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Reduced Basis Solutions of Parametrized Optimal Control Problems with Non-Affine Source Terms

Zoi Tokoutsi[†*], Martin Grepl[*], Karen Veroy[*], Marco Baragona[†], Ralph Maessen[†]

[†]Philips Research, Eindhoven, Netherlands, [*]AICES, RWTH Aachen University, Germany

Motivation

Thermal ablation treatments in cancer therapy heat a target volume, enough to cause it to burn, but leave healthy tissue and neighboring sensitive structures undamaged. Performing such treatments involves the placement of one or multiple heat sources within the target volume and the power control of each heat source. Since the final placement of the heat sources and tissue parameter estimates are only possible during treatment, real-time updates of the heat source powers are very relevant. Optimizing the power of each heat source involves the solution of optimal control problems governed by parametrized elliptic partial differential equations with non-affine source terms. We employ the reduced basis method to derive a reliable and real-time efficient surrogate model. We work on extending [3] and [4] in order to derive error bounds for the ensuing problem.

Optimal Control Problem

The optimal power for each heat source is determined by solving the following parametrized optimal control problem, given $\mu = (k, c, p^1, \cdots, p^{n_u}) \in \mathscr{P} \subset \mathbb{R}^{n_P}$, solve

$$\begin{aligned} &\min_{(y_e,u)\in Y_e\times U} J(y_e,u;\mu) = \frac{1}{2}\|y_e - y_d\|^2_{L^2(\Omega_t)} + \frac{\lambda}{2}\|u\|^2_U, \\ &\text{s.t. } (y_e,u) \in Y_e \times U := H^1(\Omega)\times\mathbb{R}^{n_u} \\ &\qquad \text{satisfies } a(y_e,v;\mu) = \langle \mathcal{B}(\mu)u, v\rangle_{Y_e',Y_e} := \sum_{i=1}^{n_u} b^i(v;\mu)u_i,\ \forall v \in Y_e \end{aligned} \tag{1}$$

The PDE constraint represents the weak formulation of the bioheat equation [6], $a(\cdot,\cdot;\mu) : Y_e \times Y_e \to \mathbb{R}$ is affine parameter dependent, continuous and coercive bilinear form and $b^i(\cdot;\mu) : Y_e \to \mathbb{R}$, $1 \le i \le n_u$ are non-affinely decomposable continuous linear forms, given by

$$\begin{aligned} a(y_e,v;\mu) &= \int_\Omega k\nabla y_e \nabla v + \int_\Omega c y_e v + \int_\Gamma h y_e v, \quad \forall y_e, v \in Y_e \\ b^i(v;\mu) &= b^i(v; g(x,\mu)) = \int_\Omega v g^i(x;\mu), \quad \forall v \in Y_e. \end{aligned} \tag{2}$$

where $y_e = y_e(\mu) \in Y_e$ is the tissue temperature, k is the tissue thermal conductivity, c is the perfusion parameter, and h is the Biot number. Furthermore $u = u(\mu) = [u_1, \cdots, u_{n_u}] \in U \subset \mathbb{R}^{n_u}$ represents the power control of each heat source $g^i(\cdot;\mu)$, $p^1, \cdots, p^{n_u}$ are the position parameters of each heat source, and $g^i(\cdot;\mu) \in L^\infty(\mathscr{P})$ for all $1 \le i \le n_u$. Finally $\Omega_t \subset \Omega$ denotes the target volume.

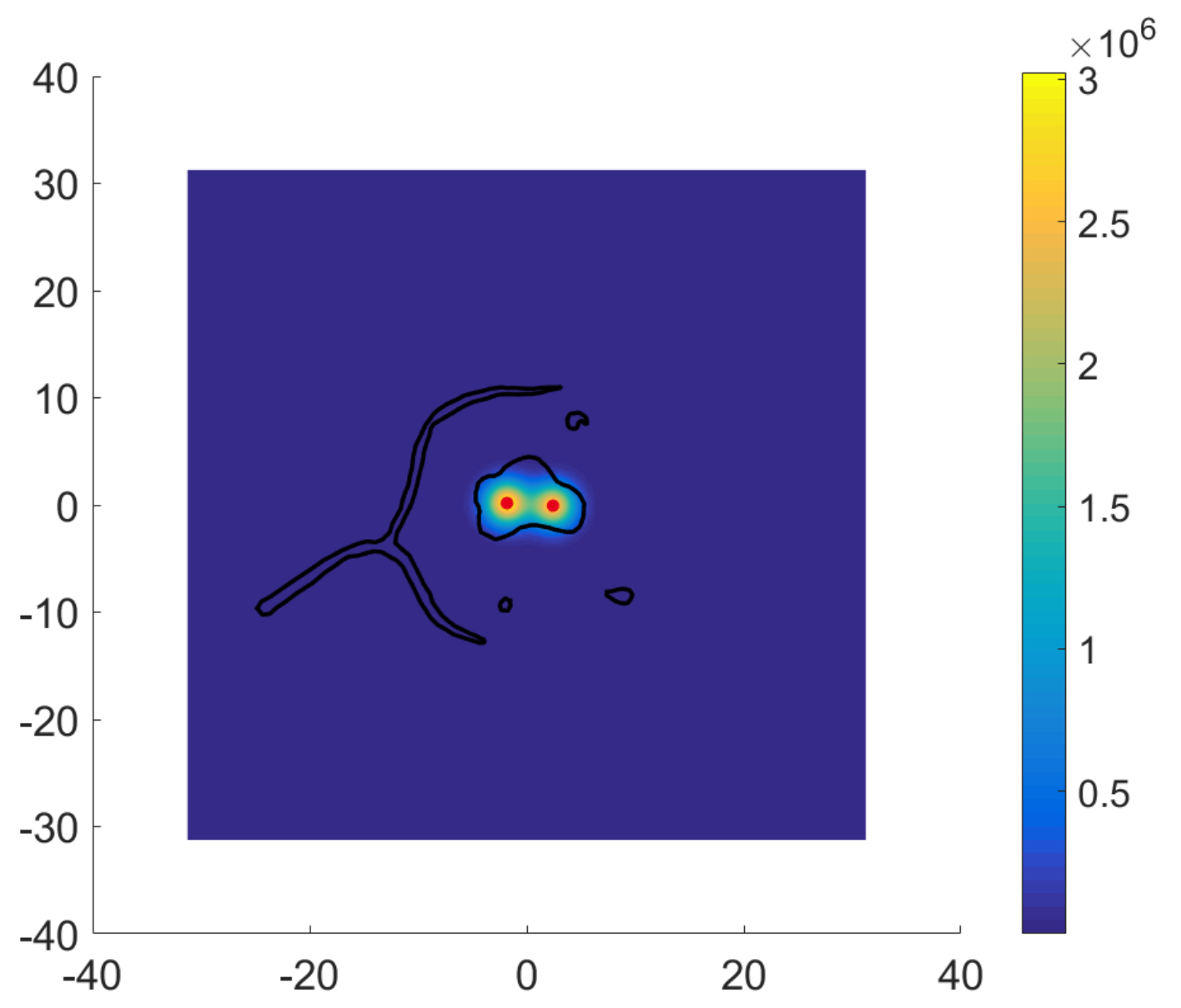

Figure 1: Heat source consisting of $u_1g^1(x;\mu) + u_2g^2(x;\mu) = u_1g(x,p^1) + u_2g(x,p^2)$.

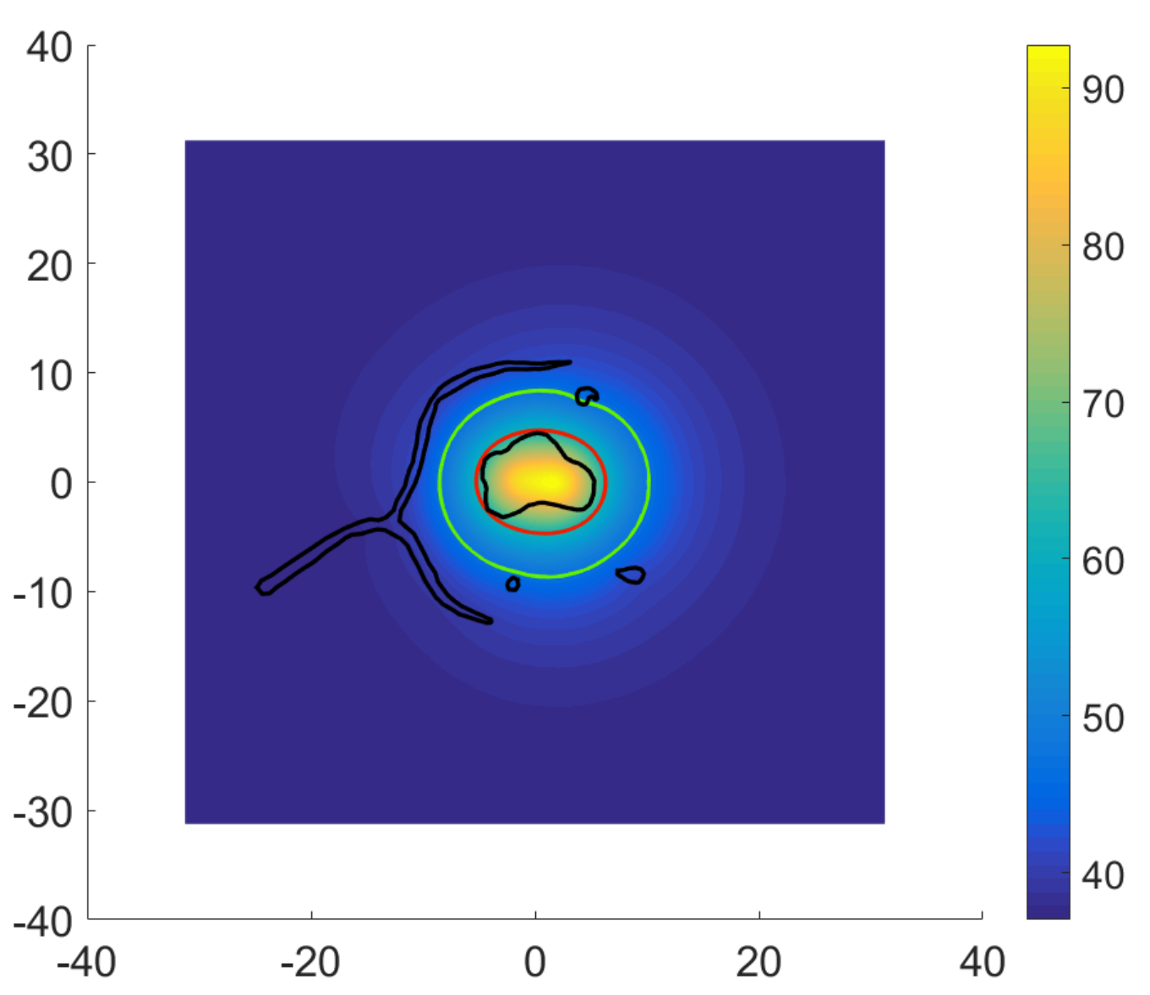

Figure 2: Resulting temperature in °C and tissue damage indication isolines $62°$C (red) and $47°$C (green).

For the numerical solution of problem (1) we use a finite element space $Y \subset Y_e$ spanned by piecewise linear continuous basis functions. By employing the Lagrangian formulation and deriving the first order optimality conditions, the unique solution to (1) can be determined by solving the following system of the state, adjoint and optimality equations. Given $\mu \in \mathscr{P}$, the optimal solution $(y^*, p^*, u^*) \in Y \times Y \times U$ satisfies

$$\begin{aligned} a(y^*,\phi;\mu) &= \big\langle \mathcal{B}(\mu)u^*, \phi\big\rangle_{Y',Y} && \forall \phi \in Y, \\ a(\varphi,p^*;\mu) &= \big(y_d(\mu) - y^*, \varphi\big)_{L^2(\Omega_t)} && \forall \varphi \in Y, \\ \big(\lambda(u^* - u_d), \psi\big)_U - \big\langle \mathcal{B}(\mu)\psi, p^*\big\rangle_{Y',Y} &= 0 && \forall \psi \in U \end{aligned} \tag{3}$$

Empirical Interpolation Method

We approximate the heat source $g(.;\mu)$ using the empirical interpolation method (EIM) [1] We construct the collateral reduced basis space $W^g_M := \text{span}\{\hat{g}^1(x) = g(x,\mu_1), \cdots, \hat{g}^M(x) = g(x,\mu_M)\}$, where $\mu_1, \cdots, \mu_M \in \mathscr{P}$ and $W^g_M \subset \{g(.;\mu) \,|\, \mu \in \mathscr{P}\} \subset L^\infty(\Omega)$. Then we define the approximation $g_M(x;\mu)$ to $g(x;\mu)$ by

$$g(.;\mu) \approx g_M(.,\mu) = \sum_{m=1}^{M} \omega_m(\mu)\hat{g}^m(x), \tag{4}$$

and the corresponding interpolation error

$$\varepsilon^g_M(\mu) := \|g(.;\mu) - g_M(.;\mu)\|_{L^\infty(\Omega)} \le \delta^g_M(\mu) \tag{5}$$

where $\delta^g_M(\mu)$ denotes either the next point based error estimate presented in detail in [5], or the rigorous error bound from [2].
Finally define $b^i_M(\cdot;\mu) : Y \to \mathbb{R}$, $1 \le i \le n_u$, s.t. $b^i_M(v;\mu) = b(v; g^i_M(x;\mu))$, and

$$\langle \mathcal{B}_M(\mu)u, v\rangle_{Y',Y} = \sum_{i=1}^{n_u} b^i_M(v;\mu)u_i \tag{6}$$

Reduced Basis Method

Given an 'integraded' reduced basis space

$$Y_N = \text{span}\{\zeta_n, 1 \le n \le N\} = \text{span}\Big\{y^*(\mu^n), p^*(\mu^n) 1 \le n \le \frac{N}{2}\Big\},\ 1 \le \frac{N}{2} \le \frac{N_{\text{max}}}{2}, \tag{7}$$

where the ζ_n, $1 \le n \le N$, are pairwise $(\cdot,\cdot)_Y$-orthogonal basis functions and N and N_{max} are even.
The reduced basis optimal control problem is given by

$$\begin{aligned} &\min_{y_{M,N}\in Y_N, u_{M,N}\in U} J(y_{M,N}, u_{M,N}),\ \text{ s.t. } (y_{M,N}, u_{M,N}) \in Y_N \times U \ \text{ solves} \\ &\qquad a(y_{M,N}, v;\mu) = \big\langle \mathcal{B}_M(\mu)u_{M,N}, v\big\rangle_{Y',Y},\ \forall v \in Y_N. \end{aligned} \tag{8}$$

Given $\mu \in \mathscr{P}$, the solution $(y^*_{M,N}, p^*_{M,N}, u^*_{M,N}) \in Y_N \times Y_N \times U$ to the associated reduced basis optimality system satisfies

$$\begin{aligned} a(y^*_{M,N},\phi;\mu) &= \big\langle \mathcal{B}_M(\mu)u^*_{M,N}, \phi\big\rangle_{Y',Y} && \forall \phi \in Y_N, \\ a(\varphi,p^*_{M,N};\mu) &= \big(y_d(\mu) - y^*_{M,N}, \varphi\big)_{L^2(\Omega_t)} && \forall \varphi \in Y_N, \\ \big(\lambda(u^*_{M,N} - u_d), \psi\big)_U - \big\langle \mathcal{B}_M(\mu)\psi, p^*_{M,N}\big\rangle_{Y',Y} &= 0 && \forall \psi \in U \end{aligned} \tag{9}$$

Define the optimality error for the state, adjoint and control to be

$$\begin{aligned} e^{y,*} &= y^*(u^*) - y^*_{M,N}(u^*_{M,N}) \\ e^{p,*} &= p^*(y^*(u^*)) - p^*_{M,N}(y^*_{M,N}(u^*_{M,N})) \\ e^{u,*} &= u^* - u^*_{M,N}. \end{aligned} \tag{10}$$

The challenge in the derivation of the error bounds for this problem is in the error residuals

$$\begin{aligned} r_y(\phi;\mu) &= a(e^{y,*},\phi;\mu) - \big\langle \mathcal{B}(\mu)u^*, \phi\big\rangle_{Y',Y} - \big\langle \mathcal{B}_M(\mu)u^*_{M,N}, \phi\big\rangle_{Y',Y} && \forall \phi \in Y, \forall \mu\mathscr{P}, \\ r_p(\varphi;\mu) &= a(\varphi, e^{p,*};\mu) + \big(e^{y,*}, \varphi\big)_{L^2(\Omega_t)} && \forall \varphi \in Y, \forall \mu\mathscr{P}, \\ r_u(\psi;\mu) &= \Big(\lambda e^{u,*}, \psi\Big) - \big\langle \mathcal{B}^\star(\mu)\psi, p^*\big\rangle_{Y',Y} - \big\langle \mathcal{B}^\star_M(\mu)\psi, p^*_{M,N}\big\rangle_{Y',Y} && \forall \psi \in U, \forall \mu\mathscr{P}. \end{aligned} \tag{11}$$

Outlook and Future Work

- Derive error bounds for the above problem
- Apply to the numerical example motivated by ablation treatments
- Extend to parabolic optimal control problems

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References

[1] M. Barrault, Y. Maday, N. C. Nguyen, and A. T. Patera. An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations. *Comptes Rendus Mathematique*, 339(9):667 – 672, 2004.

[2] J. L. Eftang, M. A. Grepl, and A. T. Patera. A posteriori error bounds for the empirical interpolation method. *Comptes Rendus Mathematique*, 348(9-10):575 – 579, 2010.

[3] M. Kärcher, Z. Tokoutsi, M. A. Grepl, and K. Veroy. Certified reduced basis methods for parametrized elliptic optimal control problems with distributed controls. *Journal of Scientific Computing*, Aug 2017.

[4] D. Klindworth, M. A. Grepl, and G. Vossen. Certified reduced basis methods for parametrized parabolic partial differential equations with non-affine source terms. *Computer Methods in Applied Mechanics and Engineering*, 209–212:144 – 155, 2012.

[5] N. C. Nguyen. A posteriori error estimation and basis adaptivity for reduced-basis approximation of nonaffine-parametrized linear elliptic partial differential equations. *Journal of Computational Physics*, 227:983–1006, Dec. 2007.

[6] H. H. Pennes. Analysis of tissue and arterial blood temperatures in the resting human forearm. *Journal of Applied Physiology*, 85(1):5–34, 1998.

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