_{0,2,2}\}$ -free or $\{P_7, H\}$ -free, then G is Hamiltonian.*

Our main result in this section is the following:

Theorem 2.11 (Feng and Guo [50]) *Let $G = (V, E)$ be a 2-connected, 2-heavy and almost distance-hereditary graph on n vertices. Then G is Hamiltonian.*

Remark 2.12 *The graph $K_2 \vee (2K_1 + K_{n-4})$ (see Figure 2.3) shows that we cannot relax 2-heavy to 1-heavy in Theorem 2.11. Moreover, the graph $K_2 \vee (K_1 + K_{\frac{n}{2}-2} + K_{\frac{n}{2}-1})$ shows that we cannot also relax the degree $\frac{n}{2}$ on the end vertices of claws in Theorem 2.11.*

Since all claw-free graphs are 2-heavy, we have the following corollary:

Corollary 2.13 (Feng and Guo [49]) *Let $G = (V, E)$ be a 2-connected, claw-free and almost distance-hereditary graph. Then G is Hamiltonian.*

The graph in Figure 2.4 shows that there are 2-connected, claw-free and almost distance-hereditary graphs which do not satisfy the conditions of Theorem 1.15, 2.7, 2.8, 2.9 and 2.10, but are Hamiltonian.

Before we give the proof of Theorem 2.11, we introduce some additional terminologies and notations. Let C be a cycle in a graph G . A path P between x and y is said to be

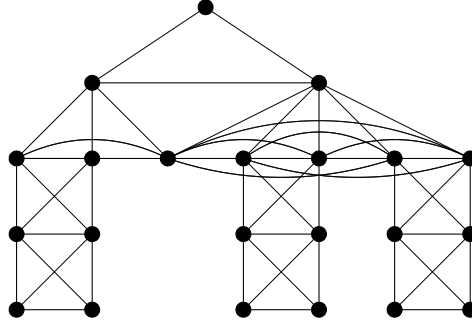


Figure 2.4: A 2-connected, claw-free and almost distance-hereditary graph

a C -bypass if $|V(P)| \geq 3$ and $V(P) \cap V(C) = \{x, y\}$. The *gap* of P with respect to C is the length of the shortest path between x and y in the cycle C .

Let G be a graph on n vertices and let C be a cycle of G . We denote by $\vec{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle with the reverse orientation. If $u, v \in V(C)$, then $u \vec{C} v$ denotes the consecutive vertices of C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. We will consider $u \vec{C} v$ and $v \overleftarrow{C} u$ both as paths and as vertex sets.

Proof of Theorem 2.11: By Lemma 2.5, G contains a heavy cycle. Let $\vec{C} = u_1 u_2 \dots u_m u_1$ be a longest heavy cycle of G with the given direction. Assume that G is not Hamiltonian. Since G is 2-connected, there exists at least one C -bypass. Let $P = v_1 v_2 \dots v_t$ be a shortest C -bypass having minimal gap among all C -bypasses. It is easy to see that $v_1 v_k \notin E(G)$ for $k = 3, \dots, t-1$ and $v_k v_t \notin E(G)$ for $k = 2, 3, \dots, t-2$ if $t \geq 4$. Without loss of generality, let $u_1 = v_1$ and $u_s = v_t$. It is clear that $\min\{s, t\} \geq 3$ and $u_i v_j \notin E(G)$ for $i = 2, 3, \dots, s-1$ and $j = 2, 3, \dots, t-1$.

If $t \geq 5$, then we consider the subgraph $H = G[V(P) \cup u_1 \vec{C} u_s] - \{u_1\}$. It is clear that $d_H(u_2, v_2) \geq 4$, which yields a contradiction to the fact that G is almost distance-hereditary and $d_G(u_2, v_2) = 2$.

For the case when $t = 4$, we consider $H = G[V(P) \cup u_1 \vec{C} u_s] - \{u_1\}$. Since $d_G(u_2, v_2) = 2$ and G is almost distance-hereditary, we have $d_H(u_2, v_2) \leq 3$. It is evident that $u_2 u_s \in E(G)$. If $u_{s-1} u_{s+1} \in E(G)$, then the cycle $u_1 P u_s u_2 \vec{C} u_{s-1} u_{s+1} \overleftarrow{C} u_m u_1$ is longer than C , a contradiction. If $u_{s-1} u_{s+1} \notin E(G)$, then u_{s-1} and u_{s+1} are heavy. By considering the graph $G + u_{s-1} u_{s+1}$, we see that $G + u_{s-1} u_{s+1}$ has a cycle C' containing all vertices of C , which is longer than C . By Lemma 2.6, there is a cycle in G containing all vertices of C' , a contradiction to the choice of C .

We now consider the remaining case when $t = 3$. Let $d_C(v_2) = \gamma$ and $N_C(v_2) = \{u_{i_1} (= u_1), u_{i_2} (= u_s), \dots, u_{i_\gamma}\}$ with $i_1 < i_2 < \dots < i_\gamma$. It is clear that $i_{k+1} - i_k \geq 2$ for $k = 1, 2, \dots, \gamma$, where $i_{\gamma+1} = i_1$.

By the choice of P , we see that $s-1 = i_2 - i_1 = \min\{i_{k+1} - i_k \mid k = 1, 2, \dots, \gamma\}$. Since either u_{i_k-1}, u_{i_k+1} are heavy or $u_{i_k-1} u_{i_k+1} \in E(G)$ for $k = 1, 2, \dots, \gamma$, it is not difficult to check that $u_{i_\alpha} u_{i_\beta \pm 1}$, $u_{i_\alpha-1} u_{i_\beta-1}$, $u_{i_\alpha+1} u_{i_\beta+1} \notin E(G)$. Moreover, if $s \geq 5$, then $u_{i_\alpha} u_{i_\beta \pm 2}$, $u_{i_\alpha-1} u_{i_\beta-2}$, $u_{i_\alpha+2} u_{i_\beta+1} \notin E(G)$ for all $\alpha, \beta \in \{1, 2, \dots, \gamma\}$ with $\alpha \neq \beta$, where $i_1 - 1 = m$.

and $i_1 - 2 = m - 1$. By Lemma 2.6, it is easy to see that there exists at most one integer $k \in \{1, 2, \dots, \gamma\}$ such that $u_{i_k-1}u_{i_k+1} \notin E(G)$.

If $3 \leq s \leq 4$, then it is not difficult to see that C is extendable, a contradiction. Thus, $s \geq 5$.

Let $H_k = G[\{v_2\} \cup u_{i_k} \xrightarrow{C} u_{i_{k+1}}] - \{u_{i_k}\}$ for $k = 1, 2, \dots, \gamma$. Note that $d_G(u_{i_{k+1}}, v_2) = 2$. Since G is almost distance-hereditary, it is clear that $2 \leq d_{H_k}(u_{i_{k+1}}, v_2) \leq 3$. Since $u_{i_k+1}u_{i_{k+1}} \notin E(G)$, we have $d_{H_k}(u_{i_{k+1}}, v_2) = 3$. We consider the following two cases:

Case 1. There exists an integer k , $1 \leq k \leq \gamma$, such that $u_{i_k} \xrightarrow{C} u_{i_{k+1}}$ contains a vertex u_ℓ with $i_k + 3 \leq \ell \leq i_{k+1} - 2$ and $u_{i_k+1}u_\ell, u_\ell u_{i_{k+1}} \in E(G)$.

Without loss of generality, suppose $i_k = i_1 = 1$ and $i_{k+1} = i_2 = s$. From the discussion above, we see that G contains at least one of the edges u_2u_m and $u_{s-1}u_{s+1}$.

Subcase 1.1. Either $u_2u_m \notin E(G)$ or $u_{s-1}u_{s+1} \notin E(G)$.

If $u_2u_m \notin E(G)$, then u_2 and u_m are heavy. It is easy to see that u_{s-1} , u_s and u_{s+1} are light. Moreover, $u_{s-1}u_{s+1} \in E(G)$.

We confirm that $\ell \leq s - 3$. If $\ell = s - 2$, then by considering the induced subgraph $H = G[\{v_2, u_1, u_2, u_{s-2}, u_{s-1}, u_{s+1}\}]$, we see that $d_H(v_2, u_{s+1}) \leq 3$ because G is almost distance-hereditary. From $u_1u_{s-2}, u_1u_{s-1}, u_1u_{s+1}, u_2u_{s+1} \notin E(G)$, it follows that $d_H(v_2, u_{s+1}) \geq 4$, a contradiction. So, we see that $\ell \leq s - 3$.

If $u_2u_{\ell+1} \in E(G)$, then we obtain a cycle $v_2u_su_\ell \xleftarrow{C} u_2u_{\ell+1} \xrightarrow{C} u_{s-1}u_{s+1} \xrightarrow{C} u_mu_1v_2$ longer than C , a contradiction. Thus, $u_2u_{\ell+1} \notin E(G)$.

Since G is 2-heavy, u_s is light and $u_2u_s, u_2u_{\ell+1} \notin E(G)$, we have $u_{\ell+1}u_s \in E(G)$. Otherwise, G has a claw $G[\{u_\ell, u_2, u_{\ell+1}, u_s\}]$ which contains two heavy vertices u_2 and $u_{\ell+1}$. With a similar discussion as above, $G + u_2u_{\ell+1}$ has a heavy cycle longer than C . By Lemma 2.6, we deduce a contradiction.

Let $H' = G[\{v_2, u_1, u_2, u_\ell, u_{\ell+1}\}]$. Since $d_G(v_2, u_{\ell+1}) = 2$, we see that $d_{H'}(v_2, u_{\ell+1}) \leq 3$. If $u_1u_\ell \in E(G)$ ($u_1u_{\ell+1} \in E(G)$, respectively), then $v_2u_1u_\ell \xleftarrow{C} u_2u_m \xleftarrow{C} u_{s+1}u_{s-1} \xleftarrow{C} u_{\ell+1}u_sv_2$ ($v_2u_1u_{\ell+1} \xleftarrow{C} u_{s-1}u_{s+1} \xleftarrow{C} u_mu_2 \xleftarrow{C} u_\ell u_sv_2$, respectively) is a cycle in $G + u_2u_m$, which is longer than C . By Lemma 2.6, we obtain a contradiction to the choice of C . Thus, $u_1u_\ell, u_1u_{\ell+1} \notin E(G)$. By recalling that $u_2u_{\ell+1} \notin E(G)$, we see that $d_{H'}(v_2, u_{\ell+1}) = 4$, a contradiction. Analogously, one can consider the case when $u_{s-1}u_{s+1} \notin E(G)$.

Subcase 1.2. $u_2u_m, u_{s-1}u_{s+1} \in E(G)$.

We confirm that $\ell \leq s - 3$. If $\ell = s - 2$, then by considering the induced subgraph $H = G[\{v_2, u_1, u_2, u_{s-2}, u_{s-1}\}]$, we see that $d_H(v_2, u_{s-1}) \leq 3$ because G is almost distance-hereditary. From $u_1u_{s-2}, u_1u_{s-1} \notin E(G)$, it follows that $u_2u_{s-1} \in E(G)$. So, we obtain a cycle $v_2u_su_{s-2} \xleftarrow{C} u_2u_{s-1}u_{s+1} \xrightarrow{C} u_mu_1v_2$ longer than C , a contradiction. Thus, $\ell \leq s - 3$.

Let $H' = G[\{v_2, u_m, u_2, u_\ell, u_s\}]$. Since $d_G(v_2, u_m) = 2$, we have $d_{H'}(v_2, u_m) \leq 3$. From $u_mu_s, u_2u_s \notin E(G)$, it follows that $u_\ell u_m \in E(G)$.

If $u_2u_{\ell+1} \in E(G)$ ($u_{\ell+1}u_s \in E(G)$, respectively), then we see that the cycle $v_2u_su_\ell \xrightarrow{\leftarrow} u_2u_{\ell+1} \xrightarrow{\leftarrow} u_{s-1}u_{s+1} \xrightarrow{\leftarrow} u_mu_1v_2$ ($v_2u_1u_2 \xrightarrow{\leftarrow} u_\ell u_m \xrightarrow{\leftarrow} u_{s+1}u_{s-1} \xrightarrow{\leftarrow} u_{\ell+1}u_sv_2$, respectively) is longer than C , a contradiction. So, $u_2u_{\ell+1}, u_{\ell+1}u_s \notin E(G)$. Since $u_2u_s \notin E(G)$, the subset $\{u_\ell, u_2, u_{\ell+1}, u_s\}$ induces a claw. Because G is 2-heavy, there are at least two heavy vertices of $\{u_2, u_{\ell+1}, u_s\}$. By Lemma 2.6, we can obtain a heavy cycle longer than C , a contradiction.

Case 2. There exists no integer k , $1 \leq k \leq \gamma$, such that $u_{i_k} \xrightarrow{\leftarrow} u_{i_{k+1}}$ contains a vertex u_ℓ with $i_k + 3 \leq \ell \leq i_{k+1} - 2$ and $u_{i_k+1}u_\ell, u_\ell u_{i_{k+1}} \in E(G)$.

From this assumption, we obtain $u_{i_k+1}u_{i_{k+1}-1} \in E(G)$ for all $1 \leq k \leq \gamma$, where $i_{\gamma+1} = i_1$. Recall that there exists at most one integer k , $1 \leq k \leq \gamma$ such that $u_{i_k-1}u_{i_{k+1}} \notin E(G)$. We consider the following two subcases:

Subcase 2.1. $\gamma \geq 3$.

Suppose that there is an integer μ , $1 \leq \mu \leq \gamma$, such that $u_{i_\mu-1}u_{i_\mu+1} \notin E(G)$, say $\mu = 1$. Then, u_2 and u_m are heavy. It is not difficult to see that u_{i_k-1}, u_{i_k} and u_{i_k+1} are light. Moreover, $u_{i_k-1}u_{i_{k+1}} \in E(G)$ for $2 \leq k \leq \gamma$. Let $\hat{H} = G[\cup_{k=1}^\gamma \{u_{i_k-1}, u_{i_k+1}\}]$. Since $d_G(u_2, u_m) = 2$, we have $d_{\hat{H}}(u_2, u_m) \leq 3$. From $u_2u_{i_k+1}, u_mu_{i_k-1} \notin E(G)$ for $2 \leq k \leq \gamma$, it follows that $d_{\hat{H}}(u_2, u_m) = 3$. So, there are $x, y \in \cup_{k=2}^\gamma \{u_{i_k-1}, u_{i_k+1}\}$ with $u_2x, xy, yu_m \in E(G)$. By the way, we see that $x = u_{i_\alpha-1}$ and $y = u_{i_\beta+1}$ for some $2 \leq \alpha, \beta \leq \gamma$.

If $\alpha = 2$ and $\beta = \gamma$, then $\{u_{i_2-1}, u_2, u_{i_\gamma+1}, u_{i_2+1}\}$ induces a claw with one heavy end vertex, a contradiction.

If $\alpha = 2$ and $\beta \neq \gamma$, then $\{u_{i_\beta+1}, u_{i_2-1}, u_m, u_{i_\beta+1}-1\}$ induces a claw with one heavy end vertex, a contradiction. Analogously, we deduce a contradiction for the case when $\alpha \neq 2$ and $\beta = \gamma$.

So, $\alpha \neq 2$ and $\beta \neq \gamma$. Now, we see that $\{u_{i_\alpha-1}, u_2, u_{i_\alpha-1+1}, u_{i_\beta+1}\}$ induces a claw with one heavy end vertex, a contradiction.

Thus, we see that $u_{i_k-1}u_{i_k+1} \in E(G)$ for all $1 \leq k \leq \gamma$. Let $\tilde{H} = G[\cup_{k=1}^\gamma \{u_{i_k-1}, u_{i_k+1}\}] - \{u_2\}$. Since $d_G(u_{i_2-1}, u_m) = 2$, we have $d_{\tilde{H}}(u_{i_2-1}, u_m) \leq 3$.

If $d_{\tilde{H}}(u_{i_2-1}, u_m) = 2$, then there is $x \in \cup_{k=2}^\gamma \{u_{i_k-1}, u_{i_k+1}\}$ with $u_{i_2-1}x, xu_m \in E(G)$. Since $u_{i_2-1}u_{i_j-1} \notin E(G)$ for $3 \leq j \leq \gamma$ and $u_mu_{i_j-1} \notin E(G)$ for $2 \leq j \leq \gamma$, we have $x = u_{i_k+1}$ for some $2 \leq k \leq \gamma$. So, we see that either $G[\{u_{i_k+1}, u_{i_2-1}, u_m, u_{i_k-1}\}]$ (if $k > 2$) or $G[\{u_{i_2+1}, u_{i_2-1}, u_m, u_{i_3-1}\}]$ (if $k = 2$) is a claw in G . Thus, $\cup_{k=2}^\gamma \{u_{i_k-1}\}$ contains at least two heavy vertices. By Lemma 2.6, we can obtain a heavy cycle longer than C , a contradiction.

So, $d_{\tilde{H}}(u_{i_2-1}, u_m) = 3$. This implies that there are $x, y \in \cup_{k=2}^\gamma \{u_{i_k-1}, u_{i_k+1}\}$ with $u_{i_2-1}x, xy, yu_m \in E(G)$. Analogously, we see that $x, y \in \cup_{k=2}^\gamma \{u_{i_k+1}\}$, say $x = u_{i_\alpha+1}$ and $y = u_{i_\beta+1}$ for some $2 \leq \alpha, \beta \leq \gamma$. This implies that $u_{i_\alpha+1}u_{i_\beta+1} \in E(G)$, a contradiction.

Subcase 2.2. $\gamma = 2$.

Recall that $u_2u_{s-1}, u_{s+1}u_m \in E(G)$. Firstly, we confirm that $u_2u_m, u_{s-1}u_{s+1} \in E(G)$.

If $u_2u_m \notin E(G)$, then u_2 and u_m are heavy. From the discussion above, we see that $u_{s-2}, u_{s-1}, u_s, u_{s+1}$ and u_{s+2} are light and $u_{s-1}u_{s+1} \in E(G)$.

If $s = 5$, then we observe the subgraph $H = G[\{u_1, u_2, u_3, u_6, u_m\}]$. Since $u_1u_3, u_2u_m, u_3u_m \notin E(G)$ and $u_iu_6 \notin E(G)$ for $i = 1, 2, 3$, we have $d_H(u_3, u_6) = 4$, a contradiction to the fact that G is almost distance-hereditary and $d_G(u_3, u_6) = 2$.

So, $s \geq 6$. Consider the subgraph $H = G[\{u_{s-1}, u_2, u_{s-2}, u_s\}]$.

If $u_{s-2}u_s \in E(G)$, then the cycle $v_2u_su_{s-2} \xrightarrow{\leftarrow} u_2u_{s-1}u_{s+1} \xrightarrow{\rightarrow} u_mu_1v_2$ is longer than C , a contradiction. So, $u_{s-2}u_s \notin E(G)$.

Since u_s and u_{s-2} are light and $u_2u_s \notin E(G)$, we have $u_2u_{s-2} \in E(G)$. Otherwise, $\{u_{s-1}, u_2, u_{s-2}, u_s\}$ induces a claw with two light end vertices, a contradiction.

If $u_{s-2}u_{s+1} \in E(G)$, then the cycle $v_2u_su_{s-1}u_2 \xrightarrow{\rightarrow} u_{s-2}u_{s+1} \xrightarrow{\rightarrow} u_mu_1v_2$ is longer than C , a contradiction. Thus, $u_{s-2}u_{s+1} \notin E(G)$.

Denote $\overline{H} = G[\{u_m, u_1, u_2, u_{s-2}, u_{s+1}\}]$. By recalling that $u_1u_{s-2}, u_1u_{s+1}, u_2u_m, u_2u_{s+1}, u_{s-2}u_m \notin E(G)$, we have $d_{\overline{H}}(u_{s-2}, u_{s+1}) = 4$, a contradiction.

So, we see that $u_2u_m \in E(G)$. Analogously, $u_{s-1}u_{s+1} \in E(G)$.

Secondly, consider the case when $m = s + 3$. By the choice of P and s , we have $s = 5$ and $m = 8$. Let $H = G[\{u_2, u_4, u_5, u_7, u_8\}]$. Recalling that $u_2u_5, u_2u_7, u_4u_7, u_4u_8, u_5u_7, u_5u_8 \notin E(G)$, we have $d_H(u_5, u_7) = 4$, a contradiction.

Finally, we consider the case when $m \geq s + 4$.

If one of $\{u_m, u_2, u_{s-1}, u_{s+1}\}$ is heavy, say u_2 , then u_s, u_{s+1} and u_{s+2} are light. Let $H' = G[\{u_{s+1}, u_s, u_{s+2}, u_m\}]$. If $u_su_{s+2} \in E(G)$, then $v_2u_su_{s+2} \xrightarrow{\rightarrow} u_mu_{s+1}u_{s-1} \xrightarrow{\leftarrow} u_1v_2$ is longer than C , a contradiction. So, we have $u_su_{s+2} \notin E(G)$. To avoid the claw H' with at least two light end vertices, we have $u_{s+2}u_m \in E(G)$.

Let $H'' = G[\{u_2, u_{s-1}, u_s, u_{s+2}, u_m\}]$. If $u_{s-1}u_{s+2} \in E(G)$, then the cycle $v_2u_1 \xrightarrow{\rightarrow} u_{s-1}u_{s+2} \xrightarrow{\rightarrow} u_mu_{s+1}u_sv_2$ is longer than C , a contradiction. So, we have $u_{s-1}u_{s+2} \notin E(G)$. From $u_2u_s, u_2u_{s+2}, u_{s-1}u_m, u_su_{s+2}, u_su_m \notin E(G)$, it follows that $d_{H''}(u_s, u_{s+2}) = 4$, a contradiction to $d_G(u_s, u_{s+2}) = 2$.

Therefore, u_m, u_2, u_{s-1} and u_{s+1} are light. With a similar discussion as above, we see that $u_{s-1}u_{s+2}, u_su_{s+2} \notin E(G)$. Moreover, we deduce that $u_{s+2}u_m \in E(G)$. Otherwise, since $u_{s-1}u_m \notin E(G)$, the subset $\{u_{s+1}, u_{s-1}, u_{s+2}, u_m\}$ induces a claw with two light end vertices, namely u_{s-1}, u_m , a contradiction. Let $H''' = G[\{u_2, u_{s-1}, u_s, u_{s+2}, u_m\}]$. It is not difficult to see that $d_{H'''}(u_s, u_{s+2}) = 4$, a contradiction to $d_G(u_s, u_{s+2}) = 2$. The proof is completed. $\blacksquare$

2.2 Graphs with Ore-type Condition

In this section, we consider a condition of Ore-type, denoted by (A), with respect to the maximal induced path.

Definition 2.14 *A graph G satisfies the condition (A) if for every maximal induced path P of length $p \geq 2$ with end vertices u and v it holds:*

$$d(u) + d(v) \geq |V(G)| - p + 2.$$

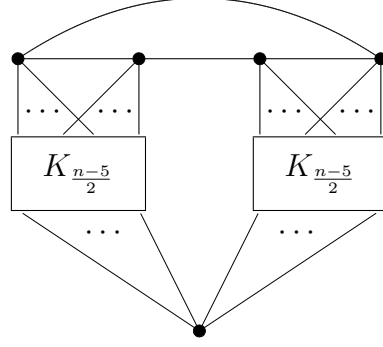


Figure 2.5: A claw-free graph satisfying (A)

Note that the hypothesis of Theorem 1.2 implies that G satisfies the condition (A). It is not difficult to see that the graph in Figure 2.5 and an arbitrary cycle satisfy the condition (A) but do not satisfy the conditions in Theorem 1.2, 1.5, 1.15. Furthermore, both graphs are Hamiltonian.

In what follows, we consider the connected claw-free graphs satisfying (A) and obtain the following:

Theorem 2.15 (Feng [46]) *Let G be a connected claw-free graph. If G satisfies (A), then each maximum induced cycle of G is 2-dominating.*

Theorem 2.16 (Feng [46]) *Let G be a connected claw-free graph. If G satisfies (A), then G is Hamiltonian.*

For the proofs of Theorem 2.15 and Theorem 2.16, we need the following results:

Theorem 2.17 (Balakrishnan and Paulraja [4]) *All 2-connected claw-free and chordal graphs are Hamiltonian.*

Theorem 2.18 (Brandstädt et al. [22]) *Every claw-free graph that contains an induced doubly dominating cycle has a Hamiltonian cycle.*

Lemma 2.19 *Let G be a connected graph satisfying (A) and P a maximal induced path of length at least 2 with the end vertices u, v . Then $N(u) \cap N(v) \neq \emptyset$.*

Proof. Let $R = G - V(P)$ and p be the length of P . It is clear that $|V(R)| = |V(G)| - |V(P)| = |V(G)| - p - 1$. According to the condition (A), we have $d(u) + d(v) \geq |V(G)| - p + 2$. Because P is an induced path, it is evident that $d_P(u) = d_P(v) = 1$. So, we obtain

$$\begin{aligned}
 d_R(u) + d_R(v) &= (d(u) - d_P(u)) + (d(v) - d_P(v)) \\
 &= d(u) + d(v) - (d_P(u) + d_P(v)) \\
 &\geq |V(G)| - p + 2 - 2 \\
 &= |V(R)| + 1.
 \end{aligned}$$

This inequality shows that $N(u) \cap N(v) \neq \emptyset$. ■

Lemma 2.20 *Let G be a connected graph on $n \geq 3$ vertices. If G satisfies the condition (A), then G is 2-connected.*

Proof. Let $u, v \in V(G)$ be arbitrary. We only need to show that there are two disjoint paths from u to v in G .

If $uv \notin E(G)$, then there is a shortest path between them. Note that any such path is an induced path, and each induced path can be enlarged to a maximal induced path. By Lemma 2.19, we are done.

Assume that $uv \in E(G)$. If $N(u) \cap N(v) \neq \emptyset$, then we are also done. For the case when $N(u) \cap N(v) = \emptyset$, we see that uv is contained in an induced path, since G is connected and $n \geq 3$. Easily, one can obtain two distinct paths between them. ■

Lemma 2.21 *Let G be a connected claw-free graph satisfying (A) and let $C = v_1v_2 \dots v_tv_1$ be an induced cycle of G with $t \geq 4$. If $N_2(C) \neq \emptyset$, then for each $x \in N_1(C) \cap N_2(C)$, there exists exactly one integer i , $1 \leq i \leq t$, such that $xv_i, xv_{i+1} \in E(G)$, where the indices are taken modulo t with $v_0 = v_t$.*

Proof. Because G is claw-free and C is an induced cycle with $t \geq 4$, it is easy to see that the statement is true. ■

Proof of Theorem 2.15: Let $C = v_1v_2 \dots v_tv_1$ be a maximum induced cycle of G . We confirm that C is a 2-dominating cycle, i.e. $d(x, C) \leq 2$ for all $x \in V(G)$.

Suppose to the contrary that $\{z \in V(G) \mid d(z, C) \geq 3\} \neq \emptyset$, say $y \in N_3(C)$. Let $yy_1y_2v_i$ be a shortest path from y to C for some i , $1 \leq i \leq t$. We consider the following two cases:

Case 1. $t \geq 4$.

By Lemma 2.21, we see that G contains exactly one of the edges y_2v_{i-1} and y_2v_{i+1} , say $y_2v_{i+1} \in E(G)$. It is clear that the path $P = yy_1y_2v_{i+1}v_{i+2} \dots v_{i-1}$ is an induced path of G .

Since $d(y, C) = 3$, it is not difficult to see by Lemma 2.19 that P is not maximal. Let $P' = u_0 \dots u_\alpha y_1 y_2 v_{i+1} v_{i+2} \dots v_{i-2} z_\beta \dots z_1 z_0$ be an enlarged maximal induced path of P , where $u_\alpha = y$ and $z_\beta = v_{i-1}$. Clearly, $\alpha + \beta \geq 1$. By Lemma 2.19, there is a vertex $w \in N(u_0) \cap N(z_0)$. We consider the following subcases:

Subcase 1.1. $\alpha = 1$ and $\beta = 0$.

From $y \in N_3(C)$, it follows that $wy \notin E(G)$. Since G is claw-free and P' is an induced path, it is not difficult to see that $wy_1, wy_2 \notin E(G)$ and $wv_j \notin E(G)$ for all j , $1 \leq j \leq t$ and $j \neq i-2, i-1, i$.

If $wv_{i-2} \notin E(G)$, then $wu_0yy_1y_2v_{i+1}v_{i+2} \dots v_{i-2}v_{i-1}w$ is an induced cycle longer than C , a contradiction.

If $wv_{i-2} \in E(G)$, then $wu_0yy_1y_2v_{i+1}v_{i+2} \dots v_{i-2}w$ is also an induced cycle longer than C , a contradiction.

Subcase 1.2. $\alpha = 0$ and $\beta = 1$.

Since G is claw-free and P' is an induced path, it is easy to see that $wy_2 \notin E(G)$. Because $d(y, C) = 3$, we have $wv_j \notin E(G)$ for all $j, 1 \leq j \leq t$.

If $wy_1 \in E(G)$, then the induced cycle $wy_1y_2v_{i+1}v_{i+2} \dots v_{i-2}v_{i-1}z_0w$ is longer than C , a contradiction.

If $wy_1 \notin E(G)$, then the induced cycle $wyy_1y_2v_{i+1}v_{i+2} \dots v_{i-2}v_{i-1}z_0w$ is longer than C , a contradiction.

Subcase 1.3. $\alpha \geq 1$ and $\beta \geq 1$.

Since G is claw-free and P' is an induced path, it is easy to see that G contains at most two edges wu_1 and wz_1 . Thus, we can easily obtain an induced cycle longer than C , a contradiction.

Case 2. $t = 3$.

Without loss of generality, let $i = 1$. Denote $P = yy_1y_2v_1$. By Lemma 2.19, we see that P is not a maximal induced path in G because $d(y, C) = 3$. Let $P' = u_0u_1 \dots u_\alpha y_1y_2z_\beta \dots z_1z_0$ be an enlarged maximal induced path of P , where $u_\alpha = y$ and $z_\beta = v_1$. It is clear that $\alpha + \beta \geq 1$. By Lemma 2.19, there is a vertex $w \in N(u_0) \cap N(z_0)$.

Since the length of the cycle $wP'w$ is $\alpha + \beta + 5 \geq 6$ and w is possibly adjacent with u_1 and z_1 , it is not difficult to see that G contains an induced cycle of length at least 4, a contradiction.

The proof of Theorem 2.15 is completed. ■

Before we give the proof of Theorem 2.16, we firstly prove the following lemma:

Lemma 2.22 *Let G be a connected claw-free graph satisfying (A) and containing a maximum induced cycle $C = v_1v_2 \dots v_tv_1$ with $t \geq 4$. If there is a vertex $y \in N_2(C)$ with $xv_1, xv_2 \in E(G)$ for some $x \in N(y)$, then t is even and there are $t/2$ vertices $x_1, x_2, \dots, x_{t/2}$ in $G - V(C) \cup \{y\}$ such that*

$$yx_i, x_iv_{2i-1}, x_iv_{2i} \in E(G), \quad i = 1, 2, \dots, t/2, \quad \text{and } G[\{x_1, x_2, \dots, x_{t/2}\}] \text{ is complete.}$$

Furthermore, there exists no vertex $y' \in N_2(C)$ containing a neighbour x'_i such that $x'_iv_{2i}, x'_iv_{2i+1} \in E(G)$ for some $1 \leq i \leq t/2$, where $v_{t+1} = v_1$.

Proof. By Lemma 2.21, we see that $N_C(x) = \{v_1, v_2\}$. Denote $x = x_1$ and $P = yy_1v_1v_t \dots v_3$. It is easy to see that P is an induced path. Firstly, we show the following claim:

Claim A *P is a maximal induced path in G .*

Proof. Suppose to the contrary that $P' = u_0u_1 \dots u_\alpha v_4 \dots v_tv_1x_1z_\beta \dots z_1z_0$ is an enlarged maximal induced path of P with $u_\alpha = v_3$, $z_\beta = y$ and $\alpha + \beta \geq 1$.

By Lemma 2.19, there is a vertex $w \in N(z_0) \cap N(u_0)$. We see that the length of the cycle $wP'w$ is

$$\alpha + t - 2 + 2 + \beta + 2 = t + 2 + \alpha + \beta \geq t + 3.$$

It is not difficult to see that the cycle $wP'w$ has at most two chords:

$$\begin{aligned} &\text{either } wu_1, wz_1 \quad \text{if } \alpha \geq 1 \text{ and } \beta \geq 1, \\ &\quad \text{or } wv_4, wz_1 \quad \text{if } \alpha = 0 \text{ and } \beta \geq 1, \\ &\quad \text{or } wu_1, wx_1 \quad \text{if } \alpha \geq 1 \text{ and } \beta = 0. \end{aligned}$$

It follows that G contains an induced cycle of length at least $t + 1$, a contradiction. $\square$

By Claim A and Lemma 2.19, there is a vertex $x_2 \in N(y) \cap N(v_3)$ with $x_2 \neq x_1$. Since G is claw-free, we have $x_2v_i \notin E(G)$ for $i = 1$ and $i = 5, \dots, t$ if $t \geq 5$. By Lemma 2.21, we see that either $N_C(x_2) = \{v_2, v_3\}$ or $N_C(x_2) = \{v_3, v_4\}$.

Assume that $x_2v_2 \in E(G)$. If $x_1x_2 \in E(G)$ ($x_1x_2 \notin E(G)$, respectively), then $x_2v_3v_4 \dots v_tv_1x_1x_2$ ($x_2v_3v_4 \dots v_tv_1x_1y$, respectively) is an induced cycle and longer than C , a contradiction. So, we see that $N_C(x_2) = \{v_3, v_4\}$. Moreover, we have $x_1x_2 \in E(G)$. Otherwise, the induced cycle $yx_2v_4v_5 \dots v_tv_1x_1y$ is longer than C , a contradiction to the choice of C .

Secondly, we show that t is even. Suppose to the contrary that $t = 2s + 1$ for some $s \in \mathbb{N}$ with $s \geq 2$. With a similar discussion as above, there are $x_1, x_2, \dots, x_s \in V(G)$ satisfying $yx_i, x_iv_{2i-1}, x_iv_{2i} \in E(G)$ for $i = 1, 2, \dots, s$, $x_ix_{i+1} \in E(G)$ for $i = 1, 2, \dots, s-1$, $x_iv_{2j-1}, x_iv_{2j} \notin E(G)$ for i, j with $1 \leq i < j \leq s$ and $x_iv_t \notin E(G)$ for all i , $1 \leq i \leq s$.

With a similar discussion as in the proof of Claim A, the path $v_tv_1 \dots v_{t-2}x_sy$ is a maximal induced path in G . By Lemma 2.19, there is a vertex $x' \in N(y) \cap N(v_t)$. It is easy to see by Lemma 2.21 that either $N_C(x') = \{v_{t-1}, v_t\}$ or $N_C(x') = \{v_1, v_t\}$.

For the case when $x'v_{t-1} \in E(G)$, we obtain either the induced cycle $x'v_tv_1 \dots v_{t-2}x_sx'$ (if $x'x_s \in E(G)$) or the induced cycle $x'v_tv_1 \dots v_{t-2}x_syx'$ (if $x'x_s \notin E(G)$). Both cycles are longer than C , a contradiction. For the other case when $x'v_1 \in E(G)$, we obtain analogously an induced cycle longer than C . Therefore, t is even.

According to the proof above, we see that there are $x_1, x_2, \dots, x_{t/2} \in V(G)$ such that $yx_i, x_iv_{2i-1}, x_iv_{2i} \in E(G)$ for $i = 1, 2, \dots, t/2$, $x_1x_{t/2} \in E(G)$, $x_ix_{i+1} \in E(G)$ for $i = 1, 2, \dots, t/2 - 1$ and $x_iv_{2j-1}, x_iv_{2j} \notin E(G)$ for all $1 \leq i, j \leq t/2$ with $i \neq j$.

Thirdly, we prove that the subgraph $G[\{x_1, x_2, \dots, x_{t/2}\}]$ is complete. If $t = 4$, then we are done. So, assume that $t \geq 6$. Because of the symmetry of the cycle $x_1x_2 \dots x_{t/2}x_1$, we only need to prove $x_1x_i \in E(G)$ for all $i = 3, 4, \dots, t/2 - 1$.

To avoid the claw $G[\{x_2, x_1, x_3, v_4\}]$, we see that G must contain the edge x_1x_3 because $x_1v_4, x_3v_4 \notin E(G)$.

Assume that $x_1x_i \in E(G)$ for some $3 \leq i \leq t/2 - 2$. Then we have $x_1x_{i+1} \in E(G)$ (otherwise, $G[\{x_i, x_1, x_{i+1}, v_{2i}\}]$ is a claw). Consequently, $x_1x_i \in E(G)$ for all $i = 3, 4, \dots, t/2 - 1$. Therefore, $G[\{x_1, x_2, \dots, x_{t/2}\}]$ is complete.

Finally, suppose without loss of generality that there are $x', y' \in V(G)$ with $y' \in N_2(C)$ and $x'y', x'v_2, x'v_3 \in E(G)$. By Lemma 2.21, we see that $x' \neq x_1$ and $x'v_i \notin E(G)$ for $i = 1$ and $i = 4, 5, \dots, t$.

By Claim A, it is clear that $P = yx_1v_1v_t \dots v_3$ is a maximal induced path of G . So, we see that either $x'x_1 \in E(G)$ or $x'y \in E(G)$. If $x'x_1 \in E(G)$, then $x'v_3v_4 \dots v_tv_1x_1x'$ is an

induced cycle longer than C , a contradiction. So, we have $x'x_1 \notin E(G)$ and $x'y \in E(G)$. Then $x'v_3v_4 \dots v_tv_1x_1yx'$ is an induced cycle longer than C , a contradiction. The lemma is proved. ■

Proof of Theorem 2.16: Let $C = v_1v_2 \dots v_tv_1$ be a maximum induced cycle of G . If $t = 3$, then G is a chordal graph. By Theorem 2.17, G is Hamiltonian. We consider the following two cases for $t \geq 4$:

Case 1. $N_2(C) = \emptyset$.

If $N_1(C) = \emptyset$, then C contains all vertices of G , i.e. G is a cycle, and we are done. So, assume that $N_1(C) \neq \emptyset$. Since C is an induced cycle and G is claw-free, we see that $d_G(z) \geq 2$ for all $z \in N_1(C)$. Therefore, C is an induced doubly dominating cycle of G . By Theorem 2.18, G has a Hamiltonian cycle.

Case 2. $N_2(C) \neq \emptyset$.

It is evident that the result of Lemma 2.22 holds. We have the following claim:

Claim A $G[N_2(C)]$ is complete.

Proof. Suppose that there are two vertices $y, y' \in N_2(C)$ with $yy' \notin E(G)$. Then $N(y) \cap N_1(C) \cap N(y') = \emptyset$. Otherwise, if $x \in N(y) \cap N_1(C) \cap N(y')$ with $xv_1 \in E(G)$, then we obtain a claw $G[\{x, y, y', v_1\}]$, a contradiction.

Let x_1, x_2 (x'_1, x'_2 , respectively) be the corresponding vertices belonging to $N(y)$ ($N(y')$, respectively) with $x_1v_1, x_1v_2, x_2v_3, x_2v_4 \in E(G)$ ($x'_1v_1, x'_1v_2, x'_2v_3, x'_2v_4 \in E(G)$, respectively). Since G is claw-free, we see that $x_ix'_i \in E(G)$ and $x_ix'_{3-i} \notin E(G)$ for $i = 1, 2$. It is clear that $P = yx_1v_1v_t \dots v_5v_4x'_2y'$ is an induced path.

We confirm that P is a maximal induced path. Suppose to the contrary that $P' = u_0 \dots u_\alpha x_1v_1v_t \dots v_5v_4x'_2z_\beta \dots z_0$ is an enlarged maximal induced path of P , where $u_\alpha = y$ and $z_\beta = y'$. Clearly, $\alpha + \beta \geq 1$. By Lemma 2.19, there is $w \in N(u_0) \cap N(z_0)$.

We see that the length of the cycle $wP'w$ is $\alpha + 2 + t - 3 + 2 + \beta + 2 = t + 3 + \alpha + \beta \geq t + 4$. It is not difficult to see that the cycle $wP'w$ has at most two chords:

$$\begin{aligned} &\text{either } wu_1, wz_1 \quad \text{if } \alpha \geq 1 \text{ and } \beta \geq 1, \\ &\quad \text{or } wx_1, wz_1 \quad \text{if } \alpha = 0 \text{ and } \beta \geq 1, \\ &\quad \text{or } wu_1, wx'_2 \quad \text{if } \alpha \geq 1 \text{ and } \beta = 0. \end{aligned}$$

This implies that G contains an induced cycle of length at least $t + 4 - 2 = t + 2$, a contradiction. Therefore, P is a maximal induced path.

By Lemma 2.19, there is a vertex $w \in N(y) \cap N(y')$. Since G is claw-free, we notice that $w \in N_2(C)$. Because of $x_1x'_2, x'_1x_2 \notin E(G)$, we see that G contains at most one edge of wx_1, wx'_2 . Then either $wy'x'_2v_4v_5 \dots v_tv_1x_1w$ or $wx'_2v_4v_5 \dots v_tv_1x_1yw$ or $wy'x'_2v_4v_5 \dots v_tv_1x_1yw$ is an induced cycle longer than C , a contradiction.

Therefore, $G[N_2(C)]$ is complete. □

Let $N'_1(C) = \{x \in N_1(C) \mid \exists y \in N_2(C) : xy \in E(G)\}$ and $R = N_1(C) \setminus N'_1(C)$ and denote $|N_2(C)| = \mu$ and $|R| = \tau$. By Lemma 2.22, we have $|N'_1(C)| \geq t/2$. We consider the following subcases:

Subcase 2.1. $|N'_1(C)| = t/2$.

If $R = \emptyset$, then there is a Hamiltonian path of $G[N'_1(C) \cup N_2(C)]$ starting at x_1 and ending at x_2 , denoted by P . So, the cycle $v_1v_2Pv_3 \dots v_tv_1$ is a Hamiltonian cycle of G .

Assume that $R \neq \emptyset$. Let $y \in N_2(C)$ and $x_1 \in N'_1(C)$ with $yx_1, x_1v_1, x_1v_2 \in E(G)$. It is easy to see that

$$\begin{aligned} d(y) &= \mu - 1 + t/2, \\ d(v_3) &= d_C(v_3) + d_{N'_1(C)}(v_3) + d_R(v_3) \\ &= 2 + 1 + d_R(v_3) \\ &= d_R(v_3) + 3 \end{aligned}$$

and the length of the path $P = yx_1v_1v_t \dots v_4v_3$ is equal to $t - 2 + 2 = t$. Furthermore, we have

$$\begin{aligned} d(y) + d(v_3) &= \mu - 1 + t/2 + d_R(v_3) + 3 = \mu + t/2 + d_R(v_3) + 2, \\ |V(G)| - |P| + 2 &= (t + t/2 + \mu + \tau) - t + 2 = \mu + t/2 + \tau + 2. \end{aligned}$$

By Claim A, P is a maximal induced path of G . According to the condition (A), we conclude that

$$\mu + t/2 + d_R(v_3) + 2 \geq \mu + t/2 + \tau + 2,$$

and then $d_R(v_3) \geq \tau$. Since $d_R(v_3) \leq \tau$, we have $d_R(v_3) = \tau$. Analogously, $d_R(v_i) = \tau$ for all $i = 1, 2, \dots, t$. It follows that $d_C(z) = t$ for all $z \in R$. Since G is claw-free and C is an induced cycle, it is clear that $|N_C(x)| \leq 4$ for each $x \in N_1(C)$. Thus, $t = 4$. By the way, we see that $N'_1(C) = \{x_1, x_2\}$ and $v_iz \in E(G)$ for $i = 1, 2, 3, 4$ and $z \in R$.

It is clear that $G[R \cup \{v_2\}]$ is connected. Let $P = z_1z_2 \dots z_\lambda$ be a longest path of $G[R \cup \{v_2\}]$. We confirm that P is a Hamiltonian path of $G[R \cup \{v_2\}]$. Suppose to the contrary that there is $z \in R \cup \{v_2\}$ with $z \notin V(P)$. It is clear that $v_2 = z_i$ for some $1 < i < \lambda$ and $z_{i-1}z_{i+1} \notin E(G)$. To avoid the claw $G[v_2, z, z_{i-1}, z_{i+1}]$, we see that either $z_1z_{i-1} \in E(G)$ or $z_1z_{i+1} \in E(G)$. So, either the path $z_1z_2 \dots z_{i-1}zv_2z_{i+1} \dots z_\lambda$ or the path $z_1z_2 \dots z_izv_{i+1} \dots z_\lambda$ is longer than P , a contradiction.

Thus, $v_1x_1P'x_2v_4v_3Pv_1$ is a Hamiltonian cycle of G , where P' is a Hamiltonian path of $G[N_2(C)]$.

Subcase 2.2. $|N'_1(C)| > t/2$.

Suppose without loss of generality that $|(N(v_3) \cup N(v_4)) \cap N'_1(C)| \geq 2$. Let $C_i = N(v_i) \setminus \bigcup_{j=1}^{i-1} N[v_j]$, $i = 2, 3, \dots, t$. It is clear that $v_{i+1} \in C_i$ for $i = 2, 3, \dots, t-1$.

Since G is claw-free, it is easy to see that $C_t = \emptyset$ and the induced subgraph $G[C_i]$ is complete, $i = 2, 3, \dots, t-1$. Let P_i be an arbitrary Hamiltonian path of $G[C_i \setminus \{v_{i+1}\}]$ for $i = 2, 4, \dots, t-1$. By Lemma 2.21 and $|(N(v_3) \cup N(v_4)) \cap N'_1(C)| \geq 2$, we have

$C_3 \setminus \{v_4\} \neq \emptyset$. Moreover, $G[(C_3 \cup N_2(C)) \setminus \{v_4\}]$ contains a Hamiltonian path by Claim A, denoted by P'' , starting at a neighbor of v_3 and ending at a neighbor of v_4 .

By Lemma 2.22, it is clear that $N(v_1) \cap N_1(C) \neq \emptyset$. Let $G' = G[N[v_1] \setminus \{v_2, v_t\}]$. We prove that G' contains a Hamiltonian path P starting at a neighbor of v_t and ending at a neighbor of v_2 . Thus, the cycle $v_t P v_2 P_2 v_3 P'' v_4 P_4 \dots v_{t-1} P_{t-1} v_t$ is a Hamiltonian cycle of G , and we are done.

Since $v_1 \in V(G')$, it is evident that G' is connected. Let $Q = z_1 z_2 \dots z_\nu$ be a longest path in G' . We show that Q is a Hamiltonian path of G' . Otherwise, there is $z \in V(G')$ with $z \notin V(Q)$. It is easy to see that $v_1 = z_i$ for some $1 < i < \nu$ and $z_{i-1} z_{i+1} \notin E(G)$. To avoid the claw $G[\{v_1, z, z_{i-1}, z_{i+1}\}]$, we have either $z z_{i-1} \in E(G)$ or $z z_{i+1} \in E(G)$. Thus, we obtain either the path $z_1 z_2 \dots z_{i-1} z z_i \dots z_\nu$ or the path $z_1 z_2 \dots z_i z z_{i+1} \dots z_\nu$. Both paths are longer than Q , a contradiction.

Since G is claw-free and C is an induced cycle, we see that either $z_1 v_t \in E(G)$ or $z_1 v_2 \in E(G)$ and either $z_\nu v_t \in E(G)$ or $z_\nu v_2 \in E(G)$. If $v_1 = z_1$ or $v_1 = z_\nu$, then Q is a desired path. So, assume that $v_1 = z_i$ for some $1 < i < \nu$ and $z_{i-1} z_{i+1} \notin E(G)$.

If one of $\{z_1, z_\nu\}$ is adjacent with one of $\{v_2, v_t\}$ and the other one of $\{z_1, z_\nu\}$ is adjacent with the other vertex of $\{v_2, v_t\}$, then Q is a desired path. So, assume without loss of generality that $z_1, z_\nu \notin N(v_t)$. Then $z_1, z_\nu \in N(v_2)$. To avoid the claw $G[\{v_1, v_t, z_1, z_\nu\}]$, we see that $z_1 z_\nu \in E(G)$.

If $v_t z_{i-1} \in E(G)$, then $z_{i-1} \dots z_1 z_\nu \dots z_{i+1} v_1$ is a desired path. For the other case when $v_t z_{i-1} \notin E(G)$, we have $v_2 z_{i-1} \in E(G)$ to avoid the claw $G[\{v_1, v_t, v_2, z_{i-1}\}]$. Then $v_1 z_{i+1} \dots z_\nu z_1 \dots z_{i-1}$ is a desired path.

The proof of Theorem 2.16 is completed. ■

We present the following conjecture:

Conjecture 2.23 *Every connected graph satisfying (A) is Hamiltonian.*

2.3 Graphs with Fan-type Condition

In this section, we consider a generalization of the Fan-type condition (see Theorem 1.5).

Let G be a connected graph with $n(\geq 3)$ vertices. A vertex $u \in V(G)$ is said to be *locally heavy* if the condition

$$\max\{d(u), d(x) \mid x \in N_2(u)\} \geq \frac{n}{2}$$

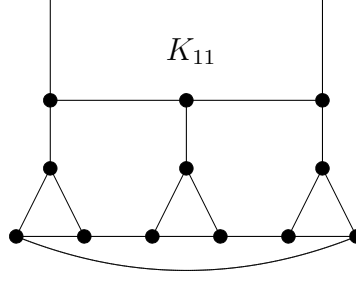
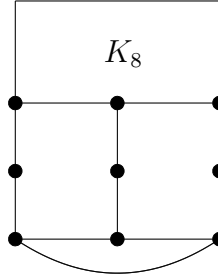
holds. A graph G is called *locally heavy* if each vertex of G is locally heavy.

Note that the condition in Theorem 1.5 implies that G is locally heavy. The graph in Figure 2.6 shows that there are $\{K_{1,3}, H\}$ -free and locally heavy Hamiltonian graphs which do not satisfy the condition in Theorem 1.5.

The graph in Figure 2.7 is a 2-connected, $\{K_{1,3}, H\}$ -free and locally heavy graph. However, it is not Hamiltonian. This implies that the connectivity at least 3 is necessary.

In what follows, we consider the $\{K_{1,3}, H\}$ -free and locally heavy graphs.

Lemma 2.24 *Let G be a claw-free graph and $u \in V(G)$. If G is not locally connected at u , then $G[N(u)]$ consists of two complete components.*

Figure 2.6: A $\{K_{1,3}, H\}$ -free and locally heavy graphFigure 2.7: A non-Hamiltonian, $\{K_{1,3}, H\}$ -free and locally heavy graph

Proof. Since G is not locally connected at u , then $G[N(u)]$ consists of at least two components. Because G is claw-free, we see that $G[N(u)]$ consists of at most two components. Thus, $G[N(u)]$ consists of two components and each of them is complete. ■

Lemma 2.25 *Let G be a locally heavy graph and let $Q = \{v \in V \mid d(v) \geq \frac{n}{2}\}$. Then $V = Q \cup N(Q) \cup N_2(Q)$.*

Proof. According to the definition of a locally heavy graph, the statement is true. ■

Theorem 2.26 *Let G be a locally heavy graph with $n \geq 3$ and let $Q = \{v \in V \mid d(v) \geq \frac{n}{2}\}$. Then $|Q| \geq 2$.*

Proof. Assume to the contrary that $Q = \{u\}$. Let $x \in N(u)$. It is clear that G is not complete. So, we see that $N(u) \setminus N(x) \neq \emptyset$. Therefore, x is not locally heavy, a contradiction. ■

Theorem 2.27 *Let G be a 2-connected, claw-free and locally heavy graph and let $Q = \{v \in V \mid d(v) \geq \frac{n}{2}\}$. If $|Q| = 2$, then G is Hamiltonian.*

Proof. Let $Q = \{u, v\}$. It is not difficult to see that $N(u) \cap N(v) = \emptyset$ and $uv \in E(G)$. This implies that u and v are not locally connected and $|N(u) \setminus \{v\}| = |N(v) \setminus \{u\}| = n/2 - 1$. By Lemma 2.24, the induced subgraphs $G[N(u) \setminus \{v\}]$ and $G[N(v) \setminus \{u\}]$ are complete. Since G is 2-connected, it is visible that G is Hamiltonian. ■

Theorem 2.28 *Let G be a non-complete, claw-free and locally heavy graph and let $Q = \{v \in V \mid d(v) \geq \frac{n}{2}\}$. If $cl(G)[Q]$ is not complete, then $3 \leq |Q| \leq 4$. Moreover, there exist $u, v, x, y \in V$ such that $u, v \in Q$, $uv, xy \notin E$, $ux, uy, vx, vy \in E$,*

$$V = \{u, v, x, y\} \cup (N(u) \cap N(x)) \cup (N(u) \cap N(y)) \cup (N(v) \cap N(x)) \cup (N(v) \cap N(y))$$

and $G[N(u) \cap N(x)]$, $G[N(u) \cap N(y)]$, $G[N(v) \cap N(x)]$, $G[N(v) \cap N(y)]$ are complete.

Proof. Since $cl(G)[Q]$ is not complete, there exist $u, v \in Q$ with $uv \notin E(cl(G))$. Then $|N(u) \cap N(v)| \geq 2$. Otherwise, we have $n \geq |N[u] \cup N[v]| = |N[u]| + |N[v]| - |N[u] \cap N[v]| \geq d(u) + 1 + d(v) + 1 - 1 \geq n + 1$, a contradiction.

Assume that $|N(u) \cap N(v)| \geq 3$, say $x, y, z \in N(u) \cap N(v)$. Since G is claw-free, G contains at least one of the edges xy, yz and zx . Without loss of generality, let $yz \in E(G)$. So, $u, v, z \in N(y)$. By Lemma 2.24, we see that G is locally connected at y . This contradicts the assumption that $cl(G)[Q]$ is not complete. Thus, $|N(u) \cap N(v)| = 2$, say $N(u) \cap N(v) = \{x, y\}$. Analogously, we have $xy \notin E(cl(G))$. Moreover, G is not locally connected at u, v, x and y .

It is not difficult to see that $N(u) = V_1 \cup V_2 \cup \{x, y\}$ and $N(v) = V_3 \cup V_4 \cup \{x, y\}$ with $V_1 \cup V_4 \subset N(x)$, $V_2 \cup V_3 \subset N(y)$ and $V_i \cap V_j = \emptyset$ for $1 \leq i \neq j \leq 4$. Denote $|V_i| = n_i$ for $i = 1, 2, 3, 4$. It is easy to see that

$$n \geq |(\cup_{i=1}^4 V_i) \cup \{u, v, x, y\}| = n_1 + n_2 + n_3 + n_4 + 4 = n_1 + n_2 + 2 + n_3 + n_4 + 2 \geq n.$$

It follows that $n_1 + n_2 + n_3 + n_4 + 4 = n$. This implies that $V = (\cup_{i=1}^4 V_i) \cup \{u, v, x, y\}$. Since $d(x) + d(y) = (n_1 + n_4 + 2) + (n_2 + n_3 + 2) = n$, we have $\max\{d(x), d(y)\} \geq \frac{n}{2}$. The proof is completed. ■

We immediately have the following theorem:

Theorem 2.29 *Let G be a claw-free and locally heavy graph and let $Q = \{v \in V \mid d(v) \geq \frac{n}{2}\}$. If $cl(G)[Q]$ is not complete, then G is Hamiltonian.*

Proof. Clearly, G is not complete. By Theorem 2.28, we see that G is Hamiltonian. ■

Theorem 2.30 (Feng and Guo [51]) *Let G be a 3-connected, $\{K_{1,3}, H\}$ -free and locally heavy graph. Then G is Hamiltonian.*

Proof. Let $Q = \{v \in V \mid d(v) \geq \frac{n}{2}\}$. By Theorem 2.28, we see that $cl(G)[Q]$ is complete.

If $Q = V$, then $cl(G)$ is complete, and hence, $cl(G)$ is Hamiltonian. By Theorem 1.18, we see that G is also Hamiltonian. So, we only need to consider the case when $V \setminus Q \neq \emptyset$. Let H be the maximal complete subgraph of $cl(G)$ with $Q \subset V(H)$. By considering the graph $cl(G)$, we obtain the following claim:

Claim A (1) $d_H(x) = 1$ for each $x \in N_1(H)$.

(2) $d_{N_1(H)}(z) \leq 1$ for each $z \in V(H)$.

(3) Assume that $x_1, x_2 \in N_1(H)$ and $x_1 x_2 \in E(cl(G))$. Let $F = (N(x_1) \cup N(x_2)) \setminus V(H)$. Then $cl(G)[F]$ is complete.

- (4) Assume that $x_1, x_2 \in N_1(H)$ and $N_{N_2(H)}(x_1) \cap N_{N_2(H)}(x_2) \neq \emptyset$. Then $x_1x_2 \in E(cl(G))$.
- (5) Each component of $cl(G) - V(H)$ has at least three neighbors in H . Analogously, each component of $cl(G) - N[H]$ has at least three neighbors in $N_1(H)$.
- (6) If $N_2(H) = \emptyset$, then each component of $cl(G) - V(H)$ contains at least 3 vertices and is complete.
- (7) Each component of $cl(G) - V(H)$ can be decomposed into paths such that the end vertices of each path are adjacent with different vertices of $V(H)$.

Proof.

- (1) It is clear that $d_H(x) \geq 1$. Suppose that $d_H(x) \geq 2$. Let $z_1, z_2 \in N_H(x)$. Since $|H| \geq 3$ and $cl(G)[H]$ is complete, we see that x is connected with all vertices of H . Then we have $x \in V(H)$, a contradiction to the fact that $x \in N_1(H)$. Therefore, $d_H(x) = 1$.
- (2) It is clear that this statement holds for the vertex $z \in V(H)$ with $N_{N_1(H)}(z) = \emptyset$. Assume that $z \in V(H)$ exists with $d_{N_1(H)}(z) \geq 2$. Then, there are at least two vertices $x_1, x_2 \in N_1(H)$ such that $x_1z, x_2z \in E(cl(G))$. By (1), we have $d_H(x_1) = d_H(x_2) = 1$. It is obvious that $x_1x_2 \in E$. So, we obtain an hourglass, a contradiction.
- (3) Let $z_1, z_2 \in V(H)$ with $x_1z_1, x_2z_2 \in E$. By (1), we see that $N_H(x_1) = \{z_1\}$, $N_H(x_2) = \{z_2\}$, $z_1 \neq z_2$. It is clear that $cl(G)[N[x_i] \setminus V(H)]$ is complete for $i = 1, 2$. Let $u_1 \in N(x_1) \setminus \{z_1, x_2\}$ and $u_2 \in N(x_2) \setminus \{z_2, x_1\}$. According to the discussion above, $u_1x_2, u_2x_1 \in E(cl(G))$, i.e. $u_1, u_2 \in N(x_i)$, $i = 1, 2$ and $u_1u_2 \in E(cl(G))$. Therefore, $cl(G)[N[x_1] \cup N[x_2]] - V(H)$ is complete.
- (4) Let $y \in N_{N_2(H)}(x_1) \cap N_{N_2(H)}(x_2)$ and let $z_1, z_2 \in V(H)$ with $x_1z_1, x_2z_2 \in E$. By (1), it is evident that $z_1 \neq z_2$. Assume to the contrary that $x_1x_2 \notin E$. Since G is 3-connected, we see that $|N(x_i) \setminus \{z_i\}| \geq 2$ for $i = 1, 2$. It is clear that $cl(G)[N(x_i)] - \{z_i\}$ is complete for $i = 1, 2$. Since G is hourglass-free and $y \in N(x_1) \cap N(x_2)$, each vertex of $N(x_i) \setminus \{z_i\}$ must be adjacent to each vertex of $N(x_{3-i}) \setminus \{z_{3-i}\}$ for $i = 1, 2$. Otherwise, without loss of generality, assume that $y_1 \in N(x_1) \setminus \{z_1\}$ and $y_2 \in N(x_2) \setminus \{z_2\}$ with $y_1y_2 \notin E(cl(G))$. Then it is easy to see that $y_i \notin N(x_{3-i})$ for $i = 1, 2$. We see that $cl(G)[\{x_1, x_2, y, y_1, y_2\}]$ is an hourglass, a contradiction. Therefore, we see that $cl(G)[N(x_1) \cup N(x_2)] - \{z_1, z_2\}$ is complete. According to the definition of $cl(G)$ and the fact that $|N(x_i) \setminus \{z_i\}| \geq 2$ for $i = 1, 2$, it is easy to see that $x_1x_2 \in E(cl(G))$, a contradiction.
- (5) Since G is 3-connected, this statement is true.
- (6) By the statements (1) – (3), it is not difficult to see that (6) is true.
- (7) For the case when $N_2(H) = \emptyset$, we are done by (1) and (6). So, assume that $N_2(H) \neq \emptyset$. Note that for each $y \in N_2(H)$, there is at least one vertex $x \in N_1(H)$ with $xy \in E(cl(G))$. Let K be an arbitrary component of $cl(G)[N_2(H)]$.

Set $X = N_1(H) \cap N(K)$ and $M = cl(G)[X \cup V(K)]$. Clearly, $|X| \geq 3$. Let $x \in X$ be a vertex such that the degree of x is maximal among all vertices of X in the subgraph $cl(G)[X]$. By (3), we see that $cl(G)[N[x] \cap (N_1(H) \cup N_2(H))]$ is complete.

Let $t = \max\{d_M(x, x') \mid x' \in X\}$. If $t = 1$, then M is complete, and we are done. So, we only need to consider the case when $t > 1$.

Denote $N_k^M(x) = \{x' \in X \mid d_M(x, x') = k\}$, $k = 1, 2, \dots, t$. By (3) and (4), we see that if $x' \notin N(x)$, then $d_M(x, x')$ is odd. So, t is odd and $N_k^M(x) = \emptyset$ for $k = 2\ell$, $\ell = 1, \dots, (t-1)/2$.

Let $L_i = cl(G)[\cup_{a \in N_i(x)} (N[a] \cap V(K))]$, $i = 1, 3, \dots, t$. It is clear that each component of L_i is connected with some component of L_{i-2} by edges for $i = 3, 5, \dots, t$. Since $cl(G)$ is claw-free and hourglass-free, it is not difficult to see that every pair of edges between L_i and L_{i+2} is not adjacent for $i = 1, 3, \dots, t-2$. Moreover, the edges between L_{i-2} and L_i are not adjacent with the edges between L_i and L_{i+2} for $i = 3, \dots, t-2$.

By applying the following algorithm, we can decompose M into desired paths.

Let $x_1^k, x_2^k, \dots, x_{t_k}^k$ be the vertices of $N_k^M(x)$ for $k = 1, 3, \dots, t$. Begin with N_{t-2}^M . Let $R_1^{t-2} = \{x_j^t \mid d_M(x_1^{t-2}, x_j^t) = 3\}$, $R_j^{t-2} = \{x_j^t \mid d_M(x_j^{t-2}, x_j^t) = 3\} \setminus (\cup_{\ell=1}^{j-1} R_\ell^{t-2})$, etc. Now, we obtain $R_1^{t-2}, \dots, R_{t-2}^{t-2}$. For each $1 \leq j \leq t-2$, we construct paths in the following way:

- For the adjacent vertices of R_j^{t-2} , we see by (3) that the neighbors of such vertices in K induce a complete subgraph. So, we choose an arbitrary Hamiltonian path of this subgraph as a desired path.
- For the remaining vertices of R_j^{t-2} , say x_i^t and x_k^t , the subgraphs $M[N_K(x_i^t)]$ and $M[N_K(x_k^t)]$ are connected through three edges, namely, an edge between $M[N_K(x_i^t)]$ and $M[N_K(x_j^{t-2})]$, an edge of $M[N_K(x_j^{t-2})]$ and an edge between $M[N_K(x_k^t)]$ and $M[N_K(x_j^{t-2})]$. So, we can construct a desired path for each pair of vertices of R_j^{t-2} and its neighbors in K . Note that if $|R_j^{t-2}|$ is odd, then there is finally a remaining vertex in R_j^{t-2} .
- Continue this process for the remaining subgraph obtained from M by taking out all the constructed paths.

It is not difficult to see that if, without loss of generality, R_j^{t-2} contains odd vertices, then the subgraph induced by the remaining vertex and its neighbors in K is connected with a constructed path in R_j^{t-4} . Consequently, we obtain the desired paths. $\square$

By Claim A (7), all vertices of $V(G) \setminus V(H)$ can be inserted into a Hamiltonian cycle of $cl(G)[H]$. So, $cl(G)$ is Hamiltonian. By Theorem 1.18, G is also Hamiltonian. $\blacksquare$

At last, we have the following conjecture:

Conjecture 2.31 *Every 3-connected, claw-free and locally heavy graph is Hamiltonian.*

Chapter 3

Properly Colored Hamiltonian Paths

In this chapter, we consider a special topic, namely, the existence of a properly colored Hamiltonian path in an edge-colored complete graph. Let the abbreviation PCHP stand for “properly colored Hamiltonian path”.

3.1 PCHP Conjecture

Let $K_n = (V, E)$ be a complete graph, and let $c : E \rightarrow \{1, 2, \dots, \chi\}$ be a fixed (not necessarily proper) edge-coloring of K_n with χ colors, $\chi \geq 2$. With a given c , K_n is called a χ -edge-colored (or, *edge-colored*) complete graph. A subgraph $H \subseteq K_n$ is called *properly colored* if c defines a proper edge-coloring of H .

The existence of properly colored Hamiltonian paths and cycles has been studied in several papers; this topic was surveyed in [6] (see also [7], Chapter 11). While there are characterizations of 2-edge-colored complete graphs with properly colored Hamiltonian cycles [10, 75] (see also [7], Chapter 11), no such characterization is known for χ -edge-colored complete graphs with $\chi \geq 3$, and it is still an open question to determine the computational complexity of this problem [15].

The most studied possibly sufficient condition for an edge-colored complete graph with n vertices to have a properly colored Hamiltonian cycle is $\Delta_{\text{mon}} < \lfloor n/2 \rfloor$, where Δ_{mon} is the maximal number of edges of the same color incident to the same vertex. This was conjectured by Bollobás and Erdős [19] in 1976, but remains unsolved. The best result so far for ‘small’ values of n is by Shearer [76]: $7\Delta_{\text{mon}} < \lfloor n/2 \rfloor$ guarantees the existence of a properly colored Hamiltonian cycle. The best result so far for large values of n is due to Alon and Gutin [2]: For every $\epsilon > 0$ and $n = n_\epsilon$ large enough, $\Delta_{\text{mon}} \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)\lfloor n/2 \rfloor$ implies the existence of a properly colored Hamiltonian cycle.

For the case of properly colored Hamiltonian paths, the situation is somewhat different. Let G be an edge-colored graph. A *properly colored cycle factor* of G is a spanning subgraph of G consisting of properly colored cycles $C_1, C_2, \dots, C_d$ such that $V(C_i) \cap V(C_j) = \emptyset$ provided $1 \leq i < j \leq d$. A *properly colored 1-path-cycle factor* of G is a spanning subgraph of G consisting of a properly colored path C_0 and a (possibly empty) collection of properly colored cycles $C_1, C_2, \dots, C_d$ such that $V(C_i) \cap V(C_j) = \emptyset$ provided $0 \leq i < j \leq d$.

Theorem 3.1 (Bang-Jensen and Gutin [6]) *It takes polynomial time to check whether an edge-colored graph has a properly colored 1-path-cycle factor.*

The following theorem gives a PCHP characterization for the case of just two colors:

Theorem 3.2 (Bang-Jensen and Gutin [6]) *A 2-edge-colored complete graph K_n has a PCHP if and only if K_n contains a properly colored 1-path-cycle factor.*

Bang-Jensen and Gutin [6] conjectured that the above theorem holds for any number of colors. In support of such conjecture, the following result was proved.

Theorem 3.3 (Bang-Jensen et al. [9]) *If a χ -edge-colored complete graph K_n ($\chi \geq 2$) contains a properly colored cycle factor, then K_n contains a PCHP.*

It is easy to see that the conjecture of Bang-Jensen and Gutin in [6] can be reduced to the following:

Conjecture 3.4 (PCHP Conjecture) *Let K_n be a χ -edge-colored, $\chi \geq 3$, complete graph. Assume that there exist $C, P \subseteq K_n$, where C is a properly colored cycle and P a properly colored path, such that $V(C) \cap V(P) = \emptyset$ and $V(C) \cup V(P) = V(G)$. Then K_n contains a PCHP.*

In the next section, we prove the PCHP conjecture and, thus, the conjecture of Bang-Jensen and Gutin.

3.2 The Proof for PCHP Conjecture

To prove Theorem 3.6, we firstly prove the following theorem:

Theorem 3.5 *Let $C = v_1 \dots v_\ell v_1$ ($\ell \geq 3$) and $P = u_1 \dots u_m$ ($m \geq 2$) be the cycle and path in PCHP Conjecture with $c(u_1 v_j) = c(u_1 u_2)$ and $c(u_m v_j) = c(u_{m-1} u_m)$ for every $j = 1, 2, \dots, \ell$. Then there exists a PCHP H in K_n with u_1 as its first vertex, such that the initial edge of H is either $u_1 u_2$ or one of the edges $u_1 v_j$ ($1 \leq j \leq \ell$), and such that u_m is the last vertex of H and the last edge of H is either $u_{m-1} u_m$ or one of the edges $v_j u_m$ ($1 \leq j \leq \ell$).*

Proof. Throughout we will perform addition and subtraction in the indices of the vertices $v_j \in C$ modulo ℓ .

Let $b(P, C) = 2(\ell - 3) + m$; we notice that $b(P, C) \geq 2$. Suppose that the statement is false, and let (K_n, P, C, c) be a counterexample with a minimal value of $b(P, C)$.

Now we have the following claim:

Claim A $m \geq 3$.

Proof. Suppose that $m = 2$. With $x = c(u_1 u_2)$, we have $c(u_1 v_j) = c(u_2 v_j) = x$ for all $j = 1, 2, \dots, \ell$, according to the condition of the theorem. Choose r such that $c(v_{r-1} v_r) \neq x \neq c(v_{r+1} v_{r+2})$; this is possible if r can be chosen with $c(v_r v_{r+1}) = x$, and otherwise it is trivial. But then the path $u_1 v_r v_{r-1} \dots v_{r+2} v_{r+1} u_2$ yields a contradiction. $\square$

We continue to prove further properties of the coloring c .

Claim B *For every i , $1 < i < m$, there exist r and s ($1 \leq r, s \leq \ell$), such that $c(u_i u_{i+1}) \neq c(u_i v_r)$, $c(u_{i-1} u_i) \neq c(u_i v_s)$.*

Proof. Suppose to the contrary that the statement is not true. Then the path $P' = P - \{u_1, u_2, \dots, u_{i-1}\}$ satisfies the condition of the theorem in $K' = K_n - \{u_1, u_2, \dots, u_{i-1}\}$ (with u_i in place of u_1). Since $b(P', C) < b(P, C)$, there exists a PCHP H' in K' that starts from u_i with one of the edges $u_i u_{i+1}$ or $u_i v_j$, $1 \leq j \leq \ell$ and finishes in u_m with $u_{m-1} u_m$ or with an edge $v_k u_m$, $1 \leq k \leq \ell$. Then $u_1 \dots u_{i-1} H'$ is a PCHP in K_n of the desired type having $u_1 u_2$ as its initial edge, a contradiction. A similar argument shows the existence of s . $\square$

Claim C *Assume that $d_C(v_s, v_{t+1}) \geq 3$. If $c(v_{s-1} v_s) = c(v_t v_{t+1}) = x$, then $c(v_s v_t) \neq x$.*

Proof. Suppose that $c(v_{i-1} v_i) = c(v_j v_{j+1}) = x$, where $d_C(v_i, v_{j+1}) \geq 3$. If $c(v_i v_j) = x$, then define $K' = K_n - v_i - v_{i+1} - \dots - v_j$ and define $c' : E(K') \rightarrow \{1, 2, \dots, \chi\}$ by

$$c'(e) = \begin{cases} x & \text{if } e = v_{i-1} v_{j+1}, \\ c(e) & \text{otherwise.} \end{cases}$$

Both P and the cycle $C' = v_{j+1} v_{j+2} \dots v_\ell v_1 \dots v_{i-1} v_{j+1}$ are properly colored by c' . Moreover, P clearly satisfies the condition of the theorem with respect to C' . Since $b(P, C') < b(P, C)$, there exists a PCHP P_1 in G' with initial edge $u_1 u_2$ or $u_1 v_r$ for some $r \in \{j+1, j+2, \dots, \ell, 1, \dots, i-1\}$.

If $v_{i-1} v_{j+1} \in P_1$, then we find the desired PCHP H in K_n by replacing the edge $v_{i-1} v_{j+1}$ by the path $v_{i-1} v_i \dots v_j v_{j+1}$. Otherwise, if $v_{i-1} v_{j+1} \notin P_1$, then P_1 is properly colored in K_n and satisfies the condition of the theorem with respect to $C'' = v_i v_{i+1} \dots v_j v_i$, which is a properly colored cycle. By $b(P_1, C'') < b(P, C)$, the desired PCHP exists in K_n , a contradiction. $\square$

Consider the path $u_1 \dots u_{p-2} u_{p-1} v_{q-1} v_{q-2} \dots v_{q+1} v_q u_p u_{p+1} \dots u_m$. As it cannot be a PCHP, we conclude the following (Figure 3.1 and 3.2):

Claim D *Let $2 \leq p \leq m$ and $1 \leq q \leq \ell$. Then at least one of the following holds:*

- (a) $p \geq 3$ and $c(u_{p-2} u_{p-1}) = c(u_{p-1} v_{q-1})$,
- (b) $c(u_{p-1} v_{q-1}) = c(v_{q-2} v_{q-1})$,
- (c) $c(u_p v_q) = c(v_q v_{q+1})$,
- (d) $p < m$ and $c(u_p v_q) = c(u_p u_{p+1})$.

Considering the path $u_1 \dots u_{p-2} u_{p-1} v_{q+1} v_{q+2} \dots v_{q-1} v_q u_p u_{p+1} \dots u_m$ similarly leads to:

Claim E *Let $2 \leq p \leq m$ and $1 \leq q \leq \ell$. Then at least one of the following holds:*

- (a) $p \geq 3$ and $c(u_{p-2} u_{p-1}) = c(u_{p-1} v_{q+1})$,
- (b) $c(u_{p-1} v_{q+1}) = c(v_{q+1} v_{q+2})$,

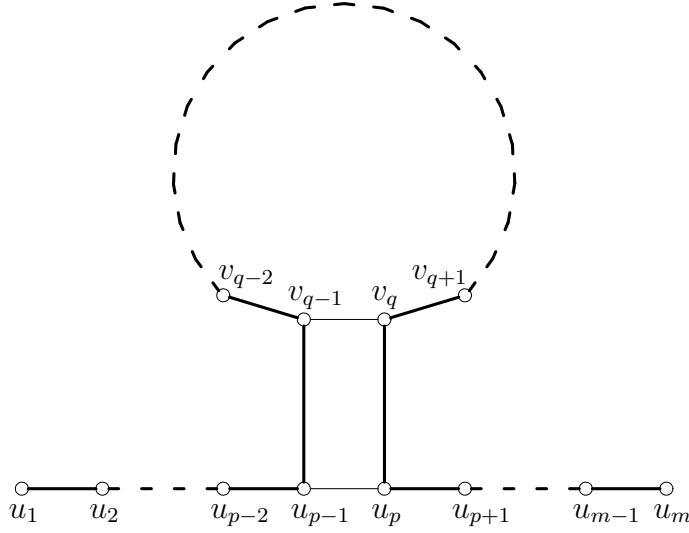


Figure 3.1: The path used to prove Claim D.

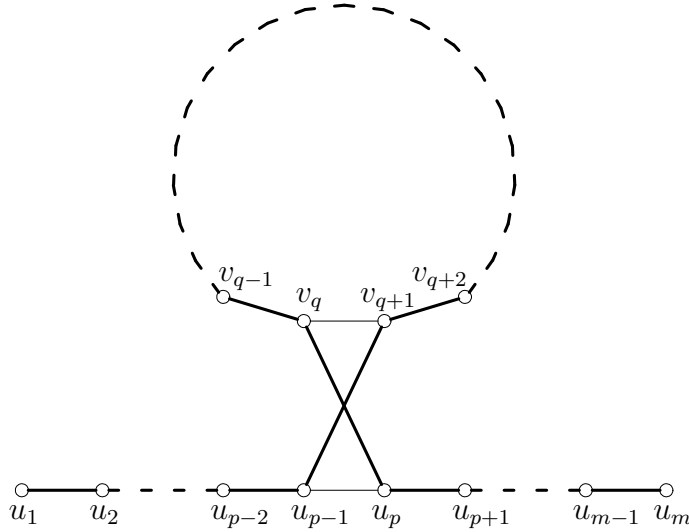


Figure 3.2: The path used to prove Claim E.

$$(c) \ c(u_p v_q) = c(v_{q-1} v_q),$$

$$(d) \ p < m \text{ and } c(u_p v_q) = c(u_p u_{p+1}).$$

In several of the following applications of Claim D and E it will be useful to note that Claim D (c) and Claim E (c) are mutually exclusive statements for any values of p and q , since C is properly colored, and that Claim D (d) and Claim E (d) are identical statements.

For the remaining part of the proof, we define $x = c(u_1 u_2)$ and $y = c(u_{m-1} u_m)$.

Claim F Assume $c(u_2 v_j) = z \neq c(u_2 u_3)$ for some $j \in \{1, 2, \dots, \ell\}$.

(a) If $c(v_j v_{j+1}) \neq z$, then $c(v_{j-2} v_{j-1}) = x$, and if $c(v_{j-1} v_j) \neq z$, then $c(v_{j+1} v_{j+2}) = x$.

- (b) If $z = x$ or $c(v_{j-1}v_j) \neq z \neq c(v_jv_{j+1})$, then $c(v_{j-2}v_{j-1}) = c(v_{j+1}v_{j+2}) = c(v_{j-1}v_{j+1}) = x$ and $\ell \in \{3, 5\}$.

Proof. If $c(v_{j-1}v_j) \neq z$, then $c(v_{j+1}v_{j+2}) = x$ follows from Claim E with $p = 2$ and $q = j$ (only (b) of Claim E is not necessarily false). If $c(v_jv_{j+1}) \neq z$, then $c(v_{j-2}v_{j-1}) = x$ similarly follows from Claim D. This shows (a).

Now assume $c(v_{j-1}v_j) \neq z \neq c(v_jv_{j+1})$ or $z = x$. In the case $z = x$ the fact that c is a proper coloring of C together with (a) implies that $c(v_{j-2}v_{j-1}) = c(v_{j+1}v_{j+2}) = x$. The same conclusion follows directly from (a) when $c(v_{j-1}v_j) \neq z \neq c(v_jv_{j+1})$. By symmetry, and since $c(v_{j-1}v_j) \neq c(v_jv_{j+1})$, we may assume $c(v_{j-1}v_{j+1}) \neq c(v_jv_{j+1})$. Since the Hamiltonian path $u_1v_{j+2}v_{j+3} \dots v_{j-2}v_{j-1}v_{j+1}v_ju_2 \dots u_m$ fails to be a PCHP, it follows that $c(v_{j-1}v_{j+1}) = x$.

Set $s = j - 1$ and $t = j + 1$. By Claim C, we conclude that $c(v_s v_t) \neq x$, i.e. $d_C(v_{j-1}, v_{j+2}) \leq 2$, which is only possible for $\ell \leq 5$. Moreover, the edges $v_{j-2}v_{j-1}$ and $v_{j+1}v_{j+2}$, both of color x , are not adjacent on C , which implies that $\ell \neq 4$. Thus (b) is proved. $\square$

Claim G Assume $c(u_{m-1}v_k) = z \neq c(u_{m-2}u_{m-1})$ for some $k \in \{1, 2, \dots, \ell\}$.

- (a) If $c(v_{k-1}v_k) \neq z$ then $c(v_{k+1}v_{k+2}) = y$, and if $c(v_kv_{k+1}) \neq z$ then $c(v_{k-2}v_{k-1}) = y$.
(b) If $z = y$ or $c(v_{k-1}v_k) \neq z \neq c(v_kv_{k+1})$, then $c(v_{k-2}v_{k-1}) = c(v_{k+1}v_{k+2}) = c(v_{k-1}v_{k+1}) = y$ and $\ell \in \{3, 5\}$.

Proof. The proof of Claim G is similar to that of Claim F. $\square$

Claim H Assume $c(u_2v_j) \neq x$ for all $j = 1, 2, \dots, \ell$. Then there exists $j \in \{1, 2, \dots, \ell\}$ such that

- (a) $c(u_2v_{j-1}) = c(u_2v_j) \neq c(u_2u_3)$, and
(b) $c(v_{j-2}v_{j-1}) = c(v_jv_{j+1}) = x$.

Proof. By Claim B, there exists $j \in \{1, 2, \dots, \ell\}$ such that $c(u_2v_j) = z \neq c(u_2u_3)$. We may assume $c(v_jv_{j+1}) \neq z$ (if not, then $c(v_{j-1}v_j) \neq z$ holds, and we may renumber the vertices on C so that $v_{j+\tau}$ becomes $v_{j-\tau}$ for all $\tau = 0, 1, 2, \dots$ without change of the conclusion). By Claim F (a), we have $c(v_{j-2}v_{j-1}) = x$.

Again by Claim B there exists $r \in \{1, 2, \dots, \ell\}$ such that if $m \geq 4$, then $c(u_3v_r) \neq c(u_3u_4)$ holds.

Suppose $c(u_2v_{j-1}) \neq z$. Since $u_1v_{r+1}v_{r+2} \dots v_{j-2}v_{j-1}u_2v_jv_{j+1} \dots v_{r-1}v_ru_3u_4 \dots u_m$ is not a PCHP, at least one of the following holds:

- (i) $c(u_1v_{r+1}) = c(v_{r+1}v_{r+2})$,
(ii) $c(u_3v_r) = c(v_{r-1}v_r)$,
(iii) $r = j$ and $c(u_2v_r) = c(u_3v_r)$.

Since $u_1v_{r-1}v_{r-2} \dots v_{j+1}v_ju_2v_{j-1}v_{j-2} \dots v_{r+1}v_ru_3u_4 \dots u_m$ is not a PCHP, at least one of the following holds:

- (iv) $c(u_1v_{r-1}) = c(v_{r-2}v_{r-1})$,
- (v) $c(u_3v_r) = c(v_rv_{r+1})$,
- (vi) $r = j - 1$ and $c(u_2v_r) = c(u_3v_r)$.

Let $p = 3$ and $q = r$, and observe that neither of Claim D (a), Claim D (d), Claim E (a) or Claim E (d) holds.

We will now show that (iii) and (vi) do not hold. If $r = j$, then $c(v_{r-2}v_{r-1}) = x \neq c(u_2v_{r-1})$, hence also Claim D (b) does not hold, and Claim D (c) must be satisfied, that is, $c(u_3v_r) = c(v_rv_{r+1})$. In particular, if $r = j$, then $c(u_3v_r) = c(v_jv_{j+1}) \neq z = c(u_2v_j)$, contrary to (iii).

If $r = j - 1$, then $c(u_2v_{r+1}) = c(u_2v_j) = z$ and $c(v_{r+1}v_{r+2}) = c(v_jv_{j+1}) \neq z$, so Claim E (b) does not hold. By Claim E (c), we have $c(u_3v_r) = c(v_{r-1}v_r) = c(v_{j-2}v_{j-1}) = x$. In particular, if $r = j - 1$, then $c(u_2v_r) \neq c(u_3v_r)$ follows from the assumption that $c(u_2v_j) \neq x$ for all $j = 1, 2, \dots, \ell$, hence also (vi) does not hold.

We deduce that (i) or (ii) is true, and that (iv) or (v) is true. Now Claim E (c) is equivalent to (ii), and Claim E (b) and (i) are not both true (according to the condition of Theorem 3.5 and our assumption), therefore (ii) holds. Similarly Claim D (c) is equivalent to (v), and Claim D (b) contradicts (iv), so also (v) holds. But (ii) contradicts (v), since C is properly colored. This establishes $c(u_2v_{j-1}) = z$.

Finally $c(v_jv_{j+1}) = x$ follows from Claim F (a). $\square$

Claim I Assume $c(u_{m-1}v_j) \neq y$ for all $j = 1, 2, \dots, \ell$. Then there exists $k \in \{1, 2, \dots, \ell\}$ such that

- (a) $c(u_{m-1}v_{k-1}) = c(u_{m-1}v_k) \neq c(u_{m-2}u_{m-1})$, and
- (b) $c(v_{k-2}v_{k-1}) = c(v_kv_{k+1}) = y$.

Proof. The proof is similar to the proof of Claim H. $\square$

Claim J Assume $z = c(u_2v_j) = c(u_2v_{j-1}) \notin \{x, c(u_2u_3)\}$ for some $j \in \{1, 2, \dots, \ell\}$. Furthermore assume $c(v_jv_{j+1}) \neq z$. Then one of the following holds:

- (a) ℓ is an even number, and $\ell/2$ edges of C have color x .
- (b) If $c(v_{j'}v_{j'+1}) = x$, then $v_{j'} \in S = \{v_{j-4}, v_{j-2}, v_j, v_{j+2}\}$.

Proof. First, we have $c(v_{j-2}v_{j-1}) = c(v_jv_{j+1}) = x$ by two applications of Claim F (a), applying Claim F to u_2v_j and u_2v_{j-1} in this order. Since C is properly colored, we conclude that $c(v_{j'}v_{j'+1}) = x$ implies that $v_{j'} \notin \{v_{j-3}, v_{j-1}, v_{j+1}\}$. Assume that (b) does not hold, and choose any $v_{j'} \in V(C) \setminus S$ satisfying $c(v_{j'}v_{j'+1}) = x$. Then $\ell \geq 8$ follows from $v_{j'} \notin S \cup \{v_{j-3}, v_{j-1}, v_{j+1}\}$. We will prove the following statement.

- (*) $c(v_{j'-2}v_{j'-1}) = c(v_{j'+2}v_{j'+3}) = x$.

First we suppose $c(v_{j'-2}v_{j'-1}) \neq x$. Then $v_{j'} \notin S \cup \{v_{j-3}, v_{j-1}, v_{j+1}\}$ and $\ell \geq 8$ imply $d_C(v_{j-1}, v_{j'+1}) \geq 3$. By Claim C with $s = j - 1$ and $t = j'$ we have $c(v_{j-1}v_{j'}) \neq x$. We consider the path $P_1 = u_1v_{j'-1}v_{j'-2} \dots v_{j+1}v_ju_2u_3 \dots u_m$ and the cycle $C_1 = v_{j'}v_{j'+1} \dots v_{j-2}v_{j-1}v_{j'}$, which are properly colored and satisfy the condition of the theorem. Since $b(C_1, P_1) < b(C, P)$ holds, our minimality assumption yields a PCHP, which is a contradiction. So $c(v_{j'-2}v_{j'-1}) = x$ holds.

Now suppose $c(v_{j'+2}v_{j'+3}) \neq x$. Then by Claim C with $s = j$ and $t = j' + 1$ we similarly have $c(v_jv_{j'+1}) \neq x$, and we consider the path $P_2 = u_1v_{j'+2}v_{j'+3} \dots v_{j-2}v_{j-1}u_2u_3 \dots u_m$ and the cycle $C_2 = v_{j'+1}v_{j'} \dots v_{j+1}v_jv_{j'+1}$ instead, again with a contradiction. Thus we have also $c(v_{j'+2}v_{j'+3}) = x$, which finishes the proof of (*).

Applying (*) recursively, it follows that $c(v_{j'+2\tau}v_{j'+2\tau+1}) = x$ holds for every $\tau \in \mathbb{N}$ with $v_{j'+2\tau} \notin S \cup \{v_{j-3}, v_{j-1}, v_{j+1}\}$. In particular, either $c(v_{j+3}v_{j+4}) = x$ or $c(v_{j+4}v_{j+5}) = x$ holds ($v_{j+3} \notin S$ follows from $\ell \geq 8$, and $v_{j+4} \in S$ only occurs if $\ell = 8$ and $v_{j'} = v_{j+3}$, in which case $c(v_{j+3}v_{j+4}) = x$ follows). However, applying (*), $c(v_{j+3}v_{j+4}) = x$ would imply $c(v_{j+1}v_{j+2}) = x$, contradicting the fact that C is properly colored. So, $c(v_{j+4}v_{j+5}) = x$ holds. Similar reasoning leads to $c(v_{j-6}v_{j-5}) = x$. It follows from (*) that all of $v_{j+4}v_{j+5}, v_{j+6}v_{j+7}, \dots, v_{j-6}v_{j-5}$ are colored x .

Since both $v_{j+4}v_{j+5}$ and $v_{j-6}v_{j-5}$ are colored x , it follows from (*) that $v_{j+2}v_{j+3}$ and $v_{j-4}v_{j-3}$ are also colored x . Combining this with the fact that v_jv_{j+1} and $v_{j-2}v_{j-1}$ are both colored x , we have shown that (a) is true, which proves Claim J. $\square$

Claim K *There is an index j , $1 \leq j \leq \ell$, such that $c(u_2v_j) = x$ or $c(u_{m-1}v_j) = y$.*

Proof. Suppose $c(u_2v_j) \neq x$ and $c(u_{m-1}v_j) \neq y$ for all $j = 1, 2, \dots, \ell$. By Claim H and Claim I we may choose $j, k \in \{1, 2, \dots, \ell\}$ such that j satisfies Claim H (a) and Claim H (b), and k satisfies Claim I (a) and Claim I (b).

By Claim H (b) we have $c(v_{j-2}v_{j-1}) = c(v_jv_{j+1}) = x$, from which $m \geq 4$ follows, as otherwise $u_1u_2v_jv_{j+1} \dots v_{j-2}v_{j-1}u_3$ would be a PCHP. Now the path

$$u_1v_{k-1}v_{k-2} \dots v_{j+1}v_ju_2u_3 \dots u_{m-2}u_{m-1}v_kv_{k+1} \dots v_{j-2}v_{j-1}u_m$$

is not a PCHP, so $y = x$ follows.

We will show that Claim J (a) holds. So suppose not; then it follows from Claim J that Claim J (b) holds. By Claim I (b) we have $c(v_{k-2}v_{k-1}) = c(v_kv_{k+1}) = x$, which by Claim J (b) implies that $v_k \in \{v_{j-2}, v_j, v_{j+2}\}$.

The case $v_k = v_j$ would lead to a contradiction, since the path

$$u_1v_{j-2}v_{j-3} \dots v_{j+1}v_ju_2u_3 \dots u_{m-2}u_{m-1}v_{j-1}u_m$$

would be a PCHP.

Suppose $v_k = v_{j-2}$. By Claim I (b) we then have $c(v_{j-4}v_{j-3}) = x$. Then $\ell \notin \{3, 5\}$ follows from the fact that C is properly colored, and $\ell \notin \{4, 6\}$ holds since Claim J (a) is not satisfied, so we deduce that $\ell \geq 7$. Applying Claim C to $s = j - 3$ and $t = j$ we have $c(v_{j-3}v_j) \neq x$. However, the path $u_1v_{j-1}u_2u_3 \dots u_{m-2}u_{m-1}v_{j-2}u_m$ and the cycle $v_jv_{j+1} \dots v_{j-4}v_{j-3}v_j$ are properly colored and satisfy the condition of the theorem, which contradicts our minimality assumption.

For $v_k = v_{j+2}$ we similarly conclude by Claim I (b) and Claim C, with $s = j - 1$ and $t = j + 2$, that $c(v_{j-1}v_{j+2}) \neq x$. Examination of the path $u_1v_ju_2u_3 \dots u_{m-2}u_{m-1}v_{j+1}u_m$

and the cycle $v_{j-1}v_{j+2}v_{j+3}\dots v_{j-2}v_{j-1}$ again leads to a contradiction, which shows that Claim J (a) does hold.

We have that ℓ is an even number, and the edges of C are alternately colored x . Let $r \in \{1, 2, \dots, n\}$ be chosen so that $c(u_3v_r) \neq c(u_3u_4)$; this is possible by Claim B. We apply Claim D and Claim E with $p = 3$ and $q = r$. Then Claim D (d) and Claim E (d) both fail by the choice of r . Neither of the edges u_2v_{r-1} and u_2v_{r+1} have color x , due to our assumption, hence Claim D (a) and Claim E (a) both fail. For the same reason one of Claim D (b) and Claim E (b) fails, because either $v_{r-2}v_{r-1}$ or $v_{r+1}v_{r+2}$ is colored x . If $v_{r-2}v_{r-1}$ is colored x , then Claim D (b) fails, so Claim D (c) holds and gives $c(u_3v_r) = c(v_rv_{r+1})$, and now $u_1u_2v_kv_{k+1}\dots v_{r-1}v_ru_3u_4\dots u_{m-2}u_{m-1}v_{k-1}v_{k-2}\dots v_{r+2}v_{r+1}u_m$ is a PCHP, a contradiction. If $v_{r+1}v_{r+2}$ is colored x , then Claim E (b) fails, and $c(u_3v_r) = c(v_{r-1}v_r)$ follows similarly. But $u_1u_2v_{k-1}v_{k-2}\dots v_{r+1}v_ru_3u_4\dots u_{m-2}u_{m-1}v_kv_{k+1}\dots v_{r-2}v_{r-1}u_m$ is a PCHP, again with a contradiction. This finishes the proof of Claim K. $\square$

Claim L Assume $c(u_2v_j) = x$ for some $j \in \{1, 2, \dots, \ell\}$, and let $w = c(v_{j-1}v_j)$ and $z = c(v_jv_{j+1})$. Then

- (a) $c(v_{j-2}v_{j-1}) = c(v_{j+1}v_{j+2}) = c(v_{j-1}v_{j+1}) = x$ and $\ell \in \{3, 5\}$.
- (b) $c(u_2v_k) \neq x$ for all $v_k \neq v_j$.
- (c) $c(u_2v_{j-1}) = c(u_2v_{j+1}) = c(u_2u_3) \in \{w, z\}$.
- (d) $m \geq 4$.
- (e) If $c(u_2u_3) = w$, then $c(u_3v_k) = c(u_3u_4) \neq c(u_3v_{j-1}) = w$ for all $v_k \neq v_{j-1}$.
- (f) If $c(u_2u_3) = z$, then $c(u_3v_k) = c(u_3u_4) \neq c(u_3v_{j+1}) = z$ for all $v_k \neq v_{j+1}$.

Proof. We may assume $j = 1$, so that $c(u_2v_1) = x$, $c(v_\ell v_1) = w$ and $c(v_1v_2) = z$. Then $c(u_2v_1) = x \neq c(u_2u_3)$ and Claim F (b) directly imply (a). Since $\ell \in \{3, 5\}$ and C is properly colored, $c(u_2v_k) = x$ now implies that $v_k = v_1$ by (a) and Claim F (b), so (b) holds.

Now $c(u_2v_2) = c(u_2u_3)$ follows from Claim D and Claim E with $p = 2$ and $q = 2$, since Claim D (a), (b), Claim E (a), (b) all fail, and Claim D (c) contradicts Claim E (c), so that Claim D (d) and the equivalent Claim E (d) hold. Similarly we deduce that $c(u_2v_\ell) = c(u_2u_3)$ from Claim D and Claim E with $p = 2$ and $q = \ell$.

Now (d) and $c(u_3v_1) = c(u_3u_4)$ follow from Claim D and Claim E with $p = 3$ and $q = 1$, as Claim D (d) or Claim E (d) is again satisfied. Moreover, for $\ell = 5$ the identity $c(u_3v_3) = c(u_3u_4)$ follows, using $c(v_2v_3) \neq c(v_3v_4)$, from the fact that neither $u_1u_2v_4v_5v_1v_2v_3u_3u_4\dots u_m$ nor $u_1v_2v_1u_2v_5v_4v_3u_3u_4\dots u_m$ is a PCHP. Similarly we have $c(u_3v_4) = c(u_3u_4)$ when $\ell = 5$.

We deduce that $c(u_3v_\ell) \in \{w, c(u_3u_4)\}$, since $u_1v_{n-1}v_{n-2}\dots v_3v_2u_2v_1v_\ell u_3u_4\dots u_m$ is not a PCHP. Moreover, $c(u_3v_\ell) \in \{c(u_2u_3), c(u_3u_4)\}$ follows, using $c(u_2v_\ell) = c(u_2u_3)$, by examining the path $u_1v_{\ell-1}v_{\ell-2}\dots v_2v_1u_2v_\ell u_3u_4\dots u_m$, and we have

$$c(u_3v_\ell) \in \{w, c(u_3u_4)\} \cap \{c(u_2u_3), c(u_3u_4)\}.$$

A similar argument shows that $c(u_3v_2) \in \{z, c(u_3u_4)\} \cap \{c(u_2u_3), c(u_3u_4)\}$. At least one of $c(u_3v_\ell)$ and $c(u_3v_2)$ is not equal to $c(u_3u_4)$, by Claim B. So, it follows that either

$c(u_3v_\ell) = c(u_2u_3) = w$ or $c(u_3v_2) = c(u_2u_3) = z$ holds. This shows that the remaining parts of (c), (e) and (f), and Claim L is proved. $\square$

Claim M Assume $c(u_{m-1}v_j) = y$ for some $j \in \{1, 2, \dots, \ell\}$, and let $w' = c(v_{j-1}v_j)$ and $z' = c(v_jv_{j+1})$. Then

- (a) $c(v_{j-2}v_{j-1}) = c(v_{j+1}v_{j+2}) = c(v_{j-1}v_{j+1}) = y$ and $\ell \in \{3, 5\}$.
- (b) $c(u_{m-1}v_k) \neq y$ for all $v_k \neq v_j$.
- (c) $c(u_{m-1}v_{j-1}) = c(u_{m-1}v_{j+1}) = c(u_{m-2}u_{m-1}) \in \{w', z'\}$.
- (d) $m \geq 4$.
- (e) If $c(u_{m-2}u_{m-1}) = w'$, then $c(u_{m-2}v_k) = c(u_{m-3}u_{m-2}) \neq c(u_{m-2}v_{j-1}) = w'$ for all $v_k \neq v_{j-1}$.
- (f) If $c(u_{m-2}u_{m-1}) = z'$, then $c(u_{m-2}v_k) = c(u_{m-3}u_{m-2}) \neq c(u_{m-2}v_{j+1}) = z'$ for all $v_k \neq v_{j+1}$.

Proof. The proof is similar to Claim L. $\square$

By Claim K, we may assume $c(u_2v_1) = x$ without loss of generality. Let $w = c(v_\ell v_1)$ and $z = c(v_1v_2)$. We will further assume $c(u_2v_\ell) = c(u_2v_2) = c(u_2u_3) = w$, which is admissible by Claim L (c) without loss of generality. Then $m \geq 4$ holds by Claim L (d) and $c(u_3v_k) = c(u_3u_4) \neq c(u_3v_\ell) = w$ for all $v_k \neq v_\ell$ by Claim L (e). These facts will be used frequently throughout the remaining part of the proof.

Claim N $\ell = 3$.

Proof. Suppose $\ell \neq 3$. By Claim L (a), we see that ℓ is equal to 5. Further we then have $c(v_4v_5) = c(v_2v_3) = c(v_2v_5) = x$ by Claim L (a), and $c(u_2v_k) \neq x$ for $v_k \in \{v_3, v_4\}$ by Claim L (b).

We will first show that there exists $j \in \{1, 2, \dots, 5\}$ with $c(u_{m-1}v_j) = y$. So suppose not. Then by Claim I there is a k such that $c(u_{m-1}v_{k-1}) = c(v_{m-1}v_k) \neq c(u_{m-2}u_{m-1})$ and $c(v_{k-2}v_{k-1}) = c(v_kv_{k+1}) = y$. The latter implies that $k \notin \{1, 2\}$, using $c(v_4v_5) \neq c(v_1v_2)$ and $c(v_5v_1) \neq c(v_2v_3)$. For $k \in \{3, 4\}$ the path $u_1v_4u_2v_1v_2v_5u_3u_4 \dots u_{m-2}u_{m-1}v_3u_m$ is a PCHP, and for $k = 5$ the path $u_1v_3u_2v_1v_2v_5u_3u_4 \dots u_{m-2}u_{m-1}v_4u_m$ is a PCHP, giving a contradiction in each case. Thus we have shown that j exists as desired. Figure 3.3 summarizes what has been shown so far about the colors of various edges.

Now Claim M (a) implies that $y \in \{x, z, w\}$, and the value of j is uniquely determined, by Claim M (b). Moreover, Claim M (c) implies that $c(u_{m-2}u_{m-1}) \in \{x, z, w\} \setminus \{y\}$.

Case 1. $y = x$.

In this case $j = 1$, and $c(u_{m-1}v_1) = x \neq c(u_{m-2}u_{m-1})$. However, we obtain a PCHP $u_1u_2v_2v_1u_{m-1}u_{m-2} \dots u_4u_3v_5v_4v_3u_m$, a contradiction.

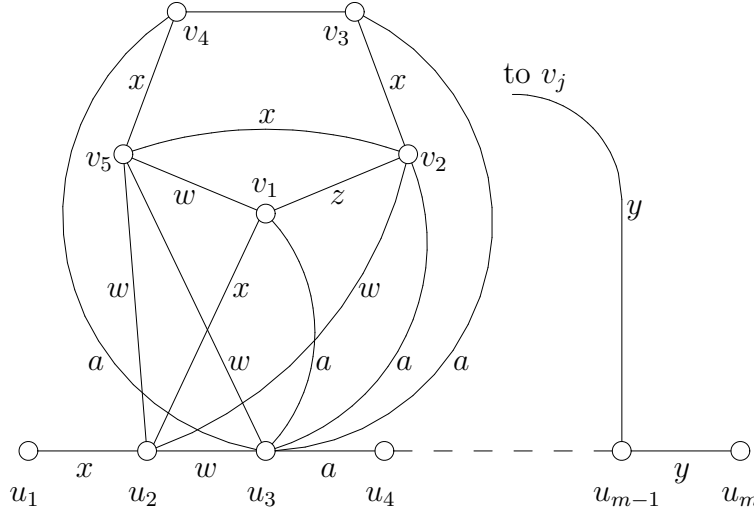


Figure 3.3: The situation in the proof of Claim N, where $a = c(u_3u_4)$.

Case 2. $y = z$.

In this case $j = 5$, $c(u_{m-1}v_5) = z$, and $c(u_{m-2}u_{m-1}) \in \{x, w\}$. Supposing that $c(u_{m-2}u_{m-1}) = x$ holds. Then the path $u_1u_2 \dots u_{m-3}u_{m-2}v_4v_3u_{m-1}u_m$ and the cycle $v_1v_5v_2v_1$ would both be properly colored, contradicting our minimality assumption. So, we have $c(u_{m-2}u_{m-1}) = w$.

By Claim L (e), we have $c(u_3v_1) = c(u_3u_4)$, and $c(u_{m-1}v_1) = c(u_{m-2}u_{m-1})$ follows from Claim M (c), so we deduce that $m \neq 4$. Since $c(u_2u_3) = w = c(u_{m-2}u_{m-1})$, it is clear that $m \neq 5$ holds, hence $m \geq 6$. By Claim M (e), we have $c(u_{m-2}v_1) = w$. Now $u_1u_2v_2v_1u_{m-2}u_{m-3} \dots u_4u_3v_5v_4v_3u_{m-1}u_m$ is a PCHP, a contradiction.

Case 3. $y = w$.

In this case $j = 2$, and $c(u_{m-1}v_2) = w$. We note by Claim M (a) that $c(v_1v_3) = w$ holds. However $u_1u_2v_4v_5u_3u_4 \dots u_m$ and $v_1v_2v_3v_1$ are now properly colored, again contradicting our minimality assumption. This finishes the last case, and Claim N is proved. $\square$

We choose $j \in \{1, 2, 3\}$ with $c(u_{m-1}v_j) = y$, which is possible since Claim I (b) fails for $n = 3$, implying that the assumption of Claim I does not hold. Then j satisfies the assumption of Claim G, hence $c(v_{j+1}v_{j+2}) = y$ follows from Claim G (b). We deduce $y \in \{x, z, w\}$ and proceed to divide into the three respective cases (see also Figure 3.4).

Case 1. $y = x$.

Then $j = 1$ follows from $c(v_2v_3) = x$, and the choice of j implies that $c(u_{m-1}v_1) = x$. But now $u_1u_2v_2v_1u_{m-1}u_{m-2} \dots u_4u_3v_3u_m$ is a PCHP, a contradiction.

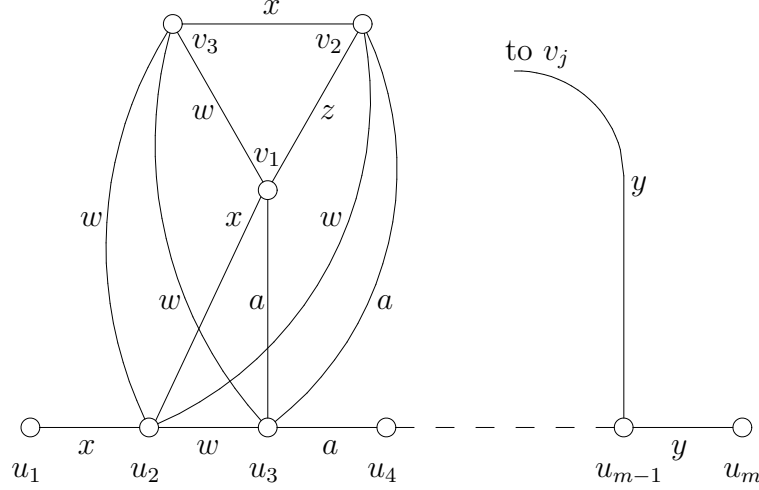


Figure 3.4: The situation in the final steps of the proof of Theorem 3.5, where $a = c(u_3u_4)$.

Case 2. $y = z$.

In this case $j = 3$ follows from $c(v_1v_2) = y$, implying $c(u_{m-1}v_3) = z$. By Claim L (e), $c(u_3v_1) = c(u_3v_2) = c(u_3u_4) \neq c(u_3v_3) = w$. For $m = 4$ all of Claim E (a)-(d) would fail for $p = 4$ and $q = 1$ (in particular Claim E (b) fails by $c(u_3v_2) = c(u_3u_4) = y \neq x = c(v_2v_3)$), so we deduce that $m \geq 5$.

Suppose $c(u_4v_1) \neq c(u_4u_5)$. Then $c(u_3v_1) = c(u_4v_1)$ holds, as otherwise the path $u_1u_2v_2v_3u_3v_1u_4u_5 \dots u_m$ would be a PCHP. Now Claim E with $p = 4$ and $q = 1$ implies that $c(u_3v_2) = x$ or $c(u_4v_1) = w$ (i.e. Claim E (b) or Claim E (c)). We have $c(u_4v_1) = c(u_3v_1) = c(u_3u_4) \neq c(u_2u_3) = w$, so $c(u_3v_2) = x$ follows. But we see that $u_1u_2v_2u_3v_3u_{m-1}u_{m-2} \dots u_5u_4v_1u_m$ is a PCHP, which is a contradiction. We conclude that $c(u_4v_1) = c(u_4u_5)$ holds.

Suppose $c(u_4v_2) \neq c(u_4u_5)$. Then Claim D (c) for $p = 4$ and $q = 2$ holds, that is $c(u_4v_2) = x$, and $u_1u_2u_3v_1v_3u_{m-1}u_{m-2} \dots u_5u_4v_2u_m$ is a PCHP, a contradiction. Hence $c(u_4v_2) = c(u_4u_5)$ follows. Now by Claim B we have $c(u_4v_3) \neq c(u_4u_5)$. With $p = 4$ and $q = 3$, either Claim D (b) or Claim D (c) holds, hence $c(u_3v_2) = z$ or $c(u_4v_3) = w$. Examination of the path $u_1v_2u_2v_1u_3v_3u_4u_5 \dots u_m$ allows us to deduce that $c(u_3v_1) = x$ or $c(u_4v_3) = w$. Then $c(u_4v_3) = w$ follows, since the two alternatives conflict due to $c(u_3v_1) = c(u_3v_2)$.

Let i be the largest number such that $i \leq m$ and such that $c(u_i'v_1) = c(u_i'v_2) = c(u_i'u_{i'+1}) \neq c(u_i'v_3) = w$ holds for every $i' = 3, 4, \dots, i-1$. We note that i exists and satisfies $i \geq 5$. It is useful to observe that Claim D (a) and Claim E (a) fail for $p = i$ and every $q = 1, 2, 3$; for Claim D (a) this follows from $c(u_{i-2}u_{i-1}) \neq w$ for $q = 1$, and it follows from $c(u_{i-1}v_{q-1}) = c(u_{i-1}u_i) \neq c(u_{i-2}u_{i-1})$ for $q \neq 1$. Similarly for Claim E (a).

For $i = m$ all of Claim E (a,b,c,d) fail for $p = i$ and $q = 1$ (in particular Claim E (b) fails by $c(u_{i-1}v_2) = c(u_{i-1}u_i) = y \neq x = c(v_2v_3)$), so we deduce that $i < m$.

Suppose $c(u_i'v_1) \neq c(u_i'u_{i'+1})$. For $p = i$ and $q = 1$, Claim D and Claim E imply $c(u_i'v_1) = z$ and $c(u_{i-1}v_2) = x$ (respectively Claim D (c) and Claim E (b)). Using $c(u_{i-1}v_1) = c(u_{i-1}v_2) = x$ and $c(u_{i-2}v_2) = c(u_{i-2}u_{i-1}) \neq c(u_{i-1}u_i) = c(u_{i-1}v_2) = x$,

the path $u_1 \dots u_{i-3} u_{i-2} v_2 v_3 u_{i-1} v_1 u_i u_{i+1} \dots u_m$ is a PCHP. This contradiction shows that $c(u_i v_1) = c(u_i u_{i+1})$.

Suppose $c(u_i v_2) \neq c(u_i u_{i+1})$. Then Claim D with $p = i$ and $q = 2$ implies that $c(u_i v_2) = x$, but $u_1 u_2 \dots u_{i-2} u_{i-1} v_1 v_3 u_{m-1} u_{m-2} \dots u_{i+1} u_i v_2 u_m$ is a PCHP, a contradiction. Therefore $c(u_i v_2) = c(u_i u_{i+1})$. By Claim B we have $c(u_i v_3) \neq c(u_i u_{i+1})$. Then $c(u_i v_3) \neq w$ follows from the choice of i , and $u_1 v_2 v_1 u_2 u_3 \dots u_{i-2} u_{i-1} v_3 u_i u_{i+1} \dots u_m$ is a PCHP, a contradiction.

Case 3. $y = w$.

This implies that $j = 2$ and $c(u_{m-1} v_2) = w$. By Claim M (c), we have $c(u_{m-2} u_{m-1}) \in \{x, z\}$. The case $c(u_{m-2} u_{m-1}) = z$ is symmetric to Case 2, so only the case $c(u_{m-2} u_{m-1}) = x$ remains. Then $m \geq 5$ follows, as P is properly colored. We can assume $c(u_2 u_{m-1}) \neq x$ without loss of generality. But $u_1 u_2 u_{m-1} u_{m-2} \dots u_4 u_3 v_3 v_2 v_1 u_m$ is a PCHP, with a contradiction. This finishes the proof of Case 3, and of Theorem 3.5. ■

Theorem 3.6 (Feng, et al. [48]) *The PCHP Conjecture holds.*

Proof. Let $C = v_1 \dots v_\ell v_1$ ($\ell \geq 3$) and $P = u_1 \dots u_m$ ($m \geq 1$). Throughout we will perform addition and subtraction in the indices of the vertices $v_j \in C$ modulo ℓ .

If $m = 1$, then either $u_1 v_1 v_2 \dots v_\ell$ or $u_1 v_1 v_\ell \dots v_2$ is a PCHP as desired. So, we need only to consider the case when $m \geq 2$.

Let $j \in \{1, 2, \dots, \ell\}$. If $c(u_1 v_j) \neq c(u_1 u_2)$, then we see that at least one of the paths $u_m u_{m-1} \dots u_1 v_j v_{j+1} \dots v_{j-1}$ and $u_m u_{m-1} \dots u_1 v_j v_{j-1} \dots v_{j+1}$ is a PCHP. Similarly, there exists a PCHP if $c(u_m v_j) \neq c(u_{m-1} u_{m-2})$.

So, we only need to consider the case when $m \geq 2$, $c(u_1 v_j) = c(u_1 u_2)$ and $c(u_m v_j) = c(u_{m-1} u_m)$ for every $j = 1, 2, \dots, \ell$. By Theorem 3.5, there exists a PCHP H in K_n with u_1 as its first vertex, such that the initial edge of H is either $u_1 u_2$ or one of the edges $u_1 v_j$ ($1 \leq j \leq \ell$), and such that u_m is the last vertex of H and the last edge of H is either $u_{m-1} u_m$ or one of the edges $v_j u_m$ ($1 \leq j \leq \ell$). The proof is completed. ■

Remark 3.7 *By Theorem 3.1, our result implies that the PCHP problem is polynomial time solvable for edge-colored complete graphs.*

Chapter 4

Complementary Cycles in Jump Graphs

Another interesting topic in graph theory is the research on the existence of a 2-factor in a graph. In this chapter, we consider a special 2-factor, the complementary cycles in jump graphs and we give a characterization for jump graphs containing complementary cycles.

The *line graph* $L(G)$ of G has the edges of G as its vertices, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G . The complement of the line graph $L(G)$ is called the *jump graph* $J(G)$ of G , i.e. the jump graph $J(G)$ is the graph defined on $E(G)$, and in which two vertices are adjacent if and only if the corresponding edges are not adjacent in G .

Note that the line graph $L(G)$ and jump graph $J(G)$ are defined on the edge set of a graph G , and isolated vertices of G (if G has) play no role in line graph transformation and jump graph transformation. Assume that the graph G under consideration is nonempty and has no isolated vertices in what follows. For convenience, we also denote the size of a graph G by $q(G)$ or by q if there is no confusion.

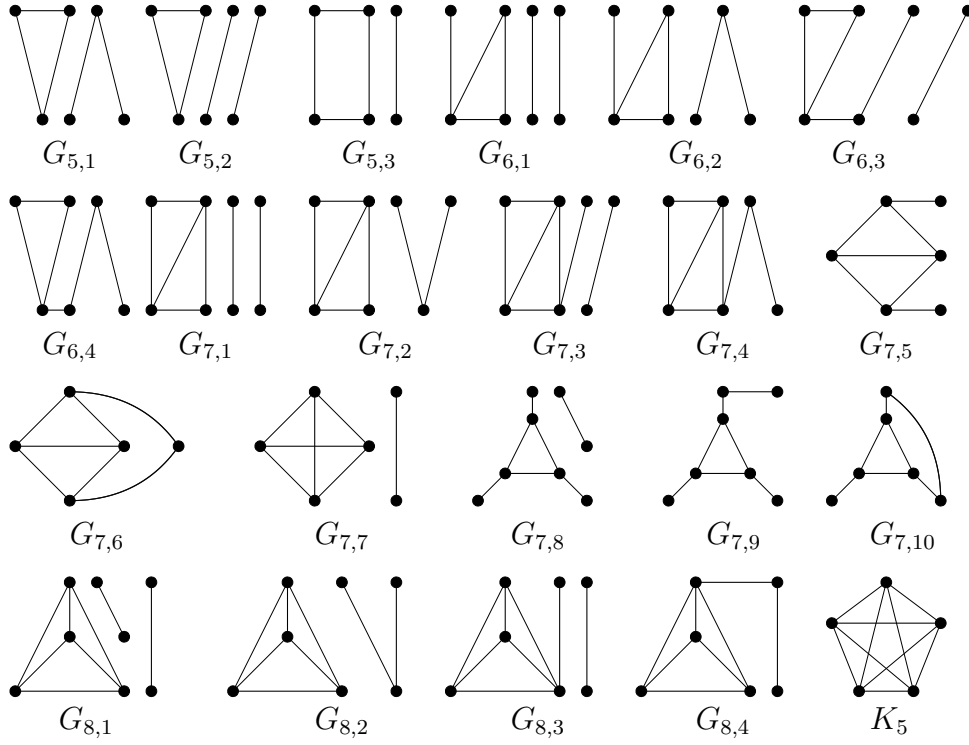
4.1 Some Results on Jump Graphs

By using a characterization for the connectivity of jump graphs ([34]), Wu and Meng gave the following theorem:

Theorem 4.1 (Wu and Meng [81]) *For a graph G , $J(G)$ is connected if and only if $G \notin \{C_4, K_4\}$ and $|q| \geq d(u) + d(v)$ for all $uv \in E(G)$.*

Harary and Nash-Williams [61] showed that the line graph $L(G)$ of a graph G is Hamiltonian if and only if $G = K_{1,n}$, where $n \geq 3$, or G contains a dominating cycle. In [34], Chartrand et al. presented some sufficient conditions for a jump graph to be Hamiltonian. In addition, they posed the following two conjectures:

Conjecture A. Let G be a graph of order at least 7 and size $q \geq 5$. If $q \geq 2\Delta(G)$, then $J(G)$ is Hamiltonian.

Figure 4.1: $\mathcal{H}_1$

Conjecture B. Let G be a Hamiltonian graph of order $n \geq 7$ and size q . If $q \geq 2n - 2$, then $J(G)$ is Hamiltonian.

In 2004, Wu and Meng presented the following theorem:

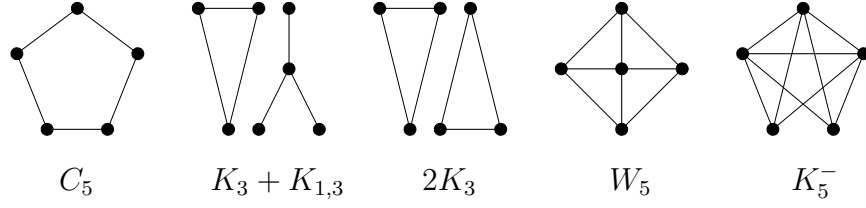
Theorem 4.2 (Wu and Meng [81]) *Let G be a graph of size $q \geq 1$. Then $J(G)$ is not Hamiltonian if and only if G satisfies one of the following conditions:*

- (1) $q < 2\Delta(G)$.
- (2) $q = 2\Delta(G)$ and G has an edge uv such that $d(u) = d(v) = \Delta(G)$.
- (3) G is isomorphic to $K_3 + P_3$, $K_3 + 2K_2$ or $C_4 + K_2$.
- (4) G has a subgraph isomorphic to

$$\begin{cases} K_3^+ & \text{if } q = 6, \\ K_4^- \text{ or } K_3 \circ K_1 & \text{if } q = 7, \\ K_4 & \text{if } q = 8. \end{cases}$$

- (5) G is isomorphic to K_5 .

By Theorem 4.2, $J(G)$ is Hamiltonian if and only if G satisfies the following conditions:

Figure 4.2: $\mathcal{H}_2$

- (1) $2\Delta(G) \leq q$,
- (2) $d(u) + d(v) \leq q - 1$ for all $uv \in E(G)$,
- (3) G is not isomorphic to any graph in Figure 4.1.

As a corollary, Conjecture B is confirmed, and Conjecture A is disproved.

A graph G is called *pancyclic* if there is a cycle of length k for each $3 \leq k \leq |V(G)|$. Recently, Liu gave a characterization of pancyclic jump graphs.

Theorem 4.3 (Liu [65]) *$J(G)$ is pancyclic if and only if G satisfies the following conditions:*

- (1) $2\Delta(G) \leq q$,
- (2) $d(u) + d(v) \leq q - 1$ for any two vertices u and v ,
- (3) G is not isomorphic to any graph of $\mathcal{H}_1 \cup \mathcal{H}_2$ (see Figure 4.1 and 4.2).

Let G be a graph. Two disjoint cycles C_1 and C_2 are a pair of *complementary cycles* of G if $V(C_1) \cup V(C_2) = V(G)$. In next section, we consider the complementary cycles in jump graphs and characterize jump graphs which have complementary cycles.

4.2 Complementary Cycles in Jump Graphs

Let $G = (V, E)$ be a graph and $u \in V$. Partition $I(u)$ into disjoint (non-empty) sets $I_1(u), I_2(u), \dots, I_{m_u}(u)$, where $1 \leq m_u \leq |I(u)|$. Let $\mathcal{P}$ denote the resulting partition. The splitting graph $split_u(G)$ of G at u formed from G using $\mathcal{P}$ has vertices $V(split_u(G)) = \{u^i \mid 1 \leq i \leq m_u\} \cup (V \setminus \{u\})$, and the edges $E(split_u(G)) = (\cup_{k=1}^{m_u} I_k(u)) \cup (E \setminus I(u))$ with $I_{split_u(G)}(u^i) = I_i(u)$, $1 \leq i \leq m_u$, and $u^i x \in E(split_u(G))$ if and only if $ux \in I_i(u)$. A graph H is called a *splitting graph* of G if there is a sequence of graphs $H_0, H_1, \dots, H_t$ and vertices $v_1, \dots, v_t$ of G such that $H_0 = G$, $H_i = split_{v_i}(H_{i-1})$, $i = 2, \dots, t$ and $H_t = H$.

Proposition 4.4 *If $J(G)$ has complementary cycles, then for each splitting graph G' of G , the jump graph $J(G')$ also contains the same complementary cycles.*

Proof. Since $J(G)$ is a subgraph of $J(G')$, the proposition follows. ■

To characterize jump graphs containing complementary cycles, we need the following:

Lemma 4.5 *If $J(G)$ has complementary cycles, then the following statements hold:*

- (1) $q \geq 6$.
- (2) *If $q = 6$, then $\Delta(G) \leq 2$ and G contains no subgraph isomorphic to C_5 or C_3 .*
- (3) $2\Delta(G) \leq q$.
- (4) *If $q = 2\Delta(G)$, then there is no edge uv with $d(u) = d(v) = \Delta(G)$.*
- (5) $G \notin \mathcal{H}_1 \cup \mathcal{H}_2 - \{K_5\}$.

Proof. It is easy to check that the statements (1), (2) and (5) are true.

- (3) Assume to the contrary that $2\Delta(G) > q$. Then there is a vertex u with $d(u) = \Delta(G) > q/2$.

Since $J(G)$ contains complementary cycles, the edge set of G has disjoint subsets E_1, E_2 such that $E_1 \cup E_2 = E(G)$ and $J(G[E_i])$ is Hamiltonian for $i = 1, 2$. It follows that $I(u) = I_1(u) \cup I_2(u)$, where $I_i(u) = I(u) \cap E_i$, $i = 1, 2$. By Theorem 4.2, we see that $|I_i(u)| \leq |E_i|/2$. Moreover, $d(u) = |I_1(u)| + |I_2(u)| \leq |E_1|/2 + |E_2|/2 = q/2$, a contradiction.

- (4) Suppose to the contrary that $q = 2\Delta(G)$ and there is an edge $uv \in E(G)$ with $d(u) = d(v) = \Delta(G)$. We only need to consider the case when $q \geq 8$.

A similar discussion as in (3) shows that G has disjoint edge subsets E_1, E_2 such that $E_1 \cup E_2 = E(G)$ and $J(G[E_i])$ is Hamiltonian for $i = 1, 2$. So, $I(u) = I_1(u) \cup I_2(u)$ and $I(v) = I_1(v) \cup I_2(v)$, where $I_i(u) = I(u) \cap E_i$, $i = 1, 2$. Since $J(G[E_1])$ and $J(G[E_2])$ are Hamiltonian, we see that $|I_i(u)| = |I_i(v)| = |E_i|/2$, and then $|I_i(u)| + |I_i(v)| = |E_i|$ for $i = 1, 2$. Without loss of generality, assume that $uv \in E_1$. By Theorem 4.2, we have $|I_1(u)| + |I_1(v)| \leq |E_1| - 1$, a contradiction to $|I_1(u)| + |I_1(v)| = |E_1|$. ■

Lemma 4.6 *Let G be a graph of size $q \geq 13$. $J(G)$ has complementary cycles if and only if the following holds:*

- (1) $2\Delta(G) \leq q$,
- (2) *if $q = 2\Delta(G)$, then there is no edge uv with $d(u) = d(v) = \Delta(G)$.*

Proof. Denote $\omega(G) = t$. Let $G_1, G_2, \dots, G_t$ be the labeled components of G such that $\Delta(G_i) \geq \Delta(G_{i+1})$, $i = 1, 2, \dots, t-1$. We consider the following cases:

Case 1. $t \geq 3$.

For $i = 1, 2, 3$, choose $u_i v_i \in E(G_i)$ satisfying

- (i) $d(u_i) = \Delta(G_i)$ and $d(u_i) + d(v_i)$ is maximal among all the edges in $I(u_i)$,
- (ii) subject to condition (i), the edge $u_i v_i$ is contained in as more as possible 3-cycles in G_i .

It is clear that $J(G)[\{u_1v_1, u_2v_2, u_3v_3\}] = K_3$. Denote $G' = G - \{u_1v_1, u_2v_2, u_3v_3\}$. In the following, we prove that $J(G')$ is Hamiltonian or $J(G)$ has complementary cycles.

Assume to the contrary that $J(G')$ is not Hamiltonian. Since $q \geq 13$, it is easy to see that $G' \notin \mathcal{H}_1$. By Theorem 4.2, we only need to consider the following subcases:

Subcase 1.1. There is a vertex $x \in V(G')$ with $d_{G'}(x) > |E(G')|/2 = (q-3)/2$.

We confirm that $x \in V(G_1)$. Otherwise, $x \in V(G_i)$, $i \geq 2$. Then we have either $q \geq d(u_1) + d(u_2) + d(u_3) > 2((q-3)/2 + 1) + 1 = q$ (if $x = u_2$) or $q \geq d(u_1) + d(u_2) + d(x) - 1 + d(u_3) > 3(q-3)/2 \geq q$ (if $x \neq u_2$), both deduce contradictions.

If $x = u_1$, then we have $q/2 \geq d_G(u_1) = d_{G'}(x) + 1 > (q-3)/2 + 1 = (q-1)/2$. So, q is even and $d_G(u_1) = q/2$. Now, we choose $u_1w_1 \in E(G_1)$ such that $d(u_1) + d(w_1)$ is maximal among the edges in $I(u_1) - \{u_1v_1\}$. As a result, $J(G)[\{u_1v_1, u_1w_1, u_2v_2, u_3v_3\}]$ is Hamiltonian. Denote $G'' = G - \{u_1v_1, u_1w_1, u_2v_2, u_3v_3\}$. By the choice of u_1, v_1 and w_1 , it is evident that $d_{G''}(u_1) = \Delta(G'') = (q-4)/2$ and $d_{G''}(a) + d_{G''}(b) \leq q-5 = |E(G'')| - 1$ for all $ab \in E(G'')$. Moreover, $G'' \notin \mathcal{H}_1$. By Theorem 4.2, we see that $J(G'')$ is Hamiltonian.

For the case when $x = v_1$, it is obvious that $d(u_1) \geq d(v_1) > (q-3)/2 + 1$. Then, $q \geq |E(G_1)| + |E(G_2)| + |E(G_3)| \geq |E(G_1)| + 2 > 2((q-3)/2 + 1) - 1 + 2 = q$, a contradiction.

If $x \notin \{u_1, v_1\}$, then $d(u_1) \geq d(x) > (q-3)/2$. For the case when $u_1x \in E(G_1)$, we see that $d(v_1) \geq d(x)$. So, $q \geq |E(G_1)| + |E(G_2)| + |E(G_3)| \geq |E(G_1)| + 2 > 3(q-3)/2 - 3 + 2$. It follows that $q < 11$, a contradiction. Thus, assume that $u_1x \notin E(G_1)$. If $q = 2k + 1$, then $d(x) \geq k$. Moreover, $q \geq d(u_1) + d(x) + 2 \geq 2k + 2 = q + 1$, a contradiction. If $q = 2k$, then $d(x) \geq k - 1$. Since $t \geq 3$, we see that $d(u_1) = d(x) = k - 1$. Furthermore, $|E(G_1)| = 2k - 2$, $|E(G_2)| = 1 = |E(G_3)|$. It is easy to see that $J(G)$ contains complementary cycles.

Subcase 1.2. $\Delta(G') = (q-3)/2$ and there is $xy \in E(G')$ with $d_{G'}(x) = d_{G'}(y) = \Delta(G')$.

Note that q is odd. Since $q \geq 13$ and $t \geq 3$, it is not difficult to see that $xy \in E(G_1)$ and $G - V(G_1) \in \{2K_2, 3K_2, P_3 + K_2\}$. From $q \geq 13$, it follows that $d_{G'}(x) = d_{G'}(y) \geq 5$. Therefore, $u_1 \in \{x, y\}$, which deduces a contradiction to the choice of u_1v_1 , since $d(a) \leq 4$ for all $a \in V(G_1) \setminus \{x, y\}$.

In this case, we see that $J(G)$ contains complementary cycles.

Case 2. $t = 2$.

Subcase 2.1. $|E(G_2)| \geq 2$.

For $i = 1, 2$, choose $u_iv_i, u_iw_i \in E(G_i)$ satisfying

- (i) $d(u_i) = \Delta(G_i)$ and $d(u_i) + d(v_i)$ is maximal among all the edges in G_i and $d(u_i) + d(w_i)$ is maximal among all the edges in $G_i - u_iv_i$,
- (ii) subject to condition (i), the edges u_iv_i and u_iw_i are contained in as more as possible 3-cycles in G_i .

It is easy to see that $J(G)[\{u_1v_1, u_1w_1, u_2v_2, u_2w_2\}]$ is Hamiltonian. Denote $G' = G - \{u_1v_1, u_1w_1, u_2v_2, u_2w_2\}$. In the following, we prove that $J(G')$ is Hamiltonian or $J(G)$ contains complementary cycles.

Assume that $J(G')$ is not Hamiltonian. Since $q \geq 13$ and $|E(G')| \geq 9$, it is easy to see that $G' \notin \mathcal{H}_1$. By Theorem 4.2, we only need to consider the following subcases:

Subcase 2.1.1. There is a vertex $x \in V(G')$ with $d_{G'}(x) > |E(G')|/2 = (q-4)/2$.

Note that $d_{G'}(x) \geq 5$. If $x \in V(G_2)$, then one of the following statements holds:

- (1) $q \geq d(u_1) + d(u_2) > 2((q-4)/2 + 2) = q$ if $x = u_2$.
- (2) $q \geq d(u_1) + d(u_2) + d(x) - 1 > (q-4)/2 + 1 + (q-4)/2 + 1 + (q-4)/2 + 1 - 1 = q + (q-8)/2 > q$ if $x \in \{v_2, w_2\}$.
- (3) $q \geq d(u_1) + d(u_2) + d(v_2) + d(w_2) - 3 > 4(q-4)/2 - 3 = q + q - 11 > q$ if $x \in N(u_2) \setminus \{v_2, w_2\}$.
- (4) $q \geq d(u_1) + d(u_2) + d(x) > (q-4)/2 + (q-4)/2 + (q-4)/2 = q + (q-12)/2 > q$ if $x \notin N(u_2)$.

Each of them deduces a contradiction. So, we have $x \in V(G_1)$.

If $x = u_1$, then $d(u_1) = d_{G_1}(u_1) + 2 > (q-4)/2 + 2 = q/2$, a contradiction.

If $x \in \{v_1, w_1\}$, then $d(x) = d_{G_1}(x) + 1 > (q-4)/2 + 1 = (q-2)/2$. Thus, $d(u_1) + d(x) > q - 2$. Moreover, $d(u_1) + d(x) = q - 1$, $d(u_1) = d(x) = (q-1)/2$, $E(G_1) = q - 2$ and $E(G_2) = 2$. Now, it is not difficult to see that $J(G)$ contains complementary cycles.

If $x \in N(u_1) \setminus \{v_1, w_1\}$, then $\min\{d(v_1), d(w_1)\} \geq d(x)$, according to the choice of u_1, v_1 and w_1 . Thus, we have $q \geq d(u_1) + d(v_1) + d(w_1) + d(x) - 6 + 2 > 4(q-4)/2 - 6 + 2 = q + q - 12 \geq q$, a contradiction.

Assume that $x \notin N(u_1)$. Then $|E(G_1)| \geq d(u_1) + d(x) > (q-4)/2 + (q-4)/2 = q - 4$, i.e. $|E(G_1)| \geq q - 3$.

If $q = 2k$, then $d(x) \geq k - 1$ and $d(u_1) \leq k$. Since $|E(G_2)| \geq 2$, we see that $d(x) = d(u_1) = k - 1$. It is not difficult to see that $J(G)$ contains complementary cycles.

If $q = 2k + 1$, then $d(x) \geq k - 1$ and $d(u_1) \leq k$. Since $|E(G_2)| \geq 2$, we see that $d(x) = k - 1$ and either $d(u_1) = k$ or $d(u_1) = k - 1$. For both cases, it is not complicated to see that $J(G)$ contains complementary cycles.

Subcase 2.1.2. $\Delta(G') = (q-4)/2$ and there is $xy \in E(G')$ with $d_{G'}(x) = d_{G'}(y) = (q-4)/2$.

This implies that q is even, say $q = 2k$, $k \geq 7$. If $u_1 \in \{x, y\}$, then $q/2 \geq d(u_1) = d_{G'}(u_1) + 2 > (q-4)/2 + 2 = q/2$, a contradiction. Thus, $u_1 \notin \{x, y\}$.

If $xy = v_1w_1$, then $d(u_1) \geq d_{G'}(x) + 1 = d_{G'}(y) + 1 = (q-4)/2 + 1 = (q-2)/2 = k - 1$. Moreover, $q \geq d(u_1) + d(x) + d(y) - 3 + 2 \geq 3k - 3 - 1 = q + (q-8)/2 > q$, a contradiction.

Consider the case when $|\{x, y\} \cap \{v_1, w_1\}| = 1$. Without loss of generality, let $x = v_1$. Then, $d(u_1) \geq d_{G'}(x) + 1$.

By the choice of v_1 and w_1 , we have $|\{x, y\} \cap N(u_1)| = 1$. Otherwise, we see that $\min\{d(v_1), d(w_1)\} \geq d(y) = (q-4)/2$. Furthermore, $q \geq d(u_1) + d(v_1) + d(w_1) + d(y) - 6 + 2 \geq 2(q-4)/2 + 2 + 2(q-4)/2 - 4 = 2q - 12 > q$, a contradiction. So, $y \notin N(u_1)$. We have

$q \geq d(u_1) + d(x) + d(y) - 2 + 2 = (q-4)/2 + 1 + (q-4)/2 + (q-4)/2 = q + (q-10)/2 > q$, a contradiction.

If $|\{x, y\} \cap \{v_1, w_1\}| = 0$, then $|\{x, y\} \cap N(u_1)| = 0$. Then, $q \geq d(u_1) + d(x) + d(y) - 1 + 2 \geq (q-4)/2 + (q-4)/2 + (q-4)/2 + 1 = q + (q-10)/2 > q$, a contradiction again.

Subcase 2.2. $|E(G_2)| = 1$.

It is clear that $|V(G_2)| = 2$ and $d(u) = 1$ for $u \in V(G_2)$. Let $x_1, x_2, \dots, x_n$ be the labeled vertices of G such that $d(x_i) \geq d(x_{i+1})$ for $i = 1, 2, \dots, n-1$ and $x_{n-1}, x_n \in V(G_2)$.

At first, we consider the case when $d(x_2) \geq 3$. Choose $v_1, v_2 \in V(G)$ such that $x_1v_1, x_2v_2 \in E(G)$, $d(x_1) + d(v_1)$ is maximal in $\{d(x_1) + d(z) \mid z \in N(x_1), z \neq x_2\}$ and $d(x_2) + d(v_2)$ is maximal in $\{d(x_2) + d(z) \mid z \in N(x_2), z \neq x_1, v_1\}$. Clearly, $v_1 \neq v_2$. It is easy to see that $J(G)[\{x_1v_1, x_2v_2, x_{n-1}x_n\}] = K_3$. Denote $G' = G - \{x_1v_1, x_2v_2, x_{n-1}x_n\}$. Now, we prove that $J(G')$ is Hamiltonian or $J(G)$ contains complementary cycles.

Assume to the contrary that $J(G')$ is not Hamiltonian. Since $q \geq 13$, we see that $G' \notin \mathcal{H}_1$. By Theorem 4.2, we only need to consider the following cases:

Subcase 2.2.1. $d_{G'}(x_i) > (q-3)/2$ for some $1 \leq i \leq n-3$.

If $i = 1$, then we see that $q/2 \geq d(x_1) > (q-3)/2 + 1 = (q-1)/2$. For $q = 2k$, we see that $k \geq d(x_1) > k - 1/2$, i.e. $d(x_1) = k = q/2$. Then we can choose the edges x_1u_1, x_1w_1 and ab such that $a, b \notin \{x_1, u_1, w_1\}$ and $d(a) = \max\{d(y) \mid y \in V(G), y \neq x_1, u_1, w_1\}$. It is evident that $J(G)[\{x_1u_1, x_1w_1, ab, x_{n-1}x_n\}]$ is Hamiltonian. Let $G'' = G - \{x_1u_1, x_1w_1, ab, x_{n-1}x_n\}$. If $d_{G''}(x'') > (q-4)/2 = k-2$ for some $x'' \in V(G'')$, then we have $d_{G''}(x'') \geq k-1$ and $x'' \neq x_1$. So, $d(x'') = k-1$ or $d(x'') = k$. It is not difficult to see that $J(G)$ has complementary cycles. If $\Delta(G'') = (q-4)/2 = k-2$ and there exists $x''y'' \in E(G'')$ such that $d_{G''}(x'') = d_{G''}(y'') = k-2$, then it is easy to see that $x_1 \notin \{x'', y''\}$. Additionally, $q \geq d(x_1) + d(x'') + d(y'') - 3 + 1 \geq k + k - 2 + k - 2 - 3 + 1 = q + (q-12)/2 > q$, a contradiction. Therefore, $J(G'')$ is Hamiltonian, and hence, $J(G)$ has complementary cycles. For $q = 2k+1$, we have $k \geq d(x_1) > k$, a contradiction.

If $i = 2$, then $q/2 \geq d(x_1) \geq d(x_2) > (q-3)/2 + 1 = (q-1)/2$. For $q = 2k$, we see that $k \geq d(x_1) \geq d(x_2) > k - 1/2$, and hence, $d(x_1) = d(x_2) = k = q/2$. Since $|E(G_1)| = q-1$, we see that $x_1x_2 \in E(G)$, a contradiction. For $q = 2k+1$, we have $k \geq d(x_1) \geq d(x_2) > k$, a contradiction.

For the case when $i \geq 3$, we see that $q/2 \geq d(x_1) \geq d(x_2) \geq \dots \geq d(x_i) > (q-3)/2$. If $i \geq 4$, then we have $q \geq d(x_1) + d(x_2) + d(x_3) + d(x_4) - 6 + 1 > 4(q-3)/2 - 5 = 2q - 11 \geq q$, a contradiction. Thus, $i = 3$. If either $x_3 = v_1$ or $x_3 = v_2$, then $q/2 \geq d(x_1) \geq d(x_2) \geq d(x_3) > (q-3)/2 + 1 = (q-1)/2$. For $q = 2k$, we see that $k \geq d(x_3) > k - 1/2$, and then $d(x_3) = k = q/2$. It follows that $q \geq d(x_1) + d(x_2) + d(x_3) - 3 + 1 = q + k - 2 > q$, a contradiction. For $q = 2k+1$, we have $k \geq d(x_3) > k$, a contradiction.

If $x_3 \in N(x_1) \cup N(x_2) \setminus \{v_1, v_2\}$, then either $d(v_1) \geq d(x_3)$ or $d(v_2) \geq d(x_3)$. Thus, $q \geq 4(q-3)/2 - 6 + 1 = 2q - 11 > q$, a contradiction.

If $x_3 \notin N(x_1) \cup N(x_2)$, then we have $q \geq d(x_1) + d(x_2) + d(x_3) - 1 + 1 \geq 3(q-3)/2 = q + (q-9)/2 > q$, a contradiction.

Subcase 2.2.2. $\Delta(G') = (q - 3)/2$ and there is $xy \in E(G)$ with $d_{G'}(x) = d_{G'}(y) = \Delta(G')$.

It is clear that $q = 2k + 1$ for some $k \geq 6$. If $xy \in \{x_1x_2, v_1v_2\} \cup \{x_1v_2, x_2v_1\}$, then $d(x) = d(y) = k$. It is easy to see that $J(G)$ has complementary cycles.

For the case when $|\{x, y\} \cap \{x_1, x_2, v_1, v_2\}| = 1$, say $x \in \{x_1, x_2, v_1, v_2\}$, we have the following:

- (1) If $x = x_1$, then $d(x_1) = k$ and $d(x_2) \geq d(v_1) \geq d(y)$. So, $q \geq d(x_1) + d(x_2) + d(v_1) + d(y) - 6 + 1 \geq k + 3(k - 1) - 6 + 1 = q + q - 10 > q$, a contradiction.
- (2) If $x = v_1$, then $d(x_1) = d(x_2) = d(v_1) = k$. Thus, $q \geq d(x_1) + d(x_2) + d(v_1) + d(y) - 5 + 1 \geq 3k + (k - 1) - 5 + 1 = q + q - 7 > q$, a contradiction.
- (3) If $x = x_2$, then $d(x_1) = d(x_2) = k$ and $d(v_2) \geq d(y)$. As a result, $q \geq d(x_1) + d(x_2) + d(v_2) + d(y) - 5 + 1 \geq 2k + 2(k - 1) - 5 + 1 = q + q - 8 > q$, a contradiction.
- (4) If $x = v_2$, then $d(x_1) = d(x_2) = d(v_2) = k$. Furthermore, $q \geq d(x_1) + d(x_2) + d(v_1) + d(y) - 5 + 1 \geq 3k + (k - 1) - 5 + 1 = q + q - 7 > q$, a contradiction.

If $x, y \notin \{x_1, x_2, v_1, v_2\}$, then $q \geq d(x_1) + d(x_2) + d(x) + d(y) - 6 + 1 \geq 4(k - 1) - 5 = q + q - 11 > q$, a contradiction.

Therefore, $J(G)$ has complementary cycles for the case when $d(x_2) \geq 3$.

At last, we consider the remaining case when $d(x_2) \leq 2$. If $d(x_2) = 1$, then $d(x_i) = 1$ for $i \geq 2$. Since G_1 is connected, we have $d(x_1) = q - 1$, a contradiction to the condition that $\Delta(G) \leq q/2$. So, $d(x_2) = 2$. It is not difficult to check that $J(G)$ has complementary cycles.

Case 3. $t = 1$.

Let $x_1, x_2, \dots, x_n$ be the labeled vertices such that $d(x_i) \geq d(x_{i+1})$ for $i = 1, 2, \dots, n - 1$.

Subcase 3.1. $d(x_3) = 1$.

It is easy to see that $x_1x_2 \notin E(G)$ (otherwise, from $\Delta(G) \leq q/2$, it follows that $d(x_1) + d(x_2) - 1 \leq q - 1$, a contradiction to $d(x_1) + d(x_2) - 1 = q$). So, we have $d(x_1) + d(x_2) = q$, and then $d(x_1) = d(x_2) = q/2$. Since $q \geq 13$, it is not difficult to see that $J(G)$ has complementary cycles.

Subcase 3.2. $d(x_3) = 2$.

Note that $d(x_i) \leq 2$ for $i \geq 4$. If $d(x_2) = 2$, then $G - x_1$ consists of some paths with at least $q/2$ edges. So, it is not complicated to see that $J(G)$ has complementary cycles. For $d(x_2) \geq 3$, the induced subgraph $G - \{x_1, x_2\}$ consists of some paths or is an empty graph. It is also not difficult to see that $J(G)$ has complementary cycles.

Subcase 3.3. $d(x_3) = 3$.

Assume to the contrary that $J(G)$ does not contain complementary cycles.

Firstly, assume that $x_3x_1, x_3x_2 \notin E(G)$. Then there is $x_iv_i \in E(G)$ for $i = 1, 2, 3$ with $|\{v_1, v_2, v_3\}| = 3$. Let $G' = G - \{v_1x_1, v_2x_2, v_3x_3\}$. If there is a vertex $x \in V(G')$ with $d_{G'}(x) > (q-3)/2$, then either $x = x_1$ or $x = x_2$. For both cases, it is easy to obtain a contradiction. If $\Delta(G') = (q-3)/2$ and there is $uv \in E(G')$ with $d_{G'}(u) = d_{G'}(v) = (q-3)/2$, then $uv = x_1x_2$ with $d(x_1) = d(x_2) = (q-1)/2$. It follows that $|I(x_1) \cup I(x_2)| = q-2$, a contradiction to the fact that $|E(G) \setminus I(x_1) \cup I(x_2)| \geq 3$. Thus, $J(G')$ is Hamiltonian, a contradiction.

Secondly, consider the case when G contains only one of the edges x_3x_1 and x_3x_2 . If $|N(x_1) \cup N(x_2) \setminus (\{x_1, x_2\} \cup N[x_3])| \geq 1$, then there is $x_iv_i \in E(G)$ for $i = 1, 2, 3$ with $|\{v_1, v_2, v_3\}| = 3$. Let $G' = G - \{v_1x_1, v_2x_2, v_3x_3\}$. With a similar discussion as above, $J(G')$ is Hamiltonian. So, $J(G)$ contains complementary cycles, a contradiction. Then $N(x_1) \cup N(x_2) \setminus (\{x_1, x_2\} \cup N[x_3]) = \emptyset$, and hence, $N(x_1) \cup N(x_2) \subset \{x_1, x_2\} \cup N[x_3]$. Since $d(x_3) = 3$, we have $x_1x_2, x_1x_3 \in E(G)$ and $N(x_2) = N(x_3)$. It is clear that $q \leq 8$, a contradiction.

Finally, assume that $x_3x_1, x_3x_2 \in E(G)$. If $|N(x_1) \cup N(x_2) \setminus (\{x_1, x_2\} \cup N[x_3])| \geq 2$, then there are $x_1u_1, x_1v_1, x_3u_3, x_3v_3 \in E(G)$ with $|\{u_1, v_1, u_3, v_3\}| = 4$ (if $|N(x_i) \setminus (N(x_j) \cup N(x_3))| \geq 2$ for $i \neq j \in \{1, 2\}$) or there is $x_iv_i \in E(G)$ for $i = 1, 2, 3$ with $|\{v_1, v_2, v_3\}| = 3$ (if $|N(x_1) \setminus (N(x_2) \cup N(x_3))| \geq 1$ and $|N(x_2) \setminus (N(x_1) \cup N(x_3))| \geq 1$). A similar discussion as above deduces that $J(G)$ contains complementary cycles, a contradiction. If $N(x_1) \cup N(x_2) \setminus (\{x_1, x_2\} \cup N[x_3]) = \emptyset$, then $N(x_1) \cup N(x_2) \subset \{x_1, x_2\} \cup N[x_3]$. So, it is easy to see that $G = K_4$, a contradiction. The case when $|N(x_1) \cup N(x_2) \setminus (\{x_1, x_2\} \cup N[x_3])| = 1$ implies that $n \leq 5$, a contradiction.

Subcase 3.4. $d(x_3) = 4$.

If $|N(x_1) \cup N(x_2) \setminus (\{x_1, x_2\} \cup N[x_3])| \geq 1$, then there are $x_1u_1, x_1v_1, x_3u_3, x_3v_3 \in E(G)$ with $|\{u_1, v_1, u_3, v_3\}| = 4$ ($|N(x_i) \setminus (N(x_j) \cup N(x_3))| \geq 2$ for $i, j \in \{1, 2\}, i \neq j$) or there is $x_iv_i \in E(G)$ for $i = 1, 2, 3$ with $|\{v_1, v_2, v_3\}| = 3$ ($|N(x_1) \setminus (N(x_2) \cup N(x_3))| \geq 1$ or $|N(x_2) \setminus (N(x_1) \cup N(x_3))| \geq 1$). With a similar discussion as above, $J(G)$ contains complementary cycles. If $(N(x_1) \cup N(x_2)) \setminus (\{x_1, x_2\} \cup N[x_3]) = \emptyset$, then $N(x_1) \cup N(x_2) \subset \{x_1, x_2\} \cup N[x_3]$, and hence, $G = K_5^-$ or $G = K_5$, a contradiction.

Subcase 3.5. $d(x_3) \geq 5$.

There are $v_1, v_2, v_3 \in V(G)$ such that $x_iv_i \in E(G)$ and $v_i \neq v_j$ for $i, j = 1, 2, 3$ and $i \neq j$. It is clear that $J(G)[\{x_1v_1, x_2v_2, x_3v_3\}] = K_3$. Denote $G' = G - \{x_1v_1, x_2v_2, x_3v_3\}$. In the following, we prove that $J(G')$ is Hamiltonian or $J(G)$ contains complementary cycles.

Suppose to the contrary that $J(G')$ is not Hamiltonian. Since $q \geq 13$, we see that $G' \notin \mathcal{H}_1$. By Theorem 4.2, we only need to consider the following cases:

Subcase 3.5.1. There is $x_i \in V(G)$ with $d_{G'}(x_i) > (q-3)/2$ for some $1 \leq i \leq n$.

If $x_i \in \{x_3, v_1, v_2, v_3\}$, then $q/2 \geq d(x_1) \geq d(x_2) \geq d(x_3) > (q-3)/2 + 1 = (q-1)/2$. Thus, $q \geq d(x_1) + d(x_2) + d(x_3) - 3 > 3(q-1)/2 - 3 = q + (q-9) \geq q$, a contradiction.

If $i = 2$, then $q/2 \geq d(x_1) \geq d(x_2) > (q-3)/2 + 1 = (q-1)/2$. For $q = 2k$, we have $k \geq d(x_1) \geq d(x_2) > k - 1/2$, and hence, $d(x_1) = d(x_2) = k = q/2$, a contradiction to the assumption that $d(x_3) \geq 5$. For $q = 2k + 1$, we see that $k \geq d(x_1) \geq d(x_2) > k$, a contradiction.

If $i = 1$, then $q/2 \geq d(x_1) > (q-3)/2 + 1 = (q-1)/2$. For $q = 2k$, we see that $k \geq d(x_1) > k - 1/2$, and hence, $d(x_1) = k = q/2$. Note that $k \geq 7$. Then there are $x_1u_1, x_1w_1, x_2u_2, x_3u_3 \in E(G)$ such that $|\{u_1, w_1, u_2, u_3\}| = 4$. It is easy to check that $J(G)[\{x_1u_1, x_1w_1, x_2u_2, x_3u_3\}]$ is Hamiltonian and $J(G) - \{x_1u_1, x_1w_1, x_2u_2, x_3u_3\}$ is Hamiltonian. For $q = 2k + 1$, we have $k \geq d(x_1) > k$, a contradiction.

Consider the case when $x_i \notin \{x_1, v_1, x_2, v_2, x_3, v_3\}$. So, $i \geq 4$ and $q/2 \geq d(x_1) \geq d(x_2) \geq \dots \geq d(x_i) > (q-3)/2$. It follows that $q \geq d(x_1) + d(x_2) + d(x_3) + d(x_4) - 6 > 4(q-3)/2 - 6 = 2q - 12 \geq q$, a contradiction.

Subcase 3.5.2. $\Delta(G') = (q-3)/2$ and there is $uv \in V(G')$ with $d_{G'}(u) = d_{G'}(v) = \Delta(G')$.

Clearly, q is odd, say $q = 2k + 1$. Then $d_{G'}(u) = d_{G'}(v) = k - 1$. According to the assumption that $d(x_3) \geq 5$, we see that $uv \notin \{x_i x_j \mid i, j = 1, 2, 3, i \neq j\}$. If $uv \in \{x_i v_j, v_i v_j \mid i, j = 1, 2, 3, i \neq j\}$, then $d(u) = d(v) = k$. We deduce that either $q \geq 4k - 6 = q + q - 8$ (if $uv = v_i v_j$) or $q \geq 3k - 3 = q + (q-9)/2$ (if $uv = x_i v_j$), a contradiction.

If $\{u, v\} \cap \{x_i, v_i \mid i = 1, 2, 3\}$ contains only one vertex, then $q \geq k + 2(k-1) - 3 + 2 = q + (q-8)/2$, a contradiction.

If $\{u, v\} \cap \{x_i, v_i \mid i = 1, 2, 3\} = \emptyset$, then $q \geq 5(k-1) - 10 = q + q - 12 + (q-11)/2$, also a contradiction.

Thus, $J(G)$ contains complementary cycles for the case when $d(x_3) \geq 5$. ■

Lemma 4.7 *Let G be a graph of size $q = 6$. $J(G)$ contains complementary cycles if and only if $\Delta(G) \leq 2$ and G contains no subgraph isomorphic to K_3 or C_5 .*

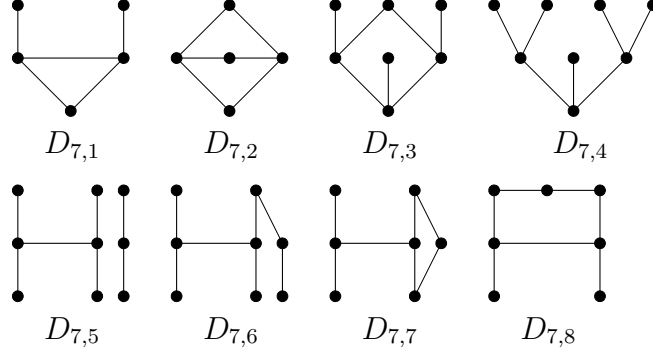
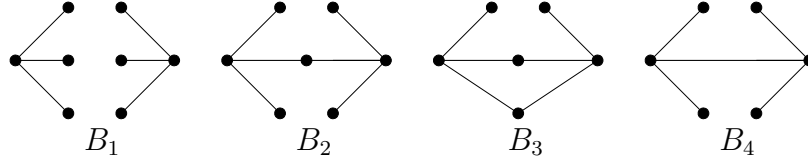
Proof. If $J(G)$ contains complementary cycles, say $C_1 = e_1 e_2 e_3 e_1$ and $C_2 = f_1 f_2 f_3 f_1$, then $G[\{e_1, e_2, e_3\}] = 3K_2$ and $G[\{f_1, f_2, f_3\}] = 3K_2$. So, we see that $\Delta(G) \leq 2$ and no subgraph isomorphic to C_3 and C_5 . Thus, the necessity holds.

Since $\Delta(G) \leq 2$, we see that G consists of paths or cycles. If G contains cycles, then such cycles must be C_4 or C_6 because G contains no C_3 and C_5 . Thus, we see that G can be easily decomposed into two subgraphs $3K_2$. Therefore, $J(G)$ contains complementary cycles. ■

Lemma 4.8 *Let G be a graph with $q = 7$. $J(G)$ contains complementary cycles if and only if $\Delta(G) \leq 3$ and G contains no subgraph isomorphic to K_4^- or to any graph in Figure 4.3.*

Proof. Let C_1 and C_2 be a pair of complementary cycles of $J(G)$. Without loss of generality, let $|V(C_1)| = 3$ and $|V(C_2)| = 4$, say $V(C_1) = \{e_1, e_2, e_3\}$ and $V(C_2) = \{f_1, f_2, f_3, f_4\}$. Then $G[\{e_1, e_2, e_3\}] = 3K_2$ and $G[\{f_1, f_2, f_3, f_4\}] \in \{4K_2, 2K_2 + P_3, 2P_3\}$. It is easy to check that $\Delta(G) \leq 3$ and G contains no subgraph isomorphic to K_4^- or to any graph in Figure 4.3.

Now, prove the sufficiency. Let $u_1, u_2, \dots, u_n$ be the vertices of G such that $d(u_1) \geq d(u_2) \geq \dots \geq d(u_n)$.

Figure 4.3: $\mathcal{H}_3$ Figure 4.4: $\mathcal{H}_4$

If $d(u_2) \geq 3$, then we have $d(u_1) = d(u_2) = 3$, since $\Delta(G) \leq 3$. To avoid some subgraph of G isomorphic to K_4^- , $D_{7,1}$ and $D_{7,2}$, we see that $G[N(u_1) \cup N(u_2) \cup \{u_1, u_2\}] = B_i$ for some $i \in \{1, 2, 3, 4\}$ (see Figure 4.4).

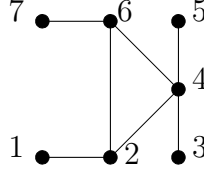
For each case when $G[N(u_1) \cup N(u_2) \cup \{u_1, u_2\}] = B_i$, $i \in \{1, 2, 3\}$, it is not difficult to find the complementary cycles of $J(G)$. If $G[N(u_1) \cup N(u_2) \cup \{u_1, u_2\}] = B_4$, then $G[E \setminus E(B_4)] = 2K_2$. Thus, $J(G)[E \setminus E(B_4 - \{u_1 u_2\})]$ and $J(G)[E(B_4 - \{u_1 u_2\})]$ are Hamiltonian.

For the other case when $d(u_2) \leq 2$, the graph G can be easily decomposed into $3K_2$ and $2P_3$ or the splitting graph of them. Thus, $J(G)$ has complementary cycles. ■

Lemma 4.9 *Let G be a graph with $q = 8$. $J(G)$ contains complementary cycles if and only if none of the following conditions holds:*

- (a) $\Delta(G) > 4$,
- (b) there is an edge uv with $d(u) + d(v) = q$ for the case when $\Delta(G) = 4$,
- (c) G has a subgraph isomorphic to K_4 ,
- (d) G has a subgraph isomorphic to K_4^- , in which there is a vertex with degree 4 in G ,
- (e) G has a subgraph isomorphic to F_8 (see Figure 4.5).

Proof. If $J(G)$ has complementary cycles, say C_1 and C_2 , then the lengths of C_1 and C_2 is either 3 and 5 or 4 and 4, respectively. By Lemma 4.5 and Theorem 4.2, it is not difficult to check that the necessity holds.

Figure 4.5: F_8

Now, prove the sufficiency. Let $u_1, u_2, \dots, u_n$ be the ordered vertices of G with $d(u_1) \geq d(u_2) \geq \dots \geq d(u_n)$. It is clear that we only need to consider the case when $d(u_1) \geq 3$.

Firstly, consider the case when $d(u_1) = 4$. If $d(u_2) = 4$, then we have $u_1 u_2 \notin E(G)$. It is clear that $J(G)$ contains complementary 4-cycles. In the following, we consider the subcase when $d(u_2) \leq 3$.

If $d(x) = 3$ for some $x \in N(u_1)$, then we see that there is neither $xy \in E(G)$ with $x, y \in N(u_1)$ and $d(x) = d(y) = 3$ nor a vertex $x \in N(u_1)$ with $d(x) = 3$ and $N(x) \setminus \{u_1\} \subset N(u_1)$, according to the conditions (c) – (e). Thus, we see that $N(u_1)$ contains at most two vertices with degree 3. If it contains two vertices with degree 3, then both vertices are not adjacent. So, we see that $J(G)$ has complementary 4-cycles. For the case when $d(x) \leq 2$ for all $x \in N(u_1)$, it is also easy to obtain complementary cycles of $J(G)$.

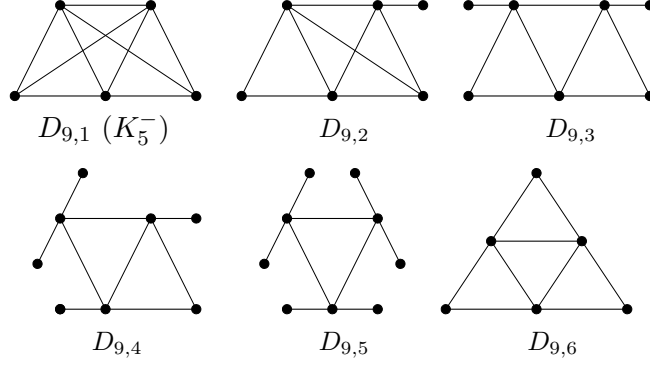
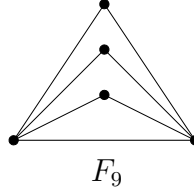
Finally, consider the case when $d(u_1) = 3$. If $d(u_2) \leq 2$, then it is easy to see that $J(G)$ has complementary cycles. For $d(u_2) = 3$, we have the following:

Assume that there is a vertex $x \in N(u_1)$ with $d(x) = 3$. Let $x \in N(u_1)$ such that $N(u_1)$ contains as many vertices of $N(x)$ as possible. If $N(u_1) \setminus \{x\} = N(x) \setminus \{u_1\} =: \{y, z\}$, then $yz \notin E(G)$ and $\max\{d(y), d(z)\} \leq 3$. So, there is $e \in E(G - N(\{u_1, x, y, z\}))$. Then, $J(G)[\{u_1 z, xy, e\}]$ and $J(G) - \{u_1 z, xy, e\}$ are Hamiltonian. For $N(u_1) \setminus \{x\} = N(x) \setminus \{u_1\} =: \{y\}$, let $z_1 \in N(u_1)$ and $z_2 \in N(x)$. Then $\max\{d(y), d(z_1), d(z_2)\} \leq 3$ and $\{yz_1, yz_2\} \notin E(G)$, according to the choice of u_1 and x . Now, we consider the case $yz_1, yz_2 \notin E(G)$. If $d(z_1) = 3$, then there is an edge f non-adjacent with y and z_2 , and hence, the Hamiltonian cycles of $J(G)[\{u_1 y, xz_2, f\}]$ and $J(G) - \{u_1 y, xz_2, f\}$ are desired. For the case when $d(z_2) = 3$ or $d(z_1) = d(z_2) = 2$ but $z_1 z_2 \notin E(G)$, we analogously obtain the complementary cycles of $J(G)$. If $d(z_1) = d(z_2) = 2$ and $z_1 z_2 \in E(G)$, then there is an edge, say e , non-adjacent with any vertex of u_1, x, y, z_1 and z_2 . It is easy to obtain the complementary cycles of $J(G)$.

For the remaining case when $d(x) \leq 2$ for all $x \in N(u_1)$, we analogously see that the statement is true. $\blacksquare$

Lemma 4.10 *Let G be a graph with $q = 9$. $J(G)$ contains complementary cycles if and only if the following holds*

- (a) $\Delta(G) \leq 4$,
- (b) G has no subgraph isomorphic to K_3 and each vertex of such a subgraph has the vertex degree 4 in G , i.e. G is not isomorphic to one of the graphs in Figure 4.6,
- (c) G contains no subgraph isomorphic to F_9 with $G - E(F_9) = P_3$, where F_9 consists of 5 vertices and there are two vertices x, y with $d_{F_9}(x) = d_{F_9}(y) = 4$ and $N_{F_9}(x) \setminus \{y\} = N_{F_9}(y) \setminus \{x\}$ (see Figure 4.7).

Figure 4.6: $\mathcal{H}_6$ Figure 4.7: $\mathcal{H}_7$

Proof. If $G - E(F_9) = P_3$, then by Theorem 4.2, it is not difficult to see that $J(G)$ contains no complementary cycle. According to Lemma 4.5, we see that the necessity is true.

Let $u_1, u_2, \dots, u_n$ be the vertices of G with $d(u_1) \geq d(u_2) \geq \dots \geq d(u_n)$. It is clear that we only need to consider the case when $3 \leq d(u_1) \leq 4$.

Consider the case when $d(u_1) = 4$. For the case when $d(u_2) = 4$ and $u_1 u_2 \notin E(G)$, it is easy to obtain the complementary cycles of $J(G)$. For $u_1 u_2 \in E(G)$, we assume without loss of generality that $N(u_1)$ contains as more as possible vertices of $N(u_2)$ among all $xy \in E(G)$ with $d(x) = d(y) = 4$.

If $N(u_1) \setminus \{u_2\} = N(u_2) \setminus \{u_1\} =: \{x, y, z\}$, then G contains a subgraph isomorphic to F_9 . From (c), it follows that $G - E(F_9) = 2P_2$, and then $J(G - E(F_9) \setminus \{u_1 u_2\})$ and $J(G)[F_9 \setminus \{u_1 u_2\}]$ are Hamiltonian.

Assume that $N(u_1) \setminus \{u_2\} = N(u_2) \setminus \{u_1\} =: \{x, y\}$. Let $z_1 \in N(u_1) \setminus \{u_2, x, y\}$ and $z_2 \in N(u_2) \setminus \{u_1, x, y\}$. From (b), we see that $d(x), d(y) \leq 3$. If $xy \in E(G)$, then $d(x) = d(y) = 3$. So, there is $e \in E(G - \{u_1, u_2, x, y\}) \neq \emptyset$. Thus $J(G)[\{u_1 u_2, xy, e\}]$ and $J(G) - \{u_1 u_2, xy, e\}$ are Hamiltonian. If $xy \notin E(G)$, then we see that $G - \{u_1 x, u_1 z_1, u_2 y, u_2 z_2\}$ is C_5 or a splitting graph of C_5 . Therefore, $J(G)$ contains complementary cycles.

For the case when $N(u_1) \setminus \{u_2\} = N(u_2) \setminus \{u_1\} =: \{x\}$ or $N(u_1) \cap N(u_2) = \emptyset$, it is not difficult to check that $J(G)$ contains complementary cycles.

If $d(u_2) = 3$, then we analogously assume that $N(u_1)$ contains as more as possible vertices x of $N(u_2)$ with $d(x) = 3$. We only need to consider the case when $u_1 u_2 \in E(G)$. So, we see that $G - \{u_1, u_2\}$ consists of either $3K_2$ or $P_3 + K_2$ or K_3 or P_4 or $K_{1,3}$. Thus, $J(G)$ contains complementary cycles.

For the remaining case when $d(u_1) = 3$, it is not difficult to check that $J(G)$ contains complementary cycles. ■

Lemma 4.11 *Let G be a graph with $q \in \{10, 11, 12\}$. $J(G)$ contains complementary cycles if and only if the following conditions hold:*

- (a) $\Delta(G) \leq q/2$,
- (b) *there is no edge uv with $d(u) + d(v) = q$ for the case when $q = 2\Delta(G)$.*

Proof. With a similar discussion as in Lemma 4.6, it is not difficult to check that $J(G)$ contains complementary cycles. ■

By Lemma 4.5 – 4.11, we have the following:

Theorem 4.12 (Feng [47]) *Let G be a graph of size q . $J(G)$ has complementary cycles if and only if the following holds*

- (1) $2\Delta(G) \leq q$,
- (2) *for $q = 2\Delta(G)$, there is no edge uv with $d(u) = d(v) = \Delta(G)$,*
- (3) *for $q = 6$, G contains no subgraph isomorphic to K_3 or C_5 ,*
- (4) *for $q = 7$, G contains no subgraph isomorphic to K_4^- or any graph in Figure 4.3,*
- (5) *for $q = 8$, G contains no subgraph isomorphic to K_4 or F_8 (see Figure 4.5) or K_4^- , in which there is a vertex with degree 4 in G ,*
- (6) *for $q = 9$, G is not isomorphic to one of the graphs in Figure 4.6 and G contains no subgraph isomorphic to F_9 with $G - E(F_9) = P_3$, where F_9 consists of 5 vertices and there are two vertices x, y with $d_{F_9}(x) = d_{F_9}(y) = 4$ and $N_{F_9}(x) \setminus \{y\} = N_{F_9}(y) \setminus \{x\}$ (see Figure 4.7).*

Part II

Out-arc Pancyclic Vertices in Tournaments

Chapter 5

Introduction

We will assume that the reader is familiar with the standard terminology on digraphs and refer to [7] for terminology not mentioned here. In this part, we only consider finite digraphs without loops and multiple arcs.

5.1 Terminology and Notation

Denote the vertex set and the arc set of a digraph D by $V(D)$ and $A(D)$, respectively. A subdigraph induced by a subset $U \subseteq V(D)$ is denoted by $D[U]$. In addition, $D - U = D[V(D) \setminus U]$.

If xy is an arc of a digraph D , then we say that x *dominates* y or xy is an *out-arc* of x , write $x \rightarrow y$. More generally, if A and B are two distinct subsets of $V(D)$ such that every vertex of A dominates all vertices of B , then we say that A dominates B and write $A \rightarrow B$. For two subdigraphs H, H' of D with $V(H) \rightarrow V(H')$, we also write $H \rightarrow H'$.

Let x be a vertex of D . The set of all vertices dominating x (dominated by x , respectively) is called *in-neighborhood* (*out-neighborhood*, respectively) of x , denoted by $N_D^-(x)$ ($N_D^+(x)$, respectively). Furthermore, the integer $d_D^-(x) = |N_D^-(x)|$ ($d_D^+(x) = |N_D^+(x)|$, respectively) is called the *in-degree* (*out-degree*, respectively) of x . Denote $\min\{d_D^-(x) \mid x \in V(D)\}$ ($\min\{d_D^+(x) \mid x \in V(D)\}$, respectively) by $\delta^-(D)$ ($\delta^+(D)$, respectively). We will omit the subscript if the digraph D is known from the context.

A digraph D is said to be *strong* if for every pair of vertices x and y , D contains a path from x to y and a path from y to x . D is called *k-strong* if $|V(D)| \geq k + 1$ and for any subset $U \subseteq V(D)$ of at most $k - 1$ vertices, $D - U$ is strong. If D is k -strong, but not $(k + 1)$ -strong, then the number $\sigma(D) = k$ is defined as the *strong connectivity* of D .

A subset $S \subseteq V(D)$ with $\sigma(D - S) = 0$ is called a *separating set* of D . Moreover, a separating subset $S \subseteq V(D)$ with $|S| = \sigma(D)$ is called a *minimum separating set* of D .

A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong.

We will use the operation of *path-contracting* introduced in [7]. Let x and y be two distinct vertices of D and let P be an (x, y) -path in D . We say that H is obtained from D by contracting P to w if the following holds: $V(H) = (V(D) \setminus V(P)) \cup \{w\}$, where w is a new vertex. Furthermore, $N_H^+(w) = N_D^+(y) \cap (V(D) \setminus V(P))$, $N_H^-(w) = N_D^-(x) \cap (V(D) \setminus V(P))$ and an arc with both end-vertices in $(V(D) \setminus V(P))$ belongs to H if and only if it belongs to D . Note that uwv is a path in H if and only if uPv

is a path in D . Moreover, if there exists a k -cycle containing w in H , then there is a $(k + |V(P)| - 1)$ -cycle containing P in D .

5.2 Some Results on Tournaments

A *tournament* is an orientation of the edges of a complete graph. In the following, we introduce some useful results on Tournaments.

Lemma 5.1 *Let T be a k -strong tournament, P a path of length $k - 1$ from u to v in T and let D_w be the digraph obtained from T by contracting the path P to w . Then D_w is strong.*

Proof. It is clear that $D_w - \{w\} = T - V(P)$. Let $T_1, \dots, T_r$ ($r \geq 1$) be the strong components of $D_w - \{w\}$ with $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_r$ if $r \geq 2$.

Since T is k -strong, we have $V(P) = N^-(T_1) = N^+(T_r)$. It follows that $d_{T_1}^+(w) = d_{T_1}^+(v) \geq 1$ and $d_{T_r}^-(w) = d_{T_r}^-(u) \geq 1$. Thus, D_w is strong. ■

Proposition 5.2 *Let $P = u_1 u_2 \dots u_s$ be a path of a tournament T with $s < |V(T)|$. If there is a vertex $v \in V(T) \setminus V(P)$ with $u_1 \rightarrow v$ and $N^+(v) \cap V(P) \neq \emptyset$, then there is an index k , $1 \leq k \leq s - 1$ such that $u_1 \dots u_k \rightarrow v$ and $v \rightarrow u_{k+1}$.*

Proof. If there is no index k , $1 \leq k \leq s - 1$ such that $u_1 \dots u_k \rightarrow v, v \rightarrow u_{k+1}$, then $P \rightarrow v$, i.e. $N^+(v) \cap V(P) = \emptyset$, a contradiction. ■

In Proposition 5.2, we call that v can be *inserted* into P .

Let T be a strong tournament with minimum out-degree at least two. A vertex v of T is called a *bridgehead* if there is a partition (A, B) of $V(T)$ such that the following conditions are satisfied:

- (1) $v \in A$, $T[A]$ is non-trivial and strong,
- (2) $B \rightarrow A \setminus \{v\}$.

It is obvious that v is a separating vertex of T and $|A| \geq 5$.

Lemma 5.3 (Yao, Guo and Zhang [82]) *Let T be a strong tournament on n vertices with minimum out-degree at least two and assume that the vertices are labeled $u_1, u_2, \dots, u_n$ such that $d^+(u_1) \leq d^+(u_2) \leq \dots \leq d^+(u_n)$. If T has a bridgehead v with respect to the partition (A, B) , then the following holds:*

- (1) $\{u_1, u_2, u_3, u_4\} \subset A$.
- (2) $\{u_1, u_2\}$ contains at most one bridgehead.

Lemma 5.4 *Let T be a tournament with $\sigma(T) = 1$ and the vertices $u_1, u_2, \dots, u_n$ are labeled as $d^+(u_1) \leq d^+(u_2) \leq \dots \leq d^+(u_n)$. The following holds:*

- (1) *There is a separating vertex s with the strong components $T_1, T_2, \dots, T_t$ of $T - \{s\}$ such that $T_i \rightarrow T_j$ for $i < j$ and T_t contains no separating vertex of T .*

- (2) Let s be an arbitrary separating vertex of T and let $T_1, T_2, \dots, T_t$ be the corresponding strong components of $T - \{s\}$ with $T_i \rightarrow T_j$ for $i < j$. Then $\{u_1, u_2\} \cap V(T_t) \neq \emptyset$.

Proof. (1) Since $\sigma(T) = 1$, there is a separating vertex in T . Let s' be a separating vertex of T and let $T'_1, T'_2, \dots, T'_t$ be the labeled strong components of $T - \{s'\}$ such that the number of separating vertices of T in the last component of $T - \{s'\}$ is as small as possible. In the following, we show that T'_t contains no separating vertex of T , i.e. s' is a desired separating vertex.

Suppose that $s \in V(T'_t)$ is another separating vertex of T . If $\sigma(T'_t - \{s\}) = 1$, then $s' \rightarrow T'_t - \{s\}$ and $s \rightarrow s'$. Let k be the index such that $s' \rightarrow T'_k \dots T'_{t-1}$ and $N^-(s') \cap V(T'_{k-1}) \neq \emptyset$. Note that $N^-(s') = \{s\}$ if $k = 1$. Set $T_1 = T[(\cup_{i=1}^{k-1} V(T'_i)) \cup \{s'\}]$, $T_2 = T'_k, \dots, T_\ell = T'_t - \{s\}$. We see that there are more separating vertices of T in T'_t than in T_ℓ , a contradiction to the choice of s' .

For the case when $\sigma(T'_t - \{s\}) = 0$, let $S'_1, \dots, S'_s$ be the strong components of $T'_t - \{s\}$. If $s' \rightarrow T'_t - \{s\}$, then we see that $s \rightarrow s'$. With a similar discussion as above, we deduce a contradiction.

If $N^-(s') \cap V(T'_t - \{s\}) \neq \emptyset$, then $s' \rightarrow S'_s$. Let ℓ be the index such that $s' \rightarrow S'_{\ell+1} \dots S'_s$, $N^-(s') \cap V(S'_\ell) \neq \emptyset$. It is clear that $\ell \geq 1$. Set $T_1 = T[(\cup_{i=1}^{\ell-1} V(T'_i)) \cup (\cup_{j=1}^\ell V(S'_j)) \cup \{s'\}]$, $T_2 = S'_{\ell+1}, \dots, T_\mu = S'_s$. We see that there are more separating vertices of T in T'_t than in T_μ , a contradiction to the choice of s' .

- (2) Denote $|V(T_t)| = n_t$. It is trivial for the case when $n_t = 1$. If $n_t \geq 3$, then $d^+(x) \geq n_t > n_t - 2 + 1 \geq d^+(y)$ for $x \in V(T - \{s\}) \setminus V(T_t)$ and $y \in V(T_t)$. ■

An arc (a vertex, respectively) in a digraph D with $n \geq 3$ vertices is said to be *k-pancyclic* if it is contained in a cycle of length ℓ for all $k \leq \ell \leq n$. For $k = 3$, we say that the arc (the vertex, respectively) is *pancyclic*. If all out-arcs of a vertex u are pancyclic, then we say that u is an *out-arc pancyclic vertex*.

In [54], Goldberg and Moon proved that every s -arc-strong tournament has at least s distinct Hamiltonian cycles. Thomassen [78] confirmed that every strong tournament contains a vertex x such that each out-arc of x is contained in a Hamiltonian cycle. In 2000, Yao, Guo and Zhang [82] improved the result of Thomassen:

Theorem 5.5 (Yao, Guo and Zhang [82]) *Let T_n be a strong tournament on n vertices and assume that the vertices of T_n are labeled $u_1, u_2, \dots, u_n$ such that $d^+(u_1) \leq d^+(u_2) \leq \dots \leq d^+(u_n)$. Let u be a vertex of T_n which can be chosen as follows:*

- (1) if $d^+(u_1) = 1$ then $u = u_1$,
- (2) if $d^+(u_1) \geq 2$ then $d^+(u) = \min\{d^+(x) \mid x \in V(T_n) \text{ and } x \text{ is not a bridgehead}\}$.

Then all out-arcs of u are pancyclic.

Remark 5.6 *If the vertices of T_n are labeled $u_1, u_2, \dots, u_n$ with $d^+(u_1) \leq d^+(u_2) \leq \dots \leq d^+(u_n)$, then by Lemma 5.3 and Theorem 5.5, one of $\{u_1, u_2\}$ can be chosen as the vertex u with the property. It is easy to check that $d^+(u) \leq \lfloor \frac{n-1}{2} \rfloor$. Therefore, every strong tournament T with $n \geq 3$ vertices contains a vertex u with $d^+(u) \leq \lfloor \frac{n-1}{2} \rfloor$ such that all out-arcs of u are pancyclic.*

Remark 5.7 Let T be a strong tournament with $\sigma(T) = 1$ and $\delta^+(T) \geq 2$. Let w be a vertex of T with minimal out-degree among all vertices of T , which are not bridgehead, and let $x (\neq w)$ be a separating vertex of T . If $T_1, T_2, \dots, T_t$ ($t \geq 2$) are the strong components of $T - \{x\}$ with $T_i \rightarrow T_j$ for all $i < j$, then by Lemma 5.4 and Theorem 5.5, the following holds:

- (1) $w \in V(T_t)$.
- (2) If $w \rightarrow x$, then $T_t - \{w\}$ contains no separating vertex of T and $d_{T_t}^+(w) \leq \lfloor \frac{n_t-1}{2} \rfloor$.
- (3) If $x \rightarrow w$, then $d^+(w) = d_{T_t}^+(w) \leq n_t - 2$.

In [82], Yao, Guo and Zhang posed the following conjecture:

Conjecture 5.8 (Yao, Guo and Zhang [82]) A k -strong tournament T has at least k vertices $v_1, v_2, \dots, v_k$ such that all out-arcs of v_i are pancyclic for $i = 1, 2, \dots, k$.

According to Theorem 5.5, this conjecture is true for $k = 1$. In the next chapter, we will prove Conjecture 5.8 is also true for $k = 2, 3$.

We conclude this section with some useful results.

Theorem 5.9 (Camion [33]) A tournament has a Hamiltonian cycle if and only if it is strong.

Lemma 5.10 (Yeo [83]) Let D be a k -strong digraph with $k \geq 1$, and let S be a separating set in D such that $T = D - S$ is a tournament. Let $T_1, T_2, \dots, T_r$ ($r \geq 2$) be the strong components of T such that $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_r$. Now the following holds:

- (i) At least k vertices in S dominate some vertices in T_1 , and at least k vertices in S are dominated by some vertices in T_r .
- (ii) For every $1 \leq \ell \leq |V(T)| - 1$, $u \in T_1$ and $v \in T_r$, there exists a (u, v) -path of length ℓ in T .
- (iii) If $S = \{x\}$, then x is pancyclic in D .

Lemma 5.11 (Yeo [83]) Let D be a strong digraph, containing a vertex x , such that $D - \{x\}$ is a tournament and $d_D^+(x) + d_D^-(x) \geq |V(D)|$. Then there is a k -cycle containing x in D for all $2 \leq k \leq |V(D)|$.

Lemma 5.12 (Yeo [83]) Let T be a 2-strong tournament, containing an arc $e = xy$, such that $d^+(y) \geq d^+(x)$. Then e is pancyclic in T .

Lemma 5.13 (Feng, Li and Li [53]) Let T be a 3-strong tournament and $u_1, x \in V(T)$ with $d^+(u_1) = \delta^+(T)$ and $x \in N^+(u_1)$. If there is $y \in N^+(u_1)$ with $y \rightarrow x$, then all out-arcs of x are 4-pancyclic in T .

Proof. For convenience, denote $x_1 = y$ and $x_2 = x$. Then $x_1 \rightarrow x_2$. Let $z \in N^+(x_2)$ and D_w be the digraph obtained from T by contracting u_1x_2z to w , where $w \notin V(T)$. It is clear that $d^+(z) \geq d^+(u_1)$. By Lemma 5.1, D_w is strong. Now, we consider the following two cases:

Case 1. $u_1z \in A(T)$.

It is easy to see that

$$\begin{aligned} d_{D_w}^+(w) + d_{D_w}^-(w) &= |N^+(z)| + |N^-(u_1)| \\ &= d^+(z) + d^-(u_1) \\ &= d^+(z) + [n - 1 - d^+(u_1)] \\ &\geq n - 1 > |V(D_w)|. \end{aligned}$$

By Lemma 5.11, w is in a j -cycle of D_w for $j = 2, 3, \dots, n-2$, and then there is a k -cycle in T containing x_2z for $k = 4, 5, \dots, n$.

Case 2. $zu_1 \in A(T)$

If $d^+(z) \geq d^+(u_1) + 1$, then with a similar discussion as above, we are done. So, we assume that $d^+(z) = d^+(u_1) = t$.

If $D_w - \{w\}$ is not strong, then by Lemma 5.10, w is pancyclic in D_w , and hence, x_2z is in a k -cycle of T , $k = 5, 6, \dots, n$. Since $u_1x_2zu_1$ is a 3-cycle and $u_1x_1x_2zu_1$ is a 4-cycle, x_2z is pancyclic in T .

If $D_w - \{w\}$ is strong, then by Theorem 5.9, $D_w - \{w\}$ contains a Hamiltonian cycle, say $C = v_1v_2 \dots v_{n-3}v_1$. In the following, all subscripts of the vertices in C appearing in related calculations are taken modulo $n-3$, where $v_0 = v_{n-3}$. For convenience, we denote the subpath $v_i v_{i+1} \dots v_j$ by $C[v_i, v_j]$ for two arbitrary vertices v_i, v_j of C .

Clearly, $u_1x_2zu_1$ is a 3-cycle and $u_1x_1x_2zu_1$ is a 4-cycle containing x_2z . In the following, we will confirm that x_2z is pancyclic in T , i.e. we prove that there is a k -cycle of T containing x_2z , $k = 5, 6, \dots, n$.

Recall that $d^+(u_1) = d^+(z) = t$. It follows that

$$\begin{aligned} d_C^+(u_1) = d_C^+(z) &= t - 1, \\ d_C^+(z) + d_C^-(u_1) &= |N^+(z) \setminus \{u_1\}| + |N^-(u_1) \setminus \{z\}| \\ &= d^+(z) - 1 + d^-(u_1) - 1 \\ &= d^+(z) - 1 + [n - 1 - d^+(u_1)] - 1 \\ &= n - 3 = |V(C)|. \end{aligned}$$

Assume to the contrary that T has no $(n-\ell)$ -cycle containing x_2z for some ℓ with $0 \leq \ell \leq n-5$. Let $\eta = t-1$ and let $N_C^+(z) = \{v_{i_1}, v_{i_2}, \dots, v_{i_\eta}\}$. Then we see that $N_C^+(u_1) = \{v_{i_1-\ell-1}, v_{i_2-\ell-1}, \dots, v_{i_\eta-\ell-1}\}$ (otherwise, $v_{i_j-\ell-1}u_1 \in A(T)$ for some $1 \leq j \leq \eta$, then $x_2zv_{i_j}C[v_{i_j+1}, v_{i_j-\ell-2}]v_{i_j-\ell-1}u_1x_2$ is an $(n-\ell)$ -cycle through x_2z , a contradiction). Note that $N_C^+(u_1) = \{v_{i_1-\ell-1}, v_{i_2-\ell-1}, \dots, v_{i_\eta-\ell-1}\} = \{x_1\} \cup \{x_3, \dots, x_t\}$ and $x_1 \rightarrow x_2$. Without loss of generality, let $x_1 = v_{i_1-\ell-1}$. We consider the following two subcases:

Subcase 2.1. $u_1 \rightarrow v_{i_1}$.

Then $x_2 z u_1 v_{i_1} C[v_{i_1+1}, v_{i_1-\ell-2}] x_1 x_2$ is a cycle of length $n - \ell$ which contains $x_2 z$, a contradiction.

Subcase 2.2. $v_{i_1} \rightarrow u_1$.

If $v_{i_1-\ell-2} \rightarrow u_1$, then $x_2 z C[v_{i_1}, v_{i_1-\ell-2}] u_1 x_1 x_2$ is a cycle of length $n - \ell$ through $x_2 z$, a contradiction.

We have $u_1 \rightarrow v_{i_1-\ell-2}$. Suppose now that $C[v_k, v_{i_1-\ell-2}]$ is the longest subpath with $u_1 \rightarrow C[v_k, v_{i_1-\ell-2}]$. Since $v_{i_1} \rightarrow u_1$, we have $V(C[v_{i_1}, v_{k-1}]) \neq \emptyset$. Thus, the cycle $x_2 z C[v_{i_1}, v_{k-1}] u_1 C[v_k, v_{i_1-\ell-1}] x_2$ is of length $n - \ell$ and through $x_2 z$, a contradiction.

Therefore, T has an $(n - \ell)$ -cycle containing $x_2 z$ for all $0 \leq \ell \leq n - 5$. So, $x_2 z$ is pancyclic. This completes the proof of Lemma 5.13. $\blacksquare$

Chapter 6

Out-arc Pancyclicity of Vertices in Tournaments

6.1 Out-arc Pancyclic Vertices in 2-strong Tournaments

For 3-strong tournaments, Yeo proved the following:

Theorem 6.1 (Yeo [83]) *Every 3-strong tournament has two distinct vertices x and y , such that all arcs out of x and all arcs out of y are pancyclic.*

Furthermore, x and y can be chosen, such that $x \rightarrow y$ and $d^+(x) \leq d^+(y)$.

In addition, Yeo posed the following conjecture:

Conjecture 6.2 (Yeo [83]) *Each 2-strong tournament has 3 vertices whose all out-arcs are pancyclic.*

In this section, we consider exactly 2-strong tournaments.

Theorem 6.3 (Li, Li and Feng [66]) *Each tournament T on n vertices with $\sigma(T) = 2$ contains at least 2 vertices v_1, v_2 such that all out-arcs of v_i are pancyclic for $i = 1, 2$.*

Proof. Let M contain all vertices which have minimum out-degree in T . By Lemma 5.12, all out-arcs of every vertex in M are pancyclic. If $|M| \geq 2$, then we are done. So, assume that $|M| = 1$ and denote $M = \{v_1\}$. Note that

$$d^+(v) > d^+(v_1) \geq 2 \text{ for all } v \neq v_1 \tag{6.1}$$

It is sufficient to show that there exists another vertex whose out-arcs are pancyclic. We consider the following two cases:

Case 1. $\sigma(T - v_1) = 1$.

Let $S = \{v_1, x\}$ be a minimum separating set of T and let $T_1, T_2, \dots, T_t$ ($t \geq 2$) be the strong components of $T - S$. It is clear that for each $i \in \{1, 2, \dots, t\}$, T_i is a single

vertex or contains a Hamiltonian cycle by Theorem 5.9. Assume without loss of generality that the strong components $T_1, T_2, \dots, T_t$ of T have been labeled such that $T_i \rightarrow T_j$ for $1 \leq i < j \leq t$. Note that $N^+(T_t) = S = N^-(T_1)$. In particular, the final strong component T_t contains at least 3 vertices by (6.1).

Assume without loss of generality that among all possible minimum separating sets $\{v_1, x'\}$ with $x' \in V(T - v_1)$, the vertex $x \in S$ has been chosen such that the final strong component of $T - S$ is minimum. This implies

$$|N^-(x) \cap V(T_t)| \geq 2. \quad (6.2)$$

For the case when $|V(T_t)| = 3$, let $T_t = x_1x_2x_3x_1$. It is easy to see from (6.1) that $T_t \rightarrow S$. Since $d^+(x) \geq 3$ and T is 2-strong, we have $n \geq 7$. Furthermore, it is easy to check that the subtournament $T - (S \cup V(T_t))$ contains a Hamiltonian path, say $u_1u_2 \dots u_{n-5}$, such that $v_1 \rightarrow u_1$ and $x \rightarrow u_\alpha$ for some α satisfying $2 \leq \alpha \leq n-5$. Now, we will confirm that the three out-arcs of x_1 are pancyclic.

For the arc x_1x_2 , we see that $x_1x_2x_3x_1, x_1x_2v_1u_1x_1, x_1x_2v_1u_1x_3x_1, x_1x_2v_1u_1 \dots u_ix_3x_1$ for $i = 2, \dots, n-5$ are cycles of length 3, 4, $\dots, n-1$, respectively. In addition, it is easy to see that $x_1x_2v_1u_1 \dots u_{\alpha-1}x_3xu_\alpha \dots u_{n-5}x_1$ is a Hamiltonian cycle of T . Hence, x_1x_2 is pancyclic.

Since $x_1v_1u_1x_1, x_1v_1u_1x_3x_1, x_1v_1u_1x_2x_3x_1, x_1v_1u_1u_2 \dots u_ix_2x_3x_1$ for $i = 2, \dots, n-5$ and $x_1v_1u_1 \dots u_{\alpha-1}x_2xu_\alpha \dots u_{n-5}x_3x_1$ are cycles in T , the arc x_1v_1 is pancyclic.

For the arc x_1x , we see that $x_1xu_\alpha x_1, x_1xu_\alpha x_3x_1, x_1xu_\alpha x_2x_3x_1, x_1xu_\alpha \dots u_ix_2x_3x_1$ for $i = \alpha, \dots, n-5$, $x_1xu_\alpha \dots u_{n-5}x_3v_1u_1x_1$ and $x_1xu_\alpha \dots u_{n-5}x_2x_3v_1u_1 \dots u_jx_1$ for $j = 1, \dots, \alpha-1$ are cycles in T . Therefore, x_1x is pancyclic.

In the following, we consider the case when $|V(T_t)| \geq 4$. Let $\ell = |V(T_t)|$ and let $C_t = x_1x_2 \dots x_\ell x_1$ be a Hamiltonian cycle of T_t .

At first, we show that $|N^-(v_1) \cap V(T_t)| \geq 3$. Suppose to the contrary that $|N^-(v_1) \cap V(T_t)| \leq 2$. Then $|N^+(v_1) \cap V(T_t)| \geq \ell - 2$. Since v_1 dominates at least one vertex in T_1 , it holds $d^+(v_1) \geq \ell - 1$. Now, we look at the vertex, say x_i , with minimum out-degree among all vertices in T_t . It is clear that $|N^+(x_i) \cap V(T_t)| \leq \lfloor \frac{\ell-1}{2} \rfloor$. Thus, we have

$$d^+(x_i) = d_{T_t}^+(x_i) + d_S^+(x_i) \leq \lfloor \frac{\ell-1}{2} \rfloor + 2 \leq \lfloor \frac{\ell-1}{2} \rfloor + \lceil \frac{\ell-1}{2} \rceil = \ell - 1 \leq d^+(v_1),$$

a contradiction to (6.1). Therefore, it holds

$$|N^-(v_1) \cap V(T_t)| \geq 3. \quad (6.3)$$

By Theorem 5.5, the subtournament T_t contains a vertex, say v_2 , whose out-arcs are pancyclic in T_t . By Remark 5.6, we can choose v_2 such that $d_{T_t}^+(v_2) \leq \lfloor \frac{\ell-1}{2} \rfloor$.

Assume without loss of generality that the vertices of the Hamiltonian cycle C_t have been labeled with $x_\ell = v_2$.

We will show that either all out-arcs of v_2 are pancyclic or there exists another vertex apart from v_1, v_2 whose out-arcs are pancyclic. Let $v_2 \rightarrow w$ be an arc of T .

Subcase 1.1. $w \in V(T_t)$.

By the choice of v_2 , the arc v_2w is in an i -cycle for $i = 3, 4, \dots, \ell$. Assume without loss of generality that C_t has been the Hamiltonian cycle of T_t which contains the arc v_2w , i.e.

$w = x_1$. Moreover, we note from (6.3) that there is an integer $\beta \in \{1, 2, \dots, \ell - 2\}$ with $x_\beta \rightarrow v_1$.

Because $\{v_1, x\} = N^-(T_1)$ and T_1 is a single vertex or it contains a Hamiltonian cycle, it is obvious that the subtournament $T - (S \cup V(T_t))$ contains a Hamiltonian path, say $u_1 u_2 \dots u_{n-2-\ell} (u'_1 u'_2 \dots u'_{n-2-\ell}, \text{ respectively})$, with $v_1 \rightarrow u_1$ ($x \rightarrow u'_1$, respectively). It is easy to see that $v_2 x_1 \dots x_\beta v_1 u_1 x_{\beta+2} \dots x_\ell$ is a cycle of length $\ell + 1$ and $v_2 x_1 \dots x_\beta v_1 u_1 \dots u_j x_{\beta+1} \dots x_\ell$ is a cycle of length $\ell + 1 + j$ for $j = 1, 2, \dots, n - 2 - \ell$. It remains to show that $v_2 w$ is also in a Hamiltonian cycle of T .

If $v_1 \rightarrow x$, then $v_2 x_1 \dots x_\beta v_1 x u'_1 \dots u'_{n-2-\ell} x_{\beta+1} \dots x_\ell$ is a Hamiltonian cycle of T . If $x \rightarrow v_1$, then, by (6.2), there is a vertex x_γ with $1 \leq \gamma \leq \ell - 1$ such that $x_\gamma \rightarrow x$. Now, we see that $v_2 x_1 \dots x_\gamma x v_1 u_1 \dots u_{n-2-\ell} x_{\gamma+1} \dots x_\ell$ is a Hamiltonian cycle of T .

Subcase 1.2. $w \in \{v_1, x\}$.

For convenience, we denote the other vertex in $\{v_1, x\}$ by w' , i.e. $\{w, w'\} = \{v_1, x\}$.

By recalling that $T - (V(T_t) \cup \{w, w'\})$ contains a Hamiltonian path starting at a vertex in $N^+(w)$, we can confirm, similar to Subcase 1.1, that the arc $v_2 w$ is pancyclic in $T - w'$. It remains to check whether the arc $v_2 w$ is in a Hamiltonian cycle of T .

Suppose that $T - (V(T_t) \cup \{w, w'\})$ contains at least two vertices. Then it is not difficult to show that $T - (V(T_t) \cup \{w, w'\})$ can be decomposed into two paths, say $w_1 w_2 \dots w_\mu$ and $w'_1 w'_2 \dots w'_\nu$ with $w \rightarrow w_1$ and $w' \rightarrow w'_1$, respectively. According to (6.2) or (6.3), there is a vertex x_j with $1 \leq j \leq \ell - 1$ such that $x_j \rightarrow w'$. Now, we see that

$$v_2 w w_1 w_2 \dots w_\mu x_1 x_2 \dots x_j w' w'_1 w'_2 \dots w'_\nu x_{j+1} \dots x_\ell$$

is a Hamiltonian cycle of T .

Suppose now that $T - (V(T_t) \cup \{w, w'\})$ contains exactly one vertex w_1 . Note that $t = 2$ and $\{w, w'\} \rightarrow w_1$.

If $w \rightarrow w'$, then we see that $v_2 w w' w_1 x_1 x_2 \dots x_\ell$ is a Hamiltonian cycle of T . So, we consider the case when $w' \rightarrow w$.

Assume that $d^+(w') \geq 3$. Let D be the digraph obtained from T by contracting the path $v_2 w w_1$ to z . Clearly, $D - z$ is a tournament. Since $\ell \geq 4$, we see that $d^+(w_1) = \ell > \lfloor \frac{\ell-1}{2} \rfloor + 2 \geq d^+(v_2)$. It follows that

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= |N^+(w_1) \setminus \{v_2\}| + |N^-(v_2) \setminus \{w_1\}| \\ &= (d^+(w_1) - 1) + (d^-(v_2) - 1) \\ &= (d^+(w_1) - 1) + [(n - 1) - d^+(v_2)] - 1 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

We will show that D is strong. It is obvious that $D - w'$ is strong. By (6.2) or (6.3), $N^-(w') \cap V(T_2 - v_2) \neq \emptyset$ holds. Because $d^+(w') \geq 3$, we have $N^+(w') \cap V(T_2) \neq \emptyset$. Therefore, D is strong. By Lemma 5.11, z is in all i -cycles for $i = 2, \dots, n - 2$, and hence, $v_2 w$ is in a Hamiltonian cycle of T .

Assume now that $d^+(w') < 3$. This implies that $d^+(w') = 2$ and $w' = v_1$ by (6.1). Now, we see that $T_2 \rightarrow w'$, and it is not difficult to check that all out-arcs of w_1 are pancyclic.

Case 2. $\sigma(T - v_1) = 2$.

Let v_2 be the vertex which has minimum out-degree among all vertices in $N^+(v_1)$. We will prove that all out-arcs of v_2 are pancyclic.

Let $x \in N^+(v_2)$ be arbitrary. If $d^+(x) \geq d^+(v_2)$, then we are done by Lemma 5.12. So, assume that $d^+(x) < d^+(v_2)$. This implies that $x \notin N^+(v_1)$, i.e. $x \rightarrow v_1$, and hence, v_2x is in the 3-cycle $v_2xv_1v_2$.

Let D be the digraph obtained by contracting v_1v_2x to w . We show at first that D is strong. Let $y \in V(D - w)$ arbitrary. It is sufficient to confirm that D contains a path from y to w and a path from w to y . In fact, from the definition of path-contracting, we only need to verify that $T - \{v_1, v_2\}$ contains a path from x to y and $T - \{v_2, x\}$ contains a path from y to v_1 .

Since $T - v_1$ is 2-strong, $T - \{v_1, v_2\}$ is strong, and hence, $T - \{v_1, v_2\}$ contains a path from x to y .

Suppose now, to the contrary, that $T - \{v_2, x\}$ contains no path from y to v_1 . Let

$$Z = \{z \in V(T - \{v_2, x, y\}) \mid T - \{v_2, x\} \text{ contains a path from } y \text{ to } z\}.$$

By (6.1), we see that $Z \neq \emptyset$. In particular, $v_1 \notin Z$ and $v_1 \rightarrow Z$. It follows that $N^+(y) \subseteq Z \cup \{v_2, x\}$ and $Z \cup \{y, v_2\} \subseteq N^+(v_1)$. This implies $d^+(y) \leq |Z| + 2 \leq d^+(v_1)$, a contradiction to (6.1). Therefore, $T - \{v_2, x\}$ contains a path from y to v_1 .

Now, for the strong digraph D , we have

$$\begin{aligned} d_D^+(w) + d_D^-(w) &= |N^+(x) \setminus \{v_1\}| + |N^-(v_1) \setminus \{x\}| \\ &= (d^+(x) - 1) + (d^-(v_1) - 1) \\ &= (d^+(x) - 1) + [(n - 1) - d^+(v_1)] - 1 \\ &\geq n - 2 = |V(D)|, \end{aligned}$$

where the inequality follows from (6.1). According to Lemma 5.11, w is in an i -cycle for all $2 \leq i \leq |V(D)|$. Thus, v_2x is in a j -cycle in T for all $4 \leq j \leq n$.

Altogether, v_2x is pancyclic in T . The proof of the theorem is completed. $\blacksquare$

Remark 6.4 1. Theorem 6.3 shows that Conjecture 5.8 also holds for $\sigma(T) = 2$.

2. According to the proof of Theorem 6.3, it is not difficult to get a polynomial algorithm to find two vertices v_1, v_2 in a tournament T with $\sigma(T) = 2$ such that all out-arcs of v_1 and all out-arcs of v_2 are pancyclic.

Combining Theorem 6.1 and Theorem 6.3, we reach the following corollary:

Corollary 6.5 Each 2-strong tournament T contains at least 2 vertices v_1, v_2 such that all out-arcs of v_i are pancyclic for $i = 1, 2$.

6.2 Out-arc Pancyclic Vertices in 3-strong Tournaments

In this section, we show that Conjecture 5.8 also holds for $\sigma(T) = 3$.

Theorem 6.6 (Feng [52]) *Every 3-strong tournament T contains 3 vertices whose all out-arcs are pancyclic.*

Proof. For convenience, denote $\delta^+(T) = q$. Note that $q \geq 3$. Let $M = \{x \mid d^+(x) = q\}$. By Lemma 5.12, all out-arcs of each $x \in M$ are pancyclic. If $|M| \geq 3$, then we are done. Assume that $|M| \leq 2$.

By Theorem 6.1, one can choose u, v_1 satisfying the following conditions:

- (1) $d^+(u) = \delta^+(T)$,
- (2) $u \rightarrow v_1$ and $d^+(v_1) = \min\{d^+(x) \mid x \in N^+(u)\}$,
- (3) all out-arcs of u and v_1 are pancyclic in T .

In the following, we find a desired vertex apart from u, v_1 . Let $N^+(u) = \{v_1, v_2, \dots, v_q\}$. If $d^+(v_i) = d^+(v_1)$ for some $2 \leq i \leq q$, then all out-arcs of v_i are also pancyclic. So, we can assume that $d^+(v_1) < d^+(v_2) \leq d^+(v_3) \leq \dots \leq d^+(v_q)$. Let $N^+(v_1) = \{w_1, w_2, \dots, w_s\}$, and assume without loss of generality that $d^+(w_i) \leq d^+(w_j)$ for $1 \leq i < j \leq s$, where $s = d^+(v_1)$.

Claim A *If $v_1 \rightarrow v_i$ for some $2 \leq i \leq q$, then there is a vertex $v_k \neq v_1$ whose out-arcs are pancyclic.*

Proof. Let k be the index such that $v_1 \rightarrow v_k$ and $v_i \rightarrow v_1$ for all $i \leq k-1$ if $k > 2$. By Lemma 5.13, all out-arcs of v_k are 4-pancyclic.

Let $x \in N^+(v_k)$. By Lemma 5.12, we only need to consider the case when $d^+(x) < d^+(v_k)$. If $x \in V(T) \setminus \{v_2, \dots, v_{k-1}\}$, then $x \rightarrow u$, and hence, $v_k x u v_k$ is a cycle of length 3. If $k > 2$ and $x \in \{v_2, \dots, v_{k-1}\}$, then $v_k x$ is in the 3-cycle $v_k x v_1 v_k$. Thus, all out-arcs of v_k are pancyclic. $\square$

By Claim A, one can assume that $v_i \rightarrow v_1$ for $i = 2, \dots, q$. We consider the vertex v_2 .

Let $x \in N^+(v_2)$. Similarly, we consider only the case when $d^+(x) < d^+(v_2)$. If $x \neq v_1$, then $x \rightarrow u$, since $x \notin N^+(u) \cup \{u\}$. Let D be the digraph obtained from T by contracting the path $u v_2 x$ to z . By Lemma 5.1, D is strong. From $d^+(x) > d^+(u)$, it follows that

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(x) - 1 + d^-(u) - 1 \\ &= d^+(x) - 1 + n - 1 - d^+(u) - 1 \\ &= n - 2 + d^+(x) - d^+(u) - 1 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

By Lemma 5.11, z is pancyclic in D . As a result, $v_2 x$ is 4-pancyclic in T , and hence $v_2 x$ is pancyclic, since the cycle $u v_2 x u$ is of length 3 and contains $v_2 x$.

For the arc $v_2 v_1$, we consider the digraph D obtained from T by contracting the path $u v_2 v_1$ to z . Since T is 3-strong, D is strong. We see that

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(v_1) + d^-(u) \\ &= d^+(v_1) + n - 1 - d^+(u) \\ &= n - 1 + d^+(v_1) - d^+(u) \\ &\geq n - 1 > |V(D)|. \end{aligned}$$

By Lemma 5.11, z is pancyclic in D , and then v_2v_1 is 4-pancyclic in T . If v_2v_1 is in a 3-cycle, then we are done. So, assume that there is no 3-cycle containing v_2v_1 . It follows that $N^+(v_1) \subset N^+(v_2)$ and $N^-(v_2) \subset N^-(v_1)$. We have the following claim:

Claim B *If $w_j \rightarrow v_i$ for some $2 \leq i \leq q$ and $1 \leq j \leq s$, then there is a vertex v_k whose all out-arcs are pancyclic.*

Proof. We choose the index k such that $N^-(v_k) \cap N^+(v_1) \neq \emptyset$, say $w_j \rightarrow v_k$, and $v_i \rightarrow N^+(v_1)$ for all $2 \leq i \leq k-1$.

Let $x \in N^+(v_k)$. Similarly, we only need to consider the case when $d^+(x) < d^+(v_k)$. If $x \in \{v_1, v_2, \dots, v_{k-1}\}$, then by considering the digraph D obtained from T by contracting the path uv_kx to z , we see by Lemma 5.11 and $d^+(x) \geq d^+(u)$ that v_kx is 4-pancyclic. Since $v_kxw_jv_k$ is a 3-cycle, the arc v_kx is pancyclic. If $x \notin \{v_1, v_2, \dots, v_{k-1}\}$, then $x \rightarrow u$, and hence, v_kxuv_k is a 3-cycle. By considering the digraph D obtained from T by contracting the path uv_kx to z , we see that v_kx is pancyclic in T . $\square$

By Claim B, one can still assume that $N^+(u) \rightarrow N^+(v_1)$. In other words, $v_i \rightarrow w_j$, $i = 1, 2, 3, \dots, q$ and $j = 1, 2, 3, \dots, s$. In the following, we prove that all out-arcs of w_1 are pancyclic in T or there is another vertex apart from u, v_1, w_1 with the desired property.

Case 1. $|M| = 2$.

It is clear that $d^+(v_1) = \delta^+(T)$. Let $x \in N^+(w_1)$. Similarly, we only need to consider the case when $d^+(x) < d^+(w_1)$. If $x \neq u$, then $x \rightarrow v_1$. So, w_1x is in the 3-cycle $w_1xv_1w_1$. Note that $d^+(x) > d^+(v_1)$. By considering the digraph D obtained from T by contracting the path v_1w_1x to z , it is not difficult to see that w_1x is pancyclic in T .

For the arc w_1u , we consider the following two subcases:

Subcase 1.1. $\sigma(T) \geq 4$.

Let D be the digraph obtained from T by contracting the path $v_1w_1uv_2$ to z . By Lemma 5.1, D is strong. If $d^+(v_2) \geq d^+(v_1) + 2$, then

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(v_2) - 2 + d^-(v_1) - 2 \\ &= d^+(v_2) - 2 + n - 1 - d^+(v_1) - 2 \\ &= n - 3 + d^+(v_2) - d^+(v_1) - 2. \\ &\geq n - 3 = |V(D)|. \end{aligned}$$

By Lemma 5.11, z is pancyclic in D , and then w_1u is 5-pancyclic in T . Since w_1u is in the cycles $w_1uv_1w_1$ and $w_1uv_2v_1w_1$, the arc w_1u is pancyclic in T .

So, assume that $d^+(v_2) \leq d^+(v_1) + 1$. Since $d^+(v_2) > d^+(v_1)$, we have $d^+(v_2) = d^+(v_1) + 1$. Recalling that $v_2 \rightarrow v_1$ and $v_2 \rightarrow w_i$, $1 \leq i \leq q$, it is clear that $N^+(v_2) = N^+(v_1) \cup \{v_1\}$. This implies that $v_i \rightarrow v_2$, $i = 3, \dots, q$. Then $d^+(v_3) \geq d^+(v_1) + 2$. By considering the digraph D obtained from T by contracting the path $v_1w_1uv_3$ to z , it is not difficult to see that w_1u is pancyclic in T . Thus, we see that all out-arcs of w_1 are pancyclic in T .

Subcase 1.2. $\sigma(T) = 3$.

Let D_2 be the digraph obtained from T by contracting the path $v_1w_1uv_2$ to z_2 and D_3 the digraph obtained from T by contracting the path $v_1w_1uv_3$ to z_3 .

If D_2 and D_3 are strong, then we see by a similar argument as in Subcase 1.1 that w_1u is pancyclic.

So, assume that either D_2 or D_3 is not strong. For $i = 2, 3$, let

$$A_i = \{x \in V(T) \setminus \{u, v_i, w_1\} \mid v_1 \text{ is reachable from } x \text{ in } T - \{u, v_i, w_1\}\},$$

$$B_i = \{y \in V(T) \setminus \{u, v_1, w_1\} \mid y \text{ is reachable from } v_i \text{ in } T - \{u, v_1, w_1\}\},$$

$$\bar{A}_i = V(T) \setminus (A_i \cup \{u, v_1, v_i, w_1\}) \quad \text{and} \quad \bar{B}_i = V(T) \setminus (B_i \cup \{u, v_1, v_i, w_1\}).$$

If D_i is not strong, then either $\bar{A}_i \neq \emptyset$ or $\bar{B}_i \neq \emptyset$.

If $\bar{A}_i \neq \emptyset$, then $A_i \cup \{v_1\} \rightarrow \bar{A}_i$. Thus, $\sigma(T - \{u, v_i, w_1\}) = 0$. By Lemma 5.10, we see that w_1u is 5-pancyclic in T . Considering the cycles $w_1uv_1w_1$ and $w_1uv_iv_1w_1$, the arc w_1u is pancyclic in T .

If $\bar{B}_i \neq \emptyset$, then $\bar{B}_i \rightarrow B_i \cup \{v_i\}$. Thus, $\sigma(T - \{u, v_1, w_1\}) = 0$. By Lemma 5.10, we see that w_1u is 5-pancyclic in T . Moreover, w_1u is pancyclic in T .

Case 2. $|M| = 1$.

Note that $d^+(v_2) > d^+(v_1) > d^+(u)$. Let $x \in N^+(w_1)$. We only need to consider the case when $d^+(x) < d^+(w_1)$. Since $N^+(u) \rightarrow N^+(v_1)$, we see that $x \notin N^+(u)$.

Subcase 2.1. $\sigma(T) \geq 4$.

Consider the case when $x = u$. With a similar argument as in Subcase 1.1 for the digraph D_i obtained from T by contracting the path $v_1w_1uv_i$ to z_i , $i = 2, 3$, we see that w_1u is pancyclic in T .

For the case when $x \neq u$, it is clear that $x \rightarrow uv_1$. Note that w_1x is in the cycles $w_1xv_1w_1$ and $w_1xuv_1w_1$.

We consider the digraph D obtained from T by contracting the path uv_1w_1x to z . By Lemma 5.1, we see D is strong. If $d^+(x) \geq d^+(u) + 2$, then

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(x) - 2 + d^-(u) - 2 \\ &= d^+(x) - 2 + n - 1 - d^+(u) - 2 \\ &= n - 3 + d^+(x) - d^+(u) - 2 \\ &\geq n - 3 = |V(D)|. \end{aligned}$$

By Lemma 5.11, z is pancyclic in D , and then w_1x is 5-pancyclic in T . Furthermore, w_1x is pancyclic in T .

So, assume that $d^+(x) \leq d^+(u) + 1$. Since $d^+(x) > d^+(u)$, we have $d^+(x) = d^+(u) + 1$. It follows that $d^+(y) \geq d^+(x)$ for all $y \in V(T) \setminus \{u\}$. By Lemma 5.12, we see that xy is pancyclic for all $y \in N^+(x) \setminus \{u\}$.

Since $d^+(v_2) \geq d^+(u) + 2$, we have $d^+(v_2) \geq d^+(x) + 1$. Considering the digraph D obtained from T by contracting the path xuv_2 to z , we see that xu is 4-pancyclic. If xu is contained in a 3-cycle, then x is a desired vertex.

For the case when there is no 3-cycle containing xu , we see that $N^+(x) = N^+(u) \cup \{u\}$ and $N^-(x) = V(T) \setminus (N^+(u) \cup \{u, x\})$. Let D_i be the digraph obtained from T by contracting the path $v_1w_1xv_i$ to z_i for $i = 2, 3$. By a similar argument as in Subcase 1.1, we see that w_1x is pancyclic.

Thus, we see that if xu is not pancyclic in T , then w_1 is a desired vertex.

Subcase 2.2. $\sigma(T) = 3$.

For the arc w_1u , we consider the digraph D_2 (D_3 , respectively) obtained from T by contracting the path $v_1w_1uv_2$ ($v_1w_1uv_3$, respectively) to z_2 (z_3 , respectively). With a similar discussion as in Subcase 1.2, we see that w_1u is pancyclic in T .

Consider the arc w_1x , $x \neq u$. It is clear that $x \rightarrow uv_1$. So, we see that w_1x is in the cycles $w_1xv_1w_1$ and $w_1xuv_1w_1$.

Let D_w be the digraph obtained from T by contracting the path uv_1w_1x to w .

At first, we consider the case when D_w is strong. If $d^+(x) \geq d^+(u) + 2$, then

$$\begin{aligned} d_{D_w}^+(w) + d_{D_w}^-(w) &= d^+(x) - 2 + d^-(u) - 2 \\ &= d^+(x) - 2 + n - 1 - d^+(u) - 2 \\ &= n - 3 + d^+(x) - d^+(u) - 2 \\ &\geq n - 3 = |V(D_w)|. \end{aligned}$$

By Lemma 5.11, w is pancyclic in D_w , and then w_1x is 5-pancyclic in T . Moreover, w_1x is pancyclic in T .

For the case when $d^+(x) \leq d^+(u) + 1$, we have $d^+(x) = d^+(u) + 1$, since $d^+(x) > d^+(u)$. It follows that $d^+(y) \geq d^+(x)$ for all $y \in V(T) \setminus \{u\}$. By Lemma 5.12, we see that xy is pancyclic for all $y \in N^+(x) \setminus \{u\}$.

Since $d^+(v_2) \geq d^+(u) + 2$, we have $d^+(v_2) \geq d^+(x) + 1$. Considering the digraph D obtained from T by contracting the path xuv_2 to z , we see that xu is 4-pancyclic. If xu is contained in a 3-cycle, then x is a desired vertex.

So, assume that there is no 3-cycle containing xu . This implies that $N^+(x) = N^+(u) \cup \{u\}$ and $N^-(x) = V(T) \setminus (N^+(u) \cup \{u, x\})$.

Let D_2 (D_3 , respectively) be the digraph obtained from T by contracting the path $v_1w_1xv_2$ ($v_1w_1xv_3$, respectively) to z_2 (z_3 , respectively). By a similar argument as in Subcase 1.2, it is not difficult to see that w_1x is pancyclic in T .

At last, consider the remaining case when D_w is not strong. Let

$$A = \{y \in V(T) \setminus \{v_1, w_1, x\} \mid u \text{ is reachable from } y \text{ in } T - \{v_1, w_1, x\}\},$$

$$B = \{y \in V(T) \setminus \{u, v_1, w_1\} \mid y \text{ is reachable from } x \text{ in } T - \{u, v_1, w_1\}\},$$

$$\bar{A} = V(T) \setminus (A \cup \{u, v_1, w_1, x\}) \quad \text{and} \quad \bar{B} = V(T) \setminus (B \cup \{u, v_1, w_1, x\}).$$

If $\bar{A} \neq \emptyset$, then $A \cup \{u\} \rightarrow \bar{A}$. Thus, $\sigma(T - \{v_1, w_1, x\}) = 0$. By Lemma 5.10, we see that w_1x is 5-pancyclic in T . With the cycles $w_1xv_1w_1$ and $w_1xuv_1w_1$, the arc w_1x is pancyclic in T .

If $\bar{B} \neq \emptyset$, then $\bar{B} \rightarrow B \cup \{x\}$. Thus, $\sigma(T - \{u, v_1, w_1\}) = 0$. Let $T_1, T_2, \dots, T_r$ ($r \geq 2$) be the strong components of $T - \{u, v_1, w_1\}$ with $T_i \rightarrow T_j$ for $1 \leq i < j \leq k$. By Lemma 5.10, we see that $N^-(T_1) = \{u, v_1, w_1\} = N^+(T_r)$.

If $x \in V(T_i)$ for some $i < r$, then it is not difficult to see that w_1x is pancyclic in T . For the case when $x \in V(T_r)$, we consider T_r . Recalling that $d^+(z) > d^+(u) \geq 3$ for all $z \in V(T) \setminus \{u\}$, it is clear that $t_r = |V(T_r)| \geq 4$.

By Theorem 5.5, T_r contains a vertex u_1 whose out-arcs are pancyclic in T_r . If $u_1z \in A(T)$ for some $z \in \{u, v_1, w_1\}$, then it is easy to check that u_1z is pancyclic in T .

For some arc $u_1y \in A(T_r)$, let $u_1u_2 \dots u_{t_r}u_1$ be a Hamiltonian cycle of T_r with $u_1u_2 = u_1y$. Since T is 3-strong, we see that $|N^-(\{u, v_1, w_1\}) \cap (V(T_r) \setminus \{u_1\})| \geq 2$. Moreover, it is not difficult to check that u_1y is pancyclic in T .

Therefore, we see that if $\sigma(T - \{u, v_1, w_1\}) = 0$, then either all out-arcs of w_1 are pancyclic in T or there is another vertex whose out-arcs are pancyclic in T . The proof is completed. $\blacksquare$

Remark 6.7 *Conjecture 6.2 is still open for $\sigma(T) = 2$.*

6.3 Out-arc 4-pancyclic Vertices

In Section 6.1, 6.2, we showed that Conjecture 5.8 is true for $\sigma(T) = 2, 3$.

In [83], Yeo gave for all $k \in \mathbb{N}$ an infinite class of k -strong tournaments, each of which contains at most 3 vertices whose out-arcs are pancyclic. Those give rise to the question: How many vertices does a k -strong tournament contain such that all out-arcs of those vertices are 4-pancyclic?

In this section, we will prove that an s -strong tournament contains $s+1$ vertices whose all out-arcs are 4-pancyclic, where $s \geq 2$. At first, we consider the case when $s \geq 3$.

Theorem 6.8 (Feng, Li and Li [53]) *Every s -strong tournament T with $s \geq 3$ contains at least $s+1$ vertices whose all out-arcs are 4-pancyclic.*

Proof. Let $|V(T)| = n$. Since T is at least 3-strong, it is clear that $n \geq 7$. Let $u_1 \in V(T)$ with $d^+(u_1) = \min\{d^+(u) \mid u \in V(T)\}$. According to Lemma 5.12, all out-arcs of u_1 are pancyclic. Let $N^+(u_1) = \{x_1, x_2, \dots, x_t\}$. It is clear that $s \leq t \leq \lfloor \frac{n-1}{2} \rfloor$.

Let $A = \{x \in N^+(u_1) \mid N^-(x) \cap N^+(u_1) = \emptyset\}$. It is easy to check that $|A| \leq 1$. If $|A| = 0$, i.e. every vertex of $N^+(u_1)$ is dominated by another vertex of $N^+(u_1)$, then by Lemma 5.13, all out-arcs of x_i are 4-pancyclic for $i = 1, 2, \dots, t$. Recalling that all out-arcs of u_1 are pancyclic, we have $t+1 (\geq s+1)$ desired vertices.

Now, consider the case when $|A| = 1$. Without loss of generality, assume that $A = \{x_1\}$. Because $T[N^+(u_1)]$ is a tournament, we have $x_1 \rightarrow x_j$ for $j = 2, 3, \dots, t$. By Lemma 5.13, we see that all out-arcs of x_j are 4-pancyclic for $j = 2, 3, \dots, t$. Considering the pancyclicity of u_1 , we have t desired vertices. In the following, we find the $(t+1)$ -th vertex whose all out-arcs are 4-pancyclic.

Let $W = N^+(x_1) \setminus N^+(u_1)$. Because of $d^+(x_1) \geq d^+(u_1)$, we have $W \neq \emptyset$. If there is a vertex $y \in W$ with $d^+(y) = d^+(u_1)$, then y is the desired vertex according to Lemma 5.12. So, assume that $d^+(y) \geq d^+(u_1) + 1$ for all $y \in W$. Now, we confirm that all out-arcs of x_1 are 4-pancyclic.

For a vertex $y \in N^+(x_1)$, let D_w be the digraph obtained from T by contracting u_1x_1y to w , where $w \notin V(T)$.

If $y \in W$, then $yu_1 \in A(T)$. From $d^+(y) \geq d^+(u_1) + 1$, it follows that

$$\begin{aligned} d_{D_w}^+(w) + d_{D_w}^-(w) &= |N^+(y) \setminus \{u_1\}| + |N^-(u_1) \setminus \{y\}| \\ &= d^+(y) - 1 + d^-(u_1) - 1 \\ &= d^+(y) - 1 + [n - 1 - d^+(u_1)] - 1 \\ &\geq n - 2 = |V(D_w)|. \end{aligned}$$

By Lemma 5.11, w is in a j -cycle of D_w for $j = 2, 3, \dots, n-2$, and then there is a k -cycle in T containing x_1y for $k = 4, 5, \dots, n$.

Now, we consider the case when $y \notin W$, i.e. $y \in N^+(u_1)$. By using an analogous argument to Case 1 in Lemma 5.13, we see that x_1y is 4-pancyclic.

Therefore, T contains at least $d^+(u_1) + 1 = t + 1 \geq s + 1$ vertices whose all out-arcs are 4-pancyclic. The proof of the theorem is completed. $\blacksquare$

In the following, we consider exactly 2-strong tournaments.

Theorem 6.9 *Each tournament T with n vertices and $\sigma(T) = 2$ contains at least 3 vertices v_1, v_2, v_3 such that all out-arcs of v_i are 4-pancyclic for $i = 1, 2, 3$.*

Proof. Since T is 2-strong, Theorem 6.3 indicates that T contains 2 vertices v_1, v_2 such that all out-arcs of v_i are pancyclic for $i = 1, 2$. Without loss of generality, let $d^+(v_1) = \min\{d^+(x) \mid x \in V(T)\}$.

If there is $z \in V(T) \setminus \{v_1, v_2\}$ with $d^+(z) = d^+(v_1)$, then by Lemma 5.12, all out-arcs of z are pancyclic, and we are done. So, we assume that

$$d^+(x) > d^+(v_1) \geq 2 \text{ for all } x \in V(T) \setminus \{v_1, v_2\}. \quad (6.4)$$

It is sufficient to find another vertex whose out-arcs are 4-pancyclic. Let $S = \{v_1, v_2\}$ and $H = T - S$. Consider the following three cases:

Case 1. $\sigma(H) = 0$.

Let $T_1, T_2, \dots, T_t$ ($t \geq 2$) be the strong components of H . It is clear that for each $i \in \{1, 2, \dots, t\}$, T_i is a single vertex or contains a Hamiltonian cycle by Theorem 5.9. Assume without loss of generality that the strong components $T_1, T_2, \dots, T_t$ of T have been labeled such that $T_i \rightarrow T_j$ for $1 \leq i < j \leq t$. Note that $N^+(T_t) = S = N^-(T_1)$. In particular, the final strong component T_t contains at least 3 vertices by (6.4).

For the case when $|V(T_t)| = 3$, let $T_t = x_1x_2x_3x_1$. It is easy to see from (6.4) that $T_t \rightarrow S$. Since $d^+(v_2) \geq 2$ and T is 2-strong, we have $n \geq 7$. Furthermore, it is easy to check that the subtournament $T - (S \cup V(T_t))$ contains a Hamiltonian path, say $u_1u_2 \dots u_{n-5}$, such that one of the following holds:

- (i) If $v_1 \rightarrow v_2$, then $v_1 \rightarrow u_1$ and $v_2 \rightarrow u_\alpha$ for some α satisfying $2 \leq \alpha \leq n-5$.
- (ii) If $v_2 \rightarrow v_1$, then $v_2 \rightarrow u_1$ and $v_1 \rightarrow u_\alpha$ for some α satisfying $2 \leq \alpha \leq n-5$.

We have the following claim:

Claim A *The three out-arcs of x_i are pancyclic for $i = 1, 2, 3$.*

Proof. Without loss of generality, we only check the vertex x_1 for the case when $v_1 \rightarrow v_2$.

For the arc x_1x_2 , we see that $x_1x_2x_3x_1$, $x_1x_2v_1u_1x_1$, $x_1x_2v_1u_1x_3x_1$, $x_1x_2v_1u_1 \dots u_ix_3x_1$ for $i = 2, \dots, n-5$ are cycles of length 3, 4, $\dots$, $n-1$, respectively. In addition,

$$x_1x_2v_1u_1 \dots u_{\alpha-1}x_3v_2u_\alpha \dots u_{n-5}x_1$$

is a Hamiltonian cycle of T . Hence, x_1x_2 is pancyclic.

Since $x_1v_1u_1x_1$, $x_1v_1u_1x_3x_1$, $x_1v_1u_1x_2x_3x_1$, $x_1v_1u_1u_2 \dots u_ix_2x_3x_1$ for $i = 2, \dots, n-5$ and $x_1v_1u_1 \dots u_{\alpha-1}x_2v_2u_\alpha \dots u_{n-5}x_3x_1$ are cycles in T , the arc x_1v_1 is pancyclic.

For the arc x_1v_2 , we see that $x_1v_2u_\alpha x_1$, $x_1v_2u_\alpha x_3x_1$, $x_1v_2u_\alpha x_2x_3x_1$, $x_1v_2u_\alpha \dots u_ix_2x_3x_1$ for $i = \alpha, \dots, n-5$, $x_1v_2u_\alpha \dots u_{n-5}x_3v_1u_1x_1$ and $x_1v_2u_\alpha \dots u_{n-5}x_2x_3v_1u_1 \dots u_jx_1$ for $j = 1, \dots, \alpha-1$ are cycles in T . Therefore, x_1v_2 is pancyclic. $\square$

Now, we consider the case when $|V(T_t)| \geq 4$. Let $\ell = |V(T_t)|$ and let $C_t = x_1x_2 \dots x_\ell x_1$ be a Hamiltonian cycle of T_t .

At first, we show that $|N^-(v_1) \cap V(T_t)| \geq 3$. Suppose to the contrary that $|N^-(v_1) \cap V(T_t)| \leq 2$. Then we have $|N^+(v_1) \cap V(T_t)| \geq \ell - 2$. Since v_1 dominates at least one vertex in T_1 , it holds: $d^+(v_1) \geq \ell - 1$. Now, we look at the vertex, say x_i , with minimum out-degree among all vertices in T_t . It is clear that $|N^+(x_i) \cap V(T_t)| \leq \lfloor \frac{\ell-1}{2} \rfloor$. Thus, we have

$$d^+(x_i) = d_{T_t}^+(x_i) + d_S^+(x_i) \leq \lfloor \frac{\ell-1}{2} \rfloor + 2 \leq \lfloor \frac{\ell-1}{2} \rfloor + \lceil \frac{\ell-1}{2} \rceil = \ell - 1 \leq d^+(v_1),$$

a contradiction to (6.4). Therefore, it holds

$$|N^-(v_1) \cap V(T_t)| \geq 3. \tag{6.5}$$

By Theorem 5.5, the subtournament T_t contains a vertex, say u , whose out-arcs are pancyclic in T_t . By Remark 5.6, we can choose u such that $d_{T_t}^+(u) \leq \lfloor \frac{\ell-1}{2} \rfloor$.

Assume without loss of generality that the vertices of the Hamiltonian cycle C_t have been labeled with $x_\ell = u$. We will show that either all out-arcs of x_ℓ are pancyclic or there exists another vertex apart from v_1, v_2, x_ℓ whose out-arcs are pancyclic. Let $x_\ell \rightarrow w$ be an arc of T .

Subcase 1.1. $w \in V(T_t)$.

By the choice of x_ℓ , the arc $x_\ell w$ is in an i -cycle for $i = 3, 4, \dots, \ell$. Assume without loss of generality that C_t has been the Hamiltonian cycle of T_t , which contains the arc $x_\ell w$, i.e. $w = x_1$. Moreover, we notice from (6.5) that there is an integer $\beta \in \{1, 2, \dots, \ell-2\}$ with $x_\beta \rightarrow v_1$.

Because $\{v_1, v_2\} = N^-(T_1)$ and T_1 is a single vertex or it contains a Hamiltonian cycle, it is obvious that the subtournament $T - (S \cup V(T_t))$ contains a Hamiltonian path, say $u_1u_2 \dots u_{n-2-\ell}$ ($u'_1u'_2 \dots u'_{n-2-\ell}$, respectively) with $v_1 \rightarrow u_1$ ($v_2 \rightarrow u'_1$, respectively). It is easy to see that $x_\ell x_1 \dots x_\beta v_1 u_1 x_{\beta+2} \dots x_\ell$ is a cycle of length $\ell + 1$ and $x_\ell x_1 \dots x_\beta v_1 u_1 \dots u_j x_{\beta+1} \dots x_\ell$ is a cycle of length $\ell + 1 + j$ for $j = 1, 2, \dots, n-2-\ell$. It remains to show that $x_\ell x_1$ is also in a Hamiltonian cycle of T .

If $v_1 \rightarrow v_2$, then $x_\ell x_1 \dots x_\beta v_1 v_2 u'_1 \dots u'_{n-2-\ell} x_{\beta+1} \dots x_\ell$ is a Hamiltonian cycle of T . So, suppose that $v_2 \rightarrow v_1$. If there is a vertex x_γ with $1 \leq \gamma \leq \ell - 1$ such that $x_\gamma \rightarrow v_2$, then $x_\ell x_1 \dots x_\gamma v_2 v_1 u_1 \dots u_{n-2-\ell} x_{\gamma+1} \dots x_\ell$ is a Hamiltonian cycle of T . Thus, assume that $v_2 \rightarrow T_t - \{x_\ell\}$. Since T is 2-strong, one obtains $d^-(v_2) \geq 2$. So, $|N^-(v_2) \cap (V(T) - (S \cup V(T_t)))| \geq 1$. Let $u_\delta \in N^-(v_2)$ such that $v_2 \rightarrow u_j$ for all $\delta + 1 \leq j \leq n - 2 - \ell$ if $\delta < n - 2 - \ell$. We obtain $x_\ell x_1 \dots x_\beta v_1 u_1 \dots u_\delta v_2 u_{\delta+1} \dots u_{n-2-\ell} x_{\gamma+1} \dots x_\ell$ which is a Hamiltonian cycle of T .

Subcase 1.2. $w \in \{v_1, v_2\}$.

For convenience, we denote the other vertex in $\{v_1, v_2\}$ by w' , i.e. $\{w, w'\} = \{v_1, v_2\}$.

By recalling that $T - (V(T_t) \cup \{w, w'\})$ contains a Hamiltonian path starting at a vertex in $N^+(w)$, we can confirm, similar to Subcase 1.1, that the arc $x_\ell w$ is pancyclic in $T - w'$. It remains to check whether the arc $x_\ell w$ is in a Hamiltonian cycle of T .

Suppose that $T - (V(T_t) \cup \{w, w'\})$ contains at least two vertices. Since T is 2-strong, we see that if $n_1 \geq 3$, then $|(N^+(w) \cup N^+(w')) \cap V(T_1)| \geq 2$ and if $n_1 = 1$, then $|(N^+(w) \cup N^+(w')) \cap V(T_2)| \geq 1$.

Similarly to Subcase 1.1, let $u_1 u_2 \dots u_{n-2-\ell}$ ($u'_1 u'_2 \dots u'_{n-2-\ell}$, respectively) with $w \rightarrow u_1$ ($w' \rightarrow u'_1$, respectively) be a Hamiltonian path of the subtournament $T - (S \cup V(T_t))$. If $w \rightarrow w'$, then we see that $x_\ell w w' u'_1 u'_2 \dots u'_{n-2-\ell} x_1 x_2 \dots x_\ell$ is a Hamiltonian cycle of T .

Now, consider the remaining case when $w' \rightarrow w$. Assume that there is a vertex x_γ with $1 \leq \gamma \leq \ell - 1$ such that $x_\gamma \rightarrow w'$. If $n_1 \geq 3$, then there is a vertex u'_η such that $w \rightarrow u'_\eta$ for some $2 \leq \eta \leq n - 2 - \ell$. We see that $x_\ell w u'_\eta \dots u'_{n-2-\ell} x_1 x_2 \dots x_\gamma w' u'_1 u'_2 \dots u'_{\eta-1} x_{\gamma+1} \dots x_\ell$ is a Hamiltonian cycle of T . If $n_1 = 1$, then either $N^+(w) \cap V(T_2) \neq \emptyset$ or $N^+(w') \cap V(T_2) \neq \emptyset$. So, we obtain a desired cycle either $x_\ell w u'_2 \dots u'_{n-2-\ell} x_1 x_2 \dots x_\gamma w' u'_1 x_{\gamma+1} \dots x_\ell$ or $x_\ell w u_1 x_1 x_2 \dots x_\gamma w' u_2 \dots u_{n-2-\ell} x_{\gamma+1} \dots x_\ell$.

Assume that $w' \rightarrow T_t - \{x_\ell\}$. Recalling that $d^-(w') \geq 2$, we have $|N^-(w') \cap (V(T) - (S \cup V(T_t)))| \geq 1$. Let $u_\delta \in N^-(w')$ such that $w' \rightarrow u_j$ for all $\delta + 1 \leq j \leq n - 2 - \ell$ if $\delta < n - 2 - \ell$. Then $x_\ell w u_1 \dots u_\delta w' u_{\delta+1} \dots u_{n-2-\ell} x_1 \dots x_\ell$ is a desired Hamiltonian cycle of T .

Suppose now that $T - (V(T_t) \cup \{w, w'\})$ contains exactly one vertex w_1 . Note that $t = 2$ and $\{w, w'\} \rightarrow w_1$.

If $w \rightarrow w'$, then $x_\ell w w' w_1 x_1 x_2 \dots x_\ell$ is a desired Hamiltonian cycle of T . So, we consider the case when $w' \rightarrow w$.

Assume that $d^+(w') \geq 3$. Let D be the digraph obtained from T by contracting the path $x_\ell w w_1$ to z . Clearly, $D - z$ is a tournament. Since $\ell \geq 4$, we see that $d^+(w_1) = \ell > \lfloor \frac{\ell-1}{2} \rfloor + 2 \geq d^+(x_\ell)$. It follows that

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= |N^+(w_1) \setminus \{x_\ell\}| + |N^-(x_\ell) \setminus \{w_1\}| \\ &= (d^+(w_1) - 1) + (d^-(x_\ell) - 1) \\ &= (d^+(w_1) - 1) + ((n - 1) - d^+(x_\ell)) - 1 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

We will show that D is strong. It is obvious that $D - w'$ is strong. By $d^-(w') \geq 2$, $N^-(w') \cap V(T_2 - x_\ell) \neq \emptyset$ holds. By the assumption that $d^+(w') \geq 3$, we have $N^+(w') \cap V(T_2) \neq \emptyset$. Therefore, D is strong. By Lemma 5.11, z is in all i -cycles for $i = 2, \dots, n - 2$, and hence, $x_\ell w$ is in a Hamiltonian cycle of T .

Assume now that $d^+(w') < 3$. This implies $d^+(w') = 2$. Since $w' \rightarrow w$ and $w' \rightarrow w_1$, we see that $T_2 \rightarrow w'$. By $w_1 \rightarrow T_2$, it is not difficult to check that all out-arcs of w_1 are pancyclic.

Case 2. $\sigma(H) = 1$.

We have the following claims.

Claim B *Let w be a vertex in H , whose all out-arcs are pancyclic in H . If $v_1 \rightarrow w$ and $v_2 \rightarrow w$, then all out-arcs of w are pancyclic.*

Proof. Let wx be an arbitrary out-arc of w . Since wx is pancyclic in H , we only need to show that T contains an $(n-1)$ -cycle through wx and a Hamiltonian cycle through wx .

For convenience, we denote one of $\{v_1, v_2\}$ by v and another by v' . Without loss of generality, let $vv' \in A(T)$. Let $u_1u_2 \dots u_{n-2}u_1$ be a Hamiltonian cycle of H with $u_1u_2 = wx$. Since T is 2-strong, we see that $d^-(v') \geq 2$, and then $d_H^-(v') \geq 1$. So, there is an index k , $2 \leq k \leq n-2$ such that $u_k \rightarrow v'$ and $v' \rightarrow u_{k+1} \dots u_{n-2}$ if $k < n-2$. Now, we obtain a desired $(n-1)$ -cycle $u_1u_2 \dots u_kv'u_{k+1} \dots u_{n-2}u_1$.

If $u_j \rightarrow v$ for some $k \leq j \leq n-2$, then $u_1u_2u_jvv'u_{j+1} \dots u_{n-2}u_1$ is a desired Hamiltonian cycle. So, assume that $v \rightarrow u_k \dots u_{n-2}$. Since $d^-(v) \geq 2$, there is an index ℓ , $2 \leq \ell \leq k-1$ such that $u_\ell \rightarrow v, v \rightarrow u_{\ell+1}$. Thus, $u_1u_2 \dots u_\ell v u_{\ell+1} \dots u_kv'u_{k+1} \dots u_{n-2}u_1$ is a desired Hamiltonian cycle. $\square$

Claim C *If H contains a vertex w with $d_H^+(w) = 1$, then either all out-arcs of w are 4-pancyclic in T or there is another vertex $w' \in V(H)$ with $d^+(w') = 3$ whose all out-arcs are pancyclic in T .*

Proof. From (6.4), it follows that $d^+(v_1) = 2$, $d^+(w) = 3$, $w \rightarrow v_1$ and $w \rightarrow v_2$. Let $x \in V(H)$ with $w \rightarrow x$. It is easy to see that x is a separating vertex of H . Let $T_1, T_2, \dots, T_t$ ($t \geq 2$) be the strong components of $H - x$ with $T_i \rightarrow T_j$ for $1 \leq i < j \leq t$. It is clear that $V(T_t) = \{w\}$. Since T is 2-strong, we see that $(N^+(v_1) \cup N^+(v_2)) \cap V(T_1) \neq \emptyset$. In what follows, we denote a Hamiltonian cycle of T_i by T_i or $C_i = x_1^i x_2^i \dots x_{n_i}^i$, $1 \leq i \leq t-1$, where $n_i = |V(T_i)|$, $1 \leq i \leq t$. We only need to check whether the arcs wx , wv_1 and wv_2 are 4-pancyclic in T .

It is evident from (6.4) that $d^+(x) \geq d^+(v_1) + 1 = 3 = d^+(w)$. By Lemma 5.12, the arc wx is pancyclic in T . Now, we check wv_1 in the following cases:

Case 1. $v_2 \rightarrow x$ and $v_1 \rightarrow v_2x$.

It is obvious that wv_1 is 4-pancyclic.

Case 2. $x \rightarrow v_2$ and $v_1 \rightarrow xv_2$.

It is easy to see that wv_1 is in a k -cycle for $k = 4, 5, \dots, n-1$. Since $d^+(v_1) = 2$, we see that $T_1 \rightarrow v_1$. This implies that $N^+(v_2) \cap V(T_1) \neq \emptyset$. So, $wv_1xv_2T_1 \dots T_{t-1}w$ is a desired Hamiltonian cycle of T . Therefore, wv_1 is 4-pancyclic.

Case 3. $v_1 \rightarrow x$ and $v_2 \rightarrow v_1x$.

Clearly, wv_1 is in a k -cycle for $k = 4, 5, \dots, n-1$. It is sufficient to show that wv_1 is in a Hamiltonian cycle of T . Assume to the contrary that T contains no Hamiltonian cycle through wv_1 .

Note that v_2v_1 is pancyclic. So, v_2v_1 is in a 3-cycle. Since $v_2v_1 \rightarrow x$, there is vertex $u \in V(T) - \{v_1, v_2, w, x\}$ with $v_1 \rightarrow u$ and $u \rightarrow v_2$.

At first, we consider the case when $N^+(v_1) \cap V(T_1) \neq \emptyset$. If $|(N^+(x) \cup N^+(v_1)) \cap V(T_1)| \geq 2$, say $x \rightarrow x_1^1$ and $v_1 \rightarrow x_i^1 (= u)$ ($i \neq 1$), then we see that $v_2 \rightarrow T_j$ for all $2 \leq j \leq t-1$ if $t \geq 3$. In fact, if $x_1^j \rightarrow v_2$ for some $2 \leq j \leq t-1$, then $wv_1x_1^1 \dots x_{n_1}^1T_2 \dots T_{j-1}x_2^j \dots x_{n_j}^jx_1^jv_2xx_1^1 \dots x_{i-1}^1T_{j+1} \dots T_{t-1}w$ is a Hamiltonian cycle, a contradiction.

If $t \geq 3$, then v_2 can be inserted into the path $x_1^1 \dots x_{n_1}^1T_2$. So, we obtain a Hamiltonian cycle through wv_1 , a contradiction.

Thus, we have $t = 2$. Moreover, $x_i^1 \neq x_{n_1}^1$ and $v_2 \rightarrow x_{n_1}^1$. a Hamiltonian cycle either $wv_1x_{n_1}^1v_2xx_1^1 \dots x_{n_1-1}^1w$ or $wv_1x_i^1 \dots x_{n_1}^1v_2xx_1^1 \dots x_{i-1}^1w$ through wv_1 , a contradiction.

Since $x_i^1 \rightarrow v_2$ and $v_2 \rightarrow x_{n_1}^1$, we see that v_2 can be inserted into the path $x_i^1 \dots x_{n_1}^1$. From the cycle wv_1xT_1w , we obtain a Hamiltonian cycle through wv_1 , a contradiction.

If $|N^+(x) \cup N^+(v_1)) \cap V(T_1)| = 1$, say $\{x_1^1\} = (N^+(x) \cup N^+(v_1)) \cap V(T_1)$, then $x_1^1 \rightarrow v_2$. Since $d^+(x) \geq 3$, we see that $t \geq 3$.

If $n_1 \geq 3$, then $N^+(v_2) \cap V(T_1) \neq \emptyset$. Analogously, v_2 can be inserted into the path $x_1^1 \dots x_{n_1}^1$. Then we obtain a Hamiltonian cycle through wv_1 , a contradiction.

If $n_1 = 1$, then $T_2 \dots T_{t-1} \rightarrow v_2$. Otherwise, v_2 can be inserted into the path $T_1T_2 \dots T_{t-1}$. Then we obtain a Hamiltonian cycle through wv_1 , a contradiction. Since T is 2-strong and $T_2 \dots T_{t-1} \rightarrow v_1$, we see that $N^+(x) \cap V(T_2) \neq \emptyset$. Thus, we obtain a Hamiltonian cycle $wv_1x_1^1v_2xT_2 \dots T_{t-1}w$ through wv_1 , a contradiction.

Now, consider the case when $N^+(v_1) \cap V(T_1) = \emptyset$. This implies that $N^+(v_2) \cap V(T_1) \neq \emptyset$. Since $d^+(v_1) = 2$, we see that $t \geq 3$ and there is another out-neighbor of v_1 in T_i for some $2 \leq i \leq t-1$, say x_1^i . Note that $u = x_1^i$ and $x_1^i \rightarrow v_2$. Then we see that $wv_1x_1^iv_2xT_1 \dots T_{i-1}x_2^i \dots x_{n_i}^iT_{i+1} \dots T_{t-1}w$ is a Hamiltonian cycle, a contradiction.

Case 4. $v_1 \rightarrow x, x \rightarrow v_2$ and $v_2 \rightarrow v_1$.

It is sufficient to show that wv_1 is in a Hamiltonian cycle. If $N^+(v_2) \cap V(T_1) \neq \emptyset$, then $wv_1xv_2T_1 \dots T_{t-1}w$ is a desired Hamiltonian cycle. So, suppose that $N^+(v_2) \cap V(T_1) = \emptyset$, i.e. $T_1 \rightarrow v_2$. Since $d^+(v_2) \geq 2$, we have $t \geq 3$ and $v_2 \rightarrow x_i^1$ for some $2 \leq i \leq t-1$. It is easy to see that v_2 can be inserted into the path $T_1 \dots T_{t-1}$. Thus, we can obtain a desired Hamiltonian cycle.

Case 5. $x \rightarrow v_1, v_1 \rightarrow v_2$ and $v_2 \rightarrow x$.

Obviously, $wv_1v_2xT_1T_2 \dots T_{t-1}w$ is a Hamiltonian cycle. From this cycle, we can easily obtain a k -cycle containing wv_1 for $k = 5, 6, \dots, n-1$. Since $d^+(v_2) \geq 2$, we see that $N^+(v_2) \cap V(T - \{v_1, w, x\}) \neq \emptyset$. Then T contains a 4-cycle through wv_1 .

Case 6. $x \rightarrow v_1$ and $xv_1 \rightarrow v_2$.

At first, consider the case when $N^+(v_2) \cap V(T_1) \neq \emptyset$. Since $wv_1v_2T_1 \dots T_{t-1}w$ is a cycle, it is easy to see that wv_1 is in a k -cycle for $k = n - 1, n - 2, \dots, 4$. It is sufficient to show that wv_1 is in a Hamiltonian cycle of T . Assume to the contrary that T has no Hamiltonian cycle containing wv_1 .

If $n_1 \geq 3$, then we confirm that $|(N^+(v_2) \cup N^+(x)) \cap V(T_1)| = 1$. Otherwise, assume that $v_2 \rightarrow x_1^1$ and $x \rightarrow x_\ell^1$ with $\ell \neq 1$. Then we see that $x \rightarrow T_j$ for $j = 2, 3, \dots, t - 1$ if $t \geq 3$. In fact, with $x_1^j \rightarrow x$ for some $2 \leq j \leq t - 1$, we obtain a Hamiltonian cycle

$$wv_1v_2x_1^1 \dots x_{\ell-1}^1T_2 \dots T_{j-1}x_2^j \dots x_{n_j}^j \dots x_1^jxx_\ell^1 \dots x_{n_1}^1T_{j+1} \dots T_{t-1}w,$$

a contradiction. Since $d^-(x) \geq 2$, we have $N^-(x) \cap V(T_1) \neq \emptyset$. It is not difficult to see that x can be inserted into the path $v_2T_1T_2$ if $t \geq 3$. Furthermore, we obtain a Hamiltonian cycle containing wv_1 for the case when $t \geq 3$, a contradiction.

We consider the case when $t = 2$. If x can be inserted into the path $v_2x_1^1 \dots x_{n_1}^1$, then T has a Hamiltonian cycle containing wv_1 , a contradiction. Since $d^-(x) \geq 2$ and $N^+(x) \cap V(T_1) \neq \emptyset$, there is a vertex x_k^1 for some $2 \leq k \leq n_1 - 1$, satisfying $x_k^1 \dots x_{n_1}^1 \rightarrow x$ and $x \rightarrow x_1^1 \dots x_{k-1}^1$. Because $d^+(v_2) \geq 2$ and $N^+(v_2) \subset V(T_1)$, we have $|N^+(v_2) \cap V(T_1)| \geq 2$, say $v_2 \rightarrow x_j^1$, $2 \leq j \leq n_1$. So, we see that $wv_1v_2x_j^1 \dots x_{n_1}^1xx_1^1 \dots x_{j-1}^1w$ is a Hamiltonian cycle containing wv_1 , a contradiction. Thus, $|(N^+(v_2) \cup N^+(x)) \cap V(T_1)| = 1$, say $(N^+(v_2) \cup N^+(x)) \cap V(T_1) = \{x_1^1\}$. Recalling that $n_1 \geq 3$ and $T - x_1^1$ is strong, there is x_k^1 with $k \neq 1$ and $v_1 \rightarrow x_k^1$. So, $wv_1x_k^1 \dots x_{n_1}^1xv_2x_1^1 \dots x_{k-1}^1T_2 \dots T_{t-1}w$ is a Hamiltonian cycle, a contradiction.

Therefore, we have $n_1 = 1$. It is clear that $t \geq 3$. Since $d^+(v_2) \geq 2$, there is an out-neighbor of v_2 in T_ℓ for some $2 \leq \ell \leq t - 1$, say x_1^ℓ . Analogously, we see that $x \rightarrow T_j$ for $j = \ell, \dots, t - 1$.

Since $d^-(x) \geq 2$, the path $Q = T_2 \dots T_{\ell-1}x_1^\ell$ contains an in-neighbor of x . Since $x \rightarrow x_1^\ell$, x can be inserted into Q . Thus, we obtain a Hamiltonian cycle containing wv_1 , a contradiction.

At last, consider the case when $N^+(v_2) \cap V(T_1) = \emptyset$, i.e. $T_1 \rightarrow v_2$. This implies that $N^+(v_1) \cap V(T_1) \neq \emptyset$, say $v_1 \rightarrow x_1^1$.

Let D be the digraph obtained from T by contracting $wv_1x_1^1$ to z . Since $d^-(x) \geq 2$, we see that all vertices are reachable from z in D . Since $d^+(v_2) \geq 2$, we see that $t \geq 3$ and there is an out-neighbor in T_i for some $2 \leq i \leq t - 1$. It is clear that z is reachable from all vertices of D . Thus, D is strong.

From $d^+(v_2) \geq 2$, it follows that $|V(T) - \{v_1, v_2, w, x\} \cup V(T_1)| \geq 2$. So, $d^+(x_1^1) \geq 4$. Furthermore, we have

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(x_1^1) - 1 + d^-(w) - 1 \\ &= d^+(x_1^1) - 1 + n - 1 - d^+(w) - 1 \\ &= n - 2 + d^+(x_1^1) - 4 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that z is in a ℓ -cycle for $\ell = 2, 3, \dots, |V(D)|$, and then wv_1 is 4-pancyclic in T .

Case 7. $x \rightarrow v_1$ and $v_2 \rightarrow xv_1$.

Since v_2v_1 is pancyclic, v_2v_1 is in a 3-cycle, i.e. there is a vertex $u \in V(T) - \{v_1, v_2, w, x\}$ with $v_1 \rightarrow u, u \rightarrow v_2$.

If $u \in V(T_i)$ for some $2 \leq i \leq t-1$, then $wv_1uv_2xT_1 \dots T_{i-1}(T_i - \{u\})T_{i+1} \dots T_{t-1}w$ is a Hamiltonian cycle containing wv_1 . From this cycle, it is easy to see that wv_1 is in a k -cycle for $k = 6, \dots, n-1$. Since $d^+(v_1) = 2$, we see that T has a 3-cycle and a 4-cycle containing wv_1 .

If $u \rightarrow x$, then one may obtain a desired 5-cycle from wv_1uxT_1w .

Assume that $x \rightarrow u$. If $N^+(v_2) \cap V(T_1) \neq \emptyset$, say $v_2 \rightarrow x_1^1$, then $wv_1uv_2x_1^1w$ is a desired cycle. Thus, we consider the case when $N^+(v_2) \cap V(T_1) = \emptyset$, i.e. $T_1 \rightarrow v_2$. Clearly, $N^+(v_1) \cap V(T_1) \neq \emptyset$, say $v_1 \rightarrow x_1^1$. For the case when $|V(T) - \{v_1, v_2, w, x\}| \geq 3$, it is easy to obtain a desired 5-cycle through wv_1 . For the remaining case when $|V(T) - \{v_1, v_2, w, x\}| = 2$, we have $d^+(u) = 2$, a contradiction to (6.4).

In the following, assume that T_i contains no vertex u such that $v_1 \rightarrow u$ and $u \rightarrow v_2$ for $i = 2, \dots, t-1$ if $t \geq 3$. This implies that $u \in V(T_1)$, say $u = x_1^1$.

Let D be the digraph obtained from T by contracting $wv_1x_1^1$ to z . Since $x_1^1 \rightarrow v_2, v_2 \rightarrow x$ and $d^+(x) \geq 3$, it is not difficult to see that D is strong. If $d^+(x_1^1) \geq 4$, then

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(x_1^1) - 1 + d^-(w) - 1 \\ &= d^+(x_1^1) - 1 + n - 1 - d^+(w) - 1 \\ &= d^+(x_1^1) - 1 + n - 1 - 3 - 1 \\ &= n - 2 + d^+(x_1^1) - 4 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that z is in ℓ -cycle of D for $\ell = 2, 3, \dots, |V(D)|$, and then wv_1 is pancyclic.

If $d^+(x_1^1) \leq 3$, then $d^+(x_1^1) = 3$ follows from (6.4). It is clear that $x \rightarrow x_1^1$ (otherwise, $N^+(x_1^1) = \{v_2, x, w\}$, and hence, $n_1 = 1$ and $t = 2$, a contradiction to the fact that $N^+(x) \cap V(T_1) \neq \emptyset$). So, we only need to consider the following cases:

- (a) $n_1 = 1, t = 3$ and $n_2 = 1$.
- (b) $n_1 \geq 3, t = 2$ and $d_{T_1}^+(x_1^1) = 1$.

For (a), we see that $xv_1 \rightarrow x_1^2$, since $d^+(v_1) = 2$ and $d^+(x) \geq 3$. So, $d^+(x_1^2) \leq 2$, a contradiction to (6.4). By considering (b), it is easy to see that wv_1 is in a γ -cycle for $\gamma = 3, 4, \dots, n-2$.

If $x \rightarrow x_1^1$, then $wv_1x_1^1v_2xx_2^1 \dots x_{n_1-1}^1w$ and $wv_1x_1^1v_2xx_2^1 \dots x_{n_1}^1w$ are desired cycles of length $n-1$ and n , respectively.

Assume that $x_2^1 \rightarrow x$. Since $d^+(x) \geq 3$, there is a vertex x_i^1 such that $2 \leq i \leq n_1-1$, $x_2^1 \dots x_i^1 \rightarrow x$ and $x \rightarrow x_{i+1}^1$. Then we obtain a desired $(n-1)$ -cycle

$$wv_1x_1^1 \dots x_i^1xx_{i+1}^1 \dots x_{n_1}^1w.$$

It is sufficient to find a Hamiltonian cycle containing wv_1 .

We only need to consider the case when $v_2 \rightarrow x_i^1$. Since $x_1^1 \rightarrow v_2$ and $i \geq 2$, we see that v_2 can be inserted into the path $x_1^1 \dots x_i^1$. Thus, wv_1 is in a Hamiltonian cycle.

Case 8. $v_2 \rightarrow v_1$ and $x \rightarrow v_2v_1$.

Since v_2v_1 is pancyclic, there is a vertex $u \in V(T) - \{v_1, v_2, w, x\}$ with $v_1 \rightarrow u$ and $u \rightarrow v_2$. Let $u \in V(T_k)$ with $k = \min\{i \mid v \in V(T_i), v_1 \rightarrow v, v \rightarrow v_2\}$.

We consider the digraph D obtained from T by contracting wv_1u to z . Since $d^+(v_2) \geq 2$ and $d^-(x) \geq 2$, all vertices of D are reachable from z . With $x \rightarrow v_2$, it is visible that z is reachable from all vertices of D . Thus, D is strong. If $d^+(u) \geq 4$, then we have

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(u) - 1 + d^-(w) - 1 \\ &= d^+(u) - 1 + n - 1 - d^+(w) - 1 \\ &= d^+(u) - 1 + n - 1 - 3 - 1 \\ &= n - 2 + d^+(u) - 4 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that z is in ℓ -cycle for $\ell = 2, 3, \dots, |V(D)|$, and then wv_1 is 4-pancyclic.

From (6.4), we have $d^+(u) \geq 3$. It remains to consider the case when $d^+(u) = 3$, i.e. to consider the following three cases:

- (a) $n_k \geq 3$, $x \rightarrow u$, $k = t - 1$ and $d_{T_k}^+(u) = 1$.
- (b) $n_k = 1$, $u \rightarrow x$ and $k = t - 1$.
- (c) $n_k = 1$, $x \rightarrow u$, $k = t - 2$ and $n_{t-1} = 1$.

If $2 \leq k \leq t-1$, then $N^+(v_2) \cap V(T_1) \neq \emptyset$. Otherwise, $T_1 \rightarrow v_2$ and $N^+(v_1) \cap V(T_1) \neq \emptyset$, a contradiction to the choice of u . It is not difficult to see that wv_1 is in a γ -cycle for $\gamma = 3, 4, \dots, n-1$. It is enough to find a Hamiltonian cycle containing wv_1 . If $u \rightarrow x$, then we are done. So, assume that $x \rightarrow u$.

For (a), without loss of generality, let $u = x_1^{t-1}$. If there is a vertex $x_j^{t-1} \rightarrow x$, then the cycle $wv_1x_1^{t-1} \dots x_j^{t-1}xv_2T_1 \dots T_{t-2}x_{j+1}^{t-1} \dots x_{n_{t-1}}^{t-1}w$ is desired. So, assume that $x \rightarrow T_{t-1}$. Since $d^-(x) \geq 2$, we see that x can be inserted into $T_1 \dots T_{t-2}x_2^{t-1}$. Then we are done.

For (c), we only need to consider $T_{t-1} \rightarrow x$. Analogously, $wv_1ux_1^{t-1}xv_2T_1 \dots T_{t-3}w$ is a desired cycle.

Now, we consider the case when $k = 1$. Without loss of generality, let $u = x_1^1$. Note that the case (b) is impossible, since $N^+(x) \cap V(T_1) \neq \emptyset$.

For (c), we have $t = 3$, $T_1 = x_1^1$, $T_2 = x_1^2$, $v_1 \rightarrow x_1^1x_1^2$ and $x_1^1 \rightarrow v_2$. From $d^-(x) \geq 2$ and $d^+(v_2) \geq 2$, it follows that $v_2 \rightarrow x_1^2$ and $x_1^2 \rightarrow x$. So, $d^+(x_1^2) = 2$, a contradiction to (6.4).

For (a), it is easy to see that wv_1 is in a γ -cycle for $\gamma = 3, 4, \dots, n-2$. Since $d^+(v_2) \geq 2$, there is a vertex x_i^1 such that $1 \leq i \leq n_1 - 1$, $x_1^1 \dots x_i^1 \rightarrow v_2$ and $v_2 \rightarrow x_{i+1}^1$. Then we obtain a desired $(n-1)$ -cycle $wv_1x_1^1 \dots x_i^1v_2x_{i+1}^1 \dots x_{n_1}^1w$. It is enough to find a Hamiltonian cycle containing wv_1 .

Assume to the contrary that T has no Hamiltonian cycle containing wv_1 . Then $x \rightarrow x_1^1 \dots x_i^1$ and there is an index j , $i+1 \leq j \leq n_1 - 1$, such that $x_{j+1}^1 \dots x_{n_1}^1 \rightarrow x$ and $x \rightarrow x_1^1 \dots x_j^1$.

If $i \geq 2$, then $wv_1x_1^1v_2x_{i+1}^1 \dots x_{n_1}^1xx_2^1 \dots x_i^1w$ is a Hamiltonian cycle containing wv_1 , a contradiction. Thus, $i = 1$. If $x_2^1 \rightarrow v_1$, then we have $v_1 \rightarrow x_\ell^1$ for some $3 \leq \ell \leq$

n_1 , and hence, $wv_1x_\ell^1 \dots x_{n_1}^1 x_1^1 v_2 x_2^1 \dots x_{\ell-1}^1 w$ is a Hamiltonian cycle containing wv_1 , a contradiction. So, $v_1 \rightarrow x_2^1$. Note that $xv_1 \rightarrow x_1^1$, $x_1^1 \rightarrow v_2$ and $v_2 v_1 \rightarrow x_2^1$.

At this point, we show that all out-arcs of x_1^1 are pancyclic. Since $d^+(x_1^1) = 3$, it is clear that $d^+(x_2^1) \geq d^+(x_1^1)$ and $d^+(w) = d^+(x_1^1)$. By Lemma 5.12, we see that $x_1^1 x_2^1$ and $x_1^1 w$ are pancyclic in T .

For the arc $x_1^1 v_2$, we see that $x_1^1 v_2 v_1 x_1^1$, $x_1^1 v_2 x_2^1 x_3^1 x_1^1$, $x_1^1 v_2 v_1 x_2^1 x_3^1 x_1^1$, $x_1^1 v_2 v_1 x_2^1 w x x_1^1$, $x_1^1 v_2 v_1 x_2^1 \dots x_\ell^1 w x x_1^1$ for $\ell = 3, \dots, n_1$ are cycles of T . Thus, $x_1^1 v_2$ is pancyclic in T .

It remains to check the arc wv_2 . If $d^+(v_2) \geq d^+(w)$, then by Lemma 5.12, we are done. So, assume that $d^+(v_2) < d^+(w)$. From $d^+(v_2) \geq d^+(v_1) = 2$ and $d^+(w) = 3$, it follows that $d^+(v_2) = 2$. With a similar discussion as above, the arc wv_2 is 4-pancyclic in T . $\square$

In what follows, we only consider the case when $\delta^+(H) \geq 2$.

Claim D Suppose that $\delta^+(H) \geq 2$. Let w be a vertex in H satisfying

$$d_H^+(w) = \min\{d_H^+(u) \mid u \text{ is not bridgehead of } H\}.$$

If $d^+(w) = d_H^+(w) + 1$, then either all out-arcs of w are 4-pancyclic in T or there is another vertex $w' \notin \{v_1, v_2, w\}$ whose out-arcs are 4-pancyclic in T .

Proof. By Theorem 5.5, it is clear that w is an out-arc pancyclic vertex of H . For convenience, we denote one of $\{v_1, v_2\}$ by v and another by v' . Without loss of generality, let $w \rightarrow v$ and $v' \rightarrow w$.

Let wx be an arbitrary out-arc of w . By Lemma 5.12, we only need to consider the case when $d^+(x) < d^+(w)$ in the following two cases:

Case 1. $x \neq v$.

It is clear that $x \in V(H)$. Since wx is pancyclic in H , we only need to show that T contains an $(n-1)$ -cycle through wx and a Hamiltonian cycle through wx .

Let $C = u_1 u_2 \dots u_{n-2} u_1$ be a Hamiltonian cycle of H , where $u_1 u_2 = wx$. Since T is 2-strong, we see that $d^-(v') \geq 2$, and then $d_H^-(v') \geq 1$. So, there is an index k , $2 \leq k \leq n-2$ such that $u_k \rightarrow v'$ and $v' \rightarrow u_{k+1} \dots u_{n-2}$ if $k < n-2$. Otherwise, we have $v' \rightarrow H$ and $d^-(v') \leq 1$, a contradiction. Now, we obtain a desired $(n-1)$ -cycle $u_1 u_2 \dots u_k v' u_{k+1} \dots u_{n-2} u_1$. We only need to show that there is a Hamiltonian cycle containing wx .

Suppose to the contrary that T contains no Hamiltonian cycle through wx . We see that $v' \rightarrow v$. Otherwise, we have $v \rightarrow u_k \dots u_{n-2}$. Since $d_C^-(v) = d^-(v) \geq 2$, with a similar discussion as in Claim B, we can obtain a Hamiltonian cycle of T through wx , a contradiction.

We confirm that x is a separating vertex of H unless all out-arcs of x are pancyclic. Suppose that x is not a separating vertex of H , and hence, x is not a bridgehead of H . According to the choice of w , we see that $d_H^+(x) \geq d_H^+(w)$.

From $d^+(w) = d_H^+(w) + 1$ and $d^+(x) \leq d^+(w) - 1$, it follows that

$$d^+(x) = d_H^+(x) = d_H^+(w) = d^+(w) - 1.$$

So, we see that x is also an out-arc pancyclic vertex of H . Since $d^+(x) = d_H^+(x)$, we are done by Claim B.

Let $T_1, \dots, T_t$ ($t \geq 2$) be the strong components of $H - \{x\}$ with $T_i \rightarrow T_j$ for all $1 \leq i < j \leq t$. Denote a Hamiltonian cycle of T_i by T_i or $C_i = u_1^i u_2^i \dots u_{n_i}^i$, where $n_i = |V(T_i)|$ for all $1 \leq i \leq t$. Since $d_H^+(w) \geq 2$, we have $n_t \geq 3$. Without loss of generality, let $u_1^t \dots u_{n_t}^t u_1^t$ be a Hamiltonian cycle of T_t with $u_1^t = w$.

From $d^+(x) < d^+(w)$, it follows that $d_H^+(x) \leq d_H^+(w)$. By Lemma 5.3 and Remark 5.7 with $w \rightarrow x$, it is not difficult to see that $d_{T_t}^+(w) \leq \lfloor \frac{n_t-1}{2} \rfloor$.

Subcase 1.1. $v \rightarrow x$ and $x \rightarrow v'$.

If $N^+(v) \cap V(T_1) \neq \emptyset$, then $w x v' v T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t w$ is a Hamiltonian cycle, a contradiction. So, we have $N^+(v) \cap V(T_1) = \emptyset$, i.e. $T_1 \rightarrow v$. This implies that $N^+(v') \cap V(T_1) \neq \emptyset$.

Since $d^+(v) \geq 2$ and $T_1 \rightarrow v$, we see that v can be inserted into $T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t$. Thus, $w x$ is in a Hamiltonian cycle, a contradiction.

Subcase 1.2. $x \rightarrow v'v$.

With a similar discussion as in Subcase 1.1, we are done.

Subcase 1.3. $v' \rightarrow x$ and $x \rightarrow v$.

If $N^+(v) \cap V(T_1) \neq \emptyset$, then $w x v T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t w$ is an $(n-1)$ -cycle containing $w x$. Since $d^-(v') \geq 2$ and $v' \rightarrow w$, we see that v' can be inserted into the path $T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t w$. So, we obtain a Hamiltonian cycle containing $w x$, a contradiction. So, $N^+(v) \cap V(T_1) = \emptyset$, i.e. $T_1 \rightarrow v$. This implies that $N^+(v') \cap V(T_1) \neq \emptyset$. Moreover, if $t_1 \geq 3$, then $|(N^+(v') \cup N^+(x)) \cap V(T_1)| \geq 2$.

Since $v'v$ is pancyclic, there is a vertex z such that $v'vzv'$ is a 3-cycle. If $z \in V(T_j)$ for some $2 \leq j \leq t-1$, then $w x v z v' T_1 \dots T_{j-1} (T_j - z) T_{j+1} \dots T_{t-1} u_2^t \dots u_{n_t}^t w$ is a Hamiltonian cycle, a contradiction. Thus, $z \in V(T_t)$, say $z = u_j^t$. It is not difficult to see that v can be inserted into the path $T_1 \dots T_{t-1} u_2^t \dots u_j^t$ and v' can be inserted into the path $u_j^t \dots u_{n_t}^t w$. Thus, we obtain a Hamiltonian cycle from the cycle $w x T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t w$, a contradiction.

Subcase 1.4. $v'v \rightarrow x$.

Let $x \rightarrow u_1^1$. If $N^+(v) \setminus \{x, u_1^1\} \neq \emptyset$, then the digraph D is strong, where D is obtained from T by contracting $w x u_1^1$ to y . If $n_t \geq 4$, then we have

$$\begin{aligned} d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 1 + d^-(w) - 1 \\ &= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\ &= n - 2 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq n - 2 + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 2 - 1 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

By Lemma 5.11, D has a Hamiltonian cycle containing y , and then there is a Hamiltonian cycle containing $w x$ in T , a contradiction.

For $n_t = 3$, analogously consider the case when $n_1 = 1$ and $t = 2$. According to the choice of w , we see that $u_2^t \rightarrow x$ and $u_3^t \rightarrow x$. It follows that $d_H^+(u_2^t) = 1$ or $d_H^+(u_3^t) = 1$, a contradiction.

Thus, $N^+(v) = \{x, u_1^1\}$. This implies that $u_1^1 \rightarrow v'$. Similarly, $N^+(x) \setminus \{u_1^1\} \subset V(T_t)$. It is easy to see that $u_2^t \rightarrow x$. Since $d^+(x) > d^+(v) = 2$, we have $d_{T_t}^+(x) \geq 2$.

Let z' be an arbitrary out-neighbor of x in T_t . We consider the digraph D' obtained from T by contracting wxz' to y' . Since $T_t \rightarrow v$, it is clear that D' is strong.

If $d^+(z') > d^+(w)$, then we obtain a Hamiltonian cycle containing wx by Lemma 5.11, a contradiction. So, $d^+(z') \leq d^+(w)$ for all $z' \in N_{T_t}^+(x)$.

Since $T_t \rightarrow v$, we see that $d^+(z') \geq d_H^+(z') + 1$. From $d^+(w) = d_H^+(w) + 1$, it follows that $d_H^+(z') \leq d_H^+(w)$. Recalling that $d_H^+(u) \geq d_H^+(w)$ for all $u \in V(T_t)$, we have $d_H^+(z') = d_H^+(w)$.

According to the assumption of x and Claim B, we see that x is a bridgehead of H with $d_H^+(x) \leq d_H^+(w)$. Thus, $d^+(z') \geq d_H^+(z') + 1 = d_H^+(w) + 1 \geq d_H^+(x) + 1 = d^+(x) + 1 > d^+(x)$ for all $z' \in N_{T_t}^+(x)$. It is clear that $d^+(u_1^1) \geq n_t + 1 \geq d^+(w) \geq d^+(x)$. By Lemma 5.12, all out-arcs of x are pancyclic in T .

Case 2. $x = v$.

At first, we consider the case when H contains only w as the separating vertex of H . Let $T_1, \dots, T_t$ be the strong components of $H - \{w\}$ with $T_i \rightarrow T_j$ for $1 \leq i < j \leq t$. Since all out-arcs of w are pancyclic in H , we have $N_H^+(w) \subset V(T_1), T_2 \dots T_t \rightarrow w$ and $n_i \geq 3$, $i = 1, t$. Recall that all out-arcs of v' are pancyclic in T . Then $N^-(v') \cap V(T_t) \neq \emptyset$ unless $v \rightarrow v', v' \rightarrow T_t$ and $T_t \rightarrow v$. For the latter case, it is easy to check that wv is pancyclic in T . Thus, we consider the case when $N^-(v') \cap V(T_t) \neq \emptyset$.

If $N^+(v) \cap V(T_i) \neq \emptyset$ for some $1 \leq i \leq t-1$, then it is not difficult to see that wv is 4-pancyclic, since $v' \rightarrow w$.

For the case when $N^+(v) \cap V(T_i) = \emptyset$ for all $i \leq t-1$, we see that $T_1 \dots T_{t-1} \rightarrow v$, $N^+(v') \cap V(T_1) \neq \emptyset$ and $N^+(v) \cap V(T_t) \neq \emptyset$, since $d^+(v) \geq 2$. Clearly, we only need to consider the case when $v' \rightarrow v$. Since $v'v$ is pancyclic, we have $N^-(v') \neq \emptyset$. Moreover, $N^-(v') \cap N^+(v) \subset V(T_t)$. Then it is not difficult to see that wv is 4-pancyclic.

At last, we consider the case when $H - \{w\}$ contains at least a separating vertex of H and denote the strong components of $H - \{s\}$ by $T_1, \dots, T_t$ for some separating vertex $s \neq w$ of H . So, we see that $w \in V(T_t)$. Note that if $T - v'$ has a Hamiltonian cycle through wv , then T contains a Hamiltonian cycle through wv .

If H contains a separating vertex s such that $v \rightarrow s$, then we see that $T - \{v'\}$ contains a k -cycle through wv for $k = 4, 5, \dots, n-1$. So, wv is 4-pancyclic.

Now, consider the case when H contains no separating vertex s such that $v \rightarrow s$, i.e. for each separating vertex s , $s \rightarrow v$ holds.

Assume that there is a separating vertex s in H with $w \rightarrow s$. Then we see that T_t contains no other separating vertex of H . By Remark 5.6 and 5.7, we have $d_{T_t}^+(w) \leq \lfloor \frac{n_t-1}{2} \rfloor$. Note that $s \rightarrow v$.

Subcase 2.1. $N^+(v) \cap V(T_1) \neq \emptyset$, say $v \rightarrow u_1^1$.

It is clear that wv is in a k -cycle for $k = 3, \dots, n-2$. If $v' \rightarrow T - \{v, s\}$, then $sv \rightarrow v'$,

since $d^-(v') \geq 2$. It is obvious that wv is pancyclic in T . Thus, we consider the case when $N^-(v') \cap V(T - \{s\}) \neq \emptyset$.

Since $v' \rightarrow w$, one can insert v' into the path $T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t w$, and hence, there is an $(n-1)$ -cycle through wv . It is sufficient to find a Hamiltonian cycle containing wv .

Subcase 2.1.1. $v \rightarrow v'$.

If $v' \rightarrow s$, then the cycle $wvv'sT_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t w$ is desired. So, assume that $s \rightarrow v'$.

Since wu_2^t is pancyclic in H , we see that $N^-(s) \cap V(T_t - w) \neq \emptyset$. It is clear that $N^+(v') \cap V(T - \{v, v', w, s\}) \neq \emptyset$.

Let D be the digraph obtained from T by contracting wvu_1^1 to z . It is evident that D is strong, since $v' \rightarrow w$.

If $n_t \geq 4$, then $n_t > \lfloor \frac{n_t-1}{2} \rfloor + 2$. It follows that $d^+(u_1^1) > d^+(w)$. By Lemma 5.11, we see that wx is pancyclic.

For the case when $n_1 \geq 3$ or $t \geq 3$ or $u_1^1 \rightarrow v'$, we are done analogously.

If $n_t = 3$, $n_1 = 1$, $t = 2$ and $v' \rightarrow u_1^1$, then $u_2^t \rightarrow s$, $s \rightarrow u_3^t$, since wu_2^t is pancyclic in H . Thus, $d_H^+(u_3^t) = 1$, a contradiction.

Subcase 2.1.2. $v' \rightarrow v$.

Since wu_2^t is pancyclic in H , we see that $N^-(s) \cap V(T_t - w) \neq \emptyset$. Let D be the digraph obtained from T by contracting wvu_1^1 to z . It is clear that D is strong.

If $n_t \geq 4$, then $n_t > \lfloor \frac{n_t-1}{2} \rfloor + 2$. It follows that $d^+(u_1^1) > d^+(w)$. By Lemma 5.11, we see that wx is pancyclic.

For $n_t = 3$, we only need to consider the case when $t = 2$, $n_1 = 1$ and $v' \rightarrow u_1^1$. Since wu_2^t is pancyclic in H , we have $u_2^t \rightarrow s$ and $s \rightarrow u_3^t$. Thus, $d_H^+(u_3^t) = 1$, a contradiction.

Subcase 2.2. $T_1 \rightarrow v$.

This implies that $N^+(v') \cap V(T_1) \neq \emptyset$, say $v' \rightarrow u_1^1$.

Subcase 2.2.1. $v \rightarrow v'$.

From the cycle $wvv'w$ and $wvv'T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t w$, it is easy to see that wv is in a k -cycle for $k = 3, \dots, n-1$.

If $v' \rightarrow s$, then wv is pancyclic in T . So, assume $s \rightarrow v'$. Since wu_2^t is pancyclic in H , we have $N^-(s) \cap V(T_t - w) \neq \emptyset$.

We consider the digraph D obtained from T by contracting $wvv'u_1^1$ to z . It is easy to see that D is strong.

Recall that $d_{T_t}^+(w) \leq \lfloor \frac{n_t-1}{2} \rfloor$. If $n_t \geq 4$, then we have

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(u_1^1) - 2 + d^-(w) - 2 \\ &\geq n_t + 1 - 2 + n - 1 - \lfloor \frac{n_t-1}{2} \rfloor - 2 - 2 \\ &\geq n - 3 + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 3 \\ &\geq |V(D)|. \end{aligned}$$

By Lemma 5.11, z is in a Hamiltonian cycle of D . Then wv is in a Hamiltonian cycle of T .

For the case $n_t = 3$, we see that $u_2^t \rightarrow s, s \rightarrow u_3^t$, and hence, $d_H^+(u_3^t) = 1$, a contradiction.

Subcase 2.2.2. $v' \rightarrow v$.

Since $v'u$ is pancyclic in T , there is a vertex $a \in V(T) - V(T_1) \cup \{s, w\}$ with $v \rightarrow a$ and $a \rightarrow v'$.

If there is a vertex $b \in T_i$ for some $2 \leq i \leq t-1$ with $v \rightarrow b$, then we consider the digraph D obtained from T by contracting wvb to z . It is clear that D is strong and $d_D^+(z) + d_D^-(z) \geq n-2 = |V(D)|$ for $n_t \geq 4$.

For the case when $n_t = 3$, we only need to consider the case when the following holds:

$$i = t-1, T_1 \dots T_{t-2} \rightarrow v, n_{t-1} = 1, s \rightarrow u_1^{t-1}, v' \rightarrow u_1^{t-1} \text{ and } T_t \rightarrow s.$$

Then we see that u_1^{t-1} is an out-arc pancyclic vertex in H . By Claim B, u_1^{t-1} is a desired vertex.

So, assume that $N^+(v) \subset V(T_t)$. Let D be the digraph obtained from T by contracting wva to z .

Since wu_2^t is pancyclic in H , we see that $N^-(s) \cap V(T_n - \{w\}) \neq \emptyset$. It is easy to see that all vertex of D are reachable from z in D .

We confirm that z is reachable from every vertex of D in D , i.e. w is reachable from every vertex of $V(T - \{v, a\})$ in $T - \{v, a\}$.

Otherwise, let $A = \{y \in V(T) \setminus \{v, a\} \mid w \text{ is reachable from } y\}$, $B = V(T - \{v, a\}) \setminus A$. It is clear that $v', s \in A$, $w \rightarrow B$ and $A \rightarrow B$. If $B \neq \emptyset$, then we see that a is a separating vertex of H , a contradiction to the assumption that H contains no separating vertex dominated by v . So, $B = \emptyset$. Therefore, D is strong.

By Lemma 5.11, we only need to consider the case when $d^+(a) \leq d^+(w)$. Since T_t contains no separating vertex of H , we have $d_H^+(c) \geq d_H^+(w)$, $c \in V(T_t)$. From $d^+(a) = d_H^+(a) + 1$ and $d^+(w) = d_H^+(w) + 1$, it follows that $d^+(a) = d^+(w)$ and $d_H^+(a) = d_H^+(w)$. So, a is also an out-arc pancyclic vertex in H . With a similar discussion as in Case 1 and above, we see that a is a desired vertex.

In the following, consider the case when H contains no separating vertex dominated by v and w , i.e. there is only such one separating vertex, say s , that $s \rightarrow wv$.

If $N^+(v) \cap V(T_1) \neq \emptyset$, then wv is in a k -cycle for $k = 3, 4, \dots, n-2$. Since $s \rightarrow w$ and $v' \rightarrow w$, it is easy to see that wv is pancyclic in T . So, assume that $N^+(v) \cap V(T_1) = \emptyset$, and then $N^+(v') \cap V(T_1) \neq \emptyset$. If $v \rightarrow v'$, then wv is in a k -cycle for $k = 4, \dots, n-1$. By $s \rightarrow w$ and the cycle $wv'w$, we see that wv is pancyclic in T .

For the case when $N^+(v) \cap V(T_1) = \emptyset$ and $v' \rightarrow v$, we have $N^+(v') \cap V(T_1) \neq \emptyset$. Since $v'u$ is pancyclic in T , there is a vertex $a \in V(T) - V(T_1) \cup \{s, w\}$ with $v \rightarrow a$ and $a \rightarrow v'$.

If $a \in V(T_i)$ for some $2 \leq i \leq t-1$, then we consider the digraph D obtained from T by contracting wva to z . Since a is not a separating vertex of H and $N^-(s) \cap V(T_t) \neq \emptyset$, it is clear that D is strong.

From $d^+(a) \geq n_t + 1$ and $d^+(w) \leq n_t - 2 + 1 = n_t - 1$, it follows that $d^+(a) > d^+(w)$, and hence, $d_D^+(z) + d_D^-(z) \geq n-2 = |V(D)|$. By Lemma 5.11, we see that wv is pancyclic.

So, suppose that $a \in V(T_t)$. Analogously, consider the digraph D obtained from T by contracting wva to z . Since a is a not separating vertex of H and $N^-(s) \cap V(T_t) \neq \emptyset$, we see that D is strong. By Lemma 5.11, we only need to consider the case $d^+(a) \leq d^+(w)$. From $d_H^+(a) + 1 = d^+(a)$ and $d_H^+(w) + 1 = d^+(w)$, it follows that $d_H^+(a) \leq d_H^+(w)$. According to the choice of w and the fact that a is not a separating vertex of H , it is clear that $d_H^+(a) = d_H^+(w)$ and all out-arcs of a are pancyclic in H . With a similar discussion as in Case 1 and above, we see that a is a desired vertex. $\square$

Now, we continue to prove Theorem 6.9. For convenience, denote one of $\{v_1, v_2\}$ by v and another by v' . Without loss of generality, let $v \rightarrow v'$. Let w be a vertex in H satisfying

$$d_H^+(w) = \min\{d_H^+(u) \mid u \text{ is not a bridgehead of } H\}.$$

By Theorem 5.5, all out-arcs of w in H are pancyclic in H . In what follows, we show that either w is a desired vertex or there is another vertex $w' \notin \{v, v', w\}$ whose out-arcs are 4-pancyclic in T . By Claim B – D, we only need to consider the case when $\delta^+(H) \geq 2$, $d^+(w) = d_H^+(w) + 2$, i.e. $w \rightarrow vv'$.

Claim E *If H contains only w as the separating vertex of H , then there is another vertex $w' \notin \{v, v', w\}$ whose out-arcs are 4-pancyclic in T .*

Proof. Let $T_1, \dots, T_t$ be the strong components of $H - w$ with $T_i \rightarrow T_j$ for $1 \leq i < j \leq t$. Then we see that $N_H^+(w) \subset V(T_1)$, $T_2 \dots T_t \rightarrow w$ and $n_i \geq 3$, $i = 1, t$.

By Theorem 5.5, T_t contains a vertex w' whose out-arcs are pancyclic in T_t . Since $T_t \rightarrow w$, it is easy to see that all out-arcs of w' are also pancyclic in H .

Let $w'x$ be an arbitrary out-arc in H . For the case when $N^+(v') \cap V(T_1) \neq \emptyset$, it is obvious that $w'x$ is contained in an $(n-1)$ -cycle and a Hamiltonian cycle of T . So, assume that $N^+(v') \cap V(T_1) = \emptyset$, and then $N^+(v) \cap V(T_1) \neq \emptyset$. It is clear that we only need to find a Hamiltonian cycle through $w'x$. Since $d^+(v') \geq 2$, such a Hamiltonian cycle can be easily obtained. Analogously, we see that $w'v$ is 4-pancyclic in T .

Now, we consider the arc $w'v'$. If $N^+(v') \cap (V(H) \setminus V(T_t)) \neq \emptyset$, then it is not hard to see that $w'v'$ is pancyclic.

So, assume that $N^+(v') \cap (V(H) \setminus V(T_t)) = \emptyset$, and then $N^+(v) \cap (V(H) \setminus V(T_t)) \neq \emptyset$ and $|N^+(v') \cap V(T_t)| \geq 2$. Since vv' is pancyclic in T , it is not difficult to see that $w'v'$ is in a ℓ -cycle, $\ell = 5, \dots, n$.

It remains to find a 4-cycle containing $w'v'$. Since vv' is in a 3-cycle, say $vv'u_j^t v$, we only need to consider the case when $w' \rightarrow v$ and $N_{T_t}^+(u_j^t) \subset N_{T_t}^+(w')$. This implies that $d_{T_t}^+(u_j^t) \leq d_{T_t}^+(w')$. If u_j^t is not a bridgehead of T_t , then all out-arcs of u_j^t in T_t are pancyclic. With a similar argumentation as above, we see that u_j^t is a desired vertex. So, assume that u_j^t is a bridgehead of T_t . From the structure of T , it is not hard to see that all out-arcs of u_j^t in T are 4-pancyclic. $\square$

By Claim E, one can assume that $H - \{w\}$ contains at least one separating vertex of H . Let wx be an arbitrary out-arc of w . By Lemma 5.12, we consider only the arc wx with $d^+(x) < d^+(w)$ in the following cases:

Subcase 2.1. $x \notin \{v, v'\}$.

At first, we consider the case when x is a separating vertex of H . Let $T_1, \dots, T_t$ be the strong components of $H - \{x\}$ with $T_i \rightarrow T_j$ for all $1 \leq i < j \leq t$. According to the choice of w , we see that $w \in V(T_t)$ and T_t contains no other separating vertex of H . Moreover, $n_t \geq 3$ and $2 \leq d_H^+(w) \leq \lfloor \frac{n_t-1}{2} \rfloor + 1$. Let $u_1^t \dots u_{n_t}^t u_1^t$ be a Hamiltonian cycle of T_t with $u_1^t = w$.

Since wx is pancyclic in H , we only need to show that T contains an $(n-1)$ -cycle through wx and a Hamiltonian cycle through wx .

Subcase 2.1.1. $x \rightarrow v$.

If $N^+(v') \cap V(T_1) \neq \emptyset$, then we are done. So, assume that $N^+(v') \cap V(T_1) = \emptyset$. This implies that $N^+(v) \cap V(T_1) \neq \emptyset$. It is clear that wx is in an $(n-1)$ -cycle.

Since $d^+(v') \geq 2$, we see that v' can be inserted into the path $vT_1 \dots T_{t-1}u_2^t \dots u_{n_t}^t$. So, wx is in a Hamiltonian cycle.

Subcase 2.1.2. $v \rightarrow x$ and $x \rightarrow v'$.

Since all out-arcs of v are pancyclic, we see that $T_t - w$ contains an in-neighbor of v , say u_k^t .

Firstly, we consider the case when $N^+(v') \cap V(T_1) \neq \emptyset$. Clearly, wv in an $(n-1)$ -cycle. We need to find a Hamiltonian cycle containing wx . Assume to the contrary that there is no Hamiltonian cycle containing wx in T .

If $|(N^+(v') \cup N^+(x)) \cap V(T_1)| \geq 2$, say $x \rightarrow u_1^1$ and $v' \rightarrow u_i^1$, then we obtain a Hamiltonian cycle $wxu_1^1 \dots u_{i-1}^1 T_2 \dots T_{t-1}u_2^t \dots u_k^t v v' u_i^1 \dots u_{n_1}^1 u_{k+1}^t \dots u_{n_t}^t w$, a contradiction.

So, $|(N^+(v') \cup N^+(x)) \cap V(T_1)| = 1$, say $xv' \rightarrow u_1^1$. If $n_1 \geq 3$, then $N^+(v) \cap V(T_1 - \{u_1^1\}) \neq \emptyset$, say $v \rightarrow u_i^1$. Thus, $wxv' u_1^1 \dots u_{i-1}^1 T_2 \dots T_{t-1}u_2^t \dots u_k^t v u_i^1 \dots u_{n_1}^1 u_{k+1}^t \dots u_{n_t}^t w$ is a Hamiltonian cycle, a contradiction.

Consider the case when $n_1 = 1$. We confirm that $t = 2$. Otherwise, $t \geq 3$ and T_2 contains at least one of the out-neighbors of v, v', x . With a similar discussion as above, a Hamiltonian cycle through wx can be easily obtained.

From $d^+(v') \geq 2$, we see that $N^+(v') \cap V(T_2) \neq \emptyset$, say $v' \rightarrow u_\ell^2$. It is evident that $v \rightarrow u_1^1 u_1^2 \dots u_{\ell-1}^2$.

Since vv' is pancyclic, there is vertex $y \in V(T_2)$ such that $v' \rightarrow y$ and $y \rightarrow v$, say $y = u_\ell^2$. So, $u_\ell^2 \dots u_{n_2}^2 \rightarrow v$. Then $u_2^2 \rightarrow xv'$. Otherwise, we obtain a Hamiltonian cycle through wx : either $wxv' u_2^2 \dots u_\ell^2 v u_1^1 u_{\ell+1}^2 \dots u_{n_2}^2 w$ or $wxu_2^2 \dots u_\ell^2 v v' u_1^1 u_{\ell+1}^2 \dots u_{n_2}^2 w$, a contradiction.

Let D be the digraph obtained from T by contracting wxu_1^1 to z . It is clear that D is strong. If $n_2 \geq 6$, then we have

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\ &\geq n_2 - 1 + n - 1 - \lfloor \frac{n_2-1}{2} \rfloor - 3 - 1 \\ &= n - 2 + n_2 - \lfloor \frac{n_2-1}{2} \rfloor - 4 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that wx is pancyclic, a contradiction.

If $n_2 = 3$, then $u_1^2 \rightarrow x$ and $x \rightarrow u_3^2$, since wu_2^2 is pancyclic in H and $\delta^+(H) \geq 2$. Thus, we have $d_H^+(u_3^2) = 1$, a contradiction.

Consider the case when $n_2 = 4$. According to the choice of w , we have $d_H^+(w) = 2$, and hence, $u_3^2 \rightarrow w$. It is easy to see that $y \neq u_2^2$ and $u_2^2 \rightarrow xv'$. So, we see that either $y = u_3^2$ or $y = u_4^2$.

We confirm that $y = u_3^2$. Otherwise, $wxv'u_4^2vu_1^2u_2^2u_3^2w$ is a Hamiltonian cycle through wx , a contradiction.

So, we see that $u_4^2 \rightarrow vv'$ and $v \rightarrow u_2^2$. Now, $wxu_1^2u_4^2vu_2^2v'u_3^2w$ is a Hamiltonian cycle through wx , a contradiction.

Consider the remaining case when $n_2 = 5$. Since $d_H^+(w) \leq \lfloor \frac{n-3}{2} \rfloor = 3$ and $d^+(u_1^2) = 5$, we see that $d_H^+(w) = 3$, and then either $w \rightarrow u_3^2$ or $w \rightarrow u_4^2$. Obviously, $u_2^2 \rightarrow xv'$.

If $w \rightarrow u_3^2$, then $u_4^2 \rightarrow w$. Since u_3^2 is not a separating vertex, we see that $d_H^+(u_3^2) \geq d_H^+(w) = 3$, and hence, $d_H^+(u_3^2) = 3$. Moreover, $u_3^2 \rightarrow x$, $u_3^2 \rightarrow u_5^2$ and $u_3^2 \rightarrow vv'$, according to the choice of w . Because every vertex of T_2 is not a separating vertex, we see that $d_H^+(u_j^2) \geq d_H^+(w) = 3$ for $j = 2, 3, 4, 5$. It is clear that $u_5^2 \rightarrow xv'$. So, $y = u_4^2$, and hence, $wxu_1^2u_2^2u_3^2vv'u_4^2u_5^2w$ is a Hamiltonian cycle through wx .

If $w \rightarrow u_4^2$, then $u_3^2 \rightarrow w$. To avoid the cycle $wxv'u_4^2u_5^2vu_1^2u_2^2u_3^2w$, we have $u_4^2 \rightarrow v'$. We confirm that $y \neq u_5^2$, i.e. $u_5^2 \rightarrow v'$.

Suppose that $v' \rightarrow u_5^2$ and $u_5^2 \rightarrow v$. We have $v \rightarrow u_2^2u_3^2u_4^2$, $u_4^2 \rightarrow xv'$ and $u_2^2 \rightarrow u_4^2$. Otherwise, $wxu_4^2u_5^2vv'u_1^2u_2^2u_3^2w$ or $wxv'u_4^2u_5^2vu_1^2u_2^2u_3^2w$ or $wxv'u_5^2vu_1^2u_4^2u_2^2u_3^2w$ is a Hamiltonian cycle, a contradiction. Since u_5^2 is not a separating vertex of H , we have $d_H^+(u_5^2) \geq d_H^+(w) + 1 = 4$, and hence, $d_H^+(u_5^2) = 4$, according to the choice of w . So, we see that $u_5^2 \rightarrow u_1^2u_2^2u_3^2$ and $u_5^2 \rightarrow x$. We have $x \rightarrow u_3^2$, since $d_H^+(x) \geq 2$. Now, we obtain $d_H^+(u_3^2) = 2$, a contradiction to the choice of w .

Therefore, $u_5^2 \rightarrow vv'$. Analogously, we see that $u_4^2 \rightarrow xv'$. Thus, $y = u_3^2$, and then $v' \rightarrow u_3^2$ and $u_3^2 \rightarrow v$. Moreover, $u_4^2 \rightarrow v$ and $v \rightarrow u_2^2$. Since u_3^2 is not a separating vertex of H , we have $d_H^+(u_3^2) \geq d_H^+(w) + 1 = 4$, and hence, $u_3^2 \rightarrow u_4^2u_5^2u_1^2x$, according to the choice of w . So, $x \rightarrow u_5^2$. As a result, $u_5^2 \rightarrow u_2^2$ and $d_H^+(u_5^2) \leq 2$, a contradiction to the choice of w .

Finally, we consider the case when $N^+(v') \cap V(T_1) = \emptyset$, i.e. $T_1 \rightarrow v'$. This implies that $N^+(v) \cap V(T_1) \neq \emptyset$. In addition, if $n_1 \geq 3$, then $|(N^+(v) \cup N^+(x)) \cap V(T_1)| \geq 2$.

Since $d^+(v') \geq 2$, it is not difficult to see that v' can be inserted into the path $T_1 \dots T_{t-1}u_2^t \dots u_{n_t}^t$. So, there is an $(n-1)$ -cycle through wx . We only need to find a Hamiltonian cycle containing wx .

Suppose to the contrary that there is no Hamiltonian cycle containing wx . If there is an out-neighbor of v' in some T_i , $2 \leq i \leq t-1$, then we deduce a contradiction, since $wxv'T_iu_2^t \dots u_k^t v T_1 \dots T_{i-1} T_{i+1} \dots T_{t-1} u_{k+1}^t \dots u_{n_t}^t w$ is a Hamiltonian cycle. So, $N^+(v') \subset V(T_t)$. We have $u_2^t \rightarrow v'$. Otherwise, $wxv'u_2^t \dots u_k^t v T_1 \dots T_{t-1} u_{k+1}^t \dots u_{n_t}^t w$ is a Hamiltonian cycle, a contradiction.

Since vv' is pancyclic, there is vertex z such that $v' \rightarrow z$ and $z \rightarrow v$, without loss of generality, say $z = u_k^t$. So, we have $u_k^t \dots u_{n_t}^t \rightarrow v, v \rightarrow u_{k-1}^t$ and $u_{k+1}^t \rightarrow v'$. Moreover, $u_{k+1}^t \dots u_{n_t}^t \rightarrow v'$.

If either $n_1 \geq 3$ or $n_1 = 1$ and $t \geq 3$, then it is not difficult to see that there is a Hamiltonian cycle through wx , a contradiction.

For the remaining case when $n_1 = 1$ and $t = 2$, we consider the digraph D obtained from T by contracting wxu_1^1 to y . It is clear that D is strong.

If $n_t \geq 4$, then we have

$$\begin{aligned}
d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 1 + d^-(w) - 1 \\
&= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\
&\geq n_t + 1 - 1 + n - 1 - \lfloor \frac{n_t-1}{2} \rfloor - 3 - 1 \\
&= n - 2 + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 3 \\
&\geq n - 2 = |V(D)|.
\end{aligned}$$

By Lemma 5.11, we see that wx is in a Hamiltonian cycle of T , a contradiction.

For the case when $n_t = 3$, we see that $v' \rightarrow u_2^t u_3^t$, since $d^+(v') \geq 2$. We deduce a contradiction to the assumption that $u_2^t \rightarrow v'$.

Subcase 2.1.3. $vv' \rightarrow x$.

Without loss of generality, let $x \rightarrow u_1^1$. Because all out-arcs of v and v' are pancyclic, we see that $N^-(v) \cap V(T_t - w) \neq \emptyset$, say $u_k^t \rightarrow v$ where $k \neq 1$. Especially, since vv' is pancyclic, there is a vertex z such that $v'zv$ is a 3-cycle.

If $z \in V(T_j)$ for some $2 \leq j \leq t-1$, say $z = u_1^j$, then we obtain a Hamiltonian cycle $wxT_1 \dots T_{j-1}u_2^t \dots u_k^t vv'T_j \dots T_{t-1}u_{k+1}^t \dots u_{n_t}^t w$ through wx . From this cycle, we can easily obtain a cycle of length $n-1$ through wx .

Assume that $z \in V(T_1)$. If $|(N^+(v') \cup N^+(x)) \cap V(T_1)| \geq 2$, say $v' \rightarrow u_i^1$, then $wxu_1^1 \dots u_{i-1}^1 u_2^t \dots u_k^t vv'u_i^1 \dots u_{n_1}^1 T_2 \dots T_{t-1}u_{k+1}^t \dots u_{n_t}^t w$ is a Hamiltonian cycle through wx . From this cycle, we can also easily obtain a cycle of length $n-1$ through wx .

If $|(N^+(v') \cup N^+(x)) \cap V(T_1)| = 1$, then $v' \rightarrow u_1^1, u_1^1 \rightarrow v$. If there is an out-neighbor of x in T_ℓ for some $2 \leq \ell \leq t-1$, say $x \rightarrow u_\ell^1$, then we obtain a Hamiltonian cycle $wxT_\ell \dots T_{t-1}u_2^t \dots u_k^t vv'T_1 \dots T_{\ell-1}u_{k+1}^t \dots u_{n_t}^t w$ through wx . From this cycle, a cycle of length $n-1$ through wx can be easily obtained.

So, assume that $(T_1 - \{u_1^1\})T_2 \dots T_{t-1} \rightarrow x$. If v' has an out-neighbor in $H - \{x, u_1^1\}$, then we see that the digraph D_y is strong, where D_y is obtained from T by contracting wxu_1^1 in y . If $n_1 \geq 3$ or $t \geq 3$ or $n_t \geq 4$, then we are done by Lemma 5.11. Thus, consider the case when $n_1 = 1$, $t = 2$ and $n_t = 3$. Since wu_2^t is pancyclic in H , it is easy to see that $u_2^t \rightarrow x$. This implies that $x \rightarrow u_3^t$, since $d_H^+(x) \geq 2$. Then $d_H^+(u_3^t) = 1$. This deduces a contradiction to the choice of w . Therefore, consider the case when v' has no out-neighbor in $H - \{x, u_1^1\}$. It follows that $d^+(v') = 2$ and $H - \{x, u_1^1\} \rightarrow v'$. Recall that $T_t - w$ contains no separating vertex of H .

Let y be an arbitrary out-neighbor of x in T_t and D be the digraph obtained from T by contracting wxy to z . It is clear that D is strong, since $T_t \rightarrow v', v' \rightarrow u_1^1$ and $u_1^1 \rightarrow v$. By Lemma 5.11, we only need to consider the case when $d^+(y) \leq d^+(w)$. Recall that $d_H^+(y) \geq d_H^+(w)$ and $d_H^+(y) + 1 \leq d^+(y) \leq d_H^+(y) + 2$. Since $d^+(x) < d^+(w)$, we see that $d^+(x) \leq d^+(w) - 1 = d_H^+(w) + 1 \leq d_H^+(y) + 1 \leq d^+(y)$. In addition, it is easy to see that $d^+(u_1^1) \geq n_t \geq d^+(w) \geq d^+(x)$. By Lemma 5.12, all out-arcs of x are pancyclic in T , and hence, x is a desired vertex.

Now, consider the remaining case when $z \in V(T_t)$ and $z \notin V(T_j)$, $j = 1, 2, \dots, t-1$. Let $z = u_\ell^t$ for some $2 \leq \ell \leq n_t$.

Let D be the digraph obtained from T by contracting wxu_1^1 to y . It is obvious that D is strong. Recall that $d_{T_t}^+(w) \leq \lfloor \frac{n_t-1}{2} \rfloor$.

If $d^+(u_1^1) \geq n_t + 2$, then $d_D^+(y) + d_D^-(y) = n - 2 + d^+(u_1^1) - d^+(w) - 1 \geq |V(D)| + n_t + 2 - \lfloor \frac{n_t-1}{2} \rfloor - 3 - 1 = |V(D)| + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 2 \geq |V(D)|$. By Lemma 5.11, we are done. For the case when $d^+(u_1^1) \leq n_t + 1$, it is clear that either $d^+(u_1^1) = n_t + 1$ or $d^+(u_1^1) = n_t$, since $d^+(u_1^1) \geq n_t$.

If $n_t \geq 6$, then $\lfloor \frac{n_t-1}{2} \rfloor + 4 \leq n_t$, and hence, $d_D^+(y) + d_D^-(y) = n - 2 + d^+(u_1^1) - d^+(w) - 1 \geq |V(D)| + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 3 - 1 = |V(D)| + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 4 \geq |V(D)|$. Then we are done by Lemma 5.11. So, assume that $n_t \leq 5$.

If $d^+(u_1^1) = n_t + 1$, then we only need to consider the case when $n_t = 3$. Otherwise, $d_D^+(y) + d_D^-(y) = n - 2 + d^+(u_1^1) - d^+(w) - 1 \geq n - 2 + n_t + 1 - \lfloor \frac{n_t-1}{2} \rfloor - 3 - 1 = n - 2 + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 3 \geq |V(D)|$, and we are done by Lemma 5.11. If $t = 2$, then either $T_2 \rightarrow x$ or $T_2 \not\rightarrow x$. If $T_2 \rightarrow x$, then z is a desired vertex, since $d_H^+(z) = d_H^+(w)$ and z is not a separating vertex of H . If $T_2 \not\rightarrow x$, then $u_2^t \rightarrow x$ and $x \rightarrow u_3^t$, and hence, $d_H^+(u_3^t) = 1$, a contradiction to the choice of w . If $t \geq 3$, then $t = 3$, $n_1 = n_2 = 1$, $vv' \rightarrow u_1^1$, $T_t \rightarrow x$ and $x \rightarrow u_2^1$. Then all out-arcs of each vertex of T_t are pancyclic in H . By Claim D, u_ℓ^t is a desired vertex.

If $d^+(u_1^1) = n_t$, then $vv' \rightarrow u_1^1$, $n_1 = 1$ and $t = 2$. Note that we can assume that $u_2^t \rightarrow x$, since $z \in V(T_2)$.

If $n_t = 3$, then $u_2^t \rightarrow x$ and $x \rightarrow u_3^t$. So, we have $d_H(u_3^t) = 1$, a contradiction to the choice of w .

For the case when $n_t = 4$, we have $d_{T_t}^+(w) \leq \lfloor \frac{n_t-1}{2} \rfloor = 1$. So, $u_3^t \rightarrow w$. If $u_2^t \rightarrow u_4^t$, then $u_4^t \rightarrow x$. Otherwise, $d_H^+(u_4^t) = 1$, a contradiction. Thus, $x \rightarrow u_3^t$. This implies that $d_H^+(u_3^t) = 2$. It is clear that u_3^t is not a separating vertex of H . Then $u_3^t \rightarrow vv'$, according to the choice of w . Now, we see that $wxu_3^t vv' u_1^1 u_2^t u_4^t w$ is a desired Hamiltonian cycle and $wxu_3^t v' u_1^1 u_2^t u_4^t w$ is a desired $(n-1)$ -cycle.

If $u_4^t \rightarrow u_2^t$, then we are done for $x \rightarrow u_4^t$, since $u_\ell^t \rightarrow v$. Thus, assume that $u_4^t \rightarrow x$. It is visible that $d^+(x) = 2$, a contradiction to the choice of v, v' .

If $n_t = 5$, then $d_{T_t}^+(w) \leq \lfloor \frac{n_t-1}{2} \rfloor = 2$. Since $d^+(u_1^1) = n_t = 5$, we have $d_{T_t}^+(w) = 2$. $d^+(x) \geq 4$ follows from $\min\{d^+(v), d^+(v')\} \geq 3$. Recalling that $u_2^t \rightarrow x$, we see that $x \rightarrow u_3^t u_4^t u_5^t$, and hence, $d^+(x) = 4$.

If $w \rightarrow u_3^t$, then $u_4^t \rightarrow w$ and $u_3^t \rightarrow u_5^t$. Moreover, $u_5^t \rightarrow u_2^t$, according to the choice of w . So, we see that $d_H^+(u_3^t) = d_H^+(u_5^t) = 2$ and all out-arcs of u_3^t and u_5^t are out-arc pancyclic in H , a contradiction to the choice of w .

If $w \rightarrow u_4^t$, then $u_3^t \rightarrow w$ and $u_4^t \rightarrow u_2^t$. We see that $d_H^+(u_4^t) = 2$ and u_4^t is pancyclic in H , a contradiction to the choice of w .

Therefore, if x is a separating vertex of H , then wx is 4-pancyclic in T or there is another vertex whose all out-arcs are 4-pancyclic in T .

Finally, we consider the remaining case when x is not a separating vertex of H with $d^+(x) = d_H^+(x) = d_H^+(w) + 1 = d^+(w) - 1 < d^+(w)$.

If each out-neighbor of x is not a separating vertex of H , then $d_H^+(y) \geq d_H^+(w)$ for all $y \in N^+(x)$. For the case when $d_H^+(y) = d_H^+(w)$, we have $d^+(y) = d_H^+(y) + 2 = d_H^+(w) + 2 = d^+(w)$. Otherwise, we see that $d^+(y) \leq d_H^+(y) + 1$ and all out-arcs of y are pancyclic in H , a contradiction to the choice of w . So, $d^+(y) > d^+(x)$. If $d_H^+(y) \geq d_H^+(w) + 1$, then $d^+(y) \geq d^+(x)$. By Lemma 5.12, we see that all out-arcs of x are pancyclic in T . By Claim B, x is a desired vertex, a contradiction to the choice of w .

So, assume that at least one of the out-neighbors of x is a separating vertex of H , say y . Let $T_1, T_2, \dots, T_t$ be the strong components of $H - y$ with $T_i \rightarrow T_j$ for $1 \leq i <$

$j \leq t$. According to the choice of w , we see that $w \in V(T_t)$. Thus, $x \in V(T_t)$ and $N^+(x) \setminus V(T_t) = \{y\}$. According to the choice of w and $w \rightarrow x$, there is no other out-neighbor of x as a separating vertex of H . With a similar discussion as above, we have $d^+(y') \geq d^+(x)$ for $y' \in N^+(x) \setminus \{y\}$. Thus, xy' is pancyclic in T .

It is not difficult to check that xy is pancyclic in H . Since $w \rightarrow vv'$ and $vv' \rightarrow x$, we see that xy is also pancyclic in T . This implies that x is a desired vertex.

Subcase 2.2. $x = v$.

If H contains a separating vertex s such that $v \rightarrow s$, then T contains a k -cycle through wv for $k = 4, 5, \dots, n-1$. If either $v' \rightarrow s$ or $s \rightarrow v'$, $N^+(v') \cap V(T_1) \neq \emptyset$, then we are done. So, consider the case when $s \rightarrow v'$ and $N^+(v') \cap V(T_1) = \emptyset$. Since $d^+(v') \geq 2$, v' can be inserted into the path $T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t$, and hence, T contains a Hamiltonian cycle through wv .

Now, assume that H contains no separating vertex dominated by v , i.e. every separating vertex of H dominates v . Suppose that there is a separating vertex s satisfying $v' \rightarrow s$. Then, we notice that wv is in a k -cycle for $k = 5, \dots, n$. If $s \rightarrow w$ or $N^+(v') \cap V(T_1) \neq \emptyset$, then wv is in 4-cycle. For the case when $w \rightarrow s$ and $N^+(v') \cap V(T_1) = \emptyset$, note that $N^+(v) \cap V(T_1) \neq \emptyset$, and hence, wv is pancyclic.

So, consider the case when H contains no separating vertex dominated by v or v' , i.e. every separating vertex of H dominates vv' and each out-neighbor of v, v' in H is not a separating vertex of H .

Let s be a separating vertex of H and $T_1, \dots, T_t$ be the strong components of $H - s$ with $T_i \rightarrow T_j$ for all $1 \leq i < j \leq t$. It is clear that $s \rightarrow vv'$.

Firstly, consider the case when $N^+(v') \cap V(T_1) \neq \emptyset$. Let $v' \rightarrow u_1^1$. It is easy to see that T has a k -cycle through wv for $k = 4, \dots, n-1$. If $s \rightarrow w$, then there is a Hamiltonian cycle through wv , since $d^-(s) \geq 2$. In what follows, we consider the case when $w \rightarrow s$.

Note that u_1^1 is not a separating vertex of H . If $n_1 \geq 3$, then $N^+(s) \cap V(T_1 - \{u_1^1\}) \neq \emptyset$. Or if $n_1 = 1$, $t \geq 3$, then $N^+(s) \cap V(T_2) \neq \emptyset$. Because $N^-(s) \cap V(T_t - \{w\}) \neq \emptyset$, it is not difficult to see that T contains a Hamiltonian cycle through wv for the case when either $n_1 \geq 3$ or $n_1 = 1$ and $t \geq 3$.

Assume that $n_1 = 1$ and $t = 2$. We consider the case when $v \rightarrow u_1^1$. Let D be the digraph obtained from T by contracting wvu_1^1 to y . It is clear that D is strong.

If $d^+(u_1^1) > d^+(w)$, then we are done by Lemma 5.11. So, suppose that $d^+(u_1^1) \leq d^+(w)$. Since u_1^1 is not a separating vertex, we see that $d_H^+(u_1^1) \geq d_H^+(w)$. According to the choice of w and $vv' \rightarrow u_1^1$, we only need to consider the case when $d_H^+(u_1^1) \geq d_H^+(w) + 1$. Indeed, if $d_H^+(u_1^1) = d_H^+(w)$, then by Claim B, u_1^1 is a desired vertex.

It is clear that $d^+(u_1^1) = n - 4$ and $d^+(w) = d_H^+(w) + 2 \leq \lfloor \frac{n-3}{2} \rfloor + 2$. For the case when $n \geq 10$, we see that $d_D^+(y) + d_D^-(y) = d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \geq n - 4 + n - 2 - \lfloor \frac{n-3}{2} \rfloor - 3 = n - 2 + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 4 \geq n - 2$. We are done by Lemma 5.11.

According to the choice of w and the fact that $d^+(w) \geq 4$ and u_1^1 is not a separating vertex, we only need to consider the case when $n \in \{8, 9\}$, i.e. $n_t \in \{4, 5\}$.

If $n_t = 4$, then $d^+(u_1^1) = d_H^+(u_1^1) = 4$ and $d_H^+(w) = 2$. Consequently, $u_3^t \rightarrow w$. If s can be inserted into $u_1^1 u_2^t u_3^t u_4^t$, then we are done. Assume that s cannot be inserted into $u_1^1 u_2^t u_3^t u_4^t$. From $d_H^+(s) \geq 2$, it follows that $u_4^t \rightarrow s$ and $s \rightarrow u_2^t$. Therefore, $wvv'u_1^1 u_4^t s u_2^t u_3^t w$ is a desired cycle.

For the case $n_t = 5$, one can assume that $u_5^t \rightarrow s$ and $s \rightarrow u_2^t$ to avoid that s can be inserted into $u_1^1 u_2^t u_3^t u_4^t u_5^t$. Since $d^+(u_1^1) = d_H^+(u_1^1) = 5$, we only need by Lemma 5.11 to consider the case when $d_H^+(w) = 3$.

Confirm that T_t contains no vertex z with $d_{T_t}^+(z) = 1$. Otherwise, it is not difficult to check that all out-arcs of z in H are pancyclic. This implies a contradiction to the choice of w .

So, $d_{T_t}^+(u_i^t) = 2$ for $i = 1, 2, 3, 4, 5$. It is easy to see that all out-arcs of each u_i^t are pancyclic in H . Since vv' is pancyclic in T , there is a vertex $z \in V(T_t)$ with $v' \rightarrow z$ and $z \rightarrow v$. By Claim D, z is a desired vertex.

Assume that $u_1^1 \rightarrow v$. By recalling that s cannot be inserted into the path $u_1^1 u_2^t \dots u_{n_t}^t$, we have $u_{n_t}^t \rightarrow s$, $s \rightarrow u_2^t$, $u_2^t \rightarrow vv'$, $w \rightarrow u_{n_t-1}^t$, $N^+(v) \cap V(T_t) \neq \emptyset$ and $N^+(v') \cap V(T_t) \neq \emptyset$.

Let D be the digraph obtained from T by contracting the path $wvv'u_1^1$ to z . It is clear that D is strong.

For $n \geq 8$, one notice that

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(u_1^1) - 2 + d^-(w) - 1 \\ &= d^+(u_1^1) - 2 + n - 1 - d^+(w) - 1 \\ &= n - 3 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq |V(D)| + n - 3 - \left(\lfloor \frac{n-3}{2} \rfloor + 2\right) - 1 \\ &= |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 3 \\ &\geq |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that z is pancyclic in D . Thus, T contains a Hamiltonian cycle through wv .

If $n = 7$, i.e. $n_t = 3$, then wu_2^t is not pancyclic in H , since $s \rightarrow u_2^t$. We deduce a contradiction to the choice of w .

Finally, consider the case when $N^+(v') \cap V(T_1) = \emptyset$. This implies that $N^+(v) \cap V(T_1) \neq \emptyset$, say $v \rightarrow u_1^1$.

It is clear that T contains a k -cycle through wv for $k = 3, 4, \dots, n-2$. Since $d^+(v') \geq 2$, the vertex v' can be inserted into the path $T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t$, and hence, there is an $(n-1)$ -cycle containing wv . It is sufficient to find a Hamiltonian cycle through wv .

Suppose to the contrary that T has no Hamiltonian cycle through wv . It is easy to see that $T_1 \dots T_{t-1} u_2^t \rightarrow v'$, i.e. $N^+(v') \subset V(T_t)$, and hence, $n_t \geq 4$. So, there is a vertex u_ℓ^t , $2 \leq \ell \leq n_t - 2$, such that $u_2^t \dots u_\ell^t \rightarrow v'$, $v' \rightarrow u_{\ell+1}^t$. Moreover, $s \rightarrow T_1 \dots T_{t-1} u_2^t \dots u_\ell^t$ and $u_2^t \rightarrow v$. This implies that there is an integer j , $\ell+1 \leq j \leq n_t$, such that $u_{j+1}^t \dots u_{n_t}^t w \rightarrow s$ and $s \rightarrow u_{\ell+1}^t \dots u_j^t$.

We only need to consider the case $t = 2$ and $n_1 = 1$. Let D be the digraph obtained from T by contracting wvu_1^1 to z . Clearly, D is strong. From $n \geq 8$, it follows that

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= d^+(u_1^1) - 1 + d^-(w) - 1 \\ &= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\ &= n - 2 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq |V(D)| + n - 3 - \left(\lfloor \frac{n-3}{2} \rfloor + 2\right) - 1 \\ &= |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 3 \\ &\geq |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that wv is pancyclic, a contradiction.

Subcase 2.3. $x = v'$.

Subcase 2.3.1. H contains a separating vertex s such that $v' \rightarrow s$.

We see that T contains a k -cycle through wv' for $k = 4, 5, \dots, n-1$. It is enough to show that T contains a Hamiltonian cycle through wv' . Suppose to the contrary that T contains no Hamiltonian cycle through wv' .

If $s \rightarrow v$, then we only need to consider the case when $T_1 \rightarrow v$. Since $d^+(v) \geq 2$, the vertex v can be inserted into the path $T_1 \dots T_{t-1}u_2^t \dots u_{n_t}^t$, and hence, there is a Hamiltonian cycle through wv' , a contradiction. So, $v \rightarrow s$.

Note $N^-(v) \cap V(T_t - \{w\}) \neq \emptyset$. It is not difficult to see that $|(N^+(v) \cup N^+(v') \cup N^+(s)) \cap V(T_1)| = 1$ (otherwise, we can obtain a Hamiltonian cycle through wv'). Since T is 2-strong, we see that $n_1 = 1$. Analogously, $t = 2$.

Firstly, assume that $v' \rightarrow u_1^1$. We consider the digraph D obtained from T by contracting $wv'u_1^1$ to y . Since $d_H^+(s) \geq 2$ and $N^-(v) \cap V(T_t - \{w\}) \neq \emptyset$, it is obvious that D is strong.

Assume that $u_1^1 \rightarrow v$. If $n \geq 8$, then

$$\begin{aligned} d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 1 + d^-(w) - 1 \\ &= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\ &= n - 2 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq |V(D)| + n - 3 - \left(\left\lfloor \frac{n-3}{2} \right\rfloor + 2\right) - 1 \\ &= |V(D)| + n - 3 - \left\lfloor \frac{n-3}{2} \right\rfloor - 3 \\ &\geq |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic, a contradiction. For the case when $n = 7$, i.e. $n_2 = 3$, there is a vertex $z \in V(H)$ with $d_H^+(z) = 1$, a contradiction to the choice of w .

So, $v \rightarrow u_1^1$. If $n \geq 10$, then

$$\begin{aligned} d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 1 + d^-(w) - 1 \\ &= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\ &= n - 2 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq |V(D)| + n - 4 - \left(\left\lfloor \frac{n-3}{2} \right\rfloor + 2\right) - 1 \\ &= |V(D)| + n - 3 - \left\lfloor \frac{n-3}{2} \right\rfloor - 4 \\ &\geq |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic, a contradiction.

So, we consider only the case when $n \leq 9$, i.e. $n_2 \in \{3, 4, 5\}$. If $T_2 \rightarrow v$, then it is easy to see that u_1^1 is a desired vertex. Thus, assume that $N^+(v) \cap V(T_2) \neq \emptyset$. Since v cannot be inserted into the path $u_2^2 \dots u_{n_2}^2$, we have $u_{n_2}^2 \rightarrow v$ and $v \rightarrow u_2^2$.

If $n_2 = 3$, then there is a vertex $z \in V(H)$ with $d_H^+(z) = 1$, a contradiction to the choice of w .

For $n_2 = 4$, we notice that $2 \leq d_H^+(w) \leq \left\lfloor \frac{5}{2} \right\rfloor = 2$, i.e. $d_H^+(w) = 2$. If $w \rightarrow u_3^2$, then $s \rightarrow w$. Clearly, $wv'u_1^1u_2^2u_3^2u_4^2vsw$ is a Hamiltonian cycle, a contradiction. For the case when $u_3^2 \rightarrow w$, we have a Hamiltonian cycle $wv'su_1^1u_4^2vu_2^2u_3^2w$, also a contradiction.

If $n_2 = 5$, then $2 \leq d_H^+(w) \leq \left\lfloor \frac{6}{2} \right\rfloor = 3$. By Lemma 5.11, we have $d_H^+(w) = 3$. So, $w \rightarrow u_3^2u_4^2$. Otherwise, we can obtain a Hamiltonian cycle: either $wv'su_1^1u_4^2u_5^2vu_2^2u_3^2w$ if

$u_3^2 \rightarrow w$ or $wv'su_1^1u_5^2vu_2^2u_3^2u_4^2w$ if $u_4^2 \rightarrow w$, a contradiction. This implies that $s \rightarrow w$. As a result, $wv'u_1^1u_2^2u_3^2u_4^2u_5^2vsw$ is a Hamiltonian cycle, a contradiction.

Secondly, assume that $u_1^1 \rightarrow v'$. This implies that $v \rightarrow u_1^1$. If $T_2 \rightarrow v$, then u_1^1 is a desired vertex, since vv' is in a 3-cycle. So, consider the case when $N^+(v) \cap V(T_2) \neq \emptyset$. Since v cannot be inserted into the path $u_2^2 \dots u_{n_2}^2$, we see that $u_{n_2}^2 \rightarrow v$ and $v \rightarrow u_2^2$.

Let D be the digraph obtained from T by contracting $wv'su_1^1$ to y . It is clear that D is strong.

If $w \rightarrow s$, then for $n \geq 8$ holds

$$\begin{aligned} d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 2 + d^-(w) - 1 \\ &= d^+(u_1^1) - 2 + n - 1 - d^+(w) - 1 \\ &= n - 3 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq |V(D)| + n - 3 - \left(\lfloor \frac{n-3}{2} \rfloor + 2\right) - 1 \\ &= |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 3 \\ &\geq |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic, a contradiction. For $n = 7$, i.e. $n_2 = 3$, there is a vertex with out-degree 1 in H , a contradiction to the choice of w .

If $s \rightarrow w$, then we have

$$\begin{aligned} d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 2 + d^-(w) - 2 \\ &= d^+(u_1^1) - 2 + n - 1 - d^+(w) - 2 \\ &= n - 3 + d^+(u_1^1) - d^+(w) - 2 \\ &\geq |V(D)| + n - 3 - \left(\lfloor \frac{n-3}{2} \rfloor + 2\right) - 2 \\ &= |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 4 \\ &\geq |V(D)| \end{aligned}$$

for $n \geq 10$. By Lemma 5.11, we see that wv' is pancyclic, a contradiction. So, we only need to consider the case $n \in \{8, 9\}$, i.e. $n_2 \in \{4, 5\}$.

If $n_2 = 4$, then it is plain that $2 \leq d_H^+(w) \leq \lfloor \frac{5}{2} \rfloor = 2$, i.e. $d_H^+(w) = 2$. It follows that $w \rightarrow u_3^2$ as well as $u_2^2 \rightarrow v's$. If $v' \rightarrow u_3^2$, then $wv'u_3^2u_4^2vu_1^1u_2^2sw$ is a Hamiltonian cycle, a contradiction. So, $u_3^2 \rightarrow v'$. This implies that $v' \rightarrow u_4^2$. If $u_3^2 \rightarrow s$, then $wv'u_4^2vu_1^1u_2^2u_3^2sw$ is a Hamiltonian cycle, a contradiction. So, $s \rightarrow u_3^2$. Therefore, $d_H^+(u_3^2) = 1$, a contradiction to the choice of w .

For $n_2 = 5$, one notice that $2 \leq d_H^+(w) \leq \lfloor \frac{6}{2} \rfloor = 3$. By Lemma 5.11, it follows that $d_H^+(w) = 3$. Similarly, we have $w \rightarrow u_3^2u_4^2$. It is easy to see that $u_2^2 \rightarrow v's$. If $v' \rightarrow u_3^2$, then $wv'u_3^2u_4^2u_5^2vu_1^1u_2^2sw$ is a Hamiltonian cycle, a contradiction. So, $u_3^2 \rightarrow v'$.

If $v' \rightarrow u_4^2$, then $s \rightarrow u_3^2$. Otherwise, $wv'u_4^2u_5^2vu_1^1u_2^2u_3^2sw$ is a Hamiltonian cycle, a contradiction. Thus, $u_3^2 \rightarrow u_5^2$, since $d_H^+(u_3^2) \geq 2$. This implies that $d_H^+(u_3^2) = 2$. Note that u_3^2 is not a separating vertex of H since $u_2^2 \rightarrow s$. We deduce a contradiction to the choice of w . So, $u_4^2 \rightarrow v'$ and $v' \rightarrow u_5^2$, since vv' is pancyclic. Analogously, we have $s \rightarrow u_4^2$ and $u_2^2 \rightarrow u_4^2$ to avoid the Hamiltonian cycle $wv'u_5^2vu_1^1u_2^2u_3^2u_4^2sw$ as well as $wv'u_3^2vu_1^1u_3^2u_4^2u_5^2sw$. It follows that $d_H^+(u_4^2) = 1$, a contradiction to the choice of w .

Subcase 2.3.2. H contains no separating vertex dominated by v' .

This implies that all separating vertices of H dominate v' and each out-neighbor of v' is not a separating vertex of H . We choose a separating vertex s of H with the labeled

strong components $T_1, \dots, T_t$ of $H - s$ such that $T_i \rightarrow T_j$ for all $1 \leq i < j \leq t$ and $T_t - w$ contains no separating vertex of H . It is clear that $d_H^+(y) \geq d_H^+(w)$ for all $y \in V(T_t)$.

Let $M = \{z \in V(H) \mid v' \rightarrow z, z \rightarrow v\}$. It is obvious that $s \notin M$. Since vv' is pancyclic, we see that $M \neq \emptyset$.

Subcase 2.3.2.1. $M \cap V(T_1) \neq \emptyset$, say $u_1^1 \in M$.

It is clear that wv' is in a k -cycle for $k = 3, \dots, n-2$. If $s \rightarrow w$ or $N^+(v) \setminus \{v', s\} \neq \emptyset$, then we see that s or v can be inserted into the path $T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t w$, and hence, wv' is in an $(n-1)$ -cycle. For the case $w \rightarrow s$ and $N^+(v) \setminus \{v', s\} = \emptyset$, it is easy to see that wv' is in an $(n-1)$ -cycle if $n_1 \geq 3$ or $t \geq 3$. So, consider the case when $n_1 = 1$ or $t = 2$. Let D be the digraph obtained from T by contracting $wv'u_1^1$ to z . It is clear that D is strong, $d^+(u_1^1) = n-3$ and $d^+(w) \leq \lfloor \frac{n-3}{2} \rfloor + 2$. It follows that $d_D^+(z) + d_D^-(z) \geq |V(D)|$ unless $n = 7$. By Lemma 5.11, we see that wv' is pancyclic in T for $n \geq 8$. If $n = 7$, then T_2 contains a vertex whose out-degree in H is 1, a contradiction to the choice of w .

It is enough to find a Hamiltonian cycle containing wv' . Assume to the contrary that T has no Hamiltonian cycle containing wv' .

At first, we consider the case when $v \rightarrow s$. If $n_1 \geq 3$, then $u_2^1 \rightarrow s$. For the case when $N_{T_1}^+(s) \setminus \{u_1^1\} \neq \emptyset$, it is easy to see that v, s can be inserted into the path $u_1^1 \dots u_{n_1}^1$, and hence, there is a Hamiltonian cycle through wv' , a contradiction. So, $N_{T_1}^+(s) = \{u_1^1\}$. For the case when $N_{T_1}^+(v) \neq \emptyset$, we perceive that v can be inserted into the path $u_1^1 \dots u_{n_1}^1$ and s can be inserted into the path $u_{n_1}^1 T_2 \dots T_{t-1} u_2^t \dots u_{n_t}^t$, and hence, there exists a Hamiltonian cycle through wv' , a contradiction. If $T_1 \rightarrow v$, then $N_{T_1}^+(v') \setminus \{u_1^1\} \neq \emptyset$, say $v \rightarrow u_i^1$ for some $i \neq 1$. Then $wv'u_i^1 \dots u_{n_1}^1 vsu_1^1 \dots u_{i-1}^1 T_2 \dots T_{t-1} u_2^t \dots u_{n_t}^t$ is also a Hamiltonian cycle, a contradiction.

If $n_1 = 1$ and $t \geq 3$, then analogously $T_2 \rightarrow s$. Consequently, either $N_{T_2}^+(v) \neq \emptyset$ or $N_{T_2}^+(v') \neq \emptyset$. For both cases, we can also obtain a Hamiltonian cycle through wv' , a contradiction. So, consider the case $n_1 = 1$ and $t = 2$. It is easy to see that $u_2^2 \rightarrow sv'$.

Let D be the digraph obtained from T by contracting $wv'u_1^1$ to y . It is clear that D is strong. If $n \geq 8$, then we have

$$\begin{aligned} d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 1 + d^-(w) - 1 \\ &= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\ &= n - 2 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq |V(D)| + n - 3 - \left(\lfloor \frac{n-3}{2} \rfloor + 2\right) - 1 \\ &= |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 3 \\ &\geq |V(D)|. \end{aligned}$$

By Lemma 5.11, the arc wv' is pancyclic, a contradiction. For the case when $n = 7$, i.e. $n_2 = 3$, there is a vertex with out-degree 1 in H , a contradiction to the choice of w .

At last, we consider the case when $s \rightarrow v$. Note that $N^-(s) \cap V(T_t) - \{w\} \neq \emptyset$, since all out-arcs of w is pancyclic in H . For the case when $N_{T_1}^+(v) \neq \emptyset$, it is easy to see that the arc sv can be inserted into the path $T_1 \dots T_{t-1} u_2^t \dots u_{n_t}^t$. So, there is a Hamiltonian cycle through wv' , a contradiction. So, $T_1 \rightarrow v$.

If $t \geq 3$, then similarly $T_2 \dots T_{t-1} \rightarrow v$. Therefore, there is an index j , $2 \leq j \leq n_t$, such that $u_2^t \dots u_{j-1}^t \rightarrow v$ and $v \rightarrow u_j^t$. To avoid that s can be inserted into the path

$T_1 \dots T_{t-1} u_2^t \dots u_{j-1}^t$, we have $s \rightarrow T_1 \dots T_{t-1} u_2^t \dots u_{j-1}^t$ if $j > 2$ and $t > 2$. Since $d^-(s) \geq 2$, there is an index k , $j \leq k \leq n_t$, such that $u_k^t \dots u_{n_t}^t \rightarrow s$, $s \rightarrow u_{j-1}^t \dots u_{k-1}^t$.

According to the discussion above, we only need to consider the case when $n_1 = 1$ and $t = 2$. Let D be the digraph obtained from T by contracting $wv'u_1^1$ to y . It is clear that D is strong. If $n \geq 8$, then we have

$$\begin{aligned}
 d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 1 + d^-(w) - 1 \\
 &= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\
 &= n - 2 + d^+(u_1^1) - d^+(w) - 1 \\
 &\geq |V(D)| + n - 3 - \left(\lfloor \frac{n-3}{2} \rfloor + 2\right) - 1 \\
 &= |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 3 \\
 &\geq |V(D)|.
 \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic, a contradiction. For the case when $n = 7$, i.e. $n_2 = 3$, H contains a vertex with positive degree 1 in H , a contradiction to the choice of w .

Subcase 2.3.2.2. $M \cap V(T_1) = \emptyset$ but $M \cap V(T_i) \neq \emptyset$ for some $2 \leq i \leq t-1$, say $u_1^i \in M$.

It is not difficult to see that $N^+(v) \cap V(T_1) \neq \emptyset$ (otherwise, we have $T_1 \rightarrow v$ and $N^+(v') \cap V(T_1) \neq \emptyset$, a contradiction to the assumption that $M \cap V(T_1) = \emptyset$). So, wv' is in a k -cycle for $k = 3, \dots, n-1$. Since $N^-(s) \cap V(T_t - w) \neq \emptyset$ and $N^-(v) \cap V(T_t - w) \neq \emptyset$ for $v \rightarrow s$, it is easy to deduce that wv' is in a Hamiltonian cycle. Therefore, wv' is pancyclic.

Subcase 2.3.2.3. $M \subset V(T_t)$, say $u_j^t \in M$ for some $2 \leq j \leq n_t$.

For an arbitrary $z \in M$, let D be the digraph obtained from T by contracting $wv'z$ to y . Since z is not a separating vertex of H and $z \rightarrow v$, we see that D is strong. By Lemma 5.11, we only need to consider the case when $d^+(z) \leq d^+(w)$. Because z is not a separating vertex of H , we see that $d_H^+(z) \geq d_H^+(w) + 1$, according to the choice of w . It follows that $d_H^+(z) = d_H^+(w) + 1$ and $d^+(z) = d^+(w)$.

We consider the case when $z \rightarrow w$. Obviously, $N^+(v) \cap V(T_1) \neq \emptyset$. Otherwise, $T_1 \rightarrow v$ and $N^+(v') \cap V(T_1) \neq \emptyset$, a contradiction to the assumption that $M \cap V(T_1) = \emptyset$.

At first, consider the case when there is an out-neighbor of v' in $H - V(T_t)$. If v' has an out-neighbor in T_i for some $2 \leq i \leq t-1$, then similarly to Case 2.3.2.2, wv' is pancyclic.

For the case when $N^+(v') \cap V(T_1) \neq \emptyset$ and $T_j \rightarrow v'$, $j = 2, \dots, t-1$, if $t \geq 3$, it is clear that wv' is in a k -cycle for $k = 3, \dots, n-2$. We only need to find an $(n-1)$ -cycle through wv' and a Hamiltonian cycle through wv' . Assume to the contrary that T contains no $(n-1)$ -cycle or no Hamiltonian cycle through wv' .

Firstly, assume that $v \rightarrow s$. Since $N^-(v) \cap V(T_t - w) \neq \emptyset$, we see that $|(N^+(v') \cup N^+(s)) \cap V(T_1)| = 1$, say $sv' \rightarrow u_1^1$.

If $n_1 \geq 3$, then $N^+(v) \cap V(T_1 - u_1^1) \neq \emptyset$, say $v \rightarrow u_k^1$. So, s can be inserted into the path $u_1^1 \dots u_{k-1}^1 u_2^t \dots u_j^t v u_k^1 \dots u_{n_1}^1 T_2 \dots T_{t-1} u_{j+1}^t \dots u_{n_t}^t w$, since $d_H^+(s) \geq 2$. It follows that T contains an $(n-1)$ -cycle and a Hamiltonian cycle through wv' , a contradiction.

So, $n_1 = 1$. Analogously, we see that $t = 2$. We consider the digraph D obtained from T by contracting $wv'u_1^1$ to y . It is clear that D is strong, since $d_H^+(s) \geq 2$.

If $n \geq 10$, then we have

$$\begin{aligned}
 d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 1 + d^-(w) - 1 \\
 &= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\
 &= n - 2 + d^+(u_1^1) - d^+(w) - 1 \\
 &\geq |V(D)| + n - 4 - \left(\lfloor \frac{n-3}{2} \rfloor + 2\right) - 1 \\
 &= |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 4 \\
 &\geq |V(D)|.
 \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic, a contradiction.

For the case when $n = 7$, i.e. $n_2 = 3$, there is a vertex with out-degree 1 in H , a contradiction to the choice of w . So, we only need to consider the case when $n \in \{8, 9\}$, namely, $n_2 \in \{4, 5\}$.

If $n_2 = 4$, then $2 \leq d_H^+(w) \leq \lfloor \frac{5}{2} \rfloor = 2$, i.e. $d_H^+(w) = 2$. This implies that either $u_3^2 \rightarrow w$ and $w \rightarrow s$ or $s \rightarrow w$ and $w \rightarrow u_3^2$.

For $u_3^2 \rightarrow w$ and $w \rightarrow s$, it is easy to see that $u_2^2 \rightarrow v'$ and $u_j^t \neq u_4^2$. So, we have $u_j^t = u_3^2$, and then $v' \rightarrow u_3^2$ and $u_3^2 \rightarrow v$. To avoid $wv'u_3^2vsu_1^1u_2^2u_4^2w$ and $wv'u_3^2vu_1^1u_2^2u_4^2w$, we deduce that $u_4^2 \rightarrow u_2^2$. At this point, it is evident that u_2^2 is not a separating vertex of H . Moreover, $u_2^2 \rightarrow vs$, according to the choice of w . Thus, we obtain $wv'u_3^2u_4^2u_2^2vsu_1^1w$ and $wv'u_3^2u_4^2u_2^2vu_1^1w$, a contradiction.

If $s \rightarrow w$ and $w \rightarrow u_3^2$, then $v \rightarrow u_4^2$. Otherwise, we have $wv'u_1^1u_2^2u_3^2u_4^2vsw$ and $wv'u_1^1u_2^2u_3^2vsw$, a contradiction. Moreover, we have $u_2^2 \rightarrow v'$. Otherwise, we obtain $wv'u_2^2 \dots u_j^2vsu_1^1u_{j+1}^2 \dots u_{n_2}^2w$ and $wv'u_2^2 \dots u_j^2vu_1^1u_{j+1}^2 \dots u_{n_2}^2w$, a contradiction. It is clear that $u_j^2 = u_3^2$ and $u_4^2 \rightarrow s$. Thus, we have $wv'u_1^1u_2^2u_3^2vu_4^2sw$ and $wv'u_1^1u_2^2u_3^2vu_4^2sw$, a contradiction.

For $n_2 = 5$, it is obvious that $2 \leq d_H^+(w) \leq \lfloor \frac{6}{2} \rfloor = 3$. By Lemma 5.11, we see that $d_H^+(w) = 3$. If $s \rightarrow w$, then $w \rightarrow u_3^2u_4^2$. It is easy to see that $u_2^2 \rightarrow v'$ and $v \rightarrow u_5^2$. So, v can be inserted into the path $u_j^2 \dots u_5^2$, and hence, we obtain an $(n-1)$ -cycle through wv' . Since $s \rightarrow w$, it is visible to obtain a Hamiltonian cycle from this cycle, a contradiction. So, $w \rightarrow s$. Then we have either $u_4^2 \rightarrow w$ and $w \rightarrow u_3^2$ or $u_3^2 \rightarrow w$ and $w \rightarrow u_4^2$.

If $u_4^2 \rightarrow w$ and $w \rightarrow u_3^2$, then $u_j^2 \neq u_5^2$. Otherwise, we obtain $wv'u_2^2vsu_1^1u_2^2u_3^2u_4^2w$ and $wv'u_2^2vsu_1^1u_2^2u_4^2w$, a contradiction. Moreover, $u_2^2 \rightarrow v'$. Otherwise, we have the cycles: $wv'u_2^2 \dots u_j^2vsu_1^1u_{j+1}^2 \dots u_{n_2}^2w$ and $wv'u_2^2 \dots u_j^2vu_1^1u_{j+1}^2 \dots u_{n_2}^2w$, a contradiction.

If $u_j^2 = u_4^2$, then $u_5^2 \rightarrow s$. In addition, $u_5^2 \rightarrow u_3^2$. Otherwise, we obtain the cycles $wv'u_4^2vsu_1^1u_2^2u_3^2u_5^2w$ and $wv'u_4^2vu_1^1u_2^2u_3^2u_5^2w$, a contradiction.

Since u_4^2 is not a separating vertex of H , we see that $d_H^+(u_4^2) \geq d_H^+(w) + 1 = 3 + 1 = 4$, and then $u_4^2 \rightarrow s$ and $u_4^2 \rightarrow u_2^2$. It is not difficult to see that u_2^2 is also not a separating vertex of H . According to the choice of w , we see that $d_H^+(u_2^2) \geq d_H^+(w) = 3$, and then $d_H^+(u_2^2) = 3$, $u_2^2 \rightarrow s$, $u_2^2 \rightarrow u_5^2$ and $u_2^2 \rightarrow v$. So, $s \rightarrow u_3^2$. Now, we obtain $wv'u_1^1u_2^2vsu_3^2u_4^2u_5^2w$ and $wv'u_1^1u_2^2su_3^2u_4^2u_5^2w$, a contradiction.

If $u_j^2 \neq u_4^2$, then $u_j^2 = u_3^2$ and $u_5^2 \rightarrow s$. Since u_3^2 is not a separating vertex of H , we see that $d_H^+(u_3^2) \geq d_H^+(w) + 1 = 3 + 1 = 4$. Thus, $d_H^+(u_3^2) \leq 3$, a contradiction.

Consider the case when $u_3^2 \rightarrow w$, $w \rightarrow u_4^2$. Analogously, $u_2^2 \rightarrow v'$ and $u_j^2 \neq u_4^2$. If $u_j^2 = u_3^2$, then $d_H^+(u_3^2) = 4$, $u_3^2 \rightarrow s$ and $u_3^2 \rightarrow u_5^2$. It is clear that u_4^2 is not a separating vertex of H with $d_H^+(u_4^2) \leq 3$. According to the choice of w , we have $d_H^+(u_4^2) = 3$ and $u_4^2 \rightarrow vv'$, and then $u_4^2 \rightarrow u_2^2$ and $u_4^2 \rightarrow s$. Notice that u_2^2 is also not a separating vertex of H , but $d_H^+(u_2^2) \leq 2$, a contradiction to the choice of w .

Finally, assume that $s \rightarrow v$. Since $N^-(s) \cap V(T_t - w) \neq \emptyset$, we see that $|(N^+(v) \cup N^+(v')) \cap V(T_1)| = 1$, say $vv' \rightarrow u_1^1$.

For the case $d^+(v) = 2$, it is evident that if $n_1 = 1$ and $t = 2$, then $T_2 \rightarrow v$. Since $v \rightarrow v', v' \rightarrow u_1^1$ and $s \rightarrow u_1^1$, it is not difficult to check that all out-arcs of u_1^1 are pancyclic in T . Thus, we consider the case either $n_1 \geq 3$ or $t \geq 3$.

Let G be the digraph obtained from T by contracting $wv'u_j^t v u_1^1$ to z . Since u_j^t is not a separating vertex of G , z is reachable from every vertex of G in G .

We confirm G is strong. Otherwise, $s \rightarrow H - \{w, u_j^t\}$, and hence, $u_j^t \rightarrow s$ and $w \rightarrow s$. Consider the digraph D obtained from T by contracting $wv'u_j^t s$ to y . It is clear that D is strong. Note that $n \geq 8$. It follows that $d_D^+(y) + d_D^-(y) \geq d^+(s) - 1 + d^-(w) - 1 = d^+(s) - 1 + n - 1 - d^+(w) - 1 = n - 3 + d^+(s) - d^+(w) \geq |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 2 \geq |V(D)|$. By Lemma 5.11, wv' is pancyclic, a contradiction.

If $s \rightarrow w$, then we have

$$\begin{aligned} d_G^+(z) + d_G^-(z) &\geq d^+(u_1^1) - 2 + d^-(w) - 2 \\ &= d^+(u_1^1) - 2 + n - 1 - d^+(w) - 2 \\ &= n - 4 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq |V(G)| + n_t + 1 - (n_t - 2 + 2) - 1 \\ &= |V(G)|. \end{aligned}$$

If $w \rightarrow s$, then we have

$$\begin{aligned} d_G^+(z) + d_G^-(z) &\geq d^+(u_1^1) - 2 + d^-(w) - 2 \\ &= d^+(u_1^1) - 2 + n - 1 - d^+(w) - 2 \\ &= n - 4 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq |V(G)| + n_t + 1 - \lfloor \frac{n_t-1}{2} \rfloor - 3 - 1 \\ &= |V(G)| + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 3 \\ &\geq |V(G)| \end{aligned}$$

for $n_t \geq 4$. By Lemma 5.11, we see that wv' is pancyclic for $n_t \geq 4$, a contradiction.

For the case $n_t = 3$, we have $d_H^+(u) = 2$ for all $u \in V(T_t)$. Otherwise, there is a $u \in V(T_t)$ with $d_H^+(u) = 1$, a contradiction to the assumption of w . Thus, every vertex $u \in V(T_t)$ is out-arc pancyclic in H . Since $v' \rightarrow u_j^t$ and $u_j^t \rightarrow v$, we see that u_j^t is a desired vertex.

Therefore, $d^+(v) \geq 3$. It is easy to see that $n_1 = 1$ and $t = 2$. Let D be the digraph obtained from T by contracting $wv'u_1^1$ to y . It is clear that D is strong.

If $n \geq 10$, then we have

$$\begin{aligned} d_D^+(y) + d_D^-(y) &= d^+(u_1^1) - 1 + d^-(w) - 1 \\ &= d^+(u_1^1) - 1 + n - 1 - d^+(w) - 1 \\ &= n - 2 + d^+(u_1^1) - d^+(w) - 1 \\ &\geq |V(D)| + n - 4 - \lfloor \frac{n-3}{2} \rfloor - 2 - 1 \\ &= |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 4 \\ &\geq |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic, a contradiction. Analogously, we only need to consider the case when $n \in \{8, 9\}$, namely, $n_t \in \{4, 5\}$.

If $n_2 = 4$, then $2 \leq d_H^+(w) \leq \lfloor \frac{5}{2} \rfloor = 2$ i.e. $d_H^+(w) = 2$. This implies that either $u_3^2 \rightarrow w$ and $w \rightarrow s$ or $s \rightarrow w$ and $w \rightarrow u_3^2$.

Assume that $u_3^2 \rightarrow w$ and $w \rightarrow s$. For the case when $u_4^2 \rightarrow u_2^2$, we see that $u_2^2 \rightarrow svv'$, according to the fact that u_2^2 is not a separating vertex of H and the choice of w . It follows that either $u_3^2 = u_j^2$ or $u_4^2 = u_j^2$. Consequently, we have either $wv'u_3^2u_4^2u_2^2svu_1^1w$ and $wv'u_3^2u_4^2u_2^2vu_1^1w$ or $wv'u_4^2u_2^2svu_1^1u_3^2w$ and $wv'u_4^2u_2^2vu_1^1u_3^2w$, a contradiction.

So, assume that $u_2^2 \rightarrow u_4^2$. It is clear that $u_4^2 \rightarrow svv'$, according to the choice of w and the fact that u_4^2 is not a separating vertex of H . Thus, $u_2^2 \rightarrow v'$. Otherwise, we obtain $wv'u_2^2u_3^2u_4^2su_1^1w$ and $wv'u_2^2u_3^2u_4^2svu_1^1w$, a contradiction. Since u_4^2 is not a separating vertex of H , we have $u_4^2 \rightarrow vv'$, according to the choice of w . Thus, $u_j^2 = u_3^2$, $u_3^2 \rightarrow s$ and $s \rightarrow u_2^2$. Now, one obtains $wv'u_1^1u_4^2svu_2^2u_3^2w$ and $wv'u_1^1u_4^2su_2^2u_3^2w$, a contradiction.

Therefore, $s \rightarrow w$ and $w \rightarrow u_3^2$. If there is an $(n-1)$ -cycle containing wv' in $T-s$, then T contains a Hamiltonian cycle through wv' , since $s \rightarrow w$. To avoid that v can be inserted into the path $u_2^2u_3^2u_4^2$, we deduce that $u_4^2 \rightarrow v$ and $v \rightarrow u_2^2$. Since u_2^2 is not a separating vertex of H , it is evident that $u_2^2 \rightarrow s$ and $u_2^2 \rightarrow u_4^2$, according to the choice of w . Analogously, $u_3^2 \rightarrow svv'$. So, $u_j^2 = u_4^2$, and then $v' \rightarrow u_4^2$. Thus, we have $wv'u_1^1u_4^2vu_2^2u_3^2sw$ and $wv'u_1^1u_4^2vu_2^2sw$, a contradiction.

If $n_2 = 5$, then $2 \leq d_H^+(w) \leq \lfloor \frac{6}{2} \rfloor = 3$. By Lemma 5.11, we have $d_H^+(w) = 3$. If $s \rightarrow w$, then $w \rightarrow u_3^2u_4^2$. It is clear that if there is an $(n-1)$ -cycle containing wv' in $T-s$, then T contains a Hamiltonian cycle through wv' , since $s \rightarrow w$. To avoid that v can be inserted into the path $u_2^2u_3^2u_4^2u_5^2$, we see that $u_5^2 \rightarrow v$ and $v \rightarrow u_2^2$. Since u_2^2 is not a separating vertex of H , we have $u_2^2 \rightarrow s$ and $u_2^2 \rightarrow u_4^2u_5^2$, according to the choice of w . Analogously, $u_3^2 \rightarrow svv'$ and $u_3^2 \rightarrow u_5^2$. It follows that $d_H^+(u_4^2) \leq 2$ and $d_H^+(u_5^2) \leq 2$, a contradiction to the choice of w . So, $w \rightarrow s$. Then we see that either $u_4^2 \rightarrow w$ and $w \rightarrow u_3^2$ or $u_3^2 \rightarrow w$ and $w \rightarrow u_4^2$.

Assume that $u_4^2 \rightarrow w$ and $w \rightarrow u_3^2$. If $v' \rightarrow u_5^2$, then $s \rightarrow u_5^2$. Otherwise, we see that $wv'u_5^2svu_1^1u_2^2u_3^2u_4^2w$ ($wv'u_5^2svu_1^1u_3^2u_4^2w$, respectively) is a Hamiltonian cycle ($(n-1)$ -cycle, respectively), a contradiction. Since u_5^2 is not a separating vertex of H , we have $d_H^+(u_5^2) \geq d_H^+(w) + 1 = 4$, according to the choice of w . From $su_1^1u_4^2 \rightarrow u_5^2$, it follows that $d_H^+(u_5^2) \leq 3$, a contradiction. So, $u_5^2 \rightarrow v'$.

Now, we see that $u_j^2 = u_4^2$, since $w \rightarrow u_2^2u_3^2$. Then $u_4^2 \rightarrow s$ and $u_4^2 \rightarrow u_2^2$. Since u_2^2 is not a separating vertex of H , we see that $d_H^+(u_2^2) = 3$, $u_2^2 \rightarrow svv'$ and $u_2^2 \rightarrow u_5^2$. Since u_5^2 is not a separating vertex of H , we see that $d_H^+(u_5^2) = 3$, $u_5^2 \rightarrow svv'$ and $u_5^2 \rightarrow u_3^2$. It is easy to see that $d_H^+(s) = 1$ or $d_H^+(u_3^2) = 1$, a contradiction to the assumption that $\delta^+(H) \geq 2$.

Consider the case when $u_3^2 \rightarrow w$ and $w \rightarrow u_4^2$. Since u_4^2 is not a separating vertex of H , we notice that $u_4^2 \rightarrow svv'$ and $u_4^2 \rightarrow u_2^2$. Analogously, $u_2^2 \rightarrow svv'$ and $u_2^2 \rightarrow u_5^2$. Moreover, $u_5^2 \rightarrow svv'$, $u_5^2 \rightarrow u_3^2$ and $u_3^2 \rightarrow svv'$. We deduce a contradiction to the existence of u_j^2 .

At last, we consider the case when $T_1 \dots T_{t-1} \rightarrow v'$. It implies that $N^+(v') \subset V(T_t)$ and $N^+(v) \cap V(T_1) \neq \emptyset$, say $v \rightarrow u_1^1$. Note that $z \rightarrow w$ for $z \in M$.

Firstly, assume that $|M| \geq 2$, say $z, z' \in M$. This implies that $n_t \geq 4$. Recall that $d^+(z) = d^+(z') = d^+(w)$ and $z \rightarrow w$, $z' \rightarrow w$. It is clear that T has 3-cycle and 4-cycle containing wv' . Let G (G' , respectively) be the digraph obtained from T by contracting $wv'zvu_1^1$ ($wv'z'vu_1^1$, respectively) to a (a' , respectively). We show that either G or G' is strong.

Since z is not a separating vertex of H , we see that a is reachable from all vertices in G unless $n_1 = 1$ and $N^+(s) \setminus \{v, v', u_1^1, z\} = \emptyset$. If $n_1 = 1$ and $N^+(s) \setminus \{v, v', u_1^1, z\} = \emptyset$, then

G' is strong. Thus, one can assume that a (a' , respectively) is reachable from all vertices in G (G' , respectively).

If $N^-(s) \setminus \{w, v', z, v\} \neq \emptyset$, then all vertices of G are reachable from a in G . This implies that G is strong. So, assume that $N^-(s) \setminus \{w, v', z, v\} = \emptyset$, i.e. $s \rightarrow T_t - \{w, z\}$. Since all out-arcs of w are pancyclic in H , we have $N^-(s) \cap V(T_t - w) \neq \emptyset$, and then $z \rightarrow s$. Now, we see that G' is strong. Without loss of generality, let G be strong.

If $s \rightarrow w$, then we have

$$\begin{aligned} d_G^+(a) + d_G^-(a) &= (d^+(u_1^1) - 3) + (d^-(w) - 2) \\ &= (d^+(u_1^1) - 3) + (n - 1 - d^+(w) - 2) \\ &= n - 4 + d^+(u_1^1) - d^+(w) - 2 \\ &= |V(G)| + d^+(u_1^1) - d^+(w) - 2 \\ &\geq |V(G)| + n_t + 1 - (n_t - 3 + 2) - 2 \\ &= |V(G)|. \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic in T .

If $w \rightarrow s$, then we have $d^+(w) \leq \lfloor \frac{n_t-1}{2} \rfloor + 3$. For $t \geq 4$, we see that $d^+(u_1^1) \geq n_t + 3$. It follows that

$$\begin{aligned} d_G^+(a) + d_G^-(a) &= (d^+(u_1^1) - 3) + (d^-(w) - 2) \\ &= (d^+(u_1^1) - 3) + (n - 1 - d^+(w) - 2) \\ &= n - 4 + d^+(u_1^1) - d^+(w) - 2 \\ &= |V(G)| + d^+(u_1^1) - d^+(w) - 2 \\ &\geq |V(G)| + n_t + 3 - \lfloor \frac{n_t-1}{2} \rfloor - 3 - 2 \\ &= |V(G)| + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 2 \\ &\geq |V(G)|. \end{aligned}$$

So, by Lemma 5.11, we see that wv' is pancyclic in T .

If $t = 3$, then analogously, we only need to consider the case when $n_2 = n_1 = 1$ and $n_3 = 3$. This allows us to deduce a contradiction to the assumption that $n_3 \geq 4$.

If $t = 2$, then we similarly only need to consider the case when $n_1 = 1$ and $n_2 \in \{4, 5\}$.

For $n_2 = 4$, note that $u_3^2 \rightarrow w$ and $d_H^+(w) = 2$. So, $M = \{u_3^2, u_4^2\}$. According to the choice of w , we have $u_3^2 u_4^2 \rightarrow s$ and $u_4^2 \rightarrow u_2^2$. It is easy to see that $\min\{d_H^+(s), d_H^+(u_2^2)\} \leq 1$, a contradiction to $\delta^+(H) \geq 2$.

For $n_2 = 5$, we only need to consider $d_H^+(w) = 3$. Then we have either $u_4^2 \rightarrow w$ and $w \rightarrow u_3^2$ or $u_3^2 \rightarrow w$ and $w \rightarrow u_4^2$. Moreover, $d_H^+(z) = d_H^+(z') = 4$. It follows that $z \rightarrow s$ and $z' \rightarrow s$.

If $u_4^2 \rightarrow w$ and $w \rightarrow u_3^2$, then $\{z, z'\} = \{u_4^2, u_5^2\}$, $u_4^2 u_5^2 \rightarrow s$. It is not difficult to see that wv' is pancyclic in T .

For the case when $u_3^2 \rightarrow w$ and $w \rightarrow u_4^2$, we analogously have $\{z, z'\} = \{u_3^2, u_5^2\}$, $u_3^2 \rightarrow s$ and $u_5^2 \rightarrow s$. Recalling that $d_H^+(u_3^2) = 4$, we have $N_H^+(u_3^2) = \{w, s, u_4^2, u_5^2\}$. This implies that $d_H^+(u_5^2) = 3$, a contradiction.

Finally, consider the case when $|M| = 1$, say $M = \{z\}$. Then $v' \rightarrow z, z \rightarrow v$ and $v \rightarrow z'$ for all $z' \in N^+(v') \setminus \{z\}$. Since z' is not a separating vertex, we see that $d_H^+(z') \geq d_H^+(w) + 1$, according to the choice of w and the fact that z' is not a separating vertex of H . Recall that $d^+(z) = d^+(w)$ and $z \rightarrow w$. This implies that wv' is in a 3-cycle.

If $z' \rightarrow w$, then we see that either $wv'z'zw$ or $wv'z'zw$ is a 4-cycle. If $w \rightarrow z'$, then one can also confirm that wv' is in a 4-cycle. Otherwise, w has at least $d_H^+(z')$ out-neighbors

in H , i.e. $d_H^+(w) \geq d_H^+(z')$. We deduce a contradiction to $d_H^+(z') \geq d_H^+(w) + 1$. Therefore, ww' is in a 4-cycle.

In the following, we show that either ww' is in k -cycle for $5 \leq k \leq n$ or there is another vertex whose out-arcs are 4-pancyclic.

(a) Consider the case when $s \rightarrow w$. Let D be the digraph obtained from T by contracting $ww'zv u_1^1$ to b .

(a.1) Assume that $N^-(s) \setminus \{w, v', z, v\} \neq \emptyset$. Then all vertices of D are reachable from b in D . Since z is not a separating vertex of H , we see that b is reachable from all vertices of D in D . So, D is strong.

If $t \geq 3$ or $n_1 \geq 3$, then we have $d^+(u_1^1) \geq n_t + 2$. It is clear that $d_H^+(w) \leq n_t - 2$. It follows that

$$\begin{aligned} d_D^+(b) + d_D^-(b) &= (d^+(u_1^1) - 3) + (d^-(w) - 2) \\ &= (d^+(u_1^1) - 3) + (n - 1 - d^+(w) - 2) \\ &= n - 4 + d^+(u_1^1) - d^+(w) - 2 \\ &= |V(D)| + d^+(u_1^1) - d^+(w) - 2 \\ &\geq |V(D)| + n_t + 2 - [(n_t - 2) + 2] - 2 \\ &= |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that ww' is pancyclic in T .

Thus, assume that $t = 2$ and $n_1 = 1$. We only need to consider the case when $n_1 = 1$ and $n \leq 9$, namely, $n_2 \in \{3, 4, 5\}$. Otherwise, for $n \geq 10$, we have

$$\begin{aligned} d_D^+(b) + d_D^-(b) &= (d^+(u_1^1) - 3) + (d^-(w) - 2) \\ &= |V(D)| + d^+(u_1^1) - d^+(w) - 2 \\ &\geq |V(D)| + n - 3 - \left(\lfloor \frac{n-3}{2} \rfloor + 2\right) - 2 \\ &= |V(D)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 4 \\ &\geq |V(D)|. \end{aligned}$$

By Lemma 5.11, we see that ww' is pancyclic in T .

For $n_3 = 3$, it is not difficult to see that there is a vertex $x \in V(H)$ with $d_H^+(x) = 1$, a contradiction to the choice of w .

For $n_3 = 4$, we see that $d_H^+(w) \leq \lfloor \frac{6-1}{2} \rfloor = 2$, and then $w \rightarrow u_3^2$. So, $z = u_5^2$, $u_5^2 \rightarrow s$ and $u_5^2 \rightarrow u_2^2$. It follows that $d_H^+(u_\ell^2) \leq 2$, $\ell = 2, 3$. This implies a contradiction to $d_H^+(z') \leq 3$.

For $n_3 = 5$, we see that $d_H^+(w) \leq \lfloor \frac{7-1}{2} \rfloor = 3$. By Lemma 5.11, one can assume that $d_H^+(w) = 3$. Then $w \rightarrow u_3^2 u_4^2$.

If $v' \rightarrow u_3^2$, then we deduce a contradiction to the choice of w , since $d_H^+(u_3^2) \leq 3 = d_H^+(w)$. So, $u_3^2 \rightarrow v'$. Analogously, $u_4^2 \rightarrow v'$. Thus, we see that $v' \rightarrow u_2^2$ and $v' \rightarrow u_5^2$. According to the choice of w and the fact that u_2^2 is not a separating vertex of H , we have $u_2^2 \rightarrow s$ and $u_2^2 \rightarrow u_4^2 u_5^2$. Now, we see that $d_H^+(u_5^2) \leq 3$. By Claim B and D, the vertex u_5^2 is a desired vertex.

(a.2) Assume that $N^-(s) \setminus \{w, v', z, v\} = \emptyset$. Then $N^-(s) = \{z, v\}$ and $d^+(s) = n - 3$. Let F be the digraph obtained from T by contracting $ww'zs$ to c . Since z is not a separating vertex of H , it is not difficult to see that c is reachable from all vertices of F in F .

If $N^-(v) \setminus \{w, z\} \neq \emptyset$, then we see that F is strong. If $n \geq 10$, then we have

$$\begin{aligned}
 d_F^+(c) + d_F^-(c) &= (d^+(s) - 2) + (d^-(w) - 2) \\
 &= (d^+(s) - 2) + (n - 1 - d^+(w) - 2) \\
 &= n - 3 + d^+(s) - d^+(w) - 2 \\
 &= |V(D)| + d^+(s) - d^+(w) - 2 \\
 &\geq |V(D)| + n - 3 - \left(\left\lfloor \frac{n-3}{2} \right\rfloor + 2\right) - 2 \\
 &= |V(D)| + n - 3 - \left\lfloor \frac{n-3}{2} \right\rfloor - 4 \\
 &= |V(D)|.
 \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic in T .

According to the assumption that $d_H^+(x) \geq 2$ for all $x \in V(H)$ and $s \rightarrow w$, we only need to consider the case when $n \in \{8, 9\}$ and $n_t \geq 4$. This implies that $t = 2$ and $n_1 = 1$.

If $n = 8$, then we have $n_2 = 4$ and $w \rightarrow u_3^2$. So, $z = u_4^2$. Now, we see that $u_4^2 \rightarrow s$ and $u_4^2 \rightarrow u_2^2$. Moreover, $d_H^+(u_\ell^2) \leq 2$, $\ell = 2, 3$, a contradiction to $d_H^+(z') \geq d_H^+(w) + 1 = 3$.

If $n = 9$, then we have $n_2 = 5$ and $d_H^+(w) \leq \left\lfloor \frac{7-1}{2} \right\rfloor = 3$. If $d_H^+(w) = 2$, then with a similar discussion as above, we have

$$d_F^+(c) + d_F^-(c) = (d^+(s) - 2) + (d^-(w) - 2) = (6 - 2) + (8 - 4 - 2) = 6 = |V(F)|.$$

By Lemma 5.11, we see that wv' is pancyclic in T . So, assume that $d_H^+(w) = 3$. Then we have $w \rightarrow u_3^2 u_4^2$ and $z = u_5^2$. Now, we see that there is at least a vertex x in T_2 with $d_H^+(x) \leq 2$, a contradiction to the choice of w .

(b) Consider the case when $w \rightarrow s$. We see that

$$d_{T_t}^+(w) \leq \left\lfloor \frac{n_t-1}{2} \right\rfloor \quad \text{and} \quad d^+(w) \leq \left\lfloor \frac{n_t-1}{2} \right\rfloor + 3.$$

Let D be the digraph obtained from T by contracting $wv'zvu_1^1$ to b .

(b.1) Suppose that $N^-(s) \setminus \{w, v', z, v\} \neq \emptyset$. Then all vertices of E are reachable from b in D .

(b.1.1) Suppose that D is strong. If $t \geq 4$, then we have $d^+(u_1^1) \geq n_t + 3$. It follows that

$$\begin{aligned}
 d_D^+(b) + d_D^-(b) &= (d^+(u_1^1) - 3) + (d^-(w) - 2) \\
 &= (d^+(u_1^1) - 3) + (n - 1 - d^+(w) - 2) \\
 &= n - 4 + d^+(u_1^1) - d^+(w) - 2 \\
 &= |V(D)| + d^+(u_1^1) - d^+(w) - 2 \\
 &\geq |V(D)| + n_t + 3 - \left\lfloor \frac{n_t-1}{2} \right\rfloor - 3 - 2 \\
 &= |V(D)| + n_t - \left\lfloor \frac{n_t-1}{2} \right\rfloor - 2 \\
 &\geq |V(D)|.
 \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic in T .

For $t = 3$, we only need to consider the case when $n_1 = 1$, $n_2 = 1$ and $n_3 = 3$. In fact, we have $d^+(u_1^1) \geq n_t + 3$ for $n_1 \geq 3$ or $n_1 = 1$, $n_2 \geq 3$. Analogously, we see that $d_D^+(b) + d_D^-(b) \geq |V(D)|$. By Lemma 5.11, we are done.

If $n_1 = 1$, $n_2 = 1$ and $n_3 \geq 4$, then we have

$$\begin{aligned}
d_D^+(b) + d_D^-(b) &= (d^+(u_1^1) - 3) + (d^-(w) - 2) \\
&= (d^+(u_1^1) - 3) + (n - 1 - d^+(w) - 2) \\
&= n - 4 + d^+(u_1^1) - d^+(w) - 2 \\
&= |V(D)| + d^+(u_1^1) - d^+(w) - 2 \\
&\geq |V(D)| + n_t + 2 - \lfloor \frac{n_t-1}{2} \rfloor - 3 - 2 \\
&= |V(D)| + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 3 \\
&\geq |V(D)|.
\end{aligned}$$

By Lemma 5.11, we are also done.

For $n_3 = 3$, we see that $T_3 \rightarrow s$ and every vertex of T_3 is out-arc pancyclic in H . Then z' is a desired vertex by Claim B.

Now, consider the case when $t = 2$. If $n_1 \geq 3$, then we only need to consider the case when $d_{T_1}^+(u_1^1) = 1$, $s \rightarrow u_1^1$ and $n_2 = 3$. We see that $T_2 \rightarrow s$ and every vertex of T_2 is out-arc pancyclic in H . So, z' is a desired vertex by Claim B.

If $n_1 = 1$, then we only need to consider the case when $n_2 \leq 5$. Otherwise, $n_2 \geq 6$. It follows that

$$\begin{aligned}
d_D^+(b) + d_D^-(b) &= (d^+(u_1^1) - 3) + (d^-(w) - 2) \\
&= (d^+(u_1^1) - 3) + (n - 1 - d^+(w) - 2) \\
&= n - 4 + d^+(u_1^1) - d^+(w) - 2 \\
&= |V(D)| + d^+(u_1^1) - d^+(w) - 2 \\
&\geq |V(D)| + n_t + 1 - \lfloor \frac{n_t-1}{2} \rfloor - 3 - 2 \\
&= |V(D)| + n_t - \lfloor \frac{n_t-1}{2} \rfloor - 4 \\
&\geq |V(D)|.
\end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic in T .

Assume that $n_2 = 5$. We have $d_{T_2}^+(w) \leq \lfloor \frac{5-1}{2} \rfloor = 2$. If $d_{T_2}^+(w) = 1$, then we have

$$\begin{aligned}
d_D^+(b) + d_D^-(b) &= |V(D)| + d^+(u_1^1) - d^+(w) - 2 \\
&= |V(D)| + 6 - 4 - 2 \\
&= |V(D)|.
\end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic in T .

For the case when $d_{T_2}^+(w) \geq 2$, we see by Remark 5.7 that $d_{T_2}^+(x) \geq 2$ for all $x \in V(T_2)$, and then $d_{T_2}^+(x) = 2$ for all $x \in V(T_2)$. Moreover, x is out-arc pancyclic in H , according to the choice of s and w . By Claim B, the vertex z' is a desired vertex.

For $n_2 = 3$, we see that H contains a vertex with out-degree 1 in H , a contradiction to the choice of w .

If $n_2 = 4$, then $d_{T_2}^+(w) \leq \lfloor \frac{4-1}{2} \rfloor = 1$. Thus, $u_3^2 \rightarrow w$. If $v' \rightarrow u_2^2$, then $d_H^+(u_2^2) \geq d_H^+(w) + 1 = 3$. This implies that $u_2^2 \rightarrow s$ and $u_2^2 \rightarrow u_4^2$. Now, we see that $d_H^+(u_4^2) = 2$, and then $u_4^2 \rightarrow s$ and $u_4^2 \rightarrow vv'$. Thus, $v' \rightarrow u_3^2$. Analogously, $d_H^+(u_3^2) \geq d_H^+(w) + 1 = 3$ and $u_3^2 \rightarrow s$. It follows that $d_H^+(s) = 1$, a contradiction to the assumption that $d_H^+(x) \geq 2$ for $x \in V(H)$.

For the case when $u_2^2 \rightarrow v'$ and $\{z, z'\} = \{u_3^2, u_4^2\}$, it is analogous to see that $u_3^2 u_4^2 \rightarrow s$ and $u_4^2 \rightarrow u_2^2$. Moreover, $u_2^2 \rightarrow s$ and $u_2^2 \rightarrow vv'$. So, $d_H^+(s) = 1$, a contradiction to the assumption that $d_H^+(x) \geq 2$ for $x \in V(H)$.

(b.1.2) Assume that D is not strong. Let $F = \{x \in V(D) \mid b \text{ is reachable from } x \text{ in } D\}$. It is clear that $F \neq \emptyset$, $\overline{F} = V(D) \setminus F = V(T) \setminus (F \cup \{v, v', z, u_1^1\}) \neq \emptyset$ and $\{w\} \cup F \rightarrow \overline{F}$.

We confirm that $s \in \overline{F}$. Assume to the contrary that $s \in F$. It is easy to see that $(T_1 - u_1^1)T_2 \dots T_{t-1} \rightarrow s$ and $\overline{F} \setminus s \subset V(T_t)$. Then $H - z$ is not strong, a contradiction to the assumption that each out-neighbors of v' is not a separating vertex of H .

If $t \geq 3$ or $t = 2$ and $n_1 \geq 3$, then there is another out-neighbor of v in $H - V(T_t)$, say u . We consider the digraph D' obtained from T by contracting $wv'zvu$ to b' .

With a similar discussion as above, we only need to consider the case when $t = 2$ and $n_1 = 1$. Let $u_i^2 = z$. It is clear that $z \neq u_2^2$, $u_j^2 \in F$ and $u_j^2 \rightarrow s$ for $j = i + 1, \dots, n_2$. Moreover, if $\overline{F} \setminus \{u_2^2, \dots, u_{i-1}^2\} \neq \emptyset$, then there is an index $2 \leq k \leq i - 1$ such that $\overline{F} = \{u_k^2, \dots, u_{i-1}^2, s\}$ and $\{u_2^2, \dots, u_{k-1}^2\} \subset F$.

We confirm that $z' \in F$. Assume to the contrary that $z' \in \overline{F}$. So, we see that $N^+(z') \subset (\overline{F} \cup \{z\})$, and then $|\overline{F}| \geq d^+(z')$. It follows that $d_H^+(w) \geq |\overline{F}| \geq d^+(z')$. Recalling that $d_H^+(z') \geq d_H^+(w) + 1$ and $d_H^+(z') = d^+(z')$, we deduce a contradiction.

By $z' \in F$, we have $z' \rightarrow s$. Let D'' be the digraph obtained from T by contracting $wv'z'su_1^1$ to c .

(b.1.2.1) Assume that $N^+(v) \setminus \{w, v', z', s, u_1^1\} \neq \emptyset$. It is not difficult to see that D'' is strong. By Lemma 5.11, we only need to consider the case when $n_2 \in \{3, 4, 5\}$.

For $n_2 = 3$, one can see that H contains a vertex with out-degree 1 in H , a contradiction to the choice of w .

For $n_2 = 4$, we have that $u_3^2 \rightarrow w$. So, $z = u_3^2$, $z' = u_4^2$, $u_3^2 u_4^2 \rightarrow s$ and $u_4^2 \rightarrow u_2^2$, according to the choice of w . Moreover, $u_2^2 \rightarrow s$ and $u_2^2 \rightarrow vv'$. Now, we have $d_H^+(s) = 1$, a contradiction.

For $n_2 = 5$, we see that $d_{T_2}^+(w) \leq 2$. For $d_{T_2}^+(w) = 2$, we see by Remark 5.7 that $d_{T_2}^+(x) \geq 2$ for all $x \in V(T_2)$, and hence, $d_{T_2}^+(x) = 2$. This implies that $d_H^+(x) = 3$ for $x \in V(T_2)$, according to the assumption that T_2 contains no separating vertex of H . So, all vertices of T_2 are out-arc pancyclic in H . By Claim B, z' is a desired vertex.

Assume that $d_{T_2}^+(w) = 1$, i.e. $d^+(w) = 4$. It follows that

$$\begin{aligned} d_{D''}^+(c) + d_{D''}^-(c) &= (d^+(u_1^1) - 3) + (d^-(w) - 2) \\ &= (d^+(u_1^1) - 3) + (n - 1 - d^+(w) - 2) \\ &= n - 4 + d^+(u_1^1) - d^+(w) - 2 \\ &= |V(D'')| + d^+(u_1^1) - d^+(w) - 2 \\ &\geq |V(D'')| + 6 - 4 - 2 \\ &= |V(D'')|. \end{aligned}$$

By Lemma 5.11, we are done.

(b.1.2.2) Assume that $N^+(v) \setminus \{w, v', z', s, u_1^1\} = \emptyset$. Then we have $T - \{v', z', s, u_1^1\} \rightarrow v$. This implies that $d^+(v') = 2$ and $T - \{z, z'\} \rightarrow v'$. Now, it is not difficult to see that all out-arcs of u_1^1 are 4-pancyclic in T .

(b.2) Suppose that $N^-(s) \setminus \{w, v', z, v\} = \emptyset$. That means $s \rightarrow T_t - \{w, z\}$. Since w is out-arc pancyclic in H , we have $N^-(s) \cap V(T_t - w) \neq \emptyset$. Thus, $z \rightarrow s$. Let D''' be the digraph obtained from T by contracting $wv'zs$ to a .

Since z is not a separating vertex of H , we see that a is reachable from all vertices of D''' in D''' .

If $N^-(v) \setminus \{w, z\} \neq \emptyset$, then all vertices are reachable from a in D''' . Thus, D''' is strong. Furthermore, we have

$$\begin{aligned}
 d_{D'''}^+(a) + d_{D'''}^-(a) &= (d^+(s) - 1) + (d^-(w) - 1) \\
 &= (d^+(s) - 1) + (n - 1 - d^+(w) - 1) \\
 &= n - 3 + d^+(s) - d^+(w) \\
 &= |V(D''')| + d^+(s) - d^+(w) \\
 &\geq |V(D''')| + n - 4 - \lfloor \frac{n-3}{2} \rfloor - 2 \\
 &= |V(D''')| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 3 \\
 &\geq |V(D''')|
 \end{aligned}$$

for $n \geq 8$. By Lemma 5.11, we see that wv' is pancyclic in T .

If $n = 7$, then $v' \rightarrow u_2^2 u_3^2$, $u_3^2 \rightarrow v$, and $v \rightarrow u_2^2$. In addition, $u_3^2 \rightarrow s$ and $s \rightarrow u_2^2$. Now, we see that z is a separating vertex of H , a contradiction.

If $N^-(v) = \{w, z\}$, then we consider the digraph D^4 obtained from T by contracting $wv'zv$ to a' . It is evident that D^4 is strong. Moreover, we have

$$\begin{aligned}
 d_{D^4}^+(a') + d_{D^4}^-(a') &= (d^+(v) - 1) + (d^-(w) - 1) \\
 &= (d^+(v) - 1) + (n - 1 - d^+(w) - 1) \\
 &= n - 3 + d^+(v) - d^+(w) \\
 &= |V(D^4)| + d^+(v) - d^+(w) \\
 &\geq |V(D^4)| + n - 3 - \lfloor \frac{n-3}{2} \rfloor - 2 \\
 &\geq |V(D^4)|.
 \end{aligned}$$

By Lemma 5.11, we see that wv' is pancyclic in T .

Case 3. $\sigma(H) \geq 2$.

In what follows, denote $v_1 \rightarrow v_2$ for the case when $d^+(v_1) = d^+(v_2)$. Recall that $d^+(v_1) \geq 2$. Let u be the vertex, which has minimum out-degree among all vertices in $N^+(v_1) \setminus \{v_2\}$. By (6.4), it is clear that $d^+(u) > d^+(v_1)$.

We will prove that either all out-arcs of u are 4-pancyclic or there is another desired vertex.

Let $x \in N^+(u)$ be arbitrary. If $d^+(x) \geq d^+(u)$, then we are done by Lemma 5.12. So, assume that $d^+(x) < d^+(u)$. From (6.4), it follows that $d^+(v_1) \leq d^+(x) < d^+(u)$.

Let D be the digraph obtained by contracting $v_1 ux$ to w . Assume that D is strong. If $v_1 \rightarrow x$, then we have

$$\begin{aligned}
 d_D^+(w) + d_D^-(w) &= |N^+(x)| + |N^-(v_1)| \\
 &= d^+(x) + d^-(v_1) \\
 &= d^+(x) + [(n - 1) - d^+(v_1)] \\
 &\geq n - 1 > |V(D)|.
 \end{aligned}$$

By Lemma 5.11, there is a k -cycle containing w in D for all $2 \leq k \leq |V(D)|$, and then the arc ux is 4-pancyclic.

If $x \rightarrow v_1$, then $d^+(x) > d^+(v_1)$. Otherwise, we see that $d^+(x) = d^+(v_1)$ and $x = v_2$, a contradiction to the assumption that $v_1 \rightarrow v_2$ if $d^+(v_1) = d^+(v_2)$. Thus, we have

$$\begin{aligned} d_D^+(w) + d_D^-(w) &= |N^+(x) \setminus \{v_1\}| + |N^-(v_1) \setminus \{x\}| \\ &= (d^+(x) - 1) + (d^-(v_1) - 1) \\ &= (d^+(x) - 1) + [(n - 1) - d^+(v_1)] - 1 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

Analogously, the arc ux is 4-pancyclic in T .

Now, consider the case when D is not strong, i.e. there is a vertex $y \in V(D - \{w\})$ such that either y is not reachable from w or w is not reachable from y in D . In other words, either y is not reachable from x in $T - \{v_1, u\}$ or v_1 is not reachable from y in $T - \{u, x\}$.

Assume that y is not reachable from x in $T - \{v_1, u\}$. It follows that $\sigma(T - \{v_1\}) = 1$. This implies that there is a vertex $s \in V(T)$ such that $\sigma(T - \{v_1, s\}) = 0$. Let $T_1, \dots, T_t$ be the components of $T - \{v_1, s\}$ with $T_i \rightarrow T_j$ for $1 \leq i < j \leq t$. By recalling that $\sigma(T - \{v_1, v_2\}) \geq 2$, we have $s \neq v_2$. Since $\sigma(T - \{v_1, s\}) = 0$, it is easy to see that $n_1 = 1$ if $v_2 \in V(T_1)$ or $n_t = 1$ if $v_2 \in V(T_t)$, and then $t = 2$.

If $V(T_2) = \{v_2\}$, then $d^+(v_2) = 2$ and $v_2 \rightarrow v_1$, a contradiction to the assumption that $v_1 \rightarrow v_2$ if $d^+(v_1) = d^+(v_2)$. So, $V(T_1) = \{v_2\}$.

Let $w \in V(T_2)$ be an out-arc pancyclic vertex in T_2 with $d_{T_2}^+(w) \leq \lfloor \frac{n_2-1}{2} \rfloor$. If w is an out-arc pancyclic vertex in T , then we are done. So, assume that not all out-arcs of w is pancyclic in T .

By recalling the proof for Case 1.2 in Theorem 6.3, we see that $T_2 \rightarrow v_1$, $w \rightarrow s$, $N^+(s) \subset N^-(w)$ and $N^+(w) \subset N^-(s)$. This implies that $d^+(v_1) = 2$, $d^+(s) \geq 3$ and $u = s$.

Let $wz_1z_2 \dots z_{n-4}w$ be an arbitrary Hamiltonian cycle of T_2 . For each z_i with $s \rightarrow z_i$ for some $1 \leq i \leq n-4$, we have $i \geq 2$ and $w \rightarrow z_{i-1}$, and hence, $z_{i-1} \rightarrow s$. It is not difficult to check that sz_i is pancyclic in T . Evidently, sv_2 is also pancyclic in T . So, s is a desired vertex.

Therefore, we can assume that $T - \{v_1, u\}$ contains a path from x to every vertex but $T - \{u, x\}$ contains no path from some vertex y to v_1 . Let

$$Z = \{z \in V(T - \{u, x, y\}) \mid T - \{u, x\} \text{ contains a path from } y \text{ to } z\}.$$

It is clear that $v_1 \rightarrow y$, and $v_1 \rightarrow Z$. It follows that $N^+(y) \subseteq Z \cup \{u, x\}$ and $Z \cup \{u, y\} \subseteq N^+(v_1)$. This implies $d^+(y) \leq |Z| + 2 \leq d^+(v_1)$, and hence, $d^+(y) = d^+(v_1)$. Since $d^+(a) > d^+(v_1)$ for all $a \in V(T) \setminus \{v_1, v_2\}$, we have $y = v_2$, $N^+(v_2) = Z \cup \{u, x\}$ and $N^+(v_1) = Z \cup \{v_2, u\}$.

It is easy to see that $T - (Z \cup \{u, x\}) \rightarrow Z$. So, the set $\{u, x\}$ is a separating set of T . Let $T_1, T_2, \dots, T_t$ be the strong components of $T - \{u, x\}$ with $T_i \rightarrow T_j$ for $1 \leq i < j \leq t$. Then we see that $t \geq 3$ and $T_i = v_1$, $T_{i+1} = v_2$ for some $1 \leq i \leq t-2$. Thus, it is not difficult to see that ux is pancyclic in T . ■

By Theorem 6.8 and 6.9, we have the following theorem:

Theorem 6.10 *Every s -strong tournament T with $s \geq 2$ contains at least $s + 1$ vertices whose all out-arcs are 4-pancyclic.*

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Index

(x, y) -path	57	$N_H(u)$	3
1-heavy	12	$N_k(H)$	3
2-factor	41	$N_k(v)$	4
2-heavy	12	$N_{i,j,k}$	4
$2C_5$ -configurations	11	P_n , path	4
$A \rightarrow B$	57	P_k , path	4
B , bull	4	$V_{LC}(G)$	3
B_x	7	$V_{LD}(G)$	3
C -bypass	14	$V_{SI}(G)$	3
$C_k(G)$, k -(degree) closure	5	Z_1	4
C_n , cycle	4	$\Delta(G)$	3
C_n -configuration	11	Δ_{mon}	29
$D - U$	57	$\alpha(G)$	3
$D[U]$	57	χ -edge-colored	29
F_8	51	$\delta(G)$	3
$G + H$	4	$\delta^+(D)$	57
G'_x , local completion	7	$\delta^-(D)$	57
$G - E'$	3	$\kappa(G)$	3
$G - V'$	3	$\omega(G)$	3
$G[E']$	3	$\sigma(D)$	57
$G[V']$	3	$\vec{C}$	14
$G_2(x)$	6	$\{F_1, F_2, \dots, F_k\}$ -free	4
H , hourglass	4	$c(G)$, circumference	7
$I_G(u)$	3	$cl(G)$	8
$J(G)$	41	$d(x, H)$	3
K_n^+	4	$d(x, y)$	3
K_n^-	4	$d_D^+(x)$	57
$K_3 \circ K_1$	4	$d_D^-(x)$	57
K_n	4	$d_G(S)$	6
$K_{1,3}$, claw	4	$d_H(u)$	3
$K_{1,s}$	4	h_v , hanging at v	4
$K_{r,s}$	4	k -(degree) closure	5
$L(G)$, line graph	41	k -dominating	4
N , net	4	k -pancyclic	59
$N[H]$	3	k -strong	57
$N[u]$	3	kG	4
$N_D^+(x)$	57	$x \rightarrow y$	57
$N_D^-(x)$	57	$\mathcal{H}_1$	42

$\mathcal{H}_2$	43	induced subgraph	3
$\mathcal{H}_3$	50	inserted	58
$\mathcal{H}_6$	53	jump graph	41
$\mathcal{H}_7$	53	light vertex	12
almost distance-hereditary	5	line graph	41
bridgehead	58	locally connected graph	3
chordal	4	locally disconnected	3
circumference	7	locally heavy	24
claw	4	maximal induced path	4
complementary cycles	43	maximum induced cycle	4
condition (A)	17	minimum separating set	57
degree	3	neighborhood	3
distance	3	out-arc	57
distance-hereditary	5	out-arc pancyclic vertex	59
dominates	57	out-degree	57
doubly dominating	4	out-neighborhood	57
edge-colored	29	pancyclic arc (vertex)	59
eligible	3	pancyclic graph	43
extendable	12	path-contracting	57
gap	14	PCHP	29
Hamiltonian cycle	4	PCHP Conjecture	30
Hamiltonian path	4	properly colored	29
hanging	4	separating set	57
heavy vertex	12	simplicial	3
in-degree	57	splitting graph	43
in-neighborhood	57	strong	57
induced cycle	4	strong component	57
induced path	4	strong connectivity	57
induced subdigraph	57	tournament	58

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit hamiltonschen Graphen und einigen verwandten Themen. Vor allem geht es hier um die Existenz eines echt gefärbten Hamiltonweges in einem vollständigen Graphen mit echter Kantenfärbung, um komplementäre Kreise im Jump-Graphen und um die Anzahl der Ecken eines Turniers, deren positive Bögen allesamt panzyklisch in dem Turnier sind.

Das Hamiltonkreisproblem ist eine der fundamentalen Problemstellungen der Graphentheorie. Es fragt, ob in einem gegebenen Graphen ein sogenannter Hamiltonkreis existiert. Ein Hamiltonkreis ist dabei ein Kreis, der alle Ecken des Graphen genau einmal enthält. Namensgeber des Problems ist der irische Astronom und Mathematiker Sir William Rowan Hamilton, der im Jahr 1857 das Spiel „The Icosian Game“ erfand (und später verbesserte zum „Traveller’s Dodecahedron or A Voyage Round The World“). Der „Traveller’s Dodecahedron“ besteht aus einem hölzernen, regulären Dodekaeder, wobei die 20 Ecken mit Namen bekannter Städte assoziiert sind. Ziel ist es, eine Reiseroute entlang der Kanten des Dodekaeders zu finden, die jede Stadt genau einmal besucht und dort aufhört, wo sie beginnt.

Obwohl das Hamiltonkreisproblem **NP**-vollständig ist, sind viele hinreichende Bedingungen für hamiltonsche Graphen in den letzten 100 Jahren gefunden worden. Daraus entwickelten sich viele interessante Themen in der Graphentheorie: zum Beispiel, die Panzyklischkeit, die Faktorthorie, die Existenz des echt gefärbten Hamiltonweges, u.s.w.

Im Allgemeinen gibt es zwei wichtige Gruppen von den hinreichenden Bedingungen für die Existenz eines Hamiltonkreises in einem Graphen. Die erste Gruppe umfasst die sogenannten numerischen Bedingungen, von welchen vermutlich die Grad-Bedingungen die bekanntesten sind; strukturelle Bedingungen sind die zweite wichtige Gruppe. Ein gutes Beispiel davon ist die „verbotene Teilgraph“-Bedingung. Im Kapitel 2 werden einige neue hinreichende Bedingungen für die Existenz eines Hamiltonkreises in einem Graphen durch das Kombinieren beider gegeben.

Es sei $G = (V, E)$ ein zusammenhängender Graph. Besitzt G keinen $K_{1,3}$ als induzierten Teilgraphen, so heißt G klauenfrei. Für $x, y \in V$ definieren wir den Abstand $d_G(x, y)$ zwischen x, y durch die Länge eines kürzesten Weges von x nach y . Ein Graph heißt fast abstandstreu falls für jeden zusammenhängenden induzierten Teilgraph H von G und alle $u, v \in V(H)$ die Eigenschaft: $d_H(u, v) \leq d_G(u, v) + 1$ erfüllt ist.

Ein Graph G heißt 2-schwer falls der Eckengrad mindestens zweier Endecken jeder Klaue in G größer oder gleich zu $\frac{|V(G)|}{2}$ ist. Im Abschnitt 2.1 zeigen wir: Ist G 2-fach zusammenhängend, 2-schwer und fast abstandstreu, so ist G hamiltonsch.

Im Abschnitt 2.2 nehmen wir an, dass G ein zusammenhängender und klauenfreier Graph ist mit folgender Bedingung: Für jeden maximal induzierten Weg P der Länge $p \geq 2$ mit Endecken u, v gilt $d(u) + d(v) \geq |V(G)| - p + 2$. Unter dieser Annahme kann man zeigen, dass G einen 2-dominierenden induzierten Kreis hat und hamiltonsch ist.

Im Abschnitt 2.3 betrachten wir einen Graphen, der die folgende Bedingung erfüllt: Für jede Ecke u in G gilt $\max\{d(x) \mid d(x, u) = 2 \text{ und } x \neq u\} \geq \frac{|V(G)|}{2}$. Ist G außerdem noch 3-fach zusammenhängend, klauenfrei und sanduhrfrei, dann ist G hamiltonsch.

Im Jahr 1997 vermuteten J. Bang-Jensen und G. Gutin, dass ein vollständiger Graph G mit einer echten Kantenfärbung einen echt gefärbten Hamiltonweg besitzt genau dann, wenn G einen spannenden Teilgraphen hat, der aus einem echt gefärbten Weg C_0 und einer

(möglicherweise leeren) Ansammlung von echt gefärbten Kreisen $C_1, C_2, \dots, C_d$ besteht, so dass $V(C_i) \cap V(C_j) = \emptyset$, $0 \leq i < j \leq d$. Im Kapitel 3 prüfen wir die Richtigkeit dieser Vermutung.

Offensichtlich ist ein Hamiltonkreis ein 2-Faktor. Folglich ist die Untersuchung der Existenz eines 2-Faktors in einem Graphen ein interessantes Thema in der Graphentheorie. Im Kapitel 4 betrachten wir einen speziellen 2-Faktor, nämlich die komplementären Kreise in Jump-Graphen, und charakterisieren die Jump-Graphen, die die komplementären Kreise enthalten.

In dem zweiten Teil dieser Arbeit betrachten wir Turniere. T. Yao, Y. Guo und K. Zhang (Discrete Appl. Math. 99, 2000, 245 – 249) bewiesen, dass jedes stark zusammenhängende Turnier eine Ecke x enthält, so dass jeder positive Bogen von x panzyklisch ist. Außerdem vermuteten sie, dass jedes k -fach stark zusammenhängende Turnier k Ecken enthält, so dass jeder positive Bogen von solchen Ecken panzyklisch ist. A. Yeo (J. Graph Theory 50, 2005, 212 – 219) zeigte, dass jedes 3-fach stark zusammenhängende Turnier zwei Ecken x, y enthält, so dass alle positiven Bögen von x und y panzyklisch sind. Im Abschnitt 6.1 verbessern wir dieses Resultat und beweisen, dass jedes 2-fach stark zusammenhängende Turnier zwei Ecken v_1, v_2 enthält, so dass jeder positive Bogen von v_i für $i = 1, 2$ panzyklisch ist.

Außerdem gab A. Yeo (J. Graph Theory 50, 2005, 212 – 219) eine unendliche Klasse von k -fach stark zusammenhängenden Turnieren für $k \in \mathbb{N}$ an, so dass jedes dieser Turniere höchstens 3 Ecken enthält, deren positive Bögen panzyklisch sind. Darüber hinaus vermutete er, dass jedes 2-fach stark zusammenhängende Turnier drei Ecken x, y, z enthält, so dass alle positiven Bögen von x, y und z panzyklisch sind. Im Abschnitt 6.2 zeigen wir, dass jedes 3-fach stark zusammenhängende Turnier mindestens drei Ecken enthält, so dass alle positiven Bögen von den drei Ecken panzyklisch sind. Daher ist die Vermutung von Yao, Guo und Zhang zutreffend für 3-fach stark zusammenhängende Turniere, aber die Vermutung von Yeo ist noch offen.

Weil es k -fach stark zusammenhängende Turniere für all $k \in \mathbb{N}$ gibt, die höchstens 3 Ecken enthalten und deren positive Bögen panzyklisch sind, führt dies zu einem interessanten Problem: Wieviele Ecken besitzt ein Turnier, so dass alle positiven Bögen der Ecken 4-panzyklisch sind? Im Abschnitt 6.3 zeigen wir, dass jedes s -fach ($s \geq 2$) stark zusammenhängende Turnier mindestens $s + 1$ Ecken enthält, so dass alle positiven Bögen der Ecken 4-panzyklisch sind.

Lebenslauf

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