Regular Factors in Graphs

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Preface

Many fields of mathematics are concerned with determining the smallest parts, or factors, of a certain kind, which make up a given object. Probably the best known examples in mathematics are the factorization of non-negative integers with prime numbers or the decomposition of polynomials. It was a problem of the second type, arising in invariant theory, which marked the beginning of the nowadays strong branch of factor theory. Julius Petersen published in 1891 his famous paper entitled "Die Theorie der regulären graphs" [31]. In his paper he considered the following problem. Let

$$P = (x_1 - x_2)^{m_{1,2}}(x_1 - x_3)^{m_{1,3}} \cdots (x_{n-1} - x_n)^{m_{n-1,n}}$$

be a homogeneous polynomial of the n variables $x_1, x_2, \ldots, x_n$ with non-negative integers $m_{i,j}$ such that P is of the same positive degree in each x_i. The question is whether P can be decomposed into a product of polynomials of the same form, but with lower degree, or not. Petersen represented the variables x_i as vertices in the plane and connected two vertices x_i, x_j with $m_{i,j}$ edges. He already used the term *graph* in his presentation and called them *regular* as every vertex was connected with the same number of edges. A regular spanning subgraph of degree k is usually called a *k-factor*, and a graph is called *k-factorizable*, if its edge set can be decomposed into edge-disjoint k-factors. So Petersens problem transforms into finding a k-factorization of a given regular graph. In his famous theorem, Petersen showed that every graph is 2-factorizable if and only if it is regular of even degree, solving above problem for the case that all variables have even degree. He also pointed out, that the theory of factorization becomes much more difficult when one considers regular graphs of odd degree, but was able to prove that every connected 3-regular graph with at most two bridges has a 1-factor.

As we can already see from Petersens result, further information about the graph is needed to give sufficient conditions for the existence of a k-factor, if the degree of regularity is odd. In the past 111 years numerous results have been proven, presenting sufficient conditions for the existence of a k-factor in a (regular) graph - good surveys including many of these results can be found in [1] of H. Akiyama, M. Kano and [39] of L. Volkmann. In 1998 T. Niessen and B. Randerath presented in [30] sufficient conditions for the existence of a k-factor in a regular graph of order n which are best possible. These are of the kind that a d-regular graph of order n has a k-factor if $n < f(d, k)$, for a function $f(d, k)$. One aim of this thesis is to present new sufficient conditions for the existence of a k-factor in a graph, if the value of certain graph invariants is known and thus improving the results given by Niessen and Randerath.

Chapter 1 will be devoted to an introduction of the terminology and the basic concepts of graph theory and the theory of factors in graphs. One of the most powerful tools in the theory of factors in graphs, the f-factor Theorem of W.T. Tutte [36] and its regular form, the k-factor Theorem of H.-B. Belck [4], builds the cornerstone of many of our results. The ingenious idea of the k-factor Theorem is the connection between the existence of a k-factor and the appearance of certain components of special subgraphs. Through furhter distinguishing between these components, we will derive information about the structure of the graph. This concept will be used in the proofs of the sufficient conditions presented in Chapters 2 to 4. We will either differ between components depending on their order and derive information about the number of edges connecting to them or vice versa.

One invariant of a graph that can be easily computed is the diameter of a graph. The *distance* of two vertices in a graph is defined as the smallest number of edges leading from one vertex to the other and the *excentricity* of a given vertex is the maximum over its distances to all other vertices. The *diameter* is just the maximal excentricity over all vertices. A diameter of one means, that all vertices are connected to each other - the graph is *complete.* It is not very hard to prove that a complete graph of order n has a k-factor if and only if nk is even. If the diameter stays rather "small", it seems possible that a regular graph of order n still has a k-factor if nk is even, as we have a "high" density of edges. But what happens when the diameter of a graph becomes larger? In Chapter 2 we will take a look at the influence of vertices of excentricity greater or equal to four on the existence of a k-factor. This enables us to show that for regular graphs with diameter less or equal to three the same statement as for complete graphs holds.

In Chapter 3 we will look at the influence of the connectivity of a regular graph on the existence of a k-factor. One generally differs between the *edge-connectivity* λ and the *vertex-connectivity* σ of a graph, denoting the minimal number c such that there exist c edges (vertices) the deletion of which destroys the connectivity of the graph. In 1938 F. Baebler [2] extended Petersens result, showing that every regular graph of odd degree which is λ-edge-connected has a $2k$-factor if $2k \leq \lambda$. A full evaluation of all quadruples (n, d, λ, k) for which a d-regular graph of order n and edge-connectivity λ has a k-factor has been given by T. Niessen and B. Randerath in [30]. Although $\sigma \leq \lambda$ holds for every graph, one cannot use the knowledge of the quadruples (n, d, λ, k) to provide sharp results in case the vertex-connectivity is known, as the difference between the two invariants can be arbitrarily high [7]. It was J. Pila who in [32] first looked at the connection between the vertex-connectivity and the existence of a 1-factor in a graph. The main result of Chapter 3 will be the extension of Pilas result for k-factors with $k \geq 2$. We will further present graphs which show that the conditions are sharp.

As famous as the result of J. Petersen on the 2-factorization of regular graphs of even degree is the theorem of D. König [25] stating that every regular bipartite graph has a 1-factorization. A *bipartite* graph is a graph where the vertex set can be partitioned into two subsets which describe sets of independent vertices. The minimal number of independent sets in which the vertex set can be partitioned, is measured with the *chromatic number* χ. In Chapter 4 we take a look at the connection between the chromatic number and the existence of a k-factor. We will first present almost regular graphs, which are graphs where the vertex degrees differ by at most one, with chromatic number χ and smallest possible order. With the help of these graphs we will then present sufficient conditions for the existence of a k-factor in a regular graph with given order and chromatic number and show that these conditions are best possible.

In Chapter 5 we will move away from sufficient conditions for the existence of a k-factor and look at graphs with a unique k-factor. We will be interested in the maximal number of edges such a graph can have. This problem belongs to extremal graph theory, a branch which was started by P. Turán 60 years ago in [35], where he considered the same question with respect to the absence of a complete subgraph of given order. The first to take a unique k-factor into account were Hetyei [27] for $k = 1$ and G.R.T. Hendry [14] for $k = 2$. In [38] L. Volkmann has extended these results for $k = 3$ but the general case with $k \geq 4$ remains unsolved. In the main part of Chapter 5 we will concentrate on extremal bipartite graphs with a unique k-factor.
The helpful tool when dealing with unique factors in graphs is the colouring of

an edge with colour red or blue, depending whether it belongs to the factor or not. With the help of this colouring we will define the concept of alternating neighbourhoods. These neighbourhoods enable us to derive several results about the structure of an extremal bipartite graph with a unique k-factor. We will, for example, show, that there exist exactly k vertices of degree k in every part. This result gives an indication that a conjecture of Volkmann [38] on the existence of k vertices of degree k in an extremal graph with a unique k-factor might hold true. With the help of the structural results we will answer the question on the maximal number of edges in a bipartite graph with a unique k-factor for $k \leq 4$ and will present graphs providing sharpness. We conclude Chapter 5 with results on extremal graphs with respect to the existence of a unique $[1, k]$-factor, where a $[1, k]$-factor is a factor F such that the degree in F of every vertex lies between 1 and k. Although a $[1, k]$-factor allows for more freedom in the choice of the vertex-degrees, we will show that the uniqueness of its existence enforces a very strict structure on the graph. This will allow us to provide sharp upper bounds for the number of edges in a graph with a unique $[1, k]$-factor.

Contents

Chapter 1

Introduction

This chapter is concerned with the basic notation and terminology of graph theory which will be used throughout this thesis. We will briefly explain the basic definitions and concepts in the first section and will discuss results realted to factors in graphs in more detail in the second section. The notation mainly follows that of L. Volkmann [40] as well as G. Chartrand and L. Lesniak [8] and we direct the reader to these books for any information not given here. Special notation and definitions will be presented where needed.

1.1 Terminology and notation

General concepts

If not stated otherwhise, the term *graph* will be used throughout the thesis to represent a finite and simple graph. The *vertex set* of a graph G will be denoted with $V(G)$ and the *edge set* with $E(G)$. The cardinalities of these sets will be the *order* $n(G)$ and the *size* $e(G)$ respectively. If two vertices $u, v \in V(G)$ are connected with an edge, we simply write $uv \in E(G)$ and say that u and v are *adjacent*. An edge $e = uv$ is called *incident* to both *endvertices* u and v. Two edges are called *incident* if they are incident to the same vertex. For disjoint $X, Y \subseteq V(G)$ let $e_G(X, Y)$ denote the number of edges in G with one endvertex in X and one endvertex in Y. The *neighbourhood* $N(v, G)$ of a vertex v is defined as $N(v, G) = \{u \in V(G) | \; uv \in E(G)\}$. More generally $N(X, G) = \bigcup_{x \in X} N(x, G)$ for a subset $X \subset V(G)$. The *degree* $d(v, G)$ of a vertex v is defined as the number of edges incident with v. Note that for a simple graph it holds $d(v, G) = |N(v, G)|$. If $d(v, G) = 1$, we call v an *endvertex* and for $d(v, G) = 0$ an *isolated vertex*. The *minimum degree* $\delta(G)$ and the *maximum degree* $\Delta(G)$ of a graph G denote the minimum and maximum over all vertex-degrees in G, respectively. If $\delta(G) = \Delta(G) = d$, we call the graph G *d-regular*. A graph is called *complete* if any two vertices are adjacent. A complete graph of order n is usually denoted with K_n. A *subgraph* P of G is a graph such that $V(P) \subseteq V(G)$ and $E(P) \subset E(G)$. The subgraph P is called *induced* if $E(P) = \{uv \in E(G)|\ u, v \in V(P)\}$. For a subset $X \subset V(G)$ we denote the subgraph induced by X with $G[X]$. The deletion of a vertex x in G, in symbols $G - x$, denotes the induced subgraph $G[V(G)\backslash\{x\}]$. The deletion of an edge e in G will be expressed by $G - e$ and denotes the subgraph P with $V(P) = V(G)$ and $E(P) = E(G)\backslash\{e\}$. The deletion of sets of vertices or edges are defined analogously. We call two graphs G, H *isomorphic*, in symbols $G \cong H$, if there exists an isomorphism $g : V(G) \to V(H)$ with $g(x)g(y) \in E(H)$ for all $xy \in E(G)$.

Distance and connectivity

The *distance* $d_G(u, v)$ of two vertices $u, v \in V(G)$ is the number of edges in a shortest path from u to v. If there exists no path from u to v we set $d_G(u, v) := \infty$. We say that u and v are *connected* if $d_G(u, v) < \infty$. With $ex(v) := \max\{d_G(v, x) :\ x \in V(G)\}$ we denote the *excentricity* of v. The *radius* $r(G)$ and the *diameter* $dm(G)$ of a graph G are the minimum and maximum excentricity, respectively. All vertices which are connected to each other induce a *component* and $\kappa(G)$ denotes the number of components of G. If $\kappa(G) = 1$, then G is called *connected*. If $e \in E(G)$ is an edge of a connected graph such that $G-e$ is disconnected, we call e a *bridge*. A connected graph G is called *l-edge-connected* if the deletion of any $l-1$ edges in G does not destroy the connectivity. The *edge-connectivity* $\lambda(G)$ is the greatest integer such that G is $\lambda(G)$-edge-connected. An analogous definition holds for the *vertex-connectivity* $\sigma(G)$, which denotes the smallest number of vertices which have to be deleted to destroy the connectivity of G.
A *pathcovering* of a graph G is a set $\{P_1, P_2, \ldots, P_t\}$ of edge-disjoint paths P_i such that $E(G) = E(P_1) \cup E(P_2) \cup \ldots \cup E(P_t)$.

Independent sets and vertex-partitions

A set $M \subseteq E(G)$ is called a *matching* if no two edges in M are incident with the same vertex. We call $X \subseteq V(G)$ *independent* if $E(G[X]) = \emptyset$. A *p-colouring* of a graph is an assignment of p colours to the vertices of the graph in such a way that the vertices of the same colour form an independent set. The *chromatic number* $\chi(G)$ is the least positive integer p for which G is p-colourable. We also say that G is *p-partite* if $\chi(G) = p$. Then there exists a *partition* $V_1, V_2, \ldots, V_p$ of $V(G)$ such that $V_i \cap V_j = \emptyset$ and $V(G) = \bigcup_{i=1\ldots p} V_i$. Without loss of generality we will always order the *parts* V_i as $|V_1| \geq |V_2| \geq \ldots \geq |V_p|$ and label the vertices of set V_i as $x_1^i, \ldots, x_{|V_i|}^i$. G is called *equipartite*, or *p-equipartite*, if there exists a partition of $V(G)$ such that $||V_i| - |V_j|| \leq 1$ for all $1 \leq i, j \leq p$. A p-partite graph is called *complete* if any two vertices from different sets V_i, V_j are adjacent. For integers p and n let $O^p(n)$ denote the *complete p-equipartite graph* of order n. In the case $n = pr$ we write O_r^p. If a subgraph is induced by parts $V_i, \ldots, V_j$, we use the notation $G[V_i, \ldots, V_j]$. If a graph G has chromatic number $\chi(G) = 2$, it is usually called *bipartite*.

If there is no chance of ambiguity, we drop the mention of G in any of the parameters.
Before we move on to factors in graphs we want to state the simple but very useful Handshake-Lemma:

Lemma 1.1 (Euler 1736) *For a multigraph G it holds*

$$2e(G) = \sum_{v \in V(G)} d(v, G).$$

An easy consequence of Lemma 1.1 is the fact that a k-regular graph of order n can only exist if $k < n$ and kn is even. Throughout this thesis we will quietly assume that these criteria are met if not stated otherwhise.

1.2 Factors in graphs

A *factor* of a graph G is a subgraph $F \neq G$ such that $V(F) = V(G)$. Let f be a function assigning every vertex $v \in V(G)$ a non-negative integer. A factor F of G is called an *f-factor* if $d(v, F) = f(v)$ for every $v \in V(F)$. For constant $f \equiv k$ we simply call F a *k-factor*. If F is a factor of G such that $0 \leq a \leq d(v, F) \leq b$ for every $v \in V(F)$, we call F an *$[a, b]$-factor*. An $[a, b]$-factor F is called *perfect*, if all components of F are regular. We call a

graph G *decomposable* if there exist edge-disjoint factors $F_1, F_2, \ldots, F_s$ with $E(G) = \bigcup_{i=1\ldots s} E(F_i)$.
The first chapters of our thesis are concerned with k-factors in regular graphs. The methods of our proofs rely on the factor theorems of H.-B. Belck [4] and W.T. Tutte [36]. In 1950 H.-B. Belck stated necessary and sufficient conditions for the existence of a k-factor in a graph, based on pairs of subsets of $V(G)$. Two years later W.T. Tutte showed that similar conditions apply for the existence of an f-factor in a graph. As we will be mainly concerned with regular graphs, we state the regular version of the f-factor Theorem.

Theorem 1.2 (Belck [4], Tutte [36]) *Let G be a d-regular graph of order n and let k be a non-negative integer such that kn is even. For a disjoint pair (D, S) with $D, S \subseteq V(G)$ define $q(D, S, k)$ as the number of components U of $G - (D \cup S)$ for which $e_G(S, V(U)) + k|V(U)|$ is odd (we call these components* odd w.r.t. (D, S)). *We further define*

$$\Theta(D, S, k) := k|D| - k|S| + d|S| - e_G(D, S) - q(D, S, k).$$

G does not have a k-factor if and only if G has a pair (D, S), called Tutte-pair*, such that*

$$\Theta(D, S, k) \leq -2. \tag{1.1}$$

If there is no chance for misunderstanding, we will write q instead of $q(D, S, k)$.

Inequality (1.1) is typically very tedious to check for an arbitrary graph. Thus, starting with W.T. Tutte, researchers have looked at special Tutte-pairs to derive more information in the case that the graph does not have a k-factor. We state two of these results which we will need later. Lemma 1.3 was proved by H. Enomoto, B. Jackson, P. Katerinis and A. Saito [11] in 1985 and holds for any graph, not necessarily regular, without a k-factor.

Lemma 1.3 (Enomoto, Jackson, Katerinis and Saito [11]) *Let G be a graph and k a positive integer with $kn(G)$ even. If (D, S) is a Tutte-pair such that $|S|$ is minimum over all Tutte-pairs, then $S = \emptyset$ or $\Delta(G[S]) \leq k - 2$.*

For connected regular graphs of even order without a k-factor, one can show that certain Tutte-pairs (D, S) meet $|D| > |S|$. We present the result together with its proof, as it has only been shown implicetely before.

Lemma 1.4 *Let n, k, d be integers such that n is even and k is odd with $n > d > k > 0$. Let further $2k \leq d$ if d is even. If a connected d-regular graph G of order n has no k-factor, then for every Tutte-pair (D, S) it holds $|D| > |S|$.*

Proof. If G does not have a k-factor, then, since kn is even, there exists a Tutte-pair (D, S). Since G is connected, $D \cup S \neq \emptyset$. Let $q := q_G(D, S, k)$ and $W := G - (D \cup S)$.
Case 1: d is even. G is connected and of even degree d, thus by Lemma 1.1 G is at least 2-edge-connected. This leads to

$$e_G(D \cup S, V(W)) \geq 2q. \tag{1.2}$$

Since $e_G(D, S) \leq \min\{d|D| - e_G(D, V(W)), d|S| - e_G(S, V(W))\}$, we have

$$2e_G(D, S) \leq d(|D| + |S|) - e_G(D \cup S, V(W)), \tag{1.3}$$

which together with (1.2) results in $2q \leq d(|D| + |S|) - 2e_G(D, S)$. Taking (1.1) into account leads to $(d - 2k)(|D| - |S|) \geq 4$. For $d > 2k$ we get the desired result. If $d = 2k$, then this case cannot occur, meaning that the graph has a k-factor.
Case 2: d is odd. We get for every odd component U w.r.t. (D, S)

$$\begin{aligned} e_G(D, V(C)) &= d|V(C)| - e_G(S, V(C)) - 2|E(C)| \\ &\equiv k|V(C)| + e_G(S, V(C)) - 2|E(C)| \equiv 1 \pmod 2. \end{aligned}$$

Thus $e_G(D, S) \leq d|D| - q$ which, substituted in (1.1), resolves to

$$k(|D| - |S|) + d|S| - q + 2 \leq e_G(D, S) \leq d|D| - q.$$

Hence

$$(d - k)(|D| - |S|) \geq 2,$$

yielding $|D| > |S|$, as $d > k$. □

Numerous results have been given to ensure the existence of a k-factor, depending on different restrictions for the class of graphs invastigated. J. Akiyama and M. Kano [1] as well as L. Volkmann [39] give good surveys of the important results of the last century. Among the first results concerning factors in graphs has been the following theorem which deals with 1-factors in complete graphs and dates back to the middle of the 19th century.

Theorem 1.5 (Kirkman [24], Reiß [33]) *Every complete graph K_{2n} is decomposable into 1-factors.*

Two profound theorems in the theory of graph factors go back to J. Petersen in 1891 where he presented a complete solution for the decomposition of a regular graph into 2-factors and further proved a sufficient condition for the existence of a 1-factor in a 3-regular graph.

Theorem 1.6 (Petersen [31]) *A graph G is decomposable into 2-factors if and only if G is $(2r)$-regular with $r \geq 2$.*

Theorem 1.7 (Petersen [31]) *Let G be a connected 3-regular graph. If G has at most two bridges, then G has a 1-factor.*

The following graph of Sylvester (see [40]) shows that Theorem 1.7 is best possible.

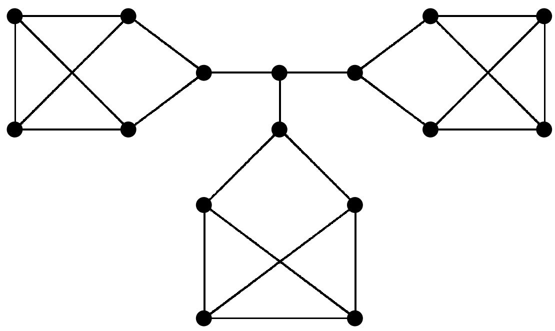

Figure 1.1: The graph of Sylvester

In 1981 W.D. Wallis [41] presented sufficient conditions for the existence of a 1-factor in a k-regular graph and showed that these are sharp.

Theorem 1.8 (Wallis [41]) *A d-regular graph G of even order n with no component of odd order has a 1-factor if $d = 2$ or if*

$$n < \begin{cases} 3d+7 & \text{if } d \geq 3 \text{ is odd,} \\ 2d+4 & \text{if } d \geq 6 \text{ is even,} \\ 22 & \text{if } d = 4. \end{cases}$$

The key to the proof of Theorem 1.8 lies in distinguishing the odd components w.r.t. a Tutte-pair, which appear in the k-factor Theorem, if the graph does not have a k-factor. W.D. Wallis distinguished between small and great components U, depending whether $|V(U)| \leq d$ or not. This provides further information about the number of edges connecting to these components, as the following lemma shows.

Lemma 1.9 *Let G be a d-regular graph and $X \subset V(G)$. If U is a component of $G - X$ such that $|V(U)| \leq d$, then $e_G(V(U), X) \geq d$.*

Proof. $e_G(V(U), X) = d|V(U)| - 2e(U) \geq d|V(U)| - |V(U)|(|V(U)| - 1) \geq d$ if and only if $1 \leq |V(U)| \leq d$. □

Counting the edges between a Tutte-pair (D, S) and $G - (D \cup S)$ results in an upper bound for the number of odd components w.r.t. (D, S). This bound leads to the conditions given in Theorem 1.8. We will later take up Wallis' method in our proofs, making the necessary adaptations corresponding to the class of graphs we examine.

Together with the following result of P. Katerinis [23], Theorem 1.8 provides sufficient conditions for the existence of a k-factor in a regular graph. However, these conditions are far from being optimal when k increases, as Theorem 1.11 of T. Niessen and B. Randerath [30] shows. They derive Theorem 1.11 as a corollary of Theorem 3.1, however, the basic idea behind the proof lies again in distinguishing the odd components as in Theorem 1.8.

Theorem 1.10 (Katerinis [23]) *Let G be a d-regular graph. If G has a 1-factor, then G has a k-factor for every $1 \leq k < d$.*

Theorem 1.11 (T. Niessen, B. Randerath [30]) *Let n, d and k be integers with $n > d > k \geq 1$ such that nd and nk are even. A d-regular graph of order n has a k-factor in the following cases:*

- *d and k are even;*
- *d is even and k is odd and $n < 2(d + 1)$;*
- *d and k are odd and $n < 1 + (k + 2)(d + 2)$;*
- *d is odd and k is even and $n < 1 + (d - k + 2)(d + 2)$.*

In all other cases there exists a d-regular graph of order n without a k-factor.

In Theorem 1.11 the graph is allowed to have components of odd order. Obviously, a disconnected graph with at least one component of odd order cannot have a factor of odd degree. Thus the bound on the order n in the case d even and $k = 1$ differs from the one given in Theorem 1.8, where components of odd order are forbidden.

Chapter 2

Regular factors in regular graphs with small diameter

As Theorem 1.11 shows, the sufficient conditions for the existence of a k-factor in an arbitrary regular graph cannot be improved unless further information on the graph is known. In this chapter we examine the connection between vertices of high excentricity and the existence of k-factors. This examination is motivated by the fact that a complete graph K_n has a k-factor for every $1 \leq k \leq d-1$ with kn even. Complete graphs have diameter 1 and it is natural to ask what influence the diameter has on the existence of a k-factor in a regular graph. To speak visually, a small diameter allows less "freedom" for possible edges and enforces a stricter structure in the graph, than a larger diameter. Thus it seems plausible that the existence of a k-factor in a graph becomes more likely the smaller the diameter of the graph is. Theorem 2.3 will show that the existence of a k-factor, with $k|V(G)|$ even, is always guaranteed in a regular graph G with diameter $dm(G) \leq 3$.

We start with the examination of the influence vertices of "high" excentricity have on the existence of a k-factor. As every connected 2-regular graph of even degree is a cycle and thus has a 1-factor, we can restrict ourselves to regular graphs with degree $d \geq 3$.

Theorem 2.1 (Hoffmann, Volkmann [19]) *Let d, k be two positive integers with $d \geq \max\{3, k+1\}$. A connected d-regular graph G has a k-factor if $kn(G)$ is even and*

- *d and k are even;*

- *d is even, k is odd and G has at most $(d+1)\cdot\min\{k+1, d-k+1\}$ vertices of excentricity ≥ 4;*
- *d and k are odd and G has at most $1+(d+2)(k+1)$ vertices of excentricity ≥ 4;*
- *d is odd and k is even and G has at most $1+(d+2)(d-k+1)$ vertices of excentricity ≥ 4.*

Proof. The first case is just Theorem 1.6. In the remaining cases let, without loss of generality, k be odd and further $2k \leq d$ if d is even, as G has a k-factor if and only if it has a $(d-k)$-factor. We are only going to prove the case that d and k are both odd. The proof for the case d even and k odd only differs in the number of vertices of excentricity ≥ 4 and uses anologous argumentation.
Assume that G does not have a k-factor. With Theorem 1.2 there exists a Tutte-pair (D, S) with $\Theta_G(D, S, k) \leq -2$. From Lemma 1.4 we know that $|D| > |S|$ and $q \geq k(|D|-|S|)+2 \geq k+2$. Let $X := \{v \in V(G) : \; ex(v) \geq 4\}$ and $C^X := V(C) \cap X$ for every odd component C w.r.t. (D, S). By the hypothesis we have $r := |X| \leq 1+(d+2)(k+1)$. Call an odd component C w.r.t. (D, S) a small component, if $|V(C)| \leq d$ and let s denote the number of small components. For every small component C it holds $e_G(D \cup S, V(C)) \geq d$.

Case 1: Assume that for an integer l with $0 \leq l \leq 2$ there exist l odd components w.r.t. (D, S) which have a vertex x with $e_G(x, D\cup S) = 0$. Then $N(x, G) \subseteq V(C)$ and these components are not small. So we have $s \leq q-l$, and it holds $e_G(V(C), D\cup S) \geq |V(C)|$ for all other odd components C w.r.t (D, S). This results in

$$\begin{aligned} e_G(V(W), D\cup S) &\geq sd + (q-s-l)(d+1) + l = q(d+1) - s - ld \\ &\geq q(d+1) - (q-l) - ld = d(q-l) + l > d(q-2). \end{aligned}$$

This together with (1.3) results in

$$d(|D|+|S|) - 2e_G(D, S) > d(q-2). \tag{2.1}$$

Inequality (1.1) and Inequality (2.1) lead to

$$(d-2k)(|D|-|S|) > (d-2)q - 2d + 4.$$

As shown above, $q \geq 2 + k(|D| - |S|)$, and so

$$(d - 2k)(|D| - |S|) > (d - 2)(2 + k(|D| - |S|)) - 2d + 4,$$

giving us the contradiction

$$0 \ \geq \ d(1 - k)(|D| - |S|) \ > \ 2(d - 2) - 2d + 4 = 0. \tag{2.2}$$

Case 2: There exist at least three odd components w.r.t. (D, S) having a vertex x with $e_G(x, D \cup S) = 0$. Assume that one of these vertices is not a member of X. Then $ex(x) \leq 3$ for this vertex and we have $e_G(V(C), D \cup S) \geq |V(C)|$ for all other odd components. Analogous to $l = 1$ in Case 1 we can show $e_G(V(W), D \cup S) > (q-2)d$ and arrive at the contradiction (2.2). Thus each vertex with $e_G(x, D \cup S) = 0$ is a member of X. Let $\mathcal{B}$ denote the set of all odd components w.r.t. (D, S) which are no small components. Then $|\mathcal{B}| \geq 3$ and $s \leq q - 3$ and it holds

$$\begin{aligned} e_G(V(W), D \cup S) &\geq sd + \sum_{C \in \mathcal{B}} (|V(C)| - |C^X|) \\ &\geq sd - r + \sum_{C \in \mathcal{B}} |V(C)| \\ &\geq sd - r + (q - s)(d + 1) = q(d + 1) - s - r. \end{aligned}$$

This combined with (1.3) and (1.1) leads to

$$(d - 2k)(|D| - |S|) \geq q(d - 1) + 4 - s - r. \tag{2.3}$$

Since $s \leq q - 3$, $q \geq k(|D| - |S|) + 2$ and $r \leq 1 + (d + 2)(k + 1)$, we can deduce the following:

$$d(1 - k)(|D| - |S|) \geq 2d + 2 - (d + 2)(k + 1). \tag{2.4}$$

Inequality (2.4) does not give us any information in the case $k = 1$. So let us first consider $k \geq 3$. Inequality (2.4) can now be rewritten as

$$|D| - |S| \leq \frac{(d + 1)(k + 1) - 2d - 3}{d(k - 1)} = 1 + \frac{k - 2}{d(k - 1)} < 2.$$

With Lemma 1.4 it follows $|D| = |S| + 1$. Let now $q = k + 2 + \eta$ with a non-negative integer η. With (2.3) and $|D| = |S| + 1$ we get

$$\begin{aligned} s &\geq (k + 2 + \eta)(d - 1) - d + 2k + 4 - 1 - (d + 2)(k + 1) \\ &= \eta(d - 1) - k - 1. \end{aligned} \tag{2.5}$$

Since $q \geq s+3$ we get $k+\eta-1 \geq \eta(d-1)-k-1$, or $2k \geq \eta(d-2)$. Thus $\eta \leq 2$ with equality if and only if $k = d-2$. So it follows $q \leq k+4$ and Inequality (1.1) yields $d|S| - e_G(D,S) \leq 2$, giving us $e_G(V(W), D \cup S) \leq d+2$. If $s \geq 1$, then there are at most 2 edges leading to a component $C \in \mathcal{B}$, which together with $q \geq s+3$ and the connectivity of G yields a contradiction.
For $\eta \geq 1$, we get $s \geq 1$ from (2.5), so it remains the case $\eta = 0$ and $s = 0$. But then $|S| = 0$ or $e_G(D,S) = d|S|$ and hence $e_G(V(W), D) \leq d$. Since $s = 0$ and from the definition of the odd components w.r.t. (D,S) in Theorem 1.2, every such component has at least $d+2$ vertices. Thus W has at least $(k+2)(d+2)$ vertices, of whom at most $r \leq 1+(d+2)(k+1)$ are not connected to D with an edge. This means $e_G(V(W), D) \geq (k+2)(d+2)-1-(d+2)(k+1) = d+1$, which yields a contradiction.
It remains the case $k = 1$. With Lemma 1.3 we have $|S| = 0$, if we choose (D,S) with $|S|$ minimum over all Tutte-pairs. Thus $q \geq |D|+2$. Again, $|V(C)| \geq d+2$ holds for every component $C \in \mathcal{B}$. This leads to

$$\begin{aligned} e_G(V(W), D) &\geq sd + (q-s)(d+2) - r \geq q(d+2) - 2s - 1 - 2(d+2) \\ &\geq qd - 2d + 1 \geq (|D|+2)d - 2d + 1 \geq d|D| + 1, \end{aligned}$$

which contradicts $e_G(V(W), D) \leq d|D|$. □

Theorem 2.1 is in the following way best possible: Let d be even and k odd with $d \geq 2k+4$. Let $\{y_1, y_2, \ldots, y_{d+1}\}$ be a set of vertices which induces a K_{d+1} with a matching of cardinality $\frac{d-2(k+1)}{2}$ removed. Further take a vertex x and $k+1$ copies of $K_{d+1} - uv$ and connect x to all vertices u, v of degree $d-1$. Last connect x to all vertices y_i of degree $d-1$. The resulting graph G is d-regular and has $(k+1)(d-1)+2k+3 = (d+1)(k+1)+1$ vertices of excentricity 4. G has no k-factor since $\Theta_G(\{x\}, \emptyset, k) = -2$.
Now let d and k be odd with $d \geq 3k+6$. For an odd integer $0 < p < d$ define $K_{d+2}(p)$ as follows. In the complete graph K_{d+2} there exists a cycle C_p of length p as well as a matching M of cardinality $\frac{d+2-p}{2}$, which is not incident to C_p. Let $K_{d+2}(p) := K_{d+2} - E(C_p) - M$. Take $k+1$ copies of $K_{d+2}(3)$, one copy of $K_{d+2}(d-3(k+1))$ as well as a vertex x. Connect x with all vertices of degree $d-1$. The resulting graph H is d-regular and has $2+(k+1)(d+2)$ vertices of excentricity 4. It further has no k-factor since $\Theta_H(\{x\}, \emptyset, k) = -2$.

Note that Theorem 2.1 holds for graphs of arbitrary radius. For a regular graph with radius ≤ 3, Theorem 2.1 provides conditions for the existence of a k-factor, which allow for a higher order than Theorem 1.11. However, if $r(G) \geq 4$, the conditions in Theorem 2.1 transform into bounds on the order of the graph, as all vertices now have excentricity greater than 3. Then Theorem 1.11 will yield better results in most cases.

Theorem 2.1 implies the following two theorems.

Theorem 2.2 (Hoffmann, Volkmann [19]) *A connected d-regular graph G , with $d \geq 2$, with at most $2d+2$ vertices of excentricity ≥ 4 has a k-factor for every $1 \leq k < d$ with $kn(G)$ even.*

Proof. If G has at most $2d + 2$ vertices of excentricity ≥ 4 it meets any of the conditions in Theorem 2.1 and thus has a k-factor. □

Theorem 2.3 (Hoffmann, Volkmann [19]) *A connected d-regular graph G, with $d \geq 2$, with diameter ≤ 3 has a k-factor for every $1 \leq k < d$ with $kn(G)$ even.*

Proof. As G does not have a vertex of excentricity ≥ 4, the statement follows from Theorem 2.2. □

Chapter 3

Regular factors in connected regular graphs

In this chapter we take a look at the connection between the connectivity of a regular graph and the existence of a k-factor. The influence of the edge-connectivity on the existence of a k-factor has been extensively studied (c.f. [2], [4], [5], [9], [13], or see [39]). T. Niessen and B. Randerath proved in [30] the following theorem, determining all quadruples (d, n, λ, k) for which a d-regular graph of order n with edge-connectivity λ has a k-factor.

Theorem 3.1 (Niessen, Randerath [30]) *Let n, d, k and λ be integers with $n > d > k > 0$, $d \geq \lambda$ such that nd and nk are even, and let λ be even, if d is even. Let $\lambda^* = 2\lfloor \lambda/2 \rfloor$ and $\hat{k} = \min\{k, d-k\}$. A d-regular graph of order n and edge-connectivity λ has a k-factor in the following cases:*

- *for all d and k even;*
- *for d even and k odd, if either*
  - *$\lambda\hat{k} \geq d$, or*
  - *$\lambda\hat{k} < d$ and $n < d_1 + (d+1)(\hat{k}d_1 + 2)$, where $d_1 = \lceil 2\lambda/(d - \lambda\hat{k}) \rceil$;*
- *for d and k odd, if either*
  - *$\lambda^* k \geq d$, or*
  - *$\lambda^* k < d$ and $n < \begin{cases} (d+2)(k+3) & \text{if } d_2 = 1 \text{ and } \lambda \text{ even,} \\ d_2 + (d+2)(kd_2 + 2) & \text{otherwise,} \end{cases}$*
    where $d_2 = \lceil 2\lambda^*/(d - \lambda^* k) \rceil$;

- *for d odd and k even, if either*
  - $\lambda^*(d-k) \geq d$, *or*
  - $\lambda^*(d-k) < d$ *and*

$$n < \begin{cases} (d+2)(d-k+3) & \text{if } d_3 = 1 \text{ and } \lambda \text{ even,} \\ d_3 + (d+2)((d-k)d_3 + 2) & \text{otherwise,} \end{cases}$$

where $d_3 = \lceil 2\lambda^*/(d - \lambda^*(d-k)) \rceil$.

In all other cases there exists a d-regular graph of order n and edge-connectivity λ without a k-factor.

With the following theorem of H. Whitney [42], Theorem 3.1 automatically yields sufficient conditions for the existence of a k-factor in case the vertex-connectivity is known.

Theorem 3.2 (Whitney [42]) *For a graph G it holds $\sigma(G) \leq \lambda(G) \leq \delta(G)$.*

As G. Chartrand and F. Harary [7] showed, there exists for any combination $0 < \sigma \leq \lambda \leq \delta$ a graph with minimum degree δ, vertex-connectivity σ and edge-connectivity λ. So it seems unlikely that Theorem 3.1 with $\lambda = \sigma$ yields best possible conditions for the existence of a k-factor in the case that the vertex-connectivity is known. In fact, the graphs providing sharpness in Theorem 3.1 do not in general fulfill $\sigma = \lambda$. So, although Theorems 3.1 and 3.5 can be used to derive conditions for the other connectivity-index, neither can be used to obtain sharpness for the other. The first to take the vertex-connectivity into account while looking for k-factors was J. Pila [32], who extended Theorem 1.8 of W.D. Wallis in 1983.

Theorem 3.3 (Pila [32]) *Let n, d, σ be integers with $n > d > 1$, n even and $d \geq \sigma \geq 1$. Define $\sigma^* \in \{\sigma, \sigma+1\}$ such that $\sigma^* \equiv d \pmod 2$. A d-regular graph of order n with vertex-connectivity σ has a 1-factor if*

- *d is even and*

  (i) $d = \sigma^* + 2$ *and* $n < \sigma^* + (\sigma^*+2)(d+1)$;

  (ii) $\sigma = 1$, $d \geq \sigma^* + 4$ *and* $n < 1 + 3(d+1)$;

(iii) $\sigma = 2$, $d \geq \sigma^* + 4$ *and* $n < 2 + 4(d+1)$;
(iv) $\sigma = 3$, $d \in \{\sigma^* + 4, \sigma^* + 6\}$ *and* $n < 3 + 5(d+1)$;
(v) $\sigma = 3$, $d \geq \sigma^* + 8$ *and* $n < 2d - l + (l+2)(d+1)$;
(vi) $\sigma \geq 4$, $d \geq \sigma^* + 4$ *and* $n < 2d - l + (l+2)(d+1)$;

- *d is odd and*

(i) $\sigma = 1$, $d \geq \sigma^* + 2$ *and* $n < \sigma + (\sigma+2)(d+2)$;
(ii) $\sigma = 2$, $d \in \{\sigma^* + 2, \sigma^* + 4\}$ *and* $n < \sigma + (\sigma+2)(d+2)$;
(iii) $\sigma = 2$, $d \geq \sigma^* + 6$ *and* $n < 3(d+2)$;
(iv) $\sigma \geq 3$, $d = \sigma^* + 2$ *and* $n < \sigma^* + (\sigma^*+2)(d+2)$;
(v) $\sigma \in \{3, 4\}$, $d = \sigma^* + 4$ *and* $n < \sigma + (l+2)(d+2) + d$;
(vi) $\sigma \in \{3, 4\}$, $d \geq \sigma^* + 6$ *and* $n < 2d - l + (l+2)(d+2)$;
(vii) $\sigma \geq 5$, $d \geq \sigma^* + 4$ *and* $n < 2d - l + (l+2)(d+2)$,

with $l := \left\lceil \frac{2\sigma^*}{d - \sigma^*} \right\rceil$.

These conditions are best possible.

With the following theorem we want to extend Theorem 3.3 to k-factors with $k \geq 2$. We restrict ourselves to connected graphs only, since the disconnected case presents no better conditions than those in Theorem 1.11. We can further restrict our discussion to $d - 1 > k > 1$ because of Theorem 3.3. For the proof of Theorem 3.5 we need the following profound theorem of K. Menger [29].

Theorem 3.4 (Menger [29]) *A graph G has connectivity $\sigma(G) = c$ if and only if there exist c paths between any two vertices x, y of G, which only have x and y in common.*

Theorem 3.5 (Hoffmann [16]) *For integers n, d, k, σ with $d - 1 > k > 1$ and $n > d \geq \sigma \geq 1$ such that nd and nk are even, let G be a d-regular graph of order n with vertex-connectivity σ. Define $\sigma^* \in \{\sigma, \sigma+1\}$ such that $\sigma^* \equiv d \pmod 2$, $p \in \{1, 2\}$ with $p \not\equiv d \pmod 2$, and*

$$\hat{k} = \begin{cases} \min\{k, d-k\}, & \text{for } d \text{ even and } k \text{ odd}; \\ k, & \text{for } d \text{ and } k \text{ odd}; \\ d-k, & \text{for } d \text{ odd and } k \text{ even}. \end{cases}$$

The graph G has a k-factor if

- *d and k are even, or else*
- *if either* $d \leq \hat{k}\sigma^*$, *or*

  (i) $d = \hat{k}\sigma^* + 2$ *and* $n < \sigma^* + (\hat{k}\sigma^* + 2)(d + p)$;

  (ii) $\sigma = 1$, $d \geq \hat{k}\sigma^* + 4$ *and* $n < 1 + (\hat{k} + 2)(d + p)$;

  (iii) $\sigma = 2$, $d = 3\hat{k} + 4$ *and* $n < 2 + (2\hat{k} + 2)(d + p)$;

  (iv) $\sigma = 2$, $d = 2\hat{k} + 4$ *and* $n < 2d - l + (\hat{k}l + 2)(d + p)$;

  (v) $\sigma = 2$, $d \geq \hat{k}\sigma^* + 6$ *and* $n < 2d - l + (\hat{k}l + 2)(d + p)$;

  (vi) $\sigma \geq 3$, $d \geq \hat{k}\sigma^* + 4$ *and* $n < 2d - l + (\hat{k}l + 2)(d + p)$,

  where $l := \left\lceil \frac{2\sigma^*}{d - \hat{k}\sigma^*} \right\rceil$.

Proof. By Theorem 1.6 the first case holds. In the remaining cases we may assume without loss of generality that $k = \hat{k} \geq 3$ is odd. This holds since G has a k-factor if and only if G has a $(d-k)$-factor. Assume that G does not have a k-factor. By Theorem 1.2 there exists a Tutte-pair (D, S). We call an odd component C of $W := G - (D \cup S)$ a small component, if $|V(C)| \leq d$. Let s denote the number of small components of W. By Theorem 3.4 and Lemma 1.1 it holds $e_G(D \cup S, V(C)) \geq \sigma^*$ for every odd component C w.r.t. (D, S), and especially $e_G(D \cup S, V(C)) \geq d$ for every small component C. For an odd component C w.r.t. (D, S), which is not a small component, we further have $|V(C)| \geq d + p$, due to the definiton in Theorem 1.2. This leads to

$$e_G(D \cup S, V(W)) \geq sd + (q - s)\sigma^* = q\sigma^* + (d - \sigma^*)s. \qquad (3.1)$$

As for (1.3) in the proof of Lemma 1.4 we get

$$2e_G(D, S) \leq d|D| + d|S| - e_G(D \cup S, V(W)). \qquad (3.2)$$

(3.1) and (3.2) yield

$$d|D| + d|S| - 2e_G(D, S) \geq q\sigma^* + (d - \sigma^*)s.$$

With (1.1) it follows

$$(d - 2k)(|D| - |S|) \geq q(\sigma^* - 2) + (d - \sigma^*)s + 4. \qquad (3.3)$$

Case 1: $\sigma^* \geq 2$. By Lemma 1.4 we have $2k < d$ and $|D| - |S| > 0$. With (1.1) this leads to

$$q \geq k(|D| - |S|) + 2 \geq k + 2 \tag{3.4}$$

and with (3.3) to

$$(d - k\sigma^*)(|D| - |S|) \geq 2\sigma^* + (d - \sigma^*)s > 0. \tag{3.5}$$

Since $|D| - |S| > 0$ we get $d \geq k\sigma^* + 2$, proving the statement that G has a k-factor if $d \leq k\sigma^*$. We can rewrite (3.5) as

$$\begin{aligned} & |D| - |S| & \geq & \ \frac{2\sigma^*}{d - k\sigma^*} + \frac{d - \sigma^*}{d - k\sigma^*}s \qquad (3.6) \\ \Rightarrow \quad & |D| - |S| & \geq & \ \left\lceil \frac{2\sigma^*}{d - k\sigma^*} \right\rceil =: l \geq 1. \end{aligned}$$

The next two subcases complete our discussion of Case 1.
Case 1.A: $d = k\sigma^* + 2$, or $d \geq k\sigma^* + 2$ for $\sigma = 1$, or $d = 3k + 4$ for $\sigma = 2$. In all three cases we have $l \geq \sigma$. As $|V(W)| \geq s + (q - s)(d + p)$ and $|D| + |S| \geq l$, we have

$$\begin{aligned} n(G) &= |D| + |S| + |V(W)| \\ &\geq l + s + (q - s)(d + p) \\ &\overset{(3.4)+(3.6)}{\geq} l + \left(\frac{k}{d - k\sigma^*}(2\sigma^* + (d - \sigma^*)s) + 2 - s \right)(d + p) \\ &\geq l + \left(2\frac{k\sigma^*}{d - k\sigma^*} + 2 \right)(d + p) \\ \Rightarrow \quad n(G) &\geq l + (kl + 2)(d + p). \end{aligned}$$

If $d = k\sigma^* + 2$, then $l = \sigma^*$ and thus $n(G) \geq \sigma^* + (k\sigma^* + 2)(d + p)$, proving statement (i) of the theorem. If $\sigma = 1$ and $d \geq 2k + 4$, we have $l + (kl + 2)(d + p) \geq 1 + (k + 2)(d + p)$, proving (ii), if d is even (d odd and $\sigma = 1$ yield Case 2). If $\sigma = 2$ and $d = 3k + 4$, then $l = 2$ $(= \sigma)$ and it follows $n(G) \geq 2 + (2k + 2)(d + p)$, giving us (iii).
Case 1.B: $\sigma \geq 2$ and $d \geq k\sigma^* + 6$, or $d \geq k\sigma^* + 4$ for $\sigma \geq 3$, or $\sigma = 2$ and $d = 2k + 4$. Inequalities (3.4) and (3.6) yield

$$\begin{aligned} q &\geq k\frac{2\sigma^*}{d - k\sigma^*} + sk\frac{d - \sigma^*}{d - k\sigma^*} - s + 2 \\ &\overset{k \geq 3}{\geq} k\frac{2\sigma^*}{d - k\sigma^*} + 2s + 2. \qquad (3.7) \end{aligned}$$

Thus we arrive at

$$
\begin{array}{rrcl}
 & |V(W)| & \geq & s + (q - s)(d + p) \\
 & & \overset{(3.7)}{\geq} & s + \left(k\dfrac{2\sigma^*}{d - k\sigma^*} + 2s + 2\right)(d + p) \\
\Rightarrow & |V(W)| & \geq & s + (kl + 2s + 2)(d + p)
\end{array}
$$

If there exists at least one small component, $|V(W)| \geq 2d+3+(kl+2)(d+p)$ and

$$
\begin{array}{rcl}
n(G) & = & |D| + |S| + |V(W)| \\
 & \overset{(3.4)}{\geq} & \sigma + 2d + 3 + (kl + 2)(d + p) \\
 & > & 2d - l + (kl + 2)(d + p).
\end{array}
$$

If W does not have any small components, then

$$|V(W)| \geq q(d + p) \overset{(1.1)}{\geq} \Big(k(|D| - |S|) + d|S| - e_G(D, S) + 2\Big)(d + p).$$

If $d|S| - e_G(D, S) \geq 2$, then

$$
\begin{array}{rcl}
n(G) & \geq & l + 2(d + p) + (kl + 2)(d + p) \\
 & > & 2d - l + (kl + 2)(d + p).
\end{array}
$$

If $e_G(D, S) = d|S| - 1$, then $|S| \geq d - 1$ and we get

$$
\begin{array}{rcl}
n(G) & \geq & |D| + |S| + (d + p) + (kl + 2)(d + p) \\
 & > & 2d - l + (kl + 2)(d + p).
\end{array}
$$

It remains the case $d|S| = e_G(D, S)$, which can only occur for either

- $|S| = 0$ and $|D| \geq \sigma$; or
- $1 \leq |S| \leq |D| - l$ with $|D| \geq d$.

Since it holds $\sigma > l$ in Subcase 1.B, a short calculation shows us that $2d - l + (kl + 2)(d + p) < \sigma + (k\sigma + 2)(d + p)$. So $|D| = d$ and $|S| = d - l$ yields the lowest possible case, and we get

$$n(G) \geq 2d - l + (kl + 2)(d + p).$$

These cases prove (iv) to (vi) of our theorem and complete the discussion of Case 1.
Case 2: $\sigma^* = 1$. In this case we have $\sigma^* = \sigma = 1$ and d odd. Using $\lambda = 1$, d odd and k odd, Theorem 3.1 tells us $d \geq k+2$ as well as $n > 1+(k+2)(d+p)$. This proves (ii) for odd d. □

Theorem 3.5 is sharp as the following examples show. Analogous to Pila [32] we first construct graphs $C(d,h)$ on $d+p$ vertices with h vertices of degree $d-1$ and $d+p-h$ vertices of degree d. This definition will hold for any $1 \leq h \leq d$ with $h \equiv d \pmod 2$. If d is even, then $p = 1$. Let $V(C(d,h)) = \{x, y_1, \ldots, y_d\}$, where $\{y_1, \ldots, y_d\}$ induces a complete graph with a matching of size $h/2$ removed. Further let x be connected to y_i for every $1 \leq i \leq d$.
If d is odd, then $p = 2$. Let $V(C(d,h)) = \{x, z, y_1, \ldots, y_d\}$ where $\{y_1, \ldots, y_d\}$ induces a complete graph with a cycle of length h removed. Further let x and z be connected to y_i for every $1 \leq i \leq d$.
In both cases $C(d,h)$ is $(d-2)$-connected with h vertices of degree $d-1$. With the help of these graphs we are now showing the sharpness of Theorem 3.5. The Cases 1 to 4 are exhaustive:

Case 1: $d = k\sigma^* + 2$. Take a set X of σ^* independent vertices and d copies of $C(d,\sigma^*)$. Connect each vertex $x \in X$ with vertices of degree $d-1$ such that $e(x,U) = 1$ holds for every vertex x and every copy U of $C(d,\sigma^*)$. By Theorem 3.4 the resulting graph is σ-connected and has order $n = \sigma^* + (k\sigma^* + 2)(d+p)$. It has no k-factor, since $\Theta(X, \emptyset, k) = -2$.
Case 2: $\sigma = 2$ and $d = 3k+4$. Here the following construction is possible (for which we need $d \geq k\sigma^* + 4$): Take a set $X = \{x_1, x_2\}$ of 2 independent vertices, $k+2$ copies of $C(d,2)$ and k copies of $C(d,4)$. Connect x_1, x_2 to the vertices of degree $d-1$ such that $e(x_i, V(C(d,2))) = 1$ and $e(x_i, V(C(d,4))) = 2$ for $i = 1,2$. The resulting graph is d-regular, 2-connected with order $2+(2k+2)(d+p)$ and has no k-factor since $\Theta(X, \emptyset, k) = -2$. For a picture see Figure 3.1.
Case 3: $\sigma = 1$ and $d \geq k\sigma^* + 4$. Here an analogous construction to that in Case 2 is possible: Take a vertex x, $k+1$ copies of $C(d,\sigma^*)$ and one copy of $C(d,h)$ with $1 \leq h = d - (k+1)\sigma^* < d$. Note that $h \equiv d \pmod 2$. Connect x to every vertex of degree $d-1$ with one edge. The resulting graph is connected, d-regular of order $1+(k+2)(d+p)$ and has no k-factor, since $\Theta(\{x\}, \emptyset, k) = -2$.

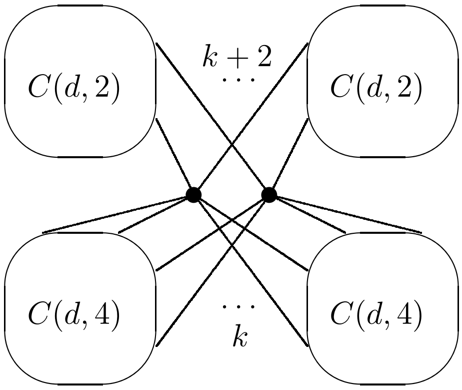

Figure 3.1: The graph G in Case 2

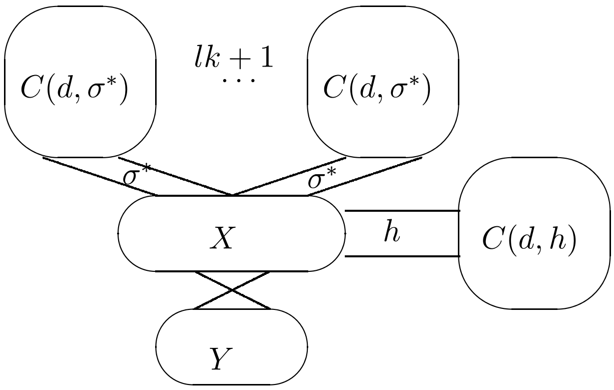

Figure 3.2: The graph G in Case 4.

Case 4: $\sigma \geq 2$ and $d \geq k\sigma^* + 6$, or $d \geq k\sigma^* + 4$ for $\sigma \geq 3$, or $\sigma = 2$ and $d = 2k + 4$. Under the conditions of Case 4 it holds $d \geq \sigma > l$. Construct a graph G as follows: Take a complete bipartite graph with partitions X and Y such that $|X| = d$ and $|Y| = d - l > 0$. Take $lk + 1$ copies of $C(d, \sigma^*)$ and one copy of $C(d, h)$ with $h = dl - \sigma^*(kl + 1)$. Note that $h \equiv d \pmod 2$ and $d \geq h \geq \sigma^*$ due to the definition of l. Connect the vertices of X with the vertices of degree $d-1$ in such a way, that each copy of $C(d,\sigma^*)$ is joined to exactly σ^* vertices of X. By Theorem 3.4 the resulting graph is σ-connected. It is d-regular of order $2d-l+(kl+2)(d+p)$ and has no k-factor, since $\Theta(X,Y,k)=-2$. For a visualization see Figure 3.2.

Chapter 4

Regular factors in regular multipartite graphs

This chapter is concerned with the influence of the chromatic number on the existence of a k-factor in a regular graph. The examination is motivated by the following well known theorem of D. König [25] and the famous theorem of R.L. Brooks [6].

Theorem 4.1 (König [25]) *A bipartite and d-regular graph, with an integer $d \geq 2$, is decomposable into 1-factors.*

Theorem 4.2 (Brooks [6]) *Let G be a connected graph which is neither complete nor a cycle of odd order. Then $\chi(G) \leq \Delta(G)$.*

Obviously, for any d-regular graph it holds $2 \leq \chi(G) \leq d + 1$. Theorem 4.1 shows, that we have complete knowledge about the existence of a k-factor for the lowest possible $\chi(G)$. With Theorem 4.2 we know about the existence of k-factors for $\chi(G) = d + 1$, as every complete graph of order n has a k-factor if nk is even and as a cylce of odd order cannot have a 1-factor. The aim of this chapter is to provide sufficient conditions for the existence of a k-factor in G, if $2 < \chi(G) \leq d$, improving the conditions given in Theorem 1.11. We will begin our examination with a look at the properties of regular graphs with a given chromatic number in Section 4.1. In Section 4.2 we are going to construct d-regular and almost d-regular graphs with given chromatic number which are of smallest order, as determined in Section 4.1. In the last section we present the proof of Theorem 4.5 as well as graphs providing its sharpness. As we have mentioned in the introduction, a graph with chromatic number p is called p-partite. We will take up that notation in this chapter rather than talking of vertex-colourings. If not cited otherwhise, most of the results of this chapter can be found in [17].

The existence of k-factors in p-partite graphs has been examined before from a different point of view. D.G. Hoffman and C.A. Rodger [15] have worked with complete p-partite graphs and derived Erdős-Gallai type conditions for the existence of a k-factor.

Theorem 4.3 (Hoffman, Rodger [15]) *Let $p \geq 2$, $1 \leq v_1 \leq v_2 \leq \ldots \leq v_p$ and $k \geq 1$ be integers. The complete p-partite graph with $|V_i| = v_i$ for each part has a k-factor if and only if*

(i) $k \leq \sum_{i=1}^{p-1} v_i$,

(ii) $k \sum_{i=1}^{p} v_i$ *is even, and*

(iii) $k \sum_{i=s+1}^{p} v_i \leq 2 \sum_{s+1 \leq i < j \leq p} v_i v_j + k \sum_{i=1}^{s} v_i$,

where $s = \max\{t | k \leq \sum_{i=t+1}^{p} v_i\}$.

Theorem 4.3 yields the following corollary for a complete p-equipartite graph O_r^p of order rp.

Corollary 4.4 *Let $p \geq 2$, $r \geq 1$ and $k \geq 1$ be integers with kpr even and $r(p-1) \geq k$. Then O_r^p has a k-factor.*

As the cases $p = 2$ and $p = d + 1$ are already covered, we will concentrate on $3 \leq p \leq d$ for the remainder of the chapter. The next sections will lead to the proof of the following theorem.

Theorem 4.5 *Let d, p, n, k be integers with $n > d > k \geq 1$ and $d + 1 \geq p \geq 2$ such that nd and nk are even. Let $d = s(p-1) + t$ with integers $s \geq 1$ and $0 \leq t \leq p - 2$. A d-regular p-partite graph of order n has a k-factor in the following cases:*

(i) p=2;

or if $p \geq 3$ *and*

(ii) d *and* k *are even;*

(iii) d is even, k is odd and

$$n < \begin{cases} 2(d+s); & s \text{ odd and } t = 0, \\ 2(d+s+2); & s \text{ odd and } t > 0, \\ 2(d+s+1); & s \text{ even and } t < p-2, \\ 2(d+s+3); & s \text{ even and } t = p-2. \end{cases}$$

(iv) d and k are odd and

$$n < 1 + \begin{cases} (k+2)(d+s+1); & \begin{array}{l} s \text{ odd and either } t < p-2 \text{ or} \\ t = p-2 \text{ and } d_s \geq k+2, \end{array} \\ (k+2)(d+s+1) - 2d_s; & s \text{ odd, } t = p-2 \text{ and } d_s < k+2, \\ (k+2)(d+s); & s \text{ even and } d_1 \geq k+2, \\ (k+2)(d+s+2) - 2d_1; & s \text{ even and } 0 \leq d_1 < k+2, \end{cases}$$

$$\text{with } d_1 := \left\lfloor \tfrac{d-(k+2)}{st+t-1} \right\rfloor \text{ and } d_s := \begin{cases} d, & s = 1; \\ \left\lfloor \tfrac{d-(k+2)}{s-1} \right\rfloor, & s > 1. \end{cases}$$

(v) d is odd, k is even and (iv) holds for $k^* = d - k$.

All bounds are sharp.

4.1 Properties of p-partite graphs

As can be seen in Theorem 4.5, the ratio s of d and $p-1$ plays a vital role. To explain this let us take a closer look at p-partite graphs. It is easy to see that for given integers n and p the graph $O^p(n)$ contains the maximal possible number of edges over all p-partite graphs with order n. This has been first recorded by P. Turán [35], whose result on the maximum number of edges in a graph without an induced K_r started the nowadays strong branch of extremal graph theory.

Lemma 4.6 (Turán [35]) *Let G be a p-partite graph with $n(G) = n$. Then*

$$e(G) \leq \; e(O^p(n)) \leq \; \frac{p-1}{2p} n^2. \tag{4.1}$$

We have equality on the left side, if and only if G is isomorphic to $O^p(n)$.

For our proof of Theorem 4.5 we will assume that G has no k-factor. As in the previous chapters, we will be interested in components U of $G-(D\cup S)$ where (D,S) denotes a Tutte-pair of G. Whereas in Chapters 2 and 3 we had been interested in the number of edges connecting to such a component, if its order is known, we will now derive lower bounds for the order of a component U, if we know the number of edges connecting to it.

Lemma 4.7 *For an integer $0 \le l \le d-1$ let U_l be a p-partite graph with $\sum_{x\in V(U_l)} d(x) = d\cdot n(U_l) - l$. Then*

$$n(U_l) \ge w(l) := \frac{p}{2(p-1)}\left(d+\sqrt{d^2-4l\frac{p-1}{p}}\right) \tag{4.2}$$

Proof. We have $2e(U_l) = nd - l$. If we substitute this in (4.1), we get $nd - l \le \frac{p-1}{p}n^2$ or $-l\frac{p}{p-1} \le n^2 - n\frac{pd}{p-1}$. Solving for n yields

$$n \ge \frac{p}{2(p-1)}\left(d+\sqrt{d^2-4l\frac{p-1}{p}}\right). \tag{4.3}$$

□

Lemma 4.8 *$w(l)$ is strictly decreasing in l with*

$$d < w(0)-1 < w(d-1) < w(0).$$

Proof. It is obvious that $w(l)$ decreases in l. Thus $w(d-1) < w(0)$. As $w(0) = \frac{p}{p-1}d = d + \frac{d}{p-1} > d+1$, we only need to show $w(0)-1 < w(d-1)$. It holds

$$\begin{array}{lrcl} & w(0)-w(d-1) & < & 1 \\ \Leftrightarrow & \frac{1}{2}-\frac{p-1}{pd} & < & \frac{1}{2}\sqrt{1-4(d-1)\frac{p-1}{pd^2}} \\ \Leftrightarrow & \frac{(p-1)^2}{p^2d^2}-\frac{p-1}{pd} & < & -(d-1)\frac{p-1}{pd^2} \\ \Leftrightarrow & 0 & < & p(d-1-(d-1))+1. \end{array}$$

□

The graphs U_l of Lemma 4.7 will play an important role in the proof of Theorem 4.5. Although the order of odd components w.r.t. to a Tutte-pair does not necessarily have to be odd, we will focus on components U_l of odd order $|V(U_l)|$ and make the following definition.

Definition 4.9 *For a real number r let $\lceil r \rceil_o$ denote the smallest odd integer greater or equal to r. For integers p, d, l with $d \geq p \geq 3$ and $0 \leq l \leq d-1$ define $n_l(p,d) := \lceil w(l) \rceil_o$. If p and d are known, we write n_l instead of $n_l(p,d)$.*

Obviously, $n(U_l) \geq n_l$ if U_l is of odd order. The next Lemma provides properties of $\lceil w(l) \rceil$ and n_l which will be needed later.

Lemma 4.10 *Let $d \geq p \geq 3$ be integers such that $d = s(p-1)+t$ with $0 \leq t \leq p-2$. Then it holds for a non-negative integer $l \leq d-1$:*

- $\lceil w(d-1) \rceil = d+s = \begin{cases} \lceil w(0) \rceil, & \text{if } t = 0, \\ \lceil w(0) \rceil - 1, & \text{if } t > 0. \end{cases}$,
- *for* $d \equiv s \pmod 2$*:* $n_l = n_0 = d+s+1$*;*
- *for* $d \not\equiv s \pmod 2$*:*
  - *if* $d = s(p-1)$ *even:* $n_{d-1} = n_0$*, and*
  - *in all other cases:* $n_l = d+s = \begin{cases} n_0, & \text{if } l < st + \lceil \frac{t^2}{p} \rceil, \\ n_0 - 2, & \text{if } l \geq st + \lceil \frac{t^2}{p} . \rceil \end{cases}$

    It holds $st + \lceil \frac{t^2}{p} \rceil \leq d-2$.

Proof. We have $w(0) = \frac{p}{p-1} d = d + s + \frac{t}{p-1}$. With Lemma 4.8 it follows $d+s+\frac{t}{p-1} > w(d-1) > d+s-1$, giving us the first statement of the lemma. The second statement is an immediate consequence of the first. So let now $d \not\equiv s \pmod 2$. The case $t = 0$ can only appear for d even and the statement follows directly. If $t > 0$, we get $n_0 = d+s+2$ and $n_{d-1} = d+s$. By Lemma 4.8 there must exist an $0 \leq l_0 \leq d-1$ such that $n_{l_0 - 1} = d+s+2$ and $n_{l_0} = d+s$. So for which l does $w(l) > d+s$ hold? This question is equivalent to $\sqrt{d^2 - 4l\frac{p-1}{p}} > 2(d+s)\frac{p-1}{p} - d$. Rearranging yields $l < d(d+s) - \frac{p-1}{p}(d+s)^2 = st + \frac{t^2}{p}$. Since $st + \frac{t^2}{p} < st + t - 1 \leq d - s - 1$, it follows $st + \left\lceil \frac{t^2}{p} \right\rceil \leq d-2$. □

4.2 Special p-partite graphs

In the previous section we have merely calculated lower bounds for the order of graphs U_l without caring for their existence. The construction of graphs U_l of odd order $\geq n_l$ is the content of this section. We distinguish two cases based on the parity of d.

4.2.1 Regular p-partite graphs of even degree d

Throughout this section let d be even. As we have seen in Theorem 1.11, a graph consisting of two components of odd order yields the smallest order for a d-regular graph without a k-factor, where k is odd. As the chromatic number and the connectivity of a graph do not influence one another, it seems plausible that a disconnected graph will again yield the smallest possible order. Our aim of this section is to construct d-regular p-partite graphs U_0 of minimal odd order. From Lemma 4.7 we know $n(U_0) \geq n_0(p, d)$. Let $d = s(p-1) + t$ with $t \leq p-2$. Lemma 4.10 tells us

$$n_0 = \begin{cases} d+s+1, & \text{if } s \text{ is even;} \\ d+s, & \text{if } s \text{ is odd and } t = 0; \\ d+s+2, & \text{if } s \text{ is odd with } t > 0. \end{cases} \tag{4.4}$$

The bound in Lemma 4.7 was derived via the p-equipartite graph $O^p(n)$. So it is a plausible assumption that we can construct the graphs U_0 p-equipartite. We will look at different cases, depending on s. It should be pointed out that the graph U_0 will differ from case to case.

Case 1: s even. We start out with the graph $O^p(n_0) = O^p(d+s+1)$. With (4.4) we have $n_0 = sp + (t+1)$ and get

$$d(x, O^p(n_0)) = \begin{cases} d, & x \in V_1 \cup \ldots \cup V_{t+1}; \\ d+1, & x \in V_{t+2} \cup \ldots \cup V_p. \end{cases}$$

All vertices in the first $t+1$ parts are of the desired degree d.

Case 1.A: $t < p-2$. Let $H := G[V_{t+2}, \ldots, V_p] \cong O_s^{p-t-1}$. Corollary 4.4 ensures the existence of a 1-factor F in H. Then $U_0 = O^p(n_0) - E(F)$ is as desired.

Case 1.B: $t = p-2$. It is easy to check that n_0 vertices do not suffice. So let us start out with $O^p(n_0+2)$. Since $n_0 + 2 = (s+1)p + 1$, we have

$$d(x, O^p(n_0+2)) = \begin{cases} d+1, & x \in V_1; \\ d+2, & x \in V_2 \cup \ldots \cup V_p. \end{cases}$$

We now find the following pathcovering $\mathbf{P^1} = \{P_1, \ldots, P_{s/2}, P^*\}$ in $O^p(n_0+2)$, illustrated in Figure 4.1:

$$\begin{aligned} P_i &: x^1_{2i-1}x^2_{2i-1}\ldots x^p_{2i-1}x^{p-1}_{2i}x^p_{2i}x^{p-2}_{2i}\ldots x^1_{2i}, \quad 1 \le i \le \frac{s}{2} \\ P^* &: x^1_{s+1}\ldots x^p_{s+1}x^1_{s+2} \end{aligned}$$

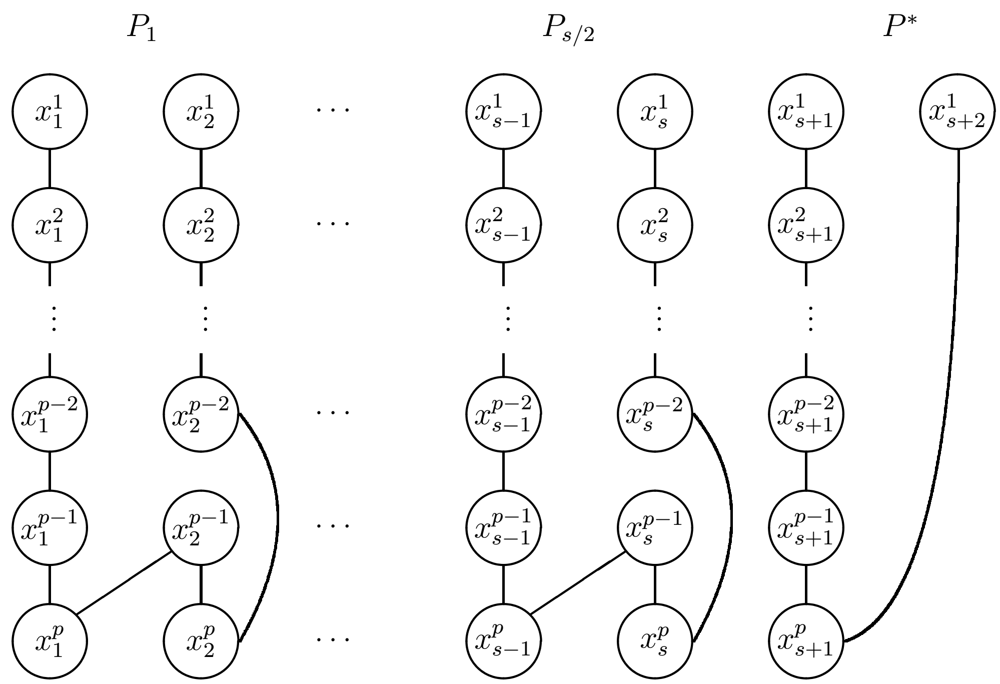

Figure 4.1: The pathcovering $\mathbf{P^1}$

With $U_0 = O^p(n_0+2) - E(\mathbf{P^1})$ we have a d-regular p-partite graph of odd order $n_0 + 2$.

Case 2: s odd.
Case 2.A: $t = 0$. Then $n_0 = d + s = sp$. Consequently $O^p(n_0) = O^p_s$ and is as such d-regular.

In the remaining cases we have $t \not\equiv p \pmod 2$ and thus $1 \le t \le p-3$. We start again with $O^p(n_0)$ and observe

$$d(x, O^p(n_0)) = \begin{cases} d+1, & x \in V_1 \cup \ldots \cup V_{t+2}; \\ d+2, & x \in V_{t+3} \cup \ldots \cup V_p. \end{cases}$$

Case 2.B: $p \geq 5$ and $t \leq p-5$. Let $H_1 = G[V_1, \ldots, V_{t+2}]$ and $H_2 = G[V_{t+3}, \ldots, V_p]$. We observe that $H_1 \cong O_{s+1}^{t+2}$ and $H_2 \cong O_s^{p-t-2}$. With Corollary 4.4 there exists a 1-factor F_1 in H_1 and a 2-factor F_2 in H_2. Thus let $U_0 = O^p(n_0) - E(F_1) - E(F_2)$.

Case 2.C: $p \geq 5$ and $t = p-3$. We have one part, V_p, of cardinality s and all remaining ones of cardinality $s+1$. Let $H_1 = G[V_1, \ldots, V_{p-3}]$ and $H_2 = G[V_{p-2},\ V_{p-1},\ V_p]$. Then $H_1 \cong O_{s+1}^{p-3}$ has a 1-factor F_1. In H_2 we find the pathcovering $\mathbf{P^2} = \{P_0, \ldots, P_{s+1}\}$, illustrated in Figure 4.2:

$$\begin{array}{lcl} P_i & : & x_i^{p-2} x_i^p x_i^{p-1}, \quad 1 \leq i \leq s \\ P_{s+1} & : & x_{s+1}^{p-1} x_{s+1}^{p-2}. \end{array}$$

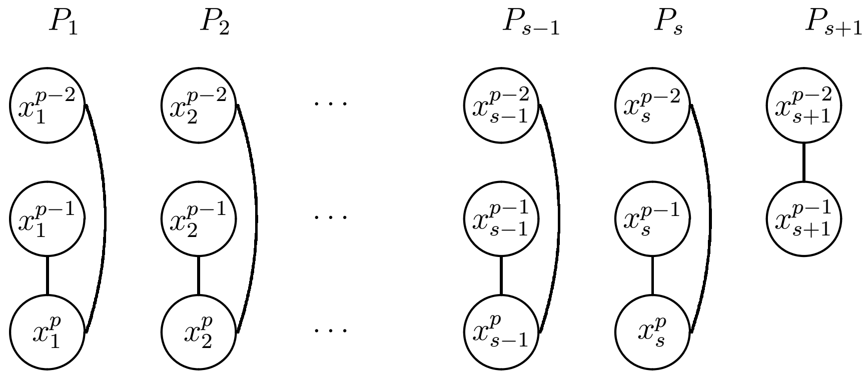

Figure 4.2: The pathcovering $\mathbf{P^2}$

With $U_0 = O^p(n_0) - E(F_1) - E(\mathbf{P^2})$ we realize the desired d-regular p-partite graph.

Case 2.D: $p = 4$. In this case we have $t = p - 3 = 1$ and $n_0 = 4s + 3$. Now $O^p(n_0)$ has the following pathcovering
$\mathbf{P^3} = \{P_0^1, \ldots, P_{(s-1)/2}^1, P_1^2, \ldots, P_{(s-1)/2}^2, P_1^3, \ldots, P_{(s-1)/2}^3, P^4, P^5\}$, illustrated in Figure 4.3:

$$\begin{array}{lll} P_i^1: \ x_{2i+1}^2 x_{2i+1}^4 x_{2i+1}^3, & 0 \leq i \leq \frac{s-1}{2}; & P^4: \ x_s^1 x_{s+1}^3; \\ P_i^2: \ x_{2i-1}^1 x_{2i}^4 x_{2i}^3, & 1 \leq i \leq \frac{s-1}{2}; & P^5: \ x_{s+1}^1 x_{s+1}^2. \\ P_i^3: \ x_{2i}^1 x_{2i}^2, & 1 \leq i \leq \frac{s-1}{2}; & \end{array}$$

Deleting the edges of $\mathbf{P^3}$ in $O^p(n_0)$ leads to the graph U_0.

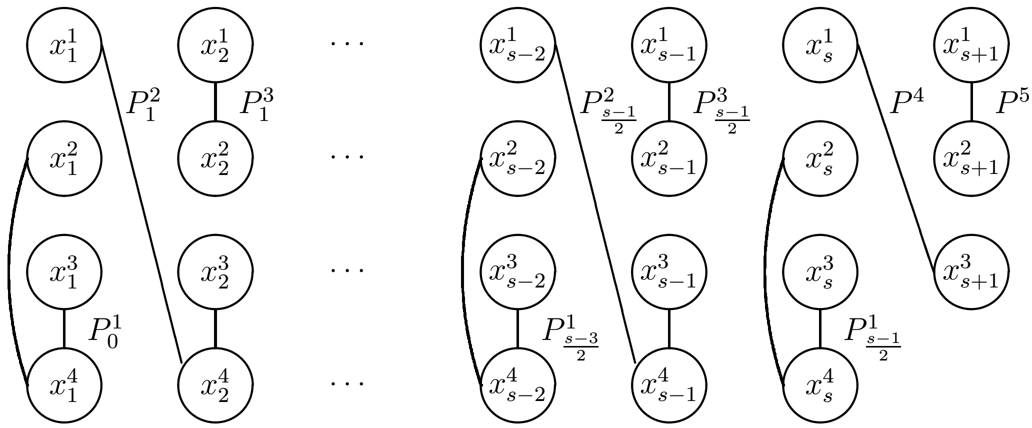

Figure 4.3: The pathcovering $\mathbf{P^3}$

This completes our discussion of the case d even. We summarize that it has always been possible to construct a d-regular p-partite graph of odd order with n_0 vertices except for the case $d = s(p-1) + p - 2$ with s even, where $n_0 + 2$ vertices are needed.

4.2.2 p-partite graphs of odd maximum degree d

In this section we are going to construct graphs U_l with $1 \leq l \leq d-2$ odd. U_l will be a p-partite graph of minimal odd order $\geq n_l$ with l vertices of degree $d-1$ and all remaining ones of degree d. We further require that there exists at least one part V_i in U_l such that $d(x, U_l) = d$ for all $x \in U_l$. This imposes no real restriction but becomes important when constructing d-regular p-partite graphs of minimum order to show the sharpness of Theorem 4.5. The graphs U_l will always be constructed in such a way that for every $x \in V_p$ it holds $d(x, U_l) = d$.
We need the following lemma on the existence of matchings in a graph. Part a) of the lemma is a simple corollary of G.A. Diracs famous theorem on the existence of a hamiltonian circuit in a graph.

Lemma 4.11 *Let G be a graph of order n and minimum degree δ.*

a) **(Dirac [10])** *If $n \leq 2\delta$, then G has a matching of cardinality $\geq \lfloor n/2 \rfloor$.*

b) **(Erdős, Pósa [12])** *If $n \geq 2\delta$, then G has a matching of cardinality $\geq \delta$.*

Let us turn to the construction of U_l. From Lemma 4.10 we know

$$n_0 = \begin{cases} d+s+1, & s \text{ odd;} \\ d+s+2, & s \text{ even.} \end{cases}$$

We again distinguish two cases depending on s. Note that always $2s \leq d$ holds since $p \geq 3$.

Case 1: s odd. Since $d = s(p-1)+t$, we have $t \equiv p \pmod 2$. We start with $O^p(n_0)$ and have, as $n_0 = d+s+1$,

$$d(x, O^p(n_0)) = \begin{cases} d, & x \in V_1 \cup \ldots \cup V_{t+1}; \\ d+1, & x \in V_{t+2} \cup \ldots \cup V_p. \end{cases}$$

Case 1.A: $t < p-2$. Let $H := G[V_{t+2}, \ldots, V_p] \cong O_s^{p-t-1}$. Then H has the connected 2-factor $x_1^{t+2}x_1^{t+3} \ldots x_1^p x_2^{t+2} \ldots x_2^p x_3^{t+2} x_s^p x_1^{t+2}$. This factor has a pathcovering consisting of a path of length 2, without loss of generality $x_{s-1}^p x_s^{t+2} x_s^{t+3}$, and $\frac{n(H)-3}{2}$ paths of length 1. Deleting the edges of this pathcovering in $O^p(n_0)$ leads to the graph U_1. If $l > 1$, consider the subgraph $U^* := U_1 - \{x_s^{t+2}\} - V_p$. With Lemma 4.11 we find a matching M of cardinality $\frac{d-1}{2}$ in U^*. Deleting $\frac{l-1}{2}$ edges of M in U_1 leads to U_l.

Case 1.B: $t = p-2$ and $l < s$. We need at least n_0+2 vertices, so let us start with $O^p(n_0+2)$. It holds

$$d(x, O^p(n_0+2)) = \begin{cases} d+1, & x \in V_1; \\ d+2, & x \in V_2 \cup \ldots \cup V_p. \end{cases}$$

We can find the pathcovering $\mathbf{P^1} = \{P_1, \ldots, P_{(s-1)/2}, P^*\}$ in $O^p(n_0+2)$, illustrated in Figure 4.4:

$$\begin{aligned} P_i &: x_{2i-1}^1 x_{2i-1}^2 \ldots x_{2i-1}^p x_{2i}^{p-1} x_{2i}^p x_{2i}^{p-2} \ldots x_{2i}^1, \quad 1 \leq i \leq \frac{s-1}{2} \\ P^* &: x_s^1 x_s^2 \ldots x_s^p x_{s+1}^{p-1} x_{s+1}^p x_{s+2}^1 x_{s+1}^{p-2} \ldots x_{s+1}^1. \end{aligned}$$

Then $U_1 := O^p(n_0+2) - E(\mathbf{P^1})$. The union

$$\bigcup_{i \in \{1,3,\ldots,s\}} \{x_i^j x_{i+1}^{j+1} \;:\; 1 \leq j \leq p-2\} \cup \{x_i^{p-1} x_{i+1}^1\}$$

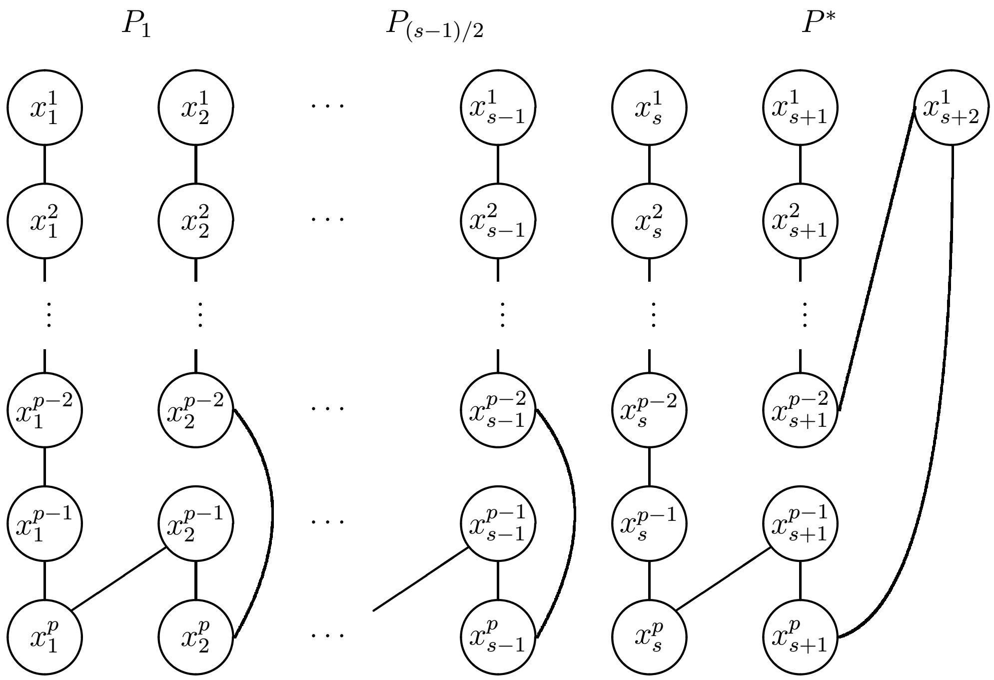

Figure 4.4: The pathcovering $\mathbf{P^1}$

denotes a matching M, with $|M| = \frac{d+1}{2}$, in U_1. Deleting $\frac{l-1}{2}$ edges of M in U_1 leads to U_l.

Case 1.C: $t = p - 2$ and $l \geq s$. Here we start with $O^p(n_0)$ and are able to find a matching M of cardinality $|V_p| = s$ in $G[V_{p-1}, V_p]$. Deleting the edges of M leads to U_s. If $p \geq 4$, then $H := G[V_1, \ldots, V_{p-2}] \cong O_{s+1}^{p-2}$ and we find a matching M_1 of cardinality $\frac{d-s}{2}$ in H. Deleting $\frac{l-s}{2}$ edges of M_1 in U_s leads to U_l.

If $p = 3$, we cannot construct U_l with $l > s + 1$ in such a way that there exists one part V_i with all vertices of degree d. However, U_l with $l > s$ can be constructed on n_0 vertices analogously as above, if we drop this extra condition.

Case 2: s even. In this case t is odd and $n_0 = d + s + 2$. We start again with $O^p(n_0)$ and get for the degrees:

$$d(x, O^p(n_0)) = \begin{cases} d+1, & x \in V_1 \cup \ldots \cup V_{t+2}; \\ d+2, & x \in V_{t+3} \cup \ldots \cup V_p. \end{cases}$$

Case 2.A: $t = p - 2$. In this case, $O^p(n_0)$ is $(d+1)$-regular and p is odd. Let $H_1 := G[\{x_j^i \mid 1 \le i \le p,\ 1 \le j \le s\}]$ be the graph induced by the first s vertices of each part V_i. Since $H_1 \cong O_s^p$ is of even order, there exists a 1-factor F_1 in H_1. Let $H_2 := G[\{x_{s+1}^1, \ldots, x_{s+1}^p\}]$. Since p is odd, we find a factor F_2 in H_2 consisting of a path of length 2 and $\frac{p-3}{2}$ paths of length 1. Then $U_1 := O^p(n_0) - E(F_1) - E(F_2)$ is as desired.
Case 2.B: $t \le p - 3$ and p odd. Since p is odd, $p - (t+2)$ is even. We look at the induced subgraphs H_1 and H_2 of $O^p(n_0)$ defined as $H_1 := G[V_1, \ldots, V_{t+2}] \cong O_{s+1}^{t+2}$ and $H_2 := G[V_{t+3}, \ldots, V_p] \cong O_s^{p-t-2}$. It holds $t < p-3$ since t and p are odd. In H_2 we find a 2-factor F_2. Analogous to Case 2.A we find a factor F consisting of a path of length 2 and $\frac{n(H_1)-3}{2}$ independent edges in H_1. Again $U_1 := O^p(n_0) - E(F_2) - E(F)$ is exactly as needed.
Case 2.C: $t \le p - 3$ and p even. We define the following subgraphs of $O^p(n_0)$:

$$\begin{aligned} H_1 &:= G[\{x_j^i \mid 1 \le j \le s,\ 1 \le i \le t+2\}], \\ H_2 &:= G[V_{t+3}, \ldots, V_p], \\ P &:= G[\{x_{s+1}^1, \ldots, x_{s+1}^{t+2}\}]. \end{aligned}$$

Now $n(H_1)$ and $n(H_2)$ are even and we find a 1-factor F_1 in H_1 and a 2-factor F_2 in H_2. In P we find a factor F_3 consisting of a path of length 2 and $\frac{t-1}{2}$ independent edges. Now $U_1 := O^p(n_0) - E(F_1) - E(F_2) - E(F_3)$ is as desired.

In all three cases 2.A, 2.B and 2.C let without loss of generality x_{s+1}^1 be the vertex of degree $d-1$ in U_1. Define $U^* := U_1 - \{x_{s+1}^1\} - V_p$. With Lemma 4.11 U^* always has a matching M of cardinality $\frac{d-1}{2}$. Deleting $\frac{l-1}{2}$ edges of M in U_1 leads to U_l.

We have thus constructed components U_l of order at least n_0 for all cases of s and t. However, we have neglected Lemma 4.10. For s even and $l \ge ts + \lceil \frac{t^2}{p} \rceil_o$ a construction of U_l could be possible with $n_0 - 2$ vertices, so far. Let us see, when this is possible. For the graph $O^p(n_0 - 2)$ we have

$$d(x, O^p(n_0 - 2)) = \begin{cases} d-1, & x \in V_1 \cup \ldots \cup V_t \\ d, & x \in V_{t+1} \cup \ldots \cup V_p. \end{cases}$$

So $O^p(n_0 - 2)$ already contains $t(s+1)$ vertices of degree $d-1$. Thus, the existence of a graph U_l with order $n_0 - 2$ can only be expected for $l \ge t(s+1) \ge st + \left\lceil \frac{t^2}{p} \right\rceil_o$. This motivates the following definition.

Definition 4.12 *For d odd, $d = s(p-1)+t$ with s even, define $l_0 := (s+1)t$.*

This paragraph will show, that for $l \geq l_0$ the construction of graphs U_l is possible. We will not look at all possible cases and rather concentrate on $l \geq l_0$ for $t < p-3$ and only at $l = l_0$, if $t = p-2$, as these graphs will be needed in the proof of Theorem 4.5. Our goal is to delete a matching of cardinality $\zeta := (l - t(s+1))/2$ in the induced subgraph $G[V_{t+1}, \ldots, V_p]$ of $O^p(n_0 - 2)$. For $t \leq p-3$ let $H_2 := G[V_{t+1}, \ldots, V_{p-1}]$. Then $H_2 \cong O_s^{p-t-1}$ is of even order and contains a 1-factor consisting of $s(p-t-1)/2$ edges. Since $l \leq d-2$, we have $\zeta < s(p-t-1)/2$ and can delete ζ edges from the 1-factor to obtain our desired graph U_l. For $t = p-2$ there remains nothing to be done since then $l = l_0 = st + t$ and thus $\zeta = 0$.

4.3 Sufficient conditions for k-factors

We proceed in proving sufficient conditions for the existence of a k-factor in a regular p-partite graph via two lemmata.

Lemma 4.13 *Let $d \geq 4$ be even, $d+1 \geq p \geq 3$ and G a d-regular p-partite graph. If k is odd, $n(G)$ even and G does not have a k-factor, then*

$$n(G) \geq \begin{cases} 2\,n_0; & s \text{ odd, or } s \text{ even and } t < p-1, \\ 2(n_0+2); & s \text{ even and } t = p-2. \end{cases}$$

with $n_0 = \left\lceil \frac{p}{p-1} d \right\rceil_o$ as in Definition 4.9.

Proof. Since G has a k-factor if and only if it has a $(d-k)$-factor, let $2k \leq d$. We first consider the case that G is connected. As d is even, G is at least 2-edge-connected. Since G does not have a k-factor there exists a Tutte-pair (D, S) by Theorem 1.2. Let $W := G - (D \cup S)$. By the definition of odd components w.r.t. (D, S), it follows that $e_G(V(U), D \cup S)$ is even. We call an odd component U w.r.t. (D, S) a big component, if

$$e_G(V(U), D \cup S) \geq d - k + 1. \tag{4.5}$$

Let b denote the number of big components. It follows

$$e_G(D \cup S, V(W)) \geq (d-k+1)b + 2(q-b) = 2q + b(d-k-1). \tag{4.6}$$

As in the proof of Lemma 1.4 we have

$$2e_G(D,S) \leq d(|D|+|S|) - e_G(D \cup S, V(W)).$$

With (1.1) and (4.6) this resolves to

$$(d-2k)(|D|-|S|) \geq b(d-k-1)+2.$$

This implies $d > 2k$ and hence we obtain $|D|-|S| \geq b+1$, leading with (1.1) to

$$q \geq k(|D|-|S|)+2 \geq k(b+1)+2. \tag{4.7}$$

By Lemma 4.7

$$|V(W)| \geq (q-b)\lceil w(d-k)\rceil + b.$$

From Lemma 4.10 we get

$$n_0 - \lceil w(d-k)\rceil \leq \begin{cases} 2, & \text{if } d < s(p-1)+(p-2), \\ 1, & \text{if } d = s(p-1)+(p-2), \end{cases}$$

since in the second case s is even and thus $n_0 = d+s+1$. Thus in the first case

$$|V(W)| \geq (q-b)(n_0-2)+b, \tag{4.8}$$

which yields

$$\begin{aligned} n(G) &= |D|+|S|+|V(W)| \\ &\overset{(4.8)}{\geq} b+1+(q-b)(n_0-2)+b \\ &\overset{(4.7)}{\geq} 2b+1+(b(k-1)+k+2)(n_0-2) \\ &\geq 1+(k+2)(n_0-2) \\ &\overset{n_0 \geq 5}{\geq} 2n_0. \end{aligned} \tag{4.9}$$

If $d = s(p-1)+(p-2)$, the proof follows the same pattern with (n_0-1) in place of (n_0-2) in (4.8). Since this case can only occur for $d \geq 8$, as s and p have to be even, we can use $n_0 \geq 9$ to get to the desired bound $n(G) \geq 2(n_0+2)$ in (4.9).
It remains the case that G is disconnected. Then G has at least two components G_1 and G_2 such that $|V(G_1)| \equiv |V(G_2)| \pmod 2$. If $|V(G_1)|$ is odd, then G_1 and G_2 each have at least n_0 vertices for $d < s(p-1)+p-2$ or n_0+2 vertices for $d = s(p-1)+p-2$ as the last section showed. If $|V(G_1)|$ is even, our result follows from the connected case for G_1. □

Lemma 4.14 *Let $d \geq 5$ be odd, $d+1 \geq p \geq 3$, and G be a d-regular p-partite graph. Let $d = s(p-1)+t$ with $s \geq 0$ and $0 \leq t \leq p-2$. If G does not have a k-factor for odd $k \geq 1$, then*

$$n \geq 1 + \begin{cases} (k+2)n_0; & s \text{ odd and either } t < p-2 \text{ or } \\ & t = p-2 \text{ and } d_s \geq k+2, \\ (k+2)(n_0+2) - 2d_s; & s \text{ odd, } t = p-2 \text{ and } d_s < k+2, \\ (k+2)(n_0-2); & s \text{ even and } d_1 \geq k+2, \\ (k+2)n_0 - 2d_1; & s \text{ even and } 0 \leq d_1 < k+2, \end{cases}$$

with $d_1 := \left\lfloor \frac{d-(k+2)}{st+t-1} \right\rfloor$ *and* $d_s := \begin{cases} d, & s = 1; \\ \left\lfloor \frac{d-(k+2)}{s-1} \right\rfloor, & s > 1. \end{cases}$

Proof. We can assume without loss of generality that G is connected, otherwise we look at one component of G that does not have a k-factor. From Theorem 1.2 we get the existence of a Tutte-pair (D, S) such that

$$k|D| + (d-k)|S| - q + 2 \leq e_G(D, S). \tag{4.10}$$

We call an odd component U w.r.t. (D, S) of $W := G - (X \cup Y)$ an

- A-component, if $e_G(V(U), D \cup S) \leq d-2$ odd,
- B_1-component, if $e_G(V(U), D \cup S) \leq d-1$ even,
- B_2-component, if $e_G(V(U), D \cup S) \geq d+1$ even.

Let a, b_1, b_2 be the number of A-, B_1- and B_2-components, respectively. If a component is a B_1- or a B_2-component, we simply call it a B-component and set $b := b_1 + b_2$.
This leads to the following bound:

$$\begin{aligned} e_G(D \cup S, V(W)) &\geq (d+1)b_2 + 2b_1 + a + (q-b-a)d \\ &= b_2 + qd - a(d-1) - b_1(d-2). \end{aligned} \tag{4.11}$$

As already shown in the proof of Lemma 1.4, every odd component U w.r.t. (D, S) satisfies

$$\begin{aligned} e_G(D, V(U)) &= d|V(U)| - e_G(S, V(U)) - 2|E(U)| \\ &\equiv k|V(U)| + e_G(S, V(U)) - 2|E(U)| \equiv 1 \pmod{2}. \end{aligned}$$

This implies $e_G(D, V(U)) \geq 1$ for each odd component U w.r.t. (D, S). Since $e_G(S, V(U)) + k|V(U)| \equiv 1 \pmod 2$ for every B-component U, it holds $e_G(S, V(U)) \geq 1$. This leads to

$$e_G(D, S) \leq d|D| - q,$$

as well as

$$e_G(D, S) \leq d|S| - b. \tag{4.12}$$

We further know from Lemma 1.4, that $|D| > |S|$. Inequality (4.12) together with (4.10) implies

$$q \geq b + 2 + k(|D| - |S|). \tag{4.13}$$

As in the proof of Lemma 1.4, we have

$$2e_G(D, S) \leq d(|D| + |S|) - e_G(D \cup S, V(W)). \tag{4.14}$$

With (4.14) and (4.10) we obtain

$$\begin{aligned}
& d(|D| + |S|) - e_G(D \cup S, V(W)) \geq 2k|D| + 2(d-k)|S| - 2q + 4 \\
\Rightarrow\quad & (d - 2k)(|D| - |S|) \geq e_G(D \cup S, V(W)) - 2q + 4 \\
\overset{(4.11)}{\Rightarrow}\quad & (d - 2k)(|D| - |S|) \geq b_2 + q(d-2) - a(d-1) - b_1(d-2) + 4 \\
\overset{(4.13)}{\Rightarrow}\quad & d(1-k)(|D| - |S|) \geq (d-1)b_2 + 2d - a(d-1) \\
\Rightarrow\quad & a \geq b_2 + \frac{d}{d-1}(2 + (k-1)(|D| - |S|)).
\end{aligned} \tag{4.15}$$

For the number of vertices of W we get

$$\begin{aligned}
|V(W)| &\geq a\lceil w(d-2)\rceil + b_1\lceil w(d-1)\rceil + b_2 + q \\
&\geq a\lceil w(d-2)\rceil + b_1(\lceil w(d-2)\rceil - 1) + b_2 + q
\end{aligned} \tag{4.16}$$

Case 1: s odd and $d \leq s(p-1) + p - 3$. Because of $|D| - |S| \geq 1$, inequality (4.15) implies $a \geq k+2$. With Lemma 4.10 we have $n_0 = \lceil w(d-2)\rceil + 1$ and hence we get

$$\begin{aligned}
n(G) &= |D| + |S| + |V(W)| \\
&\overset{(4.16)}{\geq} 1 + a(n_0 - 1) + b_1(n_0 - 2) + b_2 + q \\
&\overset{(4.13)}{\geq} 1 + (k+2)(n_0 - 1) + k + 2 \\
&= 1 + (k+2)n_0.
\end{aligned}$$

Case 2: s odd and $d = s(p-1)+p-2$. In this case we have to take greater care since the last section showed that construction of U_l with $l < s$ is not possible on n_0 vertices, whereas the construction of U_l with $l \geq s$ is possible.
Case 2.A: $d_s \geq k+2$. In this case we can proceed analogously to Case 1 and arrive at

$$n(G) \geq 1 + (k+2)n_0.$$

Case 2.B: $d_s < k+2$. Let $\alpha := |D| - |S| \geq 1$ and $\varepsilon := q - (b+2+k\alpha) \geq 0$. Using this together with (4.14) and (4.10) results in

$$a \geq \underbrace{b_2 + \alpha(k-1) + 2 + \frac{\varepsilon(d-2)+\alpha(k-1)+2}{d-1}}_{:=\zeta}.$$

Define $\eta := a - \zeta$. We observe that η is a non-negative real number. With Lemma 4.10 we have $n_0 = \lceil w(d-2) \rceil + 1 \geq d$ and get

$$\begin{aligned} n(G) &= |D| + |S| + |V(W)| \\ &\geq 2|S| + (n_0-1)(a+b_1) + 2b_2 + 2 + \varepsilon + \alpha(k+1) \\ &\geq (n_0-1)\left[b + (\alpha-1)(k-1) - 1 + \frac{\varepsilon(d-2)+2+\alpha(k-1)}{d-1} + \eta\right] \\ &\quad +\varepsilon + (k+1)(\alpha-3) + 2(b_2+d_1+|Y|-1) \\ &\quad +[1+(k+2)n_0+2(k+2-d_s)] \\ &\geq (d-1)[b+(\alpha-1)(k-1)+\eta+\varepsilon-1] + k(2\alpha-3) - 1 \\ &\quad +2(b_2+d_s+|S|-1) + [1+(k+2)n_0+2(k+2-d_s)]. \end{aligned}$$

Consequently, $n(G) \geq 1+(k+2)(n_0+2)-2d_s = 1+(k+2)n_0+2(k+2-d_s)$ holds, if $LHS \geq 0$ where LHS is defined as
$LHS := (d-1)[b+(\alpha-1)(k-1)+\eta+\varepsilon-1]+k(2\alpha-3)-1+2(b_2+d_s+|S|-1)$.
We differ between some more cases. We will be able to show either $LHS \geq 0$ or $e_G(D \cup S, \mathcal{A}) \leq d$. Here $\mathcal{A}$ denotes the vertex-set of the subgraph of G induced by all A-components of W. In the latter case $e_G(D \cup S, V(U)) \geq s$ can hold for at most d_s odd components U. Thus the remaining $q - d_s$ odd components have at least $n_0 + 2$ vertices. This will give us $|V(W)| \geq d_s n_0 + (q-d_s)(n_0+2) \geq (k+2)(n_0+2) - 2d_s$ and finally

$$n(G) \geq 1 + (k+2)(n_0+2) - 2\, d_s. \tag{4.17}$$

We keep in mind that $\alpha \geq 1$ and $\varepsilon,\ \eta \geq 0$.
Case 2.B.1: $\alpha = 1$. Then

$$LHS = (d-1)[b+\varepsilon+\eta-1] - k - 1 + 2(b_1 + d_s + |S| - 1).$$

If $|S| = 0$, then $|D| = 1$ and thus $e_G(D, V(W)) \leq d$. For $|S| \geq 1$

$$\begin{aligned} LHS &\geq (d-1)[b+\varepsilon+\eta-1] - k - 1 + 2(b_1 + d_s) \\ &\geq (d-1)[b+\varepsilon+\eta-1] - (k+1) \\ &\geq 0, \end{aligned}$$

if $b + \varepsilon + \eta \geq 2$.
Case 2.B.1.1: $1 \leq \eta < 2,\ b+\varepsilon = 0$. We automatically get the contradiction $q = k+2 \geq a \geq k+3$. Thus this case is not possible.
Case 2.B.1.2: $0 \leq \eta < 1,\ b+\varepsilon \leq 1$. If $b = \varepsilon = 0$, then $q = a = k+2$ and with (4.10) we get $d|S| \leq e_G(D, S) \leq d|S|$. This yields

$$e_G(D, V(W)) = e_G(D \cup S, V(W)) \leq d(|S|+1) - d|S| = d$$

and we are done.
If $b + \varepsilon = 1$, then $q = k + 3$ and $e_G(D, S) \in \{d|S| - 1, d|S|\}$. If $e_G(D, S) = d|S|$ we are done as above. With $e_G(D, S) = d|S| - 1$ we get $e_G(D \cup S, V(W)) \leq d + 2$. If now $b = 1$ and U is the only B-component of W, we get $e_G(S, U) \geq 1$, $e_G(D, U) \geq 1$ and $e_G(D, \mathcal{A}) \leq d$ and are done.
For $b = 0$ we have $e_G(D, V(U_i)) \geq 1$ and $e_G(S, V(U_i)) = 0$ for all A-components U_i, otherwise contradicting $e_G(D, S) = d|S| - 1$. Thus we have $e_G(D \cup S, \mathcal{A}) = e_G(D, \mathcal{A}) \leq d + 1$. We are done as $q = k + 3$.
Case 2.B.2: $\alpha \geq 2,\ k \geq 3$. In this case $LHS \geq (d-1)(k-2) + k - 3 = d(k-2) - 1 \geq 0$ holds.
Case 2.B.3: $\alpha \geq 2, k = 1$ and $|S| \geq 1$. We get

$$LHS \geq (d-1)[b+\eta+\varepsilon-1] \geq 0,$$

if $b+\eta+\varepsilon \geq 1$. For $b = \varepsilon = 0$ and $0 \leq \eta < 1$ we have $q = 2+\alpha$. With (4.10) we get $d|S| \leq e_G(D, S) \leq d|S|$. As above this yields $e_G(D \cup S, V(W)) \leq d$.
Case 2.B.4: $\alpha \geq 2,\ k = 1$ and $|S| = 0$. Then

$$LHS \geq (d-1)[b+\eta+\varepsilon-1] + 2b_2 + 2d_s - 2 \geq 0$$

holds, if either

i) $b+\eta+\varepsilon \geq 2$ or

ii) $b+\eta+\varepsilon \geq 1$ and $b_2+d_s \geq 1$.

Since $d_s = 0$ cannot occur for $k = 1$, the only cases left are $b+\varepsilon \leq 1$ and $0 \leq \eta < 1$. Here we get

$$q = \begin{cases} 2+|D|, & \text{if } b+\varepsilon = 0 \\ 3+|D|, & \text{if } b+\varepsilon = 1 \end{cases},$$

$$a = \begin{cases} 3, & \text{if } (b=0 \text{ and } \varepsilon=0) \text{ or } (b_1=1 \text{ and } \varepsilon=0) \\ 4, & \text{if } (b=0 \text{ and } \varepsilon=1) \text{ or } (b_2=1 \text{ and } \varepsilon=0) \end{cases}$$

and

$$|D|+|V(W)| \geq 2|D|+3n_s-1 \geq 1+3(n^*-2)+6 = 1+(k+2)n^*,$$

if $|D| \geq 4$. For $1 \leq |D| \leq 3$ and $b+\varepsilon \leq 1$ we always have $e_G(D,\mathcal{A}) \leq d$.
Case 3: s even and $d_1 \geq k+2$. We still have $a \geq k+2$. Since s is even, $3 \leq l_0 \leq d-2$ and Lemma 4.10 gives us $n_0 = n_{d-2}+2$. Analogous to Case 1 we arrive at

$$n(G) \geq 1+(k+2)(n_0-2).$$

Case 4: s even and $0 \leq d_1 \leq k+1$. In this case we can proceed analogously to Case 2. Just look at n_{l_0} instead of n_s and d_1 instead of d_s. Again $n_0 = n_{d-2}$ and we get

$$n(G) \geq 1+(k+2)n_0-2d_1.$$

This completes the proof of the lemma. □

Proof of Theorem 4.5.

Let G be a d-regular p-partite graph with order n. As mentioned above, the bipartite case follows from Theorem 4.1. Let $p \geq 3$.
Case 1: d even. If k is even, then Theorem 1.6 ensures the existence of a k-factor. If k is odd and G has a k-factor, Lemma 4.13 and Lemma 4.10 yield

$$n(G) < \begin{cases} 2(d+s); & s \text{ odd and } t=0, \\ 2(d+s+2); & s \text{ odd and } t>0, \\ 2(d+s+1); & s \text{ even and } t<p-2, \\ 2(d+s+3); & s \text{ even and } t=p-2. \end{cases}$$

On the other hand we can construct a d-regular p-partite graph $G(p,d)$ consisting of two components U_0 as defined in Paragraph 4.2.1. These are of odd order each and thus $G(p,d)$ cannot have a k-factor. Then $n(G(p,d)) = 2n(U_0) = 2n_0$, or in the fourth case $n(G(p,d)) = 2(n_0+2)$, sharpening our bound.

Case 2: d odd. Let us first consider k odd. For $d=3$ we have $k=1$. Since $p=4$ is only possible for the complete graph K_d, which has a 1-factor, it remains $p=d=3$. Then $s=1$, $t=1$ and $d_s=d$. It follows from Theorem 1.7 and our graphs U_1 that $n(G) < 1+(k+2)n_0$, with sharpness provided by the graph of Sylvester in Figure 1.1.

If G has a k-factor, for $d \geq 5$, we get with Lemma 4.14

$$n < 1 + \begin{cases} (k+2)n_0; & \begin{array}{l} s \text{ odd and either } t < p-2 \text{ or} \\ t = p-2 \text{ and } d_s \geq k+2, \end{array} \\ (k+2)(n_0+2) - 2d_s; & s \text{ odd, } t = p-2 \text{ and } d_s < k+2, \\ (k+2)(n_0-2); & s \text{ even and } d_1 \geq k+2, \\ (k+2)n_0 - 2d_1; & s \text{ even and } 0 \leq d_1 < k+2, \end{cases}$$

with $d_1 := \left\lfloor \frac{d-(k+2)}{st+t-1} \right\rfloor$ and $d_s := \begin{cases} d, & s=1; \\ \left\lfloor \frac{d-(k+2)}{s-1} \right\rfloor, & s>1. \end{cases}$

With Lemma 4.10 these bounds are identical to the ones in Theorem 4.5. On the other hand look at the graph $G(p,d,k)$ constructed as follows:

Case 2.A: s odd and $d < s(p-1)+p-2$. Take a vertex x, $k+1$ copies of U_1 and one $U_{d-(k+2)}$ as conctructed in Section 4.2.2. Connect all vertices of degree $d-1$ in the components U_i to x with one edge. The resulting graph $G(p,d,k)$ is d-regular p-partite with order $1+(k+2)n_0 = 1+(k+2)(d+s+1)$.

Case 2.B: s odd, $d = s(p-1)+p-2$ and $d_s < k+2$. Take a vertex x, d_s copies of U_s as well as $k-d_s+1$ copies of U_1 and one copy of U_r with $r := d-sd_s-k+d_s-1 = d-d_s(s-1)-k-1$. These have all been constructed in Section 4.2.2. Then $1 \leq r < s$ and $n_r = n_0+2$. Join each vertex of degree $d-1$ in a component U_i to x with an edge. The resulting graph $G(p,d,k)$ is d-regular p-partite with order $1+(k+2)(n_0+2)-2d_s = 1+(k+2)(d+s+3)-2d_s$. Note that this construction is possible even if $p=3$ since we only take U_s as components of $G-x$, which have been constructed in such a way that $d(x,U_s) = d$ for every $x \in V_p$.

Case 2.C: s odd, $d = s(p-1)+p-2$ and $d_s \geq k+2$. Then $p \geq 4$ and $d \geq s(k+2)$. Take $k+1$ copies of U_s and one copy of U_r with $s \leq r = d-(k+1)s \leq d-2$, as constructed in Section 4.2.2. Join all vertices

of degree $d-1$ in the copies U_i with one edge to x. Again the resulting graph $G(p,d,k)$ is d-regular p-partite of order $1+(k+2)n_0 = 1+(k+2)(d+s+1)$.
Case 2.D: s even and $0 \leq d_1 \leq k+1$. The construction of a d-regular p-partite graph $G(p,d,k)$ of order $1+(k+2)n_0-2d_1 = 1+(k+2)(d+s+2)-2d_1$ runs analogously to Case 2.B with $U_{(st+t)}$ instead of U_s and d_1 instead of d_s.
Case 2.E: s even and $d_1 \geq k+2$. In this case $t < p-3$ and $d \geq (st+t)(k+2)$ since d, k and $st+t$ are odd. The construction of a d-regular p-partite graph $G(p,d,k)$ of order $1+(k+2)(n_0-2) = 1+(k+2)(d+s)$ is accomplished analogously to Case 2.C with $U_{(st+t)}$ instead of U_s and d_1 instead of d_s.

In all five cases we have the Tutte-pair $(\{x\}, \emptyset)$ in $G(p,d,k)$. Thus $G(p,d,k)$ does not have a k-factor and our bound is sharp.

If k is even, then $d-k$ is odd and we can use the above case on $d-k$. This completes the proof of Theorem 4.5. □

Chapter 5

Unique k-factors and unique $[1, k]$-factors in graphs

In this chapter we move away from sufficient conditions for the existence of k-factors in a graph and turn towards a problem in extremal graph theory: If G is a graph with a unique k-factor, how many edges can G maximal have? We have already encountered with Lemma 4.6 a problem of similar type in Chapter 4. The first section of this chapter deals with general results on graphs with a unique k-factor. In the second section we focus on bipartite graphs with a unique k-factor and present a simple method to obtain results for the bipartite case if the corresponding general case is known. We will further present the concept of alternating neighbourhoods, which allows us to prove sharp results in the case $k \leq 4$. The last section concentrates on graphs with a unique $[1, k]$-factor and unique perfect $[1, k]$-factors. If not stated otherwise, the results of Section 5.1 and 5.2 have been shown in [20] and [18].

If a graph has a factor F, then colour the edges belonging to F red and all other ones blue and denote with $E_r(G), E_b(G)$ the set of red and blue edges, respectively. Let $N_r(v) = N(v, F)$ and $N_b(v) = N(v, G)\backslash N(v, F)$ denote the *red* and *blue neighbourhood*, respectively. Then $d_r(v) = |N_r(v)|$ and $d_b(v) = |N_b(v)|$ are the *red* and *blue degree* of v. We call a path or a circuit *alternating*, if its edges are coloured red–blue or blue–red in an alternating way. Throughout this chapter red edges will be symbolized by a thick line $x \,\textbf{—}\, y$ and blue edges will be symbolized by a thin line $x — y$.

A result on graphs with a unique 1-factor, ascribed to Hetyei [27], seems to mark the beginning of the examination of graphs with a unique factor. We present it with a short proof and afterwards provide a characterization of graphs where every edge lies in a unique 1-factor.

Lemma 5.1 (Hetyei, cf. [27]) *Let G be a graph of order n with a unique 1-factor.*
Then $|E(G)| \leq \frac{n^2}{4}$.

Proof. Let $H_1, H_2, \ldots, H_{n/2}$ be the components of the 1-factor. There can be at most 2 edges connecting H_i to H_j in G. There are $\binom{n/2}{2}$ such pairs and thus $|E(G)| \leq \frac{n}{2} + 2\binom{n/2}{2} = \frac{n^2}{4}$. □

Theorem 5.2 *Every edge of a graph G belongs to a unique 1-factor if and only if G is either 1-regular, the K_4 or a cycle of even length.*

Proof. The necessity is obvious, as every edge of G belongs to a unique 1-factor, if G belongs to one of the three families. Now let G be a graph where every edge belongs to a unique 1-factor. Then $n(G)$ is even and, since all these factors are edge-disjoint, G is d-regular. If $d = 1$, we are done. If $d = 2$, G is a cycle of even length, because if G has more than one component, then every edge belongs to more than one 1-factor. If $d = 3$ and $|V(G)| = 4$, then G is the K_4. So let now $d \geq 3$ and $|V(G)| \geq 6$. Since every edge belongs to a unique 1-factor and $d \geq 3$, G has a hamiltonian cycle C made up from two 1-factors. We remember that G and thus C are of even order. Since $d \geq 3$, there exists an edge $xy \notin E(C)$, which we call a chord, which belongs to a third 1-factor. Call xy an even (odd) chord, if $C \backslash \{x, y\}$ consists of two paths of even (odd) order. If xy is an even chord, then there exists a second 1-factor containing xy, made up of 1-factors of each of the two paths and the edge xy, which is a contradiction. Thus all chords of C are odd chords. We find a pair of two odd chords xy and vw such that $v \in N_C(x)$ and v and w lie in different components of $C - \{x, y\}$. Then $C - \{x, y, v, w\}$ consists only of paths of even length, since both chords are odd chords. However, this gives us a second 1-factor containing xy, vw and 1-factors of these even paths. Thus we get another contradiction and the proof is complete. □

In 1984, G.R.T. Hendry took up the question of edge-maximal graphs with a unique 2-factor, motivated by a result of J. Sheehan [34] on graphs with a unique hamiltonian cycle.

Theorem 5.3 (Hendry [14]) *Let G be a graph of order $n \geq 3$ with a unique 2-factor. Then $|E(G)| \leq \frac{n^2}{4} + \frac{n}{4}$.*

G.R.T. Hendry further characterized the extremal graphs and presented conjectures on graphs with a unique k-factor. For order $n = kl$ his extremal graphs consist of l copies of $K_k + O_k^1$, remember that O_k^1 denotes a set of k independent vertices. Number the copies $H_1, H_2, \ldots, H_l$ and add edges connecting all vertices of K_k in H_i with every vertex of H_j for $j > i$.

P. Johann [22] took up the work of G.R.T. Hendry and proved two of his conjectures. She also showed that Hendrys graphs provide sharpness for $k \leq \frac{n}{2}$ in the case that k divides the order of the graph and in general if $k > \frac{n(G)}{2}$. L. Volkmann [38] further improved these results for $5 \leq 2k + 1 \leq n \leq 3k$ and gave sharp results for $k = 3$. The general case for $k \geq 4$ is still open, but L. Volkmann conjectured on the graphs with a unique k-factor having maximal number of edges. These are the same graphs as Hendrys except for one component H, depending on the order n.

Theorem 5.4 (Johann [22]) *Let G be a graph of order n with a unique k-factor. It holds*

$$|E(G)| \leq \begin{cases} \frac{n^2}{4} + (k-1)\frac{n}{4}, & \text{if } k \leq \frac{n}{2}; \\ \frac{nk}{2} + \binom{n-k}{2}, & \text{if } k > \frac{n}{2}. \end{cases}$$

The first case is sharp if $n = kl$. The second case is sharp for any n.

Theorem 5.5 (Volkmann [38]) *Let G be a graph of order n with a unique k-factor such that $|E(G)|$ is maximal.*

a) For $5 \leq 2k + 1 \leq n \leq 3k$ it holds $|E(G)| = k^2 + \binom{n-k}{2}$.

b) If $k = 3$, then

$$|E(G)| = \begin{cases} \frac{n^2}{4} + \frac{n}{2}, & \text{for } n \equiv 0, 4 \pmod 6; \\ \frac{n^2}{4} + \frac{n}{2} - 1, & \text{for } n \equiv 2 \pmod 6. \end{cases}$$

Both Theorem 5.4 and Theorem 5.5 make use of the following two lemmata.

Lemma 5.6 (Johann [22]) *Let G be a graph with a unique f-factor. There exists a second f-factor in G if and only if G has an alternating circuit.*

Lemma 5.7 (Johann [22]) *Let G be a graph of order n with a unique k-factor. It holds for every $x \in V(G)$:*

$$kd_b(x) + \sum_{y \in N_r(x)} d_b(y) \leq k(n-k-1)$$

with equality holding for every x if and only if $|E(G)| = \frac{n^2}{4} + (k-1)\frac{n}{4}$.

B. Jackson and R.W. Whitty [21] took a different approach to graphs with a unique factor, following [3], [26] and [28].

Theorem 5.8 (Jackson, Whitty [21]) *If G is a 2–edge–connected graph with a unique f-factor, then there exists a vertex v such that $d(v, G) = f(v)$.*

Motivated by the structure of the graphs providing sharpness for Theorems 5.3, 5.4 and 5.5, L.Volkmann conjectured the following.

Conjecture 5.9 (Volkmann [38]) *For $k \geq 2$ let G be a graph of order $n \geq k+2$ with a unique k-factor. If G is of maximal size, then G contains exactly k vertices of degree k.*

5.1 Bipartite graphs with a unique k-factor

When one looks at the extremal graphs presented by G.R.T. Hendry in [14] and L. Volkmann in [38], one quickly sees that these have a large chromatic number, as they all have a $K_{n/2}$ as an induced subgraph. However, some of the graphs can easily be obtained from a bipartite graph by connecting all vertices of one part with each other. The following lemma makes use of this construction and provides a simple way of computing an upper bound for the number of edges in a bipartite graph with a unique f-factor, if the correlating bound for general graphs is known.

Lemma 5.10 *Let G be a bipartite graph with parts V_1, V_2 which has a unique f-factor. If G' is formed from G by adding all edges connecting two vertices in V_1, then G' has a unique f-factor, too.*

Proof. Let F be the unique f-factor of G and let us assume that G' has a second f-factor. With Lemma 5.6 this is equivalent to the existence of an alternating circuit in G' with at least one blue edge $x_1 — x_2$ of the circuit connecting two vertices of V_1. As every red edge only connects vertices of different parts, there has to be a blue edge $y_1 — y_2$ in the circuit such that $y_1, y_2 \in V_2$. This yields a contradiction, as we only added blue edges in V_1. □

Lemma 5.11 *For a non-negative integer p and a function $f : V(G) \to \mathbf{N}$ let $m(p, f)$ denote the maximal number of edges in a graph of order $2p$ with a unique f-factor. If G is a bipartite graph of order $2p$ with a unique f-factor, then $|E(G)| \leq m(p, f) - \binom{p}{2}$.*

Proof. If V_1 and V_2 are the parts of G, then $|V_1| \geq |V_2|$ and hence $|V_1| \geq p$. Add edges connecting all vertices of V_1 with each other. For the resulting graph G' it holds $|E(G')| \geq |E(G)| + \binom{p}{2}$. The result now follows with Lemma 5.10. □

Lemma 5.11 together with Theorem 5.4 and Theorem 5.5 immediately yields the following result.

Theorem 5.12 *If G is a bipartite graph of order $2p$ with a unique k-factor, then $|E(G)| \leq \frac{p^2+kp}{2}$ and for $\frac{2p}{3} \leq k < p$ it even holds $|E(G)| \leq k^2 + \binom{2p-k}{2} - \binom{p}{2}$.*

We are now going to construct bipartite graphs with a unique k-factor. For this let p and k be non-negative integers such that $p = sk + t$ with $s \geq 1$ and $0 \leq t \leq k - 1$. First define a bipartite graph $A(k, t)$ as follows: Let A_1 be a copy of $K_{t,t}$ and A_2 a bipartite $(k - t)$-regular graph on $2k$ vertices (the latter exists as a result of Theorem 4.1). Let V_j^i, with $1 \leq j \leq 2$, denote the two parts of A_i, $1 \leq i \leq 2$. Connect all vertices of V_j^1 with every vertex in $V_{(3-j)}^2$ for $1 \leq j \leq 2$. The resulting graph $A(k, t)$ is bipartite, has exactly one k-factor, consisting of the edges in A_2 and those connecting A_1 and A_2. It holds $|E(A(k, t))| = t^2 + k(k + t)$.
Next take $s-1$ copies of $K_{k,k}$ and one copy of $A(k, t)$ and number these graphs $S_1, S_2, \ldots, S_s$, respectively. Let V_1^i, V_2^i denote the partition of these graphs. For $j > i$ connect all vertices of V_1^i with all vertices in V_2^j. The resulting

graph $B(p,k)$ is bipartite of order $2p$, has exactly one k-factor, formed by the copies of $K_{k,k}$ and the unique k-factor of $A(k,t)$. For $B(p,k)$ it holds

$$\begin{aligned} |E(B(p,k))| &= (s-1)k^2 + t^2 + k(k+t) + (s-1)k(k+t) + k^2\binom{s-1}{2} \\ &= \frac{1}{2}(p^2 + kp - t(k-t)). \end{aligned}$$

Observation 5.13 *Let G be a bipartite graph of order $2p$ with a unique k-factor such that $p \equiv t \pmod{k}$, with $0 \le t < k$. If G is of maximal size, then*

$$|E(G)| \ge \frac{1}{2}(p^2 + kp - t(k-t)).$$

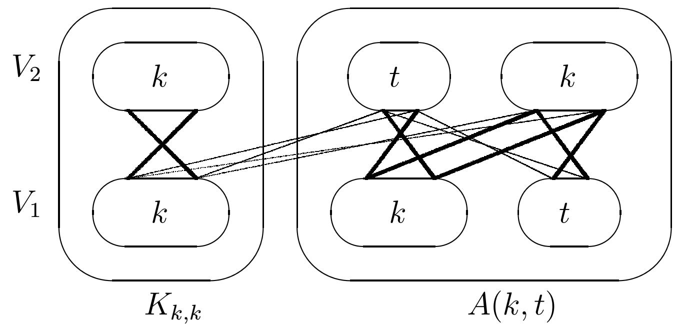

Figure 5.1: The graph $B(2k+t,k)$ with $t>0$

For $t = 0$ we get $|E(B(p,k)| = \frac{p^2}{2} + \frac{pk}{2}$, giving us the sharpness of that particular case in Theorem 5.12. The case $k = 2$ is obviously sharp, too. In most cases for $k \ge 3$ and $t > 0$, however, there is a gap between the lower bound provided by Observation 5.13 and the upper bound obtained via Theorem 5.12, which increases in k. To provide sharp bounds for the size of bipartite graphs with a unique k-factor, we are going to derive information on the structure of these graphs. For this let us look at alternating neighbourhoods of a vertex x.

Definition 5.14 *Let G be a bipartite graph with a k-factor F. For $x \in V(G)$ simultaniously define*

$$R_1(x) := N_r(x) \qquad\qquad B_1(x) := N_b(x)$$

$$R_{i+1}(x) := N_r(B_i(x)) \setminus \bigcup_{j=1}^{i} R_j(x) \qquad B_{i+1}(x) := N_b(R_i(x)) \setminus \bigcup_{j=1}^{i} B_j(x)$$

If there is no chance of ambiguity, we simply call the sets R_i and B_i.

Plainly speaking, a set $R_i \neq \emptyset$ contains all red neighbours of the vertices in B_{i-1} which are not in R_j for $j < i$. Similarily for B_i. For a visualization see Figure 5.2. Note that Figure 5.2 is not complete as there are red, as well as blue, edges missing.

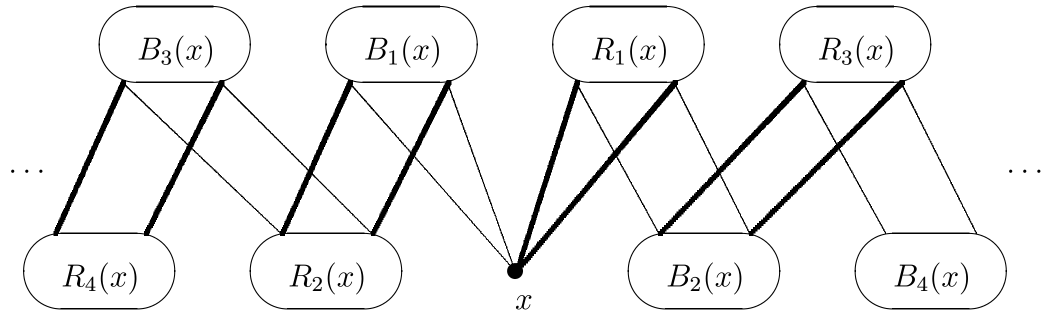

Figure 5.2: The sets $R_i(x)$ and $B_i(x)$

Observation 5.15 *Let G be a bipartite graph with a unique k-factor.*

- *If $y \in R_{2i+1}$, there exists an alternating path x ━ x_1 — x_2 ━ $\ldots$ — x_{2i} ━ y with $x_{2s} \in B_{2s}$ and $x_{2s+1} \in R_{2s+1}$.*
- *If $y \in R_{2i}$, there exists an alternating path x — x_1 ━ x_2 — $\ldots$ — x_{2i-1} ━ y with $x_{2s} \in R_{2s}$ and $x_{2s+1} \in B_{2s+1}$.*
- *If $y \in B_{2i}$, there exists an alternating path x ━ x_1 — x_2 ━ $\ldots$ ━ x_{2i-1} — y with $x_{2s} \in B_{2s}$ and $x_{2s+1} \in R_{2s+1}$.*
- *If $y \in R_{2i+1}$, there exists an alternating path x — x_1 ━ x_2 — $\ldots$ ━ x_{2i} — y with $x_{2s} \in R_{2s}$ and $x_{2s+1} \in B_{2s+1}$.*

Lemma 5.16 *Let G be a bipartite graph with a unique k-factor such that the size of G is maximal. For the sets R_i and B_i as defined in Definition 5.14 it holds* *(i) for all i and j: $R_i \cap B_j = \emptyset$;*

(ii) for all $i \geq 1$: $B_{2i+1} = R_{2i+2} = \emptyset$;

(iii) for all $i \geq 1$: $B_{2i+2} = R_{2i+3} = \emptyset$.

Proof. Let $x \in V(G)$ be fixed and consider the sets $B_i(x)$ and $R_i(x)$. By Definition 5.14 R_i and B_j lie in different parts of G if $i \not\equiv j \pmod 2$. Thus $R_i \cap B_j = \emptyset$ in this case. Assume there exist $i \equiv j \pmod 2$ such that $y \in R_i \cap B_j$. We choose i, j such that $l := \min\{i, j\}$ is minimal and $|i - j|$ is minimal over all such pairs i, j with $\min\{i, j\} = l$. Without loss of generality let i, j be even, as the proof for i, j odd runs analogously. With above observation we get the existence of a path $P_1 : x \,\text{—}\, x_1 \,\rule[0.5ex]{1em}{2pt}\, x_2 \,\text{—}\, \ldots \,\text{—}\, x_{2i-1} \,\rule[0.5ex]{1em}{2pt}\, y$ and a path $P_2 : x \,\rule[0.5ex]{1em}{2pt}\, y_1 \,\text{—}\, y_2 \,\rule[0.5ex]{1em}{2pt}\, \ldots \,\rule[0.5ex]{1em}{2pt}\, y_{2j-1} \,\text{—}\, y$. By Definition 5.14 and the way i, j were chosen, we have $x_s \neq y_t$ for $1 \leq s \leq 2i - 1$ and $1 \leq t \leq 2j - 1$. But then

$$x \,\text{—}\, x_1 \,\rule[0.5ex]{1em}{2pt}\, x_2 \,\text{—}\, \ldots \,\text{—}\, x_{2i-1} \,\rule[0.5ex]{1em}{2pt}\, y \,\text{—}\, y_{2j-1} \,\rule[0.5ex]{1em}{2pt}\, \ldots \,\rule[0.5ex]{1em}{2pt}\, y_2 \,\text{—}\, y_1 \,\rule[0.5ex]{1em}{2pt}\, x$$

forms an alternating circuit. This contradicts the uniqueness of the factor and thus statement (i) of the lemma follows.

For a proof of (ii) we just need to show that $B_3 = \emptyset$. Let us assume that $B_3 \neq \emptyset$. Then there exists a vertex $y \in B_3$. Due to the definition of B_3, the vertex y lies in a different part than x and $xy \notin E(G)$. We add the edge xy to G and colour it blue. As G was edge-maximal with respect to having a unique k-factor, there now exists an alternating circuit containing $x \,\text{—}\, y$. Choose such a circuit $C = y \,\text{—}\, x \,\rule[0.5ex]{1em}{2pt}\, x_1 \,\text{—}\, x_2 \,\rule[0.5ex]{1em}{2pt}\, \ldots \,\text{—}\, x_l \,\rule[0.5ex]{1em}{2pt}\, y$ with minimal number of edges. We know that $x_1 \in R_1$, $x_2 \in B_2$ and $x_l \in R_2 \cup R_4$. With Definition 5.14 and (i) there either exists an edge $x_j \,\rule[0.5ex]{1em}{2pt}\, x_{j+1}$ with $x_j \in B_{2s}$ and $x_{j+1} \in R_{2p}$ or an edge $x_j \,\text{—}\, x_{j+1}$ with $x_j \in R_{2s+1}$ and $x_{j+1} \in B_{2p+1}$. Assume that $s < p$. In the first case $x_{j+1} \in N_r(B_{2s}) \setminus \bigcup_{\nu=1}^{2s} R_\nu(x)$ and hence $x_{j+1} \in R_{2s+1}$. This contradicts the definition of $x_{j+1} \in R_{2p}$. The same argumentation holds for $x_j \,\text{—}\, x_{j+1}$, as well as for $s > p$ and we arrive at $s = p$. Choose j maximal. Then $d_1 := d_C(x, x_j)$ is even and $d_2 := d_C(x_{j+1}, y)$ is odd. Thus the circuit C is of length $d_1 + 1 + d_2 + 1$, which is odd. This is a contradiction as C is an alternating circuit and thus of even length. Thus y cannot exist and $B_3 = \emptyset$ holds.

For a proof of (iii) we assume that $B_4 \neq \emptyset$. Then there exists a vertex $y \in B_4$ and we can find an alternating path $x \,\rule[0.5ex]{1em}{2pt}\, v_1 \,\text{—}\, v_2 \,\rule[0.5ex]{1em}{2pt}\, v_3 \,\text{—}\, y$ with

$v_1 \in R_1(x)$, $v_2 \in B_2(x)$ and $v_3 \in R_3(x)$. By the definition of the sets B_i, $v_1y \notin E_b(G)$. Furthermore, $v_1y \notin E_r(G)$ as otherwhise we would have the alternating circuit $v_1 — v_2 \textbf{—} v_3 — y \textbf{—} v_1$. Thus $y \in B_3(v_1)$, in contradiction with (ii) for v_1. Hence, $B_4 = \emptyset$ and the last statement of the lemma follows. □

We note that statement (i) of the above lemma holds for any graph with a unique k-factor. For the following corollary and lemmata suppose that G as a bipartite graph with a unique k-factor which has maximal size.

Corollary 5.17

- *$N_b(R_2) \subseteq B_1$ and*
- *$N_b(R_3) \subseteq B_2$.*

Proof. The corollary follows immediately from Lemma 5.16 □

The following lemma gives information on certain blue edges in G. The lemma after that provides further information on blue neighbourhoods.

Lemma 5.18 *Let V_1, V_2 denote the parts of G and let $x \in V_1$.*

- *For every $v \in B_1(x)$ and every $w \in V_1 \backslash R_2(x)$ it holds $vw \in E_b(G)$;*
- *For every $v \in B_2(x)$ and every $w \in V_2 \backslash (R_1(x) \cup R_3(x))$ it holds $vw \in E_b(G)$.*

Proof. The proof is by contradiction. We assume that there exist $v \in B_1(x)$ and $w \in V_1 \backslash R_2(x)$ such that $vw \notin E_b(G)$. Obviously $w \neq x$ and $vw \notin E(G)$. We add vw to G and colour it blue. As G is edge-maximal, there now exists an alternating circuit C containing $w — v \textbf{—} z$ with $z \in R_2$. As $N_b(z) \subseteq B_1(x)$ by Corollary 5.17, and $N_r(B_1(x)) = R_2(x)$ by definition, the alternating circuit cannot leave $B_1(x) \cup R_2(x)$. As a consequence $w \in R_2(x)$, giving us the contradiction. The proof for (ii) runs analogously. □

Lemma 5.19 *Let v_1, v_2 be two vertices in the same part of G. If $d_b(v_1) \leq d_b(v_2)$, then $N_b(v_1) \subseteq N_b(v_2)$.*

Proof. If $v_1 \notin R_2(v_2)$, then Lemma 5.18 yields $N_b(v_2) \subseteq N_b(v_1)$ and $d_b(v_1) = d_b(v_2)$, giving us the statement of the lemma. If $v_1 \in R_2(v_2)$, then the statement follows directly from Corollary 5.17. □

Definition 5.20 *Let G be a graph of order $2p > 2k$ with a unique k-factor such that $e(G)$ is maximal. Let A, B denote the two parts of G. Define $A_{\geq} := \{v \in A : d_b(v) \geq 1\}$ and analogously $B_{\geq}$. We define $\bar{a}$ as a vertex in A with $d_b(\bar{a})$ minimal over all vertices in $A_{\geq}$. Define $\bar{b} \in B$ analogously. Further we define $R_2(\bar{a}) = \{a_i : 1 \leq i \leq |R_2(\bar{a})|\}$ and $R_2(\bar{b}) = \{b_i : 1 \leq i \leq |R_2(\bar{b})|\}$.*

The next Theorem is closely related to Conjecture 5.9. It shows that a bipartite graph of maximal size with a unique k-factor has exactly $2k$ vertices of degree k. As this obviously holds for a bipartite graph with order $2k$, we can restrict ourselves to order $2p > 2k$.

Theorem 5.21 *Let G be a bipartite graph of order $2p > 2k$ with a unique k-factor such that the size of G is maximal. With $\bar{a}$ and $\bar{b}$ as in Definition 5.20 it holds*

(i) $d_b(v) = 0$ *for every* $v \in R_2(\bar{a}) \cup R_2(\bar{b})$;

(ii) *G has exactly $2k$ vertices of degree k, in detail:* $|R_2(\bar{a})| = |R_2(\bar{b})| = k$;

(iii) $d_b(u) = p - k$ *for every* $u \in B_1(\bar{a}) \cup B_1(\bar{b})$.

Proof. The vertices $\bar{a}$ and $\bar{b}$ exist as $p > k$ and G is edge-maximal. Let us consider $\bar{a}$, the proof for $\bar{b}$ is the same. By their definition it holds $|B_1(\bar{a})| \geq 1$ and $|R_2(\bar{a})| \geq k$. For every $v \in R_2(\bar{a})$ it holds $N_b(v) \subseteq N_b(\bar{a})$ with Lemma 5.17, thus $d_b(v) \leq d_b(\bar{a})$. As $v \in N_r(y)$ for at least one $y \in B_1(\bar{a})$, due to the definition of R_2, it holds $d_b(v) < d_b(\bar{a})$. Since $d_b(\bar{a}) \geq 1$ minimal, statement (i) follows. For a proof of (ii) we assume that $|R_2(\bar{a})| \geq k+1$. Then there exists $w \in B_1(\bar{a})$ such that, without loss of generality, a_1 and w are not adjacent in G, where a_1 is as in Definition 5.20. We add the edge wa_1 to G and colour it blue. As G is edge-maximal, there now exists an alternating circuit containing $a_1 — w \textbf{—} y_1 — y_2$. Definition 5.14 yields $y_1 \in R_2(\bar{a})$. Then the existence of $y_1 — y_2$ contradicts (i). As $|R_2(\bar{a})| \geq k$ by definition, the second part of statement (ii) is shown. As for every vertex $z \in A \backslash R_2(\bar{a})$ it holds $d_b(z) \geq d_b(\bar{a})$ by (ii) of Lemma 5.17, (ii) is shown. Statement (iii) is a consequence of (ii) together with Lemma 5.17. □

Figure 5.3 depicts the structural results we have proved so far. Note that not all red and blue edges are shown in the figure.

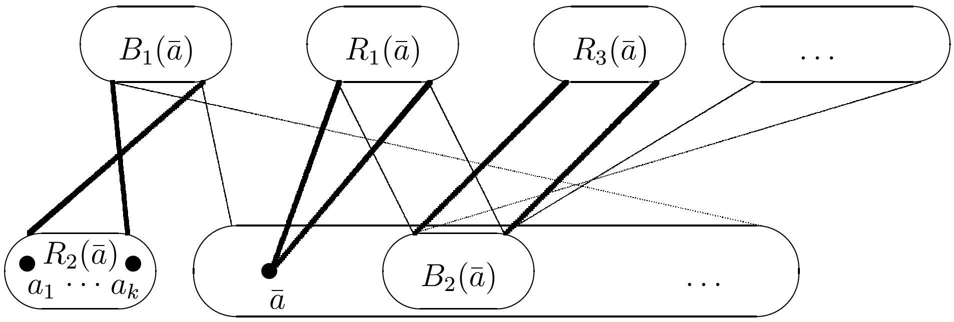

Figure 5.3: The sets $R_i(\bar{a})$ and $B_i(\bar{a})$.

Corollary 5.22 *It holds* $\max\{d_b(\bar{a}), d_b(\bar{b})\} \leq k$.

Proof. Let, without loss of generality, $d_b(\bar{a}) \geq d_b(\bar{b})$. From (ii) of Theorem 5.21 we know $|R_2(\bar{a})| = k$. Thus every $y \in B_1(\bar{a})$ is connected to every $a_i \in R_2(\bar{a})$ with a red edge. This implies $k \geq |B_1(\bar{a})| = d_b(\bar{a})$. □

Lemma 5.23 *Let G be a bipartite graph of order $2p$ with a unique k-factor such that $e(G)$ is maximal. If $p = k + t$ with $0 \leq t \leq k - 1$, then $G \cong B(p, k)$ with $B(p, k)$ as defined after Theorem 5.12.*

Proof. From Observation 5.13 we know

$$e(G) \geq e(B(p,k)) = \frac{1}{2}((k+t)^2 + k(k+t) - t(k-t)) = k(k+t) + t^2. \quad (5.1)$$

With Theorem 5.21 we know that k vertices in each part of G have degree k. Thus G can have at most t^2 blue edges, resulting in $e(G) \leq pk + t^2 = k(k+t) + t^2$. As a result we have equality in (5.1) and the statement of the lemma follows. □

Next we are going to prove the bipartite equivalent of Lemma 5.7.

Lemma 5.24 *Let G be a bipartite graph of order $2p$ with a unique k-factor F. Then it holds for every $x \in V(G)$:*

$$kd_b(x) + \sum_{y \in N_r(x)} d_b(y) \leq k(p-k) \tag{5.2}$$

with equality holding for every x if and only if $|E(G)| = \frac{p^2}{2} + \frac{kp}{2}$.

Proof. For a vertex $v \in V(G)$ define:

- $R^*(v) = N_r(R_1(v)) \backslash (R_2(v) \cup B_2(v) \cup \{v\})$,
- $r = e_r(R_1(v), B_2(v))$, the number of red edges connecting a vertex in $R_1(v)$ with one in $B_2(v)$.

By Lemma 5.16 $B_2 \cap R_2 = \emptyset$ and by definition $R^* \cap B_2 = \emptyset$. As every blue edge from R_1 connects to a vertex in B_2, there can be at most $|R_1||B_2| = k|B_2|$ such edges. As further r red edges connect R_1 to B_2, we get

$$k|B_2| \geq r + \sum_{y \in R_1(v)} d_b(y).$$

There are exactly $kd_b(v)$ red edges connecting B_1 with $R_2 \cup R^*$ and $k(k-1)-r$ red edges connecting R_1 with $R_2 \cup R^*$. As there can be at most $k|R_2 \cup R^*|$ red edges connecting to $R_2 \cup R^*$, we get

$$k|R_2 \cup R^*| \geq kd_b(v) + k(k-1) - r.$$

As v, $R_2 \cup R^*$ and B_2 are subsets of the same part of G, we deduce that

$$\begin{aligned} p &\geq |\{v\}| + |R_2 \cup R^*| + |B_2| \\ &\geq k + d_b(v) + \frac{1}{k} \sum_{y \in N_r(v)} d_b(y), \end{aligned}$$

yielding inequality (5.2). With $E_b(G)$ denote the set of blue edges in G. Now (5.2) implies

$$\begin{aligned} 4k|E_b(G)| &= \sum_{v\in V(G)} 2kd_b(v) = \sum_{v\in V(G)} \left(kd_b(v) + \sum_{y\in N_r(v)} d_b(y) \right) \\ &\leq \sum_{v\in V(G)} k(d_b(v) + p - k - d_b(v)) \\ &= 2p(kp - k^2). \end{aligned}$$

This leads to

$$|E(G)| = kp + |E_b(G)| \leq kp + \frac{p(p-k)}{2} = \frac{p^2}{2} + \frac{kp}{2},$$

and the second statement of the lemma follows. □

Lemma 5.24 motivates the following definition.

Definition 5.25 *Let G be a bipartite graph of order $2p$ with a unique k-factor. For a vertex $x \in V(G)$ define $\phi(x)$ through*

$$kd_b(x) + \sum_{y\in N_r(x)} d_b(y) = k(p-k) - \phi(x), \tag{5.3}$$

and $\phi(G) := \sum_{x\in V(G)} \phi(x)$.

Lemma 5.26 *Let G be a bipartite graph of order $2p$ with a unique k-factor such that the size of G is maximal. Let further $p = sk + t$ with $s \geq 2$ and $1 \leq t \leq k-1$. With v^* denote the vertex in $\{a_1, \ldots, a_k, b_1 \ldots, b_k\}$ with minimal ϕ. If $\phi(v^*) = 0$, then there exists a graph G' with $n(G') = 2(p-k)$ and $e(G') = e(G) - kp$ which has a unique k-factor.*

Proof. Let, without loss of generality, $a_1 = v^*$. Then equality holds for a_1 in (5.2) and thus $d_b(y) = p - k$ for every $y \in N_r(a_1)$. With Lemma 5.19 we have $N_r(a_1) = B_1(\bar{a})$. Thus the subgraph induced by $B_1(\bar{a})$ and $R_2(\bar{a})$ is isomorphic to $K_{k,k}$. Deleting this subgraph leads to a graph G' of order $2(p-k)$ and size $e(G') - k^2 - k(p-k) = e(G) - kp$. Obviously G' has a unique k-factor. □

Although we have already validated the following theorem for $k \leq 2$ as well as for $t = 0$, we are including proofs of these cases, as they follow quite easily.

Theorem 5.27 *For $1 \leq k \leq 4$ let G be a bipartite graph of order $2p$ with a unique k-factor, such that $e(G)$ is maximal. Let further $p = sk + t$ with $s \geq 1$ and $0 \leq t \leq k-1$. Then $e(G) = e(B(p,k)) = \frac{p^2+kp}{2} - \frac{t(k-t)}{2}$.*

Proof. Observation 5.13 gives us $e(G) \geq \frac{p^2+kp}{2} - \frac{t(k-t)}{2}$. So assume that there exists an integer $c \geq 1$ such that $e(G) = c + \frac{p^2+kp}{2} - \frac{t(k-t)}{2}$. Choose G minimum with respect to p. From Lemma 5.23 it follows that $s \geq 2$. Let ϕ be as in Definition 5.25. Then it holds, analogous to the proof of Lemma 5.24,

$$4k|E_b(G)| = 2kp(p-k) - \phi(G)$$

and thus

$$e(G) = \frac{p^2+kp}{2} - \frac{\phi(G)}{4k}.$$

Together with our assumption we get

$$\phi(G) = 2k(t(k-t) - 2c) \leq 2k(t(k-t) - 2). \tag{5.4}$$

For $t = 0$ as well as for $k \leq 2$ inequality (5.4) automatically yields a contradiction. So let $k \geq 3$, $t \geq 1$ and $\bar{a}, \bar{b}, a_i$ and b_i be as in Definition 5.20. Let, without loss of generality, $\phi(a_1)$ be minimum over all $\phi(a_i), \phi(b_i)$. Inequality (5.4) and the definition of $\phi(G)$ give us either

- $\phi(a_i) = \phi(b_i) = t(k-t) - 2$ for every $1 \leq i \leq 2k$ and $\phi(v) = 0$ for every other vertex v; or
- $\phi(a_1) \leq t(k-t) - 3$.

If $\phi(a_1) = 0$, then Lemma 5.26 ensures the existence of a graph G' with a unique k-factor. We further know that $n(G') = 2(p-k)$ and $e(G') = e(G) - pk = c + \frac{(p-k)^k + 4(p-k)}{2} - \frac{t(4-t)}{2}$. Thus G' meets the criteria of the assumption and is of smaller order than G, contradicting the choice of G. For $k = 3$ there remains nothing to be shown, as $\phi(G) = 0$. So let $k = 4$ and $\phi(a_1) \geq 1$.

Case 1: $t \in \{1, 3\}$ and $\phi(a_i) = \phi(b_i) = t(4-t) - 2 = 1$ for every $1 \leq i \leq 4$. With (5.2), and as $\phi(a_1) = 1$, there exists exactly one vertex $y_1 \in N_r(a_1)$ such that $d_b(y_1) = p - k - 1$. Obviously $y_1 \notin B_1(\bar{a})$. As mentioned above, it holds $\phi(v) = 0$ for every $v \notin R_2(\bar{a}) \cup R_2(\bar{b})$, so it especially holds for y_1. This leads to

$$k(p-k) = kd_b(y_1) + \sum_{v \in N_r(y_1)} d_b(v) = k(p-k-1) + \sum_{v \in N_r(y_1)} d_b(v)$$

and thus

$$\sum_{v \in N_r(y_1)} d_b(v) = k. \tag{5.5}$$

Corollary 5.19 yields $N_b(y_1) = N_b(B_1(\bar{a}))\backslash\{\bar{a}\}$ and $N_r(y_1) \subseteq R_2(\bar{a}) \cup \{\bar{a}\}$. With (5.5) it follows $\bar{a}y_1 \in E_r(G)$ and also $d_b(\bar{a}) = k$. This leads to the contradiction $k = d_r(a_1) = |B_1(\bar{a})| + |\{y_1\}| = k + 1$.

Case 2.A: $t = 2$ and $\phi(a_i) = \phi(b_i) = t(4 - t) - 2 = 2$ for every $1 \leq i \leq 4$. If there exist two vertices in $N_r(a_1)$ with blue degree $p - k - 1$, we get a contradiction analogous to Case 1. So assume that there exists exactly one vertex $y_1 \in N_r(a_1)$ with $d_b(y_1) = p - k - 2 > 0$. Hence $|B_1(\bar{a})| = k - 1$. It holds $y_1 \notin B_1(\bar{a})$, $N_b(y_1) = N_b(B_1(\bar{a}))\backslash\{\bar{a}, z\}$ and

$$\sum_{v \in N_r(y_1)} d_b(v) = 2k. \tag{5.6}$$

If $y_1 z \notin E_r(G)$, then $y_1\bar{a} \in E_r(G)$ and (5.6) leads to $d_b(\bar{a}) = 2k$. This contradicts Corollary 5.22. If $y_1\bar{a} \in E_r(G)$ and $d_b(\bar{a}) = k$ we are done as in Case 1. Thus it remains $y_1 z \in E_r(G)$ and $d_b(z) \geq k + 1$. Then there exists a vertex $z_1 \notin B_1(\bar{a})$ with $zz_1 \in E_b(G)$. Assume that z_1 has a red neighbour z_2 in $N_b(y_1)$. Then $y_1 \,\textbf{—}\, z \,—\, z_1 \,\textbf{—}\, z_2 \,—\, y_1$ is an alternating circuit, in contradiction to the uniqueness of our factor. So we have $N_r(z_1) \subseteq R_2(\bar{a}) \cup \{\bar{a}\}$, giving us $|N_r(z_1) \cap N_r(y_1) \cap R_2(\bar{a})| \geq k - 2 \geq 1$. However, as every vertex v in the intersection is adjacent to every vertex in $B_1(\bar{a})$, we have $d_r(v) = k - 1 + 2 = k + 1$, a contradiction.

Case 2.B: $t = 2$ and $\phi(a_1) \leq t(4-t)-3$. Then $\phi(a_1) = 1$. Analogous to Case 1 there exists exactly one vertex $y_1 \in N_r(a_1)\backslash B_1(\bar{a})$ with $d_b(y_1) = p - k - 1$. Remember that then $N_b(y_1) = N_b(B_1(\bar{a}))\backslash\{\bar{a}\}$, $N_r(y_1) \subseteq R_2(\bar{a}) \cup \{\bar{a}\}$ and $d_b(\bar{a}) = k - 1$. The definition of $\phi(v)$ gives us the following information for y_1:

$$\sum_{v \in N_r(y_1)} d_b(v) = k - \phi(y_1). \tag{5.7}$$

As $d_b(\bar{a}) = k - 1$, equality (5.7) yields $\phi(y_1) \in \{1, k\}$. If $\phi(y_1) = k$, then (5.7) yields $N_r(y_1) = R_2(\bar{a})$. We observe that adding the edge $y_1\bar{a}$ to G is possible without forming an alternating circuit. This contradicts the maximality of $e(G)$. It remains the case $\phi(y_1) = 1$. Then $y_1\bar{a} \in E_r(G)$ and without loss of generality $N_r(y_1) \cap R_2(\bar{a}) = \{a_1, a_2, a_3\}$. There exists a vertex $y_2 \in B$ with $N_r(a_4) = B_1(\bar{a}) \cup \{y_2\}$ and $N_r(y_2)\backslash\{a_4\} \subset N_b(B_1(\bar{a}))$. As a consequence of Lemma 5.17 we have $N_b(y_2) \subseteq N_b(B_1(\bar{a}))\backslash(N_r(y_2) \cup \{\bar{a}\})$ and hence $d_b(y_2) \leq$

$p - (2k - 1)$. Now

$$\phi(a_4) = k(p-k) - \sum_{v \in N_r(a_4)} d_b(v) = p - k - d_b(y_2) \geq k + 1. \tag{5.8}$$

A short calculation shows $\phi(a_2) = \phi(a_3) = 1$. Since $y_1 \notin R_2(\bar{b})$, as $d_b(y_1) > 0$, we get with (5.8) that

$$\sum_{i=1}^{4} \phi(b_i) \leq \phi(G) - \phi(y_1) - \sum_{i=1}^{4} \phi(a_i) \overset{(5.4)}{\leq} 16 - 1 - 3 - 5 = 7. \tag{5.9}$$

As a result we have, without loss of generality, $\phi(b_1) = 1$. An analogous discussion for b_1 and $\bar{b}$ as the one above for a_1 and $\bar{a}$ yields the existence of a vertex $w_1 \in N_r(b_1)$ with $d_b(w_1) = p-k-1$ and $\phi(w_1) = 1$ such that, without loss of generality, $N_r(w_1) = \{\bar{b}, b_1, b_2, b_3\}$. As for a_4 we get $\phi(b_4) \geq 5$. We now arrive at $\sum_{i=1}^{4} \phi(b_i) \geq 3 + 5 = 8$ in contradiction to (5.9).
Hence both Case 1 and Case 2 lead to a contradiction, showing that our assumption was wrong. □

5.2 Graphs with a unique $[1, k]$-factor

In this section we turn to graphs with a unique $[1, k]$-factor. The following observation holds in general for a graph G with a unique $[a, b]$-factor, with $a < b$.

Observation 5.28 *Let G be a graph with a unique $[a, b]$-factor F.*

(i) For $x \in V(G)$ with $d_r(x) < b$ it holds: $d_r(y) = b$ for all $y \in N_b(x)$.

(ii) For $x \in V(G)$ with $d_r(x) > a$ it holds: $d_r(y) = a$ for all $y \in N_r(x)$.

Proof. For (i) assume there exists a $y \in N_b(x)$ such that $d_r(y) < b$. Then $F \cup \{xy\}$ would be a second $[a, b]$-factor.
For (ii) assume there exists an $y \in N_r(x)$ such that $d_r(y) > a$. Then $F - xy$ would be a second $[a, b]$-factor. □

This observation has easy corollaries. From (i) it follows that every blue edge in G is incident with at least one vertex of red degree b. Thus the set $\{x \in V(G) : d_r(x) < b\}$ constitutes an independent set in $G - E(F)$. From (ii) we get that every edge xy with $d_r(x) = d_r(y) = b$ is blue.

Theorem 5.29 *For positive integers $n > k \geq 2$ with $n = q(k+1) + r$, $0 \leq r \leq k$, let G be a graph of order n with a unique $[1, k]$-factor. It holds*

- *for $r = 0$: $|E(G)| \leq \frac{q(n+k-1)}{2}$;*
- *for $r = 1$: $|E(G)| \leq \frac{q(n+k)}{2} - 1$;*
- *for $2 \leq r \leq k$: $|E(G)| \leq \frac{q(n+k+r-1)}{2} + r - 1$.*

Proof. Let G be a graph of order n with a unique $[1, k]$-factor F such that $|E(G)|$ is maximal. With (ii) of Observation 5.28 we get that every component of F is isomorphic to some $K_{1,s}$ with $1 \leq s \leq k$. Since $\frac{n}{k+1} = q + \frac{r}{k+1}$, we have at least $q + \lceil \frac{r}{k+1} \rceil$ components in F. As noted right after Observation 5.28, every blue edge in G is connected to at least one vertex of red degree k. Look at two components H_1 and H_2 of F isomorphic to $K_{1,k}$ with x_1, x_2 being the vertices of degree k in these, respectively. Assume there exists a blue edge from x_1 to a vertex $y \in V(H_2)$ with $d_r(y) = 1$. Then x_2 cannot be connected to a vertex $z \in V(H_1)$, $d_r(z) = 1$ as $z \textbf{—} x_1 — y \textbf{—} x_2 — z$ would be an alternating circuit. With (i) of Observation 5.28 there cannot be any blue edges between vertices of $d_r = 1$. So there are at most $k+1$ blue edges between H_1 and H_2. If H_1 is isomorphic to $K_{1,k}$ and H_2 is isomorphic to $K_{1,s}$, with $s < k$, then there can be at most $|V(H_2)| = s + 1$ blue edges connecting H_1 and H_2. If H_1 and H_2 are both not isomorphic to $K_{1,k}$, then again Observation 5.28 yields that there is no blue edge between H_1 and H_2 as every blue edge connects to at least one vertex with $d_r = k$. Thus if l components of F are isomorphic to $K_{1,k}$, then $|E_r(G)| \leq n - q$ and $E_b(G) \leq \binom{l}{2}(k+1) + l(n - l(k+1))$. This results in

$$|E(G)| \leq n - q + nl - \frac{l(l+1)}{2}(k+1). \tag{5.10}$$

The right–hand side of (5.10) becomes maximal for $l = q + \frac{r}{k+1} - \frac{1}{2}$. Thus $l \in \{q-1, q\}$, since $|E(G)|$ is maximal. If $r = 0$ and $l = q$, then $|E(G)| \leq qk + \binom{q}{2}(k+1) = \frac{q}{2}(n+k-1)$. If $l = q-1$, then F has at least two components not isomorphic to $K_{1,k}$ and there are no blue edges between these components in G. Thus

$$\begin{aligned} |E(G)| &\leq n - (q-1) - 2 + \binom{q-1}{2}(k+1) + (q-1)(n - (q-1)(k+1)) \\ &= \frac{(n-2)(q+1)}{2} < \frac{q}{2}(n+k-1), \end{aligned}$$

proving the case $r = 0$.
If $r = 1$, then $l = q - 1$ since otherwhise F would have an isolated vertex, in contradiction to F being a $[1, k]$-factor. Thus F has again at least two components not isomorphic to $K_{1,k}$ and we get

$$|E(G)| \leq qk + \binom{q-1}{2}(k+1) + (q-1)(k+2) = \frac{q}{2}(n+k) - 1,$$

with equality if F has exactly $q + 1$ components.
It remains the case $2 \leq r \leq k$. If $l = q$, then $|E(G)|$ becomes maximal if the remaining r vertices induce a $K_{1,r-1}$ in F. As a consequence

$$|E(G)| \leq r - 1 + qk + \binom{q}{2}(k+1) + qr = \frac{q(n+k+r-1)}{2} + r - 1.$$

If $r = k$, then $\frac{k}{k+1} - \frac{1}{2} > 0$ and (5.10) yields the solution $l = q$. For $r < k$ and $l = q - 1$, the factor F has at least two components not isomorphic to $K_{1,k}$. We thus get

$$\begin{aligned} |E(G)| &\leq qk + r - 1 + \binom{q-1}{2}(k+1) + (q-1)(k+1+r) \\ &= \frac{(q-1)(n+r) + 2qk}{2} + r - 1 \\ &< \frac{q(n+k+r-1)}{2} + r - 1. \qquad \Box \end{aligned}$$

The results of Theorem 5.29 are sharp as the following examples show. Let $n = q(k+1) + r$ with $0 \leq r \leq k$. First consider the case $r \neq 1$. Take q copies of $K_{1,k}$ and one copy of $K_{1,r-1}$ (which will be the empty graph if $r = 0$). Number the copies $C_1, C_2, \ldots, C_{p+1}$ according to decreasing order. Connect the vertex of highest degree in C_i to all vertices of C_j with $j > i \geq 1$. The resulting graph G has exactly one $[1, k]$-factor (consisting of the edges in the original copies of $K_{1,k}$ and $K_{1,r-1}$) and

$$|E(G)| = \begin{cases} \frac{q(n+k-1)}{2}, & \text{if } r = 0, \\ \frac{q(n+k+r-1)}{2} + r - 1, & \text{if } 2 \leq r \leq k. \end{cases}$$

If $r = 1$, then take $q - 1$ copies of $K_{1,k}$, one copy of $K_{1,k-1}$, as well as one copy of $K_{1,1}$. Again number the copies $C_1, C_2, \ldots, C_{p+1}$ according to their decreasing order. For $i \leq q-1$ connect the central vertex of C_i with every vertex of C_j, $i < j \leq q+1$. The resulting graph G has exactly one $[1,k]$-factor and $|E(G)| = \frac{q(n+k)}{2} - 1$.

Theorem 5.30 *Let G be a graph of order n with a unique perfect $[1,2]$-factor. Then*

$$|E(G)| \leq \begin{cases} \frac{n^2}{4}, & \text{if } n \text{ is even.} \\ \frac{n^2}{4} + \frac{3}{4}, & \text{if } n \text{ is odd,} \end{cases}$$

Proof. Let F denote the unique perfect $[1,2]$-factor. We can make the following simple observations:

1. Every cycle of F must be of odd order as every cycle of even order has a 1-factor.
2. Two odd cycles C_1, C_2 of F cannot be connected with a blue edge e, as a 1-factor would exist in $C_1 \cup C_2 \cup \{e\}$.
3. An odd cycle C cannot have a blue chord as this splits C into an odd cycle and a path of even order, thus giving us a second perfect $[1,2]$-factor.

Let G_2 be the subgraph of G induced by the 2-regular components of F, and G_1 the subgraph of G induced by the $K_{1,1}$-components of F. If there are l such $K_{1,1}$ in F, Lemma 5.1 gives us at most l^2 edges in G_1. Due to the above observation, there are $n-2l$ edges in G_2, regardless of the number of cycles. Let now C be a cycle of F and H a $K_{1,1}$–component with $V(H) = \{x,y\}$. If there exists at least one edge e joining C and H, then, without loss of generality, let $e = xv$. Assume that there exists an edge yz with $z \in V(C)$. If $z = v$, then $C - v$ is a path of even order and thus has a 1-factor. Together with the cycle $xvyx$ we get a second perfect $[1,2]$-factor in G, a contradiction. If $vz \in E(C)$, then we find a second perfect $[1,2]$-factor where $x, y, V(C)$ belong to one cycle, again a contradiction to the uniqueness. In all other cases, $C - \{v,z\}$ consists of one path P_o of odd order and one path of even order, called P_e. The path P_e obviously has a 1-factor. We further have the cycle xvP_ozyx and thus a second perfect $[1,2]$-factor in $C \cup \{x,y\}$, again a contradiction.
As a consequence, all edges between C and H have to be incident with x,

giving us at most $|V(C)|$ such edges. Thus we have at most $l(n-2l)$ edges connecting a vertex in G_1 with one in G_2 and

$$|E(G)| \leq l^2 + l(n-2l) + n - 2l = n + l(n-2) - l^2. \tag{5.11}$$

For n odd we get from (5.11) that $l = \frac{n-3}{2}$, if $|E(G)|$ is maximal, and thus $|E(G)| \leq \frac{n^2+3}{4}$. If n is even, then $|E(G)| \leq n + ln - l^2$, which becomes maximal for $l = \frac{n}{2}$. This yields $|E(G)| \leq \frac{n^2}{4}$. □

The bounds are sharp as the following examples show: For even n take $\frac{n}{2}$ copies of $K_{1,1}$ numbered $H_1, \ldots, H_{n/2}$. For $1 \leq i < \frac{n}{2}$ connect one vertex of H_i to both vertices of H_j, with $i < j \leq \frac{n}{2}$. For odd n take $\frac{n-3}{2}$ $K_{1,1}$ and one cycle of length 3 and proceed as in the case of n even. Both graphs have a unique perfect $[1,2]$–factor and the desired number of edges.

With the following result of W.T. Tutte [37] (for a proof see also [40]) we are able to easily expand Theorem 5.30 to unique perfect $[1,k]$-factors.

Theorem 5.31 (Tutte [37]) *A graph G has a perfect $[1,2]$-factor if and only if $|S| \leq |N(S,G)|$ holds for every $S \subseteq V(G)$.*

Corollary 5.32 *Let G be a graph of order n with a unique perfect $[1,k]$-factor for $k \geq 2$. Then*

$$|E(G)| \leq \begin{cases} \frac{n^2}{4}, & \textit{if } n \textit{ is even.} \\ \frac{n^2}{4} + \frac{3}{4}, & \textit{if } n \textit{ is odd,} \end{cases}$$

Proof. The result is already proved for $k \leq 2$. Let $k \geq 3$ and assume that the factor F of G has an r-regular component H for an $3 \leq r \leq k$. Since $|S| \leq |N(S,H)|$ for every $S \subseteq V(H)$, as H is a regular graph, there exists with Theorem 5.31 a perfect $[1,2]$-factor in H. This contradicts the uniqueness of F. Thus all components of F are either 1–regular or 2–regular and the result follows immediately. □

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Zusammenfassung

Schon in der Arbeit von Julius Petersen im Jahre 1891, die als Ursprung der Faktortheorie bezeichnet werden kann, ist gezeigt worden, dass jeder reguläre Graph geraden Grades 2-faktorisierbar ist. Jedoch haben sich die übrigen Fälle als schwer angreifbar herausgestellt. Im allgemeinen kann man über die Existenz eines regulären Faktors in einem regulären Graphen nur wenig aussagen, wenn man nicht weitere Informationen über den Graphen zur Verfügung hat.

Im ersten Teil dieser Arbeit untersuchen wir den Einfluss verschiedener Parameter auf die Existenz eines regulären Faktors in einem regulären Graphen. Hierbei konzentrieren wir uns auf Parameter, die in dem Graphen eine gewisse Struktur erzwingen. Dies sind zum einen der Radius und der Durchmesser eines Graphen, zum anderen die chromatische Zahl sowie der Eckenzusammenhang. Den Grundstein für unsere Beweise legt dabei der f-Faktorsatz von Tutte und Belck. Besitzt ein Graph keinen regulären Faktor, so liefert der f-Faktorsatz die Existenz bestimmter Komponenten in einem Teilgraphen. Eine Idee von Wallis, Niessen und Randerath aufgreifend, charakterisiseren wir diese Komponenten mit Hilfe des vorgegebenen Parameters, um Aussagen über den Ursprungsgraphen fällen zu können, die hinreichende Bedingungen für die Existenz eines regulären Faktors implizieren.

Im zweiten Teil dieser Arbeit beschäftigen wir uns mit einer Fragestellung, die ihre Wurzeln in der Extremalen Graphentheorie hat: Wieviele Kanten kann ein Graph maximal besitzen, wenn er einen eindeutigen regulären Faktor enthält? Diese Frage ist erstmals um 1972 von Hetjei fuer 1-Faktoren und 1984 von Hendry für 2-Faktoren untersucht worden. Neuere Arbeiten von Johann und Volkmann erweitern diese Ergebnisse für $k = 3$ und in Spezialfällen, jedoch sind für $k \geq 4$ bisher nur Vermutungen über extremale Graphen aufgestellt worden. In der vorliegenden Arbeit untersuchen wir extremale bipartite Graphen mit einem eindeutigen regulären Faktor. Eine Strukturanalyse zeigt, dass die extremalen Graphen genau $2k$ Ecken vom Grad k besitzen, wenn der Graph einen eindeutigen k-Faktor enthält. Für $k \leq 4$ ermöglichen uns diese Resultate im bipartiten Fall neue Aussagen über die maximale Kantenzahl. Abschliessend übertragen wir die Fragestellung auf Graphen mit einem eindeutigen $[1, k]$-Faktor. Obwohl die Eckengrade in diesem Fall einer grösseren Freiheit unterliegen, zeigen wir, dass die Eindeutigkeit des Faktors eine strenge Struktur erzwingt, die scharfe Aussagen über die maximale Kantenzahl ermöglicht.

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