Stabilizability, Controllability and Optimal Strategies of Linear and Nonlinear Dynamical Games

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Diplom-Mathematiker M.Sc. in Mechanical Engineering(H)

Gábor Kun

aus Budapest (Ungarn).

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Berichter: Universitätsprofessor Dr. Gerhard Jank
Associate Professor Dr. Fátima Silva Leite

Zusammenfassung

In den letzten Jahren ist die Kontrolltheorie eine wichtige Brücke geworden, die eine gemeinsame mathematische Beschreibung von technischen, wirtschaftlichen und biologischen Prozessen ermöglicht. Hierbei werden offene dynamische Systeme, das heißt Systeme, die durch die Umgebung beeinflußt werden, untersucht.

Ein Problem der Kontrolltheorie besteht darin, einen (nicht stabilen) Gleichgewichtszustand durch Anwendung geeigneter Steuerfunktionen zu stabilisieren. Darüberhinaus versucht man im Rahmen der optimalen Kontrolltheorie eine solche Steuerfunktion zu wählen, die zusätzlich ein gegebenes *Kostenfunktional* minimiert. In der Spieltheorie werden mehrere (konkurrierende) Kostenfunktionale gegeben, und die Spieler (also die äußeren Einflüsse) versuchen, ein dynamisches Gleichgewicht zu erreichen.

In dieser Arbeit werden wir neben der optimalen Kontrolltheorie, die dynamische Spieltheorie mit Hilfe linearer und nichtlinearer differentieller Systeme kennenlernen. Im Kapitel 1 werden die wichtigsten Resultate der linearen Kontrolltheorie vorgestellt, wobei auch für einige klassische Behauptungen geänderte Beweise und weiterführende Folgerungen angegeben werden. In diesem Kapitel werden die Werkzeuge, die später benutzt werden, erläutert. Kapitel 2 befaßt sich mit der Theorie linearquadratischer Nash Spiele. Neben den bekannten Sätzen über die optimale Kontrolle, die auch hier mit weiteren Bedingungen über die Existenz und Eindeutigkeit ergänzt werden, beschäftigen wir uns hier mit den Fragen der Steuerbarkeit und Stabilisierbarkeit von Spielen. Im dritten Kapitel werden Riccati Gelichungen kurz vorgestellt, und es wird auch über Approximationsmethoden gesprochen. Danach, im Kapitel 4, findet eine Untersuchung gestörter Spiele statt. Dies bedeutet, dass das System neben den optimalen Kontrollen der Spieler, auch von einem geräuschartigen Signal beeinflußt wird. Hierbei werden Strategien angegeben, die den Einfluss dieses Signals maximal unterdrücken. Untersucht werden auch die Existenz und Eindeutigkeit dieser Strategien. Schließlich wird im fünften Kapitel die Frage beantwortet, unter welchen Annahmen ein Spiel auf endlichem Zeithorizont dieselbe stabilisierenden Eigenschaften besitzt, wie ein auf unendlichem Horizont.

Kapitel 6,7,8 dieser Arbeit ist Systemen auf Mannigfaltigkeiten gewidmet. Nach einer kurzen Einführung werden im Kapitel 6 Kontrollsysteme, insbesondere invariante Kontrollsysteme über Lie Gruppen auf Steuerbarkeit und ganz kurz auch auf Stabilisierbarkeit untersucht. Danach wird eine Methode gezeigt, wie man bei steuerbaren nichtlinearen Kontrollsystemen einen optimalen Weg findet. Solche Kurven werden als "splines" bezeichnet. Es wird auch ein numerisches Verfahren zur erzeugung solcher Kurven auf verschiedenen Lie Gruppen vorgestellt und untersucht. Schließlich werden im Kapitel 8 nichtlineare differentielle Spiele auf Lie Gruppen vorgestellt und es wird die Existenz von Nash-Strategien mit und ohne Randwertproblemen untersucht.

Nagyapám emlékére

I state, that in every branch of the natural science we can only find as much actual science, as much mathematics it contains.

Imannuel Kant (1724-1804)

Dear Reader,

first of all, let me thank you heartily for reading my Thesis. The biggest appreciation for every author is when people read his works, think about it and maybe even discuss it. Of course the same holds for scientific publications and hence if you have any kind of question, comments or argument concerning this Thesis, I would be happy to discuss about it. Please, feel free to contact me at Gabor.Kun@web.de .

Since mathematics is not really the subject, where emotions can be expressed, I write here about the feelings I had during the preparation of this work. So please, if you don't like emotions, or don't care about mines, skip this section. You won't miss anything.

Why is it so emotional to me to write down this Thesis ? I have many reasons for that. First, because it covers a part of my life, that has not only been scientifically fruitful. Our first child Bálint was born in October 1997, at the very beginning of this project, whereas our son András in June 2000, which indeed coincides with the end of it.

My second reason is that during the time, I was preparing this Thesis, I somehow thought a lot about my teachers, to whom I'm really indebted to for introducing me into some chapters of mathematics. A never-ending list, I must say.

My first and surely greatest teacher was *László Bánhengyi* in the Dániel Berzsenyi Secondary School in Budapest. He showed me the first steps: the foundation, but he also showed me how to build on that. He was the one who taught me what mathematics is all about. At the Technical University Budapest *Erzsébet Horváth*, *Ádám Bosznay* and *Árpád Nágel* independently proved me that there is applied mathematics and that *there are applied mathematicians*. Unfortunately, their proofs were non-constructive and hence I still had to wait for an example.

After leaving the TU Budapest, it was *Gerhard Jank* (my master), who – together with *Gerhard Freiling* – introduced me into the wonderful world of linear control theory, noncooperative dynamical game theory and Riccati-equations. I also profited a lot from his lectures on complex function theory. Chapters in nonlinear control theory and spline-interpolation I learned from *Fátima Silva Leite* and *Peter Crouch*, while we were working together on

those problems. Essentially everything what I know about control theory on Lie groups and Riemannian manifolds originate from this joint project. Lately, it was *Vlad Ionescu*, who showed me that there is a way to treat the (robust) stabilizability of a control system in a mathematically exact manner. His lectures also helped me a lot to precisely understand the background and the limits of control systems on Hilbert spaces.

The third and last reason are all the people I met in the framework of this project. I found several real friends among them. Their collaboration was necessary to obtain the results presented here. For instance my whole research on moving horizon control policies was initiated and carried out in a joint research with *Bram van den Broek* from Tilburg. Another very important topic of this Thesis is the theory of splines curves on Riemannian manifolds. Although I learned a lot from the joint work with Fátima and Peter, the most important thing for me, which I'm very proud of, was that I met them and became friends with them. Through this friendship I also got to know *Yuri Sachkov*, who (without knowing about it) initiated and motivated my research on the controllability properties of games. And due to a recent collaboration with Prof. Schwarz at the University Duisburg, I began to work on the theory of nonlinear games. This topic was mainly motivated by two assistants of him: *Torsten Scholt* and *Jan Polzer*.

Nevertheless, no research project (not even a mathematical research) can survive without appropriate financial support. This support (and also a lot more than that) I became from Prof. Jongen and the colleagues at our department.

Finally, I should mention a real colleague and friend, *Dirk Kremer*, the third member of the 'Control Theory Group' in Aachen. Together with our master, we spent several Tuesday-afternoons on discussing results of our research to motivate each other for further and even nicer results. Just like the ancient Greeks, I believe.

I thank heartily to all my teachers and friends I mentioned above and I apologize to everyone I forgot to mention. However, the biggest and most special thanks goes to my family: my wife Krisztina and our sons Bálint and András. These last months have been a very hard time for everyone of us and without those love and appreciation I would have never done it. We did it together, indeed!

Contents

Chapter 1

Differential control systems on continuous time-horizon

The mathematical description for numerous technical, economical or biological processes leads to dynamical systems. The behavior of such processes is given by ordinary differential equations, where the state of the system is governed by the time and sometimes also by some input parameters. Here, we shall study those dynamical systems, whose state is not only determined by the time and events taking place inside, but also by outer influences.

A great number of additional questions can arise by studying such systems. For example:

- How must the influence be chosen to 'drive' the system from one state to another or along a predefined path ?
- How can the set of states that are reachable from a given state at a given time be characterized ?
- Assuming that there are some optimality constraints, how can we find an (in some sense) optimal control ?
- How can we describe systems with more than one influence ?
- What happens if the input signal is "noisy" ?

The aim of Control Theory is to study dynamical systems that are not isolated from their environment i.e. where the state of the system is not only governed by the time and by events taking place internally, but also by outer influences or controls. Such systems have received and still receive a lot of interest, since almost every technical and technological process can be regulated by some input parameters.

We control a system (machine, economical process, etc.) in order to achieve a desired behavior. In some cases it is enough if the inputs are such that a given output signal is achieved, but usually we require more, for instance low production costs or minimal times. This means that we also want to optimize our process in some sense.

In order to derive the theory of optimal control, we first need a mathematical description (see also Appendix A) of a control system and some basic definitions. Then, in Section 1.3 of this chapter we shall investigate these properties of the systems and finally, in Section 1.4 we introduce the theory for optimal control.

1.1 Preliminaries

In order to study the mathematical aspects of (usually technical) problems, we need an axiomatic approach. Here, we first give a general definition for a controlled dynamical system and then point out which restrictions we shall apply in this Thesis. We also define the most important properties, such as controllability, stabilizability and observability.

Definition 1.1 (control system) *We say that*

$$\Sigma = (\mathcal{T}, X, U, \mathcal{U}, Y, s, o)$$

is a control system*, if*

- $\mathcal{T}$ *is a nonempty subset of* $\mathbb{R}$*,*
- X*,* Y *and* U *are nonempty topological spaces,*
- $\mathcal{U}$ *is a nonempty subset of*
$$\{u | u : \mathcal{T} \to U\},$$
- s *is a mapping defined on a subset* $\mathcal{D}$ *of the set*
$$\{(t_0, t_1, x, u) \,| t_0, t_1 \in \mathcal{T}, t_0 \leq t_1, x \in X, u \in \mathcal{U}\}$$
into X *with the following properties:*
  (i) $\forall t_0 \in \mathcal{T}$ *and* $x \in X$ *there exists* $t_1 \in \mathcal{T}$ *with* $t_1 > t_0$ *and* $u \in \mathcal{U}$ *such that* $(t_0, t_1, x, u) \in \mathcal{D}$.
  (ii) $(t_0, t_0, x, u) \in \mathcal{D}$ *and it is* $s\,(t_0, t_0, x, u) = x \;\forall t_0 \in \mathcal{T}, x \in X$ *and* $u \in \mathcal{U}$.
  (iii) $\forall t_0, t_1 \in \mathcal{T}$ *with* $t_0 < t_1$ *and* $u, u^* \in \mathcal{U}$ *with* $u(t) = u^*(t)$*,* $\forall t \in [t_0, t_1] \cap \mathcal{T}$ *and for each* $x \in X$ *such that* $(t_0, t_1, x, u) \in \mathcal{D}$
$$(t_0, t_1, x, u^*) \in \mathcal{D} \text{ and}$$
$$s(t_0, t_1, x, u) = s(t_0, t_1, x, u^*)$$
  hold.
  (iv) $\forall t_0, t_1 \in \mathcal{T}$ $(t_0 < t_1)$ *and* $\forall x_0 \in X$*,* $u \in \mathcal{U}$ *with* $(t_0, t_1, x_0, u) \in \mathcal{D}$
$$(t_0, t, x_0, u) \in \mathcal{D} \text{ and}$$
$$(t, t_1, x, u) \in \mathcal{D}$$
  hold for each $t \in [t_0, t_1] \cap \mathcal{T}$ *with* $x = s(t_0, t, x_0, u)$.
  (v) $\forall t_0, t_1, t_2 \in \mathcal{T}$ *with* $t_0 \leq t_1 \leq t_2$ *and* $\forall x \in X$*,* $u \in \mathcal{U}$ *such that* $(t_0, t_2, x, u) \in \mathcal{D}$
$$s(t_1, t_2, x_1, u) = s(t_0, t_2, x, u) \text{ with } x_1 = s(t_0, t_1, x, u)$$
  holds.
- o *is a mapping from* $\mathcal{T} \times X \times U$ *into* Y*.*

Remark 1.1

- *The usual terminology for the objects appearing in Definition 1.1 is the following: $\mathcal{T}$ is called 'time-set' or 'time-horizon', X is called 'state space', U is called 'control-' or 'input-value space', $\mathcal{U}$ is called 'control space' or the set of control functions and finally, Y is called 'output-value space'. The mappings s and o are called 'system transfer' or 'transition' and 'output' or 'readout' mapping, respectively.*

- *If an initial state x_0 and a time t_0 are given, then the state of the system at time t is given for any admissible control function $u \in \mathcal{U}$ by*

$$x(t) = s(t_0, t, x_0, u). \tag{1.1}$$

Admissibility means hereby that $(t_0, t, x_0, u) \in \mathcal{D}$ holds. The output response of the system (using the same control and initial data) is then given by

$$y(t) = o(t, x(t), u(t)) = o(t, s(t_0, t, x_0, u), u(t)). \tag{1.2}$$

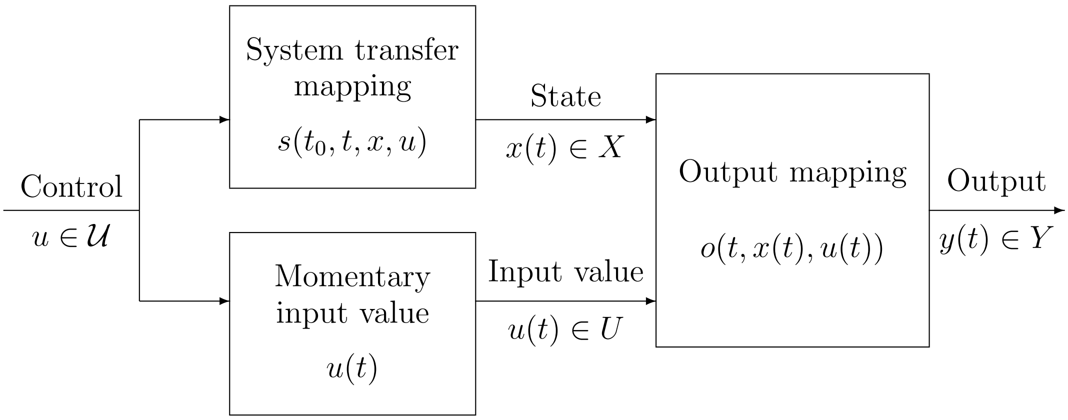

Figure 1.1: Control system

The properties of the state-transfer function s can be described as follows: Property (i) ensures that for each state the system can be controlled (at least for a short time). Property (ii) is just the definition of the initial state. Property (iii) means that the state depends deterministically on this initial state and on the control function during the time of the observation. Property (iv) is basically the connectivity of the admissible states in the time. This means that the system cannot arrive to an admissible state with a control function through some inadmissible ones. Finally, property (v) is the usual semi-group property for evolutionary systems.

Our first example illustrates a dynamical system, that doesn't fulfill Property (i) of Definition 1.1 and hence cannot be regarded as a control system:

Example 1: Let $\mathcal{T} = \mathbb{N}$, $X = \{(x_1, x_2) | x_1, x_2 \in \mathbb{Z}, x_1, x_2 \geq 0\}$, as well as

$$U = \left\{ \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$$

and $\mathcal{U} = \{u | u : \mathcal{T} \to U\}$. Furthermore, let

$$s(t_0, t_1, x, u) := x_0 + \sum_{t=t_0+1}^{t_1} u(t).$$

This leads to a system Σ that fulfills properties (ii)-(v), but not (i), since no admissible control exists if the current state is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Although discrete-time systems are very important and receive much interest in the literature, in the framework of this Thesis, we will only investigate control systems defined on continuous time-horizon:

Theorem 1.1 *Let $f : \mathcal{D}^* \to X$ be a mapping, where $\mathcal{D}^*$ is a subset of $\mathcal{T} \times X \times U$ such that the following properties hold:*

(i) For all $t \in \mathcal{T}$ and $x \in X$ there exists $u \in U$, such that $(t, x, u) \in \mathcal{D}^*$,

(ii) X and U are finite dimensional Euclidean spaces and $\mathcal{T}$ is a (bounded or unbounded) interval,

(iii) $f(t, x, u)$ is piecewise continuous in $u \in U$ and $t \in \mathcal{T}$ for each fixed $x \in X$ and finally

(iv) $f(t, x, u)$ is Lipschitz-continuous in $x \in X$ for each pair $(t, u) \in \mathcal{T} \times U$

Then, the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{1.3}$$

together with the initial-value problem

$$x(t_0) = x_0 \tag{1.4}$$

admits a unique solution for each $t_0 \in \mathcal{T}$, $x_0 \in X$ and piecewise continuous function $u : \mathcal{T} \to U$ if and only if $(t, x_0, u(t_0)) \in \mathcal{D}^*$. Furthermore, this solution can be maximally prolongated to the boundary of $\mathcal{T} \times X$.

If we then denote this maximally prolongated solution $x(t)$ of (1.3)-(1.4) for a given piecewise continuous function $u : \mathcal{T} \to U$ with $s(t_0, t, x_0, u)$ and the set of piecewise continuous mappings from $\mathcal{T}$ into U by $\mathcal{U}$, then $(\mathcal{T}, X, Y, U, \mathcal{U}, s, o)$ forms a control system (in the sense of Definition 1.1) for any output space Y and output mapping o.

Proof. The existence and uniqueness of x are direct consequences of Lindelöf's Theorem for ordinary differential equation (see [KK74] or [CL55]). Hence, we only show that s fulfills properties (i)-(v) from Definition 1.1.

Property (i) in Definition 1.1 follows from property (i) of Theorem 1.1 together with the fact that if $(t_0, x_0, \tilde{u}) \in \mathcal{D}^*$ then the solution of (1.3)-(1.4) exists for the constant control $u(t) \equiv \tilde{u}$ at least for a small neighborhood $[t_0, t_0 + \varepsilon) \in \mathcal{T}$. Property (ii) of Definition 1.1 is the initial condition (1.4). Properties (ii) and (v) of Definition 1.1 follow directly, hence x is the maximally prolongated unique solution of an ordinary differential equation. Finally, property (v) is also a consequence of the fact, that $x(t)$ is the unique solution of (1.3). □

Definition 1.2 (differential control system) *We call $\sigma = (\mathcal{T}, X, Y, U, \mathcal{U}, f, o)$ a finite dimensional differential control system if*

- $\mathcal{T}$ *is a real interval,*
- X, Y *and* U *are finite dimensional Euclidean spaces,*
- $\mathcal{U}$ *is a set of piecewise continuous functions from* $\mathcal{T}$ *into* U*,*
- $f : \mathcal{D}^* \subset \mathcal{T} \times X \times U \to X$ *fulfills the requirements of Theorem 1.1 and*
- $o : \mathcal{T} \times X \times U \to Y$ *is an output mapping.*

Definition 1.3 (infinite time horizon) *We say that a control system has* infinite time-horizon *if* $\sup \mathcal{T} = +\infty$*, i.e if* $\mathcal{T} = [t_0, +\infty)$ *for some* $t_0 \in \mathbb{R}$*, or if* $\mathcal{T} = \mathbb{R}$.

1.2 Basic properties of control systems

As mentioned before, most physical and technical processes can be described using finite dimensional Euclidean spaces (at least approximately). From now on, we shall mainly study the behavior of finite dimensional differential control systems. In order to characterize them, we first introduce some important properties, such as controllability, stabilizability and observability. Then, in Section 1.3 we examine them in a very special, but important case: considering systems governed by a *linear* differential equation.

Definition 1.4 *Let* Σ *be a (not necessarily differential) control system. We say that*

1. *the pair* $(t_1, x_1) \in \mathcal{T} \times X$ can be controlled to $(t_2, x_2) \in \mathcal{T} \times X$*, where* $t_2 > t_1$ *if there exists a control function* $u \in \mathcal{U}$ *such that* $(t_1, t_2, x_1, u) \in \mathcal{D}$ *and* $s(t_1, t_2, x_1, u) = x_2$*;*
2. *the state* $x_1 \in X$ can be controlled to the state $x_2 \in X$ in the time $T > 0$ *if for each* $t \in \mathcal{T}$ *with* $t + T \in \mathcal{T}$ *the pair* (t, x_1) *can be controlled to* $(t + T, x_2)$*;*
3. *the state* $x_1 \in X$ can be controlled to $x_2 \in X$ *if for each* $t_1 \in \mathcal{T}$ *there exists* $t_2 \in \mathcal{T}$*, with* $t_2 > t_1$ *such that the pair* (t_1, x_1) *can be controlled to* (t_2, x_2)*.*

Remark 1.2 *The pair* (t_2, x_2) *(or the state* x_2*) is sometimes said to be 'reachable' from* (t_1, x_1) *(or from* x_1*). This means that* (t_1, x_1) *(or* x_1*) can be controlled to* (t_2, x_2) *(or* x_2*, respectively).*

Definition 1.5 (controllability) *The control system* Σ *is said to be* (completely) controllable on the interval $[t_1, t_2] \subset \mathcal{T}$ *if for each* x_1 *and* $x_2 \in X$ *the pair* (t_1, x_1) *can be controlled to* (t_2, x_2)*. It is* (completely) controllable in the time T *if for each* x_1 *and* $x_2 \in X$ x_1 *can be controlled to* x_2 *in the time* T*. And finally, the system is called* (completely) controllable *if each* $x_1 \in X$ *can be controlled to any* $x_2 \in X$*.*

Definition 1.6 (equilibrium pairs) *A pair* $(x, u) \in X \times U$ *is called* equilibrium pair *of the system* Σ *if for all* $t_1, t_0 \in \mathcal{T}$ $(t_0 < t_1)$ $s(t_0, t_1, x, \tilde{u}) = x$ *holds, where* $\tilde{u} \equiv u$ *for all* $t \in \mathcal{T}$ *with* $\tilde{u} \in \mathcal{U}$.

Next, we study some asymptotic properties of equilibrium pairs:

Definition 1.7 (stability) *Let* σ *be a finite dimensional differential control system over infinite time horizon and let* $(x^*, u^*) \in X \times U$ *be an equilibrium pair. The state* x^* *is then called* (asymptotically) stable *if the corresponding ordinary differential equation*

$$\dot{x} = f(t, x, u^*)$$

has an (asymptotically) stable fixed point at x^*.

Definition 1.8 (stabilizability) *Let* σ *be a finite dimensional differentiable control system over infinite time horizon and let* $(x^*, u^*) \in X \times U$ *be an equilibrium.* x^* *is then called* stabilizable *if each state* $x \in X$ *can be asymptotically controlled to* x^**, i.e. if for each* $x \in X$ *and* $t_0 \in \mathcal{T}$ *there exists* $u \in \mathcal{U}$ *such that*

$$\lim_{t \to \infty} s(t_0, t, x, u) = x^*$$

Remark 1.3 *If a finite dimensional differential system with infinite time-horizon is controllable, then each equilibrium state is stabilizable.*

Proof. Let (x^*, u^*) be an equilibrium pair. Since σ is controllable, there exists for each $x \in X$ and $t_0 \in \mathcal{T}$ a control $u \in \mathcal{U}$ and a time $t_1 > t_0$ such that

$$s(t_0, t_1, x, u) = x^*.$$

If we then define $\tilde{u}(t) := \begin{cases} u(t) & \text{if } t \in [t_0, t_1] \\ u^* & \text{if } t > t_1 \end{cases}$ then, using properties (iii) and (v) of the state transfer mapping in Definition 1.1, we get

$$\begin{aligned} s(t_0, t, x, \tilde{u}) &= s(t_1, t, s(t_0, t_1, x, \tilde{u}), \tilde{u}) = s(t_1, t, s(t_0, t_1, x, u), u^*) \\ &= s(t_1, t, x^*, u^*) = x^* \qquad \forall t > t_1. \end{aligned}$$

□

In [Son90] a different definition (see Definition 3.1.1 there) is given for the controllability. The author calls a system *controllable* if for any $x_0, x_1 \in X$ *there exist* $t_0, t_1 \in \mathcal{T}$ such that (t_0, x_0) can be controlled to (t_1, x_1). Although this definition is equivalent to Definition 1.5 in the time-invariant case (i.e. if $f(t, x, u) = f(t+T, x, u)$ or $s(t_0, t_1, x, u) = s(t_0+T, t_1+T, x, u)$ $\forall T \in \mathbb{R}$), they are different for non-autonomous systems. The following examples illustrate this difference.

Example 2: Consider the following finite dimensional differential control system:

$$\dot{x} = x + b(t)u,$$

where $b(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$, $\mathcal{T} = \mathbb{R}$, $X = U = \mathbb{R}$ and $\mathcal{U}$ is the set of piecewise continuous mappings from $\mathbb{R}$ to $\mathbb{R}$. This system is completely controllable on the interval $[0, 1]$ (as we shall see later in Corollary 1.1), but clearly if we choose $x_0 > 0$, $t_0 > 1$ and $x_1 < x_0$, then (t_0, x_0) cannot be controlled to x_1.

The next example shows, that even using analytical coefficients the two definitions are different.

Example 3: We define the following linear control system:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} u,$$

where $\mathcal{T} = \mathbb{R}$, $X = \mathbb{R}^2$, $U = \mathbb{R}$ and finally $\mathcal{U}$ is defined as in Example 2. Then the solution $x(t)$ can be written for an initial condition (t_0, x_0) as:

$$\begin{aligned} x(t) &= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau) \begin{pmatrix} \sin \tau \\ \cos \tau \end{pmatrix} u(\tau)\, d\tau \\ &= \Phi(t, t_0)x_0 + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \int_{t_0}^{t} u(\tau)\, d\tau, \end{aligned}$$

with $\Phi(t, \tau) = \begin{pmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{pmatrix}$.

Since the function $\tilde{y}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ is a solution of

$$\dot{y} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^T y$$

with $\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}^T \tilde{y}(t) = 0\ \forall t \in \mathbb{R}$, this system is clearly not controllable (see Theorem 1.3). For applications it cannot be considered as controllable, because the control function $u(t)$ only has an averaged influence on it and it is, for instance, impossible to control the state function $x(t)$ along a given trajectory in X. On the other hand, for each x_0 and x_1 there is a t_0 and t_1 such that (t_0, x_0) can be controlled to (t_1, x_1). To show this, we use a geometric approach:

Choose t_1 such that $x_1 = R \begin{pmatrix} \sin(t_1) \\ \cos(t_1) \end{pmatrix}$ for some $R > 0$ and choose $t_0 < t_1$ such that $x_0 = r \begin{pmatrix} \sin(2t_1 - t_0) \\ \cos(2t_1 - t_0) \end{pmatrix}$ for some $r > 0$ and finally set $u(t) \equiv \frac{R-r}{t_1 - t_0}$. Then

$$x_1 = \Phi(t_1, t_0)x_0 + \begin{pmatrix} \sin t_1 \\ \cos t_1 \end{pmatrix} (t_1 - t_0)u.$$

Finally, we define a system property that shows the connection between the system state and the output:

Definition 1.9 (observability) *Let Σ be a control system. Suppose further that $\forall t_0 \in \mathcal{T}$ and $u \in \mathcal{U}$ and for each pair of different state trajectories x_1 and x_2 generated by the same control u there exists $t^* \in (-\infty, t_0] \cap \mathcal{T}$ such that*

$$o(t^*, x_1(t^*), u(t^*)) \neq o(t^*, x_2(t^*), u(t^*))$$

holds. Then Σ is called observable.

1.3 Controllability and stabilizability of linear control systems

Let us begin this section with the definition of the affine control systems. These systems, that are slightly more general then linear systems – which we also introduce here – will be investigated later in this Thesis.

Definition 1.10 (affine control system) *Let σ be a finite dimensional differential control system, where the mapping $f(t, x, u)$ is of the affine form*

$$f(t, x, u) = A(t, x) + B(t, x)u$$

for some sufficiently smooth mappings $A : \mathcal{T} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathcal{T} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$. Then σ is called finite dimensional affine differential control system *or briefly* affine control system.

Now, we define the actual subject of this section, a linear control system. Throughout the rest of this chapter, we investigate the most important properties of this class of differential control systems.

Definition 1.11 (linear control system) *Let σ be a finite dimensional differential control system, where the mappings $f(t, x, u)$ and $o(t, x, u)$ are linear in $x \in X$ and $u \in U$ for each fixed $t \in \mathcal{T}$. Then σ is called* finite dimensional linear differential control system *or briefly* linear control system.

Remark 1.4 *From now on we use the following notations for linear control systems:*

1. $\dim X = n$, $\dim U = m$ *and* $\dim Y = p$;
2. $f(t, x, u) = A(t)x + B(t)u$ *for some mappings* $A : \mathcal{T} \to \mathbb{R}^{n \times n}$ *and* $B : \mathcal{T} \to \mathbb{R}^{n \times m}$;
3. $o(t, x, u) = C(t)x + D(t)u$ *for some mappings* $C : \mathcal{T} \to \mathbb{R}^{p \times n}$ *and* $D : \mathcal{T} \to \mathbb{R}^{p \times m}$.

Using this notation, an arbitrary linear control system can be described with a linear ordinary differential equation and a linear algebraic equation:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \text{ with } x(t_0) = x_0 \in \mathbb{R}^n \tag{1.5}$$
$$y(t) = C(t)x(t) + D(t)u(t). \tag{1.6}$$

Remark 1.5 *If $A(t)$ and $B(t)$ are piecewise continuous in t, then the arising linear control system is a differential control system (in the sense of Definition 1.2). Hence, using the theory of linear ordinary differential equations, we obtain the solution of (1.5) in the following form (see for instance [KK74] or [CL55]):*

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\, d\tau, \tag{1.7}$$

where $\Phi(t, \tau)$ is the fundamental matrix of the homogenous system $\dot{x} = A(t)x$, i.e. Φ fulfills for all $t, \tau \in \mathcal{T}$ the differential equation

$$\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau), \text{ with } \Phi(t, \tau)|_{t=\tau} = id \in \mathbb{R}^{n\times n}.$$

First, we examine the controllability of a linear control system on an interval $[t_0, t_f] \subset \mathcal{T}$:

Theorem 1.2 *A linear control system σ is controllable on $[t_0, t_f] \subset \mathcal{T}$ if and only if for each state $x \in X$ the pair $(t_0, 0)$ can be controlled to (t_f, x).*

Proof. The condition is clearly necessary (see Definition 1.5) and hence we only have to show that it is also sufficient:

Let $x_0, x_f \in X$ and $\tilde{x}_f := \Phi(t_f, t_0)x_0$. Clearly, $\tilde{x}_f$ is the state, which the system would reach at the time t_f from (t_0, x_0) without any control (i.e. with $u(t) \equiv 0$). Since $x_f - \tilde{x}_f \in X$, the pair $(t_0, 0)$ can be controlled to $(t_f, x_f - \tilde{x}_f)$ for some $\tilde{u} \in \mathcal{U}$. This means, using equation (1.7):

$$x_f - \tilde{x}_f = \int_{t_0}^{t_f} \Phi(t_f, t)B(t)\tilde{u}(t)\, dt \text{ and}$$
$$\tilde{x}_f = \Phi(t_f, t_0)x_0,$$

which together admit that (t_f, x_f) is reachable from (t_0, x_0). □

We now turn our attention to the characterization of controllable linear systems by means of the coefficient matrices A and B. Most of the results of the forthcoming investigation have already been published in the late sixties and early seventies and hence are regarded as one of the few "classical" results in Control Theory. Their detailed proofs can be found in any textbook dealing with linear control systems (see for instance [KK85] or [Bro70]) and therefore are mostly omitted here. Our first condition makes use of

Lemma 1.1 (see Lemma 3.1 in [KK85]) *Let* $F : [t_0, t_f] \to \mathbb{R}^{n\times m}$ *be a piecewise continuous mapping and* $G := \int_{t_0}^{t_f} F(t)F^T(t)\,dt$. *Then, a vector* $x \in \mathbb{R}^n$ *can be written in the form*

$$x = \int_{t_0}^{t_f} F(t)u(t)\,dt$$

with a piecewise continuous function $u : [t_0, t_f] \to \mathbb{R}^m$ *if and only if* $x \in \operatorname{im}(G) := \{Gy \in \mathbb{R}^n | y \in \mathbb{R}^n\}$ *holds.*

Proof. Let x be a vector in $\mathbb{R}^n$. The product

$$\begin{aligned} x^T G x &= x^T \int_{t_0}^{t_f} F(t)F^T(t)\,dt\,x = \int_{t_0}^{t_f} x^T F(t)F^T(t)x\,dt \\ &= \int_{t_0}^{t_f} \|F^T(t)x(t)\|^2\,dt \geq 0 \end{aligned} \tag{1.8}$$

Hence, G is positive semidefinite and therefore $\{x \in \mathbb{R}^n | x^T G x = 0\} = \ker(G) := \{x \in \mathbb{R}^n | Gx = 0\}$ holds. Using (1.8) we get that

$$\ker(G) = \{x \in \mathbb{R}^n | F^T(t)x = 0 \quad \forall t \in [t_0, t_f]\}.$$

Since the set $\mathcal{F} := \{x \in \mathbb{R}^n | x = \int_{t_0}^{t_f} F(t)u(t)\,dt\, u \in \mathcal{U}\}$ forms a linear subspace of $\mathbb{R}^n$, we show that $\operatorname{im}(G) \leq \mathcal{F}$ and $\mathcal{F} \cap \ker(G) = 0$. Then, we obtain that $\mathcal{F} \leq \operatorname{im}(G)$ and hence $\mathcal{F} = \operatorname{im}(G)$.

1. Let $x \in \operatorname{im}(G)$. We show that $x \in \mathcal{F}$: For each $x \in \operatorname{im}(G)$ there exists $y \in \mathbb{R}^n$ such that
$$x = Gy = \int_{t_0}^{t_f} F(t)F^T(t)\,dt\,y = \int_{t_0}^{t_f} F(t)\left(F^T(t)y\right)\,dt.$$
This means $x = \int_{t_0}^{t_f} F(t)u(t)\,dt$ for $u(t) = F^T(t)y$ and hence $x \in \mathcal{F}$.

2. On the other hand, if $x \in \mathcal{F} \cap \ker(G)$, then:
$$\begin{aligned} \|x\|^2 &= x^T x = x^T \int_{t_0}^{t_f} F(t)u(t)\,dt = \int_{t_0}^{t_f} \left(x^T F(t)\right) u(t)\,dt \\ &= \int_{t_0}^{t_f} \left(F^T(t)x\right)^T u(t)\,dt = 0, \end{aligned}$$
hence $\mathcal{F} \cap \ker(G) = 0$.

□

Corollary 1.1 (see Theorem 3.1 in [KK85]) *The linear control system (1.5) with piecewise continuous coefficients A and B is completely controllable on the interval $[t_0, t_f] \subset \mathcal{T}$ if and only if the matrix*

$$W := \int_{t_0}^{t_f} \Phi(t_0, t)B(t)B^T(t)\Phi^T(t_0, t)\, dt$$

is positive definite.

Proof. Using (1.7) we get

$$x(t_f) = \Phi(t_f, t_0)x_0 + \int_{t_0}^{t_f} \Phi(t_f, t)B(t)u(t)\, dt.$$

Note that $\Phi(t_f, t) = \Phi(t_f, t_0)\Phi(t_0, t)$ and $\Phi(t_f, t_0)$ is regular. Hence the matrix

$$\tilde{W} := \Phi(t_f, t_0)W\Phi^T(t_f, t_0)$$

is positive definite if and only if W is positive definite.

1. Using Lemma 1.1, we get that if $x^* \notin \mathrm{im}(\tilde{W})$ then there is no function $u \in \mathcal{U}$, such that
$$x^* = \int_{t_0}^{t_f} \Phi(t_f, t)B(t)u(t)\, dt.$$
In this case the pair $(t_0, 0)$ can not be controlled to (t_f, x^*).

2. On the other hand, if $\tilde{W} > 0$, then for each $x \in X$ the equation
$$x = \tilde{W}c$$
is uniquely solvable w.r.t. $c \in \mathbb{R}^n$. Hence $(t_0, 0)$ can be controlled to (t_f, x) using the control function $\tilde{u}(t) := B^T(t)\Phi^T(t_f, t)c$.

□

For the sake of completeness, we now cite a very important result from [KK85]:

Theorem 1.3 (see Theorem 3.2 in [KK85]) *Let σ be a linear control system on infinite time horizon with piecewise continuous coefficients and $y(t)$ be a nontrivial solution of (1.9). Then σ is controllable if and only if for each $t_0 \in \mathcal{T}$ and $y(t)$ there exists $t^* \geq t_0$ such that $B^T(t^*)y(t^*) \neq 0$.*

The next theorem gives an alternative condition for the controllability over *finite time-horizon.* Its proof is based on the proof of the latter theorem:

Theorem 1.4 (see also Theorem 3.2 in [KK85]) *Let σ be a linear control system with piecewise continuous coefficients and $y(t)$ a nontrivial solution of the differential equation*

$$\dot{y}(t) = -A^T(t)y. \tag{1.9}$$

Then σ is controllable on $[t_0, t_f] \subset \mathcal{T}$ if and only if for each $y(t)$ there exists $t^* \in [t_0, t_f]$ such that $B^T(t^*)y(t^*) \neq 0$.

Proof. Since $\Phi^T(\tau, t)$ is the fundamental matrix of (1.9), where $\Phi(t, \tau)$ is the fundamental matrix of (1.5), the solution $y(t)$ is represented as

$$y(t) = \Phi^T(t_0, t)y_0$$

with $y(t_0) = y_0$. This means that the condition 'for each $y(t)$ there exists $t^* \in [t_0, t_f]$ such that $B^T(t^*)y(t^*) \neq 0$' is equivalent to: 'for each $y_0 \in \mathbb{R}\backslash\{0\}$ there exists $t^* \in [t_0, t_f]$ such that $B^T(t^*)\Phi^T(t_0, t^*)y_0 \neq 0$'.

Since the mapping $t \mapsto B^T(t)\Phi^T(t_0, t)y_0$ is piecewise continuous, the latter condition is equivalent to: 'for each $y_0 \in \mathbb{R}\backslash\{0\}$ there exists an open interval $(t_1, t_2) \subset [t_0, t_f]$ such that $\|B^T(t)\Phi^T(t_0, t)y_0\| \neq 0 \ \forall t \in (t_1, t_2)$'.

Again, this condition is equivalent to: 'for each $y_0 \in \mathbb{R}\backslash\{0\}$ the integral

$$\int_{t_0}^{t_f} \|B^T(t)\Phi^T(t_0, t)y_0\|^2 \, dt \neq 0.'$$

Finally, this last expression can be written as

$$\int_{t_0}^{t_f} \|B^T(t)\Phi^T(t_0, t)y_0\|^2 \, dt = y_0^T W y_0$$

and hence using Lemma 1.1 the proof is completed. □

The following two classical theorems give conditions for complete controllability of autonomous systems:

Theorem 1.5 (Kalman[1], 1963 – see [Kal63]) *An autonomous linear control system*

$$\dot{x} = Ax + Bu, \tag{1.10}$$

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ is controllable if and only if the so-called Kalman-matrix

$$K := \left(B, AB, A^2B, \ldots, A^{n-1}B\right)$$

has maximal rank (i.e. $\operatorname{rank} K = n$*).*

[1] Rudolf Emil Kálmán was born in Budapest (Hungary) in 1930. According to the *SIAM News (6/94)* he is "without doubt the most influential researcher of the field" and has been "the leader in the development of a rigorous theory of control systems". Among others, he is the recipient of the IEEE Medal of Honor, the Steele Prize of the AMS and Bellman Prize. He is also a member of the Hungarian, French, Russian and American Academy of Sciences.

Theorem 1.6 (Hautus, 1969 – see [Hau69]) *An autonomous linear control system*

$$\dot{x} = Ax + Bu, \tag{1.11}$$

with $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times m}$ *is controllable if and only if for each eigenvector* y *of* A^T $y^T B \neq 0$ *holds.*

We now shortly study the equilibrium pairs of linear control systems and then examine their stability:

Lemma 1.2 *Suppose that* σ *is a linear control system with infinite time-horizon. Then the equilibrium pairs of the system form a linear subspace of* $X \times U$*.*

Proof.

(i) Let (x^*, u^*) be an equilibrium pair. Then

$$\dot{x}^* = 0 = A(t)x^* + B(t)u^*$$

and hence $\forall \lambda \in \mathbb{R}$

$$0 = \lambda A(t)x^* + \lambda B(t)u^* = A(t)(\lambda x^*) + B(t)(\lambda u^*) = \frac{d}{dt}(\lambda x^*).$$

This means that $(\lambda x^*, \lambda u^*)$ is also an equilibrium pair.

(ii) Let (x_1^*, u_1^*) and (x_2^*, u_2^*) be two equilibrium pairs. Then

$$\begin{aligned} \dot{x}_1^* + \dot{x}_2^* &= A(t)x_1^* + B(t)u_1^* + A(t)x_2^* + B(t)u_2^* \\ &= A(t)(x_1^* + x_2^*) + B(t)(u_1^* + u_2^*) = 0 \end{aligned}$$

and hence $(x_1^* + x_2^*, u_1^* + u_2^*)$ is also an equilibrium pair.

□

Using this latter lemma, we can also define (similar to Definitions 1.8 and 1.7) the stabilizability and the stability of a linear control system:

Definition 1.12 (stability and stabilizability) *We call a linear control system with infinite time horizon* stabilizable *if the state* $x^* = 0$ *is stabilizable (by means of Definition 1.8). Similarly, we call the system* (asymptotically) stable *if the differential equation*

$$\dot{x} = A(t)x$$

is (asymptotically) stable.

Next, we examine the stabilizability of linear control systems with infinite time-horizon. First, we show a necessary condition similar to Theorem 1.3:

Theorem 1.7 *Let σ be a linear control system with infinite time-horizon and let $y(t)$ be a solution of the differential equation (1.9). If σ is stabilizable, then $\forall t_0 \in \mathcal{T}$ and for any bounded solution $y(t)|_{[t_0,\infty)}$ there exists $t^* \in \mathcal{T}$ such that $B^T(t^*)y(t^*) \neq 0$.*

Proof. Suppose that the system is stabilizable and that there exists a bounded solution for (1.9) $y(t)$ and $t_0 \in \mathcal{T}$ such that $B^T(t)y(t) \equiv 0 \; \forall t \geq t_0$. Then, for an arbitrary state-trajectory $x(t)$

$$\begin{aligned}\frac{d}{dt}(x^T(t)y(t)) &= \dot{x}^T(t)y(t) + x^T(t)\dot{y}(t) \\ &= (x^T(t)A^T(t) + u^T(t)B^T(t))y(t) - x^T(t)A^T(t)y(t) = 0\end{aligned}$$

holds $\forall t \in \mathcal{T}$. Thus, if we choose $x(t_0) = x_0$ such that $x_0^T y(t_0) = a \neq 0$, then $x^T(t)y(t) \equiv a$ $\forall t \geq t_0$ and hence $\lim_{t\to\infty}(x^T(t)y(t)) = a$. Since $|x^T(t)y(t)| \leq \|x(t)\|\|y(t)\|$, it follows that $\lim_{t\to\infty}\|x(t)\|\|y(t)\| > a > 0$ and hence, using the boundedness of $\|y(t)\|$ on $[t_0,\infty)$ we get that $\lim_{t\to\infty}\|x(t)\| \neq 0$. □

The following theorem is a well-known result for autonomous linear control systems. Here, its proof is carried out without explicitely introducing the canonical form of autonomous linear control systems. (For more details on this canonical form see [Bro70] or [KK85]) :

Theorem 1.8 *Let σ be an autonomous linear control system on infinite time horizon. Then σ is stabilizable if and only if there exist two subspaces V_1 and V_2 of $\mathbb{R}^n$ with the following properties:*

(i) $V_1 \oplus V_2 = \mathbb{R}^n$,

(ii) *each $x_1 \in V_1$ can be controlled to zero,*

(iii) *each $x_2 \in V_2$ is an asymptotically stable state, i.e $\lim_{t\to\infty}\xi(t) = 0$, where $\xi(t)$ is the solution of $\dot{\xi} = A\xi$ with $\xi(0) = x_2$.*

Proof.

(i) In order to prove that the three conditions are sufficient for the stabilizability, we need to show that for each $x_0 \in V_1 \oplus V_2$ and $t_0 \in \mathcal{T}$ there exists a control function $u \in \mathcal{U}$ such that $\lim_{t\to\infty} s(t_0,t,x_0,u) = 0$. Since the system is autonomous, it is $s(t_1,t_2,x,u) = s(t_1+T,t_2+T,x,u)$ for any $x \in \mathbb{R}^n$, $u \in \mathcal{U}$, $t_1,t_2 \in \mathcal{T}$ with $t_1 \leq t_2$ and $T > 0$. Hence, it is sufficient to show for only one fixed $t_0 \in \mathcal{T}$ that $\lim_{t\to\infty} s(t_0,t,x_0,u) = 0$.

Let $x_0 = x_1 + x_2$ with $x_1 \in V_1$ and $x_2 \in V_2$. Then, using property (ii), we obtain that there exist $t_1 \geq t_0$ and $u_1 \in \mathcal{U}$ such that

$$0 = e^{A(t_1-t_0)}x_1 + \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu_1(\tau)\,d\tau \tag{1.12}$$

Denote now $u = \begin{cases} u_1(t) & t \in [t_0, t_1] \\ 0 & t \in (t_1, \infty) \end{cases}$. Then for $x(t) := s(t_0, t, x_0, u)$ with $t > t_1$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau = e^{A(t-t_0)}x_0 + \int_{t_0}^{t_1} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

holds. If we now apply the semi-group property of the fundamental-matrix $e^{A(t-\tau)}$ and equation (1.12), then we obtain

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x_0 + e^{A(t-t_1)} \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)\,d\tau \\ &= e^{A(t-t_0)}x_0 - e^{A(t-t_1)}e^{A(t_1-t_0)}x_1 = e^{A(t-t_0)}(x_0 - x_1) \\ &= e^{A(t-t_0)}x_2. \end{aligned}$$

On the other hand, it is $x_2 \in V_2$ and the trajectory $e^{A(t-t_0)}x_2$ is the solution of $\dot{x} = Ax$ for $x(t_0) = x_2$, and hence $\lim_{t\to\infty} x(t) = 0$, which means that the given control function $u(t)$ stabilizes the system.

(ii) Suppose now that the system is stabilizable. Now we construct the subspaces V_1 and V_2 as the kernel and image of a symmetric matrix $X \in \mathbb{R}^{n\times n}$. Let the set of eigenvalues of A be denoted by $\Lambda(A)$. Since A is finite dimensional, there exists a positive constant q such that $q > \min\{Re(\lambda) | \lambda \in \Lambda(A)\}$. Since for each eigenvector v with eigenvalue λ of an arbitrary matrix $V \in \mathbb{R}^{n\times n}$ and $\forall s \in \mathbb{R}$

$$(V + sI)v = Vv + sIv = (\lambda + s)v,$$

v is also eigenvector of $V + sI$ with eigenvalue $\lambda + s$. This yields that the matrix $\tilde{A} := A + qI$ has only eigenvalues with positive real parts and hence the matrix $-\tilde{A}$ is stable. Using Theorem 1.1.8 and Lemma 1.1.14 in [AKFIJ], we obtain that the algebraic Lyapunov-equation

$$-\tilde{A}X - X\tilde{A}^T + BB^T = 0 \tag{1.13}$$

has a unique solution $X \in \mathbb{R}^{n\times n}$ which can be obtained using

$$X = \int_0^\infty e^{-\tilde{A}t}BB^Te^{-\tilde{A}^Tt}\,dt.$$

Hence, X is symmetric and positive semidefinite.

(iii) Using the notation $V_1 = \mathrm{im}(X)$ and $V_2 = \ker(X)$, we first obtain that, $V_1 \oplus V_2 = \mathbb{R}^n$ holds. Furthermore,

$$\begin{aligned} x_2 \in V_2 &\Leftrightarrow x_2^TXx_2 = 0 \Leftrightarrow \int_0^\infty x_2^Te^{-\tilde{A}t}BB^Te^{-\tilde{A}^Tt}x_2\,dt = 0 \\ &\Leftrightarrow \int_0^\infty \|B^Te^{-\tilde{A}^Tt}x_2\|^2\,dt = 0 \Leftrightarrow B^Te^{-\tilde{A}^Tt}x_2 = 0 \quad \forall t \geq 0 \\ &\Leftrightarrow B^Te^{-A^Tt}e^{-qt}x_2 = 0 \quad \forall t \geq 0 \Leftrightarrow x_2^Te^{-At}B = 0 \quad \forall t \geq 0 \\ &\Leftrightarrow B^Te^{-A^Tt}x_2 = 0 \quad \forall t \geq 0. \end{aligned}$$

On the other hand, the trajectory $y(t) := e^{-A^T t}x_2$ is for $x_2 \neq 0$ a non-trivial solution of $\dot{y} = -A^T y$ for $y(0) = x_2$. Since σ is stabilizable and $B^T y(t) \equiv 0$ for any $t \geq t_0$, we obtain that $\lim_{t\to\infty} \|y(t)\| = \infty$. Using now

$$\lim_{t\to\infty} \|x_2^T e^{A^T t} e^{-A^T t} x_2\| = x_2^T x_2 < \infty,$$

this latter result immediately implies that for each $x_2 \in \ker(X)$

$$\lim_{t\to\infty} \|e^{At} x_2\| = 0$$

and hence, each state $x_2 \in V_2$ is asymptotically stable w.r.t. σ.

(iv) Using the argumentation introduced in the proof of Kalman's controllability condition (Theorem 1.5), the equivalence

$$x_2 \in \ker(X) \Leftrightarrow B^T e^{-A^T t} x_2 = 0 \quad \forall t \geq 0$$

yields that $x_2 \in \ker(X) \Leftrightarrow x^T K = 0$ and hence $\ker(K^T) = \ker(X)$ holds. Furthermore, in the proof of Hautus' Theorem (Theorem 1.6) we showed that if $x \in \ker(K^T)$ then $A^T x \in \ker(K^T)$. Since $\ker(X)$ and $\mathrm{im}(X)$ are orthogonal subspaces with $\ker(X) \oplus \mathrm{im}(X) = \mathbb{R}^n$,

$$x_1 \in V_1 \Leftrightarrow \forall x_2 \in V_2 \quad x_1^T x_2 = 0$$

holds. On the other hand, using that A^T is an automorphism of V_2, we obtain that if $x_1 \in V_1$ then for any $x_2 \in V_2$ $x_1^T A^T x_2 = 0$ and hence $(Ax_1)^T x_2 = 0$ holds. This means that A is an automorphism of V_1 and hence the system that arises through the restriction of the state space of σ to V_2 is a linear control system with the same coefficients A and B. Furthermore, for any $x_1 \in V_1 \backslash \{0\}$ $x_1 \notin V_2$ and hence $x_1^T K \neq 0$ holds. Therefore the restricted control system is completely controllable, i.e. each state $x_1 \in V_1$ can be controlled to zero in finite time.

□

The following theorem shows a very useful result on stabilizable systems illustrating the connection between stability and stabilizability:

Theorem 1.9 *Suppose that σ is an autonomous stabilizable linear control system. Then there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that the matrix $A - BF$ is stable.*

We first give a construction for such a matrix F in the case of controllable autonomous systems and then generalize this construction to prove Theorem 1.9. The following calculation is based on the idea of V. Ionescu:

Lemma 1.3 *Suppose that σ is an autonomous controllable linear control system. Then there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that each eigenvalue of the matrix $A - BF$ has negative real part.*

Proof. Using the construction introduced in the proof of Theorem 1.8, we first conclude that since σ is controllable, $V_1 = \mathbb{R}^n$ and hence $\ker(X) = 0$ or equivalently $X > 0$ holds. Then, substituting

$$F := B^T X^{-1},$$

we obtain a closed-loop matrix $A - BF = A - BB^T X^{-1}$, for which

$$\begin{aligned}\left(A - BB^T X^{-1}\right) X + X \left(A - BB^T X^{-1}\right)^T &= AX - BB^T + XA^T - BB^T \\ &= \tilde{A}X + X\tilde{A}^T - 2qX - 2BB^T \\ &\overset{(1.13)}{=} -2qX - BB^T < 0\end{aligned}$$

holds. Hence, the function $V(x) := x^T X x$ is a positive definite Lyapunov-function for the closed-loop system, which means that it is asymptotically stable. □

With the help of the above construction, we can now easily prove Theorem 1.9:

Proof of Theorem 1.9. Using Theorem 1.8, we obtain the factorization $\mathbb{R}^n = V_1 \oplus V_2$ with $V_1 = \mathrm{im}(X)$ and $V_2 = \ker(X)$. Hence, X is an automorphism of V_1 with $\ker(X|_{V_1}) = 0$. With other words, the linear transformation $X|_{V_1} : V_1 \to V_1$ is invertible. We now define the following linear transformation on $\mathbb{R}^n = V_1 \oplus V_2$:

$$X^{\#}(x) = X^{\#}(x_1 + x_2) = \left(X|_{V_1}\right)^{-1}(x_1),$$

for the (unique) factorization $x = x_1 + x_2$ with $x_i \in V_i$ $(i = 1, 2)$.

Out aim now is to prove, that the matrix $A - BB^T X^{\#}$ is stable. For that, we investigate the differential equation

$$\dot{\xi} = \left(A - BB^T X^{\#}\right)\xi.$$

Again, using the factorization $\xi = \xi_1 + \xi_2$, with $\xi_i \in V_i$, we obtain that

$$\begin{aligned}\dot{\xi} = \dot{\xi}_1 + \dot{\xi}_2 &= \left(A - BB^T X^{\#}\right)\xi = A\xi - BB^T X^{\#}\xi = A\xi_1 + A\xi_2 - BB^T \left(X|_{V_1}\right)^{-1}\xi_1 \\ &= \left(A - BB^T \left(X|_{V_1}\right)^{-1}\right)\xi_1 + A\xi_2\end{aligned}$$

holds.

Moreover, as stated in point (iv) of the proof of Theorem 1.8, A is an automorphism of V_1 and hence the automorphism

$$\left.\left(A - BB^T \left(X|_{V_1}\right)^{-1}\right)\right|_{V_1}$$

is – using Lemma 1.3 – stable. Furthermore, the (asymptotical) stability of the dynamical system

$$\dot{\xi}_2 = A\xi_2$$

for $\xi_2 \in V_2$ is a consequence of Theorem 1.8.

Altogether, we obtain, that the feedback matrix $F = B^T X^{\#}$ has the desired property of stabilizing the closed-loop system. □

Theorem 1.9 directly yields the following statement:

Corollary 1.2 (stabilizing feedback-control) *Let σ be an autonomous stabilizable control system. Then there exists a* feedback-control, *i.e. a control function $\tilde{u}$ of the form $\tilde{u}(t) = F(t, x(t))$, for which the corresponding linear system*

$$\dot{x}(t) = Ax(t) + B\tilde{u}(t)$$

is asymptotically stable. Furthermore, this feedback is linear in the current state $x(t)$ and this linear mapping is autonomous.

Proof. Using Theorem 1.9, we get that there exists a matrix $F \in \mathbb{R}^{n\times m}$ such that the system

$$\dot{x} = (A - BF)x$$

is asymptotically stable. Hence, the choice

$$\tilde{u}(t) := -Fx(t)$$

yields the required property. □

Our last result on the stabilizability of a control system is a well known classical theorem of M.L.J. Hautus:

Theorem 1.10 (Hautus, 1969 – see [Hau69]) *An autonomous linear control system*

$$\dot{x} = Ax + Bu$$

is stabilizable if and only if for any eigenvector y of A^T corresponding to a non-stable eigenvalue (i.e. to an eigenvalue with non-negative real part) $B^Ty \neq 0$ holds.

To end this section, we shortly discuss the output-properties of linear control systems. A fundamental question here is the observability, which is – as we shall soon see – very strongly related to the controllability property.

Consider the following linear control system:

$$\dot{x}(t) = A(t)x + B(t)u \quad (1.14)$$
$$y(t) = C(t)x + D(t)u. \quad (1.15)$$

According to Definition 1.9 a control system is called observable if for any two solutions $x_1(.)$ and $x_2(.)$ using the same control function $u(t)$

$$y_(t) = y_2(t) \quad \forall t \leq t_0$$

implies that $x_1(t) = x_2(t)$ $\forall t \leq t_0$.

Rewriting this condition in terms of the linear mappings (1.14) and (1.15), we conclude that a linear control system Σ is observable if and only if for any *nontrivial* solution $x(.)$ of the homogeneous differential equation

$$\dot{x} = A(t)x,$$

there exists $t^* \leq t$ such that $C(t^*)x(t^*) \neq 0$ holds.

Hence, using Theorem 1.3, we obtain the following duality between observable and controllable systems:

Theorem 1.11 (duality) *A linear control system defined by the equations (1.14) and (1.15) is observable if and only if its so-called* dual *control system*

$$\dot{\xi}(t) = A^T(-t)\xi + C^T(-t)\nu$$

is controllable.

1.4 Optimal control for linear systems with quadratic costs

In this section we examine the existence and uniqueness of a control function, which minimizes a predefined cost functional. This cost functional is assumed to be given in form of a quadratic integral operator for $x(t)$ and $u(t)$. We still assume that the system is linear (see Definition 1.11) and that the initial condition, i.e. the pair $(t_0, x_0) \in \mathcal{T} \times X$ is fixed. Throughout this section we only consider systems with finite time-horizon $\mathcal{T} = [t_0, t_f]$ and we have no predefined final state $x(t_f)$, either. For results over infinite time-horizon or optimal control problems with boundary constraints, see Section 10.2 in [KK85], Section 3.5 and Chapter 7.

Definition 1.13 (optimal control system) *Let σ be a finite dimensional differential control system with finite time-horizon $\mathcal{T} = [t_0, t_f]$, as defined in Definition 1.2. We say that the pair (σ, J) forms a* finite dimensional optimal control system *(or shortly:* optimal control system*) if $J : \mathcal{X} \times \mathcal{U} \to \mathrm{I\!R}$ is defined as*

$$J(x, u) = \kappa(x(t_f)) + \int_{t_0}^{t_f} \Psi(t, x(t), u(t))\, dt, \tag{1.16}$$

where $\Psi : \mathcal{T} \times X \times U \to \mathrm{I\!R}$ is piecewise continuous in t and differentiable in x and u and $\kappa : X \to \mathrm{I\!R}$ is differentiable.

Remark 1.6 *The functional J usually is called* cost-functional *(or sometimes also* utility-functional *or* performance index*) assigned to σ.*

With the above definition, the aim of the Optimal Control Theory can be stated as follows:

Problem 1 *Let (σ, J) be an optimal control system with finite time-horizon $\mathcal{T} = [t_0, t_f]$ and let $x_0 \in X$ be given. Find $u^* \in \mathcal{U}$ such that together with the generated state trajectory $x^*(.) = s(t_0, ., x_0, u^*)$ the cost functional $J(x^*, u^*)$ is minimal.*

Definition 1.14 (optimal control and optimal state-trajectory) *Let (σ, J) be an optimal control system. We say that $u^*(t)$ is an* optimal control *and the generated state-trajectory $x^*(t)$ is an* optimal state-trajectory *if u^* is a solution of Problem 1.*

In the framework of this Thesis, we shall mainly consider optimal control systems of the following type:

Definition 1.15 (linear-quadratic control system) *The pair* (σ, J) *is called* finite dimensional linear quadratic optimal control system *(or shortly:* linear quadratic control system*) if* σ *is a finite dimensional linear control system and* $J : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ *is a mapping of the form*

$$J(x, u) = x^T(t_f)K_f x(t_f) + \int_{t_0}^{t_f} x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\, dt. \qquad (1.17)$$

Hereby denote $\mathcal{X}$ *the set of all state-trajectories in* σ*, i.e*

$$\mathcal{X} = \{s(., t_0, x_0, u) | t_0 \in \mathcal{T}, x_0 \in X, u \in \mathcal{U}\}$$

and $K_f \in \mathbb{R}^{n\times n}$ *a symmetric matrix. Further denote* $R : \mathcal{T} \to \mathbb{R}^{m\times m}$ *and* $Q : \mathcal{T} \to \mathbb{R}^{n\times n}$ *some piecewise continuous mappings with* $R(t) = R^T(t)$ *and* $Q(t) = Q^T(t)$ *for all* $t \in \mathcal{T}$*.*

There are several ways to solve Problem 1. The most common solution method makes use of the Hamilton-Jacobi equation which yields necessary conditions for a minimum. Here, we discuss two approaches, which are also advantageous later for studying different optimal control problems (see Sections 2.3.2 and 2.3.3 or Chapter 4). In the first subsection we derive a sufficient condition for the optimal control utilizing the so-called 'value-function'. Then, in the second subsection, we introduce a Hilbert-space approach. This method yields a necessary condition. It can be shown, however, that – even for differential games – these methods are equivalent and hence lead to a necessary and sufficient condition for an optimal control (see Section 2.3.1 and [JKK01a]).

Although most of the forthcoming proofs can be found in the usual literature (see for instance [Son90], [LR71] or [IOW99]) – for the convenience of the reader and for later generalization – they are included in full details.

1.4.1 Value-function approach for optimal control systems

For the sake of generality, we first introduce some main results for optimality of arbitrary control systems and then derive a sufficient condition for linear quadratic control systems.

Definition 1.16 (value-function) *Let* (σ, J) *be an optimal control system with* $\mathcal{T} = [t_0, t_f]$*. We call the mapping* $V : \mathcal{T} \times X \to \mathbb{R}$ value-function *or* Bellman-function *associated to* (σ, J) *if*

$$V(t, x) = \inf \left\{ \int_{t}^{t_f} \Psi(\tau, \hat{x}(\tau), u(\tau))\, d\tau + \kappa(\hat{x}(t_f)) | u \in \mathcal{U}, \hat{x} = s(t, ., x, u) \right\}. \qquad (1.18)$$

Clearly, $u^* \in \mathcal{U}$ is an optimal control of (σ, J) for $x(t_0) = x_0$ if and only if

$$J(x^*, u^*) = V(t_0, x_0),$$

where x^* denotes the state-trajectory generated by u^* (the so-called *optimal trajectory*), i.e. $x^* = s(t_0, ., x_0, u^*)$

Lemma 1.4 see Lemma 8.1.5 in [Son90]) *Let V denote the value-function of the optimal control system (σ, J). Then $\forall t_1, t_2 \in [t_0, t_f]$ with $t_1 \leq t_2$ and $\forall u \in \mathcal{U}$*

$$V(t_1, x_1) \leq \int_{t_1}^{t_2} \Psi(t, x(t), u(t))\, dt + V(t_2, x(t_2))$$

holds, where $x = s(t_1, ., x_1, u)$. Furthermore, if u^* is an optimal control, then

$$V(t_1, x_1) = \int_{t_1}^{t_2} \Psi(t, x^*(t), u^*(t))\, dt + V(t_2, x^*(t_2))$$

holds, where x^* is the optimal state-trajectory for $x^*(t_1) = x_1$.

Proof.

(i) We first prove that $\forall \varepsilon > 0$ and $u \in \mathcal{U}$

$$V(t_1, x_1) \leq \int_{t_1}^{t_2} \Psi(t, x(t), u(t))\, dt + V(t_2, x(t_2)) + \varepsilon$$

holds. Using (1.18) we get that there exists a control $\tilde{u} \in \mathcal{U}$ with

$$V(t_2, x(t_2)) \geq \int_{t_2}^{t_f} \Psi(t, \tilde{x}(t), \tilde{u}(t))\, dt + \kappa(\tilde{x}(t_f)) - \varepsilon, \tag{1.19}$$

where $x = s(t_1, ., x_1, u)$ and $\tilde{x} = s(t_2, ., x(t_2), \tilde{u})$. On the other hand, Properties (iii) and (v) from Definition 1.1 yield for

$$\hat{u}(t) = \begin{cases} u(t) & \text{if } t \in [t_1, t_2) \\ \tilde{u}(t) & \text{if } t \in [t_2, t_f] \end{cases}$$

that $\hat{x}(t) := s(t_1, t, x_1, \hat{u}) \equiv x(t)\ \forall t \in [t_1, t_2]$, as well as $\hat{x}(t) \equiv \tilde{x}(t)\ \forall t \in [t_2, t_f]$.

Again (1.18) and (1.19) yield that

$$\begin{aligned}
V(t_1, x_1) &\leq \int_{t_1}^{t_f} \Psi(t, \hat{x}(t), \hat{u}(t))\, dt + \kappa(\hat{x}(t_f)) \\
&= \int_{t_1}^{t_2} \Psi(t, \hat{x}(t), \hat{u}(t))\, dt + \int_{t_2}^{t_f} \Psi(t, \hat{x}(t), \hat{u}(t))\, dt + \kappa(\hat{x}(t_f)) \\
&= \int_{t_1}^{t_2} \Psi(t, x(t), u(t))\, dt + \int_{t_2}^{t_f} \Psi(t, \tilde{x}(t), \tilde{u}(t))\, dt + \kappa(\tilde{x}(t_f)) \\
&\leq \int_{t_1}^{t_2} \Psi(t, x(t), u(t))\, dt + V(t_2, x(t_2)) + \varepsilon.
\end{aligned}$$

(ii) In the second step, we prove that if u^* is optimal on $[t_1, t_f]$ then $u^*|_{[t_2,t_f]}$ is optimal on $[t_2, t_f]$ for $t_2 > t_1$, too. For this proof we use a standard method known from the Theory of Dynamic Programming (see Chapter 8 in [Son90]). Indeed, assuming that there is a control $\tilde{u}$ on $[t_2, t_f]$ with

$$\int_{t_2}^{t_f} \Psi(t, x^*(t), u^*(t)) + \kappa(x^*(t_f)) > \int_{t_2}^{t_f} \Psi(t, \tilde{x}(t), \tilde{u}(t)) + \kappa(\tilde{x}(t_f)),$$

where $x^* = s(t_1, ., x_1, u^*)$ and $\tilde{x} = s(t_2, ., x^*(t_2), \tilde{u})$ then, using a similar argumentation as in step (i), the control

$$\hat{u}(t) = \begin{cases} u^*(t) & \text{if } t \in [t_1, t_2) \\ \tilde{u}(t) & \text{if } t \in [t_2, t_f] \end{cases}$$

together with the generated state trajectory $\hat{x} = s(t_1, ., x_1, \hat{u})$ would produce a lower cost on $[t_1, t_f]$ than u^* produces. But this yields a contradiction, since u^* is minimal on this interval.

(iii) Finally, since u^* is also optimal on $[t_2, t_f]$ if it is optimal on $[t_1, t_f]$, we get

$$\begin{aligned} V(t_1, x_1) &= \int_{t_1}^{t_f} \Psi(t, x^*(t), u^*(t)) + \kappa(x^*(t_f)) \\ &= \int_{t_1}^{t_2} \Psi(t, x^*(t), u^*(t)) + \underbrace{\int_{t_2}^{t_f} \Psi(t, x^*(t), u^*(t)) + \kappa(x^*(t_f))}_{V(t_2, x^*(t_2))} \\ &= \int_{t_1}^{t_2} \Psi(t, x^*(t), u^*(t)) + V(t_2, x^*(t_2)). \end{aligned}$$

□

This latter result implies immediately the following classical theorem:

Theorem 1.12 (Hamilton-Jacobi-Bellman – see Proposition 8.1.8 in [Son90]) *Let (σ, J) be an optimal control system with $\mathcal{T} = [t_0, t_f]$, $x_0 \in X$ and value-function $V(t, x)$. Then the following statements hold:*

(i) $\forall u \in \mathcal{U}$ the function $\tilde{V}(t) := V(t, x(t))$ is continuous and piecewise differentiable for $x = s(t_0, ., x_0, u)$.

(ii)

$$\inf\left\{\frac{d}{dt}V(t, x(t)) + \Psi(t, x(t), u(t)) | u \in \mathcal{U}, x = s(t_0, ., x_0, u)\right\} = 0$$

for each $t \in \mathcal{T} \backslash T_0$, where T_0 is a finite set.

(iii) $u^* \in \mathcal{U}$ *is a solution of Problem 1 if and only if*

$$\frac{d}{dt}V(t, x^*(t)) + \Psi(t, x^*(t), u^*(t)) = 0$$

for almost every $t \in \mathcal{T}$. *Hereby,* x^* *denotes (as usual) the optimal state-trajectory for* $x^*(t_0) = x_0$, *i.e.* $x^* = s(t_0, ., x_0, u^*)$.

Proof.

(i) Clearly, if u is piecewise continuous, then, using $\dot{x} = f(t, x, u)$, the same holds for $\dot{x}$ and hence x is continuous and piecewise differentiable. Hence, using formula (1.18) and the analytic dependency of the solution of an initial value problem on the initial value, we obtain the desired smoothness.

(ii) Lemma 1.4 yields that for any $t_1 < t_2$ and $u \in \mathcal{U}$

$$V(t_1, x_1) - V(t_2, x(t_2)) \leq \int_{t_1}^{t_2} \Psi(t, x(t), u(t))\, dt$$

holds, where equality occurs if and only if u is optimal. Hence, taking $t_1 \to t_2$, we obtain that

$$\lim_{t_1 \to t_2} \Big(V(t_1, x_1) - V(t_2, x(t_2)) \Big) \leq \lim_{t_1 \to t_2} \int_{t_1}^{t_2} \Psi(t, x(t), u(t))\, dt$$

or equivalently, using $t_1 < t_2$:

$$\lim_{t_1 \to t_2} \frac{V(t_1, x_1) - V(t_2, x(t_2))}{t_1 - t_2} - \lim_{t_1 \to t_2} \frac{\int_{t_1}^{t_2} \Psi(t, x(t), u(t))\, dt}{t_1 - t_2} \geq 0$$

$$\Leftrightarrow \frac{d}{dt}V(t_2, x(t_2)) + \Psi(t_2, x(t_2), u(t_2)) \geq 0,$$

everywhere where V is differentiable. Again equality holds if and only if u is an optimal control function. Hence, points (ii) and (iii) are also proved.

□

Finally, we investigate linear quadratic control systems in order to derive a sufficient condition for the optimal control:

Suppose that (σ, J) is a linear quadratic control system. We show here that the function $\tilde{V}(t) = V(t, x(t))$ has the following form:

$$\tilde{V}(t) = x^T(t)E(t)x(t) + e^T(t)x(t) + d(t)$$

for some mappings $E : \mathcal{T} \to \mathbb{R}^{n \times n}$, $e : \mathcal{T} \to \mathbb{R}^n$ and $d : \mathcal{T} \to \mathbb{R}$, where $E(t)$ is symmetric for each $t \in \mathcal{T}$. Note, that since $x(t) = s(t, t_0, x_0, u|_{[t_0,t]})$, $\tilde{V}$ is independent of $u|_{[t,t_f]}$.

Thus,

$$\begin{aligned}
\frac{d}{dt}\tilde{V}(t) &= \dot{x}^T E x + x^T \dot{E} x + x^T E \dot{x} + \dot{e}^T x + e^T \dot{x} + \dot{d} \\
&= \left(x^T A^T + u^T B^T \dot{x}^T\right) EX + x^T \dot{E} x + x^T E \left(Ax + Bu\right) + \dot{e}^T x \\
&\quad + e^T \left(Ax + Bu\right) + \dot{d} + x^T Q x + u^T R u - \Psi \\
&= x^T \left(\dot{E} + Q + EA + A^T E\right) x + u^T R u + x^T E B u + u^T B^T E x \\
&\quad + \left(\dot{e}^T + e^T A\right) x + e^T B u + \dot{d} - \Psi \\
&= x^T \left(\dot{E} + Q + EA + A^T E\right) x + (u-y)^T R (u-y) + y^T R u \\
&\quad + u^T R y - y^T R y + x^T E B u + u^T B^T E x + \left(\dot{e}^T + e^T A\right) x \\
&\quad + e^T B u + \dot{d} - \Psi \\
&= x^T \left(\dot{E} + Q + EA + A^T E\right) x + (u-y)^T R (u-y) \\
&\quad + u^T \left(Ry + B^T E x + \frac{1}{2} B^T e\right) + \left(\frac{1}{2} e^T B + x^T E B + Ry\right) u \\
&\quad + \left(\dot{e}^T + e^T A\right) x + \dot{d} - y^T R y - \Psi
\end{aligned}$$

for any mapping $y : \mathcal{T} \to \mathbb{R}^m$. If we set, for instance

$$Ry + B^T E x + \frac{1}{2} B^T e = 0,$$

so that the terms that are linear in u cancel out then, assuming that $R(t)$ is invertible for each $t \in \mathcal{T}$, we obtain

$$\begin{aligned}
\frac{d}{dt}\tilde{V} &= x^T \left(\dot{E} + Q + EA + A^T E\right) x + (u-y)^T R (u-y) \\
&\quad + \left(\dot{e}^T + e^T A\right) x + \dot{d} - \left(\frac{1}{2} e^T + x^T E\right) \underbrace{B R^{-1} B^T}_{S} \left(EX + \frac{1}{2} e\right) - \Psi \\
&= x^T \left(\dot{E} + Q + EA + A^T E - ESE\right) x + \left(\dot{e}^T + e^T A - e^T S E\right) x \\
&\quad - \frac{1}{4} e^T S e + \dot{d} + (u-y)^T R (u-y) - \Psi
\end{aligned}$$

Assuming that the mappings E, e and d are such that they fulfill

$$\dot{E} + Q + EA + A^T E - ESE = 0 \tag{1.20}$$

$$\dot{e}^T + e^T A - e^T S E = 0 \quad \text{and} \tag{1.21}$$

$$\dot{d} - \frac{1}{4} e^T S e = 0, \tag{1.22}$$

we obtain that

$$\frac{d}{dt}\tilde{V} = (u-y)^T R (u-y) - \Psi.$$

Integrating yields

$$\tilde{V}(t_f) - \tilde{V}(t) = \int_t^{t_f} (u-y)^T R (u-y)\, d\tau - \int_t^{t_f} \Psi\Big(\tau, x(\tau), u(\tau)\Big)\, d\tau \tag{1.23}$$

If we further assume that, besides equations (1.20)-(1.22), the mappings E, e and d are chosen such that the following terminal values

$$\begin{aligned} E(t_f) &= K_f & (1.24)\\ e(t_f) &= 0 \quad \text{and} & (1.25)\\ d(t_f) &= 0 & (1.26) \end{aligned}$$

are attained, then we get that $\tilde{V}(t_f) = x^T(t_f)K_f x(t_f) = \kappa\big(x(t_f)\big)$, which together with equation (1.23) yields

$$\tilde{V}(t) = -\int_t^{t_f} (u-y)^T R(u-y)\, d\tau + \kappa\big(x(t_f)\big) + \int_t^{t_f} \Psi\big(\tau, x(\tau), u(\tau)\big)\, d\tau. \tag{1.27}$$

Note, that the r.h.s. of (1.27) is independent of $u|_{[t_0,t]}$, whereas the l.h.s. is independent of $u|_{[t,t_f]}$. Considering now the infimal values of (1.27) over all possible control function on $[t, t_f]$, we get the following

$$\begin{aligned} V\big(t, x(t)\big) &= \inf_{u|_{[t,t_f]}} \left[\int_t^{t_f} \Psi\big(\tau, x(\tau), u(\tau)\big)\, d\tau + \kappa\big(x(t_f)\big) \right] \\ &= \tilde{V}(t) + \inf_{u|_{[t,t_f]}} \int_t^{t_f} (u-y)^T R(u-y)\, d\tau, \end{aligned}$$

which equals to $\tilde{V}(t)$ if and only if $u - y \equiv 0\ \forall t \in \mathcal{T}$.

Consider now the linear differential equations (1.21) and (1.22): Since the terminal values $e(t_f)$ and $d(t_f)$ are zero, the solution of these equations are the functions that are constant zero. Hence, we obtain the following statement:

Theorem 1.13 *Suppose that (σ, J) is a linear quadratic control system with finite time-horizon $\mathcal{T} = [t_0, t_f]$ such that the coefficient $R(t)$ appearing in the cost functional J is positive definite for each $t \in \mathcal{T}$. Further assume that $E : \mathcal{T} \to \mathbb{R}^{n\times n}$ is a solution of the* matrix Riccati-equation

$$\dot{E} = -A^T(t)E - EA(t) - Q(t) + ES(t)E, \quad \textit{with } E(t_f) = K_f, \tag{1.28}$$

where $S(t)$ denotes the matrix $B(t)R^{-1}(t)B^T(t)$. Then the feedback-control

$$u^*(t) := -R^{-1}(t)B^T(t)E(t)x(t)$$

is optimal.

1.4.2 Linear quadratic control systems on Hilbert-space

Definition 1.17 *Let $\mathcal{H}^n[t_0, t_f]$ be the set of square-integrable functions mapping $[t_0, t_f]$ into $\mathbb{R}^n$. On $\mathcal{H}^n[t_0, t_f]$ we define the following scalar product:*

$$\langle f, g\rangle := f^T(t_f)g(t_f) + \int_{t_0}^{t_f} f^T(t)g(t)\, dt \quad \forall f, g \in \mathcal{H}^n[t_0, t_f].$$

In this subsection we derive a necessary condition for the optimal control of linear quadratic control systems. Throughout this calculation we shall use the notations used in Definition 1.15 and Problem 1 but, for the sake of generality, we assume that the control function can be chosen from the set of square-integrable functions defined on $\mathbb{R}^m$, i.e. that $\mathcal{U} = \mathcal{L}_2^m[t_0, t_f]$.

If we define

$$\bar{Q}(t) := \begin{cases} Q(t) & \text{if } t \in [t_0, t_f) \\ K_f & \text{if } t = t_f \end{cases} \qquad \text{and} \tag{1.29}$$

then the cost functional J can be written for each state trajectory $x \in \mathcal{X}$ and control $u \in \mathcal{U}$ as

$$J(x, u) = \left\langle x, \bar{Q}x \right\rangle_{\mathcal{H}^n[t_0,t_f]} + \langle u, Ru \rangle_{\mathcal{L}_2^m[t_0,t_f]} . \tag{1.30}$$

Furthermore, defining the linear operators

$$\mathcal{L} \; : \; \mathcal{U} \to \mathcal{H}^n[t_0, t_f] \qquad u \mapsto \int_{t_0}^{.} \Phi(., \tau) B(\tau) u(\tau)\, d\tau \qquad \text{and} \tag{1.31}$$

$$\bar{\Phi} \; : \; \mathbb{R}^n \to \mathcal{H}^n[t_0, t_f] \qquad x \mapsto \Phi(., t_0)x \tag{1.32}$$

where $\Phi(t, \tau)$ is the fundamental matrix of the equation (1.5) (see Remark 1.5), then the state trajectory x becomes (see (1.7)

$$x = \bar{\Phi}x_0 + \mathcal{L}u. \tag{1.33}$$

Our aim is now to find a control $u \in \mathcal{U}$ such that the generated state trajectory $x = \bar{\Phi}x_0 + \mathcal{L}u$ minimizes the cost functional J. Substituting (1.33) into (1.30) yields

$$\begin{aligned} J(x, u) &= \left\langle \bar{\Phi}x_0 + \mathcal{L}u, \bar{Q}\left(\bar{\Phi}x_0 + \mathcal{L}u\right) \right\rangle_{\mathcal{H}^n[t_0,t_f]} + \langle u, Ru \rangle_{\mathcal{L}_2^m[t_0,t_f]} \\ &= \langle u, \left(R + \mathcal{L}^*\bar{Q}\mathcal{L}\right) u \rangle_{\mathcal{L}_2^m[t_0,t_f]} + 2\langle u, \mathcal{L}^*\bar{Q}\bar{\Phi}x_0 \rangle_{\mathcal{L}_2^m[t_0,t_f]} + \\ &\quad \langle \bar{\Phi}x_0, \bar{Q}\bar{\Phi}x_0 \rangle_{\mathcal{H}^n[t_0,t_f]} \end{aligned} \tag{1.34}$$

Lemma 1.5 *Let $\mathcal{H}$ be an arbitrary Hilbert-space, $\alpha : \mathcal{H} \to \mathcal{H}$ a self-adjoint operator, $\beta \in \mathcal{H}$ and $\gamma \in \mathbb{R}$. Then the product $\langle x, \alpha x \rangle + 2\langle x, \beta \rangle + \gamma$ has a unique minimum if and only if α is positive definite. In this case this minimum is achieved at $x^* = -\alpha^{-1}\beta$. Furthermore, a necessary condition for the existence of a minimum is that the operator α is positive semidefinite.*

Proof.

(i) First, assume that there exists $\tilde{x} \in \mathcal{H}$ with $\langle \tilde{x}, \alpha\tilde{x} \rangle < 0$. Then the function $\varphi : \mathbb{R} \to \mathbb{R}$ with

$$\varphi(\lambda) := \langle \lambda\tilde{x}, \alpha\lambda\tilde{x} \rangle + 2\langle \lambda\tilde{x}, \beta \rangle + \gamma = \lambda^2 \langle \tilde{x}, \alpha\tilde{x} \rangle + 2\lambda \langle \tilde{x}, \beta \rangle + \gamma$$

is a real quadratic polynomial with negative leading coefficient and hence it cannot have a minimum at all. Therefore the product $\langle x, \alpha x \rangle + 2\langle x, \beta \rangle + \gamma$ does not have a minimum, either.

(ii) Supposing that $\alpha \geq 0$ and that there exists $\tilde{x} \in \mathcal{H}$ with $\langle \tilde{x}, \alpha\tilde{x} \rangle = 0$, we obtain that the function $\varphi(\lambda)$ defined in step (i) is either linear or constant and hence cannot have a unique minimum.

(iii) Assume now that α is positive definite. Then

$$\langle x, \alpha x \rangle + 2\langle x, \beta \rangle + \gamma = \left\langle x + \alpha^{-1}\beta, \alpha(x + \alpha^{-1}\beta) \right\rangle - \left\langle \alpha^{-1}\beta, \beta \right\rangle + \gamma,$$

which has a unique minimum if and only if $\langle x + \alpha^{-1}\beta, \alpha(x + \alpha^{-1}\beta) \rangle$ is minimal, but since $\alpha > 0$, it is for $x + \alpha^{-1}\beta \neq 0$ always positive, so the unique minimum is achieved for $x + \alpha^{-1}\beta = 0$.

□

Together with (1.34) this latter lemma yields some criteria for the existence and uniqueness of an optimal control:

Corollary 1.3 *Let (σ, J) be a linear quadratic optimal control system. A necessary condition for the solvability of Problem 1 is that the operator $R + \mathcal{L}^*\bar{Q}\mathcal{L}$ is positive semidefinite. Furthermore, Problem 1 is uniquely solvable if and only if $R + \mathcal{L}^*\bar{Q}\mathcal{L}$ is positive definite. In this case the optimal control is given by*

$$u^* = -\left(R + \mathcal{L}^*\bar{Q}\mathcal{L}\right)^{-1}\left(\mathcal{L}^*\bar{Q}\bar{\Phi}x_0\right). \tag{1.35}$$

Equations (1.35) and (1.33) yield

$$\begin{aligned} -\left(R + \mathcal{L}^*\bar{Q}\mathcal{L}\right)u^* &= \mathcal{L}^*\bar{Q}\bar{\Phi}x_0 = \mathcal{L}^*\bar{Q}\left(x^* - \mathcal{L}u^*\right) \\ \Rightarrow Ru^* &= -\mathcal{L}^*\bar{Q}x^*. \end{aligned}$$

Hereby, $x^* = \bar{\Phi}x_0 + \mathcal{L}u^*$ denotes the optimal state-trajectory. Altogether we obtain the following theorem:

Theorem 1.14 *Let (σ, J) be a linear quadratic optimal control system with $\mathcal{T} = [t_0, t_f]$ and $x_0 \in X$. Assume that the matrix K_f is positive semidefinite and that $\forall t \in [t_0, t_f]$ the matrix $R(t)$ is positive definite and $Q(t)$ is positive semidefinite. Then the solution of Problem 1 exists uniquely and the optimal control u^* is given by*

$$u^* = -R^{-1}\mathcal{L}^*\bar{Q}x^*, \tag{1.36}$$

where $\bar{Q}$ is defined as in equation (1.29) and $\mathcal{L}^*$ denotes the adjoint operator to $\mathcal{L}$ (see (1.31). Furthermore, x^* denotes the optimal state trajectory, i.e

$$x^* = \bar{\Phi}x_0 + \mathcal{L}u^*, \tag{1.37}$$

with Φ defined in (1.32).

Proof. Since every other statement follows using Corollary 1.3 if R and $R + \mathcal{L}^*\bar{Q}\mathcal{L}$ are positive definite, we only show these properties:

(i) Let $u \in \mathcal{U}$ be an arbitrary control function. Then

$$\left\langle u, \mathcal{L}^*\bar{Q}\mathcal{L}u \right\rangle_{\mathcal{L}_2^m[t_0,t_f]} = \left\langle \mathcal{L}u, \bar{Q}\mathcal{L}u \right\rangle_{\mathcal{H}^n[t_0,t_f]} = \langle \mathcal{L}u, Q\mathcal{L}u\rangle_{\mathcal{L}_2^n[t_0,t_f]} + \langle \mathcal{L}u(t_f), K_f\mathcal{L}u(t_f)\rangle_{\mathbb{R}^n}.$$

Since $K_f \geq 0$, $\forall x \in \mathbb{R}^n$ $\langle \mathcal{L}u(t_f), K_f\mathcal{L}u(t_f)\rangle_{\mathbb{R}^n} \geq 0$ holds.

On the other hand

$$\langle \mathcal{L}u, Q\mathcal{L}u\rangle_{\mathcal{L}_2^n[t_0,t_f]} = \int_{t_0}^{t_f} (\mathcal{L}u(t))^T Q(t) (\mathcal{L}u(t))\, dt \geq 0,$$

since $Q(t) \geq 0$ $\forall t \in [t_0, t_f]$. This yields

$$\mathcal{L}^*\bar{Q}\mathcal{L} \geq 0$$

(ii) Similarly $\forall u \in \mathcal{U}\backslash\{0\}$

$$\langle u, Ru\rangle_{\mathcal{L}_2^m[t_0,t_f]} = \int_{t_0}^{t_f} u^T(t)R(t)u(t)\, dt$$

and since $R(t) > 0$ $\forall t \in [t_0, t_f]$, it is also

$$\langle u, Ru\rangle_{\mathcal{L}_2^m[t_0,t_f]} > 0.$$

Hence the operator R is positive definite.

□

Our aim now is to describe the relation for the optimal control (1.36) more explicitely, i.e. by solutions of certain differential equations. To this end, we first need to construct the adjoint operator $\mathcal{L}^*$:

Lemma 1.6 *Let $\mathcal{L}$ be the operator defined in (1.31). Then the adjoint operator $\mathcal{L}^* : \mathcal{H}^n[t_0, t_f] \to \mathcal{L}_2^m[t_0, t_f]$ is given by*

$$\mathcal{L}^*y := B^T(.) \left[\Phi^T(t_f, .)y(t_f) + \int_{.}^{t_f} \Phi^T(t, .)y(t)\, dt \right]. \tag{1.38}$$

Proof. $\forall y \in \mathcal{H}^n[t_0, t_f]$ and $u \in \mathcal{L}_2^m[t_0, t_f]$

$$\langle y, \mathcal{L}u\rangle_{\mathcal{H}^n[t_0,t_f]} = \int_{t=t_0}^{t_f} y^T(t) \int_{\tau=t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\, d\tau\, dt$$

$$
\begin{aligned}
&+ \; y^T(t_f) \int_{\tau=t_0}^{t_f} \Phi(t_f,\tau)B(\tau)u(\tau)\,d\tau \\
&= \int_{t=t_0}^{t_f} \int_{\tau=t_0}^{t} \left(B^T(\tau)\Phi^T(t,\tau)y(t)\right)^T u(\tau)\,d\tau\,dt \\
&+ \int_{\tau=t_0}^{t_f} \left(B^T(\tau)\Phi^T(t_f,\tau)y(t_f)\right)^T u(\tau)\,d\tau
\end{aligned}
$$

holds. Interchanging the order of integration (see also Figure 1.2) yields

$$
\begin{aligned}
&\int_{t=t_0}^{t_f} \int_{\tau=t_0}^{t} \left(B^T(\tau)\Phi^T(t,\tau)y(t)\right)^T u(\tau)\,d\tau\,dt = \\
&\int_{\tau=t_0}^{t_f} \int_{t=\tau}^{t_f} \left(B^T(\tau)\Phi^T(t,\tau)y(t)\right)^T u(\tau)\,dt\,d\tau
\end{aligned}
$$

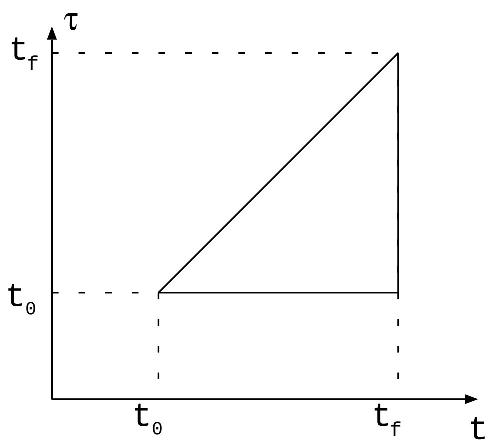

Figure 1.2: The domain of integration

Altogether we obtain

$$
\begin{aligned}
&\langle y, \mathcal{L}u\rangle_{\mathcal{H}^n[t_0,t_f]} = \langle \mathcal{L}^* y, u\rangle_{\mathcal{L}_2^m[t_0,t_f]} \\
&= \int_{\tau=t_0}^{t_f} \left[\left(B^T(\tau)\Phi^T(t_f,\tau)y(t_f)\right)^T + \int_{t=\tau}^{t_f} \left(B^T(\tau)\Phi^T(t,\tau)y(t)\right)^T dt\right] u(\tau)\,d\tau
\end{aligned}
$$

and hence

$$
\mathcal{L}^* y(\tau) = B^T(\tau)\Phi^T(t_f,\tau)y(t_f) + \int_{t=\tau}^{t_f} B^T(\tau)\Phi^T(t,\tau)y(t)\,dt.
$$

□

We can now express the optimal control function in terms of integrations. Using (1.36) and (1.38) we obtain that

$$
u^*(t) = -R^{-1}B^T \left[\Phi^T(t_f,t)K_f x^*(t_f) + \int_{t}^{t_f} \Phi^T(\tau,t)Q(\tau)x^*(\tau)\,d\tau\right].
$$

If we suppose that the expression

$$\Phi^T(t_f,t)K_f x^*(t_f) + \int_t^{t_f} \Phi^T(\tau,t)Q(\tau)x^*(\tau)\,d\tau$$

has the form $K(t)x^*(t)$ for some matrix-valued function $K : \mathcal{T} \to \mathbb{R}^n$, then we obtain that the optimal control can be written as

$$u^*(t) = -R^{-1}(t)B^T(t)K(t)x^*(t). \tag{1.39}$$

Indeed,

$$\begin{aligned} \frac{d}{dt}(K(t)x^*(t)) &= \dot{K}x^* + K\dot{x}^* = \dot{K}x^* + K(Ax^* + Bu^*) \\ &= \dot{K}x^* + KAx^* - K\underbrace{BR^{-1}B^T}_{S(t)}Kx^*. \end{aligned}$$

On the other hand,

$$\frac{d}{dt}\left[\Phi^T(t_f,t)K_f x^*(t_f) + \int_t^{t_f} \Phi^T(\tau,t)Q(\tau)x^*(\tau)\,d\tau\right] = \dot{\Phi}^T(t_f,t)K_f x^*(t_f)$$
$$+\int_t^{t_f} \frac{d}{dt}\Phi^T(\tau,t)Q(\tau)x^*(\tau)\,d\tau - \Phi^T(t,t)Q(t)x^*(t)$$

and since

$$\frac{d}{dt}\Phi^T(\tau,t) = -A^T(t)\Phi^T(\tau,t) \quad \text{and } \Phi^T(\tau,\tau) = I,$$

we obtain that

$$\frac{d}{dt}\left[\Phi^T(t_f,t)K_f x^*(t_f) + \int_t^{t_f} \Phi^T(\tau,t)Q(\tau)x^*(\tau)\,d\tau\right]$$
$$= -A^T \underbrace{\left[\Phi^T(t_f,t)K_f x^*(t_f) + \int_t^{t_f} \Phi^T(\tau,t)Q(\tau)x^*(\tau)\,d\tau\right]}_{K(t)x^*(t)} - Q(t)x^*(t)$$

Finally, the equality of the above expressions yields that

$$\left[\dot{K}(t) + K(t)A(t) - K(t)S(t)K(t)\right]x^*(t) = \left[-A^T(t)K(t) - Q(t)\right]x^*(t)$$

And hence we obtain the following necessary and sufficient condition for the optimality:

Theorem 1.15 *Let (σ, J) be a linear quadratic optimal control system with finite time horizon $\mathcal{T} = [t_0, t_f]$, such that the matrices $R(t)$ and $Q(t)$ are positive definite and positive* *semidefinite for all $t \in \mathcal{T}$, respectively. Assume further that the matrix Riccati differential equation*

$$\dot{K} = A^T(t)K - KA(t) - Q(t) + KS(t)K, \quad \text{with } K(t_f) = K_f,$$

admits a solution $K : \mathcal{T} \to \mathbb{R}^{n\times n}$ for which $K(t_f) = K_f$ holds. Then the optimal control $u^*(t)$ is

$$u^*(t) = -R^{-1}(t)B^T(t)K(t)x(t),$$

where $S(t)$ denotes the matrix $B(t)R^{-1}(t)B^T(t)$.

1.5 Notes and references

The previous chapter gave an introduction to Control Theory. We discussed the most important results and some of the classical theorems (Theorems 1.9 and 1.10). We also developed a construction for the state-space (Theorem 1.8) that is an improved consequence of the well-known *controllability canonical form* for linear systems. A more complete discussion of this canonical form and also of the other topics of Linear Control Theory can be found in the classical books [KK85] and [Bro70].

We also briefly discussed linear quadratic control systems. This is a classical and most elegant way to connect Optimization Theory and Control Theory. In the rest of this work, we shall see several further connections of these topics in the context of Game Theory. Although, the above mentioned books also contain elementary results on Optimal Control Theory, there is a huge amount of other sources covering this topic (for instance see [AM90], [Hes66] and [Son90]). Furthermore, a detailed discussion of the Theory of Differential Equations can be found in the classical textbooks [CL55] and [KK74].

Some of the topics, we didn't cover here, will be mentioned later on in this work. For a discussion on Nonlinear Control Theory see Chapter 6. The theory of Riccati Differential Equations will be presented in Chapter 3, where we also see some very important stabilizability properties of optimal control functions. Finally, the theory of disturbed control systems will be covered in Chapter 4.

Although discrete systems become a lot of interest nowadays, in the framework of this Thesis only continuous-time systems are discussed. Results on control systems defined on a discrete time-horizon can be found in the text of Kučera [Kuč79].

Chapter 2

Noncooperative dynamical games

In the previous chapter we discussed 'non-isolated' dynamical systems, i.e. systems that are controlled by external inputs. In some cases, we could also calculate an *optimal* input, by means of a cost functional. Here, we generalize this structure and investigate, what happens if more than one external input is acting on the system and all these inputs try to optimize a different cost functional. As we shall see later, the structure of differential games is much more complicated, than the one discussed for control systems. Usually, the inputs cannot fulfill their optimality constraints at the same time and hence equilibrium models become very important. On the other hand, since the inputs are independent of each other and none of them knows the strategy of the others, the state of the system cannot be reconstructed with the help of the own control law and the initial data. Hence the information structures, i.e. system-state-feedback models, play a very important role in the context of differential games. In the first section, we discuss these structures.

Then, in Section 2.2, we generalize some properties that were introduced in Section 1.3 and also derive conditions for them. Finally, we derive optimal control laws for several types of linear-quadratic differential games.

2.1 Equilibrium models and information structures

Let us introduce a simple example, before we derive abstract definitions for differential games and equilibrium strategies.

Example 4: Consider a 'black-box' governed by exactly two inputs (u_1 and u_2), that are constant real numbers. Suppose further, that according to the system state (i.e. in dependence of the two inputs, that determine this state) the corresponding cost functionals (J_1 and J_2, respectively) can be obtained by Figure 2.1. Finally assume, that the players do not communicate during the game and hence the strategy (i.e. the chosen input) of the first one is not known by the second one and vice versa.

How can we define the optimality in the sense of minimizing the costs ? The first direct way could be, that Player 1 sets u_1^0 and Player 2 u_2^0. Then the system itself goes to point (u_1^0, u_2^0) and hence the players receive higher costs than expected. Do the players regret their input ? Definitely ! Player 2 thinks: "If I had used a control $u_2^*(u_1^0)$, I would have received much less costs." And he is right, because Player 1 has no information on the choice of Player 2, and hence wouldn't have changed his choice.

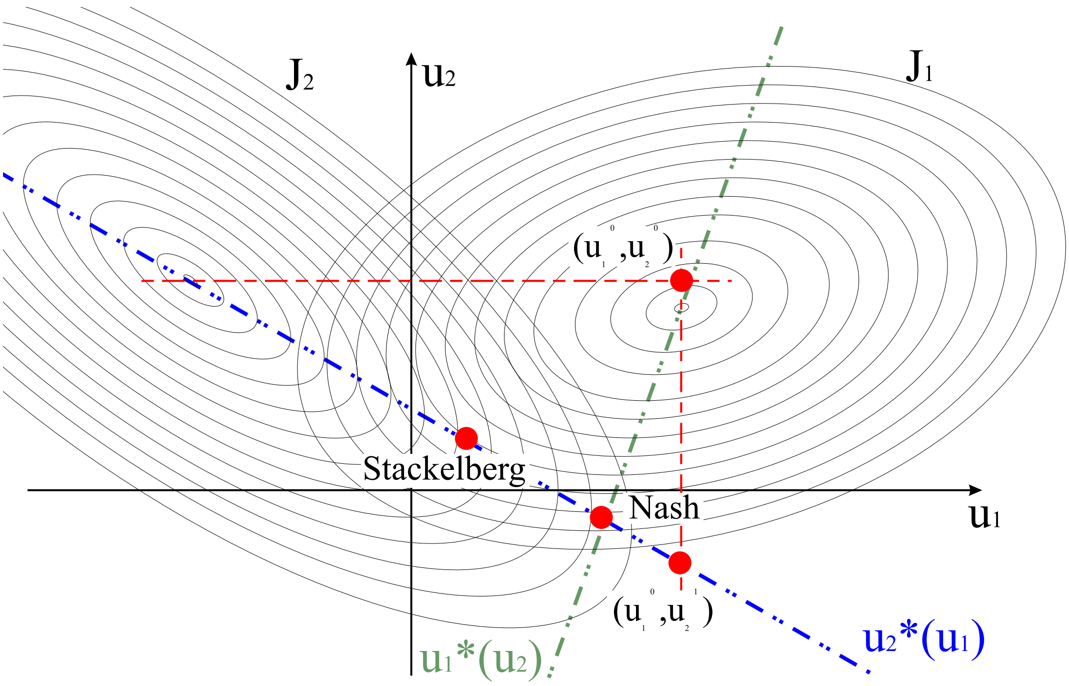

Figure 2.1: Optimality constraints of the players.

And hence Player 2 don't regret his decision, if and only if he chooses a strategy, that lies on a curve for which $\frac{\partial J_2(u_1,u_2)}{\partial u_2} = 0$. This curve is denoted by $u_2^*(u_1)$. If we suppose that there is no communication between the players, then Player 1 also chooses his strategy on the line with $\frac{\partial J_1(u_1,u_2)}{\partial u_1} = 0$ (curve denoted by $u_1^*(u_2)$). And hence the only situation, for which none of the players have a regret is at the intersection point of these lines. In this case, if the game is repeated, the players would choose the same strategies, since anything else would result more costs. This situation is called *equilibrium*. Indeed, this is the so-called *Nash equilibrium*, which we will discuss later in more details.

Suppose now, that the players have another rule in the game: The first player chooses his control law and announces it to the second player. Then the second player chooses a control using this knowledge. This is a typical situation in the economics. The government announces the tax rates for the next year and – according to these laws – the companies must meet their optimum (preferably without regretting it).

Here, the structure is more complicated. Player 2 has an easy task: using the curve for which $\frac{\partial J_2(u_1,u_2)}{\partial u_2} = 0$, he can simply find the best control for any announcement of Player 1. The question now is how to find the best control for the first player. He knows, that if he chooses u_1, Player 2 will play $u_2^*(u_1)$ and hence he is interested in a control for which $\frac{dJ_1(u_1,u_2^*(u_1))}{du_1} = 0$. He finds his optimality exactly where the curve $u_2^*(u_1)$ is tangential to one of the niveaulines of J_2. Again, if the game is now repeated, Player 1 will announce the same strategy and hence, Player 2 won't change, either. The resulting equilibrium is called *Stackelberg equilibrium*.

Now, let us try to develop – similarly to Definition 1.1 – an axiomatic approach to differential games:

Definition 2.1 (differential game) *Let N be a positive integer. Then we call the following* *object*

$$\Gamma_N = (\mathcal{T}, X, U_i, \mathcal{U}_i, \sigma_i, f, \eta_i, J_i)_{i=1,\ldots,N}$$

differential game*, if for all $i \in \{1, \ldots, N\}$ the following properties hold:*

- *$\mathcal{T}$ is a real interval,*
- *X and U_i are finite dimensional Euclidean spaces,*
- *f is a mapping defined on $\mathcal{T} \times X \times U_1 \times \ldots \times U_N$ such that $f(t, x, u_1, \ldots, u_N)$ is piecewise continuous in each $u_i \in U_i$ and $t \in \mathcal{T}$ for each fixed $x \in X$ and Lipschitz-continuous in $x \in X$ for each tuple $(t, u_1, \ldots, u_N) \in \mathcal{T} \times U_1 \times \ldots \times U_N$.*
- *σ_i is a subset of $\{\gamma_i | \gamma_i : \mathcal{T} \times \mathcal{P}(X) \to U_i\}$,*
- *η_i is a mapping $\mathcal{T} \to \mathcal{P}(X)$ with the property that $\eta_i(t) \subset \{x(s) | t_0 \leq s \leq t\}$,*
- *$\mathcal{U}_i = \{\gamma_i(., \eta_i(.)) | \gamma_i \in \sigma_i\}$ and finally*
- *J_i is a real-valued functional defined on $\mathcal{U}_1 \times \ldots \times \mathcal{U}_N \to \mathbb{R}$.*

Remark 2.1 *The usual terminology for the objects appearing in definition 2.1 is the following: similarly to control systems, X, $\mathcal{T}$, U_i and $\mathcal{U}_i$ are called state space, time horizon, control-value space and control-space, respectively. The set σ_i is called set of* possible strategies, *η_i* information structure *and finally, we say that J_i is the* cost functional *of the i-th player. Usually, we call the input function u_i* "player"*.*

Remark 2.2 *In the rest of this Thesis, we shall only consider the following three information structures*

(i) feedback *: $\eta_i(t) = \{x(t)\}$*

(ii) open-loop *: $\eta_i(t) = \{x_0\}$*

(iii) sampled-data *: $\eta_i(t) = \{x(t_k) | t_k \leq t \leq t_{k+1}\}$ for some (finite or infinite) fixed grid*

$$\tau = (t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots)$$

with $\sup \tau = \sup \mathcal{T}$*.*

Now, we investigate the dependence of the state trajectory on the actual strategy $\gamma_i \in \sigma_i$ $(i = 1, \ldots, N)$:

Theorem 2.1 (see also Theorem 1.1 and [BO95]) *Let Γ_N be an N-player differential game. Suppose further that the mapping f has the following properties*

(i) $f(t, x, u_1, \ldots, u_N)$ is continuous in t for any $x, u_1, \ldots, u_n \in X \times U_1 \times \ldots \times U_N$

(ii) f is Lipschitz continuous in $x, u_1, \ldots, u_N$,

(iii) for any $\gamma_i \in \sigma_i$, the function $\gamma_i(t, x)$ is continuous in t and Lipschitz-continuous in x.

Then, the solution of the differential equation

$$\dot{x}(t) = f(t, x(t), u_1(t), \ldots, u_N(t))$$

exists (at least for a short period of time) uniquely for any initial value $x_0 \in X$ *and set of controls* $(u_1, \ldots, u_N) \in \mathcal{U}_1 \times \ldots \times \mathcal{U}_N$. *Furthermore, these state trajectories are continuous functions of the time.*

Proof. As for differential control systems, the statement is a direct consequence of Lindelöf's Theorem (see for instance [CL55] or [KK74]) on the existence and uniqueness of the solution of differential equations. □

Now, we are very close to define a no-regret-situation, as it was already illustrated in the introductory example. For this, we first need to define the best reply of a player:

Definition 2.2 (best reply) *Let* Γ_N *be an* N*-player differential game. Assume further, that some for* $i \in 1, \ldots, N$ $\gamma_{(-i)} := (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_N) \in \sigma_1 \times \ldots \times \sigma_{i-1} \times \sigma_{i+1} \times \ldots \times \sigma_N$ *be an* $(N-1)$*-tuple of strategies. We say that* $\tilde{\gamma}_i$ *is the* best reply *against* $\gamma_{(-i)}$ *if*

$$J_i(u_1, \ldots, u_{i-1}, \tilde{u}_i, u_{i+1}, \ldots, u_N) \leq J_i(u_1, \ldots, u_N)$$

holds for any $u_i \in \mathcal{U}_i$. *Hereby denotes* u_j $(j = 1, \ldots, N)$ *the actual control induced by the strategy and the information structure:*

$$u_j = \gamma_j(t, \eta_j(t)),$$

as well as $\tilde{u}_i = \tilde{\gamma}_i(t, \eta_i(t))$. *We denote the set of all best replies by* $\mathcal{R}_i(\gamma_{(-i)})$.

Next, we define the already illustrated Nash equilibrium for general N-player differential games.

Definition 2.3 (Nash equilibrium) *Suppose that* Γ_N *is an* N*-player differential game. We say that the* N*-tuple of strategies* $(\gamma_1, \ldots, \gamma_N)$ *form a Nash equilibrium, if* $\gamma_i \in \mathcal{R}_i(\gamma_{(-i)})$ *holds for all* $i = 1, \ldots, N$.

Please note that above defined equilibrium *cannot be considered as generalization for the equilibrium of a control system* defined in Definition 1.6. In the context of differential games, *equilibria are not the constant solutions* of the autonomous dynamical system.

A very important question is the existence and uniqueness of the above defined Nash-equilibrium.

Definition 2.4 (playability – see also Section 1 in [LR71]) *We call an* N*-player differential game* (Nash)-playable *if it possesses exactly one* N*-tuple of strategies, that lead to a Nash-equilibrium.*

Remark 2.3 (see also Remark 3.2 in [Eis82]) *The fact that a game is not playable does not mean that it does not have an equilibrium state. As it is shown in [LR71] and [Eis82] in Nash games non-unique equilibrium states can also occur.*

Finally, similarly to linear-quadratic control systems (see Definition 1.11), we define linear-quadratic differential games.

Definition 2.5 (linear quadratic differential game) *Let Γ_N be an N-player differential game finite time-horizon $\mathcal{T} = [t_0, t_f]$. Suppose further, that the mapping f and the functionals J_i have the following properties:*

(i) *$f(t, x, u_1, \ldots, u_N)$ is linear for all fixed $t \in \mathcal{T}$, i.e*

$$f(t, x, u_1, \ldots, u_N) = A(t)x + \sum_{i=1}^{N} B_i(t)u_i$$

for some matrix-valued functions $A(t) \in \mathbb{R}^{n\times n}$ and $B_i(t) \in \mathbb{R}^{n\times m_i}$ for $i = 1, \ldots, N$ and $t \in \mathcal{T}$ and

(ii) *for all $i = 1, \ldots, N$*

$$J_i(u_1, \ldots, u_N) = x^T(t_f)K_{if}x(t_f) + \int_{t_0}^{t_f} x^T(t)Q_i(t)x(t) + \sum_{j=1}^{N} u_j^T(t)R_{ij}(t)u_j(t)\, dt$$

holds for symmetric matrices $K_{if} \in \mathbb{R}^{n\times n}$ and continuous symmetric matrix-valued functions $Q_i(t) \in \mathbb{R}^{n\times n}$ and $R_{ij}(t) \in \mathbb{R}^{m_j\times m_j}$

Then we call Γ_N linear-quadratic N-player differential game*.*

Differential games of this type play a crucial role in the theory of differential games. In fact, up to a recent work of Sastry et al. (see [TPS98]), one can hardly find a paper on differential games, that goes beyond this restriction. Nevertheless, in Chapter 8 of this work, we shall also consider nonlinear differential games.

2.2 Controllability and stabilizability of linear quadratic games

As far as the author is concerned, no publications have been made on a rigorous mathematical investigation of the controllability of differential (or dynamical) games. Maybe, this is due to the fact, that "everything looks like as it would be the same as for control systems". Nevertheless, as recent results on Riccati equations show, there is a big need for this investigation to treat generalized Riccati differential and algebraic Riccati equations in the same manner as standard Riccati equations are treated. (For more details see Chapter 3.) On the other hand, the discussion of well-posed boundary constraints for games with boundary conditions (as it is done in Chapter 8) also requires a more detailed treatment of this question.

Hence, in this section, we first investigate what controllability and stabilizability mean in the context of differential games, give definitions for them and finally discuss conditions to decide whether a (linear) differential game is controllable and stabilizable.

Let us recall our definition for the controllability of control systems. Roughly speaking, we said that a control system σ is controllable if for any starting time and starting point, every other point of the state space was within finite time reachable. Here, the problem is more difficult: each player chooses an independent strategy and hence there could be players with and without the ability to reach every state. The following example should illustrate this *non-cooperative* situation:

Example 5: Let Γ_2 be a two-player game with the following dynamics:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u_2,$$

with $u_1(t) \in \mathbb{R}$, $u_2(t) \in \mathbb{R}^2$. Suppose further, that each player receives feedback information from the current system state. If then u_2 uses the strategy $\gamma_2(t, \eta_2) = x(t)$, then u_1 does not have complete control on the state trajectory (to be more precise, he cannot control the second component x_2) and hence for him the system is not controllable. Nevertheless, independently of the strategy γ_1, the second player u_2 has always complete control on the system, and hence for him the system is controllable.

As this second example shows, there are even more complicated cases

Example 6: Let Γ_2 be a two-player game with the following dynamics:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2,$$

with $u_1(t) \in \mathbb{R}$ and $u_2(t) \in \mathbb{R}$. Similarly to the latter example, none of the players have complete control on the state. Nevertheless, if they act *as a team*, they can always find a pair of strategies (γ_1, γ_2) such that complete controllability is achieved. To prove this, just note that if u_1 and u_2 choose their strategies together (as if they were playing "in the same team"), we obtain the following control system for this "team":

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Which is clearly controllable by Theorem 1.5.

Now, we can give a first definition for the so-called *team-controllability* of a differential game:

Definition 2.6 (team-controllability) *Let Γ_N be an N-player differential game. We say that the game is* team-controllable *if for any initial and terminal states $x_0, x_1 \in X$ and initial time $t_0 \in \mathcal{T}$ there exist a terminal time $t_1 > t_0$ and a set of control functions $u_i \in \mathcal{U}_i$ $(i = 1, \ldots, N)$ such that for the solution of the differential equation*

$$\dot{x} = f(t, x(t), u_1(t), \ldots, u_N(t)), \quad x(t_0) = x_0$$

$x(t_1) = x_1$ holds.

Clearly, conditions on the team-controllability can be easily obtained for differential games:

Lemma 2.1 *Let Γ_N be an N-player differential game and*

$$\Sigma = (\mathcal{T}, X, U_1 \times \ldots \times U_N, \mathcal{U}_1 \times \ldots \times \mathcal{U}_N, f^*)$$

be a (differentiable) control system with $f^*(t,x,u) = f(t,x,u_1,...,u_N)$ and $u = (u_1,...,u_N)$. Then Γ_N is team controllable, if and only if Σ is controllable.

Corollary 2.1 (see also Theorem 1.5) *Let Γ_N be a differential game with autonomous linear dynamics. Then Γ_N is team-controllable if and only if the matrix*

$$K = (B_1, \ldots, B_N, AB_1, \ldots, AB_N, \ldots, A^{n-1}B_1, \ldots, A^{n-1}B_N)$$

is of rank n.

From now on, we turn our attention to the non-cooperative situation. This means, that we are interested in condition for the *individual* controllability of differential games. As we shall see later, the definitions also depend on the considered equilibrium model. Here, our only concern are Nash games, but (as it will be pointed out) similar definitions for further equilibria can be stated in the same way.

We begin our discussion with a generic definition:

Definition 2.7 (individual controllability) *Let Γ_N be an N-player differential game. Suppose that the strategies are chosen such that $(\gamma_1^*, \ldots, \gamma_N^*)$ is an equilibrium for Γ_N. Then, we say that the game is* controllable at this equilibrium point from the point of view of the i^{th} player, *if the control system*

$$\dot{x} = f(t, x(t), \gamma_1^*, \ldots, \gamma_{i-1}^*, u_i(t), \gamma_{i+1}^*, \ldots, \gamma_N^*) = f^*(t, x(t), u_i(t))$$

is controllable in $u_i \in \mathcal{U}_i = \{\gamma_i(., \eta_i(.)) | \gamma_i \in \sigma_i\}$.

Remark 2.4 *It is not necessary to suppose that $(\gamma_1^*, \ldots, \gamma_N^*)$ defines an equilibrium for the game. Nevertheless, applications usually require only those situations and hence we stick to this definition.*

To illustrate the meaning of this essentially very abstract generic definition, we present special cases for different Nash equilibria. Then, together with the results presented in Section 1.3, the characterization of *individually* controllable linear differential games becomes obvious.

Lemma 2.2 *Let Γ_N be a linear quadratic differential game with open-loop information structure. Suppose further, that the strategies $\gamma_1, \dots, \gamma_N$ form a Nash equilibrium based on solutions $K_j(t)$ of the corresponding so-called generalized open-loop matrix Riccati differential equations (2.7) (as discussed in Theorem 2.5) for $j = 1, \dots, N$. Then, Γ_N is individually controllable for the i^{th} player if and only if any triple $(t_0, x_0, x_0) \in \mathcal{T} \times X^2$ of the following linear control system*

$$\frac{d}{dt}\begin{pmatrix} x \\ x^* \end{pmatrix} = \begin{pmatrix} A(t) & -\sum\limits_{\substack{j=1 \\ j\neq i}}^{N} S_{jj}(t)K_j(t) \\ 0 & A(t) - \sum\limits_{j=1}^{N} S_{jj}(t)K_j(t) \end{pmatrix} \begin{pmatrix} x \\ x^* \end{pmatrix} + \begin{pmatrix} B_i \\ 0 \end{pmatrix} u_i$$

can be controlled to an element of the set $\{x_1\} \times X$ for all $x_1 \in X$.

Proof. Theorem 2.5 yields that the optimal controls u_j^* can be written in the form

$$u_j^*(t) = -R_{jj}^{-1}(t)B_j^T(t)K_j(t)x^*(t),$$

where K_j and x^* is the solution of (2.7) and

$$\dot{x}^* = \left(A(t) - \sum_{j=1}^{N} S_{jj}(t)K_j(t) \right) x^*, \quad x^*(t_0) = x_0,$$

respectively. Hence, the above control system represents the system trajectory for the set of controls $(u_1^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_N^*)$ under the open-loop information structure. Thus, the statement becomes an obvious consequence of the Definition 2.7. □

The next simple lemma is analogous to the latter one for feedback information structure.

Lemma 2.3 *Let Γ_N be a linear quadratic differential game with feedback information structure. Suppose further, that the strategies $\gamma_1, \dots, \gamma_N$ form a Nash equilibrium and that the conditions of Theorem 2.8 are fulfilled so that the corresponding set of so-called generalized feedback matrix Riccati equations (2.24) admit solutions $K_j(t)$ for $j = 1, \dots, N$. Then, Γ_N is individually controllable for the i^{th} player if and only if the linear control system*

$$\dot{x} = \left(A(t) - \sum_{\substack{j=1 \\ j\neq i}}^{N} S_{jj}(t)K_j(t) \right) x + B_i(t)u_i$$

is controllable.

Proof. Similarly to the latter lemma, we conclude that, using Theorem 2.8, the feedback optimal controls can be written in the form

$$u_j^*(t, x) = -R_{jj}^{-1}(t)B_j^T(t)K_j(t)x,$$

where, again, K_j is the solution of (2.24). Hence, repeating the proof of the latter lemma, the above control system represents the system trajectory for the set of controls

$$(u_1^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_N^*)$$

under the feedback information structure. □

We now briefly discuss stabilizability questions of differential games. Similarly to controllability matters, we can distinguish between the team and individual stabilizability of differential games.

Definition 2.8 (team-stabilizability) *Let Γ_N be an N-player linear differential game defined on an infinite time-horizon. We say that the game is* team-stabilizable *if for any initial state $x_0 \in X$ and initial time $t_0 \in \mathcal{T}$ there exists a set of control functions $u_i \in \mathcal{U}_i$ $(i = 1, \dots, N)$ such that for the solution of the differential equation*

$$\dot{x} = f(t, x(t), u_1(t), \dots, u_N(t)), \quad x(t_0) = x_0$$

$\lim_{t \to \infty} x(t) = 0$ *holds.*

Definition 2.9 (individual stabilizability) *Let Γ_N be an N-player linear differential game defined on an infinite time-horizon. Suppose that the strategies are chosen such that they form an equilibrium for Γ_N. Then, we say that the game is* stabilizable at this equilibrium point from the point of view of the i^{th} player*, if the control system*

$$\dot{x} = f(t, x(t), \gamma_1^*, \dots, \gamma_{i-1}^*, u_i(t), \gamma_{i+1}^*, \dots, \gamma_N^*) = f^*(t, x(t), u_i(t))$$

is stabilizable in $u_i \in \mathcal{U}_i = \{\gamma_i(., \eta_i(.)) | \gamma_i \in \sigma_i\}$.

Similarly to the results achieved for the controllability, one can easily prove the following simple statements for stabilizability of linear differential games. It is also possible to derive conditions for individual stabilizability in the manner of Lemmas 2.2 and 2.3, but since infinite time-horizon optimality constraints are not discussed in this Thesis, we omit the discussion of the corresponding stabilizability questions, too.

Lemma 2.4 *Let Γ_N be an N-player linear differential game defined on an infinite time-horizon and*

$$\Sigma = (\mathcal{T}, X, U_1 \times \ldots \times U_N, \mathcal{U}_1 \times \ldots \times \mathcal{U}_N, f^*)$$

be a (linear) control system with $f^*(t, x, u) := f(t, x, u_1, \dots, u_N)$ for $u = (u_1, \dots, u_N)$. Then Γ_N is team stabilizable, if and only if Σ is stabilizable.

2.3 Optimal strategies for linear quadratic Nash games

In this section we give, similarly to Section 1.4, conditions to determine, whether a set of strategies $(\gamma_1, \dots, \gamma_N)$ or equivalently, the induced control functions form a Nash equilibrium.

We consider three different information structures: first, we give a necessary and sufficient condition for an open-loop equilibrium, and we also discuss, under which conditions is this equilibrium unique. Then, we briefly develop a necessary condition for feedback Nash games.

Here, we are going to use a tool, that is very often mentioned in the literature concerning linear-quadratic control problems: *the Hamilton-Jacobi equations.* Finally, we discuss, how a Nash equilibrium arises, if we assume that the players can only receive information from the system at predefined times.

Our aim now is to describe the control function of the i-th player in a Nash equilibrium. According to Definition 2.3, the strategies $(\gamma_1, \ldots, \gamma_N)$ and hence the induced controls $(u_1, \ldots, u_N)$ form a Nash equilibrium if and only if for each $i = 1, \ldots, N$ $\gamma_i \in \mathcal{R}_i(\gamma_{(-i)})$ holds. Thus, we can suppose that the players $1, \ldots, i-1, i+1, \ldots, N$ already found their optimal strategies γ_j^* $(j \neq i)$ and all we have to do is to find the remaining optimal strategy u_i^* as a solution of the following variational problem:

Problem 2 *Find the control $u_i^* \in \mathcal{U}_i$, for which the following functional*

$$J_i^* = x^T(t_f)K_{if}x(t_f) + \int_{t_0}^{t_f} \left(x^T(t)Q_i(t)x(t) + \sum_{\substack{j=1 \\ j\neq i}}^{N} u_j^*(t)^T(t)R_{ij}(t)u_j^*(t) + u_i^T R_{ii}(t)u_i \right) dt$$

is minimal subject to the constraints

$$u_j^*(t) = \gamma_j^*(t, \eta_j(t))$$

and

$$\dot{x} = A(t)x(t) + \sum_{\substack{j=1 \\ j\neq i}}^{N} B_j\gamma_j^*(t, \eta_j(t)) + B_i(t)u_i, \quad x(t_0) = x_0.$$

2.3.1 Open-loop Nash games

Here, we consider the simplest situation: We discuss, how a Nash equilibrium arises if the players are *isolated* from the current state of the system, i.e. if the information structure is of open-loop type. First, we discuss the existence and uniqueness of such an equilibrium and then develop explicit formulae for the controls.

Again, we use the notation defined in Section 1.4. Let $\mathcal{H}_{t_f} := \mathcal{H}^n[t_0, t_f]$ and Φ be defined as in Definition 1.17 and let

$$\mathcal{B}_i : \mathcal{U}_i \to \mathcal{H}_{t_f}, \quad u_i \mapsto \int_{t_0}^{\cdot} \Phi(., \tau)B_i(\tau)u_i(\tau)\, d\tau.$$

Then the solution of the initial value problem

$$\dot{x} = A(x)x + \sum_{i=1}^{N} B_i(t)u_i, \quad x(t_0) = x_0$$

becomes

$$x(.) = \Phi x_0 + \sum_{i=1}^{N} \mathcal{B}_i u_i \tag{2.1}$$

Rewriting the cost functionals in the same manner as in Section 1.4 yields

$$\begin{aligned} J_i(u_1,\ldots,u_N) &= \left\langle x, \bar{Q}_i x\right\rangle_{\mathcal{H}_{t_f}} + \sum_{j=1}^{N} \langle u_j, R_{ij} u_j\rangle_{\mathcal{L}_2} \\ &= \left\langle x, \bar{Q}_i x\right\rangle_{\mathcal{H}_{t_f}} + \sum_{j=1}^{N} \left\langle u_j, \bar{R}_{ij} u_j\right\rangle_{\mathcal{H}_{t_f}}, \end{aligned}$$

with

$$\bar{Q}_i(t) = \begin{cases} Q_i(t) & t \neq t_f \\ K_{if} & t = t_f \end{cases},$$

$$\bar{R}_{ij}(t) = \begin{cases} R_{ij}(t) & t \neq t_f \\ 0 & t = t_f \end{cases}.$$

Using equation (2.1) the modified cost functionals J_i^* can be written in the following form

$$\begin{aligned} J_i^*(u_i) &= \left\langle \Phi x_0 + \mathcal{B}_i u_i + \sum_{\substack{j=1\\j\neq i}}^{N} \mathcal{B}_j u_j^*, \bar{Q}_i\left(\Phi x_0 + \mathcal{B}_i u_i + \sum_{\substack{j=1\\j\neq i}}^{N} \mathcal{B}_j u_j^*\right)\right\rangle \\ &\quad + \left\langle u_i, \bar{R}_{ii} u_i\right\rangle + \sum_{\substack{j=1\\j\neq i}}^{N} \left\langle u_j^*, \bar{R}_{ij} u_j^*\right\rangle \\ &= \left\langle u_i, \left(\mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii}\right) u_i\right\rangle + 2\left\langle u_i, \mathcal{B}_i^* \bar{Q}_i \left(\Phi x_0 + \sum_{\substack{j=1\\j\neq i}}^{N} \mathcal{B}_j u_j^*\right)\right\rangle + J_{i0}, \end{aligned}$$

where J_{i0} denotes terms independent of u_i, and all the scalar products are taken in $\mathcal{H}_{t_f}$.

Using the notations $F_i := \mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii}$ and

$$f_i := \mathcal{B}_i^* \bar{Q}_i \left(\Phi x_0 + \sum_{\substack{j=1\\j\neq i}}^{N} \mathcal{B}_j u_j^*\right),$$

the modified cost functional of the i^{th} player in an equilibrium state becomes

$$J_i^*(u_i) := J_i(u_1^*,\ldots,u_{i-1}^*, u_i, u_{i+1}^*,\ldots,u_N^*) = \langle u_i, F_i u_i\rangle + 2\langle u_i, f_i\rangle + J_{i0}.$$

And hence we can immediately obtain the following theorem

Theorem 2.2 *Suppose that Γ_N is an N-player linear quadratic game. Then, the best reply of the i^{th} player according to an open-loop information structure is uniquely defined if and only if $\mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii} > 0$ holds. In this case the i^{th} player this best reply takes the form*

$$u_i^* = -F_i^{-1} f_i = -\left(\mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii}\right)^{-1} \mathcal{B}_i^* \bar{Q}_i \left(\Phi x_0 + \sum_{\substack{j=1\\j\neq i}}^{N} \mathcal{B}_j u_j^*\right). \tag{2.2}$$

Proof. According to Lemma 1.5, there exists a unique solution of Problem 2 (or equivalently, there exists a unique best reply) if and only if the self-adjoint operator F_i is positive definite. Again, using Lemma 1.5, this best reply is the equal to the expression $-F_i^{-1} f_i$. □

Hence, we obtain the following condition for an open-loop Nash equilibrium.

Theorem 2.3 (see [LR71]) *Suppose that the differential game Γ_N is defined such that the constraints $\mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii} > 0$ are fulfilled for each $i = 1, \ldots, N$. Then, the open-loop control functions $(u_1^*, \ldots, u_N^*) \in \mathcal{U}_1 \times \ldots \times \mathcal{U}_N$ form an open-loop Nash equilibrium if and only if*

$$\begin{pmatrix} \mathcal{B}_1^*\bar{Q}_1\mathcal{B}_1 + \bar{R}_{11} & \mathcal{B}_1^*\bar{Q}_1\mathcal{B}_2 & \ldots & \mathcal{B}_1^*\bar{Q}_1\mathcal{B}_N \\ \mathcal{B}_2^*\bar{Q}_2\mathcal{B}_1 & \mathcal{B}_2^*\bar{Q}_2\mathcal{B}_2 + \bar{R}_{22} & \ldots & \mathcal{B}_2^*\bar{Q}_2\mathcal{B}_N \\ \vdots & & \ddots & \vdots \\ \mathcal{B}_N^*\bar{Q}_N\mathcal{B}_1 & \mathcal{B}_N^*\bar{Q}_N\mathcal{B}_2 & \ldots & \mathcal{B}_N^*\bar{Q}_N\mathcal{B}_N + \bar{R}_{NN} \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_N^* \end{pmatrix} = \begin{pmatrix} -\mathcal{B}_1^*\bar{Q}_1\Phi x_0 \\ -\mathcal{B}_2^*\bar{Q}_2\Phi x_0 \\ \vdots \\ -\mathcal{B}_N^*\bar{Q}_N\Phi x_0 \end{pmatrix} \tag{2.3}$$

holds.

Corollary 2.2 *An open-loop linear quadratic game admits a unique Nash equilibrium, if and only if the following conditions are fulfilled:*

(i) For any $i = 1, \ldots, N$ $\mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii} > 0$, and

(ii) the matrix

$$\begin{pmatrix} \mathcal{B}_1^*\bar{Q}_1\mathcal{B}_1 + \bar{R}_{11} & \mathcal{B}_1^*\bar{Q}_1\mathcal{B}_2 & \ldots & \mathcal{B}_1^*\bar{Q}_1\mathcal{B}_N \\ \mathcal{B}_2^*\bar{Q}_2\mathcal{B}_1 & \mathcal{B}_2^*\bar{Q}_2\mathcal{B}_2 + \bar{R}_{22} & \ldots & \mathcal{B}_2^*\bar{Q}_2\mathcal{B}_N \\ \vdots & & \ddots & \vdots \\ \mathcal{B}_N^*\bar{Q}_N\mathcal{B}_1 & \mathcal{B}_N^*\bar{Q}_N\mathcal{B}_2 & \ldots & \mathcal{B}_N^*\bar{Q}_N\mathcal{B}_N + \bar{R}_{NN} \end{pmatrix}$$

is invertible.

Although, the above theorem gives us a complete description of the open-loop Nash equilibria, we need to solve a somewhat complicated system of linear operator equations described by Theorem 2.3. Our next task is to show how the solution of this system can be given more explicitely, i.e. by means of solutions to certain differential equations.

Theorem 2.4 *Suppose that the coefficients $R_{ii}(t)$ and $Q_i(t)$ are positive definite and positive semidefinite for all $t \in [t_0, t_f]$, respectively. Suppose further, that the matrices K_{if} are positive semidefinite. Then, the controls $u_1^*, \ldots, u_N^*$ form an open-loop Nash equilibrium for the differential game if and only if for each $i = 1, \ldots, N$ the system of equations*

$$u_i^* = -R_{ii}^{-1} \mathcal{B}_i^* \bar{Q}_i x^* \tag{2.4}$$

$$x^* = \Phi x_0 + \sum_{i=1}^{N} \mathcal{B}_i u_i^* \tag{2.5}$$

admits a solution $u_i^* \in \mathcal{L}_2^{m_i}, x^* \in \mathcal{U}_{t_f}$ $(i = 1, \ldots, N)$.

Proof. Note that the positive semidefiniteness of $Q_i(t)$ and K_{if} induces directly the positive semidefiniteness of the operator $\bar{Q}_i$ according to the Hilbert-space $\mathcal{H}_{t_f}$. Furthermore, the same way the positive definiteness of $R_{ii}(t)$ implies the positive definiteness of the operator $\bar{R}_{ii}$ and hence the positive definiteness of F_i. Therefore, the uniqueness of the best reply is ensured.

Using equation (2.2), we obtain the following representation of the optimal control u_i^*:

$$\begin{aligned} F_i u_i^* &= -f_i \\ \Leftrightarrow R_{ii}u_i^* + \mathcal{B}_i^* \bar{Q}_i \mathcal{B}^i u_i^* &= \mathcal{B}_i^* \bar{Q}_i \left(\Phi x_0 + \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathcal{B}_j u_j^* \right) \\ \Leftrightarrow R_{ii}u_i^* &= -\mathcal{B}_i^* \bar{Q}_i \underbrace{\left(\Phi x_0 + \sum_{j=1}^{N} \mathcal{B}_j u_j^* \right)}_{=:x*} . \end{aligned}$$

Therefore, if the N-tuple $u_1^*, \ldots, u_N^*$ is a solution of (2.3), then together with the definition of x^*, it also solves (2.4)-(2.5) and vice versa.

Using the above calculation, the rest of the proof follows directly from Theorem 2.3.

□

Calculating now the adjoint operators (see Lemma 1.6), we obtain that

$$\mathcal{B}_i^* y = B_i^T(.) \left[\Phi^T(t_f, .)y(t_f) + \int_{.}^{t_f} \Phi^T(t, .)y(t)\, dt \right] , \tag{2.6}$$

and hence (2.4) is equivalent to

$$u_i^*(t) = -R_{ii}(t)^{-1} B_i^T(t) \left[\Phi^T(t_f, t) K_{i_f} x^*(t_f) + \int_{t}^{t_f} \Phi^T(\tau, t) Q_i(\tau) x^*(\tau)\, d\tau \right]$$

for a.e. $t \in [t_0, t_f]$ and especially for $t = t_f$.

Now, we can formulate the main result of this section. The following theorem gives a necessary and sufficient condition for the optimal Nash-control functions in terms of solutions of certain differential equations.

Theorem 2.5 *Suppose that Γ_N is a N-player linear quadratic differential game as defined in Definition 2.5 with the following properties*

(i) $R_{ii}(t) > 0$, $Q_{ii}(t) \geq 0$ $\forall t \in \mathcal{T}$ and $K_{if} \geq 0$, and

(ii) the following set of differential equations

$$\dot{K}_i = -K_i A(t) - A^T(t) K_i - Q_i(t) + K_i \sum_{j=1}^{N} S_{jj}(t) K_j, \quad K_i(t_f) = K_{if} \tag{2.7}$$

admits a set of bounded solutions K_i over the interval $[t_0, t_f]$.

Then, the set of controls $(u_1^*, \ldots, u_N^*)$ *form an open-loop Nash equilibrium if and only if for almost every* $t \in [t_0, t_f]$ *and also for* $t = t_f$

$$u_i^*(t) = -R_{ii}^{-1}(t)B_i^T(t)K_i(t)x^*(t)$$

holds for $i = 1, \ldots, N$. *Hereby denotes* $x^*(t)$ *the solution of the homogeneous linear differential equation*

$$\dot{x}^* = \left(A - \sum_{i=1}^{N} S_{ii}K_{ii}\right)x^*, \quad x(t_0) = x_0 \tag{2.8}$$

and $S_{ii}(t) = B_i(t)R_{ii}^{-1}(t)B_i^T(t)$.

Proof. The proof, that we carry out in several short steps, is mainly based on the latter calculation.

1. Observe that condition (i) coincides with the condition of Theorem 2.4 and hence the optimal controls – if they exist – are solutions of (2.4)-(2.5) and hence $u_i^* = -R_{ii}^{-1}\mathcal{B}_i^*\bar{Q}_ix^*$ holds.

2. Moreover, the terminal-values $K_i(t_f)$ is given such that

$$\Phi^T(t_f,t)K_{i_f}x^*(t_f) + \int_{t_f}^{t_f} \Phi^T(\tau,t_f)Q_i(\tau)x^*(\tau)\,d\tau = K_{i_f}x^*(t_f) = K_i(t_f)x^*(t_f)$$

   holds.

3. Furthermore, the definition of the operator Φ implies that

$$\begin{aligned}
&\frac{d}{dt}\left[\Phi^T(t_f,t)K_{i_f}\hat{x}_i(t_f) + \int_t^{t_f} \Phi^T(\tau,t)Q_i(\tau)x^*(\tau)\,d\tau\right] \\
=\;& \frac{\partial}{\partial t}\Phi^T(t_f,t)K_{if}x^*(t_f) + \int_t^{t_f} \frac{\partial}{\partial t}\Phi^T(\tau,t)Q_i(\tau)x^*(\tau)\,d\tau - Q_i(t)x^*(t) \\
=\;& A^T(t)\Phi^T(t_f,t)K_{if}x^*(t_f) + \int_t^{t_f} A^T(t)\Phi^T(\tau,t)Q_i(\tau)x^*(\tau)\,d\tau - Q_i(t)x^*(t) \\
=\;& A^T(t)\left[\Phi^T(t_f,t)K_{i_f}\hat{x}_i(t_f) + \int_t^{t_f} \Phi^T(\tau,t)Q_i(\tau)x^*(\tau)\,d\tau\right] - Q_i(t)x^*(t)
\end{aligned}$$

   holds. Hence, similarly to the calculation presented in Section 1.4, we can verify that if the Riccati equations (2.7) admit a set of solutions, then – using the uniqueness of the trajectory obtained in Theorem 2.1 –

$$\Phi^T(t_f,t)K_{i_f}\hat{x}_i(t_f) + \int_t^{t_f} \Phi^T(\tau,t)Q_i(\tau)x^*(\tau)\,d\tau = K_i(t)x^*(t)$$

   holds.

4. For that step need to prove, why the equations

$$\left(\dot{K}_i + K_i A(t) + A^T(t) K_i + Q_i(t) - K_i \sum_{j=1}^{N} S_{jj}(t) K_j\right) x^*(t) = 0 \tag{2.9}$$

   imply

$$\dot{K}_i + K_i A(t) + A^T(t) K_i + Q_i(t) - K_i \sum_{j=1}^{N} S_{jj}(t) K_j = 0.$$

   To obtain this result, observe that the – using Lindelöf's Theorem on the solvability of differential equations (see e.g. [KK74]) – the set

$$\left\{ x(t) \middle| \dot{x}(t) = \left(A - \sum_{i=1}^{N} S_{ii} K_i\right) x(t), t \in [t_0, t_f] \right\}$$

   equals to the whole state space $\mathbb{R}^n$. Since furthermore (2.9) holds for any initial value $x_0 \in \mathbb{R}^n$, the assumption is proved.

5. To complete the proof, observe that for $u_i^* = -R_{ii}^{-1}(t) B_i^T(t) K_i(t) x^*(t)$ equation (2.8) is equivalent to (2.5).

□

Using this result, we can easily obtain a very useful sufficient condition on the playability of a Nash-game:

Corollary 2.3 *A sufficient condition for an N-player linear quadratic open-loop Nash game to be playable is that the set of Riccati differential equations (2.7) admit a unique bounded solution on $[t_0, t_f]$.*

2.3.2 Feedback Nash games

Here, we consider another very important setup for linear quadratic differential games: the arising of a Nash equilibrium under feedback information structure. In this situation the optimal controls are explicite functions of the time t and also of the current system-state $x(t)$ and hence have the form

$$u_i^* = u_i^*(t, x(t))$$

In order to verify whether a set of strategies $(u_1^*, \ldots, u_N^*)$ form a feedback Nash equilibrium or not, we again make use of Problem 2. For the solution of the arising variational problem, however, we require a different technique to incorporate the dependence of u_i on $x(t)$.

This technique is the well known Hamiltonian-Jacobi Theory in the variational calculus (see for instance [Fun70], [Dus60], §3.10 in [Hes66] or §9.2 in [KK85]):

Theorem 2.6 (Hamilton-Jacobi Theory) *A necessary condition for $u : [t_0, t_f] \to \mathbb{R}^m$ to be extremal of the functional*

$$J(x, u) = \int_{t_0}^{t_f} \Psi\big(t, x(t), u(t)\big)\, dt + \kappa(x(t_f))$$

according to the constraints

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0$$

is that u fulfills the Hamilton-Jacobi partial differential equations:

$$\begin{aligned} \frac{\partial H}{\partial x} &= \dot{\lambda}^T \\ \frac{\partial H}{\partial \lambda} &= -\dot{x}^T \\ \frac{\partial H}{\partial u} &= 0, \\ \left.\frac{\partial \kappa}{\partial x}\right|_{t=t_f} &= \lambda(t_f) \end{aligned}$$

where $H(t, x, u, \lambda) := \Psi(t, x, u) - \lambda^T f(t, x, u)$ denotes the Hamilton function associated to the above variational problem.

Hence, the following set of partial differential equations

$$\frac{\partial H_i}{\partial \lambda_i} = -\dot{x}^T \tag{2.10}$$

$$\frac{\partial H_i}{\partial x} = \dot{\lambda}_i^T \tag{2.11}$$

$$\left.\frac{\partial H_i}{\partial u_i}\right|_{u_i=u_i^*} = 0 \tag{2.12}$$

$$\frac{\partial J_i^*}{\partial x(t_f)} = -\lambda_i^T(t_f) \tag{2.13}$$

for any $i = 1, 2, \ldots, N$ form a necessary condition for a feedback Nash equilibrium. Hereby denotes

$$H_i(t, x, u_i, \lambda_i) = x^T Q_i(t) x + u_i^T R_{ii}(t) u_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} u_j^{*T} R_{ij}(t) u_j^* - \lambda_i^T \left(A(t)x + B_i(t) u_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} B_j(t) u_j^* \right).$$

Thus, we obtain

$$\frac{\partial H_i}{\partial x} = \dot{\lambda}_i^T = x^T Q_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} u_j^{*T} R_{ij}(t) \frac{\partial u_j^*}{\partial x} - \lambda_i^T \Big(A(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} B_j(t) \frac{\partial u_j^*}{\partial x} \Big) \tag{2.14}$$

$$\frac{\partial H_i}{\partial u_i} = u_i^T R_{ii}(t) - \lambda_i^T B_i(t)|_{u_i=u_i^*} = 0. \tag{2.15}$$

Suppose now that $R_{ii}(t) > 0 \; \forall t \in \mathcal{T} = [t_0, t_f]$. Then, the latter equation yields the following equation about the optimal control u_i^*.

$$u_i^*(t) = R_{ii}^{-1}(t) B_i^T(t) \lambda_i(t) \tag{2.16}$$

We now restrict ourselves to use the following strategies:

Suppose that the strategies are so-called *linear feedback strategies*, i.e that

$$\sigma_i = \left\{ -F_i(t)x | t \in [t_0, t_f], x \in X, F_i(t) \in \mathbb{R}^{m_i \times n} \right\}$$

hols for all $i = 1, 2, \ldots, N$.

Remark 2.5 *It is stated in [BO95] (see Corollary 6.5 there), that under some positivity constraints, a feedback linear quadratic differential game always admits a linear feedback Nash equilibrium of the above type. However, there are – similarly to the open-loop case discussed in [Eis82] – other nonlinear feedback controls yielding a Nash equilibrium. Hence, the discussion of this general setup would be more complicated and is therefore omitted.*

Using linear feedback strategies, the optimal control functions become

$$u_i^* = -F_i^*(t)x(t)$$

and consequently,

$$\frac{\partial u_i^*}{\partial x} = -F_i^* \tag{2.17}$$

holds for $i = 1, 2, \ldots, N$.

Altogether we obtain the following result:

Theorem 2.7 *Suppose that the linear feedback control functions $(u_i^*(.) = F_i^*(.)x)_{i=1,\ldots,N}$ form a Nash equilibrium for the linear quadratic differential game Γ_N. Then the mappings $F_i^* : \mathcal{T} = [t_0, t_f] \to \mathbb{R}^{m \times n}$ fulfill the following set of equations*

$$\dot{\lambda}_i = Q_i(t)x^* + \sum_{\substack{j=1 \\ j \neq i}}^{N} F_j^{*T} R_{ij}(t) F_j^*(t) x^* - \Big(A^T(t) - \sum_{\substack{j=1 \\ j \neq i}}^{N} F_j^{*T} B_j^T(t) \Big) \lambda_i \tag{2.18}$$

$$R_{ii}(t) F_i^* x^* = -B_i^T(t) \lambda_i(t) \tag{2.19}$$

for any $i = 1, 2, \ldots, N$. Hereby denote $\lambda_i : \mathcal{T} \to \mathbb{R}^{n \times n}$ $(i = 1, \ldots, N)$ a set of sufficiently chosen smooth functions and x^* the solution of the differential equation:

$$\dot{x}^* = \left(A(t) - \sum_{i=1}^{N} B_i(t) F_i^*(t) \right) x^*, \quad x^*(t_0) = x_0 \tag{2.20}$$

Proof. Using (2.17) substituted into (2.14) and into the differential equation describing the system dynamics

$$\dot{x} = A(t)x + \sum_{i=1}^{N} B_i(t) u_i, \quad x(t_0) = x_0,$$

the statement becomes obvious. □

Using now the guess, that $\lambda_i(t)$ is a linear function of the current system state $x(t)$:

$$\lambda_i(t) = -K_i(t) x^*(t) \tag{2.21}$$

we immediately obtain that

$$F_i^*(t) = R_{ii}^{-1}(t)B_i^T(t)K_i(t)$$

holds for some mapping $K_i : \mathcal{T} \to \mathbb{R}^{n\times n}$ with $i = 1, 2, \ldots, N$. Consequently, the Hamilton-Jacobi differential equation (2.18) becomes :

$$\begin{aligned} -\dot{K}_i x^* - K_i \dot{x}^* &= -\dot{K}_i x^* - K_i\Big(Ax^* - \sum_{j=1}^{N} \underbrace{B_j R_{jj}^{-1} B_j^T}_{S_{jj}(t)} K_j x^*\Big) \\ &= Q_i x^* + \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j^T \underbrace{B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T}_{S_{ij}(t)} K_j x^* \\ &\quad + \Big(A^T - \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j^T \underbrace{B_j R_{jj}^{-1} T B_j^T}_{S_{jj}(t)}\Big) K_i x^* \end{aligned}$$

With other words,

$$\left(-\dot{K}_i - K_i A + K_i \sum_{j=1}^{N} S_{jj} K_j - Q_i - \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j^T Sij K_j - A^T K_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j^T S_{jj} K_i\right) x^*(t) = 0 \tag{2.22}$$

holds. Hereby denotes x^* the solution of the linear differential equation (2.20). Its solution $x^*(t, x_0)$ is a dense covering of $\mathbb{R}^n$ with the property that $\forall t_1 \in \mathcal{T}, x_1 \in \mathbb{R}^n\ \exists x_0 \in \mathbb{R}^n$ such that $x(t_1, x_0) = x_1$ holds. Hence

$$\{x^*(t, x_0) | t \in \mathcal{T}, x_0 \in \mathbb{R}^n\} = \mathbb{R}^n,$$

which yields that (2.22) holds for an arbitrary $x_0 \in \mathbb{R}^n$ if and only if

$$-\dot{K}_i - K_i A + K_i \sum_{j=1}^{N} S_{jj} K_j - Q_i - \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j^T Sij K_j - A^T K_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j^T S_{jj} K_i = 0. \tag{2.23}$$

Altogether, we obtain the following necessary condition for a linear feedback Nash equilibrium:

Theorem 2.8 *Suppose that the linear feedback control functions $u_i^* = -F_i^*(t)x$ $(i = 1, ..., N)$ form a Nash equilibrium for the differential game Γ_N. Suppose further that the set of* feedback matrix Riccati differential equations

$$\dot{K}_i = -K_i A - A^T K_i - Q_i + K_i \sum_{j=1}^{N} S_{jj} K_j - \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j^T S_{ij} K_j + \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j^T S_{jj} K_i, \quad K_i(t_f) = K_{if} \tag{2.24}$$

obtain a set of bounded solutions K_i $(i = 1, \ldots, N)$ on $[t_0, t_f]$. Then, the optimal feedback matrices $F_i^*(t)$ fulfill the equation

$$F_i^*(t) = R_{ii}^{-1}(t)B_i^T(t)K_i(t).$$

Hereby denotes $S_{ij}(t) = B_j(t)R_{jj}^{-1}(t)R_{ij}(t)R_{jj}^{-1}(t)B_j^T(t) \in \mathbb{R}^{n\times n}$ for $i, j = 1, \ldots, N$.

2.3.3 Nash games with sampled-data information structure

Another information structure, which we deal with in the framework of this Thesis is the so-called 'sampled-data' structure (see also Remark 2.2). Basically, this structure is nothing else, but the mathematical model of modified feedback information, where the information doesn't reach the players continuously in the time. Instead, they receive to predefined times an update and have to use a strategy knowing that the next information is only available at the next predefined time. Hence, between two sampling-times, the users control the system according to an open-loop control strategy.

Typically, such systems are given in the economics, but also every (mechanical) system controlled by digital controllers is a sampled-data system (although, sampling frequencies are nowadays extremely high and hence the feedback is almost continuous).

In this last section of this chapter, we derive the optimal control laws for a Nash-game under the sampled-data information structure. To this end, we suppose that the system state for every player is only available at the predefined times $t_0 < t_1 < t_2 < \ldots < t_h = t_f$ for some $h \in N$. Hence, the information $\eta_i(t)$ is given by

$$\eta_i(t) = \{x(t_j)|t_j \le t < t_{j+1}\}.$$

As we mentioned before, in the interval (t_j, t_{j+1}) none of the players has information on the system, and hence the optimal trajectory evolves according to an open-loop trajectory. Nevertheless, at each time t_j the players receive an update of the state, which plays the role of the initial state for the next open-loop game. Thus, this information structure yields a game that can be interpreted as a repeated open-loop game over the time-horizon $[t_j, t_{j+1}]$. (see also Proposition in [SJC73b])

The above discussion brings us very close to obtain the optimal strategies:

If we denote by x_j the system state $x(t_j)$. and if we suppose that for some $j < h$ the optimal controls u_i^* for $[t_{j+1}, t_f]$ are already calculated, then the actual costs of the game on $[t_{j+1}, t_f]$ can be obtained with the value function as in equation (5) in [SJC73b]:

$$V_i(x_{j+1}) = \frac{1}{2}x^T(t_f)K_{if}x(t_f) + \frac{1}{2}\int\limits_{t_{j+1}}^{t_f} \left(x^T Q_i(t)x + \sum_{k=1}^{N} u_k^{*T} R_{ik}(t)u_k^* \right) dt \tag{2.25}$$

In order to obtain optimal strategies for $[t_j, t_{j+1}]$, we need to investigate the arising open-loop game on that interval. To show on which interval the functions are defined, we use the notation u_i^* for the optimal control functions defined on $[t_{j+1}, t_f]$ and $\tilde{u}_i$, $\tilde{u}_i^*$ for functions on the interval $[t_j, t_{j+1}]$.

Thus, the cost functionals $\tilde{J}_i$ and the corresponding Hamilton functions $\tilde{H}_i$ become

$$\tilde{J}_i^*\left(\tilde{u}_i(.)\right) = V_i(x_{j+1}) + \frac{1}{2}\int\limits_{t_j}^{t_{j+1}} \left(\tilde{x}^T Q_i(t)\tilde{x} + \tilde{u}_i^T R_{ii}\tilde{u}_i + \sum_{\substack{k=1 \\ k\neq i}}^{N} \tilde{u}_k^{*T} R_{ik}(t)\tilde{u}_k^* \right) dt \tag{2.26}$$

and

$$\tilde{H}_i(t, \tilde{x}, \tilde{u}_i, \tilde{\lambda}_i) := \frac{1}{2}\tilde{x}^T Q_i(t)\tilde{x} + \frac{1}{2}\tilde{u}_i^T R_{ii}\tilde{u}_i + \frac{1}{2}\sum_{\substack{k=1 \\ k\neq i}}^{N} \tilde{u}_k^{*T} R_{ik}\tilde{u}_k^*$$

$$-\tilde{\lambda}_i^T \left(A(t)\tilde{x} + B_i(t)\tilde{u}_i + \sum_{\substack{k=1 \\ k\neq i}}^{N} B_k(t)\tilde{u}_k^* \right),$$

respectively.

This yields the following system of Hamilton-Jacobi differential equations (as discussed in Section 2.3.2):

$$\frac{\partial \tilde{H}_i}{\partial \tilde{\lambda}_i} = -\dot{\tilde{x}}^T \tag{2.27}$$

$$\frac{\partial \tilde{H}_i}{\partial \tilde{x}} = \dot{\tilde{\lambda}}_i^T \tag{2.28}$$

$$\frac{\partial \tilde{H}_i}{\partial \tilde{u}_i} = 0 \tag{2.29}$$

$$\frac{\partial \tilde{J}_i^*}{\partial x_{j+1}} = -\tilde{\lambda}_i^T(t_{j+1}) \tag{2.30}$$

Using the same argumentation, as in Section 2.3.2, we obtain for (2.27)-(2.29) the following set of equations:

$$\dot{\tilde{\lambda}}_i = Q_i(t)\tilde{x} + \sum_{\substack{k=1 \\ k\neq i}}^{N} \left(\frac{\partial \tilde{\gamma}_k^*}{\partial \tilde{x}}\right)^T R_{ij}^T \tilde{u}_k^* - \left(A^T(t) + \sum_{\substack{k=1 \\ k\neq i}}^{N} \left(\frac{\partial \tilde{\gamma}_k^*}{\partial \tilde{x}}\right)^T B_k^T(t) \right) \tilde{\lambda}_i \tag{2.31}$$

$$\tilde{u}_i^*(t) = R_{ii}^{-1}(t) B_i^T(t) \tilde{\lambda}_i(t) \tag{2.32}$$

As discussed before, the optimal strategies $\tilde{\gamma}_i^*$ only depend on the state x_j (since no more information is available in the interval $[t_j, t_{j+1}]$). Hence, the partial derivatives $\frac{\partial \tilde{\gamma}_k^*}{\partial \tilde{x}} = 0$.

However, it is a somewhat more complicated task to carry out the partial differentiation in equation (2.30). To do this, we assume again, that the following equations hold

$$\tilde{\lambda}_i(t) = -\tilde{K}_i(t)\tilde{x}^*(t), \tag{2.33}$$

for some matrix $\tilde{K}_i : [t_j, t_{j+1}] \to \mathbb{R}^{n\times n}$. Hereby denotes $\tilde{x}^*(t)$ the optimal (open-loop) state trajectory, i.e.

$$\dot{\tilde{x}}^* = \left(A - \sum_{i=1}^{N} S_{ii} K_i \right) \tilde{x}^*, \quad \tilde{x}^*(t_j) = x_j, \tag{2.34}$$

or equivalently

$$\tilde{x}^*(t) = \tilde{\Phi}(t) x_j \quad t_{j+1} \geq t \geq t_j$$

for some mapping $\tilde{\Phi} : [t_j, t_{j+1}] \to \mathbb{R}^{n\times n}$ with $\tilde{\Phi}(t_j) = I$ and

$$\frac{d}{dt}\tilde{\Phi}(t) = \left(A - \sum_{i=1}^{N} S_{ii} K_i \right) \tilde{\Phi}(t).$$

Hence, the dependence of the state $x_{j+1} = x^*(t_{j+1})$ on x_j becomes obvious. Moreover, the continuation $\Phi_{(j)}$ of the fundamental matrix $\tilde{\Phi}$ over the interval $[t_j, t_f]$ yields a more general form:

$$x^*(t) = \Phi_{(j)}(t) x_j \quad t \geq t_j$$

Thus, it is

$$\begin{aligned}
\frac{\partial \tilde{J}_i^*}{\partial x_{j+1}} &= \frac{\partial V_i(x_{j+1})}{\partial x_{j+1}} \\
&= x^{*T}(t_f)K_{if}\left(\frac{\partial x^*}{\partial x_{j+1}}\right)_{t=t_f} \\
&\quad + \int_{t_{j+1}}^{t_f} x^{*T}Q_i(t)\left(\frac{\partial x^*}{\partial x_{j+1}}\right) + \sum_{k=1}^{N} u_k^{*T}R_{ik}\left(\frac{\partial u_k^*}{\partial x^*}\right)\left(\frac{\partial x^*}{\partial x_{j+1}}\right) dt \\
\Rightarrow \tilde{\lambda}_i(t_{j+1}) &= \Phi_{(j+1)}^T K_{if}x^*(t_f) \\
&\quad + \int_{t_{j+1}}^{t_f} \left(\Phi_{(j+1)}^T Q_i x^*(t) + \sum_{k=1}^{N} \Phi_{(j+1)}^T K_k^T \underbrace{B_k R_{kk}^{-1} R_{ik}^T R_{kk}^{-1} B_k^T}_{S_{ik}} K_k x^*(t)\right) dt,
\end{aligned}$$

where this last equation is equivalent to the following initial value problem

$$\dot{\lambda}_i = \Phi_{(j+1)}^T Q_i x^* + \sum_{k=1}^{N} \Phi_{(j+1)}^T K_k^T S_{ik} K_k x^*(t), \qquad \lambda_i(t_f) = \Phi_{(j+1)}^T K_{if} x^*(t_f). \tag{2.35}$$

In order to obtain the value $\tilde{\lambda}_i(t_{j+1}) = \lambda_i(t_{j+1})$, we introduce the mapping

$$\Lambda_i(t) := \left(\Phi_{(j+1)}^T\right)^{-1}(t)K_i(t).$$

Hence, we conclude that $\lambda_i = -\Phi_{(j+1)}^T \Lambda_i x^*$ and thus

$$\dot{\lambda}_i = -\dot{\Phi}_{(j+1)}^T \Lambda_i x^* - \Phi_{(j+1)}^T \dot{\Lambda}_i x^* - \Phi_{(j+1)}^T \Lambda_i \dot{x}^* \tag{2.36}$$

holds. Using now

$$\dot{\Phi}_{(j+1)} = \left(A - \sum_{k=1}^{N} S_{kk}K_k\right)\Phi_{(j+1)}$$

and

$$\dot{x}^* = \left(A - \sum_{k=1}^{N} S_{kk}K_k\right)x^*,$$

we can rewrite (2.36) as follows:

$$\begin{gathered}
-\dot{\Phi}_{(j+1)}^T \Lambda_i x^* - \Phi_{(j+1)}^T \dot{\Lambda}_i x^* - \Phi_{(j+1)}^T \Lambda_i \dot{x}^* = \\
\Phi_{(j+1)}^T\left(-\left(A - \sum_{k=1}^{N} S_{kk}K_k\right)^T \Lambda_i - \dot{\Lambda}_i - \Lambda_i\left(A - \sum_{k=1}^{N} S_{kk}K_k\right)\right)x^* = \\
\Phi_{(j+1)}^T\left(Q_i + \sum_{k=1}^{N} K_k^T S_{ik} K_k\right)x^*.
\end{gathered}$$

Hence, repeating the argumentation used in Section 2.3.2, we obtain the following differential equation for $\Lambda_i(t)$:

$$\dot{\Lambda}_i = -A^T\Lambda_i - \Lambda_i A - Q_i + \Lambda_i \sum_{k=1}^{N} S_{kk}K_k + \left(\sum_{k=1}^{N} S_{kk}K_k\right)^T \Lambda_i - \sum_{k=1}^{N} K_k^T S_{ik} K_k, \tag{2.37}$$

with $\Lambda_i(t_f) = K_{if}$ (compare equation (2.35)).

Also the calculation of the matrices $\tilde{K}_i$ is similar to the technique used in Section 2.3.2 and yields

$$\dot{\tilde{K}}_i = -A^T\tilde{K}_i - \tilde{K}_iA - Q_i + \sum_{k=1}^{N} \tilde{K}_iS_{kk}\tilde{K}_k. \tag{2.38}$$

Our final task is to obtain the terminal value $\tilde{K}_i(t_{j+1})$. For this note, that it is $\Lambda_i(t_{j+1}) = {\Phi^T_{(j+1)}}^{-1}(t_{j+1})\tilde{K}_i(t_{j+1})$, with $\Phi_{(j+1)}(t_{j+1}) = I$ and hence

$$\Lambda_i(t_{j+1}) = \tilde{K}_i(t_{j+1}). \tag{2.39}$$

Altogether, we obtain the following theorem:

Theorem 2.9 *Suppose that Γ_N is a linear quadratic differential game under* sampled-data *information structure. Suppose further, that the controls $u_1^*, u_2^*, \dots, u_N^*$ form a Nash equilibrium for this game. If moreover, the following differential equations*

$$\dot{K}_i = -A^TK_i - K_iA - Q_i + \sum_{k=1}^{N} K_iS_{kk}K_k, \quad K_i(t_{j+1}) = \Lambda_i(t_{j+1}) \tag{2.40}$$

and

$$\dot{\Lambda}_i = -A^T\Lambda_i - \Lambda_iA - Q_i + \Lambda_i\sum_{k=1}^{N} S_{kk}K_k + \left(\sum_{k=1}^{N} S_{kk}K_k\right)^T \Lambda_i - \sum_{k=1}^{N} K_k^TS_{ik}K_k, \quad \Lambda_i(t_f) = K_{if} \tag{2.41}$$

admit a set of bounded solutions $(K_i(t), \Lambda_i(t))_{i=1,\dots,N}$ defined on the intervals $[t_j, t_{j+1}]$ for $j = 0, 1, \dots, h$ and $[t_0, t_f]$, respectively, then, the corresponding sampled-data Nash control functions u_i^* fulfill

$$u_i^*(t) = -R_{ii}^{-1}(t)B_i^T(t)K_i(t)\Phi_{(j)}(t)x_j,$$

where $\Phi_{(j)}$ denotes the solution of

$$\dot{\Phi}_{(j)} = \left(A - \sum_{k=1}^{N} S_{kk}K_k\right)\Phi_{(j)}, \qquad \Phi_{(j)}(t_j) = I$$

and $x_j = x(t_j)$.

Remark 2.6 *Note, that the Lyapunov equation (2.41) is in some sense independent of (2.40): its solution $\Lambda_i(t)$ only involves the values of K_i over the interval $[t, t_f]$ and hence the terminal value $K_i(t_{j+1})$ only involves the functions $K_k|_{[t_{j+1},t_f]}$. Therefore, the solution of the above set of differential equations can take place alternatively between (2.40) and (2.41) from t_f towards t_0.*

2.4 Notes and references

After the work of Isaacs on Differential Games [Isa65], this topic became one of the fastest growing area of the mathematics. Together with the neighboring topics as for instance Optimal Control Theory, Optimization Theory, Ordinary Differential Equations, Riccati Theory, Complex Function Theory and Numerical Analysis, there are up to now uncountable references dealing with general or specific properties of differential games.

Therefore, it cannot be the aim of such a Thesis to introduce differential, difference and other dynamical games detailed. Similarly to Chapter 1, this chapter gave a brief introduction to the most important results and definitions of differential games defined on Euclidean spaces. As essentially the whole first part of this work, most of these concepts concerned linear quadratic differential game, this means games where the system is governed by a linear differential equation and the cost functionals are given as quadratic functionals in $\mathcal{L}_2[t_0, t_f]$. For a more detailed introduction of noncooperative dynamical games, please refer to the text of Başar and Olsder [BO95]. If the Reader is interested in a mathematically more rigorous and detailed investigation, the book of Friedman [Fri71] is suggested.

There are several approaches known to derive optimal control functions for various equilibria. In this chapter we presented two of them. First, the generalization of the Hilbert-space treatment introduced in Section 1.4. This method was first introduced for Nash games in the marvelous but unfortunately rarely cited paper of Lukes and Russell [LR71] and later on further investigated in the similarly nice work of Eisele [Eis82]. The Hamilton-Jacobi Theory, introduced here in the context of feedback-games, is usually very often referred to by the authors (see for instance [BO95] or [Wee95]).

For results on Stackelberg equilibria, that weren't covered here, the works of Freiling, Jank et al. [FJL99], Simaan and Cruz [SJC73a] and the already mentioned text of Başar and Olsder [BO95] should be mentioned. Furthermore, various properties of sampled-data control systems are covered by Ackermann in [Ack85]. For differential games with sampled-data information structure, refer to the original paper of Simaan and Cruz [SJC73b].

Finally, as it was stated before, as far as the author is concerned, besides the (essentially trivial) team-controllability, there is no reference up to now dealing with controllability and stabilizability issues of differential games.

Chapter 3

Riccati equations

As we have seen in the previous chapters, linear-quadratic systems (i.e. linear dynamical systems with quadratic optimality constraints) essentially lead to solutions of Riccati differential equations *(shortly: RDE)*. Here, we shall investigate some properties of these solutions and also develop numerical techniques to solve these types of equations. Throughout this chapter, we shall only consider matrices with real entries. For results on complex Riccati equations, please refer to the literature cited in Section 3.5.

Usually, we distinguish between several forms of algebraic equations and differential equations, that are all called 'Riccati equation'. Since these notations are somewhat confusing in the literature, let us first present some notations that will be used in this work.

There are two main types of the matrix Riccati differential equations: The *generalized* and the *standard* types. Standard Riccati differential equations have the following form

$$\dot{K} = M_{22}(t)K - KM_{11}(t) + M_{21}(t) - KM_{12}(t)K, \quad K(t_f) = K_f, \tag{3.1}$$

with $K(t), K_f \in \mathbb{R}^{r\times s}$ for some $r, s \in \mathbb{N}$ and $M_{ij}(t)$ chosen such that the equation is well defined. Depending on the symmetry of the coefficients M_{ij}, we distinguish between symmetric (3.3) or non-symmetric (3.1) matrix Riccati differential equations.

Generalized matrix Riccati differential equations have the common form:

$$\begin{aligned} \dot{K} &= M_{22}(t)K - KM_{11}(t) + M_{21}(t) - KM_{12}(t)K \\ &\quad -f_1(t,K) + Kf_2(t,K) + f_3(t,K)K \\ K(t_f) &= K_f, \end{aligned} \tag{3.2}$$

where $K(t) \in \mathbb{R}^{r\times s}$ and $f_1(t,.)$, $f_2(t,.)$ and $f_3(t,.)$ map matrices from $\mathbb{R}^{r\times s}$ into $\mathbb{R}^{r\times s}$, $\mathbb{R}^{s\times s}$ and $\mathbb{R}^{r\times r}$ for some given r and s, respectively. Finally – as above – assume, that the matrices $M_{ij}(t)$, $(i,j = 1,2)$ are of appropriate dimensions.

In order to see the relation between optimal control laws for linear quadratic systems, we often use another variables for the definition of Riccati differential equations.

Let us first recall the equation derived in Section 1.4. The following equation is called *symmetric* Riccati differential equation (also known as *symplectic* or *control* RDE):

Definition 3.1 *The following differential equation and the corresponding initial value problem is called* symmetric matrix Riccati differential equation*.*

$$\dot{K} = -A^T(t)K - KA(t) - Q(t) + KS(t)K, \quad K(t_f) = K_f, \tag{3.3}$$

where $Q(t), S(t), K_f \in \mathbb{R}^{n\times n}$ *are symmetric and* $A(t)$ *is also of dimension* $n\times n$*. Finally, we assume that the mappings* A*,* Q *and* S *are piecewise continuous and bounded over* $[t_0, t_f]$*.*

Besides discussing the above equation, we shall also investigate properties of solutions of another types of *generalized* Riccati equations, that appear in the context of linear-quadratic differential games:

Definition 3.2 *Let* $N \in \mathbb{N}$*. Then, the following systems of differential equations for* $i = 1,\dots,N$ *are called* generalized open-loop *RDE (sometimes* non-symmetric matrix *RDE) and* generalized feedback *RDE, respectively.*

$$\dot{K}_i = -K_iA(t) - A(t)^TK_i - Q_i(t) + K_i\sum_{j=1}^{N} S_{jj}(t)K_j, \quad K_i(t_f) = K_{if} \tag{3.4}$$

and

$$\begin{aligned} \dot{K}_i &= -K_iA(t) - A(t)^TK_i - Q_i(t) \\ &\quad +K_i\sum_{j=1}^{N} S_{jj}(t)K_j - \sum_{\substack{j=1\\ j\neq i}}^{N} K_j^TS_{ij}(t)K_j + \sum_{\substack{j=1\\ j\neq i}}^{N} K_j^TS_{jj}(t)K_i, \\ K_i(t_f) &= K_{if}, \end{aligned} \tag{3.5}$$

where $Q_i(t), S_{ij}(t), K_{if} \in \mathbb{R}^{n\times n}$ *are symmetric for all* $i,j = 1,\dots,N$ *and* $A(t)$ *is also of dimension* $n\times n$*. Finally, – as for the latter case – we assume that the mappings* A*,* Q_i *and* S_{ij} *are piecewise continuous and bounded for* $i,j = 1,\dots,N$ *over* $[t_0, t_f]$*.*

Note, that equations (3.3) and (3.4) can be written in form of a standard matrix Riccati differential equation

$$\dot{K} = M_{21}(t) + M_{22}(t)K - KM_{11}(t) - KM_{12}(t)K,$$

where $M_{21} = -Q$, $M_{22} = -A^T$, $M_{11} = A$ and $M_{12} = -S$ for the symmetric RDE and

$$M_{21} = \begin{pmatrix} -Q_1 \\ -Q_2 \\ \vdots \\ -Q_N \end{pmatrix}, M_{22} = \begin{pmatrix} -A^T & & & \\ & -A^T & & \\ & & \ddots & \\ & & & -A^T \end{pmatrix}, M_{11} = A,$$

$$M_{12} = \begin{pmatrix} -S_1 & -S_2 & \dots & -S_N \end{pmatrix} \text{ and } K = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_N \end{pmatrix}$$

for the open-loop RDE, respectively.

Furthermore, note that for $N = 1$ the equations (3.3), (3.4) and (3.5) are equivalent.

In the following chapter, we first briefly mention the history of Riccati equations. Then, in Section 3.2, we investigate the qualitative behavior of the various matrix Riccati differential equations. In this section, we shall mainly concentrate on the non-autonomous case. Results on the autonomous Riccati differential equations and on the corresponding algebraic equations are presented in Section 3.3. Finally, in Section 3.4, we present an example to illustrate some neglected but very important facts discussed before.

3.1 Historical remarks

Count Jacopo Francesco Riccati was born in 1676 in Venice. He originally entered the Padua University to read law, but to the encouragement of his friends Angeli and Rizzetti, he began to study mathematics. At the beginning of the year 1721, he sent a letter to his friend Giovanni Rizzetti with a short study of the following differential equations:

$$\dot{x} = \alpha x^2 + \beta t^m$$
$$\dot{x} = \alpha x^2 + \beta t + \gamma t^2.$$

Possibly, this was the first work dealing with Riccati differential equations.

Later on, in his textbooks on the solution of ordinary differential equations, he also included the equation

$$\dot{x} = ax^2 + bx + c, \tag{3.6}$$

which he obtained by studying the following mechanical system:

Suppose that the planar motion $(x(t), y(t))$ of a point-like object is governed by a linear differential equation:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

How can we then describe the evolution of the angle between the x-axis and the given point (s. Figure 3.1) ?

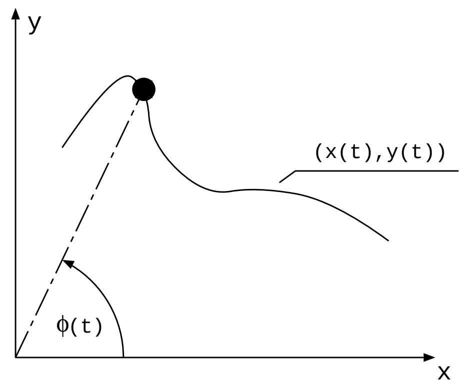

Figure 3.1: Planar motion of a point-like object

The following easy proof shows, how the answer to this question is related to equation (3.6). This short sketch is indeed a very special, one-dimensional case of the more general Radon's Lemma (Theorem 3.2).

Let the tangent of the angle $\varphi(t)$ be denoted by $k(t)$. Then it is $k(t) = \frac{y(t)}{x(t)}$ and hence

$$\dot{k}(t) = \frac{x\dot{y} - y\dot{x}}{x^2} = \frac{x(m_{21}x + m_{22}y) - y(m_{11}x + m_{12}y)}{x^2}$$

$$= \quad m_{21} + (m_{22} - m_{11})\frac{y}{x} - m_{12}\frac{y^2}{x^2} = m_{21} + (m_{22} - m_{11})k(t) - m_{12}k^2(t),$$

which is the scalar version of (3.1).

Based on his work in physics and analysis, he soon attained fame and became among others an offer from the Russian czar, Peter the Great to become President of the St. Petersburg Academy of Sciences.

Although, during the 18th century a great number of famous mathematicians (for example Daniel and Nicolaus Bernoulli, Euler, Diderot, D'Alembert and Liouville) studied the Theory of Riccati equations, until the last 35 years more or less only the classical results were known. In the last three decades Riccati's equations revived. Because of the rapid development of the theory of linear-quadratic optimal control systems and differential games, several new generalized form of the classical equation has been established and investigated.

3.2 Basic properties of solutions

First, we start with an elementary result concerning the symmetry of the solution of Riccati differential equations.

Lemma 3.1 *Let $K : [t_0, t_f] \to \mathbb{R}^{n\times n}$ be a (bounded) solution of (3.3) and $K_i : [t_0, t_f] \to \mathbb{R}^{n\times n}$ for $i = 1, \dots, N$ (bounded) solutions of the system (3.5). Then $K(t)$ and $K_i(t)$ are symmetric for all $t \in [t_0, t_f]$.*

Proof. Because of the equivalence of (3.3) and (3.5) for $N = 1$, we only show the proposition for (3.5). Rewriting (3.5) with the notation $A_{cl}(t) := A(t) - \sum_{i=1}^{N} S_{ii}(t)K_i(t)$ yields

$$\dot{K}_i = -A_{cl}^T K_i - K_i A_{cl} - Q_i - \sum_{j=1}^{N} K_j^T S_{ij}(t) K_j.$$

Using now the symmetry of Q_i and S_{ij}, we can immediately obtain a differential equation for K_i^T:

$$\dot{K}_i^T = -A_{cl}^T K_i^T - K_i^T A_{cl} - Q_i - \sum_{j=1}^{N} K_j^T S_{ij}(t) K_j.$$

Subtracting the equations yields

$$\frac{d}{dt}(K_i - K_i^T) = -A_{cl}^T (K_i - K_i^T) - (K_i - K_i^T) A_{cl},$$

with $K_i(t_f) - K_i^T(t_f) = 0$. Observe finally that the latter differential equation is homogeneous and linear and hence uniquely admits the zero solution. □

Although in this work we mainly concentrate on autonomous Riccati differential equations (i.e equations, where the mappings A, Q_i and S_{ij} are constant), for the sake of completeness, we cite here a result from [KK85] on the existence of solutions of the symmetric RDE in the non-autonomous case.

Theorem 3.1 (see Corollary 10.2 in [KK85]) *If for some t_f the mappings $Q(t) \geq 0$ and $S(t) \geq 0$ for any $t \leq t_f$ and if $K_f \geq 0$ hold, then the symmetric RDE has on $(-\infty, t_f]$* a unique positive semidefinite solution $K(t)$. *This solution is bounded from above by the solution $(X(t))$ of the following Lyapunov equation:*

$$\dot{X} = -A^T(t)X - XA(t) - Q(t), \quad X(t_f) = K_f.$$

Remark 3.1 *(s. also [Jan92]) In general it can be shown, that the (complex) solution of the (complex) matrix Riccati differential equation with polynomial coefficients is a meromorphic function, i.e. a complex-valued function without essential singularities.*

Another nice property of the matrix Riccati differential equation (3.1) is that it can be transformed into a linear differential equation as it was shown in [Rei72].

Theorem 3.2 (Radon's Lemma; see also p. 11 in [Rei72]) *Let $Id_s \in \mathbb{R}^{s \times s}$ be the identity matrix and let*

$$M(t) = \begin{pmatrix} M_{11}(t) & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{pmatrix} \in \mathbb{R}^{(s+r)\times(s+r)},$$

$Y(t) = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}$, *with $Q(t) \in \mathbb{R}^{s \times s}$ and $P(t) \in \mathbb{R}^{r \times s}$, respectively. Then the following linear differential equation*

$$\dot{Y} = M(t)Y, \quad Y(t_f) = \begin{pmatrix} Id_s \\ K_f \end{pmatrix} \tag{3.7}$$

and the matrix Riccati differential equation (3.1) are equivalent in the following sense:

(i) Let K be a solution of the matrix Riccati differential equation (3.1). Suppose further, that $Q(t)$ is the unique *solution of*

$$\dot{Q} = \big(M_{11}(t) + M_{12}(t)K(t)\big)Q, \quad Q(t_f) = Id_s.$$

Then, the mapping $Y = \begin{pmatrix} Q \\ P \end{pmatrix}$, with $P(t) = K(t)Q(t)$ is a solution of (3.7).

(ii) If $Y(t) = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}$ is a solution of (3.7) such that $Q(t)$ is regular for $t \in \mathbb{R}$, then

$$K(t) = P(t)Q^{-1}(t)$$

is a solution of (3.1).

Corollary 3.1 *Using Remark 3.1 and Theorem 3.2, it can be shown that the standard and the open-loop Riccati differential equations can be linearized on the whole domain of definition using a Grassmann-manifold of dimension $s \times (r+s)$. For more details on that, please refer to [FJ91].*

From now on, we take a closer look at the autonomous case, i.e. at Riccati differential equations with constant coefficients. Our first result –which is similar to the results obtained in [FJS97] and [FJAK96] – concerns the positive semidefiniteness of the (symmetric) solutions of the generalized feedback RDE (and hence of the symmetric RDE, too).

Theorem 3.3 *Suppose that $K_i(t)$ is the solution of the generalized feedback RDE (3.5), with $Q_i > 0$, $S_{ij} \geq 0$ and $K_{if} > 0$. Then $K_i(t) > 0$ holds for any $t \leq t_0$.*

Proof. Let $\lambda_i(t)$ be an eigenvalue of the symmetric solution $K_i(t)$ and $v_i(t)$ be the corresponding normalized eigenvector. Hence, it is $\lambda_i(t) = v_i^T(t)K_i(t)v_i(t)$. Using that K_i is analytical, we infer from Corollary 2 in Section 3.5.5 in [Bau85], that the functions $\lambda_i(t)$ and $v_i(t)$ are differentiable, too.

Hence, the following equality holds:

$$\begin{aligned}
\frac{d\lambda_i(t)}{dt} &= \frac{d}{dt}\left(v_i(t)^T K_i(t) v_i(t)\right) = \dot{v}_i^T(K_i v_i) + v_i^T \dot{K}_i v_i + (v_i^T K_i)\dot{v}_i \\
&= \dot{v}_i^T(\lambda_i v_i) + v_i^T \dot{K}_i v_i + (\lambda_i v_i^T)\dot{v}_i = 2\lambda_i(v_i^T \dot{v}_i) + v_i^T \dot{K}_i v_i \overset{|v_i|\equiv 1}{=} v_i^T \dot{K}_i v_i \\
&= v_i^T \left(-A^T K_i - K_i A - Q_i + K_i \sum_{j=1}^{N} S_{jj} K_j - \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j S_{ij} K_j + \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j S_{jj} K_i\right) v_i \\
&= \lambda_i(t)\left(-v_i^T(A + A^T)v_i + v_i^T\left(\sum_{j=1}^{N} S_{jj} K_j\right) v_i + v_i^T\left(\sum_{\substack{j=1 \\ j\neq i}}^{N} K_j S_{jj}\right) v_i\right) \\
&\quad -v_i^T Q_i v_i - v_i^T \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j S_{ij} K_j v_i
\end{aligned}$$

Using now $Q_i > 0$ and $S_{ij} \geq 0$ $Q_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} K_j S_{ij} K_j > 0$ follows. Hence, we conclude that for $\lambda_i(t) = 0$ the derivative $\frac{d\lambda_i}{dt}$ is negative. This means that the set

$$\{(\lambda, t) | \lambda > 0, t < t_f\}$$

is an invariant set of the feedback RDE. □

In the sequel, we are interested in the existence of the solution of the various matrix Riccati differential equations as time tends to $-\infty$. Our first result, which we'll try to generalize, is well known in the classical Optimal Control Theory. In order to illustrate the importance of these results, we show a very simple Riccati differential equation:

Example 7: One of the most important Riccati differential equation is the well-known (scalar) differential equation

$$\dot{x} = 1 + x^2, \quad x(0) = 0.$$

Clearly, the solution of this equation (at least for a sufficiently small neighborhood of zero) is the function $x(t) = \tan(t)$, which has a pole for each $t = k\pi$ ($k \in \mathbb{Z}$). However, the modified equation

$$\dot{x} = -1 + x^2, \quad x(0) = 0.$$

admits the solution $x(t) = \frac{1-e^{2t}}{1+e^{2t}}$ existing over the whole real axis.

Corollary 3.2 (see for instance Section 10.2 in [KK85]) *Let $A, Q, S, K_f \in \mathbb{R}^{n\times n}$ be such that S, Q and K_f are positive semidefinite and that $(A^T, \sqrt{S})$ and $(A, \sqrt{Q})$ are stabilizable. Then the matrix Riccati differential equation*

$$\dot{K} = -A^T K - KA - Q + KSK, \quad K(t_f) = K_f$$

admits a unique and bounded solution on $(-\infty, t_f]$.

Now, we turn our attention to the existence of solutions of generalized Riccati differential equations. Our first results concerns the open-loop case.

Theorem 3.4 *Suppose that the solution $Y(t) \in \mathbb{R}^{(s+r)\times r}$ of the linear differential equation (3.7) fulfills the condition, that the matrix*

$$\left(\underbrace{\begin{matrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{matrix}}_{s}\ \underbrace{\begin{matrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{matrix}}_{r}\right) Y(t) \in \mathbb{R}^{s\times s}$$

is regular for any $t \in \mathcal{T}$. Then the set of solution K_i of the generalized open-loop RDE exists and is bounded for any $t \in \mathcal{T}$.

Proof. Using Radon's Lemma (see Theorem 3.2) together with the equivalence between the open-loop RDE and the nonsymmetric matrix RDE, we conclude that the set of solutions K_i fulfills

$$\begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_N \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix} Q^{-1},$$

and hence using that $P_i(t)$ is a solution of the linear differential equation (3.7), if $Q(t)$ is regular, then $K_i(t)$ is bounded over the interval $[t_0, t_f]$. □

Now, we study a very important corollary of this latter Theorem for the 2-player open-loop RDE. To shorten our notations, in the sequel we shall use $K := \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$, $Q := \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, $S := \begin{pmatrix} S_{11} & S_{22} \end{pmatrix}$ and $\tilde{A} = \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix}$.

Hence, $Y(t) = \begin{pmatrix} E(t) \\ F(t) \end{pmatrix}$ is a solution of

$$\dot{Y} = MY \quad \text{with } M = \begin{pmatrix} A & -S_{11} & -S_{22} \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{pmatrix} \tag{3.8}$$

Denote now $V(t) := E^TCE + E^TDF + F^TD^TE$ for constant matrices $C^T = C \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{n\times 2n}$. We are now interested in a condition, under which the matrix E remains regular. Supposing that E and F are components of the solution Y of (3.8), we obtain that

$$\begin{aligned}
\tfrac{d}{dt}V(t) =&\ \dot{E}^TCE + \dot{E}^TDF + \dot{F}^TD^TE + E^TC\dot{E} + E^TD\dot{F} + F^TD^T\dot{E} \\
=&\ E^TA^TCE - F^TS^TCE + E^TA^TDF - F^TS^TDF - E^TQ^TD^TE \\
&-F^T\tilde{A}^TD^TE + F^TCAE - E^TCSF - E^TDQE \\
&-E^TD\tilde{A}F + F^TD^TAE - F^TD^TSF \\
=&\ (E^T, F^T)\underbrace{\begin{pmatrix} A^TC + CA - Q^TD^T - DQ & A^TD - D\tilde{A} - CS \\ -\tilde{A}^TD^T + D^TA - S^TC & -S^TD - D^TS \end{pmatrix}}_{L(t)}\begin{pmatrix} E \\ F \end{pmatrix} \\
=&\ Y^TL(t)Y
\end{aligned}$$

holds. Hence, the following condition for the solvability of the open-loop RDE can be proved:

Theorem 3.5 (see also Theorem 4.1 in [FJS97]) *Suppose that the matrices $C \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{n\times 2n}$ are given such that*

$$L(t) = \begin{pmatrix} A^TC + CA - Q^TD^T - DQ & A^TD - D\tilde{A} - CS \\ -\tilde{A}^TD^T + D^TA - S^TC & -S^TD - D^TS \end{pmatrix} \leq 0,$$

as well as $C + DK_f + K_f^TD^T > 0$ hold.

Hereby denote $K_f = \begin{pmatrix} K_{f1} \\ K_{f2} \end{pmatrix}$, $\tilde{A} = \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix}$, $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ and $S = \begin{pmatrix} S_1 & S_2 \end{pmatrix}$. Then, the solution of the generalized open-loop matrix RDE

$$\begin{aligned}
\dot{K}_1 &= -A^TK_1 - K_1A - Q_1 + K_1S_{11}K_1 + K_1S_{22}K_2, \quad K_1(t_f) = K_{f1} \\
\dot{K}_2 &= -A^TK_2 - K_2A - Q_2 + K_2S_{22}K_2 + K_2S_{11}K_1, \quad K_2(t_f) = K_{f2}
\end{aligned}$$

admits a bounded solution over $(-\infty, t_f]$.

Proof. Setting $Y(t_f) := \begin{pmatrix} I_n \\ K_f \end{pmatrix}$, it is $V(t_f) = C + DK_f + K_f^TD^T > 0$. Using $L \leq 0$ it is $\dot{V}(t) \leq 0$ for any $t < t_f$. Thus, the function $V(t)$ is positive definite over $(-\infty, t_f]$.

Using now $V = E^TCE + E^T(DF) + (DF)^TE$, we obtain for $W(t) := DF \in \mathbb{R}^{n\times n}$ that for any $x \in \mathbb{R}^n$

$$0 < x^TVx = x^TE^TCEx + x^TE^TWx + x^TW^TEx$$

holds. If we now assume that x is an eigenvector corresponding to an eigenvalue λ of E, then this latter formula becomes

$$0 < \lambda^2x^TCx + 2\lambda x^TWx,$$

and hence the condition $\lambda \neq 0$ follows obviously. □

After investigating the existence of the solutions of the symmetric and open-loop RDE, we now turn our attention to the 2-player feedback RDE. Despite its nice property of being symmetric, there is hardly anything known about the asymptotic behavior of the solutions of the feedback RDE. We now present a result that is based on the latter proof and then give a relaxed condition that better suits numerical purposes.

Theorem 3.6 *Denote*

$$\tilde{A}(t) := \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \tilde{Q}(t) := \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \text{ and } \tilde{S}(t) := \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}.$$

If then for some $\kappa > 0$ *and for all* $y \in \mathbb{R}^{2m}$*, as well as for* $t \in [t_0, t_f]$

$$y^T(-2\kappa\tilde{A}(t) + \kappa^2\tilde{S}(t) - \tilde{Q}(t))y > 2\kappa^2||\tilde{S}(t)|| \cdot ||y||^2 \tag{3.9}$$

holds, then the set

$$\left\{K_1, K_2 \in \mathbb{R}^{n\times n} \middle| \left\| \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \right\| < \kappa \right\}$$

is a negative invariant set for the solutions of the 2-person generalized feedback RDE on $[t_0, t_f]$.

Proof. Denote $\tilde{K} = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in \mathbb{R}^{2m\times 2m}$. Then, the feedback-RDE (3.5) becomes

$$\dot{\tilde{K}} = -\tilde{A}^T\tilde{A} - \tilde{K}\tilde{A} - \tilde{Q} + \tilde{K}\tilde{S}\tilde{K} + \tilde{K}J\tilde{S}\tilde{K}J + J\tilde{K}\tilde{S}J\tilde{K}.$$

Suppose further, that λ is the biggest eigenvalue of $\tilde{K}$ with the corresponding normalized eigenvector x. Hence

$$\begin{aligned} \tfrac{d\lambda}{dt} &= x^T\dot{\tilde{K}}x = x^T(-2\lambda\tilde{A} + \lambda^2\tilde{S} - \tilde{Q})x + 2\lambda x^T J\tilde{S}\tilde{K}Jx \\ &\geq x^T(-2\lambda\tilde{A} + \lambda^2\tilde{S} - \tilde{Q})x - 2\lambda|x^T J\tilde{S}\tilde{K}Jx| \\ &\geq x^T(-2\lambda\tilde{A} + \lambda^2\tilde{S} - \tilde{Q})x - 2\lambda||\tilde{K}|| \cdot ||\tilde{S}|| \cdot \|Jx\|^2 \\ &= x^T(-2\lambda\tilde{A} + \lambda^2\tilde{S} - \tilde{Q})x - 2\lambda^2||\tilde{S}|| \cdot ||x||^2, \end{aligned}$$

which is for $\lambda = \kappa$, because of equation (3.9), positive and hence the proof is completed. □

Repeating the latter argumentation, we can deduce a relaxed condition for the negative invariance of the solution.

Corollary 3.3 *Let* $\kappa > 0$ *and* $x(t)$ *be the normalized eigenvector corresponding to the biggest eigenvalue of* $\tilde{K}(t)$*. If then for all* $t \in [t_0, t_f]$

$$x^T(t)(-2\kappa\tilde{A}(t) + \kappa^2\tilde{S}(t) - \tilde{Q}(t))x(t) > 2\kappa^2\|\tilde{S}(t)\| \tag{3.10}$$

holds, then the set

$$\left\{K_1, K_2 \in \mathbb{R}^{n\times n} \middle| \left\| \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \right\| < \kappa \right\}$$

is a negative invariant set for the solutions of the 2-person generalized feedback RDE on $[t_0, t_f]$.

Finally, we present a convergence result for the set of Lyapunov and Riccati differential equations arising from a sampled-data Nash differential game (see Section 2.3.3):

Recall the open-loop Riccati-type (3.11) and the Lyapunov-type (3.12) differential equations defined over each sampling-interval $[t_{j-1}, t_j]$ $(j = 1 \dots h)$ of a sampled-date Nash game:

$$\dot{K}_i = -A^T K_i - K_i A - Q_i + \sum_{k=1}^{N} K_i S_{kk} K_k, \quad K_i(t_{j+1}) = \Lambda_i(t_{j+1}) \tag{3.11}$$

$$\begin{aligned} \dot{\Lambda}_i &= -A^T\Lambda_i - \Lambda_i A - Q_i + \Lambda_i \sum_{k=1}^{N} S_{kk} K_k + \left(\sum_{k=1}^{N} S_{kk} K_k\right)^T \Lambda_i - \sum_{k=1}^{N} K_k^T S_{ik} K_k, \\ \Lambda_j(t_j) &= \Lambda_{j+1}(t_j), \end{aligned} \tag{3.12}$$

by taking $\Lambda_{h+1}(t_f) = K_f$ and $t_h = t_f$.

After carefully studying (3.12) and (3.5), it is clear that for the case $\Lambda_j = K_j$ the two terminal-value problems are equivalent. If we denote the r.h.s. of (3.12) by $g(\Lambda, K)$, the r.h.s. of (3.5) by $\hat{g}(K)$, where Λ and K denote the blockdiagonal matrices $\begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_N \end{pmatrix}$ and $\begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & \ddots & \\ & & & K_N \end{pmatrix}$, respectively , then we can state the following (see also [JK99b]).

Theorem 3.7 *Suppose that the generalized feedback RDE has a unique bounded solution (i.e. no blow-up) over the interval $[t_0, t_f]$. Then the optimal control for the sampled-data information structure converges to the feedback control if the largest sampling-time tends to zero.*

Proof. Denote the length of the largest sampling-interval by δ, the blocked solution of the generalized feedback Riccati differential equation (3.5) by $\hat{K}(t)$ and the solution of the Lyapunov-type differential equation (3.12), which is continuous in the interval $[t_0, t_f]$, by $\Lambda(t)$. Furthermore denote the piecewise continuous function over $[t_0, t_f]$, being assembled from the segments K_j that solve the open-loop RDE (3.11) for $[t_{j-1}, t_j]$, by $K(t)$.

Since for each sampling interval $[t_{j-1}, t_j]$ the terminal values of both Λ and K are the same and because they are (at least for a small enough δ) analytical, we can state that

$$\|\Lambda - K\| \le \max_{t_{j-1} \le t \le t_j} \left(\|\dot{\Lambda} - \dot{K}\|\right) \cdot \delta \le M_j \cdot \delta \qquad \text{for} \quad t \in [t_{j-1}, t_j]$$

for some $M_j > 0$. Furthermore,

$$\|g(\Lambda, K) - g(\Lambda, \Lambda)\| \le \tilde{M}_j \cdot \|\Lambda - K\| \le L_j \cdot \delta \qquad \text{for} \quad t \in [t_{j-1}, t_j]$$

holds for some $L_j > 0$.

With the notation $L := \max_{j=1 \dots h} L_j$ the following holds

$$\|g(\Lambda, K) - \hat{g}(\Lambda)\| \le L \cdot \delta \qquad \text{for} \quad t \in [t_0, t_f]$$

Because the linearizable equations (3.11) and (3.12) have over a finite interval bounded solutions, the function $g(\Lambda, K)$ is bounded and furthermore

$$\limsup_{\delta \to 0} \left(\max_{j=1\dots h} \tilde{M}_j \right) < \infty.$$

This means that

$$\limsup_{\delta \to 0} \|g(\Lambda, K) - \hat{g}(\Lambda)\| = 0$$

and hence that the Lyapunov-type equation (3.12) can be considered as a perturbed generalized feedback RDE, where the perturbation term vanishes if the sampling time tends to zero. This implies (see Chapter 2.4 in [CL55]) that the function Λ and therefore also the function K converge to $\hat{K}$. □

Remark 3.2 *Simaan and Cruz (see [SJC73b]) also claim the convergence of the solution of differential games with sampled-data information structure to the solution of those with memoryless-feedback information structure. However, neither a proof of this statement is given nor it is clear what happens if the solution of the differential game using the memoryless-feedback information structure has finite escape time (i.e. a singularity).*

3.3 Stabilizing properties of solutions to the algebraic Riccati equation

In this section, we investigate questions, whether the constant solution (if there is any) of the various Riccati differential equations yields a control function, which stabilizes the closed-loop system. Although, we are not going to concern infinite time-horizon optimization problems, it can be shown (see for instance [Eng98b], [JKK01a] and Section 10.2 in [KK85]), that the methods discussed here are very strongly related to these types of questions.

Let us begin first with some definitions:

Definition 3.3 *Suppose that the coefficients A, Q, S, Q_i and S_{ij}, as well as M_{ij} of the (generalized) matrix Riccati differential equation (see equations (3.1), 3.4) and (3.5)) are constant. Then, we call the following (algebraic) equation and sets of (algebraic) equations for $i = 1, \dots, N$*

$$0 = -A^T K - KA - Q + KSK, \tag{3.13}$$

$$0 = -K_i A - A^T K_i - Q_i + K_i \sum_{j=1}^{N} S_{jj} K_j, \tag{3.14}$$

and

$$0 = -K_i A - A^T K_i - Q_i + K_i \sum_{j=1}^{N} S_{jj} K_j - \sum_{\substack{j=1 \\ j \neq i}}^{N} K_j S_{ij} K_j + \sum_{\substack{j=1 \\ j \neq i}}^{N} K_j S_{jj} K_i, \tag{3.15}$$

symmetric (or symplectic), generalized open-loop and generalized feedback algebraic Riccati equation (ARE), respectively.

Furthermore, we call the equation

$$0 = M_{21} + M_{22}K - KM_{11} - KM_{12}K, \tag{3.16}$$

nonsymmetric algebraic Riccati equation.

Remark 3.3 *Note that, using the same method as described before, the generalized open-loop ARE can be transformed into a non-symmetric ARE of dimension $N \cdot n$. Moreover, solutions of the algebraic Riccati equations are exactly the constant solutions of the corresponding autonomous Riccati differential equations.*

Let us first concentrate on the symmetric ARE. We first deal with its qualitative properties and then show, how its solutions can serve as a stabilizing feedback matrix for control systems over an infinite time-horizon. The following theorem, which is cited from [KK85] describes the most important cases for the solvability of the symmetric ARE.

Theorem 3.8 (see Theorems 10.3 and 10.5 in [KK85]) *Suppose that $Q \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$ are positive semidefinite and that the pair (A^T, Q) is stabilizable. Then one of the following possibilities occur:*

(i) The symmetric ARE has no positive semidefinite solution. Then, for any solution of the corresponding RDE with $K(t_f) \geq 0$ $\lim_{t\to-\infty} \|K(t)\| = \infty$ holds.

(ii) The symmetric ARE has exactly one positive semidefinite solution K_+. Then, for any solution of the corresponding RDE with $K(t_f) \geq 0$ $\lim_{t\to-\infty} K(t) = K_+$ holds.

Moreover, the symmetric ARE has exactly one positive semidefinite solution if and only if the pair (A, S) is stabilizable (see Definition 1.12).

Remark 3.4 *Please note, that although the proof is mostly based on the symmetry of the solutions, it cannot be generalized for the feedback ARE. Indeed, as it will be shown in the next section, generalized feedback algebraic Riccati equations can have several positive definite solutions, even under stabilizability constraints.*

Although, as it is shown in Section 3.4, results on the uniqueness of positive semidefinite solutions and on the convergence of solutions do not hold for the generalized feedback Riccati equations, the convergence results can be generalized in the following sense for non-symmetric matrix Riccati equations (and hence for the open-loop Riccati equation).

Definition 3.4 (dichotomicity) *Suppose that the eigenvalues λ_i of $M \in \mathbb{R}^{(r+s)\times(r+s)}$ are such that*

$$Re\lambda_1 \leq Re\lambda_2 \leq \ldots \leq Re\lambda_r \;<\; Re\lambda_{r+1} \leq \ldots \leq Re\lambda_{r+s} \tag{3.17}$$

Then the corresponding autonomous matrix Riccati differential equation

$$\dot{K} = M_{21} + M_{22}K - KM_{11} - KM_{12}K \tag{3.18}$$

is said to be (reverse)-dichotomically separated.

Remark 3.5 (see Lemma 2.4.1 in [AKFIJ]) *Suppose that the pairs* $(A^T, \sqrt{Q})$ *and* $(A, \sqrt{S})$ *with* $A, Q, S \in \mathbb{R}^{n\times n}$, $Q \geq 0$ *and* $S \geq 0$ *are stabilizable. Then, the corresponding symmetric matrix RDE is dichotomically separated.*

A very important property of dichotomically separated matrix Riccati equations, which is a direct generalization of Theorem 3.8, is presented by the following theorem.

Theorem 3.9 (see for instance Theorem 2 in [JK99b]) *Suppose that the autonomous non-symmetric matrix Riccati differential equation (3.18) is reverse-dichotomically separated. Then there exists a matrix* $K_+ \in \mathbb{R}^{r\times s}$ *such that for almost any initial values* $K_f \in \mathbb{R}^{r\times s}$

$$\mathrm{LIM}_{t\to-\infty} K(t) = K_+$$

holds. Hereby denotes $K(t)$ *the solution of the differential equation (3.18) with* $K(t_f) = K_f$ *and* LIM *the limit-value taken in the chordal metric. Moreover, this convergence takes place at an exponential rate.*

Finally, in order to illustrate to the stabilizing property of the constant feedback control functions arising from a solution of the algebraic Riccati equation, we refer to a technique used in Chapter 3 of [AKFIJ] and [LT85]:

Suppose that $K \in \mathbb{R}^{r\times s}$ is a solution of the non-symmetric ARE. Suppose further, that $L \in \mathbb{R}^{r\times s}$ is solution of the following (linear) equation

$$(M_{11} + M_{12}K)L - L(M_{22} - KM_{12}) + M_{12} = 0. \tag{3.19}$$

We are now interested in the stability of the autonomous dynamical system

$$\dot{\xi} = M\xi$$

To obtain that, we first apply the transformation $\tilde{\xi} = \begin{pmatrix} I_s & 0 \\ -K & I_r \end{pmatrix} \xi$. Hence, it is

$$\begin{aligned} \frac{d}{dt}\tilde{\xi} &= \begin{pmatrix} I_s & 0 \\ -K & I_r \end{pmatrix} \dot{\xi} = \begin{pmatrix} I_s & 0 \\ -K & I_r \end{pmatrix} M\xi \\ &= \begin{pmatrix} I_s & 0 \\ -K & I_r \end{pmatrix} M \begin{pmatrix} I_s & 0 \\ -K & I_r \end{pmatrix}^{-1} \tilde{\xi}, \end{aligned}$$

where $\begin{pmatrix} I_s & 0 \\ -K & I_r \end{pmatrix}^{-1} = \begin{pmatrix} I_s & 0 \\ K & I_r \end{pmatrix}$ holds and hence, using the definition of K,

$$\frac{d}{dt}\tilde{\xi} = \underbrace{\begin{pmatrix} M_{11} + M_{12}K & M_{12} \\ 0 & -KM_{12} + M_{22} \end{pmatrix}}_{\tilde{M}} \tilde{\xi}$$

follows.

Applying now a second transformation $\tilde{\tilde{\xi}} := \begin{pmatrix} I_s & -L \\ 0 & I_r \end{pmatrix} \tilde{\xi}$, we can write similarly:

$$\frac{d}{dt}\tilde{\tilde{\xi}} = \begin{pmatrix} I_s & -L \\ 0 & I_r \end{pmatrix} \frac{d}{dt}\tilde{\xi} = \begin{pmatrix} I_s & -L \\ 0 & I_r \end{pmatrix} \tilde{M} \begin{pmatrix} I_s & -L \\ 0 & I_r \end{pmatrix}^{-1} \tilde{\tilde{\xi}},$$

where again $\begin{pmatrix} I_s & -L \\ 0 & I_r \end{pmatrix}^{-1} = \begin{pmatrix} I_s & L \\ 0 & I_r \end{pmatrix}$ and hence, using (3.19),

$$\frac{d}{dt}\tilde{\tilde{\xi}} = \underbrace{\begin{pmatrix} M_{11} + M_{12}K & 0 \\ 0 & M_{22} - KM_{12} \end{pmatrix}}_{\tilde{\tilde{M}}} \tilde{\tilde{\xi}} \tag{3.20}$$

hold.

This means, that the matrix M can be transformed through the automorphism

$$\xi \mapsto \begin{pmatrix} I_s & -L \\ 0 & I_r \end{pmatrix} \begin{pmatrix} I_s & 0 \\ -K & I_r \end{pmatrix} \xi$$

into the block diagonal matrix $\tilde{\tilde{M}}$. Hence, their spectra coincide.

We can formulate this short consideration as a preliminary lemma

Lemma 3.2 (LK-transformation) *Suppose that K is a solution of the non-symmetric matrix ARE and that the solution L to the so-called algebraic Sylvester equation (3.19) exists. Then the spectrum σ of the matrix M decomposes into two sets, namely*

$$\sigma(M) = \sigma(M_{11} + M_{12}K) \cup \sigma(M_{22} - KM_{12}).$$

Remark 3.6 (see Chapter 3.I in [AKFIJ]) *It is also true, that the solution of (3.19) L exists uniquely, if and only if the two spectra $\sigma(M_{11} + M_{12}K)$ and $\sigma(M_{22} - KM_{12})$ are distinct.*

In order to obtain the stabilizability property of the algebraic solution K, we need a few more notations:

Definition 3.5 (graph subspace) *Let $X \in \mathbb{R}^{r \times s}$. Then, the mapping*

$$\phi_X : x \in \mathbb{R}^s \mapsto \begin{pmatrix} I_s \\ X \end{pmatrix} x \in \mathbb{R}^{r+s}$$

defines a linear mapping $\phi_X : \mathbb{R}^s \to \mathbb{R}^{r+s}$. We call its image

$$G(X) := \mathrm{im}(\phi_X) \subset \mathbb{R}^{r+s}$$

graph subspace *of X.*

Remark 3.7 *Since, the number of columns of the matrix* $\begin{pmatrix} I_s \\ X \end{pmatrix}$ *equals to* s*, and since it contains the matrix* I_s*, the dimension of the graph subspace* $G(X)$ *is exactly* s *for any matrix* $X \in \mathbb{R}^{r\times s}$*.*

Suppose now, that $K \in \mathbb{R}^{r\times s}$ is a solution of the non-symmetric ARE. Then, it is

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I_s \\ K \end{pmatrix} = \begin{pmatrix} M_{11} + M_{12}K \\ M_{21} + M_{22}K \end{pmatrix} = \begin{pmatrix} I_s \\ K \end{pmatrix} (M_{11} + M_{12}K) .$$

Hence, the linear transformation M is an endomorphism on $G(K)$. Choosing now a base on $G(K)$ defined by the (independent) s columns of the matrix $\begin{pmatrix} I_s \\ K \end{pmatrix}$, we conclude, using the above identity, that the action of the transformation M on this base is represented by the matrix $(M_{11} + M_{12}K)$. Consequently, the mappings $M|_{G(K)}$ and $(M_{11} + M_{12}K)$ have the same spectra.

We are now very close to the main result of this section. We formulate it as

Theorem 3.10 (stabilizing property of the solution of ARE – see Theorem 2.1.2 in [AKFIJ]) *Let* M *be semi-simple and* $\{\lambda_1, \ldots, \lambda_s\}$ *be a set of eigenvalues of* M*. Then there exists a solution* K *to the ARE (3.16), such that* $\sigma(M_{11} + M_{12}K) = \{\lambda_1, \ldots, \lambda_s\}$ *holds.*

Proof. Carrying out the same calculation as before, we conclude that K is a solution of the ARE if and only if

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I_s \\ K \end{pmatrix} = \begin{pmatrix} M_{11} + M_{12}K \\ M_{21} + M_{22}K \end{pmatrix} = \begin{pmatrix} I_s \\ K \end{pmatrix} (M_{11} + M_{12}K)$$

holds. Hence, every M-invariant graph subspace corresponds to a solution of the ARE. With other words, if V denotes the matrix formed by the column-vectors of the eigenvectors corresponding to $\lambda_1, \ldots, \lambda_s$, then there exists a solution of the ARE K such that $\mathrm{im}(V) = G(K)$. Clearly, it is

$$\sigma(M|_{\mathrm{im}(V)}) = \{\lambda_1, \ldots, \lambda_s\}$$

and therefore, using $\sigma(M|_{\mathrm{im}(V)}) = \sigma(M_{11} + M_{12}K)$, the proof is completed. □

3.4 Solution methods

In this last section numerical results are presented to obtain the solution of the feedback generalized matrix RDE and to illustrate the boundedness-criteria for that solution. We also give an example of a generalized algebraic Riccati equation having more than one positive definite solutions. In the case of the symmetric matrix Riccati differential equations, as they appear in Control Theory, under stabilizability and controllability conditions there is exactly one positive semidefinite solution (see Theorem 3.8). However, it was unnoticed by several authors (see for instance [LG95]) that this property does not hold for generalized Riccati matrix differential equations.

Example 8: Consider the autonomous generalized feedback RDE with coefficients

$$A = \begin{bmatrix} 20 & 50 \\ -25 & 15 \end{bmatrix}, Q_1 = \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} > 0, Q_2 = \begin{bmatrix} 8 & 2 \\ 2 & 14 \end{bmatrix} > 0,$$

$$S_{11} = \begin{bmatrix} 18 & 18 \\ 8 & 18 \end{bmatrix} \geq 0 \text{ and } S_{22} = \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix} \geq 0.$$

and terminal value problem $K_1(0) = \begin{bmatrix} 2.5 & 0 \\ 0 & 1 \end{bmatrix} > 0$ and $K_2(0) = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} > 0$.

Our first aim is to obtain numerical solutions for $K_1(t)$ and $K_2(t)$. In order to illustrate the difficulty of the problem, we first create numerical solutions using the classical *upwind discretization scheme* (see Figures 3.2 and 3.3).

As one can see, the convergence of the solutions remained untouched, as we decreased the time-steps. Nevertheless, using a finer grid, the solutions tend to converge more slowly, which somehow illustrates that the this behavior might only be a consequence of the solution method. As a counterpart for this approximation, let us now see the behavior of the solutions with the classical 4-5-th order Runge-Kutta method (Figures 3.4 and 3.5). No convergence can be observed. Indeed, Figure 3.5 implies that the solutions might be periodic. Since no error estimate or convergence result is known for these two methods in the case of the RDE, the qualitative behavior of the solution cannot be determined easily.

In order to characterize the qualitative behavior of the solutions, we need another solution technique, for which we already obtained a convergence result. With the help of Theorem 3.7, an approximation method for the solution of the generalized feedback RDE can be constructed as follows :

(1) Let h be a sufficiently large fixed positive integer and $T := \frac{t_f - t_0}{h}$. Further let $\tilde{\Lambda}_i = K_{f\,i}$ $(i = 1, 2)$ and set $j = h$.

(2) Define $t_k := t_0 + kT$ for each $k = 0, \ldots, h$.

(3) Let K_i be the (exact) solution of the *open-loop* RDE (3.4), according to Theorem 3.2, over the interval $[t_{j-1}, t_j]$ for the terminal-value problem $K_i(t_j) = \tilde{\Lambda}_i$.

(4) Calculate $K_{i0} = K_i\left(\frac{t_{j-1}+t_j}{2}\right)$.

(5) Determine the (exact) solution of the Lyapunov-type differential equation (2.41) over $[t_{j-1}, t_j]$ for the terminal-value problem $\Lambda_i(t_j) = \tilde{\Lambda}_i$, by substituting $K_i(t) \cong K_{i0}$.

(6) Calculate $\tilde{\Lambda}_i = \frac{1}{2}(\Lambda_i(t_{j-1}) + \Lambda_i^T(t_{j-1}))$, as well as $\hat{K}_i(t) = \frac{1}{2}(K_i(t) + K_i^T(t))$, for $t \in [t_{j-1}, t_j]$.

(7) For the case $j > 1$, decrement j by one and go to step (3).

Remark 3.8 *The symmetrization in step (6) serves for accelerating the convergence. It is based on the a priori knowledge, that the solution of the generalized feedback RDE is symmetric if the coefficients and initial data are symmetric (see Lemma 3.1).*

Remark 3.9 *It should be pointed out that the above procedure is not only a solution technique for generalized Riccati differential equations, but also a realization of an information structure used in differential systems with discrete-time feedback. (see Section 2.3.3)*

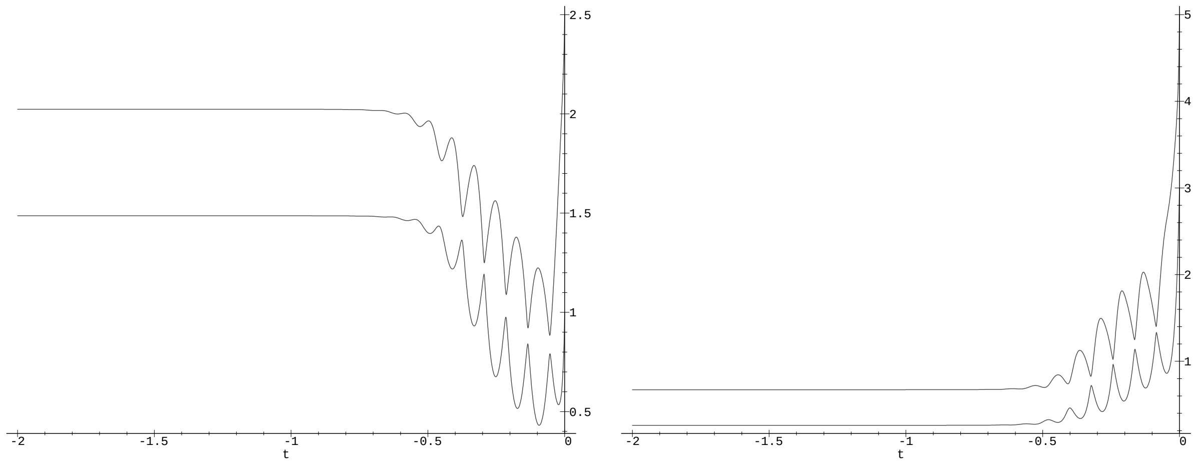

Figure 3.2: Eigenvalues of K_1 and K_2 using upwind discretization method. $\Delta t = 1.3 \cdot 10^{-3}$

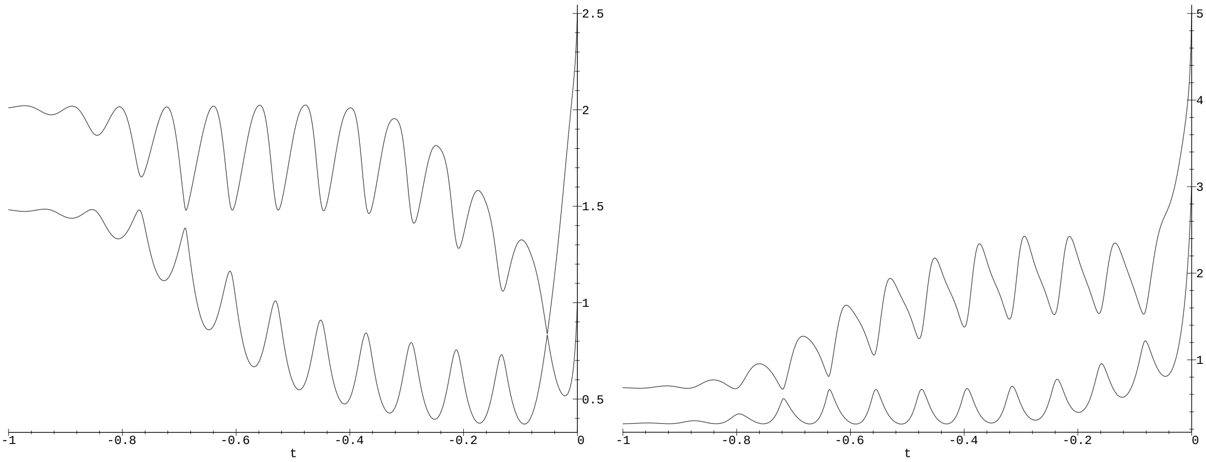

Figure 3.3: Eigenvalues of K_1 and K_2 using upwind discretization method. $\Delta t = 3.3 \cdot 10^{-4}$

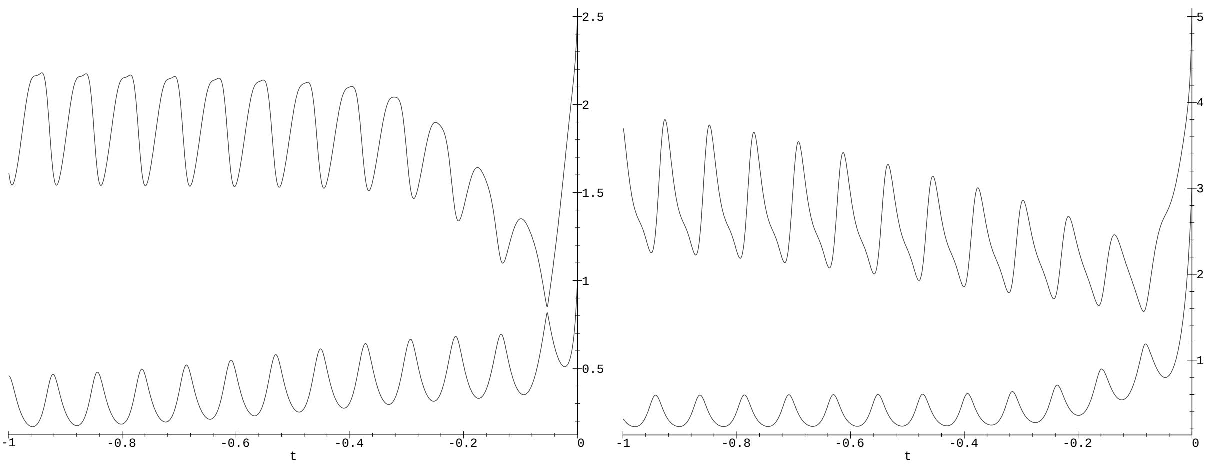

Figure 3.4: Eigenvalues of K_1 and K_2 using 4-5-th order Runge-Kutta method.

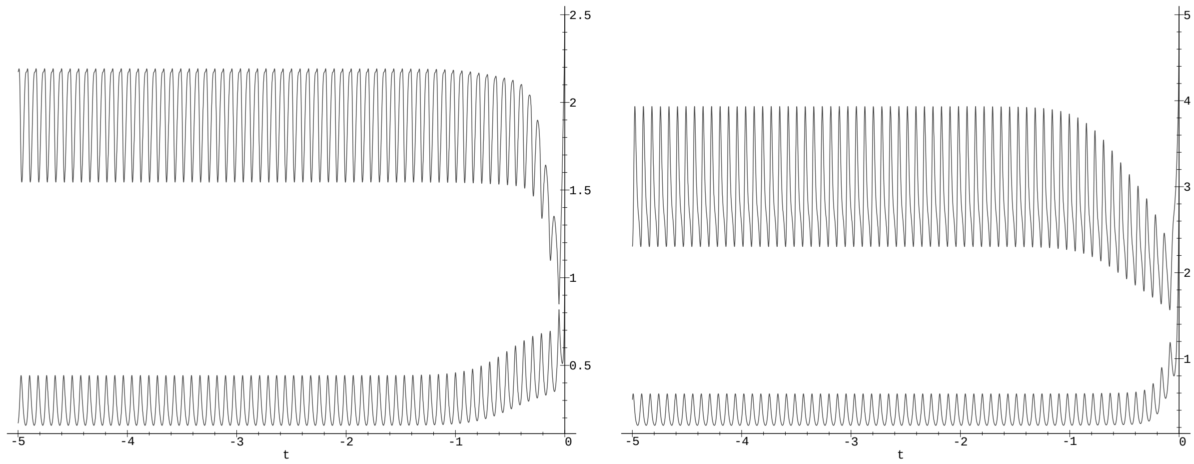

Figure 3.5: Eigenvalues of K_1 and K_2 using 4-5-th order Runge-Kutta method.

Figures 3.6-3.8 represent the approximate solutions obtained by means of the sampled-data method described above. Theorem 3.7 indicates, that *if the solution of the generalized feedback RDE exists and is bounded*, then the functions on Figures 3.6, 3.7 and 3.8 converge to it. Thus the qualitative behavior of the given solution is best reflected in Figures 3.4, 3.5 and 3.8.

Our next question is the existence of negative invariant sets. Using Theorem 3.3 we can state that *the set of positive definite matrices is negative invariant.* However, our hope for also obtaining an upper bound for the norm of the solution using Theorem 3.3 is not fulfilled.

Nevertheless, we can check, whether the assumptions of Corollary 3.3 are fulfilled. Although this weaker condition is difficult to check in general, it is more advantageous in numerical approximations. Rearranging the terms in (3.10) we get

$$\kappa^2(x^T\tilde{S}(t)x - 2\|\tilde{S}(t)\|) - 2\kappa x^T\tilde{A}(t)x - x^T\tilde{Q}(t)x \geq 0.$$

Since $\|x\| = 1$, it is $x^T\tilde{S}(t)x - 2\|\tilde{S}(t)\| < 0$ as long as $\tilde{S} \neq 0$. This shows that the possible values for κ have an upper bound. Figure 3.9 represents the maximal value of κ over $t \in [-0.14, 0]$. As it can be seen, for $t \approx -0.1$ not even the weakened condition of Corollary 3.3 is fulfilled. This means that we cannot obtain an invariant set by this theorem.

In order to illustrate how fast the solution of a generalized Riccati equation can converge to a pole, we can calculate the solutions towards $+\infty$. As Figure 3.10 shows, although the initial values are relatively small and the solution is analytic in a neighborhood of zero, the radius of this neighborhood cannot greater then about 0.008. Hence, the method of searching for the solution in form of a Taylor-series expansion is evidently impossible.

Finally, we present some solutions of the generalized (ARE) of the memoryless feedback type (3.15) with the same coefficients as before. There are up to 4 different pairs of solutions known, 2 of which are positive definite (1 and 2), 1 negative definite (3) and 1 indefinite (4):

$$K_1^{(1)} \approx \begin{bmatrix} 2.0139 & .07042 \\ .07042 & 1.4959 \end{bmatrix} \quad K_2^{(1)} \approx \begin{bmatrix} .63421 & -.11875 \\ -.11875 & .29936 \end{bmatrix}$$

$$K_1^{(2)} \approx \begin{bmatrix} .29894 & .068680 \\ .068680 & .13624 \end{bmatrix} \quad K_2^{(2)} \approx \begin{bmatrix} 3.1659 & .43560 \\ .43560 & 2.3835 \end{bmatrix}$$

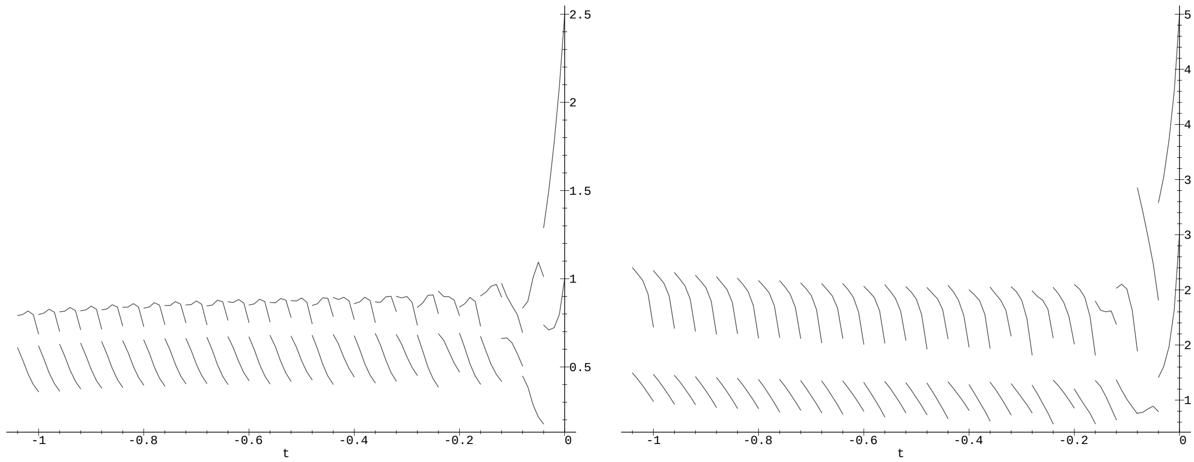

Figure 3.6: Eigenvalues using the 'sampled-data' approximation. $T = 0.04$

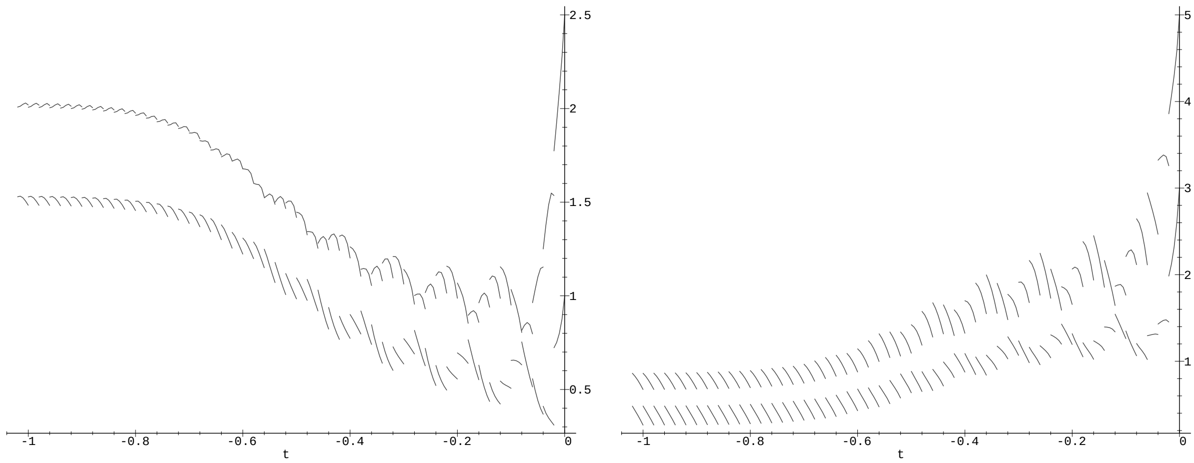

Figure 3.7: Eigenvalues using the 'sampled-data' approximation. $T = 0.02$

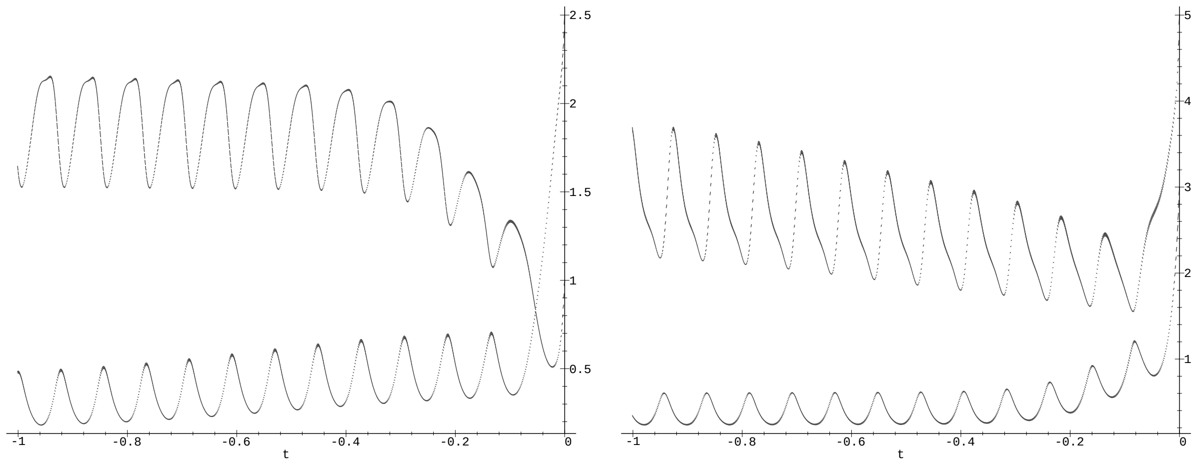

Figure 3.8: Eigenvalues using the 'sampled-data' approximation. $T = 10^{-3}$

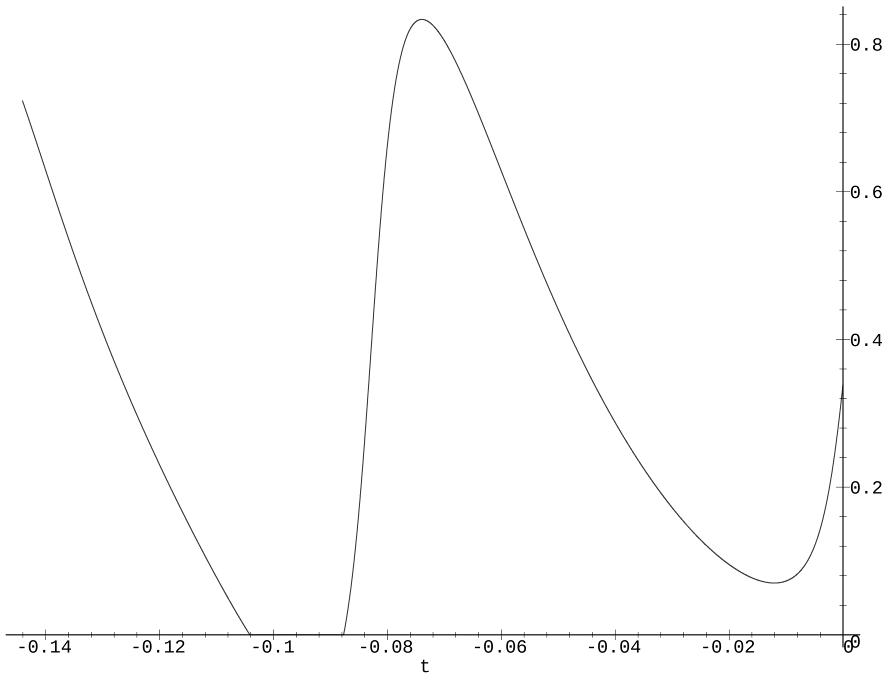

Figure 3.9: Maximal invariant set using Theorem 5.1

$$K_1^{(3)} \approx \begin{bmatrix} -.14501 & .037882 \\ .037882 & -.18551 \end{bmatrix} \quad K_2^{(3)} \approx \begin{bmatrix} -.17930 & .00018170 \\ .00018170 & -.34354 \end{bmatrix}$$

$$K_1^{(4)} \approx \begin{bmatrix} -.11991 & -.0013110 \\ -.0013110 & 2.6932 \end{bmatrix} \quad K_2^{(4)} \approx \begin{bmatrix} -.60076 & -1.2321 \\ -1.2321 & .084834 \end{bmatrix}$$

If we study the asymptotic behavior of the games being controlled by the constant controlling laws that arise from the above solutions, then we can state that *both positive definite solutions are asymptotically stabilizing* (i.e the matrix $A - S_1K_1 - S_2K_2$ is stable), the negative definite produces an unstable linear differential system with all eigenvalues having positive real parts and the indefinite solution $K_i^{(4)}$ leads to a linear differential system with two eigenvalues having positive and two having negative real parts.

3.5 Notes and references

The previous chapter covered some topics of matrix Riccati differential equations, generalized Riccati differential equations and the corresponding algebraic equations, as they appear in the context of linear-quadratic control systems and differential games. Although we presented the most important results, we didn't cover topics, that will not be further investigated in this work.

A detailed discussion on the equivalence mentioned in Theorem 3.2 in the context of the complex matrix Riccati differential equation can be found in [FJ95]. In this work the authors

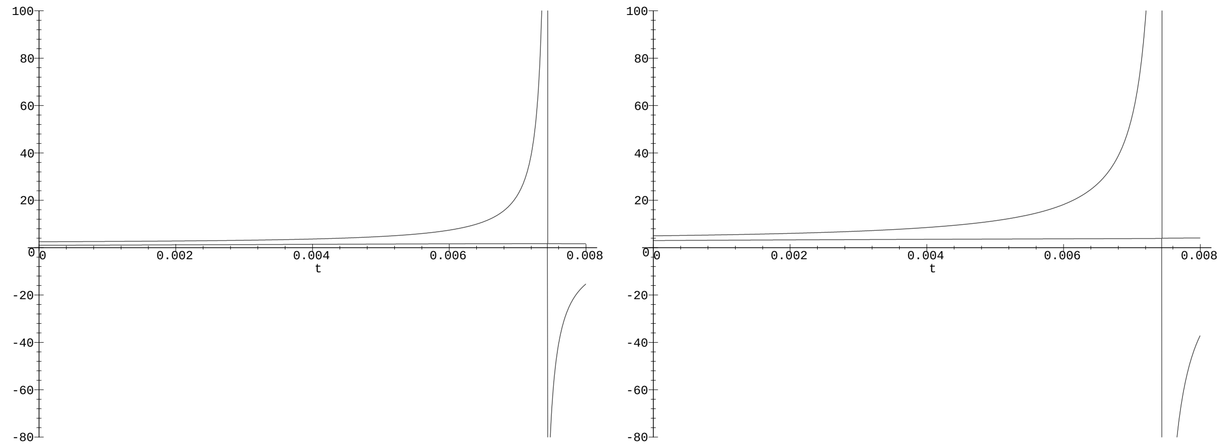

Figure 3.10: Behavior of solutions in a small neighborhood of zero.

also published a representation formula – using the mentioned equivalence – for the solution of the matrix Riccati differential equation. Using this representation formula it is also possible to construct exactly every (complex) solution of the (non-generalized) algebraic matrix Riccati equation. Unfortunately, these results cannot be generalized for the case of the feedback matrix RDE or ARE.

For further details on the boundedness criteria for solutions of generalized and non-generalized Riccati differential equations, please refer to the works of Freiling, Jank and Sarychev (see [FJS97], [FJS00]). Further global results on the (complex) matrix Riccati differential equation can be found for instance in the works of Shayman [Sha86], Jank [Jan92], Jank and Freiling [FJ96],[FJAK96], Weeren, Schumacher and Engwerda [WSE], Papavassilopoulos and Olsder [PO84], Schneider [Sch73] and Kuiper [Kui94]. For recent results concerning differential games with optimality constraint over infinity time-horizon, refer to the works of Engwerda [Eng98b] and Jank, Kremer and Kun [JKK01a].

There is a huge literature on numerical methods for solving algebraic Riccati equations and Riccati differential equations, most of them dealing with the symmetric Riccati equation (see for example [BB95] or [MV]). Besides them, only several authors published numerical methods for solutions of generalized Riccati equations (see [LG95], [Eng98a], [Kun97] and [JK99b]). It is a pity, however, that some of the authors (see [LG95]) don't even realize results presented in Section 3.4, and discuss topics on the uniqueness of the positive definite stabilizing solutions of feedback ARE.

For further reading on Riccati equations, the textbook of Reid [Rei72] is recommended. Finally, the theory of Lyapunov differential equations and algebraic Lyapunov equations, as well as further results on Riccati equations concerning for instance the LK-transformation and the representation formula are presented in the forthcoming book of Abou-Kandil, Freiling, Ionescu and Jank (see [AKFIJ]).

Chapter 4

Disturbed Systems

Similarly to the Perturbation Theory of Dynamical Systems, the question of disturbance attenuation or disturbance decoupling, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems play an important role in Control Theory.

However, there are unfortunately very few papers giving conditions on the existence and uniqueness of an equilibrium state in differential games. As it is shown in [Eis82] and [LR71] those conditions are far away from being trivial and therefore should receive more interest.

In this chapter, we first review the theory of disturbed control systems, give a condition for maximal disturbance attenuation and then, in Section 4.2, generalize this concept in the context of differential games. Finally, in Section 4.3, we discuss the difference based on a numerical calculation.

4.1 Disturbance attenuation for linear control systems

Let us consider now a class of linear control systems:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)w(t), \quad x(t_0) = x_0, \tag{4.1}$$

with piecewise continuous functions $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m_1}$ and $C(t) \in \mathbb{R}^{n\times m}$. Further assume that $u \in \mathcal{U} = \{u|u : [t_0, t_f] \to \mathbb{R}^{m_1}\} \subset \mathcal{L}_2^{m_1}[t_0, t_f]$ is a control function and $w \in \mathcal{W} = \{w|w : [t_0, t_f] \to \mathbb{R}^m\} \subset \mathcal{L}_2^m[t_0, t_f]$ is a generic disturbance of finite energy acting on the system.

Assume further that the output signal ($y(t)$) of the system can be written as

$$y(t) = M(t)x(t) + N(t)u(t),$$

with piecewise continuous functions $M(t) \in \mathbb{R}^{k\times n}$ and $N(t) \in \mathbb{R}^{k\times m}$, where for any $t \in [t_0, t_f]$ $N^T(t)N(t) > 0$ holds.

Now, we can define the following problem, that is often called in the literature as *disturbance attenuation problem of level* γ:

Problem 3 (disturbance attenuation problem) *Given a positive real constant* γ, *find a con-*

trol function $u(.) \in \mathcal{U}$*, such that for any admissible disturbance* $w \in \mathcal{W}$

$$\frac{\|y\|_{\mathcal{L}_2^k[t_0,t_f]}}{\|w\|_{\mathcal{L}_2^m[t_0,t_f]}} \leq \gamma \tag{4.2}$$

holds.

Here, we derive a solution for Problem 3, and in the next section, we generalize this problem for linear differential games.

Rewriting (4.2) yields

$$J(u,w) := \int_{t_0}^{t_f} \Psi(x,u,w)\, dt \leq 0, \tag{4.3}$$

for $\Psi(x,u,w) := x^T M^T(t) M(t) x + u^T N^T(t) N(t) u + x^T M(t) N(t) u + u^T N^T(t) M(t) x - \gamma w^T w.$

This means, that we are interested in a control function u, for which $J(u,w) \leq 0$ holds for any disturbance $w \in \mathcal{W}$. We solve this problem with the so-called *minimax*-strategy, a concept well known in the Theory of Dynamical Games.

First, we maximize $J(u,w)$ in $w \in \mathcal{W}$ for any given control function $u \in U$:

Definition 4.1 (worst-case disturbance) *Suppose that for a given* $u \in \mathcal{U}$ *there exists a disturbance function* $\hat{w} \in \mathcal{W}$ *such that*

$$J(u,w) \leq J(u,\hat{w})$$

holds for any $w \in \mathcal{W}$*. Then we say that* $\hat{w}$ *is the* worst-case disturbance *for the particular control function* u*.*

Finding the above defined worst-case disturbance, leads to a so-called *parametric optimization problem*, which we solve with the help of the generalized value-function:

Definition 4.2 (value-function) *For the disturbed linear-quadratic optimal control system (4.1)-(4.3), the value function* $V : \mathcal{T} \times X \to \mathbb{R}$ *is defined in the following way:*

$$V(t_1,x_1) = \inf_{u\in\mathcal{U}} \sup_{w\in\mathcal{W}} \left\{ \int_{t_1}^{t_f} \Psi(\hat{x},u,w)\, dt \,\middle|\, \frac{d}{dt}\hat{x} = A(t)\hat{x}(t) + B(t)u(t) + C(t)w(t),\ \hat{x}(t_1) = x_1 \right\}. \tag{4.4}$$

Remark 4.1 *Using the above definition of the value-function and of the worst-case disturbance, we immediately obtain that Problem 3 is solvable if and only if* $V(t_0,x_0) \leq 0$*. If then for some* $\tilde{u} \in \mathcal{U}$ *and for the corresponding worst-case disturbance* $\hat{w}$

$$J(\tilde{u},\hat{w}) = V(t_0,x_0)$$

holds, then $\tilde{u}$ *is a solution of Problem 3.*

Proof. To prove the above statement, just notice that $V(t_0, x_0) > 0$ implies that for any control function $u \in \mathcal{U}$ there exists a disturbance w such that $J(u, w) > 0$ and hence the disturbance attenuation problem is not solvable. Conversely, if $V(t_0, x_0) \leq 0$, then there exists a $\tilde{u} \in \mathcal{U}$ such that for any $w \in \mathcal{W}$ $J(\tilde{u}, w) \leq 0$ holds. Hence, this control fulfills the requirements of Problem 3. □

We show here that the function $\tilde{V}(t) = V(t, x(t))$ has the following form:

$$\tilde{V}(t) = x^T(t)E(t)x(t) + e^T(t)x(t) + d(t)$$

for some mappings $E : [t_0, t_f] \to \mathbb{R}^{n\times n}$, $e : [t_0, t_f] \to \mathbb{R}^n$ and $d : [t_0, t_f] \to \mathbb{R}$, where $E(t)$ is symmetric for each $t \in [t_0, t_f]$.

Using the value-function approach presented in Section 1.4, we conclude that

$$\begin{aligned}
\frac{d}{dt}\tilde{V}(t) &= \dot{x}^T Ex + x^T \dot{E}x + x^T E\dot{x} + \dot{e}^T x + e^T \dot{x} + \dot{d} \\
&= \left(x^T A^T + u^T B^T + w^T C^T\right) Ex + x^T \dot{E}x + x^T E\left(Ax + Bu + Cw\right) \\
&\quad + \dot{e}^T x + e^T \left(Ax + Bu + Cw\right) + \dot{d} + x^T M^T Mx + u^T N^T Nu \\
&\quad + x^T M^T Nu + u^T N^T Mx - \gamma w^T w - \Psi \\
&= x^T\left(\dot{E} + M^T M + EA + A^T E\right)x - \gamma(w - z)^T(w - z) - \gamma z^T w - \gamma w^T z \\
&\quad + \gamma z^T z + (u - y)^T N^T N(u - y) + y^T N^T Nu + u^T N^T Ny - y^T N^T Ny \\
&\quad + x^T\left(EB + M^T N\right)u + u^T\left(B^T E + N^T M\right)x + + x^T ECw + w^T C^T Ex \\
&\quad + \left(\dot{e}^T + e^T A\right)x + e^T Bu + e^T Cw + \dot{d} - \Psi \\
&= x^T\left(\dot{E} + M^T M + EA + A^T E\right)x - \gamma(w - z)^T(w - z) \\
&\quad + (u - y)^T N^T N(u - y) + w^T\left(C^T Ex + \frac{1}{2}C^T e - \gamma z\right) \\
&\quad + \left(\frac{1}{2}e^T C + x^T EC - \gamma z\right)w \\
&\quad + u^T\left(N^T Ny + B^T Ex + \frac{1}{2}B^T e + N^T Mx\right) \\
&\quad + \left(x^T M^T N + \frac{1}{2}e^T B + x^T EB + N^T Ny\right)u \\
&\quad + \left(\dot{e}^T + e^T A +\right)x + \dot{d} - y^T N^T Ny + \gamma z^T z - \Psi
\end{aligned}$$

for any mappings $y : [t_0, t_f] \to \mathbb{R}^{m_1}$ and $z : [t_0, t_f] \to \mathbb{R}^m$. We set now, in the manner of Section 1.4

$$\begin{aligned}
N^T Ny + B^T Ex + \frac{1}{2}B^T e + N^T Mx &= 0 \quad \text{and} \\
C^T Ex + \frac{1}{2}C^T e - \gamma z &= 0
\end{aligned}$$

so that the terms that are linear in u and w cancel out then, assuming that the matrix $N^T(t)N(t)$ is invertible for each $t \in [t_0, t_f]$, we obtain

$$\begin{aligned}
\frac{d}{dt}\tilde{V} &= x^T\left(\dot{E} + \tilde{Q} + E\tilde{A} + \tilde{A}^T E - E\tilde{S}E\right)x + \left(\dot{e}^T + e^T\tilde{A} - e^T\tilde{S}E\right)x \\
&\quad - \frac{1}{4}e^T\tilde{S}e + \dot{d} + (u - y)^T N^T N(u - y) - \gamma(w - z)^T(w - z) - \Psi,
\end{aligned}$$

with $\tilde{A} = A - B(N^TN)^{-1}N^TM$, $\tilde{Q} = M^TM - M^TN(N^TN)^{-1}N^TM$ and $\tilde{S} = B(N^TN)^{-1}B^T - \frac{1}{\gamma}CC^T$.

Additionally, if we assume that the mappings E, e and d are such that they fulfill

$$\dot{E} + \tilde{Q} + E\tilde{A} + \tilde{A}^TE - E\tilde{S}E = 0 \tag{4.5}$$

$$\dot{e}^T + e^T\tilde{A} - e^T\tilde{S}E = 0 \tag{4.6}$$

and

$$\dot{d} - \frac{1}{4}e^T\tilde{S}e = 0, \tag{4.7}$$

then we obtain

$$\frac{d}{dt}\tilde{V} = (u-y)^TN^TN(u-y) - \gamma(w-z)^T(w-z) - \Psi.$$

Integrating yields

$$\tilde{V}(t_f) - \tilde{V}(t_0) = \int_{t_0}^{t_f}(u-y)^TN^TN(u-y)\,dt - \gamma\int_{t=0}^{t_f}(w-z)^T(w-z)\,dt - \int_{t_0}^{t_f}\Psi(x,u,w)\,dt \tag{4.8}$$

Consequently, we obtain the following theorem:

Theorem 4.1 *Suppose that the coefficient $N(t)$ is given such that $N^T(t)N(t)$ is positive definite for each $t \in [t_0, t_f]$. Suppose further that the (symmetric) Riccati differential equation*

$$\dot{E} = -E\tilde{A} - \tilde{A}^TE - \tilde{Q} + E\tilde{S}E$$

admits a solution $E : [t_0, t_f] \to \mathbb{R}^{n\times n}$ with $E(t_f) = 0$. Then, Problem 3 is solvable if and only if for the control function

$$\tilde{u} = -(N^TN)^{-1}(B^TE + N^TM)\hat{x}(t)$$

$J(\tilde{u}, \hat{w}) \leq 0$ holds. If the latter inequality is true (i.e. if Problem 3 is solvable), then $\tilde{u}$ is a solution of the disturbance attenuation problem. Hereby denote

$$\hat{w} = \frac{1}{\gamma}C^TE\hat{x}(t)$$

as well as $\hat{x}(t)$ the solution of the linear differential equation

$$\frac{d}{dt}\hat{x} = (\tilde{A} - \tilde{S}E)\hat{x}, \quad \hat{x}(t_0) = x_0$$

Proof. The proof is carried out in several steps.

1. Note, that since differential equation (4.6) is homogeneous, $e(t_f) = 0$ yields that $e(t) = 0$ for any $t \in [t_0, t_f]$. Hence, $d(t_f) = 0$ implies that $d(t) = 0$ for any $t \in [t_0, t_f]$, too. Assuming now that $E(t_f) = 0$, we obtain, using (4.8) that

$$\tilde{V}(t_0) = J(x,u,w) - \int_{t_0}^{t_f}(u-y)^TN^TN(u-y)\,dt + \gamma\int_{t=0}^{t_f}(w-z)^T(w-z)\,dt \tag{4.9}$$

holds.

2. Furthermore, the l.h.s. of (4.9) is independent of u and w. Hence, considering the supremal values of (4.9) over all possible disturbance functions in $\mathcal{W}$, we obtain that

$$\sup_{w\in\mathcal{W}} J(u,w) = \tilde{V}(t_0) + \int_{t_0}^{t_f} (u-y)^T N^T N(u-y)\,dt$$
$$-\gamma \inf_{w\in\mathcal{W}} \int_{t=0}^{t_f} (w-z)^T(w-z)\,dt = \tilde{V}(t_0) + \int_{t_0}^{t_f} (u-y)^T N^T N(u-y)\,dt$$

holds if and only if $w - z \equiv 0$ almost everywhere on $[t_0, t_f]$. On the other hand,

$$\tilde{V}(t_0) = V(t_0, x_0) = \inf_{u_i\in\mathcal{U}_i} \sup_{w\in\mathcal{W}} J(u,w) = \tilde{V}(t_0) + \inf_{u\in\mathcal{U}} \int_{t_0}^{t_f} (u-y)^T N^T N(u-y)\,dt$$

if and only if $u - y \equiv 0$ almost everywhere on $[t_0, t_f]$.

3. Hence, the minimax situation is achieved if and only if the control function $\tilde{u}$ and the disturbance $\hat{w}$ are such that $\tilde{u} = -(N^T N)^{-1}(B^T E + N^T M)x$ and $\hat{w} = \frac{1}{\gamma}C^T E x$ hold together with

$$\dot{x} = Ax + B\tilde{u} + C\hat{w}, \quad x(t_0) = x_0.$$

Hence, $\tilde{u}$ and $\hat{w}$ fulfill

$$\begin{aligned} \tilde{u} &= -(N^T N)^{-1}(B^T E + N^T M)\hat{x}(t) \text{ and} \\ \hat{w} &= \frac{1}{\gamma} C^T E \hat{x}(t) \end{aligned}$$

with

$$\frac{d}{dt}\hat{x} = (\tilde{A} - \tilde{S}E)\hat{x}, \quad \hat{x}(t_0) = x_0,$$

respectively. Altogether, Remark 4.1 completes the proof.

□

In the next section, we generalize the above problem statement for differential games, where the players itself simultaneously play against each other and also try to develop a maximal disturbance attenuation.

4.2 Nash/Worst-case strategies for disturbed open-loop differential games

Throughout this section, we consider a class of linear quadratic differential games, where the game itself is controlled by N players and a disturbance term in the following way:

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{N} B_i(t)u_i(t) + C(t)w(t), \quad x(t_0) = x_0, \tag{4.10}$$

with piecewise continuous functions $A(t) \in \mathbb{R}^{n\times n}$, $B_i(t) \in \mathbb{R}^{n\times m_i}$ and $C(t) \in \mathbb{R}^{n\times m}$. Further denote $u_i(.) \in \mathcal{U}_i$ the control of the i$^{\text{th}}$ player and $w \in \mathcal{W}$ the disturbance with $\mathcal{U}_i$ and $\mathcal{W}$ denoting the Hilbert spaces $\mathcal{L}_2^{m_i}[t_0, t_f]$ and $\mathcal{L}_2^{m}[t_0, t_f]$, respectively.

We also assume that the cost functional of the i$^{\text{th}}$ player has the form:

$$J_i\Big(u_1(.), \ldots, u_N(.), w(.)\Big) = x(t_f)^T K_{if} x(t_f) + \int_{t_0}^{t_f} \Big(x^T Q_i(t) x + \sum_{j=1}^{N} u_j^T R_{ij}(t) u_j + w^T P_i w\Big)\, dt, \tag{4.11}$$

with symmetric matrices $K_{if} \in \mathbb{R}^{n\times n}$ and piecewise continuous symmetric matrix functions $Q_i(t) \in \mathbb{R}^{n\times n}$, $R_{ij}(t) \in \mathbb{R}^{m_j\times m_j}$ and $P_i(t) \in \mathbb{R}^{m\times m}$.

Although a Nash-game approach is used to select the strategies of the players, no constraints on the disturbance term are made. This means that the players have to find an equilibrium strategy without knowing anything about the disturbance. We assume that each of them independently applies Nash/worst-case strategy.

Remark 4.2 *Note that there is no cost functional assigned to the disturbance term, because no constraints can be applied on an 'unpredictable' parameter (see also Section 4.3).*

In the noncooperative case the aim of each player is to find an optimal strategy, without knowing anything about the control of the other players or the disturbance. In Nash equilibrium situations each player chooses his control assuming that also the other players use an optimal (i.e minimizing) strategy (see [BO95]). For a mixed Nash/worst-case equilibrium the strategy of each player is defined as follows:

Definition 4.3 (see Def. 2.2 in [JK98]) *We define the mixed Nash/worst-case equilibrium in two steps:*

1. *Given a set of controls $(u_1, u_2, \ldots, u_N)$, the disturbance function $\hat{w}_i(u_1, u_2, \ldots, u_N) \in \mathcal{W}$ is called* worst-case disturbance from the point of view of the i^{th} player *belonging to this set of controls if*

$$J_i\Big(u_1, u_2, \ldots, u_N, \hat{w}_i(u_1, u_2, \ldots, u_N)\Big) \geq J_i\Big(u_1, u_2, \ldots, u_N, w\Big).$$

   holds for each $w \in \mathcal{W}$.

2. *We say that the set of controls $(\tilde{u}_1, \ldots, \tilde{u}_N)$ form a* mixed Nash/worst-case equilibrium *if for all $i = 1, \ldots, N$*

   (i) *there exists* exactly one *worst-case disturbance from the point of view of the* i$^{\text{th}}$ *player according to every set of controls*

$$(\tilde{u}_1, \ldots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \ldots, \tilde{u}_N)$$

   and

(ii)

$$J_i\Big(\tilde{u}_1, \dots, \tilde{u}_N, \hat{w}_i(\tilde{u}_1, \dots, \tilde{u}_N)\Big) \leq$$
$$J_i\Big(\tilde{u}_1, \dots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \dots, \tilde{u}_N, \hat{w}_i(\tilde{u}_1, \dots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \dots, \tilde{u}_N)\Big)$$

holds for each worst-case disturbance $\hat{w}_i$ *and permissible control function* $u_i \in \mathcal{U}_i$.

Remark 4.3 *The above definition of the Nash/worst-case equilibrium strategies leads to the following rules:*

(i) Each player plays against the other players using noncooperative Nash-*strategy.*

(ii) Each player plays against the disturbance using worst-case *strategy.*

Let us now introduce a final definition before we begin our investigation concerning disturbed differential games.

Definition 4.4 (see also Definition 2.4) *We call a disturbed linear quadratic differential game defined in (4.10) and (4.11)* playable*, if there exists a unique N-tuple of controls* $\tilde{u}_1, ..., \tilde{u}_N$ *forming a mixed Nash/worst-case equilibrium.*

4.2.1 Existence and uniqueness of open-loop Nash/worst-case equilibria

Here we are interested in obtaining conditions on the playability of a disturbed Nash-game. The main tool for obtaining these conditions will be motivated by the technique involving the Hilbert-space $\mathcal{H}_{t_f}$ already introduced and discussed in Sections 1.4 and 2.3.

Denote again $\Phi(t,\tau)$ the solution of the initial value problem

$$\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau), \qquad \Phi(\tau,\tau) = I_n,$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. Furthermore, we also define the following linear operators:

$$\Phi : \mathbb{R}^n \to \mathcal{H}_{t_f}, \;\; x_0 \mapsto \Phi(.,t_0)x_0,$$
$$\mathcal{B}_i : \mathcal{U}_i \to \mathcal{H}_{t_f}, \;\; u_i \mapsto \int_{t_0}^{.} \Phi(.,\tau)B_i(\tau)u_i(\tau)\,d\tau$$

and

$$\mathcal{C} : \mathcal{W} \to \mathcal{H}_{t_f}, \;\; w \mapsto \int_{t_0}^{.} \Phi(.,\tau)C(\tau)w(\tau)\,d\tau.$$

Then the solution of (4.10) becomes

$$x(.) = \Phi x_0 + \sum_{i=1}^{N} \mathcal{B}_i u_i + \mathcal{C}w. \tag{4.12}$$

The cost functionals can be written in terms of scalar products on Hilbert spaces, too:

$$
\begin{aligned}
J_i(u_1, ..., u_n, w) &= \left\langle x, \bar{Q}_i x \right\rangle_{\mathcal{H}_{t_f}} + \sum_{j=1}^{N} \langle u_j, R_{ij} u_j \rangle_{\mathcal{L}_2} + \langle w, P_i w \rangle_{\mathcal{L}_2} \\
&= \left\langle x, \bar{Q}_i x \right\rangle_{\mathcal{H}_{t_f}} + \sum_{j=1}^{N} \left\langle u_j, \bar{R}_{ij} u_j \right\rangle_{\mathcal{H}_{t_f}} + \left\langle w, \bar{P}_i w \right\rangle_{\mathcal{H}_{t_f}},
\end{aligned}
$$

with

$$
\bar{Q}_i(t) = \begin{cases} Q_i(t) & t \neq t_f \\ K_{if} & t = t_f \end{cases},
$$

$$
\bar{R}_{ij}(t) = \begin{cases} R_{ij}(t) & t \neq t_f \\ 0 & t = t_f \end{cases} \quad \text{and}
$$

$$
\bar{P}_i = \begin{cases} P_i(t) & t \neq t_f \\ 0 & t = t_f \end{cases}.
$$

Suppose now, that the information structure for each player is of open-loop type, which means that the only information on the actual state of the system $x(t)$ is its initial value x_0 (see Remark 2.2).

Using equation (4.12) the cost functionals can also be written as

$$
\begin{aligned}
J_i &= \left\langle \Phi x_0 + \mathcal{B}_i u_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathcal{B}_j u_j + \mathcal{C} w, \bar{Q}_i \left(\Phi x_0 + \mathcal{B}_i u_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathcal{B}_j u_j + \mathcal{C} w \right) \right\rangle \\
&\quad + \left\langle u_i, \bar{R}_{ii} u_i \right\rangle + \sum_{\substack{j=1 \\ j\neq i}}^{N} \left\langle u_j, \bar{R}_{ij} u_j \right\rangle + \left\langle w, \bar{P}_i w \right\rangle \\
&= \left\langle u_i, \left(\mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii} \right) u_i \right\rangle + \left\langle w, \left(\mathcal{C}^* \bar{Q}_i \mathcal{C} + \bar{P}_i \right) w \right\rangle + 2 \left\langle u_i, \mathcal{B}_i^* \bar{Q}_i \mathcal{C} w \right\rangle \\
&\quad + 2 \left\langle u_i, \mathcal{B}_i^* \bar{Q}_i \left(\Phi x_0 + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathcal{B}_j u_j \right) \right\rangle + 2 \left\langle w, \mathcal{C}^* \bar{Q}_i \left(\Phi x_0 + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathcal{B}_j u_j \right) \right\rangle + J_{i0},
\end{aligned}
$$

where J_{i0} denotes terms independent of w and u_i, and all the scalar products are taken in $\mathcal{H}_{t_f}$.

We can now examine the question how an open-loop Nash/worst-case equilibrium arises. As explained in Section 4.2, each player minimizes his cost functional under the assumption, that the other players are using an optimal strategy and that the disturbance is the worst possible, which means, that the disturbance actually wants to maximize the cost functional of the i$^{\text{th}}$ player. Using the notations $F_i := \mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii}$, $G_i = \mathcal{C}^* \bar{Q}_i \mathcal{C} + \bar{P}_i$, $H_i := \mathcal{B}_i^* \bar{Q}_i \mathcal{C}$, as well as

$$
f_i := \mathcal{B}_i^* \bar{Q}_i \left(\Phi x_0 + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathcal{B}_j \tilde{u}_j \right) \quad \text{and}
$$

$$
g_i := \mathcal{C}^* \bar{Q}_i \left(\Phi x_0 + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathcal{B}_j \tilde{u}_j \right),
$$

the cost functional of the i$^{\text{th}}$ player in an equilibrium state becomes

$$\begin{aligned}\tilde{J}_i(u_i, w) &:= J_i(\tilde{u}_1, \dots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \dots, \tilde{u}_n, w)\\ &= \langle u_i, F_i u_i\rangle + \langle w, G_i w\rangle + 2\langle u_i, H_i w\rangle + 2\langle u_i, f_i\rangle + 2\langle w, g_i\rangle + \tilde{J}_{i0}.\end{aligned}$$

The 'worst case' disturbance maximizes this latter functional w.r.t $w \in \mathcal{W}$. In order to determine this worst-case disturbance, the functional $\tilde{J}_i$ can be written as follows:

$$\begin{aligned}\tilde{J}_i(u_i, w) &= \langle w, G_i w\rangle + 2\langle w, H_i^* u_i + g_i\rangle + \langle u_i, F_i u_i\rangle + 2\langle u_i, f_i\rangle + \tilde{J}_{i0}\\ &= \left\langle w + G_i^{-1}(H_i^* u_i + g_i), G_i\left(w + G_i^{-1}(H_i^* u_i + g_i)\right)\right\rangle + \tilde{J}_{iw0},\end{aligned}$$

where J_{iw0} is independent of w. From this equation we obtain the following result:

Theorem 4.2 *The problem of searching for a mixed Nash/worst-case equilibrium is well posed, if the operator G_i is negative definite for each $i = 1, \dots, N$. In this case the i^{th} player defines the 'worst disturbance' as*

$$\hat{w}_i(u_i) = -G_i^{-1}(H_i^* u_i + g_i). \tag{4.13}$$

Now we study, how a player chooses his strategy against the other players. To get the optimal strategy, we must derive $\tilde{u}_i$ that minimizes the functional $\tilde{J}_i(u_i, \hat{w}_i)$ with $\hat{w}_i = \hat{w}_i(u_i)$:

$$\begin{aligned}&\tilde{J}_i(u_i, \hat{w}_i(u_i))\\ &= \tfrac{1}{2}\Big[\langle u_i, F_i u_i\rangle + \langle \hat{w}_i, G_i \hat{w}_i\rangle + 2\langle u_i, H_i \hat{w}_i\rangle\\ &\quad +2\langle u_i, f_i\rangle + 2\langle \hat{w}_i, g_i\rangle\Big] + \tilde{J}_{i0}\\ &= \tfrac{1}{2}\Big[\left\langle u_i, (F_i - H_i G_i^{-1} H_i^*) u_i\right\rangle\\ &\quad +2\left\langle u_i, f_i - H_i G_i^{-1} g_i\right\rangle - \left\langle g_i, G_i^{-1} g_i\right\rangle\Big]\\ &\quad +\tilde{J}_{i0}\\ &= \tfrac{1}{2}\left\langle u_i + \alpha_i^{-1}\beta_i, \alpha_i(u_i + \alpha_i^{-1}\beta_i)\right\rangle + \tilde{J}_{iu0},\end{aligned}$$

where $\tilde{J}_{iu0}$ denotes a term independent of u_i and $\alpha_i = F_i - H_i G_i^{-1} H_i^*$ and $\beta_i = f_i - H_i G_i^{-1} g_i$. This yields

Lemma 4.1 *The mixed Nash/worst-case equilibrium is well defined if, in addition to the assumption of Theorem 4.2, the operator $F_i - H_i G_i^{-1} H_i^*$ is for each $i = 1, \dots, N$ positive definite. Then, the optimal strategy of the i^{th} player is given by*

$$\tilde{u}_i = \left(F_i - H_i G_i^{-1} H_i^*\right)^{-1}\left(H_i G_i^{-1} g_i - f_i\right). \tag{4.14}$$

This last equation can also be written in terms of the coefficients $\bar{Q}_i$, $\bar{P}_i$ and $\bar{R}_{ii}$ as follows

$$\begin{aligned}&\left[\mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii} - \mathcal{B}_i^* \bar{Q}_i \mathcal{C}\left(\mathcal{C}^* \bar{Q}_i \mathcal{C} + \bar{P}_i\right)^{-1} \mathcal{C}^* \bar{Q}_i \mathcal{B}_i\right] \tilde{u}_i\\ &= \left[\mathcal{B}_i^* \bar{Q}_i \mathcal{C}\left(\mathcal{C}^* \bar{Q}_i \mathcal{C} + \bar{P}_i\right)^{-1} \mathcal{C}^* \bar{Q}_i - \mathcal{B}_i^* \bar{Q}_i\right](\hat{x} - \mathcal{B}_i \tilde{u}_i),\end{aligned}$$

where $\hat{x}$ is the state of the *undisturbed* system, i.e

$$\hat{x} = \Phi x_0 + \sum_{i=1}^{N} \mathcal{B}_i \tilde{u}_i.$$

With this notation we obtain the equilibrium strategy from

$$\bar{R}_{ii}\tilde{u}_i = -[\mathcal{B}_i^* \bar{Q}_i - \mathcal{B}_i^* \bar{Q}_i \mathcal{C}\left(\mathcal{C}^* \bar{Q}_i \mathcal{C} + \bar{P}_i\right)^{-1} \mathcal{C}^* \bar{Q}_i]\hat{x}, \tag{4.15}$$

which differs from the optimal strategy of an *undisturbed* linear quadratic Nash-game (see [LR71]):

$$\bar{R}_{ii}\tilde{u}_i = -\mathcal{B}_i^* \bar{Q}_i x.$$

With (4.15) we obtain an explicit formula for the optimal strategies.

Theorem 4.3 *The optimal strategies of the linear quadratic differential game defined in (4.10) and (4.11) with mixed Nash/worst-case equilibrium using an open-loop information structure are given as the solution of the following system of equations for* $i = 1, 2, \ldots, N$*:*

$$\left[\mathcal{B}_i^* \bar{Q}_i - \mathcal{B}_i^* \bar{Q}_i \mathcal{C}\left(\mathcal{C}^* \bar{Q}_i \mathcal{C} + \bar{P}_i\right)^{-1} \mathcal{C}^* \bar{Q}_i\right] \sum_{j=1}^{N} \mathcal{B}_j \tilde{u}_j + \bar{R}_{ii}\tilde{u}_i =$$
$$- \left[\mathcal{B}_i^* \bar{Q}_i - \mathcal{B}_i^* \bar{Q}_i \mathcal{C}\left(\mathcal{C}^* \bar{Q}_i \mathcal{C} + \bar{P}_i\right)^{-1} \mathcal{C}^* \bar{Q}_i\right] \Phi x_0$$

Corollary 4.1 *Under the assumptions of Theorem 4.2 and Lemma 4.1, the open-loop mixed Nash/ worst-case linear quadratic differential game is* playable *if and only if the system of equations defined in Theorem 4.3 is uniquely solvable.*

4.2.2 Explicit formulation of the optimal controls

Here, we develop a more explicit formulation for the optimal controls of a Nash/worst-case equilibrium. We shall also see that the optimal controls can be described as 'virtual' affine linear feedback-controls where the feedback transfer-mappings satisfy some matrix Riccati differential equations. Virtual means hereby, that the state appearing in the feedback form is not the current state-trajectory, but the so-called worst-case trajectory.

Using equation (4.13) in equation (4.14), we obtain the following representation of the optimal control $\tilde{u}_i$:

$$\begin{aligned} F_i\tilde{u}_i - H_i G_i^{-1} H_i^* \tilde{u}_i &= H_i G_i^{-1} g_i - f_i \\ \Leftrightarrow F_i\tilde{u}_i - H_i G_i^{-1}\left(H_i^* \tilde{u}_i + g_i\right) &= -f_i \\ \Leftrightarrow F_i\tilde{u}_i + H_i \hat{w}_i(\tilde{u}_i) &= -f_i \\ \Leftrightarrow F_i\tilde{u}_i &= -H_i \hat{w}_i(\tilde{u}_i) - f_i \end{aligned}$$

Now, using the definitions of F_i, G_i, H_i, f_i and g_i, we conclude that

$$\begin{aligned} R_{ii}\tilde{u}_i + \mathcal{B}_i^*\bar{Q}\mathcal{B}^i\tilde{u}_i &= \mathcal{B}_i^*\bar{Q}_i\left(\Phi x_0 + \sum_{\substack{j=1\\j\neq i}}^N \mathcal{B}_j\tilde{u}_j\right) + \mathcal{B}_i^*\bar{Q}_i\mathcal{C}\hat{w}_i(\tilde{u}_i) \\ \Leftrightarrow R_{ii}\tilde{u}_i &= -\mathcal{B}_i^*\bar{Q}\underbrace{\left(\Phi x_0 + \sum_{j=1}^N \mathcal{B}_j\tilde{u}_j + \mathcal{C}\hat{w}_i(\tilde{u}_i)\right)}_{=:\hat{x}_i}. \end{aligned}$$

On the other hand, equation (4.13) yields

$$\begin{aligned} \hat{w}_i(\tilde{u}_i) &= -G_i^{-1}\left(H_i^*\tilde{u}_i + g_i\right) \\ \Leftrightarrow P_i\hat{w}_i(\tilde{u}_i) + \mathcal{C}^*\bar{Q}_i\mathcal{C}\hat{w}_i(\tilde{u}_i) &= \mathcal{C}^*\bar{Q}_i\mathcal{B}_i\tilde{u}_i + \mathcal{C}^*\bar{Q}_i\left(\Phi x_0 + \sum_{\substack{j=1\\j\neq i}}^N \mathcal{B}_j\tilde{u}_j\right) \\ \Leftrightarrow P_i\hat{w}_i(\tilde{u}_i) &= -\mathcal{C}^*\bar{Q}_i\hat{x}_i. \end{aligned}$$

Altogether we obtain the following theorem:

Theorem 4.4 *Suppose that the coefficients P_i and R_{ii} ($i = 1, \ldots, N$) are negative definite and positive definite, respectively. Suppose further that also the operators F_i and G_i ($i = 1, \ldots, N$) are positive and negative definite, respectively. Then, $\tilde{u}_1, \ldots, \tilde{u}_N$ form an open-loop Nash/worst-case equilibrium if and only if for each $i = 1, \ldots, N$*

$$\tilde{u}_i = -R_{ii}^{-1}\mathcal{B}_i^*\bar{Q}_i\hat{x}_i \tag{4.16}$$

holds. Moreover, in this equilibrium the worst-case disturbance from the point of view of the i^{th} player can be written as

$$\hat{w}_i(\tilde{u}_i) = -P_i^{-1}\mathcal{C}^*\bar{Q}_i\hat{x}_i, \tag{4.17}$$

where $\hat{x}_i$ denotes the 'worst-case' state-trajectory from the point of view of the i^{th} player, i.e.

$$\hat{x}_i = \Phi x_0 + \sum_{j=1}^N \mathcal{B}_j\tilde{u}_j + \mathcal{C}\hat{w}_i(\tilde{u}_i). \tag{4.18}$$

Proof. Note that if R_{ii} and G_i are positive and negative definite, respectively, then the assumptions of Theorem 1 are fulfilled. Hence, the above calculation completes the proof. □

Our aim now is to describe the relation for the optimal control (4.16) more explicitely, i.e. by solutions of certain differential equations. For this we first need to construct, using Lemma 1.6, the adjoint operators $\mathcal{B}_i^*$ and $\mathcal{C}^*$:

$$\begin{aligned} \mathcal{B}_i^*y &= B_i^T(.)\left[\Phi^T(t_f,.)y(t_f) + \int_.^{t_f}\Phi^T(t,.)y(t)\,dt\right] \text{ and} \\ \mathcal{C}^*y &= C^T(.)\left[\Phi^T(t_f,.)y(t_f) + \int_.^{t_f}\Phi^T(t,.)y(t)\,dt\right]. \end{aligned} \tag{4.19}$$

Hence, we obtain that

$$\tilde{u}_i(t) = -R_{ii}(t)^{-1}B_i^T(t)\left[\Phi^T(t_f,t)K_{i_f}\hat{x}_i(t_f). + \int_t^{t_f}\Phi^T(\tau,t)Q_i(\tau)\hat{x}_i(\tau)\,d\tau\right]$$

and similarly

$$\hat{w}_i(\tilde{u}_i)(t) = P_i^{-1}(t)C^T(t)\left[\Phi^T(t_f,t)K_{i_f}\hat{x}_i(t_f) + \int_t^{t_f}\Phi^T(\tau,t)Q_i(\tau)\hat{x}_i(\tau)\,d\tau\right].$$

If we suppose that the expression

$$\Phi^T(t_f,t)K_{i_f}\hat{x}_i(t_f) + \int_t^{t_f}\Phi^T(\tau,t)Q_i(\tau)\hat{x}_i(\tau)\,d\tau$$

can be written in the affine form $E_i(t)\hat{x}_i(t) + e_i(t)$ for some mappings $E_i : [t_0,t_f] \to \mathbb{R}^{n\times n}$ and $e_i : [t_0,t_f] \to \mathbb{R}^n$, then we get that the optimal control and the corresponding worst-case disturbance can be written as

$$\tilde{u}_i(t) = -R_{ii}^{-1}(t)B_i^T(t)\Big(E_i(t)\hat{x}_i(t) + e_i(t)\Big), \tag{4.20}$$

$$\hat{w}_i(\tilde{u}_i)(t) = -P_i^{-1}(t)C^T(t)\Big(E_i(t)\hat{x}_i(t) + e_i(t)\Big), \tag{4.21}$$

respectively. Since

$$\begin{aligned}
\frac{d}{dt}\Big(E_i(t)\hat{x}_i(t) + e_i(t)\Big) &= \dot{E}_i\hat{x}_i + E_i\frac{d}{dt}\hat{x}_i + \dot{e}_i\\
&= \dot{E}_i\hat{x}_i + E_i\left(A\hat{x}_i + \sum_{j=1}^N B_j\tilde{u}_j + C\hat{w}_i(\tilde{u}_i)\right) + \dot{e}_i\\
&= \dot{E}_i\hat{x}_i + E_iA\hat{x}_i - E_i\sum_{j=1}^N \underbrace{B_jR_{jj}^{-1}B_j^T}_{S_j(t)}(E_j\hat{x}_j + e_j) - E_i\underbrace{CP_i^{-1}C^T}_{T_i(t)}(E_i\hat{x}_i + e_i) + \dot{e}_i
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\left[\Phi^T(t_f,t)K_{i_f}\hat{x}_i(t_f) + \int_t^{t_f}\Phi^T(\tau,t)Q_i(\tau)\hat{x}_i(\tau)\,d\tau\right]\\
&= \frac{d}{dt}\Phi^T(t_f,t)K_{i_f}\hat{x}_i(t_f) - \Phi^T(t,t)Q_i(t)\hat{x}_i(t) + \int_t^{t_f}\frac{d}{dt}\Phi^T(\tau,t)Q_i(\tau)\hat{x}_i(\tau)\,d\tau,
\end{aligned}$$

then we obtain, using

$$\frac{d}{dt}\Phi^T(\tau,t) = -A^T(t)\Phi^T(\tau,t) \quad \text{and } \Phi^T(\tau,\tau) = I,$$

that

$$\begin{aligned}
&\frac{d}{dt}\left[\Phi^T(t_f,t)K_{i_f}x^*(t_f) + \int_t^{t_f}\Phi^T(\tau,t)Q(\tau)x^*(\tau)\,d\tau\right]\\
&= -A^T\left[\Phi^T(t_f,t)K_{i_f}x^*(t_f) + \int_t^{t_f}\Phi^T(\tau,t)Q(\tau)x^*(\tau)\,d\tau\right] - Q_i(t)\hat{x}_i(t)\\
&= -A^T(E_i\hat{x}_i + e_i) - Q_i\hat{x}_i
\end{aligned}$$

holds.

Suppose now that the mappings E_i and e_i fulfill the following differential equations:

$$\dot{E}_i = -A^T E_i - E_i A - Q_i + E_i(S_{ii} + T_i)E_i, \tag{4.22}$$

$$\dot{e}_i = -\Big(A^T - E_i(S_{ii} + T_i)\Big)e_i + E_i \sum_{\substack{j=1 \\ j\neq i}}^{N} S_{jj}(E_j\hat{x}_j + e_j), \tag{4.23}$$

respectively. Then we obtain the following theorem:

Theorem 4.5 *Suppose that the assumptions on the matrices R_{ii} and P_i and on the operators F_i and G_i in Theorem 4.4 are fulfilled. Further assume that there exist mappings $E_i : [t_0, t_f] \to \mathbb{R}^{n\times n}$ and $e_i : [t_0, t_f] \to \mathbb{R}^n$ and $\hat{x}_i$ such that for each $i = 1, \dots, N$ the differential equations (4.22), (4.23) and*

$$\frac{d}{dt}\hat{x}_i = A\hat{x}_i - \sum_{j=1}^{N} S_{jj}(E_j\hat{x}_j + e_j) - T_i(E_i\hat{x}_i + e_i) \tag{4.24}$$

and the boundary value problems $E_i(t_f) = K_{i_f}$, $e_i(t_f) = 0$ and $\hat{x}_i(t_0) = x_0$ are fulfilled. Then, the control functions

$$\tilde{u}_i = -R_{ii}^{-1}B_i^T(E_i\hat{x}_i + e_i)$$

form an open-loop Nash/worst-case equilibrium. Moreover, the corresponding worst-case disturbance of the i^{th} player is then given by

$$\hat{w}_i(\tilde{u}_i) = -P_i^{-1}C^T(E_i\hat{x}_i + e_i)$$

Proof. The proof, that we carry out in several short steps, is mainly based on the latter calculation.

(i) Observe that the terminal-values $E_i(t_f)$ and $e_i(t_f)$ are constructed such that

$$-R_{ii}^{-1}\mathcal{B}_i^*\bar{Q}_i\hat{x}_i(t_f) = -R_{ii}^{-1}(t_f)B_i^T(t_f)\Big(E_i(t_f)\hat{x}_i(t_f) + e_i(t_f)\Big).$$

(ii) Furthermore, using the above calculation, equations (4.22) and (4.23) imply the equality

$$\frac{d}{dt}(E_i\hat{x}_i + e_i) = \frac{d}{dt}\left[\Phi^T(t_f, t)K_{i_f}\hat{x}_i(t_f) + \int_t^{t_f} \Phi^T(\tau, t)Q_i(\tau)\hat{x}_i(\tau)\, d\tau\right]$$

and hence

$$R_{ii}^{-1}\mathcal{B}_i^*\bar{Q}_i\hat{x}_i = -R_{ii}^{-1}B_i^T(E_i\hat{x}_i + e_i)$$

holds over the interval $[t_0, t_f]$.

(iii) Finally, observe that equation (4.24) together with $\hat{x}_i(t_0) = x_0$ corresponds to equation (4.18) which means that its solution is exactly the worst-case trajectory from the point of view of the i^{th} player. Hence, Theorem 4.4 completes the proof.

□

Finally, we discuss the solvability of the coupled boundary value problem described by equations (4.23), (4.24) and the boundary values

$$x_i(t_0) = x_0$$
$$e_i(t_f) = 0$$

for $i = 1, \dots, N$. Here, for the sake of simplicity, we only discuss the two-player case ($N = 2$). Results for the more general N-player case can be obtained in the similar manner.

In order to shorten our formulae, we introduce the following notations:

$$M(t) := \begin{bmatrix} M_{11}(t) & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix} :=$$

$$\begin{bmatrix} A - (S_{11} + T_1)E_1 & -S_{22}E_2 & -S_{11} - T_1 & -S_{22} \\ -S_{11}E_1 & A - (S_{22} + T_2)E_2 & -S_{11} & -S_{22} - T_2 \\ 0 & E_1 S_{22} E_2 & -A^T + E_1(S_{11} + T_1) & E_1 S_{22} \\ E_2 S_{11} E_1 & 0 & E_2 S_{11} & -A^T + E_2(S_{22} + T_2) \end{bmatrix},$$

as well as

$$\hat{x}(t) := \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \text{ and } e(t) := \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

Our aim now, is to obtain a condition for the solvability of the following boundary value problem:

$$\frac{d}{dt} \begin{bmatrix} \hat{x} \\ e \end{bmatrix} = M(t) \begin{bmatrix} \hat{x} \\ e \end{bmatrix} \tag{4.25}$$

together with

$$\hat{x}(t_0) = \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} \text{ and } e(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.26}$$

Clearly, the function $\begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix}$ fulfills (4.25)-(4.26) if and only if the exists some $x_f \in \mathbb{R}^{2n}$ such that for all $t \in [t_0, t_f]$

$$\begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{21}(t) & \Psi_{22}(t) \end{bmatrix} \begin{bmatrix} x_f \\ 0 \end{bmatrix} \tag{4.27}$$

and

$$\Psi_{11}(t_0) x_f = \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}$$

hold with

$$\frac{d}{dt} \begin{bmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{21}(t) & \Psi_{22}(t) \end{bmatrix} = M(t) \begin{bmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{21}(t) & \Psi_{22}(t) \end{bmatrix} \tag{4.28}$$

and the terminal value problem

$$\begin{bmatrix} \Psi_{11}(t_f) & \Psi_{12}(t_f) \\ \Psi_{21}(t_f) & \Psi_{22}(t_f) \end{bmatrix} = \begin{bmatrix} I_{2n} & 0 \\ 0 & I_{2n} \end{bmatrix}, \tag{4.29}$$

where $I_{2n} \in \mathbb{R}^{2n \times 2n}$ denotes the identity matrix. Since we are interested in the solvability of (4.23)-(4.24) for any initial value x_0, we need the invertibility of the matrix $\Psi_{11}(t_0)$. Here, we present a sufficient condition for this:

Theorem 4.6 *Suppose that the matrix $M(t)$ is bounded over the interval $[t_0, t_f]$ and that the Riccati differential equation*

$$\dot{W} = M_{21} + M_{22}W - WM_{11} - WM_{12}W, \quad W(t_f) = 0 \tag{4.30}$$

admits a bounded solution for $W(t) \in \mathbb{R}^{2n\times 2n}$ in $[t_0, t_f]$. Then the boundary value problem (4.25)-(4.26) is uniquely solvable.

Proof. The proof presented here is a straightforward consequence of Radon's Lemma (see Theorem 3.2) for Riccati differential equations.

Denote by $W(t)$ the solution of (4.30). Since $M(t)$ is bounded, the terminal value problem (4.28)-(4.29) is uniquely solvable on $[t_0, t_f]$.

We now define the following terminal value problems:

$$\dot{X} = (M_{11} + M_{12}W)X, \quad X(t_f) = I_{2n} \text{ and} \tag{4.31}$$

$$\dot{Z} = -Z(M_{11} + M_{12}W), \quad Z(t_f) = I_{2n}, \tag{4.32}$$

where I_{2n} stands for the identity matrix in $\mathbb{R}^{2n\times 2n}$. Because of our boundedness assumption on $W(t)$, the solution of the above homogeneous linear differential equations $X(t)$ and $Z(t)$ are also continuous and bounded over the time horizon $[t_0, t_f]$.

Let us now define the matrix $Y(t) := W(t)X(t) \in \mathbb{R}^{2n\times 2n}$ for $t \in [t_0, t_f]$. Using equations (4.30) and (4.31), we conclude that

$$\begin{aligned} \dot{Y} &= \dot{W}X + W\dot{X} = (M_{21} + M_{22}W - WM_{11} - WM_{12}W)X + W(M_{11} + M_{12}W)X \\ &= M_{21}X + M_{22}Y. \end{aligned}$$

Altogether we obtain the following set of differential equations for the bounded mappings X and Y:

$$\begin{aligned} \dot{X} &= M_{11}X + M_{12}Y, \quad X(t_f) = I_{2n} \text{ and} \\ \dot{Y} &= M_{21}X + M_{22}Y, \quad Y(t_f) = 0, \end{aligned}$$

and hence $X(t) = \Psi_{11}(t)$ and $Y(t) = \Psi_{21}(t)$ hold for all $t \in [t_0, t_f]$. Finally, we calculate the product $Z(t)X(t) = Z(t)\Psi_{11}(t)$:

$$\frac{d}{dt}(ZX) = \dot{Z}X + Z\dot{X} = -Z(M_{11} + M_{12}WX + Z(M_{11} + M_{12}W)X = 0$$

and hence, using $Z(t_f) = X^{-1}(t_f)$, we conclude that the inverse of the matrix $\Psi_{11}(t)$ exists and is bounded for each $t \in [t_0, t_f]$. □

This latter condition (4.30) can also be written in terms of system coefficients:

Corollary 4.2 *Suppose that for the 2-player game, the assumptions on the matrices R_{ii} and P_i and on the operators G_i in Theorem 4.4 are fulfilled. Suppose further, that the symmetric matrix Riccati differential equations (4.22) for $i = 1, 2$ and the following non-symmetric matrix Riccati differential equation*

$$\dot{W} = \begin{bmatrix} 0 & E_1S_{22}E_2 \\ E_2S_{11}E_1 & 0 \end{bmatrix} + \begin{bmatrix} -A^T + E_1(S_{11} + T_1) & E_1S_{22} \\ E_2S_{11} & -A^T + E_2(S_{22} + T_2) \end{bmatrix} W$$

$$-W\begin{bmatrix} A-(S_{11}+T_1)E_1 & -S_{22}E_2 \\ -S_{11}E_1 & A-(S_{22}+T_2)E_2 \end{bmatrix} - W\begin{bmatrix} -S_{11}-T_1 & -S_{22} \\ -S_{11} & -S_{22}-T_2 \end{bmatrix} W,$$

for $W(t_f) = 0 \in \mathbb{R}^{2n\times 2n}$ *admit bounded solutions over the interval* $[t_0, t_f]$. *Then, the terminal value problem (4.23)-(4.24) is uniquely solvable. Moreover, using this solution, the (unique) optimal Nash/worst-case control function for each player can be obtained in the following form:*

$$\tilde{u}_1 = -R_{11}^{-1}B_1^T\Big((E_1+W_{11})\hat{x}_1 + W_{12}\hat{x}_2\Big) \text{ and} \tag{4.33}$$

$$\tilde{u}_2 = -R_{22}^{-1}B_2^T\Big((E_2+W_{22})\hat{x}_2 + W_{21}\hat{x}_1\Big), \tag{4.34}$$

where

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

and $\hat{x}_i$ *denotes the worst-case trajectory from the point of view of the* i^{th} *player, i.e.*

$$\frac{d}{dt}\hat{x}_1 = \Big(A-(S_{11}+T_1)(E_1+W_{11}) - S_{22}W_{21}\Big)\hat{x}_1 - \Big(S_{22}(E_2+W_{22}) - S_{11}W_{12}\Big)\hat{x}_2 \text{ and}$$

$$\frac{d}{dt}\hat{x}_2 = \Big(A-(S_{22}+T_2)(E_2+W_{22}) - S_{11}W_{12}\Big)\hat{x}_2 - \Big(S_{11}(E_1+W_{11}) - S_{22}W_{21}\Big)\hat{x}_1.$$

Proof. Considering the fact that $\Psi_{21}(t) = W(t)\Psi_{11}(t)$ we infer from equation (4.27), that for the unique solution of the boundary-value problem (4.25) $e(t) = W(t)\hat{x}(t)$ holds for each $t \in [t_0, t_f]$. And hence, using Theorems 4.5 and 4.6, the proof is completed. □

4.3 Numerical results

In this final section of this chapter, we present a numerical comparison between disturbed 2-person open-loop differential games under Nash-control laws and 2-person Nash/worst-case control-policies.

We consider the following one-dimensional system:

$$\dot{x} = 3u_1(t) - u_2(t) + w(t), \quad x(0) = 30. \tag{4.35}$$

We also assume that the cost functional of the players have the following form:

$$J_1(u_1,u_2) = x(1)^2 + \int_0^1 3u_1^2(t) - 2w^2(t)\,dt$$

$$J_2(u_1,u_2) = 3x(1)^2 + \int_0^1 u_2^2(t) - 4w^2(t)\,dt.$$

Since $R_{ii} > 0$ and $\bar{Q}_i \geq 0$, we immediately obtain the positive definiteness of F_i. Moreover, using the definition of the operator $\mathcal{C}$, we also conclude that

$$< w, (\mathcal{C}^*\bar{Q}_i\mathcal{C} + \bar{P}_i)w > = < \mathcal{C}w, \bar{Q}_i\mathcal{C}w > + < w, \bar{P}_i w > =$$

$$\int_0^1 P_i w^2(t)\,dt + K_{if}\left(\int_0^1 w(t)\,dt\right)^2 \leq \int_0^1 \Big(K_{if} + P_i\Big)\,w^2(t)\,dt < 0$$

for $i = 1, 2$ which yields the negative definiteness of G_i.

We consider now the following two setups:

First, we neglect the disturbance (i.e. we set $w(t) = 0$), calculate the open-loop Nash equilibrium (see for instance [BO95]) and then try to use the arising (open-loop) control functions for the disturbed system. Then, we calculate controls for a Nash/worst-case equilibrium of the disturbed system and compare the results.

The open-loop control functions belonging to a Nash equilibrium of the undisturbed game can be calculated with the help of the following Riccati differential equations:

$$\dot{K}_1 = 3K_1^2 + K_1K_2, \quad K_1(1) = 1$$

$$\dot{K}_2 = K_2^2 + 3K_1K_2, \quad K_2(1) = 3.$$

The state trajectory of the game using the above calculated *open-loop* controls u_1 and u_2 and a randomly chosen (but nevertheless smooth) disturbance $w(t)$ can be seen on Figure 4.2. Figure 4.1 shows the disturbance acting on the system. The actual costs, that arise using the cost functionals J_1 and J_2 are for these control functions

$$J_1^{\text{Nash}} = 27.43$$

$$J_2^{\text{Nash}} = 376.50$$

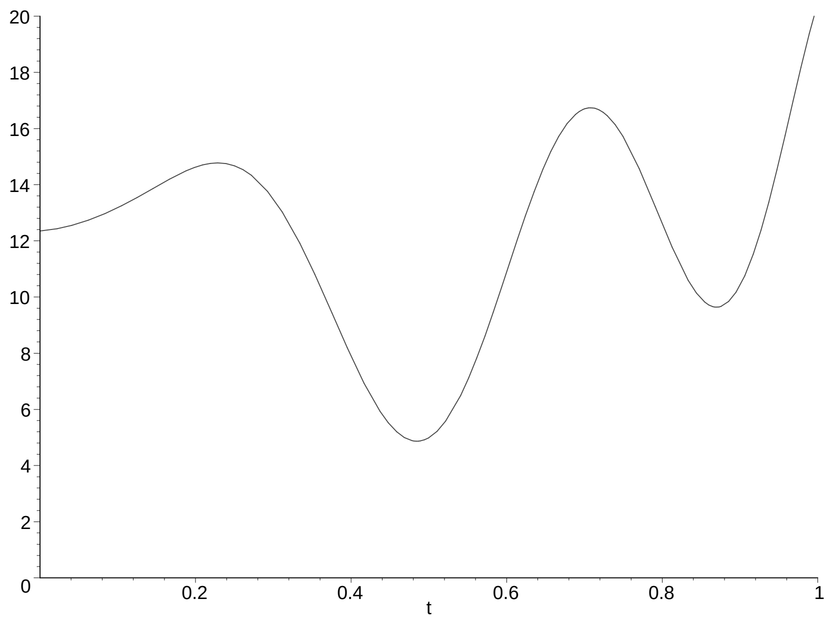

Figure 4.1: Disturbance term ($w(t)$) acting on the system

Now, we calculate, using Corollary 4.2, the controls belonging to the Nash/worst-case equilibrium of the disturbed game. The worst-case trajectories from the point of view of each player can be seen on Figure 4.3. Since they are different, the players try to independently defend ourselves against their own worst-case situation. Figure 4.4 shows the actual system trajectory using the calculated Nash/worst-case control functions.

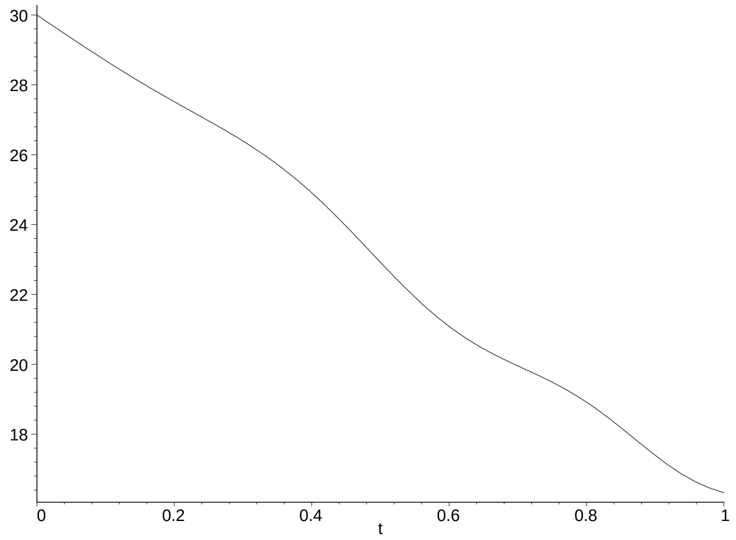

Figure 4.2: State trajectory $(x(t))$ of the system controlled by pure Nash-controls

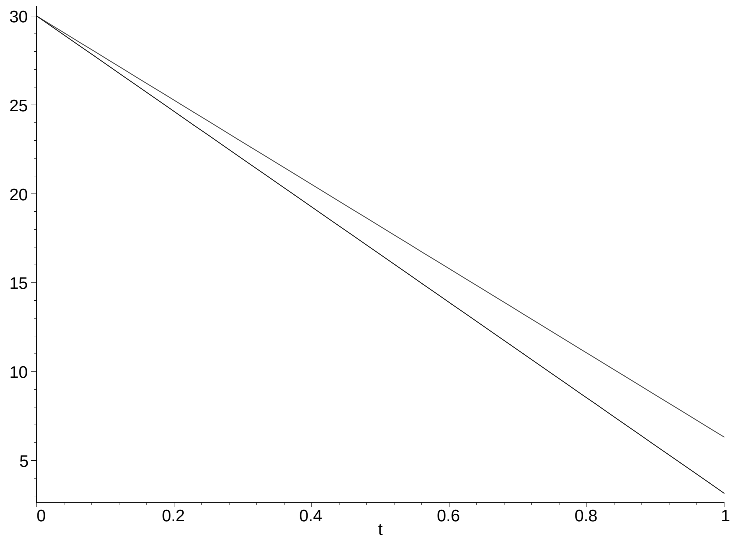

Figure 4.3: Worst-case trajectories $(\hat{x}_{1,2}(t))$ of the players

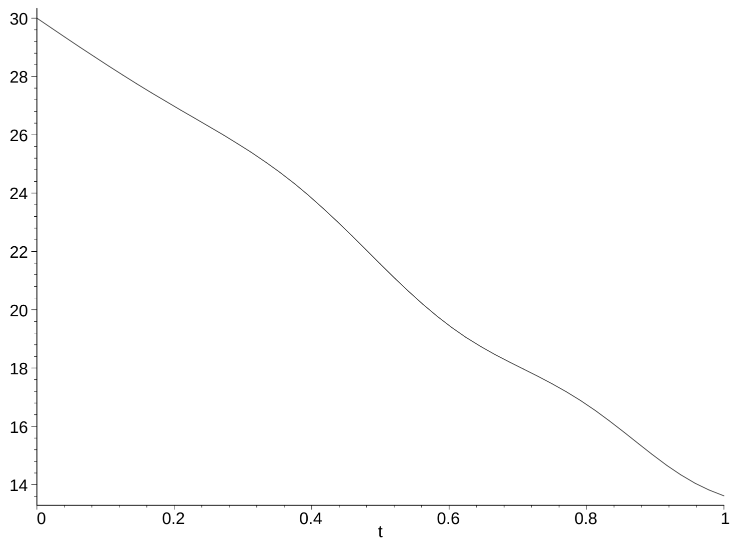

Figure 4.4: State trajectory of the system $(x(t))$ controlled by Nash/worst-case strategies

The costs, that each player receives are

$$J_1^{\text{Nash/worst-case}} = -78.80$$

$$J_2^{\text{Nash/worst-case}} = 327.19$$

Obviously, both players received less costs using Nash/worst-case controls.

4.4 Notes and references

Since control systems and differential games are typically non-isolated system, besides the output signal, there are usually noises or other types of disturbances acting on them. Hence, disturbance reduction has always been a very important topic in Control Theory.

One way to handle disturbed systems is the very popular Robust Control Theory, where disturbances are treated as perturbations of system coefficients. Clearly, most of our systems use parameters, that are determined via statistical methods (for instance by measuring and taking the average); hence no exactness can be expected. There are numerous authors dealing with those problems. For a very detailed mathematical description refer to the monography of Ionescu, Oara and Weiss [IOW99]. Another very interesting technique for robust controlling is derived by Habets with the help of the gap-topology [Hab91].

Another way to discuss stability of disturbed systems is the classical frequency-domain approach proposed by Nyquist and Bode. A very nice way to couple these problems with results known in Complex Function Theory is shown in the Lecture Note of Francis [Fra87].

In this chapter, we used a different method to investigate disturbed systems: we supposed, that the disturbance is an additional player acting on the system and that each player plays according to his own worst-case strategy against it. This idea was first presented by Limebeer et al. in [LAH94]. However, they only discussed the 1-player situation – where no competition between the players is present – and unfortunately unnoticed the fact, that if the matrices P and Q are positive (semi)definite in (4.11), then no worst-case disturbance exists. For some early results on mixed Nash/worst-case strategies see [JK98]. Further results on these types of strategies can be found in the works of Jank and Kun [JK99a], [JK00b].

Chapter 5

Moving-horizon control policies

Throughout the previous chapters, we mainly discussed differential control systems and dynamical games defined on a finite time-horizon. Nevertheless, as recent applications – for instance the modeling of governmental economics – show, it is a very difficult task to decide, whether a planning horizon is finite or not. And if it's finite, it is sometimes even more difficult to predict its length. Hence, it is assumable that players choose their strategies based on decisions for a given finite (usually very short) time-horizon, and as time goes by, they constantly *update* their decision according to the same time-horizon. Thus, as time goes by, the time horizon translates along the time-scale of the players. That's why this control policy is called *moving-horizon* (or sometimes *receding-horizon*) policy.

It is a very important question to decide, whether the system becomes stable under these controls. If not, the time horizon should be extended and the players must use strategies according to this new decision interval.

In this chapter, we first give a definition for moving-horizon controls in the context of differential control systems and games and then discuss, how the minimal length of the decision interval should be chosen to achieve stability. Then, in Section 2, we also establish results, where the stability is independent of the length of the decision interval.

5.1 Long-time stability through short-time decisions

As discussed before, moving-horizon control policies are defined in a way, that at each time $t \geq t_0$ the players make their decision based on an optimal control problem over the time horizon $[t, t+T]$. Hence the general formulation of the open-loop decision model can be formulated as follows:

Definition 5.1 (moving-horizon decision policy) *Suppose that Γ_N is a linear quadratic autonomous (i.e. the coefficients A, B_i, Q_i and R_{ij} are constant) N-player differential game defined on the finite time-horizon $[0, T]$:*

$$\dot{\xi}(t) = A\xi(t) + \sum_{i=1}^{N} B_i u_i(t), \quad \xi(0) = \xi_0, \text{ with}$$

$$J_i = \xi^T(T)K_{if}\xi(T) + \int_0^T \xi^T(t)Q_i\xi(t) + \sum_{j=1}^N u_j^T(t)R_{ij}u_j(t)\,dt.$$

Because of the autonomy of Γ_N, the time-horizon can be shifted to any starting point $t_1 \in \mathbb{R}$ without changing the system trajectory. Now, we define the so-called local game*, which is used by the players to make their decisions:*

For any $t_1 \in \mathbb{R}$ let $\Gamma_N(t_1)$ be a differential game defined on the interval $[t_1, t_1+T]$ with the same (shifted) dynamics, strategies and information structure as Γ_N and also with the same (shifted) cost functionals, i.e. defined with the variational problem

$$\dot{\xi}(t) = A\xi(t) + \sum_{i=1}^N B_i u_i(t), \quad \xi(t_1) = \xi_1, \text{ with}$$

$$J_i = \xi^T(t_1+T)K_{if}\xi(t_1+T) + \int_{t_1}^{t_1+T} \xi^T(t)Q_i\xi(t) + \sum_{j=1}^N u_j^T(t)R_{ij}u_j(t)\,dt.$$

Denote by $\nu_i^*(t; t_1, \xi_1)$ the optimal strategy of the i^{th} player according to the local game $\Gamma_N(t_1)$. We can now define the moving-horizon control policy.

We say that the players act according to a moving-horizon decision policy *defined by the local game Γ_N on the duration T, if each player uses the strategy*

$$\gamma_i(t, x(t)) = \nu_i^*(t; t, x(t))$$

to control the differential system

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^N B_i u_i(t), \quad x(t_0) = x_0$$

on the time horizon $[t_0, \infty)$.

Remark 5.1 *By taking $N = 1$, the definition of moving-horizon optimal control policy for linear-quadratic differential control systems becomes obvious.*

Throughout this chapter, we shall always assume that $R_{ii} > 0$, $Q_i \geq 0$ and $K_{if} \geq 0$ hold for any $i = 1, 2, \ldots, N$.

Note that either under feedback or open-loop information structure, the optimal strategy $\nu_i^*(t; t_1, \xi_1)$ in a small neighborhood of t_1 depends on the initial state $\xi(t_1) = \xi_1$ and hence, γ_i always depends on the current state $x(t)$. Consequently, moving-horizon differential games require feedback information structure. Further, the moving-horizon control policies depend on the chosen equilibrium (and hence on the information structure) of the local game Γ_N. In the sequel, without mentioning it, we shall always assume, that Γ_N is a *Nash-game with either open-loop or feedback information structure.*

To begin our investigation on the stability of systems governed by moving-horizon control laws, we first derive explicit formulae for the control functions.

Suppose that the local game $\Gamma_N(t_1)$ is either a feedback or an open-loop Nash-game. Suppose further that the corresponding Riccati differential equation admits a bounded solution. Then, as shown in Section 2.3, the respective optimal control policies can be given in the form:

$$u_i^{(FB)}(t,x) = -R_{ii}^{-1}B_i^T K_i^{(FB)}(t)x, \text{ or}$$

$$u_i^{(OL)}(t) = -R_{ii}^{-1}B_i^T K_i^{(OL)}(t)x^*(t),$$

where $K_i^{(FB)}$ and $K_i^{(OL)}$ fulfill the feedback and open-loop Riccati differential equations, respectively. Thus, we obtain a representation for the moving-horizon controls:

Theorem 5.1 *Let the local game Γ_N be, respectively, an open-loop or feedback linear quadratic Nash-game. Suppose further, that the respective matrix Riccati differential equations (3.4) and (3.5) admit a solution K_i over the interval $[0,T]$. Then, the arising moving-horizon control functions fulfill*

$$u_i = -\underbrace{R_{ii}^{-1}B_i^T K_i(0)}_{=:F_i}x(t),$$

for a local open-loop and feedback moving-horizon decision model, respectively.

Proof. Using Theorems 2.5 and 2.8, we obtain that for the open-loop and for the (linear) feedback control laws, respectively, $\nu_i^*(t;0;\xi_1)$ takes the form

$$\nu_i^*(t;0,\xi_1) = -R_{ii}^{-1}B_i^T K_i(t)\xi(t)$$

where the matrix $K_i(t)$ is solution of the corresponding *autonomous* generalized matrix Riccati differential equation (3.4) and (3.5) for $K_i(t_1+T) = K_{if}$. Since these equations are autonomous, their trajectories can be translated along the time-axis. Hence, for any local game $\Gamma_N(t_1)$ $\nu_i^*(t;t_1,\xi_i) = \nu_i^*(t-t_1;0,\xi_1)$ and thus

$$\gamma_i(t,x(t)) = \nu_i^*(t;t,x(t)) = \nu_i^*(0;0,x(t))$$

holds. □

Remark 5.2 *This latter theorem shows the importance of moving-horizon strategies: Although, no optimality is achieved, instead of using a time-dependent feedback matrix, by choosing this decision model, the feedback matrix becomes constant, which is an enormous advantage in mechanical and economical systems.*

Let us now review, how the linear feedback matrices F_i and hence the closed-loop matrix of the system

$$A - \sum_{i=1}^{N} B_i F_i$$

depend on the duration T of the local game Γ_N:

Lemma 5.1 *Denote by T the length of the time-horizon of the local game Γ_N. If then the solution $\tilde{K}_i$ of the following set of matrix Riccati differential equations*

$$\frac{d}{dt}\tilde{K}_i = -\tilde{K}_i A - A^T\tilde{K}_i - Q_i + \tilde{K}_i \sum_{j=1}^{N} S_{jj}\tilde{K}_j, \quad \tilde{K}_i(0) = K_{if} \tag{5.1}$$

and

$$\frac{d}{dt}\tilde{K}_i = -\tilde{K}_i A - A^T\tilde{K}_i - Q_i + \tilde{K}_i \sum_{j=1}^{N} S_{jj}\tilde{K}_j - \sum_{\substack{j=1\\ j\neq i}}^{N} \tilde{K}_j S_{ij}\tilde{K}_j + \sum_{\substack{j=1\\ j\neq i}}^{N} \tilde{K}_j S_{jj}\tilde{K}_i, \quad \tilde{K}_i(0) = K_{if}, \tag{5.2}$$

exists on $[-T, 0]$, for the open-loop and feedback local information structure, respectively, then it is

$$F_i = R_{ii}^{-1} B_i^T \tilde{K}_i(-T).$$

Proof. Again, using the autonomy of equations (5.1) and (5.2), we conclude that their solutions coincide with the respective solutions of equations (3.4) and (3.5) under the transformation $K_i(t) = \tilde{K}_i(t-T)$ for any $t \in [0, T]$. Hence Theorem 5.1 completes the proof. □

From now on, we assume that the terminal values $K_{if} = 0$ and try to obtain a bound for the decision length T such that the closed-loop system becomes stable.

Clearly, for this assumption $\tilde{K}_i(0) = 0$ and hence

$$A_{cl}(0) = A - \sum_{i=1}^{N} S_{ii} K_i(0) = A$$

follows, meaning that the closed-loop matrix is unstable for small decision intervals T, unless the system matrix A is originally stable. Hence, one needs a bound T^* such that for any duration $T \geq T^*$ stability is ensured.

To begin, we develop an existence result for differential control systems:

Lemma 5.2 *Suppose that the pairs (A, Q) and (A, B) are detectable and stabilizable, respectively. Then, there exists $T^* > 0$ such that any moving-horizon control law*

$$u = -R^{-1}B^T K(-T)x(t)$$

defined on a time duration not less than T^* yields a stable closed-loop matrix

$$A - SK(-T)$$

and hence stabilizes the system.

Proof. As shown in Section 3.3, the (unique) positive definite solution of the corresponding algebraic Riccati equation K_0 stabilizes the system, i.e. the matrix $A - SK_0$ is stable. Moreover, using Theorem 3.8, for any positive semidefinite initial value K_f

$$\lim_{t\to-\infty} K(t) = K_0$$

holds, where $K(.)$ is the corresponding solution of the symmetric matrix Riccati differential equation. Since the spectrum $\sigma(t)$ of the matrix $K(t)$ is an analytical function of t (see Corollary 2 in Section 3.5.5 in [Bau85]), we conclude that there is a neighborhood U of $-\infty$, such that for any $t \in U$ $\sigma(t) \subset \mathbb{C}_-$ holds. Hereby denotes

$$\mathbb{C}_- := \{z \in \mathbb{C} | Re(z) < 0\}.$$

□

A similar existence result can also be obtained for differential games with open-loop controls:

Lemma 5.3 *Suppose that the local game Γ_N is an open-loop Nash-game, where the dichotomy condition (3.17) is fulfilled with, such that $Re(\lambda_n) < 0$ holds. Suppose further, that the zero-matrix is contained in the basin of attraction of the dichotomic solution K_+ (which happens with the probability of 1). Then there exists $T^* > 0$ such that any moving-horizon control policy defined on a local open-loop game Γ_N with duration not less than T^* stabilizes the system.*

Proof. Using the equivalence between the generalized open-loop matrix Riccati equations and the non-symmetric matrix Riccati equation (as stated in Section 3.3), we conclude using Theorems 3.10 that the dichotomic solution stabilizes the system. Since, moreover, $\lim_{t \to -\infty} K(t) = K_+$ holds, there exists T^*, such that for any $t < -T^*$ $K(t)$ stabilizes the system. □

Now, we try to derive a slightly more explicit bound for the minimal length of the local control system.

Lemma 5.4 *Suppose that $S \geq 0$ and $Q > 0$ and that the pair (A, S) is stabilizable. Suppose further, that for some $T^* > 0$ the matrix $\dot{K}(-T^*)$ is positive definite. Hereby denotes $K(t)$ the solution of the symmetric matrix Riccati differential equation*

$$\dot{K}(t) = -A^T K - KA - Q + KSK, \quad K(0) = 0.$$

Then, the moving-horizon controls based on a decision of length not less than T^* yields a stable system.

Proof. Defining a modified local control system $\hat{\sigma}$ with the cost functional

$$\hat{J} = \xi^T(h) K(-T^*) \xi(h) + \int_0^h \xi^T(t) Q \xi(t) + u^T(t) R u(t) \, dt.$$

we conclude that the arising moving-horizon control

$$u = -R^{-1} B^T \hat{K}(-h) x$$

coincides with the moving-horizon control defined by the original local game on the duration $T^* + h$. Hence, we can restrict ourselves to the investigation of the second system.

Moreover, as it was shown in Section 1.4, the value function of the local optimal control system $\hat{\sigma}$ $\hat{V}(t,x)$ can be written as

$$\hat{V}(t,x) = x^T K(-T^* - h + t)x.$$

Denoting now by $\xi^*(t)$ an optimal trajectory of the system $\hat{\sigma}$, we conclude, using the Hamilton-Jacobi-Bellman equation, that it is

$$\begin{aligned}\frac{d}{dt}\frac{\partial}{\partial t}V(t,\xi^*(t)) &= \frac{\partial^2}{\partial t^2}V(t,\xi^*(t)) + \frac{\partial^2}{\partial t \partial x}V(t,\xi^*(t))(A\xi^* + Bu^*) \\ &= -\frac{\partial}{\partial t}(\xi^T(t)Q\xi(t) + u^{*T}(t)Ru^*(t)) = 0.\end{aligned}$$

Moreover, it is

$$\frac{\partial}{\partial t}V(t,\xi^*)|_{t=h} = \xi^*\dot{K}(-T^*)\xi^* < 0$$

for any $\xi^* \in \mathbb{R}^n$. Hence, Assumptions A1-A3 of Theorem 1 in [Gyu95] are fulfilled, which means that the local system yields a stabilizing moving-horizon controller for any $h > 0$. □

Remark 5.3 *Using the (rather technical) result achieved in Theorem 2 (ii) in [FJ95], we obtain that the convergence of the solution of the non-symmetric matrix Riccati differential equation (3.1) and hence of the open-loop matrix Riccati differential equation (3.4) takes place at an exponential rate. From Theorem 3.10 we also know, that the dichotomic solution (if it exists) stabilizes the system if $\lambda_n < 0$ holds. Thus, establishing a bound $\delta > 0$, for which any matrix with $\|K - K_+\| < \delta$ stabilizes the system, we can obtain a time-duration T^* such that for any $t > T^*$ the solution of $K(-t)$ (3.1) fulfills $\|K(t) - K_+\| < \delta$ which yields the stability of the closed-loop system.*

To end this section, we present a result for feedback local games:

Lemma 5.5 *Let Γ_N be a linear quadratic differential game with coefficients $R_{ij} = 0$ for $i \neq j$ and $Q_i > 0$ for $i = 1, \ldots, N$. Suppose further, that for some $T > 0$ and $i = 1, \ldots, N$*

$$\frac{d}{dt}\tilde{K}_i(-T) \geq 0$$

holds, where $\tilde{K}_i$ is a solution of (5.2). Then, the closed-loop system governed by feedback moving-horizon control policies over the duration T is stable.

Proof. Using the generalized feedback RDE, we conclude that

$$\begin{aligned}\dot{K}_i &= -A^T K_i = K_i A - Q_i + K_i S_{ii} K_i + \sum_{\substack{j=1\\j\neq i}}^N K_j S_{jj} K_i + \sum_{\substack{j=1\\j\neq i}}^N K_i S_{jj} K_j \\ &= -\underbrace{\left(A - \sum_{j=1}^N S_{jj}K_j\right)^T}_{A_{cl}} K_i - K_i\left(A - \sum_{j=1}^N S_{jj}K_j\right) - Q_i - K_i S_{ii} K_i\end{aligned}$$

Moreover, using $Q_i > 0$, we conclude that K_i is positive definite (see Theorem 3.3) with

$$A_{cl}^T K_i + K_i A_{cl} = -\dot{K}_i - Q_i - K_i S_{ii} K_i < -\dot{K}_i < 0.$$

Hence, A_{cl} is stable. □

Remark 5.4 *Note, that because of the periodic behavior of the solution $\tilde{K}_i$ presented in Section 3.4 it cannot be ensured that for any duration longer than T the stability remains unchanged.*

5.2 Strong stability results

In this section, we try to find an answer to the question, how to choose the initial values K_{if} so that for any $T \geq 0$ stability of the closed-loop system is ensured.

Theorem 5.2 *Let Γ_N be an N-player feedback Nash-game of the form*

$$\dot{x} = f(t, x, u_1, \dots, u_N)$$

with cost functionals

$$J_i := \kappa_i(x(t_f)) + \int_{t_0}^{t_f} \Psi_i(x, u_1, \dots, u_N)\, dt$$

Assuming further that there exists $i \in \{1, \dots, N\}$ such that the following properties hold:

(i) $\frac{\partial}{\partial t} V_i(t, x)\Big|_{t=0} \geq 0$ *for all* $x \in \mathbb{R}^n$.

(ii) $\Psi_i(0, 0, \dots, 0) = 0$ *and* $\Psi_i(x, u_1, \dots, u_N) > 0$ *for any* $x \in \mathbb{R}^n \backslash \{0\}$ *and* $u_j \in \mathbb{R}^{m_j}$ $(j = 1, \dots, N)$.

(iii) The mappings f and the value-function V_i of the local game are continuously differentiable over $[t_0, t_f]$ and $f(t, 0, 0, \dots, 0) = 0$ for any $t \in [t_0, t_f]$.

(iv) For any control functions $(u_1, ..., u_N)$ the generated trajectory $x(t)$ exists and is bounded over the time-horizon $[t_0, t_f]$.

Then the closed-loop system defined as a feedback moving-horizon game on any duration $T > 0$ yields a stable system.

Proof. Denoting $L(x) := V_i(0, x)$, we shall prove that $L(x)$ can serve as a Lyapunov-function for the closed-loop system:

1. Using $\Psi_i(x, u_1, \dots, u_N) > 0$ for any $x \in \mathbb{R}^n$ and $u_j \in \mathbb{R}^{m_j}$, we conclude that it is

$$\int_{t_0}^{t_f} \Psi_i(x, u_1^*, \dots, u_N^*)\, dt > 0$$

   for the optimal controls of the local game u_j^* and hence $L(x) > 0$ for $x \in \mathbb{R}^n \backslash \{0\}$ as well as $L(0) = 0$ follows.

2. Using the continuous differentiability of V_i, we conclude that it is

$$\begin{aligned}
\frac{d}{dt}L(x(t)) &= \lim_{\tau\to 0}\frac{L(x(t+\tau))-L(x(t))}{\tau}\\
&= \lim_{\tau\to 0}\frac{V_i(\tau,\xi(t+\tau;t,x(t)))-V_i(0,x(t))}{\tau}\\
&\quad+\lim_{\tau\to 0}\frac{V_i(0,\xi(t+\tau;t,x(t)))-V_i(\tau,\xi(t+\tau;t,x(t)))}{\tau}\\
&\quad+\lim_{\tau\to 0}\frac{V_i(0,x(t+\tau))-V_i(0,\xi(t+\tau;t,x(t)))}{\tau}\\
&= \frac{d}{dt}V_i(0,\xi(t+\tau;t,x(t)))\\
&\quad-\frac{\partial}{\partial t}V(0,x(t))\\
&\quad+\lim_{\tau\to 0}\frac{V_i(0,x(t+\tau))-V_i(0,\xi(t+\tau;t,x(t)))}{\tau}.
\end{aligned}$$

Comparing this result with the Hamilton-Jacobi-Bellman equation and with assumption (i) above, we obtain that

$$\frac{d}{dt}L(x(t)) \le -\Psi_i(x(t),u_1^*,\ldots,u_N^*)+\lim_{\tau\to 0}\frac{V_i(0,x(t+\tau))-V_i(0,\xi(t+\tau;t,x(t)))}{\tau} \quad (5.3)$$

holds with u_j^* denoting the optimal control function of the local game.

3. Supposing that $x(t) = 0$ holds, then it is with the zero-controls ($u_j^* = 0$) $\dot{x} = f(t,0,0,\ldots,0) = 0$. Hence, we can restrict ourselves to the case, where $x(t) \neq 0$ holds. Therefore, it is – using assumption (ii) – $\Psi(x(t),u_1^*,\ldots,u_N^*) > 0$ and hence, we immediately see that

$$\frac{d}{dt}L(x(t)) < \lim_{\tau\to 0}\frac{V_i(0,x(t+\tau))-V_i(0,\xi(t+\tau;t,x(t)))}{\tau}$$

holds.

4. Finally, we show that

$$\lim_{\tau\to 0}\frac{V_i(0,x(t+\tau))-V_i(0,\xi(t+\tau;t,x(t)))}{\tau} \le 0 \quad (5.4)$$

holds. For that note if V_i is continuously differentiable, that it is locally Lipschitz with the Lipschitz constant ℓ_i in the neighborhood of $(0,x(t))$ and thus we have – using the Mean Value Theorem –

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{V_i(0,x(t+\tau))-V_i(0,\xi(t+\tau;t,x(t)))}{\tau}\\
\le\ & \lim_{\tau\to 0}\frac{\ell_i\|x(t+\tau)-\xi(t+\tau;t,x(t))\|}{\tau}\\
\le\ & \lim_{\tau\to 0}\frac{\ell_i\|x(t+\tau)-x(t)+x(t)-\xi(t+\tau;t,x(t))\|}{\tau}\\
\le\ & \lim_{\tau\to 0}\ell_i\frac{\|x(t+\tau)-x(t)\|+\|\xi(t+\tau;t,x(t))-x(t)\|}{\tau}\\
\le\ & \lim_{\tau\to 0}\ell_i\|f(s_1,x(s_1),\nu_1^*(s_1;s_1,x(s_1)),\ldots,\nu_N^*(s_1;s_1,x(s_1)\|+\\
& \|f(s_2,\xi(s_2;t,x(t)),\nu_1^*(s_2;t,x(s_2)),\ldots,\nu_N^*(s_2;t,x(s_2))\|,
\end{aligned}$$

for $s_1, s_2 \in (t,t+\tau)$. From which the continuity of f and the trajectories immediately yields that inequality (5.4) holds.

□

Remark 5.5 *Note, that the condition* $\frac{d}{dt}\frac{\partial}{\partial t}V(t,x) = 0$*, that played a crucial role by transferring the condition* $\frac{\partial}{\partial t}V(t,x)\Big|_{t=0} \geq 0$ *into one containing only the terminal value* K_f*, does not hold for differential games, since the corresponding control system*

$$\dot{x} = Ax + \sum_{\substack{j=1 \\ j\neq i}}^{N} B_j u_j^*(t,x) + B_i u_i$$

is because of $\frac{\partial}{\partial t}u_j^* \neq 0$ *non-autonomous. Hence, Theorem 1 in [Gyu95] cannot be directly generalized for differential games.*

5.3 Notes and references

The concept of moving-horizon control is not new in Control Theory. It goes back to an early paper of Kleinman [Kle70] and hence, it is almost as old as the results of Kalman and Hautus on the controllability of linear control systems. Recently, Mayne and Michalska [MM90], [MM93] published (strong) stability results for nonlinear differential control systems, that were further generalized by Gyurkovics [Gyu95] for nonlinear uncertain control systems defined by differential inclusions. Further results on the stability and robustness of nonlinear time-varying control systems under receding-horizon control appeared in the recent works of Chen and Allgöwer [CA98] and De Nicolao et al. [NMS98].

The concept of moving-horizon strategies for differential games has been investigated recently by v.d. Broek [vdB98]. He also investigated scalar systems and discussed a possible application for the stabilization of the governmental monetary politics.

Finally, damped algebraic Riccati equations that appear in the context of moving horizon linear quadratic control problems were investigated by Hench, Mehrman et al. in [HHM98], [HKM98].

Chapter 6

Geometric control theory

There are numerous applications, especially in mechanics, where the configuration space is non-Euclidean. The simplest example is the motion of a rigid body: Besides translations, that can be described by a vector $x \in \mathbb{R}^3$ there is also a rotational coordinate $X \in SO(3)$ belonging to any position and hence, the configuration is defined for each pair $(x, X) \in \mathbb{R}^3 \times SO(3)$. Therefore, applications in mechanics, especially in robotics and vehicle control (see Appendix B and [Lei97], [LS90]), require a generalization of control systems over a non-Euclidean state-space X.

In order to obtain results for control systems on curved surfaces, we first have to discuss their geometric properties. Here, we will focus on two important structures: Riemannian manifolds and Lie groups. Roughly speaking, a Riemannian manifold is a set, where we also have a differential structure. We can define curves, calculate their tangentials, define a parallel translation and we can also calculate derivatives of vector fields. Lie groups have additional algebraic properties. These are the main tools, that will be used in the rest of this Thesis.

In the next section, we first give some necessary definitions for Riemannian manifolds and Lie groups, such as tangent vectors, vector fields and derivatives. Then, using this background, we investigate in Section 6.2 controllability and stabilizability properties of special types of nonlinear control systems, as we did for linear control systems in Chapter 1.

6.1 Control systems on Lie groups and smooth manifolds

In this section we first establish the background of studying nonlinear differential systems. Working on Euclidean spaces didn't require the explicit definitions of differentiation, vector fields or curves. Here, because of the complexity of the structures, the systems are defined on, we need some preliminary remarks. Then, we give a formal definition of a control system defined on a non-Euclidean space.

6.1.1 Introduction to differential geometry

To begin with, we list some of the most important definitions and properties known in the Theory of Differential Geometry. Since the following topics can be found in numerous textbooks and lecture notes (see for instance [Olv93], [Hel78] or [Pos89]), the proofs are omitted.

Definition 6.1 (chart) *Let M be a Hausdorff space. We call the pair (U, χ) n-dimensional* chart *associated to M if χ is an $\mathbb{R}^n$-valued bijective bicontinuous mapping defined on the open subset $U \subset M$.*

Definition 6.2 (compatibility) *Let M be a Hausdorff space. We call the charts (U_1, χ_1) and (U_2, χ_2)* compatible*, if one of the following holds:*

(i) $U_1 \cap U_2 = \emptyset$, or

(ii) The mapping $\chi_2|_U \circ (\chi_1|_U)^{-1} : \chi_1(U) \to \chi_2(U)$ is a (sufficiently smooth) diffeomorphism for the open subset $U = U_1 \cap U_2 \subset M$.

Definition 6.3 (atlas) *A collection of pairwise compatible $\mathbb{R}^n$-valued charts*

$$(U_\alpha, \chi_\alpha)_{\alpha \in A}$$

associated to a Hausdorff space M is called atlas *if additionally $\bigcup_{\alpha \in A} U = M$ holds.*

Now, we can define our basic differentiable structure:

Definition 6.4 (smooth manifold) *Let M be a Hausdorff space. Suppose that the atlas $(U_\alpha, \chi_\alpha)_{\alpha \in A}$ is a maximal family of pairwise compatible n-dimensional charts. Then M is said to be an* n-dimensional smooth manifold*.*

Our main tool for the discussion of differential control systems will be an infinitesimal approach, for which we first need to show how to carry out differentiations and then, how to develop the concept of a vector field on a manifold. In order to obtain this definition, we also need to define the tangent vectors, the local tangent space and the collection of the tangent spaces.

For that, we first define the following class of real-valued functions defined in a neighborhood of some point $p \in M$:

$$\mathcal{F}_p := \{f : U \to \mathbb{R} | p \in U, U \subset M \text{open}\} \tag{6.1}$$

together with the following relation $\approx_p$: $f_1 \approx_p f_2$ if and only if there exists a neighborhood U of p such that $f_1|_U = f_2|_U$.

Remark 6.1 *It is easy to see, that the above definition induces an equivalence class.*

Definition 6.5 (tangent vector) *Let M be a smooth manifold and $p \in M$. We call the mapping $X : \mathcal{F}_p \to \mathbb{R}$ tangent vector of M at the point p, if*

(i) for all $\lambda \in \mathbb{R}$ and $f, g \in \mathcal{F}_p$ $X(f + \lambda g) = X(f) + \lambda X(g)$ (i.e. $X()$ is linear) and

(ii) for $f, g \in \mathcal{F}_p$ $X(fg)=X(f)g(p)+f(p)X(g)$

hold.

Definition 6.6 (tangent space, tangent bundle) *Let M be a smooth manifold and $p \in M$. We call the set of tangent vectors* tangent space *of M at the point p. We denote this set with $T_p(M)$. Furthermore, we call the variety $T(M) = \bigcup_{p \in M} T_p(M)$* tangent bundle *of M.*

Definition 6.7 (vector field) *The (smooth) mapping $v : M \to T(M)$ is called (smooth) vector field if $v|_p \in T_p(M)$ holds for any $p \in M$. We denote the set of all vector fields over M with $D(M)$.*

Obviously, the restriction $v|_p$ is a differentiation (tangent vector) and hence for a given chart (U, χ) with $p \in U$ we conclude that

$$v|_p = \sum_{j=1}^{n} \xi_j(p) \left.\frac{\partial}{\partial x_j}\right|_p$$

holds for some (smooth) mappings $\xi_j : U \to \mathbb{R}$. Moreover, if we define the mapping $\hat{v}$ such that it maps smooth functions in U into smooth functions in U according to the rule

$$\hat{v}(f) : p \mapsto v|_p(f),$$

then we immediately obtain, that $\hat{v}$ is a differentiation, i.e. a linear mapping with $\hat{v}(fg) = \hat{v}(f)g + f\hat{v}(g)$. Indeed,

$$\hat{v}(fg)(p) = v|_p(fg) = (v|_p(f))\, g(p) + f(p)\, (v|_p(g)) = \hat{v}(f)(p)\, g(p) + f(p)\hat{v}(f)(p)$$

holds for $p \in U$. With other words, each vector field acts like a differentiation on the set of smooth functions over the manifold M.

Let us present an example of a simple smooth manifold, which we shall often investigate in the rest of this Thesis.

Example 9: (see Example 1.3 in [Olv93]) Let $\mathcal{S}^2$ be the set

$$\mathcal{S}^2 := \{(x, y, z) \in \mathbb{R} | x^2 + y^2 + z^2 = 1\}.$$

Denote by $U_1 := \mathcal{S}^2 \backslash \{(0, 0, 1)\}$ and $U_2 := \mathcal{S}^2 \backslash \{(0, 0, -1)\}$ two subsets of $\mathcal{S}^2$. If we then define, corresponding to U_1 and U_2 the mappings

$$\chi_1(x, y, z) := \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \chi_2(x, y, z) := \left(\frac{x}{1+z}, \frac{y}{1+z}\right),$$

respectively, then it is obviously

$$\chi_2 \circ \chi_1^{-1} : (x, y) \mapsto \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$

everywhere on $\chi_1(U_1 \cap U_2) = \mathbb{R}^2 \backslash \{(0,0)\}$. Moreover, the above mapping is differentiable and it is $\left(\chi_2 \circ \chi_1^{-1}\right)^{-1} = \chi_2 \circ \chi_1^{-1}$. Hence, the above charts define a two-dimensional manifold embedded into the three-dimensional Euclidean space $\mathbb{R}^3$.

Until now, we have seen, how to define derivatives of mappings between a manifolds and a Euclidean space. Next, we briefly discuss derivatives of mappings between smooth manifolds.

Definition 6.8 (differential) *Let M and M' be some smooth manifolds and let $p \in M$. Suppose further, that the mapping $f : M \to M'$ is smooth in the neighborhood of p (this means that for $p' = f(p)$ and for the two charts (U, χ), (U', χ') belonging to the manifolds M and M', respectively, with $p \in U$ and $p' \in U'$ $\chi' \circ f \circ \chi^{-1} : \chi(U) \to \chi'(U')$ is a smooth mapping). We now define the* differential *of f at point p ($df|_p$). This function maps the tangent space $T_p(M)$ into the tangent space $T_{p'}(M')$ according to the following rule:*

$$df|_p : T_p(M) \to T_{p'}(M'), \text{ with } df|_p(X)(g) = X(g \circ f)$$

for all $X \in T_p(M)$ and $g \in \mathcal{F}_{p'}(M')$.

It is easy to see, that indeed $df|_p(X)$ is a differentiation on the tangent space $T_{p'}(M')$ for any $X \in T_p(M)$. Now, we can define vector fields as special subsets of the tangent bundle and their action on smooth functions.

Before discussing the basic properties of Riemannian manifolds, we introduce our last notation concerning the geometry of general smooth manifolds:

Definition 6.9 (f-invariance) *Let $f : M \to M'$ be a smooth mapping between the smooth manifolds M and M'. A vector field v is called* f-invariant *if*

$$df|_p \left(v|_p\right) = \left(v_{f(p)}\right)$$

holds for any $p \in M$.

Hence, a vector field v is f-invariant if and only if $\hat{v}(g \circ f) = \hat{v}(g) \circ f$ for any smooth function $g : M \to \mathbb{R}$.

Suppose now that $X, Y \in T_p(M)$ are tangent vectors. We can then define the operator $X \circ Y$ in the usual manner

$$X \circ Y(f) := X(Y(f)).$$

This operator, is however not a derivation. For this simply notice the fact that

$$X \circ Y(fg) := X(Y(f)g + fY(g)) = X \circ Y(f)g + f(X \circ Y)g + Y(f)X(g) + X(f)Y(g).$$

Nevertheless, we can define a similar operator, that again belongs to the tangent space $T_p(M)$:

Definition 6.10 (Lie bracket) *Let $X, Y \in T_p(M)$. We call the operator*

$$[X, Y] := X \circ Y - Y \circ X$$

the Lie bracket *of X and Y. Similarly, the Lie bracket of two vector fields $v, w \in D(M)$ is defined by the identity*

$$[v, w]_{|_p} := v|_p \circ w|_p - w|_p \circ v|_p \quad \forall p \in M.$$

We now restrict ourselves to a (still very large) class of smooth manifolds. Additionally to the Hausdorff property of M, suppose that a metric can be defined on each tangent space the following way:

Definition 6.11 (Riemannian manifold) *Let M be an (analytic) manifold (i.e a smooth manifold, where the mapping $\chi_2|_U \circ (\chi_1|_U)^{-1} : \chi_1(U) \to \chi_2(U)$ as defined in Definition 6.2 is analytic). Suppose further, that for each $p \in M$ there exists a positive definite symmetric bilinear scalar product $g_p(.,.)$ on the vector space $T_p(M)$. If additionally, for any smooth vector fields v, w, the (bilinear) mapping $g(v, w) := p \mapsto g_p(v|_p, w|_p)$ is smooth, then we say that M is a* Riemannian manifold*. The mapping $g : D(M) \times D(M) \to \mathcal{F}$ is called* Riemannian metric *and is usually denoted simply by $< .,. >$.*

Definition 6.12 (curves and vector fields along curves) *Let $\gamma : \mathbb{R} \to M$ be a (smooth) mapping. Mappings of this type are naturally called* curves *on M. Differentiation of $\gamma(t)$ with respect to the variable t is usually denoted by $\dot\gamma(t)$ or $\frac{d}{dt}\gamma(t)$. In particular, the* velocity vector field *along the curve γ is defined by the equation*

$$\frac{d}{dt}\gamma(t) = \dot\gamma(t) := d\gamma\left(\frac{d}{dt}\right)\bigg|_t \in T_{\gamma(t)}(M).$$

In general, a vector field *along γ is a (smooth) function $v_\gamma : \mathbb{R} \to T(M)$ such that $v_\gamma(t) \in T_{\gamma(t)}(M)$ holds for any t.*

This latter definition enables us to introduce another type of differentiation, the covariant derivative of a vector field along the (smooth) curve γ:

Definition 6.13 (covariant differentiation along curves) *Let M be a smooth manifold with an affine connection ∇ (see Section 1.4 in [Hel78]) and γ be a curve on M. We say that the operator $\frac{D_\gamma}{dt}$ is a* covariant differentiation *along γ if it satisfies the conditions*

(i) $\frac{D_\gamma(v+w)}{dt} = \frac{D_\gamma v}{dt} + \frac{D_\gamma w}{dt}$,

(ii) $\frac{D_\gamma(fv)}{dt} = \frac{df}{dt}v + f\frac{D_\gamma v}{dt}$, *and finally*

(iii) if there exists a field z with $z(\gamma(t)) = v(t)$, then $\frac{D_\gamma v}{dt} = \nabla_{\frac{d\gamma}{dt}} z$

hold for any smooth vector fields v, w along γ and smooth mapping $f : M \to M$.

Now, we introduce a special family of curves, that are the generalization of straight lines on Euclidean spaces. A more detailed investigation of these curves, together with the variational problem, these curves are solutions of, will be presented in Section 7.1.

Definition 6.14 (parallel) *Let M be a smooth manifold with an affine connection, γ be a curve in M and v_γ be a vector field along γ. We say that the vector field v_γ is* parallel along *γ if $\frac{D_\gamma v_\gamma}{dt} = 0$ for all t.*

A special case of the above definition, where the vector field is the velocity field of γ yields a class of curves called geodesics:

Definition 6.15 (geodesic) *A curve defined on a smooth manifold with an affine connection is called* geodesic curve *if its velocity vector field is parallel along itself.*

Before we turn our attention to Lie groups, we present a result from [Hel78] concerning the existence and uniqueness of geodesic curves:

Theorem 6.1 (see Proposition 5.3 in [Hel78]) *Let M be a smooth manifold with an affine connection. Then there exists for any point $p \in M$ and tangent vector $X \in T_p(M)$ a unique maximally prolongated geodesic $\gamma(t)$ solving the initial value problem.*

$$\gamma(0) = p \text{ and } \dot{\gamma}(0) = X.$$

After introducing Riemannian manifolds and their most important properties, we now restrict ourselves to a class of Riemannian manifolds equipped with an algebraic structure:

Definition 6.16 (Lie group) *Let $(G, \cdot, e)$ be a group. Suppose also that G is Riemannian manifold of dimension n for which the mappings $\cdot : G \times G \to G$ and $^{-1} : G \to G$ are smooth. Then, we say that G is a* Lie group. *n is said to be the* dimension *of G. To shorten our notations, we sometimes say that G is an n-*dimensional *Lie group.*

As often in the mathematics, these marriage between an algebraic and an analytic structure leads to a powerful tool for the study of nonlinear control system. (Similar combined structures are topological groups, enabling us to define a very nice and natural property, the Haar-measure [Hal74].) From now on, we discuss the most essential details of the geometry of Lie groups and then, in the next chapter, we show the real benefits of this construction in the context of controllability of control systems.

Example 10: Let $GL(3)$ denote the set of nonsingular real-valued 3×3 matrices. Clearly, they form an algebraic group and also, the space $\mathbb{R}^{3\times 3}$ is isomorphic to $\mathbb{R}^9$. Hence, $GL(3)$ – being an open subset of this – is a 9-dimensional Lie group. However, it is not connected, which would be – as we shall see later – a very important property, though. We now intruduce two connected Lie subgroups of this group: The first group consists of matrices $X \in GL(3)$ satisfying $XX^T = I$ and $\det(X) = 1$. This set of equations define a 3-dimensional connected and compact manifold in $GL(3)$. Furthermore, if $XX^T = YY^T = I$, then it is also $XY(XY)^T = XYY^TX^T = I$. It can similarly be shown, that in fact the above manifold is a Lie group. We denote it by $SO(3)$.

A similar example of a connected (but non-compact) Lie subgroup consists of matrices of the form:

$$\left\{ \left[\begin{array}{cc} R & 0_2 \\ v & 1 \end{array} \right] \middle| R \in \mathbb{R}^{2\times 2}, RR^T = I, \det(R) = 1, v \in \mathbb{R}^2, 0_2 = [0,0]^T \right\}.$$

Again, the above set defines a submanifold of $GL(3)$, that is also an algebraic group itself. We denote this Lie group by $SE(2)$.

Similarly to the invariant vector fields for Riemannian manifolds (see Definition 6.9), we define G-invariant vector fields for Lie groups. As we shall see later, these vector fields play a crucial role in the characterization of the geometric properties.

Definition 6.17 (G- or right-invariance) *Let G be a Lie group and v be a vector field on G (as defined in Definition 6.7). We say that v is G-*invariant *(or sometimes right-invariant) if for all $g, h \in G$*

$$v|_{h\cdot g} = dR_g(v|_h)$$

holds. Hereby denotes R_g the right-multiplication with g, i.e. $R_g : G \to G$ with $x \mapsto x \cdot g$.

Note, that if v and w are right invariant vector fields on the Lie group G, so is their linear combination $v + \lambda w$ ($\lambda \in \mathbb{R}$) and hence the right invariant vector fields form a vector space. Furthermore, using the latter definition,

$$dR_g\left([v,w]|_h\right) = [dR_g(v|_h), dR_g(w|_h)] = [v|_{hg}, w|_{hg}] = [v,w]|_{hg}$$

holds. Hence the Lie bracket of any two G-invariant vector fields is again G-invariant. This facts motivate the following definition:

Definition 6.18 (Lie algebra) *Let L be a vector space with a skew-symmetric bilinear operator $[.,.] : L \times L \to L$ satisfying the Jacobi-identity, i.e.*

(i) (bilinearity) $[u, \lambda v + w] = \lambda[u,v] + [u,w]$ and $[\lambda u + v, w] = \lambda[u,w] + [v,w]$, $\forall u, v, w \in L$,

(ii) (skew-symmetry) $[u,v] = -[v,u]$, $\forall u, v \in L$,

(iii) (Jacobi-identity) $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$, $\forall u, v, w \in L$.

Then we say that L is a Lie algebra *with the corresponding* Lie bracket $[.,.]$.

The dimension of a Lie algebra is simply the dimension of the underlying vectorspace. Hence, the Lie algebra of the smooth vector fields over a Riemannian manifold is in general infinite dimensional. On the other hand, the next theorem shows that the Lie algebra of the G-invariant vector fields of a finite dimensional Lie group is always finite dimensional.

Theorem 6.2 *Suppose that G is a Lie group and $X \in T_e(G)$. Then there exists exactly one G-invariant vector field v on G such that $v|_e = X$. With other words, there exists a bijection between the Lie algebra of the right-invariant vector fields and the tangent space of the identity $T_e(G)$ and hence the dimension of this Lie algebra is equal to the dimension of the Lie group itself. Moreover, this bijection enables us to identify the Lie algebra of the right-invariant vector fields with $T_e(G)$.*

Proof. Let v be a right-invariant vector field on G with $X := v|_e \in T_e(G)$. Then it is

$$v|_g = dR_g(v|_e) = dR_g(X).$$

Hence, v is then uniquely defined for any point $g \in G$. □

Remark 6.2 *Throughout this work, according to the usual terminology in the literature, we usually call the Lie algebra of the right-invariant vector fields on G simply* the corresponding Lie algebra to G *or shorter* Lie algebra of G. *This Lie algebra is denoted by $L(G)$.*

Now we show how the above consideration yields a different definition for the concept of geodesic curves on Lie groups.

Let $\gamma : [0,1] \to G$ be a smooth curve, then the so-called velocity vector field $\dot{\gamma}(t) \in T_{\gamma(t)}(G)$ is a vector field along γ and hence the mapping

$$v_\gamma : [0,1] \to L(G) \quad t \mapsto \dot{\gamma}(t)\gamma^{-1}(t)$$

is a smooth curve on $L(G)$ representing the velocity of the curve. Supposing now, that γ is a geodesic arc we obtain, using Definition 6.15, that γ is a geodesic arc if an only if the corresponding velocity $v_\gamma \in L(G)$ is constant.

On the other hand, using Theorem 6.1, we conclude that for any $X \in T_e(G)$ (and hence, for any $v \in L(G)$) there exists a unique curve $\gamma(t)$ such that

(i) $\gamma(0) = e$,

(ii) $\dot{\gamma}(0) = X$,

(iii) $\frac{D_\gamma \dot{\gamma}(t)}{dt} = 0$.

Indeed, one can even prove more than that. Although the proof of the next theorem is not long, it is slightly technical, and hence is omitted. Nevertheless, it can be found for instance in the book of Olver (see [Olv93]).

Theorem 6.3 (see Proposition 1.48 in [Olv93]) *For any $v \in L(G)$ there exists exactly one diffeomorphism ϕ defined on the group* $(\mathbb{R}, +, 0)$ *with the following properties*

(i) The group $\phi(\mathbb{R}) \subset G$ is a 1-parameter Lie subgroup of G.

(ii) ϕ defines a curve in G with $\phi(0) = e$ and $\dot{\phi}(0) = v$.

Definition 6.19 (exponential mapping) *Let $v \in L(G)$. We call the corresponding (unique) diffeomorphism $\phi : (\mathbb{R}, 0, +) \to G$ with $\dot{\phi}(0) = v$* exponential mapping *and denote the element $\phi(1) = \exp(v)$.*

Remark 6.3

- *Obviously, using the property, that ϕ is a group-diffeomorphism, we obtain that* $\phi(t) = \exp(tv)$ *and*
$$\exp(v)^{-1} = \exp(-v), \quad \exp(0) = e$$
hold.
- *Please note, that although for any $k \in \mathbb{N}$ and $v \in L(G)$* $\exp(kv) = \exp(v)^k$ *holds, in general* $\exp(v + w) \neq \exp(v)\exp(w)$, *because the elements of the Lie group must not commute.*

To end this section, we cite a very important result that will be used later on during our investigation.

Theorem 6.4 (see [WN64] and [Mag54]) *Let G be a Lie group, $A(t)$ be a mapping $A : \mathbb{R} \to L(G)$ and $X(t)$ be a curve on G satisfying*

$$\dot{X}(t) = A(t)X(t), \quad X(0) = e. \tag{6.2}$$

Supposing now that $X_1, X_2, \ldots, X_n$ is a basis of the Lie algebra $L(G)$, the solution of (6.2) can be written (locally) as

$$X(t) = \exp\left(\sum_{i=1}^{n} g_i(t)X_i\right),$$

for some function g_i depending tacitly on A and X_i. It is also possible to give another representation of the solution of the form

$$X(t) = \prod_{i=1}^{n} \exp(f_i(t)X_i), \tag{6.3}$$

where again the functions f_i are uniquely determined by A and $L(G)$. If then the Lie algebra is solvable, then this second formula can be extended globally, i.e. there exists a basis, such that (6.3) holds for any $t \in \mathbb{R}$. In general (6.3) holds only in a neighborhood of zero, but as it was shown in [WN64] this neighborhood can be chosen relatively large.

6.1.2 Control systems on non-Euclidean spaces

In the rest of this work we shall use a class of control systems, which we define the following way:

Definition 6.20 (affine control system on Riemannian manifold) *Suppose that σ is a differential control system with the following properties:*

(i) X is a Riemannian manifold of the dimension $n < \infty$,

(ii) $U = \mathbb{R}^m$, and finally

(iii) $f(t, x, u) = a(x) + \sum_{i=1}^{m} u_i b_i(x)$, where a and b_i $(i = 1, \ldots, m)$ denote sufficiently smooth vector fields over X and $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$.

Then we call σ *finite dimensional* (differential) affine control system over the Riemannian manifold X*, or simply* nonlinear affine control system*.*

Remark 6.4 *Every autonomous affine control system defined on* $\mathbb{R}^n$ *with*

$$f(x,u) = A(x) + B(x)u$$

for sufficiently smooth mappings $A : \mathbb{R}^n \to \mathbb{R}^n$ *and* $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$*, and hence every autonomous linear control system, is also a control system over a Riemannian manifold.*

Proof. Denote by $b_i(x) \in \mathbb{R}^n$ the i$^{\text{th}}$ column of $B(x)$. Then, for $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$

$$B(x)u = \sum_{i=1}^{m} u_i b_i(x)$$

holds. Hence, using the (trivial) Riemannian structure of $X = \mathbb{R}^n$, we conclude that indeed $a(x) := A(x)$ and $b_i(x)$ define smooth vector fields over X, for which

$$f(x,u) = A(x) + B(x)u = a(x) + \sum_{i=1}^{m} u_i b_i(x).$$

□

Definition 6.21 (drift-term) *Let* σ *be a control system on a Riemannian manifold. The vector field* $a(x)$ *is usually called* drift *or* drift-term*. If* $a(x) = 0$*, then we say that* σ *is* drift-free*.*

Finally, we introduce (affine) control systems on a Lie group. Since any Lie group is a Riemannian manifold, the definition is equivalent to Definition 6.20, we include it just for future reference on the notations:

Remark 6.5 (control systems on Lie group, right-invariance) *Let* σ *be a control system over the Lie group* G*. Then, we can represent the vector fields* $a(x)$ *and* $b_i(x)$ *using elements of the Lie algebra in the following form*

$$a(x) = A(x)x, \text{ and } b_i(x) = B_i(x)x,$$

where A *and* B_i *for* $i = 1, \ldots, m$ *denote (smooth) mappings of the form* $G \to L(G)$*. If, moreover, these mappings are constant, i.e.* $A(x) \equiv A$ *and* $B_i(x) \equiv B_i$ $\forall x \in G$*, then we say that the above defined control system is* right-invariant*. Hence, the dynamics of a right-invariant control system on a Lie group can simply be written in the form*

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \tag{6.4}$$

for $x(t) \in G$*,* $u(t) = (u_1(t), u_2(t), \ldots, u_m(t))^T \in \mathbb{R}^m$ *and* $A, B_i \in L(G)$ *with* $i = 1, 2, \ldots, m$*.*

6.2 Nonlinear controllability and stabilizability

To begin our discussion on the controllability of control systems, we first review the following simple system, that has often been investigated by different authors (see for instance [Son90], [LS90] or [Lei97]).

Example 11: Consider a simple vehicle, which is well known from the world of circus: a unicycle. Suppose that the vehicle rolls without slipping on a plane. Hence, the "position" of the unicycle is uniquely determined by three coordinates (x, y, ϕ) as shown in Figure 6.1. Here, x and y denote the coordinates of the center point in the Euclidean space $\mathbb{R}^2$, while ϕ stands for the angle of the wheel. This latter coordinate could also be represented as a rotation with the given angle. Such a rotation is a bijective, isometric linear transformation of $\mathbb{R}^2$ onto itself described by the matrix $\begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$. With other words, the configuration is uniquely determined with the three coordinates $(x, y, T) \in \mathbb{R} \times \mathbb{R} \times SO(2)$, where the set $SO(2)$ denotes the Lie group of two dimensional orthogonal matrices, i.e

$$SO(2) := \{X \in \mathbb{R}^{2\times 2} | X^T X = I, det(X) = 1\}.$$

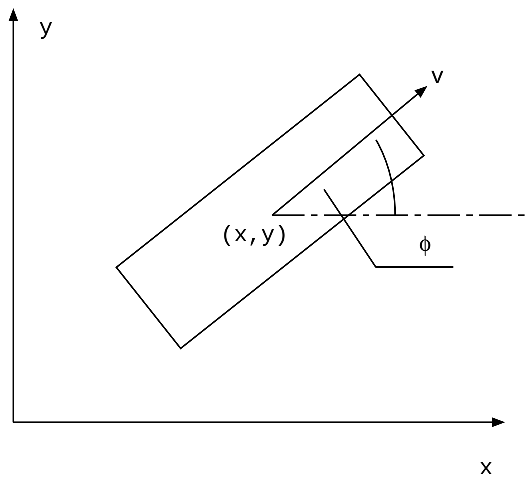

Figure 6.1: Kinematic model of the unicycle.

We can now derive the kinematic equations:

$$\begin{aligned} \dot{x} &= \cos(\phi)v \\ \dot{y} &= \sin(\phi)v, \end{aligned} \tag{6.5}$$

where v stand for the driving speed. Supposing now that the juggler controls additionally to this driving speed the velocity of the steering angle $\dot{\phi}$, we obtain the following nonlinear control system

$$\begin{aligned} \dot{x} &= \cos(\phi)u_1 \\ \dot{y} &= \sin(\phi)u_1 \\ \dot{\phi} &= u_2. \end{aligned} \tag{6.6}$$

Introducing now the matrix

$$X(t) = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ x & y & 1 \end{bmatrix},$$

we first obtain, using the kinematic equations (6.5), that

$$\dot{X} = \begin{bmatrix} -\sin(\phi)\dot{\phi} & \cos(\phi)\dot{\phi} & 0 \\ -\cos(\phi)\dot{\phi} & -\sin(\phi)\dot{\phi} & 0 \\ \dot{x} & \dot{y} & 1 \end{bmatrix} = \begin{bmatrix} 0 & \dot{\phi} & 0 \\ -\dot{\phi} & 0 & 0 \\ v & 0 & 0 \end{bmatrix} X(t)$$

holds. And hence, we immediately receive the following form for the control system (6.6):

$$\dot{X} = (u_1(t)B_1 + u_2(t)B_2)X(t), \tag{6.7}$$

where $B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Since the matrix $X(t)$ is an arbitrary element of the Lie group $SE(2)$, the above equation describes a right-invariant control system over this Lie group.

Our question is now, whether the juggler can reach every point (x, y) in every possible angle ϕ on the plane. At the first glance, one can immediately realize that the dimension of the Lie group $SE(2)$ (and hence the dimension of the configuration space) is three, and we only have a two-dimensional control. Moreover, there is no drift acting on the system, and hence our first guess (according to Kalman's controllability condition) could be NO.

Let us further simplify the system, to show why this answer is *false*. Suppose that the juggler can either turn the pedal to go forward with a velocity of 1 or turn the steering wheel with the angular velocity of 1. Then, his control function reduces to a bang-bang control between riding and turning. We can now give an explicit algorithm to choose his control function to reach any position on the plane (x_1, y_1, ϕ_1) from any initial position (x_0, y_0, ϕ_0):

1. Turn the steering wheel as long as you are facing toward the point you want to reach (x_1, y_1).
2. Go forward as long you as you reach the desired point (x_1, y_1).
3. Turn the steering again as long as you stay in the direction described by the angle ϕ_1.

This means that the nonlinear control system (6.6) is completely controllable, i.e. the juggler can reach every position on the plane in any possible angle of the unicycle.

As the latter example shows, it is an important question to investigate the controllability of nonlinear control systems. In this section, we mainly concentrate on a very special, but nevertheless extremely important, class of nonlinear control systems; on right-invariant control systems over Lie groups. We begin our discussion with some preliminary definitions:

Definition 6.22 (reachable set) *Let σ be a (generic) control system and x be an arbitrary state. We call the set*

$$\mathcal{R}(x) := \{y \in X | x \textit{ is controllable to } y\}$$

reachable set *of x w.r.t. σ.*

Definition 6.23 (orbit) *Let σ be a control system and x be an arbitrary state. We call the set*

$$\mathcal{O}(x) := \{y \in X | x \textit{ is controllable to } y \textit{ or } y \textit{ is controllable to } x\}$$

orbit *of x w.r.t. σ.*

Additionally to controllability, we introduce a similar, but weaker property that plays an important role in the investigation of nonlinear control systems:

Definition 6.24 (accessibility) *Let σ be a control system and x be an arbitrary state. We say that σ is* accessible *at the point x if the reachable set $\mathcal{R}(x)$ has nonempty interior (according to the topology of the state space).*

To begin our discussion on the controllability and stabilizability, we first need the following technical results from [Sac99]:

Remark 6.6 (see Lemmas 2.1 and 2.2 in [Sac99]) *Let σ be a right-invariant control system on the Lie group G and x be an arbitrary element of G. Then, the following statements hold:*

(i) The reachable set from x is a right-translation of the reachable set from the identity, i.e $\mathcal{R}(x) = \mathcal{R}(e)x$.

(ii) $\mathcal{R}(e)$ is an arcwise connected semigroup in G.

(iii) The orbit of x is a right-translation of the orbit of the identity, i.e $\mathcal{O}(x) = \mathcal{O}(e)x$.

(iv) The orbit of e is a connected Lie subgroup of G.

(v) The Lie algebra associated to the Lie group $\mathcal{O}(e)$ coincides with the smallest Lie algebra containing the vectors $A, B_1, \dots, B_m$. Furthermore, this Lie algebra is a Lie subalgebra of $L(G)$. In the sequel, we denote this subalgebra by $L(\sigma)$.

Using the above notation, we conclude that σ is controllable if and only if $\mathcal{R}(x) = G$ holds for any $x \in G$. On the other hand, using Remark 6.6, it is $\mathcal{R}(x) = \mathcal{R}(e)x$, and hence we can immediately obtain our first result concerning controllability of right-invariant control systems:

Corollary 6.1 *Let σ be a right-invariant control system on the Lie group G. Then σ is controllable if and only if G is connected and $\mathcal{R}(e) = G$ holds.*

Using the local representation formula of Magnus given in Theorem 6.4, we conclude that for any point of the trajectory $x(t_1) = x_1 \in G$ of a right-invariant control system, there exists $\delta > 0$ such that

$$x(t) = \exp\left(\sum_{i=1}^{n} g_i(t) X_i\right) x_1$$

holds for $t \in [t_1, t_1 + \delta)$ with smooth functions $g_i : \mathbb{R} \to \mathbb{R}$. Nevertheless, if we are only interested in the end point $x(t_1 + \delta)$, it can be obtained simply as

$$x(t_1 + \delta) = \exp\left(\sum_{i=1}^{n} g_i X_i\right) x_1,$$

with constants $g_i = g_i(t_1 + \delta) \in \mathbb{R}$.

Using now that g_i only depend on the r.h.s of (6.2) and hence on the controls $u_i(t)$ for given A and B_i, we obtain by the Mean Value Theorem that there are constant controls $u_i \in \mathbb{R}$ fulfilling the condition of driving the pair (t_1, x_1) into $(t_1 + \delta, x(t_1 + \delta))$. Hence, we can restrict ourselves to the discussion of piecewise constant control functions.

Therefore, it is possible to represent the group elements of $\mathcal{R}(e)$ with the help of the exponential mapping on G as follows:

$$\mathcal{R}(e) = \left\{ x \in G \middle| \exists t_j > 0, u^{(j)} \in \mathbb{R}^m (j = 1, \ldots, h) \text{ with } x = \prod_{j=1}^{h} \exp\left(t_j \left[A + \sum_{i=1}^{m} u_i^{(j)} B\right]\right) \right\}$$

Thus, we can state another well-known necessary condition for the controllability of a right-invariant control system, the so-called *controllability rank condition*:

Theorem 6.5 (controllability rank condition – see for instance [Son90] or [Sac99]) *A necessary condition for a right-invariant control system σ defined on the Lie group G to be controllable is that the smallest Lie algebra containing the vectors*

$$A, B_1, B_2, \ldots, B_m \in L(G)$$

coincides with the Lie algebra $L(G)$.

Proof. Clearly, if G is controllable, then $\mathcal{R}(e) = \mathcal{O}(e) = G$ holds. Hence, $L(\sigma) := L(\mathcal{O}(e)) = L(G)$ follows. □

In general, the controllability rank condition is not sufficient for the controllability of σ. Nevertheless, it is necessary and sufficient to the accessibility of σ at any point $x \in G$. Using the results obtained above, we can prove the following necessary and sufficient condition for the controllability:

Theorem 6.6 (see for instance Theorem 2.5 in [Sac99]) *Let σ be a right-invariant control system defined of the Lie Group G. Then σ is controllable, if and only if the following conditions hold:*

(i) G is connected.

(ii) $\mathcal{R}(e)$ is a subgroup of G.

(iii) The controllability rank condition holds.

Proof. The necessity is clear from Theorem 6.5, we only prove the sufficiency of the above conditions.

Suppose that $\mathcal{R}(e)$ is a subgroup of G. Then for any point $x \in \mathcal{R}(e)$ holds that both x and x^{-1} are reachable from the identity. Using Remark 6.6 (iii), we conclude that $\mathcal{R}(x) = \mathcal{R}(e)x$ and hence $e \in \mathcal{R}(x)$. With other words, if x is reachable from e, then e is reachable from x. Thus, it is $\mathcal{O}(e) = \mathcal{R}(e)$, but since $L(\mathcal{O}(e)) = L(\sigma) = L(G)$ holds, the equality $\mathcal{O}(e) = \mathcal{R}(e) = G$ follows. □

After discussing general right-invariant control systems, we now turn our attention to drift-free system. These systems are very important from the applications point of view and therefore, we include special cases of the above conditions:

Corollary 6.2 *Let σ be a drift-free right-invariant control system. Then σ is controllable if and only if G is connected and the controllability rank condition holds.*

Proof. Using that σ is drift-free, we conclude that if y is reachable from x via the control u, then x is reachable from y via $\tilde{u} = -u$. Hence similarly to the proof of Theorem 6.6 $\mathcal{O}(e) = \mathcal{R}(e)$ holds. If then, furthermore, $L(\sigma) = L(G)$ holds, then the controllability, i.e $\mathcal{R}(e) = G$ becomes obvious. □

Recall now our introductory example with the unicycle. We obtained there a drift-free control system of the form

$$\dot{X} = (u_1(t)B_1 + u_2(t)B_2)X(t), \tag{6.8}$$

with $B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It was easy to show to controllability without any further results. Nevertheless, by calculating the Lie bracket $[B_1, B_2] = B_1B_2 - B_2B_1$, it turns out to be $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Hence, the matrices $B_1, B_2, [B_1, B_2]$ are three independent elements of the (three-dimensional) Lie algebra $se(2)$. Thus, the smallest Lie algebra containing B_1 and B_2 is the Lie algebra $se(2)$ itself, which means that the above system fulfills the controllability rank condition. Furthermore, the system (6.8) is clearly drift-free and it can also be shown that $SE(2)$ is connected. Altogether, we obtain by Corollary 6.2, that the unicycle is controllable.

Finally, we discuss stabilizability matters. Although right-invariant control systems are very powerful for describing controllable mechanical systems, for stabilizability questions they are not suitable. The following short explanation illustrates this situation.

In the sequel, we always assume that G is a connected Lie group and σ is a right-invariant control system on G. Again, the property of G to be connected is clearly necessary for the stabilizability (since the state trajectory is a smooth curve in G) and hence is always assumed. For this investigation, we first have to find the equilibrium points.

Suppose now that (x^*, u^*) is an equilibrium pair of σ. Then, it is

$$\dot{x}^* = Ax^* + \sum_{i=1}^{m} u_i^* B_i x^* = \left(A + \sum_{i=1}^{m} u_i^* B_i\right) x^* = 0,$$

and hence

$$\left(A + \sum_{i=1}^{m} u_i^* B_i\right) x^* x^{*-1} = \left(A + \sum_{i=1}^{m} u_i^* B_i\right) = 0$$

follows. This means that if (x^*, u^*) is an equilibrium pair, then (e, u^*), and consequently, (x, u^*) for any $x \in G$ are equilibrium pairs. Thus, we can restrict ourselves to the stabilizability of the identity element.

Furthermore, the equation

$$\left(A + \sum_{i=1}^{m} u_i^* B_i \right) = 0$$

induces, that a necessary and sufficient condition for the existence of an equilibrium is that the vector A lies in the vector space spanned by the vectors B_i for $i = 1, \dots, m$. In the sequel, we shall always assume that this condition is fulfilled, i.e. that the identity element is an equilibrium of the system σ. Hence, we can define the stabilizability of the control system σ:

Definition 6.25 (stabilizability) *Let σ be a right-invariant control system on a connected Lie group G. Assume further, that the identity element of G is an equilibrium state of σ. We say that σ is* stabilizable, *if the identity element $e \in G$ is stabilizable.*

In the sequel, we shall also assume that σ is not controllable (otherwise, as stated in Remark 1.3 the stabilizability becomes trivial) and hence, according to Theorem 6.6, $\mathcal{R}(e) \neq \mathcal{O}(e)$ or $L(\sigma) \neq L(G)$ holds.

Let us now first discuss the situation $\mathcal{R}(e) \neq \mathcal{O}(e)$. Then, there is an element $x_0 \in G$ with $x_0 \notin \mathcal{R}(e)$ and hence $e \notin \mathcal{R}(e)x_0^{-1} = \mathcal{R}(x_0^{-1})$. This means, that the identity is not reachable from the point x_0^{-1}. Moreover, since $\mathcal{R}(e)$ is a Lie semigroup, the element $x_0 \in G$ can also be chosen such that there is a neighborhood U of x_0 with $U \not\subset \mathcal{R}(e)$ and hence there is a neighborhood V of e such that $V \not\subset \mathcal{R}(x_0^{-1})$. Hence, the identity cannot be reached (even asymptotically) from x_0^{-1}.

Thus, we obtain the following lemma:

Lemma 6.1 *Suppose that σ is a right-invariant control system with the property, that e is an equilibrium of the system. Then, a necessary condition for σ to be stabilizable is that $\mathcal{R}(e) = \mathcal{O}(e)$ holds.*

With other words, every stabilizable right invariant control system is reversible. We now use the fact that the reachable set $\mathcal{R}(e)$ is connected and is – according to Lemma 6.1 and Remark 6.6 – a Lie subgroup of G. Suppose now that the closure of $\mathcal{R}(e)$ $\overline{\mathcal{R}(e)} \neq G$. (Clearly, $\overline{\mathcal{R}(e)} = G$ would mean $\mathcal{R}(e) = G$ and hence σ would be controllable.) Then there exists a point $g \in G$ and a neighborhood U of e such that

$$Ug^{-1} \cap \mathcal{R}(e) = \emptyset \tag{6.9}$$

holds. For the stabilizability of e it is required, that there exists a control, such that the point g can be controlled in any neighborhood of e. With other words, if σ is stabilizable, then $\mathcal{R}(e)g \cap U \neq \emptyset$ holds, which contradicts with (6.9). Altogether we obtain

Theorem 6.7 *Let σ be a stabilizable right-invariant control system defined on a connected Lie group G. Then, σ is controllable.*

We finish our investigation of nonlinear controllability and stabilizability with a local theorem for the stabilizability of nonlinear control systems:

Theorem 6.8 *Suppose that σ is a control system defined on a smooth n-dimensional manifold M of the form*

$$\dot{x} = f(x,u) \tag{6.10}$$

where $f : M \times \mathbb{R}^m \to T(M)$ denotes a smooth vector field with the property $f(x,u) \in T_x(M)$ for any $u \in \mathbb{R}^m$. Suppose that the pair $(x^*, u^*) \in M \times \mathbb{R}^m$ is an equilibrium pair with the property that the corresponding linear control system

$$\dot{\xi} = A\xi + B\nu$$

is stabilizable. Hereby denote A and B the Jacobi-matrices

$$A = \left.\frac{\partial f}{\partial x}\right|_{x^*,u^*}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{x^*,u^*}$$

corresponding to a chart (U, χ) with $x^* \in U$. Then, (x^*, u^*) is locally stabilizable, i.e. there exists a neighborhood $\tilde{U}$ of x^* such that for any $x \in U$ the system can be asymptotically controlled to x^*.

Proof. Using the mapping χ, the control system can be transformed locally into $\mathbb{R}^n$. Hence, it is sufficient to discuss the transformed system

$$\frac{d}{dt}\tilde{x} = \tilde{f}(\tilde{x}, u),$$

with $\tilde{x} = \chi(x) \in \mathbb{R}^n$ and $\tilde{f} = \chi \circ f$.

Suppose now, that there exists a matrix F such that the closed-loop system

$$\dot{\xi} = (A - BF)\xi \tag{6.11}$$

is asymptotically stable. Using the control $u = u^* - F(\tilde{x} - \tilde{x}^*)$ for the nonlinear system, we obtain the nonlinear differential equation

$$\frac{d}{dt}\tilde{x} = \tilde{f}(\tilde{x}, u^* - F\tilde{x} + F\tilde{x}^*).$$

Its linearized equation around $x = x^*$ equals to

$$\frac{d}{dt}\tilde{x} = \left.\frac{\partial \tilde{f}}{\partial x}\right|_{x=\tilde{x}^*}(x - x^*) - \left.\frac{\partial \tilde{f}}{\partial u}\right|_{u=u^*-F\tilde{x}^*+F\tilde{x}^*} F(\tilde{x} - \tilde{x}^*) = (A - BF)(\tilde{x} - \tilde{x}^*),$$

which is, using $\xi = \tilde{x} - \tilde{x}^*$, equivalent to (6.11) and hence the nonlinear system can be stabilized with the above defined control function u. □

6.3 Notes and references

Besides presenting some very important properties of Riemannian manifolds and Lie groups, in this previous chapter, we summarized the main results for the controllability and stabilizability of (affine) control systems on Riemannian manifolds and Lie groups.

There are several important questions, that we didn't mention here. For instance, we didn't discuss how to choose the control variables or how to transform the control space to obtain desired properties. These questions are answered in a numerous work of Fliess (for example [Fli88] and [FLPR95]).

Questions on the controllability conditions of right-invariant control systems and control systems acting on smooth manifolds are handled in a recent survey of Sachkov [Sac99], whereas controllability and other very interesting question of (affine) control systems are discussed in the text of Nijmeijer and van der Schaft [NvdS90]. Further Lyapunov-type methods on the stabilizability of nonlinear systems are discussed in the book of Sontag [Son90].

For those, interested in geometric aspects from the point of view of application possibilities of control systems on Lie groups, the works of Lafferriere and Sussmann [LS90], Silva-Leite [Lei97], Respondek [Res96], Sørdalen [Sør93] and the text of Jurdjevic [Jur97] could be recommended.

Chapter 7

Splines and optimal control theory

In the previous chapter we discussed, how general control systems are defined for Riemannian manifolds and especially for Lie groups. We also showed conditions to decide whether a control system is controllable or not. Nevertheless, those proofs were non-constructive and hence give no help for finding the actual trajectory.

On the other hand, applications in Computer Aided Design, air traffic control, motion planning for vehicles and robots require *optimal* smooth trajectories of control systems between reachable points, that usually also fulfill higher order boundary value problems. This gives rise to the so-called dynamic interpolation problem, where, besides some interpolation properties, the actual path must also be a solution of an optimal control problem. Here, we discuss the construction of optimal trajectories under boundary constraints.

To begin with, in Section 7.1, we give an introduction into the variational calculus on Lie groups. Then, we review its direct generalization for constructing spline-curves. Finally, in Sections 7.3 and 7.4 we discuss algorithmical and computational matters.

7.1 The geodesic variational problem on Lie groups

Our aim in this section is to find solutions of the following problem:

Problem 4 *Let G be a Lie group. Find a curve $\gamma : [0,1] \to G$ such that the following functional*

$$J(\gamma) = \int_0^1 \langle \dot{\gamma}, \dot{\gamma} \rangle \; dt \tag{7.1}$$

is (locally) extremal over the set of all smooth functions fulfilling the boundary value problem

$$\gamma(0) = x_0, \quad \gamma(1) = x_1 \tag{7.2}$$

for given points $x_0, x_1 \in G$.

For the solution of this problem, we use a technique, well known in the variational calculus (see for instance [Mil69]): We examine, how the value of J changes if we go from γ to another (nearby) curve $\tilde{\gamma}$. This difference will be called the first variation of J. As we shall see later, if γ is locally extremal, then the first variation of J vanishes.

In order to put this technique into a rigorous mathematical framework, we need some preliminary definitions:

Definition 7.1 (admissible curve) *Let $\gamma : [0,1] \to G$ be a curve satisfying the boundary conditions (7.2). Then γ is called* admissible.

Definition 7.2 (admissible variation) *Suppose that $W : [0,1] \to \mathcal{T}(G)$ is a smooth vector field along an admissible curve γ satisfying the boundary conditions $W(0) = 0$ and $W(1) = 0$. Then the curve $\gamma_\varepsilon(t) := \exp_{x(t)}(\varepsilon W(t))$ is called* admissible variation of γ. *Hereby denotes the symbol $\exp_x(V)$ the exponential mapping with origin x and velocity $V \in T_x(G)$, i.e.*

$$\exp_x(V) := \exp(\log(Vx^{-1}))x.$$

Using that γ_ε is an admissible variation of the admissible curve γ, we obtain the following equalities:

$$\gamma_0(t) = \gamma(t) \tag{7.3}$$

$$\gamma_\varepsilon(i) = \gamma(i) \quad i = 0, 1 \tag{7.4}$$

$$\left.\frac{\partial}{\partial \varepsilon}\gamma_\varepsilon(t)\right|_{\varepsilon=0} = W(t) \tag{7.5}$$

$$\dot{\gamma}_\varepsilon(t)|_{\varepsilon=0} := \left.\frac{\partial}{\partial t}\gamma_\varepsilon(t)\right|_{\varepsilon=0} = \dot{\gamma}(t) \tag{7.6}$$

Hence, γ_ε is also an admissible curve.

Suppose now that γ is a critical curve for J. This means, that the function $i(\varepsilon) := J(\gamma_\varepsilon)$ is extremal for $\varepsilon = 0$. Hence, using the commutativity of the differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varepsilon}$, we obtain that

$$\frac{di}{d\varepsilon} = 2\int_0^1 \left\langle \frac{D}{\partial \varepsilon}\dot{\gamma}_\varepsilon, \dot{\gamma}_\varepsilon \right\rangle dt = 2\int_0^1 \left\langle \frac{D}{\partial t}\frac{\partial}{\partial \varepsilon}\gamma_\varepsilon, \dot{\gamma}_\varepsilon \right\rangle dt$$

Integrating by parts yields

$$\int_0^1 \left\langle \frac{D}{\partial t}\frac{\partial}{\partial \varepsilon}\gamma_\varepsilon, \dot{\gamma}_\varepsilon \right\rangle dt = \left.\left\langle \frac{\partial}{\partial \varepsilon}\gamma_\varepsilon, \dot{\gamma}_\varepsilon \right\rangle\right|_0^1 - \int_0^1 \left\langle \frac{\partial}{\partial \varepsilon}\gamma_\varepsilon, \frac{D}{\partial t}\dot{\gamma}_\varepsilon \right\rangle dt.$$

Since, according to our assumption, $\gamma = \gamma_0$ is extremal, the function $i(\varepsilon)$ has an extremum at $\varepsilon = 0$. Hence,

$$2\left[\left.\left\langle \frac{\partial}{\partial \varepsilon}\gamma_\varepsilon, \dot{\gamma}_\varepsilon \right\rangle\right|_0^1 - \int_0^1 \left\langle \frac{\partial}{\partial \varepsilon}\gamma_\varepsilon, \frac{D}{\partial t}\dot{\gamma}_\varepsilon \right\rangle dt\right]_{\varepsilon=0} = 0$$

holds. Comparing the above equality with (7.5) and (7.6) and using $W(0) = W(1) = 0$, we conclude that if γ is extremal, then for any admissible vector field $W(t)$ along γ

$$\int_0^1 \left\langle W(t), \frac{D}{\partial t}\dot{\gamma} \right\rangle dt = 0$$

holds. Choosing now the admissible variation $W(t) = f(t)\frac{D}{\partial t}\dot{\gamma}$, where $f : [0,1] \to \mathbb{R}$ is a smooth function with $f(0) = f(1) = 0$ and $f(t) > 0$ for $t \in (0,1)$, we obtain that if γ is extremal, then it fulfills

$$\int_0^1 f(t) \left\langle \frac{D}{\partial t}\dot{\gamma}, \frac{D}{\partial t}\dot{\gamma} \right\rangle dt = 0.$$

Using now the positivity of f, the equation $\frac{D}{\partial t}\dot{\gamma} = 0$ follows.

Altogether, we obtained the following theorem:

Theorem 7.1 *Suppose that γ is an extremal curve of the functional*

$$J(\gamma) = \int_0^1 \langle \dot{\gamma}, \dot{\gamma} \rangle \, dt$$

subject to the boundary constraints

$$\gamma(0) = x_0, \quad \gamma(1) = x_1$$

for given points $x_0, x_1 \in G$. Then, γ is a geodesic arc joining x_0 with x_1.

7.2 Spline curves and second order variational problems

First, we present the following optimal control problem defined on the Lie group G proposed by Crouch and Silva-Leite in [CL91]:

Problem 5 *Let G be an n-dimensional Lie group and $(X_1, X_2, \ldots, X_n)$ be an orthonormal basis on the vector space $L(G)$. We now define the following control system on $G \times \mathbb{R}^n$:*

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{n} v_i X_i x, \\ \dot{v}(t) &= u, \end{aligned} \tag{7.7}$$

for $x \in G$, $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$. Our task now is to find a smooth control function $u : [0,1] \to \mathbb{R}^n$ such that the cost functional

$$J(u) = \int_0^1 \|u(t)\|^2 \, dt \tag{7.8}$$

is minimal subject to the boundary value problem

$$\begin{aligned} x(0) = x_0 \quad x(1) = x_1 \\ v(0) = v_0 \quad v(1) = v_1 \end{aligned} \tag{7.9}$$

for given $x_0, x_1 \in G$ and $v_0, v_1 \in \mathbb{R}^n$.

Definition 7.3 (spline curve) *A smooth solution of Problem 5 is usually called* (cubic) spline.

We first show, how the above optimal control problem is related to a variational problem on G:

Theorem 7.2 *Let G be a Lie group. Assume further that $< .,. >$ denotes a Riemannian metric and ∇ the affine connection compatible with this metric. Suppose further that the curve $x : [0,1] \to G$ is twice continuously differentiable. Then, x is a solution to Problem 5 if and only if it minimizes the functional*

$$J := \int_0^1 \left\langle \frac{D^2}{dt^2} x(t), \frac{D^2}{dt^2} x(t) \right\rangle \, dt \tag{7.10}$$

over all smooth curves fulfilling the boundary-value problem

$$\begin{aligned} x(0) &= x_0 \quad x(1) = x_1 \\ \dot{x}(0) &= \xi_0 \quad \dot{x}(1) = \xi_1, \end{aligned}$$

for $\xi_0 = \sum_{i=1}^{n} (v_0)_i X_i x_0 \in T_{x_0}(G)$ and $\xi_1 = \sum_{i=1}^{n} (v_1)_i X_i x_1 \in T_{x_1}(G)$.

Hereby denotes $\frac{D^2}{dt^2}$ the covariant derivative of the vector field $\dot{x}$ along the curve x, i.e $\frac{D^2}{dt^2} x = \frac{D_x}{dt} \dot{x}$.

Proof. Note that, with (7.7), the velocity vector field of x is given by $\dot{x} = \sum_{i=1}^{n} v_i X_i x$ and hence the velocity is simply $v(t) = \sum_{i=1}^{n} v_i X_i$. Therefore, using that X_i is a basis of the Lie algebra, it is with the same basis $\frac{D}{dt}\dot{x} = \sum_{i=1}^{n} \dot{v}_i X_i x = \sum_{i=1}^{n} u_i X_i x$. Finally, the orthonormality of the chosen basis completes the proof. □

This latter theorem shows also the importance of splines. Spline curves are second-order generalizations of geodesic arcs (see Section 7.1) joining two points. This means that spline curves have the property, that the tangential component of the acceleration is minimal over the whole control interval. Hence, we can state the dynamic interpolation problem for general Riemannian manifolds, too:

Problem 6 *Let M be an n-dimensional Riemannian manifold equipped with a Riemannian metric $< .,. >$ and the corresponding affine connection ∇. Solve the following variational problem*

$$\min_{x \in C^2[0,1]} \int_0^1 \left\langle \frac{D^2}{dt^2} x(t), \frac{D^2}{dt^2} x(t) \right\rangle \, dt, \tag{7.11}$$

subject to the boundary constraints

$$\begin{aligned} x(0) &= x_0 \quad x(1) = x_1 \\ \dot{x}(0) &= \xi_0 \quad \dot{x}(1) = \xi_1, \end{aligned} \tag{7.12}$$

for given $x_0, x_1 \in M$, $\xi_0 \in T_{x_0}(M)$ and $\xi_1 \in T_{x_1}(M)$.

Now, we review how solutions to Problems 5 and 6 can be obtained.

We can now use a similar technique as in Section 7.1 – together with the definition of the curvature tensor – to obtain a necessary condition for the solutions of Problem 6. For more details concerning the proof, please refer to [CL91].

Theorem 7.3 (see Theorem 1 in [CL91]) *Suppose that $\gamma : [0,1] \to M$ is a solution of Problem 6. Then it fulfills the following differential equation:*

$$\frac{D^3\dot{\gamma}}{dt^3} + R\left(\frac{D\dot{\gamma}}{dt}, \dot{\gamma}\right)\dot{\gamma} = 0. \tag{7.13}$$

In order to rewrite the above condition for Lie groups, one needs the representation of the curvature tensor obtained in [Mil69]:

Corollary 7.1 (see Lemma 2 in [CL95]) *Suppose that G is a connected and compact Lie group. Then if γ is a solution of Problem 5 (or equivalently of Problem 6), then its velocity $v(t) := \log\left(\dot{\gamma}(t)\gamma^{-1}(t)\right)$ fulfills the following equation:*

$$\dddot{v}(t) + [v(t), \ddot{v}(t)] = 0.$$

7.3 The De Casteljau algorithm

Because of their nice property of minimizing the integral of the square of acceleration, spline curves are often used for interpolation in engineering applications, such as robotics, mechanics or fluid dynamics. However, those fields require fast algorithms to generate curves, which is usually hardly achievable by the exact solution of variational problems.

In this section, we first take a look at cubic splines on Euclidean spaces, where the curvature tensor is identically zero. On those essentially flat surfaces, one can obtain a very elegant (and fast) algorithm to solve the following generalized control problem (see also [CLK96b]):

Problem 7 *Given the vectors $x_0^{(k)}, x_1^{(k)}$ in $\mathbb{R}^n$ for $k = 0, 1, \ldots, m-1$, find a smooth control function $u : [0,1] \to \mathbb{R}^n$ such that under the constraints*

$$\begin{aligned} \dot{x}(t) &= v, \\ \tfrac{d^{m-1}}{dt^{m-1}}v(t) &= u, \end{aligned} \tag{7.14}$$

for $x : [0,1] \to \mathbb{R}^n$ and $v(t) \in \mathbb{R}^n$ the cost functional

$$J(u) = \int_0^1 \|u(t)\|^2 \, dt \tag{7.15}$$

is minimal subject to the boundary value problem

$$\frac{d^k}{dt^k}x(i) = x_i^{(k)}, \qquad i = 0, 1,\ k = 0, \ldots, m-1, \tag{7.16}$$

with $\frac{d^0}{dt^0}x(t) = x(t)$.

Similarly to Theorem 7.2, we can rewrite the above optimal control problem into a variational problem:

Problem 8 *Solve the following variational problem on* $\mathbb{R}^n$*:*

$$\min_{x \in C^m[0,1]} \int_0^1 \left\langle \frac{d^m}{dt^m} x(t), \frac{d^m}{dt^m} x(t) \right\rangle dt, \tag{7.17}$$

subject to the boundary value problem

$$\frac{d^k}{dt^k} x(i) = x_i^{(k)}, \qquad i = 0, 1,\ k = 0, \ldots, m-1, \tag{7.18}$$

with $\frac{d^0}{dt^0} x(t) = x(t)$*, for given the vectors* $x_0^{(k)}, x_1^{(k)} \in \mathbb{R}^n$ *with* $k = 0, 1, \ldots, m-1$.

First, we review the classical method proposed by De Casteljau [De 59] in 1959: Given a set of distinct points $(x_0, x_1, \ldots, x_k) \subset \mathbb{R}^n$, we successively define the following curve: Assuming that the straight line segment between the points x and y is denoted by $\ell_1(t, x, y) = (1-t)x + ty$, we set for $p > 1$

$$\ell_p(t, x_i, x_{i+1}, \ldots, x_{i+p}) := (1-t)\ell_{p-1}(t, x_i, \ldots, x_{i+p-1}) + t\ell_{p-1}(t, x_{i+1}, \ldots, x_{i+p}),$$

with $i = 0, \ldots, k-p$.

Definition 7.4 (De Casteljau curve) *Given* $k+1$ *distinct points* $(x_0, x_1, \ldots, x_k) \subset \mathbb{R}^n$*, we call the curve* $\gamma(t) = \ell_k(t, x_0, x_1, \ldots, x_k)$ k^{th} *order* De Casteljau curve *(sometimes also Bézier-spline).*

It can be easily shown, that the above defined curves are exactly the solutions of the higher order dynamic interpolation problem on $\mathbb{R}^n$.

Lemma 7.1 (see Section 2 in [CLK99a]) *The solution to the* m^{th} *order dynamic interpolation problem is a De Casteljau curve of order* $2m-1$.

Proof.

1. Let us first review the generalized Euler-Lagrange equation (see for instance [Fun70] or [Arn78]), associated to the variational problem (7.17):

$$2(-1)^m \frac{d^m}{dt^m} \frac{d^m}{dt^m} x = 0,$$

   or equivalently, $\frac{d^{2m}}{dt^{2m}} x = 0$. Hence, a necessary condition for the solution of Problem 8 is that it is a polynomial of degree at most $2m-1$ in t.

2. Consider the De Casteljau curves: it is $\ell_1(t,x,y)=(1-t)x+ty$, hence ℓ_1 is a polynomial of degree 1. Hence, by induction on i we conclude, using

$$\ell_p = t\ell_{p-1}(t,\ldots)+(1-t)\ell_{p-1}(t,\ldots),$$

that ℓ_p is a polynomial curve of degree at most p. Moreover, if we denote

$$\ell_p(t,x_0,\ldots,x_p)=a_pt^p+a_{p-1}t^{p-1}+\ldots+a_1t+a_0,$$

with some vectors $a_0,a_1,\ldots,a_p\in\mathbb{R}^n$, then

$$a_i=\binom{p}{i}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}x_j \tag{7.19}$$

holds. To prove this, just note that $\frac{d^i}{dt^i}x(0)=i!a_i$. On the other hand, using

$$\ell_p = t\ell_{p-1}(t,\ldots)+(1-t)\ell_{p-1}(t,\ldots)$$

again, we conclude that

$$\begin{aligned}\frac{d^i}{dt^i}\ell_p(t,x_0,\ldots,x_p) &= \sum_{j=1}^{i}\binom{i}{j}\frac{d^j}{dt^j}t\frac{d^{i-j}}{dt^{i-j}}\ell_{p-1}(t,x_0,\ldots,x_{p-1})\\ &\quad+\sum_{j=1}^{i}\binom{i}{j}\frac{d^j}{dt^j}(1-t)\frac{d^{i-j}}{dt^{i-j}}\ell_{p-1}(t,x_1,\ldots,x_p).\end{aligned}$$

Thus, induction on p leads to the desired equality (7.19).

3. Hence, every polynomial curve of degree p can be obtained as a De Casteljau curve of the same degree. With other words, the set of polynomial curves of degree no more then p is equivalent to the set of De Casteljau curves of order p. Thus, any solution to Problem 8 is a generalized De Casteljau curve of order $2m-1$.

□

Our next task is now to generalize the above construction for Riemannian manifolds and Lie groups. For that we need a manifold with the property, that every two distinct points of it can be joined by a geodesic arc. Such manifolds are usually called *geodesically complete.* Following the latter case on $\mathbb{R}^n$, the general construction of the De Casteljau algorithm on a geodesically complete Riemannian manifold is obvious: Given a set $(x_0,\ldots,x_k)$ of distinct points on a Riemannian manifold M, a smooth curve $\gamma(t)=g_k(t)$ on M, joining x_0 (at $t=0$) with x_k (at $t=1$) can be obtained with the help of the following successive geodesic interpolation:

Let $g_1(t,x_i,x_{i+1})$ be a geodesic curve joining x_i (at $t=0$) with x_{i+1} (at $t=1$), then we set

$$g_p(t,x_i,\ldots,x_{i+p})=g_1\left(t,g_{k-1}(t,x_i,\ldots,x_{i+k-1}),g_{k-1}(t,x_{i+1},\ldots,x_{i+k})\right),$$

for $p=1,\ldots,n$ and $i=0,\ldots,n-p$. We refer to g_p also as a *generalized polynomial curve* of degree p on M.

Now, we can give a general definition to De Casteljau curves on M:

Definition 7.5 (De Casteljau curve) *Suppose that M is a geodesically complete Riemannian manifolds. Given $k+1$ distinct points $(x_0, x_1, \ldots, x_k) \subset M$, we call the curve $\gamma(t) = g_k(t, x_0, x_1, \ldots, x_k)$ k^{th} order* De Casteljau curve *on M.*

Remark 7.1 (see Theorem 6.3) *If G is a connected and compact (and hence geodesically complete) Lie group, then the geodesic curve $g_1(t, x, y)$ can be calculated as*

$$g_1(t, x, y) = \exp(\log(yx^{-1})t)x,$$

and hence, the only difficulty to obtain De Casteljau curves on such Lie groups arise from the computation of exp *and* log*. Nevertheless, as it will be shown in Section 7.4, these difficulties are far from being trivial.*

Proof. Let V denote the element of the Lie algebra $L(G)$ with $\exp(V) = yx^{-1} \in G$. Then, the curve $g(t) = \exp(Vt)x$ is indeed a geodesic arc, that fulfills the boundary value problem $g(0) = x$ and $g(1) = y$. Hence, using Theorem 6.3, g is a geodesic arc between x and y. □

By this last remark, the construction of a (cubic) De Casteljau curve $\gamma(t)$ on a connected and compact Lie group becomes obvious:

1. Given $x_0, x_1, x_2, x_3 \in G$, calculate $V_{i-1,i} := \log\left(x_i x_{i-1}^{-1}\right) \in L(G)$ for $i = 1, 2, 3$.
2. For a given $t \in [0, 1]$ obtain the intermediate points $x_{i-1,i} := \exp(V_{i-1,i}t)x_{i-1}$ for $i = 1, 2, 3$.
3. Calculate the Lie algebra elements $V_{i-1,i+1} := \log\left(x_{i,i+1} x_{i-1,i}^{-1}\right) \in L(G)$ for $i = 1, 2$.
4. Determine the intermediate points $x_{i-1,i+1} := \exp(V_{i-1,i+1}t)x_{i-1,i}$ for $i = 1, 2$.
5. Obtain $V_{0,3} \log\left(x_{1,3} x_{0,2}^{-1}\right) \in L(G)$.
6. Set $\gamma(t) := \exp(V_{0,3}t)x_{0,2}$

To show, how De Casteljau curves are related to splines on non-Euclidean spaces, we cite from [CLK99a] the following result.

Theorem 7.4 (see Theorem 5.1 in [CLK99a]) *If G is connected and compact Abelian Lie group, the polynomial curves of degree $2m-1$ generated by the De Casteljau algorithm are also solutions of the Euler-Lagrange equation associated to the functional*

$$\int_0^1 \left\langle \frac{D^m}{dt^m} x(t), \frac{D^m}{dt^m} x(t) \right\rangle dt,$$

where $\frac{D^m}{dt^m} x(t)$ denotes the $(m-1)^{st}$ covariant derivative of the velocity vector field, i.e. $\frac{D^m}{dt^m} x = \frac{D^{m-1}}{dt^{m-1}} \dot{x}$.

If G is Abelian, the curvature tensor $R(X, Y)$ is identically zero and hence the surface is very close to a flat (Euclidean) space (although the curvature itself must not necessarily be constant as for an n-dimensional torus). Unfortunately, the latter theorem cannot be proved for surfaces with nonzero curvature tensor. The next example illustrate a numerical comparison between the solution of the variational problem and the curve generated by the De Casteljau algorithm on the surface of the unit sphere in $\mathbb{R}^3$. For more details concerning the solution method, please refer to [CLK98].

Example 12: Consider the 2-dimensional unit sphere $\mathcal{S}^2 \subset \mathbb{R}^3$ equipped with the metric induced by the Euclidean metric on $\mathbb{R}^3$. Our aim is to obtain a solution of the second-order variational problem (Problem 6) and to compare it with the corresponding De Casteljau curve.

To this end, we first require the covariant derivative along x, which is equivalent to the differentiation in $\mathbb{R}^3$ followed by an orthogonal projection to the surface, denoted by Π. This means, that

$$\frac{D_x \dot{x}}{dt} = \Pi\left(\frac{d\dot{x}}{dt}\right) = \Pi(\ddot{x}) = \ddot{x} - < x, \ddot{x} > x.$$

Furthermore, using the fact that $< x, x >= 1$ and hence $< \dot{x}, x >= 0$, we obtain

$$\frac{D^2 \dot{x}}{dt^2} = \Pi\left(\frac{d}{dt}\frac{D\dot{x}}{dt}\right) = \dddot{x} - < x, \ddot{x} > x - < x, \dddot{x} > x$$

and finally

$$\frac{D^3 \dot{x}}{dt^3} = \Pi\left(\frac{d}{dt}\frac{D^2\dot{x}}{dt^2}\right) = \ddddot{x} + < \dot{x}, \dot{x} > \ddot{x} + 5 < \dot{x}, \ddot{x} > \dot{x} + < x, \ddot{x} >^2 x - < x, \ddddot{x} > x.$$

However, since $\mathcal{S}^2$ is a manifold of constant curvature, the curvature tensor can also be obtained easily (see for instance [KN63]):

$$R(\frac{D\dot{x}}{dt}, \dot{x})\dot{x} = < \dot{x}, \dot{x} > \frac{D\dot{x}}{dt} - < \dot{x}, \frac{D\dot{x}}{dt} > \dot{x}.$$

Hence, using Theorem 7.3, a necessary condition for the extremal curve x is given by the equation

$$\ddddot{x} + 2 < \dot{x}, \dot{x} > \ddot{x} + 4 < \dot{x}, \ddot{x} > \dot{x} + 2 < \dot{x}, \dot{x} >^2 x - < x, \ddddot{x} > x = 0. \tag{7.20}$$

Now, we solve this differential equation with the following boundary value problem on $\mathcal{S}^2$

$$x(0) = \begin{pmatrix} -0.87 \\ 0.5 \\ 0 \end{pmatrix}$$

$$\dot{x}(0) = \begin{pmatrix} 0.29 \\ 0.5 \\ 0.64 \end{pmatrix}$$

$$x(1) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\dot{x}(1) = \begin{pmatrix} 0 \\ -0.52 \\ 0.74 \end{pmatrix}$$

The solution curve obtained by a finite element approximation is presented on Figure 7.1. On the other hand, however, if we compute the curve generated by the corresponding De Casteljau algorithm (see Figure 7.2) for the same boundary conditions, we can notice the difference between them. Since no error estimation for the spline curve is available, this numerical comparison cannot be considered as a counter-example for Theorem 7.4 in the general case. Nevertheless, the existence of a counter-example becomes quite obvious.

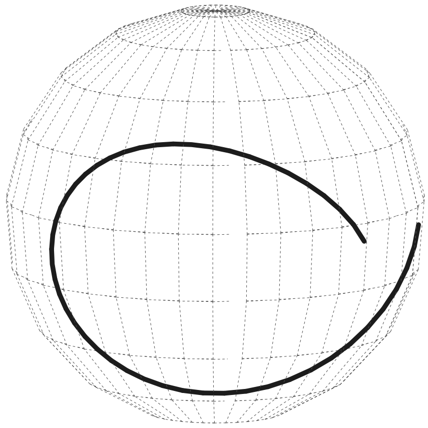

Figure 7.1: Solution of the variational problem on S^2

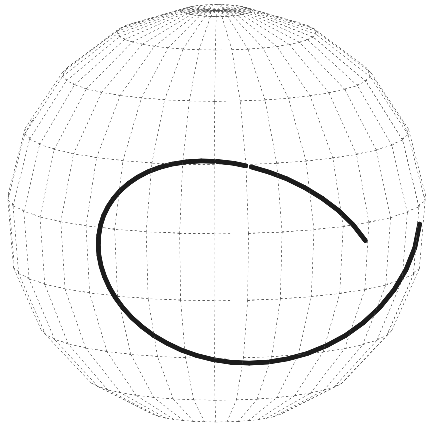

Figure 7.2: De Casteljau curve on S^2

Finally, for the sake of completeness, we cite a result from [CLK96a] to show that cubic De Casteljau curves on Lie groups interpolate between the points x_0 and x_1 in a way, that also the Hermite-type interpolation problem (7.12) is fulfilled.

Theorem 7.5 (see Theorem 2.2 in [CLK96a]) *Let G be a connected and compact Lie group and $x_0, x_1, x_2, x_3 \in G$. Then, the De Casteljau curve γ generated by the above set of points fulfills the boundary conditions*

$$\begin{aligned} \gamma(0) &= x_0, & \gamma(1) &= x_3, \\ \dot{\gamma}(0) &= 3V_{0,1}x_0, & \dot{\gamma}(1) &= 3V_{2,3}x_3. \end{aligned}$$

Hereby denote $V_{i,j} := \log\left(x_j x_i^{-1}\right) \in L(G)$.

7.4 Computational and visualizational matters

This section deals with some computational and visualizational problems that arise by investigating smooth curves on Lie groups or abstract higher dimensional manifolds. Our main goal is to derive an algorithm to obtain De Casteljau curves for the spaces $SO(3)$, $\mathcal{S}^3$ and $SE(3)$.

Besides that we also consider the question how to visualize curves evolving on these higher-dimensional abstract matrix groups.

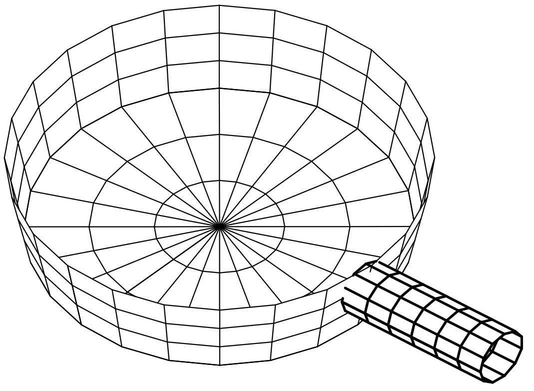

Figure 7.3: Three-dimensional object used for simulation purposes

Since any element $T \in SO(n)$ is an orthogonal matrix, if the transformation $T \in SO(n)$ is applied to each point of a subset $B \in \mathbb{R}^n$, then we get another subset B' that is exactly the rotation of B around the origin. Hence, T can be interpreted as a pure rotation of rigid bodies in $\mathbb{R}^n$. If we then fix an object $B \subset \mathbb{R}^n$ such that it is non-symmetric, we can follow through the rotational movement of B the evolution of the curve $T(t) \in SO(n)$. An example for such an object, which we shall use throughout this section is shown on Figure 7.3.

7.4.1 Smooth interpolation on $SO(n)$

Because of the computational effort for the logarithm, the De Casteljau algorithm is in general difficult to implement. However, if $G = SO(3)$, we can easily compute polynomial curves due to the fact that, that the the mappings exp and log can be explicitely given. Indeed, as it is discussed in [CLK96b], if $S_a \in so(3)$ denotes the skew-symmetric matrix defined by $S_a b = a \times b$, for a and b vectors in $\mathbb{R}^3$ and $\times$ the cross product in $\mathbb{R}^3$, we have

$$\exp(S_a) = I \cos\|a\| + \frac{\sin\|a\|}{\|a\|} S_a + \frac{1-\cos\|a\|}{\|a\|^2} aa^T. \tag{7.21}$$

Similarly, if $x = \exp S \in SO(3)$, then

$$S = \log x = \frac{\alpha}{2\sin\alpha}(x - x^T), \tag{7.22}$$

where $\cos\alpha = \dfrac{trace(x) - 1}{2}$.

Hence, using Remark 7.1, the De Casteljau algorithm on $SO(3)$ can be successively obtain for any order m.

Furthermore, as it was shown in [CLK99b], this manifold is essentially equivalent to the Lie group of unit quaternions and hence also to the Lie group $\mathcal{S}^3 \subset \mathbb{R}^4$. Therefore, there is also another way to calculate to De Casteljau curve generated on $SO(3)$, i.e. to obtain a smooth interpolation of a pure rotation of a three-dimensional rigid body (for the actual algorithm, please refer to [CLK99b]). Figure 7.4 represents such a rotation using the control points:

$$x_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, x_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$x_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

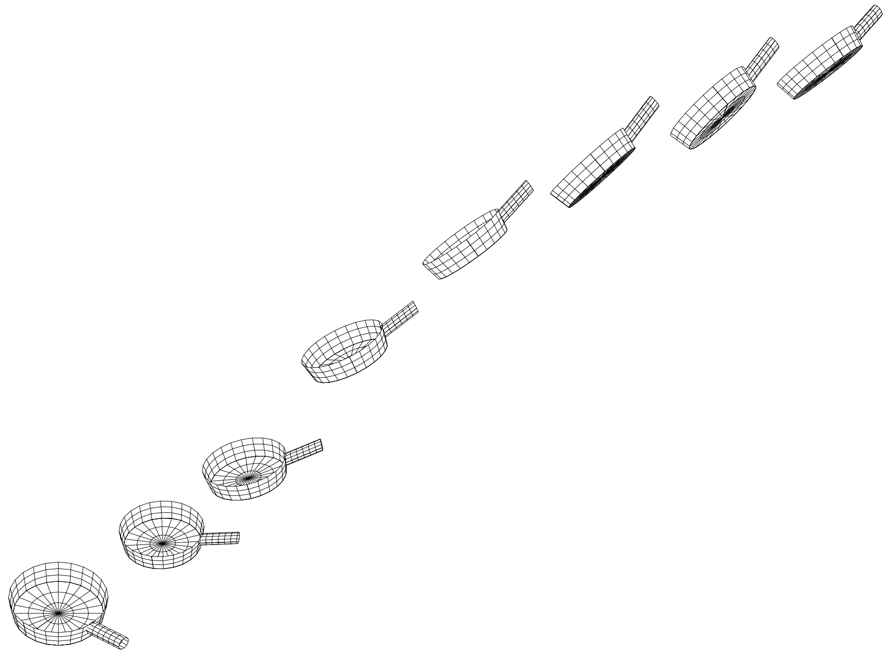

Figure 7.4: Visualization of a curve on $SO(3)$.

As it was described before, the visualization used the fact that elements of the Lie group $SO(3)$ act on a rigid body as a pure rotation. On the picture, the object is the moved in a different position, so that the actual pictures do not overlap each other.

7.4.2 Motion planning for rotation and displacement of rigid bodies

Different from the situation discussed before, the complete movement of a rigid body (i.e. rotation and displacement) is a motion with six degrees of freedom. Hence, we need the six-dimensional Lie group $SE(3)$ to represent this motion. As it was mentioned in Example 10, this Lie group consists of matrices with the structure

$$\begin{pmatrix} R & 0_3 \\ v & 1 \end{pmatrix},$$

where $0_3 \in \mathbb{R}^3$ denotes the null-vector and further $v \in \mathbb{R}^3$ and $R \in SO(3)$ are arbitrary. Moreover, it can be shown (see Examples 1.47 and 1.49 in [Olv93]) that the underlying Lie algebra $se(3)$ of the latter Lie group has the following structure:

$$se(3) = \left\{ \begin{pmatrix} r & 0_3 \\ w & 0 \end{pmatrix} | 0_3 = (0,0,0)^T \in \mathbb{R}^3, w \in \mathbb{R}^3, r \in so(3) \right\}.$$

The De Casteljau algorithm presented in [CLK96b] can be (at least theoretically) applied for every connected Lie group, the only difficulty (as pointed out before) is the calculation of $\log(g)$ and $\exp(x)$ with x and g belonging to the Lie algebra and the Lie group, respectively. Hence, our main task here is to derive explicit formulae for these functions.

Consider the Taylor-series expansion of the function $\exp(x)$ for $x \in \mathbb{R}^{4\times 4}$:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{7.23}$$

Specifically, for $x = \begin{pmatrix} r & 0_3 \\ w & 0 \end{pmatrix} \in se(3)$ we get that

$$x^2 = \begin{pmatrix} r^2 & 0_3 \\ wr & 0 \end{pmatrix},$$

or in general for $i \geq 1$

$$x^i = \begin{pmatrix} r^i & 0_3 \\ wr^{i-1} & 0 \end{pmatrix}. \tag{7.24}$$

Using (7.24) in (7.23) yields for $x = \begin{pmatrix} r & 0_3 \\ w & 0 \end{pmatrix} \in se(3)$

$$\begin{aligned} \exp(x) &= I_4 + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{pmatrix} r^n & 0_3 \\ wr^{n-1} & 0 \end{pmatrix} = I_4 + \begin{pmatrix} \displaystyle\sum_{n=1}^{\infty} \frac{r^n}{n!} & 0_3 \\ \displaystyle w\sum_{n=1}^{\infty} \frac{r^{(n-1)}}{n!} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \displaystyle\sum_{n=0}^{\infty} \frac{r^n}{n!} & 0_3 \\ \displaystyle w\sum_{n=0}^{\infty} \frac{r^n}{(n+1)!} & 1 \end{pmatrix} = \begin{pmatrix} \exp(r) & 0_3 \\ \displaystyle w\sum_{n=0}^{\infty} \frac{r^n}{(n+1)!} & 1 \end{pmatrix}, \end{aligned}$$

where I_4 denotes the identity matrix in $\mathbb{R}^{4\times 4}$ and $\exp(r)$ is calculated on the space $so(3)$ with the formula given in (7.21).

We now consider the term $\displaystyle\sum_{n=0}^{\infty} \frac{r^n}{(n+1)!}$:

Lemma 7.2 *Let $r \in \mathbb{R}^{3\times 3}$. Then*

$$\sum_{n=0}^{\infty} \frac{r^n}{(n+1)!} = \int_{t=0}^{1} \exp(rt)\, dt.$$

And the above sum converges for any matrix $r \in \mathbb{R}^{3\times 3}$. Hence the exponential of any matrix $x \in se(3)$ can be obtained by these formulae.

Proof.

$$\int_{t=1}^{1} \exp(rt)\, dt = \int_{t=0}^{1} \sum_{n=0}^{\infty} \frac{(rt)^n}{n!}\, dt = \sum_{n=0}^{\infty} \frac{r^n \int_{t=0}^{1} t^n\, dt}{n!} = \sum_{n=0}^{\infty} \frac{r^n \frac{1}{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{r^n}{(n+1)!}.$$

It can also be easily shown, that the convergence radii of the power series

$$\sum_{n=0}^{\infty} \frac{r^n}{(n+1)!}$$

and

$$\sum_{n=0}^{\infty} \frac{r^n}{n!}$$

are infinite, which yields the desired convergence. □

Altogether we obtain the following theorem

Theorem 7.6 *Let* $x = \begin{pmatrix} r & 0_3 \\ w & 0 \end{pmatrix} \in se(3)$, *with* r, w *arbitrary elements of* $so(3)$, R^3, *respectively, and* $0_3 \in \mathbb{R}^3$ *denoting the zero vector. Then the matrix* $\exp(x)$ *exists and*

$$\exp(x) = \begin{pmatrix} \exp(r) & 0_3 \\ w \int\limits_{t=0}^{1} \exp(rt)\,dt & 1 \end{pmatrix} \tag{7.25}$$

holds, where $\exp(r)$ *and* $\exp(rt)$ *are taken in the Lie algebra* $so(3)$.

Remark 7.2 *Although*

$$r \sum_{n=0}^{\infty} \frac{r^n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{r^n}{n!} = \exp(r) - I,$$

each element of the Lie algebra $se(3)$ *is singular (because of the skew-symmetric part inherited from the Lie algebra* $so(3)$*) and hence the above integral cannot be solved directly.*

Our next task is to obtain the logarithm of an arbitrary matrix in $SE(3)$. To achieve this goal, we first introduce a local result:

Theorem 7.7 *Let* A *be an arbitrary matrix in* $\mathbb{R}^{3\times 3}$ *such that* $\|A\| \leq \frac{1}{2}$ *holds. Then, the following two sums exist and converge to the matrices* $Y_1, Y_2 \in \mathbb{R}^{3\times 3}$, *respectively:*

$$\sum_{n=0}^{\infty} \frac{A^n}{(n+1)!} \tag{7.26}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(\exp(A) - I)^n}{n+1}. \tag{7.27}$$

Moreover, it is $Y_1 Y_2 = I$.

Proof. The proof is carried out in several steps. We first prove the proposition concerning the convergence of the sums and then concentrate on the identity $Y_1 Y_2 = I$.

1. Suppose that $A \neq 0$. (For $A = 0$ it is $Y_1 = Y_2 = I$ and hence the statement is true.) Denoting $a_1 := \|A\| > 0$ and $a_2 := \|\exp(A) - I\|$, we immediately obtain that $a_2 \leq a_1 e^{a_1} \leq \frac{1}{2} e^{\frac{1}{2}} < 1$ holds. Hence, it is

$$\left\| \sum_{n=0}^{\infty} \frac{A^n}{(n+1)!} \right\| \leq \sum_{n=0}^{\infty} \frac{\|A\|^n}{(n+1)!} = \sum_{n=0}^{\infty} \frac{a_1^n}{(n+1)!} = \frac{e^{a_1} - 1}{a_1}.$$

Moreover, for $A_2 = \exp(A)$ (which exists for any $A \in \mathbb{R}^{3\times 3}$), we similarly conclude that

$$\left\| \sum_{n=0}^{\infty} (-1)^n \frac{(A_2 - I)^n}{n+1} \right\| \leq \sum_{n=0}^{\infty} \frac{\|A_2 - I\|^n}{n+1} = \sum_{n=0}^{\infty} \frac{a_2^n}{n+1} = -\frac{ln(1-a_2)}{a_2},$$

whereas the above sums converge for any matrices A and A_2, with $\|A_2 - I\| = a_2 < 1$. Hence, for any given matrix $A \in \mathbb{R}^{3\times 3}$ such that $\|A\| \leq \frac{1}{2}$ holds, the above sums are absolutely convergent.

2. Let us denote the mappings defined by the power series

$$\sum_{n=0}^{\infty} \frac{A^n}{(n+1)!} \tag{7.28}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(A_2 - I)^n}{n+1}. \tag{7.29}$$

by $Y_1(A)$ and $Y_2(A_2)$, respectively. Thus, the previous consideration yields that they are continuous (in the topology induced by the norm $\|.\|$) in $\mathbb{R}^{3\times 3}$ and in the set

$$\left\{ A_2 \in \mathbb{R}^{3\times 3} | \ \|A_2 - I\| \leq \frac{1}{2} e^{\frac{1}{2}} \right\},$$

respectively.

3. Suppose now that A is regular. Then, it is using Lemma 7.2

$$AY_1(A) = A \sum_{n=0}^{\infty} \frac{A^n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{A^n}{n!} = \exp(A) - I = A_2 - I.$$

Thus $Y_1(A) = A^{-1}(A_2 - I)$. Moreover, if $\lambda \in \mathbb{C}$ is an eigenvalue of A with the corresponding eigenvector v, then it is

$$A_2 v = \left(\sum_{n=0}^{\infty} \frac{A^n}{n!} \right) v = \sum_{n=0}^{\infty} \frac{A^n v}{n!} = \sum_{n=0}^{\infty} \frac{\lambda^n v}{n!} = e^{\lambda} v,$$

and hence v is an eigenvector of A_2 with eigenvalue e^{λ}. Therefore, if A is nonsingular, than for any eigenvalue μ of $A_2 = \exp(A)$ $\mu \neq 1$ holds. (Note, that using the relation $\|A_2 - I\| \leq 1$ and hence $\|A_2\| \leq 2$, it is also $|\lambda| \leq 2$ and therefore the only point with $\exp(\lambda) = 1$ of the (complex) exponential mapping lies at $\lambda = 0$.) With other words, the matrix $A_2 - I$ is nonsingular.

Thus,

$$(A_2 - I) \sum_{n=0}^{\infty} (-1)^n \frac{(A_2 - I)^n}{n+1} = -\sum_{n=1}^{\infty} (-1)^n \frac{(A_2 - I)^n}{n} = \log(A_2) = A,$$

since $\|A_2 - I\| \leq \frac{1}{2} e^{\frac{1}{2}}$ which implies that the power series for $\log(A_2)$ converges. Hence, $Y_2(A_2) = (A_2 - I)^{-1} A$ holds. With other words, if A is regular, then $Y_1(A) Y_2(\exp(A)) = I$.

4. Finally, we show that this statement holds for any matrix A with $\|\exp(A) - I\| \leq \frac{1}{2} e^{\frac{1}{2}}$. To prove this, note that the mapping $\exp(A)$ is defined with the power series $\sum_{n=0}^{\infty} \frac{A^n}{n!}$ and hence is continuous in the topology induced by the norm $\|.\|$. Moreover, using

the continuity of the mapping Y_1 and Y_2 we conclude that the mapping $Y : A \mapsto Y_1(A)Y_2(\exp(A))$ is continuous for all $A \in \mathbb{R}^{3\times 3}$ with $\|\exp(A) - I\| \le \frac{1}{2}e^{\frac{1}{2}}$.

On the other hand, the set

$$\left\{A \in \mathbb{R}^{3\times 3} \mid \|\exp(A) - I\| \le \frac{1}{2}e^{\frac{1}{2}}, \det(A) \neq 0\right\}$$

is dense in the set

$$\left\{A \in \mathbb{R}^{3\times 3} \mid \|\exp(A) - I\| \le \frac{1}{2}e^{\frac{1}{2}}\right\}$$

and hence, using the continuity of Y, $Y(A) = I$ holds for any $A \in \mathbb{R}^{3\times 3}$ with $\|\exp(A) - I\| \le \frac{1}{2}e^{\frac{1}{2}}$.

□

Corollary 7.2 *Let* $X = \begin{pmatrix} R & 0_3 \\ v & 1 \end{pmatrix} \in SE(3)$*, where* $R \in SO(3)$*,* $v \in \mathbb{R}^3$ *are arbitrary and again,* $0_3 \in \mathbb{R}^3$ *denotes the zero vector. If we assume that* $\|X - I\| \le \frac{1}{2}e^{\frac{1}{2}}$*, then the following sum converges to a matrix* $Y \in \mathbb{R}^{3\times 3}$*:*

$$Y := \sum_{n=0}^{\infty} (-1)^n \frac{(R-I)^n}{n+1}. \tag{7.30}$$

Moreover, if we denote $a = \begin{pmatrix} \log(R) & 0_3 \\ vY & 0 \end{pmatrix}$*, where* $\log(R)$ *can be calculated by formula (7.22), then* $\log(A) = a$ *holds.*

Proof. Using a Gaussian LU-decomposition for any matrix $X \in SE(3)$, we immediately conclude, that its spectrum equals the spectrum of the upper-right matrix $R \in SO(3)$ together with a 1. Hence, it is $\|X - I_4\| = \|R - I_3\|$. Moreover, using Theorem 7.6, we obtain $r = \log(R) \in so(3)$. Thus, it is $\|\exp(r) - I\| \le \frac{1}{2}e^{\frac{1}{2}}$ and hence

$$\left(\int_0^1 \exp(rt)\, dt\right)^{-1} = Y.$$

This, together with Theorem 7.6, yields that

$$\exp\begin{pmatrix} r & 0_3 \\ vY & 0 \end{pmatrix} = \begin{pmatrix} R & 0_3 \\ vYY^{-1} & 1 \end{pmatrix}.$$

□

Unfortunately, this latter theorem cannot be applied for arbitrary matrices in $SE(3)$, since the defined sum may be divergent. In order to get more general results for global interpolation, we first use the method of computing the square root of a matrix in $SE(3)$:

Definition 7.6 *Let* $T \in \mathbb{R}^{n\times n}$ *be a real matrix. We say that the matrix* $\tilde{T} \in \mathbb{R}^{n\times n}$ *is a square root of* T *if*

$$\tilde{T}^2 = T$$

holds. If $\tilde{T}$ *is a square root of* T*, then we denote it with either* $\sqrt{T}$ *or* $T^{1/2}$*.*

As the following example shows, in general $\sqrt{T}$ is not unique:

Example 13: Let $T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\tilde{T}_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\tilde{T}_2 = \begin{pmatrix} \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Then $\tilde{T}_1^2 = \tilde{T}_2^2 = T$ holds.

Theorem 7.8 *(see Theorem 5. in [Hig87]) The square root of an invertible real matrix exists and is real if and only if every Jordan-block belonging the a negative real eigenvalue appears in the Jordan normal form an even number of times.*

Corollary 7.3 *Let A be an arbitrary matrix in $SE(3)$ then A has square root in $SE(3)$.*

Proof. As stated above, the spectrum of any matrix $A \in SE(3)$ decomposes into the spectrum of its upper-right block $R \in SO(3)$ and the set $\{1\}$. Moreover, since $R \in SO(3)$ is orthogonal, it is $|\lambda| = 1$ for any eigenvalue λ of R. Hence, for the three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of R $|\lambda_i| = 1$ and $\lambda_1\lambda_2\lambda_3 = \det(R) = 1$ hold. This leads us to the following cases

(i) if there is an $i \in \{1, 2, 3\}$ such that λ_i is complex, then $\bar{\lambda}_i$ is also an eigenvalue (because R is real). Without loss of generality assume, that $i = 1$ and $\lambda_2 = \bar{\lambda}_1$. Hence $\lambda_1\lambda_2 = |\lambda_1|^2 = 1$, which yields that $\lambda_3 = 1$. Using $Re(\lambda_1) = Re(\lambda_2)$, the condition of Theorem 7.8 is fulfilled.

(ii) Suppose now, that every eigenvalue of R is real. Thus, they are either $+1$ or -1 and hence the only possible spectrums are $\{+1, +1, +1\}$ and $\{-1, -1, +1\}$. The first one fulfills the condition from Theorem 7.8, the second one could cause a problem if R is non semi-simple.

Hence, the only case, that doesn't fulfill the condition of Theorem 7.8 is when R is not semi-simple and has the spectrum $\{-1, -1, +1\}$. We show now, that then $R \notin SO(3)$. Suppose conversely, that $R \in SO(3)$. If v_1 denotes an eigenvector corresponding to -1, then it is

$$\begin{aligned} Rv_1 &= -v_1, \text{ and} \\ Rv_2 &= -v_2 + v_1, \end{aligned}$$

for the generalized eigenvector v_2. Moreover, the subspace spanned by v_1 and v_2 is invariant under R. Hence, $R|_{sp(v_1,v_2)}$ is an orientation-preserving isomorphism. Orientation-preserving isomorphisms of the plane are exactly the rotations in $SO(2)$. Using now $Rv_1 = -v_1$, we obtain, that the angle of the rotation is exactly π. Hence, $R|_{sp(v_1,v_2)} \approx -Id_2$ and thus R is semi-simple. □

Now, we can use the inverse 'scaling-and-squaring' method to reduce the spectral radius of a matrix in $SE(3)$, such that it belongs to the set

$$\left\{ A \in SE(3) |\ \|A - I\| < \frac{1}{2} e^{\frac{1}{2}} \right\}.$$

In order to compute the square root of a matrix $A \in SE(3)$, we use the Schur decomposition method described in [Hig87] and [BH83].

Finally, we are ready to introduce an algorithm for computing the logarithm of a matrix $A \in SE(3)$:

(i) Define $A_0 := A$ and calculate $A_k = \sqrt{A_{k-1}}\ \forall k = 1, 2, 3, \dots$ using the real Schur decomposition method.

(ii) Calculate the smallest $k \in \mathbb{N}$ such that $\|A_k - I\| \leq \frac{1}{2}e^{\frac{1}{2}}$.

(iii) Choose $R \in SO(3)$ and $v \in \mathbb{R}^3$ such that $A_k = \begin{pmatrix} R & 0_3 \\ v & 1 \end{pmatrix} \in SE(3)$, with 0_3 denoting the zero vector in $\mathbb{R}^3$.

(iv) Compute Y using (7.30).

(v) Compute $a := \begin{pmatrix} \log(R) & 0_3 \\ vY & 0 \end{pmatrix}$, where $\log(R)$ is computed by the formula used in [CLK96b].

(vi) Obtain $\log(A) = 2^k a$.

In the above algorithm, the inverse 'scaling-and-squaring' method is carried out in steps (i)-(ii), whereas steps (iii)-(v) correspond to Theorem 7.7. Finally, the computation of the logarithm of the initial matrix takes place in step (vi).

Finally, we show two different motions of the same rigid body in $\mathbb{R}^3$ having the same boundary conditions:

$$\Theta(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Theta(1) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\dot\Theta(0) = \begin{pmatrix} 0 & \frac{\pi}{2} & 0 \\ \frac{-\pi}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \dot\Theta(1) = \begin{pmatrix} -\pi & 0 & 0 \\ \frac{\pi}{3} & 0 & 0 \\ 0 & -\pi & \frac{\pi}{3} \end{pmatrix}$$

$$v(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad v(1) = \begin{pmatrix} 15 \\ -2 \\ 20 \end{pmatrix}$$

$$\dot v(0) = \begin{pmatrix} -3 \\ 15 \\ 0 \end{pmatrix} \qquad \dot v(1) = \begin{pmatrix} 5 \\ 10 \\ 2 \end{pmatrix}$$

In order to carry out the generalized De Casteljau algorithm for the Lie groups $SO(3) \times \mathbb{R}^3$ and $SE(3)$, respectively, we need to obtain the tangentials at each boundary point, that is

$$\xi(i) = \dot x(i) x(i)^{-1} \tag{7.31}$$

for $i = 0, 1$ respectively. Hereby denotes $x(i)$ a point of the corresponding Lie groups, whereas $\xi(i)$ belongs to the Lie algebra.

In the case $SO(3) \times \mathbb{R}^3$ we can simply use the representation

$$x(i) := \big(\Theta(i), v(i)\big) \text{ and } \dot x(i) := \big(\dot\Theta(i), \dot v(i)\big),$$

which, together with (7.31), leads to

$$\xi(i) = \Big(\dot{\Theta}(i)\Theta(i)^{-1}, \dot{v}(i)\Big).$$

For the case of the Lie group $SE(3)$ we should use the representation formulae

$$x(i) := \begin{pmatrix} \Theta(i) & 0_3 \\ v(i) & 1 \end{pmatrix} \text{ and } \dot{x}(i) := \begin{pmatrix} \dot{\Theta}(i) & 0 \\ \dot{v}(i) & 0 \end{pmatrix}.$$

Again, using (7.31), these formulae yield

$$\xi(i) = \begin{pmatrix} \dot{\Theta}(i)\Theta(i)^{-1} & 0 \\ \dot{v}(i)\Theta(i)^{-1} & 0 \end{pmatrix}.$$

The interpolation results obtained by the discussed boundary data are presented on Figures 7.5 and 7.6 for the Lie groups $SO(3)\times\mathbb{R}^3$ and $SE(3)$, respectively. The computational effort on $SO(3)\times\mathbb{R}^3$ is slightly less than on $SE(3)$. This is mainly due to the inverse 'scaling-and-squaring' method, that could be avoided with an explicit formula for the logarithm (see [CL00]).

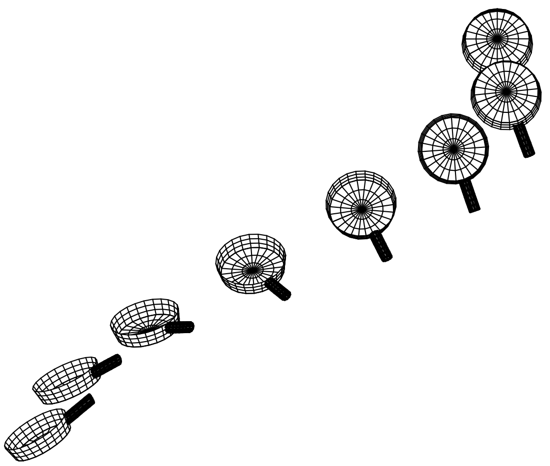

Figure 7.5: De Casteljau algorithm on $SO(3)\times\mathbb{R}^3$.

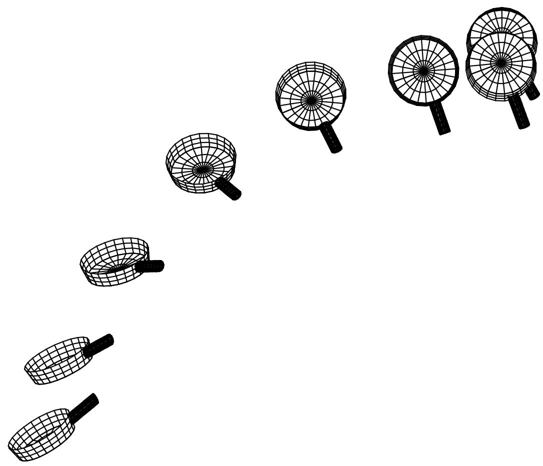

Figure 7.6: De Casteljau algorithm on $SE(3)$.

7.5 Notes and references

This section gave a brief introduction into the theory of variational problems on smooth manifolds, especially Lie groups and Riemannian manifolds. As for so many similar areas of applied mathematics, applications constantly require new methods, different solutions techniques and sometimes even more general problem settings. Hence, this subject is rapidly developing.

Therefore, this brief introduction cannot be complete. For further reading on the theoretical background of curves on smooth manifolds the textbook of Farin [Far93] and some works of Crouch and Silva-Leite (for instance [CL91], [CL95], [CLC96a] or [CLC95]) should be mentioned. For different problem settings such as elastic curves or higher order variational problems please refer to recent works of Camarinha et al. ([Cam97], [LCC95], [CLC96b] and [CLC96c]). For a more detailed discussion of variational problems on Lie groups and Riemannian manifolds the works of Griffiths [Gri83], Camarinha [Cam97] and Hestenes [Hes66] can be recommended.

The numerical treatment of the dynamic interpolation problem and similar problems can be found in the works of Maurer et al. (see for instance [MP95b] and [MP95a]). A different integration technique is presented by Crouch, et al. in [CYLB93]. Details on computing square roots and logarithms of matrices are covered by Higham [Hig87], Björck and Hammarling [BH83] and recently by Cardoso and Silva-Leite [CL00].

For facts about the De Casteljau algorithm, we should first refer to the original work of De Casteljau [De 59], but also to some recent works of Park and Ravani [PR95] and Bézier [Béz86]. Curves evolving on Lie groups and Riemannian manifolds are discussed in several papers of Crouch, Silva-Leite and Kun ([CLK96a],[CLK96b] and [CLK99a]).

Finally, to pick some works out of the enormous amount of papers, where visualization problem are discussed, we should mention works of Nielson and Heiland [NH92], Kim et al. [KKS95], Shoemake [Sho85], Barr et al. [BCGH92] and Nielson [Nie93]. Essentially they all develop algorithms that are related to curves on quaternions. A similar approach and comparison has also been used in the work of Crouch, Silva-Leite and Kun [CLK99b].

Chapter 8

Differential games on Lie groups

Recent applications in robotics and vehicle control, such as for instance multi aircraft maneuvers and robots with several controller units, require a generalization of Differential Game Theory for nonlinear dynamics. Here, we present a possible generalization scheme for the results.

Please note that – similarly to Chapter 5 – the discussion of the results and concepts only serves the presentation of the "state of the art" of an ongoing research project.

The structure of this chapter is as follows: after reviewing the framework for dynamical games on Lie groups, we shortly discuss the connections to differential games on Euclidean spaces. Then, in Section 8.2, we discuss how a Nash equilibrium arises and give formulae for the optimal controls.

8.1 Problem setting

In the sequel, we assume that G is a connected and compact Lie group of dimension n. We shall denote the corresponding bi-invariant Riemannian metric by $< .,. >$ and the induced affine connection by ∇. We shall also assume that the dynamics of the game is influenced by two players. We denote them by u and v. Hence, we can define a right-invariant differential game on G:

Definition 8.1 (right-invariant differential game) *Denote $A \in L(G)$, $B_i \in L(G)$ and $C_j \in L(G)$ two sets with $i = 1, \ldots, m_u$ and $j = 1, \ldots, m_v$. We say that the differential equation*

$$\dot{x} = Ax + \sum_{i=1}^{m_u} u_i B_i x + \sum_{i=1}^{m_v} v_i C_i x \tag{8.1}$$

Together with the cost functionals

$$J_u := \kappa_u(x(t_f)) + \int_{t_0}^{t_f} u^T R_u u \, dt, \quad J_v := \kappa_v(x(t_f)) + \int_{t_0}^{t_f} v^T R_v v \, dt \tag{8.2}$$

form a right-invariant *differential game. Hereby denote $\kappa_u : G \to \mathbb{R}$ and $\kappa_v : G \to \mathbb{R}$ smooth mappings, $u = (u_1, \ldots, u_{m_u})^T \in \mathbb{R}^{m_u}$ and $v = (v_1, \ldots, v_{m_v})^T \in \mathbb{R}^{m_v}$, as well as R_u and R_v positive definite matrices of the dimension $\mathbb{R}^{m_u \times m_u}$ and $\mathbb{R}^{m_v \times m_v}$, respectively.*

A first task could be to drive the system from a given point near to the identity element of G. Similarly to linear quadratic games, this can be reflected in the cost functionals for instance in the following manner:

$$\kappa_\alpha(x(t_f)) := \langle \log(x(t_f)), \log(x(t_f)) \rangle \,,$$

for $\alpha \in \{"u", "v"\}$.

To best match the requirements of the usual applications, we slightly modify our optimality constraints and introduce another variational problem, too. Besides fulfilling the initial condition $x(t_0) = x_0 \in G$, we require that the admissible trajectories fulfill a terminal value problem $x(t_f) = x_f$ for some given point x_f, too. Consequently, the best reply of the players changes in the following sense:

Definition 8.2 (best reply with boundary constraints) *Let Γ_2 be a right-invariant differential game on G and let $x_0, x_f \in G$ be given. Supposing that player u has chosen to play the strategy γ_u, the strategy γ_v is called* best reply *of v against γ_u if the following conditions hold.*

(i) The trajectory $x^*(t)$ with

$$\dot{x}^* = Ax^* + \sum_{i=1}^{m_u} u_i^* B_i x^* + \sum_{i=1}^{m_v} v_i^* C_i x^*$$

fulfills $x^*(t_0) = x_0$ and $x^*(t_f) = x_f$. Hereby denote $u^* = \gamma_u(t, \eta_u(t))$ and $v^* = \gamma_v(t, \eta_v(t))$.

(ii) $J_v(v^*) \leq J_v(v)$ holds for any admissible control function v fulfilling the requirements $x(t_0) = x_0$ and $x(t_f) = x_f$ for $\dot{x} = Ax + \sum_{i=1}^{m_u} u_i B_i x + \sum_{i=1}^{m_v} v_i^* C_i x$.

Now, we can define a Nash-equilibrium for differential games with boundary constraints:

Definition 8.3 (Nash-equilibrium with boundary constraints) *Suppose that Γ_2 is a right-invariant differential game (as defined in Definition 8.1) defined on a connected and compact Lie group G. Suppose further, that x_0 and x_f are given points in G. Then, we say that the strategies γ_u, γ_v form a Nash-equilibrium, if*

(i) γ_u is a best reply against γ_v and

(ii) γ_v is a best reply against γ_u

according to the boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_f$.

Remark 8.1 *Note, that the above definition is a direct generalization of the geodesic variational problem (Problem 4) discussed in Section 7.1.*

To ensure that the above problem is well-posed, the game is required to be individually controllable for any player. A weakened, but nevertheless not sufficient condition would be the team-controllability: A necessary condition for the existence of a Nash equilibrium with boundary constraints is that the Lie algebra spanned by the vectors $A, B_1, ..., B_{m_u}, C_1, ..., C_{m_v}$ equals to the Lie algebra $L(G)$.

To avoid conflicts with ill-posedness, we shall in the sequel assume that both sets of vectors B_i and C_j for $i = 1, \dots, m_u$ and $j = 1, \dots, m_v$ generate $L(G)$ as a Lie algebra.

8.2 Open-loop Nash equilibria

In this section, we derive – similarly to Section 2.3.1 – formulae for the optimal control functions and try to find an answer to the question, under which conditions a Nash equilibrium exists.

In the sequel, we first discuss Nash games with "classical" cost functionals, i.e. with end-penalty κ and without terminal value problem and then, we briefly discuss solution possibilities for games under boundary conditions.

Simplifying our investigation, we suppose that the Lie algebra $L(G)$ is commutative, i.e. that for any $X, Y \in L(G)$ $[X, Y] = 0$ holds. Hence, using Theorem 6.4, the state trajectory generated by the control functions u and v becomes

$$x(t) = \exp(w(t)),$$

with $w(t) \in L(G)$ fulfilling

$$\dot{w}(t) = A + \sum_{i=1}^{m_u} u_i B_i + \sum_{j=1}^{m_u} v_j C_j, \quad w(t_0) = \log(x_0).$$

Furthermore, as it was done in Section 2.3, we suppose that the player v already found the Nash-strategy and hence a Nash-strategy for u is to be given. Then, interchanging the players, conditions on the Nash-equilibrium are possible in the same manner as for linear quadratic differential games.

8.2.1 Nash games without boundary constraints

Rewriting the cost functional

$$J_u = \Big\langle \log\Big(x(t_f)\Big), \log\Big(x(t_f)\Big)\Big\rangle + \int_{t_0}^{t_f} u^T R_u u \, dt$$

for $\log(x(t)) = w(t)$ yields

$$J_u = \langle w(t_f), w(t_f)\rangle + \int_{t_0}^{t_f} u^T R_u u \, dt$$

$$= \left\langle w(t_0) + \int_{t_0}^{t_f} \dot{w}(t)\,dt,\ w(t_0) + \int_{t_0}^{t_f} \dot{w}(t)\,dt \right\rangle + \int_{t_0}^{t_f} u^T R_u u\,dt$$

$$= \left\langle w(t_0) + \int_{t_0}^{t_f} \left(A + \sum_{i=1}^{m_u} u_i B_i + \sum_{j=1}^{m_v} v_j^* C_j \right) dt,\ w(t_0) + \int_{t_0}^{t_f} \left(A + \sum_{i=1}^{m_u} u_i B_i + \sum_{j=1}^{m_v} v_j^* C_j \right) dt \right\rangle$$

$$+ \int_{t_0}^{t_f} u^T R_u u\,dt$$

Suppose now that the control u^* is extremal according to J_u. We now calculate the value of J_u along a modified control function $u = u^* + \varepsilon \tilde{u}$ for some fixed mapping $\tilde{u} : [t_0, t_f] \to \mathbb{R}^{m_u}$ and $\varepsilon \in \mathbb{R}$:

$$\begin{aligned} i(\varepsilon) = J_u \;=\; & \left\langle w(t_0) + \int_{t_0}^{t_f} \left(A + \sum_{i=1}^{m_u} (u_i^* + \varepsilon \tilde{u}_i) B_i + \sum_{j=1}^{m_v} v_j^* C_j \right) dt, \right. \\ & \left. w(t_0) + \int_{t_0}^{t_f} \left(A + \sum_{i=1}^{m_u} (u_i^* + \varepsilon \tilde{u}_i) B_i + \sum_{j=1}^{m_v} v_j^* C_j \right) dt \right\rangle \\ & + \int_{t_0}^{t_f} (u^* + \varepsilon \tilde{u})^T R_u (u^* + \varepsilon \tilde{u})\,dt \end{aligned}$$

and hence, using the extremality of u^*

$$\begin{aligned} 0 = \left.\frac{di}{d\varepsilon}\right|_{\varepsilon=0} \;=\; & 2 \left\langle \underbrace{w(t_0) + \int_{t_0}^{t_f} \left(A + \sum_{i=1}^{m_u} u_i^* B_i + \sum_{j=1}^{m_v} v_j^* C_j \right) dt}_{w^*(t_f)},\ \int_{t_0}^{t_f} \sum_{i=1}^{m_u} \tilde{u}_i B_i\,dt \right\rangle \\ & + 2 \int_{t_0}^{t_f} u^{*T} R_u \tilde{u}\,dt \\ =\; & 2 \int_{t_0}^{t_f} \left(\left\langle w^*(t_f), \sum_{i=1}^{m_u} \tilde{u}_i B_i \right\rangle + u^{*T} R_u \tilde{u} \right) dt \end{aligned}$$

follows. Denoting $\alpha_i^* :=< w^*(t_f), B_i >$ and $\alpha^* = (\alpha_1^*, \ldots, \alpha_{m_u}^*)^T \in \mathbb{R}^{m_u}$, we obtain the following identity

$$\int_{t_0}^{t_f} (\alpha^* + R_u u^*)^T \tilde{u}\,dt = 0.$$

Using the fact that $\tilde{u}$ is arbitrary, and thus for instance $(\alpha^* + R_u u^*)$, we obtain the following Lemma:

Lemma 8.1 *Suppose that u^* is an extremal function according to J_u. Then it fulfills the identity*

$$u^* = -R_u^{-1} \alpha^*.$$

Especially, if an optimal control function u^* exists, then it is constant.

Now, we can formulate our first result concerning Nash equilibria on non-constrained differential games on commutative Lie algebras.

Theorem 8.1 *Suppose that G is a Lie group with commutative Lie algebra $L(G)$. Suppose further that Γ_N is a right-invariant differential game on G as defined in Definition 8.1 and that the controls u^* and v^* form an open-loop Nash equilibrium. Then, the following identities hold:*

$$\text{(i)}\ u_i^*(t) \equiv \sum_{j=1}^{m_u} (R_u^{-1})_{ij} < w^*(t_f), B_j >= \left\langle w^*(t_f), \sum_{j=1}^{m_u} (R_u^{-1})_{ij} B_j \right\rangle.$$

$$\text{(ii)}\ v_i^*(t) \equiv \sum_{j=1}^{m_v} (R_v^{-1})_{ij} < w^*(t_f), C_j >= \left\langle w^*(t_f), \sum_{j=1}^{m_v} (R_v^{-1})_{ij} C_j \right\rangle,$$

for $w^*(t_f) = \log(x^*(t_f))$ with x^* fulfilling the differential equation

$$\dot{x}^* = \left(A + \sum_{i=1}^{m_u} u_i^* B_i + \sum_{i=1}^{m_v} v_i^* C_i \right) x^*, \quad x^*(t_0) = x_0.$$

8.2.2 Nash games with boundary constraints

We now turn our attention to open-loop Nash games with boundary conditions as defined in Definition 8.3. Suppose again, that the Lie algebra $L(G)$ is commutative, we can rewrite our original problem of searching for a constrained Nash equilibrium into a slightly different variational problem:

Problem 9 *Given a basis $X_1, \ldots, X_n$ of $L(G)$ and the functionals*

$$J_u^i := \int_{t_0}^{t_f} \left(a_i + \sum_{j=1}^{m_u} u_j b_{ji} + \sum_{j=1}^{m_v} v_j^* c_{ji} \right) dt$$

and

$$J_u = \int_{t_0}^{t_f} u^T R_u u \, dt$$

for $i = 1, \ldots, m_u$ with a_i, b_{ji} and c_{ji} denoting the coefficients in the representation of the vectors A, B_j, C_j according to the basis vector X_i, we are interested in the solution of the variational problem

$$\min_u J_u$$

subject to the constraints $J_u^i = w_i$ for some given numbers $w_i \in \mathbb{R}$ $(i = 1, \ldots, n)$.

Lemma 8.2 *Let $x_0 \in G$ be given. Supposing that*

$$\log(x_0) = -\sum_{i=1}^{n} w_i X_i$$

holds, then u^* is a solution of Problem 9 if and only if it is a best reply against v^* according to the boundary constraints $x(t_0) = x_0$ and $x(t_f) = e$.

Proof. Clearly, if $x(t_f)$ is fixed, then the term $\kappa_u(x(t_f))$ doesn't play a role in the optimization problem. Hence, J_u is equivalent to the cost functional of u. Furthermore, using the exponential representation of the solution, the constraints $J_u^i = -w_i$ mean that

$$\int_{t_0}^{t_f} \left(A + \sum_{i=1}^{m_u} u_i B_i + \sum_{j=1}^{m_u} v_j^* C_j \right) dt - \sum_{i=1}^{n} w_i X_i = 0$$

and therefore $w(t_f) = 0$ holds. Hence, $x(t_f) = \exp(w(t_f)) = e$ follows. □

Corollary 8.1 *Using the above lemma, it is possible to generate the Nash-trajectory according to any boundary conditions $x_0, x_f \in G$.*

Proof. Note, that the boundary value problem $x(t_0) = x_0 x_f^{-1}$, $x(t_f) = e$ can be right-translated into the boundary problem $x(t_0) = x_0$, $x(t_f) = x_f$ without changing the actual controls u, v. Hence, their optimal solutions coincide. □

Altogether, we showed, that the searching for the best reply according to boundary constraints on the Lie group G is equivalent to a variational problem on $\mathbb{R}^{m_u}$ of the form discussed in Theorem 5.1 in Chapter 5 of [Hes66].

Hence, if $u : [t_0, t_f] \to \mathbb{R}^{m_u}$ is a best reply, then it is an extremum of

$$F(t, u(t)) := \lambda_0 u^T R_u u + \sum_{i=1}^{n} \lambda_i \left(a_i + \sum_{j=1}^{m_u} u_j b_{ji} + \sum_{j=1}^{m_v} v_j^* c_{ji} \right).$$

Again, using the fact that the game is of type open-loop, we can leave out the terms that are constant according to $u(.)$. Hence, we obtain that if u is a best reply, then it is extremal according to

$$F^*(t, u(t)) = \lambda_0 u^T R_u u + \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m_u} u_j b_{ji} = \lambda_0 u^T R_u u + \lambda^T \mathcal{B} u,$$

with $\lambda := (\lambda_1, \ldots, \lambda_n)^T \in \mathbb{R}^n$ and $\mathcal{B} \in \mathbb{R}^{n \times m_u}$ such that $\mathcal{B}_{ij} = b_{ji}$ holds for $j = 1, \ldots, m_u$ and $i = 1, \ldots, n$.

A first consequence of Theorem 5.1 (Chapter 5) in [Hes66] is that again, the optimal control function u^* is constant. Hence, we can calculate the extrema of F^* by differentiation:

$$0 = \left. \frac{\partial F^*}{\partial u} \right|_{u=u^*} = 2\lambda_0 {u^*}^T R_u + \lambda^T \mathcal{B} \tag{8.3}$$

Suppose now that $\lambda_0 = 0$. Then, F^* is linear and has, therefore, no extrema. Hence, if an extremum exists, then $\lambda_0 \neq 0$ holds. Multiplying (8.3) by $\frac{1}{\lambda_0}$ yields for $\lambda^* = \frac{1}{\lambda_0}\lambda$ the equation:

$$u^* = -\frac{1}{2} {R_u}^{-1} \mathcal{B}^T \lambda^*,$$

which is somehow very much like equation (2.16). Hereby the constants $\lambda^* \in \mathbb{R}^{m_u}$ must be determined such that the interpolating condition $J_u^i = w_i$ is fulfilled. Although, this seems to be a hard task, using the fact that the optimal control is constant, we can obtain a nice formula for them and hence a representation for the optimal controls in open-loop constrained Nash games:

Theorem 8.2 *Suppose that G is a Lie group with commutative Lie algebra $L(G)$. Suppose further that Γ_N is a right-invariant differential game on G as defined in Definition 8.1 and that the controls u^* and v^* form an open-loop constrained Nash equilibrium (see Definition 8.2). Then, the corresponding optimal control functions u^* and v^* fulfill*

$$u_i^* = \sum_{j=1}^{m_u}\sum_{k=1}^{n}(R_u^{-1})_{ij} b_{kj} \lambda_k^*$$

and

$$v_i^* = \sum_{j=1}^{m_v}\sum_{k=1}^{n}(R_v^{-1})_{ij} c_{kj} \mu_k^*$$

where $\lambda^* \in \mathbb{R}^{m_u}$ and $\mu^* \in \mathbb{R}^{m_v}$ are solutions of the linear equation

$$\begin{pmatrix} \beta_{11} & \dots & \beta_{1n} & \gamma_{11} & \dots & \gamma_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \dots & \beta_{nn} & \gamma_{n1} & \dots & \gamma_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \begin{pmatrix} \frac{w_1}{t_f - t_0} - a_1 \\ \vdots \\ \frac{w_n}{t_f - t_0} - a_n \end{pmatrix}, \tag{8.4}$$

with

$$\beta_{k\ell} := \sum_{i=1}^{m_u}\sum_{j=1}^{m_u}(R_u^{-1})_{ij} b_{\ell j} b_{ik}$$

and

$$\gamma_{k\ell} := \sum_{i=1}^{m_v}\sum_{j=1}^{m_v}(R_v^{-1})_{ij} c_{\ell j} c_{ik}.$$

Further denote $w = -\log(x_0 x_f^{-1})$, as well as a_j, b_{ij}, c_{ij} and w_j are the respective coordinates of the vectors A, B_i, C_j and w according to the basis $X_1, \dots, X_n$ of $L(G)$.

Proof. Using the fact, that the control functions u^* and v^* are constant over the time-horizon $[t_0, t_f]$, the integrals J_u^i and J_v^j go over to simple multiplications by $t_f - t_0$. Hence, the above calculation completes the proof. □

8.3 Notes and references

In this last chapter of this Thesis, we discussed the mathematical description of differential games on Lie groups. Besides reviewing games with "classical" setups, i.e. without boundary conditions, we also discussed the arising of Nash equilibria under boundary constraints. For results on the existence of these equilibria, a forthcoming work [Kun01] could be suggested.

As mentioned before, games on Lie group form a new topic of Differential Game Theory. As far as the author is concerned, the only research group dealing with similar problems is at the University of California at Berkeley. Nevertheless, besides mentioning these setups by Sastry et al. in [TPS98] and discussing them very briefly in the context of an example on $SE(2)$, there is no publications known to the author on the topics covered here.

Appendix A

Mathematical description of mechanical systems

Until now, we mainly concentrated on the mathematical description of differential control systems and games. Here, we first present the background of modeling mechanical systems, i.e. a toolbox to convert mechanical problems into mathematical ones. Usually, this step is missing from most of the publications, although it is as important as any other link in the chain between the realization of a problem and the implementation of its solution.

Throughout this Appendix, we shall mainly use the ideas and results usually known in the analytical mechanics. For further reading, the text of Arnold [Arn78] as well as papers of Crouch and Bloch ([BC92], [BC94], [BC95a] and [BC95b]) are recommended. To tell the truth, the most difficult problem is usually finding a mathematical model for a given mechanical system, that describes its behavior and is simultaneously simple enough, to be treated with known methods. Although, this step cannot be algorithmized, we'll try to give some ideas here, that should be helpful for general mechanical systems.

First, we have to find a description tool for the given mechanical system. This means a choice of a set, called *configuration space*, such that every element of this set corresponds bijectively to a position of the system. For instance, for a single point in the plane, one could use the set $\mathbb{R}^2$. However, a two-dimensional rigid body can be identically described on the sets $\mathbb{R}^2 \times [0, 2\pi)$, $\mathbb{R}^2 \times SO(2)$ or $SE(2)$. Since, it is easier to describe the dynamical equations on the Lie group $SE(2)$, we shall use this latter representation of the configuration space. In the following, we refer to the elements of the configuration space as *positions* of the mechanical system.

Similarly to non-constrained mechanical systems, we can also assign a configuration set to a constrained mechanical system. For the rest of this chapter, we suppose that the system is *holonomical* (constraint are purely geometrical without involving velocities) and *autonomous* (no explicit dependence on the time parameter). The configuration space can then be obtained as a subspace (hypersurface) of the unconstrained configuration space. Considering, for instance, two points in $\mathbb{R}^3$ connected by a weightless bar of length ℓ, this yields a configuration space

$$C = \Big\{(x, y) | x \in \mathbb{R}^3, y \in \mathbb{R}^3, \|x - y\| = \ell\Big\} \cong \mathbb{R}^3 \times \mathcal{S}^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$$

Example 14: Consider a planar mechanical system described by two points p_1 and p_2. Supposing that the corresponding coordinates (x_1, y_1) and (x_2, y_2) fulfill for given $\ell_1, \ell_2 > 0$

$$x_1^2 + y_1^2 = \ell_1^2 \tag{A.1}$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell_2^2, \tag{A.2}$$

a first possibility the describe the configuration space is to give it as a surface embedded in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ fulfilling the constraints (A.1) and (A.2). Although, this technique works for every mechanical system, it usually produces a quite complicated description of the configuration space.

Consider again equations (A.1) and (A.2): These means that the point p_1 has always the same distance ($= \ell_1$) from the origin and similarly, the distance between p_1 and p_2 equals to ℓ_2. Hence, the mechanical system is a double pendulum as shown in Figure A.1.

Figure A.1: Kinematical model of the double pendulum

Regarding this kinematical model, it is clear, that the angles φ_1 and φ_2 uniquely determine the position and hence we can use the Lie group $X = SO(2) \times SO(2)$ to describe the configuration of the plant.

If springs or similar elastic elements are contained in the given mechanical system, then for the configuration, they are treated as a degree of freedom (as if there would be no connection). They will only play a role if the dynamics of the system is investigated.

In general, if M is the configuration space of the unconstrained system, then M has the structure of a smooth (Riemannian) manifold and the constraints are defined as equations on M in the form:

$$g_i(m) = 0, \qquad m \in M, i = 1, 2, \ldots$$

Note that the the equations $g_i(m) = 0$ define submanifolds of M and hence their intersection (if it isn't empty) is again a submanifold of M.

Hence, in the manner of Definition 4.3.1 from [Arn78], we treat the configuration space of *mechanical systems* as a Riemannian manifold M, where every point of M denotes a unique position of the system.

Additionally to the possible configurations (that together form the kinematically possible positions), we can also speak about dynamically possible paths. These are defined (at least for a very large class of mechanical systems) mathematically through the following theorem:

Theorem A.1 (Hamilton's principle; see also §3.2.1 and §4.3.2 in [Arn78]) *Assume that the observed mechanical system is autonomous and holonomical. Suppose further that there is a functional* $L : M \times TM \to \mathbb{R}$ *such that for every* $x \in M$ *and* $v \in \mathcal{T}_x(M)$

$$L(x, v) = T - U$$

holds, with T *denoting the kinetic and* U *the potential energy of the system being at the position* x *and having the velocity* v*. Then every dynamically possible path* $\gamma : [t_0, t_1] \to M$ *of the system is an extremal curve of the functional*

$$\int_{t_0}^{t_1} L\big(\gamma(t), \dot{\gamma}(t)\big)\, dt.$$

Corollary A.1 *If an autonomous mechanical system fulfills the above criterium, then it can be very nicely modeled as an optimal control system evolving on* M*, with a special "optimal controller", the* nature *itself.*

Spring elements, that we considered as a degree of freedom, i.e. no constraints, have an influence on the potential energy and hence on the dynamically possible paths of the system. The next simple example illustrate the above consideration:

Example 15: Given a mechanical system (see Figure A.2), we now derive the optimal control problem belonging to this plant:

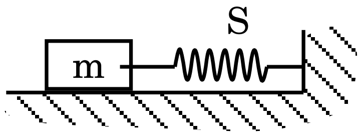

Figure A.2: Mass-spring system

Denoting by x the displacement of the mass and by s the tension of the spring, we obtain that the kinetic and the potential energy can be given as

$$T = m\dot{x}^2, \text{ and } U = Ss^2,$$

respectively. Moreover, the system dynamics are simply given by $x = -s$. Extending the dimension, we can obtain a linear quadratic optimal control problem describing the trajectories of the system:

$$\frac{d}{dt}\begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} v_s,$$

$$J(x, s, v_s) = \int_{t_0}^{t_f} m\dot{x}^2 - Ss^2\, dt = \int_{t_0}^{t_f} \begin{pmatrix} x \\ s \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & -S \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} + mv_s^2\, dt,$$

where the spring tension velocity v_s plays the role of the control variable.

Now, we turn our attention to *controlled mechanical systems.* Generally speaking, a controlled mechanical system is a mechanical system with a control parameter. This parameter usually belongs to a set of functions over the control value space. Hence, the system is given by the following definition:

Definition A.1 (controlled mechanical system) *Suppose that we are given a holonomical and autonomous mechanical system, for which the configurations is a Riemannian manifold M. Suppose also, that there is an active control that influences the system. This control takes its values from the set U. Assuming, that there exists a functional $L(x(t), \dot{x}(t), u(.))$ such that it reflects for any given control function $u(.)$ the difference between the kinetic and the potential energy of the system described by the configuration x, we can similarly to the above manner calculate the system trajectory on M as a parametric variational problem, with parameter $u(.)$. Such mechanical systems are called* controlled mechanical systems.

If the controller $u(.)$ is chosen (by us) so that it minimizes a "cost functional" J, then we can repose this problem as a (Stackelberg) differential game on M with the role of the "leader" played by the active controller and the "follower" by the nature:

1. For any given control $u(.)$ find the solution of the variational problem

$$\min L(x, \dot{x}, u(.)),$$

2. Knowing that the follower (=nature) uses an optimal strategy to play against $u(.)$ according to the above variational problem, find a strategy to minimize the functional J.

This principle is investigated in several forthcoming works of Jank, Kun et al. (see for instance [JK01], [JK00a] or [JKK$^+$01b]).

Appendix B

Application of the results for the stabilization and tracking of elastic manipulators

After considering mathematical treatment and modeling, to end this Thesis, we now discuss a possible application. There are several fields to choose examples from. Here, we deal with a mechanical example taken from an ongoing project[1] in robotics.

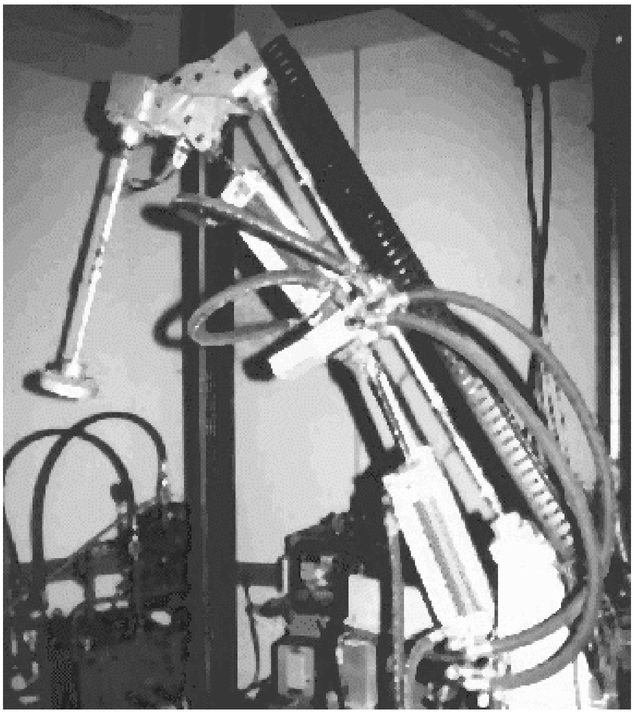

Figure B.1: Elastic robot

Using robots for wide operating ranges or for heavy loads often leads to problems where the elasticity of the plant must be taken into account. Nevertheless it is even by heavy loadings a very important task, that the robot goes along a predefined path.

In the sequel, we investigate a very interesting problem, the so-called *output tracking problem*: Given a trajectory in the output-space, find a control function, such that the output signal of the system always remains on the given trajectory.

Since this exactness is usually superfluous for technical systems, we modify the problem settings, allowing small deviations from that given trajectory, but requiring low resource-usage (i.e. slow velocities, forces, or displacements, etc.).

To illustrate solution concepts for this setup, we present the following system (see Figures B.1[2] and B.2).

[1]Special Research Area on "Elastic Manipulators for Heavy Loads in Complex Systems" – University Duisburg

[2]Picture included with the kind permission of the Department for Measurement and Control, University Duisburg

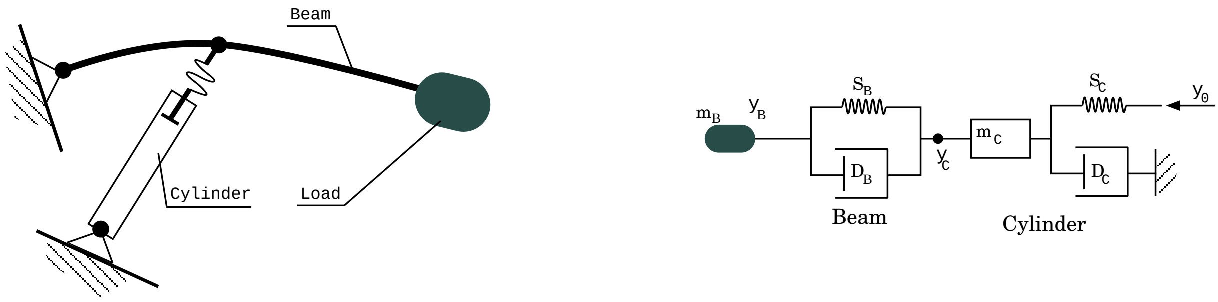

Figure B.2: Kinematical and dynamical model of the elastic robot

Our goal is to find an input function $y_0(t)$, which drives the end of the elastic beam on a given trajectory $y_B^*(t)$. To solve this problem, we first need to establish the dynamical equations:

Using Newton's law for mechanical system, we conclude that the system dynamics is given by

$$\begin{aligned} m_B\ddot{y}_B(t) &= S_B(y_C(t) - y_B(t)) + D_B(\dot{y}_C(t) - \dot{y}_B(t)) \\ m_C\ddot{y}_C(t) &= S_B(y_B(t) - y_C(t)) + S_C(y_0(t) - y_C(t)) + D_B(\dot{y}_B(t) - \dot{y}_C(t)) - D_C\dot{y}_C(t), \end{aligned} \tag{B.1}$$

where the parameters $S_B, S_C, D_B, D_C, m_B, m_C$ still have to be identified.

The above system is a typical example for a controlled mechanical system. It is of second order, that has to be first transformed into a first order system. This is possible by introducing the new variables

$$v_C(t) = \dot{y}_C(t) \text{ and } v_B(t) = \dot{y}_B(t).$$

Moreover, to avoid technical problems, we need to use a possibly low velocity of the piston and hence we introduce the piston velocity

$$v_0(t) = \dot{y}_0(t),$$

too. Now we can rewrite equations (B.1) into the well known form of an autonomous linear control system:

$$\frac{d}{dt}\underbrace{\begin{bmatrix} y_B(t) \\ y_C(t) \\ v_B(t) \\ v_C(t) \\ y_0(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{S_B}{m_B} & \frac{S_B}{m_B} & -\frac{D_B}{m_B} & \frac{D_B}{m_B} & 0 \\ \frac{S_B}{m_C} & -\frac{S_B+S_C}{m_C} & \frac{D_B}{m_C} & -\frac{D_B+D_C}{m_C} & \frac{S_C}{m_C} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_B(t) \\ y_C(t) \\ v_B(t) \\ v_C(t) \\ y_0(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{B} \underbrace{v_0(t)}_{u(t)}. \tag{B.2}$$

As stated before, our aim is to control the piston ($y_0(t)$) in a way, that the load at the end of the beam ($y_B(t)$) follows a predefined signal ($y_B^*(t)$). In order to obtain this *beam-position (y_B)-tracking*, we introduce the following "costs" over the tracking-horizon $[t_0, t_f]$:

$$\begin{aligned} J(x,u) \;:=\; & \int_{t_0}^{t_f} (x(t) - x^*(t))^T Q(x(t) - x^*(t)) + \kappa u^2(t)\, dt \\ & + (x(t_f) - x^*(t_f))^T K_f(x(t_f) - x^*(t_f)), \end{aligned}$$

with a positive semidefinite matrix $Q \in \mathbb{R}^{5\times 5}$, $\kappa > 0$ and

$$x^*(t) := \begin{bmatrix} y_B^*(t) \\ 0 \\ \dot{y}_B^*(t) \\ 0 \\ 0 \end{bmatrix}.$$

The matrix K_f is mainly introduced for technical reasons and will be specified later. For simplicity, in the sequel we use

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As one can see, minimizing this functional means a *balancing* between small deviation from the predefined trajectory and high piston velocities. Indeed, the term κ controls this *weighting*. Higher values mean that high piston velocities should be *punished* more, wheres lower values for κ result in a very robust tracking with possibly high velocities v_0.

Nevertheless, the above optimal control problem has not been handled before. Rewriting J we obtain, that it is not purely quadratic on x. Hence, we cannot expect, that the solution is a pure feedback term, as for the control problems discussed in Section 1.4.

If we assume, that the optimal feedback control u^* has the form

$$u^*(t,x) = -\frac{1}{\kappa}B^T(K_f x + \frac{1}{2}e(t))$$

for some constant matrix $K_f \in \mathbb{R}^{5\times 5}$ and some function $e(t) \in \mathbb{R}^5$, then the Hamilton-Jacobi equations (see Section 2.3.2) yield:

$$\dot{e}(t) = -(A - SK)^T e(t) + 2Qx^*(t), \quad e(t_f) = -2K_f x^*(t_f) \tag{B.3}$$

where the feedback matrix K_f is the positive semidefinite solution of the Riccati equation

$$-A^T K_f - K_f A - Q + K_f S K_f = 0. \tag{B.4}$$

Then, clearly $K(t) \equiv K_f$ is a solution of the matrix Riccati differential equation

$$\dot{K} = -A^T K - KA - Q + KSK, \quad K(t_f) = K_f,$$

which means, that we don't have to deal with a time-dependent feedback-matrix $K(t)$. Hereby denotes (as usual) $S = \frac{1}{\kappa}BB^T$. As one can see, the feedback-part of the control ($K_f x$) is independent of the tracking trajectory, whereas the open-loop driving-part ($e(t)$) is clearly influenced by it.

We can now give an algorithm to solve the *beam-position tracking problem*:

1. Calculate the matrices A, B, S and Q.

2. Find a "weighting" $\kappa > 0$.
3. Solve the algebraic Riccati equation (B.4) for $K_f \geq 0$.
4. Solve the terminal value problem (B.3).
5. Establish a feedback controller $u_1(x) = -\frac{1}{\kappa} B^T K_f x$.
6. Generate the signal $u_2(t) = -\frac{1}{2\kappa} B^T e(t)$.
7. Calculate the sum of the signals u_1 and u_2 and use it as the piston velocity $v_0(t)$.

In order to illustrate the potential of the above technique, let us consider the following system parameters (see also [PN00]):

$$S_C := 12000\, N/m,\ D_C := 5500\, Ns/m,\ S_B := 57424\, N/m,\ D_B := 300\, Ns/m,$$
$$m_B := 702.5\, kg,\ m_C := 2000\, kg$$

Furthermore, we use the weighting $\kappa = 0.1$. A simulated output signal with the above algorithm for this system is shown on Figure B.3. The line with the small circles illustrates the reference trajectory, and the solid line is the simulated one.

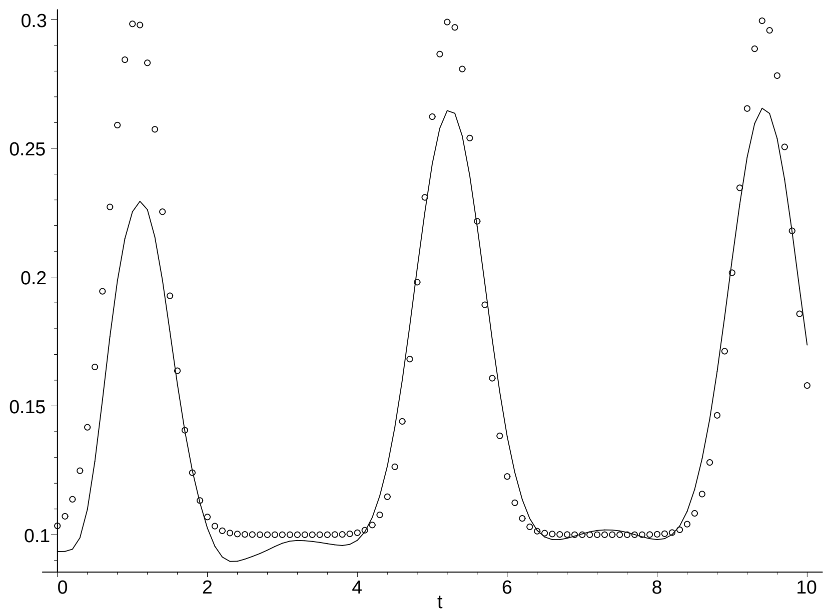

Figure B.3: Simulation of the non-disturbed tracking problem

To show the robustness of this method, we simulate a disturbed situation, too. Suppose that an unknown force acts at the end of the beam, resulting an oscillation as can be seen on Figure B.4. The simulated trajectory under the same disturbance, after switching on the controller is illustrated on Figure B.5.

Finally, it should be pointed out, that – although it seems to be very powerful – the above method is by no means the only possible solution concept. Another solution method, using the theory of flat systems, which are not handled in the framework of this Thesis, can be found in the work of Polzer and Nissing [PN00].

Besides control theoretical approaches, it is also possible to establish a model based on a game-theoretical background. However, since this model is based on a Stackelberg concept, which lies outside of the topics covered in this Thesis, it cannot be discussed here. For more details on that approach, refer to [JKK+01b].

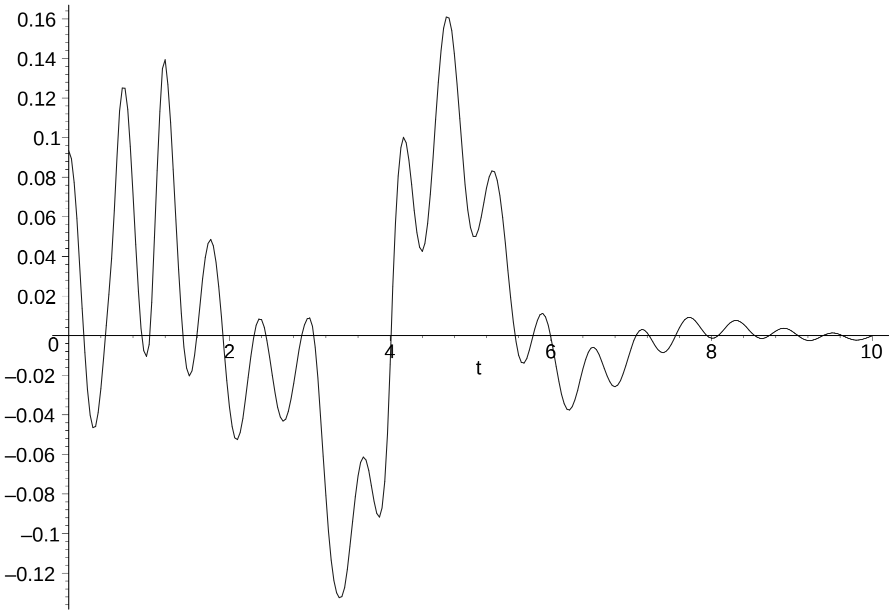

Figure B.4: Oscillation of the beam caused by the external disturbance

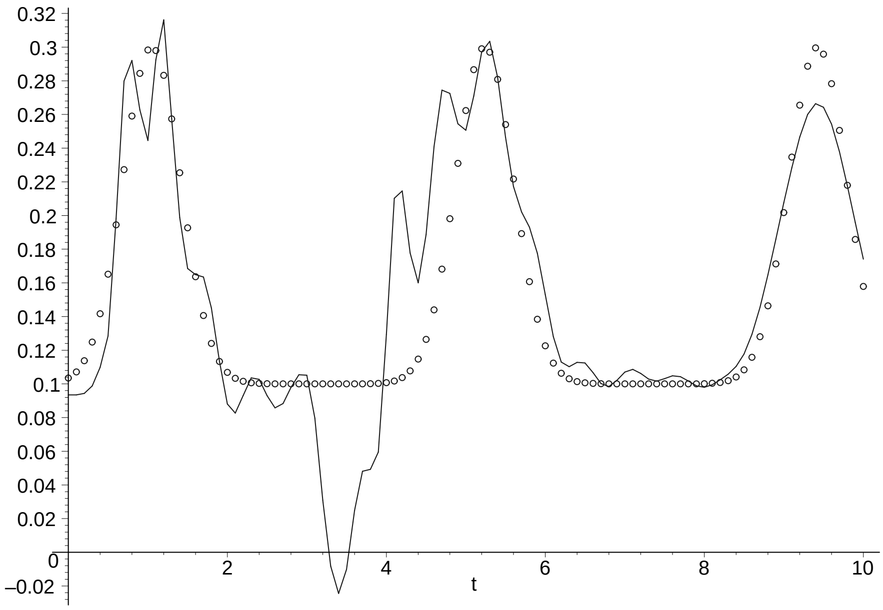

Figure B.5: Simulation of the disturbed tracking problem

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Lebenslauf

Persönliche Angaben

Anrede :	Herr
Name :	**Kun, Gábor**
Geburtsdatum :	08.06.1973
Geburtsort :	Budapest (Ungarn)

Schulausbildung

09.1979-06.1987	‘László Rajk’ Grundschule in Budapest
09.1987-06.1991	‘Dániel Berzsenyi’ Begabtförderungsgymnasium für Mathematik in Budapest
30.06.1991	Abiturprüfungen mit Gesamtnote sehr gut

Studium

09.91-06.97	Diplomstudium Maschinenbau an der TU Budapest
09.93-06.94	Stipendiat der Republik Ungarn für ”uberdurchschnittliches Studium in den ersten vier Semestern
10.94-07.1996	TEMPUS-Stipendium für Studium/Praktikum an der RWTH Aachen mit dem Ziel des *doppelten Diploms*
10.94-07.1997	Diplomstudium Mathematik an der RWTH Aachen
07.95-06.96	Studentische Hilfskraft am Lehrstuhl II für Mathematik
03.1996	ERASMUS-Stipendium zur Universidade de Coimbra (Portugal)
07.1996-10.1996	Diplomprüfungen in Mathematik an der RWTH Aachen
06.1997	Diplom-Staatsexamen in Maschinenbau an der TU Budapest
24.06.1997	*Diplom* (M.Sc.) mit Auszeichnung an der Fakultät für Maschinenbau der TU Budapest
03.07.1997	*Diplom* (Dipl.-Math.) an der Mathematisch-Naturwissenschaftlichen Fakultät der RWTH Aachen

Promotion

09.1997-07.2000	*Promotion* in Mathematik unter der Betreuung von Prof. Dr. Gerhard Jank
09.1997	Wissenschaftliche Hilfskraft am Lehrstuhl II für Mathematik
10.1997-04.2001	Wissenschaftlicher Angestellter an der Mathematisch-Naturwissenschaftlichen Fakultät der RWTH Aachen
12.1997-04.2001	Assoziiertes Mitglied im Graduiertenkolleg ‘Analysis und Konstruktion in der Mathematik’ der RWTH Aachen
07.1999-10.2000	Gastwissenschaftler im Sonderforschungsbereich SFB291 der Universität Duisburg

Berufserfahrung und Fachkenntnisse

09.1996-08.1997	Studentischer Mitarbeiter bei der Firma AG·E GmbH (1 Jahr): *Simulation und Optimierung dynamischer Systeme, Regelungsalgorithmen für verfahrenstechische Anlagen, algorithmierte statistische Versuchsplanung, nichtlineare Optimierungsverfahren, fuzzy-logic.*