

k -ordered Graphs & Out-arc Pancyclicity on Digraphs

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Preface

Graph theory as a very popular area of discrete mathematics has rapidly been developed over the last couple of decades. Numerous theoretical results and countless applications to practical problems have been discovered. The concepts of k -ordered graphs and out-arc pancyclicity are two recent topics in graph theory, which are investigated in this thesis.

Hamiltonian graphs and various Hamiltonian-related concepts such as traceable-, Hamiltonian-connected-, pancyclic-, panconnected-, and cycle extendable graphs have been studied extensively. Recently, Ng and Schulz [62] introduced a new strong Hamiltonian property: k -ordered Hamiltonian. For a positive integer k , a graph G is k -ordered if for every ordered set of k vertices, there is a cycle that encounter the vertices of the set in the given order. If the cycle is also a Hamiltonian cycle, then G is said to be k -ordered Hamiltonian. Just as many articles showed for other Hamiltonian-related properties, the condition that implies a graph to be Hamiltonian is a candidate to imply a graph to be k -ordered Hamiltonian. There has been a series of results involving degree conditions, generalized degree conditions, neighbourhood conditions and forbidden subgraph conditions that imply k -ordered or k -ordered Hamiltonian (see [23]).

In Part I, some new results on k -ordered graphs are presented.

Chapter 1 gives a general introduction to the terminology, notation and basic concepts of k -ordered (Hamiltonian) graphs. Related useful results on Hamiltonicity and k -ordered Hamiltonicity are also recalled.

In Chapter 2, connectivity properties of k -ordered graphs are investigated. Section 2.1 deals with the minimum connectivity forced by a k -ordered graph. To this aim, we introduce a new kind of connectivity: k -ordered connectivity. The concept of k -ordered graphs is related to other “connectivity” concepts, such as, linkage. In Section 2.2, relationships between connectivity, linkage and orderedness are described. Chen et al. [18] showed the absorptivity of k -linked graphs. In section 2.3, we show a similar absorptivity on k -ordered graphs.

It is well known that the Hamiltonicity problem is NP-complete. There are many conditions that imply a graph to be Hamiltonian. One of the classical theorems of this nature is due to Ore [64], who proved that a graph is Hamiltonian if the degree sum of any two nonadjacent vertices is at least n . In Chapter 3, we shall not consider “any pair of nonadjacent vertices”, but only “any pair of distance 2 vertices” and consequently give a detailed account of results concerning the Hamiltonicity and k -ordered Hamiltonicity of graphs. We shall prove that a graph is traceable, Hamiltonian or k -ordered Hamiltonian if the degree sum is sufficiently large for any pair of distance 2 vertices. We also show the sharpness of the degree sum as well as the independence of these results.

Given positive integers $k \leq m \leq n$, a graph of order n is (k, m) -pancyclic if for any set of k vertices and any integer r with $m \leq r \leq n$, there is a cycle of length r containing

the k vertices. If the additional property that the k vertices must appear on the cycle in a specified order is required, then the graph is said to be (k, m) -pancyclic ordered. Faudree et al. [24] gave the minimum sum of degree conditions of nonadjacent vertices that imply a graph is (k, m) -pancyclic or (k, m) -pancyclic ordered. In Chapter 4, we introduce a new concept: (k, m) -vertex-pancyclic ordered graphs. A graph is (k, m) -vertex-pancyclic ordered if for any specified vertex v and any ordered set S of k vertices there is a cycle of length r containing v and S and encountering the vertices of S in the specified order for each $m \leq r \leq n$. Clearly, a (k, m) -pancyclic ordered graph is (k, m) -pancyclic and a (k, m) -vertex-pancyclic ordered graph is also (k, m) -pancyclic ordered. We shall show that a graph is (k, m) -vertex-pancyclic ordered under the same minimum sum of degree conditions of nonadjacent vertices as shown in [24].

In Part II, we turn our attention to the out-pancyclicity of digraphs. Demanding that a digraph D contains an out-arc pancyclic vertex is a very strong requirement, since D would have at least $\delta^+(D)$ distinct Hamiltonian cycles, where $\delta^+(D)$ denotes the minimum out-degree. Hence it is not surprising that most results on out-arc pancyclicity only deal with tournaments and generalizations of tournaments.

Chapter 5 is devoted to an introduction of general concepts and several special classes of digraphs. We shall introduce one of the most powerful tools in the theory of the out-pancyclicity of digraphs, namely the operation of path-contraction. In the last part of this chapter, we present a survey on out-pancyclicity of digraphs, particularly tournaments.

In Chapter 6, the influence of the connectivity on the number of out-arc pancyclic vertices of tournaments is studied. In Section 6.1, we shall only investigate the number of out-arc pancyclic vertices of 1-, 2- and 3-strong tournaments since Yeo [79] presented an infinite class of k -strong tournament, such that each tournament contains at most 3 out-arc pancyclic vertices. We note that the example introduced by Yeo contains more out-arc 4-pancyclic vertices. In Section 6.2, we count the out-arc 4-pancyclic vertices in k -strong tournaments.

Recently, the out-arc pancyclicity on local tournaments was studied by S. Li, Meng and Guo [54]. As a superset of local tournaments, local in-tournaments maybe have the similar property. In Chapter 7, we shall give a structural analysis of local in-tournaments.

Path-contraction is an important tool in the proof of the results of Chapter 6. In the last chapter, we consider some applications of the path-contraction technique in digraphs, in particular in strongly Hamiltonian-connected digraphs.

I would like to express my gratitude to all those who gave me the possibility to complete this thesis. First of all, I wish to thank Prof. Dr. Shengjia Li for introducing me to graph theory and awaking my interest in this research area. I am deeply indebted to my supervisor Prof. Dr. Yubao Guo whose help, stimulating suggestions and encouragement helped me in all the time of research with writing of this thesis. I also owe a lot to Prof. Dr. Dr. h.c. Hubertus Th. Jongen for the generous support and giving me the opportunity to carry out my doctoral study. Many thanks also to Prof. Dr. Gerhard

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Last but not least thanks to my family for their support all the way from the very beginning of my doctoral study. Especially, I would like to give my thanks to my husband Xinhong whose patient love enabled me to complete this thesis.

Aachen, November 2008

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Part I

k -ordered Graphs

Chapter 1

Introduction

In this part, we shall consider finite graphs without loops and multiple arcs. In the first chapter, we present the basic terminology and notation of graph theory which will be used throughout this part. Before discussing results related to k -ordered graphs in more detail in the following sections, we will briefly explain the basic definitions and some well-known results of Hamiltonian graphs. Our main source follows that of J.A. Bondy and U.S.R. Murty [14] as well as G. Chartrand and L. Lesniak [16], and we refer the reader to these books for any information not provided here. Special notation and definitions will be defined where needed.

1.1 Terminology and notation

Let G be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The cardinalities of these sets will be the *order* $n(G)$ and the *size* $m(G)$ of G , respectively. If two vertices $u, v \in V(G)$ are connected with an edge, we simply write $uv \in E(G)$ and say that u and v are *adjacent* or u is adjacent to v . An edge $e = uv$ is called *incident* with both *end-vertices* u and v . Let U be a nonempty set of vertices of G . We call that u is *adjacent* to U if u is adjacent to at least one element of U .

For a vertex u of G , the set $N_G(u) = \{v \mid uv \in E(G)\}$ is called the *neighbourhood* of u in G . The *degree* of u in G is defined as the number of edges incident with u , denoted by $d_G(u)$ or $d(u)$. If $d_G(u) = 0$, we call u an *isolated vertex*. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. The *maximum degree* is defined similarly and is denoted by $\Delta(G)$. The graph G is said to be *k -regular*, if $\delta(G) = \Delta(G) = k$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Whenever a subgraph H of G has the same order as G , then H is called a *spanning subgraph* of G . Among the most important subgraphs we shall encounter “induced subgraphs”. If X is a nonempty subset of the vertex set $V(G)$ of a graph G , then the subgraph $G[X]$ of G *induced* by X is the graph having vertex set X and whose edge set consists of those edges of G incident with two elements of X . A subgraph H of G is called *vertex-induced* or simply *induced* if $H = G[X]$ for some subset X of $V(G)$. In addition, $G - X = G[V(G) \setminus X]$. Similarly, if Y is a nonempty subset of $E(G)$, then the subgraph $G[Y]$ *induced* by Y is the graph whose vertex set consists of those vertices of G incident with at least one edge

of Y and whose edge set is Y . A subgraph H of G is *edge-induced* if $H = G[Y]$ for some subset Y of $E(G)$.

Let U be a subgraph or a nonempty set of vertices of a graph G and let its *neighbourhood* $N(U)$ denote the set of all vertices of G adjacent with at least one vertex of U . More generally, for arbitrary subgraphs U and V of G , the *neighbourhood* of U in V is the set of all vertices of V adjacent with at least one element of U , denoted by $N_V(U)$.

Let u and v be (not necessarily distinct) vertices of a graph G . A (u, v) -walk of G is a finite, alternating sequence $u = u_0 e_1 u_1 e_2 \dots u_{k-1} e_k u_k = v$ of vertices and edges, beginning with vertex u and ending with v , such that $e_i = u_{i-1} u_i$ for $i = 1, 2, \dots, k$. The number k (the number of occurrences of edges) is called the *length* of the walk. A (u, v) -walk is *closed* or *open* depending on whether $u = v$ or $u \neq v$. A (u, v) -trail is a (u, v) -walk in which no edge is repeated, while a (u, v) -path is a (u, v) -walk in which no vertex is repeated. A nontrivial closed trail of a graph G is referred to as a *circuit* of G , and a circuit $v_1 v_2 \dots v_n v_1$ ($n \geq 3$) whose n vertices v_i are distinct is called a *cycle*. P_n and C_n denote the path and the cycle of order n .

An *internal vertex* of a (u, v) -path P is any vertex of P different from u or v . A collection $\{P_1, P_2, \dots, P_k\}$ of paths is called *internally disjoint* if each internal vertex of P_i ($i = 1, 2, \dots, k$) lies on no path P_j ($j \neq i$).

Let C be a cycle in G . A path P between two distinct vertices x and y in C is called a *C-bypass* if $|V(P)| \geq 3$ and $V(P) \cap V(C) = \{x, y\}$. The *gap* of P with respect to C is the length of the shortest path between x and y in the cycle C .

Let C be a cycle in a graph with an understood orientation, and x be a vertex of C . x^+ and x^- denote the successor and predecessor of x on C , respectively. Also, if x and y are vertices of C , then $x \overrightarrow{C} y$ denotes the path from x to y along C in the designated direction. The notation $x \overleftarrow{C} y$ denotes the path from x to y in the opposite direction. We also use $[x, y]$, $(x, y]$, $[x, y)$ and (x, y) to denote the intervals $x \overrightarrow{C} y$, $x \overrightarrow{C} y - x$, $x \overrightarrow{C} y - y$ and $x \overrightarrow{C} y - \{x, y\}$ or, when appropriate, the vertex set of these paths. Similar notation is used in the case of paths.

A vertex u is said to be *connected* to a vertex v in a graph G if there exists a (u, v) -path in G . A graph G is *connected*, if every two of its vertices are connected. A graph that is not connected is *disconnected*. The relation “is connected to” is an equivalence relation on the vertex set of every graph G . Each subgraph induced by the vertices in a resulting equivalence class is called a *connected component* or *component* of G . Equivalently, a component of a graph G is a subgraph that is maximal with respect to the property of being connected. G is said to be *k-connected*, if $|V(G)| \geq k + 1$ and $G - S$ is connected for each $S \subset V(G)$ with $|S| \leq k - 1$. If G is *k-connected*, but not $(k + 1)$ -connected, then the number $\kappa(G) = k$ is defined as the *connectivity* of G . Moreover, a subset $K \subseteq V(G)$ with $|K| = \kappa(G)$ and $\kappa(G - K) = 0$ is called a *minimal cut set* of G .

If G is connected, then for two vertices $u, v \in V(G)$, the *distance* $d(u, v)$ between u and v is the length of a shortest path between u and v in G .

A graph is *complete* if every two of its vertices are adjacent; K_n denotes the complete graph of order n . A graph G is *k-partite*, $k \geq 1$, if it is possible to partition $V(G)$ into k subsets $V_1, V_2, \dots, V_k$ (call *partite sets*) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j , $i \neq j$. For $k = 2$, such graphs are called *bipartite graphs*. A *complete k-partite graph* G is a *k-partite* graph with partite sets $V_1, V_2, \dots, V_k$, added the property

that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. If $|V_i| = n_i$, then this graph is denoted by $K(n_1, n_2, \dots, n_k)$ or $K_{n_1, n_2, \dots, n_k}$. If $k = 2$, we call G a *complete bipartite graph*.

The *complement* G^c of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in G^c if and only if these vertices are not adjacent in G . The complement K_n^c of the complete graph K_n has n vertices and no edges and is referred to as the *empty graph* of order n .

A cycle (path, respectively) in a graph is called a *Hamiltonian cycle* (*Hamiltonian path*, respectively) if it contains all vertices of G . A graph G is said to be *Hamiltonian* if it contains a Hamiltonian cycle. We call a graph G *Hamiltonian-connected* if for every pair u and v of distinct vertices of G , there exists a Hamiltonian (u, v) -path.

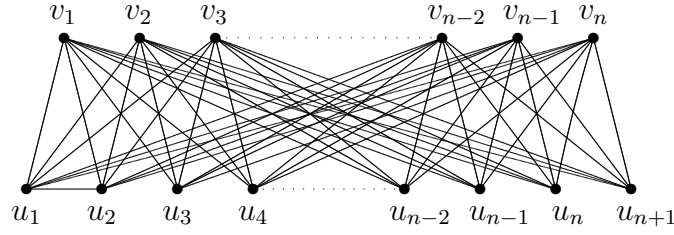
1.2 Definition of k -ordered graphs

The Hamiltonian problem is generally considered to determine conditions under which a graph contains a spanning cycle. Named after Sir William Rowan Hamilton, this problem traces its origins to the 1850s. Today, Hamiltonicity and various related properties such as traceable, Hamiltonian-connected, pancyclic, panconnected, and cycle extendable have been studied extensively. There are several surveys of results related to these properties, (see, for example, [10, 34, 35]). In 1997, a new Hamiltonian property was introduced by Ng and Schultz in [62].

Definition 1.1 (Ng and Schultz [62]) A graph G on n vertices, $n \geq 3$, is *k -ordered* for any integer k , $2 \leq k \leq n$, if for every ordered set $S = \{x_1, x_2, \dots, x_k\}$ of k distinct vertices in G , there exists a cycle that contains all the vertices of S in the designated order. A graph is *k -ordered Hamiltonian* if for every ordered set S of k vertices there exists a Hamiltonian cycle which encounter S in its designated order.

Any 2-connected graph is 2-ordered, since any pair of vertices is on a cycle and the proper orientation will give the containment. For the same reason, any 3-connected graph is 3-ordered, and any Hamiltonian graph is 3-ordered Hamiltonian, since the proper orientation of the cycle will give the required order to any collection of 3 vertices on the cycle. Thus k -ordered is really only interesting for $k \geq 4$. It is easy to see that a graph G of order n is n -ordered if and only if $G = K_n$, and if G is a k -ordered graph, then G is ℓ -ordered for every positive integer $2 \leq \ell \leq k$.

Clearly, a graph G is Hamiltonian and also k -ordered if it is k -ordered Hamiltonian. However, as shown in [23], the converse is not true. A graph G may be k -ordered and also Hamiltonian, but not k -ordered Hamiltonian. Consider the graph G_1 , which is obtained from a $K_{n, n+1}$ by adding one edge $u_1 u_2$ between two vertices in the large part of the complete bipartite graph. This graph is Hamiltonian, and if $k < n$, then it is k -ordered. However, any Hamiltonian cycle of G must contain the edge $u_1 u_2$, and so if u_1 and u_2 are in a k -set S but are not consecutive vertices in the ordered set S , then there will be no Hamiltonian cycle containing S in the required order (s. Figure 1.1).

Figure 1.1: The graph G_1 .

1.3 Survey of earlier results

There are many degree conditions that imply a graph is Hamiltonian. Two of the classical theorems of this nature are due to Dirac [19] and Ore [64].

Theorem 1.2 (Dirac [19]) *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq n/2$, then G is Hamiltonian.*

Theorem 1.3 (Ore [64]) *If a graph G of order n such that $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices u and v , then G is Hamiltonian.*

More generalized degree and neighbourhood union conditions that imply Hamiltonian properties were introduced in [25].

Just as many articles showed for Hamiltonian-connected, for instance [65], it is natural to first look at degree conditions that imply Hamiltonian and to see if they will also imply the stronger property of k -ordered Hamiltonian. In 1997, the following result was proved.

Theorem 1.4 (Ng and Schultz [62]) *Let G be a graph of order n and let k be an integer with $3 \leq k \leq n$. If*

$$d(u) + d(v) \geq n + 2k - 6$$

for every pair u and v of nonadjacent vertices of G , then G is k -ordered Hamiltonian.

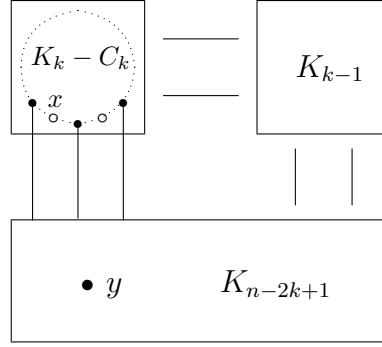
An immediate corollary is the following Dirac-type result.

Corollary 1.5 (Ng and Schultz [62]) *Let G be a graph of order n and let k be an integer with $3 \leq k \leq n$. If*

$$d(u) \geq \frac{n}{2} + k - 3$$

for every vertex u of G , then G is k -ordered Hamiltonian.

The degree conditions of Dirac and Ore would have to be strengthened as an example shown in [62]. Consider the graph G_2 of order n composed of three parts: $K_k - C_k$, K_{k-1} and K_{n-2k+1} . The graph G_2 contains all the edges between K_{k-1} and K_{n-2k+1} and all the edges between K_{k-1} and $K_k - C_k$. Between K_{n-2k+1} and $K_k - C_k$, only the edges incident to the even indexed vertices of C_k are in G . This graph is not k -ordered because

Figure 1.2: The graph G_2 , where k is even.

there is no cycle containing the vertices of C_k in the required order. For $y \in V(K_{n-2k+1})$ and $x \in V(K_k - C_k)$ an odd indexed vertex on C_k , $d(x) + d(y) = n + (3k - 10)/2$ for k even. Thus, $d(u) + d(v) \geq n + (3k - 10)/2$ for each pair of nonadjacent vertices u and v in G_2 (s. Figure 1.2).

Both bounds above for a graph to be k -ordered Hamiltonian were improved for small k with respect to n .

Theorem 1.6 (Faudree et al. [22]) *Let $k \geq 3$ be an integer and let G be a graph of order $n \geq 53k^2$. If*

$$d(u) + d(v) \geq n + \frac{3k - 9}{2}$$

for every pair u and v of nonadjacent vertices of G , then G is k -ordered Hamiltonian.

Theorem 1.7 (Kierstead et al. [44]) *Let $k \geq 2$ be an integer and let G be a graph of order $n \geq 11k - 3$. If*

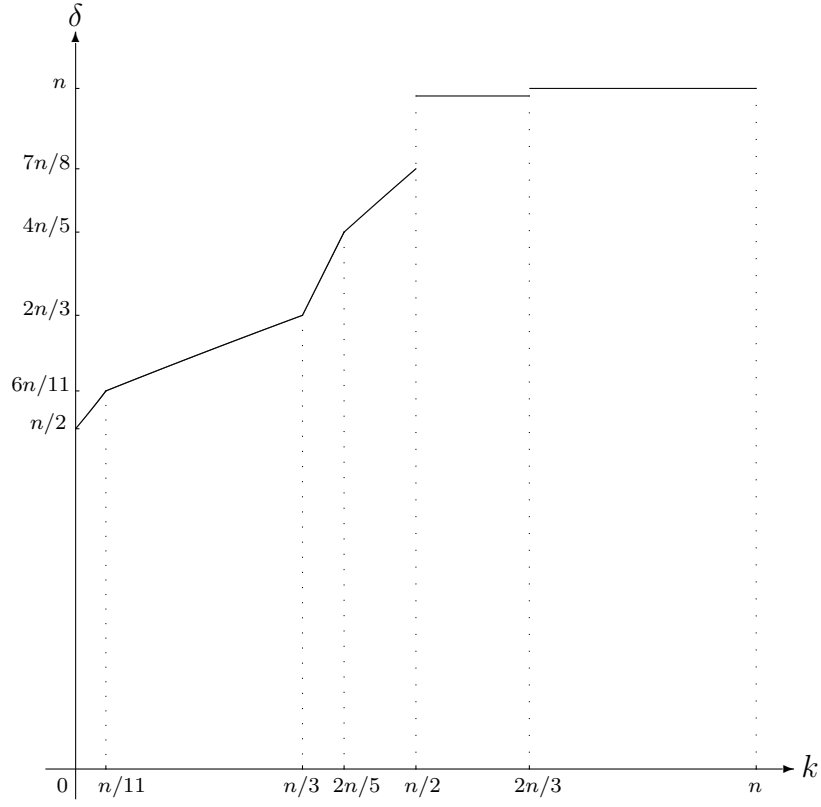
$$d(u) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1$$

for every vertex u of G , then G is k -ordered Hamiltonian.

In [26], Faudree et al. proved that the bound of Theorem 1.6 holds for $3 \leq k \leq n/2$.

Theorem 1.8 (Faudree et al. [26]) *Let k be an integer with $3 \leq k \leq n/2$ and let G be a graph of order n . If $d(u) + d(v) \geq n + (3k - 9)/2$ for every pair u and v of nonadjacent vertices of G , then G is k -ordered Hamiltonian.*

Let $\delta(n, k)$ be the smallest integer m for which any graph of order n with minimum degree at least m is k -ordered Hamiltonian. Faudree et al. gave the following summarization on Dirac-type condition.

Figure 1.3: Bounds for $\delta(n, k)$.

Theorem 1.9 (Faudree et al. [26]) *For positive integer k and n with $3 \leq k \leq n$ we have*

- (i) $\delta(n, k) = \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$, for $k \leq (n+3)/11$;
- (ii) $\delta(n, k) > n/2 + k/2 - 2$, for $(n+3)/11 < k \leq n/3$;
- (iii) $\delta(n, k) \geq 2k - 2$, for $n/3 < k < 2(n+2)/5$;
- (iv) $\delta(n, k) = \lceil n/2 + (3k-9)/4 \rceil$, for $2(n+2)/5 \leq k \leq n/2$;
- (v) $\delta(n, k) = n - 2$, for $n/2 < k \leq 2n/3$; and
- (vi) $\delta(n, k) = n - 1$, for $2n/3 < k \leq n$.

Figure 1.3 indicates the relationship between the exact values known for $\delta(n, k)$ and the bounds provided by the examples given in [26].

Chapter 2

Connectivity properties of k -ordered graphs

For the connectivity, Menger presented a well known characterization (s. [56]). With the aid of Menger's theorem, many interesting results can be directly proved for k -connected graphs. For example:

Theorem 2.1 (Whitney [74]) *A graph G is k -connected if and only if for each pair u, v of distinct vertices there are at least k internally disjoint (u, v) -paths in G .*

Theorem 2.2 (Chartrand and Lesniak [16]) *Let G be a k -connected graph, and $v, v_1, v_2, \dots, v_k$ be $k + 1$ distinct vertices of G , then there exist internally disjoint (v, v_i) -paths $(1 \leq i \leq k)$.*

Theorem 2.3 (Chartrand and Lesniak [16]) *Let G be a k -connected graph, $k \geq 2$. Then every k vertices of G lie on a common cycle of G .*

Theorem 2.4 (Menger [56]) *If G is a k -connected graph with nonempty disjoint subsets S_1 and S_2 of $V(G)$, then there exist k internally disjoint paths $P_1, P_2, \dots, P_k$ such that P_i is a (u_i, v_i) -path, where $u_i \in S_1$ and $v_i \in S_2$, for $i = 1, 2, \dots, k$, and $|S_1 \cap V(P_i)| = |S_2 \cap V(P_i)| = 1$.*

From the last two theorems, it could be said that there would be relationships among k -connected, k -ordered and k -linked (s. Definition 2.12). In this chapter, we shall deal with these relations.

2.1 Connectivity

The k -ordered property for a graph G is a strong one that forces G to have sufficient connectivity. This was observed by [62].

Proposition 2.5 (Ng and Schultz [62]) *Let G be a graph on n vertices, $n \geq 3$. If G is k -ordered, $3 \leq k \leq n$, then G is $(k - 1)$ -connected.*

Now an immediate corollary follows.

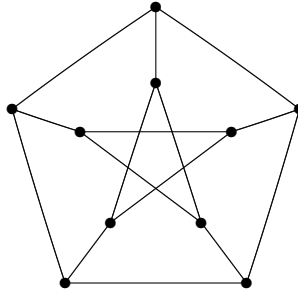


Figure 2.1: The Petersen graph.

Corollary 2.6 *If G is a k -ordered graph, then $\delta(G) \geq k - 1$.*

The previous results cannot be improved, since for any $k \geq 3$, there are exactly $(k - 1)$ -connected graphs that are k -ordered. Consider the graph G obtained from a K_{n-1} by adding a vertex that is adjacent to precisely $k - 1$ vertices of the complete graph. This graph is k -ordered (in fact, k -ordered Hamiltonian) with $\kappa(G) = k - 1$. The other example to show the sharpness of these results is due to Mészáros [57]. He constructed an infinite family of 3-regular 4-ordered graphs and specially, the Petersen graph is 4-ordered (s. Figure 2.1). Recall that the Heawood graph is the smallest 3-regular graph that is triangle, square and pentagon free [16]. He also proved:

Theorem 2.7 (Mészáros [57]) *The Heawood graph is the graph on the fewest vertices, after K_4 and $K_{3,3}$, that is 3-regular 4-ordered Hamiltonian.*

We call a graph G a *bracelet graph* if its vertex set can be partitioned into $V_1 \cup V_2 \cup \dots \cup V_m$, $m \geq 3$, with V_i nonempty for all $i \in [m]$ (we denote the set $\{1, 2, \dots, m\}$ by $[m]$), such that v is adjacent to u in G if and only if $v \in V_i$ and $u \in V_j$ and $i - j \equiv 1$ or $-1 \pmod{m}$. Alternatively, we can think about a bracelet graph as being obtained from a cycle where each vertex of the cycle is blown up to an independent set V_i . Let $G_{k,2m}$ be a bracelet graph with parts $V_1, V_2, \dots, V_{2m}$, $m \geq 2$, such that $|V_i| = k$ for $i \in [2m]$. It is clear that G is simple and $2k$ -regular by construction.

Theorem 2.8 (Mészáros [58]) *For every $k \geq 1$, there exists an infinite family of $2k$ -regular graphs that are $(2k + 1)$ -ordered Hamiltonian.*

Corollary 2.9 (Mészáros [58]) *The graph $G_{k,m}$ are $(2k + 1)$ -ordered for all $m \geq 4$.*

Theorem 2.10 (Mészáros [58]) *There exists an infinite family of $2k$ -ordered graphs $P_{k,m}$ with minimum degree $2k - 1$ and maximum degree $2k + 2$.*

Note that the graph G_2 shown in Figure 1.2 is $(\lfloor 3k/2 \rfloor - 1)$ -connected, but G_2 is not k -ordered. This means that there are non- k -ordered graphs with the high connectivity.

Chebikin (personal communication) observed that if n is the number of vertices in a 4-ordered graph, then the diameter of the graph is at most $n/4 + 2$. In general, Mészáros proved the following.

Proposition 2.11 (Mészáros [58]) (i) Given a $2k$ -ordered graph G on n vertices, the diameter d of G is at most $\lfloor n/2k \rfloor + 1$.

(ii) Given a $(2k+1)$ -ordered graph G on n vertices, the diameter d of G does not exceed $\lfloor (n-2)/2k \rfloor + 1$.

Furthermore, these bounds are tight.

2.2 k -ordered graphs and k -linked graphs

Just as we showed in the beginning of this chapter, the concept of k -ordered is related to the other “connectivity” property, linkage.

Definition 2.12 For any $1 \leq k \leq n/2$, a graph G of order n is k -linked if given any collection of k pairs of vertices $L = \{\{x_i, y_i\} \mid 1 \leq i \leq k\}$, there are k vertex disjoint paths (except possibly for end-vertices) P_i such that P_i is a path from x_i to y_i .

It is not necessary that all of the vertices in the k pairs of L are distinct. However, requiring the $2k$ vertices in the definition of k -linkage to be distinct gives an equivalent definition for k -linkage as long as the order of the graph is not too small (refer to [23]). Also, the immediate observations follow.

Observation 2.13 If a graph G is k -linked, then G is k -connected.

Observation 2.14 If a graph G is k -linked, then G is k -ordered.

Observation 2.15 If a graph G is $2k$ -ordered, then G is k -linked.

The relationship between connectivity and linkage has been investigated. The best bound by now was proved by Thomas and Wolland [69]. In fact, they presented a linear edge bound for graph linkages as follows.

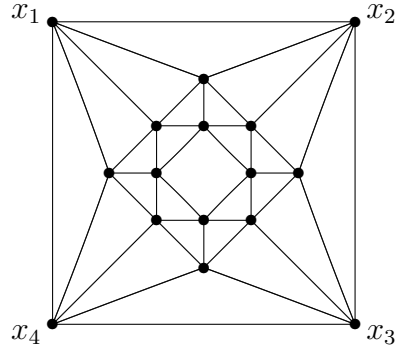
Theorem 2.16 (Thomas and Wolland [69]) If a $2k$ -connected graph G has at least $5kn$ edges, then G is k -linked.

Corollary 2.17 (Thomas and Wolland [69]) If G is $10k$ -connected, then G is k -linked.

It is generally accepted that the connectivity needed to imply k -linkage or k -orderedness is significantly less than $10k$, which is a bound that comes from just techniques. It would be interesting to have sharp results for the relationships between connectivity, linkage and orderedness. Some inequalities involving these three parameters are summarized by Faudree as below. For more details, we refer to [23].

Theorem 2.18 (Faudree [23]) For $k \geq 3$,

- (i) $(2k)$ -connected $\not\Rightarrow k$ -linked $\Rightarrow (2k-1)$ -connected;
- (ii) k -connected $\not\Rightarrow k$ -ordered $\Rightarrow (k-1)$ -connected;
- (iii) $(\lceil 4k/3 \rceil)$ -ordered $\not\Rightarrow k$ -linked $\Rightarrow k$ -ordered;
- (iv) $(\lceil k/2 \rceil + 1)$ -linked $\not\Rightarrow k$ -ordered $\Rightarrow \lfloor k/2 \rfloor$ -linked;
- (v) $(3k-3)$ -connected $\not\Rightarrow k$ -linked; $10k$ -connected $\Rightarrow k$ -linked;
- (vi) $(2k-4)$ -connected $\not\Rightarrow k$ -ordered; $10k$ -connected $\Rightarrow k$ -ordered.

Figure 2.2: The graph G_3 .

Much more is known about 2- and 3-linked graphs than k -linked for $k \geq 4$. Jung [42] proved the following.

Theorem 2.19 (Jung [42]) *If $\kappa(G) \geq 6$, then G is 2-linked.*

Jung's theorem is sharp since there are 5-connected graphs that are not 2-linked. In fact there are 5-regular and 5-connected planar graphs that are not 2-linked, such as the graph G_3 in Figure 2.2. Note that there is no linkage for the pairs $\{x_1, x_3\}$ and $\{x_2, x_4\}$ since any path from x_1 to x_3 will separate x_2 from x_4 .

The 5-connected graph G_3 in Figure 2.2 is also not 4-ordered. Little is known about the minimal connectivity that implies 4-ordered. It is natural to ask the following specialized question.

Problem 2.20 (Faudree [23]) *If G is a 6-connected graph, is G 4-ordered?*

Since being 4-linked implies being 4-ordered, we see that a 40-connected graph is 4-ordered by Theorem 2.17. In [32], the orderedness on planar graphs was investigated and the following was proved.

Lemma 2.21 (Goddard [32]) *Let G be a planar graph.*

- (i) *If G is 3-connected then G is 3-ordered.*
- (ii) *If G is 4-ordered, then G is maximal planar, 3-connected, and any cut set of cardinality 3 isolates a vertex.*
- (iii) *G is not 5-ordered.*

Theorem 2.22 (Goddard [32]) *A 4-connected maximal planar graph is 4-ordered.*

Recently, Thomas and Wollan [70] presented the extremal function for 3-linked graphs.

Theorem 2.23 (Thomas and Wollan [70]) *Every 6-connected graph G on n vertices with $5n - 14$ edges is 3-linked.*

Corollary 2.24 (Thomas and Wollan [70]) *Every 10-connected graph is 3-linked.*

The bound in Theorem 2.23 is optimal, in that there exist 6-connected graphs on n vertices with $5n - 15$ edges that are not 3-linked for arbitrarily large values of n .

A notion related to k -orderedness, that of k -edge-orderedness, has been studied in [17]. A graph G is *k -edge-ordered* (respectively, *k -edge-ordered eulerian*) if, for any sequence of k distinct edges $e_1, e_2, \dots, e_k$ of G , there exists a tour (respectively, an eulerian tour, that is, a tour containing each edge of G) in G containing these k edges in the specified order. It is natural to pose analogous questions for the relation of edge-orderedness, connectivity and linkage.

A graph G is said to be *weakly k -linked* if, given $2k$ vertices (not necessary distinct) $s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_k$, there exist edge-disjoint paths from s_i to t_i , for $i = 1, 2, \dots, k$.

Lemma 2.25 (Mészáros [58])

- (i) *If a graph G is weakly $2k$ -linked, then G is k -edge-ordered.*
- (ii) *If a graph G is $2k$ -edge-ordered, then it is weakly k -linked.*

Lemma 2.25 exhibits a relation between weak linkage and edge-orderedness, but tightness of these lemma remains uncertain.

Let $g(k)$ be the minimum edge-connectivity of a graph G that implies k -edge-orderedness of G . It is easy to see that $g(k) \geq k - 1$.

Proposition 2.26 (Mészáros [58]) *The upper bound $g(k) \leq 2k + 2$ holds.*

2.3 The absorptivity of k -ordered and k -linked graphs

In this section, some simple but interesting results are shown, which are about the absorptivity of k -ordered and k -linked graphs. The first two lemmas given by Chen, Gould and Pfender [18], are used to prove their main results (see Section 3.3).

Lemma 2.27 (Chen et al. [18]) *If a $2k$ -connected graph G has a k -linked subgraph H , then G is k -linked.*

Lemma 2.28 (Chen et al. [18]) *If G is a graph, $v \in V(G)$ with $d(v) \geq 2k - 1$, and if $G - v$ is k -linked, then G is k -linked.*

We immediately obtain the following corollary.

Corollary 2.29 (Faudree and Faudree [21]) *If G is a $2k$ -connected graph for $k \geq 1$ that contains a K_{2k} , then G is k -ordered.*

If we consider the orderedness-type condition, we have the following result.

Theorem 2.30 *Let G be a graph and $v \in V(G)$ with $d(v) \geq k - 1$. If $G - v$ is $(k + 1)$ -ordered, then G is k -ordered.*

Proof. Let $S = \{x_1, x_2, \dots, x_k\}$ be a k -ordered set of G . If $S \subseteq V(G - v)$, then we are done since $G - v$ is $(k + 1)$ -ordered. So assume that $x_k = v$. The fact that $d(v) \geq k - 1$ allows us to choose two neighbours of v in $V(G - v) \setminus \{x_2, x_3, \dots, x_{k-2}\}$, say v_1, v_2 . Since $G - v$ is $(k + 1)$ -ordered, we can find a cycle C , which contains $v_1, x_1, x_2, \dots, x_{k-1}$ and v_2 in this order (possibly, $v_1 = x_1$ or $v_2 = x_{k-1}$). $x_k v_1 \overrightarrow{C} v_2 x_k$ is a cycle encountering S in order. Therefore, G is k -ordered. ■

Using the result above several times, we have a more general result as follows.

Corollary 2.31 *Let G be a graph and ℓ, k be two positive integers with $\ell \leq k$. $v_1, v_2, \dots, v_\ell$ are distinct vertices of G such that $d(v_i) \geq k + i - 2$ for $i = 1, 2, \dots, \ell$. If $G - \{v_1, v_2, \dots, v_\ell\}$ is $(k + \ell)$ -ordered, then G is k -ordered.*

Chapter 3

Degree conditions on distance 2 vertices

Many generalizations of Theorem 1.2 and 1.3 have been found (see [25, 34, 35]). Particularly, Fan noted that the condition in Theorem 1.3 implies $N_G(u) \cap N_G(v) \neq \emptyset$ if $uv \notin E(G)$, i.e., the distance between any two nonadjacent vertices is exactly two. In [20], he did not consider “all pairs of nonadjacent vertices”, but only “all pairs of distance 2 vertices” and provided:

Theorem 3.1 (Fan [20]) *Let G be a 2-connected graph with n vertices and c be a positive integer with $3 \leq c \leq n$. If for all vertices u and v with $d(u, v) = 2$,*

$$\max\{d(u), d(v)\} \geq \frac{c}{2}$$

holds, then G contains a cycle of length at least c .

Corollary 3.2 (Fan [20]) *Let G be a 2-connected graph with $n \geq 3$ vertices. If for all vertices u and v with $d(u, v) = 2$,*

$$\max\{d(u), d(v)\} \geq \frac{n}{2}$$

holds, then G is Hamiltonian.

We shall study the degree conditions about distance (particularly, distance 2), which imply a graph to be k -ordered Hamiltonian.

3.1 Hamiltonicity

In this section, the “distance” is added to the Ore-type condition under which a graph contains a Hamiltonian path or Hamiltonian cycle.

3.1.1 Hamiltonian path

As a preparation, we firstly prove the following lemma.

Lemma 3.3 *Let $P = v_1v_2 \dots v_s$ ($s \geq 2$) and $Q = w_1w_2 \dots w_t$ ($t \geq 1$) be two disjoint paths in a graph G . If*

$$d_P(w_1) + d_P(w_t) \geq |V(P)| + 2,$$

then Q can be inserted into P (i. e., $v_1 \dots v_k Q v_{k+1} \dots v_s$ is a path in G for some $1 \leq k < s$).

Proof. From $d_P(w_1) + d_P(w_t) \geq s + 2$ and $s \geq 2$, we see that w_1 is adjacent with at least two vertices of P . Let $N_P(w_1) = \{v_{i_1}, v_{i_2}, \dots, v_{i_\alpha}\}$ with $i_1 < i_2 < \dots < i_\alpha$. Suppose to the contrary that Q can not be inserted into P . Then w_t is not adjacent with $v_{i_1+1}, v_{i_2+1}, \dots, v_{i_{\alpha-1}+1}$. This implies that $d_P(w_t) \leq s - (\alpha - 1) = s - \alpha + 1$. Now, we see $d_P(w_1) + d_P(w_t) \leq s + 1$, a contradiction. ■

Note that for $t = 1$, Lemma 3.3 states the following: If $d_P(w_1) \geq \left\lceil \frac{|V(P)|}{2} \right\rceil + 1$, then w_1 can be inserted into P .

Theorem 3.4 (S. Li, R. Li & Feng [52]) *Let G be a connected graph with $n \geq 5$ vertices. If*

$$d(u) + d(v) \geq n - d(u, v) + 1$$

for every pair of nonadjacent vertices u and v , then G contains a cycle and for every longest cycle C in G , the subgraph $G - V(C)$ is complete or empty.

Proof. Let G be a graph satisfying the conditions of Theorem 3.4. Under the assumption that G is non-Hamiltonian, we determine the structure of G by the following claims.

Claim 1. G contains a cycle.

Proof. Suppose to the contrary that G contains no cycle. Then G is a tree. Let $P = v_1v_2 \dots v_s$ be a longest path in G . It is clear that $3 \leq s \leq n$. If $s = n$, then the inequality $d(v_1) + d(v_3) = 3 < n - 1 = n - d(v_1, v_3) + 1$ yields a contradiction. In the other case when $s < n$, we have from $d(v_1, v_s) = s - 1$ that $d(v_1) + d(v_s) = 2 < n - s + 2 = n - d(v_1, v_s) + 1$, a contradiction. □

By Claim 1, G has at least one cycle. Let $C = u_1u_2 \dots u_mu_1$ be a longest cycle in G .

Claim 2. The subgraph $G - V(C)$ is connected.

Proof. We prove this claim indirectly. Let H and H' be two components of $G - V(C)$ with $|V(H)| = \ell$ and $|V(H')| = \ell'$. Since G is connected, there is at least one edge between C and every component of $G - V(C)$. Let $Q = ux_1x_2 \dots x_kv$ be a shortest path from H to H' in $G - \{xy \mid x, y \in V(C) \text{ and } xy \notin E(C)\}$. It is clear that $V(Q) \cap V(H) = \{u\}$, $V(Q) \cap V(H') = \{v\}$ and $x_1x_2 \dots x_k$ is a segment of C . Assume without loss of generality that $x_i = u_i$ for $i = 1, 2, \dots, k$. Note that $d(u, v) \leq k + 1$ and $m + \ell + \ell' \leq n$.

Since neither u nor v can be inserted into C , it is easy to check

$$d_C(u) \leq \frac{m}{2} \quad \text{and} \quad d_C(v) \leq \frac{m}{2}. \quad (3.1)$$

In the following, we show that $d(u) + d(v) < n - d(u, v) + 1$ for all $k \geq 1$, which contradicts to the assumption of the lemma.

Firstly, suppose that $k = 1$. Then, we see that $d(u, v) = 2$ and

$$\begin{aligned} d(u) + d(v) &= d_C(u) + d_C(v) + d_H(u) + d_{H'}(v) \\ &\leq \frac{m}{2} + \frac{m}{2} + (\ell - 1) + (\ell' - 1) \\ &< n - d(u, v) + 1. \end{aligned}$$

Next, suppose that $k = 2$. Then, it is easy to see that $d(u, v) = 3$. We now consider the case when m is even and $d_C(u) = d_C(v) = m/2$. From the choice of Q , it is easy to check that $N_C(u) = \{u_1, u_3, \dots, u_{m-1}\}$ and $N_C(v) = \{u_2, u_4, \dots, u_m\}$, hence, $uu_3u_2vu_4u_5 \dots u_mu_1u$ is a cycle longer than C . This contradiction to the choice of C , together with (3.1), yields $d_C(u) + d_C(v) < m$. It follows that

$$\begin{aligned} d(u) + d(v) &= d_C(u) + d_C(v) + d_H(u) + d_{H'}(v) \\ &< m + (\ell - 1) + (\ell' - 1) \\ &= n - d(u, v) + 1. \end{aligned}$$

Finally, we consider the case when $k \geq 3$.

If $d_C(u) = 1$, then we see from the choice of Q that v is not adjacent with any vertex of the segment $u_{m-(k-3)} \dots u_mu_1u_2 \dots u_{k-1}$ of C . Note that $m \geq k + (k - 2) = 2k - 2$. Furthermore, since v cannot be inserted into the segment $u_ku_{k+1} \dots u_{m-(k-2)}$ of C , we deduce from Lemma 3.3 that

$$d_C(v) \leq \left\lceil \frac{[m - (k - 2)] - k + 1}{2} \right\rceil \leq \frac{m - 2k + 4}{2}.$$

It follows that

$$\begin{aligned} d(u) + d(v) &= d_C(u) + d_C(v) + d_H(u) + d_{H'}(v) \\ &\leq 1 + \frac{m - 2k + 4}{2} + (\ell - 1) + (\ell' - 1) \\ &\leq 1 + n - \frac{m}{2} - k \\ &< n - (k + 1) + 1 \\ &\leq n - d(u, v) + 1. \end{aligned}$$

Thus, we have $d_C(u) \geq 2$. By the same argument as above, we have $d_C(v) \geq 2$, too. Define

$$\begin{aligned} \alpha &= \min\{i \mid i \geq 2 \text{ and } uu_i \in E(G)\} \quad \text{and} \\ \beta &= \max\{j \mid j \leq m \text{ and } vu_j \in E(G)\}. \end{aligned}$$

From the choice of Q , we conclude that $\alpha \geq 2k - 1$ and $\beta \leq m - k + 2$. Since u (v , respectively) cannot be inserted into the segment $u_\alpha \dots u_m u_1$ with $m + 2 - \alpha$ vertices (the segment $u_k u_{k+1} \dots u_\beta$ with $\beta - k + 1$ vertices, respectively) of C , we have

$$\begin{aligned} d_C(u) &\leq \left\lceil \frac{m+2-\alpha}{2} \right\rceil \leq \left\lceil \frac{m+2-(2k-1)}{2} \right\rceil = \left\lceil \frac{m-2k+3}{2} \right\rceil \\ \text{and } d_C(v) &\leq \left\lceil \frac{\beta-k+1}{2} \right\rceil \leq \left\lceil \frac{(m-k+2)-k+1}{2} \right\rceil = \left\lceil \frac{m-2k+3}{2} \right\rceil. \end{aligned}$$

It follows that

$$\begin{aligned} d(u) + d(v) &= d_C(u) + d_C(v) + d_H(u) + d_{H'}(v) \\ &\leq 2 \cdot \left\lceil \frac{m-2k+3}{2} \right\rceil + (\ell - 1) + (\ell' - 1) \\ &\leq [(m - 2k + 3) + 1] + (\ell - 1) + (\ell' - 1) \\ &\leq n - 2k + 2 \\ &\leq n - (k + 1) + (3 - k) \\ &< n - (k + 1) + 1 \\ &\leq n - d(u, v) + 1. \end{aligned}$$

The proof of Claim 2 is complete. $\square$

Claim 3. $G - V(C)$ is a complete graph.

Proof. By Claim 2, the subgraph $G - V(C)$ is connected. Denote $H = G - V(C)$ and $\ell = |V(H)|$. Clearly, we only need to consider the case when $\ell \geq 3$.

Firstly, we show the following statements: If uu_i is an edge of G with $u \in V(H)$ and $1 \leq i \leq m$, then we have

- 1) $d_C(u_{i-1}) \leq m - d_C(u)$, where $u_0 = u_m$ for $i = 1$,
- 2) $d_H(u) = \ell - 1$.

To prove 1), we assume that there is an integer j with $1 \leq j \leq m$ with $uu_j, u_{i-1}u_{j-1} \in E(G)$. Then, $u_{i-1}u_{j-1}u_{j-2} \dots u_i uu_j \dots u_{i-1}$ is a cycle longer than C . This contradiction to the choice of C implies that $d_C(u_{i-1}) \leq m - d_C(u)$.

The statement 2) can be confirmed indirectly. Suppose thus that $d_H(u) < \ell - 1$. From the choice of C and the fact that H is connected, we see $d_H(u_{i-1}) = 0$ and $d(u, u_{i-1}) = 2$. It follows from 1) that

$$\begin{aligned} d(u) + d(u_{i-1}) &= d_H(u) + d_C(u) + d_H(u_{i-1}) + d_C(u_{i-1}) \\ &\leq d_H(u) + d_C(u) + 0 + (m - d_C(u)) \\ &< (\ell - 1) + m \\ &= n - d(u, u_{i-1}) + 1, \end{aligned}$$

a contradiction. Therefore, $d_H(u) = \ell - 1$ holds.

Next, we show that H is complete. Suppose to the contrary that H is not complete. Then, H contains a vertex v with $d_H(v) < \ell - 1$. Since G is connected, there exists an

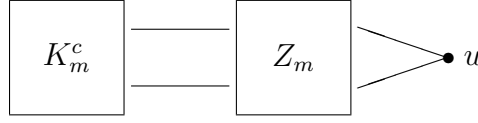


Figure 3.1: A graph $Z_m \vee (K_m^c + \{u\})$ in $\mathcal{L}_{2m+1}$.

edge uu_k with $u \in V(H)$ and $1 \leq k \leq m$. By 2) above, we have $d_H(u) = \ell - 1$ and $d_C(v) = 0$. It follows that $d(u_{k-1}, v) = 3$. Combining with 1) above, we obtain

$$\begin{aligned}
 d(u_{k-1}) + d(v) &= d_H(u_{k-1}) + d_C(u_{k-1}) + d_H(v) + d_C(v) \\
 &< 0 + (m - d_C(u)) + (\ell - 1) + 0 \\
 &= n - d_C(u) - 1 \\
 &= n - d(v, u_{k-1}) + 1,
 \end{aligned}$$

a contradiction.

The proof of the theorem is complete. ■

Corollary 3.5 (Rahman and Kaykobad [68]) *Let G be a connected graph with $n \geq 3$ vertices. If*

$$d(u) + d(v) \geq n - d(u, v) + 1$$

for every pair of nonadjacent vertices u and v , then G has a Hamiltonian path.

3.1.2 Hamiltonian cycle

We define a special class of non-Hamiltonian graphs, namely,

$$\mathcal{L}_{2m+1} = \{ Z_m \vee (K_m^c + \{u\}) \mid Z_m \text{ is a graph with } m \text{ vertices} \},$$

where K_m^c is a set of m vertices (also as the complement of the complete graph K_m) and u is another single vertex, furthermore, the edge set of $Z_m \vee (K_m^c + \{u\})$ consists of $E(Z_m)$ and $\{xy \mid x \in V(Z_m) \text{ and } y \in K_m^c \cup \{u\}\}$ (s. Figure 3.1).

It is easy to check that every graph $G \in \mathcal{L}_{2m+1}$ for $m \geq 2$ is non-Hamiltonian, but it satisfies the condition that $d(u) + d(v) \geq (2m + 1) - 1$ for each pair of vertices u and v with $d(u, v) = 2$.

Theorem 3.6 (S. Li, R. Li & Feng [53]) *Let G be a 2-connected graph with $n \geq 3$ vertices. If*

$$d(u) + d(v) \geq n - 1$$

for every pair of vertices u and v with $d(u, v) = 2$, then G is Hamiltonian, unless n is odd and $G \in \mathcal{L}_n$.

Proof. Let G be a graph satisfying the condition of Theorem 3.6. It is obvious that if $3 \leq n \leq 4$, then G is Hamiltonian. For $n \geq 5$, we prove that if G is non-Hamiltonian, then n is odd and $G \in \mathcal{L}_n$.

It is easy to see that G satisfies the conditions of Theorem 3.1 for $c = n - 1$. Therefore, G contains a cycle of length $n - 1$, denoted by $C = u_1 u_2 \dots u_{n-1} u_1$. Let $\{v\} = V - V(C)$. Since G is 2-connected, there is a C -bypass. Let $P = v_1 v v_2$ be a C -bypass with minimum gap among all C -bypasses in G , and assume without loss of generality that $v_1 = u_1$ and $v_2 = u_\gamma$.

Suppose that G is non-Hamiltonian. Then, we have $\gamma \geq 3$. From the choice of P , we see that $v \notin N(u_i)$ for $2 \leq i \leq \gamma - 1$. Let

$$C' = u_2 u_3 \dots u_{\gamma-1} \quad \text{and} \quad C'' = u_\gamma u_{\gamma+1} \dots u_{n-1} u_1.$$

We consider the following two cases.

Case 1: $\gamma \geq 4$.

Because of $d(u_2, v) = d(u_{\gamma-1}, v) = 2$, we have

$$\min\{d(u_2) + d(v), d(u_{\gamma-1}) + d(v)\} \geq n - 1. \quad (3.2)$$

Since P has the minimum gap among all C -bypasses, $|V(C'')| \geq \gamma \geq 4$ holds, and furthermore, we conclude that

$$\begin{aligned} d(v) &= d_{C''}(v) = d_C(v) \leq \frac{n-1}{\gamma-1} \\ &= \frac{|V(C'')| + (\gamma-2)}{\gamma-1} \\ &= \frac{|V(C'')|}{2} - \frac{(\gamma-3)(|V(C'')| - 2) - 2}{2(\gamma-1)} \\ &\leq \frac{|V(C'')|}{2}. \end{aligned} \quad (3.3)$$

Moreover, the following holds:

$$\begin{aligned} d(u_j) &= d_{C'}(u_j) + d_{C''}(u_j) \\ &\leq (|V(C')| - 1) + d_{C''}(u_j) \quad \text{for } j = 2, \gamma - 1. \end{aligned} \quad (3.4)$$

It follows from (3.2)–(3.4) that

$$\begin{aligned} 2(n-1) &\leq d(u_2) + d(v) + d(u_{\gamma-1}) + d(v) \\ &\leq 2(|V(C')| - 1) + d_{C''}(u_2) + d_{C''}(u_{\gamma-1}) + 2 \cdot \frac{|V(C'')|}{2} \\ &= 2n - 4 - |V(C'')| + d_{C''}(u_2) + d_{C''}(u_{\gamma-1}), \end{aligned}$$

and hence, we have $d_{C''}(u_2) + d_{C''}(u_{\gamma-1}) \geq |V(C'')| + 2$. By Lemma 3.3, the path $C' = u_2 u_3 \dots u_{\gamma-1}$ can be inserted into C'' . So, we obtain a Hamiltonian cycle of G , a contradiction.

Case 2: $\gamma = 3$.

Since u_2 (v , respectively) can not be inserted into the cycle $u_1vu_3 \dots u_{n-1}u_1$ (C , respectively), we have $\max\{d(u_2), d(v)\} \leq (n-1)/2$. By recalling $d(u_2) + d(v) \geq n-1$, we conclude that $d(u_2) = d(v) = (n-1)/2$. Clearly, n is an odd integer. Now, it is easy to check that $N_G(u_2) = N_G(v) = \{u_1, u_3, \dots, u_{n-2}\}$. Similarly, we can verify that $N_G(u_{2k}) = N_G(u_2)$ for $k = 2, \dots, (n-1)/2$. Let

$$Z_{\frac{n-1}{2}} = G[\{u_1, u_3, \dots, u_{n-2}\}].$$

Then, we see that

$$G = Z_{\frac{n-1}{2}} \vee (K_{\frac{n-1}{2}}^c + \{v\}) \in \mathcal{L}_n.$$

The proof of the theorem is complete. ■

It is easy to see that the condition of Theorem 3.6 is weaker than the condition of Ore's theorem (see Theorem 1.3).

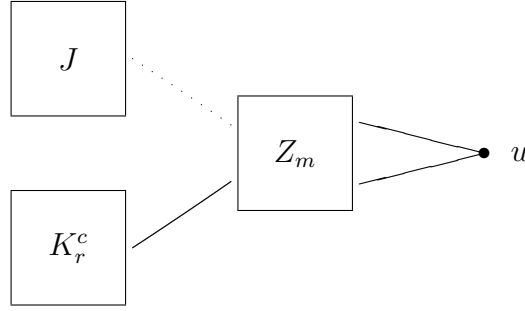
The next example shows that there are graphs, whose Hamiltonicity can be verified by Theorem 3.6, but neither by Ore's theorem nor by Fan's theorem (see Theorem 3.2).

Example 3.7 *Let G be a 2-connected, k -regular graph with $2k+1$ vertices. It is easy to see that G does not satisfy the assumption either of Theorem 1.3 or of Theorem 3.2. However, it is not difficult to check that G satisfies the assumption of Theorem 3.6 and $G \notin \mathcal{L}_{2k+1}$. Hence, G is Hamiltonian.*

3.2 k -ordered Hamiltonian graphs on lower connectivity

We first give a sufficient condition for a k -ordered graph to be k -ordered Hamiltonian. To present the result, we consider a special class $\mathcal{A}_{2m+1}$ of graphs. A graph G is said to be in $\mathcal{A}_{2m+1}$ if and only if the following holds: G contains $2m+1$ vertices and can be partitioned into four subgraphs K_r^c , Z_m , J and a single vertex u , where Z_m is an arbitrary graph with m vertices, K_r^c is a set of r independent vertices (also as the complement of the complete graph K_r) with $r \geq m-k$, and J is a graph with $m-r$ vertices such that if $|V(J)| > \lfloor k/2 \rfloor$, say $|V(J)| = \lfloor k/2 \rfloor + t$, then J has at least $2t$ pairs of nonadjacent vertices if k is even, and at least $2t-1$ pairs of nonadjacent vertices if k is odd. Furthermore, the edge set consists of $E(J)$, $E(Z_m)$, $\{xy \mid x \in V(Z_m) \text{ and } y \in K_r^c \cup \{u\}\}$ and some edges between J and Z_m (s. Figure 3.2).

Consider an ordered sequence $S = \{x_1, x_2, \dots, x_k\}$ of G , where S contains all vertices of J and some vertices of Z_m if $|V(J)| < k$, and S is arranged such that x_i, x_{i+1} are nonadjacent if $x_i, x_{i+1} \in V(J)$. In this section, we always set $x_{k+1} = x_1$. We see that G has a Hamiltonian cycle that encounters the vertices of S in order if and only if the graph G' obtained from G by deleting all edges of the subgraph J has the same property. It is easy to check that G' is non-Hamiltonian, and hence, not k -ordered Hamiltonian. So G is also not k -ordered Hamiltonian. However, it is possible that G satisfies the condition $d(u) + d(v) \geq (2m+1) - 1$ for each pair of vertices u, v with $d(u, v) = 2$.

Figure 3.2: A graph G in $\mathcal{A}_{2m+1}$.

Theorem 3.8 (R. Li, S. Li & Guo [50]) *Let $k \geq 4$ be an integer and let G be a $(k+1)$ -connected, k -ordered graph of order $n \geq 4k + 3$. If for every pair of vertices u and v in $V(G)$ with $d(u, v) = 2$,*

$$d(u) + d(v) \geq n - 1,$$

then G is k -ordered Hamiltonian unless n is odd and $G \in \mathcal{A}_n$.

Proof. Let $S = \{x_1, x_2, \dots, x_k\}$ be an ordered sequence of k vertices of G . Since G is k -ordered, there is a cycle C that encounters these vertices in order. Choose such a cycle C such that $|V(C)|$ is as large as possible. Assume $V(C) \neq V(G)$, let $L = G - V(C)$ and H be a component of L . Since G is $(k+1)$ -connected, $|N_C(H)| \geq k+1$ and hence $|N_C(H) \cap x_i \vec{C} x_{i+1}| \geq 2$ for some i , $1 \leq i \leq k$. We may assume $|N_C(H) \cap x_k \vec{C} x_1| \geq 2$. Choose a pair of distinct vertices y_1, y_2 in $N_C(H) \cap x_k \vec{C} x_1$ so that x_k, y_1, y_2 and x_1 appear in order along C , and so that $y_1 \vec{C} y_2$ is as short as possible (s. Figure 3.3). Possibly $x_k = y_1$ or $y_2 = x_1$. Let $z_i \in N_H(y_i)$ for $i = 1, 2$. Note that possibly $z_1 = z_2$. Since H is connected, there exists a path P from z_1 to z_2 in H . Then $C_1 = y_2 \vec{C} y_1 z_1 \vec{P} z_2 y_2$ is a cycle which encounters $x_1, x_2, \dots, x_k$ in order. If $y_2 = y_1^+$, then $|V(C_1)| > |V(C)|$, which contradicts the maximality of $|V(C)|$. Therefore, $y_2 \neq y_1^+$. Let

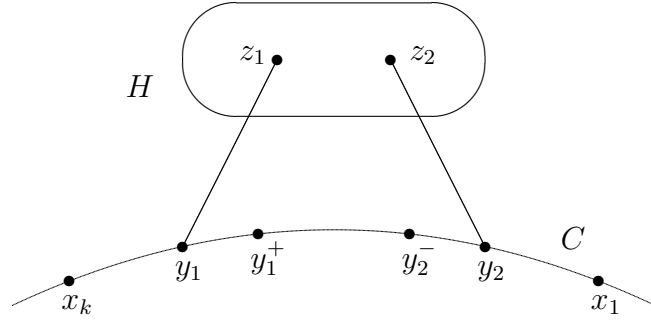
$$C' = y_1^+ \vec{C} y_2^- \quad \text{and} \quad C'' = y_2 \vec{C} y_1.$$

Note that possibly $y_1^+ = y_2^-$. By the choice of y_1 and y_2 , $N_C(H) \cap V(C') = \emptyset$. In particular, $y_1^+ z_1, z_2 y_2^- \notin E(G)$, and hence $d(y_1^+, z_1) = d(z_2, y_2^-) = 2$. By the maximality of $|V(C)|$, we have $N_L(z_1) \cap N_L(y_1^+) = \emptyset$, i.e., $d_L(z_1) + d_L(y_1^+) \leq |V(L)| - 1 = n - |V(C)| - 1$. Therefore,

$$\begin{aligned} n - 1 &\leq d(z_1) + d(y_1^+) \\ &= d_L(z_1) + d_L(y_1^+) + d_{C'}(y_1^+) + d_{C''}(z_1) + d_{C''}(y_1^+) \\ &\leq n - |V(C)| - 1 + |V(C')| - 1 + d_{C''}(z_1) + d_{C''}(y_1^+), \end{aligned} \tag{3.5}$$

which implies that $d_{C''}(z_1) + d_{C''}(y_1^+) \geq |V(C'')| + 1$. By Lemma 3.3, it follows that

$$d_{C''}(z_1) \leq \frac{|V(C'')| + 1}{2} \tag{3.6}$$

Figure 3.3: The edges between H and the interval $[x_k, x_1]$.

for otherwise z_1 can be inserted into C'' . Then we have $d_{C''}(y_1^+) \geq (|V(C'')| + 1)/2$. Similarly, $d_{C''}(z_2) \leq (|V(C'')| + 1)/2$, $d_{C''}(y_2^-) \geq (|V(C'')| + 1)/2$.

Notice from (3.5) and (3.6) that, if $d_{C'}(y_1^+) < |V(C')| - 1$ or $d_L(z_1) + d_L(y_1^+) < n - |V(C)| - 1$ or $d_{C''}(z_1) < (|V(C'')| + 1)/2$, then $d_{C''}(y_1^+) > (|V(C'')| + 1)/2$, and hence $d_{C''}(y_1^+) + d_{C''}(y_2^-) > |V(C'')| + 1$. By Lemma 3.3, C' can be inserted into C'' . We obtain a longer cycle than C , which does not change the order of $x_1, x_2, \dots, x_k$, a contradiction. So we have

$$d_{C'}(y_1^+) = |V(C')| - 1, \quad d_L(z_1) + d_L(y_1^+) = n - |V(C)| - 1,$$

$$d_{C''}(z_1) = \frac{|V(C'')| + 1}{2}, \quad d_{C''}(y_1^+) = \frac{|V(C'')| + 1}{2}.$$

Similarly,

$$d_{C'}(y_2^-) = |V(C')| - 1, \quad d_L(z_2) + d_L(y_2^-) = n - |V(C)| - 1,$$

$$d_{C''}(z_2) = \frac{|V(C'')| + 1}{2}, \quad d_{C''}(y_2^-) = \frac{|V(C'')| + 1}{2}.$$

Clearly, $|V(C'')|$ is an odd integer. Furthermore, if we suppose $C'' = w_1 w_2 \dots w_l$ with $w_1 = y_2$, $w_l = y_1$ and $l = |V(C'')|$, then

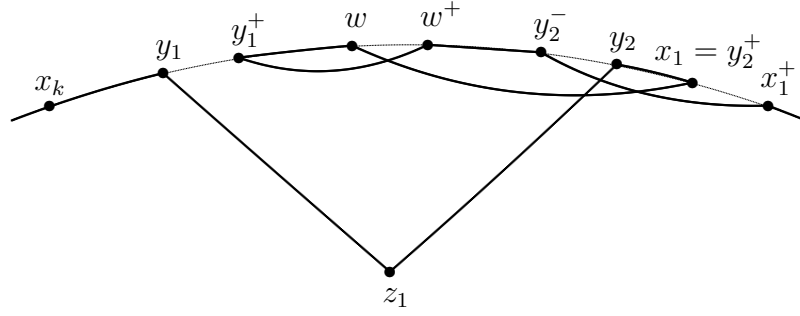
$$N(y_1^+) = N(y_2^-) = \{w_1, w_3, \dots, w_l\} = N(z_1) = N(z_2).$$

We consider the following two cases.

Case 1: $y_1^+ \neq y_2^-$.

Recall that y_1 and y_2 are chosen so that $y_1 \vec{C} y_2$ is as short as possible. Then we have $|[y_2, x_1]| \leq 2$, $|[x_k, y_1]| \leq 2$ and $|[x_i, x_{i+1}]| \leq 1$ for $1 \leq i \leq k - 1$. So, $|V(C'')| \leq 2k + 1$.

We confirm that L is connected. Suppose to the contrary that H' is another component apart from H in $V(D) - V(C)$. From the equality $d_L(z_1) + d_L(y_1^+) = n - |V(C)| - 1$ ($d_L(z_2) + d_L(y_2^-) = n - |V(C)| - 1$, respectively), we see that $N_L(z_1) \cup N_L(y_1^+) = V(L) - \{z_1\}$ and $N_L(z_1) \cap N_L(y_1^+) = \emptyset$ ($N_L(z_2) \cup N_L(y_2^-) = V(L) - \{z_2\}$ and $N_L(z_2) \cap N_L(y_2^-) = \emptyset$,

Figure 3.4: In the case when $x_1 = y_2^+$.

respectively). Let $u \in V(H')$. Since z_1 and z_2 are nonadjacent to u , then y_1^+ and y_2^- are adjacent to u , which contradicts the choice of y_1 and y_2 . Therefore, $V(D) - V(C)$ contains only one component.

We also confirm that either L is complete or L is Hamiltonian-connected. Suppose that L is neither complete nor Hamiltonian connected. Then there exist $u, v \in V(L)$ such that $uv \notin E(G)$ and $d_L(u) + d_L(v) \leq |V(L)|$. Since y_1^+ is nonadjacent to any vertex of L , $N_L(z_1) = V(L) - \{z_1\}$, and hence $d(u, v) = 2$. According to the adjacency between z_1 and C'' , we have $N_{C''}(u) \subseteq N_{C''}(z_1)$ and $N_{C''}(v) \subseteq N_{C''}(z_1)$. Specially, u is nonadjacent to y_1 , otherwise we consider u instead of z_1 in (3.5), and then $d_{C''}(y_1^+) > (|V(C'')| + 1)/2$, a contradiction. So $d_{C''}(u) < d_{C''}(z_1)$. Similarly, $d_{C''}(v) < d_{C''}(z_1)$. Therefore,

$$d(u) + d(v) < |V(L)| + 2d_{C''}(z_1) = |V(L)| + 2 \cdot \frac{|V(C'')| + 1}{2} \leq n - 1,$$

a contradiction.

Now, we continue our proof for this case. Recall that $|[y_2, x_1]| \leq 2$ i.e., $x_1 = y_2^+$ or y_2 . If $x_1 = y_2^+$, then x_1 is nonadjacent to any vertex of L , specially, x_1 is nonadjacent to z_1 , and hence $d(x_1, z_1) = 2$. Since $|V(C')| \geq |V(L)|$ and $|V(C'')| \leq 2k + 1$, we have

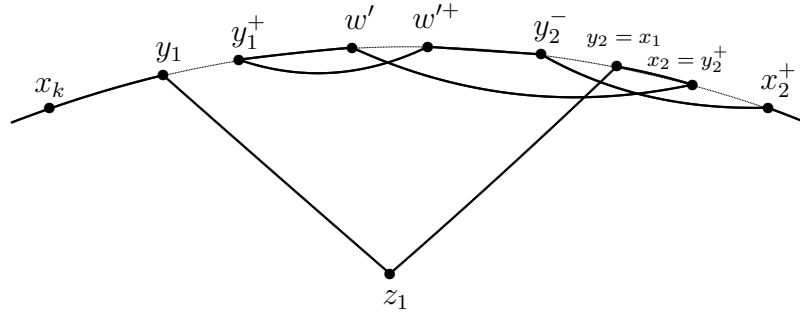
$$|V(C')| \geq \frac{n - |V(C'')|}{2} \geq \frac{4k + 3 - (2k + 1)}{2} = k + 1 \quad \text{and} \quad \frac{|V(C'')| - 1}{2} \leq k.$$

Then x_1 is adjacent to some vertex of C' , otherwise

$$\begin{aligned} d(x_1) + d(z_1) &\leq |V(C'')| - 1 + \frac{|V(C'')| + 1}{2} + n - |V(C)| - 1 \\ &= n - 1 + \frac{|V(C'')| - 1}{2} - |V(C')| \\ &< n - 1, \end{aligned} \tag{3.7}$$

which is a contradiction. Let $w \in V(C')$ with $wx_1 \in E(G)$. Specially, $w \neq y_2^-$. Then $x_1 w \overrightarrow{C} y_1^+ w^+ \overrightarrow{C} y_2^- x_1^+ \overrightarrow{C} y_1 z_1 y_2 x_1$ is a longer cycle than C (s. Figure 3.4), which contains $x_1, x_2, \dots, x_k$ in order, a contradiction.

If $x_1 = y_2$, then $y_2^+ = x_2$, otherwise we consider y_2 and $(y_2^+)^+$ instead of y_1 and y_2 , respectively. Similarly, x_2 is adjacent to some vertex of C' , otherwise it induces the

Figure 3.5: In the case when $x_1 = y_2$.

same contradiction as (3.7) by replacing x_1 with x_2 . So there is $w'(\neq y_2^-) \in V(C')$ with $w'x_2 \in E(G)$. Then $x_1x_2w'\overleftarrow{C}y_1^+w'^+\overrightarrow{C}y_2^-x_2^+\overrightarrow{C}y_1z_1x_1$ is a longer cycle than C (s. Figure 3.5), which contains $x_1, x_2, \dots, x_k$ in order, a contradiction.

Case 2: $y_1^+ = y_2^-$.

We see that $z_1 = z_2$. Analogously, $d(z_1, y_1^+) = 2$ and

$$n - 1 \leq d(z_1) + d(y_1^+) \leq \frac{|V(C)|}{2} + \frac{|V(C)|}{2} + |V(L)| - 1 = n - 1.$$

So $d_C(z_1) = d_C(y_1^+) = |V(C)|/2$ and $d_L(z_1) = |V(L)| - 1$, and hence L contains only one component H . We also see that $V(G) - V(C) = V(H) = \{z_1\}$, otherwise for any $u \in V(H) - \{z_1\}$, $N_C(u) \subseteq N_C(z_1)$ and u is nonadjacent to any vertex of $[x_k, x_1]$, and hence $G - N_C(u) \cup \{z_1\}$ is not connected, a contradiction with the connectivity of G . So $n = |V(C'')| + 2$ is odd. Let

$$Z_{\frac{n-1}{2}} = G[N(z_1)], \quad K_r^c = G - (N(z_1) \cup S),$$

$$J = G[(V(G) - N(z_1)) \cap S], \quad u = z_1.$$

For each $y \in V(K_r^c)$, y has the same property as z_1 . So $N(y) = N(z_1)$, and hence K_r^c is the complement of a complete graph. Additionally, if $x_i, x_{i+1} \in V(J)$, then $x_ix_{i+1} \notin E(G)$, otherwise $x_ix_{i+1}\overleftarrow{C}x_i^+z_1x_{i+1}^+\overrightarrow{C}x_i$ is a longer cycle than C , which contains $x_1, x_2, \dots, x_k$ in order, a contradiction. So $G \in \mathcal{A}_n$. $\blacksquare$

Corollary 3.9 *Let $k \geq 4$ be an integer. If G is a k -ordered, $(k+1)$ -connected graph of order $n \geq 4k+3$ such that $d(u) + d(v) \geq n$ for every pair of vertices u and v with $d(u, v) = 2$, then G is k -ordered Hamiltonian.*

To show Theorem 1.8, Faudree et al. presented the following to make the proof easier.

Theorem 3.10 (Faudree et al. [26]) *A graph G is k -ordered Hamiltonian if G is a k -ordered, $(k+1)$ -connected graph of order $n \geq 3$ such that $d(u) + d(v) \geq n$ for every pair u, v of nonadjacent vertices of G .*

Considering every pair of vertices with the distance 2, we have:

Lemma 3.11 *Let k be an integer with $k \geq 3$ and let G be a $(k+1)$ -connected, k -ordered graph of order n . If for every pair of vertices u and v in G with $d(u, v) = 2$,*

$$d(u) + d(v) \geq n,$$

then G is k -ordered Hamiltonian.

Proof. Let $S = \{x_1, x_2, \dots, x_k\}$ be an ordered sequence of k vertices of G . Since G is k -ordered, there is a cycle that encounters these vertices in order. Choose such a cycle C such that $|V(C)|$ is as large as possible. Assume $|V(C)| < n$ and let $H = G - V(C)$ and L be a component of H . Since G is $(k+1)$ -connected, $|N_C(L)| \geq k+1$ and hence there is an interval $[x_i, x_{i+1}]$ to have two vertices with neighbors in L , where we consider $x_{k+1} = x_1$. Without loss of generality, assume $|N_C(L) \cap [x_k, x_1]| \geq 2$. Choose a pair of distinct vertices y_1, y_2 in $N_C(L) \cap [x_k, x_1]$ so that x_k, y_1, y_2 and x_1 appear in this order along C , and so that $[y_1, y_2]$ is as short as possible. Possibly, $x_k = y_1$ or $y_2 = x_1$. Let $z_i \in N_L(y_i)$ for $i = 1, 2$. Since L is connected, there is a path P from z_1 to z_2 in L . Then $C' = z_1 P z_2 y_2 \overrightarrow{C} y_1 z_1$ is a cycle which encounters S in order. If $y_2 = y_1^+$, then C' is longer than C , which contradicts the maximality of $|V(C)|$. Therefore, $y_2 \neq y_1^+$. Furthermore $y_2^- \neq y_1^+$. Indeed, $y_2^- = y_1^+$ implies $z_1 = z_2$. Note $d(y_1^+, z_1) = 2$. Neither y_1^+ nor z_1 can be adjacent to more than half the vertices of C which forces

$$d(y_1^+) + d(z_1) \leq 2 \cdot \frac{|V(C)|}{2} + |V(H)| - 1 < n,$$

which is a contradiction.

Let $s = |[y_1^+, y_2^-]|$, $t = |V(H)|$ and $t_1 = |V(L)|$. Because z_1 and z_2 have no neighbours in $[y_1^+, y_2^-]$,

$$d(z_1) + d(z_2) \leq 2 \cdot \left((t_1 - 1) + \frac{n - s - t + 1}{2} \right).$$

If y_1^+ is adjacent to a vertex, say w , on $C - [y_1^+, y_2^-]$, y_2^- cannot be adjacent to the successor, w^+ , on C or else $y_1 z_1 P z_2 y_2 \overrightarrow{C} w y_1^+ \overrightarrow{C} y_2^- w^+ \overrightarrow{C} y_1$ is a longer cycle than C and contains S in order. Hence,

$$d(y_1^+) + d(y_2^-) \leq 2(s - 1) + n - s - t + 1 + 2(t - t_1).$$

Since $d(y_1^+, z_1) = d(y_2^-, z_2) = 2$, the initial condition implies $d(z_1) + d(z_2) + d(y_1^+) + d(y_2^-) \geq 2n$. But, by the previous two inequalities $d(z_1) + d(z_2) + d(y_1^+) + d(y_2^-) \leq 2n - 2$, which is a contradiction. The lemma follows. $\blacksquare$

To prove the main result, we firstly prove some lemmas.

Lemma 3.12 *Let G be a connected graph of order n and q be an integer with $0 \leq q \leq n - 4$. If*

$$d(u) + d(v) \geq n + q$$

for every pair of vertices u and v with $d(u, v) = 2$, then G is $(q+2)$ -connected.

Proof. We prove this lemma indirectly. Suppose that G is at most $(q + 1)$ -connected. Let S be a minimal cut set with $|S| \leq q + 1$ and H and H' be two components of $G - S$ with $|V(H)| = \ell$ and $|V(H')| = \ell'$. It is clear that $S \subseteq N(H) \cap N(H')$. Let $u \in V(H)$ and $v \in V(H')$ with $d(u, v) = 2$. Then,

$$d(u) + d(v) \leq \ell - 1 + \ell' - 1 + 2|S| \leq n + |S| - 2 < n + q,$$

a contradiction, and the lemma follows. $\blacksquare$

Corollary 3.13 *Let G be a connected graph of order n and k be an integer with $4 \leq k \leq n$. If*

$$d(u) + d(v) \geq n + \frac{3k - 9}{2}$$

for every pair of vertices u and v with $d(u, v) = 2$, then G is $(k + 1)$ -connected for $k \geq 6$ and G is k -connected for $k = 4, 5$.

Lemma 3.14 *Let G be a 4-ordered, 4-connected graph of order $n \geq 11$. If $d(u) + d(v) \geq n$ for every pair of vertices u and v with $d(u, v) = 2$, then G is 4-ordered Hamiltonian.*

Proof. Let $S = \{x_1, x_2, x_3, x_4\}$ be an ordered subset of the vertices of G . Let C be a cycle of maximum order containing all vertices of S in appropriate order. Suppose that C is not a Hamiltonian cycle of G . The four vertices of S split the cycle C into four intervals: $[x_1, x_2]$, $[x_2, x_3]$, $[x_3, x_4]$, $[x_4, x_1]$. Let $H = G - V(C)$ and L be a component of H . Assume there are vertices $x, y \in V(L)$ with distinct neighbours in one of the intervals of C determined by S , say $[x_i, x_{i+1}]$. Note that we allow $x = y$. Let z_1 and z_2 be the immediate successor and predecessor on C to the neighbours of x and y , respectively, according to the orientation of C .

Observe we can choose x and y and their neighbours in C such that none of the vertices on the intervals $[z_1, z_2]$ have neighbours in L . We can also assume that $z_1 \neq z_2$, because $z_1 = z_2$ implies $x = y$ or C is not maximal order. But neither z_1 nor x can be adjacent to more than half the vertices of C which forces

$$d(z_1) + d(x) \leq 2 \cdot \frac{|V(C)|}{2} + |V(H)| - 1 = n - 1,$$

a contradiction.

Let $s = |[z_1, z_2]|$, $t = |V(H)|$ and $t_1 = |V(L)|$. Because x and y have no neighbours in $[z_1, z_2]$,

$$d(x) + d(y) \leq 2 \left((t_1 - 1) + \frac{n - s - t + 1}{2} \right).$$

Similarly, if z_1 is adjacent to a vertex, say w , on $C - [z_1, z_2]$, then z_2 cannot be adjacent to the successor, w^+ , on C or else the segment $[z_1, z_2]$ could be inserted between w and w^+ , while replacing $[z_1, z_2]$ with a path from x to y . Hence,

$$d(z_1) + d(z_2) \leq 2(s - 1) + n - s - t + 1 + 2(t - t_1).$$

Since $d(x, z_1) = d(y, z_2) = 2$, the initial degree condition forces $d(x) + d(y) + d(z_1) + d(z_2) \geq 2n$. But, by the previous two inequalities

$$d(x) + d(y) + d(z_1) + d(z_2) \leq 2n - 2,$$

which is a contradiction. Thus on any interval $[x_i, x_{i+1}]$ of C , there exists at most one vertex with neighbours in L . The connectivity, then, requires each segment $[x_i, x_{i+1}]$ to have exactly one vertex with a neighbour in L . Let y_i be the vertex of the interval $[x_i, x_{i+1}]$ with $v_i y_i \in E(G)$ for some $v_i \in V(L)$, $i = 1, 2, 3, 4$. It is not necessary that all v_i 's are different. Let w_i be the successor of y_i on C . We have $d(v_i, w_i) = 2$. Let P be a (v_1, v_2) -path in L . We see that $w_1 w_2 \notin E(G)$, otherwise the cycle $x_1 \xrightarrow{C} y_1 v_1 \xrightarrow{P} v_2 y_2 \xrightarrow{C} w_1 w_2 \xrightarrow{C} x_1$ is longer than C and contains S in order, a contradiction. Similarly, we have $w_1 w_4 \notin E(G)$. Then $d(w_1) = d_C(w_1) \leq |V(C)| - 3$, and hence

$$4 \geq d_C(v_1) \geq d(v_1) - (|V(L)| - 1) \geq n - d(w_1) - (|V(L)| - 1) \geq 4. \quad (3.8)$$

So $d_C(v_1) = 4$ and $d_C(w_1) = |V(C)| - 3$, which means $N_C(v_1) = \{y_1, y_2, y_3, y_4\}$ and $N_C(w_1) = V(C) - \{w_1, w_2, w_4\}$. Similarly, $N_C(w_i) = V(C) - \{w_i, w_{i-1}, w_{i+1}\}$, $i = 2, 3, 4$ (set $w_5 = w_1$). Note that $w_2 = x_3$, otherwise the cycle $x_1 \xrightarrow{C} y_1 v_1 y_2 \xrightarrow{C} w_1 w_2^+ \xrightarrow{C} y_4 w_2 w_4 \xrightarrow{C} x_1$ is longer than C and contains S in order, a contradiction. Analogously, $w_3 = x_4$, $w_4 = x_1$, $w_1 = x_2$. We also claim that $x_3 = y_3^-$, because $x_3 \neq y_3^-$ implies that $x_1 \xrightarrow{C} y_1 v_1 y_2 \xrightarrow{C} w_1 (= x_2) y_3^- \xrightarrow{C} x_3 (= w_2) y_3 \xrightarrow{C} x_1$ is longer than C and contains S in order, a contradiction. Similarly, $x_1 = y_1^-$, $x_2 = y_2^-$, $x_4 = y_4^-$. We see that $|V(C)| = 8$. Thus $d(w_1, w_2) = 2$ and $d(w_1) + d(w_2) \leq 2(|V(C)| - 3) = 10$, a contradiction. The lemma follows. $\blacksquare$

Lemma 3.15 *Let G be a 5-ordered, 5-connected graph of order $n \geq 37$. If $d(u) + d(v) \geq n$ for every pair of vertices u and v with $d(u, v) = 2$, then G is 5-ordered Hamiltonian.*

Proof. Let $S = \{x_1, x_2, \dots, x_5\}$ be an ordered subset of the vertices of G and C be a cycle of maximum order containing all vertices of S in appropriate order. Let L be a component of $G - V(C)$. There exists at most one vertex with neighbours in L on any interval $[x_i, x_{i+1}]$ of C and the connectivity requires each segment $[x_i, x_{i+1}]$ to have exactly one vertex with a neighbour in L . We still use the notation y_i, v_i and w_i as the proof of Lemma 3.14 and add y_5, v_5, w_5 . Just as the inequality (3.8), we have $4 \leq d_C(v_i) \leq 5$. We firstly consider the case when $d_C(v_i) = 5$ for all $1 \leq i \leq 5$. Since $d(v_i, w_i) = 2$ and w_i is nonadjacent to w_{i-1} and w_{i+1} (set $w_6 = w_1$), then $|V(C)| - 3 \geq d_C(w_i) = d(w_i) \geq |V(C)| - 4$. If y_i is x_i or x_{i+1} , say $y_i = x_i$, then $[x_{i-1}, x_{i+1}]$ contains only one vertex x_i with neighbours in L , which contradicts the connectivity of G . So $y_i \neq x_i$ and $y_i \neq x_{i+1}$, and then $|V(C)| \geq 10$.

We claim that $|(y_i, x_{i+1})| \leq 1$ for all $1 \leq i \leq 5$. Suppose to the contrary, say $|(y_2, x_3)| \geq 2$. Then $w_2^+ \neq x_3$. Because $d_C(w_1) \geq |V(C)| - 4$ and w_1 is nonadjacent to w_2 and w_4 , w_1 is adjacent one of the vertices w_2^+ and $(w_2^+)^+$. Then

$$C_1 = x_1 \xrightarrow{C} y_1 v_1 y_2 \xrightarrow{C} w_1 w_2^+ (\text{or } (w_2^+)^+) \xrightarrow{C} x_1$$

is a cycle containing S in order. In both cases, we have $d_{C_1}(w_2) > |V(C_1)|/2$ (using the fact $|V(C)| \geq 10$). Then w_2 can be inserted in the cycle C_1 . If w_2 is adjacent to w_2^+ , then we obtain a longer cycle than C containing S in order. If w_2 is adjacent to $(w_2^+)^+$, then we obtain a cycle with the same length as C but there are two vertices y_1, y_2 with neighbours in L in the segment $[x_1, x_2]$. We can consider C_1 instead of C to force a contradiction.

We also confirm that $|(x_i, y_i)| \leq 1$ for all $1 \leq i \leq 5$. Suppose $|(x_1, y_1)| \geq 2$. Then $(y_1^-)^- \in (x_1, y_1)$. From $d(v_1, y_2^-) = 2$, we have $d_C(y_2^-) = d(y_2^-) \geq n - 5 \geq |V(C)| - 4$. Note

that y_2^- is nonadjacent to y_1^- , otherwise $x_1 \overrightarrow{C} y_1^- y_2^- \overleftarrow{C} y_1 v_1 y_2 \overrightarrow{C} x_1$ is a longer cycle than C containing S in order. Also, y_2^- is nonadjacent to y_3^- . Then y_2^- is adjacent to one of the vertices $(y_1^-)^-$ and $((y_1^-)^-)^-$. Then

$$C_2 = x_1 \overrightarrow{C} (y_1^-)^- (\text{or } ((y_1^-)^-)^-) y_2^- \overleftarrow{C} y_1 v_1 y_2 \overrightarrow{C} x_1$$

is a cycle containing S in order and y_1^- can be inserted in C_2 . Then we obtain either a longer cycle than C containing S in order or a cycle with the same length as C but containing two vertices y_1, y_2 with neighbours in L in the segment $[x_2, x_3]$, a contradiction.

Then we have $|V(C)| \leq 20$. Notice that $G' = G[V(C) - \{y_1, \dots, y_5\}]$ is complete, otherwise there exist a pair of vertices $u, v \in V(C) - \{y_1, \dots, y_5\}$ with $d(u, v) = 2$,

$$d(u) + d(v) = d_C(u) + d_C(v) \leq 2(|V(C)| - 2) \leq 36,$$

a contradiction. It is not possible since w_1 is nonadjacent to w_2 .

For the case when $d(v_i) = 4$ for some $1 \leq i \leq 5$, say v_i is nonadjacent to y_1 , there exists v_j ($j \neq i$) such that v_j is adjacent to y_1 . The proof is same as above except that we replace v_1 with the path between v_i and v_j in C_1 and C_2 . ■

To prove the next theorem, we also use the following result.

Theorem 3.16 (Znám [13]) *If a graph of order n does not contain a $K_{t,t}$, then it contains at most $\frac{1}{2}((t-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + \frac{t-1}{2}n)$ edges.*

Theorem 3.17 (R. Li, S. Li & Guo [50]) *Let $k \geq 4$ be an integer and let G be a graph of order $n \geq 27k^3$. If for every pair of vertices u and v with $d(u, v) = 2$,*

$$d(u) + d(v) \geq n + \frac{3k-9}{2},$$

then G is k -ordered Hamiltonian.

Proof. Let $S = \{x_1, x_2, \dots, x_k\}$ be an ordered set of vertices of G . Note that by Corollary 3.9, 3.13 and Lemma 3.14, 3.15, it is enough to show that G is k -ordered. The proof will be split into cases according to the connectivity of the graph.

Case 1: $\kappa(G) \geq 9k$.

We claim that there exists a shortest (x_1, x_2) -path P_1 in $G - \{x_3, \dots, x_k\}$, whose length is no more than 5. If not, we can label a path of minimum length as $u_1 u_2 \dots u_\ell$ with $x_1 = u_1$ and $x_2 = u_\ell$, where $\ell \geq 7$. Note that u_1 and u_5 , u_3 and u_7 are nonadjacent and have disjoint neighbourhoods in G , and $d(u_1, u_3) = d(u_5, u_7) = 2$. We have

$$2(n-2) \geq d(u_1) + d(u_5) + d(u_3) + d(u_7) \geq 2 \cdot \left(n + \frac{3k-9}{2} \right) = 2n + 3k - 9,$$

which implies that $3k \leq 5$, a contradiction.

In the following, find a cycle in G by finding a shortest (x_2, x_3) -path P_2 in $G - \{x_4, \dots, x_k\} - (V(P_1) \setminus \{x_2\})$, then the shortest (x_3, x_4) -path P_3 in $G - \{x_5, \dots, x_k\} -$

$V(P_1) - V(P_2) \setminus \{x_3\}$, and continue this process to generate $k-1$ internally disjoint paths P_i for $i = 1, \dots, k-1$. We will prove that the length of shortest path from x_i to x_{i+1} is no more than 9. Suppose that there is a path of minimum length $v_1 v_2 \dots v_\ell$ with $x_i = v_1$ and $x_{i+1} = v_\ell$, where $\ell \geq 11$. Then v_1, v_6 and v_{11} are all mutually nonadjacent and have mutually disjoint neighbourhoods in $G - (S \setminus \{x_i, x_{i+1}\}) - V(P_1) - \dots - V(P_{i-2}) - (V(P_{i-1}) \setminus \{x_i\})$.

If $d(v_1, v_6) = d(v_6, v_{11}) = d(v_1, v_{11}) = 2$, then

$$\begin{aligned} n + 3(9i + (k - i)) &\geq \frac{1}{2} (d(v_1) + d(v_6) + d(v_6) + d(v_{11}) + d(v_{11}) + d(v_1)) \\ &\geq \frac{3}{2} \cdot \left(n + \frac{3k - 9}{2} \right) \end{aligned}$$

is a contradiction for $n \geq 60k$.

If there is only one of $\{d(v_1, v_6), d(v_6, v_{11}), d(v_1, v_{11})\}$ more than two, say $d(v_1, v_{11}) > 2$, then $d(v_1) + d(v_{11}) \leq n - 2$, and hence $d(v_1) \leq (n - 2)/2$ or $d(v_{11}) \leq (n - 2)/2$. Anyway $d(v_6) \geq n + (3k - 9)/2 - (n - 2)/2 = n/2 + (3k - 7)/2$. Since $d(v_1, v_3) = d(v_9, v_{11}) = 2$,

$$\begin{aligned} n + 3(9i + (k - i)) &\geq d(v_6) + d(v_i) + d(v_j) \\ &\geq \left(\frac{n}{2} + \frac{3k - 7}{2} \right) + \left(n + \frac{3k - 9}{2} \right), \end{aligned}$$

where $i = 1$ or 3 and $j = 9$ or 11 . This is a contradiction for $n \geq 60k$.

If there are exactly two of $\{d(v_1, v_6), d(v_6, v_{11}), d(v_1, v_{11})\}$ more than two, say $d(v_1, v_6) > 2$ and $d(v_6, v_{11}) > 2$, then $d(v_1) + d(v_6) \leq n - 2$, $d(v_6) + d(v_{11}) \leq n - 2$ and $d(v_1) + d(v_{11}) \geq n + (3k - 9)/2$, which implies that $d(v_6) \leq (n - 2)/2$, and hence $d(v_4) \geq n/2 + (3k - 7)/2$, $d(v_8) \geq n/2 + (3k - 7)/2$. If $d(v_1) \geq (n + (3k - 9)/2)/2$, then we consider the vertices v_1, v_4 and v_8 to obtain a contradiction. Otherwise we consider the vertices v_4, v_8 and v_{11} .

We remain to consider the case when $d(v_1, v_6) > 2, d(v_1, v_{11}) > 2$ and $d(v_6, v_{11}) > 2$. Then $d(v_1) + d(v_6) \leq n - 2$, $d(v_1) + d(v_{11}) \leq n - 2$ and $d(v_6) + d(v_{11}) \leq n - 2$. If there is a vertex among v_1, v_6 and v_{11} such that its degree is not less than $(n - 2)/2$, without loss of generality, let $d(v_6) \geq (n - 2)/2$. Then, $d(v_1) < (n - 2)/2$ and $d(v_{11}) < (n - 2)/2$, and hence $d(v_3) \geq n/2 + (3k - 7)/2$ and $d(v_9) \geq n/2 + (3k - 7)/2$. We obtain the contradiction by considering the vertices v_3, v_6 and v_9 . So assume that $d(v_1) < (n - 2)/2$, $d(v_6) < (n - 2)/2$ and $d(v_{11}) < (n - 2)/2$. Then $d(v_3) > n/2 + (3k - 7)/2$, $d(v_8) > n/2 + (3k - 7)/2$ and $d(v_9) > n/2 + (3k - 7)/2$. If $d(v_6) < n/4$, then $d(v_4) > 3n/4 + (3k - 9)/2$ and $d(v_8) > 3n/4 + (3k - 9)/2$. Thus

$$\begin{aligned} n + 2(9i + (k - i)) &\geq d(v_4) + d(v_8) \\ &> 2 \cdot \left(\frac{3}{4}n + \frac{3k - 9}{2} \right) = \frac{3}{2}n + 3k - 9, \end{aligned}$$

a contradiction. So $d(v_6) \geq n/4$. However,

$$\begin{aligned} n + 3(9i + (k - i)) &\geq d(v_3) + d(v_6) + d(v_9) \\ &\geq 2 \cdot \left(\frac{n}{2} + \frac{3k - 7}{2} \right) + \frac{n}{4} = \frac{5}{4}n + (3k - 7), \end{aligned}$$

a contradiction. By the connectivity of G , (x_k, x_1) -path must exist.

Case 2: $\frac{3k-1}{2} \leq \kappa(G) \leq 9k$.

Subcase 2.1: $\delta < 70k$.

Without loss of generality, we can assume that G is edge-maximal with respect to the property of not being k -ordered (i.e., the addition of any edge makes G k -ordered). Let $d(v) = \delta$ and

$$\begin{aligned} L &= \{u \in V(G) \mid d(u) < n/2\}, \\ H &= V(G) - L, \\ F_i &= \{u \in V(G) \mid d(u, v) = i\} \text{ for } i \geq 2. \end{aligned}$$

We have the following claims about the structure of G :

Claim 1. $G[F_2]$ is complete and each $x \in H - F_2$ is adjacent to each vertex of F_2 .

Proof. Note that for $x \in F_2$, $d(x) > n - 70k$. Let x and y be nonadjacent vertices in F_2 . Then the insertion of the edge xy makes G k -ordered. Let C be the smallest cycle in $G + \{xy\}$ that contains S in order. We claim that $|V(C)| \leq n/3$. Assume otherwise, and let $|V(C)| = \alpha n$ where $\alpha > 1/3$. Choose an interval, say $[x_i, x_{i+1}]$, which contains vertices as many as possible. Then $|[x_i, x_{i+1}]| \geq \alpha n/k$ vertices. Note that there are at least two vertices of H among four continuous vertices on $[x_i, x_{i+1}]$. Thus, we can find a subset of vertices, say M , such that

$$|M| \geq \frac{|[x_i, x_{i+1}]|}{4} - 1 \geq \frac{\alpha n}{4k} - 1,$$

M is independent, and all vertices of M have degree greater than $n/2$. Since nonadjacent vertices with a distance three or more apart on an interval can have no common neighbours off the cycle, the set M must send at least $(n/2)|M| - (n - \alpha n) - 2|M| = m$ edges to vertices of $C - [x_i, x_{i+1}]$. Thus, there exists an interval different from $[x_i, x_{i+1}]$, say $[x_j, x_{j+1}]$, such that there are at least $m/(k-1)$ edges between the two intervals. Consider the subgraph G' of G induced by the vertices of $[x_j, x_{j+1}]$ and M . Let $n' = |V(G')|$ and $m' = |E(G')|$. On the one hand

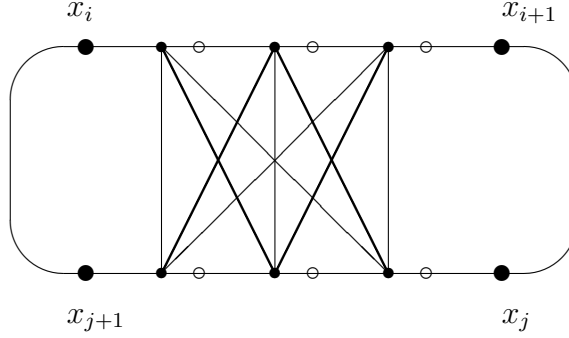
$$n' = |M| + |[x_j, x_{j+1}]| < \frac{1}{4} \cdot \frac{n}{3} + \frac{1}{2} \cdot \frac{n}{3} = \frac{n}{4},$$

and on the other hand

$$n' \leq |M| + |[x_i, x_{i+1}]| \leq |M| + 4(|M| + 1) < 6|M|.$$

We have

$$\begin{aligned} m' \geq \frac{m}{k-1} &= \frac{(\frac{n}{2} - 2)|M| - (1 - \alpha)n}{k-1} \geq \frac{(\frac{n}{2} - 2)|M| - \frac{(1-\alpha)}{\alpha}4k(|M| + 1)}{k-1} \\ &> \frac{\frac{n}{2} - 2 - 9k}{k-1} \cdot |M| \stackrel{(n \geq 27k^3)}{\geq} \frac{1}{2} \left(2^{\frac{1}{3}} \cdot \left(\frac{n}{4}\right)^{\frac{2}{3}} + 1 \right) \cdot 6|M| \\ &\geq \frac{1}{2} \left(2^{\frac{1}{3}}(n')^{\frac{5}{3}} + n' \right). \end{aligned}$$

Figure 3.6: A smaller cycle than C .

By Theorem 3.16, there exists a $K_{3,3}$ in G' . Because M is independent and C is minimal, such $K_{3,3}$ is between M and $[x_j, x_{j+1}]$. However, it can be used to produce a smaller cycle, a contradiction (s. Figure 3.6). Thus, the cycle C contains at most $n/3$ vertices.

Let $|V(C)| = c$. Because $c \leq n/3$ and x, y have no common neighbours off the cycle, we have

$$2(n - 70k) \leq d(x) + d(y) \leq n - c + 2(c - 1) < n + c, \quad (3.9)$$

which implies that $n < 210k$, a contradiction. So $G[F_2]$ is complete. Note that the same argument applies if we choose $x \in H - F_2$ and $y \in F_2$. The difference is the inequality (3.9) above becomes

$$\frac{n}{2} + n - 70k \leq d(x) + d(y) < n + c,$$

which implies that $n < 420k \leq 27k^3$, a contradiction. $\square$

Claim 2. $F_i = \emptyset$ for $i \geq 5$ and $F_4 \cap H = \emptyset$.

Proof. For a vertex $x \in F_2$, since $d(x) \geq n + (3k - 9)/2 - \delta$ and the neighbours of x are in $N(v) \cup F_2 \cup F_3$, then $V(G) - \bigcup_{i \geq 4} F_i$ contains at least $n + (3k - 9)/2 - \delta + 2$ vertices, and hence $\bigcup_{i \geq 4} F_i$ has at most $\delta - (3k - 9)/2 - 2$ vertices. Thus, $d(y) < d(v)$ for every vertex $y \in F_i$ ($i \geq 5$) if such F_i exists, a contradiction. By Claim 1, it is clear that $F_4 \cap H = \emptyset$. $\square$

In the following, we will complete the proof of this case. Let $L_1 = N[v] \cap L$ and $L_2, \dots, L_r$ be the vertex sets of the components of $G[(F_3 \cup F_4) \cap L]$. Note that $G[L_i]$ must be complete for $i = 1, 2, \dots, r$. Partition S into $S_H, S_{L_1}, \dots, S_{L_r}$, where

$$S_H = S \cap H \quad \text{and} \quad S_{L_i} = S \cap L_i.$$

Note also that every vertex in S_H is either a vertex in the complete subgraph $G[F_2]$ or is adjacent to every vertex of F_2 . Assume $|S_{L_i}| = \ell_i$ for $i = 1, 2, \dots, r$. So $|S_H| =$

$k - \ell_1 - \dots - \ell_r$. Let $\ell = \max\{\ell_i \mid i = 1, 2, \dots, r\}$. If $\ell \leq k/2$, then

$$\kappa(G - S_H) \geq \left\lceil \frac{3k-1}{2} \right\rceil - (k - \ell_1 - \dots - \ell_r) = \left\lceil \frac{k-1}{2} \right\rceil + \ell_1 + \dots + \ell_r \geq 2\ell.$$

For each S_{L_j} , we construct a graph G_j^* as follows: Create a vertex $x_{j_i}^*$ such that $N(x_{j_i}^*) = N(x_{j_i})$ for every $x_{j_i} \in S_{L_j}$. Let

$$S_{L_j}^* = S_{L_j} \cup \{x_{j_i}^* \mid i = 1, 2, \dots, \ell_j\}.$$

Add a vertex u_j such that $N(u_j) = S_{L_j}^*$. Then $G_j^* - S_H$ is still 2ℓ -connected. Let M_j be a set of $2\ell_j$ distinct vertices of $F_2 - S_H \subset V(G_j^*)$. Theorem 2.1 implies that there exist $2\ell_j$ internally vertex disjoint paths, each starting at the vertex u_j and ending at distinct vertices of M_j . This implies that, if we return to the graph G , for every vertex x_{j_i} in S_{L_j} , we can find a pair of internally vertex disjoint paths, starting at x_{j_i} and ending at distinct vertices of M_j , say an $(x_{j_i}, v_{j_i,1})$ -path and an $(x_{j_i}, v_{j_i,2})$ -path, where $v_{j_i,k} \in M_j$ ($i = 1, \dots, \ell_j$ and $k = 1, 2$) and $v_{j_i,k} \neq v_{j_{i'},k'}$. By the connectivity of G , we see that F_2 contains at least $\lceil (3k-1)/2 \rceil$ vertices. Here we assume that F_2 contains enough vertices (at least $2k$ vertices) to force all $v_{j_i,k}$'s mutually different. Indeed, if there exists a vertex $x \in N[v]$ with $d(x) \geq \delta + 2k$, then F_2 has at least $2k$ vertices since $|N[v]| = \delta + 1$. So $d(x) < \delta + 2k$ for all $x \in N[v]$. We have $d(y) \geq n - \delta - 2k$ for $y \in F_3$. Analogously to the proof of Claim 1, $G[F_2 \cup F_3]$ is complete. Then we consider $F_2 \cup F_3$ instead of F_2 . Without loss of generality, each $(x_{j_i}, v_{j_i,k})$ -path is as short as possible. We see that each $(x_{j_i}, v_{j_i,2})$ -path, after going out of L_j , either reaches directly M_j , or reaches M_j through a vertex of $N[v] \cap H$ or $F_3 \cap H$. Since the neighbourhoods of the subsets L_i and L_j ($i \neq j$) in $F_2 \cup (N[v] \cap H) \cup (F_3 \cap H)$ are disjoint, we can choose the paths from x_{j_i} to $v_{j_i,k}$ are internally vertex disjoint and all end-vertices are mutually different for all $i = 1, \dots, \ell_j$, $j = 1, \dots, r$ and $k = 1, 2$. Now all vertices of S are either in the complete subgraph $G[F_2]$ of G , or have two disjoint paths to F_2 (Note that each vertex of $F_3 \cap S_H$ or $N[v] \cap S_H$ has two distinct edges to F_2 and we may choose the end-vertices are disjoint from the vertices of all M_j 's). Thus, we can construct a cycle containing S in order using these paths to M_j and edges in F_2 .

If there exists ℓ_i such that $\ell_i = \lfloor k/2 \rfloor + t$ for $t > 0$, then $\ell_j < k/2$ for $j \neq i$ and

$$\kappa(G - S_H) \geq \frac{3k-1}{2} - \left(\left\lceil \frac{k}{2} \right\rceil - t \right) \geq 2\ell_i - t - \frac{1}{2}.$$

Since S_{L_i} has over half the vertices of S , there are at least t consecutive pairs (x_i, x_{i+1}) . Paths between these are made with edges in $G[L_i]$, leaving no more than $k - t$ end-vertices of paths. Construct the vertex disjoint paths in $G - S_H$ just as was done in the case $\ell \leq k/2$.

Subcase 2.2: $\delta \geq 70k$.

Let K be a minimal cut set and A, B be the components of $G - K$. Without loss of generality, let $|V(A)| \geq |V(B)| (\geq 70k - 9k = 61k)$. Using the fact that K is a minimal cut set, there is a matching of K to A , and respectively K to B , which together produce $\kappa(G)$ pairwise disjoint P_3 's. Of all such matchings, pick ones with the fewest intersections

with the set S . Since each point of K has large degree to at least one of A or B , we can avoid P_3 's with both end-vertices in S . Thus we can assume that if a P_3 has two vertices in S , one of these is the middle vertex of the P_3 , and we call such a P_3 a double. If a P_3 has one vertex in S , we call it a single. If a P_3 is disjoint from S , we call it free.

We claim that the subgraph A of G is $2k$ -linked. By Theorem 2.17, it is sufficient to prove that A is $20k$ -connected. For any x, y in A with $d(x, y) = 2$, we have

$$\begin{aligned} d_A(x) + d_A(y) &\geq d(x) + d(y) - 2|K| \geq n + \frac{3k-9}{2} - 2|K| \\ &\geq n - |K| - |V(B)| + \frac{3k-9}{2} + |V(B)| - |K| \\ &\geq |V(A)| + \frac{3k-9}{2} + 61k - 9k > |V(A)| + 52k. \end{aligned}$$

Just as the proof of Lemma 3.12, A is $52k$ -connected. Similarly, B is $2k$ -linked.

Now construct the k -ordered cycle. Assign every vertex in S to side A or side B as follows: If a vertex is in $A(B, \text{ respectively})$ assign it side $A(B, \text{ respectively})$. If a vertex is in K and on a double, assign it the side to which it has large degree. The last instance is when the vertex, say x_i , is in K and on a single. For every such x_i there is an (x_i, x_{i+1}) -path by A or B that does not require using a free P_3 . Assign x_i to the opposite side this path takes. We construct the cycle containing $x_1, x_2, \dots, x_k$ in order by constructing a set of disjoint (x_i, x_{i+1}) -paths. Note that in constructing each (x_i, x_{i+1}) -path, a free P_3 is only necessary if (1) x_i is not on a single and (2) x_i and x_{i+1} are assigned different sides. So we need at most $k/2$ free P_3 's. Let d be the number of doubles and s be the number of singles. The number of free P_3 's is at least

$$\kappa(G) - (d + s) \geq \frac{3k-1}{2} - (k-d) = \frac{k-1}{2} + d$$

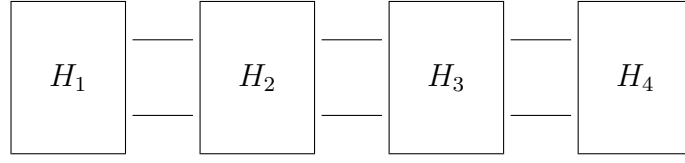
since $k \geq 2d + s$. So the free P_3 's is enough to connect the vertices assigned different sides and not on a single. Since A and B are $2k$ -linked, it is straightforward to form a cycle containing S in the appropriate order.

$$\text{Case 3: } \frac{3k-5}{2} \leq \kappa(G) \leq \frac{3k-2}{2}.$$

Pick a minimal cut set K . Let A and B be the components of $G - K$. Let A_1 (B_1 , respectively) be the subset of $V(A)$ ($V(B)$, respectively) such that $N_K(x) \neq \emptyset$ for all $x \in A_1$ (B_1 , respectively). Note that each set of A_1 , B_1 and K contains at least $\lceil (3k-5)/2 \rceil \geq k$ vertices. We confirm that a vertex x in A_1 (or respectively in B_1) is adjacent to every other vertex of $A(B)$ and K except possibly one. To show this consider $x \in A_1$. By the minimality of K , there is $y \in B_1$ with $d(x, y) = 2$. Then,

$$n + \frac{3k-9}{2} \leq d(x) + d(y) \leq (n - \kappa(G) - 2) + 2\kappa(G) - 2,$$

which forces $\kappa(G) \geq (3k-1)/2$. Also, if there is x , say $x \in A_1$, such that x is nonadjacent to one of vertices of A or K , then y is adjacent to every other vertex of B and K for all $y \in B_1$. By the structure of G and the degree condition, it is not difficult to see that G is k -ordered. ■

Figure 3.7: The graph G_4 described in Example 3.18.

We can see that the degree sum condition in Theorem 3.17 is sharp by considering the graph G_2 which is mentioned in Section 1.3, Figure 1.2. The next example shows that there are graphs, whose k -ordered Hamiltonicity can be verified by Theorem 3.17, but not by Theorem 1.6.

Example 3.18 *Let $k \geq 4$ be an integer and let G_4 be a graph of order $n \geq 27k^3$ containing four complete graphs H_1, H_2, H_3 and H_4 with $|V(H_1)| = n - 3\lceil(3k-5)/2\rceil$ and $|V(H_2)| = |V(H_3)| = |V(H_4)| = \lceil(3k-5)/2\rceil$. Each vertex of H_i is adjacent to each vertex of H_{i+1} for $i = 1, 2, 3$. For $u \in V(H_1)$ and $v \in V(H_4)$, $d(u) + d(v) = n - 2 < n + (3k-9)/2$. So G_4 does not satisfy the assumption of Theorem 1.6. However, it is not difficult to check that G_4 satisfies the assumption of Theorem 3.17. Hence G_4 is k -ordered Hamiltonian (s. Figure 3.7).*

3.3 k -ordered Hamiltonian graphs on upper connectivity

Chen et al. noted that an upper connectivity leads to the resultant effect of being able to lower the degree sum condition and gave the following theorem.

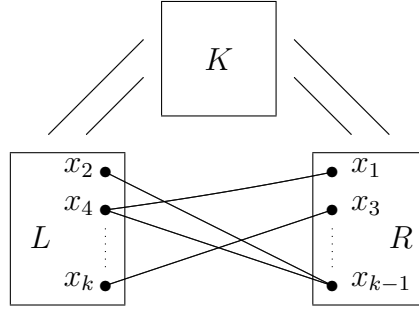
Theorem 3.19 (Chen et al. [18]) *Let G be a graph on n vertices with*

$$d(u) + d(v) \geq n$$

for any two nonadjacent vertices u and v . Let $k \leq n/176$ be an integer. If G is $\lfloor 3k/2 \rfloor$ -connected, then G is k -ordered Hamiltonian.

The connectivity in Theorem 3.19 is sharp by considering the following example G_5 , which was mentioned in [18]. Let L , K and R be complete graphs with $|V(L)| = \lfloor k/2 \rfloor$, $|V(K)| = 2\lfloor k/2 \rfloor - 1$ and $|V(R)| = n - |V(L)| - |V(K)|$. Let G'_5 be the union of the three graphs, adding all possible edges containing vertices of K . Let $x_i \in V(L)$ if i is even, $x_i \in V(R)$ otherwise. Add all edges $x_i x_j$ whenever $|i - j| \notin \{0, 1, k-1\}$. The resulting graph is G_5 . The degree sum condition is satisfied and G_5 is $(\lfloor 3k/2 \rfloor - 1)$ -connected. But this graph is not k -ordered because there is no cycle containing the x_i in the given order (s. Figure 3.8).

Instead of requiring the degree sum for any pair of nonadjacent vertices, we choose to ask for the degree condition for the pairs with distant 2. We prove the following.

Figure 3.8: The graph G_5 .

Theorem 3.20 (R. Li [46]) *Let G be a graph on n vertices with*

$$d(x) + d(y) \geq n$$

for any two vertices x, y with $d(x, y) = 2$. Let $k \leq n/100$ be an integer. If G is $\lfloor 3k/2 \rfloor$ -connected, then G is k -ordered Hamiltonian.

Clearly, Theorem 3.20 implies Theorem 3.19 and the sharpness of the connectivity can also be indicated by the graph G_5 of Figure 3.8.

The minimum degree condition to imply a graph to be k -linked was provided by Kawarabayashi, Kostochka and Yu, which will be applied to our next proof.

Theorem 3.21 (Kawarabayashi et al. [43]) *Let k be an integer and G be a graph of order n . If*

$$\delta(G) \geq \lceil (n - 3)/2 \rceil + k,$$

then G is k -linked.

Proof of Theorem 3.20. By Lemma 3.11, it is sufficient to show that G is k -ordered. Let $S = \{x_1, \dots, x_k\}$ be a set of k vertices. We will construct a cycle C including the vertices of S in the given order. Let K be a minimal cut set of G . Let L and R be two components of $G - K$ with $|V(L)| \leq |V(R)|$. If $|K| \geq 10k$, by Corollary 2.17, G is k -linked and hence k -ordered. Assume that $\lfloor 3k/2 \rfloor \leq |K| \leq 10k - 1$.

Claim 1. L and R are the only components of $G - K$.

Proof. Suppose that $G - K$ contains at least three components. Let $x \in V(L)$, $y \in V(R)$ and $z \in V(G - (K \cup L \cup R))$ such that $d(x, y) = d(y, z) = d(z, x) = 2$. Then

$$\begin{aligned} 3n \leq 2(d(x) + d(y) + d(z)) &< 2(|V(L)| + 3|K| + |V(R)| + (n - |V(L)| - |V(R)|)) \\ &\leq 2n + 6|K| < 2n + 60k, \end{aligned}$$

which implies that $n < 60k$, a contradiction. $\square$

Claim 2. R contains at least $(n-10k+1)/2$ vertices, and L contains at least $n/2-10k+2$ vertices or L is a complete subgraph.

Proof. Clearly, $|V(R)| \geq (n-|K|)/2 \geq (n-10k+1)/2$. Suppose L is not complete. Let $x, y \in L$ with $d(x, y) = 2$. Then

$$n \leq d(x) + d(y) \leq 2(|V(L)| - 1 + |K|) \leq 2(|V(L)| + 10k - 2),$$

which implies that $|V(L)| \geq n/2 - 10k + 2$. $\square$

Claim 3. For every vertex $v \in K$, at least one of the following holds:

1. $d_R(v) \geq 2k$; 2. $d_L(v) \geq 2k$; 3. $d_L(v) = |V(L)|$.

Proof. Suppose the claim is false for some vertex $v \in K$. Let $x \in L - N(v)$, $y \in R - N(v)$ with $d(x, v) = d(y, v) = 2$. Then

$$\begin{aligned} 2n \leq d(x) + 2d(v) + d(y) &< |V(L)| + |K| + 2(|K| + 4k) + |V(R)| + |K| \\ &\leq n + 3|K| + 8k < n + 38k, \end{aligned}$$

a contradiction. $\square$

Let L_1 (R_1 , respectively) be the subgraph of L (R , respectively) such that each vertex of L_1 (R_1 , respectively) has at least a neighbour in K . Let $u \in L_1$ and $v \in R_1$ with $d(u, v) = 2$. Then

$$\begin{aligned} n \leq d(u) + d(v) &\leq |V(L)| - 1 + |K| + |V(R)| - 1 + |K| \\ &\leq n + |K| - 2. \end{aligned}$$

Thus u (v , respectively) is adjacent to all but at most $|K|-2$ vertices in L (R , respectively).

Claim 4. R is $\min\{2k, |V(L)| + 2\}$ -connected and L is $2k$ -connected or complete.

Proof. Let K_R be a minimal cut set of the subgraph R , and let A_R and B_R be two components of $R - K_R$. Let $u \in A_R$, $v \in B_R$ with $d(u, v) = 2$. If both u and v are adjacent to K , then they are adjacent to all but at most $|K| - 2 \leq 10k - 3$ vertices in R , and hence K_R contains at least $|V(R)| - 20k$ vertices. So R is $2k$ -connected. Assume without loss of generality, that v is not adjacent to K . Then

$$\begin{aligned} n \leq d(u) + d(v) &\leq |V(A_R)| - 1 + |K_R| + |K| + |V(B_R)| - 1 + |K_R| \\ &\leq |V(R)| + |K| + |K_R| - 2, \end{aligned}$$

which implies that $|K_R| \geq |V(L)| + 2$. Similarly, L is $2k$ -connected or complete. $\square$

Claim 5. There exists the subgraphs $L', R'_1, R'_2, \dots, R'_\ell$ of G such that their vertex sets form a partition of $V(G)$ and the following holds:

- (a) R'_1 is k -linked, L' is k -linked or complete. If $\ell \geq 2$, then R'_i is complete for all $2 \leq i \leq \ell$;

- (b) There is no connection between L' and R'_i as well as between R'_i and R'_j for $2 \leq i \neq j \leq \ell$. Moreover, the neighbourhoods of $L', R'_2, \dots, R'_\ell$ in R'_1 are pairwise disjoint if $\ell \geq 2$.

Proof. We consider two cases.

Case 1: $|V(L)| \geq 2k$.

By Claim 4, R is $2k$ -connected, and L is $2k$ -connected or complete. We confirm that R is k -linked, and L is k -linked or complete. If $d_R(v) \geq 14k$ for some $v \in K$, then the number of the edges of R ,

$$|E(R)| \geq \frac{1}{2}(14k(|V(R)| - |K| + 2)) \geq 5k|V(R)|.$$

By Theorem 2.16, R is a k -linked subgraph. So assume that $d_R(v) < 14k$ for each $v \in K$. Let $u \in R$ with $d(u, v) = 2$. Then

$$n \leq d(u) + d(v) < d_R(u) + |V(L)| + 14k + |K| - 1,$$

which implies that $d_R(u) > |V(R)| - 14k + 1$. Note that we can find at least $|V(R)| - 10k + 3 - 14k$ vertices with distance 2 from v , since each vertex of R_1 is adjacent to all but at most $10k - 3$ vertices in R . Therefore,

$$|E(R)| \geq \frac{1}{2}((|V(R)| - 24k + 3) \cdot (|V(R)| - 14k + 1)) \geq 5k|V(R)|.$$

So, R is k -linked. Similarly, L is k -linked or complete. Here we use the fact $n \geq 100k$.

Based on Claim 3, we give a partition of K as follows:

$$\begin{aligned} K_R &= \{v \in K \mid d_R(v) \geq 2k\}, \\ K_{L1} &= \{v \in K \mid d_L(v) \geq 2k\} - K_R, \\ K_{L2} &= \{v \in K \mid d_L(v) = |V(L)|\} - (K_R \cup K_{L1}). \end{aligned}$$

Note that either $K_{L1} = \emptyset$ or $K_{L2} = \emptyset$, and that the graph induced on K_{L2} is complete, since all vertices in K_{L2} have degree less than $4k$ in L and R . Now let

$$R'_1 = G[V(R) \cup K_R] \quad \text{and} \quad L' = G[V(L) \cup K_{L1} \cup K_{L2}].$$

Then R'_1 is k -linked and L' is k -linked or complete.

Case 2: $|V(L)| \leq 2k - 1$.

Obviously, L is complete. Let K_R be a minimal cut set of R . If R is $2k$ -connected, analogously to Case 1, we are done. So assume that $|K_R| \leq 2k - 1$. From the proof of Claim 4, we see that if there are two vertices which are in different components of $R - K_R$ and are adjacent to K , then the connectivity of R is at least $2k$. So assume that there is at most one component of $R - K_R$ which is connected to K and hence K_R is a cut set of G . So $|K_R| \geq |K| \geq 3k/2$.

We confirm that R contains a k -linked subgraph R^* with at least $2|V(R)|/3$ vertices. To show it, we consider two cases.

If $|V(R_1)| > 10k$, then R_1 is a k -linked subgraph of R . Let R^* be the k -linked subgraph of R such that $|V(R^*)|$ is as large as possible. Note that the number of the edges between R^* and $R - R^*$ is at least $|V(R_1)| \cdot (|V(R - R^*)| - 2k)$. Then there is a vertex of $R - R^*$ with at least

$$\frac{|V(R_1)| \cdot (|V(R - R^*)| - 2k)}{|V(R - R^*)|}$$

adjacencies in R^* . By Lemma 2.28, the vertex of $R - R^*$ is adjacent to at most $2k - 2$ vertices of R^* . Therefore, R^* must contain at least $2|V(R)|/3$.

We consider the case when R_1 contains at most $10k$ vertices. Then for a pair of vertices $u \in K$ and $v \in V(R - R_1)$ with $d(u, v) = 2$,

$$n \leq d(u) + d(v) \leq |V(L)| + |K| - 1 + |V(R_1)| + d_R(v),$$

which implies that $d_R(v) \geq n - |V(L)| - |K| + 1 - |V(R_1)| \geq n - 14k$, i.e., v is adjacent to all but at most $14k$ vertices. Let R^* be the graph induced by $V(R_1) \cup N_R(R_1)$. Note that R^* has at least $|V(R)| - 2k$ vertices. Therefore R^* is k -linked by Theorem 3.21.

Without loss of generality, we still use R^* to denote the maximal subgraph of R . Let $R'_2, R'_3, \dots, R'_\ell$ be the components of $R - R^*$. Note that each vertex of R'_i has the lower degree. Then R'_i is complete and $N_{R'_1}(R'_i) \cap N_{R'_1}(R'_j) = \emptyset$ for all $2 \leq i \neq j \leq \ell$.

Let K_R, K_{L1} and K_{L2} be a partition of K , which is shown in Case 1. Now let

$$L' = G[V(L) \cup K_{L1} \cup K_{L2}] \quad \text{and} \quad R'_1 = G[V(R^*) \cup K_R].$$

Then R'_1 is k -linked and L' is complete. Also, $N_{R'_1}(L') \cap N_{R'_1}(R'_i) = \emptyset$ for all $2 \leq i \leq \ell$. $\square$

To complete the proof, let

$$S_{L'} = V(L') \cap S \quad \text{and} \quad S_{R'_i} = V(R'_i) \cap S \quad \text{for} \quad 1 \leq i \leq \ell.$$

Create a new graph G' as follows: If $x_i \in S_T$ for some $T \in \{L', R'_2, \dots, R'_\ell\}$ and x_{i-1}, x_{i+1} are not in this set S_T , add a new vertex x'_i with $N(x'_i) = N(x_i) \cup \{x_i\}$. Obviously, G' is $\lfloor 3k/2 \rfloor$ -connected. Therefore, $G' - S_{R'_1}$ is $(\lfloor 3k/2 \rfloor - |S_{R'_1}|)$ -connected. According to Claim 5, the neighbourhoods of $L', R'_2, \dots, R'_\ell$ in R'_1 are pairwise disjoint if $\ell \geq 2$. Also note that

$$|S_{L'} \cup \{x'_i \mid x_i \in S_{L'} \text{ and } x_{i-1}, x_{i+1} \notin S_{L'}\}| \leq \min\{k, 2|S_{L'}|\} \leq 3k/2 - |S_{R'_1}|,$$

and for all $2 \leq j \leq \ell$,

$$|S_{R'_j} \cup \{x'_i \mid x_i \in S_{R'_j} \text{ and } x_{i-1}, x_{i+1} \notin S_{R'_j}\}| \leq \min\{k, 2|S_{R'_j}|\} \leq 3k/2 - |S_{R'_1}|.$$

Using these facts, we can find independent paths in $G' - S_{R'_1}$ from each of the vertices in $S_{L'} \cup \bigcup_{2 \leq j \leq \ell} S_{R'_j} \cup \bigcup x'_i$ into $R'_1 - S_{R'_1}$.

The existence of the k -ordered cycle C is now guaranteed, since we can use the fact that R'_i and L' are k -linked or complete to find the necessary connections.

The proof of the theorem is complete. $\blacksquare$

The next example shows that there are graphs, whose k -ordered Hamiltonicity can be verified by Theorem 3.20 but not Theorem 3.19.

Example 3.22 *Let $k \geq 3$ be an integer and let H_1, H_2, H_3 and H_4 be four complete graphs such that each H_i contains at least $\lfloor 3k/2 \rfloor$ vertices. For $1 \leq i \leq 3$, add all edges between H_i and H_{i+1} . The resulting graph is G and its order is denoted by n . For $u \in V(H_1)$ and $v \in V(H_4)$, $d(u) + d(v) = n - 2 < n$. However, it is easy to check that G satisfies the condition of Theorem 3.20. Hence G is k -ordered Hamiltonian.*

Chapter 4

(k, m) -vertex-pancyclic ordered graphs

A graph G is called *pancyclic* whenever G of order n contains a cycle of each length r for $3 \leq r \leq n$. A stronger related property is *vertex pancyclic* which requires for any specified vertex v of G there are cycles of length 3 through n containing v . The following definition generalized these ideas:

Definition 4.1 Let $0 \leq k \leq m$ be fixed integers and G be a graph of order n . The graph G is (k, m) -pancyclic if $n \geq m$ and for any set S_k of k vertices there is a cycle C_r of G containing S_k for each $m \leq r \leq n$.

Note that $(0, 3)$ -pancyclic and $(1, 3)$ -pancyclic graphs are pancyclic and vertex pancyclic graphs, respectively. For convenience, let

$$\sigma_2(G) = \max\{s \mid d(u) + d(v) \geq s \text{ for all } u, v \in G \text{ with } uv \notin E(G)\}.$$

Faudree et al. [24] provided:

Theorem 4.2 (Faudree et al. [24]) Let $1 \leq k \leq m \leq n$ be integers, and G be a graph of order n . The graph G is (k, m) -pancyclic if $\sigma_2(G)$ satisfies any of the following conditions:

- | | | |
|-------|---|---|
| (i) | $\sigma_2(G) \geq n,$ | when $m = n$; |
| (ii) | $\sigma_2(G) \geq \lfloor (4n - 1)/3 \rfloor,$ | when $k = 1$ and $m = 3$; |
| (iii) | $\sigma_2(G) \geq 2n - 3,$ | when $k = 2$ or 3 and $m = 3$; |
| (iv) | $\sigma_2(G) \geq 2n - m,$ | when $k = 3$ and $m = 4$ or 5 ; |
| (v) | $\sigma_2(G) \geq 2n - 2\lceil (m - 1)/2 \rceil - 1,$ | when $4 \leq k \leq m < 2k, n > m$; |
| (vi) | $\sigma_2(G) \geq 2\lfloor n/2 \rfloor + 1,$ | when $k \geq 2, m \geq 2k,$ and $n > m$. |

Also, all of the conditions on $\sigma_2(G)$ are sharp.

Corollary 4.3 (Faudree et al. [24]) Let $1 \leq k \leq m \leq n$ be integers, and G be a graph of order n . The graph G is (k, m) -pancyclic if $\delta(G)$ satisfies any of the following

conditions:

- (i) $\delta(G) \geq n/2$, when $m = n$;
- (ii) $\delta(G) \geq (n+1)/2$, when $k = 1$ and $m = 3$;
- (iii) $\delta(G) \geq n-1$, when $k = 2$ or 3 and $m = 3$;
- (iv) $\delta(G) \geq n-2$, when $k = 3$ and $m = 4$ or 5 ;
- (v) $\delta(G) \geq n - (m/2)$, when $4 \leq k \leq m < 2k, n > m$;
- (vi) $\delta(G) \geq \lfloor (n+2)/2 \rfloor$, when $k \geq 2, m \geq 2k$, and $n > m$.

Also, all of the conditions on $\delta(G)$ are sharp.

A generalization of both k -ordered and pancyclic graphs is given in the following.

Definition 4.4 Let $0 \leq k \leq m$ be fixed integers and G be a graph of order n . The graph G is (k, m) -pancyclic ordered if $n \geq m$ and for any ordered set S_k of k vertices there is a cycle C_r containing S_k and encountering the vertices of S_k in the specified order for each $m \leq r \leq n$.

For (k, m) -pancyclic ordered graphs, Faudree et al. proved:

Theorem 4.5 (Faudree et al. [24]) Let $4 \leq k \leq m \leq n$ be positive integers, and let G be a graph of order n . Then, the graph G is (k, m) -pancyclic ordered if $\sigma_2(G)$ satisfies any of the following conditions:

- (i) $\sigma_2(G) \geq 2n-3$, when $k \leq m < \lfloor 3k/2 \rfloor$;
- (ii) $\sigma_2(G) \geq 2n-4$, when $\lfloor 3k/2 \rfloor \leq m < \lceil (5k-2)/3 \rceil$;
- (iii) $\sigma_2(G) \geq 2n-5$, when $\lceil (5k-2)/3 \rceil \leq m < 2k$;
- (iv) $\sigma_2(G) \geq n+4k-m-6$, when $2k \leq m \leq (5k-3)/2$;
- (v) $\sigma_2(G) \geq n+(3k-9)/2$, when $m > (5k-3)/2$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

In this chapter, we introduce a new generalization of both k -ordered and vertex pancyclic graphs.

Definition 4.6 Let $0 \leq k \leq m-1$ be fixed integers and G be a graph of order n . The graph G is (k, m) -vertex-pancyclic ordered if $n \geq m$ and for any specified vertex v and any ordered set S_k of k vertices there is a cycle C_r containing v and S_k and encountering the vertices of S_k in the specified order for each $m \leq r \leq n$.

Note that a (k, m) -vertex-pancyclic ordered graph is also $(k, m+1)$ -vertex-pancyclic ordered. Furthermore,

Remark 4.7 A (k, m) -vertex-pancyclic ordered graph is (k, m) -pancyclic ordered and a $(k+1, m)$ -pancyclic ordered graph is (k, m) -vertex-pancyclic ordered. A $(0, 3)$ -vertex-pancyclic ordered graph is vertex pancyclic. For $k = 1$ and 2 , a graph that is (k, m) -vertex-pancyclic ordered is also $(k+1, m)$ -pancyclic ordered.

We will first prove the following theorem, which deals with the cases when $m < 2k$.

Theorem 4.8 *Let $4 \leq k \leq m - 1 \leq n - 1$ be positive integers, and let G be a graph of order n . The graph G is (k, m) -vertex-pancyclic ordered if $\sigma_2(G)$ satisfies any of the following conditions:*

- (i) $\sigma_2(G) \geq 2n - 3$, when $k + 1 \leq m < \lfloor 3k/2 \rfloor$;
- (ii) $\sigma_2(G) \geq 2n - 4$, when $\lfloor 3k/2 \rfloor \leq m < \lceil (5k - 2)/3 \rceil$;
- (iii) $\sigma_2(G) \geq 2n - 5$, when $\lceil (5k - 2)/3 \rceil \leq m < 2k$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

Proof. Since the conditions of (i), (ii) and (iii) in Theorem 4.5 is sharp, Remark 4.7 forces the sharpness of this theorem.

Case (i): If $\sigma_2(G) \geq 2n - 3$ for a graph G of order n , then $G = K_n$ and so is (k, m) -vertex-pancyclic ordered for all $m \geq k + 1$.

Case (ii): If $\sigma_2(G) \geq 2n - 4$ for a graph G of order n , then the graph $K_n - \lfloor n/2 \rfloor K_2$ is a subgraph of G . It is easy to see that G is (k, m) -vertex-pancyclic ordered for all $m \geq \lfloor 3k/2 \rfloor$.

Case (iii): If $\sigma_2(G) \geq 2n - 5$ for a graph G of order n , then the complement G^c is a disjoint union of paths of length at most 2. Thus, G is (k, m) -vertex-pancyclic ordered for all $m \geq \lceil (5k - 2)/3 \rceil$. ■

In the remainder of this chapter we provide the following result that gives the bound of $\sigma_2(G)$ when $m \geq 2k$.

Theorem 4.9 (R. Li [47]) *Let G be a graph with n vertices. k and m are two integers with $k \geq 7$ and $2k \leq m \leq (5k - 3)/2$. If*

$$\sigma_2(G) \geq n + 4k - m - 6,$$

then G is $(k, m + 2)$ -vertex-pancyclic ordered.

To prove this theorem, we use a statement on extremal graph theory, where K_r acts as the forbidden subgraph.

Theorem 4.10 (Zarankiewicz [13]) *Let G be a graph of order n and r be an integer with $2 \leq r < n$. If G contains no the complete subgraph K_r , then $\delta(G) \leq (1 - 1/(r - 1))n$.*

Proof of Theorem 4.9: Suppose that $S = \{v_1, v_2, \dots, v_k\}$ is an ordered set of k vertices of G . Let $v_0 \in V(G) \setminus S$ be arbitrary and $S' = S \cup \{v_0\}$.

Claim 1. There exists a cycle C with at most $m + 2$ vertices that contains the vertex of S' and encounters the vertices of S in the required order.

Proof. Let C' be a cycle with m vertices and encountering S in the given order. Theorem 4.5 provides the existence of C' . Then the vertices of S split the cycle C' into k intervals

$[v_1, v_2][v_2, v_3], \dots, [v_k, v_1]$. Assume that C' contains no v_0 and v_0 cannot have two adjacencies in an interval of the cycle C' . So $d_{C'}(v_0) \leq k$. Without loss of generality, suppose that v_0 is adjacent to a vertex of $[v_i, v_{i+1}]$. Then v_0 and v_{i+1} have

$$(n + 4k - m - 6) - (n - 2) - k \geq 3k - m - 4 \geq 1$$

common adjacencies in $G - C'$, and hence we get the required cycle. $\square$

Select a cycle C of maximum length $p \leq m + 2$ that contains S' and encounters the vertices of S in the required order. Without loss of generality, assume that $v_0, v_1, v_2, \dots, v_k$ appear in this order along the cycle C and the vertices split C into $k + 1$ intervals $[v_0, v_1], [v_1, v_2], \dots, [v_k, v_0]$. Let H be the subgraph $G - C$.

Claim 2. $p \geq k + 2$.

Proof. Suppose $p = k + 1$. Since $\delta(G) \geq 4k - m - 4 \geq k + 1$, each vertex of S' has an adjacency in $G - S'$. Assume that $v_i u_i \in E(G)$ with $u_i \in G - S'$. Note that $u_1 = u_2$ or $u_1 u_2 \in E(G)$ yields a cycle of length either $k + 2$ or $k + 3$ which is less than $m + 2$. So consider $u_1 u_2 \notin E(G)$. Since u_1 and u_2 have at least $4k - m - 4$ common neighbour on C , either u_1 or u_2 can be inserted into C or they have at most $k/2$ common neighbour on C . The latter implies that u_1 and u_2 have a common adjacency in $G - C$. We get a contradiction. So $p \geq k + 2$. $\square$

Claim 3. $p = m + 2$.

Proof. Suppose to the contrary that $p \leq m + 1$. Note that each vertex of C has an adjacency in H . If it is not true, let $x \in V(C)$ with no adjacency in H . Let y be a vertex in H such that the degree of y is as small as possible. The vertex y cannot be adjacent to two consecutive vertices of C . We have the following inequality:

$$n + 4k - m - 6 \leq d(x) + d(y) \leq (p - 1) + (n - p - 1) + \frac{p}{2}. \quad (4.1)$$

Indeed, $d(x) + d(y) < (p - 1) + (n - p - 1) + p/2$. Obviously, the equality does not hold if p is odd. We only consider the case when p is even. Suppose the equality of (4.1) holds. Then x is adjacent to all vertices of C , H is a complete graph and each vertex of H is adjacent alternatively to C . Because x is not adjacent to H , all vertices of H have the same adjacencies in C . Claim 2 allows us to choose $x \in V(C)$ such that $yx^-, yx^+ \in E(G)$ and either x or x^+ is not a vertex of S' . Then $x^- y x^+ x x^{++} \overrightarrow{C} x^-$ is a required $(p + 1)$ -cycle, a contradiction. So $n + 4k - m - 6 \leq (p - 1) + (n - p - 1) + (p - 1)/2$. This forces that $8k - 2m - 7 \leq p \leq m + 1$ and hence $(8k - 8)/3 \leq m \leq (5k - 3)/2$. This yields a contradiction, unless $k = 7$, $p = m + 1 = 17$. In this special case the graph H is complete and x is adjacent to all of the other vertices of C . The adjacencies of y alternate on C except two vertices, which have the distance three on C . Also, $|V(H)| \geq 2$, since G is k -ordered Hamiltonian. Let $v_i x' x'^+ v_{i+1}$ be an interval with length three. If an end-vertex, say v_i , has a neighbour, say also y in H , then $x'^+ y$ or $v_{i+1} y \in E(G)$. The latter implies that either x' and x'^+ can be inserted into the path $v_{i+1} \overrightarrow{C} v_i$ or there is a common adjacency between x' (or x'^+) and y , both of which can induce a required C_{18} . The former

forces that there is $y' \in V(H)$ with $x'y' \in E(G)$, and v_{i+1} is not adjacent to x' . So v_{i+1} has at least one adjacency in H . In particular, $v_{i+1}y' \in E(G)$. Observe that x' and y' cannot have a common adjacency in H , since this gives a cycle of length 18 avoiding x'^+ and using y' and the common adjacency. The same argument implies that x'^+ and y do not have a common adjacencies in H . We have the following inequality involving x', y, x'^+ and y' :

$$2(n + 4k - m - 6) \leq d(x') + d(y) + d(x'^+) + d(y') \leq 2(n - p - 1) + 4 \cdot \frac{p}{2} < 2n,$$

which is a contradiction. So neither v_i nor v_{i+1} is adjacent to H . Without loss of generality, $x'y \in E(G)$. Then $v_i v_{i+1}, v_{i+1}^+ y \in E(G)$ and $v_i v_{i+1} x'^+ x' y v_{i+1}^+ \xrightarrow{C} v_i$ is a required C_{18} . Hence, we may assume that each vertex of C has an adjacency in H .

Select a vertex $x \in V(C)$ such that $x^+ \notin S'$. For convenience let $y = x^+, z = x^{++}$ and $w = x^-$. If possible, choose x not in S' . Let $u \in V(H)$ such that $xu \in E(G)$. We will show that $zu \in E(G)$. If it is not true, observe that z and u have no common adjacency in H , since this would imply the existence of a cycle of length $p + 1$ with the required properties. Hence, we have the following inequality:

$$n + 4k - m - 6 \leq d(z) + d(u) \leq (p - 1) + (n - p - 1) + \frac{p - 1}{2}.$$

This implies that $8k - 2m - 7 \leq p \leq m + 1$. Hence, $(8k - 8)/3 \leq m \leq (5k - 3)/2$, a contradiction unless $k = 7, p = m + 1 = 17$. In this special case, z is adjacent to all vertices of C and u is adjacent alternatively to C from z^+ to x . Note that z is adjacent to x . To avoid a required $(p + 1)$ -cycle, the vertices not adjacent to u are exactly in S' . Then $zz^{++}z^+uz^{+++}\xrightarrow{C}z$ is a required $(p + 1)$ -cycle. Therefore $N_H(x) = N_H(z)$, and in particular $zu \in E(G)$.

Now suppose $xz \notin E(G)$. Let $N = N_H(x)$ and $N^+ = N \cup \{y\}$. If two vertices of N^+ are adjacent, then a cycle of length $p + 1$ results. So we can assume that N^+ is an independent set. For vertices u_1 and u_2 in N^+ we have the following inequality:

$$n + 4k - m - 6 \leq d(u_1) + d(u_2) \leq p + 2 \cdot \left(n - p - \frac{n + 4k - m - 2p - 2}{2} \right).$$

So $p \geq 8k - 2m - 8$. Note that $p \leq m - 1$ forces $(8k - 7)/3 \leq m \leq (5k - 3)/2$, and hence $k \leq 5$, a contradiction. Also, $m \leq 2k + 1$ implies that $(8k - 9)/3 \leq m \leq 2k + 1$, and hence $k \leq 6$, a contradiction. So $m + 1 \geq p \geq m \geq 2k + 2$. If $p \geq 2k + 3$, we have $x \notin S'$. Let $v \in V(H - N)$ with $wv \in E(G)$. Then $uv \in E(G)$, otherwise there is a common adjacency of u and v , which induces a required $(p + 1)$ -cycle by adding these three vertices and omitting x and y . Similarly, $yv \in E(G)$. Note that x, y, u and v cannot be inserted into the path $z \xrightarrow{C} w$ and, u and x (similarly, v and y) cannot have a common adjacency in H . The following inequality follows:

$$2(n + 4k - m - 6) \leq d(x) + d(v) + d(y) + d(u) \leq 2(n - p - 1) + 4 \cdot \frac{p}{2} < 2n,$$

a contradiction. So $p = m = 2k + 2$. Then $(8k - 8)/3 \leq m \leq (5k - 3)/2$, which is a contradiction unless $k = 7$. In this case, we see that $p = m = 16$, $x(z)$ is adjacent to

all vertices except $z(x)$ and itself on C and each vertex of N^+ is adjacent to all vertices of $H - N$ as well as alternatively to the vertices of C . If $x \notin S'$, we can get the same contradiction as the case when $p \geq 2k + 3$. We assume that N is adjacent exactly to S' . In particular, w is not a vertex of S' . Let u' be a common adjacency of y and u . Then $w^-xu u'yz \vec{C} w^-$ is a required $(p + 1)$ -cycle, a contradiction. It follows that $xz \in E(G)$.

Note that $wy \notin E(G)$ and the vertex $u \in N$ is not adjacent to w . If w and u have a common adjacency in H , there exists a cycle of length $p + 1$ using the edge xz . However, no common adjacency of w and u in H implies the following inequality:

$$n + 4k - m - 6 \leq d(w) + d(u) \leq (n - p - 1) + (p - 2) + \frac{p}{2}.$$

This implies that $8k - 2m - 6 \leq p \leq m + 1$. This gives the inequality $(8k - 7)/3 \leq m \leq (5k - 3)/2$, and hence $k \leq 5$, a contradiction. Thus we can conclude that there is a cycle of length $m + 2$ that contains S' and encounters the vertices of S in the required order. $\square$

Assume there exist cycles of every length from $m + 1$ to p containing S' and encountering the vertices of S in the correct order, but there is no cycle of length $p + 1$ with this property. Let C be a required cycle of length p , and let $H = G - C$. Without loss of generality, assume that $v_0, v_1, v_2, \dots, v_k$ appear in this order along the cycle C . The $k + 1$ vertices of S' divide the vertices of C into $k + 1$ disjoint intervals except for end-vertices, each starting and ending with a vertex of S' .

Claim 4. Some vertex in H has at least two adjacencies in some interval, or every vertex of C has an adjacency in H .

Proof. Suppose there is a vertex x in C having no adjacencies in H , and every vertex $y \in V(H)$ at most one adjacency in every interval of C . It follows that y has at most $k + 1$ adjacencies in C . This gives the following inequality:

$$n + 4k - m - 6 \leq d(x) + d(y) \leq (p - 1) + (k + 1) + (n - p - 1).$$

This implies $m \geq 3k - 5$, and hence $3k - 5 \leq (5k - 3)/2$, and so $k \leq 7$. Thus, if $k > 7$ the claim follows. Thus we have a contradiction unless $k = 7$. Observe also that this implies that any vertex in C adjacent to no vertex of H must be adjacent to all of C .

In the case when $k = 7$, the vertex y has precisely eight adjacencies in C and none of these can be in S' , since vertices of S' are each in two intervals. Thus, the vertices of S have no adjacencies in H , so they are adjacent to all other vertices of C . Also, the vertices of H induce a complete graph. Further, each of the intervals in C has at least three vertices. Let $u_1 \in (v_k, v_0)$ and $u_2 \in (v_0, v_1)$ with adjacencies w_1 and w_2 in H (possibly, $w_1 = w_2$) such that no vertex in $[u_1^+, u_2^-]$ has any adjacencies in H . Note that all vertices between u_1 and u_2 are adjacent to all other vertices of C , and hence they can be inserted into $u_2 \vec{C} u_1$ step by step. Then, there is a cycle of length $p + 1$ or $p + 2$ that contains S' in the correct order. There is a chord of length two from each vertex of S' in C , so the cycle of length $p + 2$ can be shortened if necessary, a contradiction. $\square$

Claim 5. If each vertex of C has an adjacency in H , then there is a vertex of H with $k + 2$ adjacencies in C and hence two adjacencies in some interval of C .

Proof. First consider the case when there is some interval in C with at least four vertices. Let x_1, x_2, x_3, x_4 be four consecutive vertices in that interval. Let y_i be an adjacency of x_i in H for $1 \leq i \leq 4$. If $y_i = y_j, i \neq j$, then we are done, so the y_i 's are distinct. If y_1 and y_4 have a common adjacency in H , then there is a cycle of length $p+1$ with the required property, a contradiction. If $y_1y_4 \notin E(G)$, then the following inequality follows:

$$n + 4k - m - 6 \leq d(y_1) + d(y_4) \leq (n - p - 2) + \frac{p}{2} + \frac{p}{2} < n,$$

a contradiction. Hence $y_1y_4 \in E(G)$. If x_2 or x_3 is inserted in the cycle obtained from C by replacing x_2 and x_3 with y_1 and y_4 , then there is a cycle of length $p+1$, contradicting our choice of C . A cycle of length $p+1$ also results if y_1 and x_3 have a common adjacency in H . Since, neither of these occur, we have the following inequality:

$$n + 4k - m - 6 \leq d(y_1) + d(x_3) \leq (n - p - 1) + \frac{p}{2} + \frac{p}{2} < n,$$

which gives a contradiction unless $y_1x_3 \in E(G)$. However, $y_1x_3 \in E(G)$ implies there is a vertex in H with two adjacencies in an interval of C .

We may assume that there is no interval of C with at least four vertices. It follows that $p = m + 2 = 2k + 2$, the vertices of S' alternate, and each interval has precisely three vertices. Let x_1, x_2, x_3 be the three consecutive vertices in some interval. Let y_1 be an adjacency of x_1 in H . If $y_1x_3 \in E(G)$ or $y_1x_2 \in E(G)$, then we have proved the claim, so neither is an edge in G . Also, if y_1 and x_3 have a common adjacency in H , then there is a cycle of length $p+1$, a contradiction. Therefore, we have the following inequality:

$$n + 2k - 6 \leq n + 4k - m - 6 \leq d(y_1) + d(x_3) \leq (n - p - 1) + (p - 1) + \frac{p}{2} \leq n + k - 1.$$

This implies $k \leq 5$, a contradiction. Therefore, we can conclude that there is a vertex in H with at least $k+2$ adjacencies in C and hence at least two adjacencies in some interval of C , completing the proof of Claim 5. $\square$

Select two vertices x and y in one of the intervals of C which have a common adjacency, say $z \in V(H)$, that are at a minimum distance along C . Let A be the vertices of C strictly between x and y in this interval. Thus, none of the vertices A are in S .

Claim 6. Some vertex in A has an adjacency in H .

Proof. Suppose not and consider the cycle obtained from C by replacing the path with vertices in A by the path xzy . If all of the vertices of A can be inserted into this cycle, then the required cycle of length $p+1$ exists, which gives a contradiction. If not, then insert as many vertices as possible, and assume we are left with a set $\emptyset \neq B \subseteq A$ of vertices that cannot be inserted. Select a vertex $w \in B$. If $b = |B|$ and w has no adjacency in H , then we have the following inequality:

$$n + 4k - m - 6 \leq d(w) + d(z) \leq (b - 1) + \frac{p - b + 1}{2} + (n - p - 1) + \frac{p - b + 1}{2} < n,$$

a contradiction, completing the proof of Claim 6. $\square$

Claim 7. $|A| = 1$.

Proof. Let $|A| = a$. Suppose $a \geq 2$. If all of the vertices in A are insertible in the path $C - A$, then the required cycle of length $p + 1$ is obtained. Assume not, and let $x_1 x_2 \dots x_a$ be the path of C using the vertices in A . Let x_t be the first vertex of A starting from x_1 that is not insertible. Observe that x_t and z must have a common adjacency in H , since if this is not true then we get the following inequality:

$$n + 4k - m - 6 \leq d(x_t) + d(z) \leq (a - 1) + \frac{p - a + 1}{2} + (n - p - 1) + \frac{p - a + 1}{2} < n,$$

a contradiction. Let z_t be such a common adjacency. If $t > 1$, then the required cycle of length $p + 1$ is obtained by using the path $x z z_t x_t$ to replace $x x_1 \dots x_t$ and inserting all of the remaining vertices of (x_1, x_t) . Hence, we must have that x_1 is not insertible, and so $t = 1$. Likewise, x_a is not insertible, and there is a vertex $z_s \in V(H)$ that is a common adjacency of x_s and z . If $a = 2$, then the required cycle of length $p + 1$ can be obtained by using the path $x z z_s x_2 y$ and avoiding the vertex x_1 . The required cycle can also be obtained if all of the vertices of A strictly between x_1 and x_a can be inserted. Thus, we can assume that $a > 2$, and let x_r be the first vertex past x_1 that is not insertible in the path $C - A$. Associated with x_r is the vertex $z_r \in V(H)$ that is commonly adjacent to z and x_r . Again, the required cycle is obtained by using the path $x z z_r x_r$, inserting the vertices strictly between x_1 and x_r and avoiding x_1 . Therefore, $|A| = 1$. $\square$

Claim 8. No vertex of H can have three adjacencies in one interval.

Proof. Assume that there is a vertex $z \in V(H)$ with adjacencies x_1, x_2 and x_3 . By Claim 7 we know that there is precisely one vertex on C between x_1 and x_2 and between x_2 and x_3 . Denote these vertices by y_1 and y_2 . Neither y_1 and y_2 is insertible, since this would give the desired cycle of length $p + 1$. Also, $y_1 y_2 \notin E(G)$ for the same reason. Therefore, y_1 and y_2 have a common adjacency in H , which we will denote by z' , since if this did not occur the following inequality results:

$$n + 4k - m - 6 \leq d(y_1) + d(y_2) \leq (n - p - 1) + \frac{p}{2} + \frac{p}{2} < n,$$

a contradiction. This implies that x_2 is not insertible for the same reason as y_1 and y_2 . Observe that x_2 and z cannot have a common adjacency in H , since this gives a cycle of length $p + 1$ avoiding y_1 and using z and the common adjacency. The same argument implies that y_2 and z' do not have a common adjacency in H . This implies the following inequality involving x_2, y_2, z, z' :

$$2(n + 4k - m - 6) \leq d(x_2) + d(z') + d(y_2) + d(z) \leq 2(n - p - 1) + 4 \cdot \frac{p}{2} \leq 2(n - 1),$$

a contradiction. Thus, no vertex of H can have three adjacencies in an interval of C . $\square$

Claim 9. If two vertices are in the same interval of C and are at a distance three apart on the cycle C , then both cannot have adjacencies in H .

Proof. Suppose that x_1, x_2, x_3 and x_4 are in some interval of C and $u, v \in V(H)$ are the adjacencies of x_1 and x_4 , respectively. Note that $u \neq v$ since $|A| = 1$ and $uv \in E(G)$,

otherwise the degree condition implies that u and v have a common adjacency, which forces a given cycle of length $p + 1$. Similarly, u and x_3 (v and x_2) are adjacent and have no common adjacencies, which implies the following inequality:

$$2(n + 4k - m - 6) \leq d(u) + d(x_2) + d(v) + d(x_3) \leq 2(n - p - 1) + 4 \cdot \frac{p}{2} \leq 2(n - 1),$$

a contradiction. $\square$

Claim 10. If $y_1, y_2 \in V(H)$ each have two adjacencies in the same interval of C , then they have the same two adjacencies.

Proof. Assume that $x_1x_2 \dots x_t$ are the vertices of an interval, and that $y_1x_i, y_1x_{i+2}, y_2x_j$ and y_2x_{j+2} are the edges of G with $i < j$. Previous observations imply that x_{i+1} and x_{j+1} have adjacencies in H . Hence, to avoid having two vertices in the interval with adjacencies in H that are at a distance three on C , we must have $j \geq i + 6$. Either $y_1y_2 \in E(G)$ or there is a $y \in V(H)$ such that y_1yy_2 is a path in H . Let $A = \{x_{i+3}, x_{i+4}, \dots, x_{j-1}\}$, which is a set with at least three vertices, and let P be the path containing the remaining vertices of C . Starting with x_{i+3} and using the order on A , insert one at a time the vertices of A into P or the path obtained from P by inserting vertices of A . If all of the vertices of A can be inserted, then a C_{p+1} cycle can be constructed using the path from x_{i+2} to x_j containing y_1 and y_2 , and replacing P with a path with the appropriate number of vertices of A inserted. If all of the vertices of A cannot be inserted, then let x_q be the first vertex that cannot be inserted. Let $B = \{x_q, x_{q+1}, \dots, x_{j-1}\}$ with $b = |B|$. There must be some common adjacency, say $z \in V(H)$, of x_q and y_1 , for otherwise the following inequality results:

$$n + 4k - m - 6 \leq d(y_1) + d(x_q) \leq (n - p - 1) + (b - 1) + 2 \cdot \frac{p - b + 1}{2} < n,$$

a contradiction. A C_{p+1} can be constructed using the path $x_{i+2}y_1zx_q$ and by inserting all but one of the vertices of $A - B$, a contradiction. $\square$

Claim 11. There exists a vertex in C which is not adjacent to H .

Proof. Suppose that each vertex of C has at least one adjacency in H . By Claim 9, $p = m + 2 = 2k + 2$ and the vertices of S' alternate on C , and each interval has precisely three vertices. Note that the adjacencies of v_i and v_j in H are same for any two distinct vertices v_i and v_j of S' . In fact, if y is an adjacency of v_i but $v_{i+1}y \notin E(G)$, then v_{i+1} and y have $(n + 2k - 6) - (n - 2k - 3) - (k + 1) - (2k + 1) \geq 1$ common adjacencies in H . This implies a cycle of length $p + 1$ by using y and a common adjacency of y and v_{i+1} and omitting v_i^+ , a contradiction. So $N_H(v_i) \subseteq N_H(v_{i+1})$, and hence, by symmetry $N_H(v_i) = N_H(v_j)$ for any two distinct vertices v_i and v_j of S' . Let $L = N_H(v_i) \cup v_i^+$. Clearly, L is an independent set. Let $\ell = |L|$. Then $\ell \geq 2$.

We also note that v_i is not adjacent to v_{i+1} for some $i \in \{0, 1, \dots, k\}$. Suppose to the contrary that $v_iv_{i+1} \in E(G)$ for all $0 \leq i \leq k$. Let z denote an adjacency of v_i^- in H . Then z and v_i have at least $(n + 2k - 6) - (n - 2k - 3) - (k + 1) - (2k + 1) \geq 1$ common adjacencies in H . This implies a cycle of length $p + 1$ by using z and a common adjacency of z and v_i and avoiding v_i^+ , a contradiction.

Let $u, v \in L$. We have the following inequality:

$$\begin{aligned} 2(n + 2k - 6) &\leq d(v_i) + d(v_{i+1}) + d(u) + d(v) \\ &\leq 2(p - 3) + 2\ell + 2 \cdot \frac{p}{2} + 2(n - p - \ell) \\ &\leq 2n + p - 6, \end{aligned}$$

which implies $k \leq 4$, a contradiction. $\square$

For each pair of nonadjacent vertices $y, z \in V(H)$, we have:

$$d_H(y) + d_H(z) \geq n + 4k - m - 6 - p \geq n - p + 6,$$

which implies that they have at least eight common adjacencies in H .

Let y be a vertex of H . If the interval $[v_i, v_{i+1}]$ contains two vertices (one vertex, respectively) adjacent to y , we call it a *double (single, respectively) interval* of y . Let $x_1x_2x_3$ be a path in a double interval of y such that $x_1y, x_3y \in E(G)$. We call x_2 the *symmetric vertex* of y in the interval. For convenience, we give the following notation:

- u^* : a vertex of C not adjacent to H with $d_C(u^*) = p - r$;
- v^* : a vertex of H with $d_H(v^*)$ as large as possible and $d_H(v^*) = n - p - s$;
- w^* : a vertex of H with $d_H(w^*)$ as small as possible.

For $y \in V(H) \setminus v^*$, let $s(y)$ denote an integer with $d_H(y) = n - p - s(y)$. Clearly, $s(y) \geq s$. Suppose $r + s \geq 7$. Then,

$$n + 4k - m - 6 \leq d(u^*) + d(y) \leq (p - r) + (n - p - s(y)) + d_C(y), \quad (4.2)$$

which implies that $d_C(y) \geq 4k - (5k - 3)/2 + 1 = (3k + 5)/2$, and hence the number of the double intervals of y will be more than half of the $k + 1$ intervals of C . So each pair of vertices $h_1, h_2 \in V(H)$ will have two adjacencies in some common interval. Claim 10 forces that the vertices h_1 and h_2 will have the same two adjacencies in this interval and they will be at a distance two on C . Hence, $h_1h_2 \notin E(G)$, since otherwise there would be the required C_{p+1} . Thus, H has no edges, which contradicts the fact that each vertex in H has at least one adjacency in H . We assume that $r + s \leq 6$.

In the remaining parts of the proof, let C be chosen such that the number of the edges of H is as large as possible. Observe that $\delta(H) \geq (n - p)/2 + 3$ if $r + s \geq 3$. Suppose to the contrary that there is a vertex $y \in V(H)$ with $d_H(y) < (n - p)/2 + 3$. Note that $3 \leq r + s \leq r + s(y)$. From (4.2), we have $d_C(y) \geq (3k - 3)/2$ and y has a double interval on C . Let x be the symmetric vertex of y in the interval. Then, $d_H(x) < (n - p)/2 + 3$ by the choice of C , and hence we have the following contradiction:

$$n + 6 \leq n + 4k - m - 6 \leq d(x) + d(y) < 2 \cdot \frac{p}{2} + 2 \cdot \left(\frac{n - p}{2} + 3 \right).$$

We also observe that H contains enough vertices. Indeed, for each vertex $y \in V(H)$, the inequality (4.2) implies that $d_C(y) \geq (3k - 5)/2$. Then w^* contains at least one double interval on C , where we consider $[v_k, v_1]$ as an interval. Therefore, y contains at least $(n + 4k - m - 6) - p - (n - p - 1) = 4k - m - 5 \geq (3k - 7)/2$ adjacencies in H . Then H contains at least $4k - m - 4 \geq (3k - 5)/2 \geq 8$ vertices. Furthermore, $\delta(H) \geq 7$.

Claim 12. $r + s \leq 4$.

Proof. Suppose that $r + s \geq 5$. The inequality (4.2) implies that $d_C(y) \geq (3k + 1)/2 = (k + 1) + (k - 1)/2$ for each vertex $y \in V(H)$. Note that $h_1 h_2 \in E(G)$ implies that they do not have the same double interval on C for two arbitrary vertices $h_1, h_2 \in V(H)$. Since $3(k - 1)/2 > k + 1$, H does not contain 3-cycle. Theorem 4.10 implies that $\delta(H) \leq (n - p)/2$, which contradicts the fact that $\delta(H) \geq (n - p)/2 + 3$. $\square$

Claim 13. $r + s \leq 3$.

Proof. Suppose to the contrary that $r + s = 4$. From inequality (4.2), we have $d_C(y) \geq (3k - 1)/2$ for each vertex $y \in V(H)$, and hence y has at least $(k - 3)/2$ double intervals in C . Let $L = G[N_H(v^*)]$.

If $k \geq 8$, the subgraph L cannot contain a 3-cycle since $4 \cdot (k - 3)/2 > k + 1$. Theorem 4.10 implies that $\delta(L) \leq (n - p - s)/2$, and hence $\delta(H) \leq \delta(L) + s \leq (n - p + 3)/2$, a contradiction. We consider the case when $k = 7$. Since $5 \cdot (k - 3)/2 > k + 1$, H contains no K_5 , and hence $\delta(H) \leq 3/4(n - p)$ by Theorem 4.10. On the other hand, since L contains no K_4 , we have $\delta(H) \leq 2/3(n - p - s) + s$. So,

$$\delta(H) \leq \min \left\{ \frac{3}{4}(n - p), \frac{2(n - p) + 3}{3} \right\}.$$

Recall that $d_H(w^*) = \delta(H)$. Let w' be a symmetric vertex of w^* in some interval of C . If $n - p \geq 19$, then $d_H(w^*) = \delta(H) \leq n - p - 6$, and hence $r + s(w^*) \geq 7$. w^* contains at least $(k + 3)/2$ double intervals in C . Since $(k + 3)/2 + 2 \cdot (k - 3)/2 > k + 1$, $N_H(w^*)$ is an independent set. For each vertex $x \in N_H(w^*)$, we have $d_H(x) \leq n - p - \delta(H) \leq n - p - 7$ and hence x contains at least $(k + 5)/2$ double intervals in C . Then w^* and x have a common double interval in C , which implies a $(p + 1)$ -cycle with the required property, a contradiction. If $16 \leq n - p \leq 18$, then $d_H(w^*) = \delta(H) \leq n - p - 5$. If $\delta(H) \leq n - p - 6$, we get the same contradiction as above. So assume that $\delta(H) = n - p - 5$, and hence $r + s(w^*) \geq 6$. w^* contains at least $(k + 1)/2$ double intervals, which implies that the subgraph induced by $N_H(w^*)$ does not contain K_3 since $(k + 1)/2 + 3 \cdot (k - 3)/2 > k + 1$. Then there is a vertex $x \in N_H(w^*)$ such that $d_H(x) \leq (n - p - 5)/2 + 5 < n - p - 5$, a contradiction. If $13 \leq n - p \leq 15$, then $d_H(w^*) = \delta(H) \leq n - p - 4$, and hence $r + s(w^*) \geq 5$. w^* contains at least $(k - 1)/2$ double intervals, and the subgraph induced by $N_H(w^*)$ does not contain K_3 . There exists a vertex $x \in N_H(w^*)$ such that $d_H(x) \leq (n - p - 4)/2 + 5 < n - p - 4$. Thus,

$$n + 6 \leq d(w^*) + d(w') \leq 2 \cdot \frac{p}{2} + 2(n - p - 5) \leq n + 5,$$

a contradiction. If $9 \leq n - p \leq 12$, then $d_H(w^*) = \delta(H) \leq n - p - 3$. Thus,

$$n + 6 \leq d(w^*) + d(w') \leq 2 \cdot \frac{p}{2} + 2(n - p - 3),$$

which is a contradiction unless $n - p = 12$ and $\delta(H) = n - p - 3$. If $d_H(v^*) \geq n - p - 2$, then L does not contain K_3 since $(k - 1)/2 + 3 \cdot (k - 3)/2 > k + 1$, and hence $\delta(H) \leq (n - p + 2)/2 < n - p - 3$, a contradiction. So, each vertex of H has the same property of w^* . By Claim 10 and 11, the cycle C contains exactly $2k + 2$ vertices and w^* is adjacent

to each vertex of S' . This contradicts to the fact that w^* is adjacent to at least $(3k-1)/2$ vertices of C . If $n-p=8$, then $d_H(w^*) = \delta(H) = n-p-2$. Therefore,

$$n+6 \leq d(w^*) + d(w') \leq \frac{p}{2} + \frac{p}{2} + n-p-2 + n-p-2 \leq n+4,$$

a contradiction. Thus, $r+s \leq 3$. $\square$

Claim 14. $r+s \leq 2$.

Proof. Suppose to the contrary that $r+s=3$. From inequality (4.2), we have $d_C(y) \geq (3k-3)/2$ for each vertex $y \in V(H)$, and y has at least $(k-5)/2$ double intervals in C .

In the following, we consider two cases:

Case 1: $r=2, s=1$.

First, we show that each interval of C is double. In fact, we note that $5 \cdot \lceil (k-5)/2 \rceil \geq k+1$ for $k \geq 8$. If H contains a vertex $v' \in V(H-v^*)$ with the degree on H more than $n-p-3$, we are done unless the subgraph induced by $N_H(v') \setminus \{v^*\}$, say L , does not contain 3-cycle, which implies $\delta(H) \leq \delta(L) + 3 \leq (n-p-3)/2 + 3$, a contradiction. So for each vertex $y \in V(H-v^*)$, we have $d_H(y) \leq n-p-3$, and hence y has $(k-1)/2$ double intervals in C . The inequality $(k-5)/2 + 3 \cdot (k-1)/2 > k+1$ forces that $H-v^*$ does not contain 3-cycle, which implies that $\delta(H) \leq (n-p-1)/2 + 1$, a contradiction.

It is remaining to consider the case when $k=7$. Suppose that there exists an interval which is not double. Let w_0 be a vertex with a maximum number, say t , of double intervals in C . Clearly, $t \leq 7$. Let R be the set of vertices of H which are adjacent to w_0 .

If $t=7$, all of the vertices of R do not have double interval in C , which contradicts the fact that each vertex of H has at least $(k-5)/2$ double interval.

If $t=6$, all of the vertices of R would have the same double interval, and no pair of vertices of R would be adjacent. This implies that each vertex of R has degree at most $n-p-7$, and hence $t \geq 7$, a contradiction.

If $t=5$, we see that $s(w_0) \leq 5$. Note that the subgraph induced by R , say also R , does not contain 3-cycle, and hence $\delta(R) \leq (n-p-5)/2 + 5$, a contradiction.

If $t=4$, we have $s(w_0) \leq 4$, and hence $s(y) \leq 4$ for all $y \in V(H)$. Clearly, R is not complete. Let $w_1 \in R$ be a vertex with $d_H(w_1) \leq n-p-2$. We are done for $s(w_1)=3$ or 4. So assume that $s(w_1)=2$ and w_1 has two double intervals. Let R' be the subgraph induced by the vertex subset $N_H(w_0) \cap N_H(w_1) \setminus \{w_0, w_1\}$. Since each pair of nonadjacent vertices of H has at least eight common adjacencies in H , we see that R' has at least six vertices. Now each vertex of R' has the same double interval. This implies that it has the degree at most $n-p-6$, and hence $t \geq 6$, a contradiction.

If $t=3$, we have $s(w_0) \leq 3$, and hence $s(y) \leq 3$ for all $y \in V(H)$. Let w_1 and R' be defined as above. The same argument implies that R' has at least five vertices. Observe that R' does not contain 3-cycle. There is at most one vertex in R' which has the degree $n-p-1$. So R' has no edges and each vertex of R' has the degree at most $n-p-5$, a contradiction.

If $t \leq 2$, we have $s(y) \leq 2$ and $d_H(y) \geq n-p-2$ for all $y \in V(H)$. It not difficult to check that H contains at most 8 vertices, which is a contradiction unless H is a

complete subgraph with 8 vertices. In the special case, each vertex $y \in V(H)$ is adjacent alternatively to the vertices of C . With the same argument, the symmetric vertex of y has the same property. Since u^* has no adjacencies in H , all vertices of H have the same adjacencies in C . There exists a symmetric vertex y' which can be inserted into the path $y'^+ \overrightarrow{C} y'^-$, a contradiction. So each interval of C is double.

Second, there are exactly three vertices which have adjacencies on H if an interval is double. Let $y_1 \in V(H)$, and let $x_1 \dots x_\ell$ be a path in some interval of C with $y_1 x_i, y_1 x_{i+2} \in E(G)$. Previous observations imply that there is a vertex $y_2 \in V(H)$ which is adjacent to x_{i+1} . Let x_j be the first vertex of this interval such that x_j has an adjacency, say y_3 , in H . Obviously, y_3 is neither y_1 nor y_2 , and y_3 and x_{i+1} have a common adjacency y_4 in H . Then we obtain a $(p+1)$ -cycle by inserting the vertices $x_{i+3}, \dots, x_{j-1}$ into the cycle $x_j \overrightarrow{C} x_{i+1} y_4 y_3 x_j$ if these vertices exist, a contradiction.

Third, each interval contains exactly three vertices. Let $y_1 \in V(H)$, and let $x_1 x_2 x_3$ be a path in some interval of C with $y_1 x_1, y_1 x_3 \in E(G)$. Without loss of generality, say this interval (v_i, v_{i+1}) . Suppose $v_i \neq x_1$. Then v_i is not adjacent to H . Obviously, there is a common adjacency, say y_2 , between x_2 and y_1 . Since $d_C(v_i) = p-2$, we get a $(p+1)$ -cycle by omitting a vertex in the cycle $y_1 y_2 x_2 \overrightarrow{C} x_1 y_1$, a contradiction.

The last claim implies that every vertex of H has at most $k+1$ adjacencies in C . This is impossible since $k+1 < (3k-3)/2$.

Case 2: $r = 1, s = 2$.

Let $y_1 \in V(H)$ be arbitrary, $x_1 x_2 x_3$ be a path in some interval of C with $y_1 x_1, y_1 x_3 \in E(G)$ and $y_2 \in V(H)$ be the common adjacency of x_2 and y_1 . If x_1^- is in the same interval of C as x_1 , we see that x_1^- has no adjacencies in H since otherwise an adjacency of x_1^- in H and y_1 have a common adjacency in H , which can be used to form a $(p+1)$ -cycle with the required property. However, x_1^- cannot be adjacent to x_2 , which contradicts the fact that $d(x_1^-) = d_C(x_1^-) = p-1$. Hence, x_1 and likewise x_3 are in S , and if y_1 has two adjacencies in any interval, both are in S . This implies that y_1 has at most $k+1$ adjacencies in C , which is impossible since $k+1 < (3k-3)/2$. $\square$

Finally we consider the case when $r = 1, s = 1$, that is $d(u^*) = d_C(u^*) = p-1$ and $d_H(v^*) = n-p-1$. From inequality (4.2), we have $d_C(y) \geq (3k-5)/2$ for each vertex $y \in V(H)$, and hence y has at least $(k-7)/2$ double intervals in C . Each vertex of H has a double interval in C unless $k = 7$. Similarly to the proof of Case 2 in Claim 14, each vertex of H has just $k+1$ adjacencies in C . This gives a contradictions except when $k = 7$, in which case $y \in V(H)$ is adjacent to precisely $k+1$ vertices of C . If y is adjacent to a vertex of S' , all vertices of S' are precisely the adjacencies of y in C and $p = 2k+2$. A $(p+1)$ -cycle can be constructed by inserting a vertex of $V(C) \setminus S'$ into the remaining path of C . So we assume that each vertex $y \in V(H)$ is not adjacent to S' . Let $z_1 \in (v_k, v_0)$ and $z_2 \in (v_0, v_1)$ such that $yz_1, yz_2 \in E(G)$ and (z_1, v_0) has no adjacencies in H . We get a $(p+1)$ -cycle $z_1 y z_2 \overleftarrow{C} z_1^+ z_2^+ \overrightarrow{C} z_1$, a contradiction.

The proof of Theorem 4.9 is completed. $\blacksquare$

For $m = (5k - 3)/2$, the assumption of Theorem 4.9 is $\sigma_2(G) \geq n + (3k - 9)/2$, and this assumption does not involve the variable m . The example G_2 in Figure 1.2 implies the sharpness. Hence, we have the following which is a direct corollary of Theorem 4.9.

Corollary 4.11 *Let $k \geq 7$ and G be a graph of order n with $\sigma_2(G) \geq n + (3k - 9)/2$. Then, G is $(k, m + 2)$ -vertex-pancyclic ordered if $(5k - 3)/2 < m \leq n$. Also, the bound on σ_2 is sharp.*

The results of this chapter are summarized in the following.

Theorem 4.12 (R. Li [47]) *Let $7 \leq k \leq m - 1 \leq n - 1$ be positive integers, and let G be a graph of order n . Then, the graph G is $(k, m + 2)$ -vertex-pancyclic ordered if $\sigma_2(G)$ satisfies any of the following conditions:*

- (i) $\sigma_2(G) \geq 2n - 3$, when $k + 1 \leq m < \lfloor 3k/2 \rfloor$;
- (ii) $\sigma_2(G) \geq 2n - 4$, when $\lfloor 3k/2 \rfloor \leq m < \lceil (5k - 2)/3 \rceil$;
- (iii) $\sigma_2(G) \geq 2n - 5$, when $\lceil (5k - 2)/3 \rceil \leq m < 2k$;
- (iv) $\sigma_2(G) \geq n + 4k - m - 6$, when $2k \leq m \leq (5k - 3)/2$;
- (v) $\sigma_2(G) \geq n + (3k - 9)/2$, when $m > (5k - 3)/2$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

Remark 4.13 *Let $3 \leq k \leq 6$ be a positive integer and G be a graph of order n . Note that a $(k + 1, m)$ -pancyclic ordered graph is (k, m) -vertex-pancyclic ordered. Therefore, G is $(k, m + 2)$ -vertex-pancyclic ordered if $\sigma_2(G) \geq n + 4k - m - 2$ for $2k \leq m \leq (5k - 3)/2$ by Theorem 4.5. Also, G is $(k, m + 2)$ -vertex-pancyclic ordered if $\sigma_2(G) \geq n + (3k - 9)/2 + 3/2$ for $(5k - 3)/2 < m \leq n$.*

Maybe the bound on $\sigma_2(G)$ is less than the one shown in Remark 4.13 since the condition $k \geq 7$ comes from just the proof technique. We pose the following problems: Is G a $(k, m + 2)$ -vertex-pancyclic ordered graph if $\sigma_2(G) \geq n + 4k - m - 6$ for $3 \leq k \leq 6$ and $2k \leq m \leq (5k - 3)/2$? Is G a $(k, m + 2)$ -vertex-pancyclic ordered graph if $\sigma_2(G) \geq n + (3k - 9)/2$ for $3 \leq k \leq 6$ and $(5k - 3)/2 < m \leq n$?

Part II

Out-arc Pancyclicity on Digraphs

Chapter 5

Introduction

In this part, we turn towards digraphs. The main emphasis here, however, is the study of out-arc pancyclicity. In this introductory chapter, we present the relevant terminology and a background of main results. Our source is J. Bang-Jensen and G. Gutin [6] as well as G. Chartrand and L. Lesniak [16]. We refer the reader to these books for any information not mentioned here.

5.1 Terminology and notation

General concepts

We shall consider finite digraphs without loops and multiple arcs. Let D be a digraph on n vertices. We denote the vertex set and the arc set of a digraph D by $V(D)$ and $A(D)$, respectively. If xy is an arc of a digraph D , then we say that x *dominates* y or xy is an *out-arc* of x , and we write $x \rightarrow y$. If $x \rightarrow y$ but $y \not\rightarrow x$, we say that the arc xy is *ordinary* and write $x \mapsto y$. More generally, if A and B are two distinct subsets of $V(D)$ such that every vertex of A dominates all vertices of B , then we say that A *dominates* B and write $A \rightarrow B$. In the case when there is no arc from B to A , we write $A \Rightarrow B$. If both $A \rightarrow B$ and $A \Rightarrow B$ hold, then we say A *strictly dominates* B , denoted by $A \mapsto B$.

Let x be a vertex of D . The set of all vertices dominating x (dominated by x , respectively) is called the *in-neighbourhood* (*out-neighbourhood*, respectively) of x , denoted by $N_D^-(x)$ ($N_D^+(x)$, respectively). For a set $A \subseteq V(D)$, we let

$$N_D^+(A) = \bigcup_{x \in A} N_D^+(x) - A, \quad N_D^-(A) = \bigcup_{x \in A} N_D^-(x) - A.$$

We will omit the subscript if the digraph D is known from the context. Every vertex of $N^+(A)$ is called an *out-neighbour* of A and every vertex of $N^-(A)$ is an *in-neighbour* of A . Furthermore, the *neighbourhood* of A is defined by $N(A) = N^+(A) \cup N^-(A)$.

The number $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ are called the *out-degree* and *in-degree* of x , respectively. We call out-degree or in-degree of a vertex its *semi-degree*. The *degree* of x is the sum of its semi-degrees, i.e., the number $d_D(x) = d_D^+(x) + d_D^-(x)$. We will omit the subscript if the digraph D is known from the context.

The *minimum out-degree* (*minimum in-degree*) of D is

$$\delta^+(D) = \min\{d_D^+(x) \mid x \in V(D)\} \quad (\delta^-(D) = \min\{d_D^-(x) \mid x \in V(D)\}).$$

The *minimum semi-degree* of D is

$$\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}.$$

Similarly, the *maximum out-degree* (*maximum in-degree*) of D is

$$\Delta^+(D) = \max\{d_D^+(x) \mid x \in V(D)\} \quad (\Delta^-(D) = \max\{d_D^-(x) \mid x \in V(D)\}).$$

The *maximum semi-degree* of D is

$$\Delta^0(D) = \max\{\Delta^+(D), \Delta^-(D)\}.$$

The digraph D is called *k-regular* (or just, *regular*) if $\delta^0(D) = \Delta^0(D) = k$. A digraph D is *m-irregular* if there exists a natural number k such that $\max\{|k - d^+(x)|, |k - d^-(x)|\} \leq m$ for every vertex $x \in V(D)$. So, a regular digraph is 0-irregular.

Paths and cycles in a digraph are always directed. A *path* is a digraph with vertex set $\{x_1, x_2, \dots, x_k\}$ and the arc set $\{x_1x_2, x_2x_3, \dots, x_{k-1}x_k\}$. This path is called an (x_1, x_k) -*path* and is denoted $x_1x_2 \dots x_k$. A path from x to y is an (x, y) -path and a path connecting x and y is either an (x, y) -path or a (y, x) -path. The *cycle* $x_1x_2 \dots x_kx_1$ is the digraph obtained from the path $x_1x_2 \dots x_k$ by adding the arc x_kx_1 . The *length* of a path and a cycle is the number of its arcs. A *k-cycle* is a cycle of length k . An arc or a vertex in the digraph D with $|V(D)| \geq 3$ is said to be *s-pancyclic* if it is contained in a k -cycle for all k satisfying $s \leq k \leq |V(D)|$. In particular, if an arc or a vertex is 3-pancyclic, we also call it *pancyclic*. We call a vertex *out-arc s-pancyclic* if its all out-arcs are *s-pancyclic*. If a vertex is out-arc 3-pancyclic, we also call it an *out-arc pancyclic vertex*. A cycle or path C of a digraph D is *ordinary* if all arcs of C are ordinary. A *Hamiltonian path (cycle)* of the digraph D is a path (cycle) including every vertices of D .

A digraph H is a *subdigraph* of a digraph D if $V(H) \subseteq V(D)$, $A(H) \subseteq A(D)$ and every arc in $A(H)$ has both end-vertices in $V(H)$. If $V(H) = V(D)$, we say that H is a *spanning subdigraph* of D . If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that H is *induced* by $X = V(H)$, and we write $H = D[X]$. In addition, $D - X = D[V(D) \setminus X]$. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong.

A digraph D is said to be *strongly connected* (or, just *strong*), if for every pair of vertices x and y , D contains a path from x to y and a path from y to x . D is called *k-strongly connected* (or *k-strong*) if $|V(D)| \geq k + 1$ and for any set $U \subseteq V(D)$ of at most $k - 1$ vertices, $D - U$ is strong. A separating set S of D is *minimal* if for any proper subset S' of S , the subdigraph $D - S'$ is strong. If D is k -strong, but not $(k + 1)$ -strong, then the number $\sigma(D) = k$ is defined as the *strong connectivity* of D . Moreover, a subset $S \subseteq V(D)$ with $|S| = \sigma(D)$ and $\sigma(D - S) = 0$ is called a *minimum separating set* of D . A digraph D is *k-arc strong* if for every subset W of $A(D)$ of at most $k - 1$ arcs, $D - W$ is strong.

The *underlying graph* of D is the graph obtained by ignoring the orientations of arcs in D and deleting parallel edges. We say that D is *connected* if its underlying graph is connected.

Let $G = (V, E)$ be an undirected graph without loops and multiple edges. A *biorientation* of G is obtained from G by replacing every edge xy of G by the arc xy , the arc yx or the pair xy, yx of arcs. If no edge of G is replaced by a pair of arcs, we speak of an

orientation of G . The *complete biorientation* of G is a biorientation of G such that every edge xy of G is replaced by the pair xy, yx of arcs.

Classes of digraphs

A *semicomplete digraph* is a biorientation of a complete graph. The complete biorientation of a complete graph is a *complete digraph*. A *tournament* is an orientation of a complete graph. Thus tournaments are a special subclass of the semicomplete digraphs.

A digraph D is *locally in-semicomplete* (*locally out-semicomplete*, respectively) if, for every vertex x of D , the in-neighbourhood of x (its out-neighbourhood, respectively) induces a semicomplete digraph. A digraph D is *locally semicomplete* if it is both locally in- and locally out-semicomplete. Clearly, every semicomplete digraph is locally semicomplete. Similarly, we can define *locally in-tournament digraphs*, *locally out-tournament digraphs* and *locally tournament digraphs* by merely replacing the word “semicomplete” by “tournament” in the definitions above. For convenience, we shall sometimes refer to locally tournament digraphs as *local tournaments* and to locally in-tournament (out-tournament, respectively) digraphs as *local in-tournaments* (*local out-tournaments*, respectively).

A digraph D of order n is *pancyclic* if it contains a cycle of length k for each $k = 3, 4, \dots, n$ and is *vertex pancyclic* if each vertex v of D is pancyclic. A digraph D of order n is *arc- k -cyclic* for some $k \in \{3, 4, \dots, n\}$ if each arc of D is contained in a cycle of length k . A digraph D is *arc pancyclic* if it is arc- k -cyclic for every $k = 3, 4, \dots, n$.

A digraph is said to be *acyclic* if it has no cycles acting as induced subdigraphs. A digraph D is *transitive* if, for every pair of arcs xy and yz in D with $x \neq z$, the arc xz is also in D . Obviously, a tournament is transitive if and only if it is acyclic, moreover, a transitive tournament contains the unique Hamiltonian path.

Let D be a digraph with $V(D) = \{v_1, v_2, \dots, v_p\}$ and let $H_1, H_2, \dots, H_p$ be a collection of digraphs. Then $D[H_1, H_2, \dots, H_p]$ is the new digraph obtained from D by replacing each vertex v_i of D with H_i and by adding the arcs from every vertex of H_i to every vertex of H_j if $v_i v_j$ is an arc of D for all i and j satisfying $1 \leq i \neq j \leq p$.

The *converse* of a digraph D is the digraph obtained from D by reversing the directions of all arcs. A *complement* D^c of D is that digraph with vertex set $V(D)$ such that xy is an arc of D^c if and only if xy is not an arc of D . A digraph is *symmetric* if every arc is contained in a 2-cycle. If G is an undirected graph, we denote by G^* the symmetric digraph associated with G . Let K_n be a complete graph with n vertices. Then K_n^* is the *complete digraph* and K_n^c is a set of n vertices (also as the complement of K_n).

Path-contraction

We shall use the operation of *path-contraction* introduced in [6]. Let x and y be two distinct vertices of D and let P be an (x, y) -path in D . We say that H is obtained from D by *contracting* P to a new vertex w , if the following holds:

- (i) $V(H) = (V(D) \setminus V(P)) \cup \{w\}$, where w is a new vertex;
- (ii) $N_H^+(w) = N_D^+(y) \cap (V(D) \setminus V(P))$, $N_H^-(w) = N_D^-(x) \cap (V(D) \setminus V(P))$; and
- (iii) an arc with both end-vertices in $(V(D) \setminus V(P))$ belongs to H if and only if it belongs to D .

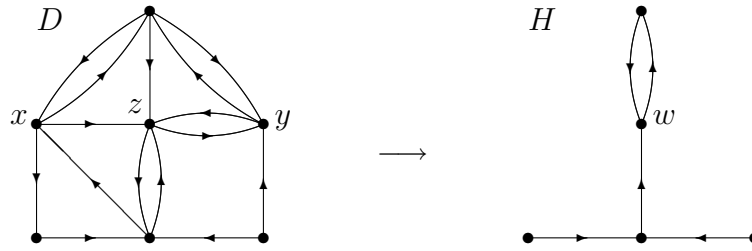


Figure 5.1: The graph H obtained from D by contracting xzy to w .

We write $H = D/P$ and simply D/e if P is an arc e . Let S be a system of pairwise disjoint paths $P_1, P_2, \dots, P_s$ of the complete digraph with vertex set $V(D)$. We say that H' is obtained from D by contracting S if $H' = (\dots((D/P_1)/P_2)/\dots)/P_s$, and write $H' = D/S$. Obviously, the operation of contraction doesn't depend on the order of path-contracting.

Figure 5.1 gives an example of “path-contraction”. Note that uwv is a path in H , if and only if uPv is a path in D . Analogously, if there exists a k -cycle containing w in H , then there is a $(k - 1 + |V(P)|)$ -cycle containing P in D .

5.2 Survey of earlier results

Demanding that a digraph D contains a out-arc pancyclic vertex is a very strong requirement, since D would have $\delta^+(D)$ distinct Hamiltonian cycles. Hence it is not surprising that most results on out-arc pancyclicity are for tournaments and generalizations of tournaments.

Tournaments are without any doubt the most well understood class of directed graphs, and they have a very rich structure, in particular with respect to cycles and paths; see the survey papers [4, 8, 12, 60, 67]. Let us begin with two well known structural properties of tournaments.

Theorem 5.1 (Rédei [66]) *Every tournament has a Hamiltonian path.*

Theorem 5.2 (Camion [15]) *A tournament is strong if and only if it has a Hamiltonian cycle.*

Moser [38] and Moon [60] later obtained the stronger results.

Theorem 5.3 (Moser [38]) *Every strong tournament is pancyclic.*

Theorem 5.4 (Moon [60]) *Every strong tournament is vertex pancyclic.*

The first result on arc pancyclic tournaments was given by Alspach.

Theorem 5.5 (Alspach [1]) *A regular tournament is arc pancyclic.*

Jacobsen [41] proved every arc of an 1-irregular tournament on $n \geq 8$ vertices is 4-pancyclic. Thomassen [71] generalized these results as follows:

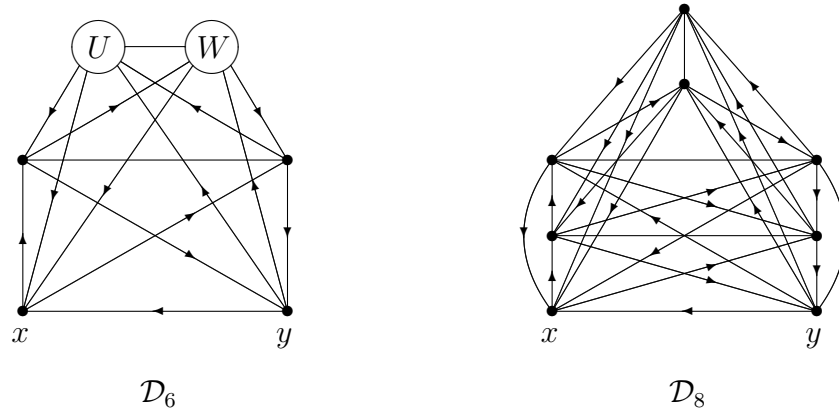


Figure 5.2: The two families of non-arc-pancyclic arc-3-cyclic tournaments. Each of the sets U and W induce an arc-3-cyclic tournament. All edges that are not already oriented may be oriented arbitrarily, but all arcs between U and W have the same direction.

Theorem 5.6 (Thomassen [71]) *Let T be an m -irregular tournament on n vertices. Let k be any integer such that $4 \leq k \leq n$. If $n \geq 5m + 9$, then the tournament T has an (x, y) -path with k vertices for every pair of vertices x and y . If $n \geq 5m + 3$, then every arc of T is contained in a cycle of length k .*

Tian, Wu and Zhang found a further generalization. See Figure 5.2 for the definition of the classes $\mathcal{D}_6$, $\mathcal{D}_8$.

Theorem 5.7 (Tian et al. [73]) *An arc-3-cyclic tournament is arc pancyclic unless it belongs to one of the families $\mathcal{D}_6$, $\mathcal{D}_8$ (in which case the arc yx does not belong to a Hamiltonian cycle).*

It is not difficult to derive the following two corollaries.

Corollary 5.8 (Tian et al. [73]) *At most one arc of an arc-3-cyclic tournament is not pancyclic.*

Theorem 5.9 (Wu et al. [77]) *A tournament is arc pancyclic if and only if it is arc-3-cyclic and arc- n -cyclic.*

Observe that each tournament in the infinite family $\mathcal{D}_6$ is 2-strong and the arc yx is not in any Hamiltonian cycle. We have the following result due to Thomassen.

Theorem 5.10 (Thomassen [71]) *There exist infinitely many 2-strong tournaments containing an arc which is not in any Hamiltonian cycle.*

The degree-type condition to imply a tournament to be arc pancyclic is given by Zhu and Tian [81] as follows.

Theorem 5.11 (Zhu and Tian [81]) *If T is a tournament of order n such that $d^+(v) + d^-(u) \geq n - 2$ for every arc uv , then T is arc pancyclic.*

For the pancyclic arcs in tournaments, some research papers deal with the following interesting questions: how many pancyclic arcs are there in a tournament and how many pancyclic arcs are there in a Hamiltonian cycle of a tournament? Let $p(D)$ denote the number of pancyclic arcs in a digraph D and let $h(D)$ denote the maximum number of pancyclic arcs belonging to the same Hamiltonian cycle of D . Clearly, $p(D) \geq h(D)$. Note that the higher connectivity of a tournament leads to the more pancyclic arcs. Havet [39] gave the definition $p_k(n)$ and $h_k(n)$ as follows:

$$p_k(n) = \min\{p(T) \mid T \text{ is a non-trivial } k\text{-strong tournament of order } n\};$$

$$h_k(n) = \min\{h(T) \mid T \text{ is a non-trivial } k\text{-strong tournament of order } n\}.$$

Theorem 5.12 (Moon [61]) $h_1(n) \geq 3$.

The equality holds if and only if the tournament is 3-cycle. Also, $p_1(n) \geq 3$. The following results are due to Havet and Yeo, respectively.

Proposition 5.13 (Havet [39]) $p_k(n) \leq 2kn - 2k^2 - k$, for all $k \geq 1$.

Proposition 5.14 (Havet [39]) $h_k(n) \leq 3k$ for all $k \geq 1$.

Theorem 5.15 (Yeo [79]) $p_k(n) \geq kn/2$, for all $k \geq 2$.

Theorem 5.16 (Yeo [79]) $h_k(n) \geq (k + 5)/2$, for all $k \geq 1$.

These results give support for the following conjecture.

Conjecture 5.17 (Havet [39]) $h_k(n) \geq 2k + 1$, and for sufficiently large n , $h_k(n) = 3k$.

Another interesting question on tournaments is whether there are vertices whose each out-arc is in a Hamiltonian cycle, or pancyclic, and how many. In Chapter 6, we shall count out-arc pancyclic vertices as well as out-arc 4-pancyclic vertices on tournaments.

Tournaments are by no means the only class of digraphs with a very rich structure. It was proved that a number of structural properties of tournaments are still valid for its various generalizations (See survey paper [5]). However, the out-arc pancyclicity on these much larger classes of digraphs has been less studied. In Chapter 7, we shall see that this problem is highly non-trivial, even for local tournaments and local in-tournaments.

In the last chapter, we prove several simple results on Hamiltonian-connected digraphs (refer to Chapter 8), which are very useful while applying path-contraction.

Chapter 6

Out-arc pancyclicity on tournaments

In [33], Goldberg and Moon proved that every s -arc strong tournament has at least s distinct Hamiltonian cycles. Thomassen [72] confirmed that every strong tournament contains a vertex x such that each out-arc of x is contained in a Hamiltonian cycle. In 2000, Yao, Guo and Zhang improved the result of Thomassen as follows:

Theorem 6.1 (Yao et al. [78]) *Every strong tournament T with $n \geq 3$ vertices contains a vertex u such that all out-arcs of u are pancyclic.*

Let T be a strong tournament with minimum out-degree at least two. A vertex v of T is called a *bridgehead* if there is a partition (A, B) of $V(T)$ such that the following conditions are satisfied:

- (i) $v \in A$, $T[A]$ is non-trivial and strong;
- (ii) $B \rightarrow A \setminus \{v\}$.

Theorem 6.1 is a corollary of the following statement:

Theorem 6.2 (Yao et al. [78]) *Let T be a strong tournament on n vertices and assume that the vertices of T are labelled $u_1, u_2, \dots, u_n$ such that $d^+(u_1) \leq d^+(u_2) \leq \dots \leq d^+(u_n)$. Let u be a vertex of T which can be chosen as follows:*

- (i) if $d^+(u_1) = 1$ then $u = u_1$;
- (ii) if $d^+(u_1) \geq 2$ then $d^+(u) = \min\{d^+(x) \mid x \in V(T) \text{ and } x \text{ is not a bridgehead}\}$.

Then all out-arcs of u are pancyclic.

Remark 6.3 In Theorem 6.2, one of $\{u_1, u_2\}$ can be chosen as the vertex u with the property by Lemma 2.2 and Theorem 3.1 in [78]. It is easy to check that $d^+(u) \leq \lfloor (n-1)/2 \rfloor$. Therefore, every strong tournament T with $n \geq 3$ vertices contains a vertex u with $d^+(u) \leq \lfloor (n-1)/2 \rfloor$ such that all out-arcs of u are pancyclic.

Immediately, we have the following corollary:

Corollary 6.4 *In a 2-strong tournament, all out-arcs of the vertices with minimum out-degree are pancyclic.*

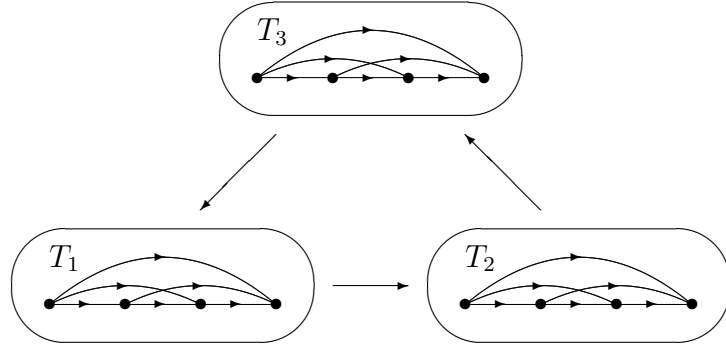


Figure 6.1: The tournament T : 4-strong, 3 out-arcs pancyclic vertices.

Yao, Guo and Zhang [78] posed the conjecture: A k -strong tournament T has at least k vertices $v_1, v_2, \dots, v_k$ such that all out-arcs of v_i are pancyclic for $i = 1, 2, \dots, k$. Theorem 6.1 indicates that the conjecture is true for $k = 1$. In Section 6.1, we will show that the conjecture is also true for $k = 2$ and 3. However, the statement is not true for $k \geq 4$ since Yeo [79] gave the following example. Let T_1, T_2 and T_3 be transitive tournaments of order at least k . Let T be the tournament obtained by adding arcs between the T_i 's, such that $T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow T_1$. Clearly, T is k -strong. Note that the only vertex in T_i such that its out-arcs are pancyclic, is the vertex with the out-degree equal to zero in T_i , as any arc totally within T_i doesn't lie on a 3-cycle. An example of a 4-strong tournament is shown in Figure 6.1. Thus, we have the following result.

Theorem 6.5 (Yeo [79]) *Let $k \geq 1$, be arbitrary. There exists an infinite class of k -strong tournaments such that each tournament contains at most 3 vertices, with the property that all arcs out of them are pancyclic.*

The number of vertices whose out-arcs are pancyclic is given by Yeo [79] as follows.

Theorem 6.6 (Yeo [79]) *Every 3-strong tournament has two distinct vertices x and y , such that all arcs out of x and all arcs out of y are pancyclic.*

Furthermore, x and y can be chosen such that $x \rightarrow y$ and $d^+(x) \leq d^+(y)$.

However, the following might hold.

Conjecture 6.7 (Yeo [79]) *If T is a 2-strong tournament, then it has three distinct vertices, $\{x, y, z\}$, such that every arc out of x, y and z , is pancyclic.*

Note that the example introduced by Yeo [79] (see also Figure 6.1), contains 6 out-arc 4-pancyclic vertices since the vertices with the out-degree 1 in each T_i are also out-arc 4-pancyclic. In Section 6.2, the number of out-arc 4-pancyclic vertices in s -strong tournaments is studied.

6.1 The number of out-arc pancyclic vertices

As shown in the beginning of this chapter, we only consider 2-strong or 3-strong tournaments. To give the number of out-arc pancyclic vertices, we introduce some easy but useful lemmas.

Lemma 6.8 (Yeo [79]) *Let D be a k -strong digraph with $k \geq 1$, and let S be a separating set in D such that $T = D - S$ is a tournament. Let $T_1, T_2, \dots, T_r$ ($r \geq 2$) be the strong components of T such that $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_r$. Now the following holds:*

- (i) *At least k vertices in S dominate some vertices in T_1 , and at least k vertices in S are dominated by some vertices in T_r ;*
- (ii) *For every $1 \leq \ell \leq |V(T)| - 1$, $u \in T_1$ and $v \in T_r$, there exists a (u, v) -path of length ℓ in T ;*
- (iii) *If $S = \{x\}$, then x is pancyclic in D .*

Lemma 6.9 (Yeo [79]) *Let D be a strong digraph containing a vertex x such that $D - x$ is a tournament and $d_D^+(x) + d_D^-(x) \geq |V(D)|$. Then there is an ℓ -cycle containing x in D for $2 \leq \ell \leq |V(D)|$.*

By contracting one arc of a tournament, it is easy to give a sufficient condition for an arc to be pancyclic.

Lemma 6.10 (Yeo [79]) *Let T be a 2-strong tournament containing an arc $e = xy$ such that $d^+(y) \geq d^+(x)$. Then e is pancyclic in T .*

The operation “path-contraction” has an influence on the strong connectivity of a tournament, which is shown as follows.

Lemma 6.11 *Let T be an s -strong tournament and $P = x_1x_2 \dots x_t$ a path in T . Let D be the digraph obtained from T by contracting the path P to w . If $s \geq t \geq 2$, then D is strong.*

Proof. Clearly, $D - w = T - V(P)$. Let $T_1, T_2, \dots, T_r$ be the strong components of $D - w$ with $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_r$ (possibly, $r = 1$).

Note that

$$\begin{aligned} d_D^+(w) &\geq d_T^+(x_t) - (t - 1) \geq s - t + 1 \geq 1 \quad \text{and} \\ d_D^-(w) &\geq d_T^-(x_1) - (t - 1) \geq s - t + 1 \geq 1. \end{aligned}$$

If $D - w$ is strong, then we are done. For the case when $r \geq 2$, we note that $N^-(T_1)$ and $N^+(T_r)$ are the subsets of $V(P)$ which implies that $|N^-(T_1)| \leq t$ and $|N^+(T_r)| \leq t$. Since T is s -strong, $|N^-(T_1)| \geq s$ and $|N^+(T_r)| \geq s$. So, we have $s = |N^-(T_1)| = |N^+(T_r)| = t$ since $s \geq t$. It follows that $d_{T_1}^+(w) = d_{T_1}^+(x_t) \geq 1$ and $d_{T_r}^-(w) = d_{T_r}^-(x_1) \geq 1$. Thus, D is strong. ■

A generalization of Lemma 6.10 is given as follows:

Theorem 6.12 *Let T be a tournament and $P = x_1x_2 \dots x_k$ be a path of T with $k \geq 2$. If T is k -strong and*

$$d^+(x_k) - d_P^+(x_k) \geq d^+(x_1) + d_P^-(x_1) - k + 2,$$

then the arc $x_i x_{i+1}$ is $(k+1)$ -pancyclic for $i = 1, 2, \dots, k-1$.

In particular, for $k = 3$, x_1x_2 and x_2x_3 are 4-pancyclic if T is 3-strong and one of the following holds:

- (i) $x_3 \rightarrow x_1$ and $d^+(x_3) \geq d^+(x_1) + 1$;
- (ii) $x_1 \rightarrow x_3$ and $d^+(x_3) \geq d^+(x_1) - 1$.

Proof. Let D be the digraph obtained from T by contracting P . Then D is strong by Lemma 6.11. Let $w \in V(D)$ be the new vertex obtained by contracting P . Then

$$\begin{aligned} d_D^+(w) + d_D^-(w) &= |N^+(x_k) \setminus N_P^+(x_k)| + |N^-(x_1) \setminus N_P^-(x_1)| \\ &= d^+(x_k) - d_P^+(x_k) + d^-(x_1) - d_P^-(x_1) \\ &= d^+(x_k) - d_P^+(x_k) + [n - 1 - d^+(x_1)] - d_P^-(x_1) \\ &= d^+(x_k) - d_P^+(x_k) - d^+(x_1) - d_P^-(x_1) + k - 2 + (n - k + 1) \\ &\geq n - k + 1 = |V(D)|. \end{aligned}$$

By Lemma 6.9, there is an i -cycle containing w in D for $2 \leq i \leq |V(D)|$. Back to T , $x_i x_{i+1}$ is $(k+1)$ -pancyclic for $i = 1, 2, \dots, k-1$.

It is direct to check the case when $k = 3$. ■

Remark 6.13 *In Theorem 6.12, the requirement for the tournament T to be k -strong is only used to prove strong connectivity of the digraph D . When later applying the path-contraction to show the out-arc pancyclicity of tournaments, we always check whether the digraph obtained by path-contraction is strong or not.*

Note that Lemma 6.10 is a corollary of Theorem 6.12 by setting $k = 2$. Now, we consider the out-arc pancyclic vertices of 2-strong tournaments.

Theorem 6.14 (R. Li, S. Li & Feng [49]) *Each 2-strong tournament T with n vertices contains at least 2 vertices v_1, v_2 such that all out-arcs of v_i are pancyclic for $i = 1, 2$.*

Proof. By Theorem 6.6, we consider only the case when $\sigma(T) = 2$. Let M be the set of all vertices in T that have minimum out-degree. All out-arcs of every vertex in M are pancyclic by Lemma 6.10. If $|M| \geq 2$, then we are done. So, assume that $|M| = 1$ and denote $M = \{v_1\}$. Note that

$$d^+(v) > d^+(v_1) \geq 2 \text{ for all } v \neq v_1. \tag{6.1}$$

It is sufficient to show that there exists another vertex, whose out-arcs are pancyclic.

We consider the following two cases:

Case 1: $\sigma(T - v_1) = 1$.

Let $S = \{v_1, x\}$ be a minimum separating set of T and let $T_1, T_2, \dots, T_t$ ($t \geq 2$) be the strong components of $T - S$. It is clear that for each $i \in \{1, 2, \dots, t\}$, T_i is a single vertex or contains a Hamiltonian cycle by Theorem 5.2. Assume without loss of generality that the strong components $T_1, T_2, \dots, T_t$ of T have been labelled such that $T_i \rightarrow T_j$ for $1 \leq i < j \leq t$. Note that $N^+(T_t) = S = N^-(T_1)$. In particular, the final strong component T_t contains at least 3 vertices by (6.1).

Assume without loss of generality that among all possible minimum separating sets $\{v_1, x'\}$ with $x' \in V(T - v_1)$, the vertex $x \in S$ has been chosen such that the final strong component of $T - S$ is minimum. This implies

$$d_{T_t}^-(x) \geq 2. \quad (6.2)$$

Let $\ell = |V(T_t)|$ and let $C_t = x_1 x_2 \dots x_\ell x_1$ be a Hamiltonian cycle of T_t . We claim that $d_{T_t}^-(v_1) \geq 3$. For the case when $\ell = 3$, we note the stronger result $T_t \rightarrow S$ holds from (6.1). For the case when $\ell \geq 4$, suppose to the contrary that $d_{T_t}^-(v_1) \leq 2$. Then we have $|d_{T_t}^+(v_1)| \geq \ell - 2$. Since v_1 dominates at least one vertex in T_1 , it holds $d^+(v_1) \geq \ell - 1$. Now, we look at the vertex, say x_i , such that $d_{T_t}^+(x_i)$ is minimum among all vertices in T_t . It is clear that $d_{T_t}^+(x_i) \leq \lfloor (\ell - 1)/2 \rfloor$. Thus, we have

$$d^+(x_i) = d_{T_t}^+(x_i) + d_S^+(x_i) \leq \left\lfloor \frac{\ell - 1}{2} \right\rfloor + 2 \leq \left\lfloor \frac{\ell - 1}{2} \right\rfloor + \left\lceil \frac{\ell - 1}{2} \right\rceil = \ell - 1 \leq d^+(v_1),$$

a contradiction to (6.1). Therefore, it holds

$$d_{T_t}^-(v_1) \geq 3. \quad (6.3)$$

By Theorem 6.1, the subtournament T_t contains a vertex, say v_2 , whose out-arcs are pancyclic in T_t . By Remark 6.3, we can choose v_2 such that $d_{T_t}^+(v_2) \leq \lfloor (\ell - 1)/2 \rfloor$.

Assume without loss of generality that the vertices of the Hamiltonian cycle C_t have been labelled with $x_\ell = v_2$.

We will show that either all out-arcs of v_2 are pancyclic or there exists another vertex apart from v_1, v_2 whose out-arcs are pancyclic. Let $v_2 \rightarrow w$ be an arc of T .

Subcase 1.1: $w \in V(T_t)$.

By the choice of v_2 , the arc $v_2 w$ is in an i -cycle for $i = 3, 4, \dots, \ell$. Assume without loss of generality that C_t has been the Hamiltonian cycle of T_t , which contains the arc $v_2 w$ (i.e., $w = x_1$). Moreover, we note from (6.3) that there is an integer $\beta \in \{1, 2, \dots, \ell - 2\}$ with $x_\beta \rightarrow v_1$.

Because $\{v_1, x\} = N^-(T_1)$ and T_1 is a single vertex or it contains a Hamiltonian cycle, it is obvious that the subtournament $T - (S \cup V(T_t))$ contains a Hamiltonian path, say $u_1 u_2 \dots u_{n-2-\ell}$ ($u'_1 u'_2 \dots u'_{n-2-\ell}$, respectively), with $v_1 \rightarrow u_1$ ($x \rightarrow u'_1$, respectively). It is easy to see that $v_2 x_1 \dots x_\beta v_1 u_1 x_{\beta+2} \dots x_\ell (= v_2)$ is a cycle of length $\ell + 1$ and $v_2 x_1 \dots x_\beta v_1 u_1 \dots u_j x_{\beta+1} \dots x_\ell (= v_2)$ is a cycle of length $\ell + 1 + j$ for $j = 1, 2, \dots, n - 2 - \ell$. It remains to show that $v_2 w$ is also in a Hamiltonian cycle of T .

If $v_1 \rightarrow x$, then $v_2 x_1 \dots x_\beta v_1 x u'_1 \dots u'_{n-2-\ell} x_{\beta+1} \dots x_\ell (= v_2)$ is a Hamiltonian cycle of T . If $x \rightarrow v_1$, then, by (6.2), there is a vertex x_γ with $1 \leq \gamma \leq \ell - 1$ such that $x_\gamma \rightarrow x$. Now, we see that $v_2 x_1 \dots x_\gamma x v_1 u_1 \dots u_{n-2-\ell} x_{\gamma+1} \dots x_\ell (= v_2)$ is a Hamiltonian cycle of T .

Subcase 1.2: $w \in \{v_1, x\}$.

For convenience, we denote the other vertex in $\{v_1, x\}$ by w' , i.e. $\{w, w'\} = \{v_1, x\}$.

By recalling that $T - (V(T_t) \cup \{w, w'\})$ contains a Hamiltonian path starting at a vertex in $N^+(w)$, we can confirm, similar to Subcase 1.1, that the arc v_2w is pancyclic in $T - w'$. It remains to check whether the arc v_2w is in a Hamiltonian cycle of T .

Suppose that $T - (V(T_t) \cup \{w, w'\})$ contains at least two vertices. Then, it is not difficult to show that $T - (V(T_t) \cup \{w, w'\})$ can be decomposed into two disjoint paths, say $w_1w_2 \dots w_\mu$ and $w'_1w'_2 \dots w'_\nu$ with $w \rightarrow w_1$ and $w' \rightarrow w'_1$, respectively. According to (6.2) or (6.3), there is a vertex x_j with $1 \leq j \leq \ell - 1$ such that $x_j \rightarrow w'$. Now, we see that

$$v_2ww_1w_2 \dots w_\mu x_1x_2 \dots x_jw'w'_1w'_2 \dots w'_\nu x_{j+1} \dots x_\ell (= v_2)$$

is a Hamiltonian cycle of T .

Suppose now that $T - (V(T_t) \cup \{w, w'\})$ contains exactly one vertex w_1 . Note that $t = 2$ and $\{w, w'\} \rightarrow w_1$.

If $w \rightarrow w'$, then we see that $v_2ww'w_1x_1x_2 \dots x_\ell (= v_2)$ is a Hamiltonian cycle of T . So, we consider the case when $w' \rightarrow w$.

Assume that $d^+(w') \geq 3$, and hence $N^+(w') \cap V(T_2) \neq \emptyset$. Recall that $T_t \rightarrow S$ if T_t contains exactly 3 vertices. So, we have $\ell \geq 4$. Let D be the digraph obtained from T by contracting the path v_2ww_1 to z . Clearly, $D - z$ is a tournament. Since $\ell \geq 4$, we see that $d^+(w_1) = \ell > \lfloor (\ell - 1)/2 \rfloor + 2 \geq d^+(v_2)$. It follows that

$$\begin{aligned} d_D^+(z) + d_D^-(z) &= |N^+(w_1) \setminus \{v_2\}| + |N^-(v_2) \setminus \{w_1\}| \\ &= (d^+(w_1) - 1) + (d^-(v_2) - 1) \\ &= (d^+(w_1) - 1) + [(n - 1) - d^+(v_2)] - 1 \\ &\geq n - 2 = |V(D)|. \end{aligned}$$

We will show that D is strong. It is obvious that $D - w'$ is strong. By (6.2) or (6.3), $N^-(w') \cap V(T_2 - v_2) \neq \emptyset$ holds. Because $d^+(w') \geq 3$, we have $N^+(w') \cap V(T_2) \neq \emptyset$. Therefore, D is strong. By Lemma 6.9, z is in all i -cycles for $i = 2, \dots, n - 2$, and hence, v_2w is in a Hamiltonian cycle of T .

Assume now that $d^+(w') < 3$. This implies $d^+(w') = 2$ and $w' = v_1$ by (6.1). Now, we see that $T_2 \rightarrow w'$, and it is not difficult to check that all out-arcs of w_1 are pancyclic.

Case 2: $\sigma(T - v_1) = 2$.

Let v_2 be a vertex with minimum out-degree among all vertices in $N^+(v_1)$. We prove that all out-arcs of v_2 are pancyclic.

Let $x \in N^+(v_2)$ be arbitrary. If $d^+(x) \geq d^+(v_2)$, then we are done by Lemma 6.10. So, assume that $d^+(x) < d^+(v_2)$. This implies that $x \notin N^+(v_1)$, i.e., $x \rightarrow v_1$, and hence, v_2x is in the 3-cycle $v_2xv_1v_2$.

Let D be the digraph obtained by contracting v_1v_2x to w . We show at first that D is strong. Let $y \in V(D - w)$ arbitrary. It is sufficient to confirm that D contains a path from y to w and a path from w to y . In fact, from the definition of path-contracting, we only need to verify that $T - \{v_1, v_2\}$ contains a path from x to y and $T - \{v_2, x\}$ contains a path from y to v_1 .

Since $T - v_1$ is 2-strong, $T - \{v_1, v_2\}$ is strong, and hence, $T - \{v_1, v_2\}$ contains a path from x to y .

Suppose now, to the contrary, that $T - \{v_2, x\}$ contains no path from y to v_1 . Let

$$Z = \{z \in V(T - \{v_2, x, y\}) \mid T - \{v_2, x\} \text{ contains a path from } y \text{ to } z\}.$$

By (6.1), we see that $Z \neq \emptyset$. In particular, $v_1 \notin Z$ and $v_1 \rightarrow Z$. It follows that $N^+(y) \subseteq Z \cup \{v_2, x\}$ and $Z \cup \{y, v_2\} \subseteq N^+(v_1)$. This implies $d^+(y) \leq |Z| + 2 \leq d^+(v_1)$, a contradiction to (6.1). Therefore, $T - \{v_2, x\}$ contains a path from y to v_1 .

Now, for the strong digraph D , we have

$$\begin{aligned} d_D^+(w) + d_D^-(w) &= |N^+(x) \setminus \{v_1\}| + |N^-(v_1) \setminus \{x\}| \\ &= (d^+(x) - 1) + (d^-(v_1) - 1) \\ &= (d^+(x) - 1) + [(n - 1) - d^+(v_1)] - 1 \\ &\geq n - 2 = |V(D)|, \end{aligned}$$

where the inequality follows from (6.1). According to Lemma 6.9, w is in an i -cycle for all $2 \leq i \leq |V(D)|$. Thus, v_2x is in a j -cycle in T for all $4 \leq j \leq n$.

Altogether, v_2x is pancyclic in T .

The proof of the theorem is complete. ■

Remark 6.15 According to the proof of Theorem 6.14, it is not difficult to get a polynomial algorithm to find two vertices v_1, v_2 in a tournament T with $\sigma(T) \geq 2$ such that all out-arcs of v_1 and all out-arcs of v_2 are pancyclic.

Remark 6.16 In the proof of Theorem 6.14, we obtain a special tournament at the end of Subcase 1.2. For convenience, we describe it, say T , again as follows: T is a tournament with $\sigma(T) = 2$ and $S = \{v_1, x\}$ is a minimum separating set of T . $T_1 = \{w_1\}$ and T_2 are the strong components of $T - S$ such that $w_1 \rightarrow T_2$. $v_2 \in V(T_2)$ is a vertex whose all out-arcs are pancyclic in the tournament T_2 , which is chosen just as described in Theorem 6.1. Set $v_1 \rightarrow x$, $\{x, v_1\} \rightarrow w_1$, $V(T_2) \rightarrow v_1$ and $v_2 \rightarrow x$. The remaining arcs are arbitrary (s. Figure 6.2). As it is mentioned in the proof of Theorem 6.14, v_1 and w_1 are two vertices whose all out-arcs are pancyclic. In fact, there exists another vertex apart from v_1 and w_1 with the same property.

Illustration to Remark 6.16: Clearly, $n \geq 6$. We will prove that x is an out-arc pancyclic vertex or there is another vertex apart from v_1, w_1 and x with the property. Note that xw_1 is pancyclic. It is sufficient to show that the out-arcs of x with the other end-vertex in T_2 are pancyclic. Let $y \in V(T_2)$ be an out-neighbour of x .

Let D be the digraph obtained from T by contracting the path xyv_1w_1 . We confirm that D is strong. Note that w_1 dominates T_2 and x has at least one in-neighbour in T_2 . If $T_2 - y$ is strong, then D is also strong. If $T_2 - y$ is not strong, let $T'_1, T'_2, \dots, T'_s$ be the components of $T_2 - y$ such that $T'_1 \rightarrow T'_2 \rightarrow \dots \rightarrow T'_s$. Recall that v_2 is one of the two vertices with the minimum degree in the tournament T_2 (s. Remark 6.3). So v_2 is in T'_s unless T'_s contains only one vertex. The latter implies that the unique vertex of T'_s dominates x or it has the out-degree 2, and hence D is strong or the vertex is out-arc pancyclic. From the former, it is easy to see that D is strong. Furthermore,

$$d^+(w_1) - 1 = |V(T_2)| - 1 = |V(T_2)| + 1 - 2 \geq d^+(x) + 1 - 2.$$

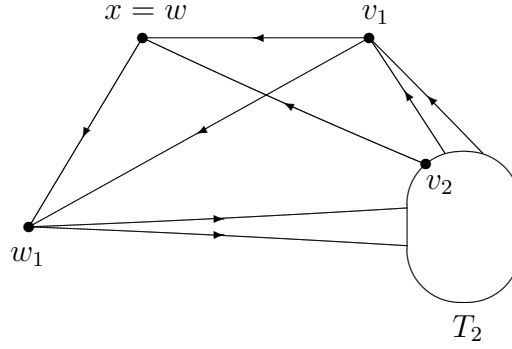


Figure 6.2: The possible structure of the tournament T in Remark 6.16. It is not difficult to check that all out-arcs of w_1 are pancyclic.

By Theorem 6.12 and Remark 6.13, xy is 5-pancyclic. Note that xyv_1x is a 3-cycle and $xyy'v_1x$ is a 4-cycle for some out-neighbour y' of y in T_2 . Therefore, xy is pancyclic. ■

For 3-strong tournaments, we have an easy observation as follows:

Proposition 6.17 *Let T be a 3-strong tournament, and let $v_1 \in V(T)$ with $d^+(v_1) = \min\{d^+(v) \mid v \in V(T)\}$ and $v_2 \in V(T) \setminus \{v_1\}$ such that all out-arcs of v_2 are pancyclic. If $N^+(v_1) \cap N^+(v_2) \neq \emptyset$, then there is another vertex apart from v_1 and v_2 , whose all out-arcs are pancyclic.*

Proof. Let u be the vertex, which has minimum out-degree among all vertices in $N^+(v_1) \cap N^+(v_2)$. Let $x \in N^+(u)$ be arbitrary. If $d^+(x) \geq d^+(u)$, then ux is pancyclic by Lemma 6.10. So assume that $d^+(x) < d^+(u)$. By the choice of u , $x \rightarrow v_1$ or $x \rightarrow v_2$. Then uxv_iu is a 3-cycle for some $i = 1$ or 2 . Let D be the digraph obtained from T by contracting v_1ux to w . By Lemma 6.11, D is strong.

If $d^+(x) = d^+(v_1)$, then x is the desired vertex. So assume that $d^+(x) \geq d^+(v_1) + 1$. By Lemma 6.12, ux is 4-pancyclic. Since uxv_iu is a 3-cycle for some $i = 1$ or 2 , ux is pancyclic. ■

Recently, Feng [28] gave the number of out-arc pancyclic vertices on 3-strong tournaments.

Theorem 6.18 (Feng [28]) *Every 3-strong tournament T contains 3 vertices whose all out-arcs are pancyclic.*

6.2 The number of out-arc 4-pancyclic vertices

Before presenting the main results of this section, we first give an interesting observation.

Lemma 6.19 *Let T be a 3-strong tournament and $u \in V(T)$ with*

$$d^+(u) = \min\{d^+(v) \mid v \in V(T)\}.$$

If $x \in N^+(u)$ such that there is $y \in N^+(u)$ with $y \rightarrow x$, then all out-arcs of x are 4-pancyclic.

Proof. Let $|V(T)| = n$ and $d^+(u) = t$. Let $z \in N^+(x)$ and D_w be the digraph obtained from T by contracting uxz to w , where w is a new vertex. It is clear that $d^+(z) \geq d^+(u)$. By Lemma 6.11, D_w is strong. Now, we consider the following two cases:

Case 1: $u \rightarrow z$.

It is easy to see that

$$\begin{aligned} d_{D_w}^+(w) + d_{D_w}^-(w) &= |N^+(z)| + |N^-(u)| \\ &= d^+(z) + d^-(u) \\ &= d^+(z) + [n - 1 - d^+(u)] \\ &\geq n - 1 > n - 2 = |V(D_w)|. \end{aligned}$$

By Lemma 6.9, w is in a j -cycle of D_w for $j = 2, 3, \dots, n - 2$, i.e., there is a k -cycle in T containing xz for $k = 4, 5, \dots, n$.

Case 2: $z \rightarrow u$.

If $d^+(z) > d^+(u)$, then, with a similar discussion as above, we are done. So we assume that $d^+(z) = d^+(u) = t$.

If $D_w - \{w\}$ is not strong, then we see from Lemma 6.8 that w is pancyclic in D_w , i.e., xz is in a k -cycles of T , $k = 5, 6, \dots, n$. Since $uxzu$ is a 3-cycle and $uyxzu$ is a 4-cycle, xz is pancyclic.

If $D_w - \{w\}$ is strong, then, by Theorem 5.2, $D_w - \{w\}$ contains a Hamiltonian cycle, say $C = v_1v_2 \dots v_{n-3}v_1$. In the following, we set $v_0 = v_{n-3}$. For convenience, we denote the subpath $v_i v_{i+1} \dots v_j$ by $C[v_i, v_j]$ for two arbitrary vertices v_i, v_j of C .

Clearly, $uxzu$ is a 3-cycle and $uyxzu$ is a 4-cycle containing xz . In the following, we will confirm that xz is pancyclic in T , i.e., we prove that there is a k -cycle of T containing xz , $k = 5, 6, \dots, n$.

Recall that $d^+(z) = d^+(u) = t$. It follows that

$$\begin{aligned} d_C^+(z) = d_C^+(u) &= t - 1, \\ d_C^+(z) + d_C^-(u) &= |N^+(z) \setminus \{u\}| + |N^-(u) \setminus \{z\}| \\ &= d^+(z) - 1 + d^-(u) - 1 \\ &= d^+(z) - 1 + [n - 1 - d^+(u)] - 1 \\ &= n - 3 = |V(C)|. \end{aligned}$$

Assume to the contrary that T has no $(n - \ell)$ -cycle containing xz for some ℓ with $0 \leq \ell \leq n - 5$. Let $\eta = t - 1$ and let $N_C^+(z) = \{v_{i_1}, v_{i_2}, \dots, v_{i_\eta}\}$. Then we see that $N_C^+(u) = \{v_{i_1 - \ell - 1}, v_{i_2 - \ell - 1}, \dots, v_{i_\eta - \ell - 1}\}$ (otherwise, $v_{i_j - \ell - 1}u \in A(T)$ for some $1 \leq j \leq \eta$, then

$xzv_{i_j}C[v_{i_j+1}, v_{i_j-\ell-2}]v_{i_j-\ell-1}ux$ is a $(n-\ell)$ -cycle through xz , a contradiction.) Without loss of generality, let $y = v_{i_j-\ell-1}$. We consider the following two cases:

Subcase 1: $u \rightarrow v_{i_1}$.

Then, $xzuv_{i_1}C[v_{i_1+1}, v_{i_1-\ell-2}]yx$ is a cycle of length $n-\ell$ containing xz , a contradiction.

Subcase 2: $v_{i_1} \rightarrow u$.

If $v_{i_1-\ell-2} \rightarrow u$, then $xzC[v_{i_1}, v_{i_1-\ell-2}]uyx$ is a cycle of length $n-\ell$ containing xz , a contradiction. We have $u \rightarrow v_{i_1-\ell-2}$. Suppose now that $C[v_k, v_{i_1-\ell-2}]$ is the longest subpath with $u_1 \rightarrow C[v_k, v_{i_1-\ell-2}]$. Since $v_{i_1} \rightarrow u$, we have $V(C[v_{i_1}, v_{k-1}]) \neq \emptyset$. Thus, the cycle $xzC[v_{i_1}, v_{k-1}]uC[v_k, v_{i_1-\ell-1}]x$ is of length $n-\ell$ containing xz , a contradiction.

Therefore, T has $(n-\ell)$ -cycle containing xz for all $0 \leq \ell \leq n-5$. So, xz is pancyclic. The lemma follows. $\blacksquare$

For an s -strong tournament T with $s \geq 3$, we counted the out-arc 4-pancyclic vertices and proved the following:

Theorem 6.20 (Feng, S. Li & R. Li [29]) *Every s -strong tournament T with $s \geq 3$ contains at least $s+1$ vertices whose all out-arcs are 4-pancyclic.*

Proof. Let $|V(T)| = n$. Since T is at least 3-strong, it is clear that $n \geq 7$. Let $u \in V(T)$ with $d^+(u) = \min\{d^+(v) \mid v \in V(T)\}$. According to Lemma 6.10, all out-arcs of u are pancyclic. Let $d^+(u) = t$ and $N^+(u) = \{x_1, x_2, \dots, x_t\}$. It is clear that $s \leq t \leq \lfloor (n-1)/2 \rfloor$.

Let $A = \{x \in N^+(u) \mid N^-(x) \cap N^+(u) = \emptyset\}$. Clearly, $|A| \leq 1$. If $|A| = 0$, i.e., every vertex of $N^+(u)$ is dominated by another vertex from $N^+(u)$, then by Lemma 6.19, all out-arcs of x_i are 4-pancyclic for $i = 1, 2, \dots, t$. Recalling that all out-arcs of u are pancyclic, we have $t+1 \geq s+1$ desired vertices.

Now consider the case when $|A| = 1$. Without loss of generality, assume that $A = \{x_1\}$. Because $T[N^+(u)]$ is a tournament, we have $x_1 \rightarrow x_j$ for $j = 2, 3, \dots, t$. By Lemma 6.19, we see that all out-arcs of x_j are 4-pancyclic for $j = 2, 3, \dots, t$. Considering the pancyclicity of u , we have t desired vertices. In the following, we find the $(t+1)$ -th vertex whose all out-arcs are 4-pancyclic.

Let $W = N^+(x_1) \setminus N^+(u)$. Because of $d^+(x_1) \geq d^+(u)$, we have $W \neq \emptyset$. If there is a vertex $y \in W$ with $d^+(y) = d^+(u)$, then y is the desired vertex according to Lemma 6.10. So, assume that $d^+(y) > d^+(u)$ for all $y \in W$. Now, we confirm that all out-arcs of x_1 are 4-pancyclic.

For a vertex $z \in N^+(x_1)$, let D_w be the digraph obtained from T by contracting ux_1z to w , where $w \notin V(T)$.

If $z \in W$, then $zu \in A(T)$. From $d^+(y) > d^+(u)$, it is easy to see that

$$\begin{aligned} d_{D_w}^+(w) + d_{D_w}^-(w) &= |N^+(z) \setminus \{u\}| + |N^-(u) \setminus \{z\}| \\ &= d^+(z) - 1 + d^-(u) - 1 \\ &= d^+(z) - 1 + [n - 1 - d^+(u)] - 1 \\ &\geq n - 2 = |V(D_w)|. \end{aligned}$$

By Lemma 6.9, w is in a j -cycle of D_w for $j = 2, 3, \dots, n-2$, i.e., there is a k -cycle in T containing x_1z for $k = 4, 5, \dots, n$.

Now, we consider the case when $z \notin W$, i.e., $z \in N^+(u)$. By using an analogous argument to the Case 1 in Lemma 6.19, we see that x_1z is 4-pancyclic.

Therefore, T contains at least $d^+(u) + 1 = t + 1 \geq s + 1$ vertices whose all out-arcs are 4-pancyclic. $\blacksquare$

In the Ph.D. thesis [27] Feng proved that Theorem 6.20 is also true for $s = 2$, i.e., each 2-strong tournament contains at least 3 out-arc 4-pancyclic vertices. His proof is very technical and consists of a long case analysis. Here we only give some observations on the out-arc 4-pancyclicity of 2-strong tournaments.

Theorem 6.21 *Let T be a 2-strong tournament, $v_1 \in V(T)$ with $d^+(v_1) = \min\{d^+(v) \mid v \in V(T)\}$ and $v_2 \in V(T) \setminus \{v_1\}$ such that all out-arcs of v_2 are pancyclic. If*

$$\sigma(T - \{v_1, v_2\}) = 0,$$

then there is another vertex apart from v_1 and v_2 , whose all out-arcs are pancyclic.

The proof of this theorem is completely similar to Case 1 of Theorem 6.14 by only replacing the vertex x in the minimum separating set S with v_2 mentioned above. Hence it makes no sense to give it again.

Note under the condition of Theorem 6.17 and 6.21, T contains 3 out-arc pancyclic vertices, and thus 3 out-arc 4-pancyclic vertices.

Theorem 6.22 *Let T be a 2-strong tournament, $v_1 \in V(T)$ with $d^+(v_1) = \min\{d^+(v) \mid v \in V(T)\}$ and $v_2 \in V(T) \setminus \{v_1\}$ such that all out-arcs of v_2 are pancyclic. If*

$$\sigma(T - \{v_1, v_2\}) \geq 2,$$

then there is another vertex apart from v_1 and v_2 , whose all out-arcs are 4-pancyclic.

Proof. In what follows, let $v_1 \rightarrow v_2$ for the case when $d^+(v_1) = d^+(v_2)$. If there is $z \in V(T) \setminus \{v_1, v_2\}$ with $d^+(z) = d^+(v_1)$, then, by Lemma 6.10, all out-arcs of z are pancyclic. So we assume that

$$d^+(z) > d^+(v_1) \geq 2, \quad \text{for all } z \in V(T) \setminus \{v_1, v_2\}.$$

Let u be the vertex, which has minimum out-degree among all vertices in $N^+(v_1) \setminus \{v_2\}$. We will prove that either all out-arcs of u are pancyclic or there is another desired vertex.

Let $x \in N^+(u)$ be arbitrary. If $d^+(x) \geq d^+(u)$, then we are done by Lemma 6.10. So assume that $d^+(u) > d^+(x) \geq d^+(v_1)$. Let D be the digraph obtained by contracting v_1ux to a new vertex w .

Case 1: D is strong.

If $x \rightarrow v_1$, then we assume that $d^+(x) > d^+(v_1)$; otherwise, $d^+(x) = d^+(v_1)$ and x is the desired vertex. Then, the arc ux is 4-pancyclic by Theorem 6.12. Since uxv_1u is a 3-cycle,

then ux is pancyclic. If $v_1 \rightarrow x$, then $x = v_2$ by the choice of u . Since $d^+(v_2) \geq d^+(v_1)$, uv_2 is 4-pancyclic by Theorem 6.12.

Case 2: D is not strong.

There is a vertex $y \in V(D - w)$ such that either y is not reachable from w or w is not reachable from y in D . In other words, either y is not reachable from x in $T - \{v_1, u\}$ or v_1 is not reachable from y in $T - \{u, x\}$. The latter implies that $T - \{u, x\}$ is not strong and the component including v_1 dominates the component including y in $T - \{u, x\}$, which is impossible since v_1 is the vertex with the minimum out-degree. So y is not reachable from x in $T - \{v_1, u\}$. Let $T_1, T_2, \dots, T_r$ be the components of $T - \{v_1, u\}$ with $T_i \rightarrow T_j$ for $1 \leq i < j \leq t$. Note that $\sigma(T - \{v_1, v_2\}) \geq 2$ implies that $\sigma(T - \{v_1, v_2, u\}) \geq 1$. So we have $t = 2$ and $|V(T_1)| = 1$ if $v_2 \in V(T_1)$ or $|V(T_2)| = 1$ if $v_2 \in V(T_2)$. If $v_2 \in V(T_2)$, then $d^+(v_2) = 2$ and $v_2 \rightarrow v_1$, a contradiction to the assumption that $v_1 \rightarrow v_2$ if $d^+(v_1) = d^+(v_2)$. So $V(T_1) = \{v_2\}$.

Let $v \in V(T_2)$ be an out-arc pancyclic vertex in T_2 with $d_{T_2}^+(v) \leq \lfloor (|V(T_2)| - 1)/2 \rfloor$. Similarly to the proof of Theorem 6.14 and by Remark 6.16, either v or another vertex apart from v_1, v_2 and v is the desired vertex. The theorem follows. ■

Chapter 7

Structural analysis of local in-tournaments

A digraph on n vertices is *round* if we can label its vertices $v_0, v_1, \dots, v_{n-1}$ so that for each i , we have $N^+(v_i) = \{v_{i+1}, \dots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, \dots, v_{i-1}\}$ (all subscripts are taken modulo n). We will refer to the ordering $v_0, v_1, \dots, v_{n-1}$ as a *round labelling* of D . A locally semicomplete digraph D is *round decomposable* if there exists a round local tournament R on $p \geq 2$ vertices such that $D = R[D_1, D_2, \dots, D_p]$, where each D_i is a strong semicomplete subdigraph of D for $i = 1, 2, \dots, p$.

Let D be a strong local tournament. The parameter g of D is defined as follows: if D is round decomposable and it has the round decomposition $D = R[D_1, D_2, \dots, D_p]$ with $V(R) = \{x_1, x_2, \dots, x_p\}$ and $x_i \in V(D_i)$ for $i = 1, 2, \dots, p$, then let $\ell(x_i)$ denote the length of the shortest cycle containing x_i in R for $i = 1, 2, \dots, p$, and $g = \max_{1 \leq i \leq p} \{\ell(x_i)\} + 1$; if D is not round decomposable, then let $g = 3$.

The out-arc pancyclicity on local tournaments was studied by S. Li, Meng and Guo [54].

Theorem 7.1 (S. Li et al. [54]) *Let D be a strong local tournament with n vertices, then the following holds:*

- (a) *If D is round decomposable, then D contains a vertex u such that all out-arcs of u are g -pancyclic, unless D is isomorphic to one of $T^{**} = \{R_n^2 \mid n \text{ is even and } n \geq 6\}$, where R_n^2 is a 2-regular, round decomposable local tournament with n vertices.*
- (b) *If D is not round decomposable, then D contains a vertex u such that all out-arcs of u are g -pancyclic, i.e., pancyclic.*

In this chapter, we shall contrast the structure of local in-tournaments with local tournaments and give a structural analysis to the out-arc pancyclicity on local in-tournaments.

Bang-Jensen, Huang and Prisner extended Theorem 5.2 to local in-tournaments. It is shown that the requirement of Hamiltonicity on local in-tournaments is the same as on tournaments (not only local tournaments).

Theorem 7.2 (Bang-Jensen et al. [7]) *A local in-tournament is Hamiltonian if and only if it is strong.*

Let us firstly recall the structure of locally semicomplete graphs. The most fundamentally structural properties of strong components of a connected non-strong locally semicomplete digraph are given in the following result.

Theorem 7.3 (Bang-Jensen [2]) *Let D be a connected locally semicomplete digraph that is not strong. Then the following holds for D .*

- (a) *If A and B are distinct strong components of D with at least one arc between them, then either $A \mapsto B$ or $B \mapsto A$.*
- (b) *If A and B are strong components of D , such that $A \mapsto B$, then A and B are semicomplete digraphs.*
- (c) *The strong components of D can be ordered in a unique way $D_1, D_2, \dots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \dots, p-1$.*

Let D be a connected, but non-strong locally semicomplete digraph. Then the unique sequence $D_1, D_2, \dots, D_p$ of the strong components of D with $N^+(D_j) \cap V(D_i) = \emptyset$ for $j > i$ and $D_i \rightarrow D_{i+1}$ for $i = 1, 2, \dots, p-1$ is called a *strong decomposition* of D . Note that every connected non-strong locally semicomplete digraph has a unique strong decomposition. Guo and Volkmann [37] found a further useful decomposition.

Theorem 7.4 (Guo and Volkmann [37]) *Let D be a connected locally semicomplete digraph that is not strong and let $D_1, D_2, \dots, D_p$ be the strong decomposition of D . Then D can be decomposed in $r \geq 2$ subdigraphs $D'_1, D'_2, \dots, D'_r$ as follows:*

$$D'_1 = D_p, \quad \lambda_1 = p,$$

$$\lambda_{i+1} = \min\{j \mid N^+(D_j) \cap V(D'_i) \neq \emptyset\}, \text{ and}$$

$$D'_{i+1} = D[V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \dots \cup V(D_{\lambda_i-1})].$$

The subdigraphs $D'_1, D'_2, \dots, D'_r$ satisfy the properties below:

- (a) *D'_i consists of some strong components of D and is semicomplete for $i = 1, 2, \dots, r$;*
- (b) *D'_{i+1} dominates the initial component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, 2, \dots, r-1$;*
- (c) *if $r \geq 3$, then there is no arc between D'_i and D'_j for i, j satisfying $|j - i| \geq 2$.*

For a connected, but not strong locally semicomplete digraph D , the unique sequence $D'_1, D'_2, \dots, D'_r$ as defined in Theorem 7.4 is called the *semicomplete decomposition* of D . The next result is a characterization of locally semicomplete digraphs which are not semicomplete and not round decomposable.

Theorem 7.5 (Bang-Jensen et al. [3]) *Let D be a strong locally semicomplete digraph which is not semicomplete. Then D is not round decomposable if and only if the following conditions are satisfied:*

- (a) *There is a minimal separating set S such that $D - S$ is not semicomplete and for each such S , $D[S]$ is semicomplete and the semicomplete decomposition of $D - S$ has exactly three components D'_1, D'_2, D'_3 ;*
- (b) *There are integers α, β, μ, ν with $\lambda \leq \alpha \leq \beta \leq p - 1$ and $p + 1 \leq \mu \leq \nu \leq p + q$ such that*

$$N^-(D_\alpha) \cap V(D_\mu) \neq \emptyset \text{ and } N^+(D_\alpha) \cap V(D_\nu) \neq \emptyset, \\ \text{or } N^-(D_\mu) \cap V(D_\alpha) \neq \emptyset \text{ and } N^+(D_\mu) \cap V(D_\beta) \neq \emptyset,$$

where $D_1, D_2, \dots, D_p$ and $D_{p+1}, \dots, D_{p+q}$ are the strong decompositions of $D - S$ and $D[S]$, respectively, and D_λ is the initial component of D'_2 .

In [3], Bang-Jensen, Guo, Gutin and Volkmann gave a classification of locally semicomplete digraphs.

Theorem 7.6 (Bang-Jensen et al. [3]) *Let D be a connected locally semicomplete digraph. Then exactly one of the following possibilities hold.*

- (a) *D is round decomposable with a unique round decomposition $R[D_1, D_2, \dots, D_\alpha]$, where R is a round local tournament on $\alpha \geq 2$ vertices and D_i is a strong semicomplete digraph for $i = 1, 2, \dots, \alpha$;*
- (b) *D is not round decomposable and not semicomplete and it has the structure as described in Theorem 7.5;*
- (c) *D is a not round decomposable, semicomplete digraph.*

Just as shown in [3], round digraphs play an important role in the study of locally semicomplete digraphs. Huang [40] proved the following:

Proposition 7.7 (Huang [40]) *Every round digraph is locally semicomplete.*

Theorem 7.8 (Huang [40]) *A connected locally semicomplete digraph D is round if and only if the following holds for each vertex x of D :*

- (a) *$N^+(x) \setminus N^-(x)$ and $N^-(x) \setminus N^+(x)$ induce transitive tournaments and*
- (b) *$N^+(x) \cap N^-(x)$ induces a (semicomplete) subdigraph containing no ordinary cycle.*

This characterization, given in Theorem 7.8, generalizes the characterization of round local tournaments, due to Bang-Jensen [2].

Theorem 7.9 (Bang-Jensen [2]) *A connected local tournament D is round if and only if, for each vertex x of D , $N^+(x)$ and $N^-(x)$ induce transitive tournaments.*

To generalize Theorem 7.8 and Theorem 7.9 to locally in-semicomplete digraphs and local in-tournaments, we introduce the generalizations of round digraphs, which are called the positive-round digraph and the negative-round digraph.

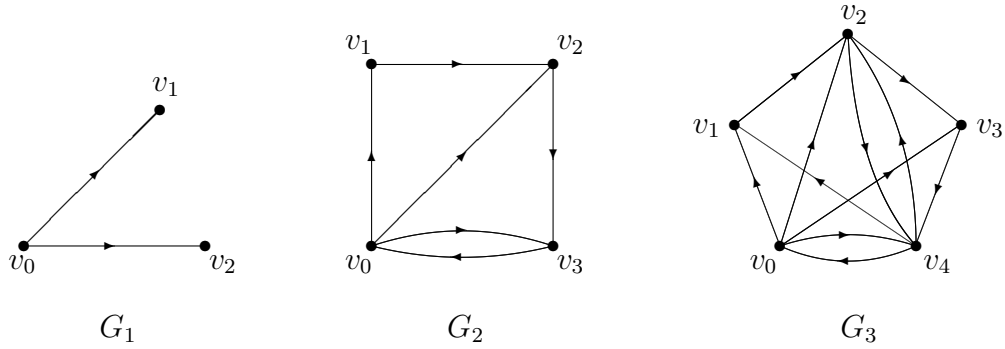


Figure 7.1: The positive-round digraphs G_1, G_2, G_3 with the positive-round labellings $v_0, v_1, \dots, v_{n-1}$, for $n = 3, 4, 5$.

Definition 7.10 A digraph D on n vertices is *positive-round* (*negative-round*, *respectively*) if we can label its vertices $v_0, v_1, \dots, v_{n-1}$ so that for each i , we have $N^+(v_i) = \{v_{i+1}, \dots, v_{i+d^+(v_i)}\}$ ($N^-(v_i) = \{v_{i-d^-(v_i)}, \dots, v_{i-1}\}$, respectively), where all subscripts are taken modulo n . The ordering $v_0, v_1, \dots, v_{n-1}$ is referred as a *positive-round labelling* (*negative-round labelling*, respectively) of D .

To illustrate these definitions, consider some simple digraphs of Figure 7.1. We see that the positive-round digraphs G_1, G_2, G_3 are not locally semicomplete digraphs, thus, they are not round digraphs by Proposition 7.7. However, it is easy to see that G_1, G_2, G_3 are locally in-semicomplete digraphs. We point out that positive-round digraphs form subclasses of the class of locally in-semicomplete digraphs.

Proposition 7.11 (R. Li et al. [51]) *Every positive-round digraph is locally in-semicomplete.*

Proof. Let D be a positive-round digraph and let $v_0, v_1, \dots, v_{n-1}$ be a positive-round labelling of D . Consider an arbitrary vertex, say v_i . Let x, y be a pair of in-neighbours of v_i . We show that x and y are adjacent. Without loss of generality, assume that v_i, x, y appear in that circular order in the positive-round labelling. Since $x \rightarrow v_i$ and the out-neighbour of x appear consecutively succeeding x , we must have $x \rightarrow y$. Therefore, D is locally in-semicomplete. ■

We shall give a sufficient condition such that a strong locally in-semicomplete digraph is positive-round. We need the following definition.

Definition 7.12 Let T be a transitive tournament and P be the unique Hamiltonian path on T . If D is a spanning subdigraph of T and D contains the path P , we say that D is an *almost transitive subdigraph* of T (or, just, *almost transitive*).

Note that the digraph D is almost transitive if and only if D is acyclic and contains a Hamiltonian path.

Theorem 7.13 (R. Li et al. [51]) *A strong locally in-semicomplete digraph D is positive-round, if the following holds for each vertex x of D :*

- (a) $N^+(x) \cap N^-(x)$ induces a (semicomplete) subdigraph containing no ordinary cycle,
- (b) $N^-(x) \setminus N^+(x)$ induces a transitive tournament and
- (c) $N^+(x) \setminus N^-(x)$ induces an almost transitive subdigraph of a transitive tournament.

Proof. To prove the theorem, we consider two cases.

Case 1: D contains an ordinary cycle.

We start by proving that D contains an ordinary Hamiltonian cycle. Let $C = x_1x_2 \dots x_kx_1$ be a longest ordinary cycle in D . Assume that $k < n$, the number of vertices in D . Since D is strong, there is a vertex $v \in V(D) \setminus V(C)$ such that v dominates some vertex of C , say x_1 .

Suppose that $v \mapsto x_1$. The vertices v and x_k are adjacent since they are in-neighbours of x_1 . The arc between v and x_k must be ordinary since $N^-(x_1) \setminus N^+(x_1)$ induces transitive tournament. Since C is a longest ordinary cycle, $x_k \not\mapsto v$. Thus $v \mapsto x_k$. Similarly, we can prove that $v \mapsto x_i$ for every $i = 2, 3, \dots, k-1$. Hence, $N^+(v) \setminus N^-(v)$ contains all vertices of C , which contradicts the assumption that $N^+(v) \setminus N^-(v)$ induces a spanning subdigraph of a transitive tournament.

Assume that vx_1v is a 2-cycle of D . Since $x_k \mapsto x_1$, the vertices v and x_k must be adjacent. If $v \mapsto x_k$, then we get the contradiction as the case above. If $x_k \mapsto v$, then the subdigraph induced by $N^+(x_k) \setminus N^-(x_k)$ contains a 2-cycle, which contradicts that $N^+(x_k) \setminus N^-(x_k)$ induces a spanning subdigraph of a transitive tournament. So we have $v \rightarrow x_k \rightarrow v$. Similarly, we can prove that $v \rightarrow x_i \rightarrow v$ for every $i = 2, 3, \dots, k-1$, i.e. $V(C) \subseteq N^+(v) \cap N^-(v)$. This contradicts the assumption that the subdigraph induced by $N^+(v) \cap N^-(v)$ contains no ordinary cycle.

Thus, we have shown that D contains an ordinary Hamiltonian cycle, which implies that $N^+(x) \setminus N^-(x) \neq \emptyset$ for every $x \in V(D)$.

We apply the following algorithm to find a positive-round labelling of D . Begin with an arbitrary vertex, say y_1 , and, for each $i = 1, 2, \dots$, let y_{i+1} be the vertex of in-degree zero in the subdigraph induced by $N^+(y_i) \setminus N^-(y_i)$. Let $y_1, y_2, \dots, y_r$ be distinct vertices produced by the algorithm such that the vertex w of in-degree zero in the subdigraph induced by $N^+(y_r) \setminus N^-(y_r)$ is in $\{y_1, y_2, \dots, y_{r-2}\}$.

We show that $w = y_1$. Suppose to the contrary that $w = y_j$ with $j > 1$. Then, $\{y_{j-1}, y_r\} \mapsto y_j$. Since $N^-(y_j) \setminus N^+(y_j)$ induces a transitive tournament, y_{j-1} and y_r are adjacent by an ordinary arc. But $y_{j-1} \mapsto y_r$ or $y_r \mapsto y_{j-1}$ contradicts the fact that y_j is the vertex of in-degree zero in the subdigraph induced by $N^+(y_{j-1}) \setminus N^-(y_{j-1})$ or $N^+(y_r) \setminus N^-(y_r)$. Thus, $w = y_1$ and $C' = y_1y_2 \dots y_r y_1$ is an ordinary cycle.

We next show that $r = n$. Suppose that $r < n$. Then, there is a vertex u , which is not in C' and dominates some vertex y_i of C' . Since the subdigraph induced by $N^+(u) \cap N^-(u)$ contains no ordinary cycle, it is not difficult to show that there is an ordinary arc between u and C' . Then, there exists $y_k \in V(C')$ such that $u \mapsto y_k$. In fact, we are done if $u \mapsto y_i$. So, assume that $y_i \in N^+(u) \cap N^-(u)$. Then, the vertices u and

y_{i-1} must be adjacent. If $u \mapsto y_{i-1}$, we are done. Since $y_{i-1} \mapsto u$ contradicts the fact that $N^+(y_{i-1}) \setminus N^-(y_{i-1})$ has no 2-cycle, then suppose that $y_{i-1} \in N^+(u) \cap N^-(u)$. Since the subdigraph induced by $N^+(u) \cap N^-(u)$ contains no ordinary cycle, there exists $y_k \in V(C')$ such that $u \mapsto y_k$. Then, u and y_{k-1} must be adjacent since $\{u, y_{k-1}\} \mapsto y_k$. As $N^-(y_k) \setminus N^+(y_k)$ induces a subdigraph containing no 2-cycle, the arc between u and y_{k-1} must be ordinary. Since $y_{k-1} \mapsto u$ contradicts the fact that y_k is the vertex of in-degree zero in the subdigraph induced by $N^+(y_{k-1}) \setminus N^-(y_{k-1})$, we have $u \mapsto y_{k-1}$. Similarly, we obtain $u \mapsto V(C')$, which contradicts the fact that $N^+(u) \setminus N^-(u)$ induces a spanning subdigraph of a transitive tournament. Thus, $r = n$, i.e. the algorithm labels all vertices of D .

To complete Case 1, it is sufficient to prove that $y_1, y_2, \dots, y_n$ is a positive-round labelling. Suppose to the contrary that there are three vertices y_a, y_b, y_c listed in the circular order in the labelling such that, without loss of generality,

$$y_a \rightarrow y_c \text{ but } y_a \not\rightarrow y_b.$$

Assume that the three vertices were chosen such that the number of vertices from y_b to y_c in the circular order is as small as possible. This implies that $c = b + 1$. Since $\{y_a, y_b\} \rightarrow y_c$ and $y_a \not\rightarrow y_b$, we have $y_b \mapsto y_a$. Moreover, $y_a \mapsto y_c$ since $y_b \mapsto \{y_a, y_c\}$ and $N^+(y_b) \setminus N^-(y_b)$ induces a subdigraph containing no 2-cycle. So y_c is not the vertex of in-degree zero in the subdigraph induced by $N^+(y_b) \setminus N^-(y_b)$, a contradiction.

Case 2: D contains no ordinary cycle.

It is not hard to see that D is complete if D has no ordinary arc. Thus, any labelling of $V(D)$ is positive-round. So assume that D contains an ordinary arc. Since D has an ordinary arc, but has no ordinary cycle, we claim that there exists a vertex z_1 with

$$N^-(z_1) \setminus N^+(z_1) = \emptyset \text{ and } N^+(z_1) \setminus N^-(z_1) \neq \emptyset.$$

Indeed, let $w_2 w_1$ be an ordinary arc in D . We may set $z_1 = w_2$ unless $N^-(w_2) \setminus N^+(w_2) \neq \emptyset$. In the last case there is an ordinary arc whose head is w_2 . Let $w_3 w_2$ be such an arc. Again, either we may set $z_1 = w_3$ or there is an ordinary arc $w_4 w_3$. Since D is finite and contains no ordinary cycle, the above process cannot repeat vertices and hence terminates at some vertex w_j such that we may set $z_1 = w_j$.

We apply the following algorithm to find an ordinary path in D . Start with z_1 and, for each $i = 1, 2, \dots$, let z_{i+1} be the vertex of in-degree zero in the subdigraph induced by $N^+(z_i) \setminus N^-(z_i)$ unless this set is empty. Since D has no ordinary cycle, this produces a path $P = z_1 z_2 \dots z_s$ with $N^+(z_s) \setminus N^-(z_s) = \emptyset$. Applying an argument similar to Case 1, we can show that $z_1, z_2, \dots, z_s$ is a positive-round labelling of the subdigraph induced by $V(P)$. Thus, if P contains all vertices of D , then a positive-round labelling of D is established. So assume that $V(D) \setminus V(P) \neq \emptyset$.

Claim 1. *For each vertex $v \in V(D) \setminus V(P)$, there is at least one arc between v and P .*

Proof. Since D is a strong digraph, there exists a path from v to P . Let $Q = v_1 v_2 \dots v_t$ be the shortest among all such paths with $v_1 = v$ and $v_t = z_i$. Obviously, $V(Q) \cap V(P) = z_i$.

If $|A(Q)| = 1$, we are done. So, assume that $|A(Q)| \geq 2$. Without loss of generality, assume that $z_{i-1} \mapsto v_{t-1}$ unless $i = 1$. When $i > 1$, the vertices z_{i-1} and v_{t-2} are adjacent since $\{z_{i-1}, v_{t-2}\} \rightarrow v_{t-1}$. Since Q is the shortest path from v to P , we have $z_{i-1} \mapsto v_{t-2}$. Similarly, we also have $z_{i-1} \mapsto v_j$ for $j = 1, 2, \dots, t-3$, if the vertices exist. When $i = 1$, $v_{t-1}z_1v_{t-1}$ is a 2-cycle by the choice of z_1 . It is easy to see that $z_1 \mapsto v_k$, for $k = 1, 2, \dots, t-2$, in the similar way. $\square$

Claim 2. *For each vertex $v \in V(D) \setminus V(P)$, there is no vertex $z_i \in V(P)$ with $v \mapsto z_i$.*

Proof. Suppose not. Then there are v and z_i with $v \mapsto z_i$. Clearly, $i \geq 2$. Since $\{z_{i-1}, v\} \rightarrow z_i$, the vertices z_{i-1} and v are adjacent. Furthermore, the arc between z_{i-1} and v is ordinary since $N^-(z_i) \setminus N^+(z_i)$ induces a transitive tournament. But $z_{i-1} \mapsto v$ contradicts the fact that z_i is the vertex of in-degree zero in the subdigraph induced by $N^+(z_{i-1}) \setminus N^-(z_{i-1})$. So, $v \mapsto z_{i-1}$. Similarly, we have $v \mapsto z_j$ for $j = 1, 2, \dots, i-2$, if the vertices exist. But this contradicts the choice of z_1 . $\square$

Claim 3. *If vz_iv is a 2-cycle, then vz_kv is a 2-cycle for all $k = 1, 2, \dots, i$.*

Proof. If $i = 1$, then we are done. Assume that $i > 1$. z_{i-1} and v must be adjacent since $\{z_{i-1}, v\} \rightarrow z_i$. By Claim 2 and the fact that the subdigraph induced by $N^+(z_{i-1}) \setminus N^-(z_{i-1})$ contains no 2-cycle, we see that $vz_{i-1}v$ is a 2-cycle. Similarly, vz_kv is a 2-cycle for all $k = 1, 2, \dots, i$. $\square$

Claim 4. *For each vertex $v \in V(D) \setminus V(P)$, there is no vertex $z_i \in V(P)$ with $z_i \mapsto v$.*

Proof. Suppose not, i.e. there is $v \in V(D) \setminus V(P)$ such that $z_i \mapsto v$ for some $z_i \in V(P)$. Without loss of generality, assume that z_i is chosen such that the subscript is as large as possible. Then, $i < s$ and v is not adjacent to any vertex of $\{z_{i+1}, z_{i+2}, \dots, z_s\}$. Otherwise, v is adjacent to z_j for some $i < j \leq s$. By Claim 2 and the definition of z_i , we see that vz_jv is a 2-cycle. Claim 3 implied that vz_iv is a 2-cycle, contradicting the fact that $z_i \mapsto v$. Recall that $N^+(z_i) \setminus N^-(z_i)$ is almost transitive, then there is an ordinary path from z_{i+1} to v in the subdigraph induced by $N^+(z_i) \setminus N^-(z_i)$ since $v \in N^+(z_i) \setminus N^-(z_i)$ and z_{i+1} is the vertex of in-degree zero in the subdigraph. Then, there exist $z_{j_1} \in (N^+(z_i) \setminus N^-(z_i)) \cap V(P)$ and $v' \in (N^+(z_i) \setminus N^-(z_i)) \cap (V(D) \setminus V(P))$ with $z_{j_1} \mapsto v'$. Since D contains no ordinary cycle, it is easy to see that $j_1 \geq i+1$. For the vertex v' , without loss of generality, let z_{j_1} be chosen such that the subscript is as large as possible. Repeat the above process, then we can obtain $u \in V(D) \setminus V(P)$ with $z_s \mapsto u$, contradicting the choice of z_s . $\square$

Followed by Claim 1-4, we can see that, for each vertex $v \in V(D) \setminus V(P)$, there is $z_i \in V(P)$ such that $v \rightarrow z_j \rightarrow v$ for all $j = 1, 2, \dots, i$.

At last, we show that D has a positive-round labelling. According to the discussion

above, we can give a partition $A_1, A_2, \dots, A_s$ of the set $V(D) \setminus V(P)$ as follows:

$$\begin{aligned}
A_s &= N^+(z_s) \cap N^-(z_s) \\
A_{s-1} &= N^+(z_{s-1}) \cap N^-(z_{s-1}) \setminus A_s \\
&\dots\dots\dots \\
A_i &= N^+(z_i) \cap N^-(z_i) \setminus (A_s \cup A_{s-1} \cup \dots \cup A_{i+1}) \\
&\dots\dots\dots \\
A_1 &= N^+(z_1) \cap N^-(z_1) \setminus (A_s \cup A_{s-1} \cup \dots \cup A_2).
\end{aligned}$$

Note that $A_i \mapsto A_j$ for any $1 \leq j < i \leq s$ if A_i, A_j are not empty. Indeed, let $x \in A_i, y \in A_j$. Since $x \in N^+(z_i) \cap N^-(z_i)$, we have $x \in N^+(z_j) \cap N^-(z_j)$ by Claim 3. So, x and y are adjacent since $\{x, y\} \rightarrow z_j$. If $y \rightarrow x$, then yz_iy is a 2-cycle since $\{y, z_i\} \rightarrow x$. This contradicts the definition of A_j . Thus, $x \mapsto y$.

We also note that $A_s \neq \emptyset$ and $D[A_s]$ is a semicomplete digraph because D is strong and all vertices of A_s dominate z_s . Furthermore, either A_i is empty or $D[A_i]$ is a transitive tournament for $i = 1, 2, \dots, s-1$ since all vertices of A_i dominate x_i and $A_s \mapsto A_i$.

Let $z_1^i z_2^i \dots z_{m_i}^i$ be the unique Hamiltonian path of $D[A_i]$ for $i = 1, 2, \dots, s-1$ unless $A_i = \emptyset$, and $z_1^s, z_2^s, \dots, z_{m_s}^s$ be the round labelling of $D[A_s]$ (the labelling exists by Theorem 7.8). Note that the subdigraph induced by the vertex set $\{z_1, z_2, \dots, z_s\}$ is transitive tournament since $A_s \neq \emptyset$. It is not difficult to verify that labelling the vertices according to the ordering

$$z_1, z_2, \dots, z_s, z_1^s, z_2^s, \dots, z_{m_s}^s, z_1^{s-1}, z_2^{s-1}, \dots, z_{m_{s-1}}^{s-1}, \dots, z_1^1, z_2^1, \dots, z_{m_1}^1$$

results in a positive-round labelling of D . ■

Lemma 7.14 *Let D be a positive-round digraph, then the following is true:*

- (a) *Every induced subdigraph of D is positive-round.*
- (b) *For each $x \in V(D)$, the subdigraphs induced by $N^-(x) \setminus N^+(x)$, $N^+(x) \setminus N^-(x)$ and $N^+(x) \cap N^-(x)$ contain no ordinary cycles.*

Proof. The statement (a) can be easily verified from the definition of a positive-round digraph.

Suppose the subdigraph induced by some $N^-(x) \setminus N^+(x)$ contains an ordinary cycle C . Let $v_1, v_2, \dots, v_n$ be a positive-round labelling of D . Without loss of generality, assume that $x = v_1$. Then C must contain an arc $v_i v_j$ such that $v_j v_i \notin A(D)$ and $i > j$. We have $v_1 \in N^+(v_j)$ but $v_i \notin N^+(v_j)$, contradicting the assumption that $v_1, v_2, \dots, v_n$ is a positive-round labelling of D . Similarly, we can verify that the subdigraphs induced by $N^+(x) \setminus N^-(x)$ and $N^+(x) \cap N^-(x)$ contain no ordinary cycles. ■

Remark 7.15 *If D is a strong positive-round local in-tournament, then the subdigraph induced by $N^+(x)$ contains a Hamiltonian path for each vertex x of D .*

By Theorem 7.13, Lemma 7.14 and Remark 7.15, we have directly the following theorem, which is a characterization of strong positive-round local in-tournaments.

Theorem 7.16 (R. Li et al. [51]) *A strong local in-tournament D is positive-round if and only if, for each vertex x of D , $N^-(x)$ induces a transitive tournament and $N^+(x)$ induces an almost transitive subdigraph of a transitive tournament.*

Let D be a positive-round digraph and H be the converse of D , and let $v_1, v_2, \dots, v_n$ be a positive-round labelling of D . It is easy to show that $v_n, v_{n-1}, \dots, v_1$ is a negative-round labelling of H . Then, H is a negative-round digraph. Observe that a locally out-semicomplete digraph is the converse of a locally in-semicomplete digraph. Therefore, we can obtain immediately the similar results about negative-round digraphs.

From Figure 7.1, we see that G_2 is a strong locally in-semicomplete digraph and satisfies the conditions of Theorem 7.13, thus it is positive-round. We also see that G_1 is not a strong digraph and doesn't satisfy (c) of Theorem 7.13, and G_3 doesn't satisfy (b) of Theorem 7.13 since $N^-(v_1) \setminus N^+(v_1)$ induces a 2-cycle $v_0v_4v_0$, but they are positive-round digraphs. Thus, the conditions given by Theorem 7.13 are not necessary. The following problem is raised.

Problem 7.17 *Find a characterization of positive-round digraphs.*

A subdigraph T of a digraph D is an *out-branching* if T is a spanning oriented tree (refer to [6]) of D and T has only one vertex s of in-degree zero.

Lemma 7.18 (Bang-Jensen and Gutin [6]) *Every connected locally in-semicomplete digraph D has an out-branching.*

Let G be a subdigraph of a digraph D . The *contraction* of G in D is a directed multigraph D/G with $V(D/G) = \{g\} \cup (V(D) \setminus V(G))$, where g is a “new” vertex not in D , and for all distinct vertices $x, y \in V(D) \setminus V(G)$, xy is an arc of D/G if and only if $xy \in A(D)$, xg is an arc of D/G if and only if $xv \in A(D)$ for some $v \in V(G)$, and gy is an arc of D/G if and only if $wy \in A(D)$ for some $w \in V(G)$. (Note that there is no loop in D/G .) The *strong component digraph* $SC(D)$ of D is obtained by contracting strong components of D and deleting any parallel arcs obtained in this process. More details about this definition can be found in [6]. For connected but non-strong locally in-semicomplete digraphs, we have also the structural properties of strong components, which is similar to Theorem 7.3.

Theorem 7.19 (Bang-Jensen and Gutin [5]) *Let D be a connected but non-strong locally in-semicomplete digraph.*

- (a) *Let A and B be distinct strong components of D . If a vertex $a \in A$ dominates some vertex in B , then $a \mapsto B$. Furthermore, $A \cap N^-(b)$ induces a semicomplete digraph for each $b \in B$.*
- (b) *If D is connected, then $SC(D)$ has an out-branching. Furthermore, if R is the root of that out-branching and A is any other component, there is a path from R to A containing all the components that can reach A .*

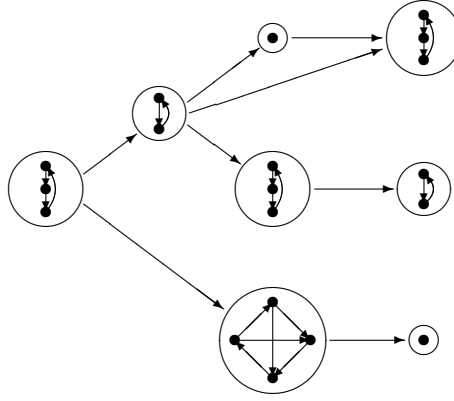


Figure 7.2: The strong decomposition of a non-strong locally in-semicomplete digraph. The big circles indicate strong components and an arc from a component A to a component B between two components indicates that there is at least one vertex $a \in A$ such that $a \mapsto B$.

Figure 7.2 is an example of non-strong locally in-semicomplete digraph. Note that the strong component digraph $SC(D)$ of a non-strong locally in-semicomplete digraph is not necessary a Hamiltonian path, which is different from locally semicomplete digraph. However, for a strong local in-tournament D and a minimal separating set S of D , the strong component digraph $SC(D - S)$ is a Hamiltonian path, which was given by Bang-Jensen, Huang and Prisner [7] as Theorem 7.20. To show the structural property, we give the following definition. Let A and B be two disjoint subdigraphs of the digraph D or subsets of $V(D)$. We say that A *weakly dominates* B if there is no arc leading from B to A and at least one arc leading from A to B .

Theorem 7.20 (Bang-Jensen et al. [7]) *Let D be a strong local in-tournament and let S be a minimal separating set of D .*

- (a) *If A and B are two distinct strong components of $D - S$, either there is no arc between them or A weakly dominates B or B weakly dominates A . Furthermore, if A weakly dominates B , the set $N_A^-(B)$ dominates B .*
- (b) *If A and B are two distinct strong components of $D - S$ such that A weakly dominates B , the set $N_A^-(b)$ induces a tournament for each $b \in B$.*
- (c) *The strong components of $D - S$ can be ordered in a unique way $D_1, D_2, \dots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i has an arc to D_{i+1} for $i = 1, 2, \dots, p - 1$.*

The unique labelling $D_1, D_2, \dots, D_p$ of the strong components of $D - S$ as described in Theorem 7.20 is called the *strong decomposition* of $D - S$. We call D_1 the *initial* and D_p the *terminal component*. We have immediately the following corollary:

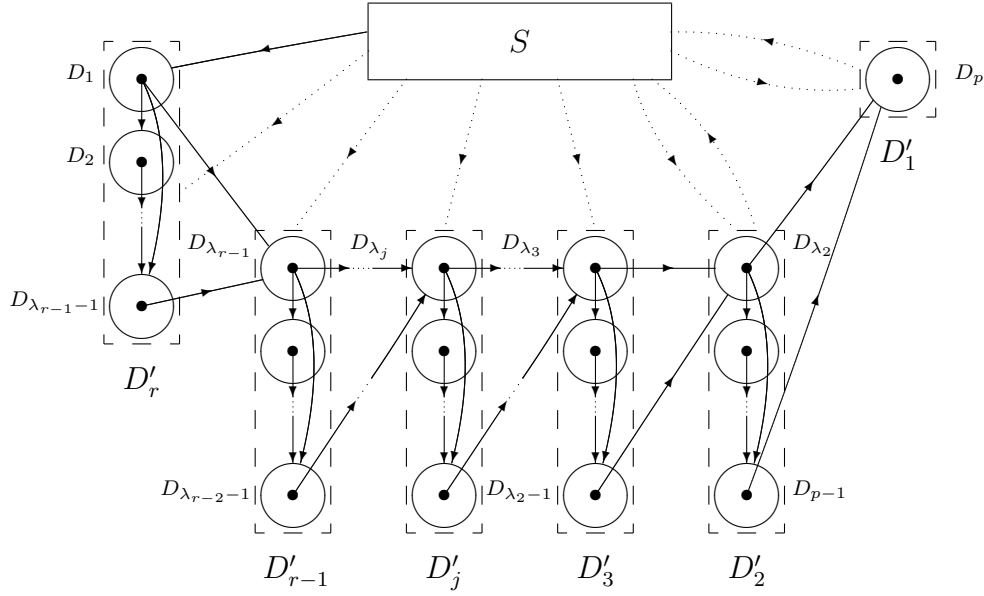


Figure 7.3: The decomposition of a strong local in-tournament.

Corollary 7.21 (Bang-Jensen et al. [7]) *Let D be a strong local in-tournament and let S be a minimal separating set of D . The strong decomposition of $D - S$ has the following properties.*

- (a) *If $x_i \rightarrow x_k$ for $x_i \in V(D_i)$ and $x_k \in V(D_k)$ with $1 \leq i \neq k \leq p$, then $x_i \rightarrow D_j$ for every $i + 1 \leq j \leq k$.*
- (b) *The digraph $D - S$ has a Hamiltonian path.*
- (c) *For every $s \in S$ we have $d_{D_1}^+(s) > 0$ and $d_{D_p}^-(s) > 0$.*

Remark 7.22 *Let $v_1, v_2, \dots, v_p$ denote the new vertices obtained by contracting the strong components $D_1, D_2, \dots, D_p$ of $D - S$, respectively. Then $v_1, v_2, \dots, v_p$ is a positive-round labelling, and hence $SC(D - S)$ is a positive-round digraph.*

From the fact that every connected non-strong local in-tournament has a unique strong decomposition, Meierling and Volkmann [55] gave a further useful decomposition, which play an important role in the proof of their main results.

Theorem 7.23 (Meierling and Volkmann[55]) *Let D be a strong local in-tournament and let S be a minimal separating set of D . There is a unique order $D'_1, D'_2, \dots, D'_r$ with $r \geq 2$ of the strong components of $D - S$ such that*

- (a) *D'_1 is the terminal component of $D - S$ and D'_i consists of some strong components of D for $i \geq 2$;*

- (b) *there exists a vertex x in the initial component of D'_{i+1} and a vertex y in the terminal component of D'_{i+1} such that $\{x, y\}$ dominates the initial component of D'_i for $i = 1, 2, \dots, r - 1$;*
- (c) *there are no arcs between D'_i and D'_j for i, j satisfying $|i - j| \geq 2$;*
- (d) *if $r \geq 3$, there exist no arcs from D'_i to S for $i \geq 3$, $S \rightarrow D_1$ and S induces a tournament in D .*

It is clear that some properties of local tournaments no longer hold for local in-tournaments. For a strong local in-tournament D and a minimal separating set S of D , the component D_i of $D - S$ is not necessarily a tournament, D_i does not strictly dominate D_{i+1} , D_p does not strictly dominate S and so on. However, its structure with respect to cycles maybe holds. For instance, Meierling and Volkmann [55] showed that every 2-strong local in-tournament D on $n \geq 6$ vertices is cycle complementary unless D is a member of three exceptional graphs. The structural properties of local in-tournaments suggest that it is possible to extend Theorem 7.1 to strong local in-tournaments, i.e., maybe every strong local in-tournament D contains an out-arc g -pancyclic vertex for some positive integer g unless D belongs to an exceptional class of graphs. It seems difficult to show this proposition, since the proof of Theorem 7.1 has already been very technical and consisted of a long case analysis. Maybe we could firstly show a possibly easier conjecture: Every strong local in-tournament D contains a vertex x such that each out-arc of x is in a Hamiltonian cycle.

Chapter 8

Applications of path-contraction technique

We shall give some applications of path-contraction technique in digraphs, in particular, in strongly Hamiltonian-connected digraphs.

As we have shown in Part I, the Hamiltonian cycle problem is NP-complete. Many sufficient conditions respect to various parameters, have been found on undirected graphs. It often happens that a result on the Hamiltonicity in undirected graphs has a natural, but more difficult, generalization to digraphs. The fundamental early results are the generalizations of Dirac's theorem that every undirected graph of order n and minimum degree at least $n/2$ is Hamiltonian.

Theorem 8.1 (Ghouila-Houri [30]) *Every strong digraph of order n and minimum degree at least $n/2$ is Hamiltonian.*

Corollary 8.2 *Let D be a strong digraph of order n . If the minimum semi-degree of D , $\delta^0(D) \geq n/2$, then D is Hamiltonian.*

Theorem 8.3 (Thomassen [72]) *Let D be a digraph of order $n = 2k + 1$ and minimum semi-degree at least k . Then D is Hamiltonian unless D has a set of $k + 1$ mutually nonadjacent vertices (which then dominate and are dominated by all the k remaining vertices), or D is isomorphic to D_5 or D_7 of Figure 8.1, or D is the symmetric digraph consisting of two disjoint copies of K_k^* plus one vertex joined to all others by 2-cycles.*

Ore proved that the Hamiltonicity of an undirected graph holds if we only assume that the sum of the degrees of any two nonadjacent vertices is at least n . This was extended to digraphs by Woodall.

Theorem 8.4 (Woodall [76]) *A digraph D of order n is Hamiltonian if, for any two vertices x and y , either x dominates y or*

$$d^+(x) + d^-(y) \geq n.$$

A common generalization of Theorem 8.1 and Theorem 8.4 was obtained by Meyniel.

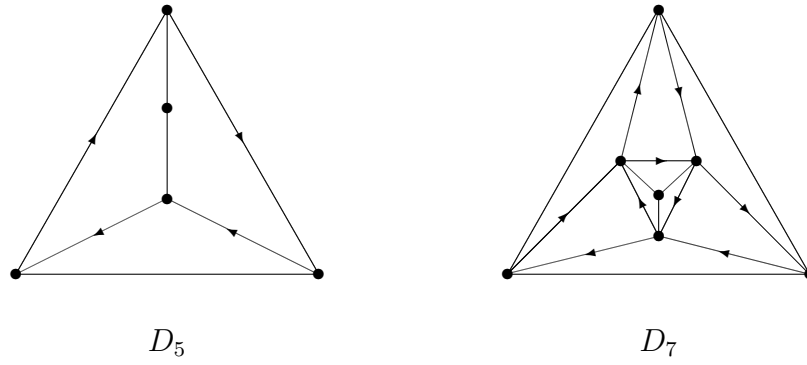


Figure 8.1: Two non-Hamiltonian digraphs. The undirected edges denote 2-cycles.

Theorem 8.5 (Meyniel [59]) *A strong digraph D of order n is Hamiltonian if for any two nonadjacent vertices x and y we have*

$$d(x) + d(y) \geq 2n - 1.$$

Let x, y be a pair of distinct vertices in a digraph D . If there is a vertex z with $z \rightarrow x$ and $z \rightarrow y$, we say that the pair $\{x, y\}$ is *dominated*. Likewise, if there is a vertex z with $x \rightarrow z$ and $y \rightarrow z$, we call the pair $\{x, y\}$ *dominating*.

Another extension of Ore's theorem was given by Zhao and Meng.

Theorem 8.6 (Zhao and Meng [80]) *Let D be a strong digraph of order $n \geq 2$. If*

$$d^+(x) + d^+(y) + d^-(u) + d^-(v) \geq 2n - 1$$

for every pair x, y of dominating vertices and every pair u, v of dominated vertices, then D is Hamiltonian.

Bondy showed that the only non-Hamiltonian graphs with $(n-1)(n-2)/2 + 1$ edges are the graphs G_5 in Figure 8.2 and $G(1, n)$ consisting of a complete graph K_{n-1} plus a vertex joined to a given vertex of this K_{n-1} . This result has been generalized to digraphs as follows.

Theorem 8.7 (Bermond et al. [11]) *The only non-Hamiltonian strong digraphs with at least $(n-1)(n-2) + 2$ arcs are the symmetric digraph $G(1, n)^*$, the digraph H_n of Figure 8.2 and its converse, G_5 and D_4 of Figure 8.2.*

If we do not require G to be strong, we have the following theorem.

Theorem 8.8 (Lewin [45]) *If D is a digraph of order n with more than $(n-1)^2$ arcs, then D is Hamiltonian.*

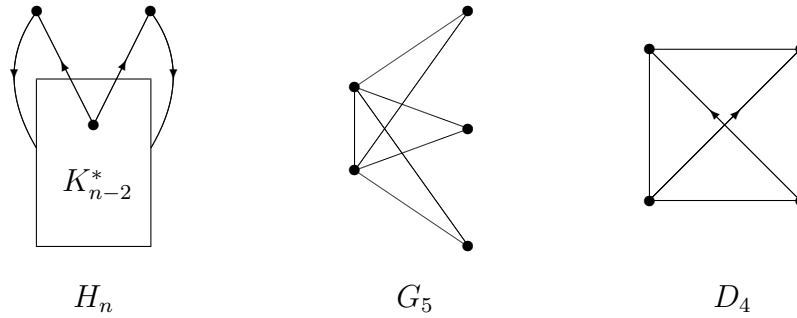


Figure 8.2: Non-Hamiltonian digraphs with $(n - 1)(n - 2) + 2$ arcs. The arc between a vertex and a boxed set indicates that there is a complete domination in the direction shown.

We say that a digraph D is *weakly Hamiltonian-connected* if for any two vertices x and y of D , there is a Hamiltonian path from x to y or from y to x , and D is *strongly Hamiltonian-connected* if for any two vertices x and y , there are Hamiltonian paths from x to y and from y to x . The sufficient conditions for a digraph to be weakly or strongly Hamiltonian-connected are not largely investigated. Ghouila-Houri proved the following result which is a generalization of Theorem 8.1.

Theorem 8.9 (Ghouila-Houri [31]) *A 2-strong digraph of order n and minimum degree at least $n + 1$ is weakly Hamiltonian-connected.*

A digraph $D = (V, A)$ is said to be *strongly q -arc Hamiltonian* if, for every system S of pairwise disjoint paths of the complete digraph with vertex set $V(D)$ of total length q , the digraph $D' = (V, A \cup S)$ has a Hamiltonian cycle containing S . We say that a digraph is *strongly (p, q) -Hamiltonian* if the digraph obtained by deleting any r vertices is strongly q -arc Hamiltonian for all r , $0 \leq r \leq p$. Clearly, a strongly $(0, 1)$ -Hamiltonian digraph is exactly strongly Hamiltonian-connected. The following results generalize Theorem 8.4 and Theorem 8.5, respectively.

Theorem 8.10 (Bermond [9]) *If a digraph D of order n has the property that, for any two vertices x and y , either x dominates y or*

$$d^+(x) + d^-(y) \geq n + p + q,$$

then D is strongly (p, q) -Hamiltonian.

Theorem 8.11 (Wojda [75]) *Let $D = (V, A)$ be a $(p + q + 1)$ -strong digraph. Suppose that, for each four vertices $u, v, w, z \in V(D)$ (not necessarily distinct) such that $\{u, v\} \cap \{w, z\} = \emptyset$, $(w, u) \notin A(D)$, and $(v, z) \notin A(D)$, we have*

$$d^-(u) + d^+(v) + d^+(w) + d^-(z) \geq 2(n + p + q) - 1.$$

Then D is strongly (p, q) -Hamiltonian.

For tournaments, Thomassen proved the following:

Theorem 8.12 (Thomassen [71]) *A tournament T with at least three vertices is weakly Hamiltonian-connected if and only if it satisfies (i), (ii) and (iii) below.*

(i) T is strong.

(ii) For each vertex x of T , $T - x$ has at most two components.

(iii) T is not isomorphic to any of T_6^s or $\overline{T}_6^s$ (refer to [71]).

Theorem 8.13 (Thomassen [71]) *Each 4-strong tournament is strongly Hamiltonian-connected.*

In 1995, Guo found in his Ph.D. Thesis an analogous and very important sufficient condition for strongly Hamiltonian-connected locally semicomplete digraphs.

Theorem 8.14 (Guo [36]) *Each 4-strong locally semicomplete digraph is strongly Hamiltonian-connected.*

In [71], the following digraph D was described by Thomassen: Let $D_1, D_2, \dots, D_9$ be nine complete digraphs such that D_5 consists of a vertex u and each other D_i has at least three vertices. Each vertex of D_i dominates each vertex of D_j where $1 \leq i < j \leq 9$ and $u \rightarrow D_4, D_6 \rightarrow u$. Add three vertices x, y and z . Let x dominate u and the vertices of D_8 and be dominated by all vertices except u . Let y be dominated by u and the vertices of D_2 and dominate all vertices except u . Let z dominate and be dominated by all other vertices. He showed that the semi-complete digraph D is 2-strong with the minimum degree at least $n + 1$, but D contains no Hamiltonian path from x to y . He conjectured that every 3-strong digraph with the degree-constrained condition above is strongly Hamiltonian-connected. Indeed, if we modify slightly the digraph above, the following can be obtained.

Theorem 8.15 (R. Li [48]) *There exist infinite 3-strong non-strongly Hamiltonian-connected digraphs with minimum degree at least $n + 1$.*

Proof. Let D be a digraph described by Thomassen. Construct a digraph D' by adding two new vertices a, b and the following arcs: Let a dominate b, x, u and be dominated by all vertices except x and let b be dominated by a, y, u and dominate all vertices except y . Let $|V(D')| = n$. It is not difficult to check that D' is 3-strong and minimum degree is at least $n + 1$.

To show that D' contains no Hamiltonian path from a to b , it is sufficient to show that D contains no Hamiltonian paths from x to y , from x to u and from u to y . Suppose to the contrary that $P = xu_1u_2 \dots u_{n-2}u$ is a Hamiltonian path of D . Then u_1 is in D_8 , and hence the segment $P[x, z]$ contains no vertex of D_7 . This implies that $D - \{x, z\}$ contains a path from D_7 to u , a contradiction. Similarly, D has no Hamiltonian paths from x to y and from u to y . ■

Probably, 4-strong digraph with the degree-constrained condition above is strongly Hamiltonian-connected.

Lemma 8.16 *Let $D = (V, A)$ be a $(q + 1)$ -strong digraph and S be a system of pairwise disjoint paths of the complete digraph with vertex set $V(D)$ of total length q . If D' is the digraph obtained from D by contracting S , then D' is strong.*

Proof. Let S contain the arcs $e_1, e_2, \dots, e_q$. Then $(\dots((D/e_1)/e_2)/\dots)/e_q = D/S = D'$. So, it is sufficient to show that D/e_1 is a q -strong digraph.

Let $e_1 = ab$ and D contract e_1 to w . Let $x, y \in V(D/e_1)$ be two distinct vertices. If neither x nor y is the new vertex w on D/e_1 , then there exist $q + 1$ internally disjoint (x, y) -paths in D by Menger's Theorem. If there are q paths among the $q + 1$ internally disjoint paths, which contain no end-vertices of e_1 , then the digraph D/e_1 has q internally disjoint (x, y) -paths. So there are two paths, say P_1 and P_2 such that $a \in P_1, b \in P_2$. Obviously, $P_1[x, w]P_2[w, y]$ is a path from x to y in D/e_1 . Combining the other $q - 1$ paths, we have q internally disjoint (x, y) -paths in D/e_1 .

So assume that either x or y is the vertex w . Without loss of generality, suppose that $x = w$. By Menger's Theorem, there are $q + 1$ internally disjoint (b, y) -paths in D . Clearly, there is at most one path containing the vertex a . Therefore, the other q internally disjoint paths are still q paths from w to y in D/e_1 .

According to Menger's Theorem and the choice of x and y , D/e_1 is q -strong. $\blacksquare$

Lemma 8.17 *Let $D = (V, A)$ be a 2-strong digraph of order n and minimum degree at least $n + 1$. Let x, y be two vertices of D . If $d^+(x) \geq d^+(y)$, then D contains a Hamiltonian path from x to y .*

Proof. Let yx be an arc of the complete digraph with vertex set $V(D)$. Construct a digraph D' from D by contracting yx to w . By Lemma 8.16, D' is strong. Let $z \in V(D')$ be arbitrary. We see that

$$\begin{aligned} d_{D'}(z) &\geq d_D(z) - 2 \geq n + 1 - 2 = n - 1, \quad \text{for } z \neq w, \\ d_{D'}(z) &\geq d_D^+(x) - 1 + d_D^-(y) - 1 \geq d_D^+(x) - 1 + (n + 1 - d_D^+(y)) - 1 \\ &\geq n - 1, \quad \text{for } z = w. \end{aligned}$$

Then D' is Hamiltonian by Theorem 8.1, which implies that D contains a Hamiltonian path from x to y . $\blacksquare$

Immediately, we see that Theorem 8.9 is a corollary of the lemma above. We have also the following result.

Theorem 8.18 (R. Li [48]) *A 4-strong digraph D of order n and minimum degree at least $n + 3$ is strongly Hamiltonian-connected if, for any two vertices x and y , either $d^+(x) \geq d^+(y)$ or $N^+(x) \cap N^-(y) \geq 2$.*

Proof. Let $x, y \in V(D)$ be arbitrary. If $d^+(x) \geq d^+(y)$, then D contains a Hamiltonian path from x to y by Lemma 8.17. We assume that $N^+(x) \cap N^-(y) \geq 2$. Let $u, v \in N^+(x) \cap N^-(y)$ and D' is the digraph obtained by deleting x and y . We see that D' is 2-strong and, for each vertex $z \in V(D')$,

$$d_{D'}(z) \geq d_D(z) - 4 \geq n + 3 - 4 = n - 1.$$

By Theorem 8.9, D' is weakly Hamiltonian-connected. Without loss of generality, let P be a Hamiltonian path from u to v of D' . Then xPy is a Hamiltonian path of D . ■

To prove the other results on strongly Hamiltonian-connected digraphs, we give the following simple but useful lemma.

Lemma 8.19 *Let $D = (V, A)$ be a digraph and S be the system of pairwise disjoint paths of the complete digraph with vertex set $V(D)$. Then the digraph $D' = (V, A \cup S)$ has a Hamiltonian cycle containing S if and only if the digraph D/S is Hamiltonian.*

The following two theorems generalize Theorem 8.3 and Theorem 8.7.

Theorem 8.20 (R. Li [48]) *Let D be a digraph of order $n = 2k$ and minimum semi-degree at least k . Then D is strongly Hamiltonian-connected, unless D is isomorphic to one of the digraphs in Figure 8.3.*

Proof. Let $u, v \in V(D)$ be arbitrary and D' be the digraph obtained by contracting uv to w . Then $|V(D')| = 2k - 1$ and minimum semi-degree at least $k - 1$. If D is isomorphic to one of the digraphs in Figure 8.3, then D has no Hamiltonian path from v to u . On the other hand, if D is not isomorphic to one of the digraphs in Figure 8.3, then D' is not any of the non-Hamiltonian digraphs shown in Theorem 8.3. So D' is Hamiltonian and hence D is strongly Hamiltonian-connected. ■

Theorem 8.21 (R. Li [48]) *The only 2-strong digraphs with at least $(n - 1)(n - 2) + 4$ arcs, which are not strongly Hamiltonian-connected, are the digraphs in Figure 8.4 and the converses of the digraphs (4), (5) and (6) of Figure 8.4.*

Proof. Let $u, v \in V(D)$ be arbitrary and D' be the digraph obtained by contracting uv to w . Then D' is strong by Lemma 8.16. Since it is possible that $u \rightarrow v \rightarrow u$, $u \rightarrow x$, $x \rightarrow v$ for each $x \in V \setminus \{u, v\}$, then the number of the arcs of D' is

$$\begin{aligned}
 |A(D')| &= \sum_{x \in V(D')} d_{D'}^+(x) = d_{D'}^+(w) + \sum_{x \in V(D') \setminus w} d_{D'}^+(x) \\
 &\geq (d_D^+(v) - 1) + \sum_{x \in V(D) \setminus \{u, v\}} (d_D^+(x) - 1) \\
 &= d_D^+(v) + d_D^+(u) + \sum_{x \in V(D) \setminus \{u, v\}} d_D^+(x) - d_D^+(u) - (n - 2) - 1 \\
 &\geq (n - 1)(n - 2) + 4 - (n - 1) - (n - 2) - 1 \\
 &= (n - 2)(n - 3) + 2.
 \end{aligned}$$

By Theorem 8.7, D' is Hamiltonian unless D' is isomorphic to one of the digraphs, $G(1, n)^*$, H_n of Figure 8.2 and its converse, G_5 and D_4 of Figure 8.2. On the other hand, D has a Hamiltonian path from v to u if and only if D' is Hamiltonian by Lemma 8.19. Therefore, D contains no Hamiltonian path from v to u if and only if D' is isomorphic to one of the digraphs shown in Theorem 8.7.

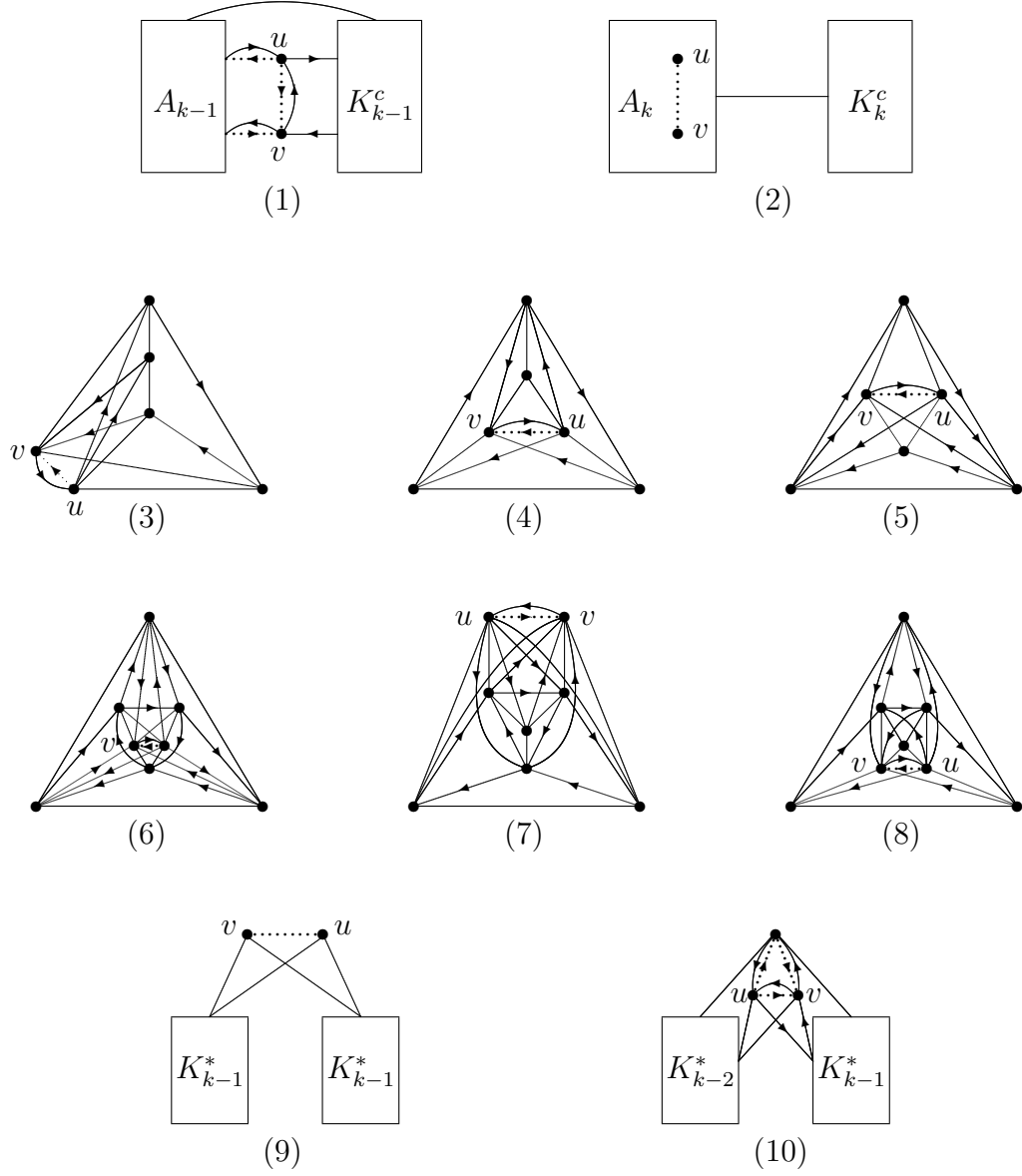


Figure 8.3: Non-strongly Hamiltonian-connected digraphs with largest possible semi-degree. The dotted line indicates that the adjacency of its end-vertices is arbitrary and the dotted arc denotes it can be omitted. The edge between different boxed sets indicates that there is a complete adjacency with 2-cycles. A_{k-1} (A_k) is a digraph with $k-1$ (k) vertices.

If D' is isomorphic to one of the digraphs, $G(1, n)^*$, H_n of Figure 8.2 and its converse, G_5 and D_4 of Figure 8.2, then D' contains exactly $(n-2)(n-3) + 2$ arcs. So the equality in the formula above holds, which implies D has exactly $(n-1)(n-2) + 4$ arcs and $uv, vu, ux, xv \in A(D)$ for each $x \in V - \{u, v\}$. It is not difficult to check that the digraphs of Figure 8.4 and the converses of (4), (5) and (6) in Figure 8.4 are the only digraphs satisfying the conditions above. ■

Corollary 8.2, Theorem 8.6 and Theorem 8.8 are extended as follows.

Theorem 8.22 (R. Li [48]) *Let D be a digraph of order n . If the minimum semi-degree of D ,*

$$\delta^0(D) \geq \frac{n+p+q}{2}, \quad \text{for } 1 \leq p+q \leq n-2,$$

then D is strongly (p, q) -Hamiltonian.

Proof. Let r be an arbitrary integer with $0 \leq r \leq p$ and D' be the digraph obtained from D by deleting any r vertices. Then D' has $n-r$ vertices. To prove D is strongly (p, q) -Hamiltonian, it is sufficient to show that D' is strongly q -arc Hamiltonian. Let S be a system of pairwise disjoint paths of the complete digraph with vertex $V(D')$ of total length q . Construct a new digraph D'' from D' by contracting S . Since D' decreases by a vertex while we contract an arc, then D'' contains $n-r-q$ vertices. Furthermore,

$$\delta^0(D'') \geq \delta^0(D') - q \geq \delta^0(D) - r - q \geq \frac{n+p+q}{2} - r - q \geq \frac{n-r-q}{2}.$$

According to Corollary 8.2, D'' is Hamiltonian, which implies D' is strongly q -arc Hamiltonian. ■

Theorem 8.23 (R. Li [48]) *Let D be a $(p+q+1)$ -strong digraph of order $n \geq p+q+2$. If*

$$d^+(x) + d^+(y) + d^-(u) + d^-(v) \geq 2(n+p+q) - 1$$

for every pair x, y of dominating vertices and every pair u, v of dominated vertices, then D is strongly (p, q) -Hamiltonian.

Proof. Let r, D', S, D'' be defined as Theorem 8.22. It is clear that D' is $(q+1)$ -strong, and hence D'' is strong by Lemma 8.16. Also, D'' has $n-r-q$ vertices. As it is shown in Theorem 8.22, it is sufficient to prove that D'' is Hamiltonian. Let $S' \subset V(D'')$ be the set of new vertices obtained by the contraction. Let x, y be a pair of dominating vertices and u, v be a pair of dominated vertices of D'' . If $x(y) \in S'$, then $x(y)$ has the same out-neighbours in $D'' - S'$ as the terminal vertex of some path of S in $D - S$. We still write the terminal vertex as $x(y)$. Similarly, if $u(v) \in S'$, then $u(v)$ has the same in-neighbours in $D'' - S'$ as the initial vertex of some path of S in $D - S$. Write the initial vertex also as $u(v)$. Note that $x, y(u, v)$ are also dominating (dominated) vertices in D , and

$$\begin{aligned} & d_{D''}^+(x) + d_{D''}^+(y) + d_{D''}^-(u) + d_{D''}^-(v) \\ & \geq d_{D'}^+(x) + d_{D'}^+(y) + d_{D'}^-(u) + d_{D'}^-(v) - 4q \\ & \geq d^+(x) + d^+(y) + d^-(u) + d^-(v) - 4q - 4r \\ & \geq 2(n-r-q) - 1. \end{aligned}$$

By Theorem 8.6, D'' is Hamiltonian. ■

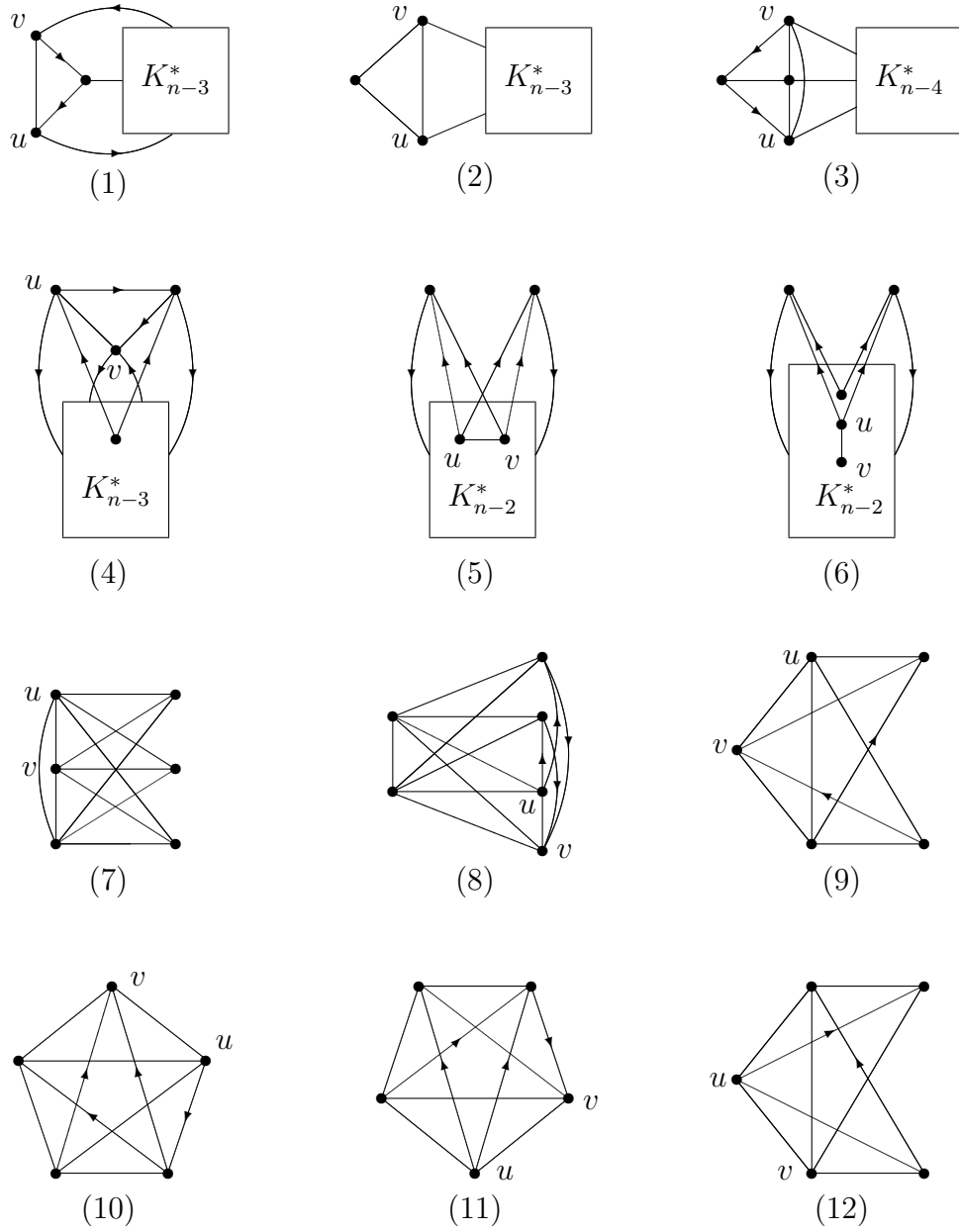


Figure 8.4: Non-strongly Hamiltonian-connected digraphs with $(n-1)(n-2)+4$ arcs.

Theorem 8.24 (R. Li [48]) *If D is a digraph of order n with more than*

$$(n-1)^2 + q^2 + p \quad (1 \leq q^2 + p \leq n-1)$$

arcs, then D is strongly (p, q) -Hamiltonian.

Proof. Let r, D', S, D'' be defined as Theorem 8.22. Then D' has $n - r$ vertices and D decreases by at most $2r(n - r) + r(r - 1)$ arcs when we delete r vertices. Also, D'' contains $n - r - q$ vertices. Suppose that P is a path of S . The digraph D' decreases by at most $2|A(P)|(n - r - 1)$ arcs if we contract P in D' . Then D' decreases by at most $2q(n - r - 1)$ arcs when we contract S . So,

$$\begin{aligned} |A(D'')| &\geq |A(D')| - 2q(n - r - 1) \\ &\geq |A(D)| - (2r(n - r) + r(r - 1)) - 2q(n - r - 1) \\ &\geq (n - 1)^2 + q^2 + p - (2r(n - r) + r(r - 1)) - 2q(n - r - 1) \\ &\geq (n - 1)^2 + q^2 + r - (2r(n - r) + r(r - 1)) - 2q(n - r - 1) \\ &= (n - r - q - 1)^2. \end{aligned}$$

By Theorem 8.8, D'' is Hamiltonian. Therefore, D is strongly (p, q) -Hamiltonian. ■

Set $p = 0, q = 1$, we get the following immediate corollaries.

Corollary 8.25 *Let D be a digraph of order n . If the minimum semi-degree of D , $\delta^0(D) \geq (n + 1)/2$, then D is strongly Hamiltonian-connected.*

Corollary 8.26 *Let D be a 2-strong digraph of order $n \geq 3$. If*

$$d^+(x) + d^+(y) + d^-(u) + d^-(v) \geq 2n + 1$$

for every pair x, y of dominating vertices and every pair u, v of dominated vertices, then D is strongly Hamiltonian-connected.

Corollary 8.27 *If D is a digraph of order n with more than $(n - 1)^2 + 1$ arcs, then D is strongly Hamiltonian-connected.*

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Summary

This thesis contains new results on two topics: k -ordered graphs and out-arc pancyclicity of digraphs.

Over the years Hamiltonian graphs have been widely studied. Various Hamiltonian-related properties have also been considered. Some of the properties are weaker, for example traceability and existence of a cycle factor in graphs, while other are stronger, for example Hamiltonian-connectivity, pancyclicity and panconnectivity. In 1997, L. Ng and M. Schultz [62] introduced the idea of cycle orderability and gave a new strong Hamiltonian property. A graph is called k -ordered Hamiltonian if for every ordered sequence of k vertices, there is a Hamiltonian cycle that encounters the vertices of the sequence in the given order. As shown in the recent survey [23], there are many conditions that imply Hamiltonicity can also imply the stronger property of k -ordered Hamiltonian if we give a slight modification.

Our first work (Chapter 3) is the new sufficient conditions in terms of degree sum on distance 2 vertices for a graph to have a k -ordered Hamiltonian cycle. These conditions are given not only on the lower connectivity but also on the upper connectivity. Additionally, the traceability and Hamiltonicity have also been proved under degree sum conditions on distance 2 vertices. We have showed the sharpness of these conditions as well as the independence of these results.

Our second work (Chapter 4) is the introduction of a new pancyclicity of graphs, “ (k, m) -vertex-pancyclic ordered” graphs which is a generalization of both k -ordered and vertex-pancyclic graphs. This idea comes from the “ (k, m) -pancyclic ordered” introduced by R. J. Faudree, R. J. Gould, M. S. Jacobson and L. Lesniak [24], which is the generalization of both k -ordered and pancyclic graphs. Note that every (k, m) -vertex-pancyclic ordered graph is (k, m) -pancyclic ordered. We have proved that a graph is (k, m) -vertex-pancyclic ordered under the same minimum sum of degree conditions of non-adjacent vertices as required by the (k, m) -pancyclic ordered graphs.

As an important branch of graph theory, the area of digraphs has developed enormously within the last four decades. There are large numbers of topics on digraphs. Out-arc pancyclicity is one of the newest and most interesting themes on digraphs, which deals with the existence of the vertices whose all arcs out of them are pancyclic.

Our third work (Chapter 6) is the investigation of the vertices, with the property that all arcs out of them are pancyclic and 4-pancyclic in a k -strong tournament. We have only considered 2- and 3-strong tournaments since A. Yeo [79] presented an infinite class of k -strong tournament, such that each tournament contains at most 3 such vertices. We have showed that every k -strong tournament contains at least k vertices whose all arcs out of them are pancyclic for $k = 2, 3$. If 3-cycles are not considered, we have proved that every s -strong tournament with $s \geq 3$ contains at least $s + 1$ vertices whose all arcs out of them are 4-pancyclic.

Our fourth work (Chapter 7) is the structural analysis of special locally in-semicomplete digraphs, namely, positive-round digraphs. We have given a sufficient condition for a strong digraph to be positive-round and a characterization of strong positive-round oriented digraphs.

Our last work (Chapter 8) is the new sufficient conditions for a digraph to be strongly Hamiltonian-connected. Path-contraction is a powerful tool in the proof of out-arc pancyclicity on tournaments. We have brought forward its additional applications on strongly Hamiltonian-connected digraphs and given the sufficient conditions involving minimum semi-degree, minimum degree sum and the number of arcs to force a digraph to be strongly Hamiltonian-connected.

Zusammenfassung

Diese Arbeit enthält neue Ergebnisse zu zwei Themen: k -geordnete Graphen und Positiv-Bogen-Panzyklizität in Digraphen.

Im Laufe der Jahre sind Hamiltongraphen intensiv studiert worden. Verschiedene Hamilton-ähnliche Eigenschaften sind zusätzlich betrachtet worden. Einige dieser Eigenschaften sind schwächer, zum Beispiel die Existenz eines Hamiltonweges und eines Kreisfaktors in einem Graphen, während andere stärker sind, zum Beispiel hamiltonsch-zusammenhängend, panzyklisch und panzusammenhängend. Im Jahr 1997 führten L. Ng and M. Schultz [62] eine neue starke hamiltonsche Eigenschaft ein. Ein Graph heißt „ k -geordnet hamiltonsch“ wenn für jede geordnete Folge von k Ecken ein Hamiltonkreis existiert, der die Ecken der Folge in der gegebenen Reihenfolge durchläuft. Wie in [23] gezeigt, gibt es viele Bedingungen, die die Eigenschaft hamiltonsch implizieren, welche auch die stärkere Eigenschaft „ k -geordnet hamiltonsch“ implizieren, wenn wir eine geringe Modifizierung vornehmen.

Unsere erste Arbeit (Kapitel 3) beschäftigt sich mit neuen hinreichenden Bedingungen in Bezug auf die Summe der Eckengrade der Ecken mit Abstand 2 für einen Graphen um einen k -geordneten Hamiltonkreis zu enthalten. Diese Bedingungen sind nicht nur für geringeren Zusammenhang sondern auch für größeren Zusammenhang gegeben. Zusätzlich wird unter den Gradbedingungen an die Ecken mit Abstand 2 die Existenz eines Hamiltonweges und eines Hamiltonkreises nachgewiesen. Wir zeigen sowohl die Schärfe dieser Bedingungen als auch die Unabhängigkeit dieser Resultate.

Unsere zweite Arbeit (Kapitel 4) ist eine Einführung neuer panzyklischer Graphen, „ (k, m) -Ecken-panzyklisch geordneter“ Graphen, welche eine Verallgemeinerung der k -geordneten und eckenpanzyklischen Graphen sind. Diese Idee entstand aus den „ (k, m) -panzyklisch geordneten“ Graphen, welche von R. J. Faudree, R. J. Gould, M. S. Jacobson and L. Lesniak [24] vorgestellt wurden und eine Verallgemeinerung der k -geordneten und panzyklischen Graphen sind. Man beachte, dass jeder „ (k, m) -Ecken-panzyklisch geordnete“ Graph auch ein „ (k, m) -panzyklisch geordneter“ Graph ist. Wir haben bewiesen, dass ein Graph unter der gleichen minimalen Gradsummenbedingung für nicht-adjazente Ecken, unter der er ein „ (k, m) -panzyklisch geordneter“ Graph ist, ein „ (k, m) -Ecken-panzyklisch geordneter“, Graph ist.

Als ein wichtiges Teilgebiet der Graphentheorie sind Digraphen in den letzten vier Jahrzehnten intensiv erforscht worden. Es gibt sehr viele Themen zu Digraphen. Positiv-Bogen-Panzyklizität ist eines der Neuesten und Interessantesten, bei dem es um die Existenz von Ecken geht, deren positive Bögen panzyklisch sind.

Unsere dritte Arbeit (Kapitel 6) ist die Untersuchung der Ecken mit der Eigenschaft, dass alle positiven Bögen panzyklisch und 4-panzyklisch in einem k -fach stark zusammenhängenden Turnier sind. Wir haben nur Ecken mit panzyklischen positiven Bögen in 2- und 3-fach stark zusammenhängenden Turnieren untersucht, nachdem A. Yeo [79] eine unendliche Klasse von k -fach stark zusammenhängenden Turnieren präsentiert hat, so dass jedes Turnier höchstens 3 Ecken mit panzyklischen positiven Bögen enthält. Wir

haben gezeigt, dass jedes k -fach stark zusammenhängende Turnier mindestens k Ecken mit panzyklischen positiven Bögen für $k = 2, 3$ enthält. Falls 3-Kreis nicht beachtet werden, haben wir bewiesen, dass jedes s -fach stark zusammenhängende Turnier mit $s \geq 3$ mindestens $s + 1$ Ecken mit 4-panzyklischen positiven Bögen enthält.

Unsere vierte Arbeit (Kapitel 7) beschäftigt sich mit der strukturellen Analyse von besonderen lokalen in-Tunieren, nämlich positiv-runden Digraphen. Wir geben eine hinreichende Bedingung für stark zusammenhängende positiv-runde Digraphen und eine Charakterisierung von starken positiv-runden orientierten Digraphen.

Unsere letzte Arbeit (Kapitel 8) beinhaltet neue hinreichende Bedingungen für starken hamiltonschen Zusammenhang eines Digraphen. Weg-Kontraktion ist ein mächtiges Werkzeug in dem Beweis von Positiv-Bogen-Panzyklizität in einem Turnier. Dies haben wir übertragen und wir geben hinreichende Bedingungen für starken hamiltonschen Zusammenhang eines Digraphen, welche minimalen semi-Grad, minimale Gradsumme und die Anzahl der Bögen beinhalten.

Lebenslauf

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