Domination Parameters and their Unique Realizations

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Preface

Graph theory has undergone a powerful development, since its beginning in the 18th century. One of the branches of graph theory that has emerged rapidly in the last decades is domination in graphs. Berge [5] and Ore [85] have introduced the concept of domination around 1962, and numerous generalizations and modifications of this concept followed, motivated by various applications and problems. The corresponding domination parameters state the minimum or maximum cardinalities of subsets of the vertex set (or of the edge set) satisfying the domination properties. There are also several graph parameters belonging to similar concepts such as independence, matching and irredundance.

We say that a parameter has a unique realization for some graph if the subset measured by this parameter is unique. For instance, the domination number indicates the minimum cardinality of a dominating set in a graph, and this number has a unique realization if the considered graph has a unique minimum dominating set.

An introduction and an extensive overview on domination in graphs and related topics is given in the book 'Fundamentals of domination in graphs' [59] by Haynes, Hedetniemi and Slater. In the sequel 'Domination in graphs, Advanced Topics' [60], edited by Haynes, Hedetniemi and Slater, several authors present a survey of the wide field of domination in graphs.

In this thesis we will consider the classical graph parameters derived from concepts as ordinary domination, total domination, distance domination, edge domination, independence and irredundance. In this context our main attention is turned to the question of unique realizations of these parameters.

In graph theory, as well as in other areas of mathematics, besides the existence of special objects also their uniqueness is of interest. For example there are numerous publications on graphs with unique k-factors, especially on graphs being uniquely Hamiltonian, on graphs being uniquely Hamiltonian-connected from a vertex, uniquely pancyclic, uniquely factorizable, uniquely intersectable or uniquely partitionable. Maybe one of the most noted topics in graph theory concerning uniqueness are uniquely colorable graphs, see e.g. [9], [16], [55], [56], [83], [84] and [97].

There succeeded a couple of publications on unique realizations of other graph parameters. For instance, Hopkins and Staton [70] and Siemes, Topp, and Volkmann [90] have investigated graphs with unique maximum independent sets. Unique minimum dominating sets and related topics have been studied e.g. in [1], [53] and [94].

In this thesis, that is subdivided into two parts, we mainly study unique realizations of domination parameters. In the first part we investigate for several graph classes and various graph parameters the case of unique realization and the structure of the corresponding unique subsets. In the second part we consider bounds on domination parameters with and without unique realizations and we characterize extremal graphs in respect of these bounds.

Chapter 1 contains an introduction of the terminology and notation used throughout this text. In Chapter 2, at the beginning of Part I, we take a look at the complexity of the decision problem whether a graph parameter has a unique realization for a given graph. We present an idea how to make this decision in polynomial time in special classes of graphs without looking at the structure of the graphs, and we apply this method to the domination number, the independence number and the chromatic number. But the structure of graphs where domination parameters have unique realizations is also of interest, since this structure can give informations about other criteria as upper bounds on the size (see Chapter 6) and on the domination parameter itself (see Part II). Furthermore, we will see that for certain graph classes the knowledge of the structure leads to more efficient and often linear time algorithms to solve the decision problem. Therefore, we study the unique realization of the ordinary domination number in Chapter 3. Gunther, Hartnell, Markus and Rall [53] have characterized unique minimum dominating sets of trees. We generalize this characterization for block graphs (see Theorem 3.5) and we present a characterization of cactus graphs with unique minimum dominating sets (see Theorem 3.11). Both characterizations imply efficient algorithms to decide whether a given graph of this class has a unique minimum dominating set. Chapter 4 deals with the parameters in the famous inequality chain $ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq \mathrm{IR}(G)$, first observed by Cockayne, Hedetniemi and Miller in [26]. Here, we study graphs where two of the parameters in this chain have the same value, and we investigate the influence of the unique realization of one parameter to the realization of the other parameter with equal value. Furthermore, we present a characterization of unique minimum independent dominating sets in trees with equal domination and independent domination number and a characterization of unique upper dominating sets in chordal graphs, both leading to efficient algorithms. The structure of graphs having a unique realization of the distance domination number, total domination number or edge domination number is considered in Chapter 5. Distance domination, introduced by Slater in [91], is an extension of the ordinary domination concept with many applications. Also useful is the concept of total domination which was introduced by Cockayne, Dawes and Hedetniemi [19], motivated by the famous *Five Queens Problem* which was posed by de Jaenisch [29] in 1862. Edge domination is the analogue to the domination concept for edge sets, and Topp has already studied graphs with unique minimum edge dominating sets e.g. in [94]. For each of these three parameters we present general necessary conditions for a unique realization and characterizations of trees for which the parameter has a unique realization. Again, these characterizations imply polynomial time algorithms to decide whether a tree has a unique minimum distance dominating set, total dominating set or edge dominating set. Some of the structural results in this chapter are used to prove results in the second part of this thesis. In Chapter 6, the last chapter of Part I, we investigate the size of graphs having unique minimum dominating sets. Firstly, we pose a conjecture on the maximum size and we present a class of graphs achieving the upper bound in this conjecture, and secondly we prove a special case and a weakened version of the conjecture.

The second part of this thesis is devoted to upper bounds on domination parameters and to extremal graphs with regard to these bounds. In Chapter 7 we consider distance domination, and we present a characterization of graphs achieving the upper bound on the distance domination number given by Henning, Oellermann and Swart in [67]. Furthermore, we give a better upper bound on the distance domination number with unique realization, and we show a necessary condition for graphs achieving this bound. Moreover, we show that for a special class of graphs, containing the trees, this condition characterizes all graphs in this class for which the distance domination number has a unique realization and attains this bound. Chapter 8 considers the same problem for total domination, and here we present an upper bound on the total domination number with unique realization and a characterization of all graphs attaining this bound. In the last chapter we study upper bounds and extremal graphs with respect to exact distance domination. This concept was introduced by Boland, Haynes and Lawson in [8]. We characterize all graphs of diameter at least $2k-1$ which have the exact distance-k domination number equal Ore's upper bound (see Theorem 9.6), whereby we give an affirmative answer to a conjecture of Boland, Haynes and Lawson in [8].

Aachen, October 2001 Miranca Fischermann

Contents

Chapter 1

Introduction

In this first chapter we present most of the terminology and notation used throughout this thesis. Since we assume some basic knowledge of graph theory, readers who are unfamiliar with it may consult the books of Volkmann [101] or Chartrand and Lesniak [17]. Some special definitions that are only relevant in particular chapters will not be declared here but in place where they are used.

We only consider finite and simple graphs. Furthermore, all results and proofs that are not due to the author and all results found by the author that are already published or submitted for publication are indicated by the corresponding reference. Results without any indication of their source are discovered by the author.

1.1 General concepts

Definition 1.1 [Graphs] For any *graph* G the *vertex set* of G is denoted by $V(G)$, and the *edge set* of G is denoted by $E(G)$. If $V(G)$ is finite, then we call the graph G a *finite graph*, and we define the *order* $n(G)$ of G by $n(G) = |V(G)|$ and the *size* $m(G)$ of G by $m(G) = |E(G)|$. If G has order $n(G) = 1$ and size $m(G) = 0$, then we call G a *trivial graph*.
Let $e \in E(G)$ be an edge in G that has the two endpoints $v, w \in V(G)$. Then, we also write $e = vw$, and we say e is *incident with* the vertices v and w. We call a graph G *simple*, if firstly every edge in G is incident with two different vertices and secondly no two different edges in G are incident with the same two vertices. If $e = vw$ and $v \neq w$, then we say v and w are *adjacent*. If two different edges $e, e' \in E(G)$ have one common endpoint, then we say e and e' are incident.

Definition 1.2 [Neighbourhood and degree] For any vertex $x \in V(G)$ the *(open) neighbourhood* $N(x,G)$ of x in G is defined as

$$N(x) = N(x,G) = \{y \in V(G) \mid xy \in E(G)\},$$

and the *closed neighbourhood* $N[x,G]$ of x in G is defined as

$$N[x] = N[x,G] = N(x,G) \cup \{x\}.$$

If $X \subseteq V(G)$, then

$$N(X) = N(X,G) = \bigcup_{x \in X} N(x,G) \quad \text{and} \quad N[X] = N[X,G] = N(X,G) \cup X$$

denote the (open) neighbourhood and the closed neighbourhood of X, respectively.
For a subset D of $V(G)$ and a vertex $x \in D$, the set

$$P(x,D,G) = P(x,D) = N[x,G] \setminus N[D \setminus \{x\},G]$$

is the *private neighbourhood* of x with regard to D and a vertex $y \in P(x,D)$ is called a *private neighbour* of x with regard to D. Furthermore, for a subset D of $V(G)$ and a subset $A \subseteq D$ we define the set $P(A,D) = \bigcup_{x \in A} P(x,D)$. The set $P(x,D) \setminus \{x\}$ is called the *private exterior neighbourhood* of x with regard to D.
In a simple graph the *degree* $d(x) = d_G(x)$ of any vertex x in G quotes the number of edges in G incident with x, i.e. $d(x) = |N(x)|$ for every vertex $x \in V(G)$. A vertex $x \in V(G)$ of degree $d(x) = 0$ is called *isolated* and a vertex $x \in V(G)$ of degree $d(x) = 1$ is called an *endvertex* of G. The *minimum degree* $\delta(G)$ and the *maximum degree* $\Delta(G)$ of G are defined by

$$\delta(G) = \min\{d(x) \mid x \in V(G)\} \quad \text{and} \quad \Delta(G) = \max\{d(x) \mid x \in V(G)\}.$$

For two subsets X and Y of $V(G)$ we denote by $m(X,Y) = m_G(X,Y)$ the number of edges in G that have one endpoint in X and the other endpoint in Y. Further, we denote $\bar{X} = V(G) \setminus X$ and $d(X) = d_G(X) = m_G(X,\bar{X})$.

Definition 1.3 [Subgraphs and graph operations] Let G be a graph. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and we write $H \subseteq G$. A subgraph H of G is called a *factor* or a *spanning subgraph* of G if $V(H) = V(G)$. For any set $X \subseteq V(G)$ we define the *subgraph* $G[X]$ *of* G *induced by* X as the graph with vertex set X and edge set $\{xy \in E(G) \mid x,y \in X\}$. Further, for two subsets X and Y of the vertex set we define $G[X,Y]$ as the graph with vertex set $X \cup Y$ and with edge set $\{xy \in E(G) \mid x \in X, y \in Y\}$.

For any vertex set $X \subseteq V(G)$ and any vertex $x \in V(G)$ we define $G - X = G[V(G) \setminus X]$ and $G - x = G - \{x\}$.
For any edge set B we denote by $V(B)$ the set of vertices that are endpoints of an edge in B. Further, for any subset $B \subseteq E(G)$ we define the subgraph $G(B)$ as the graph with edge set B and vertex set $V(B)$. We call $G(B)$ the *subgraph of G created by B*. For any set $B \subseteq E(G)$ and any edge $e \in E(G)$ we define $G - B = G(E(G) \setminus B)$ and $G - e = G - \{e\}$. Notice that in our definition the subgraphs $G(B)$, $G - B$ and $G - e$ contain no isolated vertices.
For any graph G, an arbitrary vertex set X and an arbitrary edge set B we define
$G - X = G - (V(G) \cap X)$ and $G - B = G - (E(G) \cap B)$.
If G, X and B fulfil $V(B) \subseteq V(G) \cup X$, then $G + X + B$ is the graph with vertex set $V(G) \cup X$ and edge set $E(G) \cup B$.
For any vertex x and an edge yz with $y, z \in V(G) \cup \{x\}$ we define

$$G + x + yz = G + \{x\} + \{yz\}.$$

The *corona* $G \circ K_1$ is the graph consisting of one copy of the graph G where for every vertex $v \in V(G)$ a new vertex v' and the edge vv' are added.

Definition 1.4 [Path, cycle, and distance] Let G be a graph. A *path* P in G of *length* $l = \mathcal{L}(P)$ for some positive integer l is a sequence of $l+1$ in pairs different vertices $v_0, v_1, \ldots, v_l \in V(G)$ such that $v_{i-1}v_i \in E(G)$ for every $1 \leq i \leq l$. We use the notation

$$P = v_0 v_1 \ldots v_l,$$

and v_0 and v_l are called *endvertices of the path.* A path of length 0 consists of only one vertex v_0 and is called a *trivial path.*
A *cycle* C in G of *length* $l = \mathcal{L}(C)$ for some positive integer l is a sequence of l in pairs different vertices $v_1, v_2, \ldots, v_l \in V(G)$ such that $v_i v_{i+1} \in E(G)$ for every $1 \leq i < l$ and $v_l v_1 \in E(G)$. In a simple graph any cycle has length at least 3. We use the notation

$$C = v_1 v_2 \ldots v_l v_1.$$

For two vertices $x, y \in V(G)$ the *distance* $d(x, y) = d_G(x, y)$ between x and y in G is defined as the length of a shortest path in G from x to y. If there is no path in G from x to y, then we set $d(x, y) = d_G(x, y) = \infty$. If we define the *eccentricity ecc*(x) of a vertex x in G as $\mathrm{ecc}(x) = \max_{y \in V(G)} d_G(x, y)$, then the *diameter diam(G)* of G and the *radius rad(G)* of G are defined as

$$\mathrm{diam}(G) = \max_{x \in V(G)} \mathrm{ecc}(x) = \max_{x,y \in V(G)} d_G(x, y) \quad \text{and} \quad \mathrm{rad}(G) = \min_{x \in V(G)} \mathrm{ecc}(x),$$

and the *center* $C(G)$ of the graph G is defined as

$$C(G) = \{x \in V(G) \mid \text{ecc}(x) = \text{rad}(G)\}.$$

Definition 1.5 [Connectivity] A graph G is called *connected* if there exists a path between every two vertices of G. A connected subgraph H of G is called a *component* of G if there is no connected subgraph $H' \subseteq G$ with $H \subset H'$ and $H \neq H'$. The number of components of G is denoted by $\kappa(G)$, and the parameter $\mu(G) = m(G) - n(G) + \kappa(G)$ is called the *cyclomatic number* of G. A vertex $x \in V(G)$ is called a *cutvertex* of G, if

$$\kappa(G - x) > \kappa(G).$$

A connected subgraph B of G is called a *block*, if B has no cutvertex and every subgraph $B' \subseteq G$ with $B \subseteq B'$ and $B \neq B'$ has at least one cutvertex. A block B of G is called an *endblock*, if B contains at most one cutvertex of G. An edge $e \in E(G)$ is called a *bridge* of G, if

$$\kappa(G - e) > \kappa(G).$$

Definition 1.6 [k-neighbourhood] Let G be a graph and let k be a positive integer. For any vertex $x \in V(G)$ the *(open) k-neighbourhood* $N_k(x, G)$ and the closed k-neighbourhood $N_k[x, G]$ of x are defined by

$$N_k(x) = N_k(x, G) = \{y \in V(G) \mid y \neq x \text{ and } d(x, y) \leq k\} \quad \text{and}$$
$$N_k[x] = N_k[x, G] = N_k(x, G) \cup \{x\}.$$

Analogous we define for every set $X \subseteq V(G)$ the *(open) k-neighbourhood* $N_k(X, G)$ and the *closed k-neighbourhood* $N_k[X, G]$ *of* X by

$$N_k(X) = N_k(X, G) = \bigcup_{x \in X} N_k(x) \quad \text{and} \quad N_k[X] = N_k[X, G] = N_k(X) \cup X,$$

and for any vertex $x \in X$ we define the *private k-neighbourhood of x with regard to D* by

$$P_k(x, D) = N_k[x] \setminus N_k[D \setminus \{x\}]$$

and every vertex $y \in P_k(x, D)$ is called a *private k-neighbour of x with regard to D.*

Definition 1.7 [Distance graphs and the complement of a graph] Let G be a graph and let k be a positive integer. The *k-th power* G^k of G is a graph with vertex set $V(G)$ and two distinct vertices are adjacent in G^k if and only if the distance between them in G is at most k.
The *exact distance-k graph* $D_k(G)$ has vertex set $V(G)$ and two distinct vertices are adjacent in $D_k(G)$ if and only if the distance between them in G is equal k. Briefly we name $D_k(G)$ the *ed-k graph* of G.
The complement $\bar{G}$ of a graph G has vertex set $V(G)$ and two distinct vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

If $\mathrm{diam}(G) = 2$, then $\bar{G} = D_2(G)$. Harary, Hoede and Kadleček [57] have already investigated exact distance-k graphs and, especially, the connectedness of distance-2 graphs in 1982.

Definition 1.8 [Special graph classes] We denote a path of order n by P_n and a cycle of order n by C_n. If G is a simple graph with vertex set $V(G)$ and edge set $E(G) = \{vw \mid v, w \in V(G)\}$, then we call G *complete*. We denote a complete graph of order n by K_n.
If G is a graph with vertex set $V(G) = V_1 \cup V_2 \cup \ldots \cup V_p$ for $p \geq 2$ disjoint subsets $V_1, V_2, \ldots, V_p \subseteq V(G)$ such that the induced subgraph $G[V_i]$ contains no edge for every subindex $i = 1, 2, \ldots, p$, then we call G *p-partite* with *partite sets* $V_1, V_2, \ldots, V_p$. For $p = 2$ we call the graph G also *bipartite*.
If G is a simple p-partite graph with partite sets $V_1, V_2, \ldots, V_p$ and with edge set $E(G) = \{xy \mid x \in V_i, y \in V_j, 1 \leq i < j \leq p\}$, then we call G *complete p-partite*, and if $p = 2$, *complete bipartite*. We denote a complete p-partite graph with partite sets $V_1, V_2, \ldots, V_p$ of order $|V_i| = n_i$ for every $1 \leq i \leq p$ by $K_{n_1,n_2,\ldots,n_p}$.
The complete bipartite graph $K_{1,n-1}$ is called a *star* and the star $K_{1,3}$ is called a *claw*. We say that a graph G is *claw-free* if G does not contain the graph $K_{1,3}$ as an induced subgraph.
A graph without cycles is called a *forest*, and if a forest is connected, we call it a *tree*.
A simple graph G is called a *cactus graph*, if all cycles in G are edge disjoint in pairs.
A graph G is called a *block graph*, if every block in G is complete, and a graph G is called a *block-cactus graph*, if every block in G is either complete or a cycle of order at least 3.

Observation 1.9 It is straightforward to see that a graph G is a *forest* if and only if every block in G is either the K_1 or the K_2, and G is a cactus graph if and only if every block in G is either the K_1, K_2 or a cycle of order at least 3.

The following lemma contains well-known characterizations of forests and of cactus graphs by the cyclomatic number.

Lemma 1.10

a) A simple graph G is a forest if and only if $\mu(G) = 0$.

b) A simple graph G is a cactus graph if and only if the number of cycles in G is equal $\mu(G)$.

1.2 Domination concepts

First, we define the ordinary domination concept.

Definition 1.11 [Domination] A set $D \subseteq V(G)$ is a *dominating set* of G, if every vertex $v \in V(G) \setminus D$ has at least one neighbour in D. We call a dominating set D *minimal* if there is no dominating set $D' \subseteq V(G)$ with $D' \subset D$ and $D' \neq D$. Further, we call a dominating set D *minimum* if there is no dominating set $D' \subseteq V(G)$ with $|D'| < |D|$. The cardinality of a minimum dominating set is called the *domination number*, denoted by $\gamma(G)$ and a minimum dominating set D of G is also called a γ-*set*.

Whenever we talk of ordinary domination in this thesis, we mean the domination concept in Definition 1.11.

Observation 1.12 It is easy to see that for any graph G a dominating set D is minimal if and only if $|P(v, D)| \geq 1$ for every vertex $v \in D$.

This observation is due to Ore [85], such as the well-known upper bound on the domination number in the following theorem.

Theorem 1.13 ([85]) *If a graph G has no isolated vertices, then*

$$\gamma(G) \leq n(G)/2.$$

There are many generalizations and modifications of ordinary domination. Cockayne, Dawes, and Hedetniemi [19] introduced the total domination, motivated by the famous *Five Queens Problem* posed in 1862 by de Jaenisch [29].

Definition 1.14 [Total domination] A set $D \subseteq V(G)$ is a *total dominating set* of G if every vertex in $V(G)$ has at least one neighbour in D. Note that a graph has a total dominating set if and only if it has no isolated vertices. A total dominating set D is called *minimal* if no proper subset of D is a total dominating set. If a graph G has no isolated vertices, then we define the *total domination number* of G as the minimum cardinality of a total dominating set of G and we denote the total domination number of G by $\gamma_t(G)$. A total dominating set D of G of cardinality $\gamma_t(G)$ is called a *γ_t-set* or a *minimum total dominating set*.

Another generalization of the ordinary domination concept is the k-domination introduced by Fink and Jacobson in [36].

Definition 1.15 [k-domination] Let k be a positive integer and let $D \subseteq V(G)$. A vertex in $V(G) \setminus D$ is said to be *k-dominated* by D if it has at least k neighbours in D. If every vertex in $V(G) \setminus D$ is k-dominated by D, then D is called a *k-dominating set* of G. We define the *k-domination number* of G as the minimum cardinality of a k-dominating set of G, denoted by $\gamma_k(G)$. A k-dominating set D of G of cardinality $\gamma_k(G)$ is called a *γ_k-set* or a *minimum k-dominating set*.

Volkmann [102] introduced the X-domination. This concept appears in a natural way while constructing a minimal dominating set of a graph.

Definition 1.16 [X-domination] For a subset X of the vertex set $V(G)$ a set $D \subseteq V(G)$ is an *X-dominating set* of G, if $X \subseteq N[D, G]$. An X-dominating set of minimum cardinality is a *minimum X-dominating set*. The cardinality of a minimum X-dominating set is denoted by $\gamma(G, X)$. Note that the case $X = V(G)$ leads to ordinary domination.

Further extensions are the following two distance domination concepts the first defined by Slater [91] and the second by Boland, Haynes, and Lawson [8].

Definition 1.17 [Distance-k domination] Let k be a positive integer. A set $D \subseteq V(G)$ is a *distance-k dominating set* of G if $N_k[D, G] = V(G)$. The minimum cardinality of a distance-k dominating set is called the *distance-k domination number* denoted by $\gamma_{\leq k}(G)$. A distance-k dominating set D of G with cardinality $\gamma_{\leq k}(G)$ is called a *$\gamma_{\leq k}$-set* or a *minimum distance-k dominating set*.

Definition 1.18 [Exact distance-k domination] A set $D \subseteq V(G)$ is called a *exact distance-k dominating set* or briefly a *ed-k dominating set*, if every vertex $v \in V(G) - D$ has exactly distance k to at least one vertex in D. The minimum cardinality among all ed-k dominating sets is called the *exact distance-k domination number* or briefly the *ed-k domination number* denoted by $\gamma_{=k}(G)$.

The following properties of distance domination are straightforward to see (e.g. in [8]).

Observation 1.19

a) In Definition 1.15, Definition 1.17 and Definition 1.18 the case $k = 1$ leads to ordinary domination.

b) For every positive integer k, any graph G, and any subset $D \subseteq V(G)$ we get that D is a $\gamma_{\leq k}$-set ($\gamma_{=k}$-set) of G if and only if D is a γ-set of G^k (of $D_k(G)$), which implies that

$$\gamma_{\leq k}(G) = \gamma(G^k) \text{ and } \gamma_{=k}(G) = \gamma(D_k(G)).$$

c) Note that the distance-k domination number $\gamma_{\leq k}(G) = 1$, if $\mathrm{rad}(G) \leq k$, and the exact distance-k domination number $\gamma_{=k}(G) = n(G)$, if $\mathrm{diam}(G) < k$.

A further modification of the domination concept is the edge domination.

Definition 1.20 [Edge domination] A subset F of the edge set $E(G)$ is an *edge dominating set* of G if every edge in $E(G) \setminus F$ is incident to at least one edge in F. The *edge domination number* $\gamma'(G)$ is the smallest cardinality of all edge dominating sets and an edge dominating set of cardinality $\gamma'(G)$ is called a *minimum edge dominating set* or a *γ'-set* of G.

1.3 Domination related concepts

In this section we define those of the numerous graph parameters related to the domination concept which occur in this thesis.

The first definition deals with the property of minimal dominating sets mentioned in Observation 1.12.

Definition 1.21 [Irredundance and upper irredundance] A set $D \subseteq V(G)$ is *irredundant* if every vertex in D has at least one private neighbour. An irredundant set D of G is called *maximal irredundant* if $D \cup \{v\}$ is no longer irredundant for every vertex $v \in V(G) \setminus D$. The minimum cardinality of a maximal irredundant set is called the *irredundance number* and is denoted by $ir(G)$. A maximal irredundant set of G of cardinality $ir(G)$ is called a *minimum irredundant set* or an *ir-set*. Further, the maximum cardinality of an irredundant set is called the *upper irredundance number* and is denoted by $\mathrm{IR}(G)$, and an irredundant set of G of cardinality $\mathrm{IR}(G)$ is called an *upper irredundant set* or an IR-*set*.

Irredundant sets were first defined and studied by Cockayne, Hedetniemi and Miller in [26]. As upper irredundance is the counterpart of irredundance, the counterpart of ordinary domination is the concept of upper domination.

Definition 1.22 [Upper domination] A minimal dominating set D of a graph G is called a Γ-*set* or an *upper dominating set* if there is no minimal dominating set D' of G with $|D'| > |D|$. The cardinality of a Γ-set of G is denoted by $\Gamma(G)$.

Another property of vertex sets in graphs studied a lot is the independence introduced in [23] by Cockayne and Hedetniemi.

Definition 1.23 [Independence and distance independence]
a) A set $I \subseteq V(G)$ is called *independent*, if the subgraph induced by I contains no edge. We call an independent set I *maximal* if there is no independent set $I' \subseteq V(G)$ with $I \subset I'$ and $I' \neq I$. Further, we call an independent set I *maximum* if there is no independent set $I' \subseteq V(G)$ with $|I'| > |I|$. The cardinality of a maximum independent set is called the *independence number* denoted by $\alpha(G)$ and a maximum independent set is also called an α-set.
b) Let k be a positive integer. A set $I \subseteq V(G)$ is called *distance-k independent* if every two vertices $v, w \in I$ have distance at least $k+1$ from each other. The maximum cardinality of a distance-k independent set of G is denoted by $\alpha_{\leq k}(G)$, and we call a maximum distance-k independent set of G an $\alpha_{\leq k}$-set.

Note that every maximal (distance-k) independent set of a graph G is a minimal (distance-k) dominating set of G. This leads to the independent domination introduced by Cockayne and Hedetniemi [23] and to the extension distance-k independent domination introduced by Henning, Oellermann, and Swart [67].

Definition 1.24 [Independent domination and distance-k independent domination]
a) The minimum cardinality of an independent dominating set D of G is called the *independent domination number* denoted by $i(G)$, and a minimum independent dominating set is called an i-set.

b) Let k be a positive integer. A set $D \subseteq V(G)$ is called a *distance-k independent dominating set* of G if D is a distance-k dominating set of G and D is distance-k independent. The minimum cardinality of a distance-k independent dominating set of G is denoted by $i_{\leq k}(G)$, and we call a minimum distance-k independent dominating set of G an $i_{\leq k}$-set.

Distance-k independent domination is not to confuse with the concept of *independent distance-k domination* where a distance-k dominating set only has to be independent but not distance-k independent.

Observation 1.25 For every graph G independence is equivalent to distance-1 independence and $\alpha(G) = \alpha_{\leq 1}(G)$.
Independent domination is equivalent to distance-1 independent domination and $i(G) = i_{\leq 1}(G)$ for every graph G.
For every positive integer k and any graph G we obtain

$$\alpha_{\leq k}(G) = \alpha(G^k) \quad \text{and} \quad i_{\leq k}(G) = i(G^k).$$

The following lemma contains the well-known inequality chain which was first observed by Cockayne, Hedetniemi and Miller [26].

Lemma 1.26 *Let G be a graph. Every maximal independent set of G is a minimal dominating set and every minimal dominating set is maximal irredundant, and thus,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq \mathrm{IR}(G).$$

Analogously to independence and independent domination of vertex sets there are definitions for edge sets.

Definition 1.27 [Matching and independent edge domination]
a) A subset F of the edge set $E(G)$ is called *independent*, if no two edges in F are incident. An independent edge set is also named a *matching*, and we call a matching (an independent edge set) F *maximal* if there is no matching $F' \subseteq E(G)$ with $F \subset F'$ and $F' \neq F$. Further, we call a matching (an independent edge set) F *maximum* if there is no matching $F' \subseteq E(G)$ with $|F'| > |F|$. A matching F of a graph G is called *perfect* if $n(G) = 2|F|$. The cardinality of a maximum matching is called the *matching number* or the *edge independence number*, denoted by $\alpha'(G)$, and a maximum matching is also called an α'-set.
b) The minimum cardinality of an independent edge dominating set F of G is called the *independent edge domination number* denoted by $i'(G)$, and a minimum independent edge dominating set is called an i'-set.

The last graph parameter we define is the only parameter in this section that does not quote the cardinality of a special subset of the vertex set or of the edge set.

Definition 1.28 [Coloring and chromatic number] For any graph $G = (V, E)$ we name a function $h : V \to \{1, 2, \ldots, q\}$ a *coloring* or a *q-coloring* of G if $h(x) \neq h(y)$ for every two adjacent vertices x and y. The *chromatic number* $\chi(G)$ of G is the integer q such that G has a q-coloring but no $(q-1)$-coloring.

A q-coloring of a graph G yields a partition of the vertex set in q disjoint and independent subsets.

Finally we define the expression 'unique realization' which is of great importance in this thesis.

Definition 1.29 [Unique realization] Let ν be some graph parameter that measures some property of an object derived from a graph G. We say that a graph parameter ν has a *unique realization* in a graph G if the object in the graph measured by ν is unique.

For instance, the domination number γ and the independence number α indicate the cardinalities of special subsets of the vertex set (minimum dominating set, maximum independence set) and the chromatic number χ is the minimum number of disjoint and independent subsets that partition the vertex set. Thus, γ or α have a unique realization in some graph, if the graph has a unique minimum dominating set or a unique maximum independent set, respectively, and χ has a unique realization in some graph, if the χ-partition of the vertex set of this graph into independent sets is unique up to permutations.

Remark 1.30 If it is obvious to which graph G the notation refers, then we use the shorter notation $N(x), N[x], d(x)$, and so on.

For other graph theory terminology we follow the monograph by Haynes, Hedetniemi and Slater [59].

Part I

Uniqueness of domination

Chapter 2

On the complexity of unique realizations of graph parameters

In this chapter we consider the complexity of the problem to decide whether a graph parameter ν has a unique realization (cf. Definition 1.29) for a given graph G. We sketch a general method to make this decision in polynomial time for graphs in classes $\mathcal{G}_\nu$ satisfying the property that ν can be determined in polynomial time for any graph G in $\mathcal{G}_\nu$ and for special graphs obtained from G.

General idea.
Let ν be some graph parameter, and let us assume that there is a characterization $\mathcal{C}_\nu$ of the graphs with unique realizations of ν that can be checked for any given graph G by evaluating ν for a couple of graphs that arise from G by some local changes. Furthermore, we assume that the number of these graphs we have to look at is bounded by some polynomial in $n(G)$ and $m(G)$. Let $\mathcal{G}$ be a class of graphs. If it is possible to determine ν in polynomial time for any graph in $\mathcal{G}$ and for all graphs that arise from a graph in $\mathcal{G}$ by the above-mentioned local changes, then we can decide - again in polynomial time - whether ν has a unique realization for some graph in $\mathcal{G}$ using the characterization $\mathcal{C}_\nu$.

The results in the following section show that for several parameters ν there exist very simple such characterizations $\mathcal{C}_\nu$. Another possibility to decide whether ν has a unique realization is to use deeper structural characterizations of graphs with unique realization of ν. There are several such characterizations for special classes of graphs as e.g. in [53], [55], [70], [90], [94] and [97], and we will present further structural characterizations in the following chapters. These often imply - as a byproduct - that in the considered classes it can be decided in polynomial time whether ν has a unique realization. In some cases this strategy is more efficient than the general method presented here.

On the other hand, there are several classes of graphs for which no structural characterization is known but where the method presented in this chapter works (e.g. see Example 2.5).

In the following section we apply the general idea to the domination number γ, the independence number α and the chromatic number χ, and we describe the conditions on the classes $\mathcal{G}_\gamma, \mathcal{G}_\alpha$ and $\mathcal{G}_\chi$. In the case of uniqueness this method also yields unique minimum dominating sets, unique maximum independent sets and unique minimum colorings, respectively.

It is clear that our method also works for several other graph parameters.

2.1 Examples of the strategy

First we present easy characterizations of graphs with unique realizations for γ, α and χ, respectively. In order to do this, we define for any graph G the following three sets.

$$V_\gamma(G) = \{v \in V(G) \mid v \text{ belongs to every } \gamma\text{-set of } G\},$$
$$V_\alpha(G) = \{v \in V(G) \mid v \text{ belongs to every } \alpha\text{-set of } G\}, \text{ and}$$
$$E_\chi(G) = \{uv \notin E(G) | u, v \in V(G),\ \chi(G) = \chi(G + uv)\},$$

i.e. $E_\chi(G)$ is the set of edges of $\bar{G}$ whose addition to G does not increase the chromatic number.

Lemma 2.1 (Fischermann, Rautenbach and Volkmann [40])

(i) A graph G has a unique minimum dominating set if and only if the set $V_\gamma(G)$ is a dominating set of G.

(ii) A graph G has a unique maximum independent set if and only if the set $V_\alpha(G)$ is maximal independent.

(iii) A graph G is uniquely colorable if and only if the graph $G' = G + E_\chi(G)$ is a complete $\chi(G)$-partite graph.

Proof.
(i) '⇒' (trivial). '⇐' Let D_1 and D_2 be two different minimum dominating sets of G, then the set $D_1 \cap D_2$ dominates G and $|D_1 \cap D_2| < \gamma(G)$ which is a contradiction.
(ii) '⇒' (trivial). '⇐' Let I_1 and I_2 be two different maximum independent sets of G, then the set $I_1 \cap I_2 \neq I_1$ is maximal independent which is a contradiction.

(iii) Let $V_1 \cup V_2 \cup ... \cup V_{\chi(G)} = V(G)$ be a $\chi(G)$-coloring of G. Clearly, all non-edges of G with endpoints in different sets V_i belong to $E_\chi(G)$. If G is uniquely colorable, then no edge with both endpoints in the same set V_i belongs to $E_\chi(G)$ and $G + E_\chi(G)$ is a complete $\chi(G)$-partite graph. If G is not uniquely colorable, then there is a pair of vertices x, y and a second $\chi(G)$-coloring $V_1' \cup V_2' \cup ... \cup V_{\chi(G)}' = V(G)$ of G such that without loss of generality $x, y \in V_1$, $x \in V_1'$, and $y \in V_2'$. This implies that $xy \in E_\chi(G)$ and $\chi(G + E_\chi(G)) > \chi(G)$. Thus, $G + E_\chi(G)$ is no complete $\chi(G)$-partite graph. □

Now, we describe how to determine the sets $V_\gamma(G), V_\alpha(G)$ and $E_\chi(G)$ by evaluating the corresponding parameters for G and for special graphs arising from G by some local changes. In order to do so, we firstly have to define the local changes we need to find $V_\gamma(G)$.

Definition 2.2 Let G be a graph, let $v \in V(G)$, and let $u \in N(v, G)$. Then, we define the graph $G_{v,u} = (G - v) + u' + uu'$ where $u' \notin V(G)$, i.e. $G_{v,u}$ has vertex set $V(G_{v,u}) = (V(G) \setminus \{v\}) \cup \{u'\}$ and edge set $E(G_{v,u}) = (E(G) \setminus \{vw | w \in N(v, G)\}) \cup \{uu'\}$.

Lemma 2.3 (Fischermann, Rautenbach and Volkmann [40]) *Let G be a graph and let $v \in V(G)$ and $e \in E(\bar{G})$.*

(i) The vertex v belongs to $V_\gamma(G)$ if and only if $\gamma(G) < \gamma(G_{v,u})$ for every $u \in N(v, G)$.

(ii) The vertex v belongs to $V_\alpha(G)$ if and only if $\alpha(G - N[u, G]) < \alpha(G) - 1$ for every $u \in N(v, G)$.

(iii) The edge e belongs to $E_\chi(G)$ if and only if $\chi(G) = \chi(G + e)$.

Proof. (i) Let D be a minimum dominating set of $G_{v,u}$. Since in $G_{v,u}$ the vertex u has a neighbour of degree one, we can assume without loss of generality that $u \in D \subseteq V(G)$. Hence D is also a dominating set of G and we obtain that $|D| = \gamma(G_{v,u})$ is the minimum cardinality of a dominating set of G that contains u but not v. Therefore, $\min\{\gamma(G_{v,u}) | u \in N(v, G)\}$ is the minimum cardinality of a dominating set of G that does not contain v and the result follows.
(ii) As above, $\alpha(G - N[u, G]) + 1$ is the maximum cardinality of an independent set of G that contains u. Therefore, $\max\{\alpha(G - N[u, G]) + 1 | u \in N(v, G)\}$ is the maximum cardinality of an independent set of G that does not contain v and the result follows.
(iii) Trivial, by the definition of $E_\chi(G)$. □

The following propositions show the properties of graph classes that allow to decide efficiently if γ, α, or χ have unique realizations.

Proposition 2.4 (Fischermann, Rautenbach and Volkmann [40]) *Let $\mathcal{G}_\gamma$ be a class of graphs and let $p_\gamma(n,m)$ be some polynomial such that for every $G \in \mathcal{G}_\gamma$ and every $v \in V(G)$ and $u \in N(v,G)$, it is possible to determine γ for the graphs G and $G_{v,u}$ in time $p_\gamma(n(G), m(G))$. Then, for any graph $G \in \mathcal{G}_\gamma$ it can be decided in polynomial time $n(G)^2 \cdot p_\gamma(n(G), m(G))$ whether G has a unique minimum dominating set.*

Proof. Let $G \in \mathcal{G}_\gamma$ be arbitrary. Then, we can determine $\gamma(G)$ and $\gamma(G_{v,u})$ for every $v \in V(G)$ and $u \in N(v,G)$ in time $p_\gamma(n(G), m(G))$ where p_γ is some polynomial. By Lemma 2.3, we can decide in time $n(G) \cdot p_\gamma(n(G), m(G))$ for a specific vertex $v \in V(G)$, whether v is contained in every minimum dominating set of G. We can therefore find the set $V_\gamma(G)$ in time $n(G)^2 \cdot p_\gamma(n(G), m(G))$. By Lemma 2.1(i), it is now trivial to decide whether G has a unique minimum dominating set. □

The property of $\mathcal{G}_\gamma$ is not very restrictive and many of the classes of graphs for which γ can be computed efficiently have this property.

Example 2.5 The *strongly chordal graphs* [32] contain several other well-known classes of graphs (see [76]) and for them γ can be computed in linear time (see [33]). If G is a strongly chordal graph, then $G_{v,u}$ is also strongly chordal for every $v \in V(G)$ and $u \in N(v,G)$ (note that if $v_1v_2...v_n$ is a strong elimination ordering of the vertices of G and $v = v_i$ and $u = v_j$, then $v_1v_2...v_{i-1}v_{i+1}...v_{j-1}u'v_j...v_n$ is a strong elimination ordering of $G_{v,u}$). Thus, we can decide in $O(n^3)$ time if any strongly chordal graph has a unique γ-set.

Analogously as Proposition 2.4 we can now prove the following two results for α and χ.

Proposition 2.6 (Fischermann, Rautenbach and Volkmann [40]) *Let $\mathcal{G}_\alpha$ be a class of graphs and let $p_\alpha(n,m)$ be a polynomial such that for every $G \in \mathcal{G}_\alpha$ and every $v \in V(G)$, it is possible to determine α for the graphs G and $G - N[v,G]$ in polynomial time $p_\alpha(n(G), m(G))$. Then, for any graph $G \in \mathcal{G}_\alpha$ it can be decided in polynomial time $n(G)^2 \cdot p_\alpha(n(G), m(G))$ whether G has a unique maximum independent set.*

Again, the property of $\mathcal{G}_\alpha$ is not very restrictive and there are several classes of graphs for which α can be computed in polynomial time that have this property because they are closed under taking induced subgraphs (see e.g. [18], [69], [79], [80] and [87]).

Proposition 2.7 (Fischermann, Rautenbach and Volkmann [40]) *Let $\mathcal{G}_\chi$ be a class of graphs and let $p_\chi(n, m)$ be some polynomial such that for every $G \in \mathcal{G}_\chi$ and every $e \in E(\bar{G})$, it is possible to determine χ for the graphs G and $G + e$ in polynomial time $p_\chi(n(G), m(G))$. Then, for any graph $G \in \mathcal{G}_\chi$ it can be decided in polynomial time $n(G)^2 \cdot p_\chi(n(G), m(G))$ whether G is uniquely colorable.*

We will complete our consideration by a brief description of the algorithmic methods which follow immediately from Lemma 2.1 and Lemma 2.3.

Methods. (Fischermann, Rautenbach, Volkmann [40]) Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let the complement of G have edge set $E(\bar{G}) = \{e_1, e_2, \ldots, e_t\}$.

(i) 1. Let $V_0 = \emptyset$ and $i = 1$. Compute $\gamma(G)$.
2. For every neighbour $u \in N(v_i, G)$ determine $\gamma(G_{v_i,u})$.
If $\gamma(G) < \gamma(G_{v_i,u})$ for every $u \in N(v_i, G)$, then we define $V_i = V_{i-1} \cup \{v_i\}$. Otherwise, we define $V_i = V_{i-1}$.
3. If $i < n$, then increase i by one and go to 2. If $i = n$, then go to 4.
4. If V_n dominates G, then V_n is the unique γ-set of G. If V_n is not a dominating set of G, then γ has no unique realization for G.

(ii) 1. Let $V_0 = \emptyset$ and $i = 1$. Then, compute $\alpha(G)$.
2. For every neighbour $u \in N(v_i, G)$ determine $\alpha(G - N[u, G])$.
If $\alpha(G) > \alpha(G - N[u, G]) + 1$ for every $u \in N(v_i, G)$, then we define $V_i = V_{i-1} \cup \{v_i\}$. Otherwise, we define $V_i = V_{i-1}$.
3. If $i < n$, then increase i by one and go to 2. If $i = n$, then go to 4.
4. If $|V_n| = \alpha(G)$, then V_n is the unique α-set of G. If $|V_n| < \alpha(G)$, then α has no unique realization for G.

(iii) 1. Let $E_0 = \emptyset$ and $i = 1$. Then, compute $\chi(G)$.
2. Determine $\chi(G + e_i)$.
If $\chi(G) = \chi(G + e)$, then we define $E_i = E_{i-1} \cup \{e_i\}$. Otherwise, we define $E_i = E_{i-1}$.
3. If $i < t$, then increase i by one and go to 2. If $i = t$, then go to 4.
4. If $\chi(G + E_t) = \chi(G)$, then $G + E_t$ is complete $\chi(G)$-partite and the partition of $G + E_t$ is the unique χ-partition of G. If $\chi(G + E_t) > \chi(G)$, then χ has no unique realization for G.

Chapter 3

Ordinary domination

The main topic in this chapter is the characterization of graphs with unique minimum dominating sets. Gunther, Hartnell, Markus, and Rall have considered such graphs in [53], and they have characterized trees with unique minimum dominating sets.

We firstly present necessary conditions and sufficient conditions for the unique realization of the domination number γ in arbitrary graphs.

Lemma 3.1 (Gunther, Hartnell, Markus, Rall [53]) *If a graph G without isolated vertices has a unique γ-set D, then every vertex in D has at least two private neighbours other than itself.*

Proof. Let D be the unique γ-set of G, and let $x \in D$ be arbitrary. Then, for any vertex $y \in N(x, G) \neq \emptyset$ the set $(D \setminus \{x\}) \cup \{y\}$ is not a dominating set of G. Hence, we obtain that $|P(x, D) \setminus \{x\}| \geq 2$. □

Lemma 3.2 (Gunther, Hartnell, Markus, Rall [53]) *Let D be a γ-set of a graph G. If $\gamma(G - x) > \gamma(G)$ for every vertex $x \in D$, then D is the unique γ-set of G.*

Proof. Let D be a γ-set of the graph G, such that $\gamma(G - x) > \gamma(G)$ for every $x \in D$. Suppose, there is a γ-set D' of G different from D. Then, there is at least one vertex $x \in D \setminus D'$ and the set D' dominates $G - x$. This leads to the contradiction $|D'| \geq \gamma(G - x) > \gamma(G)$. □

The following result shows that the preceding two conditions characterize unique γ-sets in trees.

Theorem 3.3 (Gunther, Hartnell, Markus, Rall [53]) *Let T be a tree of order at least 3. Then the following conditions are equivalent:*

(i) *T has the unique γ-set D.*

(ii) *T has a γ-set D for which every vertex in D has at least two private neighbours other than itself.*

(iii) *T has a γ-set D for which $\gamma(T - x) > \gamma(T)$ for every vertex $x \in D$.*

In the first section of this chapter we generalize the characterization in Theorem 3.3 for block graphs. The second section contains a generalization of Theorem 3.3 for X-domination in trees which we use in the last section of this chapter. There, we present a characterization of cactus graphs with unique minimum dominating sets which is related to the one in Theorem 3.3.

3.1 Block graphs

In order to show that the characterization of Gunther, Hartnell, Markus and Rall in Theorem 3.3 also holds for block graphs we firstly present a helpful lemma about blocks and cutvertices by König. The interested reader can find a short proof of this result in the book 'Fundamente der Graphentheorie' by Volkmann [101], p.171.

Lemma 3.4 (König [75]) *Let G be a connected graph with at least one cut-vertex. If $B_1, B_2, \ldots, B_t$ are all blocks of G, then the following conditions hold.*

(i) $|V(B_i) \cap V(B_j)| \leq 1$ *for any* $1 \leq i < j \leq t$.

(ii) $E(G) = E(B_1) \cup \ldots \cup E(B_t)$ *and* $E(B_i) \cap E(B_j) = \emptyset$ *for any* $1 \leq i < j \leq t$.

(iii) *If* $x \in V(B_i) \cap V(B_j)$ *for any* $1 \leq i < j \leq t$, *then x is a cutvertex of G.*

(iv) *Every cutvertex of G belongs to at least two different blocks of G.*

(v) *If two vertices a and b do not belong to a common block of G, then there exists a cutvertex $x \neq a, b$ of G which lies on every path from a to b in G, i.e. a and b lie in different components of $G - x$.*

Using these simple properties of blocks and cutvertices and the special structure of block graphs we now prove the generalization of Theorem 3.3 for block graphs.

Theorem 3.5 (Fischermann [38]) *Let G be a block graph without isolated vertices and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique γ-set of G.

(ii) D is a γ-set of G such that every vertex in D has at least two private neighbours that do not lie in a common block.

(iii) D is a dominating set of G such that every vertex in D has at least two private neighbours that do not lie in a common block.

(iv) D is a γ-set of G such that $\gamma(G-x) > \gamma(G)$ for every vertex $x \in D$.

Proof. First we show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and then we prove (i) $\Rightarrow$ (iv). By Lemma 3.2, we obtain immediately (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): (iii) follows immediately from (ii). Now, we prove that (ii) follows from (i). Let D be the unique γ-set of G. Assume, there is a vertex $x \in D$, such that $P(x, D) \subseteq V(B)$ for some block B. Let $w \in V(B) \setminus \{x\}$. Hence, $P(x, D) \subseteq N[w, G]$ and $(D \setminus \{x\}) \cup \{w\}$ is a γ-set of G different from D, which is a contradiction.

(iii) $\Rightarrow$ (i): Assume, there is a block graph G without isolated vertices and a dominating set D of G that fulfil (iii) but D is no unique γ-set of G. Let G be such a counterexample of minimal order. Since D is no unique γ-set, there is a γ-set $D' \neq D$ of G and at least one vertex $y \in D \setminus D'$.
Define $Q_0 = \{y\}$ and for $i = 0, 1, \ldots$ define $Q_{i+1} = \bigcup_{x \in Q_i \cap D} N[x, G]$. Obviously, there is an integer s, such that $Q_{i-1} \neq Q_i$ for all $1 \leq i \leq s$ and $Q_i = Q_s = Q$ for all $i \geq s$. Let $G_0 = G[Q]$ be the subgraph of G induced by the vertices of Q. If $Q = V(G)$, then let $\kappa = 0$, and if $Q \neq V(G)$, then let $G_1, G_2, \ldots, G_\kappa$ be the components of $G - Q$. Further, let $D_i = D \cap V(G_i)$ and $D'_i = D' \cap V(G_i)$ for $i = 0, 1, \ldots, \kappa$. For every $i = 0, 1, \ldots, \kappa$ the set D_i is a dominating set of G_i and $P(x, D) \subseteq V(G_i)$ for every $x \in D_i$. Hence, for every $i = 0, 1, \ldots, \kappa$ the graph G_i is a block graph without isolated vertices which fulfils (iii) with the corresponding dominating set D_i.
Firstly, we show that $\kappa > 0$, which implies that for every $i = 0, 1, \ldots, \kappa$ the order of G_i is less than the order of G and D_i is the unique γ-set of G_i, by the choice of G.
Let $w_1, w_2 \in P(y, D)$, such that w_1, w_2 do not lie in a common block of G. By Lemma 3.4(ii), there is a unique block B_i of G_0 that contains the edge yw_i for $i = 1, 2$.

Claim 1: B_1 and B_2 are endblocks of G_0 and y is the unique cutvertex of B_1 and B_2.

Proof of Claim 1. By Lemma 3.4(iii), $y \in V(B_1) \cap V(B_2)$ is a cutvertex of G_0. Suppose, B_i contains a further cutvertex $u' \neq y$ for some $i = 1, 2$ in G_0. Since $w_i \in P(y, D)$, we get $u' \notin D$. By Lemma 3.4(i) and (iv), there is another block $B' \neq B_i$ of G_0 that contains u' but not y. Let w be a vertex in $V(B') \setminus V(B_i) \neq \emptyset$. Since $w \in V(G_0)$, there is a vertex $w' \in D_0$ that dominates w. By the construction of Q, the induced subgraph $G[D_0]$ is connected and there is a path P in $G[D_0]$ from y to w'. Thus, we get the path $P \cup w'w$ from y to w in $G_0 - u'$, which is a contradiction to Lemma 3.4(v). This completes the proof of the claim.

Further, let $D'(w_i) = D' \cap N[w_i, G]$ for $i = 1, 2$. There are three possibilities for each $w_i, i = 1, 2$.

a) There exists a vertex $v_i \in D'(w_i) \cap Q$ with $P(v_i, D') \subseteq Q$.

b) There exists a vertex $v_i \in D'(w_i) \cap Q$ and a vertex $u \in P(v_i, D') \setminus Q$.

c) $D'(w_i) \cap Q = \emptyset$.

Suppose, w_1 and w_2 fulfil a). By Claim 1, B_1 and B_2 are endblocks of G_0. Hence, $v_i \in V(B_i) \setminus \{y\}$ and $P(v_i, D') \subseteq V(B_i) \subseteq N[y, G]$ for $i = 1, 2$. Thus, $D'' = (D' \setminus \{v_1, v_2\}) \cup \{y\}$ dominates G, but $|D''| < |D'| = \gamma(G)$, which is a contradiction. Hence, at least one of the vertices w_1 and w_2 fulfils b) or c). This implies that $G_0 \neq G$, $\kappa > 0$ and D_i is the unique γ-set of G_i for every $i = 0, 1, \ldots, \kappa$.

Claim 2: For every $1 \leq j \leq \kappa$ and for every $a, b \in Q \cap N(V(G_j), G)$ or $a, b \in N(Q, G) \cap V(G_j)$ the vertices a and b lie in a common block of G.

Proof of Claim 2. First, let $a, b \in Q \cap N(V(G_j), G)$. Let a_j and b_j be the neighbours of a and b in G_j, respectively. There exists a path P_j in G_j from a_j to b_j and a second path P in G_0 from a to b. Hence, for every $z \in V(P) \setminus \{a, b\}$ there is the path $aa_j \cup P_j \cup b_jb$ from a to b in $G - z$. Thus, in view of Lemma 3.4(v), the vertices a and b are contained in a common block. Analogous, we can prove the case $a, b \in N(Q, G) \cap V(G_j)$.

Claim 3: If w_i fulfils c) for some $i = 1, 2$, then there is a vertex $v_i \in D'(w_i) \cap V(G_j)$ for some $j \in \{1, 2, \ldots, \kappa\}$, and we get $|D'_j| > |D_j|$ and $N(D'_j, G) \setminus V(G_j) \subseteq N[y, G]$.

Proof of Claim 3. For some $i = 1, 2$ let w_i fulfil c), that means, $D'(w_i) \subseteq V(G - Q)$. Hence, there exists a vertex $v_i \in D'(w_i) \cap V(G_j)$ for some $j \in \{1, 2, \ldots, \kappa\}$. Suppose, there is a vertex $w \in V(G_j) \setminus N[D'_j, G]$. Then, a vertex

$w' \in D'_0$ dominates w. Thus, $w' \in Q \cap N(V(G_j), G)$ and $w_i \in (Q \cap N(v_i, G)) \subseteq (Q \cap N(V(G_j), G))$. By Claim 2, also w' dominates w_i and $w' \in D'(w_i) \cap Q$, which is a contradiction. Thus, D'_j dominates G_j. Since $v_i \in D'_j \setminus D$, the unique γ-set D_j of G_j is not equal D'_j and $|D'_j| > |D_j|$. Since $w_i \in Q \cap N(G_j, G)$ and $N(D'_j, G) \setminus V(G_j) \subseteq Q \cap N(G_j, G)$, we get $N(D'_j, G) \setminus V(G_j) \subseteq N[w_i, G]$, by Claim 2. By Claim 1, the set $N[w_i, G] = V(B_i) \subseteq N[y, G]$, and the proof of the claim is complete.

In the following we distinguish three cases.
Case 1: At least one of the vertices w_1 and w_2 fulfils b).
Without loss of generality let this vertex be w_1, let $v_1 \in D'(w_1) \cap Q$, and let $u \in P(v_1, D') \cap V(G_j)$ for some $j \in \{1, 2, \ldots, \kappa\}$. Suppose, there is a vertex $w \in V(G) \setminus V(G_j)$, which is not dominated by $D' \setminus D'_j$. Then $w \in N[D'_j, G] \cap Q$ and $v_1 \in N(u, G) \cap Q$. By Claim 2, we get that w and $v_1 \in D' \setminus D'_j$ are adjacent, which is a contradiction. Hence, $D' \setminus D'_j$ dominates $G - V(G_j)$. Obviously, $D \setminus D_j$ is a dominating set of $G - V(G_j)$, which satisfies (iii). Therefore, $D \setminus D_j$ is the unique γ-set of $G - V(G_j)$ and $|D' \setminus D'_j| > |D \setminus D_j|$ and $|D'_j| < |D_j|$. By Claim 2, all vertices of $N(Q, G) \cap V(G_j)$ lie in a common block of G. Hence, $D'_j \cup \{u\}$ dominates G_j and $|D'_j \cup \{u\}| \leq |D_j|$. This implies $D'_j \cup \{u\} = D_j$. The vertex $u \in D$ has the two private neighbours $u_1, u_2 \in P(u, D) \subseteq V(G_j)$, which do not lie in a common block of G. This implies by Claim 2, that at most one of the vertices u_1, u_2 can be adjacent to Q. Let $u_1 \notin N(Q, G)$. Hence, $D'_j = D_j \setminus \{u\}$ dominates u_1, which is a contradiction.
Case 2: w_1 and w_2 fulfil c).
Let $v_i \in D'(w_i) \subseteq V(G - Q)$ for $i = 1, 2$. By Claim 2, v_1 and v_2 do not belong to the same component of $G - Q$. Without loss of generality let v_i belong to the component G_i for $i = 1, 2$. By Claim 3, $|D'_i| > |D_i|$ and $N(D'_i, G) \setminus V(G_i) \subseteq N[y, G]$ for $i = 1, 2$. Hence, $D' \setminus (D'_1 \cup D'_2) \cup \{y\}$ dominates $G - (V(G_1) \cup V(G_2))$ and $D'' = D' \setminus (D'_1 \cup D'_2) \cup (D_1 \cup D_2 \cup \{y\})$ dominates G. But $|D''| < |D'| = \gamma(G)$, which is a contradiction.
Case 3: One of the vertices w_1 and w_2 fulfils a) and the other one fulfils c).
Without loss of generality let w_1 fulfil c) and let $v_1 \in D'(w_1) \subseteq V(G - Q)$ belong to the component G_1. By Claim 3, we get $|D'_1| > |D_1|$ and $N(D'_1, G) \setminus V(G_1) \subseteq N[y, G]$. Then, the vertex w_2 satisfies a). By Claim 1, the block B_2 is an endblock of G_0 and we deduce that $P(v_2, D') \subseteq V(B_2) \subseteq N[y, G]$. Therefore, the set $D' \setminus (D'_1 \cup \{v_2\}) \cup \{y\}$ dominates $G - V(G_1)$ and $D'' = D' \setminus (D'_1 \cup \{v_2\}) \cup (D_1 \cup \{y\})$ dominates G. This leads to the contradiction $\gamma(G) \leq |D''| < |D'| = \gamma(G)$, which completes the proof (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv): Let D be the unique γ-set of G and let $x \in D$ arbitrarily. We already have proved that G and D also satisfy (ii). Let D' be a γ-set of $G - x$. Let $G_1, G_2, \ldots, G_\kappa$ be the components of $G - x$ and let $D_i = D \cap V(G_i)$

and $D'_i = D' \cap V(G_i)$ for every $i = 1, 2, \ldots, \kappa$. Since $D''_i = (D \setminus D_i) \cup D'_i$ dominates G, we get either $|D''_i| > |D|$ and $|D'_i| > |D_i|$ or $D''_i = D$ and $D'_i = D_i$ for all $i \in \{1, 2, \ldots, \kappa\}$. By (ii) and Lemma 3.4(v), there are $x_1, x_2 \in P(x, D)$ that lie in different components of $G - x$. Without loss of generality, let $x_i \in V(G_i)$ for $i = 1, 2$. Thus, for $i = 1, 2$ the set D_i does not dominate G_i and $D_i \neq D'_i$, which implies $|D'_i| > |D_i|$. Hence, $\gamma(G - x) = |D'| = \Sigma_{i=1}^{\kappa}|D'_i| \geq 2 + \Sigma_{i=1}^{\kappa}|D_i| = 2 + |D \setminus \{x\}| > |D| = \gamma(G)$. □

Remark 3.6 As a corollary from Theorem 3.5 we obtain a slightly stronger version of Theorem 3.3 where the dominating set in Condition (ii) has not to be minimal.

Remark 3.7 Block graphs are strongly chordal and Farber [32, 33] has shown that the domination problem for strongly chordal graphs is linear. As mentioned in Example 2.5 we can decide in time $O(n^3)$ whether a given block graph G has a unique γ-set by using the general method in Chapter 2. Our characterization in Theorem 3.5 implies a linear algorithm to decide whether a given γ-set of a block graph is unique. Thus, we can check in linear time whether a given block graph G has a unique γ-set, by using the algorithm of Farber or any further linear algorithm determining γ-sets of block graphs, as the one given by Volkmann [100, 102], and by using Theorem 3.5(ii).

3.2 X-domination in trees

In order to prove our characterization of cactus graphs with unique γ-sets in the last section of this chapter, we need the following generalization of Theorem 3.3 for X-domination.

Theorem 3.8 (Fischermann and Volkmann [44]) *Let T be a tree of order at least 3 and let X and D be subsets of $V(T)$. Then the following conditions are equivalent:*

(i) D is the unique minimum X-dominating set of T.

(ii) D is an X-dominating set of T for which every vertex in D has at least two private neighbours in X other than itself.

(iii) D is a minimum X-dominating set of T for which $\gamma(T - x, X \setminus \{x\}) > \gamma(T, X)$ for every vertex $x \in D$.

Proof.
(i) $\Rightarrow$ (ii): Obviously.

(ii) $\Rightarrow$ (i): Let D be an X-dominating set of T such that every vertex in D has at least two private neighbours in X other than itself. Suppose, there is a minimum X-dominating set $F \neq D$ of T. Let H be the subgraph of T induced by the vertex set $(D \setminus F) \cup (F \setminus D) \cup (P(D \setminus F, D) \cap X)$. This yields

$$m_H(D\setminus F, (P(D\setminus F, D)\cap X)\setminus D) = \sum_{v \in D\setminus F} m_H(\{v\}, (P(v, D)\cap X)\setminus\{v\}) \geq 2|D\setminus F|.$$

Since F is an X-dominating set of T, we get

$$m_H(F \setminus D, (P(D \setminus F, D) \cap X) \setminus (D \cup F)) \geq |(P(D \setminus F, D) \cap X) \setminus (D \cup F)|.$$

Because of $|D \setminus F| \geq |F \setminus D|$, we obtain for the size of H

$$\begin{aligned} m(H) &\geq m_H(D \setminus F, (P(D \setminus F, D) \cap X) \setminus D) \\ &\quad + m_H(F \setminus D, (P(D \setminus F, D) \cap X) \setminus (D \cup F)) \\ &\geq |D \setminus F| + |F \setminus D| + |(P(D \setminus F, D) \cap X) \setminus (D \cup F)| \\ &= n(H). \end{aligned}$$

Since H is a forest, we know by Lemma 1.10 a), that the cyclomatic number $\mu(H) = 0$. Thus, we obtain the contradiction $m(H) = n(H) - \kappa(H) < n(H)$.

(i) $\Rightarrow$ (iii): Let D be the unique minimum X-dominating set of T, let $x \in D$ arbitrarily, let $\kappa = \kappa(T - x)$ and let $T_1, T_2, \ldots, T_\kappa$ be the components of $T - x$. Further, let D' be a minimum $(X \setminus \{x\})$-dominating set of $T - x$ and for every $1 \leq i \leq \kappa$ let $X_i = X \cap V(T_i)$, $D_i = D \cap V(T_i)$, and $D'_i = D' \cap V(T_i)$. For every $1 \leq i \leq \kappa$ the set $D''_i = (D \setminus D_i) \cup D'_i$ is an X-dominating set of T, which implies that either $D_i = D'_i$ or $|D_i| < |D'_i|$. By (i) $\Rightarrow$ (ii), the vertex x has at least two private neighbours x_1, x_2 in X other than itself. Without loss of generality, let $x_1 \in X_1$ and $x_2 \in X_2$. Then, for $i = 1, 2$, the set D_i is not an X_i-dominating set of T_i, in contrast to the set D'_i. Hence, we have $D_i \neq D'_i$ and $|D_i| < |D'_i|$ for $i = 1, 2$, which implies $\gamma(T - x, X \setminus \{x\}) = |D'| = \sum_{i=1}^{\kappa} |D'_i| \geq 2 + \sum_{i=1}^{\kappa} |D_i| = 1 + |D| > \gamma(T, X)$.

(iii) $\Rightarrow$ (i): Let D be a minimum X-dominating set of T such that for every vertex $x \in D$ yields $\gamma(T - x, X \setminus \{x\}) > \gamma(T, X)$. Suppose that there is a minimum X-dominating set $F \neq D$ of T. Since there exists at least one vertex $x \in D \setminus F$, the set F is an $(X \setminus \{x\})$-dominating set of $T - x$. Hence, $\gamma(T - x, X \setminus \{x\}) \leq |F| = \gamma(T, X)$ for some $x \in D$, which is a contradiction. $\square$

For $X = V(T)$ Theorem 3.8 yields immediately a slightly stronger version of Theorem 3.3, and our proof of this generalization is different from the way Gunther, Hartnell, and Rall have proved Theorem 3.3. The method of counting the edges of a special subgraph H to deduce the contradiction $\mu(H) \leq m(H) - n(H)$ is also useful in some other proofs in this thesis (cf. Theorem 3.11, Theorem 4.13, Theorem 5.20).

3.3 Cactus graphs

It is not possible to generalize Theorem 3.3 for cactus graphs in the same way as for block graphs. On the one hand, the third condition of Theorem 3.3 is not necessary for the uniqueness of γ-sets in cactus graphs, as one can see at the cactus graph G in Figure 3.1 where the set $\{x, y\}$ is the unique γ-set of G but $\gamma(G - x) = \gamma(G)$.

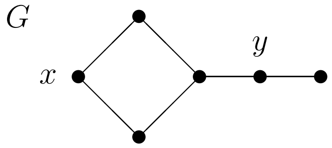


Figure 3.1

On the other hand, the second condition in Theorem 3.3 is necessary but not sufficient for the uniqueness of γ-sets in cactus graphs, as one can see considering the cycle of order $3t$ for any positive integer t. Thus, for cactus graphs we need a stronger structural condition with regard to the cycles of the graph.

We consider a cactus graph G with γ-set D and an arbitrary fixed cycle C in G. There may be some vertices in C that are already dominated from vertices in $D \setminus V(C)$. Thus, the set $D \cap V(C)$ is an X-dominating set of C where $X = V(C) \setminus N(D \setminus V(C), G)$. Furthermore, there may be some vertices in $D \cap V(C)$ that have private neighbours outside of the cycle. These vertices have to be contained in every X-dominating set F of C with the property that the set $(D \setminus V(C)) \cup F$ dominates G. This consideration leads to the following definition.

Definition 3.9 For two subsets X, Y of the vertex set $V(G)$ a set $D \subseteq V(G)$ is an (X, Y)-*dominating set* of G, if $Y \subseteq D$ and $X \subseteq N[D]$. An (X, Y)-dominating set of minimum cardinality is a *minimum* (X, Y)-*dominating set.* The cardinality of a minimum (X, Y)-dominating set is denoted by $\gamma(G; X, Y)$. Note that the case $Y = \emptyset$ leads to X-domination and the case $Y = \emptyset$ and $X = V(G)$ leads to ordinary domination.

Besides our motivation, this definition has an immediate eligibility, since in applications there can be special 'vertices' that have to be in the dominating set for some reason. (For example, if one wants to extend a net of already existing SOS-telephones in a city such that from every crossroad at least one SOS-telephone is at most one block of houses away.)

We will deduce a characterization of unique domination in cactus graphs by reducing this problem to unique (X, Y)-domination in the cycles of the cactus graphs. In order to do this, we characterize cycles having unique minimum (X, Y)-dominating sets.

Lemma 3.10 (Fischermann and Volkmann [44]) *Let the graph C be a cycle and let X, Y and D be arbitrary subsets of $V(C)$. Then the following conditions are equivalent:*

(i) The cycle C has a unique minimum (X, Y)-dominating set D.

(ii) The cycle C has an (X, Y)-dominating set D such that $N(x, C) \subseteq P(x, D) \cap X$ for every vertex x in $D \setminus Y$, and C fulfils at least one of the following three conditions.
a) $Y \neq \emptyset$.
b) The cycle C contains two adjacent vertices $a, b \notin X$.
c) The cycle C contains a path $x_1 x_2 \dots x_{3t}$ for some positive integer t such that $x_1, x_2, \dots, x_{3t} \in X$ and x_1 and x_{3t} both have their second neighbour not in X.

Proof.
(i) $\Rightarrow$ (ii): Let D be the unique minimum (X, Y)-dominating set of C. Let $x \in D \setminus Y$ be arbitrary. Assume $|(P(x, D) \setminus \{x\}) \cap X| \leq 1$. If $|(P(x, D) \setminus \{x\}) \cap X| = 1$, then choose $z \in (P(x, D) \setminus \{x\}) \cap X$, else choose $z \in N(x, C)$ arbitrarily. In both cases the set $(D \setminus \{x\}) \cup \{z\}$ is another minimum (X, Y)-dominating set of C, which is a contradiction. Hence, we have $|(P(x, D) \setminus \{x\}) \cap X| = 2$ and $N(x, C) \subseteq P(x, D) \cap X$ for every $x \in D \setminus Y$.
Suppose C fulfils neither a), nor b), nor c). Since $Y = \emptyset$, D is the unique minimum X-dominating set of C. If we have $X = V(C)$, then D is the unique γ-set of C. Since no cycle has a unique γ-set, we obtain that $V(C) \setminus X \neq \emptyset$. Let $Z = V(C) \setminus X = \{z_1, z_2, \dots, z_\eta\}$. In the following we understand $\eta + 1$ as $1 \equiv \eta + 1 \pmod{\eta}$. By not b) and not c), we get that $C - Z$ consists of η disjoint paths $W_1, W_2, \dots, W_\eta$ which fulfil for every $1 \leq i \leq \eta$ that $V(W_i) \subseteq X$ and $|V(W_i)| = n_i = 3t_i + r_i$ for some not negative integers n_i, t_i and $r_i \in \{1, 2\}$. Let $W_i = y_1^{(i)} y_2^{(i)} \dots y_{n_i}^{(i)}$ and let $z_{i+1} \in N(y_{n_i}^{(i)}) \cap N(y_1^{(i+1)})$ for every $1 \leq i \leq \eta$. Since $N(x, C) \subseteq P(x, D) \cap X$ for every $x \in D$, for every $1 \leq i \leq \eta$ we have the following.

If $r_i = 2$, then $z_i, z_{i+1} \in D$ and $\{y^{(i)}_{3j} \mid 1 \le j \le t_i\} = D \cap V(W_i)$.
If $r_i = 1$, then $\eta > 1$ and $|\{z_i, z_{i+1}\} \cap D| = 1$.
If $r_i = 1$ and $z_i \in D$, then $\{y^{(i)}_{3j} \mid 1 \le j \le t_i\} = D \cap V(W_i)$.
If $r_i = 1$ and $z_i \notin D$, then $\{y^{(i)}_{3j-1} \mid 1 \le j \le t_i\} = D \cap V(W_i)$.
Hence, $|\{1 \le i \le \eta \mid r_i = 1 \wedge z_i \in D\}| = |\{1 \le i \le \eta \mid r_i = 1 \wedge z_i \notin D\}| = \nu$ and $|D| = \sum_{i=1}^{\eta}(t_i + 1) - \nu$. Let D' be defined as follows.

$$\begin{aligned} D' &= \{z_i, y^{(i)}_3, y^{(i)}_6, \ldots, y^{(i)}_{3t_i} \mid 1 \le i \le \eta \text{ with } z_i \notin D \text{ (and } r_i = 1)\} \\ &\cup \{y^{(i)}_2, y^{(i)}_5, \ldots, y^{(i)}_{3t_i-1} \mid 1 \le i \le \eta \text{ with } z_i \in D \text{ and } r_i = 1\} \\ &\cup \{y^{(i)}_2, y^{(i)}_5, \ldots, y^{(i)}_{3t_i+2} \mid 1 \le i \le \eta \text{ with } r_i = 2\}. \end{aligned}$$

Hence, $D' \neq D$ is another (X, Y)-dominating set of C with $|D'| = |D|$, which is a contradiction.

(ii) $\Rightarrow$ (i): Let D be an (X, Y)-dominating set of C that fulfils (ii). Let F be an arbitrary minimum (X, Y)-dominating set of C.
Case 1: C fulfils a). Let $H = C - Y$ and let $P_1, P_2, \ldots, P_\kappa$ be the components of H. Further, for every $1 \le i \le \kappa$ let $X_i = (X \cap V(P_i)) \setminus N[Y]$, $D_i = D \cap V(P_i)$ and $F_i = F \cap V(P_i)$. Let $i \in \{1, 2, \ldots, \kappa\}$ be arbitrary. It is straightforward to see that D_i is an X_i-dominating set of P_i where $N(x, C) = N(x, P_i) \subseteq P(x, D) \cap X_i$ for every vertex $x \in D_i$, and F_i is a minimum X_i-dominating set of P_i. Since P_i is a path, we have $F_i = D_i$, by Theorem 3.8. Thus,

$$D = Y \cup \bigcup_{i=1}^{\kappa} D_i = Y \cup \bigcup_{i=1}^{\kappa} F_i = F$$

and D is the unique minimum (X, Y)-dominating set of C.
Case 2: C does not fulfil a) but b). Let $ab \in E(C)$ with $a, b \notin X$, and let $H = C - ab$. Then, D is an X-dominating set of the path H such that $N(x, H) \subseteq P(x, D) \cap X$, and F is a minimum X-dominating set of H. Thus, we have $F = D$, by Theorem 3.8.
Case 3: C fulfils neither a) nor b), but C fulfils c). Thus, C contains a path $x_1x_2 \ldots x_{3t}$ for some positive integer t such that $x_1, x_2, \ldots, x_{3t} \in X$ and the vertices in $N(x_1, C) \setminus \{x_2\}$ and $N(x_{3t}, C) \setminus \{x_{3t-1}\}$ are not in X. Let $R = \{x_1, x_2, \ldots, x_{3t}\}$. Since $N(x, C) \subseteq P(x, D) \cap X$ for every $x \in D$, we have $D \cap R = \{x_{3j-1} \mid 1 \le j \le t\}$. Since $F \cap R$ has to dominate at least the vertices $x_2, x_3, \ldots, x_{3t-1}$, we know that $|F \cap R| = t$. If $X = R$, then $\{x_{3j-1} \mid 1 \le j \le t\}$ is the unique minimum (X, Y)-dominating set of C which yields $F = D$. In the following let $R \neq X$, $H = C - R$, $X' = X \setminus R$, $D' = D \setminus R$, and $F' = F \setminus R$. It is straightforward to see that D' is an X'-dominating set of H where $N(x, C) = N(x, H) \subseteq P(x, D) \cap X'$ for every vertex $x \in D'$, and F' is a minimum X'-dominating set of H. Since H is a path, we have

$F' = D'$, by Theorem 3.8. Thus, F' dominates no vertex in R which implies that $F \setminus F' = \{x_{3j-1} \mid 1 \leq j \leq t\} = D \setminus D'$. This yields $F = D$, and D is the unique minimum (X, Y)-dominating set of C. □

Using Lemma 3.10 we can now prove a characterization of cactus graphs with unique γ-sets.

Theorem 3.11 (Fischermann and Volkmann [44]) *Let G be a non trivial, connected cactus graph and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique γ-set of G.

(ii) D is a dominating set of G such that every vertex in D has at least two non-adjacent private neighbours and every cycle C of G fulfils at least one of the following three conditions.
a) There exists a vertex $z \in D \cap V(C)$ such that $P(z, D) \setminus V(C) \neq \emptyset$.
b) The cycle C contains two adjacent vertices $a, b \in N(D \setminus V(C), G)$.
c) The cycle C contains a path $x_1x_2 \ldots x_{3t}$ for some positive integer t such that $x_1, x_2, \ldots, x_{3t} \notin N(D \setminus V(C), G)$ and x_1 and x_{3t} both have one neighbour in $V(C) \cap N(D \setminus V(C), G)$.

Proof.
First, we show that Condition (ii) is equivalent to

(iii) D is a dominating set of G such that every vertex in D has at least two non-adjacent private neighbours and for every cycle C the set $D \cap V(C)$ is the unique minimum (X_C, Y_C)-dominating set of C, where $X_C = V(C) \setminus N(D \setminus V(C), G)$ and $Y_C = \{z \in D \cap V(C) \mid P(z, D) \setminus V(C) \neq \emptyset\}$.

<u>(ii) ⇔ (iii):</u> Let C be an arbitrary cycle in G, let $D_C = D \cap V(C)$, let $X = V(C) \setminus N(D \setminus V(C), G)$ and $Y = \{z \in D \cap V(C) \mid P(z, D) \setminus V(C) \neq \emptyset\}$. It suffices to prove that C has one of the properties a),b), and c) in Theorem 3.11 (ii) if and only if C and D_C fulfil Condition (ii) in Lemma 3.10 where X and Y are defined as above.
It is straightforward to see that with these definitions of X and Y the properties a),b) and c) in Theorem 3.11 (ii) and in Lemma 3.10 (ii) are equivalent. Thus, it remains to prove that the set D_C fulfils its part in Condition (ii) in Lemma 3.10.
Obviously, the set D_C is an (X, Y)-dominating set of C. Since every vertex $x \in D_C \setminus Y$ has at least two non-adjacent private neighbours with regard to D and fulfils $P(x, D) \subseteq V(C)$, we get that $N(x, C) \subseteq P(x, D) = P(x, D_C, C) \cap X$.

Hence, the cycle C and its (X, Y)-dominating set D_C fulfil for every vertex $x \in D_C \setminus Y$ that $N(x, C) \subseteq P(x, D_C, C) \cap X$.

Now, we prove (i) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (iii): Let D be the unique γ-set of G and let $x \in D$ arbitrarily. By Lemma 3.1, we know that $|P(x, D)| \geq 2$. Suppose all vertices in $P(x, D)$ are adjacent. Then, we can replace x in D by any vertex in $P(x, D) \setminus \{x\}$ and we get a γ-set different from D, which is a contradiction.
Let C be an arbitrary cycle in G, and let $X_C = V(C) \setminus N(D \setminus V(C), G)$ and $Y_C = \{z \in D \cap V(C) \mid P(z, D) \setminus V(C) \neq \emptyset\}$. The set $D \cap V(C)$ is an (X_C, Y_C)-dominating set of C. Let $F \subseteq V(C)$ be an (X_C, Y_C)-dominating set of C with $|F| \leq |D \cap V(C)|$. Then, $D' = (D \setminus V(C)) \cup F$ is a dominating set of G with $|D'| \leq |D|$. Hence, $D' = D$ and $F = D \cap V(C)$.

(iii) $\Rightarrow$ (i): Suppose there is a non trivial, connected cactus graph with a dominating set that fulfils (iii) but not (i). Let G be such a counterexample of minimal size $m(G)$, let D be the dominating set of G that fulfils (iii) and let $D' \neq D$ be a γ-set of G. Further, for every cycle C in G let $Y_C = \{z \in D \cap V(C) \mid P(z, D, G) \setminus V(C) \neq \emptyset\}$ and $X_C = V(C) \setminus N(D \setminus V(C), G)$.

Claim 1: For any arbitrary edge $ab \in E(G)$ either D or D' do not dominate the graph $G - ab$.

Proof of Claim 1. Suppose D and D' dominate $G' = G - ab$. Then, D' is a γ-set of G'. Obviously, every vertex $x \in D$ fulfils $P(x, D, G) \subseteq P(x, D, G')$ and x still has at least two non-adjacent private neighbours in G'. Thus, either G' is a non trivial connected cactus graph or the two components of G' are non trivial connected cactus graphs. Let C be a cycle in G', let $Y'_C = \{z \in D \cap V(C) | P(z, D, G') \setminus V(C) \neq \emptyset\}$ and $X'_C = V(C) \setminus N(D \setminus V(C), G')$. Further, let $F \subseteq V(C)$ be an arbitrary (X'_C, Y'_C)-dominating set in G' with $|F| \leq |D \cap V(C)|$. Then, we have $Y_C \subseteq Y'_C \subseteq F$ and $X_C \subseteq X'_C \subseteq N[F, G'] \subseteq N[F, G]$ which means that F also is an (X_C, Y_C)-dominating set in G.
Since G and D fulfil Condition (iii), we get that $F = D \cap V(C)$. Thus, also G' and D fulfil Condition (iii). If G' is not connected, this also holds restricted to the components of G'. Since G is a counterexample of minimal size, every component H of G' has the unique γ-set $D \cap V(H)$, and hence, G' has the unique γ-set D. This is a contradiction to $D' \neq D$, and the proof of Claim 1 is complete.

Let $G^* \subseteq G$ be the graph with vertex set

$$V(G^*) = (D \setminus D') \cup (D' \setminus D) \cup P(D \setminus D', D)$$

and edge set

$$E(G^*) = \{ab \in E(G) \mid a \in ((D \setminus D') \cup (D' \setminus D)) \text{ and } b \in (P(D \setminus D', D) \setminus D)\}.$$

Claim 2: For any cycle C in G^* we have $|D' \cap V(C)| \leq |D \cap V(C)|$.

Proof of Claim 2. Let $C = x_1x_2 \dots x_lx_1$ be an arbitrary cycle in G^*. If $D' \cap V(C) = \emptyset$, there is nothing to prove. Therefore, let $D' \cap V(C) \neq \emptyset$ and let $x \in D' \cap V(C)$ arbitrarily. Without loss of generality, let $x = x_l$ and let $j \geq 1$ be the minimal subindex with $x_j \in D' \cap V(C)$. Since $(D \cap D') \cap V(G^*) = \emptyset$, we get that $x_l, x_j \in D' \setminus D$. If $j = 1$, then D and D' dominate $G - x_lx_1$, which is a contradiction to Claim 1. If $j = 2$, then the set D' dominates the graph $G - x_lx_1$, and by Claim 1, the set D does not dominate $G - x_lx_1$ which implies that $x_1 \in D \setminus D'$. If $j \geq 3$, then $x_1 \in D \setminus D'$ or $x_2 \in D \setminus D'$, since otherwise D and D' dominate $G - x_1x_2$ which again is a contradiction to Claim 1. Hence, between every two vertices of $D' \cap V(C)$ there is at least one vertex of $D \cap V(C)$.

Claim 3: For any cycle C in G^* and for every vertex $x \in D \cap V(C)$ we have $N(x, C) \subseteq P(x, D) \setminus \{x\}$.

Proof of Claim 3. Let $x \in D \cap V(C)$ and $y \in N(x, C)$. Hence, $xy \in E(G^*)$ and $x \notin P(D \setminus D', D) \setminus D$. By the definition of $E(G^*)$, we get $y \in P(D \setminus D', D) \setminus D$. Since y is a neighbour of $x \in D$, this implies $y \in P(x, D) \setminus \{x\}$.

Claim 4: There is a cycle C in G^* such that $|P(x, D) \setminus \{x\}| = 2$ for every vertex $x \in D \cap V(C)$.

Proof of Claim 4. By Lemma 1.10, the number of cycles in G^* is the cyclomatic number $\mu(G^*) = m(G^*) - n(G^*) + \kappa(G^*)$. Let $s = \mu(G^*)$ and let $C_1, C_2, \dots C_s$ be the cycles of G^*. Suppose either $s = 0$ or for every $i = 1, 2, \dots, s$ there exists a vertex $y_i \in D \cap V(C_i)$ such that $|P(y_i, D) \setminus \{y_i\}| \geq 3$. Let $t = |\{y_1, y_2, \dots, y_s\}|$ and $\{z_1, z_2, \dots, z_t\} = \{y_1, y_2, \dots, y_s\}$. Without loss of generality let $j_0 = 0 < j_1 < \dots < j_t = s$ such that $y_i = z_p$ for every $j_{p-1} < i \leq j_p$ and for every $1 \leq p \leq t$. This implies that for every $1 \leq p \leq t$ the vertex z_p lies on $r_p = j_p - j_{p-1}$ cycles in G^*. By Claim 3, we get for every $1 \leq p \leq t$ that

$$|P(z_p, D) \setminus \{z_p\}| \geq |\bigcup_{j_{p-1} < i \leq j_p} N(z_p, C_i)| = r_p \cdot 2$$

which yields that

$$|P(z_p, D) \setminus \{z_p\}| \geq \max\{3, 2r_p\} \geq 2 + r_p.$$

Hence, we obtain

$$\begin{aligned} m_{G^*}(\{z_1, z_2, \dots, z_t\}, P(D \setminus D', D) \setminus D) &\geq \sum_{p=1}^{t}(2 + r_p) = 2t + \sum_{p=1}^{t} r_p \\ &= 2t + (j_t - j_0) = 2t + s. \end{aligned}$$

Since every vertex in D has at least 2 private neighbours other than itself, we get

$$m_{G^*}((D \setminus D') \setminus \{z_1, z_2, \ldots, z_t\}, P(D \setminus D', D) \setminus D) \geq 2(|D \setminus D'| - t).$$

The fact that $(D' \setminus D)$ dominates $P(D \setminus D', D) \setminus D$ leads to

$$m_{G^*}(D' \setminus D, P(D \setminus D', D) \setminus (D \cup D')) \geq |P(D \setminus D', D) \setminus (D \cup D')|.$$

This yields

$$\begin{aligned} m(G^*) &\geq |P(D \setminus D', D) \setminus (D \cup D')| + 2(|D \setminus D'| - t) + (2t + s) \\ &\geq |P(D \setminus D', D) \setminus (D \cup D')| + |D \setminus D'| + |D' \setminus D| + s \\ &= n(G^*) + s. \end{aligned}$$

But $s = \mu(G^*) \geq m(G^*) - n(G^*) + 1$, which is a contradiction. Thus, Claim 4 holds.

By Claim 3 and 4, there is a cycle C in $G^* \subseteq G$ with $P(x, D) \setminus \{x\} = N(x, C)$ for every $x \in D \cap V(C)$ which yields that the set Y_C is empty. Let $D'_C = D' \cap V(C)$. It follows from Claim 2 and from the definition of G^* that D'_C is a subset of $V(C)$ different from $D \cap V(C)$ and of cardinality $|D'_C| \leq |D \cap V(C)|$. By Condition (iii), the set D'_C cannot be an (X_C, Y_C)-dominating set of C. Since D'_C contains the empty set Y_C, we obtain that D'_C is not an X_C-dominating set, and hence, there exists a vertex $v \in X_C = V(C) \setminus N(D \setminus V(C), G)$ that lies not in $N[D'_C, G]$. Thus, we obtain that $N[v, C] \cap D' = \emptyset$ and there exists a vertex $x \in N[v, C] \cap D$. Suppose x is different from v. Then, we know that $v \in N(x, C) = P(x, D) \setminus \{x\}$ which implies that v and the vertex $y \in N(v, C) \setminus \{x\}$ are not in D. Thus, the two sets D and D' dominate $G - vy$, which is a contradiction to Claim 1. Hence, we get that $v = x \in D$ and $N[v, C] = P(v, D)$. Let G_1 be that component of the graph $G'' = G - \{vw \in E(G) \mid w \notin V(C)\}$ that contains vertex v and let G_2 be the union of the remaining components of G''. At least one vertex $w' \in D' \setminus V(C)$ is adjacent to v which implies that G_2 is not empty. Since every vertex $w \in N(v, G) \setminus V(C)$ is neither in D nor in $P(v, D)$, this vertex w has at least one more neighbour in D besides v. Hence, G_1 and every component of G_2 are non trivial, connected cactus graphs. Let $D_i = D \cap V(G_i)$ and $D'_i = D' \cap V(G_i)$ for $i = 1, 2$. Obviously, the set D_i is a dominating set of G_i for $i = 1, 2$. Analogous to the proof of Claim 1 we obtain that G'' and D fulfil Condition (iii) which implies that G_1 and the set D_1 and every component of G_2 and the set D_2 restricted to this component fulfil Condition (iii). By the minimality of G, we get that D_i is the unique γ-set of G_i for $i = 1, 2$. Since $v \notin D'$, the set D'_2 dominates G_2. Since a vertex $w' \in D'_2$ dominates v but

$w' \notin D_2$, we obtain that $D'_2 \neq D_2$ and $|D'_2| > |D_2|$. Further, D'_1 dominates $V(G_1) \setminus \{v\}$ and for $y \in N(v, C)$ the set $D'_1 \cup \{y\}$ ($\neq D_1$) dominates G_1. Hence, we get $|D'_1| + 1 > |D_1|$. This yields that $|D'| = |D'_1| + |D'_2| > |D_1| + |D_2| = |D|$, which is a contradiction and completes the proof. □

Remark 3.12 There exist linear time algorithms to determine γ-sets in cactus graphs given by Hedetniemi, Laskar and Pfaff [64] and in block-cactus graphs given by Volkmann [102]. Further, Theorem 3.11(ii) implies an algorithm to decide in $O(n^2)$ time whether a given γ-set of a cactus graph is unique. These together yield an algorithm to decide in $O(n^2)$ time whether a given cactus graph has a unique γ-set. The method in Chapter 2 needs $O(n^3)$ time.

Chapter 4

Equality between graph parameters

As mentioned in the Introduction (Lemma 1.26), Cockayne, Hedetniemi and Miller [26] have found the following inequality chain

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq \mathrm{IR}(G).$$

In this chapter we consider graphs where at least two of these parameters are equal. We investigate on which conditions the unique realization of one parameter implies the unique realization of the other parameter with equal value. The answer to this question is trivial for the two parameters i and α.

Observation 4.1 If any graph G satisfies $i(G) = \alpha(G)$, then any maximal independent set $I \subseteq V(G)$ is an i-set if and only if I is an α-set. This implies that such a graph has a unique i-set if and only if it has a unique α-set. Furthermore, it is straightforward to see that the only graphs satisfying $i(G) = \alpha(G)$ and having a unique i-set are the graphs without edges.

By this observation, it is reasonable to split up the above inequality chain in first and second half

$$ir(G) \leq \gamma(G) \leq i(G) \qquad \text{and} \qquad \alpha(G) \leq \Gamma(G) \leq \mathrm{IR}(G),$$

which are called the *lower chain* and the *upper chain*, respectively. The first section of this chapter deals with the lower chain, and in the second section we consider the upper chain.

Furthermore, we present characterizations by the private neighbourhood, similar to Theorem 3.3, of unique i-sets in trees T with $\gamma(T) = i(T)$ and of unique Γ-sets in chordal graphs. These results lead to polynomial algorithms to decide whether such graphs have unique i-sets or unique Γ-sets, respectively.

4.1 The lower chain

Several publications deal with the question, for which graphs G there is equality between the parameters $ir(G)$ and $\gamma(G)$ or between $\gamma(G)$ and $i(G)$, see for example [3], [10], [20], [34], [58], [59], [60], [77], [95], [96] and [105]. There is no general characterization of such graphs. Even the characterization of trees with equal domination and independent domination numbers seems to be very difficult. In 1986, Harary and Livingston [58] have presented a first, quite complicated characterization of such trees. A second, also difficult characterization was given by Cockayne, Favaron, Mynhardt, and Puech [20] in 2000.

More is known about the related problem, which graphs are γ-perfect.

Definition 4.2 If $\gamma(H) = i(H)$ for every induced subgraph H of G, then a graph G is called *domination perfect* or *γ-perfect.*

Several publications present sufficient conditions for graphs being γ-perfect, as e.g. [3], [48], [92], [96], or characterizations of γ-perfect graphs for special graph classes, as e.g. [92]. We use the following result of Topp and Volkmann [96].

Theorem 4.3 (Topp and Volkmann [96]) *If a graph G has no induced subgraph isomorphic to one of the six graphs $H_1 - H_6$ in Figure 4.1, then G is domination perfect.*

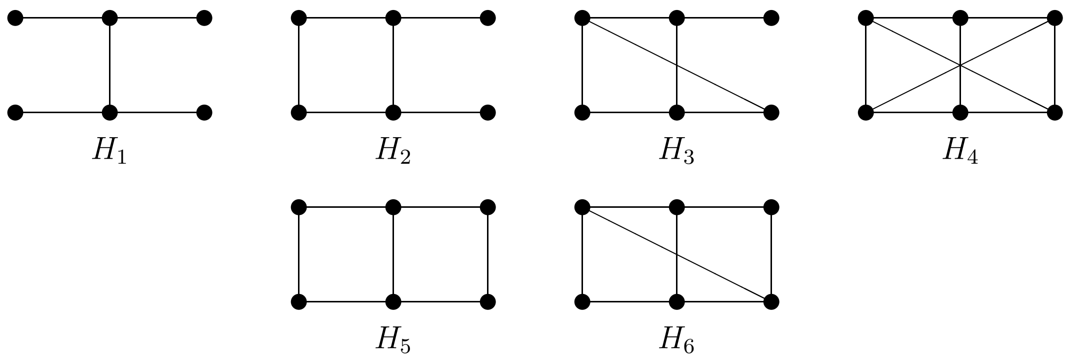


Figure 4.1

As a deep generalization of the main result in [96], very surprisingly, the brothers I. E. Zverovich and V. E. Zverovich [105] have found a complete characterization of γ-perfect graphs by forbidden induced subgraphs.

Theorem 4.4 (Zverovich and Zverovich [105]) *A graph G is domination perfect if and only if G has no induced subgraph isomorphic to one of the seventeen graphs $H_1 - H_4$ in Figure 4.1 and $G_5 - G_{17}$ in Figure 4.2.*

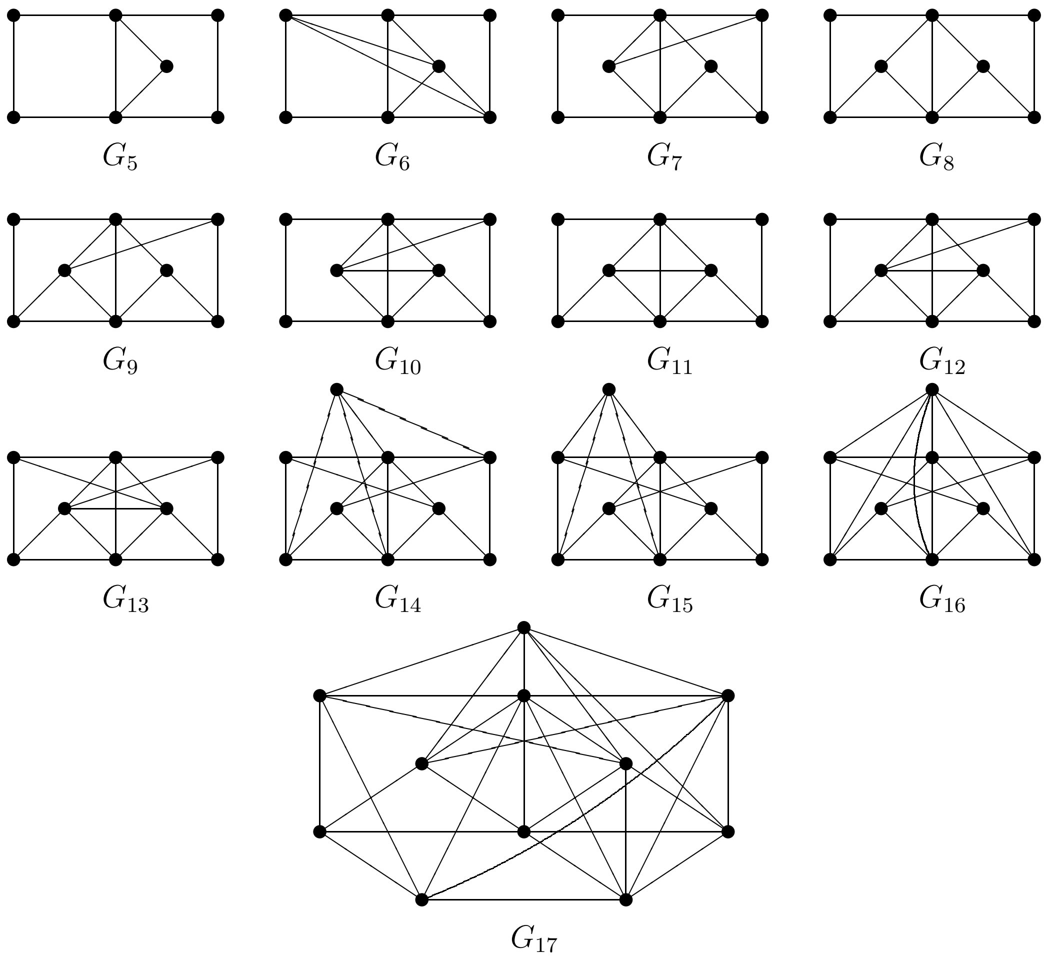


Figure 4.2

We prove in this section that any graph G with $ir(G) = \gamma(G)$ has a unique ir-set if and only if it has a unique γ-set. Furthermore, using Theorem 4.3, we prove for a special class of graphs, which contains the claw-free graphs, that every graph G in this class is γ-perfect and has a unique γ-set if and only if it has a unique i-set. For trees T with $\gamma(T) = i(T)$ we present a characterization of unique i-sets which leads to a linear time algorithm to decide whether such a tree has a unique minimum independent dominating set.

First, we present a simple observation.

Observation 4.5 (Fischermann and Volkmann [46]) Let the graph G be arbitrary.

a) If $\gamma(G) = i(G)$ and G has a unique γ-set D, then D is also the unique i-set of G.

b) If $ir(G) = \gamma(G)$ and G has a unique ir-set D, then D is also the unique γ-set of G.

Proof. Lemma 1.26 implies that every i-set of G is a γ-set of G if $\gamma(G) = i(G)$, and every γ-set of G is an ir-set of G if $ir(G) = \gamma(G)$. Hence, the required result follows. □

Our next result shows that the reversion of Observation 4.5 b) is true for arbitrary graphs with irredundance number equal domination number.

Theorem 4.6 (Fischermann and Volkmann [46]) *Let G be an arbitrary graph with $ir(G) = \gamma(G)$. Then, G has a unique γ-set if and only if G has a unique ir-set.*

Proof. By Observation 4.5 b), we know that a unique ir-set of such a graph always is a unique γ-set.

Conversely, suppose that G has the unique γ-set D and an ir-set $D' \neq D$. Let D' be an ir-set of G different from D where $|V(G) \setminus N[D']|$ is minimal. Since D is a unique γ-set and $ir(G) = \gamma(G)$, we deduce that $V(G) \setminus N[D'] \neq \emptyset$. Let $v \in V(G) \setminus N[D']$ be arbitrary. Since D' is maximal irredundant, there exists a vertex w in D' that has all its private neighbours with regard to D' in the neighbourhood $N(v)$ of v. Thus, the set $D'' = (D' \setminus \{w\}) \cup \{v\}$ dominates $N[D'] \cup N[v]$. If D'' is not irredundant, then there exists a proper subset I of D'' that is maximal irredundant and we deduce the contradiction that $ir(G) < |D''| = |D'|$. If D'' is not maximal irredundant, then there exists a vertex $x \in V(G) \setminus D''$ such that $D'' \cup \{x\}$ is irredundant, which implies the existence of a private neighbour $p_x \in N[x] \setminus N[D''] \subseteq N[x] \setminus N[D']$ contradicting the maximality of D'. Hence, also the set D'' is an ir-set of G and $|V(G) \setminus N[D'']| \leq |(V(G) \setminus N[D']) \setminus \{v\}| < |V(G) \setminus N[D']|$. By the minimality of $|V(G) \setminus N[D']|$, we obtain $D'' = (D' \setminus \{w\}) \cup \{v\} = D$. Since $v \in V(G) \setminus N[D']$ was arbitrary, this implies $V(G) \setminus N[D'] = \{v\}$. The vertex $v \in D$ has at least one private neighbour with regard to D. If the induced subgraph $G[P(v, D)]$ is complete, then we obtain the contradiction that the set $(D \setminus \{v\}) \cup \{u\}$ is a γ-set of G different from D for any private neighbour $u \in P(v, D) \setminus \{v\}$ or for any neighbour u of v if $P(v, D) = \{v\}$. Thus, there exist at least two non-adjacent private neighbours p_v and p'_v of v with regard to D. The set $F = D' \cup \{p_v\}$ is obviously a dominating set of G. Since D' is maximal irredundant, the set F is not irredundant which implies the existence of a vertex x in F such that the set $F' = F \setminus \{x\}$ also dominates G. By the equality $|D'| = |F'|$, we obtain that F' is a γ-set of G and $D = F'$. The set F does not contain the vertex v and hence, $v \in D \setminus F'$ which is a contradiction. □

A graph G is called *domistable* if any minimal dominating set of G is independent. Domistable graphs were introduced by Benzaken and Hammer in [4]. Obviously, domistable graphs G fulfil the equalities $\gamma(G) = i(G)$ and $\alpha(G) = \Gamma(G)$, and Topp [95] has proved the interesting result that domistable graphs even satisfy the equality $ir(G) = \gamma(G) = i(G)$.

This result together with Theorem 4.6 implies the following.

Corollary 4.7 (Fischermann and Volkmann [46]) *Let G be a domistable graph and let $D \subseteq V(G)$ be arbitrary. Then, the following conditions are equivalent.*

(i) D is the unique i-set of G.

(ii) D is the unique γ-set of G.

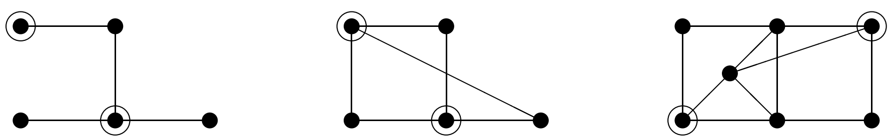
(iii) D is the unique ir-set of G.

Proof.
(i) $\Leftrightarrow$ (ii) By the definition of domistable graphs, we obtain that D is an i-set of G if and only if D is a γ-set of G, which immediately implies the equivalence of (i) and (ii).

(ii) $\Leftrightarrow$ (iii) By the result of Topp [95] mentioned above, we know that $ir(G) = \gamma(G)$. Thus, we deduce by Theorem 4.6, that D is the unique γ-set of G if and only if D is the unique ir-set of G. □

We now show that for arbitrary graphs G a unique i-set of G not necessarily is the unique γ-set of G, even not if the graph G satisfies $i(G) = \gamma(G)$.

Observation 4.8 (Fischermann and Volkmann [46]) The graphs B_1, B_2, $K_{2,t}$ for every integer $t \geq 4$, and B_3 (see Figure 4.3) have domination number equal independent domination number and they have a unique i-set but at least two γ-sets. Thus, the reversion of Observation 4.5 a) does not hold in general.



B_1 $B_2 \cong K_{2,3}$ B_3

Figure 4.3

For any graph H we say that a graph G is *H-free* if G does not contain the graph H as an induced subgraph.

Theorem 4.9 (Fischermann and Volkmann [46]) *Let G be a graph that is B_1-, B_2- and H_5-free (see Figure 4.1 and Figure 4.3). Then, G is γ-perfect and G has a unique i-set if and only if G has a unique γ-set.*

Proof. Let G be an arbitrary graph that does not contain any of the graphs B_1, B_2, H_5 as an induced subgraph. Let $H_1 - H_6$ be the graphs in Figure 4.1. It is straightforward to see that the graphs H_1 and H_2 contain the graph B_1 as an induced subgraph, and the graphs H_3, H_4 and H_6 contain the graph B_2 as an induced subgraph. Hence, the graph G does not contain any of the graphs $H_1 - H_6$ as an induced subgraph, and we deduce by Theorem 4.3, that the graph G is γ-perfect.

This implies that, if G has a unique γ-set D, then D is also the unique i-set of G, by Observation 4.5.

Now, suppose there exists a graph that is B_1-, B_2- and H_5-free and has a unique i-set and a further γ-set. Let G be such a graph of minimal order, let I be the unique i-set of G, and let $D \neq I$ be a γ-set of G such that the number of edges $m(G[D])$ in the induced subgraph $G[D]$ is minimal. Since I is unique, the set D is not independent and $|D| = |I| \geq 2$.

Claim 1: Every vertex $u \in D \setminus I$ has a neighbour w in D which has at least two non-adjacent private neighbours with regard to D.

Proof of Claim 1. Let $u \in D \setminus I$ be arbitrary. Suppose that either the set $N(u) \cap D$ is empty or for every vertex $w \in N(u) \cap D$ the induced subgraph $G[P(w, D)]$ is complete. Then, choose for every $w \in N(u) \cap D$ an arbitrary private neighbour $p_w \in P(w, D)$. Define the set $D' = (D \setminus N[u]) \cup \{p_w \mid w \in N(u) \cap D\}$. If the set $N(u) \cap D$ is empty, then $D' = D \setminus \{u\}$. The set D' is a dominating set of the induced subgraph $G_u = G - N[u]$ of G. Hence, we obtain $\gamma(G_u) = i(G_u) \leq |D'| = |D| - 1$. Let I' be an i-set of the graph G_u. The set $I' \cup \{u\}$ is an independent dominating set of G of cardinality at most $|D| = |I|$. By the uniqueness of I we obtain that $I' \cup \{u\} = I$ which is a contradiction, since I does not contain the vertex u.

Claim 2: Every vertex $v \in I$ which is not isolated in G has at least two non-adjacent private neighbours with regard to I.

Proof of Claim 2. Let $v \in I$ be arbitrary and let $w \in I \setminus \{v\}$. The set $I \setminus \{w\}$ is an independent dominating set of the graph $G_w = G - N[w]$. If the graph G_w has an i-set $I' \neq I \setminus \{w\}$, then the set $I' \cup \{w\}$ is an independent dominating set of G different from I and of cardinality $|I' \cup \{w\}| \leq |I|$, which is a contradiction to the uniqueness of I. Thus, the set $I \setminus \{w\}$ is the unique i-set of G_w and, since G_w is B_1-, B_2- and H_5-free, we obtain by the minimality of G that this set $I \setminus \{w\}$ is also the unique γ-set of G_w. The vertex $v \in I \setminus \{w\}$ is its own private neighbour with regard to $I \setminus \{w\}$. If $P(v, I \setminus \{w\}, G_w) = \{v\}$, then let u be an arbitrary neighbour of v in G, and else let $u \in P(v, I \setminus \{w\}, G_w) \setminus \{v\}$. If the induced subgraph $G[P(v, I \setminus \{w\}, G_w)]$ is complete, then we obtain the contradiction that the set $(I \setminus \{v, w\}) \cup \{u\}$ is a γ-set of G_w different from $I \setminus \{w\}$. Hence, the vertex $v \in I \setminus \{w\}$ has at least two non-adjacent private neighbours in $P(v, I \setminus \{w\}, G_w)$. We obtain the required result by the equality $P(v, I \setminus \{w\}, G_w) = P(v, I, G)$.

Claim 3: Every vertex $u \in D \setminus I$ has at least two non-adjacent private neighbours with regard to D.

Proof of Claim 3. Suppose there exists a vertex $u \in D \setminus I$ such that the induced subgraph $G[P(u, D)]$ is complete. Let p_u be a private neighbour of u and let $D' = (D \setminus \{u\}) \cup \{p_u\}$. By Claim 1, the vertex u has a neighbour in D which yields that $p_u \neq u$ and $D' \neq D$. By the assumption and by the equality $|D'| = |D|$, the set D' is a γ-set of G. Since p_u has no neighbour in $D \setminus \{u\}$ and u has at least one neighbour in $D \setminus \{u\}$, we obtain that $m(G[D']) < m(G[D])$. By the minimality of $m(G[D])$, we deduce $D' = I$. This implies that $I \setminus D = \{p_u\}$, $D \setminus I = \{u\}$ and p_u is the only private neighbour of u. We obtain the equality

$$\begin{aligned} P(p_u, I) &= V(G) \setminus N[I \setminus \{p_u\}] \\ &= V(G) \setminus N[D \setminus \{u\}] \\ &= P(u, D) \\ &= \{p_u\} \end{aligned}$$

which is a contradiction to Claim 2.

In view of Claim 1 and Claim 3, every vertex $u \in D \setminus I$ has a neighbour $w \in N(u) \cap D$ such that both vertices have two non-adjacent private neighbours $p_u, p'_u \in P(u, D)$ and $p_w, p'_w \in P(w, D)$. Consider the induced subgraph $H = G[\{u, p_u, p'_u, w, p_w, p'_w\}]$. Since the graph $H - p_u$ is not equal to B_1 or to B_2, the vertex p'_u has exactly one neighbour in $\{p_w, p'_w\}$. The same consideration for the other private neighbours in H yields that either $N(p_u) \cap \{p_w, p'_w\} = \{p_w\}$ and $N(p'_u) \cap \{p_w, p'_w\} = \{p'_w\}$ or $N(p_u) \cap \{p_w, p'_w\} = \{p'_w\}$ and $N(p'_u) \cap \{p_w, p'_w\} = \{p_w\}$. This leads to the contradiction that H is equal the graph H_5. □

We say that a graph G has property $\mathcal{P}$ if any induced subgraph H of G fulfils $\gamma(H) = i(H)$ and any subset $D \subseteq V(H)$ is a unique i-set of H if and only if it is a unique γ-set of H.

Observation 4.10 By Theorem 4.9, it is straightforward to see that any graph G that is B_1-, B_2-, and H_5-free has property $\mathcal{P}$. Since the graph H_5 fulfils property $\mathcal{P}$, the reversion of this statement does not hold.

But Observation 4.8 implies that every graph with property $\mathcal{P}$ is B_1-, B_2- and B_3-free. Since the graph B_1 is an induced subgraph of $H_1 - H_3, G_5, G_7 - G_{12}, G_{14}$ and G_{15}, the graph B_2 is an induced subgraph of H_4 and G_6, and the graph B_3 is an induced subgraph of G_{13}, G_{16} and G_{17}, we obtain by the result of Zverovich and Zverovich in Theorem 4.4, that any graph G that is B_1-, B_2- and B_3-free is γ-perfect and hence, if G has a unique γ-set, then G has also a unique i-set, by Observation 4.5.

Problem 4.11 (Fischermann and Volkmann [46]) *Any H_5-free graph G has property $\mathcal{P}$ if and only if G is B_1- and B_2-free. It remains the question if there exists a general characterization of graphs with property $\mathcal{P}$ by forbidden induced subgraphs.*

In the end of this section we consider unique i-sets in trees T which satisfy $\gamma(T) = i(T)$. The following lemma contains a general necessary condition for the uniqueness of an i-set.

Lemma 4.12 (Fischermann and Volkmann [46]) *Let G be an arbitrary graph and let $I \subseteq V(G)$ be an i-set of G. If I is the unique i-set of G, then every vertex x in I fulfils either $P(x, I) = \{x\}$ or $|P(x, I)| \geq 3$.*

Proof. Obviously, every vertex x in I is its own private neighbour with regard to I. Suppose that for a vertex x in I we have $P(x, I) = \{x, y\}$ for some vertex $y \neq x$. Then, the set $(I \setminus \{x\}) \cup \{y\}$ is a further independent dominating set of cardinality $i(G)$, which is a contradiction. □

For trees T with $\gamma(T) = i(T)$ this necessary condition is also sufficient for the uniqueness of an i-set.

Theorem 4.13 (Fischermann and Volkmann [46]) *Let T be a tree with $\gamma(T) = i(T)$ and let I be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) I is the unique i-set of T.

(ii) I is an i-set of T such that every vertex x in I fulfils either $P(x, I) = \{x\}$ or $|P(x, I)| \geq 3$.

Proof.
(i) $\Rightarrow$ (ii) Follows immediately from Lemma 4.12.

(ii) $\Rightarrow$ (i) Suppose that T has an i-set I such that every vertex x in I fulfils either $P(x,I) = \{x\}$ or $|P(x,I)| \geq 3$, and there exists a second i-set $I' \neq I$ of T. It yields that $|I| = |I'|$ and $|I \setminus I'| = |I' \setminus I|$. We define the four sets

$$\begin{aligned} I_1 &= \{x \in I \setminus I' \mid P(x,I) = \{x\}\}, \\ I_2 &= \{x \in I \setminus I' \mid |P(x,I)| \geq 3\}, \\ I_1' &= \{y \in I' \mid N(y) \cap I_1 \neq \emptyset\}, \\ I_2' &= \{y \in I' \setminus I \mid N(y) \cap I_1 = \emptyset\}. \end{aligned}$$

By the assumption, the union $I_1 \cup I_2$ is equal $I \setminus I'$, and since the set I is independent, we obtain $I_1' \subseteq I' \setminus I$ and $I_1' \cup I_2' = I' \setminus I$. If a vertex $y \in I_1'$ is adjacent to two different vertices x and x' in I_1, then the set $D = (I \setminus \{x, x'\}) \cup \{y\}$ is a dominating set of T which leads to the contradiction $\gamma(T) < i(T)$. This and the definition of I_1' imply that every vertex in I_1' has exactly one neighbour in I_1. Since the set I' is maximal independent, every vertex x in I_1 has a neighbour in I' which implies that this neighbour lies in I_1'. Thus, we obtain that

$$|I_1| = |I_1'| \quad \text{and} \quad |I_2| = |I_2'|. \tag{4.1}$$

Since every vertex $y \in I_1'$ has at least one neighbour $x \in I_1$ and since y is not a private neighbour of x with regard to I, we know that

$$|N(y) \cap I| \geq 2 \qquad \text{for every vertex } y \in I_1'. \tag{4.2}$$

Let $P = P(I \setminus I', I) \setminus (I \cup I')$ and let H be the subgraph of T induced by the vertex set $(I \setminus I') \cup (I' \setminus I) \cup P$. Note, that the subgraph H is a forest and thus, we know that $m(H) \leq n(H) - 1$. Furthermore, we know that the size of H is composed as follows

$$m(H) = m(I_1', I \setminus I') + m(I_2', I \setminus I') + m(P, I \setminus I') + m(P, I' \setminus I) + m(G[P]).$$

Now we look at the single addends of this summation. By (4.2), we obtain that

$$m(I_1', I \setminus I') = m(I_1', I) \geq 2|I_1'|.$$

Since every vertex in I_2 has at least two private neighbours with regard to I besides itself, and since these private neighbours lie in the disjoint union $P \cup I_2'$, we conclude that

$$m(I_2', I \setminus I') + m(P, I \setminus I') \geq m(I_2', I_2) + m(P, I_2) \geq 2|I_2|.$$

The set I' is a dominating set of T and thus, every vertex in P has at least one neighbour in I' and, by the definition of P, this neighbour has to lie outside the set $I' \cap I$, which leads to

$$m(P, I' \setminus I) \geq |P|.$$

These estimations together with (4.1) yield that

$$\begin{aligned} m(H) &\geq 2|I_1'| + 2|I_2| + |P| = |I_1| + |I_1'| + |I_2| + |I_2'| + |P| \\ &= |I \setminus I'| + |I' \setminus I| + |P| = n(H), \end{aligned}$$

which is a contradiction. □

Remark 4.14 If we consider the cycle C_{3t} of order $3t$ for any positive integer t, we see that Condition (ii) in Theorem 4.13 is not sufficient for the uniqueness of an i-set in arbitrary graphs with domination number equal independent domination number.

Remark 4.15 Cockayne, Goodman and Hedetniemi [22] and Beyer, Proskurowski, Hedetniemi and Mitchell [7] have found linear time algorithms to determine a γ-set and an i-set in a tree, respectively. Consequently, one can decide in linear time whether $\gamma(T) = i(T)$. Using the result in [7] and Theorem 4.13 one can compute in linear time whether a tree with equal domination and independent domination numbers has a unique i-set.

4.2 The upper chain

There are several publications on relations between independence, upper domination and upper irredundance as for example [10], [21], [25], [26], [54], [59], [60], [73] and [95]. One question that has been posed often is, which graphs G fulfil $\alpha(G) = \Gamma(G)$ or $\Gamma(G) = \mathrm{IR}(G)$, and which graphs fulfil these equations for every induced subgraph.

Definition 4.16 Let G be an arbitrary graph. If $\alpha(H) = \Gamma(H)$ for every induced subgraph H of G, then a graph G is called Γ-*perfect*. Furthermore, a graph G is called IR-*perfect* if $\Gamma(H) = \mathrm{IR}(H)$, for every induced subgraph H of G.

In 1998, Gutin and Zverovich have found the following interesting relation.

Theorem 4.17 (Gutin and Zverovich [54]) *Any* Γ-*perfect graph is* IR-*perfect.*

In opposition to γ-perfect graphs (cf. Section 4.1 and [105]), there does not exist yet a characterization of Γ-perfect graphs. But several classes of Γ-perfect graphs are known, of which four are considered here.

One class has been found by Cockayne, Favaron, Payan and Thomason [21] in 1981.

Theorem 4.18 (Cockayne, Favaron, Payan and Thomason [21]) *If G is a bipartite graph, then*

$$\alpha(G) = \Gamma(G) = \mathrm{IR}(G).$$

About ten years later Jacobson and Peters [73] have proved the same equality for chordal graphs and for a class of graphs defined by three forbidden induced subgraphs.

Theorem 4.19 (Jacobson and Peters [73]) *If G is a chordal graph, then*

$$\alpha(G) = \Gamma(G) = \mathrm{IR}(G).$$

Theorem 4.20 (Jacobson and Peters [73]) *For any graph G that does not contain either $K_{1,3}$, C_4 or the graph H in Figure 4.4 as an induced subgraph,*

$$\alpha(G) = \Gamma(G) = \mathrm{IR}(G).$$

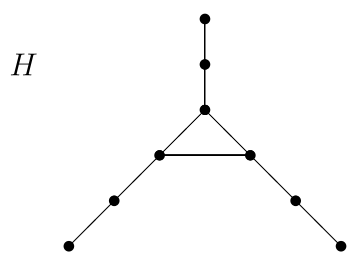


Figure 4.4

For unicyclic graphs, which are graphs containing at most one cycle, the equality of independence, upper domination and upper irredundance number has been pointed out by Topp [95].

Theorem 4.21 (Topp [95]) *If G is a unicyclic graph, then*

$$\alpha(G) = \Gamma(G) = \mathrm{IR}(G).$$

Since the main properties of the graphs in the last four theorems are hereditary with regard to induced subgraphs, all those graphs are Γ-perfect and IR-perfect.

Using these results, we prove for each graph in one of those four graph classes the equivalence of the uniqueness of an α-set, the uniqueness of a Γ-set and the uniqueness of an IR-set. One direction of this equivalence follows immediately from Lemma 1.26.

Observation 4.22 (Fischermann and Volkmann [45]) Let the graph G be arbitrary.

a) If $\Gamma(G) = \mathrm{IR}(G)$ and G has a unique IR-set D, then D is also the unique Γ-set of G.

b) If $\alpha(G) = \Gamma(G)$ and G has a unique Γ-set D, then D is also the unique α-set of G.

Proof. Lemma 1.26 implies that every α-set of G is a Γ-set of G if $\alpha(G) = \Gamma(G)$, and every Γ-set of G is an IR-set of G if $\Gamma(G) = \mathrm{IR}(G)$. Hence, the required result follows. □

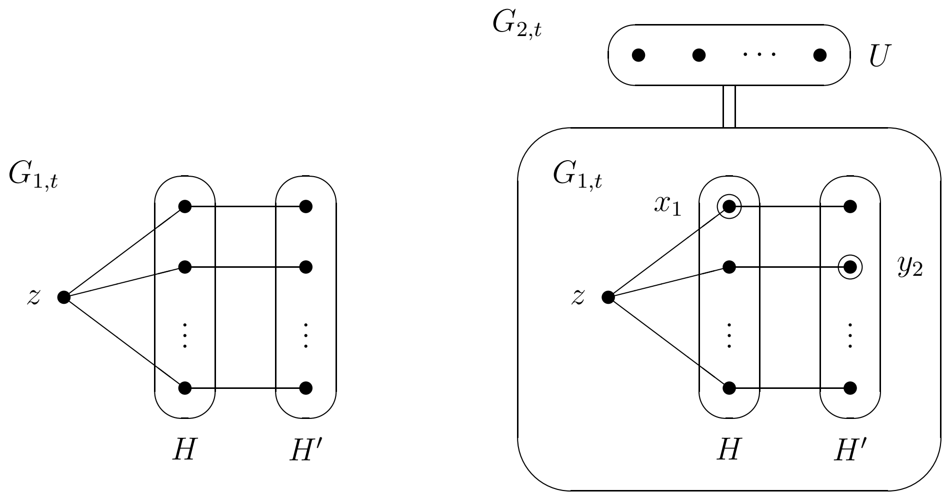
We will now construct graphs which show that the converse of Observation 4.22 does not hold in general.

For every integer $t \geq 3$, let $G_{1,t}$ be the graph consisting of two disjoint complete graphs H and H' with vertex sets $V(H) = \{x_1, x_2, \ldots, x_t\}$ and $V(H') = \{y_1, y_2, \ldots, y_t\}$ and of the additional edges $\{x_i y_i \mid 1 \leq i \leq t\}$ and of a further vertex z that is adjacent to all vertices in H (cf. Figure 4.5 a)). Further, let the graph $G_{2,t}$ consist of the graph $G_{1,t}$, of a disjoint independent vertex set U of cardinality $t-2$ and of the additional edges $uv \in E(G_{2,t})$ for every $u \in U$ and $v \in V(G_{1,t}) \setminus \{x_1, y_2\}$ (cf. Figure 4.5 b)).

Observation 4.23 (Fischermann and Volkmann [45]) The graph $G_{1,t}$ has the unique Γ-set $V(H)$ and $\Gamma(G_{1,t}) = \mathrm{IR}(G_{1,t}) = t$ but $G_{1,t}$ has no unique IR-set, since the two sets $V(H)$ and $V(H')$ are IR-sets of $G_{1,t}$.
The graph $G_{2,t}$ has the unique α-set $U \cup \{x_1, y_2\}$ and $\alpha(G_{2,t}) = \Gamma(G_{2,t}) = \mathrm{IR}(G_{2,t}) = t$ but $G_{2,t}$ has the two Γ-sets $U \cup \{x_1, y_2\}$ and $V(H)$.

Proof. First, we consider the graph $G_{1,t}$ for any $t \geq 3$. Obviously, the set $V(H)$ is a minimal dominating set of $G_{1,t}$ with $P(x_i, V(H)) = \{y_i\}$ for every $1 \leq i \leq t$. Let D be an arbitrary Γ-set of $G_{1,t}$. Then, this set D has cardinality at least $t = |V(H)|$. Suppose that the vertex z is in D. Since $\{z\} \cup V(H) =$

$N[z] \subseteq N[x_i]$ for any i, we get that $D \cap V(H) = \emptyset$ and $|D \cap V(H')| \leq 1$. This leads to the contradiction that $|D| \leq 2 < t$. Hence, we obtain that $z \notin D$ which implies that $x_i \in D$ for at least one i and $|D \cap V(H')| \leq 1$. Suppose that the vertex y_j is in D for some j. This leads to the contradiction that $D = \{x_i, y_j\}$. We receive that $D \subseteq V(H)$ which yields that $V(H)$ is the unique Γ-set of $G_{1,t}$ and $\Gamma(G_{1,t}) = t \leq \mathrm{IR}(G_{1,t})$. Suppose that there exists an irredundant set S in $G_{1,t}$ of cardinality greater than t. If $z \in S$, then $S \cap V(H) = \emptyset$ and $|S \cap V(H')| \leq 1$, which is a contradiction to $|S| > t$. Thus, the vertex z is not in S and there exists a vertex $x_i \in S \cap V(H)$ and a vertex $y_j \in S \cap V(H')$. Since for every vertex $v \in V(G_{1,t}) \setminus \{x_i, y_j\}$ the private neighbourhood $P(v, \{x_i, y_j, v\})$ is empty, we achieve the contradiction that $S = \{x_i, y_j\}$. Thus, we get that $\Gamma(G_{1,t}) = t = \mathrm{IR}(G_{1,t})$, and it is straightforward to see that the set $V(H')$ is a second IR-set of $G_{1,t}$.



$H \cong H' \cong K_t,\ t \geq 3$

U independent, $|U| = t - 2$ and
$u \in U, v \in V(G_{1,t}), uv \in E(G_{2,t}) \Leftrightarrow v \neq x_1, y_2$

a) b)

Figure 4.5

Now, we consider the graph $G_{2,t}$. The set $U \cup \{x_1, y_2\}$ is independent. Let I be an arbitrary α-set of $G_{2,t}$. Then, this set I has cardinality at least $|U| + 2 = t \geq 3$. Suppose, there exists a vertex $v \in I \cap \{z, x_2, \ldots, x_t\}$. This results in the contradiction that $S \setminus \{v\} \subseteq V(H')$ and $|S \cap V(H')| \leq 1$. Suppose, the vertex y_j is in I for some $j \neq 2$. This leads to the contradiction that $S \setminus \{y_j\} \subseteq V(H) \cup \{z\}$ and $|S \cap (V(H) \cup \{z\})| \leq 1$. We obtain that $I \subseteq U \cup \{x_1, y_2\}$ which implies that $U \cup \{x_1, y_2\}$ is the unique α-set of $G_{2,t}$ and $\alpha(G_{2,t}) = t \leq \Gamma(G_{2,t}) \leq \mathrm{IR}(G_{2,t})$. Suppose, there exists an irredundant set S in $G_{2,t}$ of cardinality greater than t. If there exists a vertex v in $S \cap (V(H) \cup \{z\})$

and a vertex y_i in $S \cap V(H')$, then every vertex $w \in S \setminus \{v, y_j\}$ has its private neighbours in U which implies the contradiction $|S| \leq 2 + |U| = t$. Hence, either $S \subseteq (\{z\} \cup V(H) \cup U)$ and $|S \cap (\{z\} \cup V(H))| \geq 3$ or $S \subseteq (V(H') \cup U)$ and $|S \cap V(H')| \geq 3$. If $S \subseteq (\{z\} \cup V(H) \cup U)$ and $|S \cap (\{z\} \cup V(H))| \geq 3$, then z is not in S and every x_i in S only has the private neighbour $y_i \in V(H')$ which leads to the contradiction that $S \subseteq V(H)$. Analogously, if $S \subseteq (V(H') \cup U)$ and $|S \cap V(H')| \geq 3$, then every y_i in S only has the private neighbour $x_i \in V(H)$ which leads to the contradiction that $S \subseteq V(H')$. Thus, we get that $\alpha(G_{2,t}) = \Gamma(G_{2,t}) = \mathrm{IR}(G_{2,t}) = t$, and it is straightforward to see that the set $V(H)$ is a second Γ-set of $G_{2,t}$. □

We will see that for the four classes of Γ-perfect graphs G considered in this section, a unique α-set I of G also is the unique Γ-set and the unique IR-set of G. In order to prove this, we use the following two lemmas.

Lemma 4.24 (Siemes, Topp and Volkmann [90]) *If any graph G has a unique α-set I, then every vertex in $V(G) \setminus I$ has at least two neighbours in I.*

Proof. Let G be an arbitrary graph that has a unique α-set I. Suppose that a vertex v in $V(G) \setminus I$ has at most one neighbour in I. Then, the set $(I \setminus N(v)) \cup \{v\}$ is independent which contradicts either the maximality or the uniqueness of I. □

Lemma 4.25 (Fischermann and Volkmann [45]) *Let G be a Γ-perfect graph. If G has a unique α-set I and an* IR*-set $D \neq I$, then*

a) for every vertex $x \in I \setminus D$ there exists a unique vertex $w_x \in D \setminus I$ such that $P(w_x, D) = \{x\}$,

b) $\{ab \in E(G) \mid a \in I \setminus D, b \in D \setminus I\} = \{xw_x \mid x \in I \setminus D\}$, and

c) there exists a cycle C in G such that $C = x_1x_2 \dots x_{4p}x_1$ for some positive integer p and for every $0 \leq i < p$ we have $x_{4i+2} \in D \setminus I$, $x_{4i+3} \in I \cap D$ and $x_{4i+1} \in P(x_{4i+2}, D) \subseteq I \setminus D$, $x_{4i+4} \in P(x_{4i+3}, D) \subseteq P(I \cap D, D) \setminus (I \cup D)$.

Proof. Let G be a Γ-perfect graph that has a unique α-set I and an IR-set $D \neq I$. By Theorem 4.17, we have $\alpha(H) = \Gamma(H) = \mathrm{IR}(H)$ for every induced subgraph H of G, and especially $|I| = |D|$. Let $x \in I \setminus D$ be arbitrary. We define the induced subgraph $G_x = G - x$ of G and the independent set $I_x = I \setminus \{x\}$. Since I is unique, the set I_x is an α-set of G_x. Thus, we get that $|D| > |I_x| = \alpha(G_x) = \Gamma(G_x) = \mathrm{IR}(G_x)$. This and the fact that D is a subset of $V(G_x)$ imply that D is not irredundant in G_x. Hence, there exists a vertex

w_x in D with $P(w_x, D) \cap V(G_x) = \emptyset$. Since the set D is irredundant in G, we obtain that $P(w_x, D) = \{x\}$, $w_x \in D \setminus I$ and $N(x) \cap (D \setminus I) = \{w_x\}$, whereby the proof of a) is complete.
The equality $|D \setminus I| = |I \setminus D|$ implies that every vertex in $D \setminus I$ has exactly one neighbour in $I \setminus D$ and this neighbour is its only private neighbour. This proves b).
It remains to prove c). Let $x_1 \in I \setminus D$ be arbitrary and let x_2 be its unique neighbour in $D \setminus I$. By Lemma 4.24, the vertex x_2 has at least two neighbours in I which yields the existence of a neighbour x_3 of x_2 in $I \cap D$. Since D is irredundant and $x_3 \notin P(x_3, D)$, there exists a fourth vertex $x_4 \in P(x_3, D) \subseteq P(I \cap D, D) \setminus D$. It even yields that $x_4 \in P(I \cap D, D) \setminus (D \cup I)$, since x_3 lies in I. By Lemma 4.24 and since $x_4 \in P(x_3, D)$, the vertex x_4 has a neighbour in $I \setminus D$. Now, we can extend the path $x_1x_2x_3x_4$ to a longest path $P = x_1x_2 \ldots x_t$, $t \geq 4$, such that for every $1 \leq i \leq t$ we have

$$\begin{aligned} x_i &\in I \setminus D && \text{if } i \equiv 1 \pmod 4, \\ x_i &\in D \setminus I && \text{if } i \equiv 2 \pmod 4, \\ x_i &\in I \cap D && \text{if } i \equiv 3 \pmod 4 \text{ and} \\ x_i &\in P(I \cap D, D) \setminus (I \cup D) && \text{if } i \equiv 0 \pmod 4. \end{aligned}$$

This implies that $x_i \in P(x_{i+1}, D)$ for every $1 \leq i < t$ with $i \equiv 1 \pmod 4$ and $x_i \in P(x_{i-1}, D)$ for every $1 \leq i \leq t$ with $i \equiv 0 \pmod 4$.
If $t \equiv 1 \pmod 4$, then the unique neighbour w of x_t in $D \setminus I$ does not belong to P, since it has no other neighbour in $I \setminus D$, and we can extend P with w, which is a contradiction.
Analogously, if $t \equiv 3 \pmod 4$, then x_t has a private neighbour u outside I and D and, since $u \in P(I \cap D, D) \setminus (I \cup D)$, this vertex u does not belong to P and we can extend P with u, which is a contradiction.
If $t \equiv 0 \pmod 4$, then $x_t \in P(x_{t-1}, D)$ and by Lemma 4.24, the vertex x_t has a second neighbour in I besides x_{t-1} that has to lie outside of D. Since P is a longest path, this neighbour has to lie in the set $(I \setminus D) \cap V(P)$. Let $x_\nu \in N(x_t) \cap (I \setminus D) \cap V(P)$. Then, we know that $\nu \equiv 1 \pmod 4$. If we define $y_{i-(\nu-1)} = x_i$ for every $\nu \leq i \leq t$, then $y_1y_2 \ldots y_{t-(\nu-1)}y_1$ is a cycle that fulfils the properties in c).
If $t \equiv 2 \pmod 4$, then x_{t-1} is the only neighbour of x_t in $I \setminus D$, and by Lemma 4.24, x_t has at least one neighbour in $I \cap D$. Since the path P is a longest path, there exists an index $1 \leq \nu < t-1$ such that the vertex $x_\nu \in N(x_t) \cap (I \cap D)$, and we know that $\nu \equiv 3 \pmod 4$. In this case, we define $y_1 = x_{t-1}$, $y_2 = x_t$ and $y_{3+(i-\nu)} = x_i$ for every $\nu \leq i \leq t-2$. Hence, the cycle $y_1y_2 \ldots y_{t-(\nu-1)}y_1$ fulfils the properties in c). □

With this lemma we are able to prove the following results.

Theorem 4.26 (Fischermann and Volkmann [45]) *Let G be a bipartite graph and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

Proof.
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Follows from Theorem 4.18 and Observation 4.22.

(iii) $\Rightarrow$ (i) Let G be a bipartite graph with partite sets A and B, and let I be the unique α-set of G. Suppose that G has an IR-set $D \neq I$. Let $D_0 = \{x \in D \mid P(x, D) = \{x\}\}$. Since G is Γ-perfect, Lemma 4.25 yields that every vertex $w \in D \setminus I$ has its unique private neighbour in $I \setminus D$ which implies that the set D_0 is a subset of $I \cap D$. We define the four subsets

$$\begin{aligned} A_1 &= (D \setminus I) \cap A, \\ A_2 &= (P(I \cap D, D) \setminus (D \cup I)) \cap A, \\ B_1 &= (I \setminus D) \cap B, \\ B_2 &= ((I \cap D) \setminus D_0) \cap B. \end{aligned}$$

Figure 4.6 illustrates these sets where $P = P(I \cap D, D) \setminus (D \cup I)$. Note, that the sets A_1 and A_2 are disjoint subsets of $A \setminus I$ and the sets B_1 and B_2 are disjoint subsets of $B \cap I$.

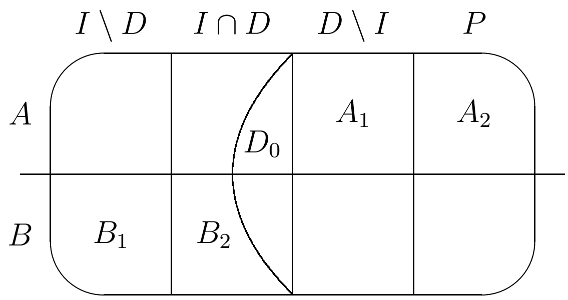


Figure 4.6

Next, we define the set

$$\begin{aligned} I' &= (I \setminus (B_1 \cup B_2)) \cup (A_1 \cup A_2) \\ &= (I \setminus B) \cup (D_0 \cap B) \cup (A_1 \cup A_2). \end{aligned}$$

Suppose that there exist two adjacent vertices a and b in I'. The union $(I \setminus B) \cup (D_0 \cap B)$ is independent as a subset of I, and the union $(I \setminus B) \cup (A_1 \cup A_2)$ is independent as a subset of A. Thus, without loss of generality, we deduce that $a \in A_1 \cup A_2$ and $b \in D_0 \cap B$.
If $a \in A_1$, then we obtain the contradiction that $b \in D_0$ is its own private neighbour with regard to D but b is adjacent to $a \in D$.
If $a \in A_2$, then a has to be a private neighbour of b, which contradicts that b belongs to D_0.
Hence, the set I' is independent, and by the uniqueness of I, we obtain that $|I'| < |I|$, which is equivalent to $|B_1 \cup B_2| > |A_1 \cup A_2|$.
By Lemma 4.25 b), we obtain $|B_1| \leq |A_1|$, since any vertex in B_1 only has neighbours in the partite set A.
Furthermore, every vertex $x \in B_2 = ((I \cap D) \setminus D_0) \cap B$ has a private neighbour p_x in $P(I \cap D, D) \setminus (D \cup I)$. Since x belongs to the partite set B, we even obtain that $p_x \in A_2$, which implies that $|B_2| \leq |A_2|$. This results in the contradiction that $|B_1 \cup B_2| \leq |A_1 \cup A_2|$. □

Corollary 4.27 (Fischermann and Volkmann [45]) *Let G be a unicyclic graph and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

Proof.
(i) ⇒ (ii) ⇒ (iii) This follows from Theorem 4.21 and Observation 4.22.

(iii) ⇒ (i) Let G be a unicyclic graph. If the only cycle in G is even, then G is bipartite and the required result follows from Theorem 4.26. Now, let the only cycle in G be odd and let I be the unique α-set of G. Suppose that G has an IR-set $D \neq I$. Since G - as a unicyclic graph - is Γ-perfect, we obtain by Lemma 4.25 c) the existence of a cycle C in G of even length. This contradiction completes the proof. □

Theorem 4.28 (Fischermann and Volkmann [45]) *Let G be a Γ-perfect, claw-free graph and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

Proof.
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Follows from Theorem 4.17 and Observation 4.22.

(iii) $\Rightarrow$ (i) Let G be a Γ-perfect, claw-free graph and let I be the unique α-set of G. By Lemma 4.24, every vertex in $V(G) \setminus I$ has at least two neighbours in I. Since G is claw-free, every vertex in $V(G) \setminus I$ has exactly two neighbours in I. Suppose that G has an IR-set $D \neq I$. Then, there exists a cycle C in G as in Lemma 4.25 c). Let $C = x_1x_2 \ldots x_{4p}x_1$ for some positive integer p such that $x_1 \in I \setminus D$ and $x_2 \in D \setminus I$. Furthermore, let

$$\begin{aligned}(I \setminus D)_C &= V(C) \cap (I \setminus D)\\ (D \setminus I)_C &= V(C) \cap (D \setminus I)\\ (I \cap D)_C &= V(C) \cap (I \cap D)\\ P_C &= V(C) \cap (P(I \cap D, D) \setminus (I \cup D))\end{aligned}$$

Suppose that there is an edge ab in the induced subgraph $G[(D \setminus I)_C \cup P_C]$. Since every vertex in P_C is a private neighbour of a vertex in $I \cap D$, there is no edge between any vertex in P_C and any vertex in $(D \setminus I)_C$. Thus, it yields either $a, b \in (D \setminus I)_C$ or $a, b \in P_C$. If $a, b \in (D \setminus I)_C$, then let, without loss of generality, $a = x_2$ and $b = x_j$ for some $j \equiv 2 \pmod 4$, $2 < j \leq 4p$. Since $N(x_2) \cap I = \{x_1, x_3\}$ and $N(x_j) \cap I = \{x_{j-1}, x_{j+1}\}$, the induced subgraph $G[\{x_1, x_2, x_3, x_j\}]$ is a claw, which is a contradiction. Analogously, if $a, b \in P_C$ and, without loss of generality, $a = x_4$ and $b = x_j$ for some $j \equiv 0 \pmod 4$, $4 < j \leq 4p$, then $N(x_4) \cap I = \{x_3, x_5\}$, $N(x_j) \cap I = \{x_{j-1}, x_{j+1}\}$, and the induced subgraph $G[\{x_3, x_4, x_5, x_j\}]$ is a claw, which is a contradiction. Hence, the set $(D \setminus I)_C \cup P_C$ is independent. Every vertex x in $(D \setminus I)_C \cup P_C$ has two neighbours in $I \cap V(C)$ and hence no neighbour in $I \setminus V(C)$. This implies that the set $I' = (I \setminus V(C)) \cup (D \setminus I)_C \cup P_C$ is independent. Since the cardinalities $|I \cap V(C)|$ and $|(D \setminus I)_C \cup P_C|$ are both equal $2p$, we obtain the contradiction that $|I'| = |I|$ and I' is a second α-set of G different from I. $\square$

By Theorem 4.20 and Theorem 4.28, we obtain the following.

Corollary 4.29 *Any graph G that does not contain either $K_{1,3}$, C_4 or the graph H in Figure 4.4 as an induced subgraph has a unique* IR*-set if and only if it has a unique Γ-set if and only if it has a unique α-set.*

The last class considered in this section is the class of chordal graphs.

Theorem 4.30 (Fischermann and Volkmann [45]) *For a chordal graph G and for any subset D of $V(G)$ the following conditions are equivalent.*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

Proof.
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Follows from Theorem 4.19 and Observation 4.22.

(iii) $\Rightarrow$ (i) Let G be a chordal graph and let I be the unique α-set of G. By Theorem 4.19, the graph G is Γ-perfect. Suppose that G has an IR-set $D \neq I$. Then, there exists a cycle C in G as described in Lemma 4.25 c). Let $C = x_1x_2 \dots x_{4p}x_1$ for some positive integer p such that $x_1 \in I \setminus D$ and $x_2 \in D \setminus I$. Furthermore, let C' be a cycle of minimal length in the induced subgraph $G[V(C)]$ that contains the edge x_1x_2. Since the induced subgraph $G[V(C)]$ is chordal, we obtain that $C' = x_1x_2yx_1$ for some vertex $y \in V(C) \setminus \{x_1, x_2\}$. The fact that $x_1 \in I$ leads to $y \in V(C) \setminus I$. Note that $V(C) \setminus I \subseteq (D \setminus I) \cup (P(I \cap D, D) \setminus (I \cup D))$. If $y \in P(I \cap D, D) \setminus (I \cup D)$, then we obtain the contradiction that y is adjacent to the vertex x_2 in $D \setminus I$. Hence, it remains that $y \in D \setminus I$. But in this case the vertex y lies in D and is adjacent to $x_1 \in P(x_2, D)$ which is a contradiction. □

Siemes, Topp, and Volkmann [90] have investigated so called k-independent sets - a generalization of unique α-sets - and they have found characterizations of k-independent sets for several classes of graphs. For $k = 1$ one of their results contains a further characterization of unique α-sets in chordal graphs (cf. Theorem 4 in [90]).

Theorem 4.31 (Siemes, Topp, and Volkmann [90]) *Let G be a graph in which every even cycle possesses a chord. Then the following statements are equivalent.*

a) D is the unique α-set of G.

b) D is an independent dominating set of G such that every vertex in $V(G) \setminus D$ has at least two neighbours in D.

The following lemma contains a simple characterization of dominating sets as in Theorem 4.31 b). The proof of this result is trivial.

Lemma 4.32 *Let G be an arbitrary graph. Then the following two conditions are equivalent.*

a) D is a dominating set of G such that $P(x, D) = \{x\}$ for every vertex $x \in D$.

b) D is an independent dominating set of G such that every vertex in $V(G) \setminus D$ has at least two neighbours in D.

Our result in Theorem 4.30 together with Theorem 4.31 and Lemma 4.32 yields a characterization of unique Γ-sets in chordal graphs by the private neighbourhood, similar to Theorem 3.3.

Corollary 4.33 (Fischermann and Volkmann [45]) *Let G be a chordal graph and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

(iv) D is a independent dominating set of G such that every vertex in $V(G) \setminus D$ has at least two neighbours in D.

(v) D is a dominating set of G such that $P(x, D) = \{x\}$ for every vertex $x \in D$.

Remark 4.34 By Lemma 4.24, Condition (iv) and (v) in Corollary 4.33 are necessary for the uniqueness of α-sets in arbitrary graphs. But they are not necessary for the uniqueness of Γ-sets in arbitrary graphs (cf. the graph $G_{1,t}$ in Figure 4.5 a)).
Furthermore, in arbitrary graphs Condition (iv) and (v) are not sufficient for the unique realization of IR, Γ or α, even not if the graph G satisfies $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$ and an α-set I of G fulfils Condition (iv) and (v). For example consider for some integer $s \geq 3$ the complete bipartite graph $K_{s,s}$ satisfying $\alpha(K_{s,s}) = \Gamma(K_{s,s}) = \mathrm{IR}(K_{s,s}) = s$, where both partite sets are α-, Γ- and IR-sets that fulfil Condition (iv) and (v). Thus, Corollary 4.33 does not even hold for bipartite graphs.

Remark 4.35 There exist polynomial time algorithms to compute the independence number α for chordal graphs by Gavril [50] and for claw-free graphs by Minty [81] and Sbihi [89].

For bipartite graphs we know by Königs Theorem and Gallai's Theorem (see Chartrand and Lesniak [17] or Volkmann [101]), that the sum of independence number and matching number equals the order of the graph ($\alpha(G) + \alpha'(G) = n(G)$). Since the matching number $\alpha'(G)$ can be computed in polynomial time by the algorithm of Edmonds [31] for any graph G, one can determine $\alpha(G)$ for every bipartite graph G also in polynomial time.

If we consider an arbitrary unicyclic graph G and an edge xy on its cycle, then it is straightforward to see that $\alpha(G) = \max\{\alpha(G-x), \alpha(G-y)\}$. Since $G-x$ and $G-y$ are forests, we can determine $\alpha(G)$ for unicyclic graphs G in linear time, by using the algorithm of Daykin and Ng [28] for trees.

Thus, every one of the four graph classes considered in this section fulfils the condition in Proposition 2.6, and we can decide in polynomial time whether any graph in these classes has a unique α-set. By our results in Theorem 4.26, Corollary 4.27, Theorem 4.28 and Theorem 4.30, this also yields an efficient algorithm for the decision problem whether a graph in these classes has a unique Γ-set or a unique IR-set.

Chapter 5

Further domination concepts

Motivated by the basic necessary condition for unique γ-sets in Lemma 3.1 and the simple characterization of unique γ-sets of trees in Theorem 3.3 by Gunther, Hartnell, Markus, and Rall we were interested in similar results for other domination concepts. The preceding two chapters contain among other things such results for X-domination, for independent domination and for upper domination.

In the first section of this chapter we generalize Lemma 3.1 and Theorem 3.3 for distance domination. In Section 2 and 3 we present similar results for total domination and edge domination. Further, we give iterative characterizations of trees having unique minimum total dominating sets and unique minimum edge dominating sets, respectively. In the end of each section we have a short look at the complexity of the inclosed problems.

5.1 Distance domination in trees

The concept of distance-k domination was introduced by Slater in [91] under the name R-domination, and it is studied in several other publications as e.g. [8], [15], [52], [67], [68] and [72]. The interested reader can find a survey on this topic by Henning in [60]. We consider for any positive integer k graphs where the distance-k domination number has a unique realization. First, we present two necessary conditions for the uniqueness of a $\gamma_{\leq k}$-set which both refer to the private k-neighbourhoods. For this purpose we define a relevant subset of the private k-neighbourhood of a vertex.

Definition 5.1 Let G be a graph, let $D \subseteq V(G)$ and let $x \in D$. Further, let k be a positive integer. We define for every vertex $x \in D$

$$\tilde{P}_k(x, D) = \{y \in P_k(x, D) \mid d(x, y) = k\}.$$

The following lemma contains two necessary conditions for the unique realization of the distance-k domination number that generalizes Lemma 3.1.

Lemma 5.2 *Let G be a non trivial connected graph, let k be a positive integer, and let D be the unique $\gamma_{\leq k}$-set of G.*

a) For every vertex x in D we have $|\tilde{P}_k(x,D)| \geq 2$, and for every vertex $v \in \tilde{P}_k(x,D)$ and every shortest path P_{xv} from x to v there exists a vertex $w \in \tilde{P}_k(x,D)$ such that for every shortest path P_{xw} from x to w we have $V(P_{xv}) \cap V(P_{xw}) = \{x\}$.

b) For every vertex x in D and for every $v \in P_k(x,D) \setminus \{x\}$ there is a vertex $w \in P_k(x,D)$ with $d(v,w) > k$.

Proof. Let D be the unique $\gamma_{\leq k}$-set of G and let $x \in D$ arbitrarily.

a) Let $\tilde{P} = \tilde{P}_k(x,D)$. If $d(v,x) < k$ for every vertex v in $P_k(x,D)$, then we can replace x in D by any $z \in N(x)$ and $(D \setminus \{x\}) \cup \{z\}$ is a $\gamma_{\leq k}$-set of G different from D, which is a contradiction. Hence, there is a vertex $v \in \tilde{P}$. Let $v \in \tilde{P}$ arbitrarily and let P_{xv} be an arbitrary shortest path from x to v. Suppose, for every vertex $w \in \tilde{P} \setminus \{v\}$ there exists a shortest path P_{xw} from x to w such that the set $V(v,w) = (V(P_{xv}) \cap V(P_{xw})) \setminus \{x\}$ is not empty. Then, let $y \in N(x) \cap V(P_{xv})$. Obviously, the distance $d(y,z) \leq d(y,x) + d(x,z) \leq k$ for every vertex $z \in P_k(x,D) \setminus \tilde{P}$. Furthermore, for every vertex $z \in \tilde{P}$ there is a vertex $a \in V(v,z)$ and we obtain $d(y,z) \leq d(y,a) + d(a,z) = d(x,a) - 1 + d(a,z) = k - 1$. Thus, the set $(D \setminus \{x\}) \cup \{y\}$ is a $\gamma_{\leq k}$-set of G different from D, which is a contradiction. Hence, we have $|\tilde{P}| \geq 2$ and there exists a vertex $w \in \tilde{P}$ such that for every shortest path P_{xw} from x to w we have $V(P_{xv}) \cap V(P_{xw}) = \{x\}$.

b) Let $v \in P_k(x,D) \setminus \{x\}$ arbitrarily. If $d(v,w) \leq k$ for every vertex $w \in P_k(x,D)$, then the set $(D \setminus \{x\}) \cup \{v\}$ is a $\gamma_{\leq k}$-set of G different from D, which is a contradiction. □

Remark 5.3 In some way the conditions in Lemma 5.2 are best possible. For example see the graph consisting of the two paths $v_0v_1 \ldots v_k \ldots v_{2k}$ and $w_0w_1 \ldots w_k \ldots w_{2k}$ and of the two additional edges w_0v_0 and w_0v_{2k}. Here, $D = \{v_k, w_k\}$ is the unique $\gamma_{\leq k}$-set and for $x \in \{v_k, w_k\}$ we have $|\tilde{P}_k(x,D)| = 2$. Furthermore, for every vertex v in $P_k(v_k,D) \setminus \{v_k\}$ there exists a unique vertex $w \in P_k(v_k,D)$ such that $d(v,w) = k+1$ and $d(v,y) \leq k$ for every vertex $y \in P_k(v_k,D) \setminus \{w\}$.

For arbitrary graphs the existence of a $\gamma_{\leq k}$-set D satisfying the conditions a) and b) in Lemma 5.2 is not sufficient for the unique realization of $\gamma_{\leq k}$. For instance, consider the cycle $C = x_0x_1 \dots x_{4k+1}x_0$ for some positive integer k. This cycle has a $\gamma_{\leq k}$-set $D = \{x_k, x_{3k+1}\}$ that satisfies a) and b) but obviously D is not the unique $\gamma_{\leq k}$-set of C.

The following theorem shows that for trees the existence of a $\gamma_{\leq k}$-set satisfying a) and b) is necessary and sufficient for the unique realization of $\gamma_{\leq k}$.

Theorem 5.4 (Fischermann and Volkmann [43]) *Let T be a tree of order at least 3, let D be a subset of $V(T)$, and let k be a positive integer. Then the following conditions are equivalent:*

(i) D is the unique $\gamma_{\leq k}$-set of T.

(ii) D is a distance-k dominating set of T such that every vertex in D has at least two private k-neighbours v and w with $d(v, w) = 2k$.

(iii) D is a $\gamma_{\leq k}$-set of T such that $\gamma_{\leq k}(T - x) > \gamma_{\leq k}(T)$ for every vertex $x \in D$.

Proof.

(i) $\Rightarrow$ (ii): Follows immediately from Lemma 5.2a).

(ii) $\Rightarrow$ (i): We prove this by induction on the order $n(T)$. If a tree T has a distance-k dominating set D as in (ii), then the diameter of T is at least $2k$ and the order $n(T)$ at least $2k+1$. First, let T be a tree of order $n(T) = 2k+1$, that has a distance-k dominating set D as in (ii). Then, the tree T is isomorphic to the path $x_1x_2 \dots x_{2k+1}$ and $D = \{x_{k+1}\}$. Obviously, D is the unique $\gamma_{\leq k}$-set of T. Assume, the claim holds for every tree T' of order $2k + 1 \leq n(T') < n$. Now, let T be a tree of order $n(T) = n$, and let D be a distance-k dominating set of T as in (ii). Suppose, there exists a $\gamma_{\leq k}$-set of T different of D. Let D' be a $\gamma_{\leq k}$-set of T such that $D' \neq D$ and $|D \cap D'|$ is maximal. There is at least one vertex $x \in D \setminus D'$ and there are two vertices $y_1, y_2 \in P_k(x, D)$ with $d(y_1, y_2) = 2k$. Hence, we have $d(x, y_1) = d(x, y_2) = k$ and x lies on the unique path from y_1 to y_2 in T. Let $T_1, T_2, \dots, T_\kappa$ be the components of $T - x$ such that $y_i \in V(T_i)$ for $i = 1, 2$. Further, let $D_i = D \cap V(T_i)$ and $D'_i = D' \cap V(T_i)$ for $i = 1, 2$. Since D_i does not distance-k dominates the vertex y_i but D'_i dominates T_i, there is a vertex $z_i \in D'_i \setminus D_i$ with $d(z_i, y_i) \leq k$ for $i = 1, 2$. The set $D'' = (D' \setminus (D'_1 \cup D'_2)) \cup (D_1 \cup D_2 \cup \{x\})$ is a distance-k dominating set of T, which implies $|D''| \geq |D'|$ and $|D_1| + |D_2| + 1 \geq |D'_1| + |D'_2|$. If $|D'_1| > |D_1|$ and $|D'_2| > |D_2|$, then we obtain a contradiction. Hence, without loss of generality, we have $|D'_1| \leq |D_1|$. Let P be the unique path in T from x to y_2 and let $T' = T[V(T_1) \cup V(P)]$. It is easy to see that $D_1 \cup \{x\}$ is a distance-k

dominating set of T' that fulfils (ii). If $n(T') = n(T)$, then $T_2 = P - x$ and $N_k[z_2, T] \subseteq N_k[x, T]$. Hence, $D'' = (D' \setminus \{z_2\}) \cup \{x\}$ is a $\gamma_{\leq k}$-set of T with $|D'' \cap D| > |D' \cap D|$. Since $z_1 \in D'' \setminus D$, we have $D'' \neq D$, and this is a contradiction to the choice of D'. Hence, we have $n(T') < n(T)$. Then, by the induction hypothesis, the set $D_1 \cup \{x\}$ is the unique $\gamma_{\leq k}$-set of T'. But $D_1' \cup \{x\}$ is also a distance-k dominating set of T' with $|D_1' \cup \{x\}| \leq |D_1 \cup \{x\}|$ and $z_1 \in D_1' \setminus D_1$, which is a contradiction.

(i) $\Rightarrow$ (iii): Let D be the unique $\gamma_{\leq k}$-set of T, let $x \in D$ arbitrarily, let $\kappa = \kappa(T - x)$ and let $T_1, T_2, \ldots, T_\kappa$ be the components of $T - x$. Further, let D' be a $\gamma_{\leq k}$-set of $T - x$ and for every $1 \leq i \leq \kappa$ let $D_i = D \cap V(T_i)$ and $D_i' = D' \cap V(T_i)$. For every $1 \leq i \leq \kappa$ the set $D_i'' = (D \setminus D_i) \cup D_i'$ is a distance-k dominating set of T, which implies that either $D_i = D_i'$ or $|D_i| < |D_i'|$. By Lemma 5.2a), the vertex x has at least two private k-neighbours x_1, x_2 in T with $d(x_1, x_2) = 2k$. Without loss of generality, let $x_1 \in V(T_1)$ and $x_2 \in V(T_2)$. Then, for $i = 1, 2$, the set D_i is not a distance-k dominating set of T_i, in contrary to the set D_i'. Hence, we have $D_i \neq D_i'$ and $|D_i| < |D_i'|$ for $i = 1, 2$, which implies $\gamma_{\leq k}(T - x) = |D'| = \sum_{i=1}^{\kappa} |D_i'| \geq 2 + \sum_{i=1}^{\kappa} |D_i| = 1 + |D| > \gamma_{\leq k}(T)$.

(iii) $\Rightarrow$ (i): Let D be a $\gamma_{\leq k}$-set of T such that $\gamma_{\leq k}(T - x) > \gamma_{\leq k}(T)$ for every vertex $x \in D$. Suppose, that there is a $\gamma_{\leq k}$-set $D' \neq D$ of T. Since there exists at least one vertex $x \in D \setminus D'$, the set D' is distance-k dominating set of $T - x$. Hence, $\gamma_{\leq k}(T - x) \leq |D'| = \gamma_{\leq k}(T)$ for some $x \in D$, which is a contradiction.

□

For $k = 1$ also Theorem 5.4 implies the slightly stronger version of Theorem 3.3.

Remark 5.5 For trees a distance-k dominating set D satisfies Condition (ii) in Theorem 5.4 if and only if D satisfies Condition a) and b) in Lemma 5.2. But this equivalence does not hold for arbitrary graphs.
Particularly Condition (ii) in Theorem 5.4 is no longer necessary for the uniqueness of a $\gamma_{\leq k}$-set if the considered graph contains cycles and $k \geq 2$. For example see the graph constructed in Remark 5.3 where any two vertices in $P_k(v_k, D)$ have distance at most $k + 1$ from each other.

Now, we have a look at the complexity of the related problems of fixing the distance-k domination number of a tree, finding a $\gamma_{\leq k}$-set in a tree, and determining whether a tree has a unique $\gamma_{\leq k}$-set.

By Observation 1.19, we know that a subset D of the vertex set of a tree T is a $\gamma_{\leq k}$-set of T if and only if D is a γ-set of the k-th power T^k. Therefore, it is helpful to characterize unique γ-sets of k-th powers T^k of trees T.

Corollary 5.6 *Let T be a tree of order at least 3, let D be a subset of $V(T)$, and let k be a positive integer. Then the following conditions are equivalent:*

(i) D is the unique γ-set of T^k.

(ii) D is a dominating set of T^k such that every vertex x in D has at least two private neighbours v and w in T^k such that $N[v,T^k] \cap N[w,T^k] = \{x\}$.

Proof. By Observation 1.19 and Theorem 5.4, it remains to prove that the conditions (ii) in Theorem 5.4 and Corollary 5.6 are equivalent. Obviously, for every vertex $x \in D$ the private k-neighbourhood in T is equal the private neighbourhood in T^k. Thus, it remains to verify that for any vertex $x \in D$ and for two vertices v and w in $P(x, D)$ with regard to the graph T^k we have $N[v,T^k] \cap N[w,T^k] = \{x\}$ if and only if $d_T(v,w) = 2k$. Since $v, w \in N_k[x,T]$, we know that $x \in N[v,T^k] \cap N[w,T^k]$ and $d_T(v,w) \leq 2k$. If $d_T(v,w) = 2k$, then T contains exactly one vertex within distance k to v and w which implies that $N[v,T^k] \cap N[w,T^k] = \{x\}$. On the other hand, if $d_T(v,w) \leq k$, then we have $v, w \in N[v,T^k] \cap N[w,T^k]$, and if $k < d_T(v,w) < 2k$, then we obtain that $k \geq 2$ and on the unique path in T from v to w there lie at least two vertices within distance k to v and w. Thus, if $d_T(v,w) < 2k$, then we have $|N[v,T^k] \cap N[w,T^k]| \geq 2$, which completes the proof. □

For arbitrary graphs the problem of fixing the distance-k domination number is NP-complete. But for trees there are several possibilities to determine $\gamma_{\leq k}$ in polynomial time. By Observation 1.19, we have $\gamma_{\leq k}(G) = \gamma(G^k)$ for the k-th power graph G^k of G. Further, Chang and Nemhauser [15] have proved that $\gamma_{\leq k}(T) = \alpha(T^{2k}) = \theta(T^{2k})$ for any tree T, where $\theta(G)$ denotes the minimum number of cliques in G covering G. Hence, in view of [15], for any tree T the following problems are equivalent:

a) Fixing the distance-k domination number of T.

b) Fixing the domination number of the graph $G = T^k$.

c) Fixing the independence number of the graph $G = T^{2k}$.

d) Fixing the clique covering number of the graph $G = T^{2k}$.

Lubiw [78] has shown that powers of strongly chordal graphs are also strongly chordal. Note, that trees are strongly chordal. Chang and Nemhauser have noticed in [15] that we can construct the strongly chordal graph T^k in $O(n^3)$ time for any tree T of order n. They have also mentioned, that therefore we can use every algorithm for finding the cardinality of a minimum dominating set, a maximum independent set, or a minimum clique covering on strongly chordal graphs to determine the distance-k domination number of a tree. There are efficient such algorithms by Farber [33], Kolen [74], Lubiw [78], Frank [47], and Gavril [50]. The algorithm of Farber [33] even determines γ-sets of strongly chordal graphs in linear time. Thus, the construction of T^k together with the algorithm of Farber lead to an algorithm which determines a $\gamma_{\leq k}$-set of a tree T in $O(n^3)$ time.

Remark 5.7 We can decide whether a given tree T has a unique $\gamma_{\leq k}$-set by constructing T^k and deciding whether T^k has a unique γ-set. By now, we know two methods to make this decision. One possibility is to use the general method in Chapter 2 where we have to fix the domination number for $2 \cdot m(T^k)$ strongly chordal graphs obtained from T^k, which in the worst case leads to nearly n^2 executions of the algorithm of Farber. The more efficient possibility is to determine a γ-set D of T^k by the algorithm of Farber [33] and to use Corollary 5.6.

5.2 Total domination in trees

In 1980, Cockayne, Dawes, and Hedetniemi [19] have introduced the concept of total domination, motivated by the *Five Queens Problem*. There are numerous publications on total domination as e.g. [2], [6], [12], [19], [35], [61], [62], [63], [66] and [93].

In this section we consider unique total domination. We present firstly a general necessary condition for a subset of the vertex set being the unique γ_t-set of a graph, and secondly a characterization for unique γ_t-sets of trees. At least we give a constructive characterization of trees having unique γ_t-sets.

Definition 5.8 For a subset D of $V(G)$ and a vertex $x \in D$ the set $P_t(x, D) = N(x) \setminus N(D \setminus \{x\})$ is called the *total private neighbourhood* of x with regard to D. We call a vertex $y \in P_t(x, D)$ a *total private neighbour* of x with regard to D, and we define $P_t(D, D) = \bigcup_{x \in D} P_t(x, D)$.

It is easy to see ([62]) that a total dominating set D of a graph G is minimal if and only if $P_t(D, D)$ dominates D which is equivalent to the property that every vertex $x \in D$ has at least one total private neighbour. Now, we present a necessary condition for a total dominating set of a graph G being a unique γ_t-set.

Lemma 5.9 (Fischermann [39]) *Let G be a connected graph of order at least 3. If D is the unique γ_t-set of G, then for every vertex $x \in D$ we have either $|P_t(x, D)| \geq 2$ or $P_t(x, D) = P_t(x, D) \setminus D = \{y\}$ for some endvertex y of G.*

Proof. Let D be the unique γ_t-set of G and let $x \in D$ arbitrarily. Since D is minimal, we have $P_t(x, D) \neq \emptyset$. If $|P_t(x, D)| \geq 2$, then there is nothing to prove. Now, let $P_t(x, D) = \{y\}$ for some vertex y in G. Suppose $y \in D$. If there is a vertex $z \in N(y) \setminus \{x\}$, then $D' = (D \setminus \{x\}) \cup \{z\} \neq D$ is also a γ_t-set of G, which is a contradiction. If $N(y) = \{x\}$, then there is a vertex $z \in N(x) \setminus \{y\}$, by $n(G) \geq 3$. This leads to the contradiction, that $D' = (D \setminus \{y\}) \cup \{z\} \neq D$ is a second γ_t-set of G. Hence, $P_t(x, D) = P_t(x, D) \setminus D = \{y\}$. Suppose, there is a vertex $z \in N(y) \setminus \{x\}$. Then, $z \in N(D) \setminus D$ and $D' = (D \setminus \{x\}) \cup \{z\}$ is a γ_t-set of G different from D, which again is a contradiction. □

The next theorem shows that for trees the necessary condition in Lemma 5.9 is also sufficient.

Theorem 5.10 (Fischermann [39]) *Let T be a tree of order at least 3 and let D be a subset of $V(T)$. Then the following conditions are equivalent:*

(i) D is the unique γ_t-set of T.

(ii) D is a total dominating set of T such that for every vertex $x \in D$ we have either $|P_t(x, D)| \geq 2$ or $P_t(x, D) = P_t(x, D) \setminus D = \{y\}$ where y is some endvertex of T.

Proof.
(i) $\Rightarrow$ (ii): Follows immediately from Lemma 5.9.

(ii) $\Rightarrow$ (i): We prove this by induction on the order $n(T)$. For any tree T let $A(T)$ be the set of endvertices of T and let $S(T) = N(A(T))$. If a tree T has a total dominating set D as in (ii), then $A(T) \cap D = \emptyset$, $S(T) \subseteq D$, and $T[D]$ has no trivial component. Hence, the diameter of T is greater or equal 3 and $n(T) \geq 4$.
First, let T be a tree of order $n(T) = 4$ that has a total dominating set D as in (ii). Then, the tree T is isomorphic to the path $x_1x_2x_3x_4$ and $D = \{x_2, x_3\}$. Obviously, D is the unique γ_t-set of T. Assume, the claim holds for every tree T' of order $4 \leq n(T') < n$. Now, let T be a tree of order $n(T) = n$, and let D be a total dominating set of T as in (ii). Let d be the diameter of T and let $P = v_0v_1 \ldots v_d$ be a longest path in T such that the first index $i(P) > 0$ with $v_{i(P)} \notin D$ is as small as possible. If $3 \leq d \leq 4$, then $T - A(T) \cong K_{1,t}$ for some positive integer t which implies that $D = V(T) \setminus A(T)$ and D is the unique γ_t-set of T. Now, let $d \geq 5$. Suppose, there exists a γ_t-set F of T different from D. If there is a vertex $x \in F \cap A(T)$, then the vertex $y \in N(x)$ is in F. Let $z \in N(y) \setminus \{x\}$. Then, also $F' = (F \setminus \{x\}) \cup \{z\}$ is a γ_t-set of T. Since $P_t(z, F') = \{y\} \subseteq F'$, the set F' does not fulfil (ii) and $F' \neq D$.

Thus, successively we can get a γ_t-set $D' \neq D$ of T such that $D' \cap A(T) = \emptyset$ which yields $S(T) \subseteq D'$. For the first three vertices of the path P we get that $v_0 \in A(T)$ and $v_1, v_2 \in D \cap D'$. Now, we choose the edge $ab \in E(T)$ as follows.
Case I: If $v_3 \notin P_t(v_2, D)$, then let $a = v_2 \in D$ and $b = v_3$.
Case II: If $v_3 \in P_t(v_2, D) \setminus D$, then let $a = v_3 \notin D$ and $b = v_4$.
Case III: If $v_3 \in P_t(v_2, D) \cap D$ and $v_4 \notin P_t(v_3, D)$, then let $a = v_3 \in D$ and $b = v_4$.
Case IV: If $v_3 \in P_t(v_2, D) \cap D$, $v_4 \in P_t(v_3, D)$, and $d(v_4) = 2$, then let $a = v_4 \notin D$ and $b = v_5$.
Case V: If $v_3 \in P_t(v_2, D) \cap D$, $v_4 \in P_t(v_3, D)$, and $d(v_4) > 2$, then there is at least one vertex $v_3' \in N(v_4) \setminus V(P)$ and, since $v_3', v_4 \notin D$, there is a second path $P' = v_0', v_1', v_2', v_3', v_4, \ldots, v_d$ of length d with $v_1', v_2' \in D$ and $v_3' \notin D$ which is a contradiction to the choice of P.

Now, let T_1 and T_2 be the two components of $T - ab$ such that $a \in V(T_1)$ and $b \in V(T_2)$. For $i = 1, 2$ let $D_i = D \cap V(T_i)$ and $D_i' = D' \cap V(T_i)$. By the choice of the edge ab, the set D_i and the tree T_i fulfil Condition (ii) for $i = 1, 2$. Since $n(T_i) < n(T)$, we get by the induction hypothesis that D_i is the unique γ_t-set of T_i for $i = 1, 2$. Further, since P is a longest path of T and D_1 fulfils Condition (ii) for T_1, we get that $T_1 - A(T_1) \cong K_{1,t}$ for some positive integer t, $D_1 = V(T_1) \setminus A(T_1)$, and $d_T(a) \geq 3$ if and only if $a \in D$. If $d_T(a) = 2$, then it is $A(T_1) = (A(T) \cap V(T_1)) \cup \{a\}$, and if $d_T(a) \geq 3$, then $A(T_1) = A(T) \cap V(T_1)$. This leads to

$$\begin{aligned} D_1 &= V(T_1) \setminus A(T) && \text{, if } a \in D \text{ (Case I, III), and} \\ D_1 &= V(T_1) \setminus (A(T) \cup \{a\}) && \text{, if } a \notin D \text{ (Case II, IV).} \end{aligned}$$

In addition, the set $S_1 = \{v_2\} \cup (S(T) \cap V(T_1)) \subseteq D_1' \subseteq V(T_1) \setminus A(T)$.
Case I: Since $a = v_2 \in D$, we have $S_1 = V(T_1) \setminus A(T) = D_1$ and $D_1' = D_1$.
Case II: Since $a = v_3 \notin D$, we have $S_1 = V(T_1) \setminus (A(T) \cup \{a\}) = D_1$ and $D_1 \subseteq D_1' \subseteq D_1 \cup \{a\}$.
Case III,IV: Since $v_3 \in D$, $v_2 \notin P_t(v_3, D)$, and $v_4 \notin A(T)$, there is at least one total private neighbour of v_3 in $N(v_3) \setminus \{v_2, v_4\} \neq \emptyset$. Since P is a longest path and $N(v_3) \cap D = \{v_2\}$, we get that $N(v_3) \setminus \{v_2, v_4\} \subseteq A(T)$. Thus, $v_3 \in S(T) \cap V(T_1) \subseteq S_1$.
In Case III where $a = v_3$ this leads to $S_1 = V(T_1) \setminus A(T) = D_1$ and $D_1' = D_1$.
In Case IV where $a = v_4$ we get that $S_1 = V(T_1) \setminus (A(T) \cup \{a\}) = D_1$ and $D_1 \subseteq D_1' \subseteq D_1 \cup \{a\}$.
Thus, in every case we obtain that $|D_2'| \leq |D_2|$ and $D_2' \neq D_2$ which implies that D_2' is not a total dominating set of T_2. Since b is the only vertex in T_2 that has a neighbour outside of T_2, we get that every vertex in $V(T_2) \setminus \{b\}$ has a neighbour in D_2', $a \in D'$, and $b \in P_t(a, D')$.

Suppose, $a \notin D$ (Case II or IV). Then, $D'_1 = D_1 \cup \{a\} \neq D_1$ and $|D'_2| < |D_2|$. Let $x \in N(b) \cap V(T_2)$. Since every vertex in $V(T_2) \setminus \{b\}$ has a neighbour in D'_2 and $b \in P_t(a, D')$, we obtain that $x \in N(D'_2) \setminus D'_2$ and the set $F_2 = D'_2 \cup \{x\}$ is a total dominating set of T_2 with $|F_2| = |D_2|$. This leads to $D_2 = D'_2 \cup \{x\}$. Since b is no endvertex of T, there is a vertex $w \in P_t(x, D) \setminus \{b\}$ that has no neighbour in $D_2 \setminus \{x\} = D'_2$, which is a contradiction.
Hence, we have $a \in D$ (Case I or III) and $D_1 = D'_1$. Let $x \in (N(b) \cap D_2)$. Since $b \in P_t(a, D')$, we have $x \notin D'_2$, $N(x) \cap A(T) = \emptyset$, and $|P_t(x, D)| \geq 2$. By $b \in V(P) \setminus P_t(x, D)$, there is at least one vertex $y_1 \in P_t(x, D) \setminus V(P)$. This vertex y_1 has at least one neighbour $y_2 \in D'_2$. This implies that $y_2 \notin A(T)$, $y_2 \neq x$, and $y_2 \notin D$, by $y_1 \in P_t(x, D)$. Thus, there is a vertex $y_3 \in N(y_2) \setminus \{y_1\}$ and a vertex $y_4 \in N(y_3) \cap D$. Thus, $y_4 \neq y_2$, $y_4 \notin A(T)$, and there is a vertex $y_5 \in N(y_4) \setminus \{y_3\}$.
In Case I where $b = v_3$ the path P' in T from y_5 to v_d has length

$$d(y_5, v_d) = d(y_5, x) + d(x, v_4) + d(v_4, v_d) \geq 5 + 0 + (d - 4) > d,$$

which is a contradiction.
In Case III where $b = v_4$ the path P' in T from y_5 to v_d has length

$$d(y_5, v_d) = d(y_5, x) + d(x, v_5) + d(v_5, v_d) \geq 5 + 0 + (d - 5) = d.$$

Hence, $P' = v'_0 v'_1 \dots v'_d$ with $v'_0 = y_5$ and $v'_d = v_d$ is a longest path of T which fulfils $v'_3 = y_2 \notin D$ and $i(P') = 3$. But, since $a = v_3 \in D$, we have $i(P) > 3$, which is a contradiction to the choice of P. □

For any positive integer s the cycle C_{4s} of order $4s$ has no unique γ_t-set but any γ_t-set of C_{4s} satisfies Condition (ii) in Theorem 5.10. Thus, Condition (ii) is no longer sufficient for the unique realization of γ_t if the considered graph contains cycles.

With support of Theorem 5.10 we are now able to give a constructive characterization of the class of trees that have unique γ_t-sets. We introduce the following shorter notation.

Definition 5.11 If a graph has a unique γ_t-set, then we call this graph a *unique total domination graph* or briefly a *utd-graph*.

We need a few more definitions in order to present a constructive characterization.

Definition 5.12 Let $\mathcal{F}_3 = \{T(s_1, s_2) \mid s_1, s_2 \geq 1\}$ be the class of trees $T(s_1, s_2)$ with vertex set

$$V(T(s_1, s_2)) = \{x_1, x_2\} \cup \{y_{1,i}, y_{2,j} \mid 1 \leq i \leq s_1, 1 \leq j \leq s_2\}$$

and edge set

$$E(T(s_1,s_2)) = \{x_1x_2\} \cup \{x_1y_{1,i}, x_2y_{2,j} \mid 1 \le i \le s_1, 1 \le j \le s_2\}.$$

Let $\mathcal{F}_4 = \{T(t; s_0, s_1, \ldots, s_t) \mid t \ge 2, s_0 \ge 0, s_1, s_2, \ldots, s_t \ge 1\}$ be the class of trees $T(t; s_0, s_1, \ldots, s_t)$ with vertex set

$$V(T(t; s_0, s_1, \ldots, s_t)) = \{x_0, x_1, \ldots, x_t\} \cup \{y_{i,j_i} \mid 1 \le j_i \le s_i, 0 \le i \le t\}$$

and edge set

$$E(T(t; s_0, s_1, \ldots, s_t)) = \{x_0x_i \mid 1 \le i \le t\} \cup \{x_iy_{i,j_i} \mid 1 \le j_i \le s_i, 0 \le i \le t\}.$$

Observation 5.13 Every tree $T(s_1,s_2) \in \mathcal{F}_3$ is of diameter 3 and has the total dominating set $\{x_1, x_2\}$, and every tree $T(t; s_0, s_1, \ldots, s_t) \in \mathcal{F}_4$ is of diameter 4 and has the total dominating set $\{x_0, x_1, \ldots, x_t\}$. By Theorem 5.10, it is straightforward to see that for $d = 3, 4$ the class $\mathcal{F}_d$ is equal the set of trees of diameter d that are utd-graphs. For these trees T the set $V(T) \setminus A(T)$ is the unique γ_t-set, where $A(T)$ is the set of endvertices of T.

Definition 5.14 For every non trivial tree T with γ_t-set D and for any tree $T_1 \in \mathcal{T}_1$ with γ_t-set D_1 we define the three graph operations

$$op^1(T_1, T),\ op^2(T_1, T),\ \text{and}\ op^3(T_1, T)$$

to be trees consisting of the disjoint union of T_1 and T and of one specific additional edge connecting T_1 and T which satisfies the following conditions.

In $op^1(T_1, T)$ the additional edge is equal xv for some $x \in D_1$ and some $v \in V(T) \setminus D$ where either $v \notin P_t(D, D)$ or $v \in P_t(w, D)$ for a vertex $w \in D$ with $|P_t(w, D)| \ge 3$ or $P_t(w, D) = \{v, v'\}$ for some endvertex v' of T.

In $op^2(T_1, T)$ the additional edge is equal xv for some $x \in D_1$ if $T_1 \in \mathcal{F}_3$, and $x = x_0$ if $T_1 \in \mathcal{F}_4$, and for some $v \in D$ where either $v \notin P_t(D, D)$ or $v \in P_t(w, D)$ for a vertex $w \in D$ with $|P_t(w, D)| \ge 3$ or $P_t(w, D) = \{v, v'\}$ for some endvertex v' of T.

In $op^3(T_1, T)$ the additional edge is equal yv for some $y \in V(T_1) \setminus D_1$ where, if $T_1 \in \mathcal{F}_4$ and $y \in N(x_i)$ for some $0 < i \le t$, then $s_i \ge 2$, and for some $v \in V(T) \setminus D$ where $v \in N(w)$ for a vertex $w \in D$ with $P_t(w, D) \ne \{v\}$.

Definition 5.15 Let $\mathcal{T}_1 = \mathcal{F}_3 \cup \mathcal{F}_4$ and for every $i \ge 1$ let

$$\mathcal{T}_{i+1} = \{op^s(T_1, T) \mid s \in \{1, 2, 3\}, T_1 \in \mathcal{T}_1, T \in \mathcal{T}_i\}.$$

We denote $\mathcal{T} = \bigcup_{i \ge 1} \mathcal{T}_i$.

Theorem 5.16 (Fischermann [39]) *Let T be a tree of order at least 3. Then, T is a utd-graph if and only if $T \in \mathcal{T}$.*

Proof. Suppose, there exists a tree that is a utd-graph but not in $\mathcal{T}$. Let T be such a tree of minimal order n. If the diameter of T is less than 5, then $T \in \mathcal{T}_1 \subseteq \mathcal{T}$, by Observation 5.13. Hence, the diameter of T is greater or equal 5. Analogous to the proof of Theorem 5.10 (ii) $\Rightarrow$ (i) we consider one of these special longest paths P in T and we choose the edge $ab \in E(P)$ by Case I–IV. Thus, we obtain the two components T_1 and T_2 of $T - ab$ such that T_1 and T_2 are utd-graphs of order less than n but at least 3. By the minimality of T and $3 \leq \mathrm{diam(T_1)} \leq 4$, we get that $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_i$ for some $i \geq 1$. Since T fulfils Condition (ii), the following is straightforward to see.
In Case I we obtain that $T = op^1(T_1, T_2)$ or $T = op^2(T_1, T_2)$.
In Case II and Case IV we have $T = op^3(T_1, T_2)$.
In Case III we get that $T = op^1(T_1, T_2)$.
Hence, $T \in \mathcal{T}_{i+1} \subseteq \mathcal{T}$, which is a contradiction.
Now, we prove by induction that for any positive integer i every tree $T \in \mathcal{T}_i$ is a utd-graph. If $i = 1$, then T is a utd-graph, by Observation 5.13. If $i > 1$, then $T = op(T_1, T_2)$ for some trees $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_{i-1}$ and for some operation $op \in \{op^1, op^2, op^3\}$. By the induction hypothesis we get that T_1 and T_2 are utd-graphs. Hence, T_1 and T_2 fulfil Condition (ii) in Theorem 5.10. It is straightforward to see that also $op^i(T_1, T_2)$ fulfils Condition (ii) for any $i = 1, 2, 3$. By Theorem 5.10, the tree $T = op(T_1, T_2)$ is a utd-graph. □

Remark 5.17 Hedetniemi, Laskar, and Pfaff [63] have found that the total domination problem is NP-complete, even when it is restricted to bipartite graphs. For trees Hedetniemi, Hedetniemi, Laskar and Pfaff [61] have found a linear time algorithm to determine γ_t-sets. Hence, with support of this algorithm and our characterization in Theorem 5.10 we can decide in linear time whether a given tree has a unique γ_t-set.

5.3 Edge domination in trees

We consider in this section unique realizations of edge domination numbers. The concept of edge domination is studied e.g. in [13], [14], [51], [65], [71], and Topp already has investigated graphs with unique minimum edge dominating sets in [94]. For this purpose one has to define edge sets corresponding to the neighbourhood and the private neighbourhood of a vertex.

Definition 5.18 For any graph G and any edge $e \in E(G)$ we define $N'(e) = \{f \in E(G) \mid f \text{ incident with } e\}$ and the set $N'[e] = N'(e) \cup \{e\}$. If $B \subseteq E(G)$, then $N'(B) = \bigcup_{e \in B} N'(e)$ and $N'[B] = N'(B) \cup B$. For a subset F of $E(G)$ and an edge $e \in F$ we define the set $P'(e, F) = N'[e] \setminus N'[F \setminus \{e\}]$, and we call an edge $f \in P'(e, F)$ a *private incident edge* of e with regard to F.

Now, we consider graphs with unique minimum edge dominating sets, and the first lemma, which is a slightly stronger version of a result of Topp (Proposition 2.8 in [94]), contains a simple necessary condition for the uniqueness of such sets.

Lemma 5.19 (Fischermann and Volkmann [43]) *Let G be a connected graph of order at least 3 and let F be a unique γ'-set of G. Then, the set F is independent and every edge $e \in F$ has two non-incident edges in $P'(e, F)$.*

Proof. Let $e \in F$ arbitrarily. Since F is minimal, we have $P'(e, F) \neq \emptyset$. If $P'(e, F) = \{e\}$, then we can take any edge f incident with e and $(F \setminus \{e\}) \cup \{f\}$ is a minimum edge dominating set of G different from F, which is a contradiction. If $f \in P'(e, F) \setminus \{e\} \neq \emptyset$ and every edge in $P'(e, F) \setminus \{f\}$ is incident with f, then again $(F \setminus \{e\}) \cup \{f\}$ is a minimum edge dominating set of G different from F, which is a contradiction. Hence, for every edge $e \in F$ the set $P'(e, F)$ contains two non-incident edges. This also implies that no two edges in F are incident. □

The next theorem is a characterization of unique minimum edge dominating sets in trees which is similar to Theorem 3.3. One part of this characterization says, that for trees the necessary condition in Lemma 5.19 is also sufficient. This inversion does not hold for graphs containing cycles, as we can see at the simple graph G with vertex set $V(G) = \{u, v, w, x\}$ and edge set $E(G) = \{uv, uw, ux, vw\}$ having the two minimum edge dominating sets $\{uv\}$ and $\{uw\}$.

Theorem 5.20 (Fischermann and Volkmann [43]) *Let T be a tree of order at least 3 and let F be a subset of $E(T)$. Then the following conditions are equivalent:*

(i) F is the unique γ'-set of T.

(ii) F is an edge dominating set of T such that every edge e in F has two non-incident edges in $P'(e, F)$.

(iii) F is an independent edge dominating set of T such that every edge e in F has two non-incident edges in $P'(e, F)$.

(iv) F is a minimum edge dominating set of T such that $\gamma'(T - e) > \gamma'(T)$ for every edge $e \in F$.

Proof.
(i) $\Rightarrow$ (iii) $\Rightarrow$ (ii): Follows immediately from Lemma 5.19.

(ii) $\Rightarrow$ (i): Let F be an edge dominating set of T as in (ii). For any subset B of the edge set of T we define $V(B) = \{u, u' \in V(T) \mid uu' \in B\}$. Thus, for every edge $e = vw \in F$ there are two edges vv' and ww' with $v' \neq w'$ and $v, v', w, w' \notin V(F \setminus \{e\})$. Hence, no two edges in F are incident. Suppose, there is a γ'-set $F' \neq F$ of T. This implies that $|F \setminus F'| \geq |F' \setminus F|$. Define the set $B = (F \setminus F') \cup (F' \setminus F)$ and the forest $H = T[V(B)]$. Furthermore, let $F'_1 = \{vw \in F' \setminus F \mid v, w \in V(F \setminus F')\}$, $F'_2 = \{vw \in F' \setminus F \mid |\{v, w\} \cap V(F \setminus F')| = 1\}$, and $F'_3 = \{vw \in F' \setminus F \mid v, w \notin V(F \setminus F')\}$. The set $F' \setminus F$ is the disjoint union of F'_1, F'_2 and F'_3. We get for the order of H

$$n(H) = |V(B)| \leq 2|F \setminus F'| + |F'_2| + 2|F'_3|.$$

By (ii), for every vertex $v \in V(F \setminus F')$ there is an edge $vw \in F \setminus F'$ and an edge $vv' \neq vw$ such that $v, v' \notin V(F \setminus \{e\})$. Since F' is an edge dominating set of T, we get that v or v' is in $V(F')$. If $v \in (V(F) \setminus V(F')) \subseteq V(F \setminus F')$, then $v' \in (V(F') \setminus V(F)) \subseteq V(F' \setminus F)$ and $vv' \in E(H) \setminus B$. This implies that

$$|E(H) \setminus B| \geq |V(F) \setminus V(F')| \geq 2|F \setminus F'| - 2|F'_1| - |F'_2|.$$

Hence, we obtain for the size of H

$$\begin{aligned} m(H) &= |F \setminus F'| + |F' \setminus F| + |E(H) \setminus B| \\ &\geq 2|F' \setminus F| + (2|F \setminus F'| - 2|F'_1| - |F'_2|) \\ &= 2(|F'_1| + |F'_2| + |F'_3|) + (2|F \setminus F'| - 2|F'_1| - |F'_2|) \\ &= |F'_2| + 2|F'_3| + 2|F \setminus F'| \\ &\geq n(H). \end{aligned}$$

But, since H is a forest, we have $m(H) = n(H) - \kappa(H) < n(H)$, which is a contradiction.

(i) $\Rightarrow$ (iv): Let F be the unique minimum edge dominating set of T, let $e = v_1v_2 \in F$ arbitrarily, and let T_1 and T_2 be the components of $T - e$ where $v_i \in V(T_i)$ for $i = 1, 2$. Further, let F' be a minimum edge dominating set of $T - e$ and for $i = 1, 2$ let $F_i = F \cap E(T_i)$ and $F'_i = F' \cap E(T_i)$. By (i) $\Rightarrow$ (ii), the edge e is incident with at least two edges $v_1w_1 \in E(T_1)$ and $v_2w_2 \in E(T_2)$ that are not incident with any other edge in F. Hence, the set F_i is not an edge dominating set of T_i in contrary to F'_i for $i \in \{1, 2\}$. Thus, we have $F_i \neq F'_i$ for $i = 1, 2$. Since the set $F''_i = (F \setminus F_i) \cup F'_i \neq F$ is an edge dominating set of T, we get $|F_i| < |F'_i|$. This yields $\gamma'(T - e) = |F'| = |F'_1| + |F'_2| \geq |F_1| + |F_2| + 2 = |F| + 1 > \gamma'(T)$.

(iv) $\Rightarrow$ (i): Let F be a minimum edge dominating set of T such that $\gamma'(T-e) > \gamma'(T)$ for every edge $e \in F$. Suppose that there is a minimum edge dominating set $F' \neq F$ of T. There exists at least one edge $e \in F \setminus F'$ and the set F' is an edge dominating set of $T-e$. Hence, $\gamma'(T-e) \leq |F'| = \gamma'(T)$ for some $e \in F$, which is a contradiction. □

As a corollary of Theorem 5.20 we obtain a characterization of caterpillars with unique γ'-sets by Topp (Corollary 3.1 in [94]) and the following iterative characterization, that also contains a result of Topp (Theorem 2.11 in [94]).

Corollary 5.21 (Fischermann and Volkmann [43]) *Let T be a tree of diameter at least 3, let F be a minimum edge dominating set of T, and let $e \in F$ be arbitrary. Then, F is the unique minimum edge dominating set of T if and only if either $F = \{e\}$ or every component of the forest $H = T - N'[e]$ is either trivial or of order at least 4 and H has the unique minimum edge dominating set $F \setminus \{e\}$.*

Proof. Let F be a minimum edge dominating set of T and let $e \in F$ be arbitrary. First, let F be unique. Hence, F fulfils (ii) in Theorem 5.20. If $F \neq \{e\}$, then the set $F \setminus \{e\}$ also fulfils (ii) for the forest H. Thus, each component of H is either trivial or of order at least 4. If we apply Theorem 5.20 to the components that are not trivial, then we obtain that H has the unique minimum edge dominating set $F \setminus \{e\}$.

Conversely, let $F = \{e\}$. Since the diameter of T is at least 3, we obtain that the edge dominating set F is unique. Now, let $F \setminus \{e\}$ be the unique edge dominating set of H and let every component of H be either trivial or of order at least 4. By Theorem 5.20, the set $F \setminus \{e\}$ fulfils (ii) with respect to the not trivial components of H. This implies that F fulfils (ii) for T. Thus, the set F is unique, by Theorem 5.20. □

Remark 5.22 There are some algorithms known to determine minimum edge dominating sets in special classes of graphs (e.g. [13], [14], [51], [65],[71]). For trees a linear time algorithm to determine γ'-sets is given by Hedetniemi and Mitchell [65], and a linear time algorithm to determine minimum independent edge dominating sets is given by Gavril and Yannakakis [51]. Further, Chang and Hwang [13] have found a linear time algorithm to determine γ'-sets in block graphs. Hence, we can inspect in linear time whether a given tree has a unique γ'-set, by using one of these algorithms and Theorem 5.20.

Chapter 6

Maximum size of graphs with unique minimum dominating sets

In the preceding chapters we have considered for several domination parameters ν, on which condition the parameter ν has a unique realization in a given graph. Here, we study how the unique realization of the domination number affects the size of the graph.

Already in 1941 Turán [98] has investigated the influence of graph parameters to the size of graphs. He has determined the maximum size of graphs by given order and clique number, and he has specified the extremal graphs of maximum size which today are called *Turán graphs*. Since the clique number of a graph is equal the independence number of its complement, he has fixed simultaneously the minimum size of graphs by given order and independence number. Obviously, the complements of the Turán graphs are the graphs of minimum size for given order and independence number. In 1994 Siemes, Topp and Volkmann [90] have solved the corresponding problem for graphs with unique α-sets.

The maximum size of graphs for given domination number γ and order n was studies e.g. by Vizing [99], Sanchis [88] and Fulman [49]. In this chapter we investigate the same problem for graphs with unique γ-sets. Firstly we present for arbitrary positive integers γ and $n \geq 3\gamma$ a class of graphs $G(n, \gamma)$ of order n that have unique minimum dominating sets of cardinality γ and large size $m(G(n, \gamma))$. In the second section we prove that in certain cases the graphs $G(n, \gamma)$ are the ones of maximum size, and in the last section of this chapter we show that these graphs are also the extremal graphs with regard to a related problem.

6.1 Graphs with large size

First we present a classical result of Vizing about the size of graphs.

Theorem 6.1 (Vizing [99]) *A graph of order n with domination number $\gamma \geq 2$ has at most $\frac{1}{2}(n-\gamma)(n-\gamma+2)$ edges.*

This result has been improved in various ways. Fulman [49] improved it having regard to the maximum degree of the graph and he was able to shorten Sanchis's proof [88] of the fact that if the graph $G = (V, E)$ has order n, domination number $\gamma \geq 2$ and maximum degree at most $n - \gamma - 1$, then G has at most $\frac{1}{2}(n-\gamma)(n-\gamma+1)$ edges (see also Theorem 2.21 in [59]).

In order to consider the analogous problem for graphs that have unique γ-sets, we define the following.

Definition 6.2 Let $m(n, \gamma)$ denote the maximum number of edges of a graph G of order n without isolated vertices that has a unique γ-set of cardinality $\gamma \geq 1$.

It is easy to see, that among all graphs G of order n with exactly $t < n$ isolated vertices that have unique γ-sets of cardinality $\gamma > t$ the maximum number of edges is equal $m(n-t, \gamma-t)$. Therefore it suffices to consider graphs without isolated vertices. If G is a graph of order n without isolated vertices and G has a unique minimum dominating set D, then, by Lemma 3.1, the private exterior neighbourhood $P(v, D) \setminus \{v\}$ contains at least two vertices for each vertex $v \in D$. This observation implies that for such graphs G

$$n \geq 3\gamma(G).$$

We propose the following conjecture.

Conjecture 6.3 (Fischermann, Rautenbach, and Volkmann [41]) *If $\gamma \geq 1$ and $n \geq 3\gamma$, then*

$$m(n, \gamma) = \begin{cases} \binom{n}{2} - \lceil \frac{n-1}{2} \rceil & , \gamma = 1 \\ \binom{n}{2} - \gamma(n + \frac{\gamma-5}{2}) = \binom{n-\gamma}{2} - \gamma(\gamma-2) & , \gamma \geq 2. \end{cases}$$

We are not able to prove this conjecture in general. Instead, we firstly show that $m(n, \gamma)$ is at least as large as stated in Conjecture 6.3. Then we verify this conjecture for the two special cases $\gamma = 1$ and $n = 3\gamma$. Finally, we prove a weakened version of Conjecture 6.3 for $\gamma \geq 2$.

Note that if a graph G of order n without isolated vertices has a unique minimum dominating set of cardinality $\gamma \geq 2$, then its maximum degree is at most $n-\gamma-1$ and Sanchis's result implies that G has at most $\binom{n-\gamma+1}{2}$ edges which is larger than the bound given in Conjecture 6.3.

First, we exhibit those graphs which we regard as the extremal graphs.

Definition 6.4 Let $\gamma = 1$ and $n \geq 3\gamma$ arbitrarily. Let K_n be the complete graph of order n and let E' be a subset of $E(K_n)$ consisting of $\frac{n-1}{2}$ independent edges if n is odd and of $\frac{n-2}{2}$ independent edges and of one additional edge which is incident with exactly one other edge in E' if n is even. Then, we define, $G(n,\gamma) = K_n - E'$.

Figure 6.1 shows the graphs $G(3,1)$, $G(4,1)$ and $G(5,1)$.

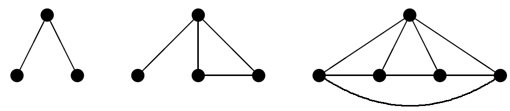


Figure 6.1

Definition 6.5 Let $\gamma \geq 2$ and $n \geq 3\gamma$ arbitrarily. Then, $G(n,\gamma) = (V,E)$ has vertex set $V = D \cup P \cup R$ for disjoint sets $D = \{x_1, x_2, ..., x_\gamma\}$, $P = \{y_i, z_i | 1 \leq i \leq \gamma\}$ and R. For $1 \leq i \leq \gamma$ we have $N[x_i, G(n,\gamma)] = D \cup \{y_i, z_i\} \cup R$, $N(y_i, G(n,\gamma)) = \{x_i\}$ and $N[z_i, G(n,\gamma)] = \{x_i\} \cup \{z_1, z_2, ..., z_\gamma\} \cup R$. Furthermore, the subgraph $G(n,\gamma)[R]$ of $G(n,\gamma)$ that is induced by the set R is a complete graph.

Figure 6.2 shows the graphs $G(7,2)$ and $G(10,3)$.

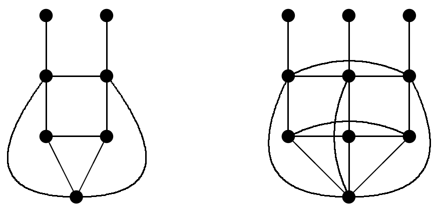


Figure 6.2

Lemma 6.6 (Fischermann, Rautenbach, and Volkmann [41])
Let $\gamma \geq 1$ and $n \geq 3\gamma$. The graph $G(n,\gamma)$ has a unique minimum dominating set of cardinality γ, and $m(G(n,\gamma)) = \binom{n}{2} - \lceil \frac{n-1}{2} \rceil$ if $\gamma = 1$ and $m(G(n,\gamma)) = \binom{n}{2} - \gamma(n + \frac{\gamma-5}{2}) = \binom{n-\gamma}{2} - \gamma(\gamma-2)$ if $\gamma \geq 2$.

Proof. It is easy to see, that for every $n \geq 3$ the graph $G(n,1)$ contains exactly one vertex of degree $n-1$ which yields that this graph has a unique minimum dominating set of cardinality 1. Furthermore, we have $m(G(n,\gamma)) = m(K_n) - |E'| = \binom{n}{2} - \lceil \frac{n-1}{2} \rceil$. Now, let $\gamma \geq 2$ and $n \geq 3\gamma$. It is straightforward to see that $\binom{n}{2} - \gamma(n + \frac{\gamma-5}{2}) = \binom{n-\gamma}{2} - \gamma(\gamma-2)$. The set D is obviously a dominating set of $G(n,\gamma)$ of cardinality γ. Let D' be an arbitrary γ-set of $G(n,\gamma)$. Since the vertex x_i is the only neighbour of y_i, we obtain that $|D' \cap \{x_i, y_i\}| \geq 1$ for every $1 \leq i \leq \gamma$. Thus, we get that $|D'| = |D| = \gamma$ and $D' = \{a_1, a_2, \ldots, a_\gamma\}$ where $a_i \in \{x_i, y_i\}$ for every $1 \leq i \leq \gamma$. If a vertex a_i is equal y_i, then D' does not dominate the vertex z_i, which is a contradiction. Hence, we obtain that $D' = D$ and the set D is the unique minimum dominating set of $G(n,\gamma)$ of cardinality γ. In order to determine the size of $G(n,\gamma)$ we consider the graph $G' = G(n,\gamma) - \{y_1, y_2, \ldots, y_\gamma\}$. It is straightforward to see that

$$m(G(n,\gamma)) = m(G') + \gamma \quad \text{and} \quad m(G') = \binom{n-\gamma}{2} - \gamma(\gamma-1)$$

which implies the desired result. □

Corollary 6.7 (Fischermann, Rautenbach, and Volkmann [41]) *If $\gamma \geq 1$ and $n \geq 3\gamma$, then*

$$m(n,\gamma) \geq \begin{cases} \binom{n}{2} - \lceil \frac{n-1}{2} \rceil & ,\gamma = 1 \\ \binom{n-\gamma}{2} - \gamma(\gamma-2) & ,\gamma \geq 2. \end{cases}$$

6.2 Some special cases

In this section we prove the two cases $\gamma = 1$ and $n = 3\gamma$ of our conjecture.

For this purpose we use the well-known *handshaking lemma*. The handshaking lemma says that in any graph $G = (V,E)$ the sum of the vertex degrees is equal two times the size ($2|E| = \sum_{x \in V} d(x,G)$) which implies that the number of vertices of odd degree is always even.

Lemma 6.8 (Fischermann, Rautenbach, and Volkmann [41]) *Let G be a graph without isolated vertices with a unique minimum dominating set of cardinality 1 and of order $n \geq 3$. Then*

$$m(G) \leq \binom{n}{2} - \left\lceil \frac{n-1}{2} \right\rceil.$$

Proof. Let $G = (V, E)$ be a graph of order n without isolated vertices that has a unique minimum dominating set of cardinality $\gamma = 1$. The graph G has a unique vertex of degree $n - 1$. This implies that

$$\sum_{x \in V} d(x, G) \leq (n-1) + (n-1)(n-2) = (n-1)^2 = n(n-1) - (n-1).$$

If n is odd, this yields

$$m(G) \leq \frac{1}{2}(n(n-1) - (n-1)) = \binom{n}{2} - \left\lceil \frac{n-1}{2} \right\rceil.$$

Now, let n be even. Then, besides the unique vertex of degree $n - 1$ there is at least one more vertex of odd degree, by the handshaking lemma, and this vertex has degree at most $n - 3$. Thus, we obtain that

$$\sum_{x \in V} d(x, G) \leq (n-1) + (n-1)(n-2) - 1 = (n-1)^2 - 1 = n(n-1) - n,$$

which yields that

$$m(G) \leq \frac{1}{2}(n(n-1) - n) = \binom{n}{2} - \left\lceil \frac{n-1}{2} \right\rceil.$$

□

The next special case that we consider is $n = 3\gamma$. (Remember that always $n \geq 3\gamma$.)

Theorem 6.9 (Fischermann, Rautenbach, and Volkmann [41]) *Let $G = (V, E)$ be a graph without isolated vertices with a unique minimum dominating set of cardinality $\gamma \geq 2$ and order $n = 3\gamma$. Then*

$$m(G) = |E| \leq \binom{n-\gamma}{2} - \gamma(\gamma - 2) = \gamma \cdot (\gamma + 1).$$

Proof. Let $D = \{x_1, x_2, ..., x_\gamma\}$ be the unique minimum dominating set of G. Let $P_i = P(x_i, D) \setminus \{x_i\}$ be the private exterior neighbourhood of x_i for $1 \leq i \leq \gamma$. By Lemma 3.1, the set P_i contains at least two vertices for each $1 \leq i \leq \gamma$ which - in view of the order $n = 3\gamma$ - implies that $|P_i| = 2$ for all $1 \leq i \leq \gamma$.

If there is some $1 \leq i \leq \gamma$ such that the two vertices p_i', p_i'' in P_i are adjacent, then $(D \setminus \{x_i\}) \cup \{p_i'\} \neq D$ is a minimum dominating set of G which is a contradiction (see Figure 6.3 a)).

If there are some $1 \le i < j \le \gamma$ such that there are at least three edges between $P_i = \{p'_i, p''_i\}$ and $P_j = \{p'_j, p''_j\}$, then we can assume without loss of generality that $p'_ip'_j, p''_ip''_j \in E$ and $(D \setminus \{x_i, x_j\}) \cup \{p'_i, p''_j\} \neq D$ is a minimum dominating set of G which is a contradiction (see Figure 6.3 b)).

If there are some $1 \le i < j \le \gamma$ such that $x_ix_j \in E$ and there are two edges between $P_i = \{p'_i, p''_i\}$ and $P_j = \{p'_j, p''_j\}$, then we can assume without loss of generality that either $p'_ip'_j, p''_ip''_j \in E$ or $p'_ip'_j, p'_ip''_j \in E$. In the first case $(D \setminus \{x_i, x_j\}) \cup \{p'_i, p''_j\} \neq D$ (see Figure 6.3 c)) and in the second case $(D \setminus \{x_j\}) \cup \{p'_i\} \neq D$ (see Figure 6.3 d)) is a minimum dominating set of G which is a contradiction.

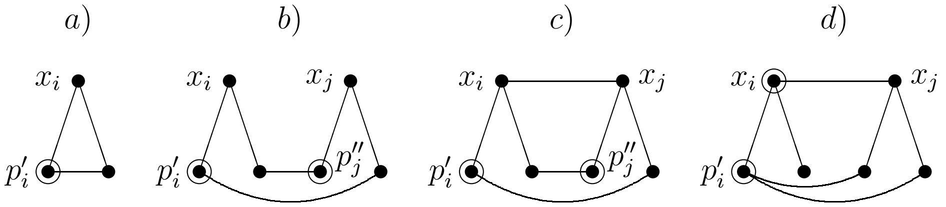


Figure 6.3

Let ν_l for $l \ge 0$ be the number of pairs $\{i, j\}$ with $1 \le i < j \le \gamma$ such that there are exactly l edges between P_i and P_j. By the above reasonings, we obtain that $\nu_l = 0$ for all $l \ge 3$ and $m(G[D]) \le \nu_0 + \nu_1$. This implies that

$$\begin{aligned} m(G) &= 2\gamma + m(G[D]) + 0 \cdot \nu_0 + 1 \cdot \nu_1 + 2 \cdot \nu_2 \\ &\le 2\gamma + \nu_0 + \nu_1 + 0 \cdot \nu_0 + 1 \cdot \nu_1 + 2 \cdot \nu_2 \\ &\le 2\gamma + 2(\nu_0 + \nu_1 + \nu_2) \\ &= 2\gamma + 2\binom{\gamma}{2} \\ &= \gamma \cdot (\gamma + 1). \end{aligned}$$

It is straightforward to see that

$$\binom{n-\gamma}{2} - \gamma(\gamma-2) = \binom{2\gamma}{2} - \gamma(\gamma-2) = \gamma \cdot (2\gamma - 1) - \gamma(\gamma - 2) = \gamma \cdot (\gamma + 1).$$

This completes the proof. □

6.3 A related problem

The graph $G(n,\gamma)$ actually satisfies a stronger property than having a unique γ-set. If D denotes the minimum dominating set of $G(n,\gamma) = (V,E)$ and

$$X_D = \bigcup_{x\in D} (P(x,D)\setminus\{x\})$$

denotes the union of the private exterior neighbourhoods of the vertices in D, then it is straightforward to see that the set D is also the unique minimum X_D-dominating set of $G(n,\gamma)$, i.e. no set of γ vertices that is different from D dominates the private exterior neighbourhoods of the vertices in D. This observation motivates the weakened version of Conjecture 6.3 that we prove in this section.

Definition 6.10 We say a graph $G = (V,E)$ has *property P(γ)* for some $\gamma \geq 2$ if G has no isolated vertices and G has a minimum dominating set D of cardinality γ such that D is the unique minimum X_D-dominating set of G, where $X_D = \bigcup_{x\in D}(P(x,D)\setminus\{x\})$. Let $\tilde{m}(n,\gamma)$ denote the maximum number of edges of a graph $G = (V,E)$ of order n that has property P(γ).

Since the graphs $G(n,\gamma)$ for $\gamma \geq 2$ have property P(γ), we know that $\tilde{m}(n,\gamma) \geq \binom{n-\gamma}{2} - \gamma(\gamma-2)$. Since a graph G that satisfies property P(γ) clearly also has a unique minimum dominating set, we deduce for $\gamma \geq 2$ that $\tilde{m}(n,\gamma) \leq m(n,\gamma)$. We will now prove the following theorem.

Theorem 6.11 (Fischermann, Rautenbach, and Volkmann [41]) *If $\gamma \geq 2$, then $\tilde{m}(n,\gamma) = \binom{n-\gamma}{2} - \gamma(\gamma-2)$.*

Proof. Since $\tilde{m}(n,\gamma) \geq \binom{n-\gamma}{2} - \gamma(\gamma-2) = \binom{n}{2} - \gamma(n + \frac{\gamma-5}{2})$ it remains to prove that $\tilde{m}(n,\gamma) \leq \binom{n}{2} - \gamma(n+\frac{\gamma-5}{2})$. Therefore, let $G = (V,E)$ be a graph of order n without isolated vertices that has a minimum dominating set D of cardinality $\gamma \geq 2$ such that there is no set $D' \subseteq V$ different from D with $|D'| = \gamma$ and

$$\bigcup_{x\in D} (P(x,D)\setminus\{x\}) \subseteq N[D',G].$$

Let $D = \{x_1, x_2, ..., x_\gamma\}$ and for $1 \leq i \leq \gamma$ let $P_i = P(x_i,D)\setminus\{x_i\}$. Let $R = V\setminus(D\cup\bigcup_{i=1}^{\gamma} P_i)$. We know that $|P_i| \geq 2$ for all $1 \leq i \leq \gamma$. Let $n_0 = |R|$ and $n_i = |P_i|$ for $1 \leq i \leq \gamma$. We may assume that $n_1 \geq n_2 \geq n_3 \geq ... \geq n_\gamma$. We will now estimate the number of edges of G.

There are exactly $\sum_{i=1}^{\gamma} n_i$ edges between D and $\cup_{i=1}^{\gamma} P_i$. There are at most $\binom{\gamma}{2} + \binom{n_0}{2} + \gamma n_0$ edges in $G[D\cup R]$. Let $1 \leq i \leq \gamma$. Since there is no vertex

$p_i \in P_i$ such that $P_i \subseteq N[p_i, G]$, there are at most $\binom{n_i}{2} - \lceil \frac{n_i}{2} \rceil$ edges in $G[P_i]$. Since there is no vertex $r \in R$ such that $P_i \subseteq N(r, G)$, there are at most $n_0(n_i - 1)$ edges between P_i and R. Now let $1 \le i < j \le \gamma$. Since there is no vertex $p_i \in P_i$ such that $P_j \subseteq N(p_i, G)$, there are at most $n_i(n_j - 1)$ edges between P_i and P_j. Furthermore, if $n_i = 2$, then also $n_j = 2$ and it is easy to see that there is at most one edge between P_i and P_j. Altogether we obtain that $m(G) \le f(n_0, n_1, ..., n_\gamma)$ for a function f defined as follows.

$$\begin{aligned} f(n_0, n_1, ..., n_\gamma) &= \sum_{i=1}^{\gamma} n_i + \binom{\gamma}{2} + \binom{n_0}{2} + \gamma n_0 + \sum_{i=1}^{\gamma} (\binom{n_i}{2} - \left\lceil \frac{n_i}{2} \right\rceil) \\ &\quad + \sum_{i=1}^{\gamma} (n_0 n_i - n_0) + \sum_{1 \le i < j \le \gamma} (n_i n_j - \max\{n_i, 3\}) \\ &= \sum_{i=1}^{\gamma} n_i + \binom{\gamma}{2} + \sum_{i=0}^{\gamma} \binom{n_i}{2} - \sum_{i=1}^{\gamma} \left\lceil \frac{n_i}{2} \right\rceil \\ &\quad + \sum_{0 \le i < j \le \gamma} n_i n_j - \sum_{i=1}^{\gamma} (\gamma - i) \max\{n_i, 3\}. \end{aligned}$$

If we regard the partition $D, R, P_1, P_2, \ldots, P_\gamma$ of V, we see that the number of possibilities to choose two vertices in V is equal the number of possibilities to choose two vertices in V that lie in the same partition set plus the number of possibilities to choose two vertices in V that do not lie in the same partition set. Hence, we obtain the identity

$$\binom{n}{2} = \binom{\gamma}{2} + \sum_{i=0}^{\gamma} \binom{n_i}{2} + \gamma \sum_{i=0}^{\gamma} n_i + \sum_{0 \le i < j \le \gamma} n_i n_j .$$

This leads to the equality

$$\begin{aligned} & f(n_0, n_1, ..., n_\gamma) \\ = & \left[\binom{n}{2} - \gamma \sum_{i=0}^{\gamma} n_i \right] + \sum_{i=1}^{\gamma} n_i - \sum_{i=1}^{\gamma} \left\lceil \frac{n_i}{2} \right\rceil - \sum_{i=1}^{\gamma} (\gamma - i) \max\{n_i, 3\} \\ = & \binom{n}{2} - \gamma n_0 - (\gamma - 1) \sum_{i=1}^{\gamma} n_i - \sum_{i=1}^{\gamma} \left\lceil \frac{n_i}{2} \right\rceil - \sum_{i=1}^{\gamma} (\gamma - i) \max\{n_i, 3\}. \qquad (6.1) \end{aligned}$$

Claim 1: If $\gamma = 2$, $n_1 = n_2 \ge 4$, n_1 and n_2 are even, then $m(G) \le f(n_0, n_1, ..., n_\gamma) - 1$.

Proof of Claim 1. Suppose, we have $\gamma = 2$, $n_1 = n_2 \ge 4$, n_1 and n_2 are even and $m(G) = f(n_0, n_1, ..., n_\gamma)$. Then, $G[P_1]$, $G[P_2]$ and $G[P_1, P_2]$ are complete graphs in which perfect matchings have been removed. If $D' = \{p_1', p_1''\}$ consists of two non-adjacent vertices in P_1, then $(P_1 \cup P_2) \subseteq N[D', G]$ which is a contradiction.

Claim 2: Let $\gamma \geq 2$, $n_i \geq 2$ for $1 \leq i \leq \gamma$ and $n_0 \geq 0$ be integers. Let $n = \gamma + \sum_{i=0}^{\gamma} n_i$ and let $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_\gamma$.

If $\gamma = 2$, $n_1 = n_2 \geq 4$, n_1 and n_2 are even, then

$$f(n_0, n_1, ..., n_\gamma) \leq \binom{n}{2} - \gamma(n + \frac{\gamma - 5}{2}) + 1.$$

Otherwise

$$f(n_0, n_1, ..., n_\gamma) \leq \binom{n}{2} - \gamma(n + \frac{\gamma - 5}{2}).$$

Proof of Claim 2. If there is some $1 \leq i \leq \gamma - 1$ such that $n_i \geq 4$ and $n_i > n_{i+1}$, then we obtain by (6.1)

$$\begin{aligned} f(n_0, n_1, ..., n_i, ..., n_\gamma) &= f(n_0 + 1, n_1, ..., n_i - 1, ..., n_\gamma) \\ &\quad + \gamma - (\gamma - 1) - \left\lceil \frac{n_i}{2} \right\rceil + \left\lceil \frac{n_i - 1}{2} \right\rceil - (\gamma - i) \\ &\leq f(n_0 + 1, n_1, ..., n_i - 1, ..., n_\gamma). \end{aligned}$$

Let l be the greatest index such that $n_l \geq 3$. Then,

$$f(n_0, n_1, ..., n_\gamma) \leq f(n'_0, n'_1, ..., n'_\gamma)$$

where $n'_1 = n'_2 = ... = n'_\gamma = n_\gamma$ if $l = \gamma$, and $n'_i = 3$ for every $1 \leq i \leq l$ and $n'_i = 2$ for every $l < i \leq \gamma$ if $l < \gamma$, and where $n'_0 = n - \gamma - \sum_{i=1}^{\gamma} n'_i$.

First, we consider the case that $n_\gamma \leq 3$. Then, $n'_i = 3$ for every $1 \leq i \leq l$, $n'_i = 2$ for every $l < i \leq \gamma$ even if $l = \gamma$. We obtain $n'_0 = n - (3\gamma + l)$ and, by (6.1),

$$\begin{aligned} f(n_0, n_1, ..., n_\gamma) &\leq f(n'_0, n'_1, ..., n'_\gamma) \\ &= \binom{n}{2} - \gamma(n - (3\gamma + l)) - (\gamma - 1)(2\gamma + l) \\ &\quad -(\gamma + l) - 3(\gamma^2 - \binom{\gamma}{2}) \\ &= \binom{n}{2} - \gamma(n - 3\gamma) - 2\gamma(\gamma - 1) - \gamma - 3(\gamma^2 - \frac{1}{2}\gamma(\gamma + 1)) \\ &= \binom{n}{2} - \gamma(n - 3\gamma + 2\gamma - 2 + 1 + 3\gamma - \frac{3}{2}(\gamma + 1)) \\ &= \binom{n}{2} - \gamma(n + \frac{\gamma - 5}{2}). \end{aligned}$$

Thus, the claim is proved for $n_\gamma \le 3$ and it remains the case $n_\gamma \ge 4$. This implies that $l = \gamma$, $n'_1 = n'_2 = ... = n'_\gamma = n_\gamma$ and $n'_0 = n - \gamma(n_\gamma + 1)$. Let $2\epsilon = n_\gamma \pmod 2$. We obtain by (6.1)

$$\begin{aligned}
f(n_0, n_1, ..., n_\gamma) &\le f(n'_0, n'_1, ..., n'_\gamma) \\
&= \binom{n}{2} - \gamma n'_0 - (\gamma-1)\gamma n_\gamma - \gamma \left\lceil \frac{n_\gamma}{2} \right\rceil - (\gamma^2 - \binom{\gamma}{2}) n_\gamma \\
&= \binom{n}{2} - \gamma(n - \gamma - \gamma n_\gamma) - (\gamma-1)\gamma n_\gamma - \gamma \frac{n_\gamma}{2} - \gamma\epsilon \\
&\quad -(\gamma^2 - \frac{1}{2}\gamma(\gamma+1)) n_\gamma \\
&= \binom{n}{2} - \gamma(n-\gamma) - \gamma n_\gamma((\gamma-1) + \frac{1}{2} - \frac{1}{2}(\gamma+1)) - \gamma\epsilon \\
&= \binom{n}{2} - \gamma(n-\gamma) - \gamma n_\gamma(\frac{1}{2}\gamma - 1) - \gamma\epsilon \\
&\le \binom{n}{2} - \gamma(n-\gamma) - 4\gamma(\frac{1}{2}\gamma - 1) - \gamma\epsilon \qquad (\gamma \ge 2, n_\gamma \ge 4) \\
&= \binom{n}{2} - \gamma(n + \gamma - 4 + \epsilon) \\
&= \binom{n}{2} - \gamma\left(n + \frac{\gamma + (\gamma - 8 + 2\epsilon)}{2}\right) \\
&\le \begin{cases} \binom{n}{2} - \gamma(n + \frac{\gamma-5}{2}) & \text{, if } \gamma \ge 3 \text{ or } n_\gamma \text{ odd} \\ \binom{n}{2} - \gamma(n + \frac{\gamma-5}{2}) + 1 & \text{, if } \gamma = 2 \text{ and } n_2 \text{ even.} \end{cases}
\end{aligned}$$

Hence, it remains to prove the claim for the case $\gamma = 2$, $n_1 > n_2 \ge 4$ and n_2 is even.

If $n_1 = n_2 + 1$, then n_1 is odd and we obtain by (6.1) and by the above result

$$\begin{aligned}
f(n_0, n_1, n_2) &= f(n_0 + 1, n_1 - 1, n_2) - \left\lceil \frac{n_1}{2} \right\rceil + \left\lceil \frac{n_1 - 1}{2} \right\rceil \\
&= f(n_0 + 1, n_2, n_2) - 1 \\
&\le \binom{n}{2} - \gamma(n + \frac{\gamma - 5}{2}) + 1 - 1.
\end{aligned}$$

If $n_1 \ge n_2 + 2$, then $f(n_0, n_1, n_2) \le f(n_0 + (n_1 - n_2 - 1), n_2 + 1, n_2)$ and as above $f(n_0 + (n_1 - n_2 - 1), n_2 + 1, n_2) \le \binom{n}{2} - \gamma(n + \frac{\gamma-5}{2}) + 1 - 1$.

This completes the proof of Claim 2.

Claim 1 and Claim 2 together imply the required result. $\square$

In view of Theorem 6.11, Conjecture 6.3 is equivalent to the following conjecture.

Conjecture 6.12 *If* $\gamma \geq 2$, *then* $\tilde{m}(n, \gamma) = m(n, \gamma)$.

Part II

Upper bounds and extremal graphs

Chapter 7

Distance domination

In this second part of the thesis we consider bounds on domination parameters, and we characterize the extremal graphs with regard to these bounds.

As mentioned in the introduction Ore [85] has found that the ordinary domination number has the upper bound $n(G)/2$ if the graph G has no isolated vertices. Independently of each other Fink, Jacobson, Kinch and Roberts as well as Payan and Xuong have given a characterization of graphs with $\gamma(G) = n(G)/2$.

Theorem 7.1 (Fink, Jacobson, Kinch, and Roberts [37], Payan and Xuong [86]) *For a graph G of even order n without isolated vertices, $\gamma(G) = n/2$ if and only if the components of G consist of the cycle C_4 of length 4 or of the corona $H \circ K_1$ for a connected graph H.*

In this chapter we generalize this result for distance domination, and we consider upper bounds and extremal graphs if the distance domination number has a unique realization. In Chapter 8 and Chapter 9 we look at total domination and exact distance domination in the same context.

Note, that the results on distance domination also contain the case of ordinary domination.

A special structure of graphs recures in some of these characterizations, therefore we introduce a notation for this class of graphs.

Definition 7.2 Let m be an arbitrary integer and let G be a graph of order n with vertex set $\{x_1, x_2, \dots, x_n\}$. The graph $G \diamond P_m$ is obtained by taking the graph G and n copies $P_m^1, P_m^2, \dots, P_m^n$ of the path P_m and connecting the vertex x_i with one endvertex of P_m^i by an edge for all $1 \le i \le n$.

For $m = 1$ the graph $G \diamond P_m$ in Definition 7.2 is equal the corona $G \circ K_1$.

7.1 Maximal distance domination number

Bollobás and Cockayne [10] have found that for any connected, non trivial graph G there exists a γ-set D such that every vertex in D has at least one private neighbour (with regard to D) other than itself. Henning, Oellermann, and Swart generalized this result for distance domination.

Theorem 7.3 (Henning, Oellermann, and Swart [68]) *For any positive integer k and any connected graph of order $n \geq k+1$ with $diam(G) \geq k$ there exists a $\gamma_{\leq k}$-set D such that every vertex $v \in D$ has at least one private k-neighbour w (with regard to D) with $d(v, w) = k$.*

Further, they have obtained the following upper bound on the distance-k domination number which is a generalization of the upper bound $n(G)/2$ on the ordinary domination number by Ore [85].

Corollary 7.4 (Henning, Oellermann, and Swart [67, 68]) *Let k be an arbitrary positive integer. If G is a connected graph of order $n \geq k+1$, then*

$$\gamma_{\leq k}(G) \leq n/(k+1).$$

For any positive integer k the authors [67] have presented the cycle C_{2k+2} and the graphs $H \diamond P_k$ for a connected graph H to show that the bound in Corollary 7.4 is sharp. We show that for every positive integer k these graphs are the only connected graphs for which equality in Corollary 7.4 holds.

Theorem 7.5 *For any positive integer k and any connected graph G of order $n \geq k+1$ we have $\gamma_{\leq k}(G) = n/(k+1)$ if and only if either $G \cong C_{2k+2}$ or $G \cong H \diamond P_k$ for a connected graph H.*

Proof. It is easy to see that $\gamma_{\leq k}(C_{2k+2}) = 2 = n(C_{2k+2})/(k+1)$ and $\gamma_{\leq k}(H \diamond P_k) = |V(H)| = n(H \diamond P_k)/(k+1)$ for any connected graph H. Now, let G be an arbitrary connected graph with $\gamma_{\leq k}(G) = n(G)/(k+1) = r$. Let $D = \{v_0^1, v_0^2, \ldots, v_0^r\}$ be a $\gamma_{\leq k}$-set of G as in Theorem 7.3 and for every $1 \leq i \leq r$ let v_k^i be a private k-neighbour of v_0^i with $d(v_0^i, v_k^i) = k$, and let $P^{(i)} = v_0^i, v_1^i, \ldots, v_k^i$ be a shortest path in G from v_0^i to v_k^i. Suppose, there are subindices $1 \leq i < j \leq r$ such that $V(P^{(i)}) \cap V(P^{(j)}) \neq \emptyset$. Let u be an arbitrary vertex in $V(P^{(i)}) \cap V(P^{(j)})$. Then, without loss of generality $u = v_s^i = v_t^j$ for some $0 \leq t \leq s \leq k$, and the distance $d(v_0^j, v_k^i) \leq (v_0^j, v_t^j) + (v_s^i, v_k^i) \leq t + (k-s) \leq k$, which is a contradiction. Hence, the sets $V(P^{(1)}), V(P^{(2)}), \ldots, V(P^{(r)})$ are pairwise disjoint and $V(G) = \bigcup_{i=1}^{r} V(P^{(i)})$, by $n(G) = r(k+1)$. Let $v_s^i v_t^j \in E(G)$ with $i \neq j$ arbitrarily. If $s \neq t$, then

we have either $d(v_0^i, v_k^j) \leq k$ or $d(v_0^j, v_k^i) \leq k$, which both is a contradiction. Thus, we obtain that $s = t$. If $0 < s < k$, then the distances $d(v_s^i, v_h^j) \leq k$ for every $0 \leq h \leq k$, and the set $(D \setminus \{v_0^i, v_0^j\}) \cup \{v_s^i\}$ is a distance-k dominating set of G, which is a contradiction. Hence, we get that $s \in \{0, k\}$. If we have $s = 0$ for every edge $v_s^i v_s^j \in E(G)$ with $i \neq j$, then $G = H \diamond P_k$ for the graph $H = G[D]$. If there is no edge in $G[D]$, then $G = H \diamond P_k$ for the graph $H = G[\{v_k^1, v_k^2, \ldots, v_k^r\}]$. If there exists an edge in $G[D]$ and an edge $v_k^i v_k^j \in E(G)$ with $i \neq j$, then we decide two cases.
If $r = 2$, then obviously $G \cong C_{2k+2}$.
If $r > 2$, then there are three pairwise different subindices i, j, h such that $v_k^i v_k^j \in E(G)$ and $v_0^j v_0^h \in E(G)$. The vertex v_k^i has distance less then or equal k to every vertex in $V(P^{(i)}) \cup (V(P^{(j)}) \setminus \{v_0^j\}$, and the set $(D \setminus \{v_0^i, v_0^j\}) \cup \{v_k^i\}$ is a distance-k dominating set of G, which is a contradiction. This completes the proof. □

Observation 7.6 The distance-k domination number of a graph G is equal the sum of the distance-k domination numbers of its components. If G is a graph which components are of order at least $k+1$, then for each component G_i of order n_i the distance-k domination number is less than or equal $n_i/(k+1)$, by Corollary 7.4. This implies that the graph G fulfils $\gamma_{\leq k}(G) = n(G)/(k+1)$ if and only if each component G_i fulfils $\gamma_{\leq k}(G_i) = n_i/(k+1)$.

Remark 7.7 Theorem 7.5 is an exact generalization of Theorem 7.1 by Payan and Xuong [86] and Fink, Jacobson, Kinch, and Roberts [37].

7.2 The influence of unique realization

If we consider graphs with unique $\gamma_{\leq k}$-sets, then the bound in Corollary 7.4 is no longer best possible. For those graphs we present an upper bound on the distance-k domination number which is sharp, and we give a necessary condition for the existence of a unique $\gamma_{\leq k}$-set attaining this bound. In order to do this, we define the following condition C1.

C1: The graph G contains a factor F that fulfils the following two conditions.

- *F consists of r disjoint paths $P^{(1)}, P^{(2)}, \ldots, P^{(r)}$ each of length $2k$.*
- *For every $1 \leq i \leq r$ let v_i be the unique vertex in the center $C(P^{(i)})$ of the path $P^{(i)}$. If $vw \in E(G) \setminus E(F)$ with $v \in V(P^{(i)})$ and $w \in V(P^{(j)})$ for some $1 \leq i, j \leq r$, then $i \neq j$ and $d_F(v_i, v) = d_F(v_j, w)$.*

Let G be any graph that fulfils Condition C1. Then, we define a *C1-partition* $\mathcal{P}(G,k)$ of G to be a set of paths $P^{(1)}, P^{(2)}, \dots, P^{(r)}$ of length $2k$ as in C1.

Observation 7.8 A set of r disjoint paths $P^{(1)}, P^{(2)}, \dots, P^{(r)}$ each of length $2k$ that form a factor F of G is a C1-partition of G if and only if for every $1 \le i, j \le r$, $j \ne i$, the path $P^{(i)}$ is an induced subgraph of G and $d_F(v_i, w_i) = d_G(v_i, w_i) < d_G(v_j, w_i)$ where $v_i \in C(P^{(i)})$, $v_j \in C(P^{(j)})$, and $w_i \in V(P^{(i)})$.

Theorem 7.9 *Let k be any positive integer and let G be a connected graph of order $n \ge 2$ that has a unique $\gamma_{\le k}$-set.*

a) Then, it applies that $\gamma_{\le k}(G) \le n/(2k+1)$.

b) If $\gamma_{\le k}(G) = n/(2k+1)$, then G has a C1-partition $\mathcal{P}(G,k)$ such that the set $\{v \in C(P) \mid P \in \mathcal{P}(G,k)\}$ is the unique $\gamma_{\le k}$-set of G.

Proof. Let $\gamma_{\le k}(G) = r$, let $D' = \{x_1, x_2, \dots, x_r\}$ be the unique $\gamma_{\le k}$-set of G, and let $\tilde{P}_i = \{w \in P_k(x_i, D') \mid d(x_i, w) = k\}$ for every $1 \le i \le r$. Lemma 5.2a) ensures for every $1 \le i \le r$ the existence of two different vertices w_i and w'_i in $\tilde{P}_i$ and the existence of a shortest path $P_{x_i w_i}$ from x_i to w_i and a shortest path $P_{x_i w'_i}$ from x_i to w'_i such that $V(P_{x_i w_i}) \cap V(P_{x_i w'_i}) = \{x_i\}$. We define the path $P^{(i)} = P_{x_i w_i} \cup P_{x_i w'_i}$ for every $1 \le i \le r$. Then, the center $C(P^{(i)}) = \{x_i\}$ for every $1 \le i \le r$.
Suppose, there are subindices $i, j \in \{1, 2, \dots, r\}$, $i \ne j$, such that there exists a vertex $u \in V(P^{(i)}) \cap V(P^{(j)}) \ne \emptyset$. Without loss of generality let $d(x_i, u) \le d(x_j, u)$ and let $u \in V(P_{x_j w_j})$. Then, we obtain the inequality $d(x_i, w_j) \le d(x_i, u) + d(u, w_j) \le d(x_j, u) + d(u, w_j) = k$, which is a contradiction to $w_j \in P_k(x_j, D')$. Hence, the intersection $V(P^{(i)}) \cap V(P^{(j)})$ is empty for every $1 \le i < j \le r$, and the order $n = |V(G)| \ge \sum_{i=1}^{r} |V(P^{(i)})| = r(2k+1)$. This completes the proof of a).
Now, let $\gamma_{\le k}(G) = r = n/(2k+1)$. Then, the cardinality $|V(G)|$ is equal the sum $\sum_{i=1}^{r} |V(P^{(i)})|$ and the subgraph F of G consisting of the r disjoint paths $P^{(1)}, P^{(2)}, \dots, P^{(r)}$ of length $2k$ is a factor of G. This implies that $\tilde{P}_i = \{w_i, w'_i\}$ for every $1 \le i \le r$. Let $vw \in E(G) \setminus E(F)$ with $v \in V(P^{(i)})$ and $w \in V(P^{(j)})$ for some $1 \le i, j \le r$. Without loss of generality let $d_F(x_i, v) \le d_F(x_j, w)$ and let $w \in V(P_{x_j w_j})$.
Suppose, $i = j$. If $w = x_j$, then $d_F(x_j, v) \le 0$ and $v = x_j = w$. This contradiction yields that $w \ne x_j$. Since $d_G(x_j, w_j) = k$, we obtain that $P_{x_j w_j}$ is an induced subgraph of G and $v \in V(P_{x_j w'_j}) \setminus V(P_{x_j w_j})$. Then, for the vertex $z \in N(x_j, P_{x_j w'_j})$ we have $d_G(z, y) \le k$ for every $y \in V(P^{(j)}) \setminus \{w_j\}$ and

$d_G(z, w_j) \le d_F(z, v) + 1 + d_F(w, w_j) = d_F(x_j, v) + d_F(w, w_j) \le d_F(x_j, w) + d_F(w, w_j) = k$, and the set $(D \setminus \{v_j\}) \cup \{z\}$ is also a $\gamma_{\le k}$-set of G. By this contradiction, we obtain that $i \ne j$.
At last suppose that $d_F(x_i, v) < d_F(x_j, w)$. Since the vertex $w \in V(P_{x_j w_j})$ and the distance $d_G(x_i, w) \le d_F(x_i, v) + 1 \le d_F(x_j, w)$, we obtain the contradiction that $d_G(x_i, w_j) \le d_G(x_i, w) + d_G(w, w_j) \le d_F(x_j, w) + d_F(w, w_j) = k$.
Thus, we have $d_F(x_i, v) = d_F(x_j, w)$. Hence, the set $\{P^{(1)}, P^{(2)}, \ldots, P^{(r)}\}$ is a C1-partition of G and the set $D = \{v \in C(P^{(i)}) \mid 1 \le i \le r\}$ is the unique $\gamma_{\le k}$-set of G. □

Remark 7.10 It is easy to see that the bound in Theorem 7.9 a) is sharp. For example consider the class of graphs G consisting of the disjoint union of r paths $P^{(1)}, P^{(2)}, \ldots, P^{(r)}$ of length $2k$ and additional edges between the vertices in $D = \{v_i \in C(P^{(i)}) \mid i = 1, 2, \ldots, r\}$ such that the induced subgraph $G[D]$ is connected. Further examples are shown in Theorem 7.12 and Theorem 7.15.

Remark 7.11 For arbitrary graphs G Condition C1 is not sufficient for the existence of a unique $\gamma_{\le k}$-set with $\gamma_{\le k}(G) = n(G)/(2k+1)$. For instance consider the cycle of order $r(2k+1)$ for any positive integer r.
Even for graphs G with unique $\gamma_{\le k}$-sets Condition C1 is not sufficient for the identity $\gamma_{\le k}(G) = n(G)/(2k+1)$. For example consider for any positive even integer $k = 2s$ the graph consisting of $k+1$ disjoint paths $P^{(0)}, P^{(1)}, \ldots, P^{(k)}$ where $P^{(i)} = x_0^{(i)} x_1^{(i)} \ldots x_{2k}^{(i)}$ for $0 \le i \le k$, and the additional edges

$$\{x_s^{(i)} x_s^{(i+1)}, x_{3s}^{(i)} x_{3s}^{(i+1)} \mid 0 \le i < k\}.$$

This graph has the unique $\gamma_{\le k}$-set $\{x_s^{(s)}, x_{3s}^{(s)}\}$ and it fulfils Condition C1 but $\gamma_{\le k}(G) = 2 < k+1$. Figure 7.1 shows this graph for $k = 2$.

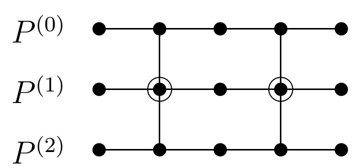


Figure 7.1

The next theorem shows that for trees T Condition C1 is necessary and sufficient for the existence of a unique $\gamma_{\le k}$-set with $\gamma_{\le k}(T) = n(T)/(2k+1)$.

Theorem 7.12 *Let k be any positive integer, let T be a tree of order $n \geq 2$, and let $D \subseteq V(T)$. Then, the following two conditions are equivalent.*

(i) D is the unique $\gamma_{\leq k}$-set of T and $\gamma_{\leq k}(T) = n/(2k+1)$.

(ii) There exists a C1-partition $\mathcal{P}(T,k)$ of T such that $D = \{v \in V(T) \mid v \in C(P), P \in \mathcal{P}(T,k)\}$.

Proof.
(i) $\Rightarrow$ (ii): Follows immediately from Theorem 7.9.

(ii) $\Rightarrow$ (i): Let T be a non trivial tree and let $D \subseteq V(T)$ such that T and D fulfil Condition (ii). Hence, T has a C1-partition $\mathcal{P}(T,k) = \{P^{(1)}, P^{(2)}, \ldots, P^{(r)}\}$ such that $D = \{v \in V(T) \mid v \in C(P^{(i)}), 1 \leq i \leq r\}$. Let F be the factor of T consisting of this r disjoint paths of length $2k$, and for every $1 \leq i \leq r$ let v_i be the unique vertex in the center of $P^{(i)}$ and let w_i and w_i' be the two endvertices of $P^{(i)}$. Since T is a tree, we obtain that $d_T(w_i, w_i') = 2k$ for every $1 \leq i \leq r$. Obviously, the set $D = \{v_1, v_2, \ldots, v_r\}$ is a distance-k dominating set of T. By Observation 7.8, we have for $u_i \in \{w_i, w_i'\}$ that $d_F(v_i, u_i) = k = d_T(v_i, u_i) < d_T(v_j, u_i)$ for every $1 \leq i, j \leq r$ with $i \neq j$. Hence, for every $1 \leq i \leq r$ the vertex $v_i \in D$ has two different private k-neighbours $w_i, w_i' \in P_k(v_i, D)$ with $d_T(w_i, w_i') = 2k$. By Theorem 5.4 and $n(T) = r(2k+1) \geq 3$, we obtain that D is the unique $\gamma_{\leq k}$-set of T and $\gamma_{\leq k}(T) = r = n/(2k+1)$. □

Now, we present a class $\mathcal{G}(k)$ of graphs G that are allowed to contain cycles and that have unique $\gamma_{\leq k}$-sets and distance-k domination number $\gamma_{\leq k}(G) = n(G)/(2k+1)$. For this purpose, we give some more definitions.

Definition 7.13 Let G be any graph that has a C1-partition $\mathcal{P}(G,k) = \{P^{(1)}, P^{(2)}, \ldots, P^{(r)}\}$ and let $v_i \in C(P^{(i)})$ for every $1 \leq i \leq r$. For every $h = 0, 1, \ldots, k$ we define the *h-th level* $L_h(G)$ of G with regard to $\mathcal{P}(G,k)$ by

$$L_h(G) = \{v \in V(G) \mid v \in V(P^{(i)}) \text{ and } d_{P^{(i)}}(v_i, v) = h \text{ for some } 1 \leq i \leq r\}.$$

For every positive integer k we define $\mathcal{G}(k)$ to be the class of graphs satisfying Condition C1 and the following condition C2.

C2: The graph G has a C1-partition $\mathcal{P}(G,k)$ such that the vertex set of every cycle C of G lies in one level of G with regard to this partition $\mathcal{P}(G,k)$, i.e. $V(C) \subseteq L_h(G)$ for some $h = 0, 1, \ldots, k$.

Let G be any graph that fulfils Condition C2. Then, we define a *C2-partition* $\mathcal{P}(G,k)$ of G to be a C1-partition of G as in Condition C2.

The following lemma contains some useful properties of the graphs in $\mathcal{G}(k)$.

Lemma 7.14 *Let $G \in \mathcal{G}(k)$ and let $L_0(G), L_1(G), \ldots, L_k(G)$ be the levels of G with regard to the C2-partition $\mathcal{P}(G,k)$.*

a) For every cycle C in G and every path $P \in \mathcal{P}(G,k)$ the cardinality $|V(C) \cap V(P)| \leq 1$.

b) If C and C' are two arbitrary cycles in G that do not lie in the same level of G, then at most one path $P \in \mathcal{P}(G,k)$ has a vertex in C and a vertex in C'.

c) For every path $P \in \mathcal{P}(G,k)$ and every two vertices $v, w \in V(P)$ the path in P from v to w is the unique path in G from v to w. Especially, every edge $e \in E(P)$ is a bridge of G.

Proof. Let $G \in \mathcal{G}(k)$ and let $L_0(G), L_1(G), \ldots, L_k(G)$ be the levels of G with regard to $\mathcal{P}(G,k)$.
a) Suppose, in G is a cycle $C = c_1c_2 \ldots c_lc_1$ such that $|V(C) \cap V(P)| \geq 2$ for some $P \in \mathcal{P}(G,k)$. Then, $V(C) \subseteq L_h(G)$ and $V(P) \cap L_h(G) = \{c_i, c_j\}$ for some $h > 0$ and for some $1 \leq i < j \leq l$. Let P_{ij} be the path in P from c_i to c_j. Then, the cycle $c_ic_{i+1} \ldots c_j \cup P_{ij}$ in G contains the vertices in $V(P) \cap L_j(G)$ for every $0 \leq j \leq h$, which is a contradiciton to Condition C2. Thus, a) is proved.
b) Suppose, there are two cycles C and C' in G with $V(C) \subseteq L_h(G)$, $V(C') \subseteq L_{h'}(G)$ and $0 \leq h < h' \leq k$, and there are two different paths $P, P' \in \mathcal{P}(G,k)$, such that $V(C) \cap V(P^*) \neq \emptyset$ and $V(C') \cap V(P^*) \neq \emptyset$ for $P^* = P, P'$. Then, there exists a cycle in G containing vertices in $V(P^*) \cap L_j(G)$ for $h \leq j \leq h'$, $P^* = P, P'$. This is a contradiction to Condition C2. Hence, the proof of b) is complete.
c) This follows immediately from a). □

Theorem 7.15 *Let k be any positive integer and let $G \in \mathcal{G}(k)$. Then, G has a unique C1-partition $\mathcal{P}(G,k)$, the set $D = \{v \in C(P) \mid P \in \mathcal{P}(G,k)\}$ is the unique $\gamma_{\leq k}$-set of G and $\gamma_{\leq k}(G) = n(G)/(2k+1)$.*

Proof. Let k be any positive integer. Suppose, there is a graph $G \in \mathcal{G}(k)$ with a C2-partition $\mathcal{P}(G,k)$ such that either $\mathcal{P}(G,k)$ is not the unique C1-partition or the set $D = \{v \in C(P) \mid P \in \mathcal{P}(G,k)\}$ with $|D| = |\mathcal{P}(G,k)| = n(G)/(2k+1)$ is not the unique $\gamma_{\leq k}$-set of this graph. Let G be such a counterexample of minimal order. Let $r = n(G)/(2k+1)$, let $\mathcal{P}(G,k) = \{P^{(1)}, P^{(2)}, \ldots, P^{(r)}\}$ be a C2-partition of G, and for every $1 \leq i \leq r$ we define $v_i \in C(P^{(i)})$, v_i^1 and v_i^2 are the two endvertices of $P^{(i)}$, and $V_i = V(P^{(i)})$. Obviously, the set $D = \{v_1, v_2, \ldots, v_r\}$ is a distance-k dominating set of G.

Claim 1: If $\mathcal{P}(G, k)$ is not the unique C1-partition, then the set D is not the unique $\gamma_{\leq k}$-set of G.

Proof of Claim 1. Let $\tilde{\mathcal{P}}(G, k) = \{\tilde{P}^{(1)}, \tilde{P}^{(2)}, \dots, \tilde{P}^{(r)}\}$ be a C1-partition of G different from $\mathcal{P}(G, k)$ and let $\tilde{D} = \{\tilde{v}_i \mid \tilde{v}_i \in C(\tilde{P}^{(i)}) \mid 1 \leq i \leq r\}$. Suppose, D is the unique $\gamma_{\leq k}$-set of G. Since $\tilde{D}$ is also a distance-k dominating set of G and $|\tilde{D}| = |D|$, we obtain that $\tilde{D} = D$. Without loss of generality let $v_i = \tilde{v}_i$ for every $1 \leq i \leq r$ and let $P^{(1)} \neq \tilde{P}^{(1)}$. By Lemma 7.14 c), the path $\tilde{P}^{(1)}$ can not contain both of the vertices v_1^1 and v_1^2. Further, $\{v_1^1, v_1^2\} \cap V(\tilde{P}^{(i)}) = \emptyset$ for every $2 \leq i \leq r$, since by Observation 7.8, $d(v_i, v_1^a) > k$ for every $2 \leq i \leq r$, $a = 1, 2$. This is a contradiction to $\tilde{\mathcal{P}}(G, k)$ being a C1-partition of G, which completes the proof of the claim.

Hence, in each case there exists a $\gamma_{\leq k}$-set $F \neq D$ of G. Without loss of generality let F be a $\gamma_{\leq k}$-set of G with $F \neq D$ and $|F \cap D|$ maximal. If $r = 1$, then $G \cong P_{2k+1}$ and $D = \{v_1\}$ is the unique $\gamma_{\leq k}$-set of G with $|D| = n(G)/(2k+1)$, which is a contradiction to $F \neq D$. Hence, we have $r > 1$. There is at least one vertex $v \in D \setminus F$. Without loss of generality let $v \in V(P^{(1)})$. Further, for $a = 1, 2$ let $x_a = v_1^a$, let $e_a = y_a y'_a \in E(P^{(1)})$ such that y_a lies on a path from x_a to a vertex $z_a \in F \cap N_k[x_a]$ and y'_a does not lie on any path from x_a to a vertex in $F \cap N_k[x_a]$. For $a = 1, 2$ let G_a be the component of $G - e_a$ that contains x_a and let $Q_a = F \cap V(G_a)$. Note, that $v \notin V(G_1) \cup V(G_2)$ and $V(G_1) \cap V(G_2) = \emptyset$. Let $a = 1, 2$ arbitrarily. For every vertex $f \in F \setminus Q_a$ we have $d(f, x_a) = d(f, y_a) + d(y_a, x_a) > k \geq d(z_a, x_a) = d(z_a, y_a) + d(y_a, x_a)$, which yields $d(f, y_a) > d(z_a, y_a)$. Thus, for every vertices $w \in V(G_a)$ and $f \in F \setminus Q_a$ we get $d(z_a, w) \leq d(z_a, y_a) + d(y_a, w) < d(f, y_a) + d(y_a, w) = d(f, w)$. Hence, the set Q_a is a distance-k dominating set of G_a for $a = 1, 2$. For every edge $e \in E(P^{(1)})$ let $H_1(e)$ and $H_2(e)$ be the components of $G - e$ such that $x_a \in V(H_a(e))$, $a = 1, 2$. We define the set

$$E^* = \{e \in E(P^{(1)}) \setminus E(G_2) \mid V(H_1(e)) \subseteq N_k[F \cap V(H_1(e)), H_1(e)]\}.$$

Note, that $H_1(e_1) = G_1$ and $e_1 \in E^*$. Let e^* be the edge in E^* with maximal distance to x_1, let $H_a = H_a(e^*)$ and $F_a = F \cap V(H_a)$ for $a = 1, 2$. By the definition of E^*, the set F_1 is a distance-k dominating set of H_1.

Claim 2: F_2 is a distance-k dominating set of H_2.

Proof of Claim 2. Suppose, the set $V(H_2) \setminus N_k[F_2, H_2]$ is not empty. Since $Q_2 \subseteq F_2$ is a distance-k dominating set of $G_2 \subseteq H_2$, we have $V(H_2) \setminus N_k[F_2, H_2] \subseteq V(H_2) \setminus V(G_2) \neq \emptyset$. Let $f \in F_1$ with minimal distance to the edge e^* and let $yy' \in E(P^{(1)})$ such that y lies on a path from f to a vertex $w \in V(H_2) \setminus N_k[F_2, H_2]$ but y' does not lie on a path from f to a vertex in $V(H_2) \setminus N_k[F_2, H_2]$. Since $w \notin V(G_2)$, the edge $yy' \notin E(G_2)$. Further, for every vertex $x \in F_2$ we obtain $d(f, y) < d(x, y)$, by $d(f, w) = d(f, y) + d(y, w) \leq$

$k < d(x, w) \leq d(x, y) + d(y, w)$. For every vertices $u \in V(H_1(yy'))$ and $f_2 \in F \setminus V(H_1(yy')) \subseteq F_2$ we obtain that $d(f, u) \leq d(f, y) + d(y, u) < d(f_2, y) + d(y, u) = d(f_2, u)$, which implies that the set $F \cap V(H_1(yy'))$ is a distance-k dominating set of $H_1(yy')$. Hence, the edge $yy' \in E^*$ and the distance of yy' to x_1 is greater than the distance of e^* to x_1, which is a contradiction.

Thus, F_a is a distance-k dominating set of H_a for $a = 1, 2$. We define the graphs $H'_a = G[V(H_a) \cup V(P^{(1)})]$ and the sets $D'_a = D \cap V(H'_a)$ and $F'_a = F_a \cup \{v\}$ for $a = 1, 2$. Note, that $D'_1 \cap D'_2 = \{v\} = F'_1 \cap F'_2$. Let $a \in \{1, 2\}$ arbitrarily. Obviously, the set F'_a is a distance-k dominating set of H'_a and the graph H'_a fulfils the conditions C1 and C2. Since the vertex $x_a \in P_k(v, D'_a) \setminus P_k(v, F'_a)$, we obtain that $F'_a \neq D'_a$. Suppose, $H'_a = G$. Then, $D'_a = D$ and F'_a is a distance-k dominating set of G with $|F'_a| = |F_a| + 1 \leq |F_a| + |F_{3-a}| = |F|$. Hence, F'_a is a $\gamma_{\leq k}$-set of G, $|F_{3-a}| = 1$ and $|F'_a \cap D| > |F \cap D|$. By the choice of F, we obtain that $F'_a = D = D'_a$, which is a contradiction. Thus, for $a = 1, 2$ we obtain that $n(H'_a) < n(G)$ and D'_a is the unique $\gamma_{\leq k}$-set of H'_a, by the minimality of G. Thus, we have $|D'_a| < |F'_a| = |F_a| + 1$. This leads to the contradiction

$$|F| \leq |D| = |D'_1| + |D'_2| - 1 \leq |F_1| + |F_2| - 1 = |F| - 1.$$

□

Chapter 8

Total domination

Several publications and some results on total domination are listed in Section 5.2. In this chapter we consider upper bounds on the total domination number and characterizations of extremal graphs achieving these bounds. First, we present some known results on this topic, and then we point out our results, which apply to the case that the total domination number has a unique realization.

A general upper bound on the total domination number is given by Cockayne, Dawes, and Hedetniemi, who have introduced the concept of total domination.

Theorem 8.1 (Cockayne, Dawes, and Hedetniemi [19]) *If G is a connected graph of order $n(G) \geq 3$, then*

$$\gamma_t(G) \leq \frac{2}{3}n(G).$$

In 2000, Brigham, Carrington and Vitray have characterized all connected graphs achieving this bound.

Theorem 8.2 (Brigham, Carrington and Vitray [11]) *Let G be a connected graph. Then, $\gamma_t(G) = \frac{2}{3}n(G)$ if and only if G is equal C_3, C_6 or $H \diamond P_2$ for a connected graph H.*

More precise the upper bound is equal $\lfloor \frac{2}{3}n(G) \rfloor$, and in [11] even all connected graphs with $\gamma_t(G) = \lfloor \frac{2}{3}n(G) \rfloor$ are characterized.

Volkmann [103] now posed the questions, if there is a better upper bound on the total domination number with unique realization, and if in this case there exists a characterization of graphs with unique γ_t-sets achieving this bound. These questions will be answered in the following section.

8.1 Extremal total domination with regard to unique realization

Theorem 8.2 implies that the bound in Theorem 8.1 is sharp. But if we consider graphs with unique minimum total dominating sets, then this bound is no longer best possible, as one can see by the following result.

Theorem 8.3 (Fischermann [39]) *Let G be a connected graph of order at least 3. If G has a unique minimum total dominating set, then*

$$\gamma_t(G) \leq \frac{3}{5}n(G).$$

Proof. Let D be the unique γ_t-set of G. Further, let $H \subseteq G$ with $V(H) = D \cup P_t(D, D)$ and $E(H) = \{ab \in E(G) \mid a \in D \text{ and } b \in P_t(a, D)\}$. For every vertex $z \in P_t(D, D) \setminus D$ we have $N_H(z) = \{x\}$ for some vertex $x \in D$. Let $H_1, H_2, \ldots, H_s$ be the components of H. For $i = 1, 2, \ldots, s$ we define $D_i = D \cap V(H_i)$. The induced subgraph $H[D_i]$ is connected for every $i = 1, 2, \ldots, s$. Suppose that some subgraph $H[D_i]$ contains a cycle $v_0v_1v_2v_0$ or a path $v_0v_1 \ldots v_d$ for some integer $d \geq 3$. Then, $v_1v_2 \in E(H)$ but $v_1 \notin P_t(v_2, D)$ and $v_2 \notin P_t(v_1, D)$, which is a contradiction. Hence, H is a forest and either $|D_i| = 1$ or $H[D_i] \cong K_{1,|D_i|-1}$ is a star for every $i = 1, 2, \ldots, s$. Without loss of generality let $|D_1| \leq |D_2| \leq \ldots \leq |D_s|$ and let $0 \leq i_1 \leq i_2 \leq s$ such that $|D_i| = 1$ for every $1 \leq i \leq i_1$, $|D_i| = 2$ for every $i_1 < i \leq i_2$ and $|D_i| > 2$ for every $i_2 < i \leq s$. By Lemma 5.9, we have $n(H_i) \geq 2$ for every $1 \leq i \leq i_1$, $n(H_i) \geq 4$ for every $i_1 < i \leq i_2$ and $n(H_i) \geq 2|D_i| - 1$ for every $i_2 < i \leq s$. Thus,

$$\frac{|D_i|}{n(H_i)} \leq \frac{1}{2} < \frac{3}{5} \quad \text{for every } 1 \leq i \leq i_2 \text{ and}$$

$$\frac{|D_i|}{n(H_i)} \leq \frac{|D_i|}{2|D_i| - 1} \leq \frac{3}{5} \quad \text{for every } i_2 < i \leq s.$$

This leads to

$$\gamma_t(G) = |D| = \sum_{i=1}^{s} |D_i| \leq \sum_{i=1}^{s} \frac{3}{5}n(H_i) = \frac{3}{5}n(H) \leq \frac{3}{5}n(G). \tag{8.1}$$

□

Now, we present a complete characterization of graphs with unique γ_t-sets for which the total domination number attains the upper bound in Theorem 8.3.

Theorem 8.4 (Fischermann [39]) *Let G be a connected graph of order at least 3. G has a unique γ_t-set of cardinality $\gamma_t(G) = \dfrac{3}{5}n(G)$ if and only if $n(G) = 5r$ for some positive integer r and G consists of the disjoint union of r paths $P^{(1)} = v_1^1 v_2^1 \dots v_5^1$, $P^{(2)} = v_1^2 v_2^2 \dots v_5^2$, …, $P^{(r)} = v_1^r v_2^r \dots v_5^r$ of length 5 and possibly of additional edges between vertices in $\{v_3^1, v_3^2, \dots, v_3^r\}$.*

Proof. Let D be the unique γ_t-set of G and let $A(G)$ be the set of endvertices of G. Furthermore, let $H_1, H_2, \dots, H_s$ and $D_1, D_2, \dots, D_s$ and i_1, i_2 be defined as in the proof of Theorem 8.3.

Since $\gamma_t(G) = 3n(G)/5$, we have identity in every inequality in (8.1). This implies that $|D_i| = 3n(H_i)/5$ for every $i = 1, 2, \dots, s$ and $n(G) = n(H)$. Hence, we have $i_1 = i_2 = 0$ and for every $1 \le i \le s$ we get

$$\frac{|D_i|}{n(H_i)} = \frac{|D_i|}{2|D_i| - 1} = \frac{3}{5},$$

which yields $|D_i| = 3$ and $n(H_i) = 5$. Then, for every $1 \le i \le s$ we obtain that $H[D_i] \cong K_{1,2} \cong x_1^i x_0^i x_2^i$ and each of the two vertices x_1^i, x_2^i has all its total private neighbours in $V(H_i) \setminus D_i$. By $|V(H_i) \setminus D_i| = 2$ and by Lemma 5.9, the vertex x_j^i has only one total private neighbour y_j^i and $y_j^i \in A(G)$ for $j = 1, 2$. This implies that $H_i \cong y_1^i x_1^i x_0^i x_2^i y_2^i$ for every $i = 1, 2, \dots, s$. Thus, $n(G) = 5s$ for the positive integer s and G consists of the disjoint union of the s paths $H_1, H_2, \dots, H_s$ of length 5 and possibly of additional edges outside $E(H)$. Now, let $ab \in E(G) \setminus E(H)$ be arbitrary. Since $D_i = \{x_0^i, x_1^i, x_2^i\}$ and $y_1^i, y_2^i \in A(G)$ for every $i = 1, 2, \dots, s$, we have $A(G) = \{y_1^i, y_2^i \mid 1 \le i \le s\}$ and $a, b \in D = V(G) \setminus A(G)$. Further, we have $P_t(x_0^i, D) = \{x_1^i, x_2^i\}$ and both vertices $a, b \notin \{x_1^i, x_2^i \mid 1 \le i \le s\}$. This yields $a, b \in \{x_0^i \mid 1 \le i \le s\}$, which completes this part of the proof.

On the other hand, it is easy to see that every graph G described above has a unique minimum total dominating set with $\gamma_t(G) = \frac{3}{5}n(G)$. □

Chapter 9

Exact distance domination

Exact distance domination is a further type of distance domination introduced by Boland, Haynes and Lawson in [8]. As defined in the introduction (see Definition 1.18), a set D is an *exact distance-k dominating set* of G if every vertex in $V(G) \setminus D$ has exact distance k to at least one vertex in D, and the *exact distance-k domination number* of a graph G is denoted by $\gamma_{=k}(G)$. In order to study exact distance domination, it is useful to consider the exact distance-k graph $D_k(G)$ (see Definition 1.7), since $\gamma_{=k}(G) = \gamma(D_k(G))$ for any graph G.

9.1 Maximal exact distance domination number

Ore's [85] upper bound $n(G)/2$ on the domination number (Theorem 1.13) applied to the exact distance-k graph $D_k(G)$ leads to an upper bound on the exact distance-k domination number.

Lemma 9.1 (Boland, Haynes, and Lawson [8]**)** *If $D_k(G)$ has no isolated vertices, then $\gamma_{=k}(G) \le n(G)/2$.*

The condition that the exact distance-k graph $D_k(G)$ has no isolated vertices is equivalent to the condition that the radius $rad(G)$ is at least k. The bound in Lemma 9.1 is sharp for every integer k as one can see considering the path P_{2k} of order $2k$.

The characterization of graphs with $\gamma(G) = n(G)/2$ by Payan and Xuong and Fink, Jacobson, Kinch and Roberts [86] (see Theorem 7.1) applied to the exact distance-k graph yields the following.

Corollary 9.2 (Boland, Haynes, and Lawson [8]) *Let k be a positive integer and let G be a graph of even order n with radius $rad(G) \geq k$. Then $\gamma_{=k}(G) = n/2$ if and only if the components of $D_k(G)$ consist of the cycle C_4 or the graph $H \circ K_1$ for a connected graph H.*

Observation 9.3 It is straightforward to see that for any integer $k \geq 2$ and any graph G the exact distance-k graph $D_k(G) \neq C_4$. Thus, if the graph $D_k(G)$ in Corollary 9.2 is connected, then $D_k(G) = H \circ K_1$ for some connected graph H.

Using Corollary 9.2, Boland, Haynes and Lawson have characterized all graphs G where G and $D_2(G)$ are connected and the exact distance-2 domination number $\gamma_{=2}(G)$ equals $n(G)/2$. For this characterization we use the notation $G \diamond P_m$ in Definition 7.2.

Theorem 9.4 (Boland, Haynes, and Lawson [8]) *Let the graphs G and $D_2(G)$ be connected of order $n \geq 4$. Then, $\gamma_{=2}(G) = n/2$ if and only if either $diam(G) = 2$ and $\bar{G} = H \diamond P_1$ or $G = G' \diamond P_3$ for a connected graph H or G'.*

These authors also posed the following conjecture on a generalization of Theorem 9.4.

Conjecture 9.5 (Boland, Haynes, and Lawson [8] 1994) *Let the graphs G and $D_k(G)$ be connected of even order n. Then, $\gamma_{=k}(G) = n/2$ if and only if $G = G' \diamond P_{2k-1}$ for a connected graph G'.*

In connection with this conjecture we observe that a graph $G = G' \diamond P_{2k-1}$ for any graph G' has diameter at least $2k - 1$. Hence, there are two ways to understand Conjecture 9.5. The first one is that the authors assume that there is no connected graph G of diameter less than $2k - 1$ such that $D_k(G)$ is connected and $\gamma_{=k}(G) = n/2$. But since there exist a lot of such graphs (cf. Examples 9.12 below), we interpret Conjecture 9.5 in the second way, that means under the natural condition that the diameter of G is at least $2k-1$. In this sense our following result provides an affirmative answer to Conjecture 9.5.

Theorem 9.6 (Fischermann and Volkmann [42]) *Let $k \geq 2$ be an integer and let G and $D_k(G)$ be connected of order n such that $diam(G) \geq 2k - 1$. Then $\gamma_{=k}(G) = n/2$ if and only if $G = G' \diamond P_{2k-1}$ for a connected graph G'.*

Proof. For any integer $k \geq 2$ let G and $D_k(G)$ be connected of order n and let $\text{diam}(G) \geq 2k - 1$. It is straightforward to see that $\gamma_{=k}(G) = n/2$ if $G = G' \diamond P_{2k-1}$ for a connected graph G'.

Conversely, let $\gamma_{=k}(G) = n/2$. From Corollary 9.2 and Observation 9.3, we deduce that $D_k(G) = H^* \circ K_1$ for a connected graph H^* with $n(H^*) = n/2$. Let $r = n/2$ and let $H = \{h_1, h_2, \ldots, h_r\}$ be the vertex set of H^*. For every $i \in \{1, 2, \ldots, r\}$ let q_i be the unique vertex in $V(G) \setminus H$ adjacent to h_i in $D_k(G)$ and let $Q = \{q_1, q_2, \ldots, q_r\}$. Then $Q = V(G) \setminus H$. Now, we obtain two obvious but important statements.

(S1) Every vertex in Q has distance k to exactly one vertex in G and this unique vertex lies in H.

(S2) There are no two vertices in Q with distance k to the same vertex.

Next we use some further definitions.

Definition 1: For any $i \in \{1, 2, \ldots, r\}$ let F_i be the component of $G - h_i$ which contains q_i. We define the set $V_i = V(F_i) \cup \{h_i\}$ and the induced subgraph $G_i = G[V_i]$. For $v = q_i \in Q$ we also use the notation $G(v) = G_i$.

Definition 2: For any path $P = v_0 v_1 \ldots v_m$ in G with $v_0 \in Q$ and $v_m \in H$ we define $i_0(P) = \max\{i \in \{0, 1, \ldots, m\} \mid v_j \in Q \ \forall\ 0 \leq j \leq i\}$ and $i_1(P) = \min\{i \in \{0, 1, \ldots, m\} \mid v_j \in H \ \ \forall\ i \leq j \leq m\}$.

Without loss of generality, we consider an arbitrary vertex $q \in Q$. Let $h \in H$ be the unique vertex with $d(q, h) = k$ and let $P = v_0 v_1 \ldots v_k$ be a shortest path from $v_0 = q$ to $v_k = h$.

Definition 3: We define $G_0 = G(q)$ and $i_0 = i_0(P)$ just as $i_1 = i_1(P)$. Furthermore, let $N_0 = \{q\}$ and for every integer $j \geq 1$ let

$$N_j = N_j(q) \setminus N_{j-1}(q) = \{x \in V(G) \mid d(x, q) = j\} \text{ and } n_j = |N_j|.$$

It is our aim to show that q and h belong to an induced path of G of order $2k$, and only one endvertex of this path has neighbours outside the path. In order to prove this, we verify the following three statements.

I) $n_{k+i} = 1$ for every $0 \leq i \leq i_0$.

II) G_0 is a path of $2k - i_0$ vertices ($G_0 \cong P_{2k-i_0}$).

III) The vertices of G_0 together with the i_0 vertices in $\bigcup_{i=1}^{i_0} N_{k+i}$ induce a path P_{2k} of order $2k$ and only the single vertex in N_{k+i_0} is adjacent to vertices outside this path.

For this purpose we prove some claims.

Claim 1: For every $i \in \{1, 2, \dots, r\}$,

a) $d(q_i, x) < k$ for all $x \in V(G_i) \setminus \{h_i\}$,
$d(q_i, h_i) = k$,
$d(q_i, x) > k$ for all $x \notin V(G_i)$, and

b) $V(G_i) \subseteq V(G_j) \setminus \{h_j\}$ for all $q_j \in V(G_i) \cap Q$ with $h_j \notin V(G_i)$.

Proof of Claim 1. Let $i \in \{1, 2, \dots, r\}$ be arbitrary.
a) By definition, $d(q_i, h_i) = k$. For every $x \notin V(G_i)$, the vertex $h_i \neq x$ belongs to every path from q_i to x, in particular to a shortest path. Therefore, $d(q_i, x) > k$. For the first inequality of a), let $x \in V(G_i) \setminus \{h_i\}$. Here, x belongs to the same component F_i of $G - h_i$ as q_i. Suppose that $d(q_i, x) \geq k$. Then every path from q_i to x contains a vertex with distance k to q_i and this vertex must be h_i. This is a contradiction.
b) Let $q_j \in V(G_i) \cap Q$ with $h_j \notin V(G_i)$ and let $y \in V(G_i)$ arbitrarily. There is a path in G_i from q_j to y, such that this path does not contain h_j. Hence, q_j and y belong to the same component F_j of $G - h_j$ and $y \in V(G_j) \setminus \{h_j\}$. Since $y \in V(G_i)$ is arbitrary, we obtain that $V(G_i) \subseteq V(G_j) \setminus \{h_j\}$ and the proof of this claim is complete.

We define $s = \max\{d(h, x) \mid x \notin V(G_0)\}$. By Claim 1a), $V(G_0) = \bigcup_{j=0}^{k} N_j$ and $V(G) \setminus V(G_0) = \bigcup_{j=k+1}^{k+s} N_j$. Next we show that $s \geq k$.

Since $D_k(G)$ and H^* are connected, h has distance k in G to a vertex $h^* \in H$. Let $q^* \in Q$ with $d(q^*, h^*) = k$.

Claim 2: $s \geq k$.

Proof of Claim 2. Suppose $s < k$. We consider two cases.
Case 1: Let $s = 0$. Then $G_0 = G$ and for every two vertices $x, y \in V(G) \setminus \{h\}$ the distance $d(x, y) \leq d(x, q) + d(q, y) \leq 2(k-1) < \operatorname{diam}(G)$. Hence, we have $x = h$ or $y = h$ for all $x, y \in V(G)$ with $\operatorname{diam}(G) = d(x, y)$. Let $x \in V(G)$ be arbitrary with $\operatorname{diam}(G) = d(x, h)$. Since $2k - 1 \leq \operatorname{diam}(G) = d(x, h) \leq d(x, q) + d(q, h) \leq 2k - 1$, the diameter $\operatorname{diam}(G) = 2k - 1$ and $d(x, q) = k - 1$, such that there is a shortest path P' from h to x, which contains q. The vertex $q^* \in V(G_0) = V(G)$ which implies that $d(p^*, q) < k = d(q^*, h^*)$ and so there is a path from q^* to q without h^*. Since the only vertex in P' with distance k to h is the vertex q but $d(h, h^*) = k$, we obtain that $h^* \notin V(P')$ and there are paths from q^* to x and to h without h^*. Hence, the two vertices x and h lie in $G(q^*)$ which leads to the contradiction that $d(x, h) \leq d(x, q^*) + d(q^*, h) \leq 2(k-1) < \operatorname{diam}(G)$.
Case 2: Let $1 \leq s < k$. If we have $V(G) \setminus V(G_0) \subseteq H$, then there is a vertex $h_s \in N_{k+s} \subseteq H$ and a vertex $q_s \in Q \subseteq V(G_0)$ with $d(q_s, h_s) = k = d(q_s, h) + s$. By Claim 1b), we obtain that $V(G_0) \subseteq V(G(q_s))$. Furthermore,

we get $d(q_s, x) \leq d(q_s, h) + s = k$ for every $x \in V(G) \setminus V(G_0)$ which implies $V(G) \subseteq V(G(q_s))$ and $G(q_s) = G$. Thus, we obtain the same contradiction as in Case 1. Hence, there is at least one vertex $q' \in (V(G) \setminus V(G_0)) \cap Q$. Let $d = d(h, q')$. We get $0 < d \leq s$ and $d(q', x) = k$ for every $x \in V(G_0)$ with $d(h, x) = k - d$. Therefore, in view of (S1), the vertex v_d is the only one in $V(G_0)$ with distance $k - d$ to h, and v_d belongs to every path from q to h. By the hypothesis $s < k$, it follows that $h^* \in V(G_0)$ and v_d belongs also to every path from h to h^*. Hence, we obtain that $d(h^*, v_d) = d$. Obviously, the vertex q^* is contained in $V(G_0)$ and there exists a path from q^* to q without h^*. Since we have $d(v, h) < k = d(h^*, h)$ for every $v \in V(P) \cap H$, we know that $h^* \notin V(P)$ and there is also a path from q^* to h without h^*. If $d(q^*, h) \geq k - s$, there is a vertex outside $V(G_0)$ with distance k to q^*, which is a contradiction to (S1). This leads to $d(q^*, h) < k - s \leq k - d$, and no shortest path from h to q^* contains v_d. Since v_d belongs to every path from h to h^*, it also belongs to every path from q^* to h^*. Therefore, we obtain that $d(q^*, v_d) = k - d(h^*, v_d) = k - d$. Since v_d belongs to every path from h to q, it also belongs to every path from q^* to q. This leads to the contradiction $d(q^*, q) = d(q^*, v_d) + d(v_d, q) = k$.

We observe that Claim 2 requires the condition $\operatorname{diam}(G) \geq 2k - 1$.

By $s \geq k$, we know that $n_{k+i} \geq 1$ for all $0 \leq i \leq k$. Since $v_i \in Q$ and $d(v_i, x) = k$ for every $x \in N_{k+i}$ and $0 \leq i \leq i_0$, we have $n_{k+i} = 1$ for every $0 \leq i \leq i_0$, by (S1). Hence, the proof of I) is completed. Next, we show $i_1 = i_0 + 1$.

According to (S2), we get $N_{2k} \subseteq H$. Let $h^* \in N_{2k}$ and let $W = w_0 w_1 \dots w_k$ be a shortest path from $h = w_0$ to $h^* = w_k$. Then $q^* \notin V(G_0)$, $w_j \in N_{k+j}$ and $d(v_j, w_j) = k$ for all $0 \leq j \leq k$.

Claim 3: $\bigcup_{j=k}^{2k} N_j \subseteq H$.

Proof of Claim 3. Suppose there is a vertex $x_i \in N_{k+i} \cap Q$ for some i with $0 \leq i \leq k$. Then v_i is the unique vertex in G with distance k to x_i. By (S1), $i \neq 0$, and by (S2), $i \neq k$. Since every path from x_i to a vertex with distance greater or equal k contains v_i, all vertices in $V(G) \setminus V(G_0)$ are within distance $k - 1$ to x_i, in particular $q^* \in V(G(x_i))$. Hence, there is a path from q^* to x_i without h^*. By $d(q, x_i) = k + i < d(q, h^*)$, there exists a path from q to x_i without h^*. Thus, we obtain a path from q^* to q without h^*, and $d(q, q^*) \leq k$, which is a contradiction to $q^* \notin V(G_0)$.

Claim 4: $i_1 = i_0 + 1$.

Proof of Claim 4. Suppose that $i_1 > i_0 + 1$. Since $v_{i_0+1} \in H$ and $v_{i_1-1} \in Q$, we conclude that $i_1 > i_0 + 2$ and that there exists an index $t \in \{i_0 + 1, \dots, i_1 - 2\}$ with $v_t \in H$ and $v_{t+1} \in Q$. Let q_t be the unique vertex in Q with $d(q_t, v_t) = k$.

In view of Claim 3, we know that $q_t \in V(G_0)$. Since $d(v_{t+1}, w_{t+1}) = k$ and $v_{t+1} \in V(G_0) \cap Q$, it follows from Claim 1b) that $V(G_0) \subseteq V(G(v_{t+1})) \setminus \{w_{t+1}\}$. In particular, $d(v_{t+1}, q_t) < k$. Hence, there is a path from q_t to v_{t+1} without v_t and consequently, there are paths from q_t to h and via h to h^* without v_t. Thus, $k > d(q_t, h^*) = d(q_t, h) + k$, which is a contradiction.

To prove II), we first show that $P_{2k-i_0} \subseteq G_0$. For this purpose, let u be the unique vertex in Q with $d(u, v_{i_1}) = k$ and let $U = u_0 u_1 \dots u_k$ be a shortest path from $u = u_0$ to $v_{i_1} = u_k$ with maximum cardinality of the intersection $V(U) \cap V(P)$. By Claim 3, we deduce that $u \in V(G_0)$, and this implies that $V(U) \subseteq V(G_0)$. The next claim shows that the vertices in U and P form a path $P_{2k-i_0} \subseteq G_0$.

Claim 5:

a) $\{v_{i_1+i} \mid 1 \le i \le k - i_1\} \cap V(G(u)) = \emptyset$,

b) $d(u_i, v_{i_1+i}) = k$ for all $0 \le i \le k - i_1$,

c) $v_i = u_{k-i_1+i}$ for all $0 \le i \le i_1$,

d) $\{u_0, u_1, \dots, u_{k-1}\} \subseteq Q$.

Proof of Claim 5. a) If $i_1 = k$, then the set $\{v_{i_1+i} \mid 1 \le i \le k - i_1\}$ is empty and it remains nothing to prove. Let $i_1 < k$. Suppose that $v_{i_1+i} \in V(G(u))$ for some $i \in \{1, 2, \dots, k - i_1\}$. Then $h, h^* \in V(G(u))$ and we obtain the contradiciton that $k \ge d(u, h^*) = d(u, h) + d(h, h^*) > k$.
b) By a), $d(u_i, v_{i_1+i}) = d(u_i, u_k) + d(v_{i_1}, v_{i_1+i}) = k$ for all $0 \le i \le k - i_1$.
c) By a), we have $V(U) \cap V(P) \subseteq \{v_0, v_1, \dots, v_{i_1}\}$. Since $d(v_i, v_{i_1}) \le i_1 < d(u_j, v_{i_1})$ for every $0 \le i \le i_1$ and $0 \le j \le k - i_1 - 1$, we get $V(U) \cap V(P) \subseteq \{u_{k-i_1}, u_{k-i_1+1}, \dots, u_k\}$. For every $0 \le i, j \le i_1, i \ne j$, we deduce $v_i \ne u_{k-i_1+j}$, by $d(v_i, u_k) \ne d(u_{k-i_1+j}, u_k)$. Suppose that $v_i \ne u_{k-i_1+i}$ for some $i \in \{0, 1, \dots, i_1\}$. Then, $V(U) \cap V(P) \subseteq \{v_0, v_1 \dots, v_{i_1}\} \setminus \{v_i\}$. If $v_0 = u_{k-i_1}$, then the path $U' = u_0 u_1 \dots u_{k-i_1} v_1 \dots v_{i_1}$ is a shortest path from u to v_{i_1} and $|V(U') \cap V(P)| = |\{v_0, v_1 \dots, v_{i_1}\}| > |V(U) \cap V(P)|$, which is a contradiction to the maximality of $|V(U) \cap V(P)|$. Hence, $v_0 \ne u_{k-i_1}$. Therefore, we deduce $u_{k-i_1} \in H$ from b) and (S2). Let q_{k-i_1} be the unique vertex in Q with distance k to u_{k-i_1}. By Claim 3, $q_{k-i_1} \in V(G_0)$ and there is a path from q_{k-i_1} to q without u_{k-i_1}. As shown above we have $u_{k-i_1} \notin \{v_1, v_2, \dots, v_k\}$. Hence, there is a path from q_{k-i_1} to h and to h^* without u_{k-i_1}. This leads to the contradiction $k \ge d(q_{k-i_1}, h^*) = d(q_{k-i_1}, h) + d(h, h^*) > k$.
d) By c), $u_{k-1} = v_{i_0} \in Q$ and $i_1(U) = k$ (cf. Definition 2). Since Claim 4 is valid for any arbitrary vertex $q \in Q$ and for any arbitrary shortest path P from q to h, it is also valid for the vertex $u \in Q$ and the path U from u to

v_{i_1}. Hence, we get that $i_0(U) = i_1(U) - 1 = k - 1$ and $\{u_0, u_1, \ldots, u_{k-1}\} \subseteq Q$ which verifies d).

Claim 5 implies that $V(U) \cap V(P) = \{v_0, v_1, \ldots, v_{i_1}\}$ and $u_0 u_1 \ldots u_{k-1} v_{i_1} \ldots v_k$ is a path of order $2k - i_0$ and a subgraph of G_0. In order to show that G_0 is equal this path of order $2k - i_0$ (II), we use the following claim.

Claim 6:

a) Every vertex $q' \in Q$ with $d(q', h') = k$ for some $h' \in (V(G_0) \cap H) \setminus V(P)$ satisfies $q' \in (V(G_0) \cap Q) \setminus V(U)$.

b) $d(q', x) > k$ for every $q' \in (V(G_0) \cap Q) \setminus V(U)$ and $x \in V(U) \cup V(P)$.

Proof of Claim 6. a) Let $h' \in (V(G_0) \cap H) \setminus V(P)$ arbitrarily and let $q' \in Q$ with $d(q', h') = k$. Claim 3 implies that $q' \in V(G_0)$. Suppose that $q' \in V(U)$. By Claim 5 c) and d), we obtain that the intersection $V(U) \cap Q$ is equal the union $\{u_i \,|\, 0 \le i \le k - i_1\} \cup \{v_i \,|\, 0 \le i \le i_0\}$. From Claim 5 b) we deduce that, if $q' \in \{u_i \,|\, 0 \le i \le k - i_1\}$, then q' has distance k to a vertex in $\{v_{i_1+i} \,|\, 0 \le i \le k - i_1\}$ besides h'. If $q' \in \{v_i \,|\, 0 \le i \le i_0\}$, then q' has distance k to a vertex in $\{w_i \,|\, 0 \le i \le i_0\}$ besides h'. Both cases contradict (S1).
b) Let $q' \in (V(G_0) \cap Q) \setminus V(U)$ and let h' be the unique vertex with distance k to q'. Suppose that $h' \notin V(G_0)$. This implies $d(q', h') = d(q', h) + d(h, h')$ and $u \in V(G_0) \subseteq V(G(q'))$, by Claim 1b). We deduce $h' \notin \{w_i \,|\, 0 \le i \le i_0\} = \bigcup_{i=0}^{i_0} N_{k+i}$ from (S2) and I). Thus, $d(h, h') > i_0$ and $d(q', h) < k - i_0$. Since $q' \in V(G(u))$ and, by Claim 5a), $h \notin V(G(u))$ or $h = u_k$, we know that $d(q', h) = d(q', u_k) + d(u_k, h) \ge 1 + (k - i_1) = k - i_0$, which is a contradiction. Hence, we get $h' \in V(G_0) \cap H$. Suppose that $h' \in \{v_{i_1+i} \,|\, 0 \le i \le k - i_1\} = (V(U) \cup V(P)) \cap H$. Then from Claim 5b), we obtain that q' and a vertex $u' \in \{u_i \,|\, 0 \le i \le k - i_1\} \subseteq Q$ have distance k to h', which is a contradiction to (S2). This yields $h' \notin (V(U) \cup V(P))$. Suppose that there is a vertex $x \in V(U) \cup V(P)$ with $d(q', x) \le k$. Then there exists a path from q' to x without h' and a second path from x via h to h^* without h'. Hence, we get $h, h^* \in V(G(q'))$ and $k \ge d(q', h^*) = d(q', h) + d(h, h^*) > k$, a contradiction.

Claim 7: $G_0 = P_{2k-i_0}$.

Proof of Claim 7. Since $d(q', q) < k$ for all $q' \in V(G_0) \cap Q$, it follows from Claim 6b) that $(V(G_0) \cap Q) \setminus V(U) = \emptyset$. This together with Claim 6a) implies that $(V(G_0) \cap H) \setminus V(P) = \emptyset$. Hence, $V(G_0) \cap Q = V(U) \cap Q$ and $V(G_0) \cap H = V(P) \cap H$. Thus, $V(G_0) = V(U) \cup V(P)$ and $u_0 u_1 \ldots u_{k-1} v_{i_1} \ldots v_k = P_{2k-i_0}$ is a spanning subgraph of G_0. By Claim 5b), there is no edge in G_0 which does not belong to U or P. Consequently, $G_0 = P_{2k-i_0}$.

It remains to prove III). By I), we have $N_{k+i} = \{w_i\}$ for all $0 \le i \le i_0$, which implies that III) is equivalent to Claim 8.

Claim 8:

a) $G[V(G_0) \cup \{w_i \mid 0 \le i \le i_0\}] = P_{2k}$

b) Every vertex $x \in (V(G_0) \setminus \{h\}) \cup \{w_i \mid 0 \le i < i_0\}$ satisfies $N(x, G) \subseteq (V(G_0) \cup \{w_i \mid 0 \le i \le i_0\})$.

Proof of Claim 8. By II) and by the definition of G_0, we obtain that $G_0 = P_{2k-i_0}$ is an induced subgraph of G and $N(x, G) \subseteq V(G_0)$ for all $x \in (V(G_0) \setminus \{h\})$. If $i_0 = 0$, then we are done. Let $i_0 > 0$. By I) and by Definition 3, the vertices $w_0, w_1, \dots, w_{i_0}$ form an induced path, and $N(h, G) = \{v_{k-1}, w_1\}$ and $N(w_i, G) = \{w_{i-1}, w_{i+1}\}$ for all $1 \le i < i_0$. Thus, $G[V(G_0) \cup \{w_i \mid 0 \le i \le i_0\}] = u_0 u_1 \dots u_{k-1} v_{i_1} \dots v_k w_1 \dots w_{i_0} = P_{2k}$ and w_{i_0} is the only vertex in this path, which may have neighbours outside the path. This completes the proof of this last claim.

So, we have shown III). Hence, every arbitrary vertex $q \in Q$ and the unique vertex $h \in H$ with distance $d(q, h) = k$ belong to an induced path P_{2k} and only the one endvertex of this path lying in the set H has neighbours outside this path. All these endvertices together induce a graph G' in G. Thus, every vertex in G' is an endvertex of at least one pendant induced path of order $2k$. Suppose that a vertex $x \in V(G') \subseteq H$ belongs to at least two such paths. Each of those paths contains a vertex in Q with distance k to x, which is a contradiction to (S2). We conclude that $G = G' \diamond P_{2k-1}$, and the proof of Theorem 9.6 is complete. □

9.2 Observations, corollaries and examples

Observation 9.7 Theorem 9.6 yields the main case $\text{diam}(G) \neq 2$ of Theorem 9.4. The case $\text{diam}(G) = 2$ of Theorem 9.4 follows immediately from Corollary 9.2 and Observation 9.3.

Observation 9.8 (Fischermann and Volkmann [42]) In Theorem 9.6 it is possible to replace the condition '$D_k(G)$ is connected' by the condition '$D_k(G)$ has no component isomorphic to C_4' (see the proof below). The hypothesis $\text{diam}(G) \ge 2k - 1$ ensures that $D_k(G)$ has no isolated vertices and consequently, Lemma 9.1 yields that $\gamma_{=k}(G)$ still has the upper bound $n(G)/2$. By Observation 9.3, the condition that $D_k(G)$ is connected implies for every $k \ge 2$ that $D_k(G)$ has no component isomorphic to C_4. Thus, the new condition is weaker than the old one.
Furthermore, with this new condition and for $k = 1$ Theorem 9.6 coincides with Theorem 7.1.

Proof that Theorem 9.6 holds with this weaker condition.
$\gamma_{=k}(G) = n/2$ follows immediately from $G = G' \diamond P_{2k-1}$ without any condition on $D_k(G)$. Hence, we only need the connectedness of $D_k(G)$ to show that $G = G' \diamond P_{2k-1}$ follows from $\gamma_{=k}(G) = n/2$. In the preceding proof this condition is used twice. Firstly in the beginning to show that $D_k(G) = H^* \diamond P_1$, and secondly after Claim 1, where we use that H^* has no isolated vertices.
Let $\gamma_{=k}(G) = n/2$. The condition that $D_k(G)$ has no component isomorphic to C_4 is sufficient to verify $D_k(G) = H^* \diamond P_1$, by $rad(G) \geq \lceil \text{diam}(G)/2 \rceil \geq k$ and by Corollary 9.2 Notice that H^* no longer has to be connected. Now, we can continue the proof of Theorem 9.6 till Claim 1 inclusive. Here we have to distinguish two cases. The first case, that H^* has no isolated vertices, is shown in Section 4. The second case, that H^* has at least one isolated vertex, leads immediately to $G = P_{2k} = P_1 \diamond P_{2k-1}$ as the following shows.
Since q is arbitrary, let q be a vertex in Q with distance k to an isolated vertex h in H^*. P is a shortest path from q to h. We define $A = \{x \in V(G) \,|d(x,h) \geq k\}$ just as $C = \{x \in V(G) \,|\, d(x,q) \geq k\}$ and $B = \{x \in V(G) \,|\, d(x,h) \leq k, d(x,q) \leq k\}$. Obviously, $d(q,x) < k$ for all $x \in A$ and $d(h,x) < k$ for all $x \in C$. Hence, $d(x,y) \leq 2k-2$ for all vertices $x,y \in V(G) \setminus A$ or $x,y \in V(G) \setminus C$. Let a and c be arbitrary with $d(a,c) = \text{diam}(G)$. Then, without loss of generality, $a \in A$ and $c \in C$ and $2k-1 \leq d(a,c) \leq 3k-2$. Let $q^* \in Q$, $q^* \neq q$ and let h^* be the unique vertex with distance k to q^*. Suppose that $q^*, h^* \in B$. Since q^* is within distance $k-1$ from q and h, no path from q^* to a vertex in A or C contains h^* and $d(a,c) \leq d(a,q^*) + d(q^*,c) \leq 2(k-1)$, which is a contradiction. Thus, no pair q^*, h^* is in B. If $q^* \in Q \setminus B$, then q^* has distance k to a vertex in P and this vertex is h^*. Hence, no pair q^*, h^* is in $V(G) \setminus B$. This implies, that there are no two different vertices in A or in C, respectively, with the same distance to q or to h, respectively. On the other hand, we deduce that $B = V(P)$. Therefore, G is a path of $n = \text{diam}(G) + 1$ vertices and $2k \leq n \leq 3k-1$. Let $G = x_1 x_2 \ldots x_n$ and suppose $n > 2k$. Then both vertices x_1 and x_{2k+1} have distance k to x_{k+1} and to no other vertex. This contradicts $D_k(G) = H^* \diamond P_1$. Hence, $G = P_{2k}$. □

Corollary 9.9 *Let $k \geq 2$ be an integer. For every integer $d = 2k-1$ and $d \geq 4k-1$ there exist connected graphs G of order n and diameter d such that $\gamma_{=k}(G) = n/2$ and $D_k(G)$ has no component isomorphic to C_4. But there exists no such graph G of diameter d where $2k-1 < d < 4k-1$.*

With the support of Observation 9.8 it is easy to see, that the characterization of Theorem 9.6 holds without any condition on the exact distance-k graph $D_k(G)$ if the diameter of the graph G is large enough. The reason for this is that the diameter of a connected graph G is at most $2(k-1)+2k$ if the exact distance-k graph of G has at least one component isomorphic to C_4.

Corollary 9.10 (Fischermann and Volkmann [42]) *Let $k \geq 2$ be an integer and let G be a connected graph of order n such that $diam(G) \geq 4k-1$. Then $\gamma_{=k}(G) = n/2$ if and only if $G = G' \diamond P_{2k-1}$ for any connected graph G'.*

The following three examples show that Corollary 9.10 (and Theorem 9.6 without a condition on the graph $D_k(G)$) is not valid for graphs of diameter less than $4k-1$.

Example 9.11 Consider graphs G of the following three types. Let $k \geq 2$ be an arbitrary integer.

1) $G = C_{4k}$.

2) G consists of two disjoint paths $x_0x_1 \dots x_k$ and $y_0y_1 \dots y_k$, along with the three additional edges $x_0y_0,\ x_0y_1,\ x_1y_0$ (cf. Figure 9.1 a)).

3) G consists of two disjoint paths $x_0x_1 \dots x_{k+1}$ and $y_0y_1 \dots y_{k+1}$, along with the six additional edges $x_0y_0,\ x_0y_1,\ x_0y_2,\ x_1y_0,\ x_1y_1,\ x_2y_0$ (cf. Figure 9.1 b)).

All these graphs have diameter $2k$ or $2k+1$ (less than $4k-1$) and the exact distance-k graphs contain at least one component isomorphic to C_4, and the remaining components are isomorphic to $P_2 = P_1 \diamond P_1$. By Corollary 9.2, we deduce $\gamma_{=k}(G) = n(G)/2$, but these graphs obviously do not satisfy $G = G' \diamond P_{2k-1}$.
Analogous to 2.) and 3.) it is possible to construct such graphs up to diameter $3k-1$ for every $k \geq 2$.

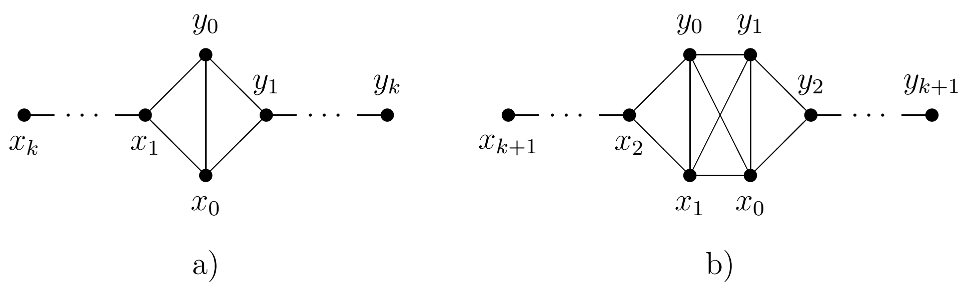


Figure 9.1

Now we give some examples of graphs G where G and $D_k(G)$ are connected, $\gamma_{=k}(G) = n(G)/2$, but $\mathrm{diam}(G) < 2k-1$ and consequently, $G \neq G' \diamond P_{2k-1}$.

Example 9.12

1) Let $k \geq 2$ arbitrarily. We consider the graph G consisting of a cycle $C_{2k-1} = x_1x_2 \ldots x_{2k-1}x_1$, the additional vertices $y_1, y_2, \ldots, y_{2k-1}$ and the additional edges y_ix_i, y_ix_{i+1} for every $1 \leq i \leq 2k-2$ and $y_{2k-1}x_{2k-1}$, $y_{2k-1}x_1$. This graph has diameter k and $D_k(G) = C_{2k-1} \diamond P_1$. Hence, G and $D_k(G)$ are connected and $\gamma_{=k}(G) = n/2$.

2) Let $k \geq 3$ arbitrarily. The graph $G = C_{2k-2} \diamond P_1$ has diameter $k+1$. If k is odd, then $D_k(G) \cong G$ and consequently G and $D_k(G)$ are connected and $\gamma_{=k}(G) = n/2$. If k is even, then $D_k(G)$ is disconnected, but it consists of two components isomorphic to $C_{k-1} \diamond P_1$, so that again $\gamma_{=k}(G) = n/2$.

3) Let $k = 3$ and let $t \geq 3$ be an arbitrary integer. For $i = 1, 2$ let G_i be a bipartite graph with partite sets $X_i = \{x_{i1}, x_{i2}, \ldots, x_{it}\}$ and $Y_i = \{y_{i1}, y_{i2}, \ldots, y_{it}\}$ and edges $x_{ir}y_{is}$ for all $1 \leq r, s \leq t$ with $r \neq s$. Consider the graph G consisting of the disjoint union of G_1 and G_2 and of additional edges connecting every vertex of X_1 with every vertex of X_2. This graph has order $n = 4t$, diameter $k = 3$ and $D_3(G) = K_{t,t} \diamond P_1$. Hence, G and $D_3(G)$ are connected and $\gamma_{=3}(G) = n/2$.

Observation 9.13 From Corollary 9.9 and Examples 9.12 we deduce that there are connected graphs G of diameter d where $D_k(G)$ has no component isomorphic to C_4 and $\gamma_{=k}(G) = n(G)/2$ for $d = k$, $k+1 (k \geq 3)$, $2k-1$ and for $d \geq 4k-1$, but not for $2k-1 < d < 4k-1$. Hence, for $k = 2, 3$ and for any given diameter we either can construct such a graph or there does not exist such a graph. For $k \geq 4$ it remains unsolved, whether there are such graphs G of diameter d where $k+1 < d < 2k-1$.

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