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*Research Article*

# Analytical Solution of the Stability Problem for the Truncated Hemispherical Shell under Tensile Loading

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Analytical solution of the problem of buckling of truncated hemispherical shell of revolution, subjected to tension loading, is obtained. Assumption of membrane prebuckling state is applied, and the range of applicability of this assumption is estimated. The developed algorithm is based on the asymptotic simplification procedure of bifurcation equations. The formula for the bifurcation tension load is derived and compared with the earlier published empirical and numerical results. It is shown that it is sufficiently accurate and can be used in engineering practice.

## 1. Introduction

The tensile instability of a truncated hemispherical shell was analyzed for the first time by Yao (1963) [1]. It was assumed that the equatorial edge of the shell was clamped and the other edge was subjected to the constant tensile loads. After buckling occurred a large number of buckling waves were formed along the circumferential coordinate. The prebuckling state was assumed as linear membrane, and the bifurcation equations were based on the theory of shallow shells, for The bifurcation equations were solved by Yao [1] using the Bubnov-Galerkin approach. An empirical formula for the critical load was proposed on the basis of the numerical results. Test data for five samples of various geometry provided by Douglas Aircraft Company were also reported in [1].

This problem was revisited by Wu and Cheng (1970) [2]. They used Sanders' shell theory and assumed that prebuckling state was nonlinearly axisymmetric. A finite difference numerical approach was used for solving the bifurcation problem. The numerical results obtained in [1, 2] were in a good agreement.

Bushnell (1967) [3] (see also [4, 5] for more details) performed the FEM numerical simulations. He showed that for a shallow spherical cap the linear membrane state assumption was not justifiable, and it could lead to the critical load levels different from those predicted by the nonlinear solution. At the same time, for close to hemisphere truncated spherical cap influence of edge effect was negligible. Application of Reissner's simplified equations did not lead to significant differences in the values of bifurcation load.

Grigolyuk and Lipovtsev (1970) [6] (see also [7]) applied finite difference approach for analysis of buckling of shells of revolution under tensile loading. They used theory of shallow shells and bifurcation theory. They took into account the momentness of the prebuckling state. Numerical results obtained in [6] for

$$\alpha > \frac{10}{\sqrt{2\sqrt{3\left(1-\nu^2\right)}}}\sqrt{\frac{h}{R}} \tag{1}$$

practically coincide with results obtained by Yao [1]. The angle $\alpha$ is defined in Figure 1.

Analysis carried out in Bushnell (1967) [3, 4] allows obtaining the following estimate of applicability of the linear theory:

$$\frac{\pi}{2}-\alpha > \frac{20}{\sqrt{2\sqrt{3\left(1-\nu^2\right)}}}\sqrt{\frac{h}{R}}. \tag{2}$$

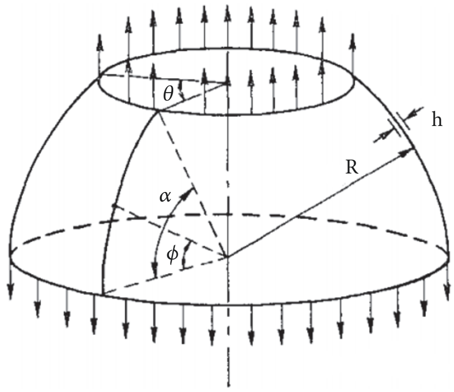


Figure 1: Truncated hemispherical shell under tensile loading.

Radhamohan and Prasad (1974) [8] used Sanders' equations and the numerical method to prove that the different boundary conditions did not affect the buckling phenomena of hemispherical shells even when the prebuckling state was represented by a general nonlinear bending state.

In the present paper a simple formula for the bifurcation load is derived for a truncated hemispherical shell under the tensile loading.

Note that the problem under study has both the fundamental analytical importance and the practical applied significance. Very large spherical tanks for transporting liquid natural gas are supported by the relatively short cylindrical shells. When the tank is filled by less than half its weight creates axial tension that might cause buckling; see, e.g., [3, 9].

## 2. Analytical Solution

The truncated shell of revolution under study is shown in Figure 1 [1].

The determining system of partial differential equations in the spherical coordinate system can be written as follows:

$$\begin{gathered}\frac{1}{12(1-\nu^2)}\left(\frac{h}{R}\right)^2\nabla^4 w-\nabla^2 F\\ -p\left(\frac{\partial^2 w}{\partial\phi^2}-\sec^2\phi\frac{\partial^2 w}{\partial\theta^2}+\tan\phi\frac{\partial w}{\partial\phi}\right)\sec^2\phi=0;\\ \nabla^4 F+\nabla^2 w=0,\end{gathered}\tag{3}$$

where

$$\begin{gathered}\nabla^4=\nabla^2\nabla^2;\\ \nabla^2=\frac{\partial^2}{\partial\phi^2}+\sec^2\phi\frac{\partial^2}{\partial\theta^2}-\tan\phi\frac{\partial}{\partial\phi};\\ w=\frac{W}{R};\\ F=\frac{\Phi}{EhR^2};\\ p=\frac{P}{Eh};\end{gathered}\tag{4}$$

$W$ is the radial displacement; $\Phi$ is the stress function; $E$ is Young's modulus; $\nu$ is Poisson's ratio; $P$ is the linear tensile force applied to the upper edge of the shell.

The shallow shell theory will be used and the membrane prebuckling state will be assumed.

The validity of the first assumption follows from the results of [2–4, 8], where more accurate Sanders' and Reissner's theories of shells were used. The second assumption is justified under the conditions given by (1) and (2). These conditions are assumed in the sequel.

System of (3) must be supplemented by boundary conditions; for example,

$$w=\frac{\partial w}{\partial\phi}=F=\frac{\partial F}{\partial\phi}=0\quad\text{at }\phi=0,\alpha.\tag{5}$$

Boundary value problem defined by (3) and (5) describes bifurcation of the shell of revolution under study under the assumption of the membrane prebuckling state. Note that as shown in [8] the buckling conditions have a small effect on the buckling load.

Considering that the loss of stability occurs with the formation of a large number of half-waves in the annular direction, (1) can be simplified as follows:

$$\begin{gathered}\frac{1}{12(1-\nu^2)}\left(\frac{h}{R}\right)^2\sec^4\phi\frac{\partial^4 w}{\partial\theta^4}-\sec^2\phi\frac{\partial^2 F}{\partial\theta^2}\\ +p\sec^4\phi\frac{\partial^2 w}{\partial\theta^2}=0;\\ \sec^2\phi\frac{\partial^4 F}{\partial\theta^4}+\frac{\partial^2 w}{\partial\theta^2}=0.\end{gathered}\tag{6}$$

Equations (6) can be reduced to the following single equation:

$$\frac{1}{12(1-\nu^2)}\left(\frac{h}{R}\right)^2\frac{\partial^4 w}{\partial\theta^4}+\cos ec^4\phi w+p\frac{\partial^2 w}{\partial\theta^2}=0.\tag{7}$$

Circumferential prebuckling stress is given by the formula

$$T_\theta^{pr}=-P\sec^2\phi\tag{8}$$

It is logical to fix the value of the variable $\phi$ in (7) at which the prebuckling stress $T_\theta^{pr}$ reaches its maximum value. It follows from (8) that this value is $\phi=\alpha$. However, due to the presence of the edge effect, this value will be shifted by an amount of the order of $\sqrt{h/R}$ characterizing the extent of the edge effect.

Using asymptotic estimates from [10] and numerical data from [3, 4], it can be assumed that the maximum value of $T_\theta^{pr}$ is reached at $\phi=\beta=\alpha-4\sqrt{h/R}$. Consequently, $\phi=\beta$ is assumed in (7).

Accordingly, $\phi=\alpha$ is substituted in (6):

$$\frac{1}{12(1-\nu^2)}\left(\frac{h}{R}\right)^2\frac{\partial^4 w}{\partial\theta^4}+co\sec^4\beta w+p\frac{\partial^2 w}{\partial\theta^2}=0.\tag{9}$$

Equation (9) describes the bifurcation loss of stability of a circular ring with an elastic filler of the Winkler type.

Writing down its solution in the form $w = \sin n\theta$ and minimizing by $n$ yields the following formula for the bifurcation tension load and number of circumferential waves:

$$p = \frac{h}{\sqrt{3(1-\nu^2)}R}\cos^2(\beta) \approx \frac{0.577h}{\sqrt{(1-\nu^2)}R}\cos^2(\beta), \quad (10)$$

$$n = \sqrt[4]{12(1-\nu^2)}\sqrt{\frac{R}{h}}\cos(\beta). \quad (11)$$

## 3. Comparison with the Empirical Formulas, Numerical and Experimental Results

Yao [1] and Bushnell [3, 4] derived simple empirical formulae for the bifurcation tension load using numerical data. For a truncated hemisphere, they can be written as follows:

according to Yao [1],

$$p = 1.57\cos^2(\alpha)\left(\frac{h}{R}\right)^{1.12}; \quad (12)$$

according to Bushnell [3, 4],

$$p = \frac{0.622h}{(1-\nu^2)R}\cos^2\left(\alpha - 3.1\frac{h}{R}\right); \quad (13)$$

$$n = 1.84\sqrt{\frac{R}{h}}\cos\left(\alpha - 4.2\frac{h}{R}\right). \quad (14)$$

The structure of formulae (12) and (14) coincides with (10) derived using the asymptotic method. The comparison of numerical results obtained using the presently derived formula (10) and the formula (12) [1] for some values of parameters is given in Table 1.

The accuracy of the asymptotic solution, as one would expect, increases with the ratio $R/h$ and is quite satisfactory.

Formula (13) [3, 4] yields values of the bifurcation load by 8-10% higher in the considered range of parameters than formula (10).

Regarding comparison with the experimental data, the test results have been reported in [1] for 5 samples. The theoretical values are found to be greater than the experimental ones. The explanation of this common mismatch in the theory of elastic stability of thin-walled systems is given in [9, 11]. It is related to a very high sensitivity to the initial geometrical imperfections and residual stresses.

## 4. Some Generalizations

One of the advantages of an asymptotic solution is that it allows a simple generalization to other thin-walled shells. As the first example of generalization, consider an orthotropic hemispherical shell. The simplified stability equation in this case can be written as follows:

$$\frac{D_2}{B_1R^2}\frac{\partial^4 w}{\partial\theta^4} + \cos ec^4\phi w + p\frac{\partial^2 w}{\partial\theta^2} = 0. \quad (15)$$

Here

$w = W/R$; $F = \Phi/B_1R^2$; $p = P/B_1$; $D_2$ is the bending stiffness in circumferential direction; $B_1$ is the membrane stiffness in meridional direction.

Note that in deriving (15), it was assumed that the orthotropy of the shell was not very strong, i.e., the flexural (membrane) stiffness in the circumferential and meridional directions is of the same order: $D_1 \sim D_2$, $B_1 \sim B_2$.

The following expressions are obtained for the bifurcation tension load and number of circumferential waves:

$$p = \sqrt{\frac{D_2}{B_1R^2}}\cos^2(\beta);$$
$$n = \sqrt[4]{\frac{B_1R^2}{D_2}}\cos(\beta); \quad (16)$$
$$\beta = \alpha - 2\sqrt[4]{\frac{B_1R^2}{D_2}}.$$

Consider now the general case of shell of revolution with radii of curvature $R_1, R_2$. The original equation can be represented as

$$\frac{1}{12(1-\nu^2)}\left(\frac{h}{R_2}\right)^2\frac{\partial^4 w}{\partial\theta^4} + co\sec^4\phi\left(\frac{R_2}{R_1}\right)^2 w + p\frac{R_2}{R_1}\frac{\partial^2 w}{\partial\theta^2} = 0. \quad (17)$$

In deriving (17), it was assumed that the variation of the radii of curvature in spatial coordinates is small: $\partial R_i/\partial\theta \sim R_i$; $\partial R_i/\partial\phi \sim R_i$, $i = 1, 2$.

The following expressions are obtained in this case for the bifurcation tension load and number of circumferential waves:

$$p = \frac{h}{\sqrt{3(1-\nu^2)}R_2(\beta)}\cos^2(\beta) \approx \frac{0.577h}{\sqrt{(1-\nu^2)}R_2(\beta)}\cos^2(\beta); \quad (18)$$
$$n = \sqrt[4]{12(1-\nu^2)}\sqrt{\frac{R_2(\beta)}{h}}\cos(\beta).$$

## 5. Conclusions

The problem of buckling of truncated hemispherical shell subjected to tension loading is solved analytically using the asymptotic simplification procedure for the bifurcation equations. The membrane prebuckling state is assumed, and the range of applicability of this assumption is estimated. The formula for the bifurcation tension load is derived and compared with the earlier published empirical and numerical results. It is shown that it is sufficiently accurate and can be used in engineering practice.

The results of the present paper demonstrate that it is possible to obtain a simple and accurate engineering formula in two ways.

TABLE 1: Comparison of numerical, asymptotic, and experimental results.

| $\alpha$ | 0.1287 | 0.1306 | 0.1889 |
|---|---|---|---|
| $R/h$ | 1600 | 480 | 757 |
| $10^4 p$; $p$ calculated from (10) | 3.78 | 12.87 | 7.48 |
| $10^4 p$; $p$ calculated from (12) [1] | 3.81 | 13.49 | 7.05 |
| $10^4 p_{ex}$; $p_{ex}$ experimental data from [1] | 1.47 | 6.15 | 4.72 |

The first one is to apply the asymptotic method [10, 12–14] for the original boundary value problem, as it is done in the present paper. In this way, after substantial simplifications, a simple analytical formula (10) for the bifurcation tension load is derived.

The second approach involves obtaining a large array of numerical data, applying standard codes and consequently reprocessing information using the empirical formulae. This approach was used in [1, 3, 4].

It is worth noting that, for the problem under consideration, both approaches led to approximately similar results, thereby confirming their reliability.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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