Paramodular Forms of Degree 2 with Particular Emphasis on Level $t = 5$

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Preface

Modular forms are important objects in Number Theory. In a simplified way, a modular form is a function on an open subset of $\mathbb{C}^n$ with an invariance property under a sufficiently large discrete group and a nice Fourier expansion. The theory of modular forms was initiated in the 19th century and was intimately related to the theory of integrals over algebraic functions (e. g. elliptic integrals). A systematic theory of modular forms with respect to the symplectic group $\mathrm{Sp}(n,\mathbb{Z})$ was developed by C. L. Siegel (1939). Modular forms can be used to determine for example numbers of representations of a quadratic form (Siegel's Hauptsatz) or Abelian extensions of (real or imaginary) number fields. Recently they were part in Wiles' proof of Fermat's last Theorem. One possible generalization of symplectic modular forms are paramodular forms.

First results about paramodular forms were shown by Christian [Ch1] and Köhler ([Kö1], [Kö2] and [Kö3]). Generators of a graded ring of paramodular forms of degree 2 were first determined by Igusa for Γ_1 (the paramodular group of degree 2 and level 1), i. e. the ring of Siegel modular forms of genus 2. Igusa's proof ([Igu]) was simplified by Freitag ([Fr4]) using a distinguished Siegel modular form Θ_5 with non-trivial multiplier system and known zeros. Then, using similar techniques, Freitag ([Fr2]) determined generators of the graded ring of modular forms of even weight with respect to $\Gamma_2^{\max}$, the maximal normal discrete extension of Γ_2 (the paramodular group of degree 2 and level 2) of index 2. Only recently, these results were extended to Γ_2 by Ibukiyama and Onodera ([IO]). Moreover, Runge [Run, Thm 2.3] described the even part of the graded ring of modular forms for paramodular groups of arbitrary degree as invariants of a certain space of theta constants. This result could so far not be used to give explicit information on generators. On the other hand Ibukiyama determined the dimension of the spaces of cusp forms ([Ib1]) from which one can deduce the Hilbert series for $\mathcal{A}(\Gamma_t)$ for small t (cf. [Ib2], [IO]).

Recent results of Borcherds ([Bor]) can be used to construct paramodular forms with known zeros, so-called Borcherds products. Note that Θ_5 is an example of a Borcherds product. Other examples of Borcherds products with respect to the paramodular group were given in [GN]. Using Borcherds products Dern ([Der]) generalized the method of Freitag ([Fr4]) and determined the generators for modular forms with respect to Γ_3. He found a Borcherds product which has a zero of order one on the divisor $\lambda^\perp$ with the lowest discriminant and no other zeros. Using generaliziations of Maaß's construction [Ma2] [Ma3] introduced by Gritsenko [Gr1], [Gr2] and Gritsenko-Nikulin [GN]– so-called "arithmetical liftings"– he was then able to lift all modular forms with respect to $\mathrm{Stab}_{\Gamma_3}\lambda^\perp$ to paramodular forms for Γ_3.

This method was extended by Dern and Krieg ([DKr1], [DKr2]) in order to determine the algebraic structure of some graded rings of hermitian modular forms of degree 2.

Given $t \in \mathbb{N}$ and $P_t = \operatorname{diag}(1,t)$, the paramodular group of level t is defined by $\widehat{\Gamma}_t = \{M \in \mathrm{GL}_4(\mathbb{Z}); M^{\mathrm{tr}} \left(\begin{smallmatrix} 0 & -P_t \\ P_t & 0 \end{smallmatrix}\right) M = \left(\begin{smallmatrix} 0 & -P_t \\ P_t & 0 \end{smallmatrix}\right)\}$. Since $\widehat{\Gamma}_t$ can be embedded into $\mathrm{Sp}_2(\mathbb{Q})$ (denoted by Γ_t), Γ_t acts on the Siegel upper half-space $\mathbb{H}_2 = \{Z = X + iY; X = X^{\mathrm{tr}}, Y^{\mathrm{tr}} = Y > 0\}$ in the usual way.

The main objects of this thesis are paramodular forms of degree 2, i. e. holomorphic functions $f : \mathbb{H}_2 \to \mathbb{C}$ with

$$f((AZ+B)(CZ+D)^{-1}) = \nu(M)\det(CZ+D)^k f(Z) \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_t,$$

where $k \in \mathbb{Z}$ is the weight and ν is an Abelian character of Γ_t. The vector space $\mathbb{M}_k(\Gamma_t, \nu)$ of these functions is finite dimensional.

The goal of this thesis was to determine the algebraic structure of (or at least a set of generators for) the $\mathbb{Z}$-graded algebra of all paramodular forms $\mathcal{A}(\Gamma_t) = \bigoplus_k \mathbb{M}_k(\Gamma_t, 1)$ for $t = 5$. In the course of the computations we were able to determine the structure of a graded ring of modular forms with respect to a certain subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})^2$ using methods from commutative algebra. The main result of this thesis yields a method to determine an invariant ring for a polynomial ring modulo a (principal) ideal $\mathcal{I}$. In order to describe $\mathcal{A}(\Gamma_5)$ we use the methods by Dern and Krieg, and thus obtain some structural results about $\mathcal{A}(\Gamma_5)$, e. g. the Hilbert series and a basis for $\mathbb{M}_k(\Gamma_5)$ for small weights k. Furthermore, we are able to determine four algebraically independent forms which are candidates for a homogeneous system of parameters. In any case their degrees match the degrees that are predicted by the Hilbert series.

Now we give a short description of the thesis:
In the first chapter we fix notations. In the second chapter we summarize results about the paramodular group – such as generators, extensions and the group of Abelian characters – and paramodular forms. Moreover, we calculate a system of representatives for the equivalence classes of $(n-1)$-cusps for the paramodular group $\Gamma(P)$ for arbitrary n and P (Theorem 2.5.13) and give an equivalent characterization of a paramodular cusp form.
Since Borcherds theory is written in the language of orthogonal groups, we translate, in Chapter 3, the paramodular group of degree 2 with square-free level t into the orthogonal setting. In the same way (Section 3.4) we can translate paramodular forms of degree 2 to orthogonal modular forms via a modular isomorphism (cf. the commutative diagram in Equation (3.16)). Following [FH] we consider Eichler transformations in order to prove that for square-free t all quadratic divisors of fixed discriminant are equivalent under $\Gamma_t^{\max}$. We determine the stabilizers for some quadratic divisors in case $t = 5$ with particular emphasis on $\lambda_9^{\perp}$. This stabilizer contains the group $\mathrm{SL}_2(\mathbb{Z})[3]^2$ as a normal subgroup with $\operatorname{Stab}\lambda_9^{\perp}/\mathrm{SL}_2(\mathbb{Z})[3]^2 \cong \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$. We complete this chapter with a characterization of $f \in \mathcal{A}(\operatorname{Stab}\lambda_9^{\perp})$ (i. e. the ring of all modular forms with respect to $\operatorname{Stab}\lambda_9^{\perp}$), which allows

us to determine $\mathcal{A}(\mathrm{Stab}\,\lambda_9^{\perp})$ by first determining $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$ and then the invariant ring of $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$ with respect to $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$.
In Chapter 4 we give some group theoretical results about the main congruence subgroup $\mathrm{SL}_2(\mathbb{Z})[3]$, e. g. generators, cusps and character table. Then we calculate generators for the graded ring of all modular forms with respect to $\mathrm{SL}_2(\mathbb{Z})[3]$ using well-known results from [Miy]. We then determine the algebraic structure of the ring of all modular forms with respect to $\mathrm{SL}_2(\mathbb{Z})[3]^2$ which is isomorphic to $\mathbb{C}[X_1,\ldots,X_4]/\langle X_1X_4 - X_2X_3\rangle$. Finally we calculate the exact representation of the invariance group.
At the beginning of Chapter 5 we collect some necessary results from Commutative Algebra. The most important structure – Cohen-Macaulay rings – will be discussed in Section 5.2, especially we generalize the Theorem of Hoechster-Eagon to Cohen-Macaulay rings. We calculate the structure of the ring

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)^{\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})}.$$

using the fact that this ring is Cohen-Macaulay and an exact sequence. More explicitly the exact sequence allows us to determine the invariants by calculating $\mathbb{C}[X_1,\ldots,X_4]^G$ and $\mathbb{C}[X_1,\ldots,X_4]^\chi$ for some character χ which can be done using computer algebra systems.

In Chapter 6 we discuss the Maaß lift which is essentially a lift from half-integral Jacobi forms with character to paramodular forms with character. The main result, Theorem 6.3.2, is essentially a reformulation of results from [Gr1] and [GN]. Using a dimension formula for these Maaß spaces from Skoruppa [Sko], we are able to give generators for the free $\mathbb{C}[g_4,g_6]$-module of Jacobi forms (where g_4 and g_6 denote the elliptic Eisenstein series of weight 4 and 6, respectively).
In Chapter 7 we construct paramodular forms of level 5 using Borcherds products. Using some properties of these forms we give a possible set of primary generators in the case that $\mathcal{A}(\Gamma_5)$ is Cohen-Macaulay. Using Ibukiyama's dimension formula ([Ib2]) for paramodular cusp forms we compute the exact Hilbert series for $\mathcal{A}(\Gamma_5)$ and determine a basis of $\mathbb{M}_k(\Gamma_5)$ for small weights. To close this chapter there are some remarks about a reduction process to determine the algebraic structure of $\mathcal{A}(\Gamma_5)$, and some ideas on further work.

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Contents

1 Notations

We denote by $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the set of natural, integral, rational, real and complex numbers, respectively, and define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\sharp A$ be the cardinality of the set A (throughout this thesis always finite). Given $z = x + iy \in \mathbb{C}$ by $x, y \in \mathbb{R}$, then $x := \mathrm{Re}\, z$ is the real part and $y := \mathrm{Im}\, z$ is the imaginary part of z. The upper half-plane is denoted by $\mathbb{H} := \{z \in \mathbb{C}; \mathrm{Im}\, z > 0\}$.

Let G be a group, H a subgroup of G (denoted by $H \leq G$) and $a, b \in G$. We write $a \equiv b \bmod H$ if $aH = bH$. Furthermore, we denote by $H \trianglelefteq G$ that H is normal in G. Given $S \subset G$, we denote by $\langle S \rangle \leq G$ the subgroup of G generated by all elements of S. The commutator of a and b is given by $[a,b] := aba^{-1}b^{-1}$. The commutator group DG is the subgroup of G generated by all commutators of elements of G. Let $G^{\mathrm{ab}} = \mathrm{Hom}(G, \mathbb{C}^\times)$ be the group of all Abelian characters of G which is isomorphic to G/DG. We denote by $G \times G$ the direct product of G with itself. We also use the abbreviation $G^2 = G \times G$.

Let n, m, k, l be elements of $\mathbb{N}$. Given a ring $\mathcal{R}$ (always assumed to be commutative and with unity), we denote by $\mathrm{Mat}_n(\mathcal{R})$ the set of n by n matrices with entries in $\mathcal{R}$. For A, $B \in \mathrm{Mat}_n(\mathcal{R})$, we denote by A^{tr} (resp. A^{-1}) the transpose (resp. the inverse provided its existence) of $A \in \mathrm{Mat}_n(\mathcal{R})$. Moreover, $B[A] := A^{\mathrm{tr}}BA$ and $A^{-\mathrm{tr}} := (A^{-1})^{\mathrm{tr}}$. The entries of A are denoted by a_{ij} for i, $j \in \{1,\ldots,n\}$, i. e. $A = (a_{ij})$. Let $I_n = (\delta_{ij}) \in \mathrm{Mat}_n(\mathcal{R})$ be the identity-matrix of rank n. We sometimes use I instead of I_n if n is obvious. The diagonal matrix with given diagonal entries $t_1,\ldots,t_n$ is denoted by $\mathrm{diag}(t_1,\ldots,t_n) = (t_j\delta_{ij})$. $E_{kl}^{(n)} = (\delta_{ki}\delta_{lj}) \in \mathrm{Mat}_n(\mathcal{R})$ is the matrix, where the (k,l)-entry is 1 and all others are 0. $e_k^{(n)} = (\delta_{ki}) \in \mathcal{R}^n$ is the column vector with k-entry 1 and 0 otherwise. We write E_{kl} for $E_{kl}^{(n)}$ and e_k for $e_k^{(n)}$ if the rank n is implicitly given by the context. If $k = l$, we use E_k instead of E_{kk}.

$\mathrm{SL}_n(\mathcal{R})$, $\mathrm{GL}_n(\mathcal{R})$ and $\mathrm{Sp}_n(\mathcal{R}) \subset \mathrm{GL}_{2n}(\mathcal{R})$ denotes the special linear group, the general linear group and the symplectic group over the ring $\mathcal{R}$, respectively. We use $(M_1, M_2) \mapsto M_1 \times M_2$ for the standard embedding $\mathrm{Sp}_m(\mathcal{R}) \times \mathrm{Sp}_n(\mathcal{R}) \hookrightarrow \mathrm{Sp}_{m+n}(\mathcal{R})$ of symplectic groups as in [Kr4, p. 44]. We denote by $\mathrm{Sym}_n(\mathcal{R})$ the set of all symmetric matrices $M = M^{\mathrm{tr}} \in \mathcal{R}^{n\times n}$. If additionally $\mathcal{R}$ is a subring of $\mathbb{R}$ and $S \in \mathrm{Sym}(\mathcal{R})$, we denote by $S > 0$ (resp. $S \geq 0$) positive definite (resp. positive semi-definite) matrices. Let $\mathcal{C}_n$ be the cyclic group of order n, i. e. $\mathcal{C}_n := \mathbb{Z}/n\mathbb{Z}$. We use $m \mid n$ if m divides n, $m \nmid n$ if m does not divide n and $m \parallel n$ if $m \mid n$ and $\gcd(m, n/m) = 1$. Given P, $U \in \mathrm{GL}_n(\mathcal{R})$ and $S \in \mathrm{Mat}_n(\mathcal{R})$, we use the abbreviations $\mathrm{rot}(U) = \begin{pmatrix} U & 0 \\ 0 & U^{-\mathrm{tr}} \end{pmatrix}$ and $\mathrm{trans}(S) = \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}$. Moreover, we define $J_P = \begin{pmatrix} 0 & -P^{-1} \\ P & 0 \end{pmatrix}$ and $\widehat{J_P} = \begin{pmatrix} 0 & -P \\ P & 0 \end{pmatrix}$. If $P = I$, we write $J := J_I = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Given matrices A, B, $N \in \mathrm{Mat}_n(\mathbb{Z})$, we write $A \equiv B \bmod N$ if $a_{j,k} \equiv b_{j,k} \bmod n_{j,k}$ for all $1 \leq j,k \leq n$, i. e. if the congruence is satisfied for every coefficient.

Given a map f, we write $\ker f$ for the kernel and $\operatorname{im} f$ for the image of f.

Given a field F we denote by $F[X_1,\dots,X_n]$ the polynomial ring in the variables $X_1,\dots,X_n$. If the variables are not necessary free, we use instead the notation $F\langle X_1,\dots,X_n\rangle$. Let $\mathcal{R}$ be a ring and $r_1,\dots,r_l \in \mathcal{R}$. Then we denote by $\langle r_1,\dots,r_l\rangle$ the ideal which is generated by the elements $r_1,\dots,r_l$.

2 Paramodular Groups

2.1 Definition and Generators

Given $P \in \mathrm{Pos}_n(\mathbb{Q}) = \{M \in \mathrm{Mat}_n(\mathbb{Q})\,;\, M = M^{\mathrm{tr}} > 0\}$, the *paramodular group of level P* is defined by

$$\Gamma(P) = \left\{ M \in \mathrm{Sp}_n(\mathbb{Q}) \,\middle|\, \left(\begin{smallmatrix} I & 0 \\ 0 & P \end{smallmatrix}\right)^{-1} M \left(\begin{smallmatrix} I & 0 \\ 0 & P \end{smallmatrix}\right) \in \mathrm{Mat}_{2n}(\mathbb{Z}) \right\}.$$

The conjugate group

$$\widehat{\Gamma}(P) = \left(\begin{smallmatrix} I & 0 \\ 0 & P \end{smallmatrix}\right)^{-1} \Gamma(P) \left(\begin{smallmatrix} I & 0 \\ 0 & P \end{smallmatrix}\right) \subset \mathrm{Mat}_{2n}(\mathbb{Z})$$

is the *integral paramodular group of level P*. Alternatively, $\widehat{\Gamma}(P)$ can be defined by

$$\widehat{\Gamma}(P) = \left\{ M \in \mathrm{Mat}_{2n}(\mathbb{Z})\,;\, \widehat{J}_P[M] = \widehat{J}_P \right\}.$$

This leads to the following *integral symplectic relations* of $\widehat{\Gamma}(P)$: Given a matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Mat}_{2n}(\mathbb{Z})$, we have

$$M \in \widehat{\Gamma}(P) \quad \Longleftrightarrow \quad A^{\mathrm{tr}}PC,\, D^{\mathrm{tr}}PB \text{ symmetric and } A^{\mathrm{tr}}PD - C^{\mathrm{tr}}PB = P. \tag{2.1}$$

For $P \neq \lambda I$, the paramodular group $\Gamma(P)$ is not invariant under transposition. Instead, $\Gamma(P)$ is invariant under the involution

$$\tau_P : \quad M \mapsto \begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix}^{-1} M^{\mathrm{tr}} \begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix}.$$

As is well-known, every paramodular group $\Gamma(P)$ is conjugate to a paramodular group $\Gamma(P')$, where $P' = \mathrm{diag}(t_1, \ldots, t_n)$ is a matrix of elementary divisors (essentially, $t_1, \ldots, t_n$ are the elementary divisors of qP, where $0 < q \in \mathbb{Q}$ is chosen in such a way, that qP is integral and primitive; see [Run]).

Thus we can assume without restriction, that P is a matrix of elementary divisors, i. e. from now on we have $P = \mathrm{diag}(t_1, \ldots, t_n)$ with $t_1 = 1$ and $t_j / t_{j-1} \in \mathbb{N}$ for all $2 \leq j \leq n$. Since the quotients t_i/t_j are used frequently in the sequel, we write $t_{i,j} := t_i^{-1} t_j$ for short. Of course, we have $t_{i,j} \in \mathbb{N}$ if $i \leq j \leq n$. For reasons explained later, it is also convenient to define $t_{0,1} := t_{n,n+1} := 12$ (see Section 2.3).

We define

$$\begin{aligned}
\Omega(P) &= \left\{U \in \mathrm{GL}_n(\mathbb{Z})\,;\, PUP^{-1} \in \mathrm{Mat}_n(\mathbb{Z})\right\},\\
\Sigma(P) &= \left\{M \in \mathrm{Mat}_n(\mathbb{Q})\,;\, M = M^{\mathrm{tr}}, MP \in \mathrm{Mat}_n(\mathbb{Z})\right\},\\
\widehat{\Sigma}(P) &= \left\{M \in \mathrm{Mat}_n(\mathbb{Z})\,;\, MP^{-1} \text{ symmetric}\right\}.
\end{aligned}$$

Typical elements of $\Gamma(P)$ are $\mathrm{rot}(U)$ for $U \in \Omega(P)$, $\mathrm{trans}(S)$ for $S \in \Sigma(P)$ and J_P.

In degree 2 we define $P_t = \mathrm{diag}(1,t)$, $\Gamma_t = \Gamma(P_t)$ and $\widehat{\Gamma}_t = \widehat{\Gamma}(P_t)$ as abbreviations. In the same way we set $\tau_t = \tau_{P_t}$, $\Omega_t = \Omega(P_t)$, $\Sigma_t = \Sigma(P_t)$ and $\widehat{J}_t = \widehat{J}_{P_t}$.

The entries of $M \in \Gamma(P)$ have bounded denominators, depending on the entries of P (cf. [Mar]). We briefly show that using both $\Gamma(P)$ and $\widehat{\Gamma}(P)$ have their advantages depending on the context: The congruence-properties of paramodular matrices can be described most easily by the integral realisation $\widehat{\Gamma}(P)$: Given $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \widehat{\Gamma}(P)$, we get by the symplectic relations

$$A, B, C, D \equiv 0 \bmod N(P) \quad \text{with} \quad N(P) = (\max\{1, t_{i,j}\})_{1 \le i,j \le n}. \tag{2.2}$$

Thus all four blocks of $M \in \widehat{\Gamma}(P)$ have the same congruence-properties, which is obviously not true for $M \in \Gamma(P)$. On the other hand $\Gamma(P)$ (as a subgroup of $\mathrm{Sp}_n(\mathbb{Q})$) acts in the usual way on the *Siegel upper half-space*

$$\mathbb{H}_n := \{Z = X + iY \in \mathrm{Mat}_n(\mathbb{C})\,;\, X, Y \in \mathrm{Sym}_n(\mathbb{R}), Y > 0\},$$

which is in turn not true for $M \in \widehat{\Gamma}(P)$. As a matter of fact, we will skip between the two realizations of the paramodular group.

Lemma 2.1.1 ([Ka1, Lem 1.2]) *Given $1 \le n < m$, let $P = \mathrm{diag}(t_1, \ldots, t_n) \in \mathrm{Pos}_n(\mathbb{Z})$ and $Q = \mathrm{diag}(u_1, \ldots, u_m) \in \mathrm{Pos}_m(\mathbb{Z})$ be matrices of elementary divisors. Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ satisfy $\sigma(i) < \sigma(j)$ if $i < j$ and $t_i = gu_{\sigma(i)}$ for all $i = 1, \ldots, n$ with a fixed $g \in \mathbb{N}$. Extend σ to a mapping $\widetilde{\sigma} : \{1, \ldots, 2n\} \to \{1, \ldots, 2m\}$ by $\widetilde{\sigma}(i) := m + \sigma(i-n)$ for $i = n+1, \ldots, 2n$. Then there is an injective homomorphism $\Lambda_\sigma : \widehat{\Gamma}(P) \to \widehat{\Gamma}(Q)$, defined by*

$$(\Lambda_\sigma(M))_{kl} = \begin{cases} m_{ij}, & \text{if } (k,l) = (\widetilde{\sigma}(i), \widetilde{\sigma}(j)), \\ \delta_{kl}, & \text{otherwise,} \end{cases}$$

for $M = (m_{ij}) \in \widehat{\Gamma}(P)$.

Example 2.1.2 *a) Let $n = 1$. In this special case every mapping $\sigma : \{1\} \to \{1, \ldots, m\}$ satisfies the conditions in the Lemma 2.1.1 (since $t_1 = 1$), and we write $\Lambda_i := \Lambda_\sigma$ for short if $\sigma(1) = i$. Thus for every $i \in \{1, \ldots, n\}$ we have an embedding $\Lambda_i : \mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \widehat{\Gamma}(P)$.*

b) Given $1 \le j < k \le n$, we define $\sigma_{j,k} : \{1,2\} \to \{1, \ldots, n\}$ by $1 \mapsto j$ and $2 \mapsto k$. Since σ fulfills the condition of Lemma 2.1.1, we get an embedding

$$\Lambda_{j,k} : \widehat{\Gamma}(P_{t_{j,k}}) \to \widehat{\Gamma}(P).$$

We will need these embeddings to calculate the number of inequivalent cusps.

c) *Obviously, we can combine these embeddings to embed direct products of paramodular groups, e. g. we can embed* $\mathrm{SL}_2(\mathbb{Z})^n$ *diagonally into* $\widehat{\Gamma}(P)$ *by*

$$(M_1, \ldots, M_n) \mapsto M_1 \times \ldots \times M_n.$$

Knowing suitable “elementary” generators of $\widehat{\Gamma}(P)$ is essential for many calculations for $\widehat{\Gamma}(P)$. The following lemma was proven by Kappler [Ka1] using results from [Ch1], [Gro] and [Kl2].

Lemma 2.1.3 ([Ka1, Thm 1.12]) $\widehat{\Gamma}(P)$ *is generated by* $\widehat{J_I}$ *and* $\mathrm{trans}(S_{ij})$ *for* $1 \leq i \leq j \leq n$ *with*

$$S_{ij} = \begin{cases} E_{ii}, & \text{if } i = j, \\ t_{i,j}E_{ij} + E_{ji}, & \text{if } i < j. \end{cases}$$

Example 2.1.4 *Let* $t \in \mathbb{N}$. *Then the group* Γ_t *is generated by*

$$J_t \text{ and } \mathrm{trans}(S) \text{ for } S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{t} \end{pmatrix}.$$

Note that Rosati ([Ros]) and Runge ([Run]) also determined (different) sets of generators for the paramodular group.

2.2 An Extension of Γ_t

The paramodular group $\Gamma(P) \subset \mathrm{Sp}_n(\mathbb{Q})$ has in general non-trivial discrete extensions in $\mathrm{Sp}_n(\mathbb{R})$ ([Kö1]). In this section we give a short description of these extensions in the case of degree 2 following [Gr1].

Let $t = \prod_{p|t} p^{r_p}$ be the prime decomposition of $t \in \mathbb{N}$ and $\nu(t)$ the number of prime factors of t. For each prime p with $p \mid t$, let $d_p = p^{r_p}$ be the highest p-power dividing t. Moreover, we define $\Delta_t = \left\{ d \mid t\,;\, d \neq 1, \gcd(d, \frac{t}{d}) = 1 \right\}$. For $d \in \Delta_t$, we choose $x, y \in \mathbb{Z}$ such that $xd - y\frac{t}{d} = 1$ and define $V_d \in \mathrm{Sp}_2(\mathbb{R})$ by

$$V_d = \begin{pmatrix} U_d & 0 \\ 0 & U_d^{-\mathrm{tr}} \end{pmatrix} \quad \text{with} \quad U_d = \tfrac{1}{\sqrt{d}} \begin{pmatrix} dx & -t \\ -y & d \end{pmatrix}. \tag{2.3}$$

Up to Γ_t-equivalence, V_d does not depend on the choice of x and y: All possible choices of (x,y) are equivalent modulo $\gcd(\frac{t}{d}, d)$. If we replace (x,y) in U_d by $(x', y') = (x + j\frac{t}{d}, y + jd)$ for some $j \in \mathbb{Z}$, we have

$$U_d' = \tfrac{1}{\sqrt{d}} \begin{pmatrix} d(x+j\frac{t}{d}) & t \\ y+jd & d \end{pmatrix} = \tfrac{1}{\sqrt{d}} \begin{pmatrix} dx & t \\ y & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} = U_d \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} \text{ and } V_d' = V_d \,\mathrm{rot}\begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix}.$$

Since the problems we are dealing with later (extensions of Γ_t generated by V_d's, characters and modular forms to these extensions etc.) are (shown to be) invariant under Γ_t-equivalence, we suppress this choice of (x,y) in the notation. Let $\Gamma_t^{\max} = \langle \Gamma_t, V_d;\, d \in \Delta_t \rangle$ be a discrete normal extension of Γ_t and $\Gamma_t^* = \langle \Gamma_t, V_t \rangle$. Since every discrete normal extension of Γ_t is contained in $\Gamma_t^{\max}$ ([Kö2, Satz 4']) it is often called a maximal discrete extension of Γ_t.

From now on let t be **square-free**. We consider

$$\Omega(P_t) = \Gamma^0(t) = \{ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z});\ \beta \equiv 0 \bmod t \}$$

and the maximal discrete extension, the so-called *Fricke-group* (cf. [Kr2]),

$$\Gamma^0(t)^{\max} = \left\langle \Gamma^0(t), U_d;\, d \in \mathbb{N},\, d \mid t \right\rangle .$$

We choose without loss of generality

$$U_t = \begin{pmatrix} 0 & \sqrt{t} \\ 1/\sqrt{t} & 0 \end{pmatrix} \quad \text{and} \quad U_1 = I.$$

The extensions in higher degrees are defined in an analogous way ([Kö1]). Let $\Gamma(P)^{\max}$ the maximal discrete normal extension of $\Gamma(P)$ as defined in [Kö2].

2.3 Abelian Characters for $\widehat{\Gamma}(P)$

In this paragraph we will summarize the result from [DM]. Keep in mind that these results were already known before in the case of degree $n \leq 2$, see for example [GH].

Since the result depends on the 2-powers dividing the quotients $t_{i,i+1}$, we define

$$s_{i,i+1} := \gcd(t_{i,i+1}, 4) \text{ for } i = 0, \dots, n$$

(recall $t_{0,1} = t_{n,n+1} = 12$, thus $s_{0,1} = s_{n,n+1} = 4$). Especially, we have to distinguish certain sub-blocks of P corresponding to distinguished patterns that appear in the sequence $(s_{i,i+1})_{0 \leq i \leq n}$:

Definition 2.3.1 *a)* $\mathrm{diag}(t_j, t_{j+1})$ *is a* sub-block of P of type 1 *if* $s_{j-1,j} > 1$, $s_{j,j+1} = 1$ *and* $s_{j+1,j+2} > 1$.
b) $\mathrm{diag}(t_j, \dots, t_{j+d-1})$ *with* $d \geq 2$ *is a* sub-block of P of type 2 *if* $s_{j-1,j} = 4$, $s_{i,i+1} = 2$ *for* $j \leq i \leq j+d-2$ *and* $s_{j+d-1,j+d} = 4$.

Note that every entry t_j belongs at most to one sub-block (and to none at all in general). Let c_j be the number of sub-blocks of P of type j and define $l_i = \gcd(t_{i-1,i}, t_{i,i+1})$ for $i = 1, \dots, n$.

Given $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \widehat{\Gamma}(P)$ and $i, k \in \mathbb{N}$ with $i+k-1 \leq n$, we define

$$\pi_{i,k}(M) = \begin{pmatrix} A_{i,k} & B_{i,k} \\ C_{i,k} & D_{i,k} \end{pmatrix}$$

where for $X = A,B,C,D$ the matrix $X_{i,k}$ is the sub-block of dimension k of X given by $X_{i,k} = (x_{lm})_{i \leq l, m \leq i+k-1}$.

We get the following homomorphisms by a straightforward calculation.

Lemma 2.3.2 *[DM, Lem 5.1]*
The following maps are homomorphisms of groups:

a) $\Psi_{i,1,l_i} : \widehat{\Gamma}(P) \to \mathrm{SL}_2(\mathcal{C}_{l_i}), M \mapsto \pi_{i,1}(M) \bmod l_i$,

b) $\Psi_{i,2,2} : \widehat{\Gamma}(P) \to \mathrm{Sp}_2(\mathcal{C}_2), M \mapsto \pi_{i,2}(M) \bmod 2$, *if* $\mathrm{diag}(t_i, t_{i+1})$ *is a sub-block of type* 1, *and*

c) $\Psi_{i,k,N} : \widehat{\Gamma}(P) \to \widehat{\Gamma}(Q_k)[N], M \mapsto \pi_{i,k}(M) \bmod N$, *if* $\mathrm{diag}(t_i, \dots, t_{i+k-1})$ *is a sub-block of type* 2, *where* $Q_k = \mathrm{diag}(1,2,4,\dots,2^{k-1})$ *and* $\widehat{\Gamma}(Q_k)[N]$ *is in principle the group* $\widehat{\Gamma}(Q_k)$ *modulo a certain matrix* N*. The exact definition is contained in [DM].*

Example 2.3.3 *Let* $t \in \mathbb{N}$.

a) We have $l_i = \gcd(12,t)$ *for* $i = 1,2$ *and therefore*

$$\Psi_{i,1,l_i} : \quad \widehat{\Gamma}_t \to \mathrm{SL}_2(\mathcal{C}_{l_i}), \qquad M \mapsto \begin{pmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{pmatrix} \bmod l_i.$$

b) P_t *itself is a sub-block of type* 1 *if and only if* $2 \nmid t$*. In this case we have*

$$\Psi_{1,2,2} : \quad \widehat{\Gamma}_t \to \mathrm{Sp}_2(\mathcal{C}_2), \qquad M \mapsto M \bmod 2.$$

c) P_t *itself is a sub-block of type* 2 *if and only if* $2 \parallel t$*. In this case we have* $N = \left(\begin{smallmatrix} \widetilde{N} & \widetilde{N} \\ \widetilde{N} & \widetilde{N} \end{smallmatrix}\right)$ *with* $\widetilde{N} = \left(\begin{smallmatrix} 4 & 4 \\ 2 & 4 \end{smallmatrix}\right)$ *and*

$$\Psi_{1,1,N} : \quad \widehat{\Gamma}_t \to \widehat{\Gamma}_2[N], \qquad M \mapsto M \bmod N.$$

Using these homomorphisms, we can define characters of $\widehat{\Gamma}(P)$ in the following way:

1) Let $\mu = v_\eta^2$ be the generator of $\mathrm{SL}_2(\mathbb{Z})^{\mathrm{ab}}$ determined by $\mu(T) = e^{2\pi i/12}$. Given $l \in \mathbb{N}$ with $l \mid 12$, one has $\ker(\mu^{12/l}) \subset \mathrm{SL}_2(\mathbb{Z})[l]$. Therefore we can think of $\mu^{12/l}$ as a character of $\mathrm{SL}_2(\mathcal{C}_l) \cong \mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})[l]$. Thus for $i \in \mathbb{N}$ with $i \leq n$ we get a character μ_i of $\widehat{\Gamma}(P)$ of order $l_i = \gcd(t_{i-1,i}, t_{i,i+1})$ defined by $\mu_i := \mu^{12/l_i} \circ \Psi_{i,1,l_i}$. Explicit values can be calculated with Corollary 4.2.5.
2) Let $\kappa : \mathrm{Sp}_2(\mathcal{C}_2) \to \mathbb{C}_1^*$ be the unique non-trivial character of $\mathrm{Sp}_2(\mathcal{C}_2)$ (see Section 8.2). If $\mathrm{diag}(t_i, t_{i+1})$ is a sub-block of P of type 1, we get a character $\kappa_{i,2}$ of $\widehat{\Gamma}(P)$ of order 2, defined by $\kappa_{i,2} := \kappa \circ \Psi_{i,2,2}$. Explicit values of $\kappa_{i,2}$ (on generators at least) can be easily derived from [Ma1, (37)].

3) Let $\lambda_k \in \widehat{\Gamma}(Q_k)[N]^{\mathrm{ab}}$ be the special character of order 4 from [DM]. If $\mathrm{diag}(t_i,\ldots,t_{i+k-1})$ is a sub-block of P of type 2, we get a character $\lambda_{i,k}$ of $\widehat{\Gamma}(P)$ of order 4, defined by $\lambda_{i,k} := \lambda_k \circ \Psi_{i,i+k-1,N}$. More explicit values of $\lambda_{i,k}$ (on generators at least) can be derived from [DM, Lem 4.2].

Example 2.3.4 *Let $t \in \mathbb{N}$.*

a) If $l = \gcd(12,t) \neq 1$, we have two characters defined by

$$\mu_i : \widehat{\Gamma}_t \to \mathbb{C}, \quad \mu_i(M) = \mu^{12/l}\left(\begin{pmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{pmatrix} \bmod l\right) \quad \text{for } i = 1,2.$$

b) If $2 \nmid t$, we get the character

$$\kappa_{1,2} : \widehat{\Gamma}_t \to \mathbb{C}, \quad \kappa_{1,2}(M) = \kappa\,(M \bmod 2)\,.$$

We are now able to state the main result of [DM]:

Theorem 2.3.5 *Given $j \in \{1,2\}$, let c_j be the number of sub-blocks of P of type j. Then $\widehat{\Gamma}(P)^{\mathrm{ab}}$ has order*

$$2^{c_1+c_2}\prod_{i=1}^{n}\gcd(t_{i-1,i},t_{i,i+1},12).$$

More precisely, if $D_1,\ldots,D_{c_1}$ are the sub-blocks of type 1 and $D_{c_1+1},\ldots,D_{c_1+c_2}$ are the sub-blocks of type 2 with $D_j = \mathrm{diag}(t_{i_j},\ldots,t_{i_j+k_j-1})$, then $\widehat{\Gamma}(P)^{\mathrm{ab}}$ is generated by

$$\mu_i, \quad 1 \le i \le n, \quad \kappa_{i_j,2}, \quad 1 \le j \le c_1, \quad \lambda_{i_j,k_j}, \quad c_1+1 \le j \le c_1+c_2.$$

Example 2.3.6 *Let $t \in \mathbb{N}$.*

a) $\widehat{\Gamma}_t^{\mathrm{ab}}$ has order $\gcd(12,t)\gcd(12,2t)$.
b) Let $t \neq 2,3$ be a prime number. Then $\widehat{\Gamma}_t^{\mathrm{ab}} \cong \mathcal{C}_2$ is generated by $\kappa_{1,2}$. Moreover, $(\widehat{\Gamma}_t^{\max})^{\mathrm{ab}}$ is isomorphic to $\mathcal{C}_2 \times \mathcal{C}_2$ and generated by $\kappa_{1,2}$ and χ_t which is defined by $\chi_t(V_t) = -1$ and $\chi_t(\widehat{\Gamma}_t) = 1$.

2.4 Paramodular Forms

Let $k \in \mathbb{Z}$ and $n \geq 2$. Since $\Gamma(P)^{\max} \subset \mathrm{Sp}_n(\mathbb{R})$, this group acts on $\{f : \mathbb{H}_n \to \mathbb{C}\}$ by

$$(f\,|_k\,M)\,(Z) = \det(CZ+D)^{-k} f\left((AZ+B)(CZ+D)^{-1}\right) \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R}).$$

Definition 2.4.1 *Let $\Gamma \subset \Gamma(P)^{\max}$ of finite index and $\nu \in \Gamma^{\mathrm{ab}}$. A holomorphic function $f : \mathbb{H}_n \to \mathbb{C}$ is a paramodular form of weight k with respect to Γ and character ν if*

$$(f\,|_k\,M)(Z) = \nu(M)\,f \quad \textit{for all } M \in \Gamma.$$

The space of paramodular forms of weight k with respect to Γ and character ν is denoted by $\mathbb{M}_k(\Gamma,\nu)$. If $\nu = 1$, then we often write $\mathbb{M}_k(\Gamma)$.

Since

$$\mathbb{M}_k(\Gamma,\nu_1)\cdot\mathbb{M}_l(\Gamma,\nu_2) \subset \mathbb{M}_{k+l}(\Gamma,\nu_1\nu_2)$$

and

$$\mathbb{M}_k(\Gamma,\nu) = \begin{cases} \{0\}, & k < 0 \text{ or } (k = 0 \text{ and } \nu \neq 1),\\ \mathbb{C}, & k = 0, \nu = 1, \end{cases}$$

we get the $\mathbb{Z}$-graded algebra $\mathcal{A}(\Gamma) := \bigoplus_{k\in\mathbb{N}_0}\mathbb{M}_k(\Gamma)$ and the $\mathcal{A}(\Gamma)$-module

$$\mathcal{A}(\Gamma,\nu) := \bigoplus_{k\in\mathbb{N}_0}\mathbb{M}_k(\Gamma,\nu) \;\text{ in the algebra }\; \bigoplus_{\nu\in\Gamma^{\mathrm{ab}}}\mathcal{A}(\Gamma,\nu).$$

Remark 2.4.2 *Since every $\Gamma \subset \Gamma_t^{\max}$ of finite index is commensurable with Γ_1, we know ([Fr1, II Thm 6.11]) that the $\mathbb{C}$-algebra $\mathcal{A}(\Gamma)$ is finitely generated. Moreover, the transcendence degree of the homogeneous field of fractions* $\mathrm{Quot}_{\mathrm{hom}}(\mathcal{A}(\Gamma))$, *which contains all quotients of paramodular forms of the same weight, is equal to 3. Especially the length of a set of algebraic independent paramodular forms of degree 2 is lower or equal to 4. For a direct proof of this statement see Section 7.*

Throughout this thesis let $\Gamma(P) \subset \Gamma \subset \Gamma(P)^{\max}$. Therefore Γ contains a congruence subgroup $\mathrm{Sp}_n(\mathbb{Z})[l] := \{M \in \mathrm{Sp}_n(\mathbb{Z}); M \equiv I \bmod l\}$ such that ν is a character mod l for every $\nu \in \Gamma^{\mathrm{ab}}$. Thus the Koecher principle holds, i. e. every $f \in \mathbb{M}_k(\Gamma,\nu)$ has a Fourier expansion of type

$$f(Z) = \sum_{T\in\Lambda_n(l),\,T\geq 0} \alpha_f(T)e^{\pi i\,\mathrm{Sp}(TZ)}$$

with

$$\Lambda_n(l) = \{T \in \mathrm{Sym}_n(\mathbb{Q});\ lT \text{ even, integral matrix}\}.$$

In the case of $\Gamma = \Gamma_t$ and $\nu = 1$ we can $\Lambda_2(l)$ substitute by

$$\Lambda_2(P_t) = \{T = \begin{pmatrix} 2\lambda_1 & \lambda_2 \\ \lambda_2 & 2t\lambda_3 \end{pmatrix};\ \lambda_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq 3\}.$$

In order to calculate the exact lattice $\Lambda(P,\nu)$ for other Abelian characters $\nu \in \Gamma(P)^{\mathrm{ab}}$, we need to determine all S with $\mathrm{trans}(S) \in \Lambda$ and $f(Z+S) = f(Z)$, which depends on P and ν only (compare with [Mar]).

2.5 Cusps of $\Gamma(P)$ and Paramodular Cusp Forms

2.5.1 Cusp Forms

We define the Siegel Φ-operator by

$$\Phi : \mathbb{M}_k(\Gamma(P), \nu) \to \mathbb{M}_k\left(\Gamma((P_{i,j})_{1\le i,j\le n-1}), \widetilde{\nu}\right), \quad f \mapsto f \mid \Phi,$$

where $(P_{i,j})_{1\le i,j\le n-1} = \operatorname{diag}(t_1, \ldots, t_{n-1})$, the character $\widetilde{\nu} : \Gamma(\operatorname{diag}(t_1, \ldots, t_{n-1})) \to \mathbb{C}$ is defined by $\widetilde{\nu}(M) = \nu(M \times I)$ for $M \in \Gamma(\operatorname{diag}(t_1, \ldots, t_{n-1}))$ and

$$f \mid \Phi : \mathbb{H}_{n-1} \to \mathbb{C}, \quad (f \mid \Phi)(Z_1) = \lim_{y\to\infty} f\begin{pmatrix} Z_1 & 0 \\ 0 & iy \end{pmatrix}.$$

Φ is well-defined and a homomorphism of vector spaces. A real matrix A is called *projetive rational* if there exists a $t \in \mathbb{R} \setminus \{0\}$ such that tA is rational. As in [Fr1, II 6.9]:

Definition 2.5.1 *$f \in \mathbb{M}_k(\Gamma(P), \nu)$ is called a* paramodular cusp form of weight k and character ν *if and only if*

$$(f \mid_k N) \mid \Phi = 0 \quad \textit{for all projective rational matrices } N \in \mathrm{Sp}_n(\mathbb{R}).$$

We denote by $\mathbb{S}_k(\Gamma(P), \nu)$ the space of all paramodular cusp forms of weight k and character ν.

Since $\mathrm{Sp}_n(\mathbb{Z})$ and $\Gamma(P)$ are commensurable, it is sufficient to take $N \in \mathrm{Sp}_n(\mathbb{Z})$ from a system of representatives of the cosets $\left(\Gamma(P) \cap \mathrm{Sp}_{n(\mathbb{Z})}\right) N$. It is possible to choose the representative in $\mathrm{Sp}_n(\mathbb{Q})$. We define the stabilizer of the standard boundary component of maximal dimension in $\mathrm{Sp}_n(\mathbb{Q})$ (compare with [Kl1, §5]) as follows

$$C(\mathbb{Q}) := C_{n,n-1}(\mathbb{Q}) = \{M \in \mathrm{Sp}_n(\mathbb{Q});\ M = \begin{pmatrix} * & * \\ 0_{2n-1}^{\mathrm{tr}} & * \end{pmatrix}\}. \tag{2.4}$$

Every $M \in C(\mathbb{Q})$ has the form

$$M = \begin{pmatrix} A_1 & 0 & B_1 & * \\ * & u^{-1} & * & * \\ C_1 & 0 & D_1 & * \\ 0 & 0 & 0 & u \end{pmatrix} \text{ with } M_1 := \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}_{n-1}(\mathbb{Q}) \text{ and } u \in \mathbb{Q} \setminus \{0\}.$$

From

$$f \mid_k NM \mid \Phi = u^{-k} f \mid_k N \mid \Phi \mid_k M_1 \ \text{ for all } N \in \mathrm{Sp}_n(\mathbb{Q}) \text{ and } M \in C(\mathbb{Q})$$

and

$$f \mid_k NM = f \mid_k M \ \text{ for all } M \in \mathrm{Sp}_n(\mathbb{Q}) \text{ and } N \in \Gamma(P),$$

we conclude

Proposition 2.5.2 *Let $f \in \mathbb{M}_k(\Gamma(P), \nu)$. Then f is a paramodular cusp form if and only if*

$$f \mid M \mid \Phi = 0 \quad \textit{for all } M \in \Gamma(P) \backslash \mathrm{Sp}_n(\mathbb{Q}) / C(\mathbb{Q}).$$

Especially, the number of matrices to be checked is finite.

2.5.2 Short Repetition on Cusps

Most of the results of this paragraph are contained in [Nam, §1 and §4]. Define $\mathbb{D}_n := \left\{Z \in \mathrm{Sym}_n(\mathbb{C});\ I_n - \overline{Z}Z > 0\right\}$ and let $\overline{\mathbb{D}}_n$ be the topological closure of $\mathbb{D}_n$ in $\mathrm{Sym}_n(\mathbb{C})$, i. e. $\overline{\mathbb{D}}_n = \{Z \in \mathrm{Sym}_n(\mathbb{C});\ I_n - \overline{Z}Z \geq 0\}$. $\mathbb{D}_n$ is isomorphic to $\mathbb{H}_n$ via the *Cayley transformation*

$$\mathcal{C} : \mathbb{H}_n \to \mathbb{D}_n,\ \tau \mapsto Z = (\tau - iI)(\tau + iI)^{-1}.$$

The inverse $\mathcal{C}^{-1}$ is given by $\mathcal{C}^{-1}(Z) = i(Z + I_n)(-Z + I_n)^{-1}$. For given $p, q \in \overline{\mathbb{D}}_n$, we call $p \sim_{\mathbb{D}} q$ if and only if there exist holomorphic maps $\xi_i : \mathbb{D}_1 \to \overline{\mathbb{D}}_n$, $i = 1, \dots, m$, such that

$$\xi_1(0) = p, \quad \xi_m(0) = q, \quad \text{and} \quad \xi_i(\mathbb{D}) \cap \xi_{i+1}(\mathbb{D}) \neq \emptyset.$$

This is an equivalence relation. Roughly speaking, $p \sim_{\mathbb{D}} q$, if they can be connected by a finite number of holomorphic curves. A maximal subset in $\overline{\mathbb{D}}_n$ of mutually equivalent points is called a *boundary component* of $\mathbb{D}_n$. Therefore $\overline{\mathbb{D}}_n$ is a disjoint union of boundary components. The action of $\mathrm{Sp}_n(\mathbb{R})$ on $\mathbb{D}_n$, which is induced from the Cayley transformation, can be extended to $\overline{\mathbb{D}}_n$ and has the following form:

$$(M, Z) \mapsto M * Z = ((A - iC)(Z + I) + (B - iD)\, i\,(Z - I)) \cdot \\ ((A + iC)(Z + I) + (B + iD)\, i\,(Z - I))^{-1}.$$

The disjoint union of $\overline{\mathbb{D}}_n$ into its boundary components is invariant under the operation of $\mathrm{Sp}_n(\mathbb{R})$. The set $\mathfrak{F}_l = \{\left(\begin{smallmatrix} Z' & 0 \\ 0 & I_{n-l}\end{smallmatrix}\right);\ Z' \in \mathbb{D}_l\} \cong \mathbb{D}_l$ is called the *Standard boundary component* of degree l and is a boundary component. We get the decomposition $\overline{\mathbb{D}}_n = \bigcup_{0 \leq l \leq n} \mathrm{Sp}_n(\mathbb{R}) * \mathfrak{F}_l$. If $\mathfrak{F} = M * \mathfrak{F}_l$ for a given $M \in \mathrm{Sp}_n(\mathbb{R})$, then $\mathfrak{F}$ is called a boundary component of degree l. A generalization of $C(\mathbb{Q})$ in Equation (2.4) is given by

Definition 2.5.3 *a) Let $F \in \{\mathbb{R}, \mathbb{Q}\}$, then we decompose the matrix X for $X = A, B, C, D$ into blocks*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

such that $X_{11} \in \mathrm{Mat}_j(F)$. We introduce the following groups:

$$C_{n,j}(F) = \{M \in \mathrm{Sp}_n(F);\ C_{21} = C_{22} = D_{21} = 0\} \ \textit{ and } \ C_{n,j}^{\mathrm{proj}} = \mathrm{Sp}_n(\mathbb{R})^{\mathrm{proj}} \cap C_{n,j}(\mathbb{R}),$$

where $\mathrm{Sp}_n(\mathbb{R})^{\mathrm{proj}}$ contains all $M \in \mathrm{Sp}_n(\mathbb{R})$ which are projective rational.

b) Given a boundary component $\mathfrak{F}$ of $\mathbb{D}_n$, we define the parabolic subgroup *to $\mathfrak{F}$ by* $\mathfrak{P}(\mathfrak{F}) := \{M \in \mathrm{Sp}_n(\mathbb{R});\, M * \mathfrak{F} = \mathfrak{F}\}$.

The parabolic subgroup of $\mathfrak{F}_l$ is $C_{n,l}(\mathbb{R})$. Given two boundary components $\mathfrak{F}_1$ and $\mathfrak{F}_2$, such that $M * \mathfrak{F}_1 = \mathfrak{F}_2$ for a matrix $M \in \mathrm{Sp}_n(\mathbb{R})$, then $\mathfrak{P}(\mathfrak{F}_2) = M\mathfrak{P}(\mathfrak{F}_1)M^{-1}$. If $\mathfrak{P}(\mathfrak{F}_1) = \mathfrak{P}(\mathfrak{F}_2)$, then $\mathfrak{F}_1 = \mathfrak{F}_2$. Given boundary components $\mathfrak{F}_1, \mathfrak{F}_2$ of $\mathbb{D}_n$, we write

$$\mathfrak{F}_1 < \mathfrak{F}_2 \quad \text{if and only if} \qquad \mathfrak{F}_1 \subset \overline{\mathfrak{F}}_2.$$

A boundary component $\mathfrak{F}$ of $\mathbb{D}_n$ is called *rational* if and only if $\mathfrak{P}(\mathfrak{F}) \subset \mathrm{Sp}_n(\mathbb{Q})$. It is known ([Nam, p. 27]) that a boundary component $\mathfrak{F}$ of $\mathbb{D}_n$ is rational if and only if there exists a matrix $M \in \mathrm{Sp}_n(\mathbb{Q})$, such that $M * \mathfrak{F} = \mathfrak{F}_l$. Moreover ([Nam, Rem. 4.16]), if $\mathfrak{F}$ is a rational boundary component of $\mathbb{D}_n$, then there exists a matrix $M \in \mathrm{Sp}_n(\mathbb{Z})$, such that $M * \mathfrak{F} = \mathfrak{F}_l$.

Definition 2.5.4 *Let $\Gamma \leq \mathrm{Sp}_n(\mathbb{R})^{\mathrm{proj}}$ be commensurable with $\mathrm{Sp}_n(\mathbb{Z})$. The* cusps *of Γ are the rational boundary components. More detailed, let $j \in \mathbb{N}$ satisfy $0 \leq j \leq n-1$. Then a* j-cusp *is a rational boundary component of degree j. Two j-cusps $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are* equivalent *if and only if there is a matrix $M \in \Gamma$, such that*

$$M * \mathfrak{F}_1 = \mathfrak{F}_2.$$

Remark 2.5.5 *a) The concept of a j-cusp can be also found in [Ch2].*
b) Let $\Gamma \leq \mathrm{Sp}_n(\mathbb{R})^{\mathrm{proj}}$ be commensurable with $\mathrm{Sp}_n(\mathbb{Z})$. Then

$$\left|\Gamma\backslash \mathrm{Sp}_n(\mathbb{R})^{\mathrm{proj}}/C_{n,j}^{\mathrm{proj}}\right|$$

is equal to the number of Γ-inequivalent j-cusps.

c) Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$ be commensurable with $\mathrm{Sp}_n(\mathbb{Z})$. Then the number of Γ-inequivalent j-cusps is equal to

$$\left|\Gamma\backslash \mathrm{Sp}_n(\mathbb{Q})/C_{n,j}(\mathbb{Q})\right|.$$

Corollary 2.5.6 *Let $\Gamma \leq \mathrm{Sp}_n(\mathbb{R})^{\mathrm{proj}}$ be commensurable with $\mathrm{Sp}_n(\mathbb{Z})$ and let $\mathfrak{F}$ be a boundary component of degree l with $l \leq n-1$. Then there exists a matrix $M \in \mathrm{Sp}_n(\mathbb{R})^{\mathrm{proj}}$ such that $M * \mathfrak{F} < \mathfrak{F}_{n-1}$.*

In view of Proposition 2.5.2 and Corollary 2.5.6 we calculate a system of representatives for the equivalence classes of $(n-1)$-cusps.

2.5.3 The Cusps of $\Gamma(P)$

Already Reefschläger ([Ree]) determined the 1-cusps for the paramodular group of degree 2:

Proposition 2.5.7 *Given $t \in \mathbb{N}$ with the prime factor decomposition $t = \prod_{i=1}^{m} p_i^{\alpha_i}$, then the number of 1-cusps is equal to*

$$\left|\Gamma_t \backslash \mathrm{Sp}_2(\mathbb{Q}) / C(\mathbb{Q})\right| = \prod_{i=1}^{m} (\alpha_i + 1),$$

i. e. the number of divisors of t.

In order to determine the number of inequivalent $(n-1)$-cusps, we need to calculate

$$\left|\Gamma(P) \backslash \mathrm{Sp}_n(\mathbb{Q}) / C(\mathbb{Q})\right|.$$

An easy calculation (see for example [Mar]) reveals

Corollary 2.5.8 *$\mathrm{Sp}_n(\mathbb{Q}) / C(\mathbb{Q})$ is isomorphic to $\mathbb{P}_{\mathbb{Q}}^{2n} = \left(\mathbb{Q}^{2n} \setminus \{0\}\right) / \mathbb{Q}^*$ via the transformation which maps $MC(\mathbb{Q})$ onto the n-th column of M.*

A vector $X = (x_1, \ldots, x_n)^{\mathrm{tr}} \in \mathbb{Z}^n$ is called *primitive* if and only if $\gcd(x_1, \ldots, x_n) = 1$. Let $V_P := \left(\begin{smallmatrix} I & 0 \\ 0 & P \end{smallmatrix}\right)$ and $\mathcal{M}_P := V_P \left\{x \in \mathbb{Z}^{2n};\ x \text{ is primitive}\right\}$. Therefore $\mathbb{P}_{\mathbb{Q}}^{2n} = \left(\mathbb{Q}^{2n} \setminus \{0\}\right) / \mathbb{Q}^* \cong \mathcal{M}_P / \{\pm 1\}$. Since $\pm I \in \Gamma(P)$ and $\Gamma(P)$ acts on $\mathcal{M}_P$ via multiplication from the left, we get

$$\Gamma(P) \backslash \mathrm{Sp}_n(\mathbb{Q}) / C(\mathbb{Q}) \cong \Gamma(P) \backslash \mathcal{M}_P.$$

Definition 2.5.9 *Let $X = (x_1, \ldots, x_4)^{\mathrm{tr}} \in \mathbb{Z}^4$ be primitive. Then we define the t-*divisor of X *as*

$$d_t(X) = \gcd(x_1, tx_2, x_3, tx_4).$$

Obviously $d_t(X) \mid t$. Moreover, Gritsenko ([Gr2, Lem 2.3]) showed that

Lemma 2.5.10 *Let $X = (x_1, \ldots, x_4)^{\mathrm{tr}} \in \mathbb{Z}^4$ be primitive. Then*

$$\widehat{\Gamma}_t\ X = \widehat{\Gamma}_t\ (d_t(X), 1, 0, 0)^{\mathrm{tr}}.$$

In order to calculate the number of cusps of the paramodular group of arbitrary degree, we generalize this result as follows:

Proposition 2.5.11 *Let $n \geq 2$, $P = \mathrm{diag}(t_1, \ldots, t_n)$ and $X = (x_1, \ldots, x_{2n})^{\mathrm{tr}} \in \mathbb{Z}^{2n}$ be primitive. Then*

$$\widehat{\Gamma}(P)\ X = \widehat{\Gamma}(P)\ (Y_1, \ldots, Y_n, 0_n)^{\mathrm{tr}}$$

with $Y_l = \gcd(R_l^{(n)} X)$ and $R_l^{(n)} = I_{2l-2} \times \mathrm{diag}(1, t_{l,l+1}, \ldots, t_{l,n}, 1, t_{l,l+1}, \ldots, t_{l,n})$.

Proof: We show this proposition by induction. The case $n = 2$ is solved by Lemma 2.5.10. Let $n \geq 3$. Before we start with the proof, we need the following definitions:

1) Let $\gcd(q) = 0$ for $q = 0$.
2) For a vector $X \in \mathbb{Z}^n$ and $M \subset \{1, \ldots, n\}$ we denote by $X|_M$ the vector $(X_i)_{i \in M, n-i \in M}$ so that $X_{1,2} = (x_1, x_2, x_{n+1}, x_{n+2})^{\mathrm{tr}}$.

If $X \in \mathbb{Z}^n$, then there exists a primitive $X' \in \mathbb{Z}^n$ such that $X = \gcd(X)X'$. Therefore we can use Lemma 2.5.10 to get a matrix $M_1 \in \widehat{\Gamma}_{t_n}$ such that

$$X_2 := \Lambda_{1,n}(M_1)X = \begin{pmatrix} \gcd(x_1, t_n x_n, x_{n+1}, t_n x_{2n}) \\ x_2 \\ \vdots \\ x_{n-1} \\ \gcd(x_1, x_n, x_{n+1}, x_{2n}) \\ 0 \\ x_{n+2} \\ \vdots \\ x_{2n-1} \\ 0 \end{pmatrix}.$$

Using the induction hypothesis there exists a matrix $M_2 \in \widehat{\Gamma}((P_{i,j})_{1 \leq i,j \leq n-1})$ such that

$$X_3 := (M_2 \times I_2)X_2 = \begin{pmatrix} \gcd(R_1^{(n)} X) \\ \gcd(R_2^{(n-1)} X|_{\{1,\ldots,n-1\}}, x_n t_n, x_{2n} t_n) \\ \vdots \\ \gcd(R_{n-1}^{(n-1)} X|_{\{1,\ldots,n-1\}}, x_n t_n, x_{2n} t_n) \\ \gcd(x_1, x_n, x_{n+1}, x_{2n}) \\ 0_n \end{pmatrix}.$$

Again by Lemma 2.5.10 we obtain a matrix $M_3 \in \widehat{\Gamma}_{t_{n-1,n}}$ such that

$$M_3 \begin{pmatrix} \gcd(R_{n-1}^{(n-1)} X|_{\{1,\ldots,n-1\}}, t_n x_n, t_n x_{2n}) \\ \gcd(x_1, x_n, x_{n+1}, x_{2n}) \end{pmatrix} = \begin{pmatrix} \gcd(R_{n-1}^{(n-1)} X|_{\{1,\ldots,n-1\}}, t_{n-1,n} x_n, t_{n-1,n} x_{2n}) \\ \gcd(X) \end{pmatrix}$$
$$= \begin{pmatrix} \gcd(R_{n-1}^{(n)} X) \\ 1 \end{pmatrix}$$

Therefore

$$X_4 := \Lambda_{n-1,n}(M_3)X_3 = \begin{pmatrix} \gcd(R_1^{(n)}X) \\ \gcd(R_2^{(n-1)}X|_{\{1,\dots,n-1\}}, x_n t_n, x_{2n} t_n) \\ \vdots \\ \gcd(R_{n-2}^{(n-1)}X|_{\{1,\dots,n-1\}}, x_n t_n, x_{2n} t_n) \\ \gcd(R_{n-1}^{(n)}X) \\ 1 \\ 0_n \end{pmatrix}.$$

If we notice that

$$\gcd(R_k^{(n-1)}X|_{\{1,\dots,n-1\}}, t_n x_n, t_n x_{2n}, t_{k,k+1}R_{k+1}^{(n)}X) = \gcd(R_k^{(n)}(X)) \quad \text{for } k = 1,\dots,n-2,$$

there exist matrices $M_4 \in \widehat{\Gamma}_{t_{n-2,n-1}}, \dots, M_{n+1} \in \widehat{\Gamma}_{t_{1,2}}$ such that we get the desired result. $\square$

It is easily checked that $(Y_i)_{1\le i\le n}$ has the following properties:

Corollary 2.5.12 *a)* $Y_i \mid t_{i,n}$ *for all* $1 \le i \le n$.
b) $1 = Y_n \mid Y_{n-1} \mid \dots \mid Y_1$.
c) $Y_i \mid Y_{i+1}t_{i,i+1}$ *for all* $1 \le i \le n-1$.
d) The numbers Y_i *are invariants of the* $\widehat{\Gamma}(P)$*-orbits. In particular the vectors* $(Y_i)_{1\le i\le n}$ *(defined as above) are a system of representatives for* $\widehat{\Gamma}(P)\backslash V_P^{-1}\mathcal{M}_P$

Now we want to determine the set of all vectors $(Y_i)_{1\le i\le n}$: Therefore we check whether every vector $l = (l_1,\dots,l_n)^{\mathrm{tr}} \in \mathbb{Z}^{2n}$ with the following conditions (*) appear as cusp:

1) $l_i \mid t_{i,n}$ for all $1 \le i \le n$.
2) $1 = l_n \mid l_{n-1} \mid \dots \mid l_1$.
3) $\frac{l_i}{l_{i+1}} \mid t_{i,i+1} = \frac{t_{i,n}}{t_{i+1,n}}$ for all $1 \le i \le n-1$.

We have to find a primitive vector $X = (x_1,\dots,x_{2n})^{\mathrm{tr}} \in \mathbb{Z}^{2n}$ such that $\gcd(R_i^{(n)}X) = l_i$ for $i = 1,\dots,n$. This gives us a equation system. Since we have $2n$ variables and only n equations we choose $x_i = x_{n+i}$ for $i = 1,\dots,n$. Moreover it is sufficient to solve this equation system for every prime p seperately. Then we get the numbers x_i by the Chinese remainder theorem. Fix a prime p and let

$$p^{\alpha_i} \parallel l_i \quad \text{and} \quad p^{\beta_i} \parallel t_{i,n}.$$

If we set $x_i = p^{\alpha_i}$ for all $i = 1,\dots,n$, then the $(n-k)$-th equation is satisfied since

$$\beta_{n-k} - \beta_{n-k+i} \ge \alpha_{n-k} - \alpha_{n-k+i} \quad \text{for all} \quad k \ge i \ge 1$$

and therefore

$$p^{\alpha_{n-k}} = \gcd(p^{\alpha_1},\dots,p^{\alpha_{n-k}}p^{\beta_{n-k}-\beta_{n-k+1}+\alpha_{n-k+1}},\dots,p^{\beta_{n-k}-\beta_{n-k+i}+\alpha_{n-k+i}},\dots,p^{\beta_{n-k}-\beta_n+\alpha_n})$$

Therefore every vector l as above represents a cusp of $\Gamma(P)$ via isomorphism:

Theorem 2.5.13 *One has*

$$\Gamma(P)\backslash \mathrm{Sp}_n(\mathbb{Q})/C(\mathbb{Q}) \cong \{l \in \mathbb{Z}^n;\ l \text{ fulfills condition (*)}\} =: \mathcal{L}_n.$$

Especially, the number of inequivalent $(n-1)$*-cusps of* $\Gamma(P)$ *is equal to*

$$\prod_{i=1}^{n-1} (\text{number of divisors of } t_{i,i+1})\,.$$

Setting

$$\mathcal{U}_{\mathcal{L}} = \{U \in \mathrm{GL}_n(\mathbb{Z}); U = \begin{pmatrix} I_{n-1} & (l_i)_{1\le i\le n-1} \\ 0 & 1 \end{pmatrix} \text{ with } l \in \mathcal{L}_n\}.$$

we get the following

Proposition 2.5.14 *Let f be in $\mathbb{M}_k(\widehat{\Gamma}(P),\nu)$. Then the following statements are equivalent:*

(i) $f \in \mathbb{S}_k(\widehat{\Gamma}(P),\nu)$.
(ii) For all $U \in \mathcal{U}_{\mathcal{L}}$ *we have* $f\,|_k\,\mathrm{rot}(U)\,|\,\Phi = 0$.
(iii) If T is not positive definite, then the Fourier coefficient $\alpha(T) = 0$.

Example 2.5.15 *a) Let $n = 2$ and t be a prime number. From Proposition 2.5.14 we get that $f \in \mathbb{M}_k(\widehat{\Gamma}_t,\nu)$ is a paramodular cusp form if and only if*

$$f|\Phi = (f|\mathrm{rot}(T))|\Phi = 0 \text{ with } T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right).$$

Comparing (ii) and (iii) we see that we can also choose a different system of representatives:

f is a paramodular cusp form if and only if $f|\Phi = (f|V_p)|\Phi = 0$.

b) Let $P = \mathrm{diag}(1,2,4)$. Then $\widehat{\Gamma}(P)$ has 4 cusps. We can choose for example

$$\mathcal{L} = \{(1,1,1)^{\mathrm{tr}},(2,1,1)^{\mathrm{tr}},(2,2,1)^{\mathrm{tr}},(4,2,1)^{\mathrm{tr}}\}.$$

3 The Paramodular Group as an Orthogonal Group

3.1 Orthogonal Groups

Let $S \in \mathrm{Sym}_n(\mathbb{R})$ be positive definite,

$$S_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -S & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}_{n+2}(\mathbb{R}) \quad \text{of signature } (1, n+1) \quad \text{and}$$

$$S_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}_{n+4}(\mathbb{R}) \quad \text{of signature } (2, n+2).$$

Then the *real orthogonal group* is defined by

$$\mathcal{O}(S_1;\mathbb{R}) := \{M \in \mathrm{Mat}_{n+4}(\mathbb{R});\ S_1[M] := M^{\mathrm{tr}} S_1 M = S_1\}.$$

Starting with the decomposition

$$M = \begin{pmatrix} \alpha & a^{\mathrm{tr}} & \beta \\ b & A & c \\ \gamma & d^{\mathrm{tr}} & \delta \end{pmatrix} \in \mathrm{Mat}_{n+4}(\mathbb{R}) \quad \text{with} \quad \alpha, \beta, \gamma, \delta \in \mathbb{R},\, a, b, c, d \in \mathbb{R}^{n+2}, \tag{3.1}$$

we calculate

$$S_1[M] = \begin{pmatrix} 2\alpha\gamma + S_0[b] & \alpha d^{\mathrm{tr}} + \gamma a^{\mathrm{tr}} + b^{\mathrm{tr}} S_0 A & \alpha\delta + \beta\gamma + b^{\mathrm{tr}} S_0 c \\ \alpha d + \gamma a + A^{\mathrm{tr}} S_0 b & a d^{\mathrm{tr}} + d a^{\mathrm{tr}} + S_0[A] & \delta a + \beta d + A^{\mathrm{tr}} S_0 c \\ \alpha\delta + \beta\gamma + c^{\mathrm{tr}} S_0 b & \delta a^{\mathrm{tr}} + \beta d^{\mathrm{tr}} + c^{\mathrm{tr}} S_0 A & 2\beta\delta + S_0[c] \end{pmatrix}. \tag{3.2}$$

Given $M \in \mathcal{O}(S_1;\mathbb{R})$, we have

$$M^{-1} = S_1^{-1} M^{\mathrm{tr}} S_1 = \begin{pmatrix} \delta & c^{\mathrm{tr}} S_0 & \beta \\ S_0^{-1} d & S_0^{-1} A^{\mathrm{tr}} S_0 & S_0^{-1} a \\ \gamma & b^{\mathrm{tr}} S_0 & \alpha \end{pmatrix}. \tag{3.3}$$

Using (3.2) we verify the following

Proposition 3.1.1 *The following matrices belong to $\mathcal{O}(S_1;\mathbb{R})$:*

a) $J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & V & 0 \\ -1 & 0 & 0 \end{pmatrix}$ *with* $V = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}$,

b) $M_\lambda = \begin{pmatrix} 1 & -\lambda^{tr} S_0 & -\frac{1}{2}S_0[\lambda] \\ 0 & I_{n+2} & \lambda \\ 0 & 0 & 1 \end{pmatrix}$ *with* $\lambda \in \mathbb{R}^{n+2}$,

c) $\widetilde{M}_\lambda = JM_\mu J = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & I_{n+2} & 0 \\ -\frac{1}{2}S_0[\lambda] & -\lambda^{tr} S_0 & 1 \end{pmatrix}$ *with* $\lambda \in \mathbb{R}^{n+2}$, $\mu = -V\lambda$,

d) $R_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *with* $A \in \mathcal{O}(S_0;\mathbb{R})$, *especially* R_V,

e) $M_D = \begin{pmatrix} D^* & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & D \end{pmatrix}$ *with* $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $D^* = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$,

f) $\widetilde{M}_D = R_V M_{D^*} R_V = \begin{pmatrix} \alpha I_2 & 0 & \beta E \\ 0 & I_n & 0 \\ \gamma E & 0 & \delta I_2 \end{pmatrix}$ *with* $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,

g) $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}$, $0 \neq \alpha \in \mathbb{R}$.

Let $\mathbb{H}_S$ be the corresponding half-space

$$\mathbb{H}_S = \{w = u + iv = (\tau, z^{tr}, \omega)^{tr} \in \mathbb{C}^{n+2};\ S_0[v] > 0; \tau, \omega \in \mathbb{H}\}.$$

$\mathcal{O}(S_1;\mathbb{R})$ acts on $\mathbb{H}_S \cup (-\mathbb{H}_S)$ via

$$w \mapsto M\langle w\rangle := (-\tfrac{1}{2}S_0[w]b + Aw + c)(M\{w\})^{-1} \quad \text{with} \quad M\{w\} := -\tfrac{\gamma}{2}S_0[w] + d^{tr}w + \delta.$$

In particular, we get the following actions for some of the matrices from Proposition 3.1.1

$$J\langle w\rangle = \frac{-1}{\frac{1}{2}S_0[w]}(\omega, -z^{tr}, \tau)^{tr}, \quad M_\lambda\langle w\rangle = w + \lambda, \qquad R_A\langle w\rangle = Aw, \tag{3.4}$$

$$M_D\langle w\rangle = \begin{pmatrix} \tau - \frac{\gamma}{2(\gamma\omega+\delta)}S[z] \\ \frac{z}{\gamma\omega+\delta} \\ \frac{\alpha\omega+\beta}{\gamma\omega+\delta} \end{pmatrix}, \qquad \widetilde{M}_D\langle w\rangle = \begin{pmatrix} \frac{\alpha\tau+\beta}{\gamma\tau+\delta} \\ \frac{z}{\gamma\tau+\delta} \\ \omega - \frac{\gamma}{2(\gamma\tau+\delta)}S[z] \end{pmatrix}. \tag{3.5}$$

We denote by $\mathcal{O}(S_1;\mathbb{R})^+$ the subgroup of $\mathcal{O}(S_1;\mathbb{R})$ which maps $\mathbb{H}_S$ onto $\mathbb{H}_S$. Given $M = \begin{pmatrix} * & * & * \\ C & * & D \end{pmatrix}$ with $C, D \in \mathrm{Mat}_2(\mathbb{R})$, we have

$$M \in \mathcal{O}(S_1;\mathbb{R})^+ \quad \Longleftrightarrow \quad \det(CQ + D) > 0, \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

Especially, we get

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 1 \end{pmatrix} \notin \mathcal{O}(S_1;\mathbb{R})^+ \quad \text{for } \alpha < 0.$$

Correspondingly, we write $\mathcal{O}(S_0;\mathbb{R})^+$ for the group of matrices $A \in \mathcal{O}(S_0;\mathbb{R})$, i. e. $S_0[A] = S_0$, for which R_A belongs to $\mathcal{O}(S_1;\mathbb{R})^+$. With the decomposition $A = \begin{pmatrix} * & * & * \\ \gamma & * & \delta \end{pmatrix}$, γ, $\delta \in \mathbb{R}$, this property is – because of Equation (3.6) – equivalent to

$$\gamma + \delta > 0. \tag{3.7}$$

For detailed proofs, we refer the reader to [Büh, §2.2].

3.2 Integral Orthogonal Group

Additionally, let S now be even and $\Gamma_S = \mathcal{O}(S_1;\mathbb{Z})^+$. Using Proposition 3.1.1 and Equation (3.6) we have

$$J \in \Gamma_S, M_\lambda \in \Gamma_S \quad \text{for all} \quad \lambda \in \mathbb{Z}^{n+2}. \tag{3.8}$$

Proposition 3.2.1 *The following matrices belong to the subgroup of Γ_S which is generated by J and M_λ, $\lambda \in \mathbb{Z}^{n+2}$:*

a) $\widetilde{M}_\lambda$ *for* $\lambda \in \mathbb{Z}^{n+2}$,
b) $\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$, $W = (I - \varepsilon\lambda\lambda^{\mathrm{tr}} S_0)V$ *for* $\lambda \in \mathbb{Z}^{n+2}$ *with* $\varepsilon = \frac{1}{2}S_0[\lambda] = \pm 1$,
c) R_A, $A = \begin{pmatrix} 1 & \mu^{\mathrm{tr}}S & \frac{1}{2}S[\mu] \\ 0 & I_n & \mu \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ \mu & I_n & 0 \\ \frac{1}{2}S[\mu] & \mu^{\mathrm{tr}}S & 1 \end{pmatrix}$, $\mu \in \mathbb{Z}^n$,
d) M_D, $\widetilde{M}_D$, $D \in \mathrm{SL}_2(\mathbb{Z})$.

Proof: a) Proposition 3.1.1 c).
b) follows directly from the proposition in [Kr5].
c) For $D_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $D_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, we have

$$M_{D_1} = M_{e_{n+2}}, \widetilde{M}_{D_1} = M_{e_1}, \quad \text{and} \quad M_{D_2} = \widetilde{M}_{-e_1}, \widetilde{M}_{D_2} = \widetilde{M}_{-e_{n+2}}, \text{respectively}$$

where $\{e_j\}$ is the standard basis of $\mathbb{R}^{n+2}$. Since $\mathrm{SL}_2(\mathbb{Z}) = \langle D_1, D_2 \rangle$, the statement is true. □

From [Büh, Satz 2.31] we adopt

Proposition 3.2.2 Γ_S *is generated by the matrices*

$$J, \quad M_\lambda, \lambda \in \mathbb{Z}^{n+2}, \quad R_A,\ A \in \mathcal{O}(S_0;\mathbb{Z})^+.$$

Defining

$$\widehat{\Gamma}_S := \{M \in \Gamma_S;\ M \in I_{n+4} + \mathrm{Mat}_{n+4}(\mathbb{Z}) \cdot S_1\},$$

we easily verify

Lemma 3.2.3 $\widehat{\Gamma}_S$ *is normal in* Γ_S *and contains the matrices*

$$J, \quad M_\lambda, \lambda \in \mathbb{Z}^{n+2}.$$

As in [Büh, Satz 2.31] we determine the generators of $\widehat{\Gamma}_S$

Corollary 3.2.4 $\widehat{\Gamma}_S$ *is generated by the matrices*

$$J, \quad M_\lambda, \lambda \in \mathbb{Z}^{n+2}, \quad R_A, A \in \widehat{\mathcal{O}}(S_0;\mathbb{Z})^+, \textit{ i. e. } A \in I_{n+2} + \mathrm{Mat}_{n+2}(\mathbb{Z})S_0.$$

Proof: Let $M \in \widehat{\Gamma}_S$ be given as in Equation (3.1). Then $\det M = \pm 1$ and

$$\gcd(\gamma, d_1, \ldots, d_{n+2}, \delta) = 1 \quad \text{for } d = (d_1, \ldots, d_{n+2})^{\mathrm{tr}} \in \mathbb{Z}^{n+2}.$$

Furthermore, let Δ be the subgroup of $\widehat{\Gamma}_S$ which is generated by J, M_λ, $\lambda \in \mathbb{Z}^{n+2}$ and R_A, $A \in \widehat{O}(S_0, \mathbb{Z})^+$. We choose $D \in \mathrm{SL}_2(\mathbb{Z})$ such that $M_1 = MM_D$ has $(0, d_1', \ldots, d_{n+2}', \delta')$ as last row. If $d' \neq 0$, then there exists a $\mu \in \mathbb{Z}^n$ such that

$$\frac{1}{2} S[\mu] d_1' + \sum_{i=1}^{n} \mu_i d_{i+1}' + d_{n+2}' \neq 0.$$

In this case, it is also possible (if necessary multiply with a appropriate R_A) to assume $d_{n+2}' \neq 0$. Given $2 \leq j \leq n+1$, we take

$$P_1 := \emptyset, \quad P_j := \{p \in \mathbb{P};\ \text{there exists } l < j \text{ with } p \nmid d_l',\ p \mid d_{n+2}'\}.$$

Defining

$$u = \prod_{\substack{p \in \mathbb{P},\, p \nmid \delta', \\ p \mid d_{n+2}'}} p \quad \text{and} \quad g_j := \prod_{p \in P_j} p \quad (\text{for } 1 \leq j \leq n+1),$$

the column $g = u(g_1, \ldots, g_{n+1}, 0)^{\mathrm{tr}} \in \mathbb{Z}^{n+2}$ so that $M_g \in \Delta$. Then $M_1 M_g$ has $(*, d_{n+2}', \delta^*)$ as last row with $\delta^* = \delta' + u \sum_{j=1}^{n+1} g_j d_j'$. Now we show that

$$\gcd(\delta^*, d_{n+2}') = 1.$$

Let p be prime with $p \mid d_{n+2}'$. Using $p \nmid \delta'$ we obtain $p \mid u$, hence the claim. In the case $p \mid \delta'$ we conclude that $p \nmid u$ and that there exists exactly one $r \in \{1, \ldots, n+1\}$ with $p \nmid g_r d_r'$, that is to say $r = \min\{l; 1 \leq l \leq n+1,\ p \nmid d_l'\}$. Furthermore, there exists a $D_1 \in \mathrm{SL}_2(\mathbb{Z})$, such that $M_2 = M_1 M_g M_{D_1}$ has the last row $(*, \widehat{d}^{\mathrm{tr}}, 1)$. Defining $h = -S_0^{-1} V \widehat{d}$ the matrix $M_h \in \Delta$ and we get

$$M_2 J M_h J = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix} \in \Delta.$$

In the case $d' = 0$, this follows directly. Altogether, we showed $M \in \Delta$, which was to be proven. $\square$

Since for $M \in \Gamma_S$, the matrix M^{-1} is integral, too, (3.3) provides already $a, d \in S_0 \mathbb{Z}^{n+2}$. Therefore, $M \in \widehat{\Gamma}_S$ is equivalent to

$$A \in I_{n+2} + \mathrm{Mat}_{n+2}(\mathbb{Z}) S_0. \tag{3.9}$$

3.3 The Case $n = 1$ and $S = (2t)$, $t \in \mathbb{N}$

Let $\mathbb{H}_2$ be the Siegel half-space and $S = (2t)$, $t \in \mathbb{N}$. Then the map

$$\Phi_t : \mathbb{H}_S \to \mathbb{H}_2, \quad w = \begin{pmatrix} \tau \\ z \\ \omega \end{pmatrix} \mapsto Z = \begin{pmatrix} \tau & z \\ z & \frac{\omega}{t} \end{pmatrix},$$

is a bijection with the property

$$S_0[w] = 2(\tau\omega - tz^2) = 2t \cdot \det Z.$$

Given $U = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$, we calculate

$$Z[U^{\mathrm{tr}}] = UZU^{\mathrm{tr}} = \begin{pmatrix} \alpha^2\tau + 2\alpha\beta z + \beta^2\frac{\omega}{t} & \alpha\gamma\tau + (\alpha\delta + \beta\gamma)z + \beta\delta\frac{\omega}{t} \\ * & \frac{\gamma^2 t\tau + 2\gamma\delta tz + \delta^2\omega}{t} \end{pmatrix}.$$

Therefore,

$$\Phi_t^{-1}\left(U\Phi_t(w)U^{\mathrm{tr}}\right) = A_U \cdot w \quad \text{with} \quad A_U = \begin{pmatrix} \alpha^2 & 2\alpha\beta & \frac{\beta^2}{t} \\ \alpha\gamma & \alpha\delta + \beta\gamma & \frac{\beta\delta}{t} \\ \gamma^2 t & 2\gamma\delta t & \delta^2 \end{pmatrix}. \tag{3.10}$$

A calculation yields

$$S_0[A_U] = (\det U)^2 \cdot S_0 \quad \text{and} \quad \det A_U = (\det U)^3. \tag{3.11}$$

Given $A = \left(\begin{smallmatrix} \alpha & a & \beta \\ b & m & c \\ \gamma & d & \delta \end{smallmatrix}\right)$, we verify analogously to (3.2)

$$S_0[A] = \begin{pmatrix} 2\alpha\gamma - 2tb^2 & \alpha d + \gamma a - 2tmb & \alpha\delta + \beta\gamma - 2tbc \\ \alpha d + \gamma a - 2tmb & 2ad - 2tm^2 & \delta a + \beta d - 2tmc \\ \alpha\delta + \beta\gamma - 2tbc & \delta a + \beta d - 2tmc & 2\beta\delta - 2tc^2 \end{pmatrix} \tag{3.12}$$

Proposition 3.3.1 *The map*

$$\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{S}\mathcal{O}(S_0;\mathbb{R})^+, \quad \pm U \mapsto A_U,$$

is an isomorphism of groups.

Proof: Since $(Z,U) \mapsto UZU^{\mathrm{tr}}$ is a group action, the map is in view of the equations (3.10) and (3.11) a group homomorphism, which is trivially injective. Let $A \in \mathrm{S}\mathcal{O}(S_0;\mathbb{R})^+$ be as above. Then the defining equations (3.12) yield

$$2\alpha\gamma = 2tb^2.$$

Hence we know that there exist $x,y \in \mathbb{R}$, $\varepsilon = \pm 1$ with

$$\alpha = \varepsilon x^2, \quad b = \varepsilon xy, \quad \text{and} \quad \gamma = \varepsilon y^2 t.$$

Because A is invertable we get $(x,y) \neq 0$. Therefore there exists a matrix $U \in \mathrm{SL}_2(\mathbb{R})$ with first column $(x,y)^{\mathrm{tr}}$. Since the first column of A_U is up to a factor ε equal to the first column of A, we get

$$B = A_U^{-1} \cdot A = \left(\begin{smallmatrix} \varepsilon & * & * \\ 0 & * & * \\ 0 & * & * \end{smallmatrix}\right) \in \mathrm{S}\mathcal{O}(S_0;\mathbb{R})^+.$$

From (3.12) we deduce

$$B = \left(\begin{smallmatrix} \varepsilon & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \varepsilon \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 2t\lambda & t\lambda^2 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \varepsilon & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \varepsilon \end{smallmatrix}\right) \cdot A_{\left(\begin{smallmatrix} 1 & t\lambda \\ 0 & 1 \end{smallmatrix}\right)}, \quad \lambda \in \mathbb{R}, \delta = \pm 1.$$

From this statement we conclude that

$$\left(\begin{smallmatrix} \varepsilon & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \varepsilon \end{smallmatrix}\right) \in \mathrm{S}\mathcal{O}(S_0;\mathbb{R})^+, \quad \text{hence } \varepsilon = \delta = 1 \quad \text{and} \quad A = A_{U\left(\begin{smallmatrix} 1 & t\lambda \\ 0 & 1 \end{smallmatrix}\right)}.$$

$\square$

From now on, let t be **square-free**. We consider, as in paragraph 2.2, $\Gamma^0(t)$ and $\Gamma^0(t)^{\max}$. Then the cosets

$$V_d \cdot \Gamma^0(t) = \Gamma^0(t) \cdot V_d = \{U = \frac{1}{\sqrt{d}} \begin{pmatrix} \alpha d & \beta t \\ \gamma & \delta d \end{pmatrix};\ \alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{Z}, \det U = 1\}$$

are independent of the choice of x,y. Above U accomplishes

$$A_U = \begin{pmatrix} \alpha^2 d & 2\alpha\beta t & \beta^2 \frac{t}{d} \\ \alpha\gamma & \alpha\delta d + \beta\gamma\frac{t}{d} & \beta\delta \\ \gamma^2\frac{t}{d} & 2\gamma\delta t & \delta^2 d \end{pmatrix} \in \mathrm{S}\mathcal{O}(S_0;\mathbb{Z})^+. \tag{3.13}$$

Corollary 3.3.2 *Given square-free t, the map*

$$\mathrm{P}\Gamma^0(t)^{\max} \to \mathrm{S}\mathcal{O}(S_0;\mathbb{Z})^+, \quad \pm U \mapsto A_U,$$

is an isomorphism of groups.

Proof: Because of Theorem 3.3.1 and (3.13), the surjectivity remains to be shown. Let $(\alpha,\ b,\ \gamma)^{\mathrm{tr}}$ be the first column of a matrix $A \in \mathrm{S}\mathcal{O}(S_0;\mathbb{Z})^+$. In view of (3.12), we obtain

$$\alpha\gamma = tb^2 \quad \text{and} \quad \gcd(\alpha, b, \gamma) = 1.$$

Since t is square-free, we get $\gcd(\alpha,\gamma) = 1$ and therefore

$$\alpha = r^2 d, \quad b = rs \quad \text{and} \quad \gamma = s^2 \frac{t}{d}$$

for appropriate $d, r, s \in \mathbb{Z}$ with $d \mid t$ and $\gcd(rd, s\frac{t}{d}) = 1$. Using Proposition 3.3.1 we get $d \in \mathbb{N}$ and there exist $u, v \in \mathbb{Z}$ with $rd \cdot u - s\frac{t}{d} \cdot v = 1$, hence

$$U = \frac{1}{\sqrt{d}} \begin{pmatrix} rd & vt \\ s & ud \end{pmatrix} \in V_d \cdot \Gamma^0(t).$$

As before, we obtain

$$B = A_U^{-1} \cdot A = \left(\begin{smallmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{smallmatrix}\right) \in \mathrm{S}\mathcal{O}(S_0; \mathbb{Z})^+.$$

With (3.12) and $\det B = 1$ we conclude that

$$B = \left(\begin{smallmatrix} 1 & 2\lambda t & \lambda^2 t \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{smallmatrix}\right) = A_{\left(\begin{smallmatrix} 1 & t\lambda \\ 0 & 1 \end{smallmatrix}\right)}, \lambda \in \mathbb{Z},$$

hence

$$A = A_{U\left(\begin{smallmatrix} 1 & t\lambda \\ 0 & 1 \end{smallmatrix}\right)} \quad \text{with} \quad U \left(\begin{smallmatrix} 1 & t\lambda \\ 0 & 1 \end{smallmatrix}\right) \in V_d \cdot \Gamma^0(t). \qquad \square$$

Using Equation (3.13) and $\alpha\delta d - \beta\gamma\frac{t}{d} = 1$ we obtain

$$\gcd(\alpha\delta d + \beta\gamma\frac{t}{d} - 1, 2t) = \gcd(2\beta\gamma\frac{t}{d}, 2t) = 2\frac{t}{d},$$

since $\gcd(\beta\gamma, d) = 1$. Therefore one has

$$A_U \in \widehat{\mathrm{S}\mathcal{O}}(S_0; \mathbb{Z})^+, \quad \text{i. e.} \quad A \in \mathrm{Mat}_3(\mathbb{Z}) \cdot S_0$$

if and only if $d = 1$, thus $U \in \Gamma^0(t)$. We obtain the following

Corollary 3.3.3 *Given square-free $t \in \mathbb{N}$, the map*

$$\mathrm{P}\Gamma^0(t) \to \widehat{\mathrm{S}\mathcal{O}}(S_0; \mathbb{Z})^+, \quad \pm U \mapsto A_U,$$

is an isomorphism of groups.

Let Γ_t denote the rational realization of the paramodular group of level $\left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$ and $\Gamma_t^{\max}$ the maximal discrete extension. It is known by Lemma 2.1.3 that Γ_t is generated by

$$J_t = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{t} \\ -1 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \end{smallmatrix}\right), \quad \mathrm{trans}(S), S = \begin{pmatrix} \alpha & \beta \\ \beta & \frac{\gamma}{t} \end{pmatrix}, \alpha, \beta, \gamma \in \mathbb{Z},$$

$$\begin{pmatrix} U & 0 \\ 0 & U^{-\mathrm{tr}} \end{pmatrix}, U \in \Gamma^0(t), \quad M_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\Gamma_t^{\max}$ is generated by Γ_t and $\begin{pmatrix} V_d & 0 \\ 0 & V_d^{-\mathrm{tr}} \end{pmatrix}$, $d \in \mathbb{N}$, $d \mid t$. Now Equation (3.4) leads to

$$J_t \langle Z \rangle = -\frac{1}{\tau\omega - tz^2} \begin{pmatrix} \omega & -z \\ -z & \frac{\tau}{t} \end{pmatrix} = \Phi_t(J \langle w \rangle)$$

$$(\mathrm{trans}\, S) \langle Z \rangle = Z + S = \begin{pmatrix} \tau+\alpha & z+\beta \\ z+\beta & \frac{\omega+\gamma}{t} \end{pmatrix} = \Phi_t(M_\lambda \langle w \rangle), \quad S = \begin{pmatrix} \alpha & \beta \\ \beta & \frac{\gamma}{t} \end{pmatrix}, \lambda = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix},$$

$$M_0 \langle Z \rangle = \begin{pmatrix} \tau & -z \\ -z & \frac{\omega}{t} \end{pmatrix} = \Phi_t(M_0^* \langle w \rangle), \quad M_0^* = \begin{pmatrix} -I_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I_2 \end{pmatrix} \in \widehat{\Gamma}_S.$$

Thus we can transfer the results for the generators and obtain the following

Corollary 3.3.4 *Let $t \in \mathbb{N}$ be square-free.*

a) The groups $\mathrm{P}\Gamma_t^{\max}$ and $\mathrm{P}\Gamma_S$ as well as $\Gamma_S \cap \mathrm{SL}_5(\mathbb{Z})$ are isomorphic.
b) The groups $\mathrm{P}\Gamma_t$ and $\mathrm{P}\widehat{\Gamma}_S$ as well as $\widehat{\Gamma}_S \cap \mathrm{SL}_5(\mathbb{Z})$ are isomorphic.

This isomorphism is denoted by Ψ_t throughout this paper.

Analogously to (3.9), we obtain that $M \in \widehat{\Gamma}_S$ if and only if $a_{2,2} \equiv 1 \bmod 2t$, thus $\widehat{\Gamma}_S = \mathcal{O}^{\mathrm{dis}}(S_1, \mathbb{Z})^+$, where the discriminant kernel of $\mathcal{O}(S_1, \mathbb{Z})^+$ is defined as in [FH, p. 222].

Remark 3.3.5 *In order to determine the image of a given matrix under this isomorphism Ψ_t, we have to decompose this matrix into a product of generators. A more explicit isomorphism (see Section 8.4) was developed in [GN, §1.3].*

3.4 Orthogonal Modular Forms and their Relationship to Paramodular Forms of Degree 2

In this section, let $n = 1$, $S = (2t)$ with $t \in \mathbb{N}$ square-free. Given $M \in \mathcal{O}(S_1; \mathbb{R})$, $k \in \mathbb{Z}$, and holomorphic functions $f : \mathbb{H}_S \to \mathbb{C}$, we define the action $f \mid M = f \mid_k M$ via

$$(f \mid M)(w) = (M\{w\})^{-k} f(M \langle w \rangle) \quad \text{for} \quad w \in \mathbb{H}_S. \tag{3.14}$$

Definition 3.4.1 *Given $k \in \mathbb{Z}$ and $\nu \in \Gamma_S^{\mathrm{ab}}$, a holomorphic function $f : \mathbb{H}_S \to \mathbb{C}$ is called an* (orthogonal) modular form of weight k with respect to Γ_S and ν *if*

$$f \mid M = \nu(M) \cdot f \quad \textit{for all} \quad M \in \Gamma_S.$$

The vector space of all modular forms of weight k with respect to Γ_S and character ν is denoted by $\mathbb{M}_k(\Gamma_S, \nu)$. If $\nu = 1$, we write $\mathbb{M}_k(\Gamma_S)$.

Let $\Lambda_1 = \mathbb{Z}^{n+4}$, $\Lambda_0 = \mathbb{Z}^{n+2}$ and $\Lambda_n = \mathbb{Z}^n$ be lattices with the associated quadratic forms belonging to S_1, S_0 and S respectively. Given $w \in \mathbb{H}_S$, $\lambda \in \Lambda_0$ and $f \in \mathbb{M}_k(\Gamma_S)$, we conclude

$$f(w) = (f \mid M_\lambda)(w) = f(w+\lambda),$$

thus f possesses an absolutely convergent Fourier expansion of type

$$f(w) = \sum_{\rho \in \Lambda_0^\sharp} \alpha_f(\rho) e^{2\pi i w^{\mathrm{tr}} S_0 \rho}.$$

Definition 3.4.2 *Given $g = (\alpha, h, \beta)^{\mathrm{tr}} \in \mathbb{R}^{n+2}$, $\alpha, \beta \in \mathbb{R}$, $h \in \mathbb{R}^n$, let*

$$\begin{aligned} g > 0 \quad &:\Leftrightarrow \quad \alpha > 0, \beta > 0, S_0[g] > 0, \\ g \geq 0 \quad &:\Leftrightarrow \quad \alpha \geq 0, \beta \geq 0, S_0[g] \geq 0. \end{aligned}$$

Given $f \in \mathbb{M}_k(\Gamma_S)$ the Koecher principle holds

$$f(w) = \sum_{\substack{\rho \in \Lambda_0^\sharp \\ \rho \geq 0}} \alpha_f(\rho) e^{2\pi i w^{\mathrm{tr}} S_0 \rho}, \quad w \in \mathbb{H}_S. \tag{3.15}$$

Moreover, the have a Fourier-Jacobi expansion (see [Büh])

$$f(w) = \sum_{m=0}^{\infty} \varphi_m(z, \tau) e^{2\pi i m w}$$

with

$$\varphi_m(z,\tau) = \sum_{l=0}^{\infty} \sum_{\substack{\lambda \in \Lambda^\sharp \\ 2ml \geq S[\lambda]}} \alpha_f(\rho) e^{2\pi i (l\tau + \lambda^{\mathrm{tr}} S_0 z)}, \quad \rho = \begin{pmatrix} m \\ \lambda \\ l \end{pmatrix}.$$

Obviously there exist analogous expansions for $f \in \mathbb{M}_k(\Gamma_S, \nu)$. As always, $f \in \mathbb{M}_k(\Gamma_S, \nu)$ is called an *(orthogonal) cusp form* if the Fourier coefficients fulfill

$$\alpha_f(\rho) \neq 0 \Rightarrow \rho > 0.$$

The vector space of (orthogonal) cusp forms is denoted by $\mathbb{S}_k(\Gamma_S, \nu)$. For generators M of Γ_t, we obtain $M\{Z\} = \Psi_t(M)\{\Phi_t(Z)\}$ and thus we get the following commutative diagram

$$\begin{array}{ccc} P\Gamma_t^{\max} \times \mathbb{H}_2 & \xrightarrow{(\Psi_t, \Phi_t)} & P\Gamma_S \times \mathbb{H}_S \\ \downarrow & & \downarrow \\ \mathbb{H}_2 & \xrightarrow{\Phi_t} & \mathbb{H}_S \end{array} \tag{3.16}$$

where the vertical arrows symbolize the action of the corresponding group on the corresponding half-space. Whenever $F \in \mathbb{M}_k(\Gamma_t^{\max})$ with Fourier expansion

$$F(Z) = \sum_{\substack{\lambda_{ij} \in \mathbb{Z} \\ T = \begin{pmatrix} 2\lambda_{11} & \lambda_{12} \\ \lambda_{12} & 2\lambda_{22} \end{pmatrix} \geq 0}} \alpha_F(T) e^{2\pi i(\lambda_{11}\tau + \lambda_{12}z + \lambda_{22}\omega)},$$

we conclude from $\rho = (\rho_1, \frac{\rho_2}{2t}, \rho_3)$ with $\rho_i \in \mathbb{Z}$ and (3.15) that

$$\alpha_f(\rho) = \alpha_F \begin{pmatrix} 2\rho_3 & -\rho_2 \\ -\rho_2 & 2t\rho_1 \end{pmatrix}.$$

3.5 Eichler Transformations

We adopt from [FH] the Eichler transformations for our special case. Let $V = \mathbb{R}^5$, $q_t(x) := x_1x_2 + x_3x_4 - tx_5^2$ of signature $(2,3)$ and Gram matrix S_{Fr} as well as $\langle x,y \rangle = q_t(x+y) - q_t(x) - q_t(y)$ the corresponding bilinear form. The associated standard basis is denoted by $f_1, \ldots, f_4$ and e_1. Thus we get the decomposition $V = H_1(\mathbb{R}) \oplus H_2(\mathbb{R}) \oplus V_0$ with hyperbolic planes

$$H_1(\mathbb{R}) := \mathbb{R}f_1 + \mathbb{R}f_2, \quad H_2(\mathbb{R}) := \mathbb{R}f_3 + \mathbb{R}f_4 \quad \text{and} \quad V_0 := \mathbb{R}e_1.$$

For $L = \mathbb{Z}^5$, this induces the decomposition

$$L = H_1 \oplus H_2 \oplus L_0$$

with

$$\begin{aligned} H_1 &= \mathbb{Z}f_1 + \mathbb{Z}f_2, \quad q(x_1f_1 + x_2f_2) = x_1x_2, \\ H_2 &= \mathbb{Z}f_3 + \mathbb{Z}f_4, \quad q(x_3f_3 + x_4f_4) = x_3x_4. \end{aligned}$$

Then L is even and the dual lattice is given by

$$L' := \{a \in V; \langle a,x \rangle \in \mathbb{Z} \text{ for all } x \in L\} = \mathbb{Z}^4 \times \tfrac{1}{2t}\mathbb{Z}.$$

As usual, $\mathcal{O}(L)$ acts on the discriminant group $\mathrm{Dis}(L) := L'/L$. The discriminant kernel is defined as $\mathcal{O}^{\mathrm{dis}}(L) := \ker(\mathcal{O}(L) \to \mathrm{Aut}(\mathrm{Dis}(L)))$. Since $4t$ is the smallest number such that q_t is integral on L', the discriminant is equal to $\Delta : L' \to \mathbb{Z}$, $a \mapsto -4tq(a)$. This induces a map

$$\overline{\Delta} : \mathrm{Dis}(L) \to \mathbb{Z}/4t\mathbb{Z}.$$

Because $\mathcal{O}(L) := \mathcal{O}(S_{\mathrm{Fr}}, \mathbb{Z})$ and $\mathcal{O}(S_1)$ have the same signature, they are isomorphic, where the isomorphism is induced by $S_{\mathrm{Fr}} = P_{\mathrm{Fr}}^{\mathrm{tr}} S_1 P_{\mathrm{Fr}}$ with

$$S_{\mathrm{Fr}} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & -2t \end{pmatrix} \quad \text{and} \quad P_{\mathrm{Fr}} = \begin{pmatrix} & & 1 & & \\ 1 & & & & \\ & & & & 1 \\ & 1 & & & \\ & & & 1 & \end{pmatrix}.$$

Let u be an isotropic element, i. e. $q(u) = 0$ and $u \neq 0$, and let v be an element, which is orthogonal to u, i. e. $\langle u, v \rangle = 0$. The *Eichler transformation*

$$E(u,v)(a) := a - \langle a, u \rangle v + \langle a, v \rangle u - q(v) \langle a, u \rangle u$$

belongs to $\mathcal{O}(L)$. In the case $u, v \in L$, one even has $E(u,v) \in \mathcal{O}^{\mathrm{dis}}(L)$.

Definition 3.5.1 *Given $L = H_1 \oplus H_2 \oplus L_0$, let $\mathrm{EO}(L_0) \subset O^{\mathrm{dis}}(L)$ be the subgroup which is generated by all Eichler transformations $E(f_i, v)$, where $1 \leq i \leq 4$ and $v \in L$ with $\langle v, f_i \rangle = 0$ for fixed i.*

A simple calculation shows

Lemma 3.5.2 *Given $u, v_1, v_2 \in V$ with $q(u) = 0$, $u \neq 0$ and $\langle u, v_1 \rangle = \langle u, v_2 \rangle = 0$, we have*

$$E(u, v_1) E(u, v_2) = E(u, v_1 + v_2).$$

The group $\mathcal{O}(S_{\mathrm{Fr}}, \mathbb{Z})$ acts on the lattice L'. A vector $a \in L'$ is called *primitive* if $\mathbb{Q}a \cap L' = \mathbb{Z}a$. Since any vector in $L' \setminus \{0\}$ is a multiple of a primitive vector, we only have to determine the orbits of primitive vectors.

We will use the following lemmata from [FH]

Lemma 3.5.3 *Let L be a lattice which contains two hyperbolic planes. Two primitive vectors $a, b \in L'$ of the same norm $q(a) = q(b)$ are contained in the same orbit of $\mathrm{EO}(L_0)$ if and only if they have the same image in the discriminant group $\mathrm{Dis}(L)$.*

Lemma 3.5.4 *Let $\Gamma \subset \mathcal{O}(S_{\mathrm{Fr}}, \mathbb{Z})$ be a group that contains $\mathrm{EO}(L_0)$. Assume that Γ acts transitively on the set of elements of $\mathrm{Dis}(L)$ with the same value of $\overline{\Delta}$. Then Γ acts transitively on the set of primitive vectors of L' of a given norm.*

3.6 Quadratic Divisors

Again, let $S = (2t)$ and q_S the quadratic form which belongs to S_1. Following [Der] we define for $\lambda = (l_1, l_2, \beta, l_3, l_4)$ with $q_S(\lambda) < 0$ the rational quadratic divisor

$$\lambda^{\perp} := \left\{ Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2;\ l_1 - l_4 t \det(Z) + \operatorname{trace}\left(\begin{pmatrix} l_3 & -t\beta \\ -t\beta & tl_2 \end{pmatrix} Z \right) = 0 \right\}. \qquad (3.17)$$

If λ is primitive, the discriminant of $\lambda^\perp$ is $\Delta(\lambda^\perp) := \Delta(\lambda) = -4tq_S(\lambda)$. Let $M\lambda^\perp := \{M\langle Z\rangle;\ Z \in \lambda^\perp\}$, then $\Gamma_t^{\max}$ acts on the set of rational quadratic divisors of fixed discriminant.

Using [FH, Lemma 6.1] we get $(\Psi_t^{-1}(M)\lambda)^\perp = M\lambda^\perp$.

Lemma 3.6.1 *Let t be square-free. Then all rational quadratic divisors of fixed discriminant are equivalent under $\Gamma_t^{\max}$.*

Proof: Using the table in Section 8.3 we know that $\Psi_t^{-1}(\mathrm{EO}(L_0)) \subset \Gamma_t^{\max}$. In view of Lemma 3.5.3 and Lemma 3.5.4, we only need to show the following
Claim: For $S = (2t)$ with t square-free, the group Γ_S acts transitively on the set of elements of $\mathrm{Dis}(L)$ with the same value $\overline{\Delta}$.
Proof: Let $x_1, y_1 \in \{0,\dots,2t-1\}$ with $x_1^2 \equiv y_1^2 \bmod 4t$. Define

$$d = \prod_{\substack{p_i|t,\, p_i|(x_1-y_1)\\ p_i\nmid(x_1+y_1)}} p_i, \quad \text{hence} \quad \gcd(\frac{t}{d}, d) = 1.$$

Therefore $2d \mid (x_1 - y_1)$ and $2\frac{t}{d} \mid (x_1 + y_1)$ (Observe that $2 \mid (x_1 + y_1)$ holds if and only if $2 \mid (x_1 - y_1)$). Then there exist $k, l \in \mathbb{Z}$ with $x_1 = y_1 + 2dk$ and $x_1 = -y_1 + 2\frac{t}{d}l$ so that

$$x_1 = dk + \frac{t}{d}l \quad \text{and} \quad y_1 = -dk + \frac{t}{d}l.$$

Using the equations (2.3) and (3.13) with $\beta = \frac{x_1}{2t}$

$$R_{A_{V_d}}(0,0,\beta,0,0)^{\mathrm{tr}} = (0, -2tx\beta, \beta(xd + \frac{t}{d}y), -2ty\beta)^{\mathrm{tr}} \tag{3.18}$$

Since $y\frac{t}{d} \equiv -1 \bmod d$,

$$2t \cdot \beta(xd + \frac{t}{d}y) \equiv (-dk - \frac{t}{d}l)(1 + 2y\frac{t}{d}) \equiv -dk - \frac{t}{d}l - 2\left(\frac{t}{d}\right)^2 yl \equiv -dk + \frac{t}{d}l \bmod 2t$$

holds. Using (the proof of) Lemma 3.5.3 the lemma is proven. □

Example 3.6.2 *If $t = 5$, then the following table lists all inequivalent, primitive divisors with $0 \geq q(\lambda) \geq -1$.*

j	$\lambda_j = (l_1, l_2, \beta, l_3, l_4)$	$q(\lambda_j)$	*defining equation*
1	$(0,0,\frac{1}{10},0,0)$	$-\frac{1}{20}$	$z = 0$
4	$(1,0,\frac{2}{10},0,0)$	$-\frac{1}{5}$	$z = \frac{1}{2}$
9	$(1,0,\frac{3}{10},0,0)$	$-\frac{9}{20}$	$z = \frac{1}{3}$
16	$(1,0,\frac{4}{10},0,0)$	$-\frac{4}{5}$	$z = \frac{1}{4}$
25	$(1,0,\frac{5}{10},0,0)$	$-\frac{5}{4}$	$z = \frac{1}{5}$
5	$(0,1,\frac{5}{10},1,0)$	$-\frac{1}{4}$	$\tau - 5z + 5\omega = 0$
20	$(0,1,0,-1,0)$	-1	$\omega = \frac{1}{5}\tau$

We would like to determine the stabilizer in $\Gamma(P)^{\max}$ of these divisors. Let $F \in \{\mathbb{Q},\mathbb{R}\}$. An easy calculation (using for example the elementary divisor algorithm) shows that all matrices which preserve $\lambda_1^\perp$ are given by

$$M_1 \times M_2 \quad \text{and} \quad M_1 \times^{\mathrm{tr}} M_2 \quad \text{with} \quad M_1, M_2 \in \mathrm{SL}_2(F),$$

where

$$M_1 \times M_2 = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}, \quad M_1 \times^{\mathrm{tr}} M_2 = \begin{pmatrix} 0 & a_1 & 0 & b_1 \\ a_2 & 0 & b_2 & 0 \\ 0 & c_1 & 0 & d_1 \\ c_2 & 0 & d_2 & 0 \end{pmatrix}.$$

Because of the special form for matrices from Γ_t (see Equation (2.2)), we get

Corollary 3.6.3 *Let $t \in \mathbb{N}$ be square-free.*

a)

$$\left\{M \in \Gamma_t; M\langle\lambda_1^\perp\rangle = \lambda_1^\perp\right\} = \mathrm{SL}_2(\mathbb{Z}) \times_t \mathrm{SL}_2(\mathbb{Z}) \cup \mathrm{SL}_2(\mathbb{Z}) \times^{\mathrm{tr}} \mathrm{SL}_2(\mathbb{Z})$$

with

$$M_1 \times_t M_2 = M_1 \times \left(P_t M_2 P_t^{-1}\right).$$

b) If t is a prime, then

$$\left\{M \in \Gamma_t^{\max}; M\langle\lambda_1^\perp\rangle = \lambda_1^\perp\right\} = \langle \mathrm{SL}_2(\mathbb{Z}) \times_t \mathrm{SL}_2(\mathbb{Z}), V_t\rangle.$$

Remark 3.6.4 *Since for $\frac{t}{d} > 1$ the variable x in $1 = xd + y\frac{t}{d}$ cannot be zero, the result in b) is also true for square-free t.*

Let $c \in \mathbb{R}$ and $\mathcal{H}_c = \{Z \in \mathbb{H}_2; z = c\}$. Using

$$\mathcal{H}_c = M_c\langle\lambda_1^\perp\rangle \quad \text{with} \quad M_c = \mathrm{trans}\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix},$$

we conclude

$$\mathrm{Stab}_{\mathrm{Sp}_2(F)} \mathcal{H}_c = M_c \left(\mathrm{Stab}_{\mathrm{Sp}_2(F)} \lambda_1^\perp\right) M_c^{-1} \quad \text{for} \quad F = \mathbb{R}, \mathbb{Q}.$$

Using the abbreviation $\mathrm{SL}_2(\mathbb{Z})^M := M\,\mathrm{SL}_2(\mathbb{Z})M^{-1}$ for $M \in \mathrm{SL}_2(\mathbb{R})$, we get

Corollary 3.6.5

$$\mathrm{Stab}_{\Gamma_5} \lambda_9^\perp \cong \left\{ \begin{pmatrix} a_1 & b_1/3 \\ 3c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2/15 \\ 15c_2 & d_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})^{P_3} \times \mathrm{SL}_2(\mathbb{Z})^{P_{15}}; \right.$$
$$\left. \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \equiv \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}^{-\mathrm{tr}} \bmod 3 \right\}.$$

Proof: An easy calculation shows that

$$\begin{aligned} M :=& M_{1/3}\left[\begin{pmatrix} a_1 & \frac{b_1}{3} \\ 3c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & \frac{b_2}{15} \\ 15c_2 & d_2 \end{pmatrix}\right] M_{1/3}^{-1} \\ =& \begin{pmatrix} a_1 & 5c_2 & -\frac{5c_2}{3}+\frac{b_1}{3} & \frac{d_2-a_1}{3} \\ c_1 & a_2 & \frac{d_1-a_2}{3} & \frac{b_2}{15}-\frac{c_1}{3} \\ 3c_1 & 0 & d_1 & -c_1 \\ 0 & 15c_2 & -5c_2 & d_2 \end{pmatrix} \in \Gamma_5 \end{aligned} \tag{3.19}$$

if and only if

$$M_1 := \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},\ M_2 := \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \quad \text{with} \quad M_1 \equiv \begin{pmatrix} d_2 & -c_2 \\ -b_2 & a_2 \end{pmatrix} \bmod 3.$$

Furthermore,

$$\begin{aligned} & M_{1/3}\left[\begin{pmatrix} a_1 & \frac{b_1}{3} \\ 3c_1 & d_1 \end{pmatrix} \times^{\mathrm{tr}} \begin{pmatrix} a_2 & \frac{b_2}{15} \\ 15c_2 & d_2 \end{pmatrix}\right] M_{1/3}^{-1} \\ =& \begin{pmatrix} 5c_2 & a_1 & \frac{d_2-a_1}{3} & \frac{b_1}{3}-\frac{5c_2}{3} \\ a_2 & c_1 & \frac{b_2}{15}-\frac{c_1}{3} & \frac{d_1-a_2}{3} \\ 0 & 3c_1 & -c_1 & d_1 \\ 15c_2 & 0 & d_2 & -5c_2 \end{pmatrix} \notin \Gamma_5. \end{aligned}$$

□

Remark 3.6.6 *In the same way we calculate the following stabilizers:*

a)

$$\begin{aligned} \mathrm{Stab}_{\Gamma_5} \lambda_4^{\perp} \cong \Bigg\{ & \begin{pmatrix} a_1 & b_1/2 \\ 2c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2/10 \\ 10c_2 & d_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})^{P_2} \times \mathrm{SL}_2(\mathbb{Z})^{P_{10}}; \\ & \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \equiv \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}^{-\mathrm{tr}} \bmod 2 \Bigg\}. \end{aligned}$$

b)

$$\begin{aligned} \mathrm{Stab}_{\Gamma_5} \lambda_{16}^{\perp} \cong \Bigg\{ & \begin{pmatrix} a_1 & b_1/4 \\ 4c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2/20 \\ 20c_2 & d_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})^{P_4} \times \mathrm{SL}_2(\mathbb{Z})^{P_{20}}; \\ & \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \equiv \begin{pmatrix} d_2 & c_2 \\ b_2 & a_2 \end{pmatrix} \bmod 4 \Bigg\}. \end{aligned}$$

The last equation cannot be written as $M_2^{-\mathrm{tr}}$.

c) The stabilizer for $\lambda_{20}^{\perp}$ *can be calculated with an adjusted method.*

We are especially interested in the quadratic divisor $\lambda_9^\perp$, since this divisor is a good candidate for a reduction process. For more details see Section 7.4.

$\mathrm{Stab}_{\Gamma_5}\,\lambda_9^\perp$ acts on $\lambda_9^\perp$ in the same way as the isomorphic subgroup of $\mathrm{SL}_2(\mathbb{Z})^{P_3} \times \mathrm{SL}_2(\mathbb{Z})^{P_{15}}$ acts on $(\tau,w) \in \mathbb{H}\times\mathbb{H}$. Let $\mathcal{G} := \{(M_1,M_2) \in \mathrm{SL}_2(\mathbb{Z})^2;\ M_1 \equiv M_2^{-\mathrm{tr}} \bmod 3\}$ which is isomorphic to $\mathrm{Stab}_{\Gamma_5}\,\lambda_9^\perp$. The group theoretical situation is described in the following, easily proven

Lemma 3.6.7 *a)*

$$\mathrm{SL}_2(\mathbb{Z})[3]^2 \trianglelefteq \mathcal{G} \leq \mathrm{SL}_2(\mathbb{Z})^2.$$

b)

$$\mathcal{G}/\mathrm{SL}_2(\mathbb{Z})[3]^2 \cong \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}).$$

c)

$$\mathcal{G} = \left\langle \mathrm{SL}_2(\mathbb{Z})[3]^2,\, (T^{-\mathrm{tr}},T),\, (J,J) \right\rangle.$$

Therefore

$$f \in \mathcal{A}(\mathcal{G}) \text{ if and only if } f \in \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2) \text{ and } f|(T^{-\mathrm{tr}},T) = f|(J,J) = f. \tag{3.20}$$

So the first step in order to determine $\mathcal{A}(\mathcal{G})$ is to determine $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$.

4 Modular Forms with Respect to $\mathrm{SL}_2(\mathbb{Z})[3] \times \mathrm{SL}_2(\mathbb{Z})[3]$

4.1 Group Theoretical Results

Let $N \in \mathbb{N}$,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\langle T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

and $\mathrm{SL}_2(\mathbb{Z})[N] := \{M \in \mathrm{SL}_2(\mathbb{Z});\ M \equiv I \bmod N\}$ the *principal congruence subgroup of level* N, which obviously is normal in $\mathrm{SL}_2(\mathbb{Z})$. A group $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is called *congruence group* if there exists an $N \in \mathbb{N}$ with $\mathrm{SL}_2(\mathbb{Z})[N] \subset \Gamma$. The group $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ acts on $(\mathbb{Z}/3\mathbb{Z})^2$ via multiplication from the left. The latter vector space has four one-dimensional subspaces, which are permuted by $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$. In that way, we obtain a homomorphism into the symmetric group $\varphi : \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \to S_4$ with $\ker\varphi = \{\pm I\}$ and $\operatorname{im}\varphi = A_4$ (the only subgroup of S_4 of index 2). In other words, we have an *exact sequence of groups*

$$1 \to \mathcal{C}_2 \to \mathrm{SL}_2\left(\mathbb{Z}/3\mathbb{Z}\right) \to A_4 \to 1. \tag{4.1}$$

Given a subgroup $G \leq \mathrm{SL}_2(\mathbb{Z})$, we set $\overline{G} := G \cup (-I) \cdot G$. Hence

$$\mathrm{SL}_2(\mathbb{Z})/\overline{\mathrm{SL}_2(\mathbb{Z})[3]} \cong A_4.$$

Because of [New, Thm VIII.7], every normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ of index $\mu \neq 2,3$ is free of rank $1 + \frac{\mu}{6}$. Therefore $\overline{\mathrm{SL}_2(\mathbb{Z})[3]}/\{\pm I\} \subset \mathrm{PSL}_2(\mathbb{Z})$ is a free group of rank 3. If there is no danger of confusion, we denote the elements of $\mathrm{PSL}_2(\mathbb{Z})$ by the same letters as the elements in $\mathrm{SL}_2(\mathbb{Z})$.

Corollary 4.1.1 *a)* $\mathrm{SL}_2(\mathbb{Z})[3]$ *is generated by*

$$T^3,\ S := -JT^{-3}J = \left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right),\ K = T^{-1}S^{-1}T = \left(\begin{smallmatrix} 4 & 3 \\ -3 & -2 \end{smallmatrix}\right) \text{ and } -JK^{-1}J = \left(\begin{smallmatrix} 4 & -3 \\ 3 & -2 \end{smallmatrix}\right).$$

b) $\overline{\mathrm{SL}_2(\mathbb{Z})[3]}/\{\pm I\} \subset \mathrm{PSL}_2(\mathbb{Z})$ *is generated by*

$$T^3, \quad S \quad \text{and} \quad L = \left(\begin{smallmatrix} -2 & 3 \\ -3 & 4 \end{smallmatrix}\right).$$

There is no algebraic relation among these generators.

Proof: a) was proven in [Der] and b) is an easy consequence or can be calculated using [GAP]. $\square$

A free group of rank r has $\mathbb{Z}^r$ as Abelian character group. We therefore determined the character group for all principal congruence subgroups. Since modular forms have a Fourier expansion (see Definition 4.2.1), we are only interested in the following characters.

Definition 4.1.2 *Let* Γ *be a congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *and* χ *a linear character of* Γ. χ *is a character* $\bmod N$ *of* Γ *if* $\mathrm{SL}_2(\mathbb{Z})[N] \subset \Gamma$ *and* $\chi(M) = 1$ *for all* $M \in \mathrm{SL}_2(\mathbb{Z})[N]$.

Every character $\bmod N$ is a finite character. Throughout this thesis, I will principally consider characters of $\mathrm{SL}_2(\mathbb{Z})$ which are obviously characters modulo 12.

Let us look at the *cusps* of $\mathrm{SL}_2(\mathbb{Z})[3]$ (i. e. the fixpoints of a parabolic matrix from $\mathrm{SL}_2(\mathbb{Z})[3]$). $\mathrm{SL}_2(\mathbb{Z})[3]$ is a *Fuchsian group* (i. e. a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$) and has finite index in $\mathrm{SL}_2(\mathbb{Z})$. Therefore $\mathbb{Q} \cup \{\infty\}$ is the set of cusps and, due to [Miy, Thm 4.2.10], $\mathrm{SL}_2(\mathbb{Z})[3]$ has four (regular) inequivalent cusps. We choose $\{\infty, -1, 0, 1\}$ as a transversal and from the following table we can read off the parabolic subgroup Γ_x associated with a cusp x as well as a transformation matrix σ_x with $\sigma_x\langle x\rangle = \infty$:

cusp x	σ_x	Γ_x
∞	I	$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^m, m \in \mathbb{Z} \right\}$
-1	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\Gamma_{-1} = \left\{ \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix}^m, m \in \mathbb{Z} \right\}$
0	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\Gamma_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^m, m \in \mathbb{Z} \right\}$
1	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\Gamma_1 = \left\{ \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}^m, m \in \mathbb{Z} \right\}$

We consider now the character table of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ and A_4. In the headline of the table, we specify the orders of the elements in the different conjugacy classes of A_4. In parenthesis we add the order of the elements in $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ if the conjugacy class does not split. In the other case, we write the order in the line below. In the first half of the table there are the characters of A_4, below the second double line there are the additional characters of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$. You can find the character table of A_4 in [JL] with a proof and more details.

ord	1 2	2(4)	3 6	3 6
1	1	1	1	1
λ	1	1	ζ_3	ζ_3^2
λ'	1	1	ζ_3^2	ζ_3
τ	3	-1	0	0
	2	0	-1	-1
ϑ	2	0	$-\zeta_3$	$-\zeta_3^2$
	2	0	$-\zeta_3^2$	$-\zeta_3$

In this table $\zeta_3 = e^{2\pi i/3} \in \mathbb{C}$ denotes a third root of unity.

4.2 Modular Forms with Respect to $\mathrm{SL}_2(\mathbb{Z})[3]$

The following results can be found in [Miy] or [Kr1, §III.7]. Let $\mathbb{H} := \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$ be the upper half-plane and $k \in \mathbb{Z}$. Then $\mathrm{SL}_2(\mathbb{Z})$ acts on $\{f : \mathbb{H} \to \mathbb{C}\}$ by

$$(f\,|_k\, M)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \quad \text{for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),\ \tau \in \mathbb{H}.$$

Definition 4.2.1 *Let $k \in \mathbb{Z}$, $\mathrm{SL}_2(\mathbb{Z})[N] \le \Gamma$ a congruence group and $\nu \in \Gamma^{\mathrm{ab}}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called an* entire modular form of weight k with respect to Γ and character ν *if*

(EF1) $f\,|_k\, M = \nu(M) \cdot f$ *for all $M \in \Gamma$,*
(EF2) $f\,|_k\, M$ *has a Fourier expansion of type*

$$(f\,|_k\, M)(\tau) = \sum_{m=0}^{\infty} \alpha_f(m, M) e^{2\pi i \frac{m}{N}\tau} \quad \textit{for all } M \in \mathrm{SL}_2(\mathbb{Z}).$$

We write $\mathbb{M}_k(\Gamma, \nu)$ for the vector space of all modular forms of weight k with respect to Γ and character ν. If $\nu = 1$, then we often write $\mathbb{M}_k(\Gamma)$. Since

$$\mathbb{M}_k(\Gamma, \nu_1) \cdot \mathbb{M}_l(\Gamma, \nu_2) \subset \mathbb{M}_{k+l}(\Gamma, \nu_1\nu_2)$$

and

$$\mathbb{M}_k(\Gamma, \nu) = \begin{cases} \{0\}, & k < 0 \text{ or } (k = 0 \text{ and } \nu \neq 1), \\ \mathbb{C}, & k = 0 \text{ and } \nu = 1, \end{cases}$$

we get the $\mathbb{Z}$-graded algebra $\mathcal{A}(\Gamma) := \bigoplus_{k\in\mathbb{N}_0} \mathbb{M}_k(\Gamma)$ and the $\mathcal{A}(\Gamma)$-module

$$\mathcal{A}(\Gamma,\nu) := \bigoplus_{k\in\mathbb{N}_0} \mathbb{M}_k(\Gamma,\nu) \quad \text{in the algebra} \quad \bigoplus_{\nu\in\Gamma^{\mathrm{ab}}} \mathcal{A}(\Gamma,\nu).$$

If for given $f \in \mathbb{M}_k(\Gamma,\nu)$ the coefficient $\alpha_f(0,M) = 0$ for every $M \in \mathrm{SL}_2(\mathbb{Z})$, then we call f a *cusp form*, denoted by $f \in \mathbb{S}_k(\Gamma,\nu)$. Given $f \in \mathbb{M}_k(\Gamma,\nu)$ and $g \in \mathbb{S}_k(\Gamma,\nu)$ (or vice versa), we define the *Petersson scalar product* by

$$(f,g) = \mu\left(\mathbb{F}_\Gamma\right)^{-1} \int_{\mathbb{F}_\Gamma} f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k d\mu(\tau),$$

where $d\mu(\tau) := y^{-2}dxdy$ for $\tau = x+iy$, $\mathbb{F}_\Gamma$ is a μ-measurable fundamental domain of $\Gamma\backslash\mathbb{H}$ and $\mu\left(\mathbb{F}_\Gamma\right) = \int_{\mathbb{F}_\Gamma} 1 d\mu(\tau)$. Defining

$$\mathcal{N}_k(\Gamma,\nu) = \{g \in \mathbb{M}_k(\Gamma,\nu); (f,g) = 0 \ \text{ for all } f \in \mathbb{S}_k(\Gamma,\nu)\},$$

we conclude

$$\mathbb{M}_k(\Gamma,\nu) = \mathbb{S}_k(\Gamma,\nu) \oplus \mathcal{N}_k(\Gamma,\nu).$$

Using the dimension formula in [Miy, §2.5] and the formula for the genus of a Riemann surface to $\mathrm{SL}_2(\mathbb{Z})[3]$ ([Miy, Thm 4.2.11]), we get

Theorem 4.2.2 *a)*

$$\dim_{\mathbb{C}} \mathbb{M}_k\left(\mathrm{SL}_2(\mathbb{Z})[3]\right) = \begin{cases} 0, & k < 0, \\ k+1, & k \geq 0. \end{cases}$$

b)

$$\dim_{\mathbb{C}} \mathbb{S}_k(\mathrm{SL}_2(\mathbb{Z})[3]) = \begin{cases} 0, & k < 3, \\ k-3, & k \geq 3. \end{cases}$$

Proof: Using $\dim_{\mathbb{C}} \mathbb{S}_2(\mathrm{SL}_2(\mathbb{Z})[3]) = 0$ we conclude that $\dim_{\mathbb{C}} \mathbb{S}_1(\mathrm{SL}_2(\mathbb{Z})[3]) = 0$, since for $f \in \mathbb{S}_1(\mathrm{SL}_2(\mathbb{Z})[3])$ the function $f^2 \in \mathbb{S}_2(\mathrm{SL}_2(\mathbb{Z})[3])$. Keep in mind that $\mathrm{SL}_2(\mathbb{Z})[3]$ has genus 0, four inequivalent cusps and no elliptic points. $\square$

We now define certain Eisenstein series with respect to $\mathrm{SL}_2(\mathbb{Z})[3]$:

Definition 4.2.3 *a) Let $N \in \mathbb{N}$. A map $\chi : \mathbb{Z} \to \mathbb{C}$ is called a* Dirichlet character $\bmod N$ *if there exists a linear character $\chi' : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ with*

$$\chi(n) = \begin{cases} \chi'(n+N\mathbb{Z}), & \gcd(n,N) = 1, \\ 0, & \gcd(n,N) \neq 1. \end{cases}$$

b) *Let χ and ψ be Dirichlet characters modulo L and modulo M, respectively. Then we define an Eisenstein series with parameter s:*

$$E_k(\tau,s;\chi,\psi) = \sum_{m,n=-\infty}^{\infty}{}' \chi(m)\psi(n)(m\tau+n)^{-k}|m\tau+n|^{-2s}.$$

The right side of this definition converges absolutely and uniformly for all s with $k + 2\,\mathrm{Re}(s) \geq 2+\varepsilon$ ($\varepsilon > 0$). Therefore this series is holomorphic for $\mathrm{Re}(s) > 1-\frac{k}{2}$. In the case of $k \leq 2$ the right side does not converge in $s=0$, but in the case $\chi\psi(-1) = (-1)^k$ the function $E_k(\tau,s;\chi,\psi)$ possesses a meromorphic continuation to the whole complex s-plane due to [Miy, Thm 7.2.9]. This meromorphic continuation – again denoted by $E_k(\tau,s;\chi,\psi)$ – is holomorphic in $s=0$, thus we define

$$E_k(\tau;\chi,\psi;L,M) := E_k(\tau,0;\chi,\psi).$$

(L, M will only be added if this seems to be necessary.) Due to [Miy, Thm 7.2.13], we calculate the following Fourier coefficients $K\cdot\sum_{n=0}^{\infty} a_n e^{2\pi i \frac{n}{3}\tau}$ (some explicit values can be found in Section 8.5)

$\frac{u}{v}$	χ	ψ	L	M	K	a_0	formula for a_n, $n\geq 1$
1	$\left(\frac{*}{3}\right)$	1	3	3	$-\frac{4\pi i}{9}$	1	$a_n = 3\cdot\sum_{0<c\mid n}\chi\left(\frac{n}{c}\right)\cdot\begin{cases}-1, & 3\nmid c,\\ 2, & 3\mid c.\end{cases}$
1	1	$\left(\frac{*}{3}\right)$	1	3	$\frac{2\pi\sqrt{3}}{9}$	1	$a_n = 6\cdot\sum_{0<c\mid n}\psi(c)$.

The mentioned normalized Eisenstein series from above (i. e. $a_0 = 1, K = 1$) will be denoted by $E_1^*(\tau;\chi,\psi;L,M)$.

Since we also would like to calculate Fourier expansions in cusps $x \neq \infty$, we need the following Eisenstein series:
Given $N \in \mathbb{N}$, $\nu,\mu \in \mathbb{N}_0$ with $\mu,\nu < N$, we set

$$E_k(\tau,s;\ \nu,\mu;\ N) = \sum_{\substack{m\equiv\mu \bmod N\\ n\equiv\nu \bmod N}}{}' (m\tau+n)^{-k}|m\tau+n|^{-2s},$$

where $\sum' = \sum_{(m,n)\neq(0,0)}$. The relation between these two kinds of Eisenstein series can be determined with the Identity Theorem. Let χ and ψ be Dirichtlet characters $\bmod L$ and $\bmod M$, respectively, for factors L and M of N, u, $v \in \mathbb{N}$, such that $uL \mid N$ and $vM \mid N$. Then

$$E_k\left(\frac{u}{v}\tau,s;\chi,\psi\right) = v^k|v|^{2s}\sum_{\substack{0\leq\mu,\nu<N\\ u\mid\mu,\,v\mid\nu}}\chi\left(\frac{\mu}{u}\right)\psi\left(\frac{\nu}{v}\right)E_k(\tau,s;\mu,\nu;N) \tag{4.2}$$

and

$$E_k(\tau,s;\mu,\nu;N)=\frac{v^{-k}|v|^{-2s}}{\varphi(L)\varphi(M)}\sum_{\substack{\chi \bmod L\\ \psi \bmod M}}\overline{\chi(\mu')\psi(\nu')}E_k\left(\frac{u}{v}\tau,s;\chi,\psi\right) \tag{4.3}$$

with $u=\gcd(\mu,N)$, $v=\gcd(\nu,N)$, $\mu'=\frac{\mu}{u}$, $\nu'=\frac{\nu}{v}$, $L=\frac{N}{u}$, $M=\frac{N}{v}$ and $\varphi(L)=\sharp(\mathbb{Z}/L\mathbb{Z})^\times$. Since the series $E_k(\tau,s;\ \mu,\nu;\ N)$ is holomorphic in $s=0$ for $k\in\mathbb{N}$, we are able to define $E_k(\tau;\mu,\nu;N):=E_k(\tau,0;\mu,\nu;N)$ and

$$\mathcal{E}_k^{(1)}(\Gamma[N]):=\langle E_k(\tau;\mu,\nu;N);0\le\nu,\mu<N\rangle,$$
$$\mathcal{E}_k^{(2)}(\Gamma[N]):=\left\langle E_k\left(\frac{u}{v}\tau;\chi,\psi\right);\chi,\psi,u,v \text{ for } \chi,\psi,u,v \text{ as above.}\right\rangle$$

The important property of these new Eisenstein series is the fact that we can easily calculate the action of matrices $M\in\mathrm{SL}_2(\mathbb{Z})$, i. e. we have

$$E_k(\tau,s;\mu,\nu;N)|_kM=E_k(\tau,s;\mu',\nu';N)\quad\text{for}\quad M\in\mathrm{SL}_2(\mathbb{Z})$$

with $0\le\mu',\nu'<N$ and

$$(\mu',\nu')\equiv(\mu,\nu)M \bmod N.$$

[Miy] has already shown the following

Theorem 4.2.4 *Given $N\in\mathbb{N}$, one has*

a) $\mathcal{E}_k^{(1)}(\Gamma[N])=\mathcal{E}_k^{(2)}(\Gamma[N])$ *for all* $k\in\mathbb{N}$,
b) $\mathcal{E}_k^{(1)}(\Gamma[N])=\mathcal{N}_k(\Gamma[N])$ *for* $k\in\mathbb{N}\setminus\{2\}$,
c) $\mathcal{E}_2^{(1)}(\Gamma[N])\cap\mathbb{M}_2(\Gamma[N])=\mathcal{N}_2(\Gamma[N])$.

The explicit Fourier expansion was determined by [Miy, §7.1] (see also Section 8.5). With these Fourier expansions it is easy to show that (powers of) $E_{3,3}=E_1(\tau;\ \left(\frac{*}{3}\right),1;3,3)$ and $E_{1,3}=E_1(\tau;\ 1,\left(\frac{*}{3}\right);\ 1,3)$ generate $\mathbb{M}_k(\mathrm{SL}_2(\mathbb{Z})[3])$ for $k\le 3$.

We are interested in the Fourier expansion of $f\in\mathbb{M}_k(\mathrm{SL}_2(\mathbb{Z})[3])$ in all inequivalent cusps. If we fix a cusp x of $\mathrm{SL}_2(\mathbb{Z})[3]$, $\sigma\in\mathrm{SL}_2(\mathbb{Z})$ with $\sigma\langle x\rangle=\infty$, then the *Fourier expansion with respect to the cusp x* can be calculated by

$$\left(f\big|_k\sigma^{-1}\right)(\tau)=\sum_{n=0}^{\infty}a_n(f,x)e^{2\pi i\frac{n}{3}\tau}.$$

Using (4.2), (4.3) and

(x,y)	$(x,y)\sigma_0^{-1}$ mod 3	$(x,y)\sigma_1^{-1}$ mod 3	$(x,y)\sigma_{-1}^{-1}$ mod 3
$(1,1)$	$(1,2)$	$(2,2)$	$(0,2)$
$(1,2)$	$(2,2)$	$(0,2)$	$(1,2)$
$(2,1)$	$(1,1)$	$(0,1)$	$(2,1)$
$(2,2)$	$(2,1)$	$(1,1)$	$(0,1)$
$(0,1)$	$(1,0)$	$(1,0)$	$(1,0)$
$(0,2)$	$(2,0)$	$(2,0)$	$(2,0)$

we get

$$\begin{aligned}\left(E_{3,3} \mid \sigma_0^{-1}\right)(\tau) &= \left(E_1(\tau;1,1;3)+E_1(\tau;1,2;3)-E_1(\tau;2,1;3)-E_1(\tau;2,2;3)\right) \mid \sigma_0^{-1}\\ &= E_1(\tau;1,2;3)+E_1(\tau;2,2;3)-E_1(\tau;1,1;3)-E_1(\tau;2,1;3)\\ &= -\tfrac{4\pi\sqrt{3}}{27}\left(E_{1,3}^*-E_{3,3}^*\right)(\tau).\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}E_{1,3} \mid \sigma_0^{-1} &= -\tfrac{2\pi i}{9}(E_{1,3}^*+2E_{3,3}^*),\\ E_{3,3} \mid \sigma_1^{-1} &= \left(-\tfrac{2\pi\sqrt{3}}{27}+\tfrac{2\pi i}{9}\right)E_{3,3}^*-\tfrac{4\pi\sqrt{3}}{27}E_{1,3}^*,\\ E_{1,3} \mid \sigma_1^{-1} &= \left(\tfrac{2\pi\sqrt{3}}{9}+\tfrac{2\pi i}{9}\right)E_{3,3}^*-\tfrac{2\pi i}{9}E_{1,3}^*,\\ E_{3,3} \mid \sigma_{-1}^{-1} &= \left(-\tfrac{2\pi\sqrt{3}}{27}-\tfrac{2\pi i}{9}\right)E_{3,3}^*-\tfrac{4\pi\sqrt{3}}{27}E_{1,3}^*,\\ E_{1,3} \mid \sigma_{-1}^{-1} &= \left(-\tfrac{2\pi\sqrt{3}}{9}+\tfrac{2\pi i}{9}\right)E_{3,3}^*-\tfrac{2\pi i}{9}E_{1,3}^*.\end{aligned} \tag{4.4}$$

Explicit Fourier coefficients of these functions can be taken from Section 8.5. Using these Fourier coefficients we can show that the matrix

$$M = \begin{pmatrix} a_0(E_{3,3},\infty)^m & a_0(E_{3,3},\infty)^{m-1}a_0(E_{1,3},\infty) & \cdots & a_0(E_{1,3},\infty)^m \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_0(E_{3,3},1)^m & a_0(E_{3,3},1)^{m-1}a_0(E_{1,3},1) & \cdots & a_0(E_{1,3},1)^m \end{pmatrix}$$

has rank 4 (for example we can show that the determinant of the matrix which consists of the first three columns and the last column is not zero). Therefore I can construct for every weight k and cusp x a polynomial F in $E_{3,3}$ and $E_{1,3}$ such that $a_0(F,x)=1$ and F vanishes at all cusps which are inequivalent to x.

Let η be the Dedekind η function defined by

$$\eta : \mathbb{H} \to \mathbb{C}, \quad \eta(\tau) = e^{\pi i \frac{\tau}{12}} \prod_{m=1}^{\infty}(1-e^{2\pi i m\tau}). \tag{4.5}$$

It is well-known (compare with [Kr1, III §6.3] and [GN, Lem 1.2]) that

Corollary 4.2.5 *a)*

$$\eta(M\tau)=v_\eta(M)(c\tau+d)^{1/2}\eta(\tau)\quad \textit{for all } M\in\mathrm{SL}_2(\mathbb{Z}),$$

where $v_\eta^{24}=1$. More explicitly, we have for D even

$$v_\eta^D=\begin{cases}\exp\left(\frac{2\pi iD}{24}((a+d)c-bd(c^2-1)-3c)\right), & c\equiv 1 \bmod 2,\\ \exp\left(\frac{2\pi iD}{24}((a+d)c-bd(c^2-1)+3(d-cd-1))\right), & d\equiv 1 \bmod 2.\end{cases}$$

Especially,

$$\eta(T\tau)=e^{\frac{\pi i}{12}}\cdot\eta(\tau)\quad \textit{and}\quad \eta(J\tau)=\sqrt{\frac{\tau}{i}}\cdot\eta(\tau),$$

where we choose the branch of the root which is positive for positive arguments.
b) $\eta(\tau)=\sum_{n\ odd}\left(\frac{12}{n}\right)e^{2\pi i\frac{n^2}{24}}$.

Example 4.2.6 *a)* $v_\eta^8(M)=1$ *for* $M\in\{T^3,S,K,-JK^{-1}J\}$.
b) $v_\eta^{12}(M)=-1$ *for* $M\in\{T,T^{-\mathrm{tr}},J,T^3,S,K,-JK^{-1}J\}$.

Because of Example 4.2.6, we have

$$\eta^8(M\tau)=(c\tau+d)^4\eta^8(\tau)\quad \text{for } M\in\mathrm{SL}_2(\mathbb{Z})[3].$$

Because of $\eta^8(\tau)=q^{\frac{1}{3}}+\sum_{n\geq 1}a_{\eta^8}(n)q^{\frac{n}{3}}$ and

$$\left(\eta^8\big|_4\sigma_0^{-1}\right)(\tau)=\eta^8(\tau),\ \left(\eta^8\big|_4\sigma_1^{-1}\right)(\tau)=e^{2\pi i/3}\eta^8(\tau),\ \left(\eta^8\big|_4\sigma_{-1}^{-1}\right)(\tau)=e^{4\pi i/3}\eta^8(\tau),$$

we conclude that $\eta^8\in\mathbb{S}_4(\mathrm{SL}_2(\mathbb{Z})[3])$.

Theorem 4.2.7 *The graded ring of modular forms with respect to* $\mathrm{SL}_2(\mathbb{Z})[3]$ *is generated by two normalized Eisenstein series of weight* 1*:*

$$\bigoplus_{k\in\mathbb{Z}}\mathbb{M}_k\left(\mathrm{SL}_2(\mathbb{Z})[3]\right)=\mathbb{C}\left[E_{3,3}^*,E_{1,3}^*\right].$$

The ideal of cusp forms is generated by η^8.

Proof: Let $f\in\mathbb{M}_k(\mathrm{SL}_2(\mathbb{Z})[3])$. Then there exists a polynomial F in $E_{3,3}^*$ and $E_{1,3}^*$ such that $f-F\in\mathbb{S}_k(\mathrm{SL}_2(\mathbb{Z})[3])$. Therefore this difference can be divided by η^8 and we get the desired result by induction because of

$$\eta^8=q^{\frac{1}{3}}-8\cdot q^{\frac{4}{3}}+20\cdot q^{\frac{7}{3}}+\mathcal{O}(q^{\frac{13}{3}})=-\tfrac{1}{27}(E_{3,3}^*)^4+\tfrac{1}{27}E_{3,3}^*\cdot(E_{1,3}^*)^3. \tag{4.6}$$

Since the homogeneous field of fractions $\mathrm{Quot}_{\mathrm{hom}}(\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]))$ – i. e. all quotients of modular forms in $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3])$ of the same weight – has a transcendence degree equal to 1 (see [Fr1, Thm 6.11]), the two functions $E_{3,3}$ and $E_{1,3}$ are algebraically independent. □

Remark 4.2.8 *The ring of modular forms with respect to* $\mathrm{SL}_2(\mathbb{Z})[3]$ *have already been determined by Ebeling ([Eb, Prop. 5.4]). His generators are* $\theta_0 = E^*_{3,3}$ *and* $\theta_1 = 3f_1^* = \frac{1}{3}(E^*_{1,3} - E^*_{3,3})$.

The proof that $E^*_{3,3}$ and $E^*_{1,3}$ are algebraically independent can also be done directly with methods from Commutative Algebra, see Example 5.3.10.
Similarly we can prove that $\eta^4 \in \mathbb{S}_2(\mathrm{SL}_2(\mathbb{Z})[3], v_\eta^{12})$ and that the $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3])$-module

$$\mathcal{A}\left(\mathrm{SL}_2(\mathbb{Z})[3],\, v_\eta^{12}\right) = \eta^4\cdot\mathbb{C}\left[E^*_{3,3}, E^*_{1,3}\right].$$

Because of Equation (4.6) we know that the algebra

$$\bigoplus_{v\in\langle v_\eta^{12}\rangle} \mathcal{A}\left(\mathrm{SL}_2(\mathbb{Z})[3],\, v\right) \cong \mathbb{C}[X_1,X_2,X_3]/\langle 27X_3^2 + X_1^4 - X_1X_2^3\rangle.$$

4.3 Lift from $\mathrm{SL}_2(\mathbb{Z})[3]$ to $\mathrm{SL}_2(\mathbb{Z})[3]\times\mathrm{SL}_2(\mathbb{Z})[3]$

Similarly to the case of elliptic modular forms, we define an operation of $\mathrm{SL}_2(\mathbb{Z})^2$ on $\{f : \mathbb{H}\times\mathbb{H}\to\mathbb{C}\}$ by

$$\left(f\,|_k\,(M_1,M_2)\right)(\tau_1,\tau_2) := (c_1\tau_1+d_1)^{-k}(c_2\tau_2+d_2)^{-k} f\left(\frac{a_1\tau_1+b_1}{c_1\tau_1+d_1}, \frac{a_2\tau_2+b_2}{c_2\tau_2+d_2}\right)$$

with $M_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ for $i = 1,2$.

Definition 4.3.1 *Let* Γ *be a group with* $\mathrm{SL}_2(\mathbb{Z})[N]^2 \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})^2$ *and* $v\in\Gamma^{\mathrm{ab}}$. *A holomorphic function* $f : \mathbb{H}\times\mathbb{H}\to\mathbb{C}$ *is called an* entire modular form of weight k with respect to Γ and character v *if*

(MF1) $f|_k M = v(M)\cdot f$ *for all* $M\in\Gamma$,
(MF2) $f\,|_k\,(M_1,M_2)$ *has a Fourier expansion of type*

$$\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \alpha_f(m_1,m_2;M_1,M_2)e^{2\pi i(\frac{m_1}{N}\tau_1 + \frac{m_2}{N}\tau_2)} \quad \textit{for all } (M_1,M_2)\in\mathrm{SL}_2(\mathbb{Z})^2.$$

Again, let $\mathbb{M}_k(\Gamma,v)$ be the vector space of all entire modular forms of weight k with respect to Γ, $\mathbb{S}_k(\Gamma,v)$ the vector space of all cusp forms (i. e. $\alpha_f(0,0,M_1,M_2) = 0$ for all $(M_1,M_2)\in\mathrm{SL}_2(\mathbb{Z})^2$) and $\mathcal{A}(\Gamma)$ the associated $\mathbb{Z}$-graded algebra. An $f\in\mathcal{A}(\Gamma)$ is called *symmetric* (*skew-symmetric*) if $f(\tau_1,\tau_2) = f(\tau_2,\tau_1)$ ($f(\tau_1,\tau_2) = -f(\tau_2,\tau_1)$). The associated modules are denoted by $\mathcal{A}(\Gamma)_{\mathrm{sym}}$ and $\mathcal{A}(\Gamma)_{\mathrm{skew}}$.
Let X,Y be free variables, $f = E^*_{3,3}$ and $g = E^*_{1,3}$. $f(\tau_1)$, $f(\tau_2)$, $g(\tau_1)$ and $g(\tau_2)$ are algebraically independent, since f, g are algebraically independent and $\mathbb{C}$ is an infinite field.

Therefore $\mathbb{C}[f(\tau_1), f(\tau_2), g(\tau_1), g(\tau_2)]$ is polynomial. Since $\{X^iY^j\otimes X^kY^l;\ i,\ j,\ k,\ l\in\mathbb{N}_0\}$ is a vector space basis for $\mathbb{C}[X,Y]\otimes\mathbb{C}[X,Y]$, a comparison of the coefficients shows that $X\otimes 1$, $Y\otimes 1$, $1\otimes X$ and $1\otimes Y$ are algebraically independent, too. Therefore the map

$$\mathbb{C}[X,Y]\otimes\mathbb{C}[X,Y]\to\mathbb{C}[f(\tau_1), f(\tau_2), g(\tau_1), g(\tau_2)],$$
$$X\otimes 1\mapsto f(\tau_1),\quad Y\otimes 1\mapsto g(\tau_1),\quad 1\otimes X\mapsto f(\tau_2),\quad 1\otimes Y\mapsto g(\tau_2),$$

is well-defined and moreover an isomorphism. Since every $F\in\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$ is an entire elliptic modular form in each variable (when the other one is fixed), we have

$$F(\tau_1,\tau_2)=\sum_{\mu,\nu=1}^{l} a_{\nu\mu}h_\nu(\tau_1)h_\mu(\tau_2)$$

with a basis $\{h_\nu;\ \nu=1,\dots,l\}$ for $\mathbb{M}_k(\mathrm{SL}_2(\mathbb{Z})[3])$. Hence the above isomorphism yields

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)\cong\mathbb{C}\langle X\otimes X, X\otimes Y, Y\otimes X, Y\otimes Y\rangle. \tag{4.7}$$

In the case that F is symmetric, we get

$$F(\tau_2,\tau_1)=\sum_{\nu,\mu=1}^{l} a_{\nu\mu}h_\nu(\tau_2)h_\mu(\tau_1)=\sum_{\nu,\mu=1}^{l} a_{\nu\mu}h_\nu(\tau_1)h_\mu(\tau_2)=F(\tau_1,\tau_2),$$

hence $a_{\nu\mu}=a_{\mu\nu}$ for $\mu,\ \nu=1,\dots,l$ and therefore

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)_{\mathrm{sym}}\cong\mathbb{C}\langle X\otimes X, X\otimes Y, Y\otimes X, Y\otimes Y\rangle_{\mathrm{sym}},$$

where $\mathbb{C}\langle X\otimes X, X\otimes Y, Y\otimes X, Y\otimes Y\rangle_{\mathrm{sym}}$ consists of all polynomials which are invariant under changing the components of the tensor product, i. e. under the homomorphism which is defined by $A\otimes B\mapsto B\otimes A$ for $A,B\in\{X,Y\}$.

Claim: $\mathbb{C}\langle X\otimes X, X\otimes Y, Y\otimes X, Y\otimes Y\rangle_{\mathrm{sym}}=\mathbb{C}\langle X\otimes X, Y\otimes Y, X\otimes Y+Y\otimes X\rangle.$ (4.8)

Let F be in the left set of homogeneous degree n, such that

$$\begin{aligned}F&=\sum_{\substack{i+j=k+l=n\\ i\le k}} a_{ijkl}\left(X^iY^j\otimes X^lY^k+X^lY^k\otimes X^iY^j\right)\\ &=\sum_{\substack{i+j=k+l=n\\ i\le k}} a_{ijkl}(X\otimes X)^{\min\{i,l\}}(Y\otimes Y)^{\min\{j,k\}}\left((X\otimes Y)^{\max\{i+k,n\}}+(Y\otimes X)^{\max\{i+k,n\}}\right)\end{aligned}$$

It is therefore sufficient to prove that $(x_1y_1)^h+(x_2y_2)^h$, $h\in\mathbb{N}$, can be described as a polynomial in x_1x_2, y_1y_2 and $x_1y_1+x_2y_2$ (the change of variables was necessary to get concise formulae):
This fact will be proven by induction: The case $h=1$ is obvious. Given $h>1$ we get

$$(x_1y_1)^h+(x_2y_2)^h-(x_1y_1+x_2y_2)^h=-\sum_{k=1}^{h-1}\binom{h}{k}(x_1y_1)^k(x_2y_2)^{h-k}.$$

Using the symmetry of binomial coefficients we get

$$-\sum_{k=1}^{\lfloor\frac{h-1}{2}\rfloor}\binom{h}{k}(x_1y_1)^k(x_2y_2)^{h-k}+(x_1y_1)^{h-k}(x_2y_2)^k-\begin{cases}0, & h \quad \text{odd},\\ \binom{h}{\frac{h}{2}}(x_1y_1)^{h/2}(x_2y_2)^{h/2}, & h \quad \text{even}.\end{cases}$$

Next,

$$\begin{aligned}(x_1y_1)^k(x_2y_2)^{h-k}+(x_1y_1)^{h-k}(x_2y_2)^k &= (x_1y_1)^{\min\{k,h-k\}}\cdot(x_2y_2)^{\min\{k,h-k\}}\\ &\cdot\left((x_1y_1)^{\max\{k,h-k\}-\min\{k,h-k\}}+(x_2y_2)^{\max\{k,h-k\}-\min\{k,h-k\}}\right)\end{aligned}$$

yields the claim by the induction hypothesis.

Remark 4.3.2 *The above given proof (in the cases of (special) modular forms) can be found in [Kl1, Prop 9.3] or [Fr1, III.1.4].*

Theorem 4.3.3

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)\cong \mathbb{C}[X_1,\dots,X_4]/(X_1X_4-X_2X_3)$$

via the isomorphism which is induced by

$$\begin{aligned} &X_1\mapsto F_1:=f(\tau_1)f(\tau_2), && X_2\mapsto F_2:=f(\tau_1)g(\tau_2),\\ &X_3\mapsto F_3:=g(\tau_1)f(\tau_2) \quad \textit{and} && X_4\mapsto F_4:=g(\tau_1)g(\tau_2),\end{aligned}$$

where $f=E^*_{3,3}$ *and* $g=E^*_{1,3}$.

Proof: Obviously we have $F_1F_4=F_2F_3$. Let $P\in\mathbb{C}[X_1,\dots,X_4]$ be a homogeneous polynomial of degree k satisfying the relation $P(F_1,F_2,F_3,F_4)=0$. Using $F_1F_4=F_2F_3$ we can assume

$$0=\sum_{j_1=0}^{k}\sum_{i_1=0}^{k-j_1}\alpha_{i_1j_1}F_1^{j_1}F_2^{i_1}F_3^{k-j_1-i_1}+\sum_{j_2=1}^{k}\sum_{i_2=0}^{k-j_2}\alpha_{i_2j_2}F_4^{j_2}F_2^{i_2}F_3^{k-j_2-i_2}.$$

Using the definition of F_i and the algebraic independency of $f(\tau_1),\dots,g(\tau_2)$, this equation holds if and only if

$$(j_1+i_1,k-i_1,k-j_1-i_1,i_1)=(i_2,k-j_2-i_2,k-i_2,i_2+j_2).$$

Hence

$$i_2=j_1+i_1=j_1+j_2+i_2, \quad \text{and therefore} \quad j_1=j_2=0$$

which provides a contradiction to $j_2\geq 1$. □

An easy calculation reveals

Corollary 4.3.4 *Via the isomorphism of Theorem 4.3.3 one has*

a)

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)_{\mathrm{sym}} \cong \mathbb{C}[X_1, X_4, X_2+X_3].$$

b)

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2, v_\eta^{12}\times v_\eta^{12})_{\mathrm{sym}} = \eta^4(\tau_1)\eta^4(\tau_2)\cdot\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)_{\mathrm{sym}}.$$

c)

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2, v)_{\mathrm{skew}} = (F_2-F_3)\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2, v)_{\mathrm{sym}} \quad \textit{for } v\in\{v_\eta^{12}\times v_\eta^{12}, 1\}.$$

Remark 4.3.5 *a) In the proof of Theorem 4.3.3 we basically determined a Gröbner basis.*
b) Corollary 4.3.4 a) and c) can also be shown using invariant theory (see for example Example 5.2.14).

4.4 The Invariant Group

The second step to determine $\mathcal{A}(\mathcal{G})$ using Equation (3.20) is to calculate the action of $\mathcal{G}/\mathrm{SL}_2(\mathbb{Z})[3]^2 \cong \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ (see Lemma 3.6.7) on the generators of $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$: Because of $\mathrm{SL}_2(\mathbb{Z})[3]\trianglelefteq\mathrm{SL}_2(\mathbb{Z})$, we know that

$$f|N\in\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]) \quad \text{for} \quad f\in\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]), N\in\mathrm{SL}_2(\mathbb{Z}),$$

hence we can describe $f|N$ by the generators $E^*_{3,3}$, $E^*_{1,3}$. Using $\sigma_0^{-1}=-J$ and the equations (4.4) and (8.1), we get

$$E^*_{3,3}|J = \tfrac{-i}{\sqrt{3}}\left(E^*_{1,3}(\tau)-E^*_{3,3}(\tau)\right) \text{ and } E^*_{1,3}|J = \tfrac{-i}{\sqrt{3}}\left(2E^*_{3,3}(\tau)+E^*_{1,3}(\tau)\right).$$

Given $f(\tau)=\sum_{n=0}^\infty a_n q^{\frac{n}{3}}$ with $q=e^{2\pi i\tau}$, we get $(f|T)(\tau)=\sum_{n=0}^\infty a_n\zeta_3^n q^{\frac{n}{3}}$. We therefore conclude from

$$\begin{aligned}(E^*_{3,3}|T)(\tau) &= 1+(\tfrac{3}{2}-\tfrac{\sqrt{3}^3}{2}i)q^{\frac{1}{3}}+\mathcal{O}(q^{\frac{2}{3}}), \text{ and}\\ (E^*_{1,3}|T)(\tau) &= 1-(3-\sqrt{3}^3 i)q^{\frac{1}{3}}+\mathcal{O}(q^{\frac{2}{3}})\end{aligned}$$

that

$$E^*_{3,3}|T = (\tfrac{1}{2}+\tfrac{\sqrt{3}}{6}i)E^*_{3,3}+(\tfrac{1}{2}-\tfrac{\sqrt{3}}{6}i)E^*_{1,3} \quad\text{and}\quad E^*_{1,3}|T=(1-\tfrac{i}{\sqrt{3}})E^*_{3,3}+\tfrac{i}{\sqrt{3}}E^*_{1,3}.$$

Since $T^{-\mathrm{tr}} = JTJ^3$, we define

$$M_J = \begin{pmatrix} \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ -\frac{2i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \end{pmatrix}, \quad M_T = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{6}i & \frac{1}{2} - \frac{\sqrt{3}}{6}i \\ 1 - \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \end{pmatrix} \quad \text{and} \quad M_{T^{-\mathrm{tr}}} = M_J M_T M_J^3.$$

Then we have

$$F_i \mapsto \sum_{j=1}^{n} m_{ij} F_j \quad \text{for } 1 \leq i \leq 4 \text{ and } M = (m_{ij})_{i,j} \in \{M_J \otimes M_J, M_{T^{-\mathrm{tr}}} \otimes M_T\},$$

where $M_1 \otimes M_2$ is the Kronecker product of two matrices, i. e. $(m_{ij}M_2)$ for $M_1 = (m_{ij})_{i,j}$.

Note that we have been choosing the row convention for these matrices, since many computer algebra systems use this convention. If we used columns, the linear transformations with respect to the basis $\{F_1, \ldots, F_4\}$ from above would be represented by the matrices M^{tr}.

To sum up, we see that $\mathcal{G}/\mathrm{SL}_2(\mathbb{Z})[3]^2$ acts on $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$ with kernel $\{\pm(I,I)\}$. Additionally, $f \in \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$ (written as a polynomial in the generators) is symmetric if

$$f|M_{\mathrm{Flip}} = f \quad \text{with} \quad M_{\mathrm{Flip}} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.9}$$

Thus we define the two groups $G := \langle M_J \otimes M_J, M_{T^{-\mathrm{tr}}} \otimes M_T \rangle$ and $\widehat{G} := \left\langle G, M_{\mathrm{Flip}} \right\rangle$. It is easy to show that $G \cong A_4$ and $\widehat{G} \cong A_4 \times \mathcal{C}_2$, e. g. with [GAP]. Notice that M_{Flip} is a matrix of order 2, but this matrix does not commute with $M_{T^{-\mathrm{tr}}} \otimes M_T$. Instead, the matrix $(M_J \otimes M_J)M_{\mathrm{Flip}}$ commutes with all matrices in $\widehat{G}$.

The character χ which is associated to the representation of G from above is four dimensional and has the values $\mathrm{trace}(M_{T^{-\mathrm{tr}}} \otimes M_T) = \zeta_3$ on a conjugacy class of order 3 (we choose the last column in the table on page 35) and $\mathrm{trace}(M_J \otimes M_J) = 0$ on the conjugacy class of order 2. It is easy to calculate by hand that there are no invariants of degree 1. Therefore the trivial character is no constituent of χ. The character is therefore uniquely determined (assuming the above mentioned choice of the conjugacy classes): $\chi = \tau + \lambda'$. Since $\vartheta \otimes \vartheta$ has the same values on the conjugacy classes, we have

$$\chi = \vartheta \otimes \vartheta = \tau + \lambda' \quad \text{and} \quad \det \vartheta = \lambda' = \lambda^{-1}. \tag{4.10}$$

The character $\vartheta \otimes \vartheta$ can be realized over $F = \mathbb{Q}(\zeta_3)$, a fact which does not follow directly from the table on page 35. Since we do not need this explicit result, we omit the proof.

The character which is associated to the representation of $\widehat{G}$ from above is determined in Section 5.6.

So we are interested in the graded algebra

$$\left(\mathbb{C}[X_1,\ldots,X_4]/\langle X_1X_4 - X_2X_3\rangle\right)^H,$$

with $H = G, \widehat{G}$, i. e. in all polynomials mod $I = \langle X_1X_4 - X_2X_3\rangle$ which remain in the same coset under the operation of H.

In the classical invariant theory this problem was only considered for polynomial rings, i. e. $F[X_1,\ldots,X_n]^H$ for a field F. In the next chapter we develop the theory to be able to calculate invariant rings like the one from above.

5 Algebraic Methods

The structural results of this chapter, especially from Section 5.3 were done in team-work with Dr. Jürgen Müller.

5.1 Commutative Algebra

In this section we give a short repetition of the structures and results of the commutative algebra we will need:

Throughout this section let F be a field, V be a finite dimensional F-vector space with $n = \dim_F V$. The main objects are finitely generated commutative F-algebras. A commutative F-algebra $\mathcal{R}$ is called $\mathbb{Z}$*-graded* (or simply *graded*) if $\mathcal{R} = \bigoplus_{d \geq 0} \mathcal{R}_d$ as F-vector spaces, $\mathcal{R}_0 \cong F$, $\dim_F \mathcal{R}_d \in \mathbb{N}_0$ for all $d \in \mathbb{N}$ and $\mathcal{R}_{d_1} \cdot \mathcal{R}_{d_2} \subset \mathcal{R}_{d_1+d_2}$ for all d_1, $d_2 \geq 0$. We call $\mathcal{R}_d$ a *homogeneous component* and set $\mathcal{R}_d = \{0\}$ for $d < 0$. Let $r = (r_d)_{d \in \mathbb{Z}} \in \mathcal{R}$ with $r_d \in \mathcal{R}_d$ for all $d \in \mathbb{Z}$, then there are only finitely many $r_d \neq 0$. If $r \neq 0$, then $\deg r = \max\{d;\ r_d \neq 0\}$ is called the *degree* of r. An example for a graded F-algebra is the *symmetric algebra*

$$\mathcal{S}[V] := \mathcal{T}[V] / \langle x \otimes y - y \otimes x;\ x, y \in \mathcal{T}[V] \rangle,$$

where $\mathcal{T}[V] = F \oplus \bigoplus_{d \geq 1} V^{\otimes d}$ denotes the tensor algebra. Obviously, $\mathcal{S}[V]$ is finitely generated and as a graded F-algebra it is isomorphic to the polynomial ring $F[X_1, \ldots, X_n]$ with the standard grading defined by $\deg X_i = 1$ for all i.

Definition 5.1.1 *Let $\mathcal{R}$ be a graded F-algebra. An $\mathcal{R}$-module M is* graded *if there exists a $d_M \in \mathbb{Z}$ such that $M = \bigoplus_{d \geq d_M} M_d$ as F-vector spaces with $\dim_F M_d \in \mathbb{N}_0$ for all $d \geq d_M$ and $\mathcal{R}_{d_1} \cdot M_{d_2} \subset M_{d_1+d_2}$ for all d_1, $d_2 \geq d_M$. Again, we set $M_d = \{0\}$ for $d < d_M$ and $\deg m = \max\{d; m_d \neq 0\}$ for $0 \neq m = (m_d)_{d \in \mathbb{Z}} \in M$.*

5.1.1 Noetherian Modules and Krull dimension

Let $\mathcal{R}$ be a commutative ring, then we call an $\mathcal{R}$-module M *Noetherian* if every ascending chain of submodules is eventually constant. A ring itself is said to be *Noetherian* if it is so as a module over itself.

The following properties are well-known (compare [Ben, §1.2])

Theorem 5.1.2 ***(Hilbert's Basis Theorem)*** *If $\mathcal{R}$ is a commutative Noetherian ring, then $\mathcal{R}[X]$ is Noetherian.*

Lemma 5.1.3 *a) A module M over a Noetherian ring $\mathcal{R}$ is Noetherian if and only if it is finitely generated.*
b) A submodule of a finitely generated module over a Noetherian ring is again finitely generated.
c) A commutative ring $\mathcal{R}$ is Noetherian if and only if every ideal $\mathcal{I}$ is finitely generated.
d) If $\mathcal{A}$ is a finitely generated commutative F-algebra, then $\mathcal{A}$ is Noetherian.

From now on we formulate the following hypothesis which describes the generality we choose to work with:

Hypothesis 5.1.4 *Let $\mathcal{R} = \bigoplus_{d\geq 0} \mathcal{R}_d$ with $\mathcal{R}_0 \cong F$ be a graded commutative ring and $\{0\} \neq M = \bigoplus_{d\geq d_M} M_d$ a finitely generated $\mathcal{R}$-module. Let the characteristic of F be $\operatorname{char} F = 0$ and let $\mathcal{R}$ be finitely generated over F with finitely many elements of positive degree. We set $\mathcal{R}^+ = \bigoplus_{d\geq 1} \mathcal{R}_d$ the* maximal homogeneous ideal.

If $\mathcal{R}$ and M fulfill the Hypothesis 5.1.4, they are in view of Lemma 5.1.3 Noetherian.

The *Krull dimension* of a commutative ring $\mathcal{R}$ is the maximal length n of a chain of proper inclusions of prime ideals $\mathcal{P}_0 \supset \mathcal{P}_1 \supset \ldots . \supset \mathcal{P}_n$, or ∞ if the maximum does not exist. If M is a $\mathcal{R}$-module, then the Krull dimension of M is the Krull dimension of the ring $\mathcal{R}/\operatorname{Ann}_{\mathcal{R}}(M)$, where

$$\operatorname{Ann}_{\mathcal{R}}(M) = \{r \in \mathcal{R}; rm = 0 \text{ for all } m \in M\},$$

is the *annihilator* of M in $\mathcal{R}$. We write $\dim \mathcal{R}$ and $\dim M$, respectively. Whenever we refer to the dimension of a vector space, we keep the field as a subscript.

The *height* of a prime ideal $\mathcal{P}$ – denoted by $\operatorname{height} \mathcal{P}$ – is the maximal length n of a chain of proper inclusions of prime ideals $\mathcal{P} = \mathcal{P}_0 \supset \mathcal{P}_1 \supset \ldots \supset \mathcal{P}_n$, or ∞, if the maximum does not exist.

Let $\mathcal{R}_1, \mathcal{R}_2$ be two commutative rings with $\mathcal{R}_1 \subset \mathcal{R}_2$. We call $r \in \mathcal{R}_2$ *integral* over $\mathcal{R}_1$ if there exists a polynomial p with coefficients in $\mathcal{R}_1$ and leading coefficient 1 such that $p(r) = 0$. We call $\mathcal{R}_2$ a *finite extension* if every element in $\mathcal{R}_2$ is integral over $\mathcal{R}_1$ and if $\mathcal{R}_2$ (as a module) is finitely generated over $\mathcal{R}_1$.

Lemma 5.1.5 *([Ben, Cor 1.4.5])*
If $\mathcal{R}_2 \supset \mathcal{R}_1$ is a finite extension of rings, then the Krull dimensions of $\mathcal{R}_1$ and $\mathcal{R}_2$ are equal.

Now we can cite Krull's Principal Ideal Theorem ([Ben, Thm 2.3.3])

Theorem 5.1.6 *Suppose that $\mathcal{R}$ is a commutative Noetherian ring and $\mathcal{I} = \langle x \rangle$ is a principal ideal in $\mathcal{R}$. Then any prime ideal $\mathcal{P}$, which is minimal among the prime ideals containing $\mathcal{I}$ has height at most one.*

5.1.2 Nakayama Lemma and Noether Normalization

Lemma 5.1.7 *([DKe, Lem 3.5.1])*
Let $\mathcal{R}$ and M satisfy Hypothesis 5.1.4 with $d_M \geq 0$. Then for a subset $S \subset M$ of homogeneous elements the two conditions are equivalent:

(i) S generates M as an $\mathcal{R}$-module.
(ii) S generates $M/_{\mathcal{R}^+M}$ as a vector space over F. Here $\mathcal{R}^+M$ is the submodule of M generated by the elements $r \cdot m$ with $r \in \mathcal{R}^+$ and $m \in M$.

In particular, a generating set S for M is of minimal cardinality if no proper subset of S generates M.

In the situation that Hypothesis 5.1.4 is fulfilled we define the the *Hilbert series* of M as

$$\mathrm{H}(M,t) := \sum_{d \geq d_M} (\dim_F M_d)\, t^d.$$

The Hilbert series is sometimes also called *Poincaré series*. A consequence of the Theorem of Hilbert-Serre ([Ben, Thm 2.1.1]) is that $\mathrm{H}(M,t)$ converges in the pointed open unit disc $\{z \in \mathbb{C}; 0 < |z| < 1\}$ and has at most a pole in $t = 1$. The order of this pole is denoted by $\operatorname{ord} M$. Moreover the Hilbert series is of the form

$$\mathrm{H}(M,t) = \frac{p(t)}{\prod_{j=1}^{r}(1-t^{k_j})} \quad \text{with } p(t) \in \mathbb{Z}[t,t^{-1}]. \tag{5.1}$$

We cite the Noether normalization in the graded case:

Theorem 5.1.8 *([Ben, Thm 2.2.7])*
Let $\mathcal{R}$ and M satisfy Hypothesis 5.1.4. Then there exist homogeneous elements $f_1, \ldots, f_n$ of positive degree in $\mathcal{R}$, which generate a polynomial subring $F[f_1,\ldots,f_n]$ in $\mathcal{R}/\operatorname{Ann}_{\mathcal{R}}(M)$ over which M is finitely generated as a module. Furthermore,

$$n = \dim M = \operatorname{ord} M.$$

If $\mathcal{R} = M$ is an integral domain, the number n is also equal to the transcendence degree of the field of fractions Quot *of $\mathcal{R}$, i. e.* $\operatorname{trdeg}\operatorname{Quot}(\mathcal{R}) = n$.

Therefore we define (cf. [DKe, §2.4.2])

Definition 5.1.9 *Let $\mathcal{R}$ and M satisfy Hypothesis 5.1.4.*
A subset $\{f_1,\ldots,f_n\} \subset \mathcal{R}/\operatorname{Ann}_{\mathcal{R}}(M)$ of homogeneous elements of positive degree is called a homogeneous system of parameters *for M if*

(hsop1) *$f_1,\ldots,f_n$ are algebraically independent and*
(hsop2) *M is finitely generated as module over $F[f_1,\ldots,f_n]$.*

By Theorem 5.1.8 the length of a homogeneous system is uniquely determined. Therefore we get

Remark 5.1.10 *Let $\mathcal{R}$ and M satisfy Hypothesis 5.1.4 with $d_M \geq 0$ and let $f_1, \dots, f_n \in \mathcal{R}/\mathrm{Ann}_{\mathcal{R}}(M)$ be homogeneous of positive degree. Then*

$$\dim_F M/\textstyle\sum_{i=1}^n M f_i < \infty \quad \textit{if and only if} \quad \dim M/\textstyle\sum_{i=1}^n M f_i = 0.$$

Proof: Let $N = M/\sum_{i=1}^n M f_i$. If $\dim N = 0$, then by Theorem 5.1.8 N is finitely generated as F-module. If $\dim_F N < \infty$, then $\mathrm{H}(N,t)$ is a polynomial and we get $\operatorname{ord} N = \dim N = 0$ by Theorem 5.1.8. $\square$

We show now a characterization that $\{f_1, \dots, f_n\} \subset \mathcal{R}/\mathrm{Ann}_{\mathcal{R}}(M)$ is a homogeneous system of parameters for M:

Proposition 5.1.11 *Let $\mathcal{R}$ and M satisfy Hypothesis 5.1.4 with $d_M \geq 0$ and let $f_1, \dots, f_n \in \mathcal{R}/\mathrm{Ann}_{\mathcal{R}}(M)$ be homogeneous of positive degree.*
$\{f_1, \dots, f_n\} \subset \mathcal{R}/\mathrm{Ann}_{\mathcal{R}}(M)$ is a homogeneous system of parameters for M if and only if $\dim_F(M/\sum_{i=1}^n M f_i) < \infty$ and the Krull dimension $n = \dim M$.

Proof: $S := F\langle f_1, \dots, f_n\rangle$. By Nakayama's Lemma 5.1.7 M is a finitely generated S-module if and only if $M/\sum_{i=1}^n M f_i$ is a finitely generated F-module. If $\{f_1, \dots, f_n\}$ is a homogeneous system of parameters, we get using Theorem 5.1.8 that $n = \operatorname{ord} M = \dim M$. Let vice versa $n = \dim M$. In the case that $f_1, \dots, f_n$ would be algebraically dependent, we would get that S is a finite extension of $F[\mathcal{X}]$, where $\mathcal{X}$ is a subset of $\{f_1, \dots, f_n\}$ with a cardinality at most $n-1$. Thus $\operatorname{ord} S = \operatorname{ord}(F[\mathcal{X}]) = \sharp\mathcal{X} \leq n-1$. Since M is a finitely generated $\mathcal{S}$-module, M is also a finitely generated $F[\mathcal{X}]$-module. Therefore

$$n = \dim M \neq \sharp\mathcal{X}.$$

which provides a contradiction to Theorem 5.1.8. $\square$

5.1.3 Invariants

Let $\mathcal{R}$ satisfy Hypothesis 5.1.4 and G be a group of graded algebra automorphisms of $\mathcal{R}$. We denote the corresponding action by r^g for $r \in \mathcal{R}$ and $g \in G$. This leads to the definition of the *invariant ring*

$$\mathcal{R}^G := \{f \in \mathcal{R}; f^g = f \quad \text{for all } g \in G\}.$$

Obviously, this is a graded F-subalgebra of $\mathcal{R}$. Moreover, let $\lambda : G \to F$ be an Abelian character of G, then we denote by

$$\mathcal{R}^\lambda := \{f \in \mathcal{R}; f^g = \lambda(g)\cdot f \quad \text{for all } g \in G\}$$

the $\mathcal{R}^G$-module of λ*-relative invariants*.

Given a group G and an F-representation $D_V : G \to \mathrm{GL}(V) \cong \mathrm{GL}_n(F)$, then V is an FG-module by $v^g := v \cdot D_V(g)$ for $v \in V$, $g \in G$. By diagonal operation of G on $V^{\otimes d}$, i. e. $(\bigotimes_i v_i)^g = \bigotimes_i v_i^g$, the symmetric algebra is an FG-module, too. Thus $\mathcal{S}[V]^G$ and $\mathcal{S}[V]^\lambda$ are well-defined.

Note that we can define the relative fixmodule V^λ for every FG-module V in the same way. In the literature V^λ is often called an *isotypic component*. Since $\mathrm{char} F = 0$, we could use representation theory, i. e. the decomposition of V into irreducible FG-modules, to define λ-relative invariants also in the case that λ is a non-linear character.

We want to prove that $\mathcal{R} = \mathcal{S}[V]^G$ and $M = \mathcal{S}[V]^\lambda$ satisfy the Hypothesis 5.1.4. Therefore we need the Reynolds operator.

5.2 The Generalized Reynolds Operator

Throughout this section let $\mathcal{R}$ fulfill Hypothesis 5.1.4 and G be a finite group of graded algebra automorphisms of $\mathcal{R}$ and $\lambda : G \to F$ a linear character. Then the *generalized Reynolds operator* is defined by

$$\mathrm{R}^\lambda := \frac{1}{|G|} \sum_{g \in G} \lambda(g^{-1}) g \in \mathrm{FG},$$

where $\mathrm{FG} := \{\sum_{g\in G} \alpha_g g; \alpha_g \in F\}$ is the *group algebra*. Given $\lambda = 1$, we use R^G instead of R^λ. Obviously, the Reynolds operator acts on $\mathcal{R}$ by

$$\mathrm{R}^\lambda f := \frac{1}{|G|} \sum_{g \in G} \lambda(g^{-1}) f^g \quad \text{for} \quad f \in \mathcal{R}$$

and is an $\mathcal{R}^G$-homomorphism, since

$$\mathrm{R}^\lambda(f_1 \cdot f_2) = f_1 \cdot \mathrm{R}^\lambda(f_2) \quad \text{for} \quad f_1 \in \mathcal{R}^G \text{ and } f_2 \in \mathcal{R}.$$

We need the following easily proven result:

Lemma 5.2.1 *Let $\lambda : G \to F$ be a linear character. Then the generalized Reynolds operator is a projection from $\mathcal{R}$ onto $\mathcal{R}^\lambda$. Especially, one has:*

a) $g \cdot \mathrm{R}^\lambda = \mathrm{R}^\lambda \cdot g$ *for all $g \in G$.*
b) $\mathrm{R}^\lambda f = f$ *for all $f \in \mathcal{R}^\lambda$ and $\mathrm{R}^\lambda(\mathcal{R}) = \mathcal{R}^\lambda$, thus $(\mathrm{R}^\lambda)^2 = \mathrm{R}^\lambda$.*

Proof: a) Using the index transformation $h \mapsto ghg^{-1}$ we get

$$g \cdot \mathrm{R}^\lambda = \frac{1}{|G|} \sum_{h \in G} \lambda(h^{-1}) gh = \frac{1}{|G|} \sum_{h \in G} \lambda(h^{-1}) hg = \mathrm{R}^\lambda \cdot g$$

b) If $f \in \mathcal{R}^\lambda$, then

$$\mathrm{R}^\lambda(f) = \frac{1}{|G|}\sum_{g\in G}\lambda(g^{-1})f^g = \frac{1}{|G|}\sum_{g\in G} f = f.$$

In particular $\mathrm{R}^\lambda(\mathcal{R}) \supset \mathcal{R}^\lambda$. For $f \in \mathcal{R}$ we use the index transformation $h \mapsto hg$ and get

$$(\mathrm{R}^\lambda(f))^g = \frac{1}{|G|}\sum_{h\in G}\lambda(h^{-1})f^{hg} = \lambda(g)\cdot\frac{1}{|G|}\sum_{h\in G}\lambda((hg)^{-1})f^{hg} = \lambda(g)\,\mathrm{R}^\lambda(f),$$

so that $\mathrm{R}^\lambda(\mathcal{R}) = \mathcal{R}^\lambda$. □

For an FG-module homomorphism α we have

$$(\mathrm{R}^\lambda \circ \alpha) = \frac{1}{|G|}\sum_{h\in G}\lambda(h^{-1})\alpha(f)^g = \alpha(\frac{1}{|G|}\sum_{h\in G}\lambda(h^{-1})f^g) = \alpha\circ\mathrm{R}^\lambda\,.$$

so that R^λ commutes with every FG-module homomorphism α.

Again our main example will be $\mathcal{R} = \mathcal{S}[V]$.

Given F with $\operatorname{char} F = 0$, we call $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ an *exact sequence* of FG-modules if $\alpha : U \to V$ is an injective FG-homomorphism, $\beta : V \to W$ is a surjective FG-homomorphism and $\ker\beta = \operatorname{im}\alpha$.

Lemma 5.2.2 *Let G be a finite group, F a field with $\operatorname{char} F = 0$ and $\lambda : G \to F$ a linear character. For every exact sequence $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ of FG-modules the sequence of the relative fixmodules*

$$0 \to U^\lambda \xrightarrow{\alpha} V^\lambda \xrightarrow{\beta} W^\lambda \to 0$$

is also exact.

Proof: Given $u \in U^\lambda$, we have

$$\mathrm{R}^\lambda(\alpha(u)) = (\mathrm{R}^\lambda \circ\alpha)(u) = (\alpha\circ\mathrm{R}^\lambda)(u) = \alpha(\mathrm{R}^\lambda(u)) = \alpha(u),$$

hence $\alpha(u) \in V^\lambda$ and therefore $\alpha|_{U^\lambda} : U^\lambda \to V^\lambda$ and in the same way $\beta|_{V^\lambda} : V^\lambda \to W^\lambda$ is well-defined. Obvioulsy, $\alpha : U^\lambda \to V^\lambda$ is injective. Let now $w \in W^\lambda$. Thus there exists a $v \in V$ with $\beta(v) = w$. It follows that $\mathrm{R}^\lambda(v) \in V^\lambda$ and

$$\beta(\mathrm{R}^\lambda(v)) = \mathrm{R}^\lambda(\beta(v)) = \mathrm{R}^\lambda(w) = w,$$

hence $\beta : V^\lambda \to W^\lambda$ is surjective.

It remains to be shown that $\ker\beta|_{V^\lambda} = \operatorname{im}\alpha|_{U^\lambda}$. Let $v \in V^\lambda$ with $\beta(v) = 0$. Then there exists an $u \in U$ with $\alpha(u) = v$. Hence $\mathrm{R}^\lambda(u) \in U^\lambda$ and

$$\alpha(\mathrm{R}^\lambda(u)) = \mathrm{R}^\lambda(\alpha(u)) = \mathrm{R}^\lambda(v) = v,$$

which proves $\ker\beta|_{V^\lambda} \subset \operatorname{im}\alpha|_{U^\lambda}$. The other inclusion follows directly from $\beta\circ\alpha = 0$. □

Later on we will need the following generalization: Let U,V be FG-modules and $\chi : G \to F$ be a linear character. Let $\alpha : U \to V$ be a linear transformation which satisfies $(\alpha(u))^g = \chi(g)\alpha(u^g)$ for all $g \in G$ and $u \in U$. In particular α is not a FG-homomorphism. We call this a χ-*twisted* FG-*homomorphism*. Let $F_\chi = F$ be the FG-module where an element $g \in G$ acts as follows: $1 \mapsto \chi(g)\cdot 1$. Via the isomorphism $U \otimes_F F \cong U$, $u \otimes 1 \mapsto u \cdot 1$, we can translate the operation of G on U to $U \otimes_F F_\chi$:

$$(u \otimes a)^g = u^g \otimes \chi(g)a \quad \text{for } u \in U,\ a \in F \text{ and } g \in G.$$

Hence $\alpha^\chi : U \otimes_F F_\chi \to V$, $\alpha^\chi(u \otimes 1) = \alpha(u)$ fulfills

$$(\alpha^\chi(u \otimes 1))^g = \alpha(u)^g = \alpha(u^g)\chi(g) = \alpha(u^g\chi(g)) = \alpha^\chi((u \otimes 1)^g)$$

and therefore α^χ is a FG-homomorphism.
Let $\alpha : U \to V$ be an injective χ-twisted FG-homomorphism and $\beta : V \to W$ a surjective FG-homomorphism such that $\ker\beta = \operatorname{im}\alpha$. We call this a χ-*twisted exact sequence* and denote it by

$$0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0.$$

Then

$$0 \to U \otimes_F F_\chi \xrightarrow{\alpha^\chi} V \xrightarrow{\beta} W \to 0$$

is an exact sequence of FG-modules. Using Lemma 5.2.2 for a linear character $\lambda : G \to F$ we get that

$$0 \to (U \otimes_F F_\chi)^\lambda \xrightarrow{\alpha^\chi} V^\lambda \xrightarrow{\beta} W^\lambda \to 0$$

is an exact sequence. Using $V^\lambda = \mathrm{R}^\lambda(V)$ and

$$\begin{aligned}\mathrm{R}^\lambda(\alpha^\chi(u\otimes 1)) &= \frac{1}{|G|}\sum_{g\in G}\lambda(g^{-1})(\alpha^\chi(u\otimes 1))^g = \frac{1}{|G|}\sum_{g\in G}\lambda(g^{-1})\alpha(u^g)\chi(g) \\ &= \alpha\left(\frac{1}{|G|}\sum_{g\in G}\lambda(g^{-1})\chi(g)u^g\right) = \alpha(\mathrm{R}^{\chi^{-1}\lambda}(u)),\end{aligned}$$

we get

$$\begin{aligned}\operatorname{im}\alpha^\chi|_{(U\otimes_F F_\chi)^\lambda} &= \operatorname{im}(\alpha^\chi \circ \mathrm{R}^\lambda)|_{U\otimes_F F_\chi} = \operatorname{im}(\mathrm{R}^\lambda \circ \alpha^\chi)|_{U\otimes_F F_\chi} \\ &= \operatorname{im}(\alpha \circ R^{\chi^{-1}\lambda})|_U\end{aligned}$$

and therefore obtain the following χ-twisted exact sequence

$$0 \to U^{\chi^{-1}\lambda} \xrightarrow{\alpha} V^\lambda \xrightarrow{\beta} W^\lambda \to 0.$$

We summarize this result in the following

Lemma 5.2.3 *Let G be a finite group, F a field with* $\operatorname{char} F = 0$ *and* $\chi, \lambda : G \to F$ *two linear characters. Let* $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ *be a* χ*-twisted exact sequence, then*

$$0 \to U^{\chi^{-1}\lambda} \xrightarrow{\alpha} V^{\lambda} \xrightarrow{\beta} W^{\lambda} \to 0$$

is also a χ*-twisted exact sequence.*

Remark 5.2.4 *a) For* $\chi = 1$ *we get the same result as in Lemma 5.2.2.*
b) Lemma 5.2.2 and Lemma 5.2.3 can be generalized to groups which possess a Reynolds operator. These groups are called linearly reductive.

5.2.1 More on Invariants

From ([Ben, Thm 1.3.1]) we cite the well-known

Theorem 5.2.5 ***(Hilbert-Noether)***
Let G be a finite group. Then $\mathcal{S}[V]^G$ *is a finitely generated F-algebra which is integrally closed. Moreover* $\mathcal{S}[V]^G \subset \mathcal{S}[V]$ *is a finite ring extension.*

Using this theorem it is possible to prove

Corollary 5.2.6 *([Ben, §1.4])*
Let G be a finite group, F a field and V a finite dimensional FG*-module. Then*

$$\dim \mathcal{S}[V] = \dim_F V = \dim \mathcal{S}[V]^G.$$

We want to generalize these results to relative-invariants $\mathcal{R}^{\lambda}$ for rings $\mathcal{R}$ which satisfies Hypothesis 5.1.4:

Proposition 5.2.7 *Let $\mathcal{R}$ be an integral domain which fulfills Hypothesis 5.1.4 and let G be a faithful finite group of graded algebra automorphisms of $\mathcal{R}$. Then for the invariant field* $(\operatorname{Quot}\mathcal{R})^G$ *we have* $(\operatorname{Quot}(\mathcal{R}))^G = \operatorname{Quot}(\mathcal{R}^G)$ *and the field extension* $(\operatorname{Quot}(\mathcal{R}))^G \subset \operatorname{Quot}(\mathcal{R})$ *is finite Galois with Galois group G.*

Proof: We clearly have $\operatorname{Quot}(\mathcal{R}^G) \subset \operatorname{Quot}(\mathcal{R})^G$. Conversely let $f = \frac{f_1}{f_2} \in \operatorname{Quot}(\mathcal{R})^G$ for $f_i \in \mathcal{R}$. By extending with $\prod_{1 \neq g \in G} f_2^g$ we may assume that $f_2 \in \mathcal{R}^G$ and hence $f_1 \in \mathcal{R}^G$ as well, thus $f \in \operatorname{Quot}(\mathcal{R}^G)$. By Artin's Theorem, see [Lan, Thm VI.1.8], the field extension $(\operatorname{Quot}(\mathcal{R}))^G \subset \operatorname{Quot}(\mathcal{R})$ is Galois. $\square$

Proposition 5.2.8 *Let $\mathcal{R}$ fulfill Hypothesis 5.1.4 and let G be a finite group of graded algebra automorphisms of $\mathcal{R}$.*

a) $\mathcal{R}$ is a finite ring extension of $\mathcal{R}^G$.

b) $\dim \mathcal{R}^G = \dim \mathcal{R}$.
c) $\mathcal{R}^G$ *is a finitely generated* F*-algebra.*
d) Let $\mathcal{R}$ *have no zero-divisors and* $\mathcal{R}^\lambda \neq \{0\}$. *Then* $\dim \mathcal{R}^\lambda = \dim \mathcal{R}$.
e) $\mathcal{R}^\lambda$ *is a finitely generated* $\mathcal{R}^G$*-module.*

Proof: a) For $f \in \mathcal{R}$ let $p_f := \prod_{g\in G}(X - f^g) \in \mathcal{R}^G[X]$. Hence we have $p_f(f) = 0$ and f is monic so that f is integral over $\mathcal{R}^G$. Let $\{f_1,\dots,f_r\} \subset \mathcal{R}$ be an F-algebra generating set of $\mathcal{R}$ and let $\mathcal{S} \subset \mathcal{R}^G \subset \mathcal{R}$ be the F-algebra generated by the coefficients of the polynomials $\{p_{f_1},\dots,p_{f_r}\} \subset \mathcal{R}^G[X]$. As all f_i are integral over $\mathcal{S}$, the F-algebra $\mathcal{R}$ is integral over $\mathcal{S}$. As $\mathcal{R}$ is a finitely generated F-algebra, it is also a finitely generated $\mathcal{S}$-algebra and hence the ring extension $\mathcal{S} \subset \mathcal{R}$ is finite. In particular, the ring extension $\mathcal{R}^G \subset \mathcal{R}$ is finite.

b) Follows directly from a) using Lemma 5.1.5.

c) $\mathcal{S}$ is a finitely generated F-algebra and therefore Noetherian due to Lemma 5.1.3. As the $\mathcal{S}$-module $\mathcal{R}$ is finitely generated, it is a Noetherian $\mathcal{S}$-module by the same lemma Thus the $\mathcal{S}$-submodule $\mathcal{R}^G \subset \mathcal{R}$ is also Noetherian and therefore a finitely generated $\mathcal{S}$-module. As $\mathcal{S}$ is a finitely generated F-algebra, $\mathcal{R}^G$ is a finitely generated F-algebra as well.

d) One has

$$\dim \mathcal{R}^\lambda = \dim \mathcal{R}^G/\mathrm{Ann}_{\mathcal{R}^G}(\mathcal{R}^\lambda) = \dim \mathcal{R}^G/\{0\} = \dim \mathcal{R}^G.$$

e) In c) we showed that $\mathcal{R}$ is a finitely generated $\mathcal{S}$-module, therefore it is also a finitely generated $\mathcal{R}^G$-module. Since $\mathcal{R}^G$ is Noetherian we get by Lemma 5.1.3 that $\mathcal{R}^\lambda$ is a finitely generated $\mathcal{R}^G$-submodule of $\mathcal{R}$. □

We have already seen in Section 5.1.2 that if M fulfills Hypothesis 5.1.4 that the Krull dimension of M is equal to the order of the rational function $\mathrm{H}(M,t)$ which was abbreviated by $\operatorname{ord} M$. Therefore

$$\deg M := \left(\mathrm{H}(M,t)(1-t)^{\operatorname{ord} M}\right)(1) \in \mathbb{R}$$

is well-defined. Moreover, it is a rational number because of Equation (5.1).

Example 5.2.9 *Let* $\mathcal{R} = F[f_1,\dots,f_n]$ *be a polynomial ring with* $\deg f_i = k_i$ *for* $1 \leq i \leq n$. *Then*

$$\mathrm{H}(\mathcal{R},t) = \prod_{i=1}^{n} \frac{1}{1-t^{k_i}}$$

and therefore we get

$$\operatorname{ord} \mathcal{R} = n \textit{ and } \deg \mathcal{R} = \prod_{i=1}^{n} \frac{1}{k_i}.$$

Theorem 5.2.10 *([Ben, Prop 2.4.2])*
Let $\mathcal{R}$ fulfill Hypothesis 5.1.4 and let $\mathcal{R} \subset \mathcal{S}$ be a finite extension of graded F-algebras such that $\mathcal{S}$ is a domain. Then we have

$$\deg \mathcal{S} = [\mathrm{Quot}(\mathcal{S}) : \mathrm{Quot}(\mathcal{R})] \deg(\mathcal{R}).$$

$\mathcal{S}[V]$ is a finitely generated graded algebra. Therefore, if $\mathcal{S}[V]^\lambda \neq \{0\}$, then the F-algebra $\mathcal{R} = \mathcal{S}[V]^G$ and the $\mathcal{R}$-module $M = \mathcal{S}[V]^\lambda$ fulfill the Hypothesis 5.1.4 by Proposition 5.2.8.

In the case of $M = \mathcal{S}[V]^\lambda$ we can calculate the Hilbert series using

Theorem 5.2.11 ***(Molien's formula**, [Ben, Thm. 2.5.2, 2.5.3])*
Let F be a field of $\mathrm{char} F = 0$. *Then*

$$\mathrm{H}(\mathcal{S}[V]^\lambda, t) = \frac{1}{|G|} \cdot \sum_{g \in G} \lambda(g)^{-1} \cdot \left(\prod_{i=1}^{m} \frac{1}{1 - \lambda_i(g) \cdot T} \right) \in \mathbb{C}(T),$$

where $m := \dim_{\mathbb{C}}(V)$ and $\lambda_1(g), \ldots, \lambda_m(g) \in \mathbb{C}$ are the eigenvalues of the action of $g \in G$ on V.

Hence if the character afforded by V is known, this formula can be evaluated from the character table of G, see [Ben, Ch.2.5]. This method and the necessary character theoretic data are available in GAP.

Another very useful concept is the Jacobi-criterion:

Definition 5.2.12 *For $\{f_1, \ldots, f_n\} \subset F[X_1, \ldots, X_n]$ the* Jacobian matrix *is defined as*

$$\mathrm{J}(f_1, \ldots, f_n) := \left(\frac{\partial f_i}{\partial X_j} \right)_{1 \leq i,j \leq n} \in \mathrm{Mat}_n(F[X_1, \ldots, X_n]).$$

Its determinant $\det \mathrm{J}(f_1, \ldots, f_n) \in F[X_1, \ldots, X_n]$ *is called* Jacobian determinant.

Proposition 5.2.13 ***(Jacobian criterion)***

Let F be a field with $\mathrm{char} F = 0$ *and $\{f_1, \ldots, f_n\} \subset F[X_1, \ldots, X_n]$.*

a) If $\mathrm{rank}\, \mathrm{J}(f_1, \ldots, f_n) = r$, *then there exist r algebraically independent elements in the algebra $F\langle f_1, \ldots, f_n \rangle$. Moreover these elements can be chosen from the set $\{f_1, \ldots, f_n\}$.*
b) $\{f_1, \ldots, f_n\}$ are algebraically independent if and only if $\det \mathrm{J}(f_1, \ldots, f_n) \neq 0$.

Proof: a) If $\mathrm{rank}\, \mathrm{J}(f_1, \ldots, f_n) \neq 0$, we can achieve by changing the order of $f_1, \ldots, f_n$ and $X_1, \ldots, X_n$ that $\det\left(\mathrm{J}(f_1, \ldots, f_n)_{i,j}\right)_{1 \leq i,j \leq r} \neq 0$. Let assume that there exists a $0 \neq h \in F[X_1, \ldots, X_r]$ of minimal degree such that $h(f_1, \ldots, f_r) = 0$. Differentiation $\frac{\partial}{\partial X_j}$ using the chain rule yields the following system of linear equations over $\mathrm{Quot}(F[X_1, \ldots, X_r])$:

$$\left(\frac{\partial h}{\partial X_i}(f_1, \ldots, f_r) \right)_{i=1,\ldots,r} \cdot \left(\mathrm{J}(f_1, \ldots, f_n)_{i,j} \right)_{1 \leq i,j \leq r} = 0$$

Since $\deg h > 0$, there exists an $i \in \{1,\dots,r\}$ such that $\frac{\partial h}{\partial X_i} \neq 0$. As $\deg \frac{\partial h}{\partial X_i} < \deg h$ we have $\frac{\partial h}{\partial X_i}(f_1,\dots,f_r) \neq 0$. Hence the above system of linear equation has a non-trivial solution and $\det (\mathrm{J}(f_1,\dots,f_n)_{i,j})_{1\le i,j\le r} = 0$ providing a contradiction.

b) "$\Leftarrow$" follows directly by a). Let $\{f_1,\dots,f_n\}$ be algebraically independent. As the transcendence degree $\operatorname{trdeg}(F[X_1,\dots,X_n]) = n$, see [Lan, §8.1], the sets $\{X_k, f_1,\dots,f_n\}$ are algebraically dependent for $k \in \{1,\dots,n\}$. Let $0 \neq h_k \in F[X_0,X_1,\dots,X_n]$ of minimal degree such that $h_k(X_k, f_1,\dots,f_n) = 0$. Differentiation $\frac{\partial}{\partial X_j}$ yields

$$\left(\frac{\partial h_k}{\partial X_i}(X_k, f_1,\dots,f_n)\right)_{k,i=1,\dots,n} \cdot \mathrm{J}(f_1,\dots,f_n) = \operatorname{diag}\left(-\frac{\partial h_k}{\partial X_0}(X_k, f_1,\dots f_n)\right)_{k=1,\dots,n}$$

As $\{f_1,\dots,f_n\}$ are algebraically independent, the indeterminate X_0 occurs in h_k, hence we have $\deg_{X_0} h_k > 0$. Since $\operatorname{char} F = 0$, we get $\frac{\partial h_k}{\partial X_0} \neq 0$, and in $F[X_0,X_1,\dots,X_n]$ we get

$$\deg \frac{\partial h_k}{\partial X_0} < \deg h_k.$$

Therefore we have $\frac{\partial h_k}{\partial X_0}(X_k, f_1,\dots,X_n) \neq 0$. Hence we conclude

$$\det\operatorname{diag}\left(-\frac{\partial h_k}{\partial X_0}(X_k, f_1,\dots f_n)\right)_{k=1,\dots,n} \neq 0 \quad \text{and thus } \det\mathrm{J}(f_1,\dots,f_n) \neq 0. \qquad \square$$

Example 5.2.14 *Let F_i be as in Theorem 4.3.3, then we have the isomorphism of algebras*

$$\mathbb{C}\langle Y_1Y_2, Y_1Z_2, Y_2Z_1, Z_1Z_2\rangle \cong \mathbb{C}\langle F_1,F_2,F_3,F_4\rangle.$$

The Jacobian matrix of $\{Y_1Z_1, Y_1Z_2+Y_2Z_1, Y_1Z_2-Y_2Z_1, Z_2Z_2\}$ is given as

$$\begin{pmatrix} Z_1 & 0 & Y_1 & 0 \\ Z_2 & Z_1 & Y_2 & Y_1 \\ Z_2 & -Z_1 & -Y_2 & Y_1 \\ 0 & Z_2 & 0 & Y_2 \end{pmatrix} \in \operatorname{Mat}_4(\mathbb{C}[Y_1,Y_2,Z_1,Z_2])$$

which has vanishing determinant, but all (3×3)-minors are non-vanshing. Hence all subalgebras of $\mathbb{C}\langle Y_1Y_2, Y_1Z_2, Y_2Z_1, Z_1Z_2\rangle$ generated by three of $\{Y_1Z_1, Y_1Z_2+Y_2Z_1, Y_1Z_2-Y_2Z_1, Y_2Z_2\}$ are by the Jacobian criterion polynomial. Thus F_1, F_2+F_3 and F_4 are algebraically independent. From Equation (4.8) and the isomorphism of Theorem 4.3.3 we get

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)_{\mathrm{sym}} \cong \mathbb{C}[X_1, X_2+X_3, X_4].$$

Since $\{F_1, F_2+F_3, F_2-F_3, F_4\}$ generate $\mathbb{C}\langle F_1,F_2,F_3,F_4\rangle$, we have

$$\begin{aligned} \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2) &= \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)_{\mathrm{sym}} \oplus \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)_{\mathrm{skew}} && (5.2)\\ &= \mathbb{C}[F_1, F_2+F_3, F_4] \oplus (F_2-F_3)\mathbb{C}[F_1, F_2+F_3, F_4]. && (5.3) \end{aligned}$$

and $\dim \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2) = 3$ *by Theorem 5.1.8.*

5.3 Cohen-Macaulay Rings and Modules

Definition 5.3.1 *Under the Hypothesis 5.1.4 we define*

a) A homogeneous element $f \in \mathcal{R}^+$ *is called* regular (non-zero divisor) *for M if* $\ker_M(\cdot f) = \{0\}$, *where* $\cdot f : M \to M, m \mapsto mf$. *A sequence* $f_1, \ldots, f_r \subset \mathcal{R}^+$ *of homogeneous elements is called* regular *for M if* f_j *is regular for* $M/\sum_{i=1}^{j-1} M f_i$ *for* $j = 1, \ldots, r$. *An element or a sequence is called regular for* $\mathcal{R}$ *if it is regular for* $M = \mathcal{R}$.

b) The depth *of M – denoted by* $\operatorname{depth} M \in \mathbb{N}_0 \cup \{\infty\}$ *– is the length of a longest regular sequence for M.*

c) A ring or a module $\mathcal{R}$ *is called* Cohen-Macaulay *if its depth is equal to its Krull dimension.*

Some properties for these concepts are listed in the following

Proposition 5.3.2 *Let* $\mathcal{R} = M$ *fulfill Hypothesis 5.1.4. Then*

a) If $f \in \mathcal{R}$ *is homogeneous, then we have* $\dim M/_{Mf} \geq \dim M - 1$. *If* $f \in \mathcal{R}$ *is regular for M, then we have* $\dim M/_{Mf} = \dim M - 1$.

b) Let M be Cohen-Macaulay. Then $f \in \mathcal{R}^+$ *homogeneous is regular for M if and only if* $\dim M/_{Mf} = \dim M - 1$.

c) Every regular sequence is algebraically independent.

d) Each regular sequence $f_1, \ldots, f_r \subset \mathcal{R}^+$, *for* $r \leq \dim \mathcal{R}$, *can be extended to a homogeneous system of parameters. In particular, a regular sequence of length* $r = \dim \mathcal{R}$ *is a homogeneous system of parameters.*

e) $\operatorname{depth} M \leq \dim M$

Proof: a) [Smi, Prop 6.7.1] b) [Smi, Thm 6.7.6] c) [Smi, Prop 6.2.1] d) [Smi, Cor 6.7.2] e) [Smi, Cor 6.2.2]
Except for c) you find these results also in [Mül, Prop. 8.2 and Prop. 8.3]. □

Benson ([Ben, Thm 4.3.5]) showed the following

Theorem 5.3.3 *Under Hypothesis 5.1.4 the following is equivalent:*

(i) M is Cohen-Macaulay.

(ii) There exist homogeneous elements $f_1, \ldots, f_n$ *(*$n = \dim M$*) generating a polynomial subring* $F[f_1, \ldots, f_n] \subset \mathcal{R}/_{\operatorname{Ann}_{\mathcal{R}} M}$ *such that M is a finitely generated free module over* $F[f_1, \ldots, f_n]$.

(iii) Whenever $f_1, \ldots, f_n \in \mathcal{R}$ *are homogeneous elements generating a polynomial subring over* $F[f_1, \ldots, f_n] \subset \mathcal{R}/_{\operatorname{Ann}_{\mathcal{R}} M}$ *over which M is finitely generated, then M is a free* $F[f_1, \ldots, f_n]$*-module.*

We already know an example for a Cohen-Macaulay ring, i. e. $\mathcal{R} = \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$ (cf. Example 5.2.14) with $\dim \mathcal{R} = 3$.

Using Theorem 5.3.3 we can generalize the decomposition in (5.3).

Example 5.3.4 *Let $\mathcal{R}$ and M fulfill Hypothesis 5.1.4 with* $\dim\mathcal{R} = n$.

a) *If $\mathcal{R}$ is a Cohen-Macaulay ring, then there exist a homogeneous system of parameters $\{f_1,\ldots,f_n\} \subset \mathcal{R}$ and other homogeneous elements $g_1,\ldots,g_r \in \mathcal{R}$, such that $\mathcal{R} = \bigoplus_{j=1}^r g_j F[f_1,\ldots,f_n]$. We call the polynomials f_i* primary generators *and the g_i* secondary generators. *If $\mathcal{R}$ is an invariant ring, we call the generators also (primary or secondary) invariants.*
b) *If $M = \mathcal{R}^\lambda$ is a Cohen-Macaulay module for $\mathcal{R}^G$, then there exist a homogeneous system of parameters $\{f_1,\ldots,f_n\} \subset \mathcal{R}^G$ and other homogeneous elements $g_1,\ldots,g_r \in \mathcal{R}^\lambda$ such that $\mathcal{R}^\lambda = \bigoplus_{j=1}^r g_j F[f_1,\ldots,f_n]$. In that case, we call the polynomials f_i* relatively primary invariants *and the polynomials g_i* secondary relative-invariants.

In both cases the associated decomposition is called Hironaka decomposition.

Remark 5.3.5 *In view of Theorem 5.3.3 we know that every homogeneous system of parameters is a set of primary invariants.*

Corollary 5.3.6 *Let $\mathcal{R}$ and M satisfy Hypothesis 5.1.4.*

a) *Let $\mathcal{R}$ be a Cohen-Macaulay ring with given Hironaka decomposition*

$$\bigoplus_{j=1}^r g_j F[f_1,\ldots,f_n] \text{ with } \deg f_i = k_i \text{ and } \deg g_j = l_j \text{ for } 1 \le i \le n \text{ and } 1 \le j \le r.$$

Then

$$\mathrm{H}(\mathcal{R},t) = \frac{\sum_{j=1}^r t^{l_j}}{\prod_{i=1}^n (1-t^{k_i})}.$$

In particular, we have

$$\deg\mathcal{R} = \frac{r}{\prod_{i=1}^n k_i} \quad \text{with} \quad r = \mathrm{rank}_{F[f_1,\ldots,f_n]}\mathcal{R}.$$

b) *Let M be a Cohen-Macaulay $\mathcal{R}$-module with Hironaka decomposition as in a), then*

$$\mathrm{H}(M,t) = \frac{\sum_{j=1}^r t^{l_j}}{\prod_{i=1}^n (1-t^{k_i})}.$$

Proof: Using Example (5.2.9) the Hilbert series for a polynomial ring $F[f_1,\ldots,f_n]$ is given by

$$\mathrm{H}(F[f_1,\ldots,f_n],t) = \frac{1}{\prod_{i=1}^n (1-t^{k_i})}.$$

The assertion follows directly from the Hironaka decomposition and comparison of the dimension as F-vector spaces. The degree is easily calculated if we use $\frac{1-t^k}{1-t} = \sum_{j=0}^{k-1} t^j$. □

We now prove two generalizations for the Theorem of Hochster-Eagon (see [Ben, Thm 4.3.6]).

Proposition 5.3.7 *Let $\mathcal{R} = M$ satisfy Hypothesis 5.1.4. If $\mathcal{R}$ is a Cohen-Macaulay ring, so is $\mathcal{R}^G$.*

Proof: Let $n = \dim \mathcal{R} = \dim \mathcal{R}^G$ (equality holds because of Proposition 5.2.8). Because of the Noether normalization 5.1.8 ($M = \mathcal{R}^G$), there exists a set $\{f_1, \dots, f_n\} \subset \mathcal{R}^G$ with

$$F[f_1, \dots, f_n] \underset{\text{finite}}{\subset} \mathcal{R}^G \underset{\text{finite}}{\subset} \mathcal{R}.$$

Since $\mathcal{R}$ is a Cohen-Macaulay ring, it follows from Theorem 5.3.3 that $\mathcal{R}$ is a finitely generated graded free $F[f_1, \dots, f_n]$-module. $\mathcal{R}^G$ is a graded $F[f_1, \dots, f_n]$-module because $F[f_1, \dots, f_n] \subset \mathcal{R}^G$. From Proposition 5.2.8 we get that $\mathcal{R}^G$ is a finitely generated graded $F[f_1, \dots, f_n]$-module. Since it is the image of a projection it is also projective. Since every graded $F[f_1, \dots, f_n]$-module is projective if and only if it is free (by Lemma [Ben, Lem 4.1.1]), we therefore know that $\mathcal{R}^G$ is a free finitely generated graded $F[f_1, \dots, f_n]$-module. □

Proposition 5.3.8 *Let $\mathcal{R}$ be a Cohen-Macaulay ring without zero-divisors and $\mathcal{R}^\lambda \neq \{0\}$, then $\mathcal{R}^\lambda$ is a Cohen-Macaulay $\mathcal{R}^G$-module. Moreover we have $\dim \mathcal{R} = \dim \mathcal{R}^G = \dim \mathcal{R}^\lambda$ and any system of primary invariants is also a system of relatively primary invariants.*

Proof: Without loss of generality let $\lambda \neq 1$, hence $\mathcal{R}^\lambda \cap \mathcal{R}^G = \{0\}$. Since $\mathcal{R}$ has no zero-divisors (and $\mathcal{R}^\lambda \neq \{0\}$), we know that $\mathrm{Ann}_{\mathcal{R}^G}(\mathcal{R}^\lambda) = \{0\}$. As in the proof of Proposition 5.3.7 we get a homogeneous system of parameters $\{f_1, \dots, f_n\} \subset \mathcal{R}^G$, such that $\mathcal{R}$ is a finitely generated $F[f_1, \dots, f_n]$-module. Using Proposition 5.2.8 we get that $\mathcal{R}^\lambda$ is a finitely generated projective $F[f_1, \dots, f_n]$-module. Hence it is free by [Ben, Lem 4.1.1] (note that $\mathcal{R}$ is a free $F[f_1, \dots, f_n]$-module). Therefore $\{f_1, \dots, f_n\}$ is also a homogeneous system of parameters for $\mathcal{R}^\lambda$. Since $\mathcal{R}^\lambda \neq \{0\}$, Proposition 5.2.8 yields $\dim \mathcal{R}^\lambda = \dim \mathcal{R}^G = \dim \mathcal{R} = n$ and therefore $\mathcal{R}^\lambda$ is a Cohen-Macaulay module. □

Remark 5.3.9 *Is the condition "$\mathcal{R}$ has no zero-divisors" necessary, or is it only of technical nature? In the general setting this is an interesting question, but in the case that $\mathcal{R}$ is a graded algebra of modular forms this condition is always fulfilled.*

We will call relatively primary invariants again primary invariants. For these structures we will look at the relation between the Hilbert series and the associated Hironaka decomposition:

If $\lambda = 1$, then $g_1 = 1$ is always a secondary invariant. Suppose that $\lambda \neq 1$, then $g = 1$ is never a secondary relative-invariant.

Let $\mathcal{S}[V]^G$ be a Cohen-Macaulay ring and let the Hilbert series – which can be calculated using Molien's formula 5.2.11 which is for example available in [GAP] – be given by $\frac{\sum_{j=1}^{r} t^{l_j}}{\prod_{i=1}^{n}(1-t^{k_i})}$. Suppose we found a homogeneous system of parameters $\{f_1, \ldots, f_n\}$ with degrees $\deg f_i = k_i$, then the numerator polynomial determines, in which homogeneous components we have to look for secondary relative-invariants, i. e. let $p(t) = \sum_i a_i t^i \in \mathbb{Z}[t]$ be the numerator polynomial, then we need a_i linear independent secondary relative-invariants of degree i. If the denominator does not match with the degrees in the homogeneous system of parameters, we have to extend the fraction with a suitable polynom, e. g. see [DKe, p. 84].

Example 5.3.10 *Let* $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

a) Since $\mathcal{R} = \mathcal{A}(\Gamma) \cong \mathbb{C}[g_4, g_6]$*, the ring* $\mathcal{R}$ *is Cohen-Macaulay with Hilbert series*

$$\mathrm{H}(\mathcal{R}, t) = \frac{1}{(1-t^4)(1-t^6)}.$$

b) In Theorem 4.2.7 we showed that $\mathcal{A}(\Gamma[3]) = \mathbb{C}\langle E_{3,3}^*, E_{1,3}^*\rangle$*. Since* $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma[3])^{\Gamma/\Gamma[3]}$ *is an integral extension we have*

$$2 = \dim \mathcal{A}(\Gamma) = \dim \mathcal{A}(\Gamma[3])$$

by Lemma 5.1.5. Therefore $E_{3,3}^*$ *and* $E_{1,3}^*$ *are algebraically independent.*

5.4 A Twisted Exact Sequence

We want to consider the more complicated example

$$\big(\mathbb{C}[X_1, \ldots, X_4]/\langle X_1X_4 - X_2X_3\rangle\big)^G \quad \text{with the group} \quad G = \langle M_J \otimes M_J, M_{T^{-\mathrm{tr}}} \otimes M_T\rangle \cong A_4.$$

Let $\mathcal{R} := \mathbb{C}[X_1, \ldots, X_4]/\langle X_1X_4 - X_2X_3\rangle$. We have already shown in Equation (5.3) that

$$\mathcal{R} := \mathbb{C}[F_1, F_2 + F_3, F_4] \oplus (F_2 - F_3)\mathbb{C}[F_1, F_2 + F_3, F_4]$$

and that $\dim \mathcal{R} = 3$. This is the Hironaka decomposition and therefore $\mathcal{R}$ is Cohen-Macaulay with Hilbert series

$$\mathrm{H}(\mathcal{R}, t) = \frac{(1+t)}{(1-t)^3}. \tag{5.4}$$

Using Proposition 5.2.8 and Proposition 5.3.7 we get that $\dim \mathcal{R}^G = 3$ and that $\mathcal{R}^G$ is Cohen-Macaulay. In order to be able to calculate the invariants for $\mathcal{R}$, we need a λ-twisted exact sequence, where λ is the character of the group A_4 which was defined in Section 4.1.

In Section 4.3 and in Example 5.2.14 we showed that

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2) \cong \mathbb{C}\langle Y_1Z_1, Y_1Z_2, Y_2Z_1, Y_2Z_2\rangle \subset \mathbb{C}[Y_1, Y_2, Z_1, Z_2].$$

The isomorphism of Theorem 4.3.3 translates into the surjective algebra homomorphism

$$\pi : \mathbb{C}[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}] \to \mathbb{C}\langle Y_1Z_1, Y_1Z_2, Y_2Z_1, Y_2Z_2\rangle \tag{5.5}$$

given by $\pi : X_{i,j} \mapsto Y_iZ_j$. Because of Theorem 4.3.3 we know that $\ker\pi = \langle F_\pi\rangle$ with $F_\pi = X_{1,1}X_{2,2} - X_{1,2}X_{2,1}$. Note that the ideal $\langle F_\pi\rangle$ is related to the Segre-variety. This relation is explained in the Appendix 8.7. There you find also a different (more general) proof for this result using algebraic geometry. In this section you also find a remark about determinental rings. We easily observe that π is a $\mathbb{C}G$-module homomorphism if we define on both sides of Equation (5.5) the G-action by the isomorphism from Example 5.2.14 and the representation of G in Chapter 4.4.

Let $V \cong \mathbb{C}^2$ and $\{v_1, v_2\}$ be a basis of V, then we see that $\mathbb{C}[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}] \cong \mathcal{S}[V \otimes V]$ and $\operatorname{im}\pi \cong \bigoplus_{d\geq 0}\mathcal{S}[V]_d \otimes \mathcal{S}[V]_d$. Moreover, F_π translates into $v_\pi = (v_1 \otimes v_1)(v_2 \otimes v_2) - (v_1 \otimes v_2)(v_2 \otimes v_1)$. If we define $\sigma : \mathcal{S}[V \otimes V] \to \mathcal{S}[V \otimes V]$ by $\sigma(1) = v_\pi$, then σ is obviously a linear transformation.

Given $h = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}(V)$, we get

$$\begin{aligned} v_\pi^{(h,1)} &= (v_1^h \otimes v_1)(v_2^h \otimes v_2) - (v_1^h \otimes v_2)(v_2^h \otimes v_1) \\ &= ((v_1a + v_2b) \otimes v_1)((v_1c + v_2d) \otimes v_2) - ((v_1a + v_2b) \otimes v_2)((v_1c + v_2d) \otimes v_1) \\ &= \det(h) \cdot v_\pi \end{aligned} \tag{5.6}$$

Analogously, we have $v_\pi^{(1,h)} = \det(h) \cdot v_\pi$. Since G operates diagonal on $\mathcal{S}[V \otimes V]$ and G operates on $V \otimes V$ with character $\vartheta \otimes \vartheta$, we get using Equation (4.10)

$$v_\pi^g = (\det\vartheta(g))^2 v_\pi = \lambda^{-2}(g)v_\pi = \lambda(g)v_\pi,$$

where λ is the character from the table on page 35. Hence σ is a λ-twisted $\mathbb{C}G$-homomorphism and we get the λ-twisted exact sequence

$$0 \to \mathcal{S}[V \otimes V] \xrightarrow{\sigma} \mathcal{S}[V \otimes V] \xrightarrow{\pi} \bigoplus_{d\geq 0}(\mathcal{S}[V]_d \otimes \mathcal{S}[V]_d) \to 0. \tag{5.7}$$

Note that this is a little abuse of notation since π is defined in a different way. Because of Lemma 5.2.3 the twisted sequence

$$0 \to \mathcal{S}[V \otimes V]^{\lambda^{-1}} \xrightarrow{\sigma|_{\mathcal{S}[V\otimes V]^{\lambda^{-1}}}} \mathcal{S}[V \otimes V]^G \xrightarrow{\pi|_{\mathcal{S}[V\otimes V]^G}} (\operatorname{im}\pi)^G \to 0$$

is also exact. R^χ is a projection for $\chi = 1, \lambda^{-1}$ so that $(\operatorname{im}\pi)^G = \operatorname{im}(\pi|_{\mathcal{S}[V\otimes V]^G})$ and $(\operatorname{im}\sigma)^G = \operatorname{im}(\sigma|_{\mathcal{S}[V\otimes V]^{\lambda^{-1}}})$. Hence

$$\begin{aligned} \big(\mathcal{S}[V \otimes V]/\operatorname{im}\sigma\big)^G &\cong (\operatorname{im}\pi)^G \cong \operatorname{im}\pi|_{\mathcal{S}[V\otimes V]^G} \\ &\cong \mathcal{S}[V \otimes V]^G/\operatorname{im}\sigma|_{\mathcal{S}[V\otimes V]^{\lambda^{-1}}} \cong \mathcal{S}[V \otimes V]^G/(\operatorname{im}\sigma)^G. \end{aligned} \tag{5.8}$$

5.5 Invariants with Respect to G

Let $\varphi_d : \mathcal{S}[V\otimes V] \to \mathcal{S}[V\otimes V]_d$ be the restriction onto the d-th homogeneous component. Since φ_d commutes with π, σ and the action of G, we get using Equation (5.8)

$$\left(\mathcal{S}[V\otimes V]/\ker\pi\right)_d^G \cong \mathcal{S}[V\otimes V]_d^G/\left(v_\pi\cdot\mathcal{S}[V\otimes V]_{d-2}^{\lambda^{-1}}\right). \tag{5.9}$$

Using Molien's formula 5.2.11 which is for example implemented in [GAP] (see Section 8.9), we can calculate the following Hilbert series

$$\begin{aligned}
\mathrm{H}(\mathcal{S}[V\otimes V]^G,t) &= \frac{1-t^2+t^3+2t^4}{(1-t^3)^2(1-t^2)^2} &= \frac{1+t^3+t^4+t^5+2t^6}{(1-t^4)(1-t^3)^2(1-t^2)},\\
\mathrm{H}(\mathcal{S}[V\otimes V]^\lambda,t) &= \frac{2t^2+t^3-t^4+t^6}{(1-t^3)^2(1-t^2)^2} &= \frac{2t^2+t^3+t^4+t^5+t^8}{(1-t^4)(1-t^3)^2(1-t^2)},\\
\mathrm{H}(\mathcal{S}[V\otimes V]^{\lambda'},t) &= \frac{t+t^2-t^3+t^4+t^5}{(1-t^3)^2(1-t^2)^2} &= \frac{t+t^2+2t^4+t^6+t^7}{(1-t^4)(1-t^3)^2(1-t^2)}.
\end{aligned}$$

Comparing the dimension as F-vector spaces we deduce from Equation (5.9) that

$$\mathrm{H}_{\mathrm{tot}} = \mathrm{H}((\mathcal{S}[V\otimes V]/\ker\pi)^G,t) = \mathrm{H}(\mathcal{S}[V\otimes V]^G,t) - t^2\cdot\mathrm{H}(\mathcal{S}[V\otimes V]^{\lambda'},t) \tag{5.10}$$

$$= \frac{1+t^4+t^5}{(1-t^3)^2(1-t^2)} = \frac{1+t^3+t^5+t^6}{(1-t^2)(1-t^3)(1-t^4)}. \tag{5.11}$$

For $\mathcal{S}[W]^H$ (W an FG-module with $\mathrm{char} F = 0$) there exist algorithms to determine primary and secondary invariants. The algorithms are well-known (see e. g. [DKe, Alg 3.3.4 and Alg 3.5.2]) and are implemented in [MAGMA]. Using the instructions from Section 8.10 we can determine primary and secondary invariants for $\mathcal{S}[V\otimes V]^G$ denoted by p_i and s_j, respectively. The explicit polynomials are written out in Section 8.8 and have the following degrees:

invariant	p_1	p_2	p_3	p_4	s_1	s_2	s_3	s_4	s_5	s_6
degree	2	3	3	4	0	3	4	5	6	6

The map π is easily implemented in the computer and therefore we can calculate relations of subsets of $\{\pi(p_i),\pi(s_j); i=1,\dots,4,\ j=1,\dots,6\}$ by determining the relation ideal with a Gröbner basis algorithm ([DKe, §1.2 and §3.6]) using the embedding into the polynmomial ring $\mathbb{C}[Y_1,Y_2,Z_1,Z_2]$. This algorithm is implemented in [MAGMA] and we refer the interested reader to Section 8.9.

For the above example we get

$$\pi(s_2)=0,\quad \pi(s_5)=0 \quad\text{and}\quad (\pi(p_1))^2 = \pi(p_4)+\tfrac{3}{4}\pi(s_3).$$

In our case we know that $\operatorname{im}\pi$ and $\mathcal{S}[V \otimes V]$ are Cohen-Macaulay. Therefore both rings have primary invariants. Note that every homogeneous system of parameters is by Remark 5.3.5 a set of primary invariants.

Because of the isomorphism (5.8), we know that $\{f_1, f_2, f_3\}$ is a homogeneous system of parameters for $\operatorname{im}\pi$ if and only if $\{\pi^{-1}(f_1), \pi^{-1}(f_2), \pi^{-1}(f_3)\}$ is a homogeneous system of parameters for $\mathcal{S}[V \otimes V]/\ker\pi$. Since $\mathcal{S}[V \otimes V]$ is a domain, v_π is regular for $\mathcal{S}[V \otimes V]$. By Proposition 5.3.2 part d) it can be extended to a homogeneous system of parameters for $\mathcal{S}[V \otimes V]$. Therefore we have shown that

$\{f_1, f_2, f_3\}$ is a set of primary invariants for $\operatorname{im}\pi$ if and only if
$\{v_\pi, \pi^{-1}(f_1), \pi^{-1}(f_2), \pi^{-1}(f_3)\}$ is a set of primary invariants for $\mathcal{S}[V \otimes V]$.

In view of Remark 5.1.10 this fact is equivalent to the fact that

$$\mathrm{H}(\mathcal{S}[V \otimes V]/\mathcal{I}, t) \text{ is a polynomial for } \mathcal{I} = \langle v_\pi, \pi^{-1}(f_1), \pi^{-1}(f_2), \pi^{-1}(f_3)\rangle.$$

Note that for every homogeneous ideal $\mathcal{I}$ in $\mathcal{S} = \mathcal{S}[V \otimes V]$ the quotient $\mathcal{S}/\mathcal{I}$ is a graded $\mathcal{S}$-module so that the Krull-dimension and the Hilbert-series are well-defined. In this case, the Hilbert series of $\mathcal{S}/\mathcal{I}$ can be calculated with [MAGMA].

Let $\mathcal{S} = \mathcal{S}[V \otimes V]$ and $\mathcal{R} = \mathcal{S}/\ker\pi$. Using the method mentioned above we get that $\mathcal{S}/\langle v_\pi, p_1, p_2, p_3\rangle$ has dimension 0, since the associated Hilbert series is a polynomial. Thus $\mathcal{R}/\langle \pi(p_1), \pi(p_2), \pi(p_3)\rangle$ has dimension 0 and therefore the set $\{\pi(p_1), \pi(p_2), \pi(p_3\}$ is a homogeneous system of parameters for $\mathcal{R}^G$. In the same way we recognize that

$$\{\pi(p_1), \pi(p_3), \pi(p_4)\} \quad \text{and} \quad \{\pi(p_2), \pi(p_3), \pi(p_4)\}$$

are also homogeneous systems of parameters. Only the set $\{\pi(p_1), \pi(p_2), \pi(p_4)\}$ is not a homogeneous system of parameters. If we fix $\{\pi(p_1), \pi(p_2), \pi(p_3)\}$ as a homogeneous system of parameters, we have to find – in view of the Hilbert series – secondary invariants of degree 4 and 5. Because of the above relations, we can choose $\pi(s_3)$ of degree 4. Since the homogeneous component of degree 5 is generated by $\{\pi(p_1)\pi(p_2), \pi(p_1)\pi(p_3), \pi(s_4)\}$ we get the following Hironaka decomposition

$$\begin{aligned}\mathcal{R}^G = \mathbb{C}[\pi(p_1), \pi(p_2), \pi(p_3)] \oplus \pi(s_3)\mathbb{C}[\pi(p_1), \pi(p_2), \pi(p_3)]& \\ \oplus\, \pi(s_4)\mathbb{C}[\pi(p_1), \pi(p_2), \pi(p_3)]&,\end{aligned}$$

with

$$
\begin{aligned}
\pi(p_1) = {} & Y_1^2Z_1^2 - \tfrac{1}{2}Y_1^2Z_1Z_2 - \tfrac{1}{2}Y_1^2Z_2^2 - \tfrac{1}{2}Y_1Y_2Z_1^2 + Y_1Y_2Z_1Z_2 - \tfrac{1}{2}Y_1Y_2Z_2^2 - \tfrac{1}{2}Y_2^2Z_1^2 \\
& - \tfrac{1}{2}Y_2^2Z_1Z_2 - \tfrac{1}{8}Y_2^2Z_2^2, \\
\pi(p_2) = {} & Y_1^3Z_1^3 + \tfrac{3}{2}Y_1^3Z_1^2Z_2 + \tfrac{3}{4}Y_1^3Z_1Z_2^2 + \tfrac{1}{8}Y_1^3Z_2^3 + \tfrac{3}{2}Y_1^2Y_2Z_1^3 - \tfrac{9}{8}Y_1^2Y_2Z_1Z_2^2 - \tfrac{3}{8}Y_1^2Y_2Z_2^3 \\
& + \tfrac{3}{4}Y_1Y_2^2Z_1^3 - \tfrac{9}{8}Y_1Y_2^2Z_1^2Z_2 + \tfrac{3}{8}Y_1Y_2^2Z_2^3 + \tfrac{1}{8}Y_2^3Z_1^3 - \tfrac{3}{8}Y_2^3Z_1^2Z_2 + \tfrac{3}{8}Y_2^3Z_1Z_2^2 - \tfrac{1}{8}Y_2^3Z_2^3, \\
\pi(p_3) = {} & Y_1^3Z_1^2Z_2 - \tfrac{1}{2}Y_1^3Z_1Z_2^2 + \tfrac{1}{4}Y_1^3Z_2^3 - Y_1^2Y_2Z_1^3 + \tfrac{3}{4}Y_1^2Y_2Z_1Z_2^2 + \tfrac{1}{4}Y_1^2Y_2Z_2^3 + \tfrac{1}{2}Y_1Y_2^2Z_1^3 \\
& - \tfrac{3}{4}Y_1Y_2^2Z_1^2Z_2 + \tfrac{1}{4}Y_1Y_2^2Z_2^3 - \tfrac{1}{4}Y_2^3Z_1^3 - \tfrac{1}{4}Y_2^3Z_1^2Z_2 - \tfrac{1}{4}Y_2^3Z_1Z_2^2, \\
\pi(s_3) = {} & 4Y_1^3Y_2Z_1^3Z_2 + \tfrac{1}{2}Y_1^3Y_2Z_2^4 + \tfrac{1}{2}Y_2^4Z_1^3Z_2 + \tfrac{1}{16}Y_2^4Z_2^4, \\
\pi(s_4) = {} & Y_1^4Y_2Z_1^5 - \tfrac{5}{2}Y_1^4Y_2Z_1^3Z_2^2 + \tfrac{5}{4}Y_1^4Y_2Z_1^2Z_2^3 + \tfrac{1}{4}Y_1^4Y_2Z_2^5 - Y_1^3Y_2^2Z_1^5 - \tfrac{5}{4}Y_1^3Y_2^2Z_1^3Z_2^2 \\
& - \tfrac{5}{4}Y_1^3Y_2^2Z_1^2Z_2^3 + \tfrac{1}{8}Y_1^3Y_2^2Z_2^5 + \tfrac{1}{8}Y_1Y_2^4Z_1^5 - \tfrac{5}{16}Y_1Y_2^4Z_1^3Z_2^2 + \tfrac{5}{32}Y_1Y_2^4Z_1^2Z_2^3 \\
& + \tfrac{1}{32}Y_1Y_2^4Z_2^5 - \tfrac{1}{8}Y_2^5Z_1^5 - \tfrac{5}{32}Y_2^5Z_1^3Z_2^2 - \tfrac{5}{32}Y_2^5Z_1^2Z_2^3 + \tfrac{1}{64}Y_2^5Z_2^5.
\end{aligned}
$$

Using the isomorphisms 4.3.3 and Example 5.2.14 we can express these invariants in a different way:

$$
\begin{aligned}
\pi(p_1) \mathrel{\widehat{=}} {} & F_1^2 - \tfrac{1}{2}F_1(F_2+F_3) + 2F_1F_4 - \tfrac{1}{2}(F_2+F_3)^2 - \tfrac{1}{2}(F_2+F_3)F_4 - \tfrac{1}{8}F_4^2 \\
\pi(p_2) \mathrel{\widehat{=}} {} & F_1^3 + \tfrac{3}{2}F_1^2(F_2+F_3) - \tfrac{3}{2}F_1^2F_4 + \tfrac{3}{4}F_1(F_2+F_3)^2 - \tfrac{3}{2}F_1(F_2+F_3)F_4 \\
& + \tfrac{3}{4}F_1F_4^2 + \tfrac{1}{8}(F_2+F_3)^3 - \tfrac{3}{8}(F_2+F_3)^2F_4 + \tfrac{3}{8}(F_2+F_3)F_4^2 - \tfrac{1}{8}F_4^3 \\
\pi(p_3) \mathrel{\widehat{=}} {} & (F_2-F_3)\left(F_1^2 - \tfrac{1}{2}F_1(F_2+F_3) + \tfrac{1}{2}F_1F_4 + \tfrac{1}{4}(F_2+F_3)^2 + \tfrac{1}{4}(F_2+F_3)F_4 + \tfrac{1}{4}F_4^2\right) \\
\pi(s_3) \mathrel{\widehat{=}} {} & 4F_1^3F_4 - \tfrac{3}{2}F_1(F_2+F_3)F_4^2 + \tfrac{1}{2}(F_2+F_3)^3F_4 + \tfrac{1}{16}F_4^4 \\
\pi(s_4) \mathrel{\widehat{=}} {} & \tfrac{1}{2}F_1^4(F_2+F_3) + F_1^4F_4 - \tfrac{1}{2}F_1^3(F_2+F_3)^2 - \tfrac{5}{4}F_1^3(F_2+F_3)F_4 - \tfrac{19}{8}F_1^3F_4^2 \\
& + \tfrac{3}{8}F_1^2(F_2+F_3)^2F_4 - \tfrac{5}{4}F_1^3F_4(F_2+F_3) + \tfrac{9}{16}F_1^2F_4^3 + \tfrac{1}{16}F_1(F_2+F_3)^4 \\
& + \tfrac{5}{16}F_1(F_2+F_3)^3F_4 - \tfrac{21}{32}F_1(F_2+F_3)^2F_4^2 + \tfrac{1}{8}F_1(F_2+F_3)F_4^3 + \tfrac{5}{32}F_1F_4^4 \\
& - \tfrac{1}{16}(F_2+F_3)^5 + \tfrac{1}{8}(F_2+F_3)^4F_4 - \tfrac{1}{64}(F_2+F_3)^3F_4^2 - \tfrac{5}{64}(F_2+F_3)^2F_4^3 \\
& + \tfrac{1}{64}(F_2+F_3)F_4^4 + \tfrac{1}{64}F_4^5 \\
& + (F_2-F_3)\left(-\tfrac{1}{2}F_1^4 + \tfrac{1}{2}F_1^3(F_2+F_3) - \tfrac{5}{4}F_1^3F_4 + \tfrac{3}{4}F_1^2(F_2+F_3)F_4 - \tfrac{9}{16}F_1^2F_4^2\right. \\
& - \tfrac{1}{16}F_1(F_2+F_3)^3 - \tfrac{3}{16}F_1(F_2+F_3)^2F_4 - \tfrac{3}{32}F_1(F_2+F_3)F_4^2 - \tfrac{7}{32}F_1F_4^3 \\
& \left. + \tfrac{1}{16}(F_2+F_3)^4 + \tfrac{1}{8}(F_2+F_3)^3F_4 + \tfrac{9}{64}(F_2+F_3)^2F_4^2 + \tfrac{5}{64}F_4^3(F_2+F_3) + \tfrac{1}{64}F_4^4\right)
\end{aligned}
$$

The exact algebraic structure of this ring will be calculated in the next section (cf. Equation (5.19)).

5.6 Invariants with Respect to $\widehat{G}$

We will now determine generators for the graded algebra of symmetric modular forms. Therefore, we need the group $\widehat{G} = \langle M_J \otimes M_J, M_{T^{-\mathrm{tr}}} \otimes M_T, M_{\mathrm{Flip}} \rangle$. We already know that $(M_J \otimes M_J)\, M_{\mathrm{Flip}}$ commutes with all elements of G.

More detailed, we have

$$(M_J \otimes M_J) M_{\mathrm{Flip}} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix} \text{ and } F_1F_4 - F_2F_3 \xrightarrow{(M_J\otimes M_J)M_{\mathrm{Flip}}} F_1F_4 - F_2F_3.$$

Furthermore, $((M_J \otimes M_J) M_{\mathrm{Flip}})^2 = I$ and $V \otimes V$ is a $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \times \mathcal{C}_2$-module, since these matrices define a (non-faithful) representation of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \times \mathcal{C}_2$ in $\mathrm{GL}_4(\mathbb{C})$.

Let $\mathrm{Flip}: V \otimes V \to V \otimes V$, $v \otimes v' \mapsto v' \otimes v$. Moreover, let $H = \langle \mathrm{Flip} \rangle$, 1 the trivial character of H and 1^- the non-trivial character of H. Then the 1-component of $V \otimes V\big|_{\mathrm{Flip}}$ has the basis $\{v_i \otimes v_i, v_i \otimes v_j + v_j \otimes v_i, i \neq j\}$ and the 1^- component has the basis $\{v_i \otimes v_j - v_j \otimes v_i, i \neq j\}$, hence we get the following decomposition into irreducible $\mathbb{C}H$-modules

$$V \otimes V = \bigoplus_{i=1}^{2} \mathrm{sp}(v_i \otimes v_i) \oplus \mathrm{sp}(v_1 \otimes v_2 + v_2 \otimes v_1) \oplus \mathrm{sp}(v_1 \otimes v_2 - v_2 \otimes v_1).$$

Therefore the associated character is equal to $3 \cdot 1 + 1 \cdot 1^-$. Since the ideal $\ker \pi$ is fixed under the operation of $(M_J \otimes M_J) M_{\mathrm{Flip}}$, we get a λ-twisted exact sequence of $\mathbb{C}\widehat{G}$-modules

$$0 \to \mathcal{S}[V \otimes V] \xrightarrow{\sigma} \mathcal{S}[V \otimes V] \xrightarrow{\pi} \bigoplus_{d \geq 0} (\mathcal{S}[V]_d \otimes \mathcal{S}[V]_d) \to 0.$$

Obviously, we have $\mathcal{R}^G_{\mathrm{sym}} = \mathcal{R}^{1_G \times 1}$ and $\mathcal{R}^G_{\mathrm{skew}} = \mathcal{R}^{1_G \times 1^-}$ with the trivial character 1_G on G. If we understand $V \otimes V$ as a $\widehat{G}$-module, we conclude with Schur's Lemma (e. g. [JL, Lem 9.1]) that $(M_J \otimes M_J) M_{\mathrm{Flip}}$ acts as a scalar on the τ- and λ'-components, respectively. Because $\mathrm{trace}(M_J \otimes M_J) M_{\mathrm{Flip}} = -2$, $\tau(1) = 3$ and $\lambda'(1) = 1$, we conclude that the character associated to the representation of $\widehat{G}$ is equal to $\tau \otimes 1^- + \lambda' \otimes 1$. The action of $\widehat{G}$ on v_π has the character $\lambda \otimes 1$ with inverse $\lambda' \otimes 1$.

Once more, we can determine the Hilbert series with [GAP]

$$\mathrm{H}_{\mathrm{sym}} = \mathrm{H}((\mathcal{S}[V \otimes V]/\ker \pi)^{\widehat{G}}, t) = \mathrm{H}(\mathcal{S}[V \otimes V]^{1_G \otimes 1}, t) - t^2\, \mathrm{H}(\mathcal{S}[V \otimes V]^{\lambda' \otimes 1}, t) \tag{5.12}$$

$$= \frac{1 + t^3 + t^4 + t^5 + 2t^6}{(1-t^2)(1-t^3)(1-t^4)(1-t^6)} - \frac{1 + t + 2t^3 + t^5 + t^6}{(1-t^2)(1-t^3)(1-t^4)(1-t^6)} \tag{5.13}$$

$$= \frac{1 + t^4 + t^5}{(1-t^2)(1-t^3)(1-t^6)}. \tag{5.14}$$

Using

$$\left(\mathcal{S}[V\otimes V]/\ker\pi\right)^{1_G\otimes 1^-}=\mathcal{S}[V\otimes V]^{1_G\otimes 1^-}/\left(v_\pi\mathcal{S}[V\otimes V]^{\lambda'\otimes 1^-}\right)$$

we get

$$\begin{aligned}\mathrm{H}_{\text{skew}}&=\mathrm{H}((\mathcal{S}[V\otimes V]/\ker\pi)^{1_G\otimes 1^-})=\mathrm{H}(\mathcal{S}[V\otimes V]^{1_G\otimes 1^-},t)-t^2\,\mathrm{H}(\mathcal{S}[V\otimes V]^{\lambda'\otimes 1^-},t)\\&=\frac{t^3+t^7+t^8}{(1-t^2)(1-t^3)(1-t^6)}.\end{aligned}$$

As a trial, we compute

$$\mathrm{H}_{\text{sym}}+\mathrm{H}_{\text{skew}}=\frac{1+t^4+t^5}{(1-t^2)(1-t^3)^2}=\mathrm{H}_{\text{tot}}\,.$$

With [MAGMA] we determine the primary and secondary invariants of $\mathcal{S}[V\otimes V]^{\widehat{G}}$ which have the following degrees:

invariant	$\widehat{p}_1$	$\widehat{p}_2$	$\widehat{p}_3$	$\widehat{p}_4$	$\widehat{s}_1$	$\widehat{s}_2$	$\widehat{s}_3$	$\widehat{s}_4$	$\widehat{s}_5$	$\widehat{s}_6$
degree	2	3	4	6	0	3	4	5	6	6

Again, we take map these invariants with π and get the following relations:

$$\pi(\widehat{s}_2)=0,\quad \pi(\widehat{s}_5)=0\quad\text{and}\quad(\pi(\widehat{p}_1))^2=\pi(\widehat{p}_3)+\tfrac{3}{4}\pi(\widehat{s}_3).$$

With the same method as for G we get that

$$\{\pi(\widehat{p}_1),\pi(\widehat{p}_2),\pi(\widehat{p}_4)\},\quad\{\pi(\widehat{p}_1),\pi(\widehat{p}_3),\pi(\widehat{p}_4)\}\quad\text{and}\quad\{\pi(\widehat{p}_2),\pi(\widehat{p}_3),\pi(\widehat{p}_4)\}$$

are homogeneous systems of parameters. This time $\{\pi(\widehat{p}_1),\pi(\widehat{p}_2),\pi(\widehat{p}_3)\}$ cannot be chosen as homogeneous system of parameters. Because of the above relations, we have the following Hironaka decomposition of $\mathcal{R}^G_{\text{sym}}$:

$$\begin{aligned}\mathcal{R}^G_{\text{sym}}=\mathbb{C}[\pi(\widehat{p}_1),\pi(\widehat{p}_2),\pi(\widehat{p}_4)]\oplus\pi(\widehat{s}_3)\mathbb{C}[\pi(\widehat{p}_1),\pi(\widehat{p}_2),\pi(\widehat{p}_4)]&\\ \oplus\,\pi(\widehat{s}_4)\mathbb{C}[\pi(\widehat{p}_1),\pi(\widehat{p}_2),\pi(\widehat{p}_4)].&\end{aligned}\tag{5.15}$$

Thereby we have

$$\pi(\widehat{p}_1)=\pi(p_1),\quad\pi(\widehat{p}_2)=\pi(p_2),\quad\pi(\widehat{p}_3)=\pi(p_4),\tag{5.16}$$

$$\pi(\widehat{p}_4)=\tfrac{114}{157}\pi(p_1)^3+\tfrac{43}{157}\pi(p_2)^2+\tfrac{117}{628}\pi(p_3)^2-\tfrac{333}{314}\pi(s_3)\pi(p_1),\tag{5.17}$$

$$\pi(\widehat{s}_3)=\pi(s_3),\quad\pi(\widehat{s}_4)=2\pi(s_4)+\pi(p_1)\pi(p_3).\tag{5.18}$$

Because of the above decomposition (5.15), we know that the ideal of all relations is generated by the relations of $\pi(\widehat{s}_3)^2$, $\pi(\widehat{s}_4)^2$ and $\pi(\widehat{s}_3)\pi(\widehat{s}_4)$. Using the Fourier coefficients we get:

$$\begin{aligned}
\pi(\widehat{s}_3)^2 &= -\tfrac{16}{81}\pi(\widehat{p}_1)^4 + \tfrac{16}{81}\pi(\widehat{p}_1)\pi(\widehat{p}_2)^2 - \tfrac{8}{9}\pi(\widehat{s}_4)\pi(\widehat{p}_2) + \tfrac{8}{9}\pi(\widehat{s}_3)\pi(\widehat{p}_1)^2. \\
\pi(\widehat{s}_3)\pi(\widehat{s}_4) &= -\tfrac{8104}{1053}\pi(\widehat{p}_1)^3 - \tfrac{3200}{1053}\pi(\widehat{p}_2)^3 + \tfrac{1256}{117}\pi(\widehat{p}_4)\pi(\widehat{p}_2) + \tfrac{1514}{117}\pi(\widehat{s}_3)\pi(\widehat{p}_2)\pi(\widehat{p}2) \\
&\quad + \tfrac{4}{9}\pi(\widehat{s}_4)\pi(\widehat{p}_1)^2, \\
\pi(\widehat{s}_4)^2 &= -\tfrac{1024}{1053}\pi(\widehat{p}_1)^5 + \tfrac{628}{117}\pi(\widehat{p}_1)^2\pi(\widehat{p}_4) - \tfrac{356}{81}\pi(\widehat{p}_1)^2\pi(\widehat{p}_2)^2 + \tfrac{256}{117}\pi(\widehat{s}_3)\pi(\widehat{p}_1)^3 \\
&\quad + \tfrac{400}{117}\pi(\widehat{s}_3)\pi(\widehat{p}_2)^2 - \tfrac{157}{13}\pi(\widehat{s}_3)\pi(\widehat{p}_4) + \tfrac{1540}{117}\pi(\widehat{s}_4)\pi(\widehat{p}_2)\pi(\widehat{p}_1).
\end{aligned}$$

The above results can also be calculated using computational commutative algebra. Therefore we get the following

Theorem 5.6.1

$$\mathcal{A}(\mathcal{G})_{\mathrm{sym}} \cong \mathcal{R}^G_{\mathrm{sym}} \cong \mathbb{C}[X_1,X_2,X_3,X_4,X_5]/\mathcal{I}_{\mathrm{sym}},$$

where $\mathcal{I}_{\mathrm{sym}}$ is the ideal which is generated by the polynomials

$$\begin{gathered}
X_4^2 + \tfrac{16}{81}X_1^4 - \tfrac{16}{81}X_1X_2 + \tfrac{8}{9}X_5X_2 - \tfrac{8}{9}X_4X_1^2, \\
X_4X_5 + \tfrac{8104}{1053}X_1^3X_2 + \tfrac{3200}{1053}X_2^3 - \tfrac{1256}{117}X_3X_2 - \tfrac{1514}{117}X_4X_1X_2 - \tfrac{4}{9}X_5X_1^2 \;\; \textit{and} \\
X_5^2 + \tfrac{1024}{1053}X_1^5 - \tfrac{628}{117}X_1^2X_3 + \tfrac{356}{81}X_1^2X_2^2 - \tfrac{256}{117}X_4X_1^3 - \tfrac{400}{117}X_4X_2^2 + \tfrac{157}{13}X_4X_3 - \tfrac{1540}{117}X_5X_1X_2.
\end{gathered}$$

In the same way, we can choose $\{\pi(\widehat{p}_1),\pi(\widehat{p}_2),\pi(\widehat{p}_4)\}$ as a homogeneous system of parameters for $\mathcal{R}^G_{\mathrm{skew}}$. In view of the Hilbert series, we are looking for skew-symmetric forms of weight 3, 7 and 8. $\pi(p_3)$, $\pi(p_3)\pi(\widehat{s}_3)$ and $\pi(p_3)\pi(\widehat{s}_4)$ are skew-symmetric. Hence $\mathcal{R}^G_{\mathrm{skew}} = \pi(p_3)\mathcal{R}^G_{\mathrm{sym}}$.

$\pi(p_3)$ fulfills a relation which is already written down in (5.17) so that

$$\mathcal{A}(\mathcal{G}) \cong \mathbb{C}[X_1,\ldots,X_6]/\big\langle \mathcal{I}_{\mathrm{sym}}, X_6^2 + \tfrac{152}{39}X_1^3 + \tfrac{172}{117}X_2^2 - \tfrac{628}{117}X_3 - \tfrac{74}{13}X_4X_1 \big\rangle. \tag{5.19}$$

At the end of this section, we want to compare these results with modular forms for $\mathrm{SL}_2(\mathbb{Z})^2$. Because of Equation (3.19), we know that for every symmetric modular form $g \in [\mathrm{SL}_2(\mathbb{Z})^2, k]$ the function $g(3z)$ is a modular form of weight k for $\mathrm{Stab}_{\Gamma_5}\lambda_9^\perp$, e. g. we have

$$\begin{gathered}
g_4(3\tau_1)g_4(3\tau_2) = \tfrac{16}{81}\pi(\widehat{s}_3) \quad \text{and} \\
g_6(3\tau_1)g_6(3\tau_2) = \tfrac{8960}{28431}\pi(\widehat{p}_1)^3 + \tfrac{3584}{28431}\pi(\widehat{p}_2)^2 - \tfrac{10048}{28431}\pi(\widehat{p}_4) - \tfrac{1600}{3159}\pi(\widehat{p}_1)\pi(\widehat{s}_3).
\end{gathered}$$

The other generator of $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)$ (see Theorem 8.6.1) can also be expressed in the generators of $\mathcal{R}^G$. The explicit relations can be found in Section 8.6.

5.7 Further Interesting Questions

As far as I know there is not much known about the calculation of invariant rings of type $F[X_1,\ldots,X_n]/\mathcal{I}$ with an ideal $\mathcal{I}$. It would be very interesting to generalize the results of this chapter and develop a generalized theory.

I would like to mention two points which could be interesting:

- Our examples are very easy in some sense, e. g. the ideal is a principal one. We only had to deal with short exact sequences. If you want to generalize this result, you have to compute syzygies and consider (longer) exact sequences (compare with [DKe, §1.3]). If you could get some general result about these (longer) exact sequences and the relation to the action of G with respect to $\mathcal{I}$, you would be able to compute more complicated examples with the computer. For example, one could calculate the relations of the invariants of $\mathcal{R}^G_{\text{sym}}$ with computer algebra systems. As you have seen in Section 5.5 I have done it quite easily.
- You could get some constraints about the degrees of generators:
  For example in the non-modular case we know that in the classical theory the product of the degrees of primary invariants is divisible by |G| (see [DKe, Prop 3.3.5]). This is not always true for $\mathcal{R}^G$ and $\mathcal{R}^{\widehat{G}}$ since
  $$\begin{aligned} |G| &= 12 \nmid 18 = 2\cdot 3\cdot 3 = \deg(p_1)\deg(p_2)\deg(p_3) \quad \text{and} \\ |\widehat{G}| &= 24 \nmid 36 = 2\cdot 3\cdot 6 = \deg(\widehat{p}_1)\deg(\widehat{p}_2)\deg(\widehat{p}_4). \end{aligned}$$
  In both cases you have to multiply the right side with at least two.

We used arguments in Section 5.3 which are well-known in the classical invariant theory. If you are lucky, you could prove some results – e. g. the ones mentioned above – with the same ideas as in the classical theory. I want to illustrate this idea considering the second item as example (we refer the reader to the proof of [DKe, Thm 3.7.1]):

Let $\mathcal{R}$ be an integral domain which satisfies the Hypothesis 5.1.4 and let G be a finite group of graded algebra automorphisms. Then by Theorem 5.1.8 there exists a homogeneous system of parameters $\{f_1,\ldots,f_n\}$ with $\deg f_i = k_i$ for $\mathcal{R}^G$. Let $\mathcal{S} := F[f_1,\ldots,f_n]$. Using Proposition 5.2.7 we get

$$\deg\mathcal{R} = [\mathrm{Quot}(\mathcal{R}) : \mathrm{Quot}(\mathcal{R}^G)]\deg\mathcal{R}^G = |G|\deg\mathcal{R}^G = |G|\,[\mathrm{Quot}(R^G) : \mathrm{Quot}(\mathcal{S})]\deg\mathcal{S}$$

In Example 5.2.9 we showed that $\deg\mathcal{S} = \prod_{i=1}^n \frac{1}{k_i}$ so that we get

$$|G|[\mathrm{Quot}(\mathcal{R}^G) : \mathrm{Quot}(\mathcal{S})] = \deg\mathcal{R}\prod_{i=1}^n k_i$$

and since $[\mathrm{Quot}(\mathcal{R}^G) : \mathrm{Quot}(\mathcal{S})] \in \mathbb{N}$ we get

$$|G| \;\Big|\; \deg(\mathcal{R}) \prod_i k_i.$$

If $\mathcal{R}$ is Cohen-Macaulay, we have $\deg \mathcal{R} = \frac{\mathrm{rank}_{\mathcal{S}} \mathcal{R}}{\prod_i k_i}$ by Corollary 5.3.6.

In view of Proposition 5.3.7 $\mathcal{R}^G$ is also Cohen-Macaulay and $f_1, \dots, f_n$ is also a homogeneous system for $\mathcal{R}$ since

$$F[f_1, \dots, f_n] \underset{\text{finite}}{\subset} \mathcal{R}^G \underset{\text{finite}}{\subset} \mathcal{R}.$$

Then $\mathrm{Quot}(S) = F(f_1, \dots, f_n)$. Let $\{h_1, \dots, h_r\} \subset \mathcal{R}^G$ be a minimal set of secondary generators. As the $\mathcal{S}$-module $\mathcal{R}$ is free of rank r, the set $\{h_1, \dots, h_r\}$ is a $\mathrm{Quot}(S)$-linearly independent generating set of $\mathrm{Quot}(\mathcal{R}^G)$, hence $[\mathrm{Quot}(\mathcal{R}^G) : \mathrm{Quot}(\mathcal{S})] = \mathrm{rank}_S \mathcal{R}^G$ which leads to

$$\mathrm{rank}_{\mathcal{S}} \mathcal{R}^G = \frac{\deg \mathcal{R} \prod_{i=1}^n k_i}{|G|}.$$

Hence we proved the following

Proposition 5.7.1 *Let $\mathcal{R}$ be an integral domain which satisfies the Hypothesis 5.1.4 and let G be a finite group of graded algebra automorphisms. Then*

$$|G| \;\Big|\; \deg(\mathcal{R}) \prod_i k_i.$$

If $\mathcal{R}$ is additionally Cohen-Macaulay, we have

$$\mathrm{rank}_{\mathcal{S}} \mathcal{R}^G = \frac{\deg \mathcal{R} \prod_{i=1}^n k_i}{|G|}.$$

Remark 5.7.2 *We did not prove the generalization for [DKe, Thm 3.7.1] if $\mathcal{R}$ is not Cohen-Macaulay.*

In the example $\mathcal{R} \cong \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$ the ring is Cohen-Macaulay and we get using Corollary 5.3.6 and the Hilbert series in Equation (5.4) that $\deg \mathcal{R} = 2$ which explains the missing factor 2.

This generalized invariant theory could then be used to calculate the algebraic structure of certain rings of modular forms. It will give new possibilities to calculate generators for rings of modular forms. Before we can use this method effectively we have to determine rings of modular forms with respect to smaller groups, e. g. main congruence subgroups or embeddings from other rings of modular forms.

6 Maaß lift for $\Gamma_5^{\max}$

The Maaß lift, defined by Gritsenko ([Gr1], [Gr2]), and Gritsenko, Nikulin ([GN]), is a fundamental method to construct paramodular forms with multiplier systems. In this chapter, there is a short description of this method following [Der, §3].

6.1 Half-integral Jacobi Forms with Character

The *metaplectic group* $\mathrm{Mp}_2(\mathbb{R})$ consists of all pairs (M,α) with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\alpha : \mathbb{H} \to \mathbb{C}$, $\tau \mapsto \alpha(\tau)$ a holomorphic root of $j(M,\tau) = c\tau + d$, i. e. $\alpha(\tau)^2 = j(M,\tau)$. The multiplication of two elements is defined by

$$(M_1,\, \alpha_1(\tau)) \cdot (M_2,\, \alpha_2(\tau)) = (M_1 M_2,\, \alpha_1(M_2\tau)\alpha_2(\tau))\,.$$

Let $k \in \frac{1}{2}\mathbb{Z}$ and V be a $\mathbb{C}$-vector space. Then $\mathrm{Mp}_2(\mathbb{R})$ acts on $\{f : \mathbb{H} \to V\}$ via

$$\left(f\big|_k M\right)(\tau) = \alpha(\tau)^{-2k} f(M\tau).$$

Often, we write $(c\tau+d)^{-k}$ instead of $\alpha(\tau)^{-2k}$. The group $\mathrm{Mp}_2(\mathbb{Z}) = \{(M,\alpha) \in \mathrm{Mp}_2(\mathbb{R});$ $M \in \mathrm{SL}_2(\mathbb{Z})\}$ is generated by

$$\widehat{T} := (T,1) \quad \text{and} \quad \widehat{J} = \left(J, \sqrt{\tau}\right),$$

where $\sqrt{\tau}$ is the principal value of the square root of τ, determined by $\mathrm{Re}(\sqrt{\tau}) > 0$ or $\mathrm{Re}(\sqrt{\tau}) = 0$ and $\mathrm{Im}(\sqrt{\tau}) \geq 0$. More explicitly, we have the following defining relations:

$$\widehat{J}^2 = (\widehat{J}\widehat{T})^3 = \widehat{E} \quad \text{with} \quad \widehat{E} = (-I, i)\,, \quad \widehat{E}^4 = 1.$$

Given $N \in \mathbb{N}$, the *(metaplectic) principal congruence group of level* N is defined by

$$\mathrm{Mp}_2(\mathbb{Z})[N] := \{(M,\alpha) \in \mathrm{Mp}_2(\mathbb{Z}); M \in \mathrm{SL}_2(\mathbb{Z})[N]\}.$$

Let

$$\mathrm{H}(\mathbb{Z}) = \left\{[\lambda,\nu;\kappa] = \mathrm{rot}\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \mathrm{trans}\begin{pmatrix} 0 & \mu \\ \mu & \kappa - \lambda\mu \end{pmatrix}; \lambda,\mu,\kappa \in \mathbb{Z}\right\}$$

be the *integral Heisenberg group* and $\mathrm{MJ}_2(\mathbb{Z}) := \mathrm{Mp}_2(\mathbb{Z}) \ltimes \mathrm{H}(\mathbb{Z})$ be the *metaplectic Jacobi group*, where the action of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathrm{H}(\mathbb{Z})$ is given by the action of the first component $M \in \mathrm{SL}_2(\mathbb{Z})$, i. e.

$$((M_1,\alpha_1),[\lambda_1,\mu_1;\kappa_1])\cdot((M_2,\alpha_2),[\lambda_2,\mu_2;\kappa_2]) = \\ \big((M_1M_2,\alpha_2(\tau)\alpha_1(M_2\tau)),[(\lambda_1,\mu_1)+(\lambda_2,\mu_2)M_1^{-1};\kappa_1+\kappa_2+\lambda_1\mu_2-\mu_1\lambda_2]\big).$$

Given $k \in \frac{1}{2}\mathbb{Z}$, the group $\mathrm{MJ}_2(\mathbb{Z})$ acts on functions $f : \mathbb{H}_2 \to V$ (here again V is a $\mathbb{C}$-vector space) via

$$f\big|_k((M,\alpha),[\lambda,\mu;\kappa])(Z) := \alpha(\tau)^{-2k} f\left((M\times I_2)\,\mathrm{rot}\left(\begin{smallmatrix}1&0\\ \lambda&1\end{smallmatrix}\right)\mathrm{trans}\begin{pmatrix}0&\mu\\ \mu&\kappa-\lambda\mu\end{pmatrix}\langle Z\rangle\right),$$

with $Z = \left(\begin{smallmatrix}\tau&z\\ z&\omega\end{smallmatrix}\right)$. Furthermore, let ν_{H} be the character of $\mathrm{H}(\mathbb{Z})$ defined by $\nu_{\mathrm{H}}([\lambda,\mu;\kappa]) := (-1)^{\lambda+\mu+\lambda\mu+\kappa}$. It was proven in [GN] that all characters of $\mathrm{MJ}_2(\mathbb{Z})$ are given by $\nu_{a,b} := \nu_\eta^a \times \nu_{\mathrm{H}}^b$ with $a \in \mathbb{Z}/24\mathbb{Z}$ and $b \in \mathbb{Z}/2\mathbb{Z}$.

Given $\Phi : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$, we abbreviate $\widetilde{\Phi}_m : \mathbb{H}_2 \to \mathbb{C}$, $Z \mapsto \Phi(\tau,z)e^{2\pi i m\omega}$. As in [GN, Def 1.4] and [Der, Def 3.1] we need the following generalized definition of a Jacobi form (compare with [EZ, §I.1]):

Definition 6.1.1 *Let $\nu_{a,b} \in \mathrm{MJ}_2(\mathbb{Z})^{\mathrm{ab}}$. A holomorphic function $\Phi : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ is called a* Jacobi form of weight $k \in \frac{1}{2}\mathbb{Z}$, index $m \in \frac{1}{2}\mathbb{Z}$ with respect to $\nu_{a,b}$ *if*

$$\widetilde{\Phi}_m\big|_k M = \nu_{a,b}(M)\widetilde{\Phi}_m \quad \text{for all } M \in \mathrm{MJ}_2(\mathbb{Z}) \tag{6.1}$$

and if Φ has a Fourier expansion of the form

$$\Phi(\tau,z) = \sum_{\substack{n,l\in\mathbb{Q},\,n\geq 0\\ 4mn-l^2\geq 0}} c(n,l)e^{2\pi i(n\tau+lz)},$$

where n,l have bounded denominators depending on $\nu_{a,b}$. Moreover, Φ is a cusp form *if $c(n,l)\neq 0$ implies $4mn-l^2>0$. The space of Jacobi forms of weight k, index m with respect to $\nu_{a,b}$ is denoted by $[\mathrm{MJ}_2(\mathbb{Z}),k,m,\nu_{a,b}]$. The subspace of cusp forms is written as $[\mathrm{MJ}_2(\mathbb{Z}),k,m,\nu_{a,b}]_{\mathrm{cusp}}$. If we drop the condition $4mn-l^2<0 \Rightarrow c(n,l)=0$, we call Φ a* weak Jacobi form of weight k, index m with respect to $\nu_{a,b}$. *The associated space will be abbreviated by $[\mathrm{MJ}_2(\mathbb{Z}),k,m,\nu_{a,b}]_{\mathrm{weak}}$.*

Using Equation (6.1) we conclude

Lemma 6.1.2 *Let $k,m \in \frac{1}{2}\mathbb{Z}$, $a,b\in\mathbb{Z}$ and $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}),k,m,\nu_{a,b}]$ with Fourier coefficients $c(n,l)$.*

a) If $2m \not\equiv b \bmod 2$, then $[\mathrm{MJ}_2(\mathbb{Z}),k,m,\nu_{a,b}] = \{0\}$.
b) If $n \not\equiv \frac{a}{24} \bmod 1$ or $l \not\equiv \frac{b}{2} \bmod 1$, then $c(n,l) = 0$.

c) *Let* $n_i, l_i \in \mathbb{Q}$ *with* $4mn_1 - l_1^2 = 4mn_2 - l_2^2$ *and* $l_1 - l_2 \in 2m\mathbb{Z}$*, then*

$$e^{-\pi i b \frac{l_1}{2m}} c(n_1, l_1) = e^{-\pi i b \frac{l_2}{2m}} c(n_2, l_2),$$

hence $e^{-\pi i b \frac{l}{2m}} c(n,l)$ *depends only on* $4nm - l^2$ *and* $l \bmod 2m$*.*

From now on, we assume $2m \equiv b \bmod 2$. Before we mention a few explicit examples which we will need later on, we cite two methods to construct Jacobi forms:

Lemma 6.1.3 *a) If* $\varphi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, 1]_{\mathrm{cusp}}$ *(hence* $m \in \mathbb{Z}$*), then*

$$\varphi\eta^{-j} \in [\mathrm{MJ}_2(\mathbb{Z}), k - \frac{j}{2}, m, \nu_{-j,0}] \quad \text{for all } j \in \mathbb{N} \text{ with } jm \leq 18.$$

b) *Let* $\Phi_i \in [\mathrm{MJ}_2(\mathbb{Z}), k_i, m_i, \nu_i]$*,* $i = 1, 2$*, be two Jacobi forms with* $m_i, k_i \in \frac{1}{2}\mathbb{Z}$ *and* $\nu_i \in \mathrm{MJ}_2(\mathbb{Z})^{\mathrm{ab}}$*. We define*

$$[\Phi_1, \Phi_2] := \frac{1}{2\pi i}(m_2 \Phi_1' \Phi_2 - m_1 \Phi_1 \Phi_2'),$$

where $\Phi'(\tau, z) = \frac{\partial \Phi(\tau, z)}{\partial z}$ *for a Jacobi form* Φ*.*
Then

$$[\Phi_1, \Phi_2] \in [\mathrm{MJ}_2(\mathbb{Z}), k_1 + k_2 + 1, m_1 + m_2, \nu_1 \cdot \nu_2].$$

Proof: a) [Der, Eq (3.4)].

b) The differential operator from [EZ, Thm 9.5] was generalized by [GN, Lem 1.23]. In that source, the Jacobi function is used to prove that result. Since I had some difficulties with different root-branches which I could not solve, I give a different proof here:
Obviously $[\Phi_1, \Phi_2]$ is a holomorphic function and has the correct Fourier expansion. Given $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we calculate

$$\widetilde{M} := (M \times I) \operatorname{rot}\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \operatorname{trans}\begin{pmatrix} 0 & \mu \\ \mu & \kappa - \lambda\mu \end{pmatrix} = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and therefore, we conclude

$$\begin{pmatrix} \widetilde{\tau} & \widetilde{z} \\ * & \widetilde{\omega} \end{pmatrix} := \widetilde{M}\langle Z \rangle = \begin{pmatrix} M\langle\tau\rangle & \frac{\lambda\tau + z + \mu}{c\tau + d} \\ * & -\frac{\lambda\tau + z + \mu}{c\tau + d}(cz + c\mu - d\lambda) + \lambda z + \omega + \kappa \end{pmatrix}.$$

Since $\widetilde{\omega} = \frac{1}{c\tau + d}(z(-2c\mu + 2d\lambda) + \mathcal{O}(\tau, \omega, 1))$, we calculate by differentiation of Equation 6.1

$$\begin{aligned} \alpha(\tau)^{-2k_i} \frac{\partial}{\partial z}\left(\Phi_i(\widetilde{\tau}, \widetilde{z}) e^{2\pi i m_i \widetilde{\omega}}\right) &= \alpha(\tau)^{-2k_i - 2} \frac{\partial \Phi_i}{\partial z}(\widetilde{\tau}, \widetilde{z}) e^{2\pi i m_i \widetilde{\omega}} \\ &\quad + \nu_i(M) \alpha(\tau)^{-2k_i - 2} 2\pi i m_i (2d\lambda - 2c\mu) \Phi_i(\tau, z) e^{2\pi i m_i \omega} \\ &= \nu_i(M)(\frac{\partial}{\partial z}\Phi_i(\tau, z)) e^{2\pi i m_i \omega}, \end{aligned}$$

hence

$$\alpha(\tau)^{-2k_i-2}\Phi_i'(\widetilde{\tau},\widetilde{z})e^{2\pi i m_i\widetilde{\omega}} = \nu_i(M)e^{2\pi i m_i\omega}\left(\Phi_i'(\tau,z) - \frac{2\pi i m_i}{c\tau+d}(2d\lambda-2c\mu)\Phi_i(\tau,z)\right).$$

This leads to

$$\begin{aligned}
&(\widetilde{[\Phi_1,\Phi_2]})_{m_1+m_2}|_{k_1+k_2+1}\,((M,\alpha),[\lambda,\mu;\kappa])\,(Z)\\
&= \frac{\alpha(\tau)^{-2k_1-2k_2-2}}{2\pi i}e^{2\pi i(m_1+m_2)\widetilde{\omega}}[\Phi_1,\Phi_2](\widetilde{\tau},\widetilde{z})\\
&= (\nu_1\nu_2)(M)e^{2\pi i(m_1+m_2)\omega}\frac{1}{2\pi i}\left\{m_2\Phi_1'\Phi_2 - \frac{2\pi i m_1 m_2}{c\tau+d}(2d\lambda-2c\mu)\Phi_1\Phi_2\right.\\
&\qquad\qquad\left. -m_1\Phi_1\Phi_2' + \frac{2\pi i m_1 m_2}{c\tau+d}(2d\lambda-2c\mu)\Phi_1\Phi_2\right\}(\tau,z)\\
&= (\nu_1\nu_2)(M)\,[\Phi_1,\Phi_2]\,(\tau,z)e^{2\pi i(m_1+m_2)\omega}. \qquad \square
\end{aligned}$$

Example 6.1.4 *a) The first example is the Dedekind η-function $\eta \in [\mathrm{MJ}_2(\mathbb{Z}),\frac{1}{2},0,\nu_{1,0}]$, which was defined in Equation (4.5).*

b) We need the non-trivial (weak) Jacobi forms (see for example [EZ])

$$E_{k,m} \in [\mathrm{MJ}_2(\mathbb{Z}),k,m,1] \quad (\textit{for } m\in\mathbb{Z}, k\in 2\mathbb{Z}, k\geq 4)$$
$$\Phi_{10,1}\in[\mathrm{MJ}_2(\mathbb{Z}),10,1,1]_{\mathrm{cusp}},\quad \Phi_{12,1}\in[\mathrm{MJ}_2(\mathbb{Z}),12,1,1]_{\mathrm{cusp}},$$
$$\widehat{\Phi}_{-2,1} := \frac{\Phi_{10,1}}{\Delta}\in[\mathrm{MJ}_2(\mathbb{Z}),-2,1,1]_{\mathrm{weak}} \quad \textit{and} \quad \widehat{\Phi}_{0,1} := \frac{\Phi_{12,1}}{\Delta}\in[\mathrm{MJ}_2(\mathbb{Z}),0,1,1]_{\mathrm{weak}}.$$

Using the abbreviations $q=e^{2\pi i\tau}$ and $r=e^{2\pi iz}$, we get the following Fourier expansions

$$\Phi_{10,1}(\tau,z) = (r-2+r^{-1})q + (-2r^{\pm2}-16r^{\pm1}+36)q^2+\mathcal{O}(q^3) \quad \textit{and}$$
$$\Phi_{12,1}(\tau,z) = (r+10+r^{-1})q + (10r^{\pm2}-88r^{\pm1}-132)q^2+\mathcal{O}(q^3).$$

c) Moreover, we need the two theta series (compare with [Der, §3] or [GN, Ex 1.5, Lem 1.6])

$$\vartheta_{1/2}\in[\mathrm{MJ}_2(\mathbb{Z}),\tfrac{1}{2},\tfrac{1}{2},\nu_{3,1}] \textit{ and } \vartheta_{3/2}\in[\mathrm{MJ}_2(\mathbb{Z}),\tfrac{1}{2},\tfrac{3}{2},\nu_{1,1}].$$

They satisfy

$$\begin{aligned}
\vartheta_{1/2}(\tau,z) &= \sum_{n\in\mathbb{Z}}\left(\tfrac{-4}{n}\right)q^{n^2/8}r^{n/2} = \sum_{n\equiv 1 \bmod 2}(-1)^{\frac{n-1}{2}}q^{n^2/8}r^{n/2}\\
&= -q^{1/8}r^{-1/2}\prod_{n\in\mathbb{N}}(1-q^{n-1}r)(1-q^nr^{-1})(1-q^n) \quad \textit{and}\\
\vartheta_{3/2}(\tau,z) &= \sum_{n\in\mathbb{Z}}\left(\tfrac{12}{n}\right)q^{n^2/24}r^{n/2} = \frac{\eta(\tau)\vartheta_{1/2}(\tau,2z)}{\vartheta_{1/2}(\tau,z)}\\
&= q^{1/24}r^{-1/2}\prod_{n\in\mathbb{N}}(1+q^{n-1}r)(1+q^nr^{-1})(1-q^{2n-1}r^2)(1-q^{2n-1}r^{-2})(1-q^n)
\end{aligned}$$

where $\left(\frac{n}{m}\right)$ is the Kronecker symbol of n and m.

d) *Since* $\dim_{\mathbb{C}} [\mathrm{MJ}_2(\mathbb{Z}), 10, 1, 1]_{\mathrm{cusp}} = 1$, *we get* $\Phi_{10,1} = \vartheta_{1/2}^2 \eta^{18}$.
e) *Using the isomorphism ([EZ, p. 108])*

$$\begin{aligned} \mathcal{P}: \quad \mathbb{M}_k \oplus \mathbb{M}_{k+2} \oplus \ldots \oplus \mathbb{M}_{k+2m} &\longrightarrow [\mathrm{MJ}_2(\mathbb{Z}), k, m, 1]_{\mathrm{weak}}, \\ (f_0, f_1, \ldots, f_m) &\longmapsto \sum_i f_i \widehat{\Phi}^i_{-2,1} \widehat{\Phi}^{m-i}_{0,1}, \end{aligned}$$

we define a cusp form

$$\Phi_{6,5} := \Delta \widehat{\Phi}^3_{-2,1} \widehat{\Phi}^2_{0,1} - g_4 \Delta \widehat{\Phi}^5_{-2,1} \in [\mathrm{MJ}_2(\mathbb{Z}), 6, 5, 1]_{\mathrm{cusp}}.$$

Given $\nu \in \mathbb{N}_0$, $k \in \mathbb{N}$, let $\mathcal{D}_{2\nu}$ be the operator

$$\begin{aligned} \mathcal{D}_{2\nu} : [\mathrm{MJ}_2(\mathbb{Z}), k, m, 1]_{\mathrm{weak}} &\rightarrow \mathbb{M}_{k+2\nu}(\mathrm{SL}_2(\mathbb{Z})), \\ \Phi = \sum_{n,l} c(n,l) q^n r^l &\mapsto \sum_{n=0}^{\infty} \sum_l P^{(k-1)}_{2\nu}(l, mn) c(n,l) q^n, \end{aligned}$$

where

$$\frac{(k+\nu-2)!}{(2\nu)!(k-2)!} P^{(k-1)}_{2\nu}(l,n) = \text{coefficient of } t^{2\nu} \text{ in } (1 - lt + nt^2)^{-k+1}.$$

The operator $\mathcal{D}_\nu$ can be defined in an analogous way for ν odd ([EZ, p. 29]).

From [EZ, Thm 9.2] we know that every $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, 1]_{\mathrm{weak}}$ for $k \in \mathbb{Z}$ is uniquely determined by the image under the operator $\bigoplus_{\nu=0}^{2m} \mathcal{D}_{2\nu}$ if k is even, or $\bigoplus_{\nu=0}^{m-2} \mathcal{D}_{2\nu+1}$ if k odd.

Using the fact that every elliptic modular form $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$ fulfills

$$f \equiv 0 \text{ if } a_n = 0 \text{ for all } 0 \leq n \leq \frac{k}{12}$$

leads to

Lemma 6.1.5 *Let* $k \in \mathbb{Z}$, $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, 1]_{\mathrm{weak}}$ *with Fourier coefficients* $c(n,l)$. *Then* Φ *is uniquely determined by the Fourier coefficients* $c(n,l)$ *with* $n \leq \frac{k+2m}{12}$ *(in the case that* k *is odd, an upper bound is actually* $\frac{k+2m-3}{12}$*).*

6.2 Vector-valued Meromorphic Modular Forms

Definition 6.2.1 *Let* $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ *be a finite dimensional representation, such that* ρ *factors over* $\mathrm{Mp}_2(\mathbb{Z})[N]$. *A holomorphic function* $f : \mathbb{H} \to V$ *is a* (V-valued) meromorphic modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ with multiplier system ρ, *if*

$$f\big|_k M = \rho(M) f \quad \textit{for all } M \in \mathrm{Mp}_2(\mathbb{Z}),$$

and f has at most a pole at the cusp $i\infty$. If in addition f has no pole at $i\infty$, then f is a (holomorphic) modular form. *Moreover, if* $\lim_{\mathrm{Im}(\tau)\to\infty} f(\tau) = 0$, *then f is a* cusp form. *The space of meromorphic modular forms of weight k and multiplier system ρ is denoted by* $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]_{\mathrm{mer}}$, *the subspace of (holomorphic) modular forms is denoted by* $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$ *and the subspace of cusp forms by* $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]_{\mathrm{cusp}}$.

Given $m, x \in \frac{1}{2}\mathbb{Z}$ and $b \in \mathbb{Z}$ with $m \equiv x \equiv \frac{b}{2} \bmod 1$, we define

$$\theta_{m,x,b} : \mathbb{H} \times \mathbb{C} \to \mathbb{C}, \quad (\tau, z) \mapsto \sum_{l \equiv x \bmod 2m} e^{\pi i b l/2m} e^{2\pi i (l^2\tau/4m + lz)}$$

and $\mathbb{V}_m := \{f : {}^{(m+\mathbb{Z})}/_{2m\mathbb{Z}} \to \mathbb{C}\}$ the vector space of all complex valued functions on ${}^{(m+\mathbb{Z})}/_{2m\mathbb{Z}}$. A typical basis of $\mathbb{V}_m$ are the *characteristic functions* $f_x^c := c\delta_{x,c} \in \mathbb{V}_m$, $x \in {}^{(m+\mathbb{Z})}/_{2m\mathbb{Z}}$. Then

$$\Theta_{m,b} : \mathbb{H} \times \mathbb{C} \to \mathbb{V}_m, \quad (\tau, z) \mapsto (\theta_{m,x,b})_{x \in (m+\mathbb{Z})/2m\mathbb{Z}}.$$

Dern ([Der, §3]) showed that $\Theta_{m,b}$ is a $\mathbb{V}_m$-valued metaplectic Jacobi form of weight $\frac{1}{2}$ with a multiplier system $\rho_{m,b}$ which is defined by

$$\begin{aligned} \left(\rho_{m,b}\left(\widehat{T}\right) f\right)(x) &= e^{2\pi i x^2/4m} f(x), \\ \left(\rho_{m,b}\left(\widehat{J}\right) f\right)(x) &= e^{2\pi i b x/4m} \frac{1}{\sqrt{2mi}} \sum_{y : (m+\mathbb{Z})/2m\mathbb{Z}} e^{-2\pi i y x/2m} e^{-2\pi i b y/4m} f(y). \end{aligned}$$

Furthermore, up to isomorphy, $\rho_{m,b}$ does not depend on b.

Given $f : \mathbb{H} \to \mathbb{V}_m$, we define the *components* $f_x : \mathbb{H} \to \mathbb{C}$ for $x \in {}^{(m+\mathbb{Z})}/_{2m\mathbb{Z}}$ by $f(\tau) = \sum_{x \in (m+\mathbb{Z})/2m\mathbb{Z}} f_x(\tau) f_x^c$. There exists a *scalar product* on $\mathbb{V}_m$ defined by

$$(f, g) = \sum_{x \in (m+\mathbb{Z})/2m\mathbb{Z}} f_x g_x,$$

which respects holomorphy instead of being hermitian. For a given representation $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathbb{V}_m$, let ρ^* denote the dual representation with respect to the above pairing, i. e.

$$(\rho^*(M) f, \rho(M) g) = (f, g) \quad \text{for all } f, g \in \mathbb{V}_m \text{ and } M \in \mathrm{Mp}_2(\mathbb{Z}).$$

The matrix representation which belongs to $\rho(M)$ is denoted by $A_\rho(M)$. Obviously $A_{\rho^*} = A_\rho^{-\mathrm{tr}}$. Using [Der, Lem 3.5] we get

Lemma 6.2.2 *Given $k \in \frac{1}{2}\mathbb{Z}$, $\nu_{a,b} \in \mathrm{MJ}_2(\mathbb{Z})^{\mathrm{ab}}$ and $m \in \frac{1}{2}\mathbb{Z}$ with $2m \equiv b \bmod 2$. Then*

$$\begin{aligned} \left[\mathrm{Mp}_2(\mathbb{Z}), k - \tfrac{1}{2}, \nu_\eta^a \rho_{m,b}^*\right] &\to \left[\mathrm{MJ}_2(\mathbb{Z}), k, m, \nu_{a,b}\right] \\ f = (f_x)_{x \in (m+\mathbb{Z})/2m\mathbb{Z}} &\mapsto (f, \Theta_{m,b}) = \sum_{x \in (m+\mathbb{Z})/2m\mathbb{Z}} f_x \theta_{m,x,b} \end{aligned}$$

is an isomorphism of vector spaces.

Since this map respects also multiplication with $\mathcal{R} := \mathbb{C}[g_4, g_6]$, it is even an isomorphism of $\mathcal{R}$-modules.

We can calculate the Fourier expansion of the image under the above isomorphism explicitly which can be found in the proof in [Der]: Let $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, \nu_{a,b}]$ with Fourier expansion

$$\Phi(\tau, z) = \sum_{\substack{n \equiv n_0, l \equiv l_0 \bmod \mathbb{Z} \\ 4nm - l^2 \geq 0}} c(n,l) e^{2\pi i(n\tau + lz)},$$

then we get $\Phi(\tau, z) = (f, \Theta_{m,b})(\tau, z)$ with

$$f_x(\tau) = \sum_{0 \leq N \equiv 4mn_0 - x^2 \bmod \mathbb{Z}} c_x(N) e^{2\pi i \frac{N}{4m}\tau}, \quad x \in (m + \mathbb{Z})/2m\mathbb{Z}, \tag{6.2}$$

where

$$c_l(4mn - l^2) := e^{-\pi i b \frac{l}{2m}} c(n,l),$$

(which depends (cf. Lemma 6.1.2) on $4nm - l^2$ and l mod $\mathbb{Z}$ only.)

Lemma 6.2.3 *Let $f \in [\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \nu_\eta^a \rho_{m,b}^*]$. If 1 is not a eigenvalue of $\nu_\eta^a \rho_{m,b}^*(\widehat{T})$, then f is a cusp form. Especially in this case, we have*

$$[\mathrm{Mp}_2(\mathbb{Z}), k - \tfrac{1}{2}, \nu_\eta^a \rho_{m,b}^*] = [\mathrm{Mp}_2(\mathbb{Z}), k - \tfrac{1}{2}, \nu_\eta^a \rho_{m,b}^*]_{\mathrm{cusp}}.$$

Proof: $f(\tau + 1) = \left(f|_k \widehat{T}\right)(\tau) = \nu_\eta^a(T) \rho_{m,b}^*(\widehat{T}) f(\tau)$. Since all eigenvalues of $A_{\rho_{m,b}^*}(\widehat{T})$ have the form $e^{2\pi i r_1}$ with $r_1 \in \mathbb{Q}$, we know that $f(\tau + 1) = e^{2\pi i r_2} f(\tau)$ with $r_2 \in \mathbb{Q} \cap [0;1)$. Hence

$$f(\tau) = \sum_{N \equiv r_2 \bmod 1,\, N \geq 0} a_N e^{2\pi i N\tau},$$

which is a cusp form for $r_2 \neq 0$. □

Skoruppa's dimension formula ([ES], [Sko]) for $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$ holds only if $\rho(\widehat{J}^2)$ acts as a scalar. This is only true for $\rho = \rho_{m,b}$ in the case $m \leq 1$. Therefore, we decompose $\rho_{m,b}$ with $\mathcal{W} := \rho_{m,b}(\widehat{J}^2)$. Let $\mathbb{V}_{m,s}$ be the eigenspace with eigenvalue $s \in \{\pm i\}$ with respect to $\mathcal{W}$ and let $\rho_{m,b,s}$ be the restriction of $\rho_{m,b}$ to $\mathbb{V}_{m,s}$. Then we get (compare with [Der]) the decomposition $\rho_{m,b} = \rho_{m,b,i} \oplus \rho_{m,b,-i}$. In general, $\rho_{m,b,s}$ is not irreducible (see [Sko], [EZ]), but in the later considered examples $m \in \{\frac{5}{2}, \frac{7}{2}, 5, 7\}$ the representation $\rho_{m,b,s}$ is irreducible, which can be shown using Schur's Lemma.

Before we define the arithmetical lifting, we introduce the following construction method for metaplectic modular forms which we will need later on. It is a generalization of the differential calculus for elliptic modular forms.

Lemma 6.2.4 *Let $k \in \frac{1}{2}\mathbb{Z}$ and $f \in [\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$. Then one has*

$$\frac{1}{\Delta}[f,\Delta] := 12f' - kf \cdot \frac{\Delta'}{\Delta} \in [\mathrm{Mp}_2(\mathbb{Z}), k+2, \rho].$$

Proof: Obviously, the function $\frac{1}{\Delta}[f,\Delta] : \mathbb{H} \to V$ is holomorphic. Let $(M,\alpha) \in \mathrm{Mp}_2(\mathbb{Z})$, then

$$\frac{\partial M\tau}{\partial \tau} = \alpha(\tau)^{-4} \quad \text{and} \quad \frac{\partial \alpha}{\partial \tau} = \frac{c}{2}\alpha(\tau)^{-1}.$$

Using [Kr1, III.6 (1)] we have

$$\left(\frac{\Delta'}{\Delta}\Big|_2(M,\alpha)\right)(\tau) = \frac{\Delta'}{\Delta}(M\tau)\alpha(\tau)^{-4} = \frac{12c}{\alpha(\tau)^2} + \frac{\Delta'}{\Delta}(\tau).$$

Differentiating $\alpha(\tau)^{-2k} f(M\tau) = \rho(M) f(\tau)$ leads to

$$\begin{aligned}\left(f'|_{k+2}(M,\alpha)\right)(\tau) &= \alpha(\tau)^{-2k-4} f'(M\tau) = \rho(M) f'(\tau) + kc\alpha(\tau)^{-2k-2} f(M\tau) \\ &= \rho(M) f'(\tau) + kc\alpha(\tau)^{-2}\rho(M) f(\tau).\end{aligned}$$

Additionally,

$$\left(\left(f\frac{\Delta'}{\Delta}\right)\Big|_{k+2}(M,\alpha)\right)(\tau) = \rho(M) f(\tau) \cdot \left(\frac{12c}{\alpha(\tau)^2} + \frac{\Delta'}{\Delta}(\tau)\right).$$

Hence

$$\left(12f' - kf \cdot \frac{\Delta'}{\Delta}\right)\Big|_{k+2}(M,\alpha) = 12f' - kf \cdot \frac{\Delta'}{\Delta}. \qquad \square$$

6.3 Hecke Operators and the Arithmetical Lift

First of all, we define *Hecke operators* as in [GN, (1.12)]: Given $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, \nu \times \nu_{\mathrm{H}}^b]$ with integral weight k, $b \in \mathbb{Z}$ and $\nu \in \mathrm{SL}_2(\mathbb{Z})^{\mathrm{ab}}$ as well as $Q \in \mathbb{N}$ with $\mathrm{SL}_2(\mathbb{Z})[Q] \subset \ker(\nu)$, we define $\mathcal{T}^{(Q)}(l)$ by

$$\left(\widetilde{\Phi}_m|_k \mathcal{T}^{(Q)}(l)\right)\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix} := l^{2k-3} \sum_{\substack{ad=l\\ b \bmod d}} d^{-k}\nu(\sigma_a)\Phi\left(\frac{a\tau+bQ}{d}, az\right)e^{2\pi i l m\omega},$$

where $\sigma_a \in \mathrm{SL}_2(\mathbb{Z})$ is chosen such that $\sigma_a \equiv \left(\begin{smallmatrix}a^{-1} & 0\\ 0 & a\end{smallmatrix}\right) \bmod Q$. Given $l \equiv 1 \bmod Q$ with $\gcd(l, 2^b Q) = 1$ we have

$$\Phi|_k \mathcal{T}^{(Q)}(l) \in [\mathrm{MJ}_2(\mathbb{Z}), k, lm, \nu \times \nu_{\mathrm{H}}^b].$$

Furthermore, if $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, 1]$ with $k \in \mathbb{Z}$ (hence $m \in \mathbb{Z}$, too), then the Hecke operators $\mathcal{T}_-(l)$ from [Gr1] match the Hecke operators $\mathcal{T}^{(1)}(l)$. This is an easy consequence of [Koh, p. 64] and [Gr1, (2.6)].

Theorem 6.3.1 *Let $k,t \in \mathbb{Z}$ be non-negative and let $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}),k,t,1]$ be a Jacobi form with Fourier expansion*

$$\Phi(\tau,z) = \sum_{\substack{n,l\in\mathbb{Z}\\ 4nt\geq l^2}} c(n,l)e^{2\pi i(n\tau+lz)}.$$

In the case that $c(0,0)\neq 0$ we assume $k\geq 4$. Define $\mathrm{Lift}_\Phi : \mathbb{H}_2 \to \mathbb{C}$ by

$$\mathrm{Lift}_\Phi(Z) := -\frac{B_k}{2k}c(0,0)g_k(\tau) + \sum_{m=1}^{\infty} m^{2-k}\widetilde{\Phi}_t\Big|_k \mathcal{T}^{(1)}(m)(Z),$$

where $g_k(\tau)$ is the elliptic Eisenstein series

$$g_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n\tau}.$$

Then Lift_Φ satisfies:

a) The series converges absolutely and uniformly on subsets of $\mathbb{H}_2$ of type $Y \geq \delta I_2$ for all $\delta>0$.
b) $\mathrm{Lift}_\Phi \in \mathbb{M}_k(\Gamma_t)$ and it fulfills $\mathrm{Lift}_\Phi|_k V_t = (-1)^k\,\mathrm{Lift}_\Phi$.
c) If Φ is cuspidal, then $\mathrm{Lift}_\Phi \in \mathbb{S}_k(\Gamma_t)$.
d) The mapping $\mathrm{Lift}_t : \Phi \mapsto \mathrm{Lift}_\Phi$ is injective and linear.
e) Lift_Φ has the Fourier expansion

$$\mathrm{Lift}_\Phi = -\frac{B_k}{2k}c(0,0)\left(g_k(\tau)+g_k(t\omega)-1\right) + \sum_{\substack{m\geq1,n\geq1\\ l\in\mathbb{Z},4nmt\geq l^2}}\left(\sum_{a|\gcd(n,m,l)} a^{k-1}c(\frac{nm}{a^2},\frac{l}{a})\right)e^{2\pi i(n\tau+lz+mt\omega)}.$$

Proof: [Gr1, Hauptsatz 2.1] or more detailed in [Koh, Thm 4.6]. □

Theorem 6.3.2 *Let $k\in\mathbb{Z}$, $t\in\frac{1}{2}\mathbb{Z}$, and $Q\in\{1,2\}$, such that $tQ\in\mathbb{Z}$ is odd. For a given Jacobi form $\Phi\in[\mathrm{MJ}_2(\mathbb{Z}),k,t,v_\eta^{\frac{24}{Q}}\times v_{\mathrm{H}}^{\frac{2}{Q}}]_{\mathrm{cusp}}$ with Fourier expansion*

$$\Phi(\tau,z) = \sum_{\substack{n\equiv\frac{1}{Q} \bmod 1\\ n>0}}\ \sum_{\substack{l\equiv\frac{1}{Q} \bmod 1\\ 4nt>l^2}} c(n,l)e^{2\pi i(n\tau+lz)},$$

let

$$\mathrm{Lift}_\Phi(Z) := \sum_{\substack{m\equiv 1 \bmod Q\\ m>0}} m^{2-k}(\widetilde{\Phi}_t\Big|_k \mathcal{T}^{(Q)}(m))(Z).$$

Then

a) *The series converges absolutely for* $k \geq 0$*.*
b) $\mathrm{Lift}_\Phi \in \mathbb{S}_k(\Gamma_{Qt}, \kappa_{1,2}^Q)$ *and it fulfills* $\mathrm{Lift}_\Phi |V_{Qt} = (-1)^k \mathrm{Lift}_\Phi$*.*
c) *If* $\Phi \not\equiv 0$*, then* $\mathrm{Lift}_\Phi(Z) \not\equiv 0$*, too.*
d) Lift_Φ *has the Fourier expansion*

$$\mathrm{Lift}_\Phi(Z) = \sum_{\substack{N,M>0 \\ N,M\equiv 1 \bmod Q}} \sum_{\substack{L \equiv \frac{2}{Q} \bmod 2 \\ 4NM\frac{t}{Q} > \left(\frac{L}{2}\right)^2}} \left(\sum_{a|\gcd(N,L,M)} a^{k-1} c\left(\frac{NM}{Qa^2}, \frac{L}{2a}\right) \right) e^{2\pi i\left(\frac{N}{Q}\tau + \frac{L}{2}z + Mt\omega\right)}.$$

Proof: Most of this result can be found in a more general version in [GN, Thm 1.12]. We just have to check that $\mathrm{Lift}_\Phi |V_{Qt} = (-1)^k \mathrm{Lift}_\Phi$ which follows directly from the symmetry of the Fourier expansion with respect to $(\tau, \frac{\omega}{Qt}) \mapsto (\omega, \frac{\tau}{Qt})$. □

Remark 6.3.3 *a)* *The restriction that Qt is odd was only chosen so that we can write down the character explicitly.*
b) *For Qt prime b) is equivalent to the fact that* $\mathrm{Lift}_\Phi \in \mathbb{S}_k(\Gamma_{Qt}^{\max}, \kappa_{1,2}^Q \chi_{Qt}^k)$*.*

Since we want to identify later on Borcherds products and Maaß lifts, we need a result from [GN, Lem 1.16] about constraint zeros on divisors:

Corollary 6.3.4 *Let Q and* Lift_Φ *be defined as in Theorem 6.3.2.*

a) *If* $\Phi(\tau,z)|_{z=0}$ *has a zero of order m, then* Lift_Φ *has at least a zero of order m in* $\mathcal{H}_0 = \{Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathbb{H}_2; z = 0\}$*.*
b) *Let* $M \subset \mathbb{Q}/\mathbb{Z}$ *be a subset which is invariant with respect to multiplication with* $\{a \in \mathbb{Z}; \gcd(a,Q) = 1\}$*. If* $\Phi(\tau,z)|_{z=\alpha}$ *has a zero of order m for all* $\alpha \in M$*, then* Lift_Φ *has at least a zero of order m in* $\mathcal{H}_\alpha = \{Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathbb{H}_2; z = \alpha\}$ *for all* $\alpha \in M$*.*

Proof:

$$\left(\widetilde{\Phi}_m|_k \mathcal{T}^{(Q)}(l)\right)(Z) := l^{2k-3} \sum_{\substack{ad=l \\ b \bmod d}} d^{-k} \Phi\left(\frac{a\tau + bQ}{d}, az\right) e^{2\pi i l m \omega} = 0$$

for all $l \equiv 1 \bmod Q$, which yields the assertion. □

Remark 6.3.5 *Let* $t = 5$*. In view of Example 3.6.2 we therefore have a result about constraint zeros on the divisors* $\lambda_{i^2}^\perp$ *for* $i = 1, \ldots, 5$*. Does there exist an analogous statement for* $\lambda_5^\perp$ *and* $\lambda_{20}^\perp$*?*

Now we have all the methods we need to prove a generalized version of [Der, Prop 3.6]:

Proposition 6.3.6 *Let $t \in \mathbb{N}$ be prime with $t \neq 2,3$, $k \in \mathbb{N}$ and $d \in \{1,2\}$. Then there exists an injective homomorphism*

$$\mathcal{M} : [\mathrm{Mp}_2(\mathbb{Z}), k - \tfrac{1}{2}, v_\eta^{\frac{24}{d}} \rho^*_{\frac{t}{d}, 2\frac{t}{d}}] \to \mathbb{M}_k(\Gamma_t^*, \kappa_{1,2}^{\frac{2}{d}} \chi_t^k)$$

defined by

$$\mathcal{M}(f)(Z) = c_0(0) \frac{-B_k}{2k} g_k(\tau) + \sum_{\substack{m \in \mathbb{N} \\ m \equiv 1 \bmod d}} m^{2-k} \left(f, \Theta_{\frac{t}{d}, 2\frac{t}{d}} \right)^{\sim}_{\frac{t}{d}} \Big|_k \mathcal{T}^{(d)}(m)(Z),$$

where $c_l(n)$ is the n^{th} Fourier coefficient of $\tau \mapsto f_l(\tau)$. Moreover, $\mathcal{M}$ maps cusp forms onto cusp forms.

Proof: In view of Lemma 6.2.2 we define

$$\Phi := (f, \Theta_{\frac{t}{d}, 2\frac{t}{d}}) \in [\mathrm{MJ}_2(\mathbb{Z}), k, \tfrac{t}{d}, v_{\frac{24}{d}, 2\frac{t}{d}}].$$

1. case $d = 2$: The representation $v_\eta^{12}(T) \rho^*_{\frac{t}{2}, t}(\widehat{T}) = -\rho^*_{\frac{t}{2}, t}(\widehat{T})$ has the eigenvalues

$$-e^{-2\pi i (t+2j-2)^2/8t} \quad \text{with} \quad j = 1, \ldots, t.$$

None of these values equals 1, since

$$e^{-2\pi i (t+2j-2)^2/8t} = -1 \quad \iff \quad \frac{(t+2j-2)^2}{8t} = \frac{1}{2} \bmod 1,$$

which can only be satisfied if $2 \mid t$.

Therefore Φ is a cusp form by Lemma 6.2.3 so that $\mathcal{M}(f) = \mathrm{Lift}_\Phi$, and the claim follows directly from Theorem 6.3.2.

2. case $d = 1$: Thus $v_\eta^{\frac{24}{d}} = 1$. The claim for cusp forms follows analogously to the first case. Since t is square-free, we know from [EZ, p. 24], that Φ is cuspidal if and only if $\Phi(\tau, 0) \in \mathbb{S}_k(\mathrm{SL}_2(\mathbb{Z}))$. If Φ is not a cusp form, we therefore know that $k \in 2\mathbb{Z}$ with $k \geq 4$, so that $\mathcal{M}(f) = \mathrm{Lift}_\Phi$. The claim follows now from Theorem 6.3.1. □

Remark 6.3.7 *$\mathcal{M}$ is called* Maaß lift *and the image*

$$\mathcal{M}_{k,d} := \mathcal{M}([\mathrm{Mp}_2(\mathbb{Z}), k - \tfrac{1}{2}, v_\eta^{\frac{24}{d}} \rho^*_{\frac{t}{d}, 2\frac{t}{d}}])$$

is the Maaß space with character $\kappa_{1,2}^{\frac{2}{d}} \chi_t^k$.

6.4 Dimension Formula for $\mathcal{M}_{k,d}$

Using Lemma 6.2.2, the Maaß lift 6.3.6 and the definition

$$\mathcal{M}_{k,d,s} := \mathcal{M}([\mathrm{Mp}_2(\mathbb{Z}), k-\tfrac{1}{2}, v_\eta^{\frac{2}{d}} \rho_{\frac{m}{d},2\frac{m}{d},s}]),$$

the decomposition $\rho_{m,b} := \rho_{m,b,i} \oplus \rho_{m,b,-i}$ leads to

$$\begin{aligned}
[\mathrm{Mp}_2(\mathbb{Z}), k, v_\eta^a \rho_{m,b}^*] &= [\mathrm{Mp}_2(\mathbb{Z}), k, v_\eta^a \rho_{m,b,i}^*] \oplus [\mathrm{Mp}_2(\mathbb{Z}), k, v_\eta^a \rho_{m,b,-i}^*],\\
[\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}] &= [\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}, i] \oplus [\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}, -i],\\
\mathcal{M}_{k,d} &= \mathcal{M}_{k,d,i} \oplus \mathcal{M}_{k,d,-i},
\end{aligned}$$

With these decompositions we are able to use Skoruppa's dimension formula ([ES, Thm 4.2], [Sko, Satz 5.1]):

Lemma 6.4.1 *Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be a representation of dimension n of $\mathrm{Mp}_2(\mathbb{Z})$ with $\mathrm{Mp}_2(\mathbb{Z})[N] \subset \ker(\rho)$ for some $N \in \mathbb{N}$ and $\rho(\widehat{J}^2) = \zeta\, \mathrm{id}_V$ with a fourth root of unity ζ. Let $l_j \in \mathbb{R}$, $j = 1, \ldots, n$, such that $e^{2\pi i l_j}$ runs through the eigenvalues of $\rho(\widehat{T})$. Define*

$$A(\rho) := \sharp\{j; l_j \equiv 0 \bmod 1\} \quad \textit{and} \quad B(\rho) = \sum_{j=1}^{n} \mathbb{B}_1(l_j)$$

with

$$\mathbb{B}_1(x) = \begin{cases} 0, & x \in \mathbb{Z},\\ x - \lfloor x \rfloor - \frac{1}{2}, & x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$$

Then, for $k \in \frac{1}{2}\mathbb{Z}$, one has

$$\begin{aligned}
&\dim_{\mathbb{C}} [\mathrm{Mp}_2(\mathbb{Z}), k, \rho] - \dim_{\mathbb{C}} [\mathrm{Mp}_2(\mathbb{Z}), 2-k, \rho^*]_{\mathrm{cusp}}\\
&\quad = \begin{cases} 0, & \rho(\widehat{E}) \neq i^{-2k}\, \mathrm{id}_V,\\ \frac{k-1}{12} \dim(\rho) + \frac{A(\rho)}{2} - B(\rho) + \frac{1}{4} \mathrm{Re}(e^{2\pi i \frac{k}{4}} \operatorname{trace} \rho(\widehat{J})) & \\ \qquad + \frac{2}{3\sqrt{3}} \mathrm{Re}(e^{2\pi i (\frac{k}{6} + \frac{1}{12})} \operatorname{trace} \rho(\widehat{J}\widehat{T})), & \rho(\widehat{E}) = i^{-2k}\, \mathrm{id}_V. \end{cases}
\end{aligned}$$

In the case $k \geq 2$, we have $\dim_{\mathbb{C}} [\mathrm{Mp}_2(\mathbb{Z}), 2-k, \rho^*]_{\mathrm{cusp}} = 0$, hence we get an explicit formula. In the cases $k = \frac{1}{2}$ and $k = \frac{3}{2}$, there exist explicit expressions (see [Sko]), too.

Let $\mathcal{R} := \mathbb{C}[g_4, g_6]$ be again the graded ring of elliptic modular forms. As in [Sko] or [Der], we see that

$$[\mathrm{Mp}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, \rho] := \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} [\mathrm{Mp}_2(\mathbb{Z}), k, \rho] \tag{6.3}$$

is always a free module of rank $\dim(\rho)$ over $\mathcal{R}$. By Lemma 6.2.2, we transfer this fact to the analogously defined spaces $[\mathrm{MJ}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, m, \nu_{a,b}]$.

Let $s \in \{\pm i\}$, $s_0 := -i(-1)^{2m}$, $f^c_{x,s} := f^c_x - s\mathcal{W} f^c_x$ and $\mathcal{B}_s := \mathcal{B}_{0,s} \cup \mathcal{B}_{1,s}$ with

$$\mathcal{B}_{0,s} := \begin{cases} \{ f^c_x; x \in {(m+\mathbb{Z})}/{2m\mathbb{Z}}, 2x \equiv 0 \bmod 2m\}, & s = s_0, \\ \emptyset, & \text{else}, \end{cases}$$
$$\mathcal{B}_{1,s} := \{ f^c_{x,s}; x \in [0;m] \cap (m+\mathbb{Z}), 2x \not\equiv 0 \bmod 2m\}.$$

Then $\mathcal{B}_s$ is a basis of $\mathbb{V}_{m,s}$. We would like to calculate the parameters needed in Lemma 6.4.1 for $\rho = \nu_\eta^{\frac{24}{d}} \rho^*_{m,b,s}$.

Given $f \in \mathbb{V}_{m,s}$, we know that

$$\rho_{m,b,s}(\widehat{M}) f = \sum_{g \in \mathcal{B}_s} c_{f,g}(\widehat{M}) g.$$

The coefficients $c_{f,g}$ for $f, g \in \mathcal{B}_s$ can be calculated easily, e. g.

$$c_{f,f}(\widehat{J}) = \frac{e^{-2\pi i x^2/2m}}{\sqrt{2mi}} + \begin{cases} is\frac{e^{2\pi i x^2/2m}}{\sqrt{2mi}}, & f \in \mathcal{B}_{1,s}, \\ 0, & f \in \mathcal{B}_{0,s_0}, \end{cases}$$
$$c_{f,f}(\widehat{JT}) = \frac{e^{-2\pi i x^2/4m}}{\sqrt{2mi}} + \begin{cases} is\frac{e^{2\pi i x^2 3/4m}}{\sqrt{2mi}}, & f \in \mathcal{B}_{1,s}, \\ 0, & f \in \mathcal{B}_{0,s_0}. \end{cases}$$

Therefore, we determine all the parameters in the dimension formula for the representation $\rho = \nu_\eta^{\frac{24}{d}} \rho^*_{m,b,s}$.

Example 6.4.2 *Let $t = m = 5$. Then these parameters are given by*

d	s	ρ	$\dim(\rho)$	$A(\rho)$	$B(\rho)$	$\operatorname{trace}\rho(\widehat{J})$	$\operatorname{trace}\rho(\widehat{JT})$
1	i	$\rho^*_{5,i}$	4	0	$\frac{1}{2}$	0	i
1	$-i$	$\rho^*_{5,-i}$	6	1	$\frac{3}{4}$	0	0
2	i	$\nu_\eta^{12}\rho^*_{\frac{5}{2},i}$	3	0	$\frac{1}{8}$	$\frac{1-i}{\sqrt{2}}$	0
2	$-i$	$\nu_\eta^{12}\rho^*_{\frac{5}{2},-i}$	2	0	$-\frac{1}{4}$	0	i

This leads to the following dimensions of $\mathcal{M}_{k,d,s}$. Because of

$$\dim_{\mathbb{C}} [\mathrm{Mp}_2(\mathbb{Z}), k+12, \rho] = \dim \rho + \dim_{\mathbb{C}} [\mathrm{Mp}_2(\mathbb{Z}), k, \rho], \tag{6.4}$$

we only list the dimensions for $k \leq 14$:

m	s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	ν
5	i	0	0	0	0	1	0	1	0	2	0	3	0	3	0	1
5	$-i$	0	0	0	1	0	2	0	3	0	4	0	5	0	6	1
$\frac{5}{2}$	i	0	0	0	0	1	0	1	0	2	0	2	0	3	0	$\kappa_{1,2}$
$\frac{5}{2}$	$-i$	0	0	0	1	0	1	0	1	0	2	0	2	0	2	$\kappa_{1,2}$

The most cases can be calculated directly by Lemma 6.4.1 or the fact that depending on the choice of s the space $\mathcal{M}_{l,d,s} = \{0\}$ for all even or odd l. The only remaining cases are the following four cases:

1) $\dim_{\mathbb{C}} \mathcal{M}_{1,1,i} = 0$, *because of [EZ, Thm 5.7].*
2) $\dim_{\mathbb{C}} \mathcal{M}_{2,1,-i} = 0$, *because of [EZ, p. 118].*
3) $\dim_{\mathbb{C}} \mathcal{M}_{1,2,i} = 0$, *since for every $f \in \mathcal{M}_{1,2,i}$ the function $f^2 \in \mathcal{M}_{2,1,i} = \{0\}$.*
4) *Now let $f \in [\mathrm{MJ}_2(\mathbb{Z}), 2, 5/2, \nu_{12,1}, -i]$. Then f has only Fourier coefficients unequal to zero if $n \equiv \frac{1}{2} \bmod 1$, $n \geq 0$ and $l \equiv \frac{1}{2} \bmod 1$. Hence $f^2 \in [\mathrm{MJ}_2(\mathbb{Z}), 4, 5, 1, -i]_{\mathrm{cusp}}$, which leads to $f = 0$ so that $\dim_{\mathbb{C}} \mathcal{M}_{2,2,-i} = 0$.*

6.5 Generators for $[\mathrm{MJ}_2(\mathbb{Z}), k, m, \nu]$ with $2m \mid 5$

Case 1: $k \in 2\mathbb{Z}$, $m = \frac{5}{2}$, $\nu = \nu_{12,1}$, hence $s = -i$.
The $\mathbb{C}[g_4, g_6]$-module $[\mathrm{MJ}_2(\mathbb{Z}), 2\mathbb{Z}, \frac{5}{2}, \nu_{12,1}]$ is free (because of (6.3)), so that it has a basis: Due to Lemma 6.1.5 we only need to consider the Fourier coefficients with $n \leq \frac{2*14+10}{12} = \frac{19}{6}$ at most. The following table lists the generators and dimensions for $[\mathrm{MJ}_2(\mathbb{Z}), k, m, \nu]$:

k	$\dim_{\mathbb{C}}$	generators
4	1	$\Phi_{4,5/2} := \vartheta_{1/2}^2 \vartheta_{3/2} \eta^5$
6	1	$\Phi_{6,5/2} := \vartheta_{3/2} \Phi_{12,1} \eta^{-13}$
8	1	$g_4 \Phi_{4,5/2}$
10	2	$g_4 \Phi_{6,5/2}, g_6 \Phi_{4,5/2}$
12	2	$g_6 \Phi_{6,5/2}, g_4^2 \Phi_{4,5/2}$
14	2	$g_4^2 \Phi_{6,5/2}, g_4 g_6 \Phi_{4,5/2}$

Lemma 6.5.1 *a) The graded $\mathbb{C}[g_4, g_6]$-module $[\mathrm{MJ}_2(\mathbb{Z}), 2\mathbb{Z}, \frac{5}{2}, \nu_{12,1}]$ has a basis*

$$\{\Phi_{4,5/2}, \Phi_{6,5/2}\}.$$

b) For even $k \geq 6$, we get

$$\dim_{\mathbb{C}} [\mathrm{MJ}_2(\mathbb{Z}), k, \tfrac{5}{2}, \nu_{12,1}] = \left[\frac{k+2}{6}\right].$$

Proof: $\Phi_{10,1}$ and $\Phi_{12,1}$ are linear independent over $\mathcal{R}=\mathbb{C}[g_4,g_6]$ and $\vartheta_{1/2}^2\eta^{18}=\Phi_{10,1}$. Thus the linear independency of $\Phi_{4,5/2}$ and $\Phi_{6,5/2}$ over $\mathcal{R}$ follows. Using Equation (6.3) with $\dim\rho=2$, we get a). Given $k\geq 6$, we get

$$\begin{aligned}\dim_{\mathbb{C}}\,[\mathrm{MJ}_2(\mathbb{Z}),k,\tfrac{5}{2},\nu_{12,1}] &= \sharp\{x,y\in\mathbb{N}_0, 4x+6y=k-4\}+\sharp\{x,y\in\mathbb{N}_0,4x+6y=k-6\}\\ &= \begin{cases}\left[\frac{k-4}{12}\right]+\left[\frac{k-6}{12}\right]+1, & k\equiv 6,8 \bmod 12,\\ \left[\frac{k-4}{12}\right]+\left[\frac{k-6}{12}\right]+2, & k\not\equiv 6,8 \bmod 12.\end{cases}\end{aligned}$$

It is easily verified that this is equal to the desired result. □

Using the same method we can determine the generators in the remaining cases:

Case 2: $k+1\in 2\mathbb{Z}, m=\frac{5}{2}, \nu=\nu_{12,1}$, hence $s=i$.

k	$\dim_{\mathbb{C}}$	generators
5	1	$\Phi_{5,5/2}:=[\vartheta_{1/2},\vartheta_{3/2}]\vartheta_{1/2}\eta^5$
7	1	$\Phi_{7,5/2}:=\vartheta_{1/2}^3E_{4,1}\eta^3=\left[\frac{\Phi_{12,1}}{\eta^{18}},\vartheta_{3/2}\right]\eta^5$
9	2	$g_4\Phi_{5,5/2}$, $\Phi_{9,5/2}:=\vartheta_{1/2}\Phi_{12,1}\eta^{-15}E_{4,1}$
11	2	$g_6\Phi_{5,5/2}, g_4\Phi_{7,5/2}$
13	3	$g_4\Phi_{9,5/2}, g_4^2\Phi_{5,5/2}, g_6\Phi_{7,5/2}$

Lemma 6.5.2 *a) The graded* $\mathbb{C}[g_4,g_6]$*-module* $[\mathrm{MJ}_2(\mathbb{Z}),2\mathbb{Z}+1,\frac{5}{2},\nu_{12,1}]$ *has a basis*

$$\left\{\Phi_{5,5/2},\,\Phi_{7,5/2},\,\Phi_{9,5/2}\right\}.$$

b) Let $k\geq 9$ *be odd, then*

$$\dim_{\mathbb{C}}\,[\mathrm{MJ}_2(\mathbb{Z}),k,\tfrac{5}{2},\nu_{12,1}]=\begin{cases}\left[\frac{k-5}{12}\right]+\left[\frac{k-7}{12}\right]+\left[\frac{k-9}{12}\right]+2, & k\equiv 7,9,11 \bmod 12,\\ \left[\frac{k-5}{12}\right]+\left[\frac{k-7}{12}\right]+\left[\frac{k-9}{12}\right]+3, & k\not\equiv 7,9,11 \bmod 12.\end{cases}$$

Case 3: $k\in 2\mathbb{Z}, m=5, \nu=1$, hence $s=-i$.

k	$\dim_{\mathbb{C}}$	generators
4	1	$E_{4,5}$
6	2	$E_{6,5}$, $\Phi_{6,5}:=\Delta\widehat{\Phi}_{-2,1}^3\widehat{\Phi}_{0,1}^2-g_4\Delta\widehat{\Phi}_{-2,1}^5$
8	3	$\Phi_{4,5/2}^2, g_4E_{4,5}, E_{4,2}E_{4,3}$
10	4	$g_4E_{6,5}, g_4\Phi_{6,5}, g_6E_{4,5}, \Phi_{4,5/2}\Phi_{6,5/2}$
12	5	$g_6E_{6,5}, g_6\Phi_{6,5}, g_4^2E_{4,5}, g_4\Phi_{4,5/2}^2, g_4E_{4,2}E_{4,3}$
14	6	$g_4\cdot$ (generators of weight 10), $g_6E_{4,2}E_{4,3}$, $g_6\Phi_{4,5/2}^2$

Lemma 6.5.3 *a) The graded* $\mathbb{C}[g_4, g_6]$*-module* $[\mathrm{MJ}_2(\mathbb{Z}), 2\mathbb{Z}, 5, 1]$ *has a basis*

$$\left\{E_{4,5},\, E_{6,5},\, \Phi_{6,5},\, \Phi_{4,5/2}^2,\, E_{4,2}E_{4,3},\, \Phi_{4,5/2}\Phi_{6,5/2}\right\}.$$

b) Let $k \geq 10$ *be even, then*

$$\dim_{\mathbb{C}}[\mathrm{MJ}_2(\mathbb{Z}), k, 5, 1] = \begin{cases} \left[\frac{k-4}{12}\right] + 2\left[\frac{k-6}{12}\right] + 2\left[\frac{k-8}{12}\right] + \left[\frac{k-10}{12}\right] + 4, & k \equiv 8, 10 \bmod 12, \\ \left[\frac{k-4}{12}\right] + 2\left[\frac{k-6}{12}\right] + 2\left[\frac{k-8}{12}\right] + \left[\frac{k-10}{12}\right] + 5, & k \equiv 6, 12 \bmod 12, \\ \left[\frac{k-4}{12}\right] + 2\left[\frac{k-6}{12}\right] + 2\left[\frac{k-8}{12}\right] + \left[\frac{k-10}{12}\right] + 6, & \text{otherwise}. \end{cases}$$

Case 4: $k + 1 \in 2\mathbb{Z}, m = 5, \nu = 1$, hence $s = i$.

k	$\dim_{\mathbb{C}}$	generators
5	1	$\Phi_{5,5} := \vartheta_{3/2}\vartheta_{1/2}^7\eta^2$
7	1	$\Phi_{7,5} := \vartheta_{3/2}\vartheta_{1/2}^5\Phi_{12,1}\eta^{-16}$
9	2	$g_4\Phi_{5,5},\, \Phi_{4,5/2}\Phi_{5,5/2}$
11	3	$g_4\Phi_{7,5},\, g_6\Phi_{5,5},\, \Phi_{11,5} := \vartheta_{3/2}^3\vartheta_{1/2}\eta^{18}$
13	3	$g_4\Phi_{4,5/2}\Phi_{5,5/2},\, g_6\Phi_{7,5},\, g_4^2\Phi_{5,5}$

Lemma 6.5.4 *a) The graded* $\mathbb{C}[g_4, g_6]$*-module* $[\mathrm{MJ}_2(\mathbb{Z}), 2\mathbb{Z}+1, 5, 1]$ *has a basis*

$$\left\{\Phi_{5,5},\, \Phi_{7,5},\, \Phi_{4,5/2}\Phi_{5,5/2},\, \Phi_{11,5}\right\}.$$

b) Let $k \geq 11$ *be odd, then*

$$\dim_{\mathbb{C}}[\mathrm{MJ}_2(\mathbb{Z}), k, 5, 1] = \begin{cases} \left[\frac{k-5}{12}\right] + \left[\frac{k-7}{12}\right] + \left[\frac{k-9}{12}\right] + \left[\frac{k-11}{12}\right] + 3, & k \equiv 1, 7, 9, 11 \bmod 12, \\ \left[\frac{k-5}{12}\right] + \left[\frac{k-7}{12}\right] + \left[\frac{k-9}{12}\right] + \left[\frac{k-11}{12}\right] + 4, & \text{otherwise}. \end{cases}$$

Using Example 6.1.4, we easily get that

$$\vartheta_{1/2}(\tau, z)|_{z=m} \quad \text{and} \quad \vartheta_{3/2}(\tau, z)|_{z=(2m+1)/2}$$

have a zero of order 1 for all $m \in \mathbb{Z}$. Using Corollary 6.3.4 with $(M, Q) = (0, 1)$, $(0, 2)$, $(\frac{1}{2}\mathbb{Z}, 1)$ or $(\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, 2)$, we get the following zeros of $F = \mathrm{Lift}_\Phi$:

Φ	$\Phi_{4,5/2}$	$\Phi_{5,5/2}$	$\Phi_{6,5/2}$	$\Phi_{7,5/2}$	$\Phi_{9,5/2}$	$\Phi_{5,5}$	$\Phi_{7,5}$	$\Phi_{11,5}$
F	$2\lambda_1^\perp \cup \lambda_4^\perp$	$\lambda_1^\perp$	$\lambda_4^\perp$	$3\lambda_1^\perp$	$\lambda_1^\perp$	$7\lambda_1^\perp \cup \lambda_4^\perp$	$5\lambda_1^\perp \cup \lambda_4^\perp$	$\lambda_1^\perp \cup \lambda_4^\perp$

In the next chapter we will introduce another fundamental method to construct paramodular forms.

7 Borcherds products to $\Gamma_5^{\max}$

R. Borcherds ([Bor]) gave a method to construct orthogonal modular forms with known zeros on quadratic divisors, so called *Borcherds products*. Using Section 3.4 we can construct paramodular forms of degree 2 with certain multiplier systems.

7.1 Definition and First Results

In [Der, Cor 4.3] and [GN, Thm 2.1], Borcherds Theorem in the paramodular setting was already given. We use the first formulation:

Theorem 7.1.1 *Let $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho^*_{t,0}]_{\mathrm{mer}}$ with Fourier coefficients $c(l,n)$ for $l \in L'/L$ and $-\infty \ll n \in q_t(l) + \mathbb{Z}$. Assume $c(l,n) \in \mathbb{Z}$ for all $n \leq 0$ and $l \in L'/L$.*

a) There exists a (meromorphic) modular form $\mathcal{B}(f)$ of weight $\frac{c(0,0)}{2}$ for Γ_t^* with some multiplier system, such that all zeros and poles of $\mathcal{B}(f)$ are along rational quadratic divisors $\lambda^\perp$, $\lambda \in L'$ primitive, $q_t(\lambda) < 0$, with multiplicity

$$\sum_{r\in\mathbb{N}} c(r\lambda, r^2 q_t(\lambda)).$$

b) Given $n_0 := -\min\{n \in \frac{1}{4t}\mathbb{Z}; c(l,n) \neq 0 \text{ for some } l \in L'/L\}$, we define

$$A = \frac{1}{24}\sum_{l\in\mathbb{Z}} c(\tfrac{l}{2t}, -\tfrac{l^2}{4t}), \qquad B = \frac{1}{2}\sum_{l\in\mathbb{N}} l\, c(\tfrac{l}{2t}, -\tfrac{l^2}{4t}),$$
$$C = \frac{1}{4}\sum_{l\in\mathbb{Z}} l^2 c(\tfrac{l}{2t}, -\tfrac{l^2}{4t}), \qquad D = \sum_{n\in\mathbb{N}, l\in\mathbb{Z}} \sigma_1(n) c(\tfrac{l}{2t}, -n - \tfrac{l^2}{4t}),$$

and set $\lambda_W = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$ (this is essentially the Weyl vector from [Bor, Thm 13.3]). Then $\mathcal{B}(f)$ has a product expansion, converging for $\mathrm{Im}(Z) > n_0 I_2$, of the form

$$\mathcal{B}(f)(Z) = e^{2\pi i \,\mathrm{trace}(\lambda_W Z)} \prod_{(m,n,l)>0} (1 - e^{2\pi i(n\tau + lz + tm\omega)})^{c(\frac{l}{2t}, mn - \frac{l^2}{4t})}$$

(here $(m,n,l) > 0$ means $m,n,l \in \mathbb{Z}$ with $m > 0$, ($m = 0$, $n > 0$) or ($m = n = 0$, $l < 0$). Moreover, the Borcherds products satisfy

$$\mathcal{B}(f)(V_t\langle Z\rangle) = (-1)^D \mathcal{B}(f)(Z)$$

Remark 7.1.2 *a) We corrected the formulation of* $(m,n,l)>0$ *in [Der, Cor 4.3]. Especially* n *is not always in* $\mathbb{N}_0$.

b) Let L *be as in Section 3.5. Then* $\rho^*_{t,0}$ *is essentially the Weil-representation* ρ_L *with respect to the quadratic module* $(L'/L, q_t \bmod \mathbb{Z})$ *as in [Bor]. This relation is explained in more detail in [Der].*

c) Borcherds products are multiplicative, i. e. let f, g *be two functions satifying the above assumptions, then* $\mathcal{B}(f+g)=\mathcal{B}(f)\cdot\mathcal{B}(g)$.

d) Every $f\in[\mathrm{Mp}_2(\mathbb{Z}),k,\rho^*_{t,0}]_{\mathrm{mer}}$ *has a Fourier expansion of the form*

$$f=\sum_{l\in\frac{1}{2t}\mathbb{Z}/\mathbb{Z}}\ \sum_{-\infty\ll n\in q_t(l)+\mathbb{Z}} c(l,n)e^{2\pi i n\tau} f^c_l.$$

e) We only have to consider $[\mathrm{Mp}_2(\mathbb{Z}),-\frac{1}{2},\rho^*_{t,0,-i}]_{\mathrm{mer}}$ *since* $[\mathrm{Mp}_2(\mathbb{Z}),-\frac{1}{2},\rho^*_{t,0,i}]_{\mathrm{mer}}=\{0\}$.

Example 7.1.3 *Let* f *have a pole at the cusp* $i\infty$ *of order less or equal to one. Then the following table lists the order* ord_Ψ *of the Borcherds product* $\Psi=\mathcal{B}(f)$ *associated to the form* $f\in[\mathrm{Mp}_2(\mathbb{Z}),-\frac{1}{2},\rho^*_{5,0,-i}]_{\mathrm{mer}}$ *with Fourier coefficients* $c(l,n)$*, along the divisors* $\lambda_j^\perp$ *from Example 3.6.2:*

j	Z	$\mathrm{ord}_\Psi(\lambda_j^\perp)$
1	$z=0$	$c(\frac{1}{10},-\frac{1}{20})+c(\frac{2}{10},-\frac{4}{20})+c(\frac{3}{10},-\frac{9}{20})+c(\frac{4}{10},-\frac{16}{20})$
4	$z=\frac{1}{2}$	$c(\frac{2}{10},-\frac{4}{20})+c(\frac{4}{10},-\frac{16}{20})$
5	$\tau-5z+5\omega=0$	$c(\frac{5}{10},-\frac{5}{20})+c(0,-1)$
9	$z=\frac{1}{3}$	$c(\frac{3}{10},-\frac{9}{20})$
16	$z=\frac{1}{4}$	$c(\frac{4}{10},-\frac{16}{20})$
20	$\tau=5\omega$	$c(0,-1)$

In order to construct Borcherds products explicitly, we need to find modular forms f in the vector space $[\mathrm{Mp}_2(\mathbb{Z}),\,-\frac{1}{2},\,\rho^*_{t,0,-i}]_{\mathrm{mer}}$ with given singularities:

1) We can use the fact

$$[\mathrm{Mp}_2(\mathbb{Z}),-\tfrac{1}{2},\rho^*_{t,0,-i}]_{\mathrm{mer}}=\sum_{n\in\mathbb{N}}\Delta_{12}^{-n}[\mathrm{Mp}_2(\mathbb{Z}),12n-\tfrac{1}{2},\rho^*_{t,0,-i}].$$

Using the isomorphism 6.2.2 and methods as in Section 6.5 it is possible to calculate the vector space $[\mathrm{Mp}_2(\mathbb{Z}),\,12n-\frac{1}{2},\,\rho^*_{t,0,-i}]$. Later we will use this fact to calculate the Fourier expansion of some Borcherds products.

2) First of all, we will instead use a method described by Borcherds ([Bor, Thm 3.1]) which gives a simple criterion for a given singularity at $i\infty$ of type

$$\Big(\sum_{\substack{n\in q_t(l)+\mathbb{Z}\\ -\infty\ll n\le 0}} h(l,n)q^n\Big)_{l\in L'/L}$$

to be extensible to a meromorphic form in $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho^*_{t,0}]_{\mathrm{mer}}$:
There exists an $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho^*_{t,0}]_{\mathrm{mer}}$ (with Fourier coefficients $c_f(l,n)$) such that

$$h(l,n) = c_f(l,n) \qquad \text{for all} \quad l \in \tfrac{1}{2t}\mathbb{Z}/\mathbb{Z} \text{ and } 0 \geq n \in q_t(l) + \mathbb{Z},$$

if and only if

$$\sum_{l \in \frac{1}{2t}\mathbb{Z}/\mathbb{Z}} \sum_{0 \geq n \in q_t(l)+\mathbb{Z}} h(l,n) c_g(l,-n) = 0 \quad \text{for all } g \in [\mathrm{Mp}_2(\mathbb{Z}), \tfrac{5}{2}, \rho_{t,0}]. \tag{7.1}$$

Note that one direction is trivial:
Given $f \in [\mathrm{Mp}_2(\mathbb{Z}), 2-k, \rho]_{\mathrm{mer}}$ and $g \in [\mathrm{Mp}_2(\mathbb{Z}), k, \rho^*]$, we have that the scalar product $(f,g) \in [\mathrm{SL}_2(\mathbb{Z}), 2, 1]_{\mathrm{mer}}$ and therefore $\sum_{l,n \leq 0} c_f(l,n) c_g(l,-n) = 0$. We call the vector space $[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_{t,0}]$ the *obstruction space* for $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho^*_{t,0}]_{\mathrm{mer}}$.

The following table lists the dimension of this obstruction space for small t:

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\dim_{\mathbb{C}}$	1	1	1	2	2	2	2	3	3	3	3	4	3	4	4	5	4	5

In view of [Bru], the Eisenstein series in the obstruction space determines the weight of a Borcherds product. As explained in [Der], finding generators for $\mathcal{A}(\Gamma_t^*)$ gets more and more involved, as the dimension of obstruction space increases. The complexity of this problem does not depend on this dimension only since it also depends on the fact how twisted the equation system (7.1) for the generators of the obstruction space is.

The rest of this chapter we consider the example $t = 5$ which is the first interesting example, where the dimension of the obstruction space is greater or equal to 2. Keep in mind that Γ_4 is conjugated to a subgroup of $\mathrm{Sp}_2(\mathbb{Z})$. An element of the obstruction space $[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_{t,0}]$ is the vector-valued Eisenstein series $E_{\frac{5}{2}}$ as given in [BK, (8)]. The Fourier coefficients (see [BK, Prop 11]) of $E_{\frac{5}{2}}$ are given by

(γ,n)	$(0,0)$	$(0,1)$	$(\frac{1}{10},\frac{1}{20})$	$(\frac{1}{5},\frac{1}{5})$	$(\frac{3}{10},\frac{9}{20})$	$(\frac{2}{5},\frac{4}{5})$	$(\frac{1}{2},\frac{1}{4})$
$\alpha(\gamma,n)$	1	$-\frac{264}{13}$	$-\frac{5}{13}$	$-\frac{35}{13}$	$-\frac{125}{13}$	$-\frac{275}{13}$	$-\frac{24}{13}$

Using $\Theta_{t,0} \in [\mathrm{Mp}_2(\mathbb{Z}), \frac{1}{2}, \rho_{t,0}]$ and Lemma 6.2.4 we know that

$$F := \frac{1}{\Delta}[\Theta_{t,0}, \Delta] \in [\mathrm{Mp}_2(\mathbb{Z}), \tfrac{5}{2}, \rho_{t,0}].$$

The first Fourier coefficients are given by

$$\begin{array}{rcl} F_0(\tau) &=& -\frac{1}{2} + 12q^1 + 36q^2 + 48q^3 + \mathcal{O}(q^4),\\ F_1(\tau) &=& \frac{1}{10}q^{\frac{1}{20}} + 12q^{\frac{21}{20}} + 36q^{\frac{41}{20}} + \mathcal{O}(q^{\frac{61}{20}}),\\ F_2(\tau) &=& \frac{19}{10}q^{\frac{1}{5}} + 12q^{\frac{6}{5}} + 36q^{\frac{11}{5}} + \mathcal{O}(q^{\frac{16}{5}}),\\ F_3(\tau) &=& \frac{49}{10}q^{\frac{9}{20}} + 12q^{\frac{29}{20}} + \frac{649}{10}q^{\frac{49}{20}} + \mathcal{O}(q^{\frac{69}{20}}),\\ F_4(\tau) &=& \frac{91}{10}q^{\frac{4}{5}} + \frac{331}{10}q^{\frac{9}{5}} + 48q^{\frac{14}{5}} + \mathcal{O}(q^{\frac{19}{5}}),\\ F_5(\tau) &=& 0q^{\frac{1}{4}} + 29q^{\frac{5}{4}} + 24q^{\frac{9}{4}} + \mathcal{O}(q^{\frac{13}{4}}). \end{array}$$

The following table lists some Borcherds products with weight at most 17 which we calculated using method 2 restricting the order of the pole at $i\infty$ to be less or equal to one.

weight	$\lambda_1^\perp$	$\lambda_4^\perp$	$\lambda_5^\perp$	$\lambda_9^\perp$	$\lambda_{16}^\perp$	$\lambda_{20}^\perp$	ν	A	B	C	D
4	2	1	1	0	0	0	$\kappa_{1,2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	0
5	7	1	0	0	0	0	χ_5	1	4	5	0
10	2	0	0	1	0	0	1	1	2	5	0
11	1	3	4	0	0	0	χ_5	1	2	5	0
12	0	0	3	0	0	1	χ_5	1	0	0	1
13	5	0	2	0	0	1	$\kappa_{1,2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	1
14	10	0	1	0	0	1	χ_5	2	5	5	1
15	15	0	0	0	0	1	$\kappa_{1,2}$	$\frac{5}{2}$	$\frac{15}{2}$	$\frac{15}{2}$	1
17	1	2	3	1	0	0	$\kappa_{1,2}\chi_5$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{15}{2}$	0

These Borcherds products will be denoted by Ψ_k with the weight k as index.

You can find some more Borcherds products in the Appendix 8.11. The multiplier-system can be determined directly by Theorem 7.1.1 as follows:
1) $\Psi_k|_k V_5 = (-1)^{k+D}\Psi_k$, hence $\nu(V_5) = (-1)^{k+D} = \chi_5(V_5)^{k+D}$.
2) From the product expansion we deduce $\Psi_k(Z + \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)) = e^{2\pi iA}\Psi_k(Z)$ and therefore we get $\nu(\text{trans}\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)) = e^{2\pi iA} = \kappa_{1,2}(\text{trans}\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right))^{2A}$.
Altogether the character is uniquely determined by these conditions and we get

$$\nu = \chi_5^{k+D}\kappa_{1,2}^{2A}.$$

Remark 7.1.4 *a) First of all, we recognize that we constructed Borcherds products for every character* $\nu \in (\Gamma_5^{\max})^{\text{ab}}$*, e. g.* Ψ_4 *for* $\kappa_{1,2}$*,* Ψ_5 *for* χ_5*,* Ψ_{10} *for* 1 *and* Ψ_{17} *for* $\kappa_{1,2}\chi_5$*.*
b) There are two skew-symmetric forms of even weight, e. g. Ψ_{12} *and* Ψ_{14}*.*

c) *Given any divisor $\lambda^\perp$ of discriminant $0 > q(\lambda^\perp) \geq -1$ we have a Borcherds product Ψ_λ with $\mathrm{ord}_{\lambda^\perp}\Psi_\lambda = 1$, e. g. Ψ_4 for $\lambda_4^\perp$ and $\lambda_5^\perp$, Ψ_{10} for $\lambda_9^\perp$, Ψ_{11} for $\lambda_1^\perp$, Ψ_{12} for $\lambda_{20}^\perp$ and Ψ_{29} for $\lambda_{16}^\perp$. Unfortunately these Borcherds products always have other zeros which forces a reduction process to be more complicated.*

Now we want to identify the input of some Borcherds products. Therefore we recall that one can think of these Borcherds products as a lift from $[\mathrm{MJ}_2(\mathbb{Z}),12,5,1]$ (via the isomorphism from Lemma 6.2.2 and method 1): If $f \in [\mathrm{Mp}_2(\mathbb{Z}),-\frac{1}{2},\rho_{t,0}^*]$ has a pole of order less or equal to one at the cusp $i\infty$, then

$$f\Delta \in [\mathrm{Mp}_2(\mathbb{Z}),12-\tfrac{1}{2},\rho_{t,0}^*] \quad \Longleftrightarrow \quad (f\Delta,\Theta_{t,0}) \in [\mathrm{MJ}_2(\mathbb{Z}),12,5,1]. \tag{7.2}$$

The latter $\mathbb{C}$-vector space has a basis $\{g_6E_{6,5}, g_6\Phi_{6,5}, g_4^2E_{4,5}, g_4^2\Phi_{4,5/2}^2, g_4E_{4,2}E_{4,3}\}$ (compare with Section 6.4). Using Example 7.1.3 we get for the input function f of Ψ_{10} the Fourier coefficients $c(\frac{1}{10},-\frac{1}{20})=1$, $c(\frac{3}{10},-\frac{9}{20})=1$, $c(0,0)=20$ and all other $c(l,n)=0$ for $n \leq 0$. Thus

$$(f\Delta,\Theta_{t,0}) = q(r^{\pm 3}+r^{\pm 1}+20)+\mathcal{O}(q^2).$$

Moreover, the Fourier coefficient at $q^2r^{\pm 5}$ is zero. Therefore, we can identify

$$(f\Delta,\Theta_{t,0}) = -\tfrac{1}{72}g_6E_{6,5}-\tfrac{1}{1563}g_6\Phi_{6,5}+\tfrac{1}{72}g_4^2E_{4,5}+\tfrac{2}{3}g_4\Phi_{4,5/2}^2.$$

With this method we can also easily identify the following inputs for Ψ_k:

weight	$g_6E_{6,5}$	$g_6\Phi_{6,5}$	$g_4^2E_{4,5}$	$g_4\Phi_{4,5/2}^2$	$g_4E_{4,2}E_{4,3}$
11	$-\frac{1}{72}$	$-\frac{713}{300096}$	$\frac{1}{18}$	$-\frac{5}{6}$	$-\frac{1}{24}$
12	$\frac{5}{12}$	$-\frac{685}{100032}$	$\frac{5}{24}$	$\frac{1}{2}$	$\frac{3}{8}$
13	$\frac{59}{144}$	$-\frac{815}{150048}$	$\frac{25}{144}$	$\frac{5}{6}$	$\frac{5}{12}$
14	$\frac{29}{72}$	$-\frac{1205}{300096}$	$\frac{5}{36}$	$\frac{7}{6}$	$\frac{11}{24}$
15	$\frac{19}{48}$	$-\frac{65}{25008}$	$\frac{5}{48}$	$\frac{3}{2}$	$\frac{1}{2}$
17	$-\frac{1}{48}$	$-\frac{809}{300096}$	$\frac{1}{16}$	0	$-\frac{1}{24}$

Using the Fourier expansion of these functions, the (explicit) isomorphism in Equation (6.2), and Equation (7.2) we can calculate the exact Fourier expansion of the input $f_\Psi \in$

$[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho^*_{t,0}]$, e. g. for Ψ_{10} we get

$$\begin{aligned}
f_0(\tau) &= 20 + 1200q^1 + 20720q^2 + 204160q^3 + \mathcal{O}(q^4),\\
f_1(\tau) &= q^{-\frac{1}{20}} - 320q^{\frac{19}{20}} - 5633q^{\frac{39}{20}} - 56768q^{\frac{59}{20}} + \mathcal{O}(q^{\frac{79}{20}}),\\
f_2(\tau) &= - 492q^{\frac{4}{5}} - 10044q^{\frac{9}{5}} - 107616q^{\frac{14}{5}} + \mathcal{O}(q^{\frac{19}{5}}),\\
f_3(\tau) &= q^{-\frac{9}{20}} + 192q^{\frac{11}{20}} + 5188q^{\frac{31}{20}} + 61824q^{\frac{51}{20}} + \mathcal{O}(q^{\frac{71}{20}}),\\
f_4(\tau) &= + 20q^{\frac{1}{5}} + 688q^{\frac{6}{5}} + 10480q^{\frac{11}{5}} + \mathcal{O}(q^{\frac{16}{5}}),\\
f_5(\tau) &= - 510q^{\frac{3}{4}} - 10880q^{\frac{7}{4}} - 119300q^{\frac{11}{4}} + \mathcal{O}(q^{\frac{15}{4}}).
\end{aligned}$$

Note that in this example we have $c_f(0,-1) = 0$. We can now calculate the Fourier coefficients for the Borcherds product Ψ if we use the following facts:

- The condition $(m,n,l) > 0$ allows only negative n if $m > 0$, so that there are only finitely many cases in which n is negative. In the particular case that f has a pole of order at most 1 at $i\infty$, we only have one possibility: $(m,n,l) = (1,-1,0)$.
- In the case that $c(\frac{l}{2t}, mn - \frac{l^2}{4t}) < 0$ we use the geometric series $(1-q)^{-1} = \sum_{n=0}^{\infty} q^n$.
- We recognize that all $c(n,l) \in \mathbb{Z}$ for all $n \in \mathbb{N}$ and $l \in L'/L$ which follows from $c(n,l) \in \mathbb{Z}$ for all $n < 0$ and $l \in L'/L$.

In the above example we get (using the abbreviations $q_1 = e^{2\pi i\tau}$, $q_2 = e^{2\pi i 5\omega}$, $r = e^{2\pi i z}$)

$$\begin{aligned}
\Psi_{10}(Z) = {} & q_1q_2(r^{\pm 2} - r^{\pm 1}) + (q_1q_2^2 + q_1^2q_2)(-r^{\pm 5} + r^{\pm 4} - r^{\pm 3} - 18r^{\pm 2} + 18r^{\pm 1} + 2)\\
& + q_1^2q_2^2(r^{\pm 8} - r^{\pm 7} - 18r^{\pm 6} - 135r^{\pm 5} + 648r^{\pm 5} - 117r^{\pm 3})\\
& + q_1^2q_2^2(-1022r^{\pm 2} + 509r^{\pm 1} + 270) + \mathcal{O}(q_1^3, q_2^3),
\end{aligned}$$

where $r^{\pm k} = r^k + r^{-k}$ and $\mathcal{O}(q_1^3, q_2^3)$ means that every remaining term can be divided by one of these polynomials in q_1 and q_2. Furthermore,

$$\begin{aligned}
\Psi_{11}(Z) = {} & q_1q_2(\pm r^{\pm 2} \pm 2r^{\pm 1}) + (q_1^2q_2 - q_1q_2^2)(\pm 3r^{\pm 4} \pm 4r^{\pm 3} \pm 18r^{\pm 2} + 36r^{\pm 1})\\
& + \mathcal{O}(q_1^3, q_1^2q_2^2, q_2^3),\\
\Psi_{12}(Z) = {} & (q_1 - q_2) - 24(q_1^2 - q_2^2) + (q_1q_2^2 - q_2q_1^2)(2r^{\pm 5} + 275r^{\pm 4})\\
& + (q_1q_2^2 - q_2q_1^2)(4050r^{\pm 3} + 19450r^{\pm 2} + 45100r^{\pm 1} + 58830) + \mathcal{O}(q_1^3, q_1^2q_2^2, q_2^3),\\
\Psi_{14}(Z) = {} & (q_1q_2^2 - q_1^2q_2)(-r^{\pm 5} + 10r^{\pm 4} - 45r^{\pm 3} + 120r^{\pm 2} - 210r^{\pm 1} + 252)\\
& + \mathcal{O}(q_1^3, q_1^2q_2^2, q_2^3),
\end{aligned}$$

where we use the abbreviation $\pm r^{\pm k} = r^k - r^{-k}$.

If we look at the above table of Borcherds products a little bit closer, we get conjectures about relations between the character, the weight and zeros on certain quadratic divisors:

7.2 More on Zeros for Paramodular Forms of Level $t = 5$

Note that modular forms of weight k with multiplier-system v necessarily have zeros along certain quadratic divisors $\lambda^\perp$ if this divisor is fixed pointwise by a transformation $M_\lambda \in \Gamma_5^{\max}$, such that $j_2(M_\lambda, Z)^{-k} \neq v(M_\lambda)$ for all $Z \in \lambda^\perp$. Here are some examples of such transformations for some of the rational quadratic divisors of discriminant less or equal to one:

$$\begin{aligned}
Z \in \lambda_1^\perp \quad &\Longrightarrow \quad Z = Z\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right], \\
Z \in \lambda_4^\perp \quad &\Longrightarrow \quad Z = Z\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
Z \in \lambda_{20}^\perp \quad &\Longrightarrow \quad Z = Z\left[\begin{pmatrix} 0 & 1/\sqrt{5} \\ \sqrt{5} & 0 \end{pmatrix}\right].
\end{aligned}$$

In all cases we have $j_2(M_\lambda, Z) = -1$. Thus any $f \in \mathbb{M}_k(\Gamma_5^{\max}, \chi_5^j \kappa_{1,2}^l)$ satisfies

$$f(Z) = 0 \quad \text{on} \quad \begin{cases} \lambda_1^\perp, & \text{if } (-1)^k \neq 1, \\ \lambda_4^\perp, & \text{if } (-1)^k \neq (-1)^l, \\ \lambda_{20}^\perp, & \text{if } (-1)^k \neq (-1)^j. \end{cases} \tag{7.3}$$

Especially, we have

$$\begin{aligned}
f = 0 \text{ on } \lambda_1^\perp &\quad \text{for } f \in \mathbb{M}_{2k+1}(\Gamma_5^{\max}, v) \text{ and all } k \in \mathbb{Z},\ v \in (\Gamma_5^{\max})^{\mathrm{ab}}, \\
f = 0 \text{ on } \lambda_4^\perp &\quad \text{for } f \in \mathbb{M}_{2k}(\Gamma_5^{\max}, \kappa_{1,2}\chi_5^j) \text{ and all } k,\ j \in \mathbb{Z}, \\
f = 0 \text{ on } \lambda_4^\perp &\quad \text{for } f \in \mathbb{M}_{2k+1}(\Gamma_5^{\max}, \chi_5^j) \text{ and all } k,\ j \in \mathbb{Z}, \\
f = 0 \text{ on } \lambda_{20}^\perp &\quad \text{for } f \in \mathbb{M}_{2k}(\Gamma_5^{\max}, \chi_5 \kappa_{1,2}^l) \text{ and all } k,\ l \in \mathbb{Z}, \\
f = 0 \text{ on } \lambda_{20}^\perp &\quad \text{for } f \in \mathbb{M}_{2k+1}(\Gamma_5^{\max}, \kappa_{1,2}^l) \text{ and all } k,\ l \in \mathbb{Z}.
\end{aligned}$$

Moreover, we know that the other divisors with discriminant less or equal to one do not have such a fixpoint transformation. Using Borcherds products we detect that all paramodular forms have zeros on certain divisors:

Lemma 7.2.1 *Let $f \in \mathbb{M}_k(\Gamma_5^{\max}, \kappa_{1,2}^l \chi_5^j)$. Then*

$$\operatorname{ord}_f \lambda_1^\perp \equiv k \bmod 2, \quad \operatorname{ord}_f \lambda_4^\perp \equiv k + l \bmod 2 \quad \textit{and} \quad \operatorname{ord}_f \lambda_{20}^\perp \equiv k + j \bmod 2. \tag{7.4}$$

Proof: 1) Given $\operatorname{ord}_f \lambda_1^\perp = m$, we consider

$$f^* = f \cdot \left(\Phi_5^{-1}\Phi_4^3\right)^m \in \mathbb{M}_{k+7m}(\Gamma_5^{\max}, v\kappa_{1,2}^m \chi_5^m).$$

Due to $\operatorname{ord}_{f^*} \lambda_1^\perp = 0$ and Equation (7.3), we get $(-1)^{k+7m} = 1$. This leads to $k \equiv m$ mod 2.

2) Given $\operatorname{ord}_f \lambda_4^\perp = m$, we consider

$$f^* = f\cdot\left(\Phi_{10}^4\Phi_5^{-1}\right)^m \in \mathbb{M}_{k+35m}(\Gamma_5^{\max}, \nu\chi_5^m).$$

Due to $\operatorname{ord}_{f^*} \lambda_4^\perp = 0$ and Equation (7.3), we get $(-1)^{k+35m} = (-1)^l$. This leads to $k+l \equiv m \bmod 2$.

3) Given $\operatorname{ord}_f \lambda_{20}^\perp = m$, we consider

$$f^* = f\cdot(\Phi_{12}^{-1}\Phi_{11})^m \in \mathbb{M}_{k-m}(\Gamma_5^{\max}, \nu).$$

Due to $\operatorname{ord}_{f^*} \lambda_{20}^\perp = 0$ and Equation (7.3), we get $(-1)^{k-m} = (-1)^j$. This leads to $k+j \equiv m \bmod 2$. $\square$

Using the Table 6.5 we get $\mathrm{Lift}_{\Phi_{5,5}} = c\cdot\Psi_5$ with some $c \neq 0$. Without loss of generality, we normalize Ψ_5 such that $c=1$. In [GN, p. 53] it is shown that $\mathrm{Lift}_{\Phi_{4,5/2}} = \Psi_4$ by identifying the exact zero divisor of $\mathrm{Lift}_{\Phi_{4,5/2}}$. The divisor $\lambda_5^\perp$ is twisted in such a way that I could not calculate the Fourier expansion of $\mathrm{Lift}_{\Phi_{4,5/2}}|_{z\in\lambda_5^\perp}$, since you need an infinite number of Fourier coefficients for the Lift to calculate a Fourier coefficient of the restriction. Therefore this divisor is in this sense far more complicated than the divisors $\{Z \in \mathbb{H}_2; z = \alpha\}$.

7.3 The Hilbert Series $\mathrm{H}(\mathcal{A}(\Gamma_5), s)$

Given $t \in \mathbb{N}$, we define the Witt-operator W_t on functions $f : \mathbb{H}_2 \to \mathbb{C}$ by

$$\mathrm{W}_t(f)(\tau_1,\tau_2) := f\begin{pmatrix}\tau_1 & 0\\ 0 & \frac{\tau_2}{t}\end{pmatrix}.$$

From the embedding $\times_t$: $\mathrm{SL}_2(\mathbb{Z})^2 \to \Gamma_t$ defined by

$$M_1 \times_t M_2 := M_1 \times \begin{pmatrix} a_2 & \frac{b_2}{t}\\ tc_2 & d_2\end{pmatrix} \quad \text{for} \quad M_2 = \begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix},$$

we get that $f \in \mathbb{M}_k(\Gamma_t, \nu)$ implies that $\mathrm{W}_t(f) \in \mathbb{M}_k(\mathrm{SL}_2(\mathbb{Z})^2, \widetilde{\nu})$ with $\widetilde{\nu}(M_1, M_2) = \nu(M_1 \times_t M_2)$. Therefore

$$\mathrm{W}_t : \bigoplus_{\nu\in\Gamma_t^{\mathrm{ab}}} \mathcal{A}(\Gamma_t,\nu) \to \bigoplus_{\nu\in\Gamma_t^{\mathrm{ab}}} \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2, \widetilde{\nu})$$

is an homomorphism of graded algebras. Moreover, we have

$$\begin{aligned}\mathrm{W}_t(f|_k V_t)(\tau_1,\tau_2) &= (-1)^k f\left(V_t\cdot\begin{pmatrix}\tau_1 & 0\\ 0 & \frac{\tau_2}{t}\end{pmatrix}\right) = (-1)^k f\begin{pmatrix}\tau_2 & 0\\ 0 & \frac{\tau_1}{t}\end{pmatrix}\\ &= (-1)^k \mathrm{W}_t(f)(\tau_2,\tau_1).\end{aligned}$$

We recognize that

$$\mathrm{W}_t(f)(\tau_1,\tau_2) = \left(f|_{\lambda_1^\perp}\right)(\tau_1,\tfrac{\tau_2}{t}). \tag{7.5}$$

Since there are no modular forms of odd weight in $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)$, we get for all $t\in\mathbb{N}$

$$\mathrm{ord}_f\,\lambda_1^\perp \geq 1 \quad \text{for } k \text{ odd and } f\in\mathbb{M}_k(\Gamma_t),$$

which is in the case of $t=5$ a weaker form of Lemma 7.2.1.

In the case $t=5$, we have the following

Example 7.3.1 *a) Because of*

$$(\Psi_{12}|V_5)(Z) = (-1)^{12}\Psi_{12}(Z[U_5^{\mathrm{tr}}]) = -\Psi_{12}(Z),$$

we know that $(\Psi_{12}|_{\lambda_1^\perp})(\tau_1,\frac{\tau_2}{5})$ *is a skew-symmetric form. Using Theorem 8.6.1 we know that*

$$\mathrm{W}_5(\Psi_{12})(\tau_1,\tau_2) = (\Psi_{12}|_{\lambda_1^\perp})(\tau_1,\tfrac{\tau_2}{5}) = c_{12}\left(g_4^3(\tau_1)g_6^2(\tau_2)-g_6^2(\tau_1)g_4^3(\tau_2)\right)$$

with $c_{12}\neq 0$ *(since* $\mathrm{trace}(\lambda_W Z)=\tau$*). Without loss of generality, we renormalize* Ψ_{12} *such that* $c_{12}=1$.

b) Using the Fourier coefficients or Equation (7.5) we determine $\mathrm{W}_5(\Psi_5)=0$ *and*

$$\mathrm{W}_5(\mathrm{Lift}_{E_{4,5}})(\tau_1,\tau_2) = \tfrac{1}{240}g_4(\tau_1)g_4(\tau_2), \quad \mathrm{W}_5(\mathrm{Lift}_{E_{6,5}})(\tau_1,\tau_2) = -\tfrac{1}{504}g_6(\tau_1)g_6(\tau_2).$$

c) Using Fourier coefficients we get

$$\mathrm{W}_5\left(\tfrac{2496}{7}\,\mathrm{Lift}_{g_6E_{6,5}} - \tfrac{7}{5}\,\mathrm{Lift}^3_{E_{4,5}} - \tfrac{53}{147}\,\mathrm{Lift}^2_{E_{6,5}}\right)(\tau_1,\tau_2) = g_4^3(\tau_1)g_6^2(\tau_2)+g_6^2(\tau_1)g_4^3(\tau_2).$$

d) Using $\Phi_{6,5/2} = q^{1/2}r^{\pm 3/2}+11q^{1/2}r^{\pm 1/2}+\mathcal{O}(q^{3/2})$ *we determine*

$$\mathrm{W}_5(\mathrm{Lift}_{\Phi_{6,5/2}})(\tau_1,\tau_2) = 24\eta^{12}(\tau_1)\eta^{12}(\tau_2).$$

Therefore we get

Corollary 7.3.2 *a) The Witt-operator* $\mathrm{W}_5:\mathcal{A}(\Gamma_5)\to\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)$ *is a surjective homomorphism of graded algebras and* $\mathrm{W}_5(\mathcal{A}(\Gamma_5,\kappa_{1,2})) = \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2,\nu_\eta^{12}\times\nu_\eta^{12})$.

b) The Φ*-operator* $\Phi:\mathcal{A}(\Gamma_5)\to\mathcal{A}(\mathrm{SL}_2(\mathbb{Z}))$, $f\mapsto f|\Phi$ *is a surjective homomorphism of vector spaces.*

Proof: We only have to check b): We consider the Φ-operator which maps from the algebra $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)$ to $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z}))$ (since this is a standard notation and since the arguments make clear which of the two operators is meant, we do not make a distinction between these operators). This operator is an homomorphism of algebras and obviously surjective, since

$$(g_4^a(\tau_1)g_6^b(\tau_1)g_4^a(\tau_2)g_6^b(\tau_2))|\Phi = g_4^a(\tau_1)g_6^b(\tau_1).$$

Using a) and $f|\Phi = \mathrm{W}_t(f)|\Phi$ we get the desired result. $\square$

Proposition 7.3.3

$$\dim_{\mathbb{C}} \mathbb{M}_k(\Gamma_5) - \dim_{\mathbb{C}} \mathbb{S}_k(\Gamma_5) = \begin{cases} \dim_{\mathbb{C}} \mathbb{M}_k(\mathrm{SL}_2(\mathbb{Z})) + \dim_{\mathbb{C}} \mathbb{S}_k(\mathrm{SL}_2(\mathbb{Z})), & k \textit{ even,} \\ 0, & k \textit{ odd.} \end{cases}$$

Proof: Since there are no modular forms of odd weight in $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)$, we get $f|\Phi = \mathrm{W}_5(f)|\Phi = 0$ for $f \in \mathbb{M}_{2k+1}(\Gamma_5)$. Given $f \in \mathcal{A}(\Gamma_5)$, we have $f|V_5 \in \mathcal{A}(\Gamma_5)$ and therefore $f|\Phi = (f|V_5)|\Phi = 0$ for $f \in \mathbb{M}_{2k+1}(\Gamma_5)$. From Example 2.5.15 we know that

$$f \text{ is a cusp form} \quad \Longleftrightarrow \quad f|\Phi = 0 \quad \text{and} \quad (f|V_5)|\Phi = 0,$$

so that $f \in \mathbb{M}_{2k+1}(\Gamma_5)$ is always a cusp form.
Let k be even. Then we can decompose f in $f = f_{\mathrm{sym}} + f_{\mathrm{skew}} + f_0$, where $\mathrm{W}_5(f_0) = 0$, $\mathrm{W}_5(f_{\mathrm{sym}})$ is symmetric and $\mathrm{W}_5(f_{\mathrm{skew}})$ is skew-symmetric. This decomposition is uniquely if we fix a transversale of preimages of W_5, e. g. the one from Example 7.3.1. If $\mathrm{W}_5(f)$ is (skew)-symmetric, then $\mathrm{W}_5(f) = \pm \mathrm{W}_5(f|V_5)$. Hence in this case,

$$f|\Phi \neq 0 \text{ or } (f|V_5)|\Phi \neq 0 \text{ if and only if } W_5(f)|\Phi \neq 0.$$

Since $\mathrm{W}_5 : \mathcal{A}(\Gamma_5) \to \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)$ is surjective and

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)_{\mathrm{sym}} \cap \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)_{\mathrm{skew}} = \{0\},$$

Theorem 8.6.1 leads to

$$\begin{aligned} \dim_{\mathbb{C}} \mathbb{M}_k(\Gamma_5) - \dim_{\mathbb{C}} \mathbb{S}_k(\Gamma_5) &= \dim_{\mathbb{C}} \mathbb{M}(\mathrm{SL}_2(\mathbb{Z})^2)_{\mathrm{sym}}|\Phi + \dim_{\mathbb{C}} \mathbb{M}(\mathrm{SL}_2(\mathbb{Z})^2)_{\mathrm{skew}}|\Phi \\ &= \dim_{\mathbb{C}} \mathbb{M}_k(\mathrm{SL}_2(\mathbb{Z})) + \dim_{\mathbb{C}} \mathbb{S}_k(\mathrm{SL}_2(\mathbb{Z})). \end{aligned}$$ □

Ibukiyama ([Ib1]) calculated a dimension formula for paramodular cusp forms for weight $k \geq 5$, so that we are able to determine the following table

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{C}} \mathbb{M}_k(\Gamma_5)$	1	0	0	0	1	1	2	1	3	2	5	4	9
$\dim_{\mathbb{C}} \mathbb{S}_k(\Gamma_5)$	0	0	0	0	0	1	1	1	2	2	4	4	6
$\dim_{\mathbb{C}} \mathbb{M}_{k+13}(\Gamma_5)$	5	10	8	16	12	21	16	28	22	36	30	48	37
$\dim_{\mathbb{C}} \mathbb{S}_{k+13}(\Gamma_5)$	5	9	8	13	12	18	16	25	22	33	30	43	37
$\dim_{\mathbb{C}} \mathbb{M}_{k+26}(\Gamma_5)$	56	48	72	60	87								
$\dim_{\mathbb{C}} \mathbb{S}_{k+26}(\Gamma_5)$	53	48	67	60	82								

Ibukiyama ([Ib2]) used some ad-hoc arguments to determine the dimension for small weights and showed that either ($\dim_{\mathbb{C}} \mathbb{M}_4(\Gamma_5) = 1$ and $\dim_{\mathbb{C}} \mathbb{S}_4(\Gamma_5) = 0$) or ($\dim_{\mathbb{C}} \mathbb{M}_4(\Gamma_5) = 2$ and

$\dim_{\mathbb{C}} \mathbb{S}_4(\Gamma_5) = 1)$. Since we constructed all forms of weight 8, we are able to decide which case is true:
The set $\{\mathrm{Lift}_{\Phi^2_{4,5/2}}, \mathrm{Lift}_{g_4 E_{4,5}} \mathrm{Lift}_{E_{4,2}E_{4,3}}\}$ is a basis of $\mathbb{M}_8(\Gamma_5)$ since the Maaß lift is injective and linear for trivial character. Let $f_1 \in \mathbb{S}_4(\Gamma_5)$, then

$$f_1(Z) = \sum_{\substack{n,m,l \\ 4nmt > l^2}} \alpha_{f_1}(m,n,l) e^{2\pi i(n\tau + mt\omega + lz)}.$$

Since $f_1^2 \in \mathbb{S}_8(\Gamma_5)$, we can express f_1^2 in the above basis. Moreover,

$$\alpha_{f_1^2}(1,m,l) = \sum_{n_1=0}^{n} \sum_{4n_1 t > l_1^2} \alpha_{f_1}(1,n_1,l_1)\alpha_{f_1}(0,n-n_1,0) = 0.$$

Therefore we have to solve the following equation system:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_{f_1^2}(0,0,0) \\ \alpha_{f_1^2}(1,1,-4) \\ \alpha_{f_1^2}(1,1,-3) \end{pmatrix} = \begin{pmatrix} \frac{1}{480} & 0 & \frac{1}{480} \\ 1 & 0 & 0 \\ 300 & 4 & 158 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

which leads to $\alpha = \beta = \gamma = 0$, hence $f_1 = 0$.

In this case (i. e. dimension $\mathbb{M}_4(\Gamma_5) = 1$ which we showed above) Ibukiyama ([Ib2]) calculated the Hilbert series for $\mathcal{A}(\Gamma_5)$ summing the dimensions of $\mathbb{M}_k(\Gamma_5)$.

Corollary 7.3.4

$$\mathrm{H}(\mathcal{A}(\Gamma_5),s) = \sum_{k=0}^{\infty} \dim_{\mathbb{C}} \mathbb{M}_k(\Gamma_5) s^k = \frac{P(s)}{(1-s^4)(1-s^5)(1-s^6)(1-s^{12})}$$

with

$$P(s) = 1 + s^6 + s^7 + 2s^8 + s^9 + 2s^{10} + s^{11} + 2s^{12} + 2s^{14} \\ + 2s^{16} + 2s^{18} + s^{19} + 2s^{20} + s^{21} + 2s^{22} + s^{23} + s^{24} + s^{30}.$$

Remark 7.3.5 *a) Due to notation conflicts in this section we use s (instead of t) as variable in the Hilbert series.*

b) The above numerator polynom is a palindrome – i. e. a polynomial $Q(s) = \sum_{i=0}^{n} a_i s^i$ with $a_{n-i} = a_i$ for $1 \le i \le n$ – which probably implies some additional information about the structure of $\mathcal{R} = \mathcal{A}(\Gamma_5)$. More explicitly we get

$$\mathrm{H}(\mathcal{R}, 1/s) = s^3\, \mathrm{H}(\mathcal{R}, s)$$

By [DKe, p. 102] a Noetherian graded domain $\mathcal{R}$ over a field $F = \mathcal{R}_0$ is Gorenstein *if and only if it is Cohen-Macaulay and the Hilbert series satisfies the identity*

$$\mathrm{H}(\mathcal{R}, 1/s) = (-1)^{\dim \mathcal{R}} s^l\, \mathrm{H}(\mathcal{R}, s) \quad \textit{for some } l \in \mathbb{Z}.$$

If additionally $l = \dim \mathcal{R}$ we speak of strongly Gorenstein. *Therefore, if $\mathcal{R}$ is Cohen-Macaulay, we have that it is Gorenstein, but not strongly Gorenstein. For more information on Gorenstein rings I refer the interested reader to [Eis, Chap 21].*

c) *If $\mathcal{A}(\Gamma_5)$ is a Cohen-Macaulay ring, we could use the Hilbert series to determine primary and secondary generators, e. g. we could hope that we find a set of primary generators in the weights $(4,5,6,12)$ and therefore only have to calculate modular forms up to weight 30. The paramodular group is isomorphic to $\mathrm{P}\Gamma_S$ (Corollary 3.3.4), hence in particular is not compact, and it seems unlikely that algebraic techniques as described in Section 5 could be apllied successfully. Γ_5 contains the main congruence group $\mathrm{Sp}_2(\mathbb{Z})[5]$ with finite index. In the case that $\mathcal{A}(\mathrm{Sp}_2(\mathbb{Z})[5])$ is Cohen-Macaulay, we could use Proposition 5.3.7 to prove that $\mathcal{A}(\Gamma_5)$ is Cohen-Macaulay. Unfortunately, this problem has not been solved yet. Shigeaki Tsuyumine ([Tsu1], [Tsu2]) showed that $\mathcal{A}(\mathrm{Sp}_2(\mathbb{Z})[l])$ is not Cohen-Macaulay for $l \geq 6$ and that $\mathcal{A}(\mathrm{Sp}_2(\mathbb{Z}))$ is Cohen-Macaulay. Unfortunately his method did not work for $l = 5$.*

Every set of primary generators is algebraically independent. Therefore we are looking for an algebraic independent set of paramodular forms of weight 4, 5, 6 and 12:

Corollary 7.3.6 *The paramodular forms* $\mathrm{Lift}_{E_{4,5}}$, Ψ_5, $\mathrm{Lift}_{E_{6,5}}$ *and* Ψ_{12} *are algebraically independent.*

Proof: Assume that $Q \in \mathbb{C}[X_1, \dots, X_4]$, $Q \neq 0$ with minimal degree satisfies

$$Q(\Psi_5, \mathrm{Lift}_{E_{4,5}}, \mathrm{Lift}_{E_{6,5}}, \Psi_{12}) = 0 \quad \text{on} \quad \mathbb{H}_2.$$

We write $Q = \sum_{j=0}^{\deg Q} X_1^j R_j$ with $R_j \in \mathbb{C}[X_2, X_3, X_4]$ for all $0 \leq j \leq \deg Q$. Restricting this equation to $\mathbb{H} \times \mathbb{H}$ we find

$$0 = \mathrm{W}_5(Q(\Psi_5, \mathrm{Lift}_{E_{4,5}}, \mathrm{Lift}_{E_{6,5}}, \Psi_{12})) = R_0(\mathrm{W}_5(\mathrm{Lift}_{E_{4,5}}), \mathrm{W}_5(\mathrm{Lift}_{E_{6,5}}), \mathrm{W}_5(\Psi_{12})).$$

Since $\mathrm{W}_5(\mathrm{Lift}_{E_{4,5}})$, $\mathrm{W}_5(\mathrm{Lift}_{E_{6,5}})$ and $\mathrm{W}_5(\Psi_{12})$ are algebraically independent (see Theorem 8.6.1) on $\mathbb{H} \times \mathbb{H}$ we get $R_0 = 0$. Then $Q = X_1(\sum_{j=0}^{\deg Q - 1} X_1^j R_{j+1}) = X_1 \widetilde{Q}$ is divisible by X_1 and

$$\widetilde{Q}(\Psi_5, \mathrm{Lift}_{E_{4,5}}, \mathrm{Lift}_{E_{6,5}}, \Psi_{12}) = 0 \quad \text{on} \quad \mathbb{H}_2$$

which is a contradiction to the assumption that Q has minimal degree. □

Remark 7.3.7 *In the case that the $\mathcal{A}(\Gamma_5)$-module and ideal* $\ker W_5$ *would be a principal ideal, we could easily show with a similar proof that the above sequence is regular: You have to use the Cohen-Macaulay property of $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)$ which is known by Remark 8.6.2. Furthermore, we note that the regularity of a sequence does not depend on the order.*

Unfortunately, it is easily shown that this ideal is not a principal one, e. g. take Ψ_4 and $\mathrm{Lift}_{\Phi_{11,5}}$. *(compare with case weight* 11 *on page 100 to see that* $\mathrm{Lift}_{\Phi_{11,5}}$ *is not a multiple of*

Ψ_5.) *Moreover, if we substract the two Hilbert series* $\mathrm{H}(\mathcal{A}(\Gamma_5),s)$ *and* $\mathrm{H}(\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2),s)$, *we get*

$$\frac{P(s)-(1-s^5)(1+s^{12})}{(1-s^4)(1-s^5)(1-s^6)(1-s^{12})}$$

so that the ideal seems to be quite complicated. But perhaps it is possible to adopt the above proof to prove that the sequence from Lemma 7.3.6 is regular. In any case, it seems to be worthwile to calculate the ideal $\ker \mathrm{W}_5$.

$\mathcal{A}(\Gamma_5) \cong \mathcal{A}(\widehat{\Gamma}_5)$ contains no zero-divisors and is finitely generated graded $\mathbb{C}$-algebra. Let Γ be the group $\Gamma = \mathrm{Sp}_2(\mathbb{Z})[5]$ and $\mathcal{R} = \mathcal{A}(\Gamma)$. Since $\Gamma \trianglelefteq \widehat{\Gamma}_5$ and $\Gamma \trianglelefteq \mathrm{Sp}_2(\mathbb{Z}) = \widehat{\Gamma}_1$ with

$$[\widehat{\Gamma}_5 : \Gamma] < \infty \quad \text{and} \quad [\widehat{\Gamma}_1 : \Gamma] < \infty$$

the rings $\mathcal{A}(\widehat{\Gamma}_1) = \mathcal{R}^{\mathrm{Sp}_2(\mathbb{Z}/5\mathbb{Z})}$ and $\mathcal{A}(\widehat{\Gamma}_5) = \mathcal{R}^{\widehat{\Gamma}_5/\Gamma}$ are finite extensions of $\mathcal{R}$. Therefore we get by Proposition 5.2.8

$$\dim \mathcal{A}(\widehat{\Gamma}_5) = \dim \mathcal{A}(\Gamma) = \dim \mathcal{A}(\widehat{\Gamma}_1) = 4$$

Since $\dim \mathcal{A}(\widehat{\Gamma}_5) = 4$, each subalgebra of it has dimension at most 4, and hence an algebraically independent subset has cardinality at most 4. By Noether Normalisation there exist a algebraically independent subset of cardinality 4.

If we assume that $\mathcal{A}(\Gamma_5)$ is a Cohen-Macaulay ring and that the elements given in Corollary 7.3.6 are a set of primary generators, then the Hilbert series given in Corollary 7.3.4 tells us the degrees of the elements of any associated set of secondary generators. Hence we have calculated bases for $\mathbb{M}_k(\Gamma_5)$ for small k, and under the above assumptions we get the following subset of a set of secondary generators:

Since the Maaß lift for trivial character is injective and linear, we can choose in **weight 6** the form $\mathrm{Lift}_{\Phi_{6,5}}$ and in **weight 7** the form $\mathrm{Lift}_{\Phi_{7,5}}$ as secondary generators.

weight 8 Using the Fourier coefficients we can show that $\mathrm{Lift}^2_{\Phi_{4,5/2}}$, $\mathrm{Lift}^2_{E_{4,5}}$ and $\mathrm{Lift}_{E_{4,2}E_{4,3}}$ are linear independent: Given $f \in \mathbb{M}_k(\Gamma_t)$, we have the Fourier expansion

$$\sum_{\substack{n,m\in\mathbb{N}_0 \\ l\in\mathbb{Z},4nmt\geq l^2}} \alpha(n,m,l)e^{2\pi i(n\tau+lz+mtw)}.$$

We calculate the Fourier coefficients $\alpha(n,m,l)$

$\alpha(n,m,l)$	$(0,0,0)$	$(1,1,-4)$	$(1,1,-3)$
$\mathrm{Lift}^2_{\Phi_{4,5/2}}$	0	0	1
$\mathrm{Lift}^2_{E_{4,5}}$	$\frac{1}{57600}$	$\frac{1}{120}$	$\frac{1}{10}$
$\mathrm{Lift}_{E_{4,2}E_{4,3}}$	$\frac{1}{480}$	0	2

and get the desired result, since the above matrix has full rank. We therefore choose as secondary generators $\mathrm{Lift}^2_{\Phi_{4,5/2}}$ and $\mathrm{Lift}_{E_{4,2}E_{4,3}}$. Comparing the Fourier coefficients (for example the same $\alpha(n,m,l)$ as above) we get the following relations:

$$\mathrm{Lift}_{\Phi^2_{4,5/2}} = \mathrm{Lift}^2_{\Phi_{4,5/2}} \quad \text{and} \quad \mathrm{Lift}_{g_4 E_{4,5}} = 120\,\mathrm{Lift}^2_{E_{4,5}}\,.$$

Using the same method we get the following results:

weight 9
- secondary generator: $\mathrm{Lift}_{\Phi_{4,5/2}\Phi_{5,5.2}}$.
- relations:

$$\mathrm{Lift}_{g_4\Phi_{5,5}} = \tfrac{1}{240}\,\mathrm{Lift}_{E_{4,5}}\,\mathrm{Lift}_{\Phi_{5,5}} \quad \text{and} \quad \mathrm{Lift}_{\Phi_{4,5/2}}\,\mathrm{Lift}_{\Phi_{5,5/2}} = \mathrm{Lift}_{\Phi_{4,5/2}\Phi_{5,5/2}}\,.$$

- possible list of needed Fourier coefficients: $(1,1,-4\ldots-3)$.

weight 10
- secondary generators: $\mathrm{Lift}_{g_4E_{6,5}}$ and $\mathrm{Lift}_{\Phi_{4,5/2}\Phi_{6,5/2}}$.
- relations:

$$\begin{aligned}
\mathrm{Lift}_{g_4\Phi_{6,5}} &= 240\,\mathrm{Lift}_{E_{4,5}}\,\mathrm{Lift}_{\Phi_{6,5}} + 5760\,\mathrm{Lift}^2_{\Phi_{5,5}}\,,\\
\mathrm{Lift}_{g_6E_{4,5}} &= -504\,\mathrm{Lift}_{E_{4,5}}\,\mathrm{Lift}_{E_{6,5}} + \tfrac{290304}{521}\,\mathrm{Lift}^2_{\Phi_{5,5}} + \tfrac{21}{10}\,\mathrm{Lift}_{g_4E_{6,5}}\,,\\
\mathrm{Lift}_{\Phi_{4,5/2}}\,\mathrm{Lift}_{\Phi_{6,5/2}} &= -12\,\mathrm{Lift}^2_{\Phi_{5,5}} + \mathrm{Lift}_{\Phi_{4,5/2}\Phi_{6,5/2}}\,,\\
\Psi_{10} &= \tfrac{7}{2}\,\mathrm{Lift}_{E_{4,5}}\,\mathrm{Lift}_{E_{6,5}} + \tfrac{40}{521}\,\mathrm{Lift}_{E_{4,5}}\,\mathrm{Lift}_{\Phi_{6,5}} - \tfrac{1056}{521}\,\mathrm{Lift}^2_{\Phi_{5,5}}\\
&\quad - \tfrac{11}{1440}\,\mathrm{Lift}_{g_4E_{6,5}} + \tfrac{1}{6}\,\mathrm{Lift}_{\Phi_{4,5/2}\Phi_{6,5/2}}\,.
\end{aligned}$$

- possible list of needed Fourier coefficients: $(0,0,0)$, $(1,1,-4\ldots-2)$, $(2,2,-8)$

weight 11
- secondary generators: $\mathrm{Lift}_{\Phi_{11,5}}$.
- relations:

$$\begin{aligned}
\mathrm{Lift}_{g_4\Phi_{7,5}} &= 240\,\mathrm{Lift}_{E_{4,5}}\,\mathrm{Lift}_{\Phi_{7,5}} + 30\,\mathrm{Lift}_{\Phi_{5,5}}\,\mathrm{Lift}_{\Phi_{6,5}}\,,\\
\mathrm{Lift}_{g_6\Phi_{5,5}} &= -504\,\mathrm{Lift}_{\Phi_{5,5}}\,\mathrm{Lift}_{E_{6,5}} + \tfrac{20727}{521}\,\mathrm{Lift}_{\Phi_{5,5}}\,\mathrm{Lift}_{\Phi_{6,5}}\,,\\
\Psi_{11} &= \tfrac{1}{24}\,\mathrm{Lift}_{\Phi_{5,5}}\,\mathrm{Lift}_{\Phi_{6,5}} + \mathrm{Lift}_{\Phi_{11,5}}\,.
\end{aligned}$$

- possible list of needed Fourier coefficients: $(1,1,-4\ldots-3)$, $(2,2,-8)$.

weight 12 Every Maaß lift of even weight is symmetric so that Ψ_{12} (as a skew-symmetric form) cannot be identified with a Maaß lift. Because of this phenomenon we distinguish between symmetric and skew-symmetric paramodular forms:

- secondary generators: $\mathrm{Lift}_{g_6E_{6,5}}, \mathrm{Lift}_{g_6\Phi_{6,5}}$.

- relations:

$$\begin{aligned}
\mathrm{Lift}_{g_4^2 E_{4,5}} &= 200\,\mathrm{Lift}^2_{E_{6,5}} + \tfrac{4080000}{171409}\,\mathrm{Lift}_{E_{6,5}}\,\mathrm{Lift}_{\Phi_{6,5}} - \tfrac{44236800}{171409}\,\mathrm{Lift}_{\Phi_{5,5}}\,\mathrm{Lift}_{\Phi_{7,5}} \\
&\quad + 19200\,\mathrm{Lift}^3_{E_{45}} + \tfrac{50}{63}\,\mathrm{Lift}_{g_6 E_{6,5}} + \tfrac{170000}{3599589}\,\mathrm{Lift}_{g_6 \Phi_{6,5}}, \\
\mathrm{Lift}_{g_4 E_{4,2} E_{4,3}} &= 200\,\mathrm{Lift}^2_{E_{6,5}} + \tfrac{1365590}{171409}\,\mathrm{Lift}_{E_{6,5}}\,\mathrm{Lift}_{\Phi_{6,5}} - \tfrac{7897760}{171409}\,\mathrm{Lift}_{\Phi_{5,5}}\,\mathrm{Lift}_{\Phi_{7,5}} \\
&\quad - 9600\,\mathrm{Lift}^3_{E_{45}} + 240\,\mathrm{Lift}_{E_{4,5}}\,\mathrm{Lift}_{E_{4,2}E_{4,3}} + \tfrac{50}{63}\,\mathrm{Lift}_{g_6 E_{6,5}} \\
&\quad + \tfrac{682795}{43195068}\,\mathrm{Lift}_{g_6 \Phi_{6,5}}, \\
\mathrm{Lift}_{g_4 \Phi^2_{4,5/2}} &= 240\,\mathrm{Lift}_{E_{4,5}}\,\mathrm{Lift}_{\Phi^2_{4,5/2}}, \\
\mathrm{Lift}_{\Phi^2_{6,5}} &= \tfrac{4168}{329}\,\mathrm{Lift}_{E_{6,5}}\,\mathrm{Lift}_{\Phi_{6,5}} - \tfrac{200064}{329}\,\mathrm{Lift}_{\Phi_{5,5}}\,\mathrm{Lift}_{\Phi_{7,5}} + \tfrac{521}{20727}\,\mathrm{Lift}_{g_6 \Phi_{6,5}}.
\end{aligned}$$

- possible list of needed Fourier coefficients for symmetric forms:

$$(0,0\ldots1,0),\ (1,1,-4\ldots-1),\ \text{and}\ (2,2,-8\ldots-7).$$

In [GN, Rem 4.4], it is explained that there exists a skew-symmetric form of weight 12 with respect to Γ_t for every $t > 1$. This fact is related to a Borcherds product of weight 12 with respect to the orthogonal group to a lattice L which is the orthogonal sum of two hyperplanes and the Leech lattice – the only even unimodular lattice in 24 dimensions with minimal norm 4 (cf. [CS, §16.1]).

The additional generators of weight 13 and 14 are written down in Section 8.12. In weight 15 we do not need additional generators. In weight 16, there is a skew-symmetric paramodular form $\mathrm{Lift}_{E_{4,5}}\Psi_{12}$. I can also show that there exists one additional generator, e. g. we can choose $\mathrm{Lift}_{g_4^3 E_{4,5}}$. Using all Fourier coefficients up to $n \le 3$ and $m \le 3$ we could not find the second additional generator, so perhaps I have not yet found all necessary paramodular forms. The other possibility is that we can distinguish these paramodular forms only with more Fourier coefficients. In weight 18 there are at least three linear independent skew-symmetric paramodular forms, i. e. $\Psi_{14}\,\mathrm{Lift}_{E_{4,5}}$, $\Psi_{12}\,\mathrm{Lift}_{E_{6,5}}$ and $\Psi_{12}\,\mathrm{Lift}_{\Phi_{6,5}}$ which can be shown using the Fourier coefficients at $(0,1,0)$ and $(1,2,-4\ldots-5)$.

7.4 Some Remarks on a Reduction Process

With Borcherds products it is sometimes possible to calculate generators for the ring of modular forms using a reduction process analogous to $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z}))$, where Δ is in fact a Borcherds product. In most cases, which have been treated so far, the dimension of the obstruction space was one (or at least there exists a Borcherds product which had only a zero on one divisor), but a few other examples (e. g. [DKr1]) also exist. For the reduction process we need a Borcherds product (or a quotient/polynomial of Borcherds products with positive weight) such that we can divide every modular form with respect to Γ with a zero on a certain divisor $\lambda^\perp$. To guarantee this zero we have to determine all modular forms with respect to $\mathrm{Stab}_\Gamma \lambda^\perp$ which can be lifted to a modular form with respect to Γ. Sometimes it was necessary to look at more than one divisor.

In general, the lifting from a certain divisor $\lambda^\perp$ gets more and more involved as the discriminant of $\lambda^\perp$ increases.

In the case of paramodular forms with respect to Γ_5 we can look at the table of Borcherds products on page 121. The divisor $\lambda^\perp$ with the lowest discriminant which admits a Borcherds product with the above mentioned property, is $\lambda_9^\perp$. The associated Borcherds product ist $\Psi_{10}\Psi_4^{-1}$. Before we consider this situation more closely we determine the restriction of the character $\kappa_{1,2}$ to $\mathrm{Stab}_{\Gamma_5} \lambda_9^\perp$:
Because of Corollary 4.1.1 and Lemma 3.6.7 we know the generators of $\mathcal{G} \cong \mathrm{Stab}_{\Gamma_5} \lambda_9^\perp$. Comparing the values of $v_\eta^{12} \times v_\eta^{12}$ on these generators with the values of $\kappa_{1,2}$ (Corollary 8.2.1) on these generators (embedded into Γ_5 via the isomorphism in Equation (3.19)) we get

$$\kappa_{1,2}|_{\lambda_9^\perp} = v_\eta^{12} \times v_\eta^{12}.$$

More detailed, let $f \in \mathbb{M}_k(\mathrm{D}\Gamma_5^{\max})$. Then $f = \sum_{v \in (\Gamma_5^{\max})^{\mathrm{ab}}} f_v$ with $f_v \in \mathbb{M}_k(\Gamma_5^{\max}, v)$. Therefore we can restrict ourselves without loss of generality to $f = f_v$.

case 1: $k \equiv 0 \bmod 2, v(V_t) = -1$. Since f has a zero in $\lambda_{20}^\perp$, we can multiply with $\frac{\Psi_{11}}{\Psi_{12}}$ and get

$$f \cdot \frac{\Psi_{11}}{\Psi_{12}} \in \mathbb{M}_{k-1}(\Gamma_5^{\max}, v).$$

case 2: $k \equiv 1 \bmod 2, v(V_t) = 1$. Since f has a zero in $\lambda_{20}^\perp$, we can multiply with $\frac{\Psi_{11}}{\Psi_{12}}$ and get

$$f \cdot \frac{\Psi_{11}}{\Psi_{12}} \in \mathbb{M}_{k-1}(\Gamma_5^{\max}, v).$$

case 3: $k \equiv 0 \bmod 2, v = \kappa_{1,2}$. Hence

$$f|_{\lambda_9^\perp} : \mathbb{H} \times \mathbb{H} \to \mathbb{C} \quad \in \mathbb{M}_k(\mathcal{G}, v_\eta^{12} \times v_\eta^{12})_{\mathrm{sym}} \quad \text{and} \quad \mathrm{ord}_{\lambda_4^\perp} f|_{\lambda_4^\perp} \geq 1.$$

case 4: $k \equiv 0 \bmod 2, v = 1$. Hence

$$f|_{\lambda_9^\perp} : \mathbb{H} \times \mathbb{H} \to \mathbb{C} \quad \in \mathbb{M}_k(\mathcal{G}, 1)_{\mathrm{sym}}.$$

case 5: $k \equiv 1 \bmod 2, v = \kappa_{1,2}\chi_5$. Hence

$$f|_{\lambda_9^\perp} : \mathbb{H} \times \mathbb{H} \to \mathbb{C} \quad \in \mathbb{M}_k(\mathcal{G}, v_\eta^{12} \times v_\eta^{12})_{\mathrm{sym}} \quad \text{and} \quad \mathrm{ord}_{\lambda_1^\perp} f|_{\lambda_1^\perp} \geq 1.$$

case 6: $k \equiv 1 \bmod 2, v = \chi_5$. Hence

$$f|_{\lambda_9^\perp} : \mathbb{H} \times \mathbb{H} \to \mathbb{C} \quad \in \mathbb{M}_k(\mathcal{G}, 1)_{\mathrm{sym}} \quad \text{and} \quad \mathrm{ord}_{\lambda_i^\perp} f|_{\lambda_i^\perp} \geq 1 \ \text{ for i=1,4.}$$

If we knew a (finite) set of generators for the ideals

$$\mathcal{J}_1 = \{ f \in \mathcal{A}(\mathcal{G})_{\text{sym}};\ \text{there exists an } F \in \mathcal{A}(\Gamma_5^{\max}, \chi_5^k) \text{ with } F|_{\lambda_9^\perp} = f\} \quad \text{and}$$
$$\mathcal{J}_2 = \{ f \in \mathcal{A}(\mathcal{G}, \nu_\eta^{12} \times \nu_\eta^{12})_{\text{sym}};\ \text{there exists an } F \in \mathcal{A}(\Gamma_5^{\max}, \kappa_{1,2}\chi_5^k) \text{ with } F|_{\lambda_9^\perp} = f\},$$

we could determine the ring $\mathcal{A}(\mathrm{D}\,\Gamma_5^{\max})$ quite explicitly:
Let G_i be a finite set of generators of $\mathcal{J}_i$. In the cases 3 to 6 there would exist a polynomial P in these generators such that $f - P$ would be zero on the divisor $\lambda_9^\perp$. Therefore

$$(f-P)\cdot\frac{\Psi_4}{\Psi_{10}} \in \mathbb{M}_{k-6}(\Gamma_5^{\max}, \nu\kappa)$$

which switches the cases 3 and 4 as well as 5 and 6. We can even get rid of the ideal $\mathcal{J}_2$ if we use $\Psi_4^2\Psi_{10}^{-1}$ in the reduction process and multiply the cases 3 and 5 once with Ψ_4. Moreover, if we multiply the function in case 6 with Ψ_5, we obtain case 1 and therefore we only have to determine symmetric paramodular forms of even weight and character $\nu = 1$ with no constraint zeros.

The constraint zeros in the other cases include at least some information on the constraint zeros for modular forms with respect to the stabilizer (with the restricted character) and some information on the values in the cusps.

So we have to determine the ideal

$$\mathcal{J}_1^{\text{new}} = \mathcal{J}_1 \cap \mathcal{A}(\Gamma_5)_{\text{sym}}^{\text{even}}|_{\lambda_9^\perp}, \text{ where } \mathcal{A}(\Gamma_5)_{\text{sym}}^{\text{even}} := \bigoplus_k \mathbb{M}_{2k}(\Gamma_5^{\max})_{\text{sym}}:$$

First of all, we recognize that we cannot lift every modular form with respect to the stabilizer $\mathrm{Stab}_{\Gamma_5^{\max}}\lambda_9^\perp \cong \mathcal{G}$, since $\dim_{\mathbb{C}}\mathbb{M}_2(\mathcal{G}) = 1 \neq 0 = \dim_{\mathbb{C}}\mathbb{M}_2(\Gamma_5,1)$. So far, I have not been able to determine this ideal.

Needless to say, we can give some elements of the ideal $\mathcal{J}_1$ respectively $\mathrm{J}_1^{\text{new}}$:

$$\mathrm{Lift}_{E_{4,5}}|_{\lambda_9^\perp} = \tfrac{8}{243}\pi(\widehat{p}_1)^2 - \tfrac{1}{135}\pi(\widehat{s}_3),$$
$$\mathrm{Lift}_{\Phi_{5,5}}|_{\lambda_9^\perp} = -\tfrac{2}{81}(\zeta_3-\zeta_3^2)\pi(\widehat{p}_1)\pi(\widehat{p}_2) + \tfrac{1}{81}(\zeta_3-\zeta_3^2)\pi(\widehat{s}_4),$$
$$\mathrm{Lift}_{E_{6,5}}|_{\lambda_9^\perp} = -\tfrac{785056}{4937517}\pi(\widehat{p}_1)^3 - \tfrac{2221504}{44437653}\pi(\widehat{p}_2)^2 + \tfrac{232360}{1280097}\pi(\widehat{p}_4) + \tfrac{764584}{3840291}\pi(\widehat{p}_1)\pi(\widehat{s}_3),$$
$$\mathrm{Lift}_{\Phi_{6,5}}|_{\lambda_9^\perp} = -\tfrac{14336}{3159}\pi(\widehat{p}_1)^3 - \tfrac{6400}{3159}\pi(\widehat{p}_2)^2 + \tfrac{2512}{351}\pi(\widehat{p}_4) + \tfrac{2560}{351}\pi(\widehat{p}_1)\pi(\widehat{s}_3).$$

In the same way we can determine

$$\Psi_{12}|_{\lambda_9^\perp} = \pi(p_3)\Big(\tfrac{57344}{767637}\pi(\widehat{p}_1)^3\pi(\widehat{p}_2) - \tfrac{10240}{85293}\pi(\widehat{p}_1)\pi(\widehat{p}_2)\pi(\widehat{s}_3)$$
$$+ \tfrac{664576}{20726199}\pi(\widehat{p}_2)^3 - \tfrac{10048}{85293}\pi(\widehat{p}_2)\pi(\widehat{p}_4)\Big).$$

7.5 Outlook

In this section we summarize some of the possibilities to get more information on the algebraic structure of $\mathcal{A}(\Gamma_5)$:

1) Determine the ideal $\mathcal{J}_1^{\text{new}}$ and use the reduction process in Section 7.4.
2) Calculate the ring $\mathcal{A}(\mathrm{Sp}_2(\mathbb{Z})[5])$ and use the methods from Chapter 5.
3) The easiest problem which is not solved in this thesis is to show that $\mathrm{Lift}_{E_{4,5}}$, Ψ_5, $\mathrm{Lift}_{E_{6,5}}$, Ψ_{12} is a homogeneous system of parameters. In order to prove this statement we have to find an argument that $\mathcal{A}(\Gamma_5)/\sum_{i=1}^{4}\mathcal{A}(\Gamma_5)f_i$ for $f = (\mathrm{Lift}_{E_{4,5}}, \Psi_5, \mathrm{Lift}_{E_{6,5}}, \Psi_{12})^{\mathrm{tr}}$ is finite-dimensional over $\mathbb{C}$ (cf. Remark 7.3.7).
4) Determine the kernel of the algebra homomorphism W_5 (cf. Remark 7.3.7)
5) We could determine generators for all $\mathbb{M}_k(\Gamma_5)$ for k less or equal to an upper bound $k_{\max}$ and find some argument that all paramodular forms of higher degree are polynomials in these generators. For example we could choose $k_{\max} = 30$ and show that $\mathcal{A}(\Gamma_5)$ is a Cohen-Macaulay ring and that $\{\mathrm{Lift}_{E_{4,5}}, \Psi_5, \mathrm{Lift}_{E_{6,5}}, \Psi_{12}\}$ is a set of primary invariants. The latter fact can be shown directly or by using the fact that $\mathcal{A}(\mathrm{Sp}_2(\mathbb{Z})[5])$ is Cohen-Macaulay (compare with Remark 7.3.5). In order to calculate these generators you have to determine at least a lot of Fourier coefficients, but there is also a chance that the paramodular forms which are mentioned in this thesis are not sufficient. Then you have to determine other paramodular forms for example by reduction from orthogonal forms of higher degree, i. e. $\mathcal{O}(2,n)$ for $n \geq 4$. For weight 16 I tried the modular embedding from [Kö4] with $\mathbb{Q}(i)$. This ring was determined explicitly in [DKr2], but I did not find any new form. One other possibility would be to raise the restriction of the order of the pole for the input in Borcherds Theorem. Keep in mind that the Fourier coefficients of the Eisenstein series $E_{\frac{5}{2}}$ require often long calculations with the computer.

8 Appendix

8.1 General Remark on Fourier Coefficients

Most of the Fourier coefficients in this thesis (especially in chapter 6 and 7) were determined using [MAPLE]. Only in a few cases a faster calculation was necessary, e. g. for the Fourier coefficients of the Eisenstein series $E_{\frac{5}{2}}$ on page 89. Then we wrote a short programm in [C++]. Some of these Fourier coefficients are available on my homepage.

8.2 The Non-trivial Abelian Character of Γ_1

We consider the action of Γ_1 on theta characteristics (compare with [Fr1, I.3.2]): Given $a, b \in (\mathbb{Z}/2\mathbb{Z})^2$ and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_1$, then

$$\begin{pmatrix} \widetilde{a} \\ \widetilde{b} \end{pmatrix} := M \begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} (CD^{\mathrm{tr}})_0 \\ (AB^{\mathrm{tr}})_0 \end{pmatrix} \bmod 2,$$

where T_0 denotes the column vector consisting of the diagonal entries of T. Let

$$\mathcal{T}_0 := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in (\mathbb{Z}/2\mathbb{Z})^4; a^{\mathrm{tr}} b \equiv 1 \bmod 2 \right\}$$

be the set of odd theta characteristics.

Corollary 8.2.1 *Let* sgn *denote the sign-character of the symmetric group S_6 of* 6 *elements. Then we get the non-trivial character of* Γ_1 *via*

$$\kappa : \Gamma_1 \to \{\pm 1\}, \quad M \to \operatorname{sgn} \pi_M,$$

where $\pi_M \in S_6$ is the induced permutation of $\mathcal{T}_0 \to \mathcal{T}_0$, $g \mapsto M\{g\}$.

A detailed proof can be found in [Kr3].

8.3 Eichler Transformations

Eichler transformation	orthogonal matrix M in the modell Γ_S	$\Psi_t^{-1}(M) \in \mathrm{P}\Gamma_t^{\max}$
$E(f_1, f_3)$	M_λ for $\lambda = (1,0,0)^{\mathrm{tr}}$	$\operatorname{trans}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$
$E(f_1, f_4)$	$\widetilde{M}_\lambda$ for $\lambda = (1,0,0)^{\mathrm{tr}}$	$J_t \operatorname{trans}\begin{pmatrix}0 & 0\\ 0 & \frac{1}{t}\end{pmatrix} J_t$
$E(f_2, f_3)$	M_λ for $\lambda = (0,0,1)^{\mathrm{tr}}$	$\operatorname{trans}\begin{pmatrix}0 & 0\\ 0 & \frac{1}{t}\end{pmatrix}$
$E(f_2, f_4)$	$\widetilde{M}_\lambda$ for $\lambda = (0,0,1)^{\mathrm{tr}}$	$J_t \operatorname{trans}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} J_t$
$E(f_1, h)$	R_A for $A = \begin{pmatrix}1 & -2th & th^2\\ 0 & 1 & -h\\ 0 & 0 & 1\end{pmatrix}$	$\operatorname{rot}\begin{pmatrix}1 & -th\\ 0 & 1\end{pmatrix}$
$E(f_2, h)$	R_A for $A = \begin{pmatrix}1 & 0 & 0\\ -h & 1 & 0\\ th^2 & -2th & 1\end{pmatrix}$	$\operatorname{rot}\begin{pmatrix}1 & 0\\ -h & 1\end{pmatrix}$
$E(f_3, h)$	M_λ for $\lambda = (0,-h,0)^{\mathrm{tr}}$	$\operatorname{trans}\begin{pmatrix}0 & -h\\ -h & 0\end{pmatrix}$
$E(f_4, h)$	$\widetilde{M}_\lambda$ for $\lambda = (0,-h,0)^{\mathrm{tr}}$	$J_t \operatorname{trans}\begin{pmatrix}0 & h\\ h & 0\end{pmatrix} J_t$

Thereby $\operatorname{rot}(U) := \begin{pmatrix}U & 0\\ 0 & U^{-\mathrm{tr}}\end{pmatrix}$.

8.4 Explicit Isomorphism

Following [GN] we will give a more explicit isomorphism between the paramodular group and the orthogonal group of signature $(3,2)$.

Let $L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$ be a lattice in $\mathbb{R}^4$ and $L \wedge L$ the exterior algebra (see for example [Eis, p. 575]), i. e. $L \otimes L / \langle l \otimes l,\, l \in L\rangle$. The elements in $L \wedge L$ are sometimes called integral bivectors and can be written as $\alpha = \sum_{i<j} \alpha_{ij} e_i \wedge e_j$. For $u, v \in \mathbb{Z}^6$ with $u = (u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34})$ let

$$(u,v) := u_{12}v_{34} - u_{13}v_{24} + u_{14}v_{23} + u_{23}v_{14} - u_{24}v_{13} + u_{34}v_{12}.$$

The corresponding Gram matrix is given by

$$\begin{pmatrix} &&&&& 1\\ &&&& -1 &\\ &&& 1 &&\\ && 1 &&&\\ & -1 &&&&\\ 1 &&&&& \end{pmatrix}.$$

This is an even, unimodular, symmetric bilinear form of signature $(3,3)$ and via the substitution homomorphism we get for $u,v \in L \wedge L$

$$u \wedge v = (u,v)e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

This bilinear form is invariant under the diagonal action

$$\begin{array}{lcl} \mathrm{SL}(L) \times L \wedge L & \to & L \wedge L, \\ (M, l_1 \wedge l_2) & \mapsto & Ml_1 \wedge Ml_2, \end{array}$$

because the generators from $\mathrm{SL}_4(\mathbb{Z})$ are – due to [New, Thm VII.3] – the matrices

$$P = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ -1 & & & \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \times I_2.$$

Given $u \in L \wedge L$, we get

$$Tu = (u_{12}, u_{13} + u_{23}, u_{14} + u_{24}, u_{23}, u_{24}, u_{34}), \quad Pu = (u_{23}, u_{24}, u_{12}, u_{34}, u_{13}, u_{14})$$

and therefore

$$(Tu, Tv) = (Pu, Pv) = (u,v) \quad \text{for all } u,v \in L \wedge L.$$

Let $\omega_t = te_1 \wedge e_3 + e_2 \wedge e_4$ and $\Omega_t = \mathbb{Z}\omega_t$. Given $x,y \in L$, let $\widehat{J}_t(x,y) = x^{\mathrm{tr}}\widehat{J}_t y$ with

$$\widehat{J}_t = \begin{pmatrix} 0 & P_t \\ -P_t & 0 \end{pmatrix} \quad \text{and} \quad P_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.$$

Then $\widehat{J}_t$ is skew-symmetric and fulfills via the substitution homomorphism

$$\widehat{J}_t(x,y)e_1 \wedge e_2 \wedge e_3 \wedge e_4 = -x \wedge y \wedge \omega_t.$$

Thus the integral realisation of the paramodular group is representable as

$$\widehat{\Gamma}_t = \left\{ M \in \mathrm{SL}(L);\ \widehat{J}_t(Mx, My) = \widehat{J}_t(x,y) \text{ for all } x,y \in L \right\}$$

and $\widehat{\Gamma}_t$ acts on $L_t = \Omega_t^{\perp}$. The lattice L_t has the $\mathbb{Z}$-basis

$$f_1 = e_1 \wedge e_2,\ f_2 = e_2 \wedge e_3,\ f_3 = te_1 \wedge e_3 - e_2 \wedge e_4,\ f_4 = e_4 \wedge e_1,\ f_5 = e_4 \wedge e_3.$$

For given $f = \sum_{i=1}^{5} \alpha_i f_i$ with $\alpha^{\mathrm{tr}} = (\alpha_1, \ldots, \alpha_5)$ and analogous definition of g and β, we get

$$(f,g) = \alpha^{\mathrm{tr}} S_t \beta \quad \text{with} \quad S_t = \begin{pmatrix} & & & & -1 \\ & & & -1 & \\ & & 2t & & \\ & -1 & & & \\ -1 & & & & \end{pmatrix}.$$

Since $\mathcal{O}(S_t,\mathbb{Z}) = \mathcal{O}(-S_t,\mathbb{Z})$, we consider $\mathcal{O}(L_t) := \mathcal{O}(-\widehat{S}_t)$.

Let $M \in \widehat{\Gamma}_t$ and $f,g \in L_t$ as above. From $(Mf,Mg) = (f,g)$ we deduce that in these new coordinates the corresponding $\widetilde{M} \in \mathcal{O}(L_t)$. Since $\widehat{\Gamma}_t$ is conjugated to Γ_t, we obtain a map

$$\Phi : \widehat{\Gamma}_t \to \mathcal{O}(L_t), \quad M \mapsto \widetilde{M}.$$

Because of $(Mf,Mg) = (f,g)$ this map is an homomorphism.

8.5 Fourier Coefficients

The first Fourier coefficients of $E_k(uz,s;\chi,\psi) = K\sum_{n=0}^{\infty} a_n e^{2\pi i n z/3}$ are given by

u	χ	ψ	L	M	K	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1	$\left(\frac{*}{3}\right)$	1	3	3	$-\frac{4\pi i}{9}$	1	-3	0	6	-3	0	0	-6	0	6	0
1	1	$\left(\frac{*}{3}\right)$	1	3	$\frac{2\pi\sqrt{3}}{9}$	1	6	0	6	6	0	0	12	0	6	0
1	1	$\left(\frac{*}{3}\right)$	3	3	$\frac{4\pi}{\sqrt{3}}$	0	1	0	0	1	0	0	2	0	0	0
1	$\left(\frac{*}{3}\right)$	1	3	1	$-\frac{2\pi}{3}$	1	0	0	6	0	0	0	0	0	6	0
$\frac{1}{3}$	$\left(\frac{*}{3}\right)$	1	3	1	$-\frac{2\pi}{3}$	1	6	0	6	6	0	0	12	0	6	0
3	1	$\left(\frac{*}{3}\right)$	1	3	$\frac{2\pi\sqrt{3}}{9}$	1	0	0	6	0	0	0	0	0	6	0

We denote the first two normalized series by $f = E^*_{3,3}$ and $g = E^*_{1,3}$. The other less important normalized series are denoted by f^*_k where $k+2$ is the number of the line it is written in this table. Therefore we have the following relations:

$$9f^*_1 = E^*_{1,3} - E^*_{3,3}, \qquad 3f^*_2 = 2E^*_{3,3} + E^*_{1,3}, \tag{8.1}$$

$$f^*_3 = E^*_{1,3}, \qquad 3f^*_4 = 2E^*_{3,3} + E^*_{1,3}. \tag{8.2}$$

The next table contains the Fourier coefficents for polynomials in $f = E^*_{3,3}$ and $g = E^*_{1,3}$:

function	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
f^2	1	-6	9	12	-42	18	36	-48	45	12
g^2	1	12	36	12	84	72	36	96	180	12
fg	1	3	-18	12	21	-36	36	24	-90	12
f^3	1	-9	27	-9	-117	216	27	-450	459	-9
g^3	1	18	108	234	234	864	756	900	1836	2178
f^2g	1	0	-27	72	0	-216	270	0	-459	720
fg^2	1	9	0	-90	117	0	-216	450	0	-738
f^4	1	-12	54	-84	-147	756	-756	-1212	3510	-2028
g^4	1	24	216	888	1752	3024	7992	8256	14040	24216
f^2g^2	1	6	-27	-84	438	-378	-756	2064	-1755	-2028
f^3g	1	-3	-27	159	-219	-378	1431	-1032	-1755	4533
fg^3	1	15	54	-84	-363	756	-756	-672	3510	-2028

The next table contains the first Fourier coefficients for $f = E_{3,3}$ and $g = E_{1,3}$ with respect to the cusps -1, 0 and 1. Note that this table should be read line by line so that the abbreviation a_i represents the element a_i in the same line.

	x	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
f	-1	$-\frac{2\pi i}{9}-\frac{2\pi\sqrt{3}}{9}$	$\frac{2\pi(i-\sqrt{3})}{3}$	0	$6a_0$	a_1	0	0	$2a_1$	0	$6a_0$	0
f	0	0	$-\frac{4\pi}{\sqrt{3}}$	0	$6a_0$	a_1	0	0	$2a_1$	0	$6a_0$	0
f	1	$\frac{2\pi i}{9}-\frac{2\pi\sqrt{3}}{9}$	$-\frac{2\pi(i+\sqrt{3})}{3}$	0	$6a_0$	a_1	0	0	$2a_1$	0	$6a_0$	0
g	-1	$-\frac{2\pi\sqrt{3}}{9}$	$-\frac{2\pi(3i-\sqrt{3})}{3}$	0	$6a_0$	a_1	0	0	$2a_1$	0	$6a_0$	0
g	0	$-\frac{2\pi i}{3}$	0	0	$6a_0$	a_1	0	0	$2a_1$	0	$6a_0$	0
g	1	$\frac{2\pi\sqrt{3}}{9}$	$-\frac{2\pi(3i+\sqrt{3})}{3}$	0	$6a_0$	a_1	0	0	$2a_1$	0	$6a_0$	0

8.6 Generators for $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)$

Using the same methods as in Theorem 4.3.3 we get

Theorem 8.6.1 *Let $l \in \mathbb{Z}$ with $1 \le l \le 24$. Then*

a)

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)_{\mathrm{sym}} = \mathbb{C}\left[g_4(\tau_1)g_4(\tau_2), g_6(\tau_1)g_6(\tau_2), g_4^3(\tau_1)g_6^2(\tau_2) + g_6^2(\tau_1)g_4^3(\tau_2)\right].$$

b)

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2, v_\eta^l \times v_\eta^l)_{\mathrm{sym}} = \eta^l(\tau_1)\eta^l(\tau_2) \cdot \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)_{\mathrm{sym}}.$$

c)

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2, v_\eta^l \times v_\eta^l)_{\mathrm{skew}} = \left(g_4^3(\tau_1)g_6^2(\tau_2) - g_6^2(\tau_1)g_4^3(\tau_2)\right) \cdot \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2, v_\eta^l \times v_\eta^l)_{\mathrm{sym}}.$$

d)

$$\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2) \cong \mathbb{C}[X_1, X_2, X_3, X_4]/\langle X_1^3 X_4^2 - X_2 X_3\rangle.$$

The isomorphism is given by mapping X_i to F_i with the following modular forms

$$\begin{aligned} &F_1 := g_4(\tau_1)g_4(\tau_2), && F_2 := g_4^3(\tau_1)g_6^2(\tau_2), \\ &F_3 := g_6^2(\tau_1)g_4^3(\tau_2) \quad \textit{and} && F_4 := g_6(\tau_1)g_6(\tau_2). \end{aligned}$$

Remark 8.6.2 *Especially*

$$\mathcal{R} = \mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2) \cong \mathbb{C}[F_1, F_4, F_2 + F_3] \oplus (F_2 - F_3)\mathbb{C}[F_1, F_4, F_2 + F_3]$$

is a Cohen-Macaulay ring with $\dim \mathcal{R} = \operatorname{depth} \mathcal{R} = 3$.

Using the Fourier expansion we can express some modular forms in $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})^2)$ in the generators of $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z})[3]^2)$:

$$\begin{aligned} g_4^3(3\tau_1)g_6^2(3\tau_2) + g_6^2(3\tau_1)g_4^3(3\tau_2) &= \tfrac{1048576}{43046721}\pi(\widehat{p}_1)^6 + \tfrac{64028672}{559607373}\pi(\widehat{p}_2)^2\pi(\widehat{p}_1)^3 \\ &+ \tfrac{4980736}{186535791}\pi(\widehat{p}_2)^4 - \tfrac{41156608}{559607373}\pi(\widehat{p}_4)\pi(\widehat{p}_1)^3 - \tfrac{51445760}{559607373}\pi(\widehat{p}_4)\pi(\widehat{p}_2)^2 \\ &- \tfrac{917504}{20726199}\pi(\widehat{s}_3)\pi(\widehat{p}_1)^4 - \tfrac{3080192}{20726199}\pi(\widehat{s}_3)\pi(\widehat{p}_2)^2\pi(\widehat{p}_1) + \tfrac{2572288}{20726199}\pi(\widehat{s}_3)\pi(\widehat{p}_4)\pi(\widehat{p}_1) \\ &- \tfrac{950272}{6908733}\pi(\widehat{s}_4)\pi(\widehat{p}_2)\pi(\widehat{p}_1)^2 \end{aligned}$$

and

$$\begin{aligned} &g_4^3(3\tau_1)g_6^2(3\tau_2) - g_6^2(3\tau_1)g_4^3(3\tau_2) = \\ &\quad \pi(p_3) \cdot \Big(\tfrac{11632640}{186535791}\pi(\widehat{p}_2)\pi(\widehat{p}_1)^3 + \tfrac{3801088}{186535791}\pi(\widehat{p}_2)^3 - \tfrac{5144576}{62178597}\pi(\widehat{p}_2)\pi(\widehat{p}_4) \\ &\qquad - \tfrac{2031616}{20726199}\pi(\widehat{s}_3)\pi(\widehat{p}_2)\pi(\widehat{p}_1) + \tfrac{16384}{1594323}\pi(\widehat{s}_4)\pi(\widehat{p}_1)^2\Big). \end{aligned}$$

8.7 Segre Variety

In this section let F be an algebraically closed field with $\operatorname{char} F = 0$ and V_1 and V_2 be two F-vector spaces of dimension n_1 and n_2. Let $\{v_i^{(j)}, i = 1, \ldots, n_j\}$ be a basis for V_j. Let V_i^* denote the dual vector space associated to V_i and fix for V_i^* the dual basis associated to $\{v_i^{(j)}, i = 1, \ldots, n_j\}$. Then we have $\mathcal{S}[V_i] \cong F[v_1^{(i)}, \ldots, v_{n_i}^{(i)}]$ and

$$\mathcal{S}[V_1] \otimes \mathcal{S}[V_2] \cong F\left[v_1^{(1)}, \ldots, v_{n_1}^{(1)}, v_1^{(2)}, \ldots, v_{n_2}^{(2)}\right] \cong \mathcal{S}[V_1 \oplus V_2]$$

via the identification

$$v_j^{(1)} \mapsto \left(v_j^{(1)}, 0\right) \in V_1 \oplus V_2, \quad v_j^{(2)} \mapsto \left(0, v_j^{(2)}\right) \in V_1 \oplus V_2.$$

We get an F-algebra homomorphism

$$\pi : \begin{array}{ccc} \mathcal{S}[V_1 \otimes V_2] & \to & \mathcal{S}[V_1] \otimes \mathcal{S}[V_2], \\ v_i^{(1)} \otimes v_j^{(2)} & \mapsto & v_i^{(1)} v_j^{(2)}. \end{array}$$

Obviously,

$$\operatorname{im} \pi = \bigoplus_{d \geq 0} \mathcal{S}[V_1]_d \otimes \mathcal{S}[V_2]_d \subset \mathcal{S}[V_1] \otimes \mathcal{S}[V_2].$$

Then there exists (e. g. by [CLO, §5.4, Prop 8]) a unique polynomial mapping (sometimes also called *morphism of affine varieties*) $\pi^* : V_1^* \oplus V_2^* \to V_1^* \otimes V_2^*$, This map is called the *pullback* of π. If we identify $V_i^* \cong \mathbb{C}^{n_i \times 1}$ with their coordinate (column) vectors, we get

$$\pi^*(v_1, v_2) = v_1 \cdot v_2^{\mathrm{tr}} \quad \text{for } v_1 \in \mathbb{C}^{1 \times n_1},\ v_2 \in \mathbb{C}^{n_2 \times 1}.$$

Given $f_1, \ldots, f_s \in F[X_1, \ldots, X_n]$, we set

$$\operatorname{var}(f_1, \ldots, f_s) := \left\{(a_1, \ldots, a_n)^{\mathrm{tr}} \in F^n;\, f_i(a_1, \ldots, a_n) = 0 \text{ for all } i\right\},$$

the *affine variety* defined by $f_1, \ldots, f_s$. Given an affine variety $W \subset F^n$, let

$$\operatorname{ideal}(W) = \{f \in F[X_1, \ldots, X_n];\, f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in W\}$$

the ideal associated to W. We need later on also projective varieties:
Let $\mathbb{P}^n(F) = (F^{n+1} \setminus \{0\})/F^*$ where $x, y \in F^{n+1} \setminus \{0\}$ are equivalent if there exists a $\lambda \in F^*$ such that $x = \lambda y$. We write $[x_0 : \ldots : x_n]$ for the equivalence class which contains the element $(x_0, \ldots, x_n) \in F^{n+1} \setminus \{0\}$. If $f_1, \ldots, f_s \in F[X_0, \ldots, X_n]$ are homogeneous polynomials, we set

$$\operatorname{var}(f_1, \ldots, f_s) := \{[a_0 : \ldots : a_n] \in \mathbb{P}^n(F);\, f_i(a_0, \ldots, a_n) = 0 \text{ for all } i\},$$

the *projective variety* defined by $f_1, \ldots, f_s$. Given a projective variety $W \subset \mathbb{P}^n(F)$, let

$$\operatorname{ideal}(W) = \{f \in F[X_0, \ldots, X_n];\, f(a_0, \ldots, a_n) = 0 \text{ for all } [a_0 : \ldots : a_n] \in W\}$$

the ideal associated to W.

Now we are able to calculate the image of π^*

Lemma 8.7.1 *Let*

$$\mathcal{I}_\pi = \{(v_i^{(1)} \otimes v_j^{(2)})(v_k^{(1)} \otimes v_l^{(2)}) - (v_i^{(1)} \otimes v_l^{(2)})(v_k^{(1)} \otimes v_j^{(2)}); 1 \leq i < k \leq n_1, 1 \leq j < l \leq n_2\},$$

then

$$\operatorname{im} \pi^* \cong \operatorname{var}(\mathcal{I}_\pi).$$

Proof:

$$\operatorname{im} \pi^* \cong \left\langle A \in F^{n_1 \times n_2};\ \operatorname{rank} A \leq 1 \right\rangle.$$

This is fulfilled if and only if all (2×2)-minors of A vanish. This leads directly to the claim. □

Let $\mathcal{R} := F[X_1, \dots, X_n]$ and $\mathcal{R}^* := F[X_0, X_1, \dots . X_n]$. For given $f \in \mathcal{R}$, let

$$f^*(X_0, X_1, \dots, X_n) = X_0^{\deg f} f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) \in \mathcal{R}^*$$

be the *homogenization* of f. For $g \in \mathcal{R}^*$, we define the *dehomogenization* as

$$g_*(X_1, \dots, X_n) = g(1, X_1, \dots, X_n) \in \mathcal{R}.$$

Moreover,we define for $\mathcal{I} \trianglelefteq \mathcal{R}$ the ideal $\mathcal{I}^* := \langle f^*; f \in \mathcal{I} \rangle$ and for homogeneous $\mathcal{J} \trianglelefteq \mathcal{R}^*$ the ideal $\mathcal{J}_* := \langle f_*; f \in \mathcal{J} \rangle$.

Lemma 8.7.2 *Let $\mathcal{J} := \langle f_1, \dots, f_s \rangle \trianglelefteq \mathcal{R}^*$ a homogeneous ideal. Then*

a) $\mathcal{J}_* = \langle (f_1)_*, \dots, (f_s)_* \rangle$.
b) If $\mathcal{J}$ is a prime ideal, then $\mathcal{J}_* = \mathcal{R}$ or $\mathcal{J}_*$ is prime.
c) If $((f_i)_*)^* = f_i$ for $1 \leq i \leq s$ and $\mathcal{J}_* = \mathcal{R}$ or $\mathcal{J}_*$ is a prime ideal, then $\mathcal{J} = \mathcal{R}^*$ or $\mathcal{J}$ is a prime ideal.

Proof: a) For given $f \in \mathcal{J}$, there exists a set $\{h_i\} \subset \mathcal{R}^*$ such that $f = \sum_i h_i f_i$. Hence

$$f_* = f(1, X_1, \dots, X_n) = \sum_i (h_i)_* (f_i)_*.$$

b) Let $f, g \in \mathcal{R}$ with $f \cdot g \in \mathcal{J}_*$. Assume $f, g \notin \mathcal{J}_*$, which means especially $\mathcal{J}_* \neq \mathcal{R}$. Thus $f^*, g^* \notin \mathcal{J}$, since otherwise $f = (f^*)_*$, $g = (g^*)_* \in \mathcal{J}_*$. Since $\mathcal{J}$ is prime, we know that $f^* \cdot g^* = (f \cdot g)^* \notin \mathcal{J}$. Without restriction we can choose the f_i homogeneous. Given $f \cdot g = \sum_i h_i (f_i)_*$ with $h_i \in \mathcal{R}$, we get

$$(fg)^* = \sum_i (h_i)^* ((f_i)_*)^* X_0^{n_i} \quad \text{with } n_i \in \mathbb{N}_0.$$

For every i there exist $c_i \in \mathbb{N}_0$ such that $((f_i)_*)^* X_0^{c_i} = f_i$. Then there exists an $N \in \mathbb{N}$ such that $X_0^N (fg)^* = \sum_i (h_i)^* X_0^{N - c_i + n_i} f_i \in \mathcal{J}$. Since $\mathcal{J}$ is a prime ideal, $X_0^N \in \mathcal{J}$. Hence $1 = (X_0^N)_* \in \mathcal{J}_*$ providing a contradiction to $f \notin \mathcal{J}_*$. Hence $\mathcal{J}_*$ is prime.

c) Let $f,g \in \mathcal{R}^*$, $f \cdot g \in \mathcal{J}$, $\mathcal{J} \neq \mathcal{R}^*$ and without restriction let f,g be homogeneous. Therefore $(f \cdot g)_* \in \mathcal{J}_*$. Since $\mathcal{J}_*$ is a prime ideal, $f_* \in \mathcal{J}_*$ or $g_* \in \mathcal{J}_*$. Without restriction we choose $f_* \in \mathcal{J}_*$. Thus $f_* = \sum_i h_i (f_i)_*$ with $h_i \in \mathcal{R}$ which leads to

$$(f_*)^* = \sum_i (h_i)^* ((f_i)_*)^* \cdot X_0^{n_i} = \sum_i (h_i)^* f_i X_0^{n_i} \in \mathcal{J}.$$

Since f is homogeneous, we know that there exists an $N \geq 0$ such that $f = X_0^N (f_*)^* \in \mathcal{J}$, hence $\mathcal{J}$ is prime. □

Proposition 8.7.3 *Given $m,n \in \mathbb{N}_0$, let*

$$\mathbb{P}^q(F) = \left\{ [z_{i,j}] := [z_{0,0} : \ldots : z_{0,m} : z_{1,0} : \ldots : z_{1,m} : \ldots : z_{n,m}] \right\} \quad \textit{with} \quad q = nm + n + m$$

be the q-dimensional projective space *over the field F.*

a) The mapping

$$\sigma_{\mathrm{Seg}} : \mathbb{P}^n(F) \times \mathbb{P}^m(F) \to \mathbb{P}^q(F), \quad [x_0 : \ldots : x_n] \times [y_0 : \ldots : y_m] \mapsto [z_{i,j} = x_i y_j]$$

is an injective mapping.

b) Let $\mathrm{Seg} := \{Z_{i,j}Z_{r,s} - Z_{i,s}Z_{r,j}; 0 \leq i,r \leq n, 0 \leq j,s \leq m\} \subset F[Z_{0,0}, \ldots, Z_{n,m}]$. *Then*

$$\operatorname{im} \sigma_{\mathrm{Seg}} = \operatorname{var}(\mathrm{Seg}) \subset \mathbb{P}^q(F).$$

c) $\langle \mathrm{Seg} \rangle \lhd F[Z_{0,0}, \ldots, Z_{n,m}]$ *is a prime ideal.*

The mapping σ_{Seg} is called Segre-mapping *and the projective variety* var(Seg) *is called* Segre-variety.

Proof: a) Let $x = [x_0 : \ldots : x_n] \in \mathbb{P}^n(F), y = [y_0 : \ldots : y_n] \in \mathbb{P}^m(F)$. We take $\lambda(x_0, \ldots, x_n) \in [x_0 : \ldots : x_n]$ and $y = \mu[y_0, \ldots, y_m] \in (y_0 : \ldots : y_n)$. Hence $\lambda\mu(x_0y_0, \ldots, x_iy_j, \ldots, x_ny_m) \in [x \times y]$ and σ_{Seg} is well-defined. If

$$[x_0^{(1)} y_0^{(1)} : \ldots : x_n^{(1)} y_m^{(1)}] = [x_0^{(2)} y_0^{(2)} : \ldots : x_n^{(2)} y_m^{(2)}],$$

then there exists a $\lambda \in F^*$ such that

$$(x_0^{(1)} y_0^{(1)}, \ldots, x_n^{(1)} y_m^{(1)}) = \lambda (x_0^{(2)} y_0^{(2)}, \ldots, x_n^{(2)} y_m^{(2)}).$$

Since $x^{(1)} \in [x^{(1)}] \in \mathbb{P}^n$, there exists an $x_i^{(1)} \neq 0$. This leads to $y_j^{(1)} = \lambda \frac{x_i^{(2)}}{x_i^{(1)}} y_j^{(2)}$, therefore $[y^{(1)}] = [y^{(2)}]$. Analogously $[x^{(1)}] = [x^{(2)}]$.

b) Since $\langle \mathrm{Seg} \rangle$ is a homogeneous ideal, the projective variety $\mathrm{var}(\mathrm{Seg})$ is well-defined. The inclusion $\mathrm{im}\, \sigma_{\mathrm{Seg}} \subset \mathrm{var}(\mathrm{Seg})$ is trivial, since the coordinates of $[x \times y]$ satisfy

$$z_{i,j} z_{r,s} - z_{i,s} z_{r,j} = x_i y_j x_r y_s - x_i y_s x_r y_j = 0.$$

Let $[w_{i,j}] \in \mathrm{var}(\mathrm{Seg})$. Without loss of generality $w_{0,0} \neq 0$. Set

$$x := [1, \tfrac{w_{1,0}}{w_{0,0}} : \ldots : \tfrac{w_{n,0}}{w_{0,0}}] \quad \text{and} \quad y := [w_{0,0} : w_{0,1} : \ldots : w_{0,m}].$$

Then $[w_{i,j}] = [(x_i \times y_j)_{i,j}] \in \mathrm{im}\, \sigma_{\mathrm{Seg}}$.

c) All elements of Seg are homogeneous and $Z_{0,0} \nmid a$ for all $a \in \mathrm{Seg}$ so that $(a_*)^* = a$ for all $a \in \mathrm{Seg}$. Using Lemma 8.7.2 it is enough to show that $\langle \mathrm{Seg}_* \rangle = \langle \mathrm{Seg} \rangle_* \trianglelefteq \mathcal{R} := F[Z_{0,1}, \ldots, Z_{n,m}]$ is a prime ideal. We get $(Z_{0,0} Z_{r,s} - Z_{0,s} Z_{r,0})_* = Z_{r,s} - Z_{0,s} Z_{r,0} \in \langle \mathrm{Seg} \rangle_*$ for $r, s \neq 0$.
We compute in $\mathcal{R}/\langle Z_{r,s} - Z_{0,s} Z_{r,0}; r,s \neq 0 \rangle$ that

$$Z_{0,j} Z_{r,s} - Z_{0,s} Z_{r,j} = 0, \quad Z_{i,0} Z_{r,s} - Z_{i,s} Z_{r,0} = 0 \quad \text{and} \quad Z_{i,j} Z_{r,s} - Z_{i,s} Z_{r,j} = 0$$

hold for $i, j, r, s \neq 0$. Therefore $\langle \mathrm{Seg} \rangle_* = \langle B \rangle \trianglelefteq \mathcal{R}$ with $B := \{Z_{r,s} - Z_{0,s} Z_{r,0}; r,s \neq 0\}$. Moreover, if we take the lexicographic ordering with $Z_{r,s} > Z_{r,0} > Z_{0,s}$, then B is a Gröbner basis for $\langle \mathrm{Seg} \rangle_*$, since the leading terms are different (e. g. by [CLO, II.6 Thm 6]), hence

$$F[Z_{r,0}, Z_{0,s}; r,s \neq 0] \cap \langle \mathrm{Seg} \rangle_* = \{0\} \quad \text{and} \quad F[Z_{r,0}, Z_{0,s}; r,s \neq 0] + \langle \mathrm{Seg} \rangle_* = \mathcal{R}.$$

Hence $\mathcal{R}/\langle \mathrm{Seg} \rangle_* \cong F[Z_{r,0}, Z_{0,s}; r,s \neq 0]$ is an integral domain so that $\langle \mathrm{Seg} \rangle_*$ is prime. □

Remark 8.7.4 *Via the projective algebra-geometry dictionary (e. g. [CLO, VIII.3]) c) is equivalent to the fact that* $\mathrm{var}(\mathrm{Seg})$ *is irreducible. For another proof for the irreducibility of the Segre-variety we refer the interested reader to [Mum, p. 50/51].*

With this result and some results of the algebra-geometry dictionary we can prove the following

Lemma 8.7.5 *a)* $\mathcal{I}_\pi = \ker \pi$, *hence*

$$\mathrm{im}\, \pi = \bigoplus_{d \geq 0} (\mathcal{S}[V_1]_d \otimes \mathcal{S}[V_2]_d) \cong S[V_1 \otimes V_2]/\mathcal{I}_\pi.$$

b) $\mathrm{height}\, \mathcal{I}_\pi = (n_1 - 1)(n_2 - 1)$.

Proof: a) Using the definition of π^* we conclude that for $(v_1, v_2) \in V_1^* \oplus V_2^*$ and $f \in \mathrm{ideal}(\mathrm{im}\, \pi^*)$ the equation $f^\pi((v_1, v_2)) = (f \circ \pi^*)(v_1, v_2) = 0$ holds. Thus $f^\pi = 0$ and $f^\pi \in \ker \pi$. Vice versa, $0 = (f \circ \pi^*)(v_1, v_2)$ for $f \in \ker \pi$ which leads to $f \in \mathrm{ideal}(\mathrm{im}\, \pi^*)$. Altogether, we have

$$\mathrm{ideal}(\mathrm{im}\, \pi^*) = \ker \pi.$$

Therefore we get an injective embedding $\mathcal{S}[V_1 \otimes V_2]/\ker\pi \to \mathcal{S}[V_1] \otimes \mathcal{S}[V_2]$. The last structure is an integral domain (since $F[X_1, \dots, X_n]$ is an integral domain), thus $\ker\pi$ is a prime ideal in $\mathcal{S}[V_1 \otimes V_2]$ by [CLO, V.1 Prop 4] and $\operatorname{im}\pi^*$ is an irreducible variety. Using the Strong Nullstellensatz [CLO, IV.2 Thm 6] we get

$$\operatorname{rad}\mathcal{I}_\pi = \operatorname{ideal}(\operatorname{var}(\mathcal{I}_\pi)) = \operatorname{ideal}(\operatorname{im}\pi^*) = \ker\pi,$$

where $\operatorname{rad}\mathcal{I}$ denotes the *radical* of $\mathcal{I} \trianglelefteq \mathcal{S}[V]$, i. e. the ideal $\{f : f^m \in \mathcal{I} \text{ for some } m \in \mathbb{N}\}$. Because of Proposition 8.7.3, we know that $\mathcal{I}_\pi$ is a prime ideal. Since every prime ideal is radical, we get

$$\mathcal{I}_\pi = \operatorname{rad}\mathcal{I}_\pi = \ker\pi.$$

b) In a) we have already shown that $\mathcal{I}_\pi$ is a prime ideal. Therefore the concept height is well-defined. Let us now determine the fibres of π^*. Let $A \in \operatorname{im}\pi^*$. If $A \neq 0$, we get $v_i \in V_i^*$ so that

$$(\pi^*)^{-1}(A) = \left\{(\lambda v_1, \lambda^{-1} v_2) \quad \text{for } \lambda \in F^*\right\}.$$

Hence $\dim(\pi^*)^{-1}(A) = 1$ and $(\pi^*)^{-1}(A)$ is irreducible. If $A = 0$, we get

$$(\pi^*)^{-1}(A) = (0 \oplus V_2^*) \cup (V_1^* \oplus 0).$$

These components are irreducible and $\dim(\pi^*)^{-1}(A) = \max(n_1, n_2)$. Therefore we get – due to [Mum, I Thm 8.3] –

$$\dim\operatorname{im}\pi^* = \dim_F V_1^* \oplus V_2^* - 1 = n_1 + n_2 - 1 = \dim\operatorname{im}\pi$$

and

$$\operatorname{height}\mathcal{I}_\pi = \dim(\mathcal{S}[V_1 \otimes V_2]) - \dim\operatorname{im}\pi = (n_1 - 1)(n_2 - 1). \qquad \square$$

Remark 8.7.6 *We want to close this section with a remark on determinantal rings:*
A ring $\mathcal{R}$ is determinantal *if it can be written in the form $\mathcal{S}/\mathcal{I}$, where $\mathcal{S}$ is a Cohen-Macaulay ring and $\mathcal{I}$ is the ideal generated by the $r \times r$-minors of a $p \times q$ matrix M, for some p, q, r such that the codimension of $\mathcal{I}$ in $\mathcal{S}$ is exactly $(p - r + 1)(q - r + 1)$. Determinantal rings are Cohen-Macaulay by Theorem [Eis, Thm 18.18]. $\mathcal{S}[V_1 \otimes V_2]$ is a determinantal ring with $\mathcal{S} = \mathcal{S}[V_1] \otimes \mathcal{S}[V_2]$, $p = n_1$, $q = n_2$ and $r = 2$ by Lemma 8.7.5. Therefore* $\operatorname{im}\pi$ *is a Cohen-Macaulay ring. For more information we refer the interested reader to [BV, §5.D].*

8.8 Invariants for $\mathcal{S}[V \otimes V]^H$ for $H = G$

A set of primary invariants for $\mathcal{S}[V \otimes V]^G$ is given by

$$
\begin{aligned}
p_1 &= x_1^2 - \tfrac{1}{2}x_1x_2 - \tfrac{1}{2}x_1x_3 + \tfrac{1}{2}x_1x_4 - \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_2x_3 - \tfrac{1}{2}x_2x_4 - \tfrac{1}{2}x_3^2 - \tfrac{1}{2}x_3x_4 - \tfrac{1}{8}x_4^2,\\
p_2 &= x_1^3 + \tfrac{3}{2}x_1^2x_2 + \tfrac{3}{2}x_1^2x_3 + \tfrac{3}{4}x_1x_2^2 - \tfrac{3}{4}x_1x_2x_4 + \tfrac{3}{4}x_1x_3^2 - \tfrac{3}{4}x_1x_3x_4 + \tfrac{1}{8}x_2^3 - \tfrac{3}{8}x_2^2x_3 - \tfrac{3}{8}x_2^2x_4\\
&\quad - \tfrac{3}{8}x_2x_3^2 + \tfrac{3}{8}x_2x_4^2 + \tfrac{1}{8}x_3^3 - \tfrac{3}{8}x_3^2x_4 + \tfrac{3}{8}x_3x_4^2 - \tfrac{1}{8}x_4^3,\\
p_3 &= x_1^2x_2 - x_1^2x_3 - \tfrac{1}{2}x_1x_2^2 + \tfrac{1}{2}x_1x_2x_4 + \tfrac{1}{2}x_1x_3^2 - \tfrac{1}{2}x_1x_3x_4 + \tfrac{1}{4}x_2^3 + \tfrac{1}{4}x_2^2x_3 + \tfrac{1}{4}x_2^2x_4 - \tfrac{1}{4}x_2x_3^2\\
&\quad + \tfrac{1}{4}x_2x_4^2 - \tfrac{1}{4}x_3^3 - \tfrac{1}{4}x_3^2x_4 - \tfrac{1}{4}x_3x_4^2,\\
p_4 &= x_1^4 - x_1^3x_2 - x_1^3x_3 - \tfrac{1}{8}x_1^3x_4 - \tfrac{3}{4}x_1^2x_2^2 - \tfrac{3}{8}x_1^2x_2x_3 - \tfrac{3}{4}x_1^2x_2x_4 - \tfrac{3}{4}x_1^2x_3^2 - \tfrac{3}{4}x_1^2x_3x_4 + \tfrac{3}{8}x_1^2x_4^2\\
&\quad + \tfrac{1}{2}x_1x_2^3 - \tfrac{3}{4}x_1x_2^2x_3 - \tfrac{3}{8}x_1x_2^2x_4 - \tfrac{3}{4}x_1x_2x_3^2 + \tfrac{3}{2}x_1x_2x_3x_4 - \tfrac{3}{16}x_1x_2x_4^2 + \tfrac{1}{2}x_1x_3^3\\
&\quad - \tfrac{3}{8}x_1x_3^2x_4 - \tfrac{3}{16}x_1x_3x_4^2 + \tfrac{1}{16}x_1x_4^3 + \tfrac{1}{4}x_2^4 - \tfrac{1}{8}x_2^3x_3 + \tfrac{1}{8}x_2^3x_4 + \tfrac{3}{8}x_2^2x_3^2 - \tfrac{3}{16}x_2^2x_3x_4\\
&\quad + \tfrac{3}{8}x_2^2x_4^2 - \tfrac{1}{8}x_2x_3^3 - \tfrac{3}{16}x_2x_3^2x_4 + \tfrac{3}{16}x_2x_3x_4^2 + \tfrac{1}{8}x_2x_4^3 + \tfrac{1}{4}x_3^4 + \tfrac{1}{8}x_3^3x_4 + \tfrac{3}{8}x_3^2x_4^2 + \tfrac{1}{8}x_3x_4^3\\
&\quad - \tfrac{1}{32}x_4^4.
\end{aligned}
$$

A set of secondary invariants for $\mathcal{S}[V \otimes V]^G$ is given by

$$
\begin{aligned}
s_1 &= 1,\\
s_2 &= x_1^2x_4 - x_1x_2x_3 + \tfrac{1}{2}x_1x_2x_4 + \tfrac{1}{2}x_1x_3x_4 - \tfrac{1}{2}x_1x_4^2 - \tfrac{1}{2}x_2^2x_3 - \tfrac{1}{2}x_2x_3^2 + \tfrac{1}{2}x_2x_3x_4,\\
s_3 &= x_1^3x_4 + 3x_1^2x_2x_3 + \tfrac{1}{2}x_2^3x_4 + \tfrac{1}{2}x_3^3x_4 + \tfrac{1}{16}x_4^4,\\
s_4 &= x_1^4x_3 - x_1^3x_2x_4 - x_1^3x_3^2 - \tfrac{1}{8}x_1^3x_4^2 - \tfrac{3}{2}x_1^2x_2^2x_3 + \tfrac{3}{4}x_1^2x_2^2x_4 - \tfrac{3}{4}x_1^2x_2x_3x_4 - \tfrac{3}{8}x_1^2x_2x_4^2\\
&\quad + \tfrac{1}{2}x_1x_2^3x_3 - \tfrac{3}{8}x_1x_2^2x_3^2 - \tfrac{3}{4}x_1x_2^2x_3x_4 + \tfrac{1}{8}x_1x_3^4 - \tfrac{3}{16}x_1x_3^2x_4^2 + \tfrac{1}{16}x_1x_3x_4^3 + \tfrac{1}{4}x_2^4x_4 - \tfrac{1}{8}x_2^3x_3^2\\
&\quad + \tfrac{1}{8}x_2^3x_4^2 - \tfrac{1}{8}x_2x_3^3x_4 + \tfrac{3}{32}x_2x_3^2x_4^2 + \tfrac{1}{32}x_2x_4^4 - \tfrac{1}{8}x_3^5 - \tfrac{5}{32}x_3^3x_4^2 - \tfrac{5}{32}x_3^2x_4^3 + \tfrac{1}{64}x_4^5,\\
s_5 &= x_1^4x_4^2 - 2x_1^3x_2x_3x_4 + x_1^3x_2x_4^2 + x_1^3x_3x_4^2 - x_1^3x_4^3 + x_1^2x_2^2x_3^2 - 2x_1^2x_2^2x_3x_4 + \tfrac{1}{4}x_1^2x_2^2x_4^2\\
&\quad - 2x_1^2x_2x_3^2x_4 + \tfrac{5}{2}x_1^2x_2x_3x_4^2 - \tfrac{1}{2}x_1^2x_2x_4^3 + \tfrac{1}{4}x_1^2x_3^2x_4^2 - \tfrac{1}{2}x_1^2x_3x_4^3 + \tfrac{1}{4}x_1^2x_4^4 + x_1x_2^3x_3^2\\
&\quad - \tfrac{1}{2}x_1x_2^3x_3x_4 + x_1x_2^2x_3^3 - 2x_1x_2^2x_3^2x_4 + x_1x_2^2x_3x_4^2 - \tfrac{1}{2}x_1x_2x_3^3x_4 + x_1x_2x_3^2x_4^2 - \tfrac{1}{2}x_1x_2x_3x_4^3\\
&\quad + \tfrac{1}{4}x_2^4x_3^2 + \tfrac{1}{2}x_2^3x_3^3 - \tfrac{1}{2}x_2^3x_3^2x_4 + \tfrac{1}{4}x_2^2x_3^4 - \tfrac{1}{2}x_2^2x_3^3x_4 + \tfrac{1}{4}x_2^2x_3^2x_4^2,
\end{aligned}
$$

$$\begin{aligned}
s_6 = {} & x_1^6 + \tfrac{6}{47}x_1^5x_2 - \tfrac{84}{47}x_1^5x_3 - \tfrac{6}{47}x_1^5x_4 - \tfrac{15}{47}x_1^4x_2^2 - \tfrac{30}{47}x_1^4x_2x_3 - \tfrac{15}{47}x_1^4x_2x_4 - \tfrac{105}{47}x_1^4x_3^2 \\
& + \tfrac{15}{94}x_1^4x_3x_4 + \tfrac{15}{94}x_1^4x_4^2 + \tfrac{140}{47}x_1^3x_2^3 - \tfrac{30}{47}x_1^3x_2^2x_3 + \tfrac{30}{47}x_1^3x_2^2x_4 + \tfrac{15}{47}x_1^3x_2x_3^2 \\
& + \tfrac{60}{47}x_1^3x_2x_3x_4 + \tfrac{15}{47}x_1^3x_2x_4^2 + \tfrac{5}{2}x_1^3x_3^3 + \tfrac{15}{94}x_1^3x_3^2x_4 - \tfrac{15}{94}x_1^3x_3x_4^2 + \tfrac{35}{94}x_1^3x_4^3 - \\
& \tfrac{15}{47}x_1^2x_2^4 + \tfrac{30}{47}x_1^2x_2^3x_3 + \tfrac{60}{47}x_1^2x_2^3x_4 + \tfrac{45}{47}x_1^2x_2^2x_3^2 + \tfrac{45}{47}x_1^2x_2^2x_3x_4 - \tfrac{45}{94}x_1^2x_2^2x_4^2 + \tfrac{15}{94}x_1^2x_2x_3^3 \\
& - \tfrac{45}{94}x_1^2x_2x_3^2x_4 + \tfrac{315}{94}x_1^2x_2x_3x_4^2 - \tfrac{15}{94}x_1^2x_2x_4^3 - \tfrac{105}{188}x_1^2x_3^4 - \tfrac{15}{94}x_1^2x_3^3x_4 - \tfrac{45}{376}x_1^2x_3^2x_4^2 \\
& + \tfrac{15}{188}x_1^2x_3x_4^3 + \tfrac{15}{376}x_1^2x_4^4 + \tfrac{6}{47}x_1x_2^5 + \tfrac{30}{47}x_1x_2^4x_3 + \tfrac{15}{47}x_1x_2^4x_4 + \tfrac{15}{47}x_1x_2^3x_3^2 - \tfrac{30}{47}x_1x_2^3x_3x_4 \\
& - \tfrac{30}{47}x_1x_2^3x_4^2 - \tfrac{15}{94}x_1x_2^2x_3^3 + \tfrac{315}{94}x_1x_2^2x_3^2x_4 - \tfrac{45}{94}x_1x_2^2x_3x_4^2 + \tfrac{15}{94}x_1x_2^2x_4^3 - \tfrac{15}{188}x_1x_2x_3^4 \\
& - \tfrac{15}{94}x_1x_2x_3^3x_4 + \tfrac{45}{188}x_1x_2x_3^2x_4^2 + \tfrac{15}{47}x_1x_2x_3x_4^3 + \tfrac{15}{376}x_1x_2x_4^4 - \tfrac{21}{188}x_1x_3^5 + \tfrac{15}{752}x_1x_3^4x_4 \\
& + \tfrac{15}{188}x_1x_3^3x_4^2 + \tfrac{15}{376}x_1x_3^2x_4^3 - \tfrac{15}{752}x_1x_3x_4^4 - \tfrac{3}{376}x_1x_4^5 - \tfrac{7}{47}x_2^6 + \tfrac{3}{47}x_2^5x_3 - \tfrac{3}{47}x_2^5x_4 \\
& - \tfrac{15}{188}x_2^4x_3^2 - \tfrac{15}{47}x_2^4x_3x_4 - \tfrac{15}{188}x_2^4x_4^2 + \tfrac{35}{94}x_2^3x_3^3 - \tfrac{15}{94}x_2^3x_3^2x_4 + \tfrac{15}{94}x_2^3x_3x_4^2 - \tfrac{35}{94}x_2^3x_4^3 \\
& - \tfrac{15}{752}x_2^2x_3^4 + \tfrac{15}{188}x_2^2x_3^3x_4 + \tfrac{45}{188}x_2^2x_3^2x_4^2 + \tfrac{15}{188}x_2^2x_3x_4^3 - \tfrac{15}{752}x_2^2x_4^4 + \tfrac{3}{752}x_2x_3^5 \\
& + \tfrac{15}{376}x_2x_3^4x_4 + \tfrac{15}{376}x_2x_3^3x_4^2 - \tfrac{15}{376}x_2x_3^2x_4^3 - \tfrac{15}{376}x_2x_3x_4^4 - \tfrac{3}{752}x_2x_4^5 - \tfrac{157}{752}x_3^6 + \tfrac{21}{376}x_3^5x_4 \\
& - \tfrac{105}{752}x_3^4x_4^2 - \tfrac{5}{16}x_3^3x_4^3 - \tfrac{105}{752}x_3^2x_4^4 + \tfrac{21}{376}x_3x_4^5 + \tfrac{1}{64}x_4^6.
\end{aligned}$$

8.9 [GAP]-Code for Section 5.5 and 5.6

a) First the code for the invariants with respect to G:

```
# Definition of the character table and the character lambda
gap> a4:=CharacterTable("a4");
gap> lambda:=Sum(Irr(a4){[3,4]});
# Calculation of the  Hilbert series without character
gap> Hil:=MolienSeries(lambda);
(1-z^2+z^3+2*z^4 ) / ( (1-z^3)^2*(1-z^2)^2 )
gap> MolienSeriesWithGivenDenominator(Hil,[2,3,3,4]);
( 1+z^3+z^4+z^5+2*z^6 ) / ( (1-z^4)*(1-z^3)^2*(1-z^2) )
# Calculation of two Hilbert series with character
gap> Hil2:=MolienSeries(lambda,Irr(a4)[2]);
( 2*z^2+z^3-z^4+z^6 ) / ( (1-z^3)^2*(1-z^2)^2 )
gap> Hil3:=MolienSeries(lambda,Irr(a4)[3]);
( z+z^{2}-z^3+z^4+z^5 ) / ( (1-z^3)^2*(1-z^2)^2 )
```

b) Now we give the additional instructions for invariants with respect to $\widehat{G}$. We used the abbreviation char for the character $\tau \otimes 1^- + \lambda' \otimes 1$.

```
# Definition of the character table
gap> c2:=CharacterTable("Cyclic",2);
CharacterTable( "C2" )
gap> c2a4:=CharacterTableDirectProduct(c2,a4);
CharacterTable( "C2xa4" )

# Display  the character table
gap> Display(c2a4);
C2xa4

     2  3  3  1  1  3  3   1   1
     3  1  .  1  1  1  .   1   1

       1a 2a 3a 3b 2b 2c  6a  6b
    2P 1a 1a 3b 3a 1a 1a  3b  3a
    3P 1a 2a 1a 1a 2b 2c  2b  2b

X.1     1  1  1  1  1  1   1   1
X.2     1  1  A /A  1  1   A  /A
X.3     1  1 /A  A  1  1  /A   A
X.4     3 -1  .  .  3 -1   .   .
X.5     1  1  1  1 -1 -1  -1  -1
X.6     1  1  A /A -1 -1  -A -/A
X.7     1  1 /A  A -1 -1 -/A  -A
X.8     3 -1  .  . -3  1   .   .

A = E(3)
  = (-1+ER(-3))/2 = b3

# Definion of the character char
gap> char:=Sum(Irr(c2a4){[3,8]});
# Definition of two Hilbert series
gap> Hil:=MolienSeries(char);
( 1-z^2+z^3+2*z^4 ) / ( (1-z^6)*(1-z^3)*(1-z^2)^2 )
gap> MolienSeriesWithGivenDenominator(Hil,[2,3,4,6]);
( 1+z^3+z^4+z^5+2*z^6 ) / ( (1-z^6)*(1-z^4)*(1-z^3)*(1-z^2) )
gap> Hil2:=MolienSeries(lambda,Irr(c2a4)[3]);
( 1+z-z^2+z^3+z^4 ) / ( (1-z^6)*(1-z^3)*(1-z^2)^2 )
gap>  MolienSeriesWithGivenDenominator(Hil2,[2,3,4,6]);
( 1+z+2*z^3+z^5+z^6 ) / ( (1-z^6)*(1-z^4)*(1-z^3)*(1-z^2) )
```

8.10 [MAGMA]-Code for Section 5.5 and 5.6

I deleted some of the output of [MAGMA] so that you can extract the idea more easily.

a) First we give the instructions for the invariant ring with respect to G. Since we want to shorten the instructions, we use the abbreviations

$$\mathrm{MJ} := M_J \otimes M_J \quad \mathrm{MT} := M_{T^{-\mathrm{tr}}} \otimes M_T \quad \text{and} \quad \mathrm{Flip} := M_{\mathrm{Flip}}.$$

```
# Get more output for Gröbner basis calculations
> SetVerbose ("Groebner",1);
# Definitions
> F<w>:=CyclotomicField(3);
> MJ:=[ -1/3, 1/3, 1/3, -1/3, 2/3, 1/3, -2/3, -1/3, 2/3,
>       -2/3, 1/3, -1/3, -4/3, -2/3, -2/3, -1/3 ];
> MT:=[ 2/3*w+1/3*w^2, 1/3*w-1/3*w^2, 0, 0, 2/3*w-2/3*w^2,
>       1/3*w+2/3*w^2, 0, 0, 0, 0, -1/3*w-2/3*w^2,
>       -2/3*w-1/3*w^2,0, 0, -4/3*w-2/3*w^2, 1/3*w-1/3*w^2 ] ;
>
> g:=MatrixGroup<4,F|[MJ,MT]>;
> invring:=InvariantRing(g);
> hsinvring:=HilbertSeries(invring);
> prim:=PrimaryInvariants(invring);
> sec:=SecondaryInvariants(invring);
```

With the instruction `>invring;` we can get a lot of information on the invariant ring, e. g. Hilbert series and a set of primary and secondary invariants. The Hilbert series can be found in Section 5.5 and a set of primary and secondary invariants are written out in Section 8.8.

b) Now we give the instructions to decide whether a given set is a homogeneous system of parameters in $\mathcal{S}[V \otimes V]/_{\ker \pi}$.

```
# Definition of the kernel of pi.
> k:=RationalField();
> P<x1,x2,x3,x4>:=PolynomialRing(k,4);
> id:= hom<P->P|[x1,x2,x3,x4]>;
> prim:=id(prim);
> sec:=id(sec);
> ker:=[x1*x4-x2*x3];
#  Testing whether the set test is a hsop
> test:=[ker[1],prim[2],prim[3],prim[4]];
> hs:=HilbertSeries(Ideal(test));
> hs;
```

```
t^8 + 4*t^7 + 9*t^6 + 14*t^5 + 16*t^4 + 14*t^3 + 9*t^2 + 4*t + 1
# Further examples
> test1:=[ker[1],prim[1],prim[2],prim[3]];
> hs1:=HilbertSeries(Ideal(test1));
> test2:=[ker[1],prim[1],prim[3],prim[4]];
> hs2:=HilbertSeries(Ideal(test2));
> test3:=[ker[1],prim[1],prim[2],prim[4]];
> hs3:=HilbertSeries(Ideal(test3));
> hs1;
t^6 + 4*t^5 + 8*t^4 + 10*t^3 + 8*t^2 + 4*t + 1
> hs2;
t^7 + 4*t^6 + 8*t^5 + 11*t^4 + 11*t^3 + 8*t^2 + 4*t + 1
> hs3;
(t^6 + 2*t^5 - 3*t^3 - 4*t^2 - 3*t - 1)/(t - 1)
```

c) We also give the instructions to determine further relations. The images of the set of primary and secondary invariants can be found in Section 5.5.

```
# Definition of pi
> Q<y1,y2,z1,z2>:=PolynomialRing(k,4);
> gens:=[y1*z1, y1*z2, y2*z1, y2*z2];
> pi:=hom<P->Q|gens>;
> primpi:=pi(prim);
> secpi:=pi(sec);
# More relations can be calculated in the following way
> R<t1,t2,t3>:=PolynomialRing(k,3);
> RelationIdeal([primpi[1],primpi[4],secpi[3]],R);
Ideal of Polynomial ring of rank 3 over Rational Field
Lexicographical Order
Variables: t1, t2, t3
Basis:
[
    t1^2 - t2 - 3/4*t3
]
```

d) Since the instructions to determine the results of Section 5.6 can be determined in an analogous way, we only write out the definition of the new group.

```
> l3:=[1,0,0,0,0,0,1,0,0,1,0,0,0,0,0,1];
> g:=MatrixGroup<4,F|[l1,l2,l3]>;
```

8.11 More Borcherds Products

The following table lists more Borcherds products with weight at most 100:

weight	$\lambda_1^\perp$	$\lambda_4^\perp$	$\lambda_5^\perp$	$\lambda_9^\perp$	$\lambda_{16}^\perp$	$\lambda_{20}^\perp$	ν	A	B	C	D
4	2	1	1	0	0	0	$\kappa_{1,2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	0
5	7	1	0	0	0	0	χ_5	1	4	5	0
10	2	0	0	1	0	0	1	1	2	5	0
11	1	3	4	0	0	0	χ_5	1	2	5	0
12	0	0	3	0	0	1	χ_5	1	0	0	1
13	5	0	2	0	0	1	$\kappa_{1,2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	1
14	10	0	1	0	0	1	χ_5	2	5	5	1
15	15	0	0	0	0	1	$\kappa_{1,2}$	$\frac{5}{2}$	$\frac{15}{2}$	$\frac{15}{2}$	1
17	1	2	3	1	0	0	$\kappa_{1,2}\chi_5$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{15}{2}$	0
18	0	5	7	0	0	0	$\kappa_{1,2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{15}{2}$	0
23	1	1	2	2	0	0	χ_5	2	3	10	0
24	0	4	6	1	0	0	1	2	3	10	0
29	1	0	1	3	0	0	$\kappa_{1,2}\chi_5$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{25}{2}$	0
29	1	4	1	0	1	0	$\kappa_{1,2}\chi_5$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{25}{2}$	0
30	0	3	5	2	0	0	$\kappa_{1,2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{25}{2}$	0
30	0	1	0	0	1	1	$\kappa_{1,2}\chi_5$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{15}{2}$	1
30	6	4	0	0	1	0	1	3	6	15	0
35	1	3	0	1	1	0	χ_5	3	4	15	0
36	0	2	4	3	0	0	1	3	4	15	0
36	0	6	4	0	1	0	1	3	4	15	0
39	3	0	1	0	1	2	$\kappa_{1,2}\chi_5$	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{15}{2}$	2
40	8	0	0	0	1	2	1	4	5	10	2
42	0	1	3	4	0	0	$\kappa_{1,2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{35}{2}$	0
42	0	5	3	1	1	0	$\kappa_{1,2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{35}{2}$	0
48	0	0	2	5	0	0	1	4	5	20	0
48	0	0	2	1	1	2	1	4	2	10	2
48	0	4	2	2	1	0	1	4	5	20	0

weight	$\lambda_1^\perp$	$\lambda_4^\perp$	$\lambda_5^\perp$	$\lambda_9^\perp$	$\lambda_{16}^\perp$	$\lambda_{20}^\perp$	ν	A	B	C	D
54	0	3	1	3	1	0	$\kappa_{1,2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{45}{2}$	0
54	0	7	1	0	2	0	$\kappa_{1,2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{45}{2}$	0
55	5	7	0	0	2	0	χ_5	5	8	25	0
60	0	2	0	4	1	0	1	5	6	25	0
60	0	6	0	1	2	0	1	5	6	25	0
65	1	0	0	4	1	1	$\kappa_{1,2}$	$\frac{11}{2}$	$\frac{11}{2}$	$\frac{45}{2}$	1
65	1	0	0	0	2	3	$\kappa_{1,2}$	$\frac{11}{2}$	$\frac{5}{2}$	$\frac{25}{2}$	3
80	4	10	0	0	3	0	1	7	10	35	0
84	0	0	1	2	2	3	χ_5	7	7	30	1
84	0	0	1	6	1	1	χ_5	7	4	20	3
95	1	1	0	8	1	0	χ_5	8	10	40	0

8.12 Paramodular Forms

weight 13
- secondary generator: none
- relation:

$$\mathrm{Lift}_{\Phi_{6,5}}\mathrm{Lift}_{\Phi_{7,5}} = 2880\,\mathrm{Lift}^2_{E_{4,5}}\mathrm{Lift}_{\Phi_{5,5}} - 24\,\mathrm{Lift}_{\Phi_{5,5}}\mathrm{Lift}_{E_{4,2}E_{4,3}}\,.$$

- possible list of needed Fourier coefficients: $(1,1,-4\ldots-2)$, $(2,2,-8\ldots-7)$.

weight 14 As in weight 12, there exists a skew-symmetric Borcherds product: Ψ_{14}. We therefore need only nine symmetric paramodular forms:

- secondary generators: $\mathrm{Lift}_{g_4^2\Phi_{6,5}}, \Psi_{14}$.
- relations:

$$\begin{aligned}
\mathrm{Lift}^2_{\Phi_{7,5}} &= 240\,\mathrm{Lift}^2_{\Phi_{5,5}}\mathrm{Lift}_{E_{4,5}} + 48\,\mathrm{Lift}_{\Phi_{5,5}}\mathrm{Lift}_{\Phi_{4,5/2}\Phi_{6,5/2}},\\
\mathrm{Lift}_{\Phi_{6,5}}\mathrm{Lift}_{\Phi^2_{4,5/2}} &= 48\,\mathrm{Lift}_{\Phi_{5,5}}\mathrm{Lift}_{\Phi_{4,5/2}\Phi_{6,5/2}},\\
\mathrm{Lift}_{\Phi_{6,5}}\mathrm{Lift}_{E_{4,2}E_{4,3}} &= 6528\,\mathrm{Lift}^2_{\Phi_{5,5}}\mathrm{Lift}_{E_{4,5}} + 192\,\mathrm{Lift}_{\Phi_{5,5}}\mathrm{Lift}_{\Phi_{4,5/2}\Phi_{6,5/2}}\\
&\quad + 136\,\mathrm{Lift}^2_{E_{4,5}}\mathrm{Lift}_{\Phi_{6,5}} - \frac{1}{3600}\mathrm{Lift}_{g_4^2\Phi_{6,5}}\,.
\end{aligned}$$

- possible list of needed Fourier coefficients:

$$(0,0,0),\ (1,1,-4\ldots-1),\ (1,2,-5),\ \text{and}\ (2,2,-8\ldots-6).$$

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