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On the Structure of Degree Vector Sets

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Summary

A graph is a suitable tool to describe the structure of a network. Standard examples are road networks, supply networks or telecommunication networks. There are many more theoretical models which can be represented as graphs. A graph consists of a set of points (or vertices) and a set of links (or edges) where each edge connects two points. In the case of a road network each point can be seen as a junction and each edge represents a road.

Suppose that each edge in a given graph has a fixed direction. Similar to a network of one-way streets every link connecting two vertices x and y is either directed from x to y or from y to x. Such a directed graph is called an orientation of the underlying (undirected) graph. Usually, we are interested in orientations of a given graph which satisfy additional conditions. For example, orientations which guarantee certain properties on the connectivity are studied intensively. A further well-known type of condition involves the number of links which start at each point. More precisely, we search for an orientation of a given graph which has in each vertex a prescribed number of outgoing directed edges. If such an orientation exists, we call the assignment of the prescribed numbers a degree vector of the given undirected graph.

Let G be an arbitrary graph. There is a well-known result which describes sufficient conditions satisfied by each degree vector of G. Moreover, there is an efficient procedure which decides whether a given integral vector is a degree vector of G or not. On the other hand, it is much more difficult to determine the complete set of degree vectors for G efficiently. One approach to solve this problem uses a partial order ($\preccurlyeq$) on the set of integral vectors which is closely related to the dominance order. Starting with the (unique) minimal degree vector $\underline{s}$ of G we are able to determine all integral vectors succeeding $\underline{s}$ with respect to $\preccurlyeq$. Subsequently, we apply the previous step to the successors of $\underline{s}$ and obtain further vectors. We continue this procedure until we reach the maximal degree vector of G. Finally, we consider only those vectors which are constructed this way and whose entries are between 0 and the degree of the corresponding vertex of G. A graph is called degree complete, if the set of integral vectors obtained by this approach is equal to the set of degree vectors. The concept of degree complete graphs was introduced by Qian [23] in 2006. Qian also characterized degree complete graphs in terms of two forbidden configurations which must not be contained in any of these graphs. It turns out that the degree completeness of a graph depends on the order in which the vertices of the graph are considered, that is, it depends on how the vertices are labeled. In this thesis we investigate the property of degree completeness more detailed. Among others we solve a problem stated by Qian asking for a characterization of those graphs which do not have a degree complete labeled version. Furthermore, we generalize the concept of degree complete graphs in the sense that we extend it to certain classes of partial orders.

Particularly, we investigate partial orders which are induced by cross-free sets or convex cones.

The structure of this thesis is the following. The first chapter includes all important definitions and notations. In chapter 2 we start with a brief survey of results concerning orientations with degree constraints. In particular, we present a famous result by Landau [19] and a well-known result due to Hakimi [14]. The second section of this chapter contains several properties of the polytope defined as the convex hull of all degree vectors of a given graph.
In chapter 3 we introduce a formal definition of degree complete graphs and give an alternative proof of the characterization by Qian. The structure used in the proofs is essential for our later investigations. We also focus on the motivation of the concept of degree completeness. Subsequently, we characterize all (unlabeled) graphs which have a so-called degree complete labeling of their vertices and obtain a solution to the labeling problem of degree complete graphs. Furthermore, we evaluate this new result in the light of a concept which is slightly stronger than the concept of degree completeness. We underline the particular importance of the partial order $\preccurlyeq$ for the structure of graphs with degree complete labeling.
In the remaining part of this thesis we study classes of partial orders including $\preccurlyeq$ to extend the concept of degree completeness. In chapter 4 we define the class of partial orders (CFPO) induced by cross-free sets. These orders form a subset of the class of so-called cone orders. We characterize the vectors generating the associated cone of every CFPO. After these investigations on different classes of partial orders we return to degree completeness. In chapter 5 we consider graphs which are degree complete with respect to a CFPO. Particularly, we generalize most of our results from chapter 3. We also obtain more insight into the initial concepts of degree complete graphs and labelings.
Finally, in chapter 6 we generalize the concept of degree completeness to all cone orders in $\mathbb{R}^n$. A geometric interpretation of the results from the previous sections and the structure of degree vector polytopes yield a characterization of those graphs which are degree complete with respect to a cone order.

Zusammenfassung

Ein Graph ist ein geeignetes Hilfsmittel, um beispielsweise den Aufbau eines Netzwerkes zu beschreiben. Bei einem solchen Netzwerk kann es sich - je nach Anwendung - um ein Straßen-, Versorgungs- oder Telekommunikationsnetz handeln. Ebenso lassen sich viele theoretische Modelle durch Graphen darstellen. Allen Graphen ist gemein, dass es eine Menge von Punkten (oder Knoten) und eine Menge von Kanten (oder Verbindungen) gibt, die jeweils zwei Punkte miteinander verbinden. In dem Beispiel eines Verkehrsnetzes können die Punkte als Abzweigungen und die Kanten als Straßen interpretiert werden.
Stellen wir uns vor, dass in einem gegebenen Graphen jede Kante eine vorher festgelegte Richtung besitzt. D.h. wie in einem Netz von Einbahnstraßen darf jede Verbindung zwischen zwei Knoten x und y entweder nur von x nach y oder nur von y nach x durchlaufen werden. In diesem Zusammenhang sprechen wir auch von einer Orientierung des gegebenen Graphen. Häufig werden Orientierungen eines vorgegebenen Graphen, die weitere Eigenschaften besitzen, gesucht. So sind zum Beispiel Bedingungen an die Erreichbarkeit der einzelnen Punkte in einem solchen orientierten Graphen von Interesse. Eine weitere wohlbekannte Zusatzbedingung betrachtet die Anzahl der ausgehenden Verbindungen in jedem Punkt. Genauer gesagt, stellt sich die Frage, ob zu einem gegebenen ungerichteten Graphen eine Orientierung existiert, welche in jedem Knoten eine vorgegebene Anzahl von ausgehenden Verbindungen besitzt. Falls es eine solche Orientierung gibt, so bezeichnen wir die Zuordnung der vorgegebenen Anzahlen ausgehender Verbindungen als sogenannten Gradvektor des nicht-orientierten Graphen.
Sei G ein beliebiger Graph. Die Frage, unter welchen Bedingungen ein gegebener Vektor ein Gradvektor zu G ist, ist gut untersucht. Es gibt effiziente Methoden mit deren Hilfe sich diese Frage beantworten lässt. Ungleich aufwendiger ist es jedoch alle zu G gehörenden Gradvektoren zu bestimmen. Ein Ansatz zur Lösung dieses Problems nutzt eine partielle Ordnung ($\preccurlyeq$), die in enger Beziehung zur Dominanzordnung steht. Zuerst betrachten wir den eindeutig bestimmten minimalen Gradvektor $\underline{s}$ von G und bestimmen die Nachfolger von $\underline{s}$ bzgl $\preccurlyeq$. Anschließend wenden wir diese Prozedur auf die so entstandenen Vektoren an. Wir führen dieses Verfahren solange fort bis wir den maximalen Gradvektor von G erreichen. Schließlich verwenden wir nur jene Vektoren, deren Einträge zwischen 0 und dem Grad des entsprechenden Punktes aus G liegen. Ein Graph heißt gradvollständig, falls die so konstruierte Menge von Vektoren genau der Menge aller Gradvektoren des Graphen entspricht. Dieses Konzept wurde 2006 von Qian [23] eingeführt. Zudem charakterisierte er die gradvollständigen Graphen mit Hilfe zweier verbotener Konfigurationen, die in keinem solchen Graphen enthalten sein dürfen. Es zeigt sich, dass die Eigenschaft der Gradvollständigkeit eines Graphen, davon abhängt in welcher Reihenfolge die Knoten des Graphen betrachten werden, d.h. wie die Knoten gelabelt sind. Diese Tatsache wurde bereits von Qian erkannt und als Problemstellung formuliert.

In dieser Arbeit untersuchen wir gradvollständige Graphen im Detail. Zunächst lösen wir das oben genannte Problem von Qian, indem wir die Graphen charakterisieren, für die jede Knotenreihenfolge des Graphen nicht gradvollständig ist. Außerdem verallgemeinern wir das Konzept eines gradvollständigen Graphen in dem Sinne, dass wir es auf verschiedene Klassen von partiellen Ordnungen erweitern. Dazu betrachten wir partielle Ordnungen, die durch kreuzungsfreie Menge oder konvexe Kegel induziert werden.

Die vorliegende Arbeit ist wie folgt gegliedert. Nach einer Vorstellung der wichtigsten Definitionen und Notationen in Kapitel 1 betrachten wir in Kapitel 2 eine kurze Übersicht der Ergebnisse zu Orientierungen mit Gradbedingungen. Hervorzuheben sind hier ein Resultat von Landau [19] und eines von Hakimi [14]. Anschließend untersuchen wir die geometrische Struktur der Menge aller Gradvektoren eines gegebenen Graphen.
In Kapitel 3 geben wir zunächst eine formale Definition der gradvollständigen Graphen und präsentieren einen alternativen Beweis für die Charakterisierung von Qian. Die dabei verwendete Beweisstruktur ist grundlegend für weitere Resultate der vorliegenden Arbeit. Außerdem widmen wir uns in einem Abschnitt der Frage nach der Motivation des Konzepts der gradvollständigen Graphen. Danach charakterisieren wir alle Graphen, die ein sogenanntes gradvollständiges Label besitzen und erhalten auf diese Weise eine Lösung des von Qian aufgestellten Problems. Zusätzlich werten wir dieses neue Ergebnis unter Zuhilfenahme eines Konzeptes, das aus dem der Gradvollständigkeit abgeleitet ist, aus. Daraus schließen wir auf die besondere Bedeutung der partiellen Ordnung $\preccurlyeq$ für die Struktur der Graphen mit gradvollständigem Label.
Anschließend betrachten wir verschiedene Klassen partieller Ordnungen, die $\preccurlyeq$ enthalten, um das Konzept der Gradvollständigkeit zu erweitern. In Kapitel 4 definieren wir die Klasse der partiellen Ordnungen (CFPO), die durch kreuzungsfreie Mengen induziert werden. Diese Ordnungen bilden eine Teilmenge in der Klasse der sogenannten Kegelordnungen. Für jede CFPO $\preceq$ bestimmen wir die Vektoren, welche den zu $\preceq$ gehörenden Kegel erzeugen. Im Anschluss wenden wir uns wieder der Untersuchung gradvollständiger Graphen zu. In Kapitel 5 betrachten wir Graphen, die bzgl. einer CFPO gradvollständig sind. Außerdem verallgemeinern wir die meisten Ergebnisse aus Kapitel 3 auf diese Klasse partieller Ordnungen. Diese Resultate liefern eine tiefere Einsicht zum ursprünglichen Konzept gradvollständiger Graphen und Graphen mit gradvollständigem Label.
In Kapitel 6 verallgemeinern wir das Konzept der gradvollständigen Graphen schließlich auf alle Kegelordnungen im $\mathbb{R}^n$. Dazu interpretieren wir unter anderem die vorangegangenen Ergebnisse aus einer geometrischen Perspektive. Zusätzlich verwenden wir die erlangten Kenntnisse über das Polytop, welches die konvexe Hülle der Gradvektoren eines gegebenen Graphen darstellt, um die Graphen zu charakterisieren, die bzgl. einer Kegelordnung gradvollständig sind.

Contents

1 Definitions and notations

In this chapter we introduce all fundamental notations and definitions used in this work. We need basic concepts from graphs and digraphs, cones and polytopes as well as partially ordered sets.

1.1 Graphs and digraphs

For the notations of graphs, digraphs and networks we follow Bang-Jensen and Gutin [4] and Korte and Vygen [18].

An *undirected graph* or *graph* is a triple (V, E, Ψ), where V and E are finite sets and Ψ is a function from E to $\binom{V}{2}$. An element of V is called *vertex* and an element of E is an *edge*. Two edges e and e' are *parallel* if $\Psi(e) = \Psi(e')$. A *simple* graph is a graph without parallel edges.

Let G be a graph. By $V(G)$ and $E(G)$ we refer to the vertex set and the edge set of G, respectively. For an edge $e \in E(G)$ with $\Psi(e) = \{v, w\}$ we use the notations vw and $\{v, w\}$. In this case we say that e *joins* v and w. Moreover, v and w are called *adjacent* and v is a *neighbor* of w (and vice versa). The vertices v and w are the *endpoints* of e. If v is an endpoint of an edge e, we say that v and e are *incident*. For a vertex $v \in V(G)$ we define $N_G(v)$ as the set of neighbors of v.

We call a set $X \subseteq V(G)$ *independent* if X does not contain any pair of adjacent vertices of G. Similarly, an edge set $M \subseteq E(G)$ is *independent* if it does not include any pair of distinct edges that have a common endpoint.

A *subgraph* H of G is a graph satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We also say that G *contains* H. Moreover, if $V(H) = V(G)$, then H is a *spanning subgraph* of G. Considering a set $X \subseteq V(G)$ we call a subgraph H of G *induced* by X if $V(H) = X$ and $E(H) = \{uv \in E(G) \,|\, u, v \in X\}$. In this case we write $H = G[X]$. Furthermore, we use the notation $G - X$ for the subgraph of G induced by $V(G) \setminus X$. We write $G - x$ instead of $G - \{x\}$. For any subset $F \subseteq \binom{V(G)}{2}$ we define $G - F$ as the subgraph with vertex set $V(G)$ and edge set $E(G) \setminus F$. If F consists of a single element e we write $G - e$ instead of $G - \{e\}$. The addition of a new edge $e \notin E(G)$ to G abbreviated by $G + e$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e\}$. For two graphs G and H the graph $G + H$ is defined by $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$.

Two graphs G and H are called *isomorphic* if there are two bijections $\Phi_V : V(G) \to V(H)$ and $\Phi_E : E(G) \to E(H)$ such that $\Phi_E(vw) = \Phi_V(v)\Phi_V(w)$ for all $vw \in E(G)$. A *labeling* of a graph G on n vertices is a bijective function $f : V(G) \to \{1, \ldots, n\}$. A graph G together with a labeling f is called *labeled graph* and denoted by G_f. For simplicity we

usually omit f and identify the vertex set of a labeled graph with $\{1, \dots, n\}$.
For a graph G denote $n(G) = |V(G)|$ and $m(G) = |E(G)|$ the *order* and the *size* of G, respectively. Considering two subsets X and Y of $V(G)$ we define $[X, Y]_G$ as the set of edges of G which have one endpoint in X and the other endpoint in Y. By $m_G(X, Y)$ we refer to the number of edges in $[X, Y]_G$. If $Y = V(G) \setminus X$, we use the notation $\delta_G(X) = [X, V \setminus X]_G$. Moreover, if $X = \{v\}$ we write $\delta_G(v)$ instead of $\delta_G(\{v\})$. The *degree* of a vertex $v \in V(G)$ is defined as $|\delta_G(v)|$ and denoted by $d_G(v)$. It is equal to the number of edges in G which are incident to v. A vertex with degree equal to zero is called *isolated*. For a labeled graph of order n we interpret $d_G \in \mathbb{Z}^n$ as the vector whose i-th entry is the degree of vertex i.
For $k \geq 0$ let $\{v_1, \dots, v_{k+1}\}$ be a set of vertices and $\{e_1 \dots, e_k\}$ a set of edges of G. The sequence $v_1 e_1 v_2 e_2 \dots e_k v_{k+1}$ is a *path* if the vertices $v_1, \dots, v_{k+1}$ are pairwise distinct and $e_i = v_i v_{i+1}$ for $1 \leq i \leq k$. The sequence $v_1 e_1 v_2 e_2 \dots e_k v_{k+1}$ is called a *cycle* if $v_1, \dots, v_k$ are pairwise distinct, $v_1 = v_{k+1}$, and $e_i = v_i v_{i+1}$ for $1 \leq i \leq k$. Since every edge of G appears at most once in a path or a cycle we sometimes omit the edges and write a sequence of vertices. For two vertices v and w in $V(G)$ a path with $v_1 = v$ and $v_k = w$ is called a vw-path. The *length* of a path or a cycle is equal to the number of its edges. A cycle of length k is also denoted as a k-cycle.
We say G is *connected* if for every pair of distinct vertices $v, w \in V(G)$ there is a vw-path. By a *component* we refer to a maximal connected induced subgraph of G. A *forest* is a graph without any cycles and a connected forest is called *tree*. Considering a tree T we say that a vertex $v \in V(T)$ is a *leaf* if its degree is equal to 1. A tree containing at most one vertex which is not leaf is called a *star*. A graph of order n consisting of a single path, cycle or star is denoted by P_n, C_n or $K_{1,n-1}$. Moreover, by a *complete graph* we refer to a simple graph G with $\Psi(E(G)) = \binom{V(G)}{2}$. The complete graph on n vertices is denoted by K_n.
A *directed graph* or *digraph* is a triple (V, A, Ψ), where V and A are finite sets and Ψ is a function from A to $\{(v, w) \in V \times V \mid v \neq w\}$. An element of V is called *vertex* and an element of A is an *arc*. Two arcs a and a' are *parallel* if $\Psi(a) = \Psi(a')$. By *reversing the arc* $a = (v, w)$, we mean that we replace $\Psi(a) = (v, w)$ by $\Psi(a) = (w, v)$. For a digraph D we define the *underlying (undirected) graph* as a graph G with $V(G) = V(D)$ and $E(G) = \{uv \mid (u, v) \in A(D)\}$. In this case we call D an *orientation* of G. Hence we obtain an orientation of a graph G by assigning each edge of G with a direction. An orientation of a forest or a tree is called *directed forest* or *directed tree*, respectively.
Let D be a digraph. By $V(D)$ and $A(D)$ we refer to the set of vertices and the set of arcs of D. For an arc $a \in A(D)$ with $\Psi(a) = (v, w)$ we usually write $a = (v, w)$. We call v the *tail* and w the *head* of a. Alternatively, we say *a leaves* v and *enters* w. Furthermore, w is a *positive neighbors* of v and v is a *negative neighbors* of w. For a vertex $v \in V(D)$ denote $N_D^+(v)$ (respectively $N_D^-(v)$) the set of positive (respectively negative) neighbors of v.
The notations for order, size, (induced) subdigraphs, deleting vertices or arcs of a digraph, and adding vertices or arcs to a digraph are completely analogue to the definitions for undirected graphs. The same holds for isomorphic digraphs.
Considering two subsets X and Y of $V(D)$ we define $(X, Y)_D$ as the set of arcs of D which have its tail in X and its head in Y. We also say that an arc in $(X, Y)_D$ leaves

X and enters Y or the arc *points* from X to Y. By $m_D(X,Y)$ we refer to the number of arcs in $(X,Y)_D$. If $Y = V(D) \setminus X$, we use the notations $\delta^+_D(X) = (X, V \setminus X)_D$ and $\delta^-_D(X) = (V \setminus X, X)_D$. Moreover, if $X = \{v\}$ we write $\delta^+_D(v)$ and $\delta^-_D(v)$ instead of $\delta^+_D(\{v\})$ and $\delta^-_D(\{v\})$, respectively. For a vertex $v \in V(D)$ the *out-degree* of v is defined by $d^+_G(v) = |\delta^+_D(v)|$ and the *in-degree* of v by $d^-_G(v) = |\delta^-_D(v)|$. A directed tree is an *in-tree* if it contains exactly one vertex with out-degree zero.
For $k \geq 0$ let $\{v_1, \ldots, v_{k+1}\}$ be a set of vertices and $\{a_1 \ldots, a_k\}$ a set of arcs of D. The sequence $v_1a_1v_2a_2 \ldots a_kv_{k+1}$ is a *(directed) path* if the vertices $v_1, \ldots, v_{k+1}$ are pairwise distinct and $a_i = (v_i, v_{i+1})$ for $1 \leq i \leq k$. The vertex v_1 is the *initial vertex* and v_{k+1} the *terminal vertex* of the path. A path with initial vertex v and terminal vertex w is called a (v,w)-path. We also use the notation that v *reaches* w. In particular, a vertex reaches itself. For two subsets X and Y of $V(D)$ a (X,Y)-path is a directed path (x,y)-path where $x \in X$ and $y \in Y$. The sequence $v_1a_1v_2a_2 \ldots a_kv_{k+1}$ is a *(directed) cycle* if $v_1, \ldots, v_k$ are pairwise distinct, $v_1 = v_{k+1}$, and $a_i = (v_i, v_{i+1})$ for $1 \leq i \leq k$. The *length* of a path or a cycle is equal to the number of arcs. We call a digraph *acyclic* if it does not contain a directed cycle.
A digraph D is *(weakly) connected* if its underlying graph G is connected. Furthermore, by *weak components* of D we refer to the components of G. On the other hand a digraph D is *strongly connected* or *strong* if for every pair of distinct vertices $v, w \in V(D)$ there exists a (v,w)-path and a (w,v)-path. Therefore, a digraph is strong if and only if every vertex reaches every other vertex. A *strong component* is a maximal strong induced subdigraph of D.
The *incidence matrix* of a digraph D is a matrix $B_G = (b_{u,a})_{u \in V(D), a \in A(D)}$ with

$$b_{u,(v,w)} = \begin{cases} 1, & \text{if } u = v, \\ -1, & \text{if } u = w, \\ 0, & \text{else.} \end{cases}$$

A matrix M is *totally unimodular* if the determinant of every square submatrix of M is in $\{0, 1, -1\}$. It is well-known that the incidence matrix of every digraph is totally unimodular. By a famous result of Hoffman and Kruskal [16] implies that for every totally unimodular $(m \times n)$-matrix M and every vector $b \in \mathbb{Z}^m$ the polyhedron $\{x \in \mathbb{R}^n \mid Mx \leq b,\ x \geq 0\}$ is integral.
Finally, we define flows in networks. Let D be a digraph, $c : A(D) \to \mathbb{R}_+$ a *capacity function*, and $p, q \in V(D)$ two specified vertices. The capacity of a set $F \subseteq A(D)$ is given by $c(F) = \sum_{a \in F} c(a)$. For a set $X \subsetneq V(D)$ with $p \in X$ and $q \notin X$ the arc set $\delta^+_D(X)$ is called a (p,q)-*cut*. The quadruple $\mathcal{N} = (D, c, p, q)$ is called a *network*. A (p,q)-*flow* is a function $f : A(D) \to \mathbb{R}_+$ satisfying

$$\sum_{a \in \delta^+_D(v)} f(a) - \sum_{a \in \delta^-_D(v)} f(a) = 0 \quad \text{for all } v \in V(D) \setminus \{p, q\},$$

$$\sum_{a \in \delta^+_D(p)} f(a) - \sum_{a \in \delta^-_D(p)} f(a) \geq 0,$$

$$\sum_{a \in \delta^+_D(q)} f(a) - \sum_{a \in \delta^-_D(q)} f(a) \leq 0,$$

and $0 \leq f(a) \leq c(a)$ for all $a \in A(D)$. The *value* of a (p,q)-flow f in $\mathcal{N}$ is defined as

$$\mathrm{val}(f) = \sum_{a \in \delta_D^+(p)} f(a) - \sum_{a \in \delta_D^-(p)} f(a).$$

A famous result by Ford and Fulkerson [10] shows the following.

Theorem 1.1
For a network $\mathcal{N} = (D, c, p, q)$ the maximum value of a (p,q)-flow in $\mathcal{N}$ is equal to the minimum capacity of a (p,q)-cut.

Moreover, it is well-known that, if the capacities in $\mathcal{N}$ are integers, then there is an integral maximum flow f, that is, $f(a) \in \mathbb{Z}$ for all $a \in A(D)$.

1.2 Cones and polytopes

All vectors unless otherwise stated are column vectors. For a vector $x \in \mathbb{R}^n$ by $x(i)$ we refer to the i-th entry of the vector x where $1 \leq i \leq n$. Moreover, for a set $Z \subseteq \{1, \ldots, n\}$ we use the notation

$$x(Z) = \sum_{i \in Z} x(i).$$

The *Minkowski sum* of two subsets X and Y from $\mathbb{R}^n$ is defined by

$$X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}.$$

If $X = \{x\}$, then we write $x + Y$ instead of $\{x\} + Y$. Furthermore, $X - Y$ is equal to $X + (-Y)$. We define $\mathbf{e}_i \in \mathbb{R}^n$ as the i-th unit vector of $\mathbb{R}^n$ and $\mathbf{z}_{(i,j)} = \mathbf{e}_i - \mathbf{e}_j$.
A nonempty translate of a linear subspace is called *affine space*. In this thesis the linear space

$$\mathcal{M}_n = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x(i) = 0 \right\}$$

and the affine space

$$\mathcal{M}_G = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x(i) = m \right\},$$

where G is a labeled graph of order n and size m are of particular interest. The *dimension* of an affine space is the dimension of the corresponding linear space and abbreviated by dim. For a nonempty set $Y \subseteq \mathbb{R}^n$ we define the *affine hull* of Y as the smallest affine space which contains Y. A set of vectors $\{v_0, v_1, \ldots, v_k\}$ is *affinely independent* if the affine hull of $\{v_0, v_1, \ldots, v_k\}$ has dimension k. An equivalent definition is that the vectors $v_1 - v_0, \ldots, v_k - v_0$ are linearly independent.
Let $x_1, \ldots, x_k$ be a collection of vectors from $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. A combination

$$\lambda_1 x_1 + \ldots + \lambda_k x_k \tag{1.1}$$

is called a *conic combination* if the coefficients $\lambda_1, \dots, \lambda_k$ are nonnegative. Let Y be a nonempty set of vectors (or points) in $\mathbb{R}^n$. If every conic combination of two arbitrary vectors $x, y \in Y$ is contained in Y, then Y is a *convex cone*. Obviously, by its definition a convex cone is closed. The *conical hull* of Y is defined as the intersection of all convex cones which contain Y. Alternatively, we could define the conical hull of Y as the set which consists of all vectors obtained by a conic combination of finitely many vectors from Y. In this case, we use the notation $\mathcal{C}(Y)$ and say that Y generates the cone $\mathcal{C} = \mathcal{C}(Y)$.
Let $\mathcal{C}$ be a convex cone in $\mathbb{R}^n$. If $\mathcal{C}$ is the finite intersection of closed halfspaces, then $\mathcal{C}$ is polyhedral. In other words, a cone $\mathcal{C}$ is polyhedral if there is a matrix $M \in \mathbb{R}^{k\times n}$ such that

$$\mathcal{C} = \{x \in \mathbb{R}^n \mid Mx \geq 0\}.$$

A famous result by Weyl and Minkowski (see [8]) proves that $\mathcal{C}$ is polyhedral if and only if it is generated by a finite set of vectors. For a convex cone $\mathcal{C}$ the set $\mathcal{C} \cap (-\mathcal{C})$ is the largest linear subspace which is contained in $\mathcal{C}$. If $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$, then we speak of a *pointed cone*. In this thesis, by a cone we always refer to a closed convex pointed cone.
A conic combination as in (1.1) satisfying $\lambda_1 + \ldots + \lambda_k = 1$ is called a *convex combination*. Let Y be a nonempty set of vectors from $\mathbb{R}^n$. The set Y is *convex* if every convex combination of two arbitrary vectors $x, y \in Y$ is contained in Y. In analogy to the conical hull we define the *convex hull* of Y as the intersection of all convex sets containing Y. We use the notation $\text{conv}(Y)$ for the convex hull of Y. A set is called a *polyhedron* if it is the (finite) intersection of closed halfspaces. Thus for every polyhedron Y there is a matrix $M \in \mathbb{R}^{k\times n}$ and a vector $b \in \mathbb{R}^k$ such that $Y = \{x \in \mathbb{R}^n \mid Mx \leq b\}$. The set Y is a *rational polyhedron* if the entries of M and b are rational numbers. A bounded polyhedron is also called a *polytope*. There is a well-known result on polytopes which is similar to the mentioned characterization by Weyl and Minkowski on polyhedral cones. A set Y is the convex hull of a finite set of vectors if and only if Y is a bounded intersection of closed halfspaces. This result implies that a polytope can be described in two different ways. Firstly, by a finite set of vectors yielding the convex hull. Secondly, by a finite set of linear inequalities which are obtained from the halfspaces.
Let $P \subseteq \mathbb{R}^n$ be a polytope. The dimension of P is the dimension of the affine hull of P. Considering $c \in \mathbb{R}^n$ and $b \in \mathbb{R}$ we call a linear inequality $c^\top x \leq b$ *valid* for P if it holds for all $x \in P$. A *face* of P is any set of the form

$$P \cap \{x \in \mathbb{R}^n \mid c^\top x = b\},$$

where $c^\top x \leq b$ is a valid inequality of P. The dimension of a face is the dimension of its affine hull. A face of dimension 0, 1, and $\dim(P) - 1$ is called *vertex*, *edge*, and *facet*, respectively. The set of all vertices of P is denoted by $\text{vert}(P)$. Since graphs also have vertices and edges we always specify whether we refer to the graph or the polytope context if necessary.
For notations and results on cones and polytopes not mentioned here we refer to Ziegler's book [27].

1.3 Partially ordered sets

Let S be a set. A binary relation $\preceq$ on S is called

- *reflexive* if $s \preceq s$ for all $s \in S$,
- *transitive* if from $r \preceq s$ and $s \preceq t$ follows $r \preceq t$ for all $r, s, t \in S$,
- *antisymmetric* if $s \preceq t$ and $t \preceq s$ implies $s = t$ for all $s, t \in S$.

A *quasi-order* on S is a reflexive and transitive relation. Moreover, a *partial order* on S is a quasi-order which is antisymmetric. If $\preceq$ is a partial order on S, then we call the pair $(S, \preceq)$ a *partially ordered set* or *poset*.
Let $(S, \preceq)$ be a poset and s and t two elements of S. We define the (s,t)-*interval* as the set of all elements $r \in S$ satisfying $s \preceq r \preceq t$. A subset T of a poset S is called *convex* if for every pair $r, t \in T$ each element of the (r,t)-interval is contained in T.
If $s \preceq t$ or $t \preceq s$ we say that s and t are *comparable*. Otherwise s and t are *incomparable*. A set $T \subseteq S$ which consists of pairwise comparable (respectively incomparable) elements is called a *chain* (respectively *anti-chain*). We write $s \prec t$ if $s \preceq t$ and $s \neq t$. An element $s \in S$ is a *predecessor* of $t \in S$ if $s \prec t$ and for all $r \in S$ with $s \preceq r \prec t$ follows $r = s$. Similarly, t is a *successor* of s if $s \prec t$ and for all $r \in S$ with $s \prec r \preceq t$ follows $r = t$.
A well-known visualization for a (finite) poset is its *Hasse diagram*. It is defined as a digraph with vertex set S and an arc from t to s if s is a predecessor of t. We usually draw the predecessors of each element $s \in S$ below the vertex s and replace each arc by an (undirected) edge. Thus for $s, t \in S$ we have $s \preceq t$ if and only if there is a downward-path from t to s in the corresponding Hasse diagram.
An element $s \in S$ is called *minimal* if it has no predecessor. Similarly, an element s is *maximal* if it not the predecessor for any element of G. An *extremal* element is a minimal or maximal element of S with respect to $\preceq$. A poset containing an unique minimal element $\underline{s}$ and an unique maximal element $\overline{s}$ is *bounded*. If for $s, t \in S$ the set $\{r \in S \,|\, s \preceq r \text{ and } t \preceq r\}$ has an unique minimal element, then we call this minimal element the *supremum* or *join* of s and t in S. Analogously, if $\{r \in S \,|\, r \preceq s \text{ and } r \preceq t\}$ has an unique maximal element, then we denote it as *infimum* or *meet* of s and t in S. If it exists, we use the notations $\sup(s,t)$ and $\inf(s,t)$ for the supremum and infimum, respectively. A *lattice* is a bounded poset such that $\sup(s,t)$ and $\inf(s,t)$ exist for all $s, t \in S$.
In the following we consider partial orders on $S = \mathbb{R}^n$. Each nonempty convex cone $\mathcal{C} \subseteq \mathbb{R}^n$ induces a binary relation $\preceq$ on $\mathbb{R}^n$ by

$$x \preceq y \quad \text{if and only if} \quad y - x \in \mathcal{C}.$$

Obviously, $\preceq$ is a quasi-order. Moreover, $\preceq$ is antisymmetric if and only if $\mathcal{C}$ is pointed. We call such an order a *cone order*. Since we only consider pointed cones in this thesis we always assume that a cone order is a partial order. Let x, y and z be vectors in $\mathbb{R}^n$ and $\lambda \geq 0$. For every cone order $x \preceq y$ implies $x + z \preceq y + z$ and $\lambda x \preceq \lambda y$. Hence a cone order is compatible with the vector space structure of $\mathbb{R}^n$. Such orders are sometimes called

linear orders and $(\mathbb{R}^n, \preceq)$ is an *ordered vector space*.
A basic example is the component-wise relation on $\mathbb{R}^n$ defined by

$$x \leq y \quad \text{if and only if} \quad x(i) \leq y(i) \text{ for all } i \in \{1, \ldots, n\}.$$

In this case the corresponding cone is generated by the unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$.
A more advanced example is the order "$\preccurlyeq$" which is closely related to the well-known dominance order (see [20]). It is defined by

$$x \preccurlyeq y \quad \text{if and only if} \quad \begin{cases} \sum_{i=1}^{k} x(i) \leq \sum_{i=1}^{k} y(i), & \text{for all } 1 \leq k \leq n-1, \\ \sum_{i=1}^{n} x(i) = \sum_{i=1}^{n} y(i). \end{cases}$$

In Section 3.2 we prove that the corresponding cone is a subset of $\mathcal{M}_n$ and it is generated by the vectors in

$$\{\mathbf{z}_{(1,2)}, \mathbf{z}_{(2,3)}, \ldots, \mathbf{z}_{(n-1,n)}\}.$$

It is possible to reverse the correspondence between a cone and a linear order. For any linear order $\preceq$ the set $S^+ = \{x \in \mathbb{R}^n \mid 0 \preceq x\}$ is a cone which is also known as *positive cone*. Thus there is an one-to-one correspondence between the cones in $\mathbb{R}^n$ and the linear orders (see [17]).
In this thesis, by "$\leq$" and "$\preccurlyeq$" we always refer to the component-wise order and the relation derived from the dominance order $\mathbb{R}^n$, respectively. For any other partial order we use the notation "$\preceq$".

2 Orientations and degree vectors

In the first section of this chapter we give a brief survey on orientations of undirected graphs with degree constraints and important results on degree vectors of graphs. Particularly, we prove well-known theorems from Landau [19] and Hakimi [14]. The ideas in both proofs and several other facts concerning these results are adapted in many parts of this thesis. In the second section we present some properties of the polytope $\mathcal{P}_G$ which is defined as the convex hull of the degree vectors of a given graph G. All results can be found in the literature in the more general setting of base polytopes of (contra-)polymatroids. The main part of the second section is a characterization of the vertices, edges and facets of $\mathcal{P}_G$. Especially, in chapters 4 and 6 the statements from this result are applied.

2.1 Orientations with degree constraints

Finding an orientation of a given undirected graph which satisfies certain conditions is a well-known task in the theory of graphs. There is a huge list of various conditions that are studied. Among others orientations with special connectivity properties or degree constraints are considered.
Let G be a labeled graph with vertex set $\{1, \dots, n\}$ and D an orientation of G. A vector of nonnegative integers $s \in \mathbb{Z}^n$ is the *out-degree vector* (respectively *in-degree vector*) of D, if $d_D^+(v) = s(v)$ (respectively $d_D^-(v) = s(v)$) holds for all $v \in V(G)$. Notice that the orientation $\tilde{D}$ of G obtained from D by reversing the direction of every arc in $A(D)$ satisfies $d_{\tilde{D}}^-(v) = d_D^+(v)$ for all $v \in V(G)$. Hence, if s is the out-degree vector of an orientation of G, then there is an orientation of G with in-degree vector s and vice versa. Therefore, it suffices to focus either on out-degree vectors or in-degree vectors. In this thesis we consider out-degree vectors. Moreover, we call $s \in \mathbb{Z}^n$ a *degree vector* of G, if there is an orientation of D with out-degree vector s. We also say that s is *realized* by D.

One of the first results on degree vectors of a given graph is due to Landau [19]. It yields a characterization for degree vectors of the complete graph K_n.

Theorem 2.1 (Landau [19] 1953)
A vector of nonnegative integers $s \in \mathbb{Z}^n$ is a degree vector of K_n if and only if

$$s(X) \geq \binom{|X|}{2} \quad \text{for all } X \subseteq V(K_n) \quad \text{and} \quad s(V(K_n)) = \binom{n}{2}.$$

An orientation of a complete graph is also known as a *tournament*. The *out-degree sequence* of a digraph D is given by arranging the entries of the out-degree vector of D in a nondecreasing way. For tournaments such a sequence is sometimes called *score sequence*. Since all labeled versions of complete graph K_n are isomorphic as labeled graphs the following theorem is equivalent to Theorem 2.1.

Theorem 2.2
A sequence of nonnegative integers $s = (s_1 \leq s_2 \leq \ldots \leq s_n)$ is the out-degree sequence of a tournament if and only if

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} \quad \text{for all } k \in \{1, \ldots, n\}, \tag{2.1}$$

where equality holds for $k = n$.

An interesting fact about this result is that there are numerous different proofs in the literature. There is an extensive survey by Reid [24] which includes several of them. Other proofs are for example by Griggs and Reid [13], Brualdi and Kiernan [7] and Reid and Sanatan [25]. Here we follow an argumentation from [13].

Proof (Theorem 2.2)
The necessity of the statement follows from the fact that every induced subdigraph of a tournament is again a tournament. Hence for every collection X of k vertices there are at least $k(k-1)/2$ arcs with initial vertex in X.
To prove sufficiency let s be a sequence satisfying condition (2.1). Starting with the transitive tournament of order n, that is an acyclic orientation of K_n, we construct step by step a tournament with out-degree sequence s. Suppose that at one stage of this procedure we have a tournament U with out-degree sequence u such that $u \preccurlyeq s$. By (2.1) we deduce that for the out-degree sequence $t = (0, 1, \ldots, n-1)$ of the transitive tournament holds $t \preccurlyeq s$. Thus we initialize the procedure with the transitive tournament. If $u = s$, we are done since s is the out-degree sequence of U. Hence we assume that $u \neq s$. There is a smallest index α such that

$$\sum_{i=1}^{\alpha} u_i < \sum_{i=1}^{\alpha} s_i.$$

We deduce that $u_\alpha < s_\alpha$. Denote β the maximal index with $u_\beta = u_\alpha$. Since $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} s_i$ there is a minimal index $\gamma > \beta$ satisfying $u_\gamma > s_\gamma$. The choices of β and γ yield $u_\beta < u_{\beta+1}$ and $u_\gamma > u_{\gamma-1}$. Furthermore, by $u_\beta < s_\beta$ we obtain

$$u_\gamma \geq s_\gamma + 1 \geq s_\beta + 1 \geq u_\beta + 2. \tag{2.2}$$

Denote $v_i \in V(U)$ the vertex with out-degree u_i. From (2.2) we conclude that U contains a vertex v_λ with $\lambda \notin \{\beta, \gamma\}$ and $(v_\gamma, v_\lambda), (v_\lambda, v_\beta) \in A(U)$. By reversing this directed

(v_γ, v_β)-path in U we obtain a tournament U' with out-degree sequence u' where

$$u'_i = \begin{cases} u_\gamma - 1, & \text{if } i = \gamma, \\ u_\beta + 1, & \text{if } i = \beta, \\ u_i, & \text{else.} \end{cases}$$

It is easy to check that $u'_1 \leq \ldots \leq u'_n$ and $u \prec u' \preccurlyeq s$ holds.
Finally, notice that in each stage the value of $\sum_{i=1}^{n} |u_i - s_i|$ is reduced by 2. Hence after

$$\frac{1}{2} \sum_{i=1}^{n} |t_i - s_i|$$

steps of the described procedure we obtain a tournament with out-degree sequence s. □

An important tool in this proof is the dominance or majorization order that yields an argumentation which can also be applied to other graph theoretical questions, for example degree sequences of simple undirected graphs (see [2]). Particularly, Theorem 2.2 shows that a nondecreasing sequence of nonnegative integers s is the score sequence of a tournament of order n if and only if s majorizes $(0, 1, \ldots, n-1)$, that is $(0, 1, \ldots, n-1) \preccurlyeq s$. Moreover, the procedure in the proof generates a chain of sequences with minimal element $(0, 1, \ldots, n-1)$ and maximal element s. Aigner [1] proved that the set consisting of the score sequences of all tournaments on n vertices forms a lattice with respect to $\preccurlyeq$. This lattice structure can be used to determine all elements of this set.
Every score sequence of length n is often interpreted as a number partition of $n(n-1)/2$. Such number partitions are usually visualized by Ferrers diagrams. For a score sequence $s = (s_1 \leq \ldots \leq s_n)$ its corresponding *Ferrers diagram* consists of n rows of boxes with s_i boxes in row i. By "lifting" boxes in the Ferrers diagram it is possible to determine all score sequences which majorize s. The successors of a score sequence s with respect to $\preccurlyeq$ are obtained by the following step. We consider an index α with $1 < \alpha \leq n$ satisfying $s_{\alpha-1} < s_\alpha$. If $s_{\alpha-1} + 2 \leq s_\alpha$, then set $\gamma = \alpha - 1$. If $s_{\alpha-1} + 1 = s_\alpha$ and there is an index $\beta < \alpha - 1$ satisfying $s_\beta + 2 = s_\alpha$, then let $\gamma < \alpha - 1$ be the largest index with $s_\gamma + 2 = s_\alpha$. For the sequence s' with

$$s'_i = \begin{cases} u_\alpha - 1, & \text{if } i = \alpha, \\ u_\gamma + 1, & \text{if } i = \gamma, \\ u_i, & \text{else,} \end{cases}$$

we deduce that $s \prec s'$ and $s'_1 \leq \ldots \leq s'_n$ by the choices of α and γ. Moreover, there does not exist a score sequence t such that $s \prec t \prec s'$. Thus every index satisfying one of the two conditions mentioned above yields a successor of s. Figure 2.1 shows the lattice of the score sequences of tournaments with 5 vertices where each sequence is represented by its Ferrers diagram.
The *strong arc-connectivity* $\lambda(D)$ of a digraph D is the minimum cardinality of an arc set F such that $D - F$ is not strong. For a tournament T of order n its score sequence s can be used to compute $\lambda(T)$. Beineke and Bagga [5] proved that

$$\lambda(T) = \min_{1 \leq k \leq n-1} \left(\sum_{i=1}^{k} s_i - \binom{k}{2} \right).$$

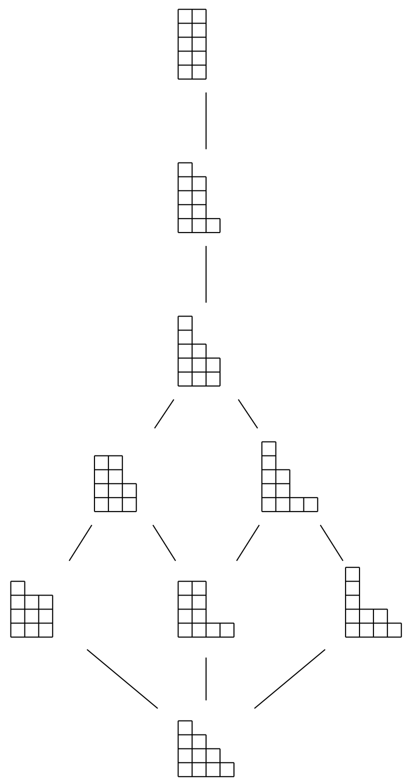


Figure 2.1: Ferrers diagrams for the score sequences of tournaments on 5 vertices.

There are several results which generalize the theorem of Landau in different ways (see for example [3, 21, 22]). In 1965 Hakimi [14] proved a result that extends Theorem 2.1 to arbitrary undirected graphs.

Theorem 2.3 (Hakimi [14] 1965)
A vector $s \in \mathbb{Z}^n$ is a degree vector of a labeled graph G if and only if

$$s(X) \geq m(G[X]) \text{ for all } X \subseteq V(G) \quad \text{and} \quad s(V(G)) = m(G). \tag{2.3}$$

Proof
Let $V = \{1, \ldots, n\}$ be the vertex set of G. Obviously, for every orientation D of G holds $m(D) = m(G)$. Additionally, for any set $X \subseteq V(D)$ there are at least $m(G[X])$ arcs in D with tail in X. Hence for every degree vector s of G holds $s(X) \geq m(G[X])$ for all $X \subseteq V$ and $s(V) = m(G)$.
To prove sufficiency consider a vector s satisfying (2.3). Let D be an arbitrary orientation of G and t its out-degree vector. If $s = t$ we are done. Thus we assume that $s \neq t$. Hence there is a vertex $v \in V$ such that $t(v) > s(v)$ since $t(V) = s(V)$. Denote X_v the set of vertices in V which can be reached from v via a directed path in D. Because there does not exist an arc leaving X_v in D we obtain $s(X_v) = m(G[X_v])$. Suppose for all $u \in X_v$

holds $t(u) \geq s(u)$. In this case we deduce

$$m(G[X_v]) = t(X_v) > s(X_v)$$

contradicting the fact that s satisfies (2.3). Hence there is a vertex $w \in V$ such that $t(w) < s(w)$. By reversing a directed (v,w)-path in D we obtain an orientation D' of G with out-degree vector t' where

$$t'(u) = \begin{cases} t(v) - 1, & \text{if } u = v, \\ t(w) + 1, & \text{if } u = w, \\ t(u), & \text{else.} \end{cases}$$

If we have $t' = s$, then s is an out-degree vector of D' and therefore a degree vector of G. Otherwise we set $D = D'$ and repeat the last step.
In each step the value of $\sum_{v \in V} |s(v) - t(v)|$ decreases by two. Hence after a finite amount of such steps we arrive at an orientation of G with out-degree vector s. □

For a given graph G of order n and a nonnegative vector $s \in \mathbb{Z}^n$ the proof of Theorem 2.3 provides a procedure that constructs an orientation of G with out-degree vector s or finds a set X which violates (2.3). With basically the same argumentation Frank and Gyárfás [11] proved a result on orientations with upper and lower bounds on the out-degree of each vertex.
An alternative procedure to construct an orientation of a graph with prescribed out-degrees uses flows in networks and is described in [4]. First we define a fixed orientation $\tilde{D}$ of G as a reference orientation with out-degree vector t. By adding two vertices $p, q \notin V(\tilde{D})$ and several arcs to this reference orientation we obtain a network $\mathcal{N} = (D, c, p, q)$ by

$$V(D) = V(\tilde{D}) \cup \{p, q\},$$

$$A(D) = E(\tilde{D}) \cup \{(p,v), (v,q) \mid v \in V(\tilde{D})\},$$

$$c(a) = \begin{cases} 1, & \text{if } a \in A(\tilde{D}), \\ t(v), & \text{if } a = (p,v) \text{ and } v \in V(\tilde{D}), \\ s(v), & \text{if } a = (v,q) \text{ and } v \in V(\tilde{D}). \end{cases}$$

Now, s is a degree-vector of G if and only if there is a (p,q)-flow f in $\mathcal{N}$ with $\text{val}(f) = m(G)$. For a proof of this statement consider a degree vector s of G. From Theorem 2.3 follows $s(X) \geq m(G[X]) = m(\tilde{D}[X])$ for every $X \subseteq V(\tilde{D})$ and thus we deduce

$$\begin{aligned} c\left(\delta^+_{\mathcal{N}}(X \cup \{p\})\right) &= c\left(\delta^+_{\tilde{D}}(X)\right) + t(V(\tilde{D}) \setminus X) + s(X) \\ &= t(X) - m(\tilde{D}[X]) + t(V(\tilde{D}) \setminus X) + s(X) \\ &= t(V(\tilde{D})) + s(X) - m(\tilde{D}[X]) \\ &\geq t(V(\tilde{D})) \\ &= m(G). \end{aligned}$$

Hence every (p, q)-cut in $\mathcal{N}$ has a capacity of at least $m(G)$. By Theorem 1.1 there exists a (p, q)-flow f with $\mathrm{val}(f) = m(G)$. On the other hand suppose f is a (p, q)-flow in $\mathcal{N}$ with $\mathrm{val}(f) = m(G)$. The definition of $\mathcal{N}$ and the flow-conservation in every vertex $v \in V(\tilde{D})$ imply

$$s(v) + \sum_{a \in \delta^+_{\tilde{D}(v)}} f(a) = \sum_{a \in \delta^+_{\mathcal{N}}(v)} f(a) = \sum_{a \in \delta^-_{\mathcal{N}}(v)} f(a) = t(v) + \sum_{a \in \delta^-_{\tilde{D}}(v)} f(a). \tag{2.4}$$

Denote D' the orientation of G we obtain from $\tilde{D}$ by reversing every arc a with $f(a) = 1$. By (2.4) we conclude that each $v \in V(D')$ has out-degree

$$d^+_{D'}(v) = t(v) + \sum_{a \in \delta^-_{\tilde{D}(v)}} f(a) - \sum_{a \in \delta^+_{\tilde{D}}(v)} f(a) = s(v).$$

Therefore, s is an out-degree vector of D'. This deduction shows that the problem to decide whether a given vector is a degree vector of a given graph can be solved in polynomial time.
For a labeled graph G we define $\mathrm{DEG}^+(G)$ as the set of all degree vectors of G. It is not difficult to see that there is a additive property between the sum of to graphs and the Minkowski sum of the corresponding sets of degree vectors, that is

$$\mathrm{DEG}^+(G + H) = \mathrm{DEG}^+(G) + \mathrm{DEG}^+(H). \tag{2.5}$$

Hence we obtain

$$\mathrm{DEG}^+(G) = \sum_{uv \in E(G)} \{\mathbf{e}_u, \mathbf{e}_v\}.$$

We close this section with a simple but useful observation.

Observation 2.4
Let D_1 and D_2 be two distinct orientations of a labeled graph whose out-degree vectors are equal. The subdigraph H of D_1 with arc set $A(H) = A(D_1) \setminus A(D_2)$ consists of arc-disjoint, directed cycles.

Proof
Denote v an arbitrary vertex of D_1 and suppose that there are k arcs in $A(H)$ with head v. Since $d^+_{D_1}(v) = d^+_{D_2}(v)$ there are k arcs in $A(D_2) \setminus A(D_1)$ with head v. These arcs have reverse direction in H and thus we deduce that $d^+_H(v) = d^-_H(v)$. Hence every component of H is Eulerian. It is well-known that the arc set of every Eulerian digraph can be partitioned into arc-disjoint cycles. □

This observation implies directly that two distinct acyclic orientations of a labeled graph have distinct out-degree vectors. On the other hand we observe that a degree vector is realized by an unique orientation if and only if the orientation is acyclic.

2.2 Degree vector polytope

Let G be a labeled graph of order n with vertex set V. In this section we consider the set

$$\mathcal{P}_G = \left\{ z \in \mathbb{R}^n \mid z(X) \geq m(G[X]) \text{ for all } X \subseteq V \text{ and } z(V) = m(G) \right\}. \tag{2.6}$$

Obviously, $\mathcal{P}_G$ is a polyhedron. Since $z(v) \geq m(G[\{v\}]) = 0$ for every $v \in V$ and $z(V) = m(G)$ the polyhedron is bounded, that is, $\mathcal{P}_G$ is a polytope. Theorem 2.3 implies that every degree vector of G is an integral point in $\mathcal{P}_G$ and vice versa. Furthermore, every vector $z \in \mathcal{P}_G$ can be seen as a fractional degree vector of G. In particular, we consider the following situation. Let D be a fixed orientation D of G and $\varrho : A(D) \to [0,1]$ any function of arc weights. Now, for every arc $a = (v,w) \in A(D)$ the value of $\varrho(a)$ can be interpreted as the ratio of direction pointing from v to w and $1 - \varrho(a)$ as the ratio from w to v. Let y_ϱ be the vector with entries

$$y_\varrho(v) = \sum_{a \in \delta^+(v)} \varrho(a) + \sum_{a \in \delta^-(v)} (1 - \varrho(a)).$$

It is easy to check that $y_\varrho(V) = m(G)$ and for every vertex set $X \subseteq V$ holds $y_\varrho(X) \geq m(D[X]) = m(G[X])$. Hence y_ϱ is contained in $\mathcal{P}_G$. Moreover, if ϱ maps into $\{0,1\}$, then y_ϱ is a degree vector of G. We call $\mathcal{P}_G$ the *degree vector polytope* of G.

For our studies on degree vectors it is useful to have some properties of $\mathcal{P}_G$. Particularly, we are interested in the dimension of $\mathcal{P}_G$ and a description of the vertices, edges and facets of this polytope. One possibility to achieve this goal is to use some theory on polymatroids.

The following definitions can be found in [26]. Denote V a finite set with n elements and 2^V the power set of V. A function $f : 2^V \to \mathbb{R}$ is called *submodular*, if for all subsets $X, Y \in 2^V$ holds

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y).$$

For a submodular function f the polyhedron EP_f defined by

$$EP_f = \{z \in \mathbb{R}^n \mid z(X) \leq f(X) \text{ for all } X \subseteq V\}$$

is known as the *extended polymatroid* (associated with f). Furthermore, a vector $z \in EP_f$ satisfying $z(V) = f(V)$ is a *base vector* of EP_f. The *base polytope* of EP_f is defined as the set of all base vectors of EP_f. A similar concept to submodular functions are supermodular functions. A function $g : 2^V \to \mathbb{R}$ is called *supermodular*, if for all subsets $X, Y \in 2^V$ holds

$$g(X) + g(Y) \leq g(X \cap Y) + g(X \cup Y).$$

Obviously, g is supermodular if $-g$ is submodular and vice versa. In analogy to an extended polymatroid an *extended contrapolymatroid* is defined by

$$EQ_g = \{z \in \mathbb{R}^n \mid z(X) \geq g(X) \text{ for all } X \subseteq V\},$$

where g is a supermodular function. Again a vector $z \in EQ_g$ satisfying $z(V) = g(V)$ is a base vector of EQ_g.

For our investigations let V be the vertex set of a labeled graph G of order n. We consider the functions $f_G : 2^V \to \mathbb{N}_0$ with $f_G(X) = m(G) - m(G - X)$ and $g_G : 2^V \to \mathbb{N}_0$ with $g_G(X) = m(G[X])$ for all $X \subseteq V$. Thus $f_G(X)$ is equal to the number of edges in G which are incident to at least one vertex in X. For any sets $X, Y \subseteq V$ holds

$$\begin{aligned} f_G(X) + f_G(Y) &= m(G) - m(G - X) + m(G) - m(G - Y) \\ &= m(G) - m(G - (X \cap Y)) + m(G) - m(G - (X \cup Y)) \\ &\quad + m_G(X \setminus Y, Y \setminus X) \\ &\geq f_G(X \cap Y) + f_G(X \cup Y) \end{aligned}$$

and thus f_G is submodular. Similarly, $g_G(X)$ is the number of edges which have both endpoints in X. For all vertex sets $X, Y \subseteq V$ we deduce that

$$\begin{aligned} g_G(X) + g_G(Y) &= m(G[X]) + m(G[Y]) \\ &= m(G[X \cap Y]) + m(G[X \cup Y]) - m_G(X \setminus Y, Y \setminus X) \\ &\leq f_G(X \cap Y) + f_G(X \cup Y). \end{aligned}$$

Hence g_G is a supermodular function. Therefore, on the one hand $\mathcal{P}_G$ contains all base vectors of the extended contrapolymatroid EQ_{g_G}. On the other hand notice that for every vector $z \in \mathcal{P}_G$ and any set $X \subseteq V$ holds

$$z(X) = z(V) - z(V \setminus X) \leq m(G) - m(G - X) = f_G(X).$$

Thus $\mathcal{P}_G$ can also be seen as the base polytope of the extended polymatroid EP_{f_G}.
Many important properties and applications of polymatroids can be found in the books of Fujishige [12] and Schrijver [26]. In particular, there is a whole subsection on the structures of base polytopes in [12] including results on the dimension and characterizations of the faces. Hence most of the statements in the remaining part of this section can be deduced from these investigations on base polytopes of polymatroids. Nevertheless to our knowledge the following results are not mentioned in the literature explicitly. Because $\mathcal{P}_G$ is one special case of a base polytope we omit further details on polymatroids. In fact, we present proofs of the following results that are based on graph theory.
We already mentioned that $\mathcal{P}_G$ contains all degree vectors of G. Hence we deduce that the convex hull of all degree vectors of G is a subset of $\mathcal{P}_G$, i.e.

$$\text{conv}(\text{DEG}^+(G)) \subseteq \mathcal{P}_G. \tag{2.7}$$

The interpretation of $\mathcal{P}_G$ as fractional degree vectors gives us the hint that every vector in $\mathcal{P}_G$ is a convex combination of degree vectors. The following theorem shows that this is true and therefore $\mathcal{P}_G$ is an integral polytope.

Proposition 2.5
For every labeled graph G holds $\mathcal{P}_G = \text{conv}(\text{DEG}^+(G))$.

Proof
Let G be a graph of order n with vertex set V and edge set $\{e_1, \ldots, e_m\}$. By (2.7) we only

have to prove that $\mathcal{P}_G \subseteq \mathrm{conv}(\mathrm{DEG}^+(G))$. Furthermore, it suffices to show that every vertex of the polytope $\mathcal{P}_G$ is a convex combination of degree vectors of G.
Let z be any vertex of $\mathcal{P}_G$. It is easy to see that $\mathcal{P}_G$ is a rational polytope. Thus z is a vector in $\mathbb{Q}^n$. For $1 \leq i \leq n$ there are nonnegative integers p_i and q_i with $q_i \neq 0$ such that $z(i) = p_i/q_i$. Denote N the least common multiple of $q_1, \ldots, q_n$. Hence, $\tilde{z} = N \cdot z$ is an integral vector. Now, we consider the graph H which we obtain from G by replacing each edge $e_j \in E(G)$ by N copies $e_j^1, \ldots, e_j^N$ of this edge. For any $X \subseteq V$ we conclude

$$\tilde{z}(X) = N \cdot z(X) \geq N \cdot m(G[X]) = m(H[X]).$$

Furthermore, we have $\tilde{z}(V) = N \cdot m(G) = m(H)$. Therefore, from Theorem 2.3 we deduce that $\tilde{z}$ is a degree vector of an orientation D of H. For $e_j \in E(G)$ and $1 \leq k \leq N$ let a_j^k be the arc of D that corresponds to the edge e_j^k of H. Denote D^k the subdigraph of D consisting of the arcs $a_1^k, \ldots, a_m^k$. Notice that D^k is an orientation of G and denote with $s^k \in \mathrm{DEG}^+(G)$ its out-degree vector. Since D is the sum of $D^1, \ldots, D^N$ we obtain $s^1 + \ldots + s^N = \tilde{z}$. Therefore, we conclude

$$z = \frac{1}{N}\tilde{z} = \frac{1}{N}\sum_{k=1}^{N} s^k$$

and z is a convex combination of the vectors in $\mathrm{DEG}^+(G)$. □

The following example presents some facts on the degree vector polytope of a cycle of length 3.

Example 2.6
Let G be a labeled version of C_3. The set of degree vectors of G is given by

$$\mathrm{DEG}^+(G) = \{(2,1,0)^\top, (2,0,1)^\top, (1,2,0)^\top, (1,0,2)^\top, (0,1,2)^\top, (0,2,1)^\top, (1,1,1)^\top\}.$$

The polyhedral description of $\mathcal{P}_G$ is given by

$$\mathcal{P}_G = \left\{ (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 \geq 0, \\ x_2 \geq 0, \\ x_3 \geq 0, \\ x_1 + x_2 \geq 1, \\ x_1 + x_3 \geq 1, \\ x_2 + x_3 \geq 1, \\ x_1 + x_2 + x_3 = 3 \end{array} \right\}. \tag{2.8}$$

It is not difficult to see that this is equivalent to

$$\mathcal{P}_{C_3} = \left\{ (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid 0 \leq x_i \leq 2, i = 1,2,3 \text{ and } x_1 + x_2 + x_3 = 3 \right\}.$$

From Figure 2.2 we observe that $\mathcal{P}_G$ is a regular hexagon. The polytope is a subset of the affine space (dotted lines in the figure)

$$\mathcal{M}_G = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3\}.$$

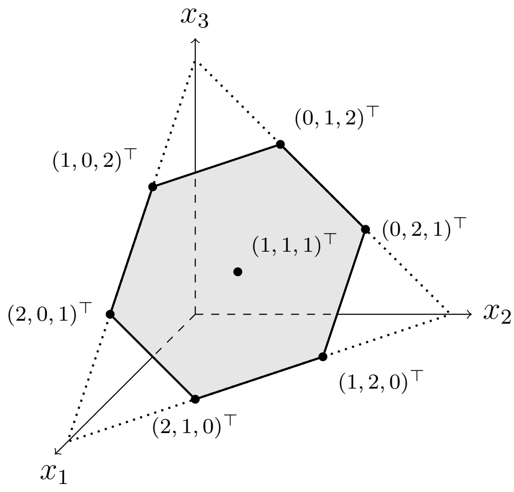


Figure 2.2: Degree vector polytope for C_3

Hence the dimension of P_{C_3} equals two. Considering the vertices of $\mathcal{P}_G$ we observe that each vertex of the degree vector polytope is the out-degree vector of an acyclic orientation of G. For example, the degree vector $(2,1,0)^\top$ is realized by the digraph D with

$$V(D) = \{1,2,3\}, \qquad \text{and} \qquad A(D) = \{(1,2),(1,3),(2,3)\}.$$

Furthermore, $(1,1,1)^\top$ is the only degree vector corresponding to a strong orientation of G and it is also the only degree vector in the relative interior of $\mathcal{P}_G$. Two vertices of this degree vector polytope are connected by an edge if the corresponding (unique) orientations of these degree vectors only differ in one arc. Additionally, each edge of $\mathcal{P}_G$ is parallel to $\mathbf{z}_{(i,j)} = \mathbf{e}_i - \mathbf{e}_j$ for $1 \leq i < j \leq 3$. For example, $(2,1,0)^\top$ to $(1,2,0)^\top$ are connected by an edge of $\mathcal{P}_G$ which is parallel to $\mathbf{z}_{(1,2)}$ and we obtain the orientation realizing $(1,2,0)^\top$ by a reversing the arc $(1,2)$ in D. Since the polytope has dimension two the edges of $\mathcal{P}_G$ are already its facets. It is not difficult to check that each inequality in (2.8) defines a facet. Finally, notice that a polytope $\mathcal{Q}$ obtained from $\mathcal{P}_G$ by omitting exactly one of the inequalities in (2.8) contains an integral vector which is not a degree vector of G. In this case such a vector is even a vertex of the new polytope $\mathcal{Q}$. For example, if we delete $x_1 \geq 0$ from (2.8), then $(-1,2,2)^\top$ satisfies all other (in-)equalities.

The definition of $\mathcal{P}_G$ contains an equality. Hence the dimension of $\mathcal{P}_G$ is smaller than the order n of G. For connected graphs we obtain that $\dim(\mathcal{P}_G)$ is equal to $n-1$. Moreover, the dimension of $\mathcal{P}_G$ is even smaller if G is disconnected.

Proposition 2.7
For a labeled graph G with n vertices and k components holds $\dim(\mathcal{P}_G) = n-k$.

Proof
Let $E = \{e_1, \dots, e_m\}$ be the edge set of G. First we show that the dimension of $\mathcal{P}_G$ is at

most $l = |V(G)| - k$. Consider any collection $s^0, s^1, \dots, s^h$ of $h+1$ affinely independent degree vectors of G. For $0 \le i \le h$ denote D_i an orientation of G realizing s^i. Moreover, we define vectors $q^i \in \mathbb{R}^m$ by

$$q^i(j) = \begin{cases} 1, & \text{if } e_j \text{ has not the same directions in } D_0 \text{ and } D_i, \\ 0, & \text{else,} \end{cases}$$

and a matrix $Q \in \mathbb{R}^{m \times h}$ with columns $q^1, \dots, q^h$. Let $B \in \mathbb{R}^{n \times m}$ be the incidence matrix of D_0. For $1 \le i \le h$ we observe that

$$s^0 - s^i = Bq^i.$$

Since $s^0, s^1, \dots, s^h$ are affinely independent the vectors $s^0 - s^1, \dots, s^0 - s^h$ are linearly independent and we deduce that

$$h = \operatorname{rank}(BQ) \le \operatorname{rank}(B) = l.$$

Therefore, $\dim(\mathcal{P}_G) \le l$.
Now we show that $\mathcal{P}_G$ contains l affinely independent points. Consider a spanning forest T and an arbitrary orientation D_0 of G. Without loss of generality we assume that $e_1, \dots, e_l$ are the edges of T. For $1 \le i \le l$ define D_i as the orientation of G we obtain from D_0 by reversing the direction of edge e_i . Furthermore, denote s^i the out-degree vector of D_i. Hence the vectors $s^1 - s^0, \dots, s^l - s^0$ form the columns of the incidence matrix B of a directed version of T. Since T does not contain a cycle the columns of B are linearly independent. Therefore, $s^0, s^1, \dots, s^l$ are affinely independent and $\dim(\mathcal{P}_G) \ge l$. □

We continue with a description of some of the faces of the degree vector polytope. Let G be a labeled graph of order n. For every nonempty, proper subset of vertices $X \subsetneq V(G)$ there is an orientation D of G such that no arc leaves X in D. Hence there is a degree vector s satisfying $s(X) = m(G[X])$. Therefore, every inequality in (2.6) defines a face of $\mathcal{P}_G$. Of course, the opposite is not true. There are faces of $\mathcal{P}_G$ which are not of this form. However, all faces which are necessary to describe this polytope are induced by a set $X \subsetneq V(G)$. We define

$$F_G(X) = \{z \in \mathbb{R}^n \mid z(X) = m(G[X])\}$$

as the affine space containing a face that is generated by X. Suppose G consists of components $G_1, \dots, G_k$ and denote V_i the vertex set of G_i. For $1 \le i \le k$ and every $z \in \mathcal{P}_G$ we have the equality $z(V_i) = m(G_i)$. Thus, if $k \ge 2$ there might be different vertex sets defining the same face of $\mathcal{P}_G$. For example, suppose X is a nonempty, proper subset of V_1. In this case the sets $X \cup V_i$ with $2 \le i \le k$ yield the same face of $\mathcal{P}_G$, that is

$$\mathcal{P}_G \cap F_G(X) = \mathcal{P}_G \cap F_G(X \cup V_2) = \ldots = \mathcal{P}_G \cap F_G(X \cup V_k).$$

To avoid this non-uniqueness we consider only those sets which contain vertices of the same component of G. Thus we obtain

$$\mathcal{P}_G = \{\, z \in \mathbb{R}^n \mid z(X) \ge m(G_i[X]) \ \text{ for all } X \subseteq V_i \text{ and } z(V_i) = m(G_i),\ i = 1, \dots, k \,\}.$$

as an equivalent formulation for $\mathcal{P}_G$. The following theorem summarizes a characterization of the vertices, edges and facets of the degree vector polytope for a given labeled graph.

Theorem 2.8
Let G be a labeled graph of order n with components $G_1, \dots, G_k$.

1. The set of vertices of $\mathcal{P}_G$ is equal to the set of degree vectors of G which correspond to acyclic orientations of G.

2. Two vertices s and t of $\mathcal{P}_G$ are connected by an edge of $\mathcal{P}_G$ if and only if there is an orientation D of G with out-degree vector s and an arc $(u,v) \in A(D)$ such that $s - t = \alpha \mathbf{z}_{(u,v)}$ for some positive integer α.

3. For a nonempty set $X \subsetneq V(G_i)$ with $1 \leq i \leq k$ the face $F_G(X)$ is a facet of $\mathcal{P}_G$ if and only if $G_i[X]$ and $G_i - X$ are connected.

Proof

1. Let $s \in \mathrm{DEG}^+(G)$ and D an orientation of G realizing s. Suppose D has a directed cycle $C = v_1a_1v_2a_2 \dots a_kv_1$. We define the vectors $r = s - \mathbf{z}_{(v_1,v_2)}$ and $t = s + \mathbf{z}_{(v_1,v_2)}$. Obviously, r is the out-degree vector of the orientation of G we obtain from D by reversing the arc a_1. Similarly, a reorientation of a (v_2, v_1)-path in D yields an orientation that realizes t. Thus r and t are degree vectors of G. From $s = (r+t)/2$ follows that s is a proper convex combination of r and t and not a vertex of $\mathcal{P}_G$. Hence every vertex of $\mathcal{P}_G$ is the out-degree vector of an acyclic orientation of G.
Now, suppose $s \in \mathrm{DEG}^+(G)$ belongs to an acyclic orientation D_s of G and s is not a vertex of $\mathcal{P}_G$. By Proposition 2.5 there are two distinct degree vectors r and t of G such that $s = \lambda r + (1-\lambda)t$ holds for some $\lambda \in (0,1)$. Let D_r and D_t be orientations of G realizing r and t, respectively. Since $r \neq t$ there is at least one edge $uv \in E(G)$ such that $(u,v) \in A(D_r)$ and $(v,u) \in A(D_t)$. Considering an arbitrary vertex set $X \subseteq V(D_s)$ with $u \in X$ and $v \notin X$ we observe the following. From $r(X) > m(G[X])$ we deduce that
$$s(X) = \lambda r(X) + (1-\lambda)t(X) > m(G[X]).$$
Analogously, by $t(V \setminus X) > m(G - X)$ we have
$$s(V \setminus X) = \lambda r(V \setminus X) + (1-\lambda)t(V \setminus X) > m(G - X).$$
Therefore, for every vertex set $X \subseteq V(D_s)$ with $u \in X$ and $v \notin X$ there is both an arc leaving and an arc entering X in D_s. Hence u and v are contained in a directed cycle in D_s contradicting the fact that D_s is acyclic.

2. Denote D_s and D_t two orientations of G with out-degree vectors s and t, respectively. The digraphs D_s and D_t do not contain any directed cycles since every vertex of $\mathcal{P}_G$ is the out-degree vector of an acyclic orientation of G. Let B be the set of arcs from $A(D_s)$ which have to be reversed to obtain D_t from D_s. Thus we deduce
$$s - t = \sum_{a \in B} \mathbf{z}_a.$$

First suppose that s and t are connected by an edge of $\mathcal{P}_G$. Hence there is a valid inequality $c^\top x \leq b$ of $\mathcal{P}_G$ with $c \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\{x \in \mathbb{R}^n \,|\, c^\top x = b\} \cap P_G = \{s - \lambda(s-t) \,|\, 0 \leq \lambda \leq 1\}.$$

For every $a \in B$ the vector $s - \mathbf{z}_a$ is the out-degree vector of the orientation of G we obtain from D_s by reversing a. Thus $s - \mathbf{z}_a \in \mathcal{P}_G$ and we conclude

$$c^\top \mathbf{z}_a = c^\top s - c^\top (s - \mathbf{z}_a) \geq b - b = 0.$$

Furthermore, we have

$$\sum_{a \in B} c^\top \mathbf{z}_a = c^\top \left(\sum_{a \in B} \mathbf{z}_a \right) = c^\top (s-t) = 0.$$

Therefore, for every $a \in B$ holds $c^\top \mathbf{z}_a = 0$ and $s - \mathbf{z}_a = s - \lambda'(s-t)$ with $0 < \lambda' \leq 1$. Hence there is a pair of vertices $u, v \in V(D_s)$ such that B consists of all parallel arcs from u to v. Thus we have $s - t = |B|\mathbf{z}_{(u,v)}$.

Now suppose there is an arc $(u, v) \in V(D_s)$ such that $s - t = \alpha \mathbf{z}_{(u,v)}$ for some positive integer α. Thus B consists of all parallel arcs from u to v in D_s. It suffices to prove the existence of a set of valid inequalities of $\mathcal{P}_G$ such that s and t are the only vertices of $\mathcal{P}_G$ for which all of these inequalities hold with equality.

Since D_s is acyclic there is an acyclic vertex ordering, that is an ordering $v_1, \ldots, v_n$ of its vertices such that every arc $(v_i, v_j) \in A(D_s)$ satisfies $i < j$. Denote α and β the indices of an acyclic ordering of D_s such that $u = v_\alpha$ and $v = v_\beta$. Hence we have $\alpha < \beta$. Under all acyclic ordering of D_s choose an ordering such that $\beta - \alpha$ is minimal. We show that $\beta - \alpha = 1$. Suppose to the contrary that $\beta - \alpha > 1$. Thus there are vertices v_γ with $\alpha < \gamma < \beta$. If there exists v_γ such that D_s neither contains a directed (v_α, v_γ)-path nor a (v_γ, v_β)-path, then we can find an acyclic ordering of D_s in which v_γ has a smaller index than v_α or a larger index than v_β. Both cases contradict the minimality of the chosen acyclic ordering. On the other hand, if there is both a (v_α, v_γ)-path and a (v_γ, v_β)-path, then D_s contains a (u, v)-path and thus D_t has a directed cycle. This yields a contradiction to the fact that D_t is acyclic. For $\alpha_1 < \ldots < \alpha_l$ denote $v_{\alpha_1}, \ldots, v_{\alpha_l}$ the vertices which can be reached by a directed path from v_α in D_s. Moreover, for $\beta_1 < \ldots < \beta_{l'}$ denote $v_{\beta_1}, \ldots, v_{\beta_{l'}}$ the vertices which v_γ reaches v_β via a directed path in D_s. We conclude that every vertex v_γ with $\alpha < \gamma < \beta$ is either contained in $W_\alpha = \{v_{\alpha_1}, \ldots, v_{\alpha_l}\}$ or in $W_\beta = \{v_{\beta_1}, \ldots, v_{\beta_{l'}}\}$. Additionally, all edges between W_α and W_β are directed from W_β to W_α. Therefore,

$$v_1, \ldots, v_{\alpha-1}, v_{\beta_1}, \ldots, v_{\beta_{l'}}, v_\alpha, v_\beta, v_k, v_{\alpha_1}, \ldots, v_{\alpha_l}, v_{\beta+1}, \ldots, v_n$$

is an acyclic ordering of D_s satisfying $\beta = \alpha + 1$.

For $1 \leq k \leq n$ we define $X_k = \{v_k \ldots, v_n\}$ and notice that from the acyclic ordering follows

$$s(X_k) = m(G[X_k]).$$

Since B consists of all arcs from v_α to $v_{\alpha+1}$ we observe that

$$t(X_k) = m(G[X_k])$$

holds for $k \neq i+1$. Now consider $\mathcal{X} = \{X_k \mid 1 \leq k \leq n,\ k \neq i+1\}$ and notice that every $X_k \in \mathcal{X}$ induces a valid inequality of $\mathcal{P}_G$ which holds with equality for s and t. Furthermore, for every degree vector r which corresponds to an orientation of G obtained from D_s by reversing any arcs different from (u, v) there is a set $X' \in \mathcal{X}$ such that

$$r(X') > m(G[X']).$$

Thus s and t are the only vertices of $\mathcal{P}_G$ for which all inequalities induced by a set from $\mathcal{X}$ hold with equality. Therefore, s and t are connected by an edge of $\mathcal{P}_G$.

3. By Proposition 2.7 the polytope $\mathcal{P}_G$ has dimension $n-k$. Consider an arbitrary set $X \subseteq V$ and define $X_i = X \cap V(G_i)$ for $i = 1, \ldots, k$. First we show that $F_G(X)$ has dimension $n-k-1$ if and only if there is exactly one component, say G_i, such that $\emptyset \neq X_i \neq V(G_i)$ and $G_i[X_i]$ and $G_i - X_i$ are connected.
Let s be any degree vector of G which is contained in $F_G(X)$ and denote D an orientation of G realizing s. Hence $s(X) = m(G[X])$ and there does not exist an arc in D directed from X to $V \setminus X$. Moreover, any orientation of G we obtain from D by reversing a collection of arcs from $A(D) \setminus \delta_D^-(X)$ corresponds to a degree vector of G in $F_G(X)$ and vice versa. Together with Proposition 2.7 we deduce that the dimension of $F_G(X)$ equals $n-l$, where l denotes the number of components in $G - \delta_G(X)$. Therefore, $\dim(F_G(X)) \leq n-k-1$ and equality holds if and only if there is exactly one index i $(1 \leq i \leq k)$ such that $\emptyset \neq X_i \neq V(G_i)$ and $G_i[X_i]$ and $G_i - X_i$ are connected. Finally, the statement follows because we only have to consider those sets for X which are included in the vertex set of a component of G.

□

Theorem 2.8 implies that for simple graphs all edges of the degree vector polytope have the same length. In particular, if G is a simple graph every edge of $\mathcal{P}_G$ has length $\sqrt{2}$. Additionally, we obtain further characterizations for the edge of $\mathcal{P}_G$. Two vectors s and t in $\mathrm{vert}(\mathcal{P}_G)$ are connected by an edge of $\mathcal{P}_G$ if and only if $s - t = \alpha \mathbf{z}_{(u,v)}$ for some positive integer α and the orientation D of G realizing s has an acyclic ordering in which u and v are consecutive. An equivalent formulation for this condition on (u, v) is that (u, v) is the only directed (u, v)-path in D.
Theorem 2.8 also yields the following formulation of $\mathcal{P}_G$. Under the above mentioned conditions we have

$$\mathcal{P}_G = \left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{l} z(X) \geq m(G_i[X]) \text{ for all } X \in \mathcal{X}_{G_i}, \text{ for all } 1 \leq i \leq k \text{ and} \\ z(V(G_i)) = m(G_i), \text{ for all } 1 \leq i \leq k \end{array} \right\}, \tag{2.9}$$

where $\mathcal{X}_{G_i} = \{X \subsetneq V(G_i) \mid X \neq \emptyset,\ G_i[X] \text{ and } G_i - X \text{ are connected}\}$. By $\mathcal{X}_G$ we refer to the union of $\mathcal{X}_{G_i}$ for all components $G_1, \ldots, G_k$. We call an element of $\mathcal{X}_G$ a *facet defining set* of $\mathcal{P}_G$. Hence $\mathcal{X}_G$ contains an unique collection of subsets which define the facets of $\mathcal{P}_G$. An inequality which does not define a facet of a polytope can be omitted without loosing information about the polytope. The next result shows that the polyhedron we obtain from $\mathcal{P}_G$ by dropping a single facet defining inequality from the formulation (2.9) contains an integral vector which is not included in $\mathcal{P}_G$.

Theorem 2.9
Let G be a labeled graph of order n with components $G_1, \dots, G_k$. For every $X \in \mathcal{X}_{G_i}$ with $1 \le i \le k$ there is a vector $t \in \mathbb{Z}^n \setminus \mathcal{P}_G$ satisfying $t(Y) \ge m(G[Y])$ for all $Y \in \mathcal{X}_G \setminus \{X\}$. Moreover, if $2 \le |X| \le |V(G_i)| - 2$, then the vector t can be chosen such that $0 \le t \le d_G$.

Proof
Let X be an arbitrary set from $\mathcal{X}_G$. From the definition of $\mathcal{X}_G$ follows that there is exactly one component G_i of G with $X \subseteq V(G_i)$. Furthermore, $G_i[X]$ and $G_i - X$ are connected and there is an edge $uv \in E(G_i)$ with $u \notin X$ and $v \in X$. We construct an acyclic orientation D of G as follows.
Start with an acyclic orientation of $G_i[X]$ such that every vertex in X can be reached by a directed path starting in v. To see that there is such an orientation take a spanning tree T of $G_i[X]$. Now, direct the edges of T such that every vertex $w \ne u$ in $V(T)$ has exactly one negative neighbor in T. The remaining edges from $E(G_i[X]) \setminus E(T)$ can be oriented such that the resulting digraph is acyclic. Similarly, we can find an acyclic orientation of $G_i - X$ such that every vertex reaches u via a directed path. Moreover, all edges connecting a vertex in X to a vertex in $V(G_i) \setminus X$ are directed towards X. Finally, we choose arbitrary acyclic orientations for all other components of G. Let s be the out-degree vector of D and consider a set $Y \in \mathcal{X}_G$. By Theorem 2.3 we have $s(Y) \ge m(G[Y])$. Next, we prove the following claim.
Claim: If $s(Y) = m(G[Y])$, then $Y = X$ or $u, v \in Y$ or $u, v \notin Y$.
Suppose Y satisfies $s(Y) = m(G[Y])$ and either u or v is in Y. If $u \in Y$ and $v \notin Y$, then the arc $(u, v) \in A(D)$ leaves Y and thus we deduce $s(Y) > m(G[Y])$. This contradiction implies $u \notin Y$ and $v \in Y$. Consider a vertex $w \in X \setminus Y$. By its construction D contains a directed (v, w)-path P. Since P has an arc leaving Y we obtain a contradiction by $s(Y) > m(G[Y])$. Hence we deduce that $X \subseteq Y$. An analog argumentation yields $Y \subseteq X$ and therefore $Y = X$. This proves the claim.
Now, consider the integral vector $t = s + \mathbf{z}_{(u,v)}$. Obviously, t is not a degree vector of G since

$$t(X) = s(X) - 1 = m(G[X]) - 1.$$

For $Y \in \mathcal{X}_G \setminus \{X\}$ we observe the following. If $s(Y) > m(G[Y])$, then we obtain

$$t(Y) \ge s(Y) - 1 \ge m(G[Y]).$$

If $s(Y) = m(G[Y])$, then from the claim follows that $u, v \in Y$ or $u, v \notin Y$. In both cases we deduce

$$t(Y) = s(Y) \ge m(G[Y]).$$

Furthermore, if $2 \le |X| \le |V(G_i)| - 2$, then from the construction of D follows that u has a negative neighbor and v has a positive neighbor in D. Hence $s(u) \le d_G(u) - 1$ and $s(v) \ge 1$ yielding $0 \le t \le d_G$. □

Of course, Theorem 2.9 implies that every polyhedron we obtain from $\mathcal{P}_G$ by omitting a collection of facet defining inequalities contains an integral vector which is not a degree vector of G.

Corollary 2.10
Let G be a graph of order n with components $G_1, \dots, G_k$. Moreover, let $\mathcal{Y} \subsetneq \mathcal{X}_G$ and denote $\mathcal{Y}_i = \mathcal{Y} \cap \mathcal{X}_{G_i}$ for $1 \le i \le k$. There exists a vector $t \in \mathbb{Z}^n \setminus \mathrm{DEG}^+(G)$ which is contained in the polyhedron

$$\left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{l} z(X) \ge m(G_i[X]) \text{ for all } X \in \mathcal{Y}_i, \text{ for all } 1 \le i \le k \text{ and} \\ z(V(G_i)) = m(G_i), \text{ for all } 1 \le i \le k \end{array} \right\}.$$

We close this section with two examples. First we take a look at the degree vector polytope of P_4 the path of length 3.

Example 2.11
We consider the labeled version G of P_4 defined by

$$V(G) = \{1,2,3,4\}, \qquad \text{and} \qquad E(G) = \{\{1,2\},\{2,3\},\{3,4\}\}.$$

The degree vector polytope of $\mathcal{P}_G$ in the form of (2.6) is

$$\mathcal{P}_G = \left\{ (x_1,x_2,x_3,x_4)^\top \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 \ge 0, \\ x_2 \ge 0, \\ x_3 \ge 0, \\ x_4 \ge 0, \\ x_1 + x_2 \ge 1, \\ x_1 + x_3 \ge 0, \\ x_1 + x_4 \ge 0, \\ x_2 + x_3 \ge 1, \\ x_2 + x_4 \ge 0, \\ x_3 + x_4 \ge 1, \\ x_1 + x_2 + x_3 \ge 2, \\ x_1 + x_2 + x_4 \ge 1, \\ x_1 + x_3 + x_4 \ge 1, \\ x_2 + x_3 + x_4 \ge 2, \\ x_1 + x_2 + x_3 + x_4 = 3 \end{array} \right\}.$$

The characterization of the facets of $\mathcal{P}_G$ in Theorem 2.8 shows that we can reduce this system of linear (in-)equalities to

$$\mathcal{P}_G = \left\{ (x_1,x_2,x_3,x_4)^\top \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 \ge 0, \\ x_4 \ge 0, \\ x_1 + x_2 \ge 1, \\ x_3 + x_4 \ge 1, \\ x_1 + x_2 + x_3 \ge 2, \\ x_2 + x_3 + x_4 \ge 2, \\ x_1 + x_2 + x_3 + x_4 = 3 \end{array} \right\}.$$

Now, the degree vector polytope of G is a subset of $\mathbb{R}^4$ but has dimension 3. For a visualization of $\mathcal{P}_G$ we wish to find a mapping from $\mathbb{R}^4$ to $\mathbb{R}^3$ not changing the appearance of $\mathcal{P}_G$. Hence we consider the linear mapping Θ defined by the orthogonal matrix

$$M = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Every vector of the affine space $\{(x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^4 \,|\, x_1 + x_2 + x_3 + x_4 = \alpha\}$ with $\alpha \in \mathbb{R}$ is mapped by Θ into $\{(x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^n \,|\, 2x_4 = \alpha\}$. Denote $\Phi : \mathbb{R}^4 \to \mathbb{R}^3$ the mapping we obtain from Θ by omitting the last row of M. Figure 2.3 shows a projection of $\Phi(\mathcal{P}_G)$. Since G is a tree every orientation is acyclic and has a different out-degree

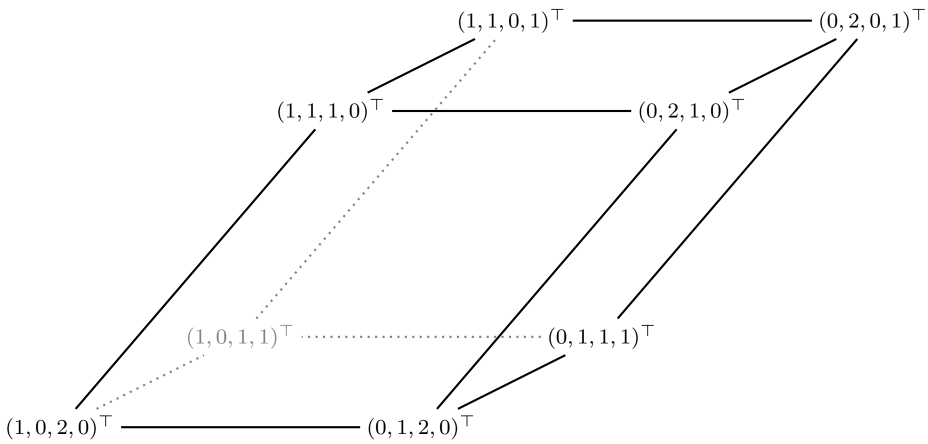


Figure 2.3: Degree vector polytope of a labeled version of P_4

vector. Therefore, every degree vector of G is a vertex of $\mathcal{P}_G$. Each vertex of the polytope is connected to three other vertices. Moreover, the surface of $\mathcal{P}_G$ consists of two squares (bottom and top) and four congruent rhombi (sides). All edges of $\mathcal{P}_G$ have the same side length. Thus $\mathcal{P}_G$ is a rhombohedron.

In the last example we prove that the degree vector polytope of every tree of order n is affinely isomorphic to the cube with dimension $n-1$.

Example 2.12
Let G be the labeled version of a tree on n vertices. Obviously, every of the 2^{n-1} orientations of G is acyclic. Thus we cannot find two orientation with the same out-degree vector. Therefore, $\mathrm{DEG}^+(G)$ consists of 2^{n-1} degree vectors of G. Each degree vector is a vertex of $\mathcal{P}_G$ and it is incident to exactly $n-1$ edges of this polytope. For every edge $e \in E(G)$ we obtain exactly two facet defining sets by taking the vertex set of the two components in $G - e$. Hence $\mathcal{P}_G$ has $2n-2$ facets. The same properties hold for the $(n-1)$-dimensional cube denoted by cube_{n-1}. In particular, we can prove that $\mathcal{P}_G$ is affinely isomorphic to cube_{n-1}.

To see this consider an arbitrary degree vector $s \in \mathrm{DEG}^+(G)$. There is an unique orientation D_s of G realizing s. Notice that we obtain every other orientation of G by reversing the arcs of a subset $A' \subseteq A(D_s)$. For example, if D_t is the orientation of G satisfying $A(D_s) \setminus A(D_t) = A'$ and t its out-degree vector, then we deduce that

$$t = s - \sum_{a \in A'} \mathbf{z}_a.$$

Thus for every degree vector $\tilde{s}$ of G there are (unique) coefficients $\lambda_a \in \{0, 1\}$ such that

$$\tilde{s} = s - \sum_{a \in A(D_s)} \lambda_a \mathbf{z}_a.$$

Now, every point in $\mathcal{P}_G$ is a convex combination of degree vectors of G. Therefore, for every $x \in \mathcal{P}_G$ there are (unique) coefficients $\lambda_a \in [0, 1]$ such that

$$x = s - \sum_{a \in A(D_s)} \lambda_a \mathbf{z}_a.$$

Let $a_1, \ldots, a_{n-1}$ be an enumeration of the arc set of D_s and consider the affine mapping $\Upsilon : \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by

$$\left(\lambda_{a_1}, \ldots, \lambda_{a_{n-1}}\right)^\top \mapsto s - \sum_{i=1}^{n-1} \lambda_{a_i} \mathbf{z}_{a_i}.$$

Since G is a tree the vectors in $\{\mathbf{z}_{a_1}, \ldots, \mathbf{z}_{a_{n-1}}\}$ are linearly independent in $\mathbb{R}^n$ and hence Υ is injective with $\Upsilon([0, 1]^{n-1}) = \mathcal{P}_G$. Thus $\mathcal{P}_G$ is affinely isomorphic to cube_{n-1}.

3 Degree complete graphs and labelings

In the first section of this chapter we define the concept of degree complete graphs which was introduced by Qian [23]. Qian also gave a characterization of degree complete graphs. We present an alternative proof for this result. The second section starts with a brief discussion on the motivations for the concept of degree complete graphs. Furthermore, we point out some properties concerning lattices of degree vectors which are analogue to a majorization argument for score sequences of tournaments. Section 3.3 starts with the important observation that the degree completeness of a graph is not independent from the chosen labeled version. This was already mentioned in [23]. Thus Qian formulated the problem to characterize the graphs which are not degree complete no matter how we label its vertices. As a solution to this problem we present two characterizations of those graphs which have a so called degree complete labeling. One characterization is in terms of forbidden subgraphs. The other implies a polynomial procedure to recognize graphs with a degree complete labeling. In the final section we consider graphs which are strongly degree complete. A characterization of the graphs which have a strongly degree complete labeling helps us to evaluate the result on graphs with degree complete labeling.

3.1 Degree complete graphs

Let G be a labeled graph. Consider two orientations D^r and D^l of G with arc sets $A_G^r = \{(i,j) \,|\, ij \in E(G), i < j\}$ and $A_G^l = \{(i,j) \,|\, ij \in E(G), i > j\}$. We define s_G^r and s_G^l as the out-degree vectors D^r and D^l, respectively. To visualize these orientations suppose all vertices of G are written on a horizontal line with increasing labels from left to right. Now, A_G^r is the arc set of the orientation of G, where all arcs are directed from left to right. In the same arrangement of vertices all arcs in A_G^l point from right to left. Hence we have

$$s_G^l(\{1,\dots,k\}) = m\big(G[\{1,\dots,k\}]\big)$$

and

$$s_G^r(\{1,\dots,k\}) = m\big(G[\{1,\dots,k\}]\big) + m_G\big(\{1,\dots,k\},\{k+1,\dots,n\}\big)$$

for $1 \le k \le n$. Thus, on the one hand, for every degree vector s of G holds

$$s_G^l \preccurlyeq s \preccurlyeq s_G^r \quad \text{and} \quad 0 \le s \le d_G. \tag{3.1}$$

On the other hand, there may be nonnegative integral vectors satisfying (3.1) which are not realized by an orientation of G. We define the following set of nonnegative integral vectors

$$\mathcal{S}(G) = \left\{ s \in \mathbb{Z}^n \mid s_G^l \preccurlyeq s \preccurlyeq s_G^r \quad \text{and} \quad 0 \le s \le d_G \right\}.$$

Since every degree vector of G fulfills (3.1) we have

$$\mathrm{DEG}^+(G) \subseteq \mathcal{S}(G). \tag{3.2}$$

In [23] Qian introduced the concept of degree complete graphs. A labeled graph G is *degree complete*, if every nonnegative integral vector s satisfying (3.1) is a degree vector of G. In other words the degree complete graphs are exactly those graphs for which equality holds in (3.2).

Qian also proved a characterization of labeled graphs that are degree complete. Here we give an alternative proof to this result.

The following lemma describes a kind of hereditary property of degree complete graphs. In particular, we show that every spanning subgraph of a degree complete graph is degree complete, too.

Lemma 3.1

Let G be a labeled graph and H a spanning subgraph of G. If G is degree complete, then H is degree complete.

Proof

We prove the statement by induction on $k = |E(G) \setminus E(H)|$. If $k = 0$, then the assertion trivially holds. Thus we assume that the statement is true for $k-1$ with $k \ge 1$. It suffices to show that an arbitrary vector $s \in \mathcal{S}(H)$ is a degree vector of H. Let uv be an edge in $E(G) \setminus E(H)$ and consider the spanning subgraph $\tilde{H} = H + uv$ of G. Denote D^l and D^r the orientations of H with out-degree vectors s_H^l and s_H^r, respectively.

We define the vectors p, p^l, p^r, q, q^l and q^r by

$$p = s + \mathbf{e}_u,\ p^l = s_H^l + \mathbf{e}_u,\ p^r = s_H^r + \mathbf{e}_u, \quad q = s + \mathbf{e}_v,\ q^l = s_H^l + \mathbf{e}_v,\ q^r = s_H^r + \mathbf{e}_v.$$

From this definition follows that p^l and p^r are out-degree vectors of $D^l + (u,v)$ and $D^r + (u,v)$, respectively. Since $D^l + (u,v)$ and $D^r + (u,v)$ both are orientations of $\tilde{H}$ the vectors p^l and p^r are degree vectors of $\tilde{H}$. Hence we deduce

$$s_{\tilde{H}}^l \preccurlyeq p^l \preccurlyeq p \preccurlyeq p^r \preccurlyeq s_{\tilde{H}}^r.$$

Additionally, it is easy to see that $0 \le p \le d_{\tilde{H}}$ holds and thus $p \in \mathcal{S}(\tilde{H})$. By an analogue deduction we prove that $q \in \mathcal{S}(\tilde{H})$.

Now, from the induction hypothesis follows that $\mathcal{S}(\tilde{H}) = \mathrm{DEG}^+(\tilde{H})$. Hence there exist orientations D^p and D^q of $\tilde{H}$ with out-degree vectors p and q, respectively.

If $(u,v) \in A(D^p)$, then $D^p - (u,v)$ is an orientation of H with out-degree vector $p - \mathbf{e}_u = s$. Otherwise we have $(v,u) \in A(D^p)$. If there is a directed path from u to v in D^p, then the arc (v,u) belongs to a directed cycle in D^p. Reorienting this cycle yields an orientation of $\tilde{H}$ which contains (u, v) and has out-degree vector p. Thus we suppose that D^p does not have an (u, v)-path. Denote $U \subseteq V$ the set of vertices that can be reached from u via a directed path in D^p. Obviously, we have $u \in U$ and $v \notin U$. By Theorem 2.3 we conclude

$$s(U) = q(U) \geq m(G[U]) = p(U) = s(U) + 1.$$

From this contradiction follows that D^p contains a path from u to v. This completes the proof. □

We use Lemma 3.1 in the following way. If we want to show that a graph is not degree complete, then we only have to check that there is a spanning subgraph of G which is not degree complete. Additionally, in chapter 5 we prove that this hereditary property still holds in a more general setting.
The following theorem characterizes all labeled graphs which are degree complete in terms of two forbidden subgraphs.

Theorem 3.2 (Qian [23] 2006)
A labeled graph G is degree complete if and only if G does not contain one of the two subgraphs H_1 and H_2, where

$$V(H_1) = V(H_2) = \{k_1, k_2, k_3, k_4\}, \quad k_1 < k_2 < k_3 < k_4$$

and

$$E(H_1) = \{k_1k_3, k_2k_4\}, \quad E(H_2) = \{k_1k_4, k_2k_3\}.$$

As in [23], we use the notation *forbidden configurations* for H_1 and H_2.
Reminding the embedding of a labeled graph G along a horizontal line with respect to the vertex labels these forbidden configurations can be visualized (see Figure 3.1) in the following way. If we draw all edges on one site (above or below) of the line, the subgraph

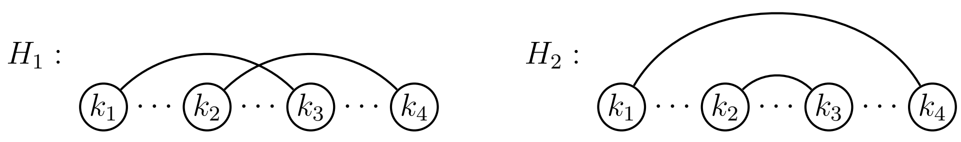


Figure 3.1: Forbidden configurations H_1 and H_2 from Theorem 3.2.

H_1 consists of a pair of crossing independent edges. Similarly, H_2 refers to a pair of overleaping independent edges in G.

The following proof of Theorem 3.2 uses Lemma 3.1 and a majorization argument. The latter point is motivated by the proof of Theorem 2.2 presented in chapter 2.

Proof (Theorem 3.2)
Let V be the vertex set of G. To prove necessity suppose that G contains a spanning

subgraph H whose edge set consists of two independent edges v_1v_3 and v_2v_4 with $v_1 < v_2 < v_3 < v_4$. Hence H contains a forbidden configuration H_1. Obviously, we have

$$\mathrm{DEG}^+(H) = \{\mathbf{e}_{v_1} + \mathbf{e}_{v_2}, \mathbf{e}_{v_1} + \mathbf{e}_{v_4}, \mathbf{e}_{v_3} + \mathbf{e}_{v_2}, \mathbf{e}_{v_3} + \mathbf{e}_{v_4}\},$$

where $\mathbf{e}_{v_3} + \mathbf{e}_{v_4} = s^l_H$ and $\mathbf{e}_{v_1} + \mathbf{e}_{v_2} = s^r_H$. From $\mathbf{e}_{v_3} \preccurlyeq \mathbf{e}_{v_2}$ and $\mathbf{e}_{v_4} \preccurlyeq \mathbf{e}_{v_1}$ we deduce that

$$s^l_H \preccurlyeq \mathbf{e}_{v_2} + \mathbf{e}_{v_4} \preccurlyeq s^r_H.$$

It is easy to check that $\mathbf{e}_{v_2} + \mathbf{e}_{v_4}$ is an element of $\mathcal{S}(H)$ but not a degree vector of H. Hence H is not degree complete. By Lemma 3.1 the graph G is not degree complete.
Now, suppose that G contains a spanning subgraph H consisting of a forbidden configuration H_2 with edges v_1v_4 and v_2v_3 where $v_1 < v_2 < v_3 < v_4$ and some isolated vertices. By similar arguments as seen before we show that $\mathbf{e}_{v_1} + \mathbf{e}_{v_4}$ is a vector in $\mathcal{S}(H)$ but not a degree vector of H. Therefore, G is not degree complete by Lemma 3.1.
To prove sufficiency suppose G does not contain one of the two subgraphs H_1 and H_2 as defined in 3.2. Let $s \in \mathcal{S}(G)$ be an arbitrary vector. We prove that there is an orientation of G with out-degree vector s. Obviously, $s = s^r_G$ is a degree vector of G. Thus we assume $s^l_G \preccurlyeq s \prec s^r_G$. Let t be a vector from $\mathrm{DEG}^+(G)$ satisfying $s^l_G \preccurlyeq s \prec t \preccurlyeq s^r_G$. Denote D an orientation of G with out-degree vector t.
Since $s \prec t$ there is a minimal vertex $v \in V$ such that $s(v) < t(v)$. Furthermore, the equation $\sum_{i=1}^n s(i) = \sum_{i=1}^n t(i)$ yields the existence of a minimal vertex $w \in V$ with $s(w) > t(w)$. Observe that $v < w$.
Suppose there does not exist a (v, w)-path in D. Denote X the set of vertices in V that can be reached by a directed path in D with initial vertex v. Analogously, we define Y as the set of vertices reaching w via a directed path in D. Notice that $v \in X$, $w \in Y$, and $X \cap Y = \emptyset$. Because $0 \leq s(v) < t(v)$ the vertex v has a positive neighbor in D. Hence we deduce $|X| \geq 2$. Similarly, by $d_G(w) \geq s(w) > t(w)$ we obtain $|Y| \geq 2$. Therefore, $G[X]$ and $G[Y]$ contain at least one edge, respectively. Now, let x be the largest vertex in X and y the smallest vertex in Y. If $x > y$, then there are edges $e \in E(G[X])$ and $e' \in E(G[Y])$ such that $\{e, e'\}$ is the edge set of a subgraph isomorphic to H_1 or H_2. This contradicts our assumption on G. Thus we have $v \leq x < y \leq w$. By the choice of $x \in X$ the orientation D does not contain an arc from $\{1, \ldots, x\}$ to $\{x+1, \ldots, n\}$. This yields

$$s(\{1, \ldots, x\}) < t(\{1, \ldots, x\}) = m(G[1, \ldots, x]) = s^l_G(\{1, \ldots, x\})$$

and therefore a contradiction to $s^l_G \preccurlyeq s$. Thus there is a directed path P from v to w in D. Now, we define the vector $\tilde{t} = t - \mathbf{z}_{(v,w)}$. Obviously, $\tilde{t}$ is the out-degree vector of the orientation of G we obtain from D by reorienting P. It is easy to check that $s \preccurlyeq \tilde{t} \prec t$ holds and $\tilde{t} \in \mathcal{S}(G)$. By a finite number of iterations we finally obtain an orientation of G with out-degree vector s. □

In the next section we continue with a brief discussion on the motivations for the concept of degree complete graphs.

3.2 Motivations for degree complete graphs

In this section we focus on three aspects which motivate the concept of degree complete graphs. The first reason includes the idea of a convex subset in a partially ordered set. Let G be a labeled graph of order n and size m. Obviously, the set of integral vectors in

$$\mathcal{M}_G = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x(i) = m \right\}$$

is partially ordered by $\preccurlyeq$. This is also true, if we consider the finite set

$$\hat{\mathcal{M}}_G = \mathcal{M}_G \cap \{x \in \mathbb{Z}^n \mid 0 \leq x \leq d_G\}.$$

The set of degree vectors $\mathrm{DEG}^+(G)$ is a subset of $\hat{\mathcal{M}}_G$ and $\mathcal{M}_G$, respectively. Now, $\mathrm{DEG}^+(G)$ is a convex set in $(\hat{\mathcal{M}}_G, \preccurlyeq)$ if and only if G is degree complete.

Another fact which motivates the consideration of degree complete graphs is related to the determination of the set $\mathrm{DEG}^+(G)$. Two approaches seem to be natural but also not very efficient. On the one hand we could iterate over all orientations of G and store the corresponding out-degree vector. For general graphs this approach is not efficient since different orientations may yield the same out-degree vector. Hence the number of iterations would be unnecessary large. On the other hand we could define a set of vectors which includes $\mathrm{DEG}^+(G)$ and use the idea from the proof of Theorem 2.3 or the associated network-flow-problem to decide for each vector whether it is a degree vector of G or not. The disadvantage of this approach is that we do not use any relations between different degree vectors although they exist.

At this point a partial order like $\preccurlyeq$ can be used to obtain a more efficient procedure. Starting at a maximal element we construct in each step the predecessors of a selected vector with respect to $\preccurlyeq$. This idea should be seen in analogy to the majorization argument concerning score sequences of tournaments. Now, for degree complete graphs we are in the comfortable situation that each vector which is constructed this way is already a degree vector, if we choose $\hat{\mathcal{M}}_G$ or a suitable subset as the ground set.

A further reason to consider degree complete graphs is the structure of the partially ordered sets $(\mathcal{S}(G), \preccurlyeq)$ and $(\mathrm{DEG}^+(G), \preccurlyeq)$. In the following we show that for an arbitrary graph G the poset $(\mathcal{S}(G), \preccurlyeq)$ is a lattice. Hence if G is degree complete, then $(\mathrm{DEG}^+(G), \preccurlyeq)$ is a lattice, too. Notice that it is not trivial to decide whether the set of degree vectors of a given (labeled) graph has a lattice structure with respect to $\preccurlyeq$ or not. To prove that $(\mathcal{S}(G), \preccurlyeq)$ is a lattice we show that for each pair of degree vectors $s, t \in \mathcal{S}(G)$ there is a supremum and an infimum in $\mathcal{S}(G)$. For our definitions of $\sup(s,t)$ and $\inf(s,t)$ we require an equivalent condition for $s \preccurlyeq t$. The following lemma also implies that $\preccurlyeq$ is a cone order. Furthermore, it determines the vectors generating the corresponding cone.

Lemma 3.3
For $s, t \in \mathbb{R}^n$ holds $s \preccurlyeq t$ if and only if there are (unique) coefficients $\lambda_i \geq 0$ such that

$$t - s = \sum_{i=1}^{n-1} \lambda_i \mathbf{z}_{(i,i+1)}.$$

Moreover, if s and t are integral vectors, then each λ_i is a nonnegative integer.

Proof
Suppose $s \preccurlyeq t$ and define $\gamma_k = \sum_{i=1}^{k} (t(i) - s(i))$ for $1 \leq k \leq n$. By the definition of $\preccurlyeq$ we deduce that $\gamma_n = 0$ and $\gamma_k \geq 0$ for $1 \leq k \leq n-1$. Moreover, if $s, t \in \mathbb{Z}^n$, then γ_k is a nonnegative integer. We consider the vector

$$r = s + \sum_{j=1}^{n-1} \gamma_j \mathbf{z}_{(j,j+1)}.$$

Notice that for $1 \leq k \leq n$ and $1 \leq j \leq n-1$ we have

$$\sum_{i=1}^{k} \mathbf{z}_{(j,j+1)}(i) = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{else.} \end{cases} \tag{3.3}$$

Hence we obtain

$$\begin{aligned} \sum_{i=1}^{k} r(i) &= \sum_{i=1}^{k} s(i) + \sum_{i=1}^{k} \left(\sum_{j=1}^{n-1} \gamma_j \mathbf{z}_{(j,j+1)} \right)(i) \\ &= \sum_{i=1}^{k} s(i) + \sum_{j=1}^{n-1} \gamma_j \sum_{i=1}^{k} \mathbf{z}_{(j,j+1)}(i) \\ &= \sum_{i=1}^{k} s(i) + \gamma_k = \sum_{i=1}^{k} t(i). \end{aligned}$$

It is easy to check that from $\sum_{i=1}^{k} r(i) = \sum_{i=1}^{k} t(i)$ for all $1 \leq k \leq n$ follows $r = t$ Therefore, we conclude

$$t - s = \sum_{j=1}^{n-1} \gamma_j \mathbf{z}_{(j,j+1)}.$$

Now, suppose there are coefficients $\lambda_i \geq 0$ such that $t - s = \sum_{i=1}^{n-1} \lambda_i \mathbf{z}_{(i,i+1)}$. For $1 \leq k \leq n$ we deduce by (3.3)

$$\begin{aligned} \sum_{i=1}^{k} t(i) &= \sum_{i=1}^{k} s(i) + \sum_{i=1}^{k} \left(\sum_{j=1}^{n-1} \lambda_j \mathbf{z}_{(j,j+1)} \right)(i) \\ &= \sum_{i=1}^{k} s(i) + \sum_{j=1}^{n-1} \lambda_j \sum_{i=1}^{k} \mathbf{z}_{(j,j+1)}(i) \\ &= \sum_{i=1}^{k} s(i) + \lambda_k \geq \sum_{i=1}^{k} s(i), \end{aligned}$$

where equality holds for $k = n$. Hence we have $s \preccurlyeq t$. □

The vectors $\mathbf{z}_{(1,2)}, \ldots, \mathbf{z}_{(n-1,n)}$ form a basis of the linear subspace $\mathcal{M}_n$. Thus the coefficients λ_i in Lemma 3.3 are unique. Moreover, the cone $\mathcal{C} \subseteq \mathbb{R}^n$ generated by the vectors $\mathbf{z}_{(1,2)}, \ldots, \mathbf{z}_{(n-1,n)}$ satisfies $s \preccurlyeq t$ if and only if $t - s \in \mathcal{C}$. Hence Lemma 3.3 also implies that $\preccurlyeq$ is a cone order.

Definition 3.4
Let $s, t \in \mathbb{R}^n$ and denote λ_i the unique coefficients with $s - t = \sum_{i=1}^{n-1} \lambda_i \mathbf{z}_{(i,i+1)}$. We define

$$\sup(s,t) = t + \sum_{i=1}^{n-1} \max(0, \lambda_i) \cdot \mathbf{z}_{(i,i+1)} = s - \sum_{i=1}^{n-1} \min(0, \lambda_i) \cdot \mathbf{z}_{(i,i+1)}$$

and

$$\inf(s,t) = t + \sum_{i=1}^{n-1} \min(0, \lambda_i) \cdot \mathbf{z}_{(i,i+1)} = s - \sum_{i=1}^{n-1} \max(0, \lambda_i) \cdot \mathbf{z}_{(i,i+1)}.$$

From this definition follows immediately

$$\inf(s,t) \preccurlyeq s, t \preccurlyeq \sup(s,t).$$

Furthermore, if s and t are integral vectors, then $\sup(s,t)$ and $\inf(s,t)$ are integral vectors, too. The next lemma shows that, as desired, $\sup(s,t)$ is the supremum and $\inf(s,t)$ is the infimum of s and t in $\mathbb{R}^n$ with respect to $\preccurlyeq$.

Lemma 3.5
Let s and t be vectors in $\mathbb{R}^n$.

- For every $p \in \mathbb{R}^n$ with $s \preccurlyeq p$ and $t \preccurlyeq p$ holds $\sup(s,t) \preccurlyeq p$.
- For every $q \in \mathbb{R}^n$ with $q \preccurlyeq s$ and $q \preccurlyeq t$ holds $q \preccurlyeq \inf(s,t)$.

Proof
Let $p \in \mathbb{R}^n$ be an arbitrary vector with $s \preccurlyeq p$ and $t \preccurlyeq p$. By Lemma 3.3 there are nonnegative coefficients μ_i and ν_i for $1 \leq i \leq n-1$ satisfying

$$p - s = \sum_{i=1}^{n-1} \mu_i \mathbf{z}_{(i,i+1)} \qquad \text{and} \qquad p - t = \sum_{i=1}^{n-1} \nu_i \mathbf{z}_{(i,i+1)}.$$

Hence from

$$s - t = (p - t) - (p - s) = \sum_{i=1}^{n-1} (\nu_i - \mu_i) \mathbf{z}_{(i,i+1)}$$

we obtain

$$\sup(s,t) = t + \sum_{i=1}^{n-1} \max(0, \nu_i - \mu_i) \cdot \mathbf{z}_{(i,i+1)}.$$

Therefore, we deduce

$$p - \sup(s,t) = p - t - \sum_{i=1}^{n-1} \max(0, \nu_i - \mu_i) \cdot \mathbf{z}_{(i,i+1)} = \sum_{i=1}^{n-1} \big(\nu_i - \max(0, \nu_i - \mu_i)\big)\mathbf{z}_{(i,i+1)}.$$

Now, for $1 \leq i \leq n-1$ we have

$$\nu_i - \max(0, \nu_i - \mu_i) = \begin{cases} \mu_i, & \text{if } \nu_i > \mu_i \\ \nu_i, & \text{if } \nu_i \leq \mu_i \end{cases}.$$

Thus $\nu_i - \max(0, \nu_i - \mu_i) \geq 0$ and we conclude $\sup(s,t) \preccurlyeq p$. A similar deduction yields that for every $q \in \mathbb{R}^n$ with $q \preccurlyeq s$ and $q \preccurlyeq t$ satisfies $q \preccurlyeq \inf(s,t)$. □

Now, we can prove that $(\mathcal{S}(G), \preccurlyeq)$ is a lattice. Using Lemma 3.5 we only have to show that $\sup(s,t)$ and $\inf(s,t)$ are elements of $\mathcal{S}(G)$.

Theorem 3.6
For every labeled graph G the poset $(\mathcal{S}(G), \preccurlyeq)$ is a lattice.

Proof
Denote n the order of G and consider a pair of vectors $s, t \in \mathcal{S}(G)$. Obviously, we have

$$s_G^l \preccurlyeq s \preccurlyeq \sup(s,t).$$

Moreover, from Lemma 3.5 follows immediately that $\sup(s,t) \preccurlyeq s_G^r$. Hence we conclude that $s_G^l \preccurlyeq \sup(s,t) \preccurlyeq s_G^r$. It suffices to prove $0 \leq \sup(s,t) \leq d_G$. Let $\lambda_1, \ldots, \lambda_{n-1}$ be integers such that $s - t = \sum_{i=1}^{n-1} \lambda_i \mathbf{z}_{(i,i+1)}$. Without loss of generality we assume that the smallest index k with $\lambda_k \neq 0$ satisfies $\lambda_k > 0$. Otherwise we interchange s and t. If $\lambda_i > 0$ holds for all $k \leq i \leq n-1$, then we have $t \preccurlyeq s$ and $\sup(s,t) = s$. In this case we are done. Hence we consider the case that there is a smallest index $l > k$ with $\lambda_l \leq 0$. Starting with t we construct a vector $\tilde{t}$ with $t \prec \tilde{t} \preccurlyeq \sup(s,t)$ and $\tilde{t} \in \mathcal{S}(G)$ such that $\sup(s,t) = \sup(s,\tilde{t})$. Notice that by a finite iteration of this step we obtain $\sup(s,t)$. In particular, we show that $\tilde{t} = t + \mathbf{z}_{(k,l)}$ satisfies the desired conditions. From the choices of k and l we deduce $1 \leq k < l < n$. Firstly, it is easy to see that we have

$$\tilde{t} - t = \mathbf{z}_{(k,l)} = \sum_{i=k}^{l-1} \mathbf{z}_{(i,i+1)}$$

and thus $t \prec \tilde{t}$. Secondly, from $\lambda_i > 0$ for $k \leq i \leq l-1$ follows that μ_i defined by

$$\mu_i = \begin{cases} \max(0, \lambda_i) - 1, & \text{for } k \leq i \leq l-1, \\ \max(0, \lambda_i), & \text{else,} \end{cases}$$

satisfies $\mu_i \geq 0$ for $1 \leq i \leq n-1$. Hence

$$\sup(s,t) - \tilde{t} = t + \sum_{i=1}^{n-1} \max(0, \lambda_i)\mathbf{z}_{(i,i+1)} - t - \mathbf{z}_{(k,l)} = \sum_{i=1}^{n-1} \mu_i \mathbf{z}_{(i,i+1)}$$

implies $\tilde{t} \preccurlyeq \sup(s,t)$. Moreover, we have to prove that $0 \leq \tilde{t} \leq d_G$. Notice that we have

$$\lambda_{k-1} \leq 0 < \lambda_k \qquad \text{and} \qquad \lambda_{l-1} > 0 \geq \lambda_l,$$

where we set $\lambda_0 = 0$, if $k = 1$. Now, suppose to the contrary that $0 \not\leq \tilde{t}$ or $\tilde{t} \not\leq d_G$. By the definition of $\tilde{t}$ we deduce that $t(k) = d_G(k)$ or $t(l) = 0$. If $t(k) = d_G(k)$, then we have

$$0 \leq t(k) - s(k) = -\sum_{i=1}^{n-1} \lambda_i \mathbf{z}_{(i,i+1)}(k) = \lambda_{k-1} - \lambda_k < 0,$$

and for $t(l) = 0$ we obtain

$$0 \leq s(l) - t(l) = \sum_{i=1}^{n-1} \lambda_i \mathbf{z}_{(i,i+1)}(l) = \lambda_l - \lambda_{l-1} < 0.$$

These contradictions prove that $0 \leq \tilde{t} \leq d_G$ and thus $\tilde{t} \in \mathcal{S}(G)$. Finally, we observe that

$$s - \tilde{t} = s - t - \mathbf{z}_{(k,l)} = \sum_{i=1}^{n-1} \lambda_i \mathbf{z}_{(i,i+1)} - \sum_{i=k}^{l-1} \mathbf{z}_{(i,i+1)}.$$

Since $\lambda_i > 0$ for $k \leq i \leq l-1$ we deduce

$$\begin{aligned} \sup(s,\tilde{t}) &= t + \mathbf{z}_{(k,l)} + \sum_{i=1}^{n-1} \max(0,\lambda_i) \cdot \mathbf{z}_{(i,i+1)} - \sum_{i=k}^{l-1} \mathbf{z}_{(i,i+1)} \\ &= t + \sum_{i=1}^{n-1} \max(0,\lambda_i) \cdot \mathbf{z}_{(i,i+1)} \\ &= \sup(s,t). \end{aligned}$$

Therefore, $\tilde{t}$ has the desired properties and we conclude that $\sup(s,t) \in \mathcal{S}(G)$. By an analogue deduction we show that $\inf(s,t) \in \mathcal{S}(G)$. □

As a direct consequence we obtain the following corollary.

Corollary 3.7
Let G be a labeled graph. If G is degree complete, then $(\mathrm{DEG}^+(G), \preccurlyeq)$ is a lattice.

In general the reverse statement is not true. Considering the graph G with $V(G) = \{1,2,3,4\}$ and $E(G) = \{\{1,3\},\{2,4\}\}$. By Theorem 3.2 the graph G is not degree complete but we observe that

$$\mathrm{DEG}^+(G) = \{(1,1,0,0)^\top, (1,0,0,1)^\top, (0,1,1,0)^\top, (0,0,1,1)^\top\}$$

is a lattice with respect to $\preccurlyeq$. This is remarkable since for $s = (1,0,0,1)^\top$ and $t = (0,1,1,0)^\top$ the vectors $\sup(s,t) = (1,0,1,0)^\top$ and $\inf(s,t) = (0,1,0,1)^\top$ are not included in $\mathrm{DEG}^+(G)$. The following example shows that there are graphs G for which $(\mathrm{DEG}^+(G), \preccurlyeq)$ is not a lattice.

Example 3.8
Denote G any labeled version of the complete graph K_5. By Theorem 2.1 $s = (0, 2, 3, 4, 1)^\top$ and $t = (0, 4, 1, 2, 3)^\top$ are degree vectors of G. From

$$s - t = (0, -2, 2, 2, -2)^\top = -2\mathbf{z}_{(2,3)} + 2\mathbf{z}_{(4,5)}$$

we deduce that

$$\sup(s, t) = t + 2\mathbf{z}_{(4,5)} = (0, 4, 1, 4, 1)^\top.$$

Obviously, there does not exist an orientation of G with out-degree vector $(0, 4, 1, 4, 1)^\top$ and thus $(0, 4, 1, 4, 1)^\top \notin \mathrm{DEG}^+(G)$. Furthermore, $p = (1, 3, 1, 4, 1)^\top$ and $q = (0, 4, 2, 3, 1)^\top$ are degree vectors of G such that $\sup(s, t) \preccurlyeq p, q$. Hence, on the one hand, we notice that p and q are both elements of the set

$$\{s \in \mathrm{DEG}^+(G) \mid \sup(s, t) \preccurlyeq s\}.$$

On the other hand, since $p - q = (1, -1, -1, 1, 0)^\top = \mathbf{z}_{(1,2)} - \mathbf{z}_{(3,4)}$ the vectors are incomparable with respect to $\preccurlyeq$ and therefore s, t have no supremum in $\mathrm{DEG}^+(G)$.

Let G be a labeled graph of order n with vertex set V. According to the visualization of score sequences of tournaments by Ferrers diagrams we present a similar approach for degree vectors. The use of Ferrers diagrams yield a procedure to determine the score sequences which majorize a given sequence. Hence for a given labeled graph G and an element $s \in \mathcal{S}(G)$ with $s \neq s_G^r$ we are interested in the predecessors of s with respect to $\preccurlyeq$ in $\mathcal{S}(G)$. Here we assume that G does not contain any isolated vertices. Let $s' \in \mathcal{S}(G)$ be a successor of s with respect to $\preccurlyeq$. By Lemma 3.3 there are nonnegative integers λ_i for $1 \leq i \leq n-1$ with $\lambda_1 + \ldots + \lambda_{n-1} > 0$ such that

$$s' - s = \sum_{i=1}^{n-1} \lambda_i \mathbf{z}_{(i,i+1)}. \tag{3.4}$$

Moreover, there is a minimal vertex $u \in V$ with $s(u) < s'(u)$ and a minimal vertex $v \in V$ with $v > u$ such that $s(v) > s'(v)$. From the minimality of u and v follows that $\lambda_1 = \ldots = \lambda_{u-1} = 0$ and $\lambda_u, \ldots, \lambda_{v-1} \geq 1$. Suppose there is an index j satisfying $\lambda_j > 2$ for $u \leq j < v$ or $\lambda_j > 0$ for $v \leq j \leq n-1$. Considering

$$t = s + \sum_{i=u}^{v-1} \mathbf{z}_{(i,i+1)} = s + \mathbf{z}_{(u,v)}$$

we observe that $s \prec t \prec s'$. Since $s(u) < d_G(u)$ and $s(v) > 0$ we deduce that $t \in \mathcal{S}(G)$ contradicting the choice of s'. Therefore, we assume that the only positive coefficients in (3.4) are $\lambda_u = \ldots = \lambda_{v-1} = 1$. Finally, we show that $v = u + 1$. Suppose to the contrary that $v > u + 1$ holds. If $s(u + 1) = 0$, then the fact that G does not contain an isolated vertex implies $s(u + 1) < d_G(u + 1)$. Considering

$$t' = s + \sum_{i=u+1}^{v-1} \mathbf{z}_{(i,i+1)}$$

we notice that $s \prec t' \prec s'$ and $t' \in \mathcal{S}(G)$ yielding again a contradiction. Thus we deduce $s(u+1) > 0$. Now observe that for $t''(s) = s + \mathbf{z}_{(v,v+1)} \in \mathcal{S}(G)$ holds $s \prec t'' \prec s'$. From this contradiction follows that $v = u+1$ and finally $s' - s = \mathbf{z}_{(u,u+1)}$.
For $s \in \mathcal{S}(G)$ denote $\lambda_1^s, \ldots, \lambda_{n-1}^s \in \mathbb{Z}$ the unique coefficients with

$$s_G^r - s = \sum_{i=1}^{n-1} \lambda_i^s \mathbf{z}_{(i,i+1)}$$

and $\Lambda^s \in \mathbb{Z}^{n-1}$ the vector with i-th entry λ_i^s. The deduction above shows that we obtain all successors of s with respect to $\preccurlyeq$ in $\mathcal{S}(G)$ by $s + \mathbf{z}_{(v,v+1)}$ where $1 \le v \le n-1$, $s(v) < d_G(v)$, $s(v+1) > 0$, and $\Lambda^s(v) > 0$ hold.
For a visualization we represent an element $s \in \mathcal{S}(G)$ by n rows of boxes where row v with $1 \le v \le n$ consists of $d_G(v)$ boxes and the first $s(v)$ of these boxes are marked (by a cross). Therefore, the number of marked boxes in row v equals the number of arcs with tail in $v \in V$. The total number of boxes equals the sum of vertex degrees of G and there are as many marked as unmarked boxes in the diagram.
To construct $(\mathcal{S}(G), \preccurlyeq)$ we start with $s = s_G^l$ and determine $\Lambda^{s_G^l}$. While $s \neq s_G^r$ the following operation on the diagram gives us the successors of s. Check for each $w \in \{1, \ldots, n-1\}$ whether row w contains an unmarked box, row $w+1$ has a marked box, and $\Lambda^s(w) > 0$. If all these conditions are satisfied, then mark the first unmarked box in line w and remove the mark of the last box in line $w+1$. Finally, we set $\Lambda^t = \Lambda^s - \mathbf{e}_v$ for the new element $t = s + \mathbf{z}_{(v,v+1)}$ and continue.
Figure 3.2 shows a labeled version of C_3 and a path of length 3. Below it depicts the lattices of degree vectors where each degree vector is represented by a diagram as described above.

3.3 Degree complete labelings

An important fact concerning the concept of degree complete graphs is illustrated by the following example.

Example 3.9 (Qian [23] (2006))
Consider the labeled graphs G_1 and G_2 from Figure 3.3. Obviously, G_1 does not contain any of the subgraphs H_1 and H_2. Thus from Theorem 3.2 follows that G_1 is degree complete. On the other hand, in G_2 the edges $\{1,3\}$ and $\{2,4\}$ form a forbidden configuration H_1. Therefore, G_2 is not degree complete.

Since G_1 and G_2 are both labeled versions of P_4 we observe that the property of being degree complete depends on the vertex labeling of the graph. Qian also noticed this fact and stated the following problem.

Problem 3.10 (Qian [23] (2006))
Characterize the graphs which are not degree complete no matter how we label its vertices.

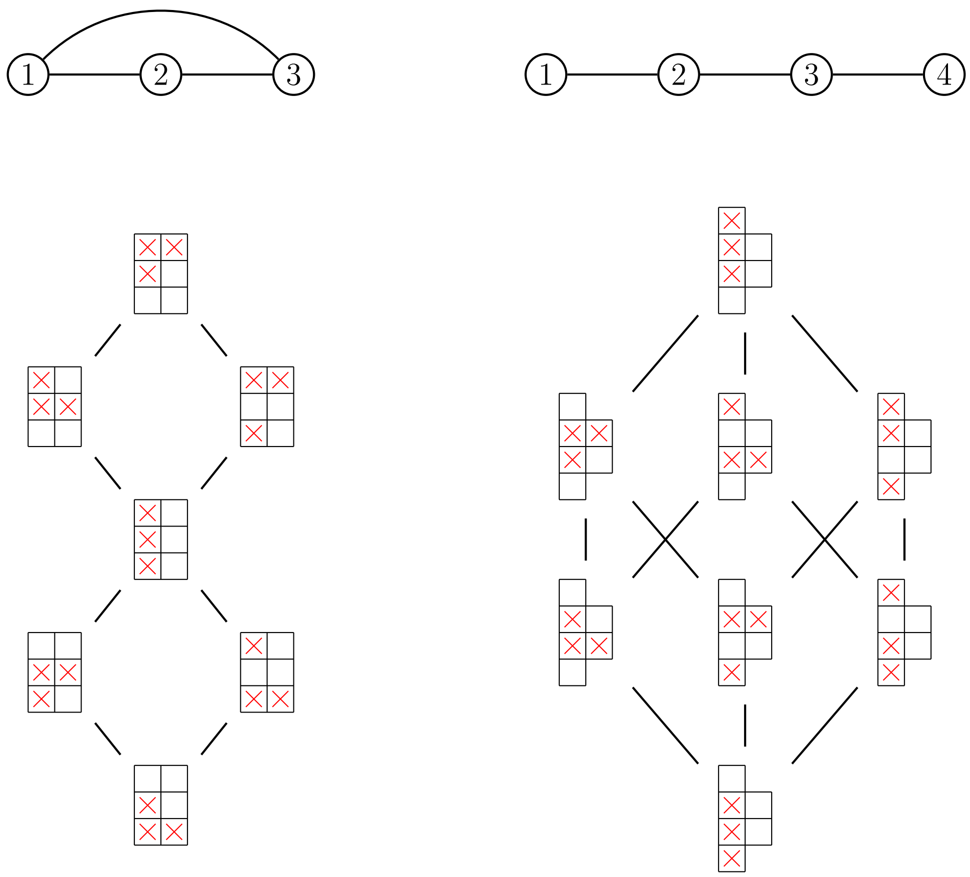


Figure 3.2: Hasse diagram of the lattice of degree vectors for a labeled version of C_3 and P_4.

To solve this problem we say that an unlabeled graph G has a *degree complete labeling*, if there exists a labeling f of its vertices such that the labeled graph G_f is degree complete. Thus Problem 3.10 asks for a characterization of the graphs which do not have any degree complete labeling.

In addition to the above mentioned problem two more questions arise. Firstly, can we find an efficient procedure to recognize graphs having a degree complete labeling? Secondly, if we know that a given graph has such a labeling, how can we determine it?

In general, an unlabeled graph has a huge number of vertex labelings (even if we just count the labelings modulo the automorphism group of the graph). Therefore, it is not useful to test different labelings by applying Theorem 3.2 and we need a new approach for this problem.

The main theorem of this section gives us three characterizations of unlabeled graphs which have a degree complete labeling. The first equivalent formulation describes the structure of these graphs in terms of forbidden subgraphs. The other characterizations yield a polynomial procedure to recognize unlabeled graphs with degree complete labeling. Furthermore, these characterizations can be used as starting points for two algorithms which determine a desired labeling.

Figure 3.3: Graphs G_1 and G_2 from Example 3.9.

We continue with two simple but important observations. The first observation is a direct consequence of Lemma 3.1. Moreover, it motivates a characterization of graphs with degree complete labeling that is based on forbidden subgraphs.

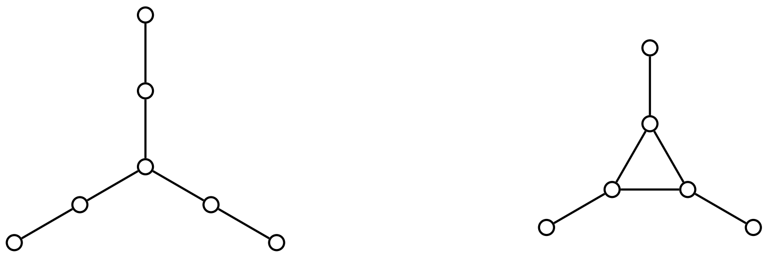
Observation 3.11
Let G be a graph and H a subgraph of G. If G has a degree complete labeling, then H has a degree complete labeling.

Proof
Every labeling f of G induces a labeling $\tilde{f}$ of the vertices of H. Considering a degree complete labeling of G from Lemma 3.1 follows that G_f does not contain a forbidden configuration. Hence there is no forbidden configuration $H_{\tilde{f}}$. Therefore, $H_{\tilde{f}}$ is degree complete and $\tilde{f}$ a degree complete labeling of H. □

Since Theorem 3.2 holds for arbitrary graphs it is particularly true for graphs with parallel edges. Suppose there is a vertex labeling f such that G_f is degree complete. From Theorem 3.2 follows that G_f does not contain a forbidden configuration H_1 or H_2. It is easy to see that for every edge $uv \in E(G)$ the graph $G_f + uv$ also does not contain any forbidden configuration. Furthermore, if $G_f + uv$ does not contain H_1 or H_2, then the same holds for G_f. Therefore, we make the following observation.

Observation 3.12
Let G be a graph and $uv \in E(G)$. The graph G has a degree complete labeling if and only if $G + uv$ has a degree complete labeling.

We can formulate this statement in a more general way. A graph with parallel edges has a degree complete labeling if and only if its related simple graph has degree complete labeling. It turns out that some results can be formulated much easier for simple graphs than for general graphs. By Observation 3.12 it is not a restriction to study only simple graphs which are degree complete.
We continue with some necessary conditions for unlabeled graphs with degree complete labeling. Thus we consider graphs which contain at least one of the forbidden configurations H_1 and H_2 in every labeled version. In particular, we are interested in a set of pairwise not including graphs with this property. Recall that C_k denotes the graph consisting of a cycle of length $k \geq 3$. Furthermore, we study the graphs from Figure 3.4. In [6] these graphs are called T_2 and *net*, respectively.



T_2 *net*

Figure 3.4: Graphs T_2 and *net*

Lemma 3.13
Let $G \in \{T_2, net\} \cup \{C_k \,|\, k \geq 4\}$. For every vertex labeling f of G there exists a forbidden configuration H_1 or H_2 in G_f.

Proof
Consider an arbitrary vertex labeling of G. Without loss of generality we identify each element in $V(G)$ with one of the integers from 1 to $n(G)$. There are three cases.

Case 1: $G = T_2$.
Denote v_1 the unique vertex of degree 3. There are three distinct vertices in $V(G)$ which are adjacent to v_1. Furthermore, we can choose two of these vertices v_2 and v_3 such that exactly one the following two conditions holds. Either we have $v_1 < v_2 < v_3$ or $v_3 < v_2 < v_1$. In both cases there is a vertex $v_4 \in V(G)$ which has v_2 as its unique neighbor. If $v_1 < v_4 < v_3$ (respectively $v_3 < v_4 < v_1$), then $\{v_1v_3, v_2v_4\}$ forms a forbidden configuration H_2. Otherwise, $\{v_1v_3, v_2v_4\}$ is the edge set of a copy of H_1.

Case 2: $G = net$.
Denote v_1, v_2, v_3 the three vertices of degree 3 in *net* such that $v_1 < v_2 < v_3$. There is a vertex $v_4 \in V(G)$ which has v_2 as unique neighbor. We consider two cases. If either $v_1 < v_4 < v_2$ or $v_2 < v_4 < v_3$ holds, then $\{v_1v_3, v_2v_4\}$ is the edge set of a forbidden configuration H_2 in G. Otherwise we have $v_4 < v_1$ or $v_4 > v_3$. In this case G contains the edges v_1v_3 and v_2v_4, that is a copy of H_1.

Case 3: $G = C_k$, $k \geq 4$.
We observe that vertex 1 has a neighbor v_1 with $2 < v_1 \leq k$. Moreover, vertex 2 is adjacent to a vertex v_2 satisfying $2 < v_2 \leq k$ and $v_1 \neq v_2$. If $v_1 < v_2$, then the edges $\{1, v_1\}$ and $\{2, v_2\}$ form a forbidden configuration H_1. In the case $v_1 > v_2$ the same edges yield H_2 as a subgraph of G. □

Combining Theorem 3.2, Observation 3.11, and Lemma 3.13 we deduce that every graph containing a subgraph isomorphic to T_2, *net* or a cycle of length $k \geq 4$ does not have a degree complete labeling. This shows that graphs with degree complete labeling have a structure that is similar to forests since they may only have cycles of length 3.

Now suppose for a moment that G is a tree. Obviously, G cannot contain a subgraph isomorphic to *net* or a cycle but it might have a copy of T_2 as a subgraph. A tree without a copy of T_2 has a path P such that all vertices which are not included in this path have degree 1 and are adjacent to a vertex of P. These trees are also known as caterpillars. A *caterpillar* is defined (see [15]) as a tree G such that $G - X_1$ is a path, where X_1 denotes the set of vertices of degree 1 in G. It is not difficult to see that caterpillars can be characterized as such trees not containing a subgraph isomorphic to T_2.

Using the above mentioned definition of caterpillars it is possible to construct a degree complete labeling for any graph of this class. Let G be a caterpillar of order $n \geq 3$ and denote $P = v_1 e_1 v_2 e_2 \ldots e_p v_{p+1}$ the path of length p in $G - X_1$. Obviously, the vertex labeling f defined by $f(v_i) = i$ is a degree complete labeling of P. We extend f to a degree complete labeling of G by a repetition of the following step. Consider an unlabeled vertex $x \in X_1$ and denote $u \in V(P)$ the unique neighbor of x in G. We add 1 to the label of every labeled vertex w with $f(w) > f(u)$ and set $f(x) = f(u) + 1$. Continuing with this procedure we terminate with a labeling of all vertices that fulfills the following condition. For every edge $uv \in E(P)$ and every vertex $w \in X_1$ satisfying $f(u) < f(w) < f(v)$ the vertex w is adjacent to u. Hence G_f does not contain a forbidden configuration H_1 or H_2.

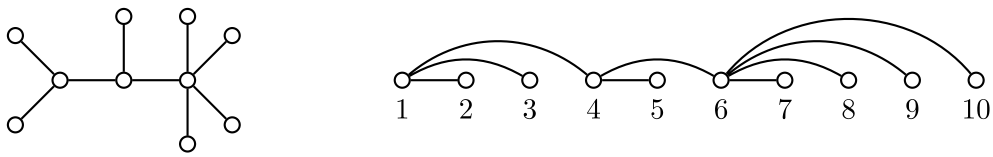


Figure 3.5: An unlabeled caterpillar (left) and a labeled version of the same caterpillar with degree complete labeling (right).

By Theorem 3.2 the labeling f yields a degree complete labeling of G.
On the left side of Figure 3.5 there is a caterpillar. The graph to the right refers to a labeled version of the same caterpillar. The sequence of vertices of the labeled graph is with respect to the labeling and shows that it is a degree complete labeling.

In the following we adapt the recognition of caterpillars by deleting certain vertices to construct a procedure recognizing arbitrary graphs with degree complete labeling. Let G be a graph not containing T_2, *net* as a subgraph or a cycle of length $k \geq 4$. Additional to vertices with an unique neighbor we have to delete a vertex in every triangle to obtain a path. Therefore, we define the following sets X_1 and X_2 by two procedures.

Procedure 3.14
Input: Graph $G = (V, E)$.
Output: $X_1 \subseteq V$.

1) Set $X_1 := \emptyset$.

2) **For** $u \in V$ **do:**
If $|N_G(u)| = 1$ **and** $N_G(u) \cap X_1 = \emptyset$
then: $X_1 := X_1 \cup \{u\}$.

3) **Return** X_1.

Let X_1 be a set constructed by Procedure 3.14 for an input graph G. Obviously, X_1 includes all vertices of G which have an unique neighbor with the exception that X_1 contains only one vertex of a component which consists of a single edge.

Procedure 3.15
Input: Graph $G = (V, E)$.
Output: $X_2 \subseteq V$.

1) Set $X_2 := \emptyset$.

2) **For** $u \in V$ **do:**
   If $N_G(u) = \{v, w\}$ **and** $\{v, w\} \cap X_2 = \emptyset$ **and** $vw \in E$
   and $N_G(v) \cap N_G(w) = \{u\}$
   then: $X_2 := X_2 \cup \{u\}$.

3) **Return** X_2.

Suppose X_2 is a set constructed by Procedure 3.15 with input graph G. For an arbitrary vertex $w \in X_2$ we notice the following properties. Firstly, w has exactly two neighbors in G and is part of a triangle. Moreover, there does not exist a further triangle in G containing both neighbors of w. We also deduce that for every triangle in G at most one of its vertices is a part of X_2. In general the choice of vertices in X_2 depends on the order in which the vertices of G are considered in the procedure.
Instead of deleting a vertex it is possible to delete an edge in each triangle. The following procedure constructs an edge set F.

Procedure 3.16
Input: Graph $G = (V, E)$.
Output: $F \subseteq E$.

1) Set $F := \emptyset$.

2) **For** $uv \in E$ **do:**
   Set $N := N_G(u) \cap N_G(v)$.
   If $N = \{w\}$ **and** $\{uw, vw\} \cap F = \emptyset$ **and** $N_G(w) = \{u, v\}$
   then: $F := F \cup \{uv\}$.

3) **Return** F.

Let F be an edge set returned by Procedure 3.16 with input G. Now, consider a triangle in G where at least one vertex w has degree 2. By its construction F contains the edge of the triangle opposing w, if this edge is not contained in a further triangle. Similar to X_2 there does not exist a triangle which has two edges in F.
The above mentioned constructions show the following. The procedures 3.14, 3.15 and

3.16 are basically greedy algorithms which additionally use some graph search elements. Thus, the sets X_1, X_2 and F can be determined in polynomial time with respect to the order of G.

We are now able to formulate and prove the main theorem of this section.

Theorem 3.17
Let G be a simple graph. Furthermore, let X_1, X_2 and F be the outputs of the procedures 3.14, 3.15 and 3.16 with input G, respectively. The following statements are equivalent:

(i) G has a degree complete labeling.

(ii) G does not contain a subgraph isomorphic to T_2, *net* or C_k $(k \geq 4)$.

(iii) $G - X_1 - X_2$ is a disjoint union of paths.

(iv) $G - X_1 - F$ is a disjoint union of paths.

Proof
From (i) to (ii): Follows from Theorem 3.2, Observation 3.11, and Lemma 3.13.

From (ii) to (iii) and (ii) to (iv): Suppose G does not contain a subgraph isomorphic to T_2, *net* or C_k $(k \geq 4)$. If G is not connected we consider each component separately. Thus assume that G is connected. There does not exist an edge in G which is part of two triangles. Otherwise G contains a cycle of length 4. Furthermore, every triangle has a vertex v of degree 2 since *net* is not a subgraph in G. From its construction follows that X_2 includes at least one vertex of every triangle. Hence $G - X_2$ is a tree. Also notice that $G - v$ is connected. Thus $G - X_2$ is connected because X_2 contains at most one vertex in a triangle. Taking into account that T_2 is not a subgraph in $G - X_2$ we deduce that $G - X_2$ is a caterpillar. From the definition of caterpillars follows that $G - X_1 - X_2$ is a path.
Considering F we observe by the same arguments as before that every triangle of G contains exactly one edge of F. Therefore, $G - F$ is a connected subgraph of G without a cycle. Hence it is a tree not including T_2, that is a caterpillar. Again, $G - X_1 - F$ is a path.

From (iii) to (i): Let G be a graph such that $G - X_1 - X_2$ is a disjoint union of paths. Obviously, G has a degree complete labeling if and only if every component of G has a degree complete labeling. Furthermore, by the constructions of X_1 and X_2 every component of G corresponds to exactly one of the paths in $G - X_1 - X_2$. In the following we give a labeling procedure for an arbitrary component of G and prove that the obtained labeled graph is degree complete.
Suppose $\tilde{G}$ is a component of G and P the corresponding path in $G - X_1 - X_2$. We define

$$\tilde{X}_1 = X_1 \cap V(\tilde{G}) \quad \text{and} \quad \tilde{X}_2 = X_2 \cap V(\tilde{G}).$$

Let $P = v_1 v_2 \ldots v_{p+1}$ with $p \geq 0$. First we initialize the labeling function f by $f(v_i) = i$. Next, we extend f step by step to the whole vertex set of $\tilde{G}$ as follows. By the construction

of $\tilde{X}_2$ we observe that every unlabeled vertex $v \in \tilde{X}_2$ has exactly two neighbors, say v_j and v_{j+1}, in P which satisfy $f(v_j) < f(v_{j+1})$. Now, we add 1 to the label of every labeled vertex w with $f(w) > f(v_j)$ and set $f(v) = f(v_j) + 1$. If all elements in $\tilde{X}_2$ are labeled this way, we finally consider an unlabeled vertex in $x \in \tilde{X}_1$. Notice that x has a labeled vertex $u \in V(P)$ as its unique neighbor. Again, we add 1 to the label of every labeled vertex w with $f(w) > f(u)$ and set $f(x) = f(u) + 1$.
It is not difficult to see that f is a bijective function from $V(\tilde{G})$ to $\{1, \ldots, |V(\tilde{G})|\}$, that is a labeling of $\tilde{G}$. To verify that f is a degree complete labeling we show that G_f does not contain a forbidden configuration. It is easy to check that two edges from $E(P)$ cannot generate a forbidden configuration. Hence we consider the edges incident to a vertex in $\tilde{X}_1 \cup \tilde{X}_2$. The labeling procedure above implies that there is no edge $uv \in E(G)$ such that $f(u) < f(v_i) < f(v)$ holds for some index i with $1 \leq i \leq p+1$. Thus we consider an arbitrary vertex y with $f(v_i) < f(y) < f(v_{i+1})$ for $1 \leq i \leq p$. Obviously, we have $y \in \tilde{X}_1 \cup \tilde{X}_2$. If $y \in \tilde{X}_2$, then y is adjacent to both v_i and v_{i+1}. Furthermore, by its construction there does not exist a further vertex in $\tilde{X}_2$ with this property of y. Thus every other vertex z with $f(v_i) < f(z) < f(v_{i+1})$ is in $\tilde{X}_1$. From the labeling procedure we deduce that z has v_i as its unique neighbor and $f(z) < f(y)$. Therefore, the subgraph induced by all vertices with labels from $f(v_i)$ to $f(v_{i+1})$ does not contain a forbidden configuration from Theorem 3.2. Hence G_f is degree complete.

From (iv) to (i): Let G be a graph such that $G - X_1 - F$ is a disjoint union of paths. By similar arguments as mentioned above it suffices to show that each component of G has a degree complete labeling. Every component corresponds to exactly one of the paths in $G - X_1 - F$.
Let $\tilde{G}$ be a component of G and P the corresponding path in $G - X_1 - F$ with vertices $v_1, \ldots, v_{p+1}$ such that v_i and v_{i+1} are consecutive in P for $1 \leq i \leq p$. We define

$$\tilde{X}_1 = X_1 \cap V(\tilde{G}) \quad \text{and} \quad \tilde{F} = F \cap E(\tilde{G}).$$

Furthermore, we initialize f by $f(v_i) = i$ for every $1 \leq i \leq p+1$. Now, consider an unlabeled vertex $x \in \tilde{X}_1$ and denote u the unique neighbor of x in P. Again, we extend f by adding 1 to the label of every labeled vertex w with $f(w) > f(u)$ and set $f(v) = f(u) + 1$.
Obviously, f is a labeling of $\tilde{G}$. Two edges from $E(P)$ cannot generate a forbidden configuration. By its construction an edge from F is incident to v_{i-1} and v_{i+1} for $2 \leq i \leq p$. Moreover, v_i has v_{i-1} and v_{i+1} as unique neighbors. Hence two edges from $E(\tilde{G} - X_1)$ do not form a forbidden configuration in $\tilde{G}_f$. Similar to the deduction from (iii) to (i) we can show that the remaining edges are not contained in a forbidden configuration of $\tilde{G}_f$. Therefore, $\tilde{G}_f$ is degree complete. □

We finish this section with an example of the labeling procedures from the proof of Theorem 3.17.

Example 3.18
Let G be the graph in Figure 3.6. By procedures 3.14, 3.15, and 3.16 we construct the

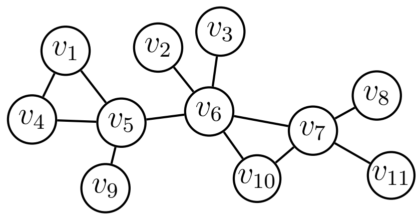


Figure 3.6: Graph G

following sets

$$X_1 = \{v_2, v_3, v_8, v_9, v_{11}\}, \quad X_2 = \{v_1, v_{10}\} \quad \text{and} \quad F = \{v_4v_5, v_6v_7\}.$$

We observe that $G - X_1 - X_2$ is a path with vertex set $\{v_4, v_5, v_6, v_7\}$. Similarly, $G - X_1 - F$ consists of the path $v_4v_1v_5v_6v_{10}v_7$. Thus, G has a degree complete labeling.
Next, as described in the proof from (iii) to (i), we initialize the labeling f by

$$f(v_4) = 1, \quad f(v_5) = 2, \quad f(v_6) = 3, \quad \text{and} \quad f(v_7) = 4.$$

Since $v_1 \in X_2$ is unlabeled and $N_G(v_1) = \{v_4, v_5\}$ we set

$$f(v_1) = 2, \quad f(v_5) = 2 + 1 = 3, \quad f(v_6) = 3 + 1 = 4, \quad \text{and} \quad f(v_7) = 4 + 1 = 5.$$

Analogously, for $v_{10} \in X_2$ the procedure yields

$$f(v_{10}) = 5 \quad \text{and} \quad f(v_7) = 5 + 1 = 6.$$

For the unlabeled vertex $v_2 \in X_1$ we continue with

$$f(v_2) = 5, \quad f(v_{10}) = 5 + 1 = 6, \quad \text{and} \quad f(v_7) = 6 + 1 = 7.$$

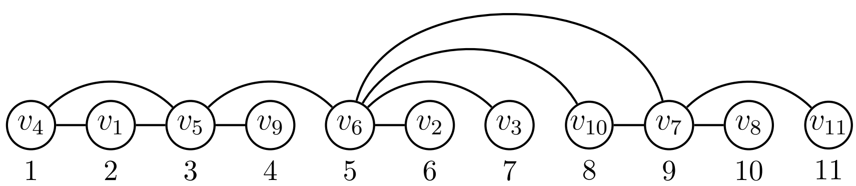


Figure 3.7: Labeled Graph G_f. The integers below each vertex denote the corresponding labels.

Repeating this step until all vertices are labeled we arrive at the labeling f from Figure 3.7. This also shows that G_f is degree complete. Finally, it is not difficult to prove that the procedure from (iv) to (i) yields the same labeling.

3.4 Strongly degree complete graphs and labelings

Let G be a labeled graph of order n. For a better understanding of Theorem 3.2, Theorem 3.17 and the properties that cause the structure of graphs with a degree complete labeling we consider the set

$$\mathcal{S}'(G) = \left\{ s \in \mathbb{Z}^n \mid s_G^l \preccurlyeq s \preccurlyeq s_G^r \right\}.$$

Thus $\mathcal{S}'(G)$ is a relaxation of the set $\mathcal{S}(G)$, where the vertex degree condition is omitted. We remind that G is degree complete if $\mathrm{DEG}^+(G) = \mathcal{S}(G)$. Obviously, we can extend (3.2) to

$$\mathrm{DEG}^+(G) \subseteq \mathcal{S}(G) \subseteq \mathcal{S}'(G)$$

for every labeled graph G. In this short section we study labeled graphs which satisfy $\mathrm{DEG}^+(G) = \mathcal{S}'(G)$. Notice that this condition is stronger than being degree complete. In particular, we call a labeled graph G *strongly degree complete* if $\mathrm{DEG}^+(G) = \mathcal{S}'(G)$ although Qian used this notation for a different property. We say that a graph has a *strongly degree complete labeling* if there is a labeled version of this graph which is strongly degree complete. The class of graphs with strongly degree complete labeling forms a subclass of graphs with degree complete labeling.
Our goal is to prove two theorems for these two conditions which are similar to Theorems 3.2 and 3.17.
The following result shows an interesting property of the set $\mathcal{S}'(G)$. Basically, it gives the main idea to further properties of $\mathcal{S}'(G)$ and yields directly to a complete characterization for labeled strongly degree complete graphs.

Lemma 3.19
Let G be a labeled graph of order n. For $1 \leq u < v \leq n$ holds

$$\mathcal{S}'(G + uv) = \mathcal{S}'(G) + \{s \in \mathbb{Z}^n \mid \mathbf{e}_v \preccurlyeq s \preccurlyeq \mathbf{e}_u\}.$$

Proof
From the definitions of s_G^l, s_G^r, s_{G+uv}^l, and s_{G+uv}^r follows that

$$s_{G+uv}^l = s_G^l + \mathbf{e}_v \quad \text{and} \quad s_{G+uv}^r = s_G^r + \mathbf{e}_u.$$

First consider any $r \in \mathcal{S}'(G)$ and $s \in \mathbb{Z}^n$ such that $\mathbf{e}_v \preccurlyeq s \preccurlyeq \mathbf{e}_u$. We deduce that

$$s_{G+uv}^l = s_G^l + \mathbf{e}_v \preccurlyeq r + s \preccurlyeq s_G^r + \mathbf{e}_u = s_{G+uv}^r$$

and therefore $r + s \in \mathcal{S}'(G + uv)$.
Now, let t be any vector in $\mathcal{S}'(G + uv)$. We show that there is a vector $s \in \mathbb{Z}^n$ satisfying $\mathbf{e}_v \preccurlyeq s \preccurlyeq \mathbf{e}_u$ and $t - s \in \mathcal{S}'(G)$. If $t - \mathbf{e}_v \in \mathcal{S}'(G)$, then $s = \mathbf{e}_v$ is a desired vector. Thus suppose that $t - \mathbf{e}_v \notin \mathcal{S}'(G)$. Since

$$s_G^l = s_{G+uv}^l - \mathbf{e}_v \preccurlyeq t - \mathbf{e}_v \tag{3.5}$$

we observe that $t - \mathbf{e}_v \not\preccurlyeq s^r_G$. Hence there is an index $i \in \{1, \dots, n-1\}$ such that

$$(t - \mathbf{e}_v)(\{1, \dots, i\}) > s^r_G(\{1, \dots, i\}). \tag{3.6}$$

Furthermore, we have

$$(t - \mathbf{e}_v)(\{1, \dots, i\}) \le (s^r_{G+uv} - \mathbf{e}_v)(\{1, \dots, i\}) = (s^r_G + \mathbf{e}_u - \mathbf{e}_v)(\{1, \dots, i\}).$$

Hence we deduce $u \le i < v - 1$ and

$$(t - \mathbf{e}_v)(\{1, \dots, i\}) = t(\{1, \dots, i\}) = s^r_G(\{1, \dots, i\}) + 1.$$

Let I be the set of all integers i with $u \le i \le v - 1$ such that (3.6) holds. We define the integral vector

$$s = \mathbf{e}_v + \sum_{i \in I} \mathbf{z}_{(i,i+1)}.$$

Now we have

$$\mathbf{e}_v \preccurlyeq \mathbf{e}_v + \sum_{i \in I} \mathbf{z}_{(i,i+1)} = s \preccurlyeq \mathbf{e}_v + \sum_{j=u}^{v-1} \mathbf{z}_{(j,j+1)} = \mathbf{e}_u.$$

Moreover, for any integer $j \notin I$ with $1 \le j \le n$ follows

$$(t - s)(\{1, \dots, j\}) = (t - \mathbf{e}_v)(\{1, \dots, j\}) - \sum_{i \in I} \mathbf{z}_{(i,i+1)}(\{1, \dots, j\}) = (t - \mathbf{e}_v)(\{1, \dots, j\}).$$

By (3.5) and the definition of I we conclude

$$s^l_G(\{1, \dots, j\}) \le (t - s)(\{1, \dots, j\}) \le s^r_G(\{1, \dots, j\}).$$

If $j \in I$, then we have

$$\begin{aligned}(t - s)(\{1, \dots, j\}) &= (t - \mathbf{e}_v)(\{1, \dots, j\}) - \sum_{i \in I} \mathbf{z}_{(i,i+1)}(\{1, \dots, j\}) \\ &= t(\{1, \dots, j\}) - 1 \\ &= s^r_G(\{1, \dots, j\}).\end{aligned}$$

Therefore, we deduce $s^l_G \preccurlyeq t - s \preccurlyeq s^r_G$ and $t - s \in \mathcal{S}'(G)$. □

The following corollary is a direct consequence of Lemma 3.19.

Corollary 3.20
If G is a labeled graph of order n, then

$$\mathcal{S}'(G) = \sum_{uv \in E(G)} \{s \in \mathbb{Z}^n \mid \mathbf{e}_v \preccurlyeq s \preccurlyeq \mathbf{e}_u\}.$$

This gives us an additive property between the sum of two graphs $G+H$ and the Minkowski sum $\mathcal{S}'(G) + \mathcal{S}'(H)$.

Corollary 3.21
For all labeled graphs G and H with $V(G) = V(H)$ holds

$$\mathcal{S}'(G+H) = \mathcal{S}'(G) + \mathcal{S}'(H).$$

Proof
By Corollary 3.20 we have

$$\begin{aligned}\mathcal{S}'(G+H) &= \sum_{uv \in E(G)\cup E(H)} \{s \in \mathbb{Z}^n \mid \mathbf{e}_v \preccurlyeq s \preccurlyeq \mathbf{e}_u\} \\ &= \sum_{uv \in E(G)} \{s \in \mathbb{Z}^n \mid \mathbf{e}_v \preccurlyeq s \preccurlyeq \mathbf{e}_u\} + \sum_{uv \in E(H)} \{s \in \mathbb{Z}^n \mid \mathbf{e}_v \preccurlyeq s \preccurlyeq \mathbf{e}_u\} \\ &= \mathcal{S}'(G) + \mathcal{S}'(H).\end{aligned}$$

□

Since the sets $\mathcal{S}(G)$ and $\mathcal{S}'(G)$ are closely related we might think of an analogous result to Corollary 3.21 for $\mathcal{S}(G)$. The following example shows that $\mathcal{S}(G+H) = \mathcal{S}(G) + \mathcal{S}(H)$ does not hold for all graphs G and H.

Example 3.22
Let G_1 and G_2 with vertex set $V(G_1) = V(G_2) = \{1,2,3,4\}$ and edge sets $E(G_1) = \{\{1,3\}\}$ and $E(G_2) = \{\{2,4\}\}$. Now, on the one hand we have

$$\mathcal{S}(G_1) = \{(1,0,0,0)^\top, (0,0,1,0)^\top\} \quad \text{and} \quad \mathcal{S}(G_2) = \left\{(0,1,0,0)^\top, (0,0,0,1)^\top\right\}.$$

On the other hand it is not difficult to show that

$$\mathcal{S}(G_1+G_2) = \left\{ \begin{array}{c} (1,1,0,0)^\top, (1,0,1,0)^\top, (1,0,0,1)^\top, \\ (0,1,1,0)^\top, (0,1,0,1)^\top, (0,0,1,1)^\top \end{array} \right\}.$$

Therefore, $\mathcal{S}(G_1+G_2) \neq \mathcal{S}(G_1) + \mathcal{S}(G_2)$.

Although we do not have an analogue for general graphs the statement is true if $G+H$ is a degree complete graph. By Lemma 3.1 we deduce $\mathrm{DEG}^+(G) = \mathcal{S}(G)$ and $\mathrm{DEG}^+(H) = \mathcal{S}(H)$. Thus from (2.5) follows that

$$\mathcal{S}(G+H) = \mathrm{DEG}^+(G+H) = \mathrm{DEG}^+(G) + \mathrm{DEG}^+(H) = \mathcal{S}(G) + \mathcal{S}(H).$$

There are also examples of graphs G and H where at least one graph is not degree complete and still $\mathcal{S}(G+H) = \mathcal{S}(G) + \mathcal{S}(H)$ holds. It is an interesting question which conditions on G and H imply this property.
We continue with the investigation of the set $\mathcal{S}'$. Corollary 3.21 yields us a simple proof of a result on a kind of hereditary property for labeled graphs. In contrast to Lemma 3.1 the following Theorem shows that a graph is strongly degree complete if and only if each subgraph is strongly degree complete.

Theorem 3.23
Let G and H be labeled graphs with $V(G) = V(H)$. The graph $G + H$ is strongly degree complete if and only if G and H are strongly degree complete.

Proof
Suppose $\mathcal{S}'(G) = \mathrm{DEG}^+(G)$ and $\mathcal{S}'(H) = \mathrm{DEG}^+(H)$ holds. By Corollary 3.21 we deduce

$$\mathcal{S}'(G+H) = \mathcal{S}'(G) + \mathcal{S}'(H) = \mathrm{DEG}^+(G) + \mathrm{DEG}^+(H) = \mathrm{DEG}^+(G+H).$$

Now, suppose $G + H$ satisfies $\mathcal{S}'(G + H) = \mathrm{DEG}^+(G + H)$. Let s be an arbitrary vector in $\mathcal{S}'(G)$. It suffices to show that s is a degree vector of G. Obviously, we have $s(V(G)) = m(G)$. Considering any nonempty, proper subset X of $V(G)$ we define t as the out-degree vector of an orientation of H where all edges in $\delta_H(X)$ are directed from $V(G) \setminus X$ to X. Thus we have $t(X) = m(H[X])$. Since

$$s + t \in \mathcal{S}'(G) + \mathcal{S}'(H) = \mathcal{S}'(G + H) = \mathrm{DEG}^+(G + H)$$

we deduce $(s + t)(X) \geq m((G + H)[X])$ by Theorem 2.3. Hence we conclude

$$s(X) \geq m((G + H)[X]) - t(X) = m((G + H)[X]) - m(H[X]) = m(G[X]).$$

Therefore, we have $s(X) \geq m(G[X])$ for all $X \subseteq V$ and $s(V(G)) = m(G)$. Thus, s is a degree vector of G by Theorem 2.3. $\square$

Theorem 3.23 points out an important difference between the sets of degree complete and strongly degree complete graphs. In Lemma 3.1 we proved that every subgraph of a labeled degree complete graph is also degree complete. Furthermore, by Example 3.22 there are graphs which are not degree complete although all of their proper subgraph are degree complete. An explanation of this phenomenon is obtained by a comparison between Theorem 3.2 and the following result.

Theorem 3.24
A labeled graph G is strongly degree complete if and only if G does not contain an edge $uv \in E(G)$ with $|u - v| > 1$.

Proof
First suppose that G contains an edge uv with $|u - v| > 1$. We consider the subgraph $H = (V(G), \{uv\})$ of G. Obviously, $s^r_H = \mathbf{e}_u$ and $s^l_H = \mathbf{e}_v$ are the only degree vectors of H. On the one hand we have $\mathbf{e}_v \preccurlyeq \mathbf{e}_w \preccurlyeq \mathbf{e}_u$ for $u < w < v$ and therefore $\mathbf{e}_w \in \mathcal{S}'(H)$. On the other hand $\mathbf{e}_w$ is not a degree vector of H since $d_H(w) = 0$. By Theorem 3.23 we conclude that $\mathrm{DEG}^+(G) \neq \mathcal{S}'(G)$.
Now suppose that for every edge $uv \in E(G)$ holds $|u - v| \leq 1$. Hence we deduce

$$\{s \in \mathbb{Z}^n \,|\, \mathbf{e}_v \preccurlyeq s \preccurlyeq \mathbf{e}_u\} = \{\mathbf{e}_u, \mathbf{e}_v\} = \mathrm{DEG}^+\big((V(G), \{uv\})\big).$$

By Corollary 3.21 we conclude

$$\mathcal{S}'(G) = \sum_{uv \in E(G)} \{s \in \mathbb{Z}^n \mid \mathbf{e}_v \preccurlyeq s \preccurlyeq \mathbf{e}_u\} = \sum_{uv \in E(G)} \mathrm{DEG}^+\big((V(G), \{uv\})\big) = \mathrm{DEG}^+(G).$$

□

We finish this section with an analogue to Theorem 3.17. We notice the subsequent implication of Theorem 3.23.

Observation 3.25
Let G be a graph H a subgraph of G and $uv \in E(G)$. A strongly degree complete labeling f of G induces a strongly degree complete labeling of H and $G + uv$.

Similar to graphs with degree complete labeling we formulate the result only for simple graphs. We obtain a result for graphs with parallel edges by applying the following theorem to the related simple graphs.

Theorem 3.26
A simple graph G has a strongly degree complete labeling if and only if G is a disjoint union of paths.

Proof
To prove necessity suppose there is a vertex labeling f such that $\mathrm{DEG}^+(G_f) = \mathcal{S}'(G_f)$. By Theorem 3.24 for every edge $e \in E(G_f)$ there is an element of $v \in \{1, \ldots, n-1\}$ such that $e = \{v, v+1\}$. Therefore, every component of G_f is a path.
To show sufficiency suppose G is a disjoint union of paths $P_1, \ldots, P_k$. For $1 \leq j \leq k$ denote $v_1^j, \ldots, v_{l_j}^j$ the vertex set of P_j, where v_i^j and v_{i+1}^j are adjacent $(1 \leq i \leq l_j - 1)$. We define the labeling f by

$$f(v_i^j) = i + l_1 + \ldots + l_{j-1}.$$

It is not difficult to see that if $v, w \in V(G)$ are adjacent, then $|f(v) - f(w)| \leq 1$. Hence by Theorem 3.24 we deduce $\mathrm{DEG}^+(G_f) = \mathcal{S}'(G_f)$. □

A comparison between Theorem 3.17 and Theorem 3.26 shows the following. Adding the degree condition $0 \leq s \leq d_G$ to $\mathcal{S}'(G)$ gives us the set $\mathcal{S}(G)$. This additional condition causes the vertex sets X_1 and X_2 in the characterization (iii) of Theorem 3.17. On the other hand we observe that the path-like structure of the graphs with degree complete labeling is basically effected by the partial order $\preccurlyeq$.
From this insight the following questions arise. Are there other partial orders that have similar properties concerning (strong) degree completeness as $\preccurlyeq$? How can we characterize the labeled graphs which are degree complete with respect to these alternative partial orders? What are analogous results for unlabeled graphs?
In chapter 4 we discuss certain classes of partial orders generalizing $\preccurlyeq$. These classes

are motivated by the observation that for arbitrary trees we are able to construct partial orders on $\mathbb{Z}^n$ which provide a degree completeness property. In chapter 5 we characterize the labeled graphs which are degree complete with respect to these new partial orders. Moreover, we prove several extensions of results from this chapter. Finally, in chapter 6 we study graphs which are degree complete with respect to a cone order. Since all partial orders in this thesis belong to this class we obtain more detailed informations on certain structures in graphs preventing degree completeness.

4 Partial orders generalizing $\preccurlyeq$

A fundamental conclusion on the results of chapter 3 is that the structure of the graphs which have a degree complete labeling is basically influenced by the partial order $\preccurlyeq$. Therefore, it is reasonable to study further partial orders which might yield similar degree completeness results for graphs that do not have a degree complete labeling with respect to $\preccurlyeq$.
In particular, we start with an example on trees showing that there are such partial orders for arbitrary trees. These partial orders are related to so called cross-free set families. In the first section we present a characterization of such cross-free families due to Edmonds and Giles [9]. A partial order is a cross-free family partial order (CFPO) if it is induced by a cross-free family. The second section consists of a description of basic properties of CFPOs. In the main theorem of this section we prove that every CFPO is a cone order. Moreover, we determine the vectors which generate the corresponding positive cone.

4.1 Cross-free families

For an informal idea which kind of partial orders are useful we consider trees. By Theorem 3.17 a tree has a degree complete order if and only if it is a caterpillar graph. Hence it would be an improvement to find partial orders that can be used for arbitrary trees.
Let G be a tree on n vertices with edge set $\{e_1, \ldots, e_{n-1}\}$. By Theorem 2.8 we obtain the following description for the degree vector set

$$\mathrm{DEG}^+(G) = \{s \in \mathbb{Z}^n \,|\, s(X) \geq m(G[X]) \text{ for all } X \in \mathcal{X}_G \text{ and } s(V(G)) = m(G)\},$$

where $\mathcal{X}_G = \{X \subsetneq V(G) \,|\, G[X] \text{ and } G - X \text{ are connected}\}$. Since G is a tree the set $\mathcal{X}_G$ consists of exactly $2n-2$ subsets of $V(G)$. In particular, for every edge $e_j = x_j y_j \in E(G)$ the graph $G - e_j$ is a forest with exactly two components. We define X_j (respectively Y_j) as the vertex set of the component of $G - e_j$ containing x_j (respectively y_j). Moreover, if there is a vertex set X such that $G[X]$ and $G - X$ are connected, then $G[X]$ and $G - X$ are subtrees of G which are connected by an edge. Hence

$$\mathcal{X}_G = \{X_j, Y_j \,|\, 1 \leq j \leq n-1\}.$$

Additionally, we make the following observation. Because X_j and Y_j are disjoint sets with $X_j \cup Y_j = V(G)$ for all $j = 1, \ldots, n-1$ and $s(V(G)) = m(G)$ we have

$$s(X_j) = s(V(G)) - s(Y_j) \leq m(G) - m(G[Y_j]) = m(G) - m(G - X_j).$$

Similarly, we deduce

$$s(Y_j) = s(V(G)) - s(X_j) \geq m(G) - (m(G) - m(G - X_j)) = m(G[Y_j]).$$

Thus $s(Y_j) \geq m(G[Y_j])$ and $s(X_j) \leq m(G) - m(G - X_j)$ are equivalent conditions. Hence the sets $X_1, \ldots, X_n$ contain all information to describe the set of degree vectors of G, that is

$$\mathrm{DEG}^+(G) = \left\{ s \in \mathbb{Z}^n \;\middle|\; \begin{array}{c} m(G[X_j]) \leq s(X_j) \leq m(G) - m(G - X_j) \\ \text{for all } j = 1, \ldots, n-1 \text{ and } s(V(G)) = m(G) \end{array} \right\}. \tag{4.1}$$

Considering X_j and X_k with $j \neq k$ we notice that the vertices x_j and x_k are connected by an unique path P in in G. There are four different cases for the relation between X_j and X_k depending which of the vertices y_j and y_k are contained in P. If y_j and y_k are both included in P, then X_j and X_k are disjoint. Furthermore, for $y_j \in V(P)$ and $y_k \notin V(P)$ (respectively $y_j \notin V(P)$ and $y_k \in V(P)$) we deduce that $X_j \subseteq X_k$ (respectively $X_j \supseteq X_k$). If y_j and y_k are not contained in P, then $X_j \cup X_k = V(G)$. Hence for all X_j and X_k holds

$$X_j \subseteq X_k \quad \text{or} \quad X_j \supseteq X_k \quad \text{or} \quad X_j \cap X_k = \emptyset \quad \text{or} \quad X_j \cup X_k = V(G). \tag{4.2}$$

For a formulation of $\mathrm{DEG}^+(G)$ in terms of an interval with respect to some partial order we define vectors $s^{\min}$, $s^{\max} \in \mathbb{Z}^n$ and a matrix $M \in \{0, 1, -1\}^{(n+1)\times n}$. Denote $s^{\min}$ (respectively $s^{\max}$) the out-degree vector of the orientation of G where every edge $e_j \in E(G)$ is directed from Y_j to X_j (respectively from X_j to Y_j). It is not difficult to see that $m(G[X_j]) = s^{\min}(X_j)$ and $m(G) - m(G - X_j) = s^{\max}(X_j)$ hold for all $j = 1, \ldots, n-1$. Now, the matrix M is defined by

$$M_{jk} = \begin{cases} 1, & \text{if } 1 \leq j \leq n-1 \text{ and } k \in X_j \text{ or } j = n, \\ -1, & \text{if } j = n+1, \\ 0, & \text{else.} \end{cases}$$

Therefore, by (4.1) we deduce

$$\mathrm{DEG}^+(G) = \{ s \in \mathbb{Z}^n \mid M s^{\min} \leq M s \leq M s^{\max} \}. \tag{4.3}$$

The last equation leads us to a desired partial order. For $s, t \in \mathbb{R}^n$ a relation $\preceq$ is defined by

$$s \preceq t \quad \text{if and only if} \quad M(t - s) \geq 0. \tag{4.4}$$

It is easy to see that $\preceq$ is a quasi-order. Moreover, if the matrix M has rank n, then $\preceq$ defines a partial order. With this definition of $\preceq$ we reformulate (4.3) to

$$\mathrm{DEG}^+(G) = \{ s \in \mathbb{Z}^n \mid s^{\min} \preceq s \preceq s^{\max} \}. \tag{4.5}$$

The right side of this equation can be seen as an analogue to the set $\mathcal{S}'(G)$ of the last section. Furthermore, we notice that if $G = \big(\{1, \ldots, n\}, \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}\}\big)$

and $X_j = \{1, \ldots, j\}$, then $\preceq$ is equivalent to $\preccurlyeq$. An equivalent formulation of $\preceq$ is the following

$$s \preceq t \quad \text{if and only if} \quad \begin{cases} s(X_j) \le t(X_j), & \text{for all } 1 \le j \le n-1, \\ s(V(G)) = t(V(G)). \end{cases}$$

In the following we give a formal definition for some classes of partial orders extending the idea of the introductory deduction on trees. Moreover, we collect some basic properties that are essential for our further studies on degree completeness in the following sections.

The first class of partial orders we consider generalizes $\preccurlyeq$ in the sense that we compare sums of certain entries of the given vectors. Let $V = \{1, \ldots, n\}$ and $\mathcal{F} = \{W_1, \ldots, W_p\}$ be a family of subsets of V. We define a binary relation on the elements of $\mathbb{Z}^n$ as follows. For two vectors $s, t \in \mathbb{Z}^n$ we define

$$s \preceq_{\mathcal{F}} t \quad \text{if and only if} \quad \begin{cases} s(W) \le t(W), \text{ for all } W \in \mathcal{F}, \\ s(V) = t(V). \end{cases}$$

Additionally, a matrix $M_{\mathcal{F}} = (M_{ij}) \in \mathbb{R}^{(p+2) \times n}$ is defined by

$$M_{ij} = \begin{cases} 1, & \text{if } 1 \le i \le p \text{ and } j \in W_i \text{ or } i = p+1, \\ -1, & \text{if } j = p+2, \\ 0, & \text{else.} \end{cases}$$

Similar to the example on trees notice that

$$s \preceq_{\mathcal{F}} t \Leftrightarrow M_{\mathcal{F}} \cdot (t - s) \ge 0.$$

Because the set $\{x \in \mathbb{R}^n \mid M_{\mathcal{F}} \cdot x \ge 0\}$ is a cone we deduce that $\preceq_{\mathcal{F}}$ is a cone order on $\mathbb{R}^n$. Hence it follows immediately that $\preceq_{\mathcal{F}}$ is a reflexive and transitive relation, that is a quasi-order. We also refer to the notation

$$\mathcal{C}_{\mathcal{F}} = \{x \in \mathbb{R}^n \mid M_{\mathcal{F}} \cdot x \ge 0\}.$$

Since in general $\preceq_{\mathcal{F}}$ is not a partial order we have to make further assumptions to $\mathcal{F}$. To ensure that $s \preceq_{\mathcal{F}} t$ and $s \succeq_{\mathcal{F}} t$ imply $s = t$ the resulting system of linear equations $M_{\mathcal{F}} \cdot x = 0$ has to have an unique solution. An equivalent condition is that the rank of $M_{\mathcal{F}}$ is n. Therefore, it is necessary (but not sufficient) that $\mathcal{F}$ contains at least $n - 1$ distinct nonempty subsets of V.
The above defined class of partial orders has some important subclasses. The following subclass of partial orders of the type $\preceq_{\mathcal{F}}$ picks up the idea on trees from the beginning of this section. Let V be a finite set. A family $\mathcal{F}$ of sets from V is called *cross-free* if for each pair U, W from $\mathcal{F}$ holds

$$U \subseteq W \quad \text{or} \quad U \supseteq W \quad \text{or} \quad U \cap W = \emptyset \quad \text{or} \quad U \cup W = V.$$

Additionally, $\mathcal{F}$ is *laminar* if for all $U, W \in \mathcal{F}$ follows

$$U \subseteq W \quad \text{or} \quad U \supseteq W \quad \text{or} \quad U \cap W = \emptyset.$$

Obviously, every laminar family is also cross-free. By (4.2) we observe that the sets X_j for $1 \leq j \leq n-1$ form a cross-free family of $V(G)$. There is an interesting characterization of cross-free and laminar families by Edmonds and Giles [9] that underlines the relation to trees.
Consider a directed tree T and a function $\pi : V \to V(T)$. For any arc $a = (v, w) \in A(T)$ we define X_a as the set of elements $x \in V$ such that $\pi(x)$ and v are contained in the same weak component of $T - a$. The pair (T, π) is called a *tree-representation* of $\mathcal{F}$ if

$$\mathcal{F} = \{X_a \mid a \in A(T)\}.$$

Example 4.1
Figure 4.1 (a) shows a tree-representation of

$$\mathcal{F} = \{\{1\}, \{6\}, \{1,2,3\}, \{4,5,6\}, \{1,2,3,4,6\}, \{1,2,3,4,5\}, \{1,2,3,4,5,6\}\}$$

as a cross-free family of subsets of $V = \{1,2,3,4,5,6\}$. Similarly, Figure 4.1 (b) depicts a tree-representation of the laminar family

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{6\}, \{1,2,4\}, \{1,2,3,4,6\}, \{1,2,3,4,5,6\}\}.$$

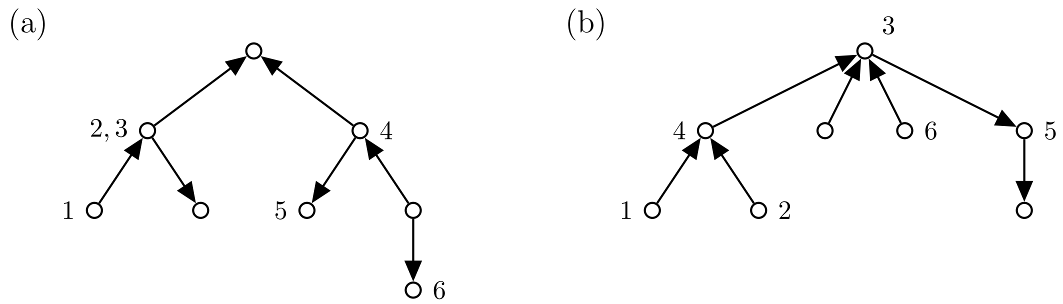


Figure 4.1: Tree-representation of a cross-free (a) and a laminar (b) family of subsets of $V = \{1,2,3,4,5,6\}$

Using this notation we observe that in our deduction on trees we constructed such a tree-representation. For every edge $e_j = x_j y_j$ of the tree G we choose the connected component of $G - e_j$ including x_j. Hence we implicitly define a direction to each edge. Moreover, the function π is the identity on $V(G)$. Analogue to (4.2) it is easy to check that for every directed tree T and any function $\pi : V \to V(T)$ the set $\{X_a \mid a \in A(T)\}$ is a cross-free family of V. Furthermore, if T is an in-tree, then we obtain a laminar family. The following theorem shows that even the converse is true. The presented proof can be found in [18].

Theorem 4.2 (Edmonds, Giles [9] (1977))
Let V be a finite set. A family $\mathcal{F}$ of subsets of V is cross-free if and only if $\mathcal{F}$ has a tree-representation. Moreover, $\mathcal{F}$ is laminar if and only if $\mathcal{F}$ has a tree-representation (T, π) such that T is an in-tree.

Proof
Let (T, π) be a tree-representation of a family $\mathcal{F}$ of subsets of V. Consider two arcs $a_1 = (x_1, y_1)$ and $a_2 = (x_2, y_2)$ in $A(T)$. There is an unique path P in the underlying (undirected) graph of T which connects x_1 and x_2. If y_1 and y_2 are also contained in P, then X_a and X_b are disjoint. Moreover, from $y_1 \in V(P)$ and $y_2 \notin V(P)$ we deduce that $X_a \subseteq X_b$. Similarly, $y_1 \notin V(P)$ and $y_2 \in V(P)$ imply $X_a \supseteq X_b$. If $y_1, y_2 \notin V(P)$, then $X_a \cup X_b = V$. Hence $\mathcal{F}$ is a cross-free family. If T is an in-tree, then the case $y_1, y_2 \notin V(P)$ cannot occur and $\mathcal{F}$ is a laminar family.
Now, let $\mathcal{F}$ be a laminar family of subsets of V. Choose $r \notin V$ and define a directed tree T by $V(T) = \mathcal{F} \cup \{r\}$ and

$$\begin{aligned} A(T) = \{(X,Y) \in \mathcal{F} \times \mathcal{F} \,|\, X \supsetneq Y \neq \emptyset \text{ and there is no } Z \in \mathcal{F} \text{ with } X \supsetneq Z \supsetneq Y\} \\ \cup \{(X, r) \,|\, X = \emptyset \text{ or } X \text{ is an (inclusion-wise) maximal element of } \mathcal{F}\}. \end{aligned}$$

For $v \in V$ we set $\pi(v) = X$, where X is the minimal element in $\mathcal{F}$ including v. If v is not contained in any set of $\mathcal{F}$, then we define $\pi(v) = r$. Obviously, r is the only vertex of T without positive neighbors. Hence T is an in-tree. Furthermore, it is not difficult to see that (T, π) is a tree-representation of $\mathcal{F}$.
Finally, let $\mathcal{F}$ be a cross-free family of subsets of V. The family we obtain from $\mathcal{F}$ by adding $V \setminus X$ for every element $X \in \mathcal{F}$ is also cross-free. Additionally, for any fixed $r \in V$ the family $\{X \in \mathcal{F} \,|\, r \notin X\}$ is laminar. We conclude that

$$\mathcal{F}^* = \{X \in \mathcal{F} \,|\, r \notin X\} \cup \{V \setminus X \,|\, X \in \mathcal{F} \text{ and } r \in X\}$$

is a laminar family. By our previous deduction there is a tree-representation (T, π) of $\mathcal{F}^*$ where T is an in-tree. In the following we construct a directed tree T'. We initialize T' by $V(T') = V(T)$ and $A(T') = A(T)$. For an arc $a = (x, y) \in A(T)$ we consider three different cases. If X_a and $V \setminus X_a$ are elements of $\mathcal{F}$, then add a new vertex z to T', delete (x, y), and add the arcs (z, x) and (z, y). Moreover, if $X_a \notin \mathcal{F}$ and $V \setminus X_a \in \mathcal{F}$, then replace (x, y) by (y, x). If $X_a \in \mathcal{F}$ and $V \setminus X_a \notin \mathcal{F}$, then we do nothing. Again we easily check that (T', π) is a tree-representation of $\mathcal{F}$. □

4.2 Partial orders induced by cross-free families

In this section we study partial orders of the type $\preceq_{\mathcal{F}}$ where $\mathcal{F}$ is a cross-free or laminar family. The tree-representation (T, π) of a cross-free or laminar family $\mathcal{F}$ helps us to decide whether $\preceq_{\mathcal{F}}$ is a quasi-order or a partial order. The following lemma shows that the injectivity of the mapping π characterizes these families.

Lemma 4.3
Let $\mathcal{F}$ be a cross-free family of subsets of $V = \{1, \ldots, n\}$ with tree-representation (T, π). The relation $\preceq_{\mathcal{F}}$ is a partial order on $\mathbb{R}^n$ if and only if π is injective.

Proof
We already mentioned that $\preceq_{\mathcal{F}}$ is a quasi-order on $\mathbb{R}^n$. Hence it suffices to show that $s \preceq_{\mathcal{F}} t$ and $s \succeq_{\mathcal{F}} t$ imply $s = t$ for all $s, t \in \mathbb{R}^n$. An equivalent condition is that the matrix $M_{\mathcal{F}}$ has rank n. We prove that $\text{rank}(M_{\mathcal{F}}) = n$ if and only if π is injective. If $\mathcal{F}$ contains $\emptyset$ or V, then the matrix $M_{\mathcal{F} \setminus \{\emptyset, V\}}$ has the same rank as $M_{\mathcal{F}}$. Thus we assume that $\mathcal{F}$ only contains nonempty, proper subsets of V.
First suppose that π is not injective. There are $v, w \in V$ such that for every $X \in \mathcal{F}$ either $v, w \in X$ or $v, w \notin X$ holds. Thus the columns of $M_{\mathcal{F}}$ corresponding to v and w are equal and therefore $\text{rank}(M_{\mathcal{F}}) < n$.
Now, suppose that π is injective. Hence the tree T has at least $n-1$ arcs and $\mathcal{F}$ includes at least $n-1$ different sets. We show by induction on n that $\text{rank}(M_{\mathcal{F}}) = n$. The case $n = 1$ is trivial. Suppose $n \geq 2$ and consider a leaf $x \in V(T)$. Moreover, denote $a \in A(T)$ the unique arc leaving or entering x. Since π is injective and $\emptyset, V \notin \mathcal{F}$ there is exactly one element $v \in V$ satisfying $\pi(v) = x$. Without loss of generality we assume that x is the tail of a and therefore $X_a = \{v\}$. Otherwise we define the cross-free family $\mathcal{F}^*$ where we replace X_a by $V \setminus X_a$, that is, we reverse a in the tree-representation of $\mathcal{F}$. In this case we obtain the matrix $M_{\mathcal{F}^*}$ by replacing the row r_a in $M_{\mathcal{F}}$ which corresponds to X_a by the difference of the row of all 1's and r_a. Obviously, $M_{\mathcal{F}^*}$ and $M_{\mathcal{F}}$ have the same rank.
We define $V' = V \setminus \{v\}$ and $\mathcal{F}' = \{X \setminus \{v\} \,|\, X \in \mathcal{F} \setminus \{X_a\}\}$. Furthermore, denote $T' = T - x$ and π' the restriction of π on V'. Notice that π' is injective and (T', π') is a tree-representation of $\mathcal{F}'$ as a family of subsets of V'. From the induction hypothesis we deduce that the matrix $M_{\mathcal{F}'}$ has rank $n-1$. Since $M_{\mathcal{F}'}$ is a $(|\mathcal{F}|+1) \times (n-1)$-submatrix of $M_{\mathcal{F}}$ and the row corresponding to X_a in $M_{\mathcal{F}}$ has its only nonzero entry in the column associated to v we conclude that $\text{rank}(M_{\mathcal{F}}) = n$. □

We continue with some examples of $\preceq_{\mathcal{F}}$ and corresponding matrices $M_{\mathcal{F}}$.

Example 4.4
$V = \{1, \ldots, n\}$, $\mathcal{F} = \{\{1\}, \{1, 2\}, \ldots, \{1, \ldots, n-1\}\}$:

$$M_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & \cdots & \cdots & 1 \\ -1 & \cdots & \cdots & -1 \end{pmatrix} \in \mathbb{Z}^{(n+1) \times n}$$

In this case $\preceq_{\mathcal{F}}$ is equal to $\preccurlyeq$. Furthermore $\mathcal{F}$ is a laminar family with the tree-representation (T, π), where T is a directed path on n vertices.

$V = \{1, \ldots, n\}$, $\mathcal{F} = \{\{2\}, \{3\}, \ldots, \{n\}\}$:

$$M_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & \cdots & \cdots & \cdots & 1 \\ -1 & \cdots & \cdots & \cdots & -1 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times n}$$

Again $\mathcal{F}$ is a laminar family and the tree-representation includes a directed star $K_{1,n-1}$ where $\pi(1)$ is the center that is incident to all arcs.

$V = \{1, \ldots, 6\}$, $\mathcal{F} = \{\{1\}, \{5\}, \{1,2,3\}, \{1,2,3,4,5\}, \{1,2,4,5,6\}\}$:

$$M_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

Here $\mathcal{F}$ is a cross-free but not a laminar family. Figure 4.2 shows the tree T of a corresponding tree-representation.

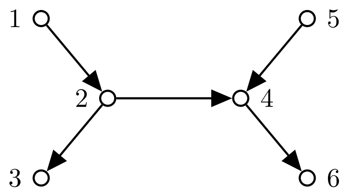


Figure 4.2: Tree-representation of $\mathcal{F} = \{\{1\}, \{5\}, \{1,2,3\}, \{1,2,3,4,5\}, \{1,2,4,5,6\}\}$ as subsets of $V = \{1,2,3,4,5,6\}$

We already mentioned that $\preceq_{\mathcal{F}}$ is a cone order on $\mathbb{R}^n$ for any family $\mathcal{F}$ of subsets of $V = \{1, \ldots, n\}$. For our further investigations it is very helpful to know the vectors generating the cone $\mathcal{C}_{\mathcal{F}}$. In general, this might be a difficult task. Nevertheless, for those cross-free families which induce a partial order the main result of this section gives us a solution. If (T, π) is a tree-representation of $\mathcal{F}$ such that π is bijective it is possible to identify V and $V(T)$. For this case we prove that $\mathcal{C}_{\mathcal{F}} = \mathcal{C}\left(\{\mathbf{z}_a\}_{a \in A(T)}\right)$. In the example corresponding to Figure 4.2 this yields

$$\mathcal{C}_{\mathcal{F}} = \mathcal{C}\left(\mathbf{z}_{(1,2)}, \mathbf{z}_{(2,3)}, \mathbf{z}_{(2,4)}, \mathbf{z}_{(5,4)}, \mathbf{z}_{(4,6)}\right).$$

In the case that π is injective but not surjective we need the following definition.

Definition 4.5
Let $\mathcal{F}$ be a cross-free family of $V = \{1, \ldots, n\}$ with tree-representation (T, π) such that π is injective. For a pair of elements $v, w \in V$ denote P_{vw} the unique (not necessary directed) path in T connecting $\pi(v)$ and $\pi(w)$. Moreover, a digraph $D_{\mathcal{F}}$ is defined by the vertex set $V(D_{\mathcal{F}}) = V$ and the set of arcs

$$A(D_{\mathcal{F}}) = \big\{(v, w) \in V^2 \,\big|\, P_{vw} \text{ is a directed } (\pi(v), \pi(w))\text{-path and } V(P_{vw}) \cap V = \{v, w\}\big\}.$$

Now, an arc $(v, w) \in A(D_{\mathcal{F}})$ implies that there is a directed $(\pi(v), \pi(w))$-path in T such that every internal vertex is not contained in the image of π. Hence if π is bijective, then $D_{\mathcal{F}}$ and T are isomorphic. Our goal is to show that the vectors $\mathbf{z}_a$ with $a \in A(D_{\mathcal{F}})$ generate the cone $\mathcal{C}_{\mathcal{F}}$.
The following lemma is useful.

Lemma 4.6
Let D be a digraph with vertex set $\{v_1, \ldots, v_n\}$, $c : A(D) \to \mathbb{Q}$ a capacity function and $s, t \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} (t(i) - s(i)) = 0$. A network $\mathcal{N}_D = \mathcal{N}_D(s, t) = ((V_{\mathcal{N}_D}, A_{\mathcal{N}_D}), c_{\mathcal{N}_D}, p, q)$ is defined by

$$V_{\mathcal{N}_D} = V(D) \cup \{p, q\},$$

$$A_{\mathcal{N}_D} = A(D) \cup \{(p, v), (v, q) \mid v \in V(D)\},$$

$$c_{\mathcal{N}_D}(a) = \begin{cases} c(a), & \text{if } a \in A(D), \\ t(v_i), & \text{if } a = (p, v_i) \text{ and } 1 \le i \le n, \\ s(v_i), & \text{if } a = (v_i, q) \text{ and } 1 \le i \le n. \end{cases}$$

There is a (p, q)-flow f in $\mathcal{N}_D$ with $\text{val}(f) = \sum_{i=1}^{n} t(i)$ if and only if there are coefficients λ_a with $0 \le \lambda_a \le c(a)$ for all $a \in A(D)$ such that

$$t - s = \sum_{a \in A(D)} \lambda_a \mathbf{z}_a.$$

Proof
Let f be a (p, q)-flow in $\mathcal{N}_D$ with $\text{val}(f) = \sum_{i=1}^{n} t(i)$. Since

$$c_{\mathcal{N}_D}(\delta^+_{\mathcal{N}_D}(p)) = \sum_{i=1}^{n} t(i) = \sum_{i=1}^{n} s(i) = c_{\mathcal{N}_D}(\delta^-_{\mathcal{N}_D}(q))$$

we deduce that $f((p, v_i)) = t(i)$ and $f((v_i, q)) = s(i)$ for $1 \le i \le n$. For all $a \in A(D)$ we define $\lambda_a = f(a) \le c(a)$. Moreover, for every $v_i \in V(D)$ we have

$$\sum_{a \in \delta^+_{\mathcal{N}_D}(v_i)} f(a) = \sum_{a \in \delta^-_{\mathcal{N}_D}(v_i)} f(a).$$

Hence for $1 \leq i \leq n$ we conclude

$$\begin{aligned} 0 &= \sum_{a \in \delta^+_{\mathcal{N}_D}(v_i)} f(a) - \sum_{a \in \delta^-_{\mathcal{N}_D}(v_i)} f(a) \\ &= s(i) - t(i) + \sum_{a \in \delta^+_D(v_i)} \lambda_a - \sum_{a \in \delta^-_D(v_i)} \lambda_a \\ &= (s-t)(i) + \sum_{a \in A(D)} \lambda_a \mathbf{z}_a(i). \end{aligned}$$

This implies $(t-s)(i) = \left(\sum_{a \in A(D)} \lambda_a \mathbf{z}_a\right)(i)$ for all $1 \leq i \leq n$. Thus there are $0 \leq \lambda_a \leq c(a)$ for all $a \in A$ such that

$$t - s = \sum_{a \in A(D)} \lambda_a \mathbf{z}_a.$$

Now, assume there are coefficients $0 \leq \lambda_a \leq c(a)$ for all $a \in A(D)$ such that $t - s = \sum_{a \in A(D)} \lambda_a \mathbf{z}_a$. We define a function f on the arc set of $\mathcal{N}_D$ and show that f is a desired (p,q)-flow. For $1 \leq i \leq n$ we set $f((p, v_i)) = t(i)$ and $f((v_i, q)) = s(i)$. Furthermore, for all $a \in A(D)$ we define $f(a) = \lambda_a$. Then $0 \leq f(a) \leq c_{\mathcal{N}_D}(a)$ holds for all $a \in A(D)$. Finally, for every $v_i \in V(D)$ we deduce

$$\begin{aligned} \sum_{a \in \delta^+_{\mathcal{N}_D}(v_i)} f(a) - \sum_{a \in \delta^-_{\mathcal{N}_D}(v_i)} f(a) &= s(i) - t(i) + \sum_{a \in \delta^+_D(v_i)} \lambda_a - \sum_{a \in \delta^-_D(v_i)} \lambda_a \\ &= - \sum_{a \in A(D)} \lambda_a \mathbf{z}_a(i) + \sum_{a \in A(D)} \lambda_a \mathbf{z}_a(i) \\ &= 0. \end{aligned}$$

Therefore, f is a (p,q)-flow and $\mathrm{val}(f) = f(\delta^+_{\mathcal{N}_D}(p)) = \sum_{i=1}^n t(i)$. □

Now we are able to prove the main theorem of this section.

Theorem 4.7
Let $\mathcal{F}$ be a cross-free family of $V = \{1, \ldots, n\}$ with tree-representation (T, π) such that π is injective. For $s, t \in \mathbb{R}^n$ holds $s \preceq_{\mathcal{F}} t$ if and only if there are coefficients $\lambda_a \geq 0$ satisfying

$$t - s = \sum_{a \in A(D_{\mathcal{F}})} \lambda_a \mathbf{z}_a. \tag{4.6}$$

Moreover, if s and t are integral vectors, then there are nonnegative integers λ_a with this property.

Proof
First we consider the case that π is bijective. Hence we identify $V(T)$ and V. Moreover, we obtain $D_{\mathcal{F}}$ as a labeled version of T, that is, a vertex $v \in V(T)$ is labeled with $\pi^{-1}(v)$. For all $(v, w) \in A(D_{\mathcal{F}})$ we define

$$Y_{(v,w)} = \{u \in V \mid u \text{ and } v \text{ are in same weak component of } D_{\mathcal{F}} - (v, w)\}.$$

Notice that $\mathcal{F} = \{Y_a \mid a \in A(D_\mathcal{F})\}$. Now consider $M = M_\mathcal{F}$ and denote m_a the row of M which corresponds to $Y_a \in \mathcal{F}$.
Suppose that $s \preceq_\mathcal{F} t$ and define $\lambda_a = t(Y_a) - s(Y_a) = m_a \cdot (t - s)$ for all $a \in A(D_\mathcal{F})$. From the definition of $\preceq_\mathcal{F}$ follows that $\lambda_a \geq 0$ for all $a \in A(D_\mathcal{F})$. For an arbitrary $b \in A(D_\mathcal{F})$ we deduce that

$$m_b \cdot \mathbf{z}_a = \mathbf{z}_a(Y_b) = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{else.} \end{cases}$$

Hence we obtain

$$m_b \cdot \left(\sum_{a \in A(D_\mathcal{F})} \lambda_a \, \mathbf{z}_a \right) = \sum_{a \in A(D_\mathcal{F})} \lambda_a \, m_b \cdot \mathbf{z}_a = \lambda_b = m_b \cdot (t - s).$$

Moreover, $s(V) = t(V)$ implies $\mathbf{1}^\top \mathbf{z}_b = 0 = \mathbf{1}^\top (t - s)$, where $\mathbf{1} \in \mathbb{R}^n$ denotes the vector whose entries are all equal to 1. Therefore, we conclude that

$$M \cdot \left(\sum_{a \in A(D_\mathcal{F})} \lambda_a \, \mathbf{z}_a \right) = M \cdot (t - s).$$

By Lemma 4.3 the matrix $M \in \mathbb{R}^{(|\mathcal{F}|+2) \times n}$ has rank n and thus we obtain $t - s = \sum_{a \in A(D_\mathcal{F})} \lambda_a \, \mathbf{z}_a$.
Now, suppose there are coefficients $\lambda_a \geq 0$ satisfying (4.6). For any $b \in A(D_\mathcal{F})$ we deduce that

$$t(Y_b) = s(Y_b) + \sum_{a \in A(D_\mathcal{F})} \lambda_a \, \mathbf{z}_a(Y_b) = s(Y_b) + \lambda_b \geq s(Y_b) \quad \text{and} \quad s(V) = t(V).$$

This implies $s \preceq_\mathcal{F} t$. Thus the statement from Theorem 4.7 holds if π is bijective.
We continue with the case that π is injective but not necessary surjective. Our goal is to deduce this case from the bijective case. Denote k the number of vertices in $V(T) \setminus \pi(V)$. We define $\tilde{V} = V \cup W = V \cup \{n+1, \ldots, n+k\}$ and choose a bijective function $\tilde{\pi}$ such that $\tilde{\pi}(v) = \pi(v)$ for all $v \in V$. For simplicity we identify $V(T)$ and $\tilde{V}$. From the tree-representation $(T, \tilde{\pi})$ we define a cross-free family $\tilde{\mathcal{F}}$ of $\tilde{V}$. Denote Y_a the set of $\tilde{\mathcal{F}}$ corresponding to $a \in A(T)$. The restriction of $\tilde{\mathcal{F}}$ on V is exactly $\mathcal{F}$, that is, we have $X_a = Y_a \cap V$ for all $a \in A(T)$. Furthermore, we define vectors $\tilde{s}, \tilde{t} \in \mathbb{R}^{n+k}$ by

$$\tilde{s}(v) = \begin{cases} s(v), & \text{if } v \in V, \\ \ \ 0 \ \ , & \text{if } v \in W, \end{cases} \qquad \text{and} \qquad \tilde{t}(v) = \begin{cases} t(v), & \text{if } v \in V, \\ \ \ 0 \ \ , & \text{if } v \in W. \end{cases}$$

For any $a \in A(T)$ these definitions yield $\tilde{s}(Y_a) = s(X_a)$ and $\tilde{t}(Y_a) = t(X_a)$. Moreover, we obtain $\tilde{s}(\tilde{V}) = s(V)$ and $\tilde{t}(\tilde{V}) = t(V)$. Thus we conclude that $s \preceq_\mathcal{F} t$ if and only if $\tilde{s} \preceq_{\tilde{\mathcal{F}}} \tilde{t}$. Hence the bijective case implies that $s \preceq_\mathcal{F} t$ is equivalent to the existence of coefficients $\tilde{\lambda} \geq 0$ satisfying

$$\tilde{t} - \tilde{s} = \sum_{a \in A(T)} \lambda_a \, \mathbf{z}_a. \tag{4.7}$$

Referring to the notation of Lemma 4.6 denote $\mathcal{N}_T = \mathcal{N}_T(\tilde{s}, \tilde{t})$ the network corresponding to the vectors $\tilde{s}, \tilde{t}$ and the digraph T, where the capacity function is defined by $c(a) = s(V) + 1$ for all $a \in A(T)$. Now, by Lemma 4.6 there is a (p,q)-flow f in $\mathcal{N}_T$ with $\mathrm{val}(f) = \tilde{s}(V)$ if and only if (4.7) holds. On the other hand there is a similar network corresponding to the vectors s,t and the digraph $D_\mathcal{F}$ denoted by $\mathcal{N}_{D_\mathcal{F}} = \mathcal{N}_{D_\mathcal{F}}(s,t)$. Again the capacity function is defined by $c(a) = s(V) + 1$ for all $a \in A(D_\mathcal{F})$. From Lemma 4.6 follows that there is a (p,q)-flow f in $\mathcal{N}_{D_\mathcal{F}}$ with $\mathrm{val}(f) = s(V)$ if and only if (4.6) holds. Therefore, it remains to show that $\mathcal{N}_T$ has a desired (p,q)-flow if and only if $\mathcal{N}_{D_\mathcal{F}}$ has one. To prove necessity suppose there does not exist a (p,q)-flow f in $\mathcal{N}_{D_\mathcal{F}}$ with $\mathrm{val}(f) = s(V)$. By Theorem 1.1 there is a vertex set $U \subseteq V(\mathcal{N}_{D_\mathcal{F}})$ with $p \in U$ and $q \notin U$ such that

$$c_{\mathcal{N}_{D_\mathcal{F}}}\big(\delta^+_{\mathcal{N}_{D_\mathcal{F}}}(U)\big) < s(V).$$

Thus there does not exist an arc in $A(D_\mathcal{F})$ leaving U . Hence we cannot find a directed path from $U \setminus \{p\}$ to $V \setminus U$ in T. The set U is also a vertex set in $\mathcal{N}_T$. Now add to U all vertices in W which can be reached via a directed path from a vertex of U in T and denote this set by $\tilde{U} \subseteq V(\mathcal{N}_T)$. The only arcs with positive capacity in $\mathcal{N}_T$ leaving $\tilde{U}$ are (u,q) with $u \in \tilde{U}$ and (p,u) for $u \in V \setminus \tilde{U}$. This is the same arc set as $\delta^+_{\mathcal{N}_{D_\mathcal{F}}}(U)$. Since the capacities of these arcs are equal in $\mathcal{N}_T$ and $\mathcal{N}_{D_\mathcal{F}}$ we conclude that

$$c_{\mathcal{N}_T}\big(\delta^+_{\mathcal{N}_T}(\tilde{U})\big) < \tilde{s}(V).$$

Therefore, there does not exist a (p,q)-flow f in $\mathcal{N}_T$ with $\mathrm{val}(f) = \tilde{s}(V)$.
Finally, assume there is a (p,q)-flow f in $\mathcal{N}_{D_\mathcal{F}}$ with $\mathrm{val}(f) = \tilde{s}(V)$. We define a function g on the arc set of $\mathcal{N}_T$ and show that g is a desired (p,q)-flow. For $v \in V$ we set $g((p,v)) = f((p,v))$ and $g((v,q)) = f((v,q))$. Moreover, define $f((p,w)) = 0 = f((w,q))$ for all $w \in W$. For $a \in A(T)$ and $b = (x,y) \in A(D_\mathcal{F})$ the coefficient $\mu_{a,b}$ indicates whether a is contained in the (x,y)-path P_{xy} in T. If $a \in A(P_{xy})$, then $\mu_{a,b}$ equals $f(b)$ otherwise $\mu_{a,b} = 0$. Now, for all $a \in A(T)$ we set

$$g(a) = \sum_{b \in A(D_\mathcal{F})} \mu_{a,b}.$$

For any fixed $b = (x,y) \in A(D_\mathcal{F})$ an arbitrary vertex $w \in W$ is either an internal vertex of P_{xy} or $w \notin V(P_{xy})$. In the first case exactly one arc in $\delta^+_T(w)$ and exactly one arc in $\delta^+_T(w)$ are included in P_{xy}. Hence we deduce for all $b \in A(D_\mathcal{F})$ and $w \in W$ that

$$\sum_{a \in \delta^+_T(w)} \mu_{a,b} - \sum_{a \in \delta^-_T(w)} \mu_{a,b} = 0.$$

In the other case the same equality trivially holds. Therefore, for all $w \in W$ we obtain

$$\begin{aligned}\sum_{a \in \delta^+_{\mathcal{N}_T}(w)} g(a) - \sum_{a \in \delta^-_{\mathcal{N}_T}(w)} g(a) &= \sum_{a \in \delta^+_T(w)} \sum_{b \in A(D_\mathcal{F})} \mu_{a,b} - \sum_{a \in \delta^-_T(w)} \sum_{b \in A(D_\mathcal{F})} \mu_{a,b} \\ &= \sum_{b \in A(D_\mathcal{F})} \left(\sum_{a \in \delta^+_T(w)} \mu_{a,b} - \sum_{a \in \delta^-_T(w)} \mu_{a,b} \right) \\ &= 0.\end{aligned}$$

Now consider a vertex $v \in V$. For every $b = (x, y) \in A(D_{\mathcal{F}})$ the vertex v is an initial or a terminal vertex of $V(P_{xy})$ or it is not contained in such a path. This implies

$$\sum_{a \in \delta_T^+(v)} \mu_{a,b} - \sum_{a \in \delta_T^-(v)} \mu_{a,b} = \begin{cases} f(b), & \text{if } b \in \delta^+_{D_{\mathcal{F}}}(v), \\ -f(b), & \text{if } b \in \delta^-_{D_{\mathcal{F}}}(v), \\ 0 \ , & \text{else.} \end{cases}$$

Thus for all $v \in V$ we conclude

$$\begin{aligned} \sum_{a \in \delta^+_{\mathcal{N}_T}(v)} g(a) - \sum_{a \in \delta^-_{\mathcal{N}_T}(v)} g(a) &= f(v,q) - f(p,v) + \sum_{a \in \delta_T^+(v)} \sum_{b \in A(D_{\mathcal{F}})} \mu_{a,b} - \sum_{a \in \delta_T^-(v)} \sum_{b \in A(D_{\mathcal{F}})} \mu_{a,b} \\ &= f(v,q) - f(p,v) + \sum_{b \in A(D_{\mathcal{F}})} \left(\sum_{a \in \delta_T^+(v)} \mu_{a,b} - \sum_{a \in \delta_T^-(v)} \mu_{a,b} \right) \\ &= f(v,q) - f(p,v) + \sum_{b \in \delta^+_{D_{\mathcal{F}}}(v)} f(b) - \sum_{b \in \delta^-_{D_{\mathcal{F}}}(v)} f(b) \\ &= \sum_{b \in \delta^+_{\mathcal{N}_{D_{\mathcal{F}}}}(v)} f(b) - \sum_{b \in \delta^-_{\mathcal{N}_{D_{\mathcal{F}}}}(v)} f(b) \\ &= 0. \end{aligned}$$

Altogether, we deduce that g is a (p, q)-flow in $\mathcal{N}_T$ with $\text{val}(g) = g(\delta^+_{\mathcal{N}_T}(p)) = \tilde{s}(V)$. This completes the proof of (4.6).
Denote $\Lambda \in \mathbb{R}^{m(D_{\mathcal{F}})}$ the vector of all λ_a with $a \in A(D_{\mathcal{F}})$. Because the vectors in $\{\mathbf{z}_a\}_{a \in A(D_{\mathcal{F}})}$ are the columns of the incidence matrix $B = B_{D_{\mathcal{F}}}$ we deduce that B is totally unimodular. Hence for $s, t \in \mathbb{Z}^n$ the system $B \cdot \Lambda = t - s$ and $\Lambda \geq 0$ has an integral solution, that is, λ_a is a nonnegative integer, if the system has a solution. □

For a cross-free family $\mathcal{F}$ of $V = \{1, \ldots, n\}$ with tree-representation (T, π) we collect some helpful facts implied by the last proof. Firstly, from the case that π is bijective we obtain the following remark.

Remark 4.8
Let $\mathcal{F}$ be a cross-free family of $V = \{1, \ldots, n\}$ with tree-representation (T, π) and $s, t \in \mathbb{R}^n$. If π is bijective, then $D_{\mathcal{F}}$ is a directed tree isomorphic to T. Hence $\{\mathbf{z}_a\}_{a \in A(D_{\mathcal{F}})}$ is a basis of the subspace $\{x \in \mathbb{R}^n \,|\, x(V) = 0\}$. Thus for every $x \in \mathbb{R}^n$ there are unique coefficients $\lambda_a \in \mathbb{R}$ satisfying

$$x = \sum_{a \in A(D_{\mathcal{F}})} \lambda_a \, \mathbf{z}_a.$$

Moreover, if $x \in \mathbb{Z}^n$, then from the total unimodularity of B follows that $\lambda_a \in \mathbb{Z}$.

Additionally, we make the following observation. Let $r, s, t \in \mathbb{R}^n$ with $r \preceq_{\mathcal{F}} s$ and $r \preceq_{\mathcal{F}} t$.

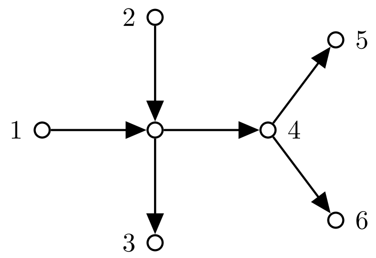


Figure 4.3: Tree-representation of the cross-free family (4.8)

If π is bijective, then there are unique $\lambda_a, \mu_a \geq 0$ such that

$$s - r = \sum_{a \in A(D_{\mathcal{F}})} \lambda_a \mathbf{z}_a \quad \text{and} \quad t - r = \sum_{a \in A(D_{\mathcal{F}})} \mu_a \mathbf{z}_a.$$

Notice that $s \preceq_{\mathcal{F}} t$ holds if and only if $\lambda_a \leq \mu_a$ for all $a \in A(D_{\mathcal{F}})$. This shows that it is the same to order a set of vectors by $\preceq_{\mathcal{F}}$ or by the component-wise order of the coefficients (λ_a and μ_a) in the linear combinations with respect to one fixed vector. Hence we conclude that for vectors $\underline{s}, \overline{s} \in \mathbb{Z}^n$ with $\underline{s} \preceq_{\mathcal{F}} \overline{s}$ and $\overline{s} - \underline{s} = \sum_{a \in A(D_{\mathcal{F}})} \mu_a \mathbf{z}_a$ we obtain

$$\{ s \in \mathbb{Z}^n \mid \underline{s} \preceq_{\mathcal{F}} s \preceq_{\mathcal{F}} \overline{s} \} = \left\{ \underline{s} + \sum_{a \in A(D_{\mathcal{F}})} \lambda_a \mathbf{z}_a \;\middle|\; \lambda_i \in \{0, \ldots, \mu_a\} \text{ for all } a \in A(D_{\mathcal{F}}) \right\}.$$

Secondly, considering two vectors $s, t \in \mathbb{R}^n$ the introduced networks $\mathcal{N}_T$ and $\mathcal{N}_{D_{\mathcal{F}}}$ give an interesting interpretation of $s \preceq_{\mathcal{F}} t$. We proved that $s \preceq_{\mathcal{F}} t$ is equivalent to the existence of nonnegative coefficients λ_a for $a \in A(T)$ such that

$$\tilde{t} - \tilde{s} = \sum_{a \in A(D_T)} \lambda_a \, \mathbf{z}_a.$$

This holds if and only if there is a (p, q)-flow f in $\mathcal{N}_T$ with $\mathrm{val}(f) = \tilde{s}(V)$. A vertex $v \in V$ is contained in V^+ if $t(v) - s(v) > 0$ and in V^- if $t(v) - s(v) < 0$. The existence of a (p, q)-flow f with $\mathrm{val}(f) = s(V)$ implies that for every $v \in V^+$ there is a directed (v, w)-path in T to some $w \in V^-$. Analogously, every $w \in V^-$ can be reached from a $v \in V^+$ via a directed (v, w)-path in T. The same is true if we replace $\mathcal{N}_T$ by $\mathcal{N}_{D_{\mathcal{F}}}$. Additionally, if we consider $\mathcal{N}_{D_{\mathcal{F}}}$, then the coefficients λ_a with $a \in A(D_{\mathcal{F}})$ can be seen as the value of the flow f on the arc a. We give a brief example of this fact.

Example 4.9
Consider the cross-free family

$$\mathcal{F} = \{\{1\}, \{2\}, \{1,2,3\}, \{1,2,4,5,6\}, \{1,2,3,4,6\}, \{1,2,3,4,5\}\} \tag{4.8}$$

of subsets of $V = \{1, \ldots, 6\}$ and the vectors

$$r = (0,0,1,2,1,3)^\top, \quad s = (1,2,0,3,1,0)^\top, \quad t = (1,2,1,0,1,2)^\top.$$

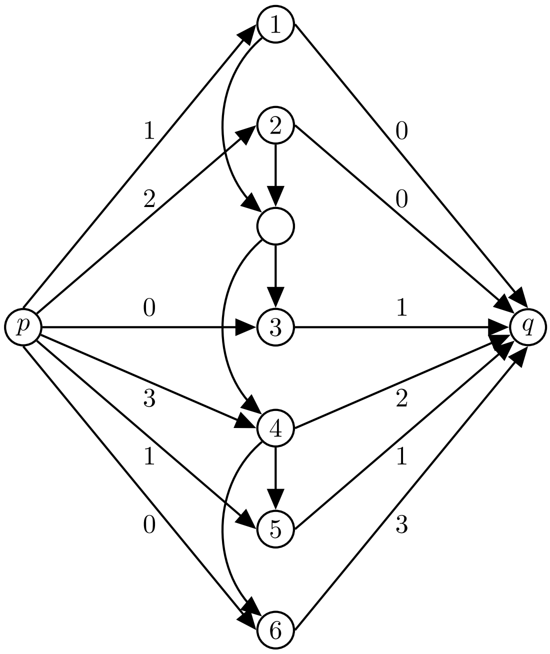


Figure 4.4: The networks $\mathcal{N}_T(r,s)$ and $\mathcal{N}_T(t,s)$. The labels on the arcs with tail p or head q are the capacities. All other arcs have a capacity of $s(V)+1$ which is omitted.

Figure 4.3 shows the directed tree T from a tree-representation of $\mathcal{F}$. Moreover, Figure 4.4 depicts the networks $\mathcal{N}_T(r,s)$ and $\mathcal{N}_T(t,s)$. We observe that there is a feasible (p,q)-flow in $\mathcal{N}_T(r,s)$ with $\mathrm{val}(f) = s(V) = 7$. Hence we deduce that

$$\begin{aligned} s - r &= (1,2,-1,1,0,-3)^\top \\ &= (1,0,-1,0,0,0)^\top + 2\cdot(0,1,0,-1,0,0)^\top + 3\cdot(0,0,0,1,0,-1)^\top \end{aligned}$$

and therefore $r \preceq_{\mathcal{F}} s$. On the other hand for s and t there is no directed path in T which reaches vertex 3 and starts at a vertex w with $s(w) - t(w) > 0$. In particular, for $X = \{p,4,5,6\}$ we deduce that $\delta^+(X)$ has capacity 6 Therefore, there is no feasible (p,q)-flow in $\mathcal{N}_T(t,s)$ with $\mathrm{val}(f) = s(V) = 7$ and we obtain that $t \not\preceq s$.

It is possible to use any directed acyclic digraph D to define a partial order by the network $\mathcal{N}_D$. We do not consider this class of partial orders but notice that every relation constructed this way is a cone order where the cone is generated by the vectors $\mathbf{z}_a$ for $a \in A(D)$.

The following definitions are essential for the next chapter.

Definition 4.10
Let $\mathcal{F}$ be a cross-free family of subsets of $V = \{1,\dots,n\}$ with tree-representation (T,π). If π is bijective, then we call $\preceq_{\mathcal{F}}$ a *cross-free family partial order* or *CFPO* on $\mathbb{R}^n$. If $\mathcal{F}$

is a laminar and π is bijective, then $\preceq_{\mathcal{F}}$ is a *laminar family partial order* or *LFPO* on $\mathbb{R}^n$. In both cases the *tree-representation* corresponding to $\preceq_{\mathcal{F}}$ is the tree-representation of the family $\mathcal{F}$. Since π is bijective we omit π and always assume that $V = \{1, \ldots, n\}$ is the vertex set of T. Furthermore, denote $\mathcal{C}_{\preceq_{\mathcal{F}}}$ the cone corresponding to $\mathcal{F}$, that is, $\mathcal{C}_{\preceq_{\mathcal{F}}} = \mathcal{C}_{\mathcal{F}}$. Notice that from Theorem 4.7 follows that

$$\mathcal{C}_{\preceq_{\mathcal{F}}} = \mathcal{C}\big(\{\mathbf{z}_a\}_{a \in A(T)}\big).$$

Every directed labeled tree T can be seen as the tree-representation of a cross-free family. Hence we notice that T induces a CFPO. Moreover, the corresponding cone is generated by the conic hull of the columns of the incidence matrix of T. For a CFPO $\preceq$ on $\mathbb{R}^n$ with tree-representation T and a labeled graph G of order n we always assume that T and G have the same vertex set.
In the following chapter we study those graphs which are degree complete with respect to some CFPO $\preceq$. At first sight this seems to be an unnecessary restriction because we do not consider partial orders that are given by cross-free families which are injective but not surjective. The results of chapter 6 show that it is not a disadvantage to focus on families whose tree-representation have a bijective function. Furthermore, we are able to use implications like Remark 4.8 for this class of partial orders and obtain a simpler argumentation.

5 Degree completeness for CFPOs and LFPOs

In the previous chapter we discussed some generalizations of the partial order $\preccurlyeq$. Now we want to apply these generalizations (more precisely cross-free family and laminar family partial orders) to the concept of degree complete graphs. Our goal is to give characterizations extending Theorems 3.2 and 3.17 to cross-free family partial orders (CFPOs).

In the first section of this chapter we focus on results covering labeled graphs. Particularly, we are able to generalize Theorems 3.2 and 3.24. The investigations on degree completeness for an arbitrary CFPO reveal the following problem. The partial order $\preccurlyeq$ is the only CFPO for which the set of degree vectors of any labeled graph has unique extremal elements. Hence we consider a more general setting in which we allow several extremal elements. In a separate result we prove a characterization of labeled graphs which have unique extremal elements and are degree complete with respect to a given CFPO. Finally, this leads to an useful definition of the concept of degree completeness with respect to a CFPO.

In the second section we describe the labeled graphs which we are degree complete with respect to some CFPO. Obviously, for every graph G with a degree complete labeling there is a CFPO $\preceq$ such that the degree complete labeled version of G is also degree complete with respect to $\preceq$. Thus our characterization of graphs which are degree complete with respect to a CFPO can be seen as a generalization of Theorem 3.17.

The third section contains some results which can be seen as direct analogues to Theorem 3.17. In contrast to the setting in the second section of this chapter we consider some predefined CFPOs and try to characterize the graphs which have a labeled version that is degree complete with respect to this CFPO. For a few CFPOs we obtain a complete characterization of those graphs which have a desired labeled version. Moreover, we are able to detect certain forbidden subgraphs similar to T_2 from Theorem 3.17 which cannot be contained in a graph that is degree complete with respect to a given CFPO. Particularly, we show that there is a class of trees $\mathcal{T}$ such that every $T \in \mathcal{T}$ is solely degree complete with respect to a CFPO whose tree-representation is an orientation of T.

In the last section we prove that if G is degree complete with respect to a CFPO $\preceq$, then the poset $(\mathrm{DEG}^+(G), \preceq)$ is a lattice. This result underlines the consideration of graphs which are degree complete with respect to a CFPO since we are able to apply the ideas on the construction of the degree vector set from section 3.2 to these graphs.

5.1 Labeled graphs

Let G be a labeled graph. We start with a result on the extremal elements of $\mathrm{DEG}^+(G)$ with respect to some CFPO $\preceq$.

Proposition 5.1
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T and G a labeled graph of order n. If there is an edge $vw \in E(G)$ such that P_{vw} is not a directed path in T, then there do not exist degree vectors $\underline{s}, \overline{s}$ of G satisfying $\underline{s} \preceq s \preceq \overline{s}$ for all $s \in \mathrm{DEG}^+(G)$.

Proof
Suppose there is a degree vector $\overline{s}$ of G such that $s \preceq \overline{s}$ holds for all $s \in \mathrm{DEG}^+(G)$. Denote D an orientation of G with out-degree vector $\overline{s}$ and $b \in A(D)$ an arc between v and w. Let s be the out-degree vector of the orientation of G we obtain from D by reversing b. By Remark 4.8 for all $a \in A(T)$ there are unique coefficients $\lambda_a \in \mathbb{Z}$ such that

$$\overline{s} - s = \mathbf{z}_b = \sum_{a \in A(T)} \lambda_a \, \mathbf{z}_a.$$

Since P_{vw} is not a directed path in T we deduce that there are $a', a'' \in A(T)$ with $\lambda_{a'} > 0$ and $\lambda_{a''} < 0$. Therefore, $\overline{s} - s$ and $s - \overline{s}$ are not contained in $\mathcal{C}_{\preceq}$. Thus $\overline{s}$ and s are incomparable with respect to $\preceq$ what contradicts our choice of $\overline{s}$. An analog deduction holds for a degree vector $\underline{s} \in \mathrm{DEG}^+(G)$ satisfying $\underline{s} \preceq s$ for all $s \in \mathrm{DEG}^+(G)$. □

If a tree-representation T of a given CFPO $\preceq$ is not isomorphic to a directed path, then we always find a pair of vertices that is not connected by a directed path. Hence, the last proposition implies that $\preccurlyeq$ is the only CFPO such that every graph has an unique maximal and an unique minimal degree vector. In this case we have seen that $\overline{s} = s_G^r$ and $\underline{s} = s_G^l$.
The next proposition shows the reverse direction of the previous result. Therefore, for any CFPO $\preceq$ a graph G has unique maximal and minimal degree vectors if for every edge $vw \in E(G)$ the path P_{vw} is a directed path in T.

Proposition 5.2
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T. Furthermore, let G be a labeled graph of order n such that for every $vw \in E(G)$ the vertices v and w are connected by a directed path in T. If $\overline{s}_{\preceq,G}$ and $\underline{s}_{\preceq,G}$ denote the out-degree vectors of the orientations of G with arc sets

$$\overline{A}_{\preceq,G} = \big\{(v,w) \,|\, vw \in E(G) \text{ and } P_{vw} \text{ is a directed } (v,w)\text{-path in } T\big\}$$

and

$$\underline{A}_{\preceq,G} = \big\{(w,v) \,|\, vw \in E(G) \text{ and } P_{vw} \text{ is a directed } (v,w)\text{-path in } T\big\},$$

then every $s \in \mathrm{DEG}^+(G)$ satisfies $\underline{s}_{\preceq,G} \preceq s \preceq \overline{s}_{\preceq,G}$.

Proof
Let $s \in \mathrm{DEG}^+(G)$ and denote D an orientation of G with out-degree vector s. Moreover, define the digraphs $\overline{D} = (V(G), \overline{A}_{\preceq,G})$ and $\underline{D} = (V(G), \underline{A}_{\preceq,G})$ which are orientations of G, too. From these definitions follow that for every arc $a \in A(\overline{D})$ there are $\lambda_{a,b} \in \{0,1\}$ such that

$$\mathbf{z}_a = \sum_{b \in A(T)} \lambda_{a,b}\, \mathbf{z}_b.$$

Obviously, $\lambda_{a,b} = 1$ holds if and only if $b \in A(P_{uv})$ where $uv \in E(G)$ is an edge connecting the head and tail of a. Hence we deduce that

$$\overline{s}_{\preceq,G} - s = \sum_{a \in A(\overline{D}) \setminus A(D)} \mathbf{z}_a = \sum_{a \in A(\overline{D}) \setminus A(D)} \left(\sum_{b \in A(T)} \lambda_{a,b}\, \mathbf{z}_b \right) = \sum_{b \in A(T)} \underbrace{\left(\sum_{a \in A(\overline{D}) \setminus A(D)} \lambda_{a,b} \right)}_{\geq 0} \mathbf{z}_b.$$

This shows that $\overline{s}_{\preceq,G} - s \in \mathcal{C}_{\preceq}$ and therefore $s \preceq \overline{s}_{\preceq,G}$. Similarly, for every arc $a \in A(\underline{D})$ we find $\mu_{a,b} \in \{-1,0\}$ such that

$$\mathbf{z}_a = \sum_{b \in A(T)} \mu_{a,b}\, \mathbf{z}_b.$$

We obtain that

$$\underline{s}_{\preceq,G} - s = \sum_{a \in A(\underline{D}) \setminus A(D)} \mathbf{z}_a = \sum_{a \in A(\underline{D}) \setminus A(D)} \left(\sum_{b \in A(T)} \mu_{a,b}\, \mathbf{z}_b \right) = \sum_{b \in A(T)} \underbrace{\left(\sum_{a \in A(\underline{D}) \setminus A(D)} \mu_{a,b} \right)}_{\leq 0} \mathbf{z}_b.$$

Thus we conclude that $\underline{s}_{\preceq,G} - s \in -\mathcal{C}_{\preceq}$ and therefore $\underline{s}_{\preceq,G} \preceq s$. □

Altogether Propositions 5.1 and 5.2 show that for a fixed CFPO $\preceq$ a graph G has unique extremal degree vectors with respect to $\preceq$ if and only if for every edge $vw \in E(G)$ there is a directed path in T connecting v and w. At this point we do not want to restrict our investigations to these combinations of a CFPO and a graph. Thus we also consider the case that a given graph may have several maximal and minimal degree vectors with respect to a predefined CFPO. For our studies the following definitions and notations are useful.

Definition 5.3
Let $\preceq$ be any partial order on $\mathbb{R}^n$ and G a labeled graph with n vertices. Denote $\overline{S}_{\preceq,G}$ (respectively $\underline{S}_{\preceq,G}$) the set of all degree vectors of G that are maximal (respectively minimal) with respect to $\preceq$. In analogy to $\mathcal{S}(G)$ and $\mathcal{S}'(G)$ we define the sets

$$\mathcal{S}_{\preceq}(G) = \left\{ s \in \mathbb{Z}^n \;\middle|\; \text{there are } \underline{s} \in \underline{S}_{\preceq,G},\ \overline{s} \in \overline{S}_{\preceq,G} \text{ with } \underline{s} \preceq s \preceq \overline{s} \text{ and } 0 \leq s \leq d_G \right\}$$

and

$$\mathcal{S}'_{\preceq}(G) = \left\{ s \in \mathbb{Z}^n \;\middle|\; \text{there are } \underline{s} \in \underline{S}_{\preceq,G},\ \overline{s} \in \overline{S}_{\preceq,G} \text{ with } \underline{s} \preceq s \preceq \overline{s} \right\}.$$

These definitions yield

$$\mathrm{DEG}^+(G) \subseteq \mathcal{S}_{\preceq}(G) \subseteq \mathcal{S}'_{\preceq}(G). \tag{5.1}$$

Furthermore, observe that $\mathcal{S}_{\preccurlyeq}(G) = \mathcal{S}(G)$ and $\mathcal{S}'_{\preccurlyeq}(G) = \mathcal{S}'(G)$.
Now let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T and G an arbitrary graph of order n. We define E_1 as the set of edges $vw \in E(G)$ such that P_{vw} is a directed path in T and by $E_2 = E(G) \setminus E_1$. For $i = 1, 2$ denote H_i the spanning subgraph of G with edge set E_i. By Proposition 5.2 $\mathrm{DEG}^+(H_1)$ has unique extremal elements $\overline{s}_{\preceq,H_1}$ and $\underline{s}_{\preceq,H_1}$. From

$$\mathrm{DEG}^+(G) = \mathrm{DEG}^+(H_1 + H_2) = \mathrm{DEG}^+(H_1) + \mathrm{DEG}^+(H_2)$$

we deduce that

$$\overline{S}_{\preceq,G} = \overline{s}_{\preceq,H_1} + \overline{S}_{\preceq,H_2} \text{ and } \underline{S}_{\preceq,G} = \underline{s}_{\preceq,H_1} + \underline{S}_{\preceq,H_2}.$$

Therefore, the number of maximal (respectively minimal elements) in $\mathrm{DEG}^+(G)$ is equal to the number of maximal (respectively minimal elements) in $\mathrm{DEG}^+(H_2)$. We also observe that $|\overline{S}_{\preceq,G}| = |\underline{S}_{\preceq,G}|$ This follows from the fact that for two degree vectors s, t of G holds $s \preceq t$ if and only if $d_G - t \preceq d_G - s$.
Finally, we make two remarks on Definition 5.3 concerning lattices and convex sets in partial orders. Obviously, the posets $(\mathcal{S}_{\preceq}(G), \preceq)$ and $(\mathcal{S}'_{\preceq}(G), \preceq)$ are lattices only if $|\overline{S}_{\preceq,G}| = 1$. In Section 5.4 we show further results on this topic. Considering convex sets we notice that if $\mathcal{S}'_{\preceq}(G) = \mathrm{DEG}^+(G)$, then $\mathrm{DEG}^+(G)$ can be interpreted as an union of convex sets.

In this chapter we often consider subgraphs of a given graph. For a graph G of order n and a subgraph H of order k of G it is not clear how a CFPO $\preceq$ on $\mathbb{R}^n$ has to be restricted to a CFPO on $\mathbb{R}^k$. Hence we always assume that H is a spanning subgraph of G. This also guarantees that a degree vector of H has the same dimension as a degree vector of G.
One of our main goals in this chapter is to give generalizations of the Theorems 3.2 and 3.24. We formulate a version for the general case, that is $s_{\preceq}(G) = \mathrm{DEG}^+(G)$ (respectively $s'_{\preceq}(G) = \mathrm{DEG}^+(G)$), and the special case where we additionally have that $|\overline{S}_{\preceq,G}| = 1$.
The next lemma generalizes Lemma 3.1 to an analogue result on cone orders. It shows that if every vector in $\mathcal{S}_{\preceq}(G)$ (respectively $\mathcal{S}'_{\preceq}(G)$) is a degree vector, then the same holds for every spanning subgraph of G.

Lemma 5.4
Let $\preceq$ be cone order on $\mathbb{R}^n$ and G be labeled graph of order n. For every spanning subgraph H of G the following holds.

(i) If $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$, then $\mathrm{DEG}^+(H) = \mathcal{S}_{\preceq}(H)$.

(ii) If $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$, then $\mathrm{DEG}^+(H) = \mathcal{S}'_{\preceq}(H)$.

Proof
We prove statement (i) by induction on $k = |E(G) \setminus E(H)|$. If $k = 0$, then the assertion trivially holds. Thus we assume that the statement is true for $k - 1$ with $k \geq 1$. Let

$vw \in E(G) \setminus E(H)$ and define the spanning subgraph $\tilde{H}$ of G by $\tilde{H} = H + vw$. Considering an arbitrary vector $s \in \mathcal{S}_{\preceq}(H)$ by (5.1) it suffices to show that s is a degree vector of H. There are degree vectors $\underline{s} \in \underline{S}_{\preceq,H}$ and $\overline{s} \in \overline{S}_{\preceq,H}$ satisfying $\underline{s} \preceq s \preceq \overline{s}$. Furthermore, H has orientations $\underline{D}$ and $\overline{D}$ with out-degree vectors $\underline{s}$ and $\overline{s}$, respectively. We define the vectors $r, \underline{r}, \overline{r}, t, \underline{t}$ and $\overline{t}$ by

$$r = s + \mathbf{e}_v,\ \underline{r} = \underline{s} + \mathbf{e}_v,\ \overline{r} = \overline{s} + \mathbf{e}_v, \quad t = s + \mathbf{e}_w,\ \underline{t} = \underline{s} + \mathbf{e}_w,\ \overline{t} = \overline{s} + \mathbf{e}_w.$$

Thus $\underline{r}$ and $\overline{r}$ are the out-degree vectors of $\underline{D} + (v, w)$ and $\overline{D} + (v, w)$ which are orientations of $\tilde{H}$. Since $\preceq$ is a cone order we observe that $\underline{s} \preceq s \preceq \overline{s}$ implies $\underline{r} \preceq r \preceq \overline{r}$. Moreover, it is easy to see that $0 \leq r \leq d_{\tilde{H}}$ holds and thus $r \in \mathcal{S}_{\preceq}(\tilde{H})$. By an analog deduction we prove that $\underline{t}, \overline{t} \in \mathrm{DEG}^+(\tilde{H})$ and $t \in \mathcal{S}_{\preceq}(\tilde{H})$.
Now, from the induction hypothesis follows that r and t are degree vectors of $\tilde{H}$. From Theorem 2.3 we obtain

$$t(X) \geq m(\tilde{H}[X]) \text{ for all } X \subseteq V. \tag{5.2}$$

Let D be an orientation of $\tilde{H}$ with out-degree vector r. If $(v, w) \in A(D)$, then $D - (v, w)$ is an orientation of H with out-degree vector $r - e_v = s$. Otherwise we have $(w, v) \in A(D)$. If there is a directed path from v to w in D, then we obtain an orientation of $\tilde{H}$ with arc (v, w) by reorienting a directed cycle of D. Thus suppose that D does not contain any directed (v, w)-path. Denote $U \subseteq V$ the set of vertices which can be reached from v by a directed path in D. Notice that $v \in U$ and $w \notin U$ and $r(U) = m(\tilde{H}[U])$. Therefore, by (5.2) we conclude

$$s(U) = t(U) \geq m(\tilde{H}[U]) = r(U) = s(U) + 1.$$

From this contradiction follows that D contains a directed (v, w)-path. Hence s is a degree vector of H and statement (i) holds. By the same arguments we prove statement (ii). □

Similar to Lemma 3.1 the previous result can be used to show $\mathrm{DEG}^+(G) \neq \mathcal{S}_{\preceq}(G)$ for a combination of a CFPO $\preceq$ and a graph G by simply proving this fact for a suitable spanning subgraph of G. Moreover, we expect that a graph G with $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$ can be characterized in terms of forbidden subgraphs analogue to the forbidden configurations H_1 and H_2 from Theorem 3.2.
We are also interested in the converse situation to Lemma 5.4. Particularly, we investigate a situation implying that from $\mathrm{DEG}^+(H) = \mathcal{S}_{\preceq}(H)$ for all subgraphs H of a given graph G follows $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$. The subsequent example show that in general this statement is not true.

Example 5.5
Consider the cross-free family $\mathcal{F} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ of subsets of $\{1, 2, 3, 4\}$ and the graph G defined by

$$V(G) = \{1, 2, 3, 4\} \quad \text{and} \quad E(G) = \{\{1, 3\}, \{2, 4\}\}.$$

Theorem 3.2 implies that every proper subgraph H of G satisfies $\mathrm{DEG}^+(H) = \mathcal{S}_{\preceq_{\mathcal{F}}}(H)$ but $\mathrm{DEG}^+(G) \neq \mathcal{S}_{\preceq_{\mathcal{F}}}(G)$.

Nevertheless, for certain CFPOs we obtain a converse result to Lemma 5.4, if G is a disjoint union of graphs.

Lemma 5.6
Let G_1 and G_2 be two labeled graphs of order n on the same vertex set V such that for every $v \in V$ holds $d_{G_1}(v) = 0$ or $d_{G_2}(v) = 0$. For $i = 1, 2$ denote $V_i = \{v \in V \mid d_{G_i}(v) > 0\}$ the set of vertices which are incident to an edge in G_i. Furthermore, let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T such that there is no directed (V_2, V_1)-path in T. If $\mathrm{DEG}^+(G_i) = \mathcal{S}_\preceq(G_i)$ holds for $i = 1, 2$, then $\mathrm{DEG}^+(G_1 + G_2) = \mathcal{S}_\preceq(G_1 + G_2)$.

Proof
Denote G the sum of G_1 and G_2. We define the following partition of $A(T)$ into three subsets $A_1 = A(T[V_1])$, $A_2 = A(T[V_2])$ and $A_3 = A \setminus (A_1 \cup A_2)$. Our assumption on T implies that all arcs in A_3 are directed from V_1 to V_2.
Let s be an arbitrary vector in $\mathcal{S}_\preceq(G)$. By (5.1) it suffices to show that $s \in \mathrm{DEG}^+(G)$. The definition of $\mathcal{S}_\preceq(G)$ implies that there are degree vectors r and t of G such that $r \preceq s \preceq t$. For $i = 1, 2$ we define $r_i \in \mathbb{R}^n$ by

$$r_i(v) = \begin{cases} r(v), & \text{if } v \in V_i, \\ 0, & \text{else }. \end{cases}$$

Notice that $r = r_1 + r_2$. Analogously, we define s_i and t_i with $s = s_1 + s_2$ and $t = t_1 + t_2$, respectively. Now, r_i and t_i are degree vectors of G_i. By Remark 4.8 there are unique $\lambda_a \in \mathbb{Z}$ such that

$$s_1 - r_1 = \sum_{a \in A} \lambda_a \mathbf{z}_a.$$

Since $s_1(v) = r_1(v) = 0$ holds for every $v \in V \setminus V_1$ and there is no directed (V_2, V_1)-path in T we deduce that $\lambda_a = 0$ for all $a \in A_2 \cup A_3$. Thus we obtain

$$s_1 - r_1 = \sum_{a \in A_1} \lambda_a \mathbf{z}_a.$$

Similarly, there are unique $\mu_a \in \mathbb{Z}$ such that

$$s_2 - r_2 = \sum_{a \in A_2} \mu_a \mathbf{z}_a.$$

From $r \preceq s$ and Theorem 4.7 we conclude that there are unique nonnegative integers ν_a satisfying

$$\sum_{a \in A_1} \nu_a \mathbf{z}_a + \sum_{a \in A_2} \nu_a \mathbf{z}_a + \sum_{a \in A_3} \nu_a \mathbf{z}_a = s - r = s_1 - r_1 + s_2 - r_2 = \sum_{a \in A_1} \lambda_a \mathbf{z}_a + \sum_{a \in A_2} \mu_a \mathbf{z}_a.$$

By Remark 4.8 the vectors in $\{\mathbf{z}_a\}_{a \in A(T)}$ are linear independent. Hence we have $\lambda_a = \nu_a \geq 0$ for all $a \in A_1$, $\mu_a = \nu_a \geq 0$ for all $a \in A_2$ yielding $r_1 \preceq s_1$ and $r_2 \preceq s_2$. An analogue deduction gives us $s_1 \preceq t_1$ and $s_2 \preceq t_2$.

Therefore, s_i is also a degree vector of G_i by the assumption $\text{DEG}^+(G_i) = \mathcal{S}_{\preceq}(G_i)$ for $i = 1, 2$. Finally, $s = s_1 + s_2$ is a degree vector of G what completes the proof. □

For the CFPO $\preccurlyeq$ Lemma 5.6 implies the following. Suppose a labeled graph G consists of components $G_1, \ldots, G_k$ for $k \geq 2$. Denote $\overline{v}_i$ (respectively $\underline{v}_i$) the vertex of component G_i with maximum (respectively minimum) label. Then G is degree complete, if for each $i \in \{1, \ldots, k\}$ the graph G_i is degree complete and every $v \in V(G)$ with $\underline{v}_i \leq v \leq \overline{v}_i$ is only adjacent to vertices in G_i.
Now we are able to prove a result that generalizes Theorem 3.24 to any CFPO.

Theorem 5.7
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T and denote $\tilde{T}$ the underlying (undirected) graph of T. A labeled graph G of order n satisfies $\text{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ and $|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1$ if and only if G is a spanning subgraph of $\tilde{T}$.

Proof
First suppose that G is not a spanning subgraph of $\tilde{T}$. There is an edge $vw \in E(G) \setminus E(\tilde{T})$. If the vertices v and w are not connected by a directed path in T, then by Proposition 5.1 we have $|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| > 1$. This contradicts our assumptions on G. Hence let P_{vw} be a directed (v, w)-path in T. From our choice of vw follows that P_{vw} contains at least two arcs of T and there is a vertex $u \in V(P_{vw}) \setminus \{v, w\}$. Considering the spanning subgraph $H = (V(G), \{vw\})$ of G we observe that $\mathbf{e}_v$ and $\mathbf{e}_w$ are the only degree vectors of H. Moreover, $\mathbf{e}_u$ is not a degree vector of H but it satisfies $\mathbf{e}_v \preceq \mathbf{e}_u \preceq \mathbf{e}_v$. Thus, $\text{DEG}^+(H) \neq \mathcal{S}'_{\preceq}(H)$ and by Lemma 5.4 we deduce $\text{DEG}^+(G) \neq \mathcal{S}'_{\preceq}(G)$.
Now suppose that G is a spanning subgraph of $\tilde{T}$ and consider any vector in $s \in \mathcal{S}'_T(\tilde{T})$. By (5.1) and Lemma 5.4 it suffices to show that $s \in \text{DEG}^+(\tilde{T})$. Denote $\overline{s}$ the out-degree vector of T and $\underline{s}$ the out-degree vector of the orientation T^* of $\tilde{T}$ where all arcs in T are reversed. From Proposition 5.2 follows that $\underline{s} \preceq s \preceq \overline{s}$. Therefore, we deduce $\overline{s} - s \in \mathcal{C}_{\preceq}$ and $s - \underline{s} \in \mathcal{C}_{\preceq}$. Denote λ_a the unique nonnegative integers with

$$\overline{s} - s = \sum_{a \in A(T)} \lambda_a \, \mathbf{z}_a.$$

Since every arc of T is reversed exactly once to obtain T^* we have

$$\overline{s} - \underline{s} = \sum_{a \in A(T)} \mathbf{z}_a.$$

We conclude that $\lambda_a \in \{0, 1\}$ and define an orientation D of $\tilde{T}$ by reversing every arc $a \in A(T)$ with $\lambda_a = 1$. It is easy to check that D has out-degree vector s and therefore $s \in \text{DEG}^+(\tilde{T})$. □

Since the tree-representation corresponding to $\preccurlyeq$ is a directed path Theorem 3.24 is a special case of the last theorem. We continue with an analogue result for the case where we allow $\text{DEG}^+(G)$ to have more than one maximal element with respect to a CFPO.

For its formulation the following definition is important. It generalizes the idea of the forbidden configurations H_1 and H_2 from Theorem 3.2 to an arbitrary CFPO.

Definition 5.8
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T and G a labeled graph on n vertices. A subgraph F of G is a *critical configuration* (with respect to $\preceq$) if there are components $F_1, \ldots, F_k$ of F with $k \geq 2$ and vertices $x_1, \ldots, x_k, y_1, \ldots, y_k \in V(F)$ such that $x_i, y_i \in V(F_i)$ for $1 \leq i \leq k$ and there is a directed (y_i, x_{i+1})-path Q_i in T (indices modulo k). The components $F_1, \ldots, F_k$ are called *essential*.

Since F_i is connected there is a path R_i in F_i connecting x_i and y_i. Obviously, $R_1, \ldots, R_k$ are pairwise disjoint. Now, every critical configuration contains an alternating sequence $R_1Q_1R_2Q_2 \ldots R_kQ_k$ that can be interpreted as a closed trail in the mixed graph we obtain as an union of G and T. The following example shows that H_1 and H_2 from Theorem 3.2 are critical configurations with respect to $\preccurlyeq$.

Example 5.9
The tree-representation T corresponding to $\preccurlyeq$ as a CFPO on $\mathbb{R}^n$ is a directed path such that $i+1$ is a positive neighbor of i for $1 \leq i \leq n-1$. First consider a forbidden configuration H_1 (see Theorem 3.2) of a labeled graph G of order n, that is, a subgraph of G consisting of two edges k_1k_3 and k_2k_4, where $k_1 < k_2 < k_3 < k_3$. We define F_1 as the subgraph of H_1 only containing the edge k_1k_3 and F_2 as the subgraph containing k_2k_4 Furthermore, set $x_1 = k_3$, $y_1 = k_1$, $x_2 = k_4$ and $y_2 = k_2$. Since $k_1 < k_4$ and $k_2 < k_3$ there is a (y_1, x_2)-path Q_1 and a (y_2, x_1)-path Q_2 in T. Hence H_1 is a critical configuration with respect to $\preccurlyeq$. Each edge of H_1 forms an essential component.
Analogously, for H_2 from Theorem 3.2 we define F_1 such that $E(F_1) = k_1k_4$ and F_2 such that $E(F_2) = k_2k_3$. Moreover, denote $x_1 = k_4$, $y_1 = k_1$, $x_2 = k_2$ and $y_2 = k_3$. Again there exists a (y_1, x_2)-path Q_1 and a (y_2, x_1)-path Q_2 in T showing that H_2 is a critical configuration, too.

It is also important to emphasize that our definition of a critical configuration allows the following. Firstly, for $1 \leq i, j \leq k$ and $i \neq j$ the two paths Q_i and Q_j in T may have common vertices and arcs. Secondly, a critical configuration H can include components that are not essential. Thirdly, it is possible that $x_i = y_i$ for some $1 \leq i \leq k$ even if F_i is nontrivial. In particular, since T is a tree we observe that there is at least one component F_i such that x_i and y_i are distinct. Thus at least one component of H is not trivial but it is possible that every other component consists of a single vertex.
Now, we are able to formulate a characterization of those labeled graphs G satisfying $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ for a CFPO $\preceq$.

Theorem 5.10
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T. A labeled graph G on n vertices satisfies $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ if and only if G does not contain a critical configuration.

Proof
We show that $\mathrm{DEG}^+(G) \neq \mathcal{S}'_{\preceq}(G)$ holds if and only if G contains a critical configuration. First suppose that G is a labeled graph containing a critical configuration F. Let $F_1, \dots, F_k$ be the essential components of F and denote the vertices $x_i, y_i \in V(F_i)$ as in Definition 5.8. For $1 \le i \le k$ let Q_i be the directed (y_i, x_{i+1})-path in T and R_i an $x_i y_i$-path in F_i. Without loss of generality we assume $V(F) = V(G)$ and $E(F_i) = E(R_i)$. Let r be the out-degree vector of the orientation of F where every edge in $E(F_i)$ is directed from x_i to y_i. Analogously, we define t as the out-degree vector of the orientation of F with opposite direction for every edge. From these definitions we obtain

$$t - r = \sum_{i=1}^{k} \sum_{a \in A(Q_i)} \mathbf{z}_a$$

and we deduce $r \preceq t$. Moreover, notice that $r(V(F_i)) = m(F_i)$ holds for $1 \le i \le k$. Considering the vector

$$s = r + \mathbf{z}_{(y_1,x_2)} = r + \sum_{a \in A(Q_1)} \mathbf{z}_a$$

we observe that $r \preceq s \preceq t$ and thus s is contained in $\mathcal{S}'_{\preceq}(G)$. Since

$$s(V(F_2)) = r(V(F_2)) - 1 < m(F_2)$$

we conclude that s is not a degree vector of F. Therefore, by Lemma 5.4 we have $\mathrm{DEG}^+(G) \neq \mathcal{S}'_{\preceq}(G)$.
Now suppose G is a labeled graph satisfying $\mathrm{DEG}^+(G) \neq \mathcal{S}'_{\preceq}(G)$. There are vectors $r, t \in \mathrm{DEG}^+(G)$ and $s \in \mathcal{S}'_{\preceq}(G) \setminus \mathrm{DEG}^+(G)$ such that $r \preceq s \preceq t$. We make two further assumptions which hold without loss of generality. Firstly, applying Lemma 5.4 we suppose that every spanning subgraph F of G satisfies $\mathrm{DEG}^+(F) = \mathcal{S}'_{\preceq}(F)$. Otherwise we consider a suitable spanning subgraph of G. Secondly, we assume that r is maximal and t is minimal in $\mathrm{DEG}^+(G)$ such that $r \preceq s \preceq t$ holds. In other words, if I is the (r,t)-interval of $\mathcal{S}'_{\preceq}(G)$ concerning $\preceq$, then r and t are the only degree vectors of G in I.
Denote D_r and D_t the orientations of G with out-degree vectors r and t, respectively. We show that G contains a forbidden configuration by proving five claims.

Claim 1: We obtain D_t by reversing all arcs of D_r.
Suppose there is an arc $(u,v) \in A(D_r) \cap A(D_t)$. We define vectors $\tilde{r}, \tilde{s}, \tilde{t}$ by

$$\tilde{r} = r - \mathbf{e}_u, \quad \tilde{s} = s - \mathbf{e}_u, \quad \text{and} \quad \tilde{t} = t - \mathbf{e}_u.$$

Obviously, we have $\tilde{r} \preceq \tilde{s} \preceq \tilde{t}$. Furthermore, $\tilde{r}$ and $\tilde{t}$ are out-degree vectors of $D_r - (u,v)$ and $D_t - (u,v)$, respectively. Hence $\tilde{r}$ and $\tilde{t}$ are degree vectors of $G - uv$. Since $\mathrm{DEG}^+(G - uv) = \mathcal{S}'_{\preceq}(G - uv)$ we deduce that $\tilde{s}$ is a degree vector of $G - uv$, too. Thus we have $s = \tilde{s} + \mathbf{e}_u \in \mathrm{DEG}^+(G)$ what contradicts our choice of s.

Claim 2: D_r is acyclic.
Suppose D_r contains a directed cycle C. By Claim 1 the digraph D_t also has a cycle on $V(C)$. We define

$$\tilde{r} = r - \sum_{v \in V(C)} \mathbf{e}_v, \quad \tilde{s} = s - \sum_{v \in V(C)} \mathbf{e}_v, \quad \text{and} \quad \tilde{t} = t - \sum_{v \in V(C)} \mathbf{e}_v.$$

Again we have $\tilde{r} \preceq \tilde{s} \preceq \tilde{t}$ and $\tilde{r}$ and $\tilde{t}$ are degree vectors of $G - \tilde{C}$, where $\tilde{C}$ denotes the underlying graph of C. Therefore, s is a degree vector of G what contradicts the choice of s.

From $r \preceq t$ and Remark 4.8 follows that there are unique integers $\lambda_a \geq 0$ satisfying

$$t - r = \sum_{a \in A(T)} \lambda_a \, \mathbf{z}_a. \tag{5.3}$$

Denote $\tilde{A}$ the set of arcs $a \in A(T)$ with $\lambda_a > 0$ in (5.3). We define $\tilde{T}$ as the spanning subdigraph of T with $A(\tilde{T}) = \tilde{A}$.

Claim 3: $|\tilde{A}| \geq 2$.
Suppose $\tilde{A} = \{(u,v)\}$ for some arc $(u,v) \in A(T)$. We deduce that $t - r = \lambda_{(u,v)} \mathbf{z}_{(u,v)}$ and we obtain D_t by reversing $\lambda_{(u,v)}$ arc-disjoint, directed (v,u)-paths of D_r. Since $r \prec s \prec t$ there is a positive integer $\mu < \lambda_{(u,v)}$ such that $s = \mu \, \mathbf{z}_{(u,v)}$. If we reverse μ arc-disjoint, directed (v,u)-paths in D_r, then we obtain an orientation of G with out-degree vector s. Hence s is a degree vector of G contradicting our choice of s.

Claim 4: If $(u,v) \in \tilde{A}$, then D_r does not contain a directed (v,u)-path.
Suppose that D_r contains a directed (v,u)-path. Considering the vector $p = r + \mathbf{z}_{(u,v)}$ we observe that $p \in \mathrm{DEG}^+(G)$ since p is the out-degree vector of the orientation of G obtained from D_r by reversing a directed (v,u)-path. Notice that $p \neq r$ and from Claim 3 follows $p \neq t$. Moreover, we have $p - r = \mathbf{z}_{(u,v)}$ and there are coefficients $\tilde{\lambda}_a \in \mathbb{Z}$ such that

$$t - p = \sum_{a \in A(T)} \tilde{\lambda}_a \mathbf{z}_a.$$

Thus we deduce that $p - r \in \mathcal{C}_{\preceq}$ and therefore $r \prec p$. Furthermore, $\tilde{\lambda}_a = \lambda_a$ for every $a \in A(T) \setminus \{(u,v)\}$ and $\tilde{\lambda}_{(u,v)} = \lambda_{(u,v)} - 1 \geq 0$ imply that $t - p \in \mathcal{C}_{\preceq}$ and hence $p \prec t$ holds. This contradicts our choice of r and t.

Claim 5: Every vertex $v \in V(G)$ with $d_G(v) > 0$ satisfies $d^+_{\tilde{T}}(v) > 0$ or $d^+_{D_r}(v) > 0$.
Let $v \in V(G)$ be a vertex with $d_G(v) > 0$. Suppose that $d^+_{D_r}(v) = 0$. Thus, we have $d^+_{D_t}(v) = d^-_{D_r}(v) > 0$ and therefore $t(v) - r(v) > 0$. Since

$$t - r = \sum_{a \in \tilde{A}} \lambda_a \mathbf{z}_a$$

and $\lambda_a > 0$ for all $a \in \tilde{A}$ there is an arc $\tilde{a} \in \tilde{A}$ such that $\mathbf{z}_{\tilde{a}}(v) = 1$. Hence $d^+_{\tilde{T}}(v) > 0$.
Now suppose $v \in V(\tilde{D})$ is a vertex with $d^+_{\tilde{T}}(v) = 0$. Then $d^-_{\tilde{T}}(v) > 0$ and there is an arc $\tilde{a} \in \tilde{A}$ such that $\mathbf{z}_{\tilde{a}}(v) = -1$. Furthermore, for any arc $a \in \tilde{A}$ holds $\mathbf{z}_a(v) \leq 0$. Thus, from (5.3) follows $d^+_{D_r}(v) = r(v) > t(v) \geq 0$.

Now, by Claim 5 we conclude that there is a collection of arcs in $\tilde{A} \cup A(D_r)$ which form a directed cycle C. Denote F the subgraph of G consisting of all edges of G that correspond to arcs in $A(D_r) \cap A(C)$. By Claim 2 and Claim 4 at least two arcs from $\tilde{A}$ are contained in C. Hence the graph F is a critical configuration of G with respect to $\preceq$. □

The following corollary shows that there is a CFPO $\preceq$ on $\mathbb{R}^n$ such that $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ holds for every graph G on n vertices.

Corollary 5.11
Let $\mathcal{F} = \{\{1\}, \{2\}, \ldots, \{n-1\}\}$ be a family of subsets of $\{1, \ldots, n\}$. For every labeled graph G on n vertices holds $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq_\mathcal{F}}(G)$.

Proof
Obviously, $\preceq_\mathcal{F}$ is a CFPO on $\mathbb{R}^n$ and the directed tree T with vertex set $\{1, \ldots, n\}$ and edge set $\{(i, n) \,|\, 1 \leq i \leq n-1\}$ is a tree-representation. Let G be a graph of order n. If G contains a critical configuration of H, then T has at least two arcs with distinct heads. Since all arcs of T enter vertex n we deduce that G cannot contain a critical configuration with respect to $\preceq_\mathcal{F}$. Hence by Theorem 5.10 follows $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq_\mathcal{F}}(G)$. $\square$

We compare the results from Theorem 5.7 and the previous corollary. Let $\mathcal{F}$ be defined as in Corollary 5.11. By Theorem 5.7 we observe that if a graph G satisfies $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq_\mathcal{F}}(G)$ and $|S^{\max}_{\preceq_\mathcal{F}}(G)| = 1$, then G is isomorphic to the star $K_{1,l}$ with $1 \leq l \leq n-1$. On the other hand, if we omit the condition $|S^{\max}_{\preceq}(G)| = 1$, then $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq_\mathcal{F}}(G)$ holds for every graph G of order n. Hence the number of extremal degree vectors has a significant influence and we only obtain $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq_\mathcal{F}}(G)$ for all graphs because the number of extremal degree vectors can be very large. A worst case would be that all degree vectors of a given graph are maximal and minimal. In this case the set of degree vectors forms an anti-chain with respect to $\preceq_\mathcal{F}$. Also notice that by Propositions 5.1 and 5.2 we can efficiently decide whether a given G has unique extremal degree vectors as well as determine them. For graphs with several extremal degree vectors this is not clear. Hence it might even be difficult to determine all extremal degree vectors.
We continue with a characterization of the labeled graphs G which satisfy $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$ for a fixed CFPO $\preceq$. First we consider the general case, that is, G is allowed to have several extremal degree vectors with respect to $\preceq$.

Theorem 5.12
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T. A labeled graph G on n vertices satisfies $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$ if and only if G does not contain a critical configuration consisting of at least two nontrivial essential components.

Proof
Suppose G contains a critical configuration H which consists of at least two nontrivial essential components. Let F be the spanning subgraph of G whose edge set is equal to $E(H)$. Denote $F_1, \ldots, F_k$ the components of F which correspond to the essential components of H. For $1 \leq i \leq k$ the vertices $x_i, y_i \in V(F_i)$ are defined as in Definition 5.8. Furthermore, denote Q_i the directed (y_i, x_{i+1})-path in T and R_i an (undirected) $x_i y_i$-path in F_i. Since a trivial essential component can be seen as a vertex of a path Q_i in T we assume without loss of generality that H only contains nontrivial essential components. From the fact that T is a tree we deduce that there is an index i satisfying

$x_i \neq y_i$. We assume that $x_1 \neq y_1$.
We define two orientations D_r and D_t of F with out-degree vectors r and t, respectively. In particular, in D_r every edge contained in a path R_i for $1 \leq i \leq k$ is directed from x_i to y_i. Moreover, every edge that is incident to exactly one vertex in $V(R_i)$ is directed such that the corresponding arc has its tail in R_i. All remaining edges of F are directed arbitrarily. The orientation D_t is obtained from D_r by reversing every arc which is included in a path R_i. From these definitions follows

$$t - r = \sum_{i=1}^{k} \sum_{a \in A(Q_i)} \mathbf{z}_a.$$

Hence we deduce $r \preceq t$. The vector s is defined by

$$s = r + \mathbf{z}_{(y_1,x_2)} = r + \sum_{a \in A(Q_1)} \mathbf{z}_a.$$

Since $r - s$ and $t - s$ are the sums of vectors $\mathbf{z}_a$ with $a \in A(T)$ we conclude that $r - s$ and $t - s$ are elements of $\mathcal{C}_{\preceq}$. Thus $r \preceq s \preceq t$ holds. Notice that x_2 has a positive neighbor in D_r. Furthermore, y_1 has a negative neighbor in D_r because it is the terminal vertex of P_1 which is a path of positive length. Hence we deduce $r(x_2) > 0$ and $r(y_1) < d_F(y_1)$ and therefore $0 \leq s \leq d_F$. Altogether this implies $s \in \mathcal{S}_{\preceq}(F)$. Now, from

$$m(F_1) = r(V(F_1)) > s(V(F_1))$$

follows that s is not a degree vector of F. By Lemma 5.4 we obtain $\mathrm{DEG}^+(G) \neq \mathcal{S}_{\preceq}(G)$.
Now, suppose G satisfies $\mathrm{DEG}^+(G) \neq \mathcal{S}_{\preceq}(G)$. Let $H_1, \ldots, H_b$ be the nontrivial components of G and denote $G_1, \ldots, G_b$ spanning subgraphs of G satisfying $E(G_i) = E(H_i)$ for $1 \leq i \leq b$. We consider two cases.

Case 1: For every G_i with $1 \leq i \leq b$ holds $\mathrm{DEG}^+(G_i) = \mathcal{S}_{\preceq}(G_i)$.
Since $\mathrm{DEG}^+(G) \neq \mathcal{S}_{\preceq}(G)$ we observe that G has at least two nontrivial components. There is a collection of nontrivial components $G_{i_1}, \ldots, G_{i_\alpha}$ with $\alpha \geq 2$ and vertices $x_{i_j}, y_{i_j} \in V(G_{i_j})$ such that T contains a directed $(y_{i_j}, x_{i_{j+1}})$-path. Otherwise there is an ordering of $G_1, \ldots, G_b$ with indices $k_1, \ldots, k_b$ such that there is no directed $(V(H_{k_\beta}), V(H_{k_\gamma}))$-path in T if $\beta > \gamma$. Hence Lemma 5.6 implies $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$, that is, a contradiction. Thus $H_{i_1}, \ldots, H_{i_\alpha}$ are the essential components of a critical configuration of G.

Case 2: There is a component G_i satisfying $\mathrm{DEG}^+(G_i) \neq \mathcal{S}_{\preceq}(G_i)$.
There are vectors $r, t \in \mathrm{DEG}^+(G_i)$ and $s \in \mathcal{S}_{\preceq}(G_i) \setminus \mathrm{DEG}^+(G_i)$ with $r \preceq s \preceq t$. Since s is not the out-degree vector of an orientation of G_i Theorem 2.8 yields that there is a nonempty vertex set $U \subsetneq V(G_i)$ such that

$$s(U) < m(G_i[U])$$

where $G_i[U]$ and $G_i - U$ are connected. By Remark 4.8 from $r \preceq s$ follows that there are unique nonnegative integers λ_a satisfying

$$s - r = \sum_{a \in A(T)} \lambda_a \mathbf{z}_a.$$

Remind that for $\mathbf{z}_a$ holds

$$\mathbf{z}_a(U) = \begin{cases} 1, & \text{if } a \in \delta_T^+(U), \\ -1, & \text{if } a \in \delta_T^-(U), \\ 0, & \text{else,} \end{cases}$$

and that Theorem 2.3 implies $r(U) \geq m(G_i[U])$. Hence we conclude that

$$0 > s(U) - r(U) = \sum_{a \in A(T)} \lambda_a \mathbf{z}_a(U) = \sum_{a \in \delta_T^+(U)} \lambda_a - \sum_{a \in \delta_T^-(U)} \lambda_a.$$

Since we have $r(v) = s(v) = 0$ for $v \in V \setminus V(G_i)$ we deduce that T contains a directed $(V(G_i) \setminus U, U)$-path. Analogously, from $s \preceq t$ follows the existence of unique nonnegative integers μ_a which satisfy

$$t - s = \sum_{a \in A(T)} \mu_a \mathbf{z}_a$$

and we obtain

$$0 < t(U) - s(U) = \sum_{a \in A(T)} \mu_a \mathbf{z}_a(U) = \sum_{a \in \delta_T^+(U)} \mu_a - \sum_{a \in \delta_T^-(U)} \mu_a.$$

Therefore, T also includes a directed $(U, V(G_i) \setminus U)$-path. We define F as the spanning subgraph of G with edge set $E(F) = E(G_i[U]) \cup E(G_i - U)$. Then F is a critical configuration of G with essential components $G_i[U]$ and $G_i - U$. Moreover, from $0 \leq s(v) \leq d_{G_i}(v)$ for all $v \in V(G_i)$ we deduce that $2 \leq |U| \leq n(G_j) - 2$ and thus $G_i[U]$ and $G_i - U$ are nontrivial. □

We compare Theorem 5.12 and Theorem 5.10. Let G be a labeled graph with $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ and H a labeled graph with $\mathrm{DEG}^+(H) = \mathcal{S}_{\preceq}(H)$. Observe that G does not contain a critical configuration with at least one nontrivial essential component. In contrast to this H does not include a critical configuration with at least two nontrivial essential components. Furthermore, notice that Theorem 5.7 also can be formulated in terms of critical configurations. In particular, it is equivalent to the following statement.

Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T. A labeled graph G of order n satisfies $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ and $|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1$ if and only if G does not contain a critical configuration consisting of exactly one edge.

Remember that $|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1$ holds if and only if for each edge vw the path P_{vw} in is directed (see Propositions 5.1 and 5.2). Hence an edge $vw \in E(G)$ with $m(P_{vw}) > 1$ yields a critical configuration. Moreover, to prove sufficiency in the result above we could apply Theorem 5.10 to show that G includes a single edge that forms a critical configuration.
This argument is the key for the proof of the following Theorem that is an analogue to Qian's Theorem 3.2 for an arbitrary CFPO.

Theorem 5.13
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T. A labeled graph G of order n satisfies $\mathrm{DEG}^+(G) = \mathcal{S}_\preceq(G)$ and $|\overline{S}_\preceq(G)| = |\underline{S}_\preceq(G)| = 1$ if and only if G does not contain a critical configuration consisting of exactly two independent edges of G as essential components.

Proof
Since $|\overline{S}_\preceq(G)| = |\underline{S}_\preceq(G)| = 1$ the endpoints of every edge are connected by a directed path in T.
First suppose G contains a pair of independent edges that are the essential components of a critical configuration. Hence this configuration has two nontrivial essential components. By Theorem 5.12 we obtain $\mathrm{DEG}^+(G) \neq \mathcal{S}_\preceq(G)$.
Now, let G be a labeled graph not containing a critical configuration which has two independent edges of G as essential components. We have to show $\mathrm{DEG}^+(G) = \mathcal{S}_\preceq(G)$. Suppose to the contrary that G satisfies $\mathrm{DEG}^+(G) \neq \mathcal{S}_\preceq(G)$. By Theorem 5.12 there is a critical configuration F in G which has at least two nontrivial essential components. From our assumption on G follows that $|E(F)| \geq 3$. Under all critical configuration in G which have at least two nontrivial essential components we choose F such that it includes a minimum number of edges. Hence every nonessential component of F is trivial.
Denote $F_1, \ldots, F_k$ the essential components of F and let F_1 and F_l be nontrivial. By the definition of a critical configuration for $1 \leq i \leq k$ each F_i contains vertices x_i, y_i such that there is an unique directed (y_i, x_{i+1})-path Q_i in T. Let R_i be a path in F_i connecting x_i and y_i. Since T is a directed tree there is at least one index i satisfying $x_i \neq y_i$. Without loss of generality we assume that $x_1 \neq y_1$.
Consider any nontrivial essential component F_i containing more than one edge. If $x_i = y_i$, then we can delete one edge in $E(F_i)$ from F and obtain a critical configuration with at least two nontrivial essential components and less edges than F, that is, a contradiction to our choice of F. With the same argument we deduce that $E(F_i) = E(R_i)$. Consider a path $R_i = v_0 v_1 \ldots v_\alpha$ where $v_0 = x_i$ and $v_\alpha = y_i$. For any β with $1 \leq \beta \leq \alpha$ there does not exist a $(v_{\beta-1}, v_\beta)$-path in T. Otherwise $F - v_{\beta-1}v_\beta$ is a critical configuration contradicting the choice of F. Thus, T contains a $(v_\beta, v_{\beta-1})$-path for every index β. A combination of these paths yields a (y_i, x_i)-path P_i in T.
Suppose F consists of at least three nontrivial essential components. Then we show by the same argument as before that there is a (y_i, x_i)-path P_i in T even in the case $m(F_i) = 1$. From the minimality of F also follows that the paths $Q_1, \ldots, Q_k$ are pairwise internally disjoint. Moreover, for $1 \leq i \leq k$ the path P_i is disjoint from Q_j if $j \notin \{i-1, i\}$. Hence T contains a cycle contradicting the fact that T is a tree.
Therefore, it remains to consider the case that F consists of exactly two nontrivial essential components F_1 and F_2. Let $w_0 w_1 \ldots w_\gamma$ be the vertices of R_1, where $w_0 = x_1$ and $w_\gamma = y_1$. Notice that for $\gamma \geq 2$ there is a directed $(w_{\gamma-1}, x_1)$-path P^* in T. If $x_2 = y_2$, then $m(F_1) \geq 2$ and T contains a directed (y_1, x_1)-path P including x_2. Thus y_1 and $w_{\gamma-1}$ are connected via a path consisting of arcs from P^* and P in T. This path cannot be directed contradicting the fact that $|\overline{S}_\preceq(G)| = |\underline{S}_\preceq(G)| = 1$. Analogously, we deduce that if $x_2 \neq y_2$ and $m(F_2) = 1$, then T does not include a directed (x_2, y_2)-path. Altogether we conclude that in every case T contains a directed (y_i, x_i)-path P_i for $i = 1, 2$.

Now, if Q_1 and Q_2 are disjoint, then the subgraph of T including P_1, Q_1, P_2, Q_2 contains a cycle. This again yields a contradiction. Thus we assume that $V(Q_1) \cap V(Q_2) \neq \emptyset$. Let Q^* be the subgraph of T induced by $V(Q_1) \cap V(Q_2)$. It is easy to see that Q^* is a directed path connecting two vertices, say u^* to v^*. Furthermore, denote Q_i' the (y_i, u^*)-path and Q_i'' the (v^*, x_{i+1})-path in T for $i = 1, 2$. Finally, a combination of Q_1', Q^* and Q_2'' yields a directed (y_1, x_1)-path in T. Hence again the endpoints of the edge $y_1 w_{\gamma-1}$ are not connected via a directed path in T. This contradicts $|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1$ what completes the proof. □

In Example 5.9 we observed that H_1 and H_2 from Theorem 3.2 are also critical configurations. Of course, each has two nontrivial, essential components. Hence for every labeled graph G containing H_1 or H_2 holds $\mathrm{DEG}^+(G) \neq \mathcal{S}_{\preccurlyeq}(G)$. Additionally, since the tree-representation of $\preccurlyeq$ is a directed path we deduce that a critical configuration consisting of two independent edges from G is isomorphic to H_1 or H_2. Therefore, Theorem 5.13 implies Theorem 3.2.
According to the results in this section we make the following definition which generalizes the concept to degree complete graphs.

Definition 5.14
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ and G a labeled graph of order n. The graph G is *degree complete with respect to* $\preceq$ if it satisfies

$$|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1 \quad \text{and} \quad \mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G).$$

Similarly, G is *strongly degree complete with respect to* $\preceq$ if we have

$$|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1 \quad \text{and} \quad \mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G).$$

Alternatively, we could think of a definition of degree complete graphs with respect to a CFPO in which we omit the condition $|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1$. There are some reasons for including this restriction. We already have mentioned that $(\mathcal{S}_{\preceq}(G), \preceq)$ is a lattice only if there are unique extremal degree vectors of G with respect to $\preceq$. Furthermore, we have observed (see Corollary 5.11) that there exists a CFPO $\preceq$ on $\mathbb{R}^n$ such that for every labeled graph G of order n holds $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$. Obviously, in this case we have $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$, too. In some sense this fact does not fit to our results on graphs with degree complete labelings. This class only contains graphs that are "path-like". On the other hand the alternative definition would imply that for every graph G we could find a CFPO $\preceq$ such that G is degree complete with respect to $\preceq$. Moreover, we notice that if we have non-unique extremal degree vectors, then it is not clear how to determine these degree vectors efficiently. Probably, this aspect is related to the fact that Theorem 5.12 does not provide an efficient procedure to test whether a labeled graph G satisfies $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$ or not. In particular, an important question is: "How many subgraphs of G do we have to consider if we want to check whether there does not exist a critical configuration with at least two nontrivial essential components?" In contrast to this we obtain a polynomial procedure to check whether a labeled graph G

satisfies $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$ and $|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1$ by Theorem 5.13. Basically, the following corollary yields such a procedure. The result reformulates the condition that a graph contains a critical configuration consisting of two independent edges.

Corollary 5.15
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T. A labeled graph G is degree complete with respect to $\preceq$ if and only if the following two conditions hold:

1.) For every edge $vw \in E(G)$ there is a directed path P_{vw} connecting v and w in T.

2.) For every pair of independent edges $vw, xy \in E(G)$ the paths P_{vw} and P_{xy} are vertex disjoint.

Proof
By Propositions 5.1 and 5.2 the first condition holds if and only if there are unique extremal degree vectors of G with respect to $\preceq$. The second condition is a reformulation of the statement in Theorem 5.13 since uv and wx form a critical configuration if and only if P_{uv} and P_{wx} are not disjoint. □

5.2 A structural result

Now that we have a suitable definition for degree complete graphs with respect to some CFPO two questions arise. Firstly, which (unlabeled) graphs are degree complete with respect to some CFPO? Secondly, which (unlabeled) graphs are degree complete with respect to a predefined CFPO? In this section we give an answer to the first question and we obtain a result which generalizes Theorem 3.17 in some sense. However, in section 5.3 we take a closer look at the second question.
In the previous section we have defined degree complete graphs only for labeled graphs. A well-known interpretation of an unlabeled graph is that it is a representative of the class containing all labeled graphs which are pairwise isomorphic to each other. Thus we could say that an unlabeled graph is degree complete with respect to some CFPO if there is a labeled version of the graph which has this property. Actually, this definition is not necessary to characterize unlabeled graphs which are degree complete with respect to some CFPO. The reason for this arises from the tree-representation of a CFPO. Consider an unlabeled graph G which is degree complete with respect to some CFPO and denote T the corresponding tree-representation. Instead of finding a suitable labeled version of G we take any labeled version of G and change the labels in T. In this way we obtain a new CFPO $\preceq$ with the property that the chosen labeled version of G is degree complete with respect to $\preceq$. Hence in this section we only consider labeled graphs.
We use a further simplification. Theorem 5.13 and Corollary 5.15 hold for graphs with multiple edges. In analogy to Observation 3.12 we obtain the following as a directed consequence of these results.

Observation 5.16
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ and G a labeled graph of order n. For every edge $vw \in E(G)$ the graph $G+vw$ is degree complete with respect to $\preceq$ if and only if G is degree complete with respect to $\preceq$.

Hence in the next theorem we consider only simple graphs. This yields a result which is easier to formulate and easier to compare with Theorem 3.17. Of course we obtain a similar result for general graphs with multiple edges by Observation 5.16. The following theorem includes three characterizations of graphs that are degree complete with respect to some CFPO $\preceq$. One (iii) is in terms of forbidden subgraphs while the remaining two ((iv),(v)) imply procedures to recognize these graphs. Moreover, the result shows that for all those graph we even find a LFPO with the same properties.

Theorem 5.17
Let G be a simple labeled graph. Furthermore, let X_2 and F be the outputs of the procedures 3.15 and 3.16 with input G, respectively. The following statements are equivalent:

(i) There is a LFPO $\preceq$ such that G is degree complete with respect to $\preceq$.

(ii) There is a CFPO $\preceq$ such that G is degree complete with respect to $\preceq$.

(iii) G does not contain a subgraph isomorphic to *net* or C_k $(k \geq 4)$.

(iv) $G - X_2$ is a forest.

(v) $G - F$ is a forest.

Proof
From (i) to (ii): Trivial, since every LFPO is a CFPO.

From (ii) to (iii): Let $\preceq$ be an arbitrary CFPO on $\mathbb{R}^n$ and denote T the tree-representation of $\preceq$. Moreover, let G be a graph of order n which contains of a copy of *net* or C_k $(k \geq 4)$. We prove that G is not degree complete with respect to $\preceq$. If there is an edge $vw \in E(G)$ such that P_{vw} is not a directed path in T, then G is not degree complete by Corollary 5.15. Thus we assume that P_{vw} is a directed path in T for every edge $vw \in E(G)$ and we have to show that $\mathrm{DEG}^+(G) \neq \mathcal{S}_{\preceq}(G)$. Consider a spanning subgraph H of G which consists of exactly one copy of *net* or C_k $(k \geq 4)$ and some isolated vertices. By Lemma 5.4 it suffices to prove $\mathrm{DEG}^+(H) \neq \mathcal{S}_{\preceq}(H)$. We distinguish between two cases.

Case 1: H contains a copy of net.
Denote u, v, w the three vertices of degree 3 in H. Thus u, v, w induce a triangle in H. The edges uv and vw are incident. Hence $A(P_{uv}) \cup A(P_{vw})$ is the arc set of a directed subtree T' of T. Obviously, v and w are vertices in T' and thus P_{uw} is included in T'. We show that T' is a directed path in T.
Without loss of generality assume that $P_{uv} = x_0x_1 \ldots x_l$ is a (u, v)-path, that is, $x_0 = u$ and $x_l = v$. If w is a vertex of P_{uv}, then P_{uw} and P_{wv} are contained in P_{uv} and hence $T' = P_{uv}$ is a directed path. If P_{uv} and P_{vw} do not share a common arc, then P_{vw} is

a (v, w)-path. Otherwise P_{uw} is not a directed path in T. In this case we deduce that $T' = P_{uw}$ is a directed path. Now suppose we have $w \notin V(P_{uv})$ and $A(P_{uv}) \cap A(P_{vw}) \neq \emptyset$. Since v is a vertex of $V(P_{uv})$ and $V(P_{vw})$ we deduce that P_{vw} is a (w, v)-path. If P_{vw} includes P_{uv}, then $T' = P_{vw}$ is a directed path. Otherwise, if P_{uv} is not contained in P_{vw}, then there is minimal index i with $1 \leq i \leq l-1$ such that $x_i \in V(P_{uv}) \cap V(P_{vw})$. Moreover, there is an (u, x_i)-path and a (w, x_i)-path in T'. Because T' is a tree there is an unique path for every pair of vertices. Therefore, u and w are not connected by a directed path in T contradicting the fact that P_{uw} is directed. Thus in all feasible cases T' is a directed path. Without loss of generality we assume that T' is a (u, w)-path.
Now, denote y the vertex of degree 1 in H which is adjacent to v. Obviously, the edges uw and vy are independent and v is contained in P_{uw} and P_{vy}. By Corollary 5.15 the graph H is not degree complete with respect to $\preceq$ and therefore $\mathrm{DEG}^+(H) \neq \mathcal{S}_{\preceq}(H)$.

Case 2: H contains a copy of C_k with $k \geq 4$.
Let $C_k = v_1 v_2 \dots v_k v_1$. The tree T contains a directed path $P_{v_i v_{i+1}}$ for every i with $1 \leq i \leq k$ (indices equal modulo k). Since C_k is connected the arcs in $A(P_{v_1 v_2}) \cup \dots \cup A(P_{v_{k-1} v_k})$ form a directed subtree T' of T. Obviously, the vertices v_1 and v_k are in $V(T')$ and hence $P_{v_1 v_k}$ is included in T'. If the paths $P_{v_1 v_k}$ and $P_{v_i v_{i+1}}$ with $2 \leq i \leq k-2$ have a common vertex, then $v_1 v_k$ and $v_i v_{i+1}$ form a critical configuration in H with two nontrivial essential components. Similarly, if $P_{v_1 v_2}$ and $P_{v_{k-1} v_k}$ have a common vertex, then $v_1 v_2$ and $v_{k-1} v_k$ form a critical configuration in H with two nontrivial essential components. In both situations we deduce that $\mathrm{DEG}^+(H) \neq \mathcal{S}_{\preceq}(H)$ by Theorem 5.12. Now, suppose $P_{v_1 v_k}$ is vertex disjoint to $P_{v_i v_{i+1}}$ for all $2 \leq i \leq k-2$ and $V(P_{v_1 v_2}) \cap V(P_{v_{k-1} v_k}) = \emptyset$. Since $P_{v_1 v_k}$ is connected $P_{v_1 v_k}$ is a subdigraph of either $P_{v_1 v_2}$ or $P_{v_{k-1} v_k}$. Thus, we conclude that either $v_k \in V(P_{v_1 v_2})$ or $v_1 \in V(P_{v_{k-1} v_k})$ what contradicts the fact that $P_{v_1 v_2}$ and $P_{v_{k-1} v_k}$ are vertex disjoint.

From (iii) to (iv) and (iii) to (v): Let G be a graph not containing a subgraph isomorphic to *net* or C_k with $k \geq 4$. Hence all cycles in G are triangles. Furthermore, every edge is contained in at most one triangle. Otherwise G has a cycle of length 4. Since *net* is not a subgraph of G every triangle has a vertex v of degree 2. From the construction of X_2 follows that v is contained in this set. Therefore, $G - X_2$ does not contain a cycle, that is, $G - X_2$ is a forest.
Considering F we deduce by the same arguments that every triangle of G contains exactly one edge of F. Therefore, the removal of F from G destroys all cycles in G and hence $G - X_2$ is a forest.

From (iv) to (i): Let G be a graph such that $G - X_2$ is a forest. We have to show that there is a LFPO $\preceq$ such that G is degree complete with respect to $\preceq$. Hence we construct step by step an in-tree D with $V(D) = V(G)$ such that for each edge $vw \in E(G)$ there is a directed path P_{vw} in D and for every pair of independent edges $vw, xy \in E(G)$ the paths P_{vw} and P_{xy} are vertex disjoint. First we initialize D with an orientation of $G - X_2$ which induces an in-tree on each component of $G - X_2$. Afterwards, for every $u \in X_2$ we repeat the following step. By the construction of X_2 there are exactly two vertices $v, w \in V(G) \setminus X_2$ which are adjacent to u in G. Moreover, u, v, and w induce a triangle in G. Hence v and w are connected by an arc of D. Without loss of generality we assume that $(v, w) \in A(D)$. Now, we add u to $V(D)$. Furthermore, we replace the arc (v, w)

by the pair of arcs $(v,u),(u,w)$ in D. We repeat this step until $V(D) = V(G)$. Since the vertices in X_2 which are added to D have in-degree and out-degree equal to 1 we deduce that the resulting digraph D is a directed forest where each weak component is an in-tree. Hence if the resulting directed forest D consists of one weak component, then D is an in-tree. Otherwise denote $D_1,\ldots,D_l$ the weak components of D and let $v_i \in V(D_i)$ be the unique vertex with $d^+_{D_i}(v_i) = 0$. By adding the arcs $(v_1,v_2),(v_1,v_3),\ldots,(v_1,v_l)$ to $A(D)$ we obtain a desired in-tree.
Now, it remains to show that for each edge $vw \in E(G)$ there is a directed path P_{vw} in D and for every pair of independent edges $vw, xy \in E(G)$ the paths P_{vw} and P_{xy} are vertex disjoint. Let vw be an edge in $E(G)$. It is not difficult to see that two vertices from X_2 are not adjacent. Thus v or w are not contained in X_2. Suppose that $v \in X_2$. From the second step of the procedure which constructs D we deduce that (v,w) or (w,v) is an arc of D. Thus P_{vw} is directed. Additionally, we notice that $d^+_D(v) = d^-_D(v) = 1$ holds for all $v \in X_2$. If $v,w \in V(G) \setminus X_2$, then P_{vw} has length 1 or 2. More precisely, P_{vw} has length 2 if and only if there exists a vertex $u \in X_2$ such that $uvwu$ is a triangle in G. This follows from Procedure 3.15 because u is the only common neighbor of v and w and $N_G(u) = \{v,w\}$. Altogether there is a directed path P_{vw} in D connecting v and w which has a length of at most 2. Now, suppose there is a pair of independent edges $uv, wx \in E(G)$ such that P_{uv} and P_{wx} have a common vertex. Then at least one path, say P_{uv}, has length 2. Hence there is a vertex $y \in X_2$ such that $P_{uv} = uyv$ and $y \in V(P_{wx})$ hold. We conclude $d^+_D(y) \geq 2$ or $d^-_D(y) \geq 2$, that is a contradiction. Therefore, for every pair of independent edges $uv, wx \in E(G)$ the corresponding paths P_{uv} and P_{wx} are vertex disjoint. By Corollary 5.15 the graph G is degree complete with respect to the LFPO with tree-representation D.

From (v) to (i): Let G be a graph such that $G - F$ is a forest. Again, we have to prove that G is degree complete with respect to some LFPO $\preceq$. Hence we construct step by step an in-tree D with $V(D) = V(G)$ which satisfies conditions 1.) and 2.) from Corollary 5.15. Let $G_1,\ldots,G_l$ be the components of $G - F$. For each $1 \leq i \leq l$ choose a vertex $v_i \in V(G_i)$ which is either not contained in a triangle of G or incident to an edge of F. The existence of such vertices follows from the construction of F. Now, initialize D by an orientation of $G - F$ that induces an in-tree on each G_i such that v_i is the unique vertex of G_i with out-degree equal to 0. Finally, if D has more than one weak component we add $(v_1,v_2),(v_1,v_3),\ldots,(v_1,v_l)$ to $A(D)$. Obviously, the construction guarantees that D is an in-tree with $V(D) = V(G)$. It remains to show that for each edge $vw \in E(G)$ there is a directed path P_{vw} in D and for every pair of independent edges $vw, xy \in E(G)$ the paths P_{vw} and P_{xy} are vertex disjoint. Let vw be an edge in $E(G)$. If $vw \notin F$, then we deduce $P_{vw} = vw$ or $P_{vw} = wv$. Otherwise, if $vw \in F$, then there is a vertex u such that $\{vu, uw\} \subseteq E(G) \setminus F$. Since u is not equal to v_i for some $1 \leq i \leq l$ we observe that $P_{vw} = vuw$ or $P_{vw} = wuv$. Now, consider a pair of independent edges $uv, wx \in E(G)$. If $\{uv, wx\} \subseteq E(G) \setminus F$, then P_{uv} and P_{wx} are obviously vertex disjoint because both paths have length 1. In the case $uv \in F$ notice that there is a vertex $y \in V(D)$ such that $V(P_{uv}) = \{u,v,y\}$. Furthermore, y has u and v as its only neighbors in G. Therefore, P_{uv} and P_{wx} are disjoint, too. By Corollary 5.15 the graph G is degree complete with respect to the LFPO with tree-representation D. □

Obviously, the graphs with a degree complete labeling in Theorem 3.17 also satisfy the conditions from Theorem 5.17. Moreover, there are some other similarities between these results. Both contain a structural characterization in terms of forbidden subgraphs and both include characterizations which yield procedures to recognize the corresponding graphs. On the other hand there seem to be a difference in the described conditions. In Theorem 3.17 we consider graphs with degree complete labeling and in Theorem 5.17 graphs which are degree complete with respect to a CFPO. A reformulation of the degree complete labeling property shows that we could interpret it as a class of partial orders similar to the classes of CFPOs and LFPOs. We already discussed that for a CFPO we do not need the concept of labelings because we change the labels in the tree-representation of a CFPO. Thus we deduce that an unlabeled graph G has a degree complete labeling if and only if G (or any labeled version of this graph) is degree complete with respect to a CFPO where the corresponding tree-representation is a directed path.

5.3 Degree complete labelings

In the previous section we have characterized the graphs which are degree complete with respect to a CFPO. In contrast to this we change the task in this section to the effect that we want to have a description of all graphs which are degree complete with respect to a predefined CFPO.

Now, for our investigations it is useful to consider labelings of unlabeled graphs.

Definition 5.18
Let $\preceq$ be a CFPO on $\mathbb{R}^n$. An unlabeled graph G of order n has a *degree complete labeling with respect to* $\preceq$ if there is a labeled version of G which is degree complete with respect to $\preceq$.

Let $\preceq$ be a CFPO on $\mathbb{R}^n$ and G be a labeled graph of order n which is degree complete with respect to $\preceq$. By Corollary 5.15 every spanning subgraph of G is also degree complete with respect to $\preceq$. We deduce that every spanning subgraph of a graph with degree complete labeling with respect to $\preceq$ has the same property. Therefore, for every CFPO $\preceq$ we can find a set of forbidden subgraphs which characterizes the set of graphs that have a degree complete labeling with respect to $\preceq$. For example, by Theorem 3.17 the CFPO $\preccurlyeq$ yields that a set of forbidden subgraphs is $\{T_2, net\} \cup \{C_k \mid k \geq 4\}$. Furthermore, Theorem 5.17 implies that for every CFPO a set of forbidden subgraphs contains *net*, C_k with $k \geq 4$ or subgraphs of them.

Unfortunately, it seems to be difficult to determine such a set of forbidden subgraph completely for an arbitrary CFPO. Hence, we collect some relations and properties between the tree-representation of a CFPO on the one hand and the graphs which have a degree complete labeling with respect to $\preceq$ on the other hand. We start with a simple implication of Corollary 5.15. A *matching* of a graph is a set of pairwise independent edges in G. Denote $\nu(G)$ the maximum cardinality of a matching in G.

Proposition 5.19
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T and denote $\tilde{T}$ the underlying graph of T. Moreover, let G be an unlabeled graph of order n. If G has a degree complete labeling with respect to $\preceq$, then $\nu(G) \leq \nu(\tilde{T})$.

Proof
Let $M = \{e_1, \ldots, e_l\}$ be matching of maximum cardinality in G. By Corollary 5.15 there is a directed path P_e in T for every $e \in M$ and we deduce that P_{e_i} and P_{e_j} are disjoint if $i \neq j$. For $1 \leq i \leq l$ denote $a_i \in A(T)$ an arbitrary arc of P_{e_i}. The set of edges in $\tilde{T}$ which correspond to $a_1, \ldots, a_l$ form a matching of cardinality l in $\tilde{T}$. Hence we conclude $\nu(G) \leq \nu(\tilde{T})$. □

Especially, for CFPOs where $\nu(\tilde{T})$ is small the set of graphs with degree complete labelings is strongly restricted. The following two results characterize the graphs which have a degree complete labeling with respect to a CFPO where the tree-representation is an orientation of the star $K_{1,n-1}$.

Corollary 5.20
Let $\preceq$ be the CFPO with tree-representation $T = (V, A)$ where $V = \{1, \ldots, n\}$ and $A = \{(1, n), (2, n), \ldots, (n-1, n)\}$ for $n \geq 2$. For an unlabeled graph G of order n the following statements are equivalent:

(i) G has a degree complete labeling with respect to $\preceq$.

(ii) G does not contain a subgraph isomorphic to $2K_2$ or C_3.

(iii) G is isomorphic to a subgraph of $K_{1,n-1}$.

Proof
Obviously, T is an orientation of $K_{1,n-1}$. Suppose G has a degree complete labeling with respect to $\preceq$. From (i) follows (ii) since a maximum matching in $K_{1,n-1}$ consists of a single edge. Hence Corollary 5.19 implies that G does not contain a subgraph isomorphic to $2K_2$. Moreover, considering three distinct vertices u, v, and w we observe that at least one of the pairs uv, vw and uw is not connected via a directed path in T. Thus G does not contain a triangle. Furthermore, the subgraphs of $K_{1,n-1}$ are the only graphs not containing any pair of independent edges or a triangle. Hence from (ii) we obtain (iii). Finally, every subgraph of $K_{1,n-1}$ has a degree complete labeling with respect to $\preceq$ by Theorem 5.7. □

Let $\mathcal{F}$ be the cross-free family with tree-representation T from Corollary 5.20, that is $\mathcal{F} = \big\{\{1\}, \{2\}, \ldots, \{n-1\}\big\}$. For the CFPO of $\mathcal{F}$ where we replace all set by its complement we obtain the same result. In the case that we replace only a some of the sets in $\mathcal{F}$ by its complement the following statement holds.

Corollary 5.21
Let $\preceq$ be the CFPO with tree-representation $T = (V, A)$ where $V = \{1, \ldots, n\}$ and $A = \{(1, n), (2, n), \ldots, (k, n)\} \cup \{(n, k + 1), (n, k + 2), \ldots, (n, n - 1)\}$ for $1 \leq k \leq n - 2$. For an unlabeled graph G of order n the following statements are equivalent:

(i) G has a degree complete labeling with respect to $\preceq$.

(ii) G does not contain a subgraph isomorphic to $2K_2$.

(iii) G is isomorphic to a subgraph of $K_{1,n-1}$ or C_3.

Proof
Obviously, T is an orientation of $K_{1,n-1}$. Suppose G has a degree complete labeling with respect to $\preceq$. From (i) follows (ii) since a maximum matching in $K_{1,n-1}$ consists of a single edge. Hence Corollary 5.19 implies that G does not contain a subgraph isomorphic to $2K_2$. Furthermore, the subgraphs of $K_{1,n-1}$ and C_3 are the only graphs not containing any pair of independent edges. Hence from (ii) we obtain (iii). Finally, every subgraph of $K_{1,n-1}$ or C_3 has a degree complete labeling with respect to $\preceq$. □

We continue with sets of forbidden subgraphs for graphs of order n that have a degree complete labeling with respect to any given CFPO on $\mathbb{R}^n$. The following example shows that for $n = 7$ the graphs *net* and C_k with $4 \leq k \leq 7$ do not form a complete set of forbidden subgraphs.

Example 5.22
Let $\preceq$ be an arbitrary CFPO on $\mathbb{R}^7$ with tree-representation T. Suppose $P_7 = v_1v_2 \ldots v_7$, that is the path of length 6, has a degree complete labeling with respect to $\preceq$ and denote G the corresponding labeled version of P_7. For $3 \leq i \leq 6$ the edges v_1v_2 and v_iv_{i+1} are independent. Hence by Corollary 5.15 the paths $P_{v_1v_2}$ and $P_{v_iv_{i+1}}$ are vertex disjoint. Therefore, $V(P_{v_1v_2}) = \{v_1, v_2\}$, that is, $P_{v_1v_2}$ consists of the single arc (v_1, v_2) or (v_2, v_1). Moreover, we deduce that T contains (v_1, v_2) or (v_2, v_1), too. Similarly, a directed version of the edge v_6v_7 is an arc of T. Considering v_3v_4 we observe that all other vertices of G are incident to an edge which is independent from v_3v_4 in G. Hence by the same argument as before we conclude $V(P_{v_3v_4}) = \{v_3, v_4\}$. Analogously, we have $V(P_{v_4v_5}) = \{v_4, v_5\}$. Finally, the edges v_2v_3, v_4v_5, v_6v_7 are pairwise independent. Hence $V(P_{v_2v_3}) \subseteq \{v_1, v_2, v_3\}$ and since T is a directed tree we conclude that a directed version of either v_2v_3 or v_1v_3 is in $A(T)$. An analog deduction for v_5v_6 shows that T is isomorphic to an orientation of P_7.
We want to compare this result with a similar assertion on T_2 from Figure 3.4. Hence suppose that

$$T_2 = (\{u, v_1, v_2, v_3, w_1, w_2, w_3\}, \{uv_1, uv_2, uv_3, v_1w_1, v_2w_2, v_3w_3\})$$

has a degree complete labeling with respect to $\preceq$ and denote H the corresponding labeled version of T_2. For $i = 1, 2, 3$ every vertex $x \in V(H)$ with $x \notin \{v_i, w_i\}$ is incident to an edge of H which is independent from v_iw_i. Thus we obtain $V(P_{v_iw_i}) = \{v_i, w_i\}$. Similarly,

we deduce that $V(P_{uv_i}) \subseteq \{u, v_i, w_i\}$. We conclude that T is isomorphic to an orientation of T_2.
Therefore, a P_7 and T_2 cannot have a degree complete labeling with respect to the same CFPO. Hence for every CFPO on $\mathbb{R}^7$ the class of forbidden subgraphs contains beside *net* and C_k with $4 \leq k \leq 7$ an element which is isomorphic to a subgraph of P_7 or T_2.

It is possible to generalize this observation. For this purpose we define a class of trees $\mathcal{T}$ by

$$\mathcal{T} = \{T \mid T \text{ is a tree and every leaf is adjacent to a vertex with degree } 2\}.$$

Moreover, denote $\mathcal{T}_n$ the set of trees in $\mathcal{T}$ of order n. We deduce that $\mathcal{T}_7$ consists of the graphs P_7 and T_2. For $3 \leq n \leq 6$ the path of length $n-1$ is the unique element in $\mathcal{T}_n$. The following result implies that for every CFPO $\preceq$ on $\mathbb{R}^n$ with $n \geq 4$ there is at most one graph in $\mathcal{T}_n$ which has a degree complete labeling with respect to $\preceq$.

Theorem 5.23
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with $n \geq 4$. If $T \in \mathcal{T}_n$ has a degree complete labeling with respect to $\preceq$, then the tree-representation of $\preceq$ is isomorphic to an orientation of T.

Proof
Let S be the tree-representation of $\preceq$ and $\tilde{S}$ be the underlying undirected tree of S. Furthermore, denote T' the tree we obtain from T by removing all of its leafs. Without loss of generality we identify T with its labeled version which is degree complete and we have $V(T) = V(S)$. By Corollary 5.15 there is a directed path P_{uv} connecting u and v in S for every edge $uv \in E(T)$. Moreover, $V(P_{uv})$ and $V(P_{wx})$ are disjoint for each pair of independent edges $uv, wx \in E(T)$.
Let u be a leaf of T and v its unique neighbor. Considering a vertex $w \in V(T) \setminus \{u, v\}$ we deduce that w is adjacent to a further vertex $x \in V(T) \setminus \{u, v, w\}$ since $n \geq 4$. From the fact that uv and wx are independent edges follows that $w \notin V(P_{uv})$. Since $V(T) = V(S)$ we conclude $V(P_{uv}) = \{u, v\}$ and therefore $uv \in E(\tilde{S})$.
For any edge $uv \in E(T')$ which is not incident to a leaf in T' we observe that every vertex in $V(T) \setminus \{u, v\}$ is an endpoint of an edge independent from uv. Thus $V(P_{uv}) = \{u, v\}$ and we deduce that $uv \in E(\tilde{S})$.
Finally, consider an edge $uv \in E(T')$ that is incident to a leaf u of T'. From the definition of $\mathcal{T}$ we obtain that $d_T(u) = 2$ and u is adjacent to a leaf $w \notin \{u, v\}$ of T. Denote x any vertex in $V(T) \setminus \{u, v, w\}$. If x has a further neighbor $y \in V(T') \setminus \{u, v\}$, then uv and xy are independent edges in T'. Otherwise x is adjacent to v and a leaf of T'. Hence $d_T(x) = 2$ and we deduce that there is a vertex $y \in V(T) \setminus \{u, v, w\}$ such that uv and xy are independent edges in T. Therefore, we conclude that $x \notin V(P_{uv})$ and $V(P_{uv}) \subseteq \{u, v, w\}$. Since $uw \in E(\tilde{S})$ and $\tilde{S}$ is a tree this implies either $vw \in E(\tilde{S})$ or $uw \in E(\tilde{S})$. In both cases the vertices u, v, and w induce a path of length 2 in $\tilde{S}$. Hence $\tilde{S}$ and T are isomorphic. □

Let G be a graph of order $n \geq 3$ which is degree complete with respect to some CFPO.

If $n \leq 6$, then Theorem 5.17 yields that G does not contain a subgraph isomorphic to *net* or C_k with $k \geq 4$. Since T_2 has 7 vertices we observe that G has a degree complete labeling with respect to $\preccurlyeq$ by Theorem 3.17. Therefore, G has a degree complete labeling with respect to a CFPO with a tree-representation that is isomorphic to an orientation of $P_n \in \mathcal{T}_n$. Particularly, we observe that all graphs of order n with $3 \leq n \leq 6$ have such a degree complete labeling with respect to the same CFPO. On the other hand Theorem 5.23 implies that for graphs with at least 7 vertices we obtain a different result.

Corollary 5.24
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with $n \geq 7$. There is at most one graph in $\mathcal{T}_n$ which has a degree complete labeling with respect to $\preceq$.

Hence for a given CFPO $\preceq$ with tree-representation T the graphs in $\mathcal{T}_n$ which are not isomorphic to the underlying undirected graph of T (or their subgraphs) yield candidates for further forbidden subgraphs. Notice that a path of length $n-1$ is the unique graph of order n in $\mathcal{T}$ that does not contain T_2 as a subgraph.
A further result underlines the observation that CFPOs whose tree-representations are isomorphic to an orientation of a graph in $\mathcal{T}_n$ play an important role in the class of CFPOs on $\mathbb{R}^n$. The next theorem can also be seen as an extension to Theorem 5.17 in the following sense. All graphs of order $n \geq 3$ which are characterized in this result have a degree complete labeling with respect to a LFPO of this particular class.

Theorem 5.25
Let G be a labeled graph of order $n \geq 3$ which is degree complete with respect to a LFPO. Then G is degree complete with respect to a LFPO $\preceq$ such that the tree-representation of $\preceq$ is isomorphic to an orientation of a graph in $\mathcal{T}_n$.

Proof
Let $\preceq$ be a LFPO on $\mathbb{R}^n$ such that G is degree complete with respect to $\preceq$. Denote T the tree-representation of $\preceq$ and $\tilde{T}$ its underlying undirected tree. Hence T is an in-tree. Moreover, Corollary 5.15 implies that for all $e \in E(G)$ there is a directed path P_e in T connecting the endpoints of e. If $\tilde{T} \in \mathcal{T}_n$ the assertion holds. Otherwise, if $\tilde{T} \notin \mathcal{T}_n$ there is a leaf u of $\tilde{T}$ which is adjacent to a vertex $v \in V(\tilde{T})$ with $d_{\tilde{T}}(v) \geq 3$. Since T is an in-tree we deduce that $d_T^-(v) \geq 2$.
First we show that we can assume that u has a positive neighbor in T. If u is the unique vertex of T with out-degree equal to 0, then we can construct an in-tree S satisfying conditions 1.) and 2.) from Corollary 5.15 and $d_S^+(u) = 1$ or $d_S^-(u) = 2$. Suppose that u is an isolated vertex in G or v is the only neighbor of u in G. We define a directed tree by $S = T - (v, u) + (u, v)$. It is not difficult to see that S is an in-tree with $d_S^+(v) = 0$ which fulfills conditions 1.) and 2.) of Corollary 5.15. Hence we suppose that u is adjacent to a vertex $w \in V(G - v)$. Let $x_1, \ldots, x_l$ be the negative neighbors of v in T. The directed path P_{uw} in T contains one of this vertices, say v_1. For every $e \in E(G)$ the directed path P_e contains an arc in $\{(x_2, v) \ldots, (x_l, v)\}$ only if e and u are incident. Hence $S = T - (x_i, v) + (x_i, u)$ with $2 \leq i \leq l$ is an in-tree with $d_S^-(u) = 2$. Moreover, S satisfies

conditions 1.) and 2.) of Corollary 5.15. Therefore, without loss of generality we assume that $(u, v) \in A(T)$.
In the following step we construct an in-tree T^* whose underlying undirected tree has less leafs that are not adjacent to a vertex of degree 2 than $\tilde{T}$ and which satisfies conditions 1.) and 2.) from Corollary 5.15. Since $d_T^+(v) \leq 1$ there is a negative neighbor $w \neq u$ of v in T. We define a directed tree by $T^* = T - (w, v) + (w, u)$. Obviously, T^* is an in-tree. For all $e \in E(G)$ denote Q_e the path in T^* connecting the endpoints of e. If (w, v) is not an arc in P_e we have $Q_e = P_e$. For every e with $(w, v) \in A(P_e)$ follows that (w, v) is replaced by $(w, u), (u, v)$ in Q_e. Therefore, Q_e is a directed path for all $e \in E(G)$. Suppose there is a pair of independent edges $e_1, e_2 \in E(G)$ such that both Q_{e_1} and Q_{e_2} are not vertex disjoint. From Corollary 5.15 follows that P_{e_1} and P_{e_2} are vertex disjoint in T. Hence Q_{e_1} and Q_{e_2} include an arc from $\{(u, v), (w, u)\}$. By the construction of T^* the arc (w, u) is contained in Q_{e_i} only if $(w, v) \in A(P_{e_1})$. Thus P_{e_1} and P_{e_2} have an arc in $\{(u, v), (w, v)\}$ contradicting the fact that P_{e_1} and P_{e_2} are vertex disjoint. Therefore, for every pair of independent edges $e_1, e_2 \in E(G)$ the paths Q_{e_1} and Q_{e_2} are vertex disjoint. We conclude that T^* fulfills conditions 1.) and 2.) of Corollary 5.15. Now, if the underlying tree of T^* is in $\mathcal{T}_n$ we are done. Otherwise we repeat the this step. In each step the number of leafs which are not adjacent to a vertex of degree 2 in the underlying undirected tree is reduced.
Altogether after a finite number of steps we obtain an in-tree T' such that its underlying undirected tree is in $\mathcal{T}_n$ and T' satisfies conditions 1.) and 2.) from Corollary 5.15. Denote $\preceq'$ a LFPO with tree-representation T'. By Corollary 5.15 the graph G is degree complete with respect to the LFPO $\preceq'$. □

The results in this section focus on the relation between the graphs which have a degree complete labeling with respect to a given CFPO $\preceq$ and the underlying tree of the tree-representation of $\preceq$. The following example shows that the orientation of the arcs in a tree-representation plays a significant role, too.

Example 5.26
Let $\tilde{T}$ be the tree defined in Figure 5.1. We consider the class of CFPOs on $\mathbb{R}^n$ whose tree-

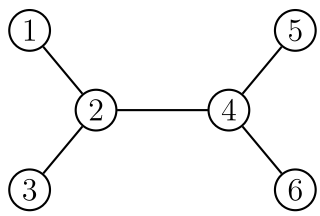


Figure 5.1: Tree $\tilde{T}$

representation is an orientation of $\tilde{T}$ and denote it by $\mathcal{A}$. Moreover, we define G_1 as the disjoint union of two triangles C and $\hat{C}$ with $V(C) = \{v_1, v_2, v_3\}$ and $V(\hat{C}) = \{w_1, w_2, w_3\}$. Suppose that G_1 has a degree complete labeling with respect to some CFPO $\preceq$ in $\mathcal{A}$. Let T be the tree-representation of $\preceq$. If the set of labels of $V(C)$ in a degree complete labeling

of G_1 with respect to $\preceq$ is not $\{1,2,3\}$ or $\{4,5,6\}$, then we can find edges $e \in E(C)$ and $\hat{e} \in E(\hat{C})$ such that P_e and $P_{\hat{e}}$ in $\tilde{T}$ are not disjoint. By Corollary 5.15 this contradicts the fact that the chosen labeling is degree complete with respect to $\preceq$. Hence $\{1,2,3\}$ and $\{4,5,6\}$ are the labels of C and $\hat{C}$ in a degree complete labeling of G_1. Therefore, both vertex sets induce directed paths in T. Particularly, the orientation T with arc set $A(T) = \{(1,2),(2,3),(2,4),(5,4),(4,6)\}$ induces such an CFPO. Furthermore, we deduce that the tree-representation of every CFPO with the property that G_1 has a degree complete labeling with respect to this CFPO is isomorphic to T.
Since T does not contain a vertex which is connected to every other vertex via a directed path in T we conclude that the star $K_{1,5}$ has not a degree complete labeling with respect to $\preceq$. On the other hand we observe that $K_{1,5}$ has a degree complete labeling with respect to any LFPO in $\mathcal{A}$.
This implies that G_1 and $K_{1,5}$ do not have a degree complete labeling with respect to the same CFPO in $\mathcal{A}$. Hence we cannot find a single CFPO $\preceq'$ in $\mathcal{A}$ such that every graph which has a degree complete labeling with respect to any CFPO in $\mathcal{A}$ has a degree complete labeling with respect to $\preceq'$.

It seems difficult to obtain a general result which takes into account the tree-representation as a directed graph. Therefore, we do not have a complete description of the graphs with degree complete labeling with respect to a given CFPO.

5.4 Results on lattices

In this section we show a further similarity between degree complete graphs in the sense of chapter 3 and graphs which are degree complete with respect to a CFPO. Consider a labeled graph G of order n and a CFPO $\preceq$ on $\mathbb{R}^n$. In analogy to Section 3.2 we prove that $(\mathcal{S}_{\preceq}(G), \preceq)$ is a lattice if G is degree complete with respect to $\preceq$. The following example shows that the restriction to degree complete graphs is necessary.

Example 5.27
Let G be the graph with $V(G) = \{1,2,3,4,5\}$ and $E(G) = \{\{1,2\},\{2,3\},\{3,4\}\}$. Moreover, let $\preceq$ be the CFPO with tree-representation T where $V(T) = V(G)$ and $A(T) = \{(1,5),(5,2),(3,5),(5,4)\}$. Hence we have $s \preceq t$ if and only if

$$t - s \in \mathcal{C}(\mathbf{z}_{(1,5)}, \mathbf{z}_{(5,2)}, \mathbf{z}_{(3,5)}, \mathbf{z}_{(5,4)})$$

by Theorem 4.7. For $a \in A(T)$ denotes λ_a the unique coefficient in $s - t = \sum_{a \in A(T)} \lambda_a \mathbf{z}_a$. Similar to Lemma 3.5 we show that there is an unique vector $r = t + \sum_{a \in A(T)} \max(0, \lambda_a)\mathbf{z}_a$ which satisfies $r \preceq \tilde{r}$ for all $\tilde{r}$ with $s \preceq \tilde{r}$ and $t \preceq \tilde{r}$. Thus r is the supremum of s and t with respect to $\preceq$ in $\mathbb{R}^5$. Considering the vectors $s = (0,1,1,1,0)^\top$ and $t = (0,2,1,0,0)^\top$ we observe that

$$s - t = (0,-1,0,1,0)^\top = \mathbf{z}_{(5,2)} - \mathbf{z}_{(5,4)}$$

and therefore $r = t + \mathbf{z}_{(5,2)} = (0,1,1,0,1)^\top$. It is easy to check that s and t are degree vectors of G but r is not an element of $\mathcal{S}_\preceq(G)$. On the other hand

$$p = (1,1,1,0,0)^\top = r + \mathbf{z}_{(1,5)} \qquad \text{and} \qquad q = (0,1,2,0,0)^\top = r + \mathbf{z}_{(3,5)}$$

are both elements of $\mathrm{DEG}^+(G)$ and hence in $\mathcal{S}_\preceq(G)$. Furthermore, we have $s, t \preceq p, q$. Since $\{\mathbf{z}_a\}_{a \in A(T)}$ is a set of linear independent vectors and $p - q = \mathbf{z}_{(1,5)} - \mathbf{z}_{(3,5)}$ and we deduce that $p \not\preceq q$ and $q \not\preceq p$. Therefore, the supremum of s and t in $\mathrm{DEG}^+(G)$ does not exist and we deduce that $(\mathrm{DEG}^+(G), \preceq)$ and $(\mathcal{S}_\preceq(G), \preceq)$ are not lattices. By Corollary 5.15 we conclude that G is not degree complete with respect to $\preceq$ since $P_{\{1,2\}}$ and $P_{\{3,4\}}$ both contain vertex 5.

In the remaining part of this section we consider graphs which are degree complete with respect to some CFPO. Moreover, by Observation 5.16 it is sufficient to study only simple graphs with this property.
Unfortunately, we cannot simply apply the results from Section 3.2. The next example shows that we have to find another definition for the supremum and the infimum of two degree vectors.

Example 5.28
Let G be the graph with $V(G) = \{1,2,3,4\}$ and $E(G) = \{\{1,3\},\{2,3\}\}$. Denote $\preceq$ the CFPO with tree-representation T where $V(T) = V(G)$ and $A(T) = \{(1,4),(2,4),(4,3)\}$. Thus for $s, t \in \mathbb{R}^4$ we have $s \preceq t$ if and only if $t - s \in \mathcal{C}(\mathbf{z}_{(1,4)}, \mathbf{z}_{(2,4)}, \mathbf{z}_{(4,3)})$. Similar to the previous example let r denote the supremum of $s = (1,0,1,0)^\top$ and $t = (0,1,1,0)^\top$ in $\mathbb{R}^4$ with respect to $\preceq$. From

$$s - t = (1,-1,0,0)^\top = \mathbf{z}_{(1,4)} - \mathbf{z}_{(2,4)}$$

we obtain

$$r = t + \mathbf{z}_{(1,4)} = (1,1,1,-1)^\top$$

which is obviously not a degree vector of G. On the other hand it is not difficult to show that $(\mathrm{DEG}^+(G), \preceq)$ is a lattice and $(1,1,0,0)^\top$ is the supremum of s and t in $\mathrm{DEG}^+(G)$.

Our next goal is to give a definition for the supremum and the infimum of two degree vectors which is based on the corresponding orientations of these vectors. Therefore, we need some preparation. The following two results are necessary to show that our definition does not depend on the orientations we choose. The next lemma contains an useful observation which simplifies the proofs.

Lemma 5.29
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ with tree-representation T and G a simple graph which is degree complete with respect to $\preceq$. Moreover, let D be an orientation of G and $(u,v) \in A(D)$ such that $\mathbf{z}_{(u,v)} \in \mathcal{C}_\preceq$. For $a \in A(D)$ and $b \in A(T)$ denotes $\lambda_{a,b}$ the unique integer with

$$\mathbf{z}_a = \sum_{b \in A(T)} \lambda_{a,b} \cdot \mathbf{z}_b.$$

If there is a set of arcs $\tilde{A} \subseteq A(D)$ such that for all $b \in A(P_{uv})$ exists an $a \in \tilde{A}$ satisfying $\lambda_{a,b} < 0$, then D has a (v,u)-path of length 2 which consists of arcs from $\tilde{A}$. Similarly, if there is a set of arcs $\tilde{A} \subseteq A(D)$ such that for all $b \in A(P_{uv})$ exists an $a \in \tilde{A}$ satisfying $\lambda_{a,b} > 0$, then D has a (u,v)-path of length 2 which consists of arcs from $\tilde{A}$.

Proof
Let $P_{uv} = v_0 a_1 v_1 a_2 \dots a_k v_k$ with $v_0 = u$ and $v_k = v$. For each index l with $1 \leq l \leq k$ consider an arc $(x,y) \in \tilde{A}$ such that $\lambda_{(x,y),a_l} < 0$. Corollary 5.15 implies that $\lambda_{(x,y),a} \leq 0$ holds for all $a \in A(P_{xy})$ because otherwise P_{xy} is not directed. Thus P_{xy} is a (y,x)-path in T containing the arc a_l. From Corollary 5.15 we also deduce that (u,v) and (x,y) are incident in D. Since P_{uv} and P_{xy} are unique in T we either have $x = v$ or $y = u$. Denote i the largest index such that there is $w \in V(D)$ satisfying $(w,u) \in \tilde{A}$ and $\lambda_{(w,u),a_i} < 0$. Similarly, let j be the smallest index such that there is $z \in V(D)$ satisfying $(v,z) \in \tilde{A}$ with $\lambda_{(v,z),a_j} < 0$. From the fact that G is simple we obtain that such indices exist. Furthermore, we conclude that $i + 1 \geq j$ because for every l with $1 \leq l \leq k$ there is an $a \in \tilde{A}$ satisfying $\lambda_{a,a_l} < 0$. Hence P_{uw} and P_{vz} are not vertex disjoint. By Corollary 5.15 the arcs (w,u) and (v,z) are incident. Therefore, we have $w = z$ and $(w,u),(v,w) \in \tilde{A}$. Thus $\tilde{A}$ contains the arcs of a (v,u)-path of length 2 in D.
If there is a set of arcs $\tilde{A} \subseteq A(D)$ such that for all $b \in A(P_{uv})$ exists an $a \in \tilde{A}$ satisfying $\lambda_{a,b} > 0$, then an analogue deduction yields the existence of a vertex $w \in V(D)$ with $(u,w),(w,v) \in \tilde{A}$. □

For a degree vector $s \in \mathrm{DEG}^+(G)$ denote $\mathcal{D}_G(s)$ the set of all orientations of G which have out-degree vector s. Let s and t be two degree vectors of a simple graph G that is degree complete with respect to some CFPO $\preceq$. Suppose there are orientations $D_s \in \mathcal{D}_G(s)$ and $D_t \in \mathcal{D}_G(t)$ such that for all $a \in A(D_s) \setminus A(D_t)$ holds $\mathbf{z}_a \in -\mathcal{C}_{\preceq}$, that is, there are coefficients $\lambda_{a,b} \leq 0$ with $\mathbf{z}_a = \sum_{b \in A(T)} \lambda_{a,b} \mathbf{z}_b$. From

$$t - s = -\sum_{a \in A(D_s) \setminus A(D_t)} \mathbf{z}_a = \sum_{b \in A(T)} \underbrace{\left(\sum_{a \in A(D_s) \setminus A(D_t)} (-\lambda_{a,b}) \right)}_{\geq 0} \mathbf{z}_b$$

we deduce $s \preceq t$. The following Lemma implies that the reverse statement holds.

Lemma 5.30
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ and G a simple graph which is degree complete with respect to $\preceq$. Moreover, let s and t be two degree vectors of G such that $s \preceq t$. For every $D_t \in \mathcal{D}_G(t)$ there is an orientation $D_s \in \mathcal{D}_G(s)$ such that for all $a \in A(D_s) \setminus A(D_t)$ holds $\mathbf{z}_a \in -\mathcal{C}_{\preceq}$.

Proof
Denote T the tree-representation of $\preceq$. For a shorter notation we define $\tilde{A} = A(D_s) \setminus A(D_t)$. By Corollary 5.15 every arc $a \in \tilde{A}$ is either in

$$\tilde{A}^+ = \{a \in \tilde{A} \,|\, \mathbf{z}_a \in \mathcal{C}_{\preceq}\} \quad \text{or} \quad \tilde{A}^- = \{a \in \tilde{A} \,|\, \mathbf{z}_a \in -\mathcal{C}_{\preceq}\}.$$

In particular, A^+ contains those arcs (u,v) such that P_{uv} is a (u,v)-path in T. Analogously, (u,v) is contained in A^- if there is a (v,u)-path in T. Let $D_s \in \mathcal{D}_G(s)$ such that $|\tilde{A}^+|$ is minimal. We prove that $\tilde{A}^+$ is empty. Suppose to the contrary that there is an arc $(u,v) \in \tilde{A}^+$ and denote $P = v_0 a_1 v_1 a_2 \dots a_k v_k$ the (u,v)-path in T with $v_0 = u$ and $v_k = v$. For every arc $a \in A(D)$ there are unique integers $\lambda_{a,b}$ such that

$$\mathbf{z}_a = \sum_{b \in A(T)} \lambda_{a,b} \cdot \mathbf{z}_b.$$

We obtain D_t from D_s by reorienting all arcs in $\tilde{A}$ hence we have

$$t - s = -\sum_{a \in \tilde{A}} \mathbf{z}_a = -\sum_{a \in \tilde{A}} \sum_{b \in A(T)} \lambda_{a,b} \cdot \mathbf{z}_b = \sum_{b \in A(T)} \left(-\sum_{a \in \tilde{A}} \lambda_{a,b} \right) \cdot \mathbf{z}_b.$$

From $s \preceq t$ follows that $\sum_{a \in \tilde{A}} \lambda_{a,b} \leq 0$ for all $b \in A(T)$. Furthermore, for all $a_i \in A(P)$ we deduce $\lambda_{(u,v),a_i} > 0$. Thus for all $a_i \in A(P)$ there is an arc $a \in \tilde{A} \setminus \{(u,v)\}$ such that $\lambda_{a,a_i} < 0$. By Lemma 5.29 there is a vertex $w \in V(D_s)$ such that $(v,w),(w,u) \in \tilde{A}$. Therefore, u,v,w induce a directed cycle C of length 3 in D_s. Similarly, there is directed 3-cycle in D_t with vertex set $V(C)$ but a different orientation. Consider the digraph D'_s we obtain from D_s be reorienting the arcs in $A(C)$. Obviously, we have $D'_s \in \mathcal{D}_G(s)$ and

$$A(D'_s) \setminus A(D_t) = \tilde{A} \setminus \{(u,v),(v,w),(w,u)\}.$$

Since an arc $a \in A(D'_s) \setminus A(D_t)$ with $\mathbf{z}_a \in \mathcal{C}_{\preceq}$ is also included in $\tilde{A}$ we have a contradiction to our choice of D_s. □

From Observation 2.4 follows that we obtain every orientation in $\mathcal{D}_G(s)$ by choosing an orientation with degree vector s where some directed cycles are reoriented. For $s,t \in \mathrm{DEG}^+(G)$ a pair of orientations $(D_s,D_t) \in \mathcal{D}_G(s) \times \mathcal{D}_G(t)$ is called *minimal* if

$$|A(D_s) \setminus A(D_t)| \leq |A(\tilde{D}_s) \setminus A(\tilde{D}_t)|$$

holds for all pairs $(\tilde{D}_s,\tilde{D}_t) \in \mathcal{D}_G(s) \times \mathcal{D}_G(t)$. The next lemma implies that the set $A(D_s) \setminus A(D_t)$ is equal for every minimal pair of orientations $(D_s,D_t) \in \mathcal{D}_G(s) \times \mathcal{D}_G(t)$.

Lemma 5.31
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ and G a simple graph which is degree complete with respect to $\preceq$. Moreover, let s and t be two degree vectors of G. Two minimal pairs of orientations $(D_s,D_t),(\tilde{D}_s,\tilde{D}_t) \in \mathcal{D}_G(s) \times \mathcal{D}_G(t)$ satisfy

$$A(D_s) \setminus A(D_t) = A(\tilde{D}_s) \setminus A(\tilde{D}_t).$$

Proof
First, notice that by Theorem 5.17 every cycle in G is a triangle and every edge is part of at most one triangle. We use the notations $B = A(D_s) \setminus A(D_t)$ and $\tilde{B} = A(\tilde{D}_s) \setminus A(\tilde{D}_t)$.

Hence we obtain D_t (respectively $\tilde{D}_t$) from D_s (respectively $\tilde{D}_s$) by reorienting the arcs in B (respectively $\tilde{B}$). Suppose $A(D_s) \neq A(\tilde{D}_s)$. Every arc $a \in A(D_s) \setminus A(\tilde{D}_s)$ is contained in a directed cycle of D_s and therefore a is part of a directed triangle C in D_s. Moreover, the fact that (D_s, D_t) is a minimal pair yields that at most one arc of C is included in B. Denote V_C the vertex set of C. Hence $D_t[V_C]$ is an acyclic orientation of a triangle. Observe that we obtain $\tilde{D}_t$ from D_t by reorienting the arcs of a collection of arc-disjoint directed 3-cycles. Thus $\tilde{D}_t$ has the same orientation of the subgraph induced by V_C as in D_t. We conclude that $\tilde{B}$ contains two arcs of $\tilde{D}_s[V_C]$. This is a contradiction to the minimality of the pair $(\tilde{D}_s, \tilde{D}_t)$ because a reorientation of the arcs in $\tilde{D}_s[V_C]$ yields an orientation $D \in \mathcal{D}_G(s)$ with

$$|\tilde{B}| - 1 = A(D) \setminus A(\tilde{D}_t).$$

Therefore, we have $A(D_s) = A(\tilde{D}_s)$.
Now, suppose $A(D_t) \neq A(\tilde{D}_t)$. An analogue deduction shows that every arc in $A(D_t) \setminus A(\tilde{D}_t)$ is part of a directed 3-cycle C' in D_t. Furthermore, in D_s and $\tilde{D}_s$ the vertices of C' induce the same acyclic orientation of a triangle. Again we deduce that $(\tilde{D}_s, \tilde{D}_t)$ is not a minimal pair and thus we have a contradiction. We conclude that $A(D_t) = A(\tilde{D}_t)$. □

For $s, t \in \mathrm{DEG}^+(G)$ we define $\tilde{A}_{s,t} = A(D_s) \setminus A(D_t)$ for a minimal pair (D_s, D_t) of orientations in $\mathcal{D}_G(s) \times \mathcal{D}_G(t)$. By Lemma 5.31 the elements in $\tilde{A}_{s,t}$ only depend on the choices of s and t and not on the orientations themselfes. As in the proof of Lemma 5.30 we use the notations

$$\tilde{A}^+_{s,t} = \{a \in \tilde{A}_{s,t} \mid z_a \in \mathcal{C}_\preceq\} \quad \text{and} \quad \tilde{A}^-_{s,t} = \{a \in \tilde{A}_{s,t} \mid z_a \in -\mathcal{C}_\preceq\}.$$

Finally, we define

$$\sup(s,t) = t + \sum_{a \in \tilde{A}^+_{s,t}} \mathbf{z}_a = s - \sum_{a \in \tilde{A}^-_{s,t}} \mathbf{z}_a$$

and

$$\inf(s,t) = t + \sum_{a \in \tilde{A}^-_{s,t}} \mathbf{z}_a = s - \sum_{a \in \tilde{A}^+_{s,t}} \mathbf{z}_a.$$

With these definitions we are able to prove that $(\mathrm{DEG}^+(G), \preceq)$ is a lattice if G is degree complete with respect to a CFPO $\preceq$.

Theorem 5.32
Let $\preceq$ be a CFPO on $\mathbb{R}^n$ and G a simple graph of order n. If G is degree complete with respect to $\preceq$, then $(\mathcal{S}_\preceq(G), \preceq)$ is a lattice.

Proof
Denote T the tree-representation of $\preceq$. Let s and t be two degree vectors of G and $(D_s, D_t) \in \mathcal{D}_G(s) \times \mathcal{D}_G(t)$ a minimal pair of orientations of G. From the definitions of $\sup(s,t)$ and $\inf(s,t)$ immediately follows that

$$\inf(s,t) \preceq s, t \preceq \sup(s,t).$$

Moreover, there are orientations $D_{\sup}$ and $D_{\inf}$ of G with out-degree vectors $\sup(s,t)$ and $\inf(s,t)$. It suffices to show that for every $r \in \mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$ satisfying $s \preceq r$ and $t \preceq r$ (respectively $r \preceq s$ and $r \preceq t$) holds $\sup(s,t) \preceq r$ (respectively $r \preceq \inf(s,t)$). Let $D_{\sup}$ be the orientation of G which is obtained by reorienting the arcs from $\tilde{A}^-_{s,t}$ in D_s. Hence reversing all arcs from $\tilde{A}^+_{s,t}$ in $D_{\sup}$ yields D_t. Denote D_r an orientation of G with out-degree vector r. If for all $a \in A(D_{\sup}) \setminus A(D_r)$ holds $\mathbf{z}_a \in -\mathcal{C}_{\preceq}$ we deduce that

$$\sup(s,t) - r = \sum_{a \in A(D_{\sup}) \setminus A(D_r)} \mathbf{z}_a \in -\mathcal{C}_{\preceq}$$

and therefore $\sup(s,t) \preceq r$. Suppose there is an arc $(u,v) \in A(D_{\sup}) \setminus A(D_r)$ such that $\mathbf{z}_{(u,v)} \in \mathcal{C}_{\preceq}$. Since $\tilde{A}_{s,t}$ only contains arcs which have different directions in D_s and D_t we deduce that $(u,v) \in A(D_s)$ or $(u,v) \in A(D_t)$. Without loss of generality assume that $(u,v) \in A(D_s)$. We consider the set of arcs $A(D_r) \setminus A(D_s)$ which have to be reversed to obtain D_s from D_r. Notice that $s \preceq r$ implies

$$r - s = \sum_{a \in A(D_r) \setminus A(D_s)} \mathbf{z}_a \in \mathcal{C}_{\preceq}. \tag{5.4}$$

Because $(v,u) \in A(D_r) \setminus A(D_s)$ and $\mathbf{z}_{(v,u)} = -\mathbf{z}_{(u,v)} \in -\mathcal{C}_{\preceq}$ we conclude that (v,u) is contained in a directed cycle C in D_r. By Theorem 5.17 the cycle C is a directed 3-cycle and thus there is a vertex $w \in V(D_r)$ such that (u,w) and (w,v) are both included in $A(D_r)$. Observe that $\{(v,u),(u,w),(w,v)\} \subseteq A(D_r) \setminus A(D_s)$ by (5.4). Therefore, the set $V(C)$ induces a directed 3-cycle C' in D_s with opposite direction to C in D_r.
Now, consider two cases. If $(u,v) \in A(D_t)$, then by the same argument as before we deduce that $V(C)$ induces a directed 3-cycle in D_t with opposite direction to C in D_r. Hence the arcs in $\{(u,v),(v,w),(w,u)\}$ are contained in $A(D_s)$ and $A(D_t)$. From the definitions of $\tilde{A}_{s,t}$ and $D_{\sup}$ follows that $\{(u,v),(v,w),(w,u)\} \subseteq A(D_{\sup})$.
If $(v,u) \in A(D_t)$, then (v,w) and (w,u) are arcs of D_t since (D_s, D_t) is a minimal pair of orientations of G and thus (u,v) is the unique arc from C' in $\tilde{A}_{s,t}$. Hence we deduce $\{(u,v),(v,w),(w,u)\} \subseteq A(D_{\sup})$.
Altogether we conclude that $V(C)$ induces a directed 3-cycle in $D_{\sup}$ with opposite direction to C in D_r. Now, Theorem 5.17 implies that all cycles in G are edge-disjoint. Therefore, by reversing all these cycles in D_r we obtain an orientation D'_r of G with out-degree vector r and $A(D_{\sup}) \setminus A(D'_r) = \emptyset$. Thus $\sup(s,t) \preceq r$. By an analogue deduction we show that $r \preceq s$ and $r \preceq t$ imply $r \preceq \inf(s,t)$. Hence $(\mathcal{S}_{\preceq}(G), \preceq)$ is a lattice. □

Let G be a labeled graph that is degree complete with respect to a CFPO $\preceq$. Beside this lattice property of $(\mathcal{S}_{\preceq}(G), \preceq)$ we also notice that there is a procedure constructing the predecessors of an element of $\mathcal{S}_{\preceq}(G)$ similar to the construction in section 3.2.

6 Degree completeness for cone orders

In the previous sections we have defined CFPOs, that is a class of partial orders which are induced by cross-free families. Furthermore, we have proved a characterization of those graphs which are degree complete with respect to some CFPO. In particular, Theorem 5.17 implies that for every tree T there is a CFPO $\preceq$ such that T is degree complete with respect to $\preceq$. On the other hand we notice that for graphs containing *net* or C_k, $k \geq 4$ we still have not found a partial order with this property. In this chapter we characterize the graphs which are degree complete with respect to some cone order. Remind at this point that all partial orders appeared so far are cone orders. Moreover, many useful partial orders which are not considered here belong to the class of cone orders.
In the first section we reformulate the concept of degree complete graph as a property of the degree vector polytope. Furthermore, we obtain a result which allows us to reduce our investigations on particular cone orders. In the second section we prove that only forests are strongly degree complete with respect to a cone order. Finally, the main theorem of the third section shows that the class of graphs which are degree complete with respect to some cone order is only a slight extension of the class of graphs which are characterized in Theorem 5.17. This fact is remarkable since the class of cone orders includes every partial order which is compatible with the vector space structure of $\mathbb{R}^n$.

6.1 Degree completeness as a property of the degree vector polytope

In analogy to the definition of degree completeness with respect to a CFPO we define graphs that are degree complete with respect to a cone order. We use the notations of $\mathcal{S}_{\preceq}(G)$ and $\mathcal{S}'_{\preceq}(G)$ from Definition 5.3.

Definition 6.1
Let $\preceq$ be a cone order on $\mathbb{R}^n$ and G a labeled graph of order n. The graph G is *degree complete with respect to* $\preceq$ if it satisfies

$$|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1 \quad \text{and} \quad \mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G).$$

Analogously, G is *strongly degree complete with respect to* $\preceq$ if we have

$$|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| = 1 \quad \text{and} \quad \mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G).$$

The arguments in this chapter have a geometric motivation and we apply several properties of the degree vector polytope. Especially, the characterizations of the vertices, edges and facets from Theorem 2.8 are important. We start with a geometric interpretation of the concept of degree completeness.
Let G be a labeled graph of order n. Since all degree vectors are included in the affine subspace

$$\mathcal{M}_G = \{x \in \mathbb{R}^n \mid x(V(G)) = m(G)\}$$

we only have to consider those cones which are contained in the corresponding linear subspace $\mathcal{M}_n$. Let $\preceq$ be a partial order induced by a cone $\mathcal{C} \subseteq \mathcal{M}_n$. Suppose there are unique degree vectors $s_G^{\min}, s_G^{\max} \in \mathrm{DEG}^+(G)$ such that $s_G^{\min} \preceq s \preceq s_G^{\max}$ holds for all $s \in \mathrm{DEG}^+(G)$. Thus we have

$$\mathcal{S}'_{\preceq}(G) = \left\{s \in \mathbb{Z}^n \,|\, s_G^{\min} \preceq s \preceq s_G^{\max}\right\}.$$

Now, consider the set of vectors we obtain by omitting the integrality condition in $\mathcal{S}'_{\preceq}(G)$, that is, we define

$$\mathcal{R}'_{\preceq}(G) = \left\{s \in \mathbb{R}^n \,|\, s_G^{\min} \preceq s \preceq s_G^{\max}\right\}.$$

From the definition of cone orders follows that for every $s \in \mathcal{R}'_{\preceq}(G)$ holds $s - s_G^{\min} \in \mathcal{C}$ and $s_G^{\max} - s \in \mathcal{C}$. Therefore, we obtain an equivalent formulation by

$$\mathcal{R}'_{\preceq}(G) = \left(s_G^{\min} + \mathcal{C}\right) \cap \left(s_G^{\max} - \mathcal{C}\right).$$

Here $+$ and $-$ denote the Minkowski sum. By $\mathrm{DEG}^+(G) = \mathcal{P}_G \cap \mathbb{Z}^n$ and $\mathcal{S}'_{\preceq}(G) = \mathcal{R}'_{\preceq}(G) \cap \mathbb{Z}^n$ we make the following observation. The assumptions on G and $\mathcal{C}$ yield $\mathrm{DEG}^+(G) \subseteq \mathcal{S}'_{\preceq}(G)$ and $\mathcal{P}_G = \mathrm{conv}(\mathrm{DEG}^+(G)) \subseteq \mathcal{R}'_{\preceq}(G)$. Hence the degree vector polytope is included in a set which can be described as the intersection of two opposed affine cones where the point of each cone is a vertex of $\mathcal{P}_G$. Moreover, we obtain a geometric idea of the concept of strongly degree complete graphs. In particular, we have $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ if and only if $\mathcal{P}_G = \mathrm{conv}(\mathcal{R}'_{\preceq}(G) \cap \mathbb{Z}^n)$. Because $s_G^{\min}$ and $s_G^{\max}$ are integral vectors the sets

$$\mathrm{conv}\left(\left(s_G^{\min} + \mathcal{C}\right) \cap \mathbb{Z}^n\right) \quad \text{and} \quad \mathrm{conv}\left(\left(s_G^{\max} - \mathcal{C}\right) \cap \mathbb{Z}^n\right)$$

are closed, convex, pointed cones. Hence there is a cone $\mathcal{C}' \subseteq \mathcal{C}$ which induces a cone order $\preceq'$ such that $\mathcal{P}_G = \mathcal{R}'_{\preceq'}(G)$. Therefore, the existence of a cone order $\preceq$ such that $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ can be translated into the property that $\mathcal{P}_G$ can be completely described by the intersection of two opposed affine cones.
For our investigations on graphs which are degree complete with respect to some cone order we have to consider the set

$$\mathcal{S}_{\preceq}(G) = \mathcal{S}'_{\preceq}(G) \cap \{s \in \mathbb{Z}^n \,|\, 0 \le s \le d_G\}.$$

The corresponding relaxation in analogy to $\mathcal{S}'_{\preceq}(G)$ and $\mathcal{R}'_{\preceq}(G)$ is

$$\mathcal{R}_{\preceq}(G) = \mathcal{R}'_{\preceq}(G) \cap \{s \in \mathbb{R}^n \,|\, 0 \le s \le d_G\}.$$

Hence if a graph G satisfies $\mathrm{DEG}^+(G) = \mathcal{S}_{\preceq}(G)$, then we can find a cone order $\preceq'$ satisfying $\mathcal{P}_G = \mathcal{R}_{\preceq'}(G)$ and $\mathcal{P}_G$ is the intersection of two opposed affine cones and a box. Therefore, the graph property degree completeness with respect to $\preceq$ translates into a property of the "shape" of the degree vector polytope of G. The advantage of a consideration of $\mathcal{P}_G$ is that its shape does not depend on the labeling of the vertices in G. Different vertex labelings only lead to a permutation of the coordinates. Thus it suffices to consider a single labeled version of every unlabeled graph.
We formalize this idea. Especially, we have to study how the degree vectors $s_G^{\min}$ and $s_G^{\max}$ should be chosen. We need the following definitions.

Definition 6.2
Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a cone and $\underline{x}, \overline{x}, d \in \mathbb{R}^n$ with $d \geq 0$ We define the sets

$$\mathcal{R}'_{\mathcal{C}}(\underline{x}, \overline{x}) = (\underline{x} + \mathcal{C}) \cap (\overline{x} - \mathcal{C})$$

and

$$\mathcal{R}_{\mathcal{C},d}(\underline{x}, \overline{x}) = \mathcal{R}'_{\mathcal{C}}(\underline{x}, \overline{x}) \cap \{x \in \mathbb{R}^n \,|\, 0 \leq x \leq d\}.$$

Suppose $\preceq$ is the cone order induced by a cone $\mathcal{C} \subseteq \mathbb{R}^n$. The definition implies that for $x \in \mathbb{R}^n$ holds $x \in \mathcal{R}'_{\mathcal{C}}(\underline{x}, \overline{x})$ if and only if $\underline{x} \preceq x \preceq \overline{x}$.
Let G be a labeled graph of order n and $\mathcal{C} \subseteq \mathcal{M}_n$ a cone. In the following results we assume that there are degree vectors $s_G^{\min}$ and $s_G^{\max}$ in $\mathrm{DEG}^+(G)$ such that $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}}(s_G^{\min}, s_G^{\max})$. This assumption ensures that every degree vector s satisfies $s_G^{\min} \preceq s \preceq s_G^{\max}$, where $\preceq$ is the cone order induced by $\mathcal{C}$.
The first proposition gives us useful informations on the extremal degree vectors $s_G^{\min}$ and $s_G^{\max}$.

Proposition 6.3
Let G be a labeled graph of order n and $\mathcal{C} \subseteq \mathcal{M}_n$ a cone. If there are $s_G^{\min}$ and $s_G^{\max}$ in $\mathrm{DEG}^+(G)$ such that $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}}(s_G^{\min},\, s_G^{\max})$, then the following statements hold.

(i) The vectors $s_G^{\min}$ and $s_G^{\max}$ are vertices of $\mathcal{P}_G$. By Theorem 2.8 this assertion implies that an orientation of G with out-degree vector $s_G^{\min}$ or $s_G^{\max}$ is acyclic and thus unique.

(ii) Denote $D_G^{\min}$ and $D_G^{\max}$ orientations of G realizing $s_G^{\min}$ and $s_G^{\max}$, respectively. Every spanning subgraph H of G induces a spanning subdigraph $D_H^{\min}$ (respectively $D_H^{\max}$) of $D_G^{\min}$ (respectively $D_G^{\max}$). The out-degree vectors of $D_H^{\min}$ and $D_H^{\max}$ denoted by $s_H^{\min}$ and $s_H^{\max}$ satisfy
$$\mathcal{P}_H \subseteq \mathcal{R}'_{\mathcal{C}}(s_H^{\min},\, s_H^{\max}).$$

Proof

(i) Since $s_G^{\min} \in \mathrm{DEG}^+(G)$ there are x and y in $\mathrm{vert}(\mathcal{P}_G)$ and $\lambda \in [0,1]$ such that $s_G^{\min} = \lambda x + (1-\lambda) y$. Hence we deduce

$$0 = s_G^{\min} - s_G^{\min} = \lambda(x - s_G^{\min}) + (1-\lambda)(y - s_G^{\min}).$$

Now, $\mathcal{P}_G \subseteq (s_G^{\min} + \mathcal{C})$ implies that $x - s_G^{\min}$ and $y - s_G^{\min}$ are contained in $\mathcal{C}$. Since $\mathcal{C}$ is a pointed cone we conclude that $x = y = s_G^{\min}$ and thus $s_G^{\min} \in \operatorname{vert}(\mathcal{P}_G)$. An analog deduction holds for $s_G^{\max}$.

(ii) We define the (spanning) subgraph $\tilde{H} = G - E(H)$ of G, that is $G = H + \tilde{H}$. Similar to $D_H^{\min}$ and $D_H^{\max}$ the subgraph $\tilde{H}$ induces subdigraphs $D_{\tilde{H}}^{\min}$ and $D_{\tilde{H}}^{\max}$ of $D_G^{\min}$ and $D_G^{\max}$, respectively. Let $s_{\tilde{H}}^{\min}$ and $s_{\tilde{H}}^{\max}$ be the corresponding out-degree vectors. From $\mathcal{P}_G = \mathcal{P}_H + \mathcal{P}_{\tilde{H}}$ we deduce that for every $x \in \mathcal{P}_H$ the vectors $x + s_{\tilde{H}}^{\min}$ and $x + s_{\tilde{H}}^{\max}$ are contained in $\mathcal{P}_G$. Since $\mathcal{P}_G \subseteq (s_G^{\min} + \mathcal{C})$ we obtain

$$x - s_H^{\min} = (x + s_{\tilde{H}}^{\min}) - (s_H^{\min} + s_{\tilde{H}}^{\min}) = (x + s_{\tilde{H}}^{\min}) - s_G^{\min} \in \mathcal{C}.$$

Similarly, from $\mathcal{P}_G \subseteq (s_G^{\max} - \mathcal{C})$ follows that

$$x - s_H^{\max} = (x + s_{\tilde{H}}^{\max}) - (s_H^{\max} + s_{\tilde{H}}^{\max}) = (x + s_{\tilde{H}}^{\max}) - s_G^{\max} \in -\mathcal{C}.$$

We conclude that $x \in (s_H^{\min} + \mathcal{C}) \cap (s_H^{\max} - \mathcal{C}) = \mathcal{R}'_{\mathcal{C}}(s_H^{\min},\, s_H^{\min})$.

□

The next proposition shows that the vector $s_G^{\max}$ is determined by $s_G^{\min}$.

Proposition 6.4
Let G be a labeled graph of order n and $\mathcal{C} \subseteq \mathcal{M}_n$ a cone. If there are vectors $s_G^{\min}$ and $s_G^{\max}$ in $\mathrm{DEG}^+(G)$ such that $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}}(s_G^{\min},\, s_G^{\max})$, then $s_G^{\min} + s_G^{\max} = d_G$.

Proof
Since $s_G^{\min} \in \mathrm{DEG}^+(G)$ there is an orientation D of G realizing $s_G^{\min}$. The vector $d_G - s_G^{\min}$ is the out-degree vector of the orientation of G we obtain by reversing every arc of D. Hence we deduce $d_G - s_G^{\min} \in \mathrm{DEG}^+(G)$ and $d_G - s_G^{\min} \in (s_G^{\max} - \mathcal{C})$. Analogously, $d_G - s_G^{\max}$ is a degree vector of G and we have $d_G - s_G^{\max} \in (s_G^{\min} + \mathcal{C})$. Thus $d_G - (s_G^{\min} + s_G^{\max})$ is contained in $\mathcal{C} \cap (-\mathcal{C})$. Since $\mathcal{C}$ is pointed we conclude $s_G^{\min} + s_G^{\max} = d_G$. □

In Lemma 5.4 we proved that for every cone order $\preceq$ the property $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ implies $\mathrm{DEG}^+(H) = \mathcal{S}'_{\preceq}(H)$ for every spanning subgraph H of G. Furthermore, the same holds if we replace $\mathcal{S}'_{\preceq}(G)$ by $\mathcal{S}_{\preceq}(G)$. The following results can be seen as a geometric version of this result. In particular, it shows that from $\mathcal{P}_G = \mathcal{R}'_{\mathcal{C}}(s_G^{\min}, s_G^{\max})$ we obtain $\mathcal{P}_H = \mathcal{R}'_{\mathcal{C}}(s_H^{\min}, s_H^{\max})$ and analogously from $\mathcal{P}_G = \mathcal{R}_{\mathcal{C},d_G}(s_G^{\min}, s_G^{\max})$ follows $\mathcal{P}_H = \mathcal{R}_{\mathcal{C},d_H}(s_H^{\min}, s_H^{\max})$. Remind that $s_H^{\min}$ (respectively $s_H^{\max}$) is determined by $s_G^{\min}$ (respectively $s_G^{\max}$) if $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}}(s_G^{\min}, s_G^{\max})$.

Lemma 6.5
Let G be a labeled graph and H a spanning subgraph of G and $\mathcal{C} \subseteq \mathcal{M}_n$ a cone. Moreover, let $s_G^{\min}$ and $s_G^{\max}$ be vectors in $\operatorname{vert}(\mathcal{P}_G)$ such that $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}}(s_G^{\min}, s_G^{\max})$.

(i) If $\mathcal{P}_G = \mathcal{R}_{\mathcal{C},d_G}(s_G^{\min}, s_G^{\max})$, then $\mathcal{P}_H = \mathcal{R}_{\mathcal{C},d_H}(s_H^{\min}, s_H^{\max})$ holds.

(ii) If $\mathcal{P}_G = \mathcal{R}'_{\mathcal{C}}(s_G^{\min}, s_G^{\max})$, then $\mathcal{P}_H = \mathcal{R}'_{\mathcal{C}}(s_H^{\min}, s_H^{\max})$ holds.

Proof
Let V be the vertex set of G. We prove statement (i) by induction on $k = |E(G) \setminus E(H)|$. If $k = 0$, then the assertion trivially holds. Thus we assume that the statement is true for $k - 1$ with $k \geq 1$.
Let $uv \in E(G) \setminus E(H)$ and consider the spanning subgraph $\tilde{H} = H + uv$ of G. By induction hypothesis the vectors $s_{\tilde{H}}^{\min}$ and $s_{\tilde{H}}^{\max}$ satisfy the condition $\mathcal{P}_{\tilde{H}} = \mathcal{R}_{\mathcal{C},d_{\tilde{H}}}(s_{\tilde{H}}^{\min}, s_{\tilde{H}}^{\max})$. Denote $D_{\tilde{H}}^{\min}$ and $D_{\tilde{H}}^{\max}$ the orientations of $\tilde{H}$ with out-degree vectors $s_{\tilde{H}}^{\min}$ and $s_{\tilde{H}}^{\max}$, respectively. By Propositions 6.3 and 6.4 we obtain $D_{\tilde{H}}^{\max}$ from $D_{\tilde{H}}^{\min}$ by reversing all arcs in $A(D_{\tilde{H}}^{\min})$. Without loss of generality we assume that $(u, v) \in A(D_{\tilde{H}}^{\min})$. Otherwise, we interchange u and v in the subsequent argumentation. Now, consider the vectors $t^{\min}$ and $t^{\max}$ defined by

$$t^{\min} = s_{\tilde{H}}^{\min} - \mathbf{e}_u \quad \text{and} \quad t^{\max} = s_{\tilde{H}}^{\max} - \mathbf{e}_v.$$

Obviously, $t^{\min}$ is the out-degree vector of $D_{\tilde{H}}^{\min} - (u, v)$ and $t^{\max}$ is the out-degree vector of $D_{\tilde{H}}^{\max} - (v, u)$. Hence $t^{\min}$ and $t^{\max}$ are degree vectors of H. In the following we prove that $\mathcal{P}_H = \mathcal{R}_{\mathcal{C},d_H}(t^{\min}, t^{\max})$. Let t be an arbitrary vector from $\mathcal{P}_H$. We consider $t + \mathbf{e}_u$. For $X \subseteq V$ follows that, if $u \in X$, then

$$(t + \mathbf{e}_u)(X) = t(X) + 1 \geq m(H[X]) + 1 \geq m(\tilde{H}[X])$$

and if $u \notin X$, then

$$(t + \mathbf{e}_u)(X) = t(X) \geq m(H[X]) = m(\tilde{H}[X]).$$

Moreover, we obtain

$$(t + \mathbf{e}_u)(V) = t(V) + 1 = m(H) + 1 = m(\tilde{H}).$$

Thus, $t + \mathbf{e}_u$ is an element of $\mathcal{P}_{\tilde{H}}$. Since $\mathcal{P}_{\tilde{H}} = \mathcal{R}_{\mathcal{C},d_{\tilde{H}}}(s_{\tilde{H}}^{\min}, s_{\tilde{H}}^{\max})$ we have $t + \mathbf{e}_u \in (s_{\tilde{H}}^{\min} + \mathcal{C})$. Hence we conclude

$$t - t^{\min} = t - (s_{\tilde{H}}^{\min} - \mathbf{e}_u) \in \mathcal{C}. \tag{6.1}$$

An analog deduction shows that $t + \mathbf{e}_v \in \mathcal{P}_{\tilde{H}}$ and thus

$$t - t^{\max} = t - (s_{\tilde{H}}^{\max} - \mathbf{e}_v) \in -\mathcal{C}. \tag{6.2}$$

A combination of (6.1) and (6.2) yields

$$t \in \left(t^{\min} + \mathcal{C}\right) \cap \left(t^{\max} - \mathcal{C}\right).$$

Furthermore, for every $t \in \mathcal{P}_H$ holds $0 \leq t \leq d_H$ and we obtain $\mathcal{P}_H \subseteq \mathcal{R}_{\mathcal{C},d_H}(t^{\min}, t^{\max})$. Now, consider any vector $t \in \mathcal{R}_{\mathcal{C},d_H}(t^{\min}, t^{\max})$. We define the vectors $p, p^{\max}, p^{\min}$ by

$$p = t + \mathbf{e}_u, \; p^{\max} = t^{\max} + \mathbf{e}_u, \; p^{\min} = t^{\min} + \mathbf{e}_u$$

and $q, q^{\max}, q^{\min}$ by

$$q = t + \mathbf{e}_v,\ q^{\max} = t^{\max} + \mathbf{e}_v,\ q^{\min} = t^{\min} + \mathbf{e}_v.$$

It is not difficult to see that $p \in \mathcal{R}'_{\mathcal{C}}(p^{\min}, p^{\max})$ and $q \in \mathcal{R}'_{\mathcal{C}}(q^{\min}, q^{\max})$. Furthermore, $p^{\max}, p^{\min}, q^{\max}$ and $q^{\min}$ are out-degree vectors of orientations of $\tilde{H}$ and thus contained in $\mathcal{P}_{\tilde{H}} = \mathcal{R}_{\mathcal{C},d_{\tilde{H}}}(s_{\tilde{H}}^{\min}, s_{\tilde{H}}^{\max})$. Hence we conclude that $p, q \in \mathcal{R}'_{\mathcal{C}}(s_{\tilde{H}}^{\min}, s_{\tilde{H}}^{\max})$. Moreover, p and q satisfy $0 \le p, q \le d_{\tilde{H}}$. Therefore, we deduce that p and q are vectors in $\mathcal{R}_{\mathcal{C}}(s_{\tilde{H}}^{\min}, s_{\tilde{H}}^{\max}) = P_{\tilde{H}}$, too. Now, for $X \subseteq V$ the following holds. If $u \notin X$, then

$$t(X) = (p - \mathbf{e}_u)(X) = p(X) \ge m(\tilde{H}[X]) = m(H[X]).$$

Analogously, if $v \notin X$, then

$$t(X) = (q - \mathbf{e}_v)(X) = q(X) \ge m(\tilde{H}[X]) = m(H[X]).$$

For $u, v \in X$ we observe that $\tilde{H}[X]$ contains an edge uv which is not in $E(H[X])$. Hence we deduce

$$t(X) = (p - \mathbf{e}_u)(X) = p(X) - 1 \ge m(\tilde{H}[X]) - 1 = m(H[X]).$$

Similarly, we obtain

$$t(V) = (p - \mathbf{e}_u)(V) = p(V) - 1 = m(\tilde{H}) - 1 = m(H).$$

Altogether we conclude that $t \in \mathcal{P}_H$ and therefore $\mathcal{R}_{\mathcal{C},d_H}(t^{\min}, t^{\max}) \subseteq \mathcal{P}_H$. With an analogue deduction we show that statement (ii) holds. □

Let G be a labeled graph of order n and $\mathcal{C} \subseteq \mathcal{M}_n$ a cone. For two vertices $s_G^{\min}$ and $s_G^{\max}$ of $\mathcal{P}_G$ satisfying $\mathcal{P}_G \subseteq \mathcal{R}_{\mathcal{C},d_G}(s_G^{\min}, s_G^{\max})$ we deduce the following. By Propositions 6.3 and 6.4 the vector $s_G^{\min}$ is realized by an acyclic orientation $D_G^{\min}$ of G. Especially, $s_G^{\min}$ is a vertex of $\mathcal{P}_G$. Furthermore, $s_G^{\max}$ is the out-degree vector of the orientation of G obtained by reversing every arc in $D_G^{\min}$. Considering a spanning subgraph H of G we also deduce that H induces a spanning subdigraph $D_H^{\min}$ of $D_G^{\min}$ which has out-degree vector $s_H^{\min}$. Reversing every arc of $D_H^{\min}$ yields a digraph with out-degree vector $s_H^{\max}$ such that $\mathcal{P}_H \subseteq \mathcal{R}_{\mathcal{C},d_H}(s_H^{\min}, s_H^{\max})$. Now, if $\mathcal{P}_H \subsetneq \mathcal{R}_{\mathcal{C},d_H}(s_H^{\min}, s_H^{\max})$, then Lemma 6.5 implies that $\mathcal{P}_G \subsetneq \mathcal{R}_{\mathcal{C},d_G}(s_G^{\min}, s_G^{\max})$. The following useful remark describes the same situation with focus on the subdigraphs of $D_G^{\min}$.

Remark 6.6
Let G be a labeled graph of order n and H a spanning subgraph of G. Moreover, let $\mathcal{C} \subseteq \mathcal{M}_n$ be a cone, $\{s_G^{\min}, s_G^{\max}\} \subseteq \mathrm{vert}(\mathcal{P}_G)$ such that $\mathcal{P}_G \subseteq \mathcal{R}_{\mathcal{C},d_G}(s_G^{\min}, s_G^{\max})$ and $\{s_H^{\min}, s_H^{\max}\} \subseteq \mathrm{vert}(\mathcal{P}_H)$ such that $\mathcal{P}_H \subsetneq \mathcal{R}_{\mathcal{C},d_H}(s_H^{\min}, s_H^{\max})$. If the orientation of H with out-degree vector $s_H^{\min}$ is a subdigraph of the orientation of G which realizes $s_G^{\max}$, then $\mathcal{P}_G \subsetneq \mathcal{R}_{\mathcal{C},d_G}(s_G^{\min}, s_G^{\max})$.

Our next goal is to prove that the following statement holds for every labeled graph G of order n. For every cone $\mathcal{C} \subseteq \mathcal{M}_n$ and every vector $s_G^{\min} \in \operatorname{vert}(\mathcal{P}_G)$ satisfying $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}}(s_G^{\min}, d_G - s_G^{\min})$ we have

$$\mathcal{P}_G \neq \mathcal{R}'_{\mathcal{C}}(s_G^{\min}, d_G - s_G^{\min})$$

if and only if G contains a cycle of length $k \geq 3$. In particular, we show that this assertion is valid for every subgraph of G which consists of a cycle of length k and possibly of some isolated vertices. For graphs containing such a cycle we even find an integral vector in $\mathcal{R}'_{\mathcal{C}}(s_G^{\min}, d_G - s_G^{\min})$ which is not included in $\mathcal{P}_G$.
We need a further result to simplify the proof of this statement on all cone orders satisfying the above mentioned conditions. In particular, we obtain both a description of the smallest cone $\mathcal{C}$ with the property $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}}(s_G^{\min}, d_G - s_G^{\min})$ in terms of generating vectors of the associated cone and a polyhedral formulation of $\mathcal{R}'_{\mathcal{C}}(s_G^{\min}, d_G - s_G^{\min})$.
For a given vertex s of the polytope $\mathcal{P}_G$ denotes $\mathcal{C}_{\text{hull}}$ the conical hull of the set $\mathcal{P}_G - s$. By its definition the conical hull $\mathcal{C}_{\text{hull}}$ itself is a cone and every cone $\mathcal{C}$ satisfying $\mathcal{P}_G \subseteq s + \mathcal{C}$ contains $\mathcal{C}_{\text{hull}}$. Since $s \in \mathcal{P}_G \subseteq \mathcal{M}_n$ notice that $\mathcal{C}_{\text{hull}} \subseteq \mathcal{M}_n$. Now, we are interested in the generating vectors of $\mathcal{C}_{\text{hull}}$. We expect that each generating vector is a vector that points from s along an edge of $\mathcal{P}_G$. Applying the characterization of the edges of $\mathcal{P}_G$ in Theorem 2.8 yields the following definition.

Definition 6.7
Let G be a labeled graph, $s \in \operatorname{vert}(\mathcal{P}_G)$ and D the orientation of G with out-degree vector s. An arc $(u, v) \in A(D)$ is included in the set $A^*(D)$, if $D - (\{u\}, \{v\})_D$ does not contain a directed (u, v)-path. We define the cone $\mathcal{C}^*_{G,s} \subseteq \mathcal{M}_n$ by

$$\mathcal{C}^*_{G,s} = \mathcal{C}\big(\{-\mathbf{z}_a\}_{a \in A^*(D)}\big).$$

From Proposition 6.3 follows that $d_G - s$ is realized by the digraph obtained from D by reversing all arcs. Thus we observe that

$$\mathcal{P}_G \subseteq \mathcal{C}^*_{G, d_G - s} = -\mathcal{C}^*_{G,s}.$$

Therefore, $\mathcal{P}_G$ is included in $\mathcal{R}'_{\mathcal{C}^*_{G,s}}(s, d_G - s)$. Now, we apply the idea of the conical hull to the set $\mathcal{R}'_{\mathcal{C}}(s, d_G - s)$ which is the intersection of two opposed cones. We define $\mathcal{Q}'_G(s)$ as the intersection of all sets $\mathcal{R}'_{\mathcal{C}}(s, d_G - s)$ where $\mathcal{C} \subseteq \mathcal{M}_n$ is a cone satisfying $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}}(s, d_G - s)$. Since $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}^*_{G,s}}(s, d_G - s)$ we deduce that $Q'_G(s)$ is a subset of $\mathcal{R}'_{\mathcal{C}^*_{G,s}}(s, d_G - s)$. The next lemma shows that $\mathcal{Q}'_G(s)$ equals $\mathcal{R}'_{\mathcal{C}^*_{G,s}}(s, d_G - s)$. Furthermore, we obtain the following description of this polyhedron by linear (in-)equalities

$$\tilde{\mathcal{Q}}'_G(s) = \{z \in \mathbb{R}^n \mid z(X) \geq m(G[X]) \text{ for all } X \in \mathcal{Z}_G(s) \text{ and } z(V(G)) = m(G)\},$$

where $\mathcal{Z}_G(s)$ is defined by

$$\mathcal{Z}_G(s) = \{X \subsetneq V(G) \mid X \neq \emptyset \text{ and } s(X) = m(G[X]) \text{ or } s(X) = m(G) - m(G - X)\}.$$

Lemma 6.8
For every labeled graph G and every $s \in \mathrm{vert}(\mathcal{P}_G)$ holds

$$\mathcal{Q}'_G(s) = \mathcal{R}'_{\mathcal{C}^*_{G,s}}(s, d-s) = \tilde{\mathcal{Q}}'_G(s).$$

Proof
Let V be the vertex set of G and $d = d_G$. Moreover, let D be the orientation of G with out-degree vector s and define $A^*(D)$ as in Definition 6.7. For a shorter notation we write $\mathcal{C}^*$ instead of $\mathcal{C}^*_{G,s}$.
We already noticed that $\mathcal{Q}'_G(s) \subseteq \mathcal{R}'_{\mathcal{C}^*}(s, d-s)$. Next, we show that $\mathcal{R}'_{\mathcal{C}^*}(s, d-s) \subseteq \tilde{\mathcal{Q}}'_G(s)$. Let t be an arbitrary vector in $\mathcal{R}'_{\mathcal{C}^*}(s, d-s) = (s + \mathcal{C}^*) \cap (d - s - \mathcal{C}^*)$. Considering the vector $\mathbf{z}_a$ for some $a \in A^*(D)$ and any set $X \subseteq V$ we observe that $\mathbf{z}_a(X) \neq 0$ if and only if a leaves or enters X. Now, for $X \in \mathcal{Z}_G(s)$ with $s(X) = m(G[X])$ all arcs between X and $V \setminus X$ in D point from $V \setminus X$ to X. Hence $\mathbf{z}_a(X)$ is equal to 0 or -1. Since $t \in (s + \mathcal{C}^*)$ there are coefficients $\lambda_a \geq 0$ such that

$$t = s + \sum_{a \in A^*(D)} \lambda_a(-\mathbf{z}_a)$$

and we obtain

$$t(X) = s(X) + \sum_{a \in A^*(D)} \lambda_a(-\mathbf{z}_a(X)) \geq m(G[X]). \tag{6.3}$$

If $X \in \mathcal{Z}_G(s)$ with $s(X) = m(G) - m(G - X)$, then every arc between X and $V \setminus X$ in D points from X to $V \setminus X$. Thus we deduce that $\mathbf{z}_a(X)$ is 0 or 1. From $t \in (d - s) - \mathcal{C}^*$ follows that there are coefficients $\mu_a \geq 0$ such that

$$t = (d - s) - \sum_{a \in A^*(D)} \mu_a(-\mathbf{z}_a).$$

Thus we conclude

$$t(X) = d(X) - s(X) + \sum_{a \in A^*(D)} \mu_a \mathbf{z}_a(X) \geq d(X) - (m(G) - m(G-X)) = m(G[X]). \tag{6.4}$$

Furthermore, we have $t(V) = m(G)$. Therefore, from (6.3) and (6.4) we obtain $t \in \tilde{\mathcal{Q}}'_G(s)$. Finally, we prove that $\tilde{\mathcal{Q}}'_G(s)$ is a subset of $\mathcal{Q}'_G(s)$. Let t be a vector in $\tilde{\mathcal{Q}}_G(s)$. Obviously, if $t \in \mathcal{P}_G$, then t is also an element of $\mathcal{Q}'_G(s)$. Thus assume that t is not in $\mathcal{P}_G$. There is a set $Y \subsetneq V$ such that $t(Y) < m(G[Y])$. Denote $\mathcal{Y}$ the family of sets which induce valid inequalities of $\mathcal{P}_G$ violated by t.
Suppose t is not an element of $\mathcal{Q}'_G(s)$. Hence there exists a cone $\mathcal{C} \subseteq \mathcal{M}_n$ with $\mathcal{P}_G \subseteq \mathcal{R}'_{\mathcal{C}}(s, d-s)$ such that $t \notin \mathcal{R}'_{\mathcal{C}}(s, d-s)$. Therefore, t is not an element of $(s + \mathcal{C})$ or $(d - s - \mathcal{C})$. If $t \notin (s + \mathcal{C})$, then for

$$L = \{\lambda s + (1 - \lambda)t \,|\, 0 \leq \lambda \leq 1\}$$

holds $L \cap \mathcal{P}_G = \{s\}$. Now, consider a vector $\tilde{t}(\lambda) \in L$. For $X \subseteq V$ with $X \notin \mathcal{Y}$ we deduce

$$\tilde{t}(\lambda)(X) = \lambda s(X) + (1 - \lambda)t(X) \geq m(G[X]).$$

If $Y \in \mathcal{Y}$, then follows

$$\tilde{t}(0)(Y) = t(Y) < m(G[Y]).$$

This fact implies $Y \notin \mathcal{Z}_G(s)$ and we obtain

$$\tilde{t}(1)(Y) = s(Y) > m(G[Y]).$$

Since $\tilde{t}(\lambda)(Y)$ is a continuous function in λ there is an $0 < \varepsilon_Y < 1$ such that $\tilde{t}(\varepsilon_Y)(Y) \geq m(G[Y])$. Therefore, choosing ε as the minimum value of ε_Y for all $Y \in \mathcal{Y}$ we conclude that $\tilde{t}(\varepsilon) \in \mathcal{P}_G$. This contradicts $\mathcal{P}_G \cap L = \{s\}$.
Analogously, we obtain a contradiction if $t \notin (d - s - \mathcal{C})$. Hence $t \in \mathcal{Q}'_G(s)$ what completes the proof. □

Since Lemma 6.8 shows the equality of $\mathcal{Q}'_G(s)$, $\mathcal{R}'_{\mathcal{C}^*_{G,s}}(s, d - s)$ and $\tilde{\mathcal{Q}}'_G(s)$ we simply use the notation $\mathcal{Q}'_G(s)$ in the remaining part of this chapter. In analogy to $\mathcal{R}_{\mathcal{C}}(s, d_G - s)$ we define

$$\mathcal{Q}_G(s) = \mathcal{Q}'_G(s) \cap \{x \in \mathbb{R}^n \mid 0 \leq x \leq d_G\}.$$

Trivially, for all $s \in \text{vert}(\mathcal{P}_G)$ we have $\mathcal{Q}_G(s) = \mathcal{R}_{\mathcal{C}^*_{G,s}}(s, d_G - s)$ and in analogy to (5.1) holds

$$\mathcal{P}_G \subseteq \mathcal{Q}_G(s) \subseteq \mathcal{Q}'_G(s). \tag{6.5}$$

An important consequence of Lemma 6.8 is that $\mathcal{Q}_G(s)$ is a polytope which we obtain from $\mathcal{P}_G$ by omitting some facet defining inequalities. In particular, we can reduce the sets in $\mathcal{Z}_G(s)$ to the ones which are also included in $\mathcal{X}_G$. Therefore, if $\mathcal{Q}'_G(s) \neq \mathcal{P}_G$, then by Theorem 2.9 there is an integral point in $\mathcal{Q}'_G(s)$ which is not in $\mathcal{P}_G$. If $\mathcal{Q}'_G(s) \neq \mathcal{P}_G$ holds for all $s \in \text{vert}(\mathcal{P}_G)$, then G is not strongly degree complete with respect to any cone order. Conversely, if there is a vector $s \in \text{vert}(\mathcal{P}_G)$ such that $\mathcal{Q}'_G(s) = \mathcal{P}_G$, then G is strongly degree complete with respect to the cone order induced by $\mathcal{C}^*_{G,s}$. Similarly, from $\mathcal{Q}_G(s) \neq \mathcal{P}_G$ follows that we can find an integral vector $t \in \mathcal{Q}_G(s)$ with $t \notin \mathcal{P}_G$. Thus G is degree complete with respect to a cone order if and only if there is a vector $s \in \text{vert}(\mathcal{P}_G)$ such that $\mathcal{Q}_G(s) = \mathcal{P}_G$.

6.2 Strongly degree complete graphs

The formulation of $\mathcal{Q}'_G(s)$ by linear (in-)equalities and the characterization of the facets of $\mathcal{P}_G$ in Theorem 2.8 yield a description of the graphs which satisfy $\mathcal{Q}'_G(s) = \mathcal{P}_G$ for some $s \in \text{vert}(\mathcal{P}_G)$. The next theorem gives an equivalent graph theoretic condition to this property.

Theorem 6.9
Let G be a labeled graph, $s \in \text{vert}(\mathcal{P}_G)$ and D the orientation of G with out-degree vector s. The equation $\mathcal{Q}'_G(s) = \mathcal{P}_G$ holds if and only if for every $X \in \mathcal{X}_G$ follows $\delta^+_D(X) = \emptyset$ or $\delta^-_D(X) = \emptyset$.

Proof
First suppose that $\mathcal{Q}'_G(s) = \mathcal{P}_G$. Thus every facet defining set $X \in \mathcal{X}_G$ is included in $\mathcal{Z}_G(s)$, that is $s(X) = m(G[X])$ or $s(X) = m(G) - m(G - X)$. If $s(X) = m(G[X])$, then there is no arc in $A(D)$ leaving X. Hence $\delta^+_D(X)$ does not contain any arcs. Similarly, if $s(X) = m(G) - m(G - X)$, then all arcs between X and $V \setminus X$ point from X to $V \setminus X$. Therefore, $\delta^-_D(X)$ is empty.
To prove sufficiency suppose $\mathcal{Q}'_G(s) \neq \mathcal{P}_G$. By (6.5) the polytope $\mathcal{P}_G$ is a proper subset of $\mathcal{Q}'_G(s)$. Thus there is a vector $t \in \mathcal{Q}'_G(s)$ and a set $Y \in \mathcal{X}_G$ such that $t(Y) < m(G[Y])$. From Lemma 6.8 follows that $t - s$ and $d - s - t$ are in $\mathcal{C}^*_{G,s}$, that is, there are coefficients $\lambda_a, \mu_a \geq 0$ such that

$$t - s = \sum_{a \in A^*(D)} \lambda_a(-\mathbf{z}_a) \qquad \text{and} \qquad d - s - t = \sum_{a \in A^*(D)} \mu_a(-\mathbf{z}_a).$$

Since $s \in \mathcal{P}_G$ we have $s(Y) \geq m(G[Y]) > t(Y)$ and we obtain

$$0 > (t - s)(Y) = \sum_{a \in A^*(D)} \lambda_a(-\mathbf{z}_a(Y)).$$

Thus there is an arc $(u, v) \in A^*(D)$ with $u \in Y$ and $v \notin Y$ yielding $\delta^+_D(Y) \neq \emptyset$. On the other hand by $s(Y) \leq m(G[Y]) + m_G(Y, V \setminus Y)$ we deduce

$$(d - s)(Y) = d(Y) - s(Y) = 2m(G[Y]) + m_G(Y, V \setminus Y) - s(Y) \geq m(G[Y]).$$

Therefore, we conclude

$$0 < m(G[Y]) - t(Y) \leq (d - s)(Y) - t(Y) = \sum_{a \in A^*(D)} \mu_a(-\mathbf{z}_a(Y)).$$

Hence there is an arc in $A^*(D)$ with head in Y and tail in $V \setminus Y$. Thus we obtain $\delta^-_D(Y) \neq \emptyset$. □

Now, we have all tools to prove the following theorem.

Theorem 6.10
Let G be a labeled graph. There exists a degree vector $s \in \mathrm{vert}(\mathcal{P}_G)$ such that $\mathcal{Q}'_G(s) = \mathcal{P}_G$ if and only if G does not contain a cycle of length $k \geq 3$.

Proof
First suppose that G does not contain a cycle of length $k \geq 3$. Consider any acyclic orientation D of G and denote the out-degree vector of D by s. Let X be an arbitrary set in $\mathcal{X}_G$. By the definition of $\mathcal{X}_G$ there is an unique component G_0 of G such that $X \subsetneq V(G_0)$. Furthermore, $G_0[X]$ and $G_0 - X$ are connected. Let D_0 be the weak component of D corresponding to G_0. Since G_0 does not contain a cycle of length at least 3 there is exactly one pair of vertices $u, v \in V(D_0)$ with $u \in X$ and $v \notin X$ which are connected by an edge of $E(G_0)$. Therefore, we deduce that all arcs of D_0 between X and

$V(D_0) \setminus X$ have the same direction. Thus $\delta_D^+(X)$ or $\delta_D^-(X)$ does not contain any arcs. Hence from Theorem 6.9 follows that $\mathcal{Q}'_G(s) = \mathcal{P}_G$.
To prove sufficiency suppose that G contains a cycle C of length $k \geq 3$. Denote H the spanning subgraph of G with $E(H) = E(C)$. Thus H contains C and possibly a collection of isolated vertices. Moreover, let s_H be an arbitrary vertex of $\mathcal{P}_H$ and denote D the acyclic orientation of H with out-degree vector s_H. Now, consider a directed path P with positive length in D which is not included in a longer directed path. Let u and v be the initial and terminal vertex of P, respectively. Hence we have $d_D^+(u) = 2$ and $d_D^+(v) = 0$. If $V(P) \neq V(C)$, then define Y as the vertex set of P. Otherwise we set $Y = \{u, v\}$. By its definition $\mathcal{X}_H$ contains the vertex set of any path in C and therefore $Y \in \mathcal{X}_H$. Furthermore, notice that in D the vertex u has a positive neighbor and v has a negative neighbor in $V(H) \setminus Y$. Thus, $\delta_D^+(X)$ and $\delta_D^-(X)$ contain an arc of D. From Theorem 6.9 follows that $\mathcal{Q}'_H(s_H) \neq \mathcal{P}_H$ holds for all $s_H \in \text{vert}(\mathcal{P}_H)$. Finally, by Lemma 6.5 we conclude $\mathcal{Q}'_G(s) \neq \mathcal{P}_G$ for all $s \in \text{vert}(\mathcal{P}_G)$. □

We continue with three important remarks on Theorem 6.10. Firstly, a graph without any cycle of length at least 3 can be seen as a forest with multiple edges. Hence we can recognize these graphs efficiently. Secondly, the proof of Theorem 6.10 actually shows that if G does not contain a cycle of length at least 3, then we have $\mathcal{Q}'_G(s) = \mathcal{P}_G$ for all $s \in \text{vert}(\mathcal{P}_G)$. A combination of this observation and Theorem 2.9 implies the following corollary.

Corollary 6.11
For a labeled graph G the following statements hold.

(i) For all $s \in \text{vert}(\mathcal{P}_G)$ there is an integral vector in $\mathcal{Q}'_G(s) \setminus \mathcal{P}_G$ if and only if G contains a cycle of length $k \geq 3$.

(ii) For all $s \in \text{vert}(\mathcal{P}_G)$ holds $\mathcal{Q}'_G(s) = \mathcal{P}_G$ if and only if G does not contain a cycle of length $k \geq 3$.

Proof

(i) By Theorem 6.10 it suffices to show that $\mathcal{Q}'_G(s) \neq \mathcal{P}_G$ implies the existence of desired integral vector. Consider any $s \in \text{vert}(\mathcal{P}_G)$ with $\mathcal{Q}'_G(s) \neq \mathcal{P}_G$. Hence there is a set $Y \in \mathcal{X}_G$ such that $t(Y) < m(G[Y])$ holds for some $t \in \mathcal{Q}'_G(s) \setminus \mathcal{P}_G$. Thus Y is not a facet defining set of $\mathcal{Q}'_G(s)$. By Theorem 2.9 there is an integral vector t' satisfying $t'(X) \geq m(G[X])$ for all $X \in \mathcal{X}_G \setminus \{Y\}$. Since the facet defining sets of $\mathcal{Q}'_G(s)$ are included in $\mathcal{X}_G \setminus \{Y\}$ we deduce $t' \in \mathcal{Q}'_G(s)$.

(ii) If G does not contain a k-cycle with $k \geq 3$, then from the proof of Theorem 6.10 follows that $\mathcal{Q}'_G(s) = \mathcal{P}_G$ holds for every degree vector of G which is realized by an acyclic orientation. Hence it holds for every element in $\text{vert}(\mathcal{P}_G)$. If G has a k-cycle with $k \geq 3$, then we have $\mathcal{Q}'_G(s) \neq \mathcal{P}_G$ for all $s \in \text{vert}(\mathcal{P}_G)$ by (i) of this corollary.

□

If G is simple and connected, then the second statement in the corollary implies that $\mathcal{Q}'_G(s) = \mathcal{P}_G$ holds for all $s \in \mathrm{vert}(\mathcal{P}_G)$ if and only if G is a tree. This coincides with our result on $\mathcal{P}_G$ in this case. In Example 2.12 we showed that for every tree of order n the degree vector polytope is affinely isomorphic to the cube of dimension $n-1$.
In the last remark we take a closer look at the consequences for cone orders. Obviously, Theorem 6.10 immediately yields the following corollary.

Corollary 6.12
A labeled graph G is strongly degree complete with respect to a cone order if and only if G does not contain a cycle of length $k \geq 3$.

In particular, by Corollary 6.11 we find such a cone order for any choice of $s_G^{\min} \in \mathrm{vert}(\mathcal{P}_G)$ and $s_G^{\max} = d_G - s_G^{\min}$ if G does not contain a cycle of length $k \geq 3$. A comparison of this result and Theorem 5.7 shows that in this case we also find a CFPO with this property. The proof of Theorem 6.10 suggests a connection to CFPOs that is formulated in the following corollary.

Corollary 6.13
Let G be a labeled graph of order n which does not contain a cycle of length $k \geq 3$ and $s \in \mathrm{vert}(\mathcal{P}_G)$. Furthermore, denote $\preceq$ the cone order which is induced by $\mathcal{C}^*_{G,s}$. If G is connected, then $\preceq$ is a CFPO. Otherwise, if G is not connected, then $\preceq$ can be extended to a CFPO in the following sense. There is a finite collection W of vectors from $\mathcal{M}_n$ such that the partial order induced by the conical hull of $\mathcal{C}^*_{G,s}$ and W is a CFPO.

Proof
Denote D an orientation of G with out-degree vector s. By its definition the cone $\mathcal{C}^*_{G,s}$ is generated by the vectors in $\{-\mathbf{z}_a\}_{a \in A^*(D)}$. Now, for every pair of adjacent vertices $u, v \in V(G)$ the set $A^*(D)$ contains exactly one of the arcs (u,v) and (v,u). Since D is acyclic and its underlying graph G does not contain any cycle of length at least 3 we observe that $A^*(D)$ is the arc set of a directed forest T with $V(T) = V(G)$. If G is connected, then T is weakly connected and thus a directed tree. We define a CFPO $\preceq$ on $\mathbb{R}^n$ with tree-representation T and observe that $\mathcal{C}_{\preceq} = \mathcal{C}^*_{G,s}$.
If G is not connected, then T is also not connected. Denote $\tilde{T}$ any directed tree with $V(\tilde{T}) = V(T)$ such that T is a subdigraph of $\tilde{T}$. Moreover, define $B = A(\tilde{T}) \setminus A(T)$ and $W = \{-\mathbf{z}_a\}_{a \in B}$. Obviously, W is a subset of $\mathcal{M}_n$. The conical hull $\mathcal{C}_{\mathrm{hull}}$ of $A^*(D) \cup W$ is equal to $\mathcal{C}(\{-\mathbf{z}_a\}_{a \in A(\tilde{T})})$. In analogy to case that G is connected the partial order induced by $\mathcal{C}_{\mathrm{hull}}$ is a CFPO. □

The last result underlines the significance of CFPOs. If there is a cone order $\preceq$ for which we can describe $\mathrm{DEG}^+(G)$ as a convex set in $(\mathbb{Z}^n, \preceq)$, then there exists a CFPO with this property, too. Corollary 6.12 also gives us a precise answer to the question: “For which graph G does not exist a cone order $\preceq$ such that $\mathrm{DEG}^+(G) = \mathcal{S}'_{\preceq}(G)$ holds?” The result is an extension of Theorem 5.7.

6.3 Degree complete graphs

In the final section of this chapter we prove results similar to Theorem 6.9 and Theorem 6.10, where we replace $\mathcal{Q}'_G(s)$ by $\mathcal{Q}_G(s)$. Finally, we are able to characterize the graphs which are degree complete with respect to some cone order.

Theorem 6.14
Let G be a labeled graph with components $G_1, \dots, G_k$ and $s \in \text{vert}(\mathcal{P}_G)$. Moreover, let D be the orientation of G with out-degree vector s. The equation $\mathcal{Q}_G(s) = \mathcal{P}_G$ holds if and only if for all $i \in \{1, \dots, k\}$ and every $X \in \mathcal{X}_{G_i}$ with $2 \leq |X| \leq |V(G_i)| - 2$ follows $\delta^+_D(X) = \emptyset$ or $\delta^-_D(X) = \emptyset$.

Proof
First suppose that $\mathcal{Q}_G(s) = \mathcal{P}_G$. For all $i \in \{1, \dots, k\}$ we observe that every facet defining set $X \in \mathcal{X}_{G_i}$ with $2 \leq |X| \leq |V(G_i)| - 2$ is an element of $\mathcal{Z}_G(s)$, that is $s(X) = m(G[X])$ or $s(X) = m(G) - m(G - X)$. If $s(X) = m(G[X])$, then there is no arc in $A(D)$ leaving X. Hence $\delta^+_D(X)$ does not contain any arcs. Similarly, if $s(X) = m(G) - m(G - X)$, then all arcs between X and $V \setminus X$ leave X. Therefore, $\delta^-_D(X)$ is empty.
Now, suppose $\mathcal{Q}_G(s) \neq \mathcal{P}_G$. This implies that $\mathcal{P}_G$ is a proper subset of $\mathcal{Q}_G(s)$. Thus there is a vector $t \in \mathcal{Q}_G(s)$ and a set $Y \in \mathcal{X}_{G_i}$ such that $t(Y) < m(G_i[Y])$ for some $i \in \{1, \dots, k\}$. By the definition of $\mathcal{Q}_G(s)$ we have $0 \leq t \leq d_G$. Suppose we have $|Y| = 1$ or $|Y| = |V(G_i)| - 1$. There is a vertex $u \in V(G)$ such that $Y = \{u\}$ or $Y = V(G_i) \setminus \{u\}$. If $Y = \{u\}$, then

$$m(G_i[Y]) = 0 \leq t(u) = t(Y)$$

contradicts our choice for t and Y. Analogously, if $Y = V(G_i) \setminus \{u\}$, then we obtain a contradiction by

$$m(G_i[Y]) = m(G_i - u) = m(G_i) - d_G(u) \leq t(V(G_i)) - t(u) = t(Y).$$

Hence we deduce that $2 \leq |Y| \leq |V(G_i)| - 2$. Because t is an element of $\mathcal{Q}_G(s)$ from Lemma 6.8 follows that $t-s$ and $d-s-t$ are contained in $\mathcal{C}^*_{G,s}$, that is, there are coefficients $\lambda_a, \mu_a \geq 0$ such that

$$t - s = \sum_{a \in A^*(D)} \lambda_a(-\mathbf{z}_a) \qquad \text{and} \qquad d - s - t = \sum_{a \in A^*(D)} \mu_a(-\mathbf{z}_a).$$

Since $s \in \mathcal{P}_G$ we have $s(Y) \geq m(G[Y]) > t(Y)$ and we obtain

$$0 > (t - s)(Y) = \sum_{a \in A^*(D)} \lambda_a(-\mathbf{z}_a(Y)).$$

Thus there is an arc $(u, v) \in A^*(D)$ with $u \in Y$ and $v \notin Y$ yielding that $\delta^+_D(Y)$ is not empty. On the other hand, by $s(Y) \leq m(G[Y]) + m_G(Y, V \setminus Y)$ we deduce

$$(d - s)(Y) = d(Y) - s(Y) = 2m(G[Y]) + m_G(Y, V \setminus Y) - s(Y) \geq m(G[Y]).$$

Therefore, we conclude

$$0 < m(G[Y]) - t(Y) \leq (d-s)(Y) - t(Y) = \sum_{a \in A^*(D)} \mu_a(-\mathbf{z}_a(Y)).$$

Hence there is an arc in $A^*(D)$ with head in Y and tail in $V \setminus Y$ and thus $\delta^-_D(Y) \neq \emptyset$. □

We use Theorem 6.14 to show that $Q_G(s) \neq P_G$ for a given $s \in \text{vert}(P_G)$. For example, suppose that G is a connected graph and denote D the orientation of G with out-degree vector s. It suffices to find a set $X \subsetneq V(D)$ with $2 \leq |X| \leq |V(D)| - 2$ such that $D[X]$ and $D - X$ are weakly connected and there exists both an arc leaving and an arc entering X.
In the following example we consider C_4 the cycle of length 4. By Theorem 5.17 there is no labeled version of C_4 which is degree complete with respect to a CFPO. Now, we show that C_4 is degree complete with respect to a cone order.

Example 6.15
Consider the labeled graph G with vertex set

$$V(G) = \{1,2,3,4\} \quad \text{and} \quad E(G) = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}.$$

It is not difficult to check that G has 16 orientations and 15 of them have different out-degree vectors. Thus the set of degree vectors of G is

$$\text{DEG}^+(G) = \left\{ \begin{array}{l} (2,1,1,0)^\top, (2,1,0,1)^\top, (2,0,1,1)^\top, (1,2,1,0)^\top, (1,2,0,1)^\top, \\ (0,2,1,1)^\top, (1,1,2,0)^\top, (1,0,2,1)^\top, (0,1,2,1)^\top, (1,1,0,2)^\top, \\ (1,0,1,2)^\top, (0,1,1,2)^\top, (2,0,2,0)^\top, (0,2,0,2)^\top, (1,1,1,1)^\top \end{array} \right\}.$$

Only the vector $(1,1,1,1)^\top$ corresponds to an orientation of G which is not acyclic. Hence all other vectors are vertices of $\mathcal{P}_G$. By Theorem 2.8 a polyhedral description of $\mathcal{P}_G$ is given by

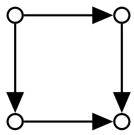
$$\mathcal{P}_G = \left\{ (x_1,x_2,x_3,x_4)^\top \in \mathbb{R}^4 \;\middle|\; \begin{array}{lr} x_1 \geq 0, & \text{(F1)} \\ x_2 \geq 0, & \text{(F2)} \\ x_3 \geq 0, & \text{(F3)} \\ x_4 \geq 0, & \text{(F4)} \\ x_1 + x_2 \geq 1, & \text{(F5)} \\ x_2 + x_3 \geq 1, & \text{(F6)} \\ x_3 + x_4 \geq 1, & \text{(F7)} \\ x_1 + x_4 \geq 1, & \text{(F8)} \\ x_1 + x_2 + x_3 \geq 2, & \text{(F9)} \\ x_1 + x_2 + x_4 \geq 2, & \text{(F10)} \\ x_1 + x_3 + x_4 \geq 2, & \text{(F11)} \\ x_2 + x_3 + x_4 \geq 2, & \text{(F12)} \\ x_1 + x_2 + x_3 + x_4 = 4 & \text{(F13)} \end{array} \right\}.$$

Accordingly, we obtain

$$\mathcal{X}_G = 2^{V(G)} \setminus \{\emptyset, \{1,3\}, \{2,4\}, \{1,2,3,4\}\}.$$

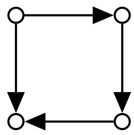
By Theorem 6.14 the important sets of $\mathcal{X}_G$ are the ones that consists of two elements. Now, consider the acyclic orientations of G. Figure 6.1 shows the three nonisomorphic acyclic orientations of C_4 and below the degree vectors of G which are realized by an orientation of G isomorphic to the digraph above, respectively. It is easy to

D_1:



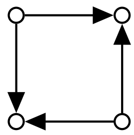
$(2,1,0,1)^\top, (1,2,1,0)^\top, (0,1,2,1)^\top, (1,0,1,2)^\top$

D_2:



$(2,1,1,0)^\top, (2,0,1,1)^\top, (1,2,0,1)^\top, (0,2,1,1)^\top, (1,1,2,0)^\top, (1,0,2,1)^\top, (1,1,0,2)^\top, (0,1,1,2)^\top$

D_3:



$(2,0,2,0)^\top, (0,2,0,2)^\top$

Figure 6.1: Three nonisomorphic acyclic orientations of C_4 and corresponding degree vectors of G

see that for an orientation D of G isomorphic to D_2 or D_3 we have $\delta_D^+(X) \neq \emptyset$ and $\delta_D^-(X) \neq \emptyset$ if $X = \{1,2\}$ or $X = \{1,4\}$. Hence Theorem 6.14 yields $\mathcal{Q}_G(s) \neq \mathcal{P}_G$ for $s \in \mathrm{DEG}^+(G) \setminus \{(2,1,0,1)^\top, (1,2,1,0)^\top, (0,1,2,1)^\top, (1,0,1,2)^\top\}$. On the other hand we observe that if D is isomorphic to D_1, then $\delta_D^+(X) = \emptyset$ or $\delta_D^-(X) = \emptyset$ holds for all $X \in \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}$. In this case $\mathcal{C}^*_{G,s}$ induces a cone order $\preceq$ such that every $t \in \mathbb{Z}^4$ which satisfies $s \preceq t \preceq d_G - s$ and $0 \leq t \leq d_G$ is a degree vector of G. Hence we investigate the case $(2,1,0,1)^\top$ more detailed.
First notice that for $s = (2,1,0,1)^\top$ the inequalities (F3), (F6), (F7) and (F12) hold with equality. Similarly, for $(0,1,2,1)^\top = (2,2,2,2)^\top - (2,1,0,1)^\top$ the inequalities (F1), (F5), (F8) and (F10) are satisfied with equality. Hence we deduce that

$$\mathcal{Z}_G(s) = \big\{\{1\}, \{3\}, \{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{1,2,4\}, \{2,3,4\}\big\}.$$

Since $\mathcal{Z}_G(s)$ contains all facet defining set of $\mathcal{P}_G$ with 2 elements we prove $\mathcal{Q}_G(s) = \mathcal{P}_G$ directly via Lemma 6.8. The corresponding cone $\mathcal{C}^*_{G,s}$ is generated by the columns of the incidence matrix of the orientation of G that realizes $d_G - s = (0,1,2,1)$, that is

$$\mathcal{C}^*_{G,s} = \mathcal{C}\big((-1,1,0,0)^\top, (0,-1,1,0)^\top, (-1,0,0,1)^\top, (0,0,1,-1)^\top\big). \tag{6.6}$$

Let $\preceq$ be the cone order induced by $\mathcal{C}^*_{G,s}$. Every integral vector $t \in \mathbb{Z}^4$ with $(2,1,0,1)^\top \preceq t \preceq (0,1,2,1)^\top$ and $0 \leq t \leq d_G$ is contained in $\mathrm{DEG}^+(G)$. The Hasse diagram of $\mathrm{DEG}^+(G)$ with respect to this order $\preceq$ is given by Figure 6.2. This diagram also shows that $(\mathrm{DEG}^+(G), \preceq)$ is not a lattice. For example, the vectors $(1,2,0,1)^\top$ and $(2,1,1,0)^\top$ do not have a supremum since $(1,2,1,0)^\top$ and $(1,1,1,1)^\top$ are minimal upper bounds.
Finally, observe that the properties noticed above are recognized in a visualization of $\mathcal{P}_G$.

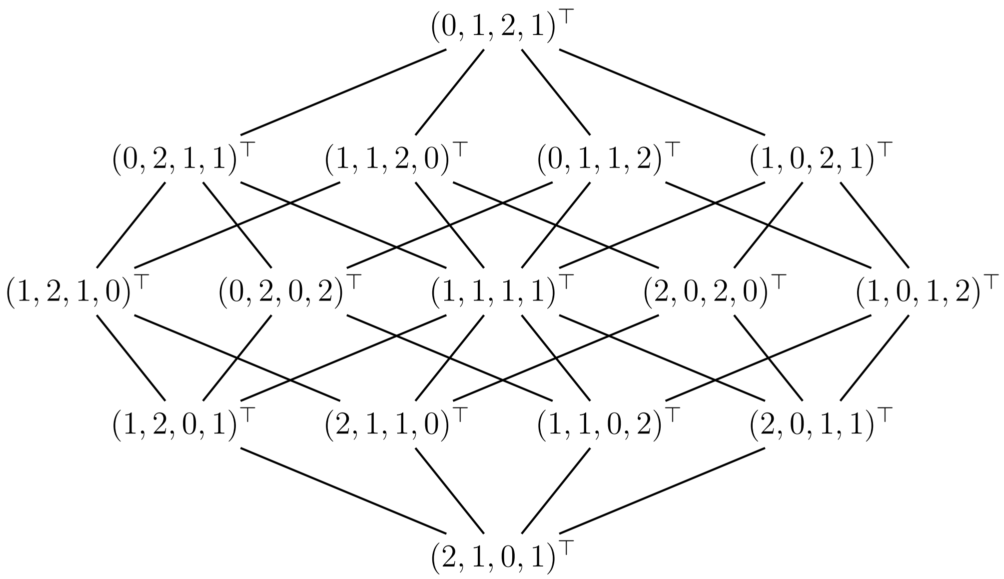


Figure 6.2: Hasse diagram of $(\mathrm{DEG}^+(G), \preceq)$

We use the same affine mapping $\Phi : \mathbb{R}^4 \to \mathbb{R}^3$ as in Example 2.11. A projection of $\Phi(P_G)$ is given by Figure 6.3. The degree vector polytope of G has 14 vertices. All of the 24 edges of $\mathcal{P}_G$ have the same length. Moreover, there are two types of facets. Each facet containing vertex $(2,0,2,0)^\top$ or $(0,2,0,2)^\top$ is a rhombus. The remaining four facets (not containing $(2,0,2,0)^\top$ or $(0,2,0,2)^\top$) are squares.
Another way to describe of $\mathcal{P}_G$ is the following. The intersection of $[0,2]^4$ with the affine space $\mathcal{M}_G$ yields an octahedron with vertices $(2,0,2,0)^\top$, $(2,0,2,0)^\top$, $(0,0,2,2)^\top$, $(0,2,2,0)^\top$, $(2,0,0,2)^\top$ and $(2,2,0,0)^\top$. Notice that only $(2,0,2,0)^\top$ and $(2,0,2,0)^\top$ are degree vectors of G. Now, we obtain $\mathcal{P}_G$ by cutting off four vertices of the octahedron (namely $(0,0,2,2)^\top$, $(0,2,2,0)^\top$, $(2,0,0,2)^\top$ and $(2,2,0,0)^\top$). The inequalities that define these four cuts are (F5) to (F9) and the resulting facets are the four squares.
With this description of $\mathcal{P}_G$ it is easy to see that $\mathcal{Q}_G((2,0,2,0)^\top)$ is equal to the octahedron $\mathcal{M}_G \cap [0,2]^4$. Hence $(0,0,2,2)^\top$, $(0,2,2,0)^\top$, $(2,0,0,2)^\top$ and $(2,2,0,0)^\top$ are integral vectors in $\mathcal{Q}_G((2,0,2,0)^\top)$ which are not degree vectors. Remember that by Theorem 2.9 the existence of at least one such integral vector is implied by $\mathcal{Q}_G(s) \neq \mathcal{P}_G$.
Considering $s = (2,1,1,0)^\top$ we observe that $\mathcal{Q}_G(s)$ contains only two facets of $\mathcal{P}_G$ which are not included in a facet of $\mathcal{M}_G \cap [0,2]^4$. Hence there are integral vectors (namely $(2,0,0,2)^\top$ and $(0,2,2,0)^\top$) in $\mathcal{Q}_G(s)$ which are not degree vectors of G. For $s = (2,1,0,1)^\top$ all four facets of $\mathcal{P}_G$ which are not facets of $\mathcal{M}_G \cap [0,2]^4$ are facets of $\mathcal{Q}_G(s)$. Thus we conclude $\mathcal{Q}_G(s) = \mathcal{P}_G$.

In contrast to Corollary 6.11 this example illustrates that for some graphs $\mathcal{Q}_G(s) = \mathcal{P}_G$ only holds for a few $s \in \mathrm{vert}(\mathcal{P}_G)$. The next example shows that the cone order induced by (6.6) also yields a similar result for the diamond graph.

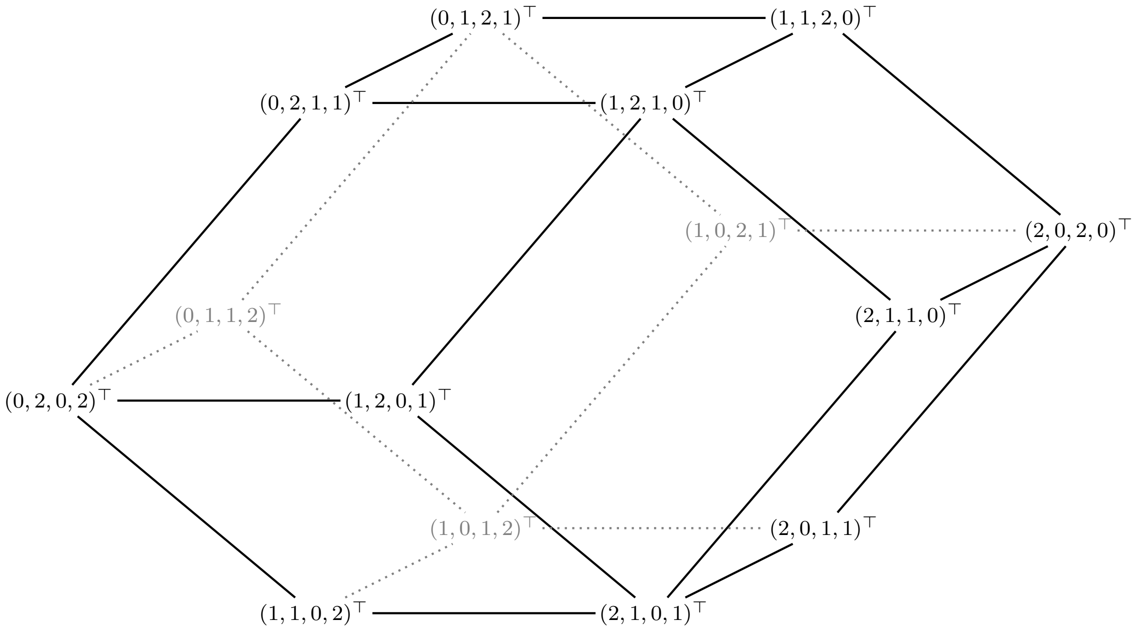


Figure 6.3: Degree vector polytope of a labeled version of $\mathcal{C}_4$

Example 6.16
We consider a labeled version G of the diamond graph defined by

$$V(G) = \{1,2,3,4\} \quad \text{and} \quad E(G) = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,3\}\}.$$

It is easy to see that $\mathcal{X}_G$ is the same as in Example 6.15. Again we list up the nonisomorphic acyclic orientations of G and the corresponding degree vectors in Figure 6.4. Notice that $\tilde{D}_2$, $\tilde{D}_3$, $\tilde{D}_4$ and $\tilde{D}_5$ contain subdigraphs which are isomorphic to D_2 or D_3 of Example 6.15, respectively. Hence Remark 6.6 implies that $\mathcal{Q}_G(s) \neq \mathcal{P}_G$ for all $s \in \operatorname{vert}(\mathcal{P}_G)$ with

$$s \notin \{(3,1,0,1)^\top, (0,1,3,1)^\top, (2,2,1,0)^\top, (2,0,1,2)^\top, (1,0,2,2)^\top, (1,2,2,0)^\top\}.$$

For an orientation D of G isomorphic to $\tilde{D}_6$ we use the argument that for $X = \{1,2\}$ or $X = \{1,4\}$ holds $\delta^+_D(X) \neq \emptyset$ and $\delta^-_D(X) \neq \emptyset$. On the other hand we observe that if D is isomorphic to $\tilde{D}_1$, then we have $\delta^+_D(X) = \emptyset$ or $\delta^-_D(X) = \emptyset$ for all $X \in \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$. Hence for $s = (3,1,0,1)^\top$ the cone order $\preceq$ induced by $\mathcal{C}^*_{G,s}$ yields that every integral vector with $(3,1,0,1)^\top \preceq t \preceq (0,1,3,1)^\top$ and $0 \leq t \leq d_G$ is a degree vector of G. Particularly, $\mathcal{C}^*_{G,s}$ is equal to (6.6).
Again a visualization of $\mathcal{P}_G$ (Figure 6.5) gives a valuable insight of the conclusions above. Notice that the surface of this polytope has 12 facets like the polytope in Example 6.15 since $\mathcal{X}_G$ is the same set in both cases. For the labeled diamond graph the surface of $\mathcal{P}_G$ consists of four regular hexagons, four rhombi and four squares. Each square contains one of the points $(2,2,1,0)^\top$, $(1,2,2,0)^\top$, $(1,0,2,2)^\top$ and $(2,0,1,2)^\top$ and they are the facets of $\mathcal{P}_G$ which are not induced by the facets of $\mathcal{M}_G \cap \{x \in \mathbb{R}^4 \mid 0 \leq x \leq d_G\}$. Hence only if we choose $s = (3,1,0,1)^\top$ or $s = (0,1,3,1)^\top$, then $\mathcal{Q}_(s)$ has the same facets like $\mathcal{P}_G$.

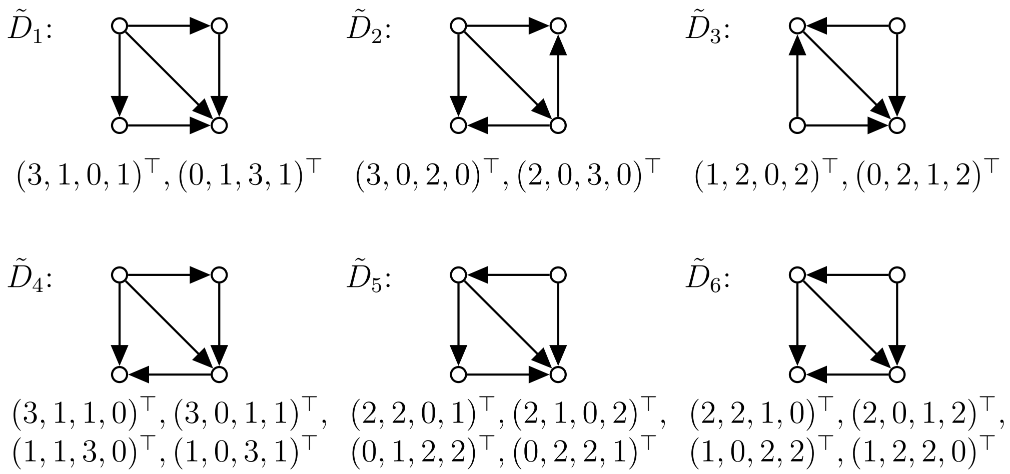


Figure 6.4: Six nonisomorphic acyclic orientations of the diamond graph and corresponding degree vectors of G

For every other vertex s of $\mathcal{P}_G$ at least one of the two pairs of "opposed" squares are not facets of $\mathcal{Q}_G(s)$. Thus $\mathcal{Q}_G(s)$ includes an integral point which is not a degree vector of G by Theorem 2.9.

In final last example we consider the complete graph on four vertices K_4. An analysis of $\mathcal{P}_{K_4}$ shows that for every $s \in \text{vert}(\mathcal{P}_G)$ we have $\mathcal{Q}_G(s) \neq \mathcal{P}_G$. In the next theorem we prove this fact.

Example 6.17
For the (labeled) graph K_4 we observe that each $X \subseteq V(K_4)$ with $1 \leq |X| \leq 3$ is in $\mathcal{X}_{K_4}$ This fact can be observed in the visualization of $\mathcal{P}_{K_4}$ in Figure 6.6. It shows a truncated octahedron. The surface consists of 8 regular hexagons and 6 squares. In contrast to Figures 6.3 and 6.5 this polytope has 14 facets instead of 12. Particularly, $\{1,3\}$ and $\{2,4\}$ are additional facet defining sets. The set $\mathcal{M}_{K_4} \cap [0,3]^4$ is an octahedron with vertices

$$(3,3,0,0)^\top,\ (0,3,3,0)^\top,\ (0,0,3,3)^\top,\ (3,0,0,3)^\top,\ (3,0,3,0)^\top,\ (0,3,0,3)^\top$$

and each hexagon is contained in a facet of $\mathcal{M}_{K_4} \cap [0,3]^4$. Furthermore, each square belongs to a facet with valid inequality

$$s(\{i,j\}) \geq 1, \quad \text{for } 1 \leq i < j \leq 4.$$

Since every vertex of $\mathcal{P}_G$ is contained in exactly one square it is not possible to find a degree vector $s \in \text{vert}(\mathcal{P}_G)$ satisfying $\mathcal{Q}_{K_4}(s) = \mathcal{P}_{K_4}$. In particular, four of the six vertices of the octahedron $\mathcal{M}_{K_4} \cap [0,3]^4$ are elements in $\mathcal{Q}_{K_4}(s)$.

We continue to show that if a labeled graph G of order n contains a cycle of length 5 or a subgraph isomorphic to K_4, *net*, *A-graph*, or *kite* (see Figure 6.7), then $\mathcal{Q}_G(s)$ contains

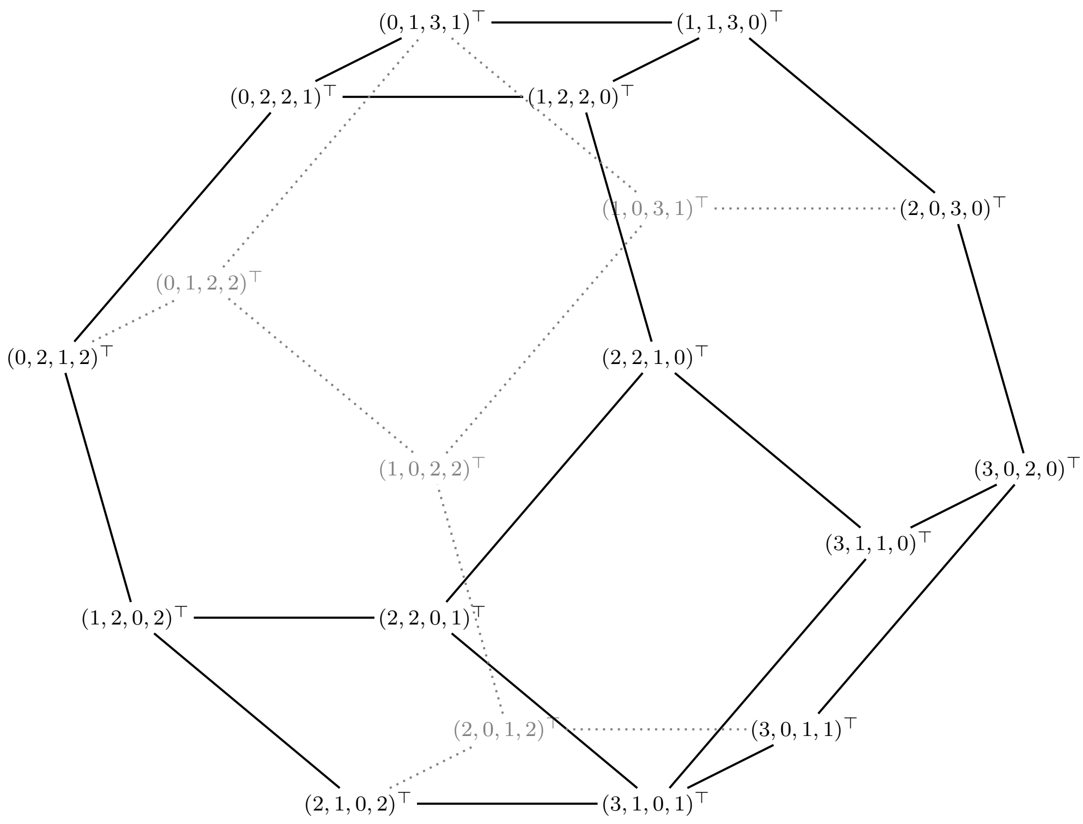


Figure 6.5: Degree vector polytope of a labeled version of the diamond graph

a vector $t \in \mathbb{Z}^n$ with $0 \leq t \leq d_G$ which is not a degree vector of G. Before we are able formulate the main theorem we need certain sets of vertices which help us to decide, whether there is a vector $s \in \text{vert}(\mathcal{P}_G)$ such that $\mathcal{Q}_G(s) = \mathcal{P}_G$. The following procedure determines such a vertex set and can be seen as an extension of Procedure 3.15.

Procedure 6.18
Input: Labeled graph $G = (V, E)$.

Output: $W \subseteq V$ (see Proposition 6.19 for properties of W).

1) Set $W := \emptyset$.

2) **For** every pair of distinct vertices $u, v \in V$ **do:**
Set $W_{u,v} := N_G(u) \cap N_G(v)$.
If $\big(\{u,v\} \cup W_{u,v}\big) \cap W = \emptyset$ **and** $uv \in E$ or $|W_{u,v}| \geq 2$
and w is isolated in $G - \{u, w\}$ for all $w \in W_{u,v}$
and $(G - W_{u,v}) - uv$ does not contain a uv-path
then: $W := W \cup W_{u,v}$.

3) **Return** W.

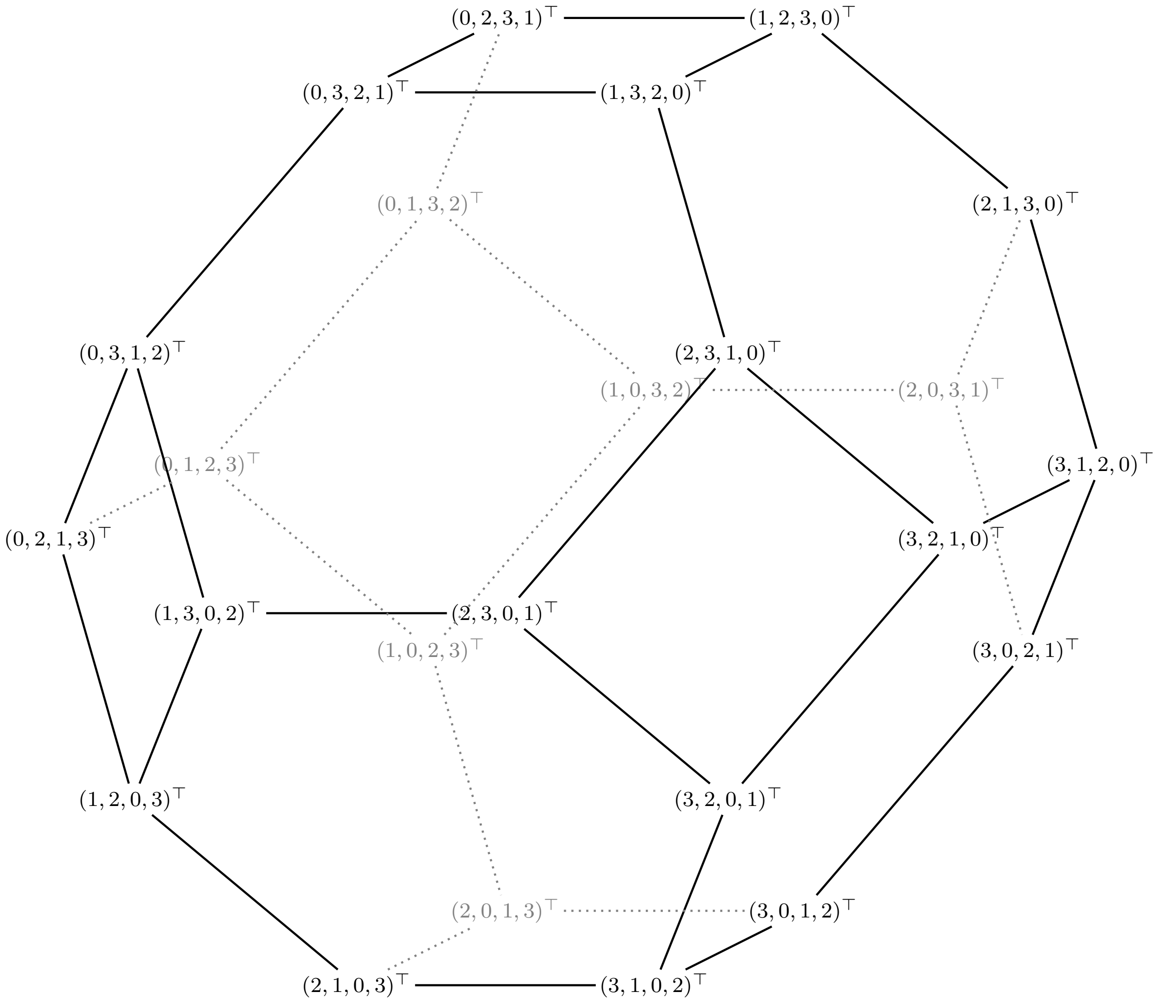


Figure 6.6: Degree vector polytope of K_4

An output of this Procedure has the following properties.

Proposition 6.19
Let G be a labeled graph and W the output of Procedure 6.18 with input G. The following statements hold

(i) W is a set of independent vertices of G.

(ii) Every $w \in W$ is contained in a 3-cycle or 4-cycle of G.

(iii) Every $w \in W$ is not contained in a k-cycle of G with $k \geq 5$.

Proof

(i) Obviously, if $W_{u,v} \subseteq W$, then $W_{u,v}$ is a set of isolated vertices in $G - \{u, v\}$. It suffices to show that from $\emptyset \neq W_{u,v} \subseteq W$ follows $u, v \notin W$. Suppose to the contrary that $u \in$

W. Thus there are vertices u' and v' in G such that $u \in W_{u',v'}$. Furthermore, u' and v' are the only neighbors of u in G since u is an isolated vertex in $G - W_{u',v'}$. Because each vertex in $W_{u,v}$ is a neighbor of u we deduce that either $uv \in E(G)$ and $|W_{u,v}| = 1$ or $uv \notin E(G)$ and $|W_{u,v}| = 2$. In both cases we obtain $\big(\{u,v\} \cup W_{u,v}\big) \cap W \neq \emptyset$. This yields a contradiction to the construction of W.

(ii) Consider an arbitrary vertex $w \in W$. From the construction of W follows that w has exactly two neighbors u and v. Additionally, u and v are adjacent or $W_{u,v}$ contains a further vertex x. In the first case w is part of the triangle $uvwu$. In the second case $uwvxu$ is a 4-cycle including w.

(iii) Let w be a vertex in W. Suppose to the contrary that w is contained in a cycle C of length $k \geq 5$. From the construction of W follows that w has exactly two neighbors u and v. Obviously, we have $u, v \in V(C)$. Since there is no uv-path in $(G - W_{u,v}) - uv$ we deduce that $W_{u,v}$ contains a further vertex $x \in V(C)$, $x \neq w$. We conclude that x is adjacent to at least 3 vertices because C has length $k \geq 5$ and x is a common neighbor of u and v. This implies $x \notin W$ and we obtain a contradiction.

□

Note that two distinct labeled versions of the same unlabeled graph may yield to different sets as outputs of Procedure 6.18. Additionally, if a graph G is not connected, then an output set W induces an output set for each component, respectively.
The following theorem gives two characterizations for those labeled graphs G satisfying $\mathcal{Q}_G(s) = \mathcal{P}_G$ for a vector $s \in \text{vert}(\mathcal{P}_G)$. The first equivalent description is in terms of forbidden subgraphs. The second characterization implies that we can recognize these

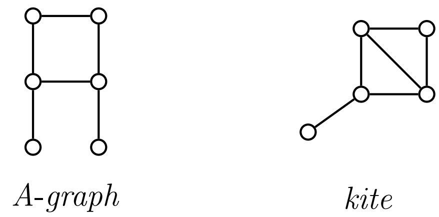


Figure 6.7: *A-graph* and *kite* (names by [6])

graphs in polynomial time with respect to the order of G.

Theorem 6.20
Let G be a labeled graph of order n and W the output of Procedure 6.18 with input G. The following statements are equivalent:

(i) There exists a vector $s \in \text{vert}(\mathcal{P}_G)$ such that $\mathcal{Q}_G(s) = \mathcal{P}_G$.

(ii) G does not contain a subgraph isomorphic to K_4, *net*, *A-graph*, *kite* or C_k with $k \geq 5$.

(iii) $G - W$ does not contain a cycle of length $k \geq 3$.

Proof
From (i) to (ii): Suppose G contains a subgraph isomorphic to K_4, *net*, *A-graph*, *kite* or C_k with $k \geq 5$. By Lemma 6.5 it suffices to show that there is a spanning subgraph H of G such that $\mathcal{Q}_H(s) \neq \mathcal{P}_H$ holds for every $s \in \text{vert}(\mathcal{P}_H)$. In the following let H be a spanning subgraph of G which consists of exactly one copy of K_4, *net*, *A-graph*, *kite* or C_k with $k \geq 5$ and some isolated vertices. Denote H_0 the unique nontrivial component of H. We consider an arbitrary $s \in \text{vert}(\mathcal{P}_H)$ and define D as the (acyclic) orientation of H with out-degree vector s. If there is a set $Y \in \mathcal{X}_{H_0}$ with $2 \leq |Y| \leq |V(H_0)| - 2$ such that both $\delta_D^+(Y)$ and $\delta_D^-(Y)$ are nonempty, then from Theorem 6.14 follows that $\mathcal{Q}_H(s) \neq \mathcal{P}_H$. We distinguish between five cases depending on which graph is isomorphic to H_0.

Case 1: H_0 is isomorphic to K_4.
Let $\{v_1, v_2, v_3, v_4\}$ be the vertex set of H_0. The subdigraph of D induced by H_0 is a tournament on 4 vertices. Without loss of generality we have $(v_i, v_j) \in A(D)$ for every pair i, j with $1 \leq i < j \leq 4$. Considering $Y = \{v_1, v_4\}$ we notice that $2 = |Y| \leq |V(H_0)| - 2$ and $Y \in \mathcal{X}_{H_0}$ since H_0, $H_0[Y]$ and $H_0 - Y$ are connected. Because $\delta_D^+(Y) = \{(v_1, v_2)\}$ and $\delta_D^-(Y) = \{(v_3, v_4)\}$ we deduce $\mathcal{Q}_H(s) \neq \mathcal{P}_H$.

Case 2: H_0 is isomorphic to net.
Denote $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ the vertex set of H_0, where v_1, v_2, v_3 are the vertices which induce a triangle in H_0. Moreover, we choose w_1, w_2 and w_3 such that $v_i w_i \in E(H)$ for $1 \leq i \leq 3$. Without loss of generality the subdigraph of D induced by $\{v_1, v_2, v_3\}$ has the arc set $\{(v_1, v_2), (v_1, v_3), (v_2, v_3)\}$ because all acyclic orientations of a triangle are isomorphic. The set $Y = \{v_2, w_2\}$ satisfies $2 = |Y| \leq |V(H_0)| - 2$ and it is an element of $\mathcal{X}_{H_0}$ since H_0, $H_0[Y]$ and $H_0 - Y$ are connected. We observe that $\delta_D^+(Y) = \{(v_2, v_3)\}$ and $\delta_D^-(Y) = \{(v_1, v_2)\}$ and therefore $\mathcal{Q}_H(s) \neq \mathcal{P}_H$ holds.

Case 3: H_0 is isomorphic to A-graph.
Let $\{v_1, v_2, v_3, v_4, w_1, w_2\}$ be the vertex set of H_0, where v_1, v_2, v_3, v_4 denote the vertices which induce a 4-cycle in H_0. If D induces an orientation of the 4-cycle which is isomorphic to D_2 or D_3 in Example 6.15, then from Remark 6.6 follows $\mathcal{Q}_H(s) \neq \mathcal{P}_H$. Thus we assume that the oriented 4-cycle in D is isomorphic to D_1 of the same example. Let the vertices v_1, v_2, v_3, v_4 be as in Figure 6.8. Furthermore, exactly one of the vertices

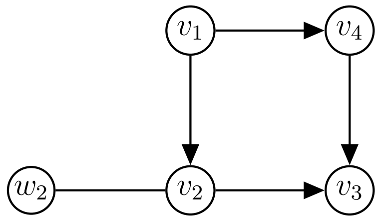


Figure 6.8: Partial orientation of a subgraph of *A-graph*

v_2 and v_4 is adjacent to w_1 or w_2 in H_0. Without loss of generality we assume that $v_2 w_2 \in E(H_0)$ and w_1 is a neighbor of v_1 or v_3. Now, the set $Y = \{v_2, w_2\}$ satisfies $2 = |Y| \leq |V(H_0)| - 2$ and it is an element of $\mathcal{X}_{H_0}$ since H_0, $H_0[Y]$ and $H_0 - Y$ are connected. Because $\delta_D^+(Y) = \{(v_2, v_3)\}$ and $\delta_D^-(Y) = \{(v_1, v_2)\}$ we deduce $\mathcal{Q}_H(s) \neq \mathcal{P}_H$.

Case 4: H_0 is isomorphic to kite.
Let $\{v_1, v_2, v_3, v_4, w_1\}$ be the vertex set of H_0, where v_1, v_2, v_3, v_4 denote the vertices which induce a diamond graph in H_0. If D induces an orientation of the diamond graph in H_0 which is not isomorphic to $\tilde{D}_1$ from Example 6.16, then $\mathcal{Q}_H(s) \neq \mathcal{P}_H$ holds by Remark 6.6. Hence we suppose that the diamond graph in H_0 is oriented as $\tilde{D}_1$ in Figure 6.4. Let the vertices v_1, v_2, v_3, v_4 be as in Figure 6.9. Exactly one of the vertices v_2 and v_4 is adjacent

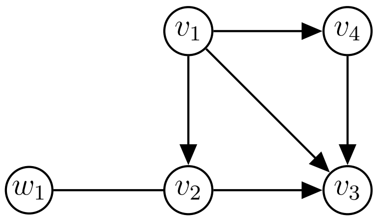


Figure 6.9: Partial orientation of *kite*

to w_1 in H_0. Without loss of generality let $v_2w_1 \in E(H)$. Now, the set $Y = \{v_2, w_1\}$ satisfies $2 = |Y| \leq |V(H_0)| - 2$ and it is an element of $\mathcal{X}_{H_0}$ since H_0, $H_0[Y]$ and $H_0 - Y$ are connected. Because $\delta_D^+(Y) = \{(v_2, v_3)\}$ and $\delta_D^-(Y) = \{(v_1, v_2)\}$ we deduce $\mathcal{Q}_H(s) \neq \mathcal{P}_H$.

Case 5: H_0 is isomorphic to a cycle of length $k \geq 5$.
Let $\{v_1, \dots, v_k\}$ be the vertex set of H_0 where vertices with consecutive indices are adjacent. By its definition $\mathcal{X}_{H_0}$ contains the vertex set of any path in H_0. Let P be a maximal, directed path in D, that is, P is not part of a longer directed path in D. Without loss of generality we assume $P = v_1v_2 \dots v_l$ for $2 \leq l \leq k$. If $l \leq k - 2$, then define $Y = V(P)$. It is not difficult to see that H_0, $H_0[Y]$ and $H_0 - Y$ are connected and $|Y| = l$. Thus we deduce $Y \in \mathcal{X}_{H_0}$ and $2 \leq |Y| \leq |V(H_0)| - 2$. From the maximality of P follows $d_D^+(v_1) = 2$ and $d_D^+(v_l) = 0$. Therefore, we obtain that $\delta_D^+(Y) = \{(v_1, v_k)\}$ and $\delta_D^-(Y) = \{(v_{l+1}, v_l)\}$. If $l > k - 2 \geq 3$, then we set $Y = \{v_2, v_3\}$. We also observe that $Y \in \mathcal{X}_{H_0}$ and $2 = |Y| \leq |V(H_0)| - 2$. Since P consists of at least 4 vertices we have $\delta_D^+(Y) = \{(v_3, v_4)\}$ and $\delta_D^-(Y) = \{(v_1, v_2)\}$. In both cases we deduce $\mathcal{Q}_H(s) \neq \mathcal{P}_H$.

From (ii) to (iii): Suppose G does not contain a subgraph isomorphic to K_4, *net*, *A-graph*, *kite* or C_k with $k \geq 5$. Since every cycle in G has length 2, 3 or 4 it is sufficient to prove that W contains a vertex of every 3-cycle or 4-cycle in G. A vertex u is included in W if and only if it has exactly two neighbors $v, w \in V(G) \setminus W$ and every common neighbor of v and w is not adjacent to a further vertex.
Now, consider a triangle $C = v_1v_2v_3v_1$ in G. There is a vertex, say v_1, which has exactly two distinct neighbors in G. Otherwise G contains a subgraph isomorphic to *net*, that is a contradiction. If $W_{v_2,v_3} = \{v_1\}$ and $v_2, v_3 \notin W$, then we deduce $v_1 \in W$. Hence suppose $|W_{v_2,v_3}| \geq 2$, that is, v_2 and v_3 have further common neighbors in G. If one of these common neighbors is adjacent to a vertex different from v_2 and v_3, then G includes a copy of *kite*. This contradiction yields that W_{v_2,v_3} is contained in W, if $v_2, v_3 \notin W$. Hence W contains exactly one vertex of each triangle in G.
Now, let $C = v_1v_2v_3v_4v_1$ be cycle of length 4 in G. The fact that G does not contain a subgraph isomorphic to K_4 implies that at most one of the vertex pairs v_1, v_3 and v_2, v_4

is adjacent. If G contains either an edge v_1v_3 or v_2v_4, then every vertex of C is part of a triangle. By the previous deduction W includes a vertex of C. Hence we assume that C is an induced 4-cycle in G. Since G does not have a subgraph isomorphic to A-*graph* there are two vertices in C, say v_2 and v_4, which are not adjacent and whose neighbors are elements of $V(C)$. If $W_{v_1,v_3} = \{v_2, v_4\}$, then we conclude that either v_1 and v_3 or v_2 and v_4 are contained in W. Suppose there is any further vertex $u \in W_{v_1,v_3}$ which is also adjacent to a vertex $v \in V(G) \setminus \{v_1, v_3\}$. In this case G includes a subgraph isomorphic to A-*graph* or *kite*, that is a contradiction. Therefore, we deduce that W_{v_1,v_3} is a subset of W, if $v_1, v_3 \notin W$. Hence W contains a vertex of each 4-cycle in G and we conclude that $G - W$ does not contain a cycle of length $k \geq 3$.

From (iii) to (i): Suppose $G-W$ does not contain a cycle of length $k \geq 3$. By Proposition 6.19 there is no k-cycle with $k \geq 5$ in G which contains a vertex from W. Hence every cycle in G has length 2, 3 or 4.
Denote $G_1, \dots, G_p$ the components of G and set $W_i = V(G_i) \cap W$. Our goal is to construct an acyclic orientation D of G such that for all $i \in \{1, \dots, p\}$ and every $X \in \mathcal{X}_{G_i}$ with $2 \leq |X| \leq |V(G_i)| - 2$ one of the arc sets $\delta^+_D(X)$ and $\delta^-_D(X)$ is empty. First consider any acyclic orientation $\tilde{D}$ of $G - W$. We extend $\tilde{D}$ to an orientation of G in the following way. For every vertex pair u, v with $\emptyset \neq W_{u,v} \subseteq W$ and each $w \in W_{u,v}$ we direct the edges incident to w from u to w and from w to v, respectively. Thus every uv-path in G corresponds to a directed (u, v)-path in D. In particular, if $uv \in E(G)$, then we choose u and v such that $(u, v) \in A(\tilde{D})$. Since every vertex in $W_{u,v}$ is only adjacent to u and v this construction yields an orientation D of G. The following two facts imply that D is acyclic. Firstly, D is defined such that it does not contain any directed 2-cycle. Secondly, every cycle of length 3 or 4 is part of a subgraph of G induced by $\{u, v\} \cup W_{u,v}$ for some $u, v \in V(G)$ with $\emptyset \neq W_{u,v} \subseteq W$. Let X be an arbitrary set from $\mathcal{X}_{G_i}$ with $2 \leq |X| \leq |V(G_i)| - 2$. We consider two cases.

Case 1: For every pair $u, v \in V(G_i)$ with $\emptyset \neq W_{u,v} \subseteq W_i$ holds $\{u, v\} \subseteq X$ or $\{u, v\} \subseteq V(G_i) \setminus X$.
In this case for any pair $u, v \in V(G_i)$ with $\emptyset \neq W_{u,v} \subseteq W_i$ we observe that $\{u, v\} \subseteq X$ (respectively $\{u, v\} \subseteq V(G_i) \setminus X$) implies that $W_{u,v}$ a subset of X (respectively $V(G_i) \setminus X$). Otherwise $G_i[X]$ or $G_i - X$ is not connected what contradicts our choice of X. Thus every edge in $\delta_{G_i}(X)$ is contained in $E(G_i - W_i)$.
Now, we show that there is an unique pair of vertices $x \in X$ and $y \in V(G_i) \setminus X$ such that every edge in $\delta_{G_i}(X)$ is adjacent to x and y. Suppose to the contrary that there are edges $xy, x'y' \in E(G_i)$ with $x, x' \in X$ and $y, y' \in V(G_i) \setminus X$ which satisfy $x \neq x'$ or $y \neq y'$. There is a xx'-path P in $G_i[X]$ and a yy'-path Q in $G_i - X$. Hence G_i contains a cycle C of length $k \geq 3$ consisting of P and Q and the edges $xy, x'y'$. Notice that C is not included in $G_i - W_i$. Thus there is a vertex $w \in V(C)$ which is contained in W_i. Since $\delta_{G_i}(X)$ only contains edges from $E(G_i - W_i)$ we deduce that $w \notin \{x, x', y, y'\}$. Moreover, w has exactly two neighbors z and z' in G_i such that $w \in W_{z,z'} \subseteq W_i$. This implies $z, z' \in V(C)$. Furthermore, $\{z, z'\} \cup W_{z,z'}$ is either included in X or in $V(G_i) \setminus X$. Because G has no k-cycle with $k \geq 5$ we observe that $\{z, z'\}$ is equal to $\{x, x'\}$ or $\{y, y'\}$. We conclude that $(G - W_{z,z'}) - zz'$ has a zz'-path via exactly one of the paths P and Q and the edges xy and $x'y'$. This contradicts the construction of W_i and we obtain that

$x = x'$ and $y = y'$. Now, from the construction of D follows that all parallel edges in G have the same orientation in D. Hence either $\delta_D^+(X)$ or $\delta_D^-(X)$ is empty.

Case 2: There is a pair $u, v \in V(G_i)$ with $\emptyset \neq W_{u,v} \subseteq W_i$ such that $u \in X$ and $v \in V(G_i) \setminus X$.
Denote H the subgraph of G induced by $\{u, v\} \cup W_{u,v}$. Suppose there is an edge $u'v' \in \delta_{G_i}(X)$ with $u' \in X$ and $v' \in V(G_i) \setminus X$ which does not connect two vertices in H. Thus u' or v' is not in $\{u, v\} \cup W_{u,v}$. Since $G_i[X]$ is connected there is an uu'-path P in $G_i[X]$. Analogously, v and v' are connected by a path Q in $G_i - X$. For any $w \in W_{u,v}$ there is a k-cycle C in G_i with $k \geq 4$ which consists of P, Q, uwv and $u'v'$. Because G only contains cycles of length at most 4 we obtain that $u' = u$ and $v' \in W_{u,v}$ or $v' = v$ and $u' \in W_{u,v}$. This contradicts the choice of u' and v'. Therefore, every arc in $\delta_{G_i}(X)$ is also included in H. Notice that D induces an orientation of H where either all arcs are directed from u to v or all arcs are directed from v to u. Since $u \in X$ and $v \in V(G_i) \setminus X$ it is not difficult to check that either $\delta_D^+(X)$ or $\delta_D^-(X)$ is empty.

We conclude that for all $i \in \{1, \ldots, p\}$ and every $X \in \mathcal{X}_{G_i}$ with $2 \leq |X| \leq |V(G_i)| - 2$ either $\delta_D^+(X)$ or $\delta_D^-(X)$ is empty. If s denotes the out-degree vector of D, then Theorem 6.14 implies $\mathcal{Q}_G(s) = \mathcal{P}_G$. □

For a labeled graph G with components $G_1, \ldots, G_k$ and $s \in \text{vert}(\mathcal{P}_G)$ we obtain $\mathcal{Q}_G(s)$ by omitting some facet defining inequalities in $\mathcal{P}_G$. In particular, we only use the facet defining sets $X \subseteq V(G_i)$ with $X \in \mathcal{Z}_G(s)$ or $|X| = 1$ or $|X| = |V(G_i)| - 1$. Hence by Theorem 2.9 we deduce the following corollary.

Corollary 6.21
Let G be labeled graph. For all $s \in \text{vert}(\mathcal{P}_G)$ there is an integral vector in $\mathcal{Q}_G(s) \setminus \mathcal{P}_G$ if and only if G contains a subgraph isomorphic to K_4, *net*, *A-graph*, *kite* or C_k with $k \geq 5$.

Proof
Denote $G_1, \ldots, G_k$ the components of G. It suffices to show that $\mathcal{Q}_G(s) \neq \mathcal{P}_G$ implies the existence of desired integral vector. Consider any $s \in \text{vert}(\mathcal{P}_G)$ with $\mathcal{Q}_G(s) \neq \mathcal{P}_G$. Hence there is a set $Y \in \mathcal{X}_{G_i}$ with $2 \leq |Y| \leq |V(G_i)| - 2$ such that $t(Y) < m(G[Y])$ holds for some $t \in \mathcal{Q}_G(s) \setminus \mathcal{P}_G$. By Theorem 2.9 there is an integral vector t' satisfying $t'(X) \geq m(G[X])$ for all $X \in \mathcal{X}_G \setminus \{Y\}$ and $0 \leq t' \leq d_G$. Now, t' is an element of $\mathcal{Q}_G(s)$ since its facet defining sets are included in $\mathcal{X}_G \setminus \{Y\}$. □

Finally, by Theorem 6.20 we are able to characterize all graphs which are degree complete with respect to a cone order.

Corollary 6.22
A labeled graph is degree complete with respect to a cone order if and only if G does not contain a subgraph isomorphic to K_4, *net*, *A-graph*, *kite* or C_k with $k \geq 5$.

7 Conclusion and future work

In this thesis we have studied degree complete graphs and characterized all graphs which have a degree complete labeling. The structure of a connected graph G with such a degree complete labeling is similar to a caterpillar graph. That is, G contains a path P and every vertex of G not included in P is only adjacent to one or two vertices of P. By comparing these graphs to the structure of graphs with a strongly degree complete labelings we have observed two aspects. Firstly, the phenomenon that certain vertices are adjacent to a central path is generated by the degree condition $0 \leq s \leq d_G$ where s is a degree vector of G. Secondly, the path-like structure is basically a result of the used partial order $\preccurlyeq$. Our observations on partial orders induced by cross-free sets (CFPOs) underline this fact. These CFPOs have a close relation to trees. Thus we have proved that for every tree T there is a CFPO $\preceq$ such that T is degree complete with respect to $\preceq$. Particularly, we have characterized all graphs which are degree complete with respect to a CFPO. In this sense we have obtained a generalization of our result on graphs which have a degree complete labeling. During our investigations on labeled graphs we have considered a property which is weaker than the concept of degree completeness with respect to a CFPO. More precisely, we have allowed several extremal elements in the degree sets in this case. We have discussed that this weaker concept is not useful in general. Furthermore, we have proved that the poset $(\mathcal{S}_{\preceq}(G), \preceq)$ is a lattice for every graph G which is degree complete with respect to a CFPO $\preceq$. Finally, we have translated the concept of degree completeness to a geometric property of the degree vector polytope. This approach has led us to an extension of our results on degree complete graphs to the class of cone orders on $\mathbb{R}^n$.

Due to these results we make the following conclusions. Even for the most general setting we considered in this thesis, that is, graphs which are degree complete with respect to a cone order, we have to notice that this class of graphs is quite restrictive. For example, cycles of length at least 5 or complete subgraphs on at least 4 vertices prohibit the existence of a cone order with the desired property. A central motivation for degree complete graphs comes from the determination of the set of degree vectors of a given graph. Thus we suggest two adaptations of the concept of degree complete graphs which yield to further problems that should be investigated.
A first adaptation picks up the results on the concept we mentioned as a weaker version of degree completeness. We have dropped this idea because the number of extremal elements could rise rapidly for arbitrary large graphs. We have noticed that there is basically one CFPO for which all labeled graphs are degree complete in this weaker sense. Unfortunately, in this case nearly all degree vectors are not comparable to each other. Nevertheless, it might be interesting to consider graphs for which the number of extremal elements is bounded by a small integer compared to the order of the given graph. This

seems to be a compromise between the restrictive property of degree completeness on the one hand and the inappropriate fact of many incomparable degree vectors on the other hand. Of course, it is necessary to obtain an idea of sensible choices for such a bound on the extremal elements. A starting point could be the following problem.

Problem 7.1
Characterize all labeled graphs G for which there is a CFPO $\preceq$ satisfying

$$|\overline{S}_{\preceq}(G)| = |\underline{S}_{\preceq}(G)| \leq 2 \quad \text{and} \quad \mathrm{DEG}^{+}(G) = \mathcal{S}_{\preceq}(G).$$

It is likely that the answer to this problem gives a characterization similar to Theorem 5.13, that is, a condition based on critical configurations fulfilling some additional properties.
A second adaptation focuses on additional inequalities which might be added to $\mathcal{S}_{\preceq}(G)$ for a given graph G. We have seen that the additional inequality $0 \leq s \leq d_G$ leads from strongly degree complete graphs to degree complete graphs. Furthermore, the properties concerning lattices are preserved. Now, what could be a suitable inequality? Let G be a labeled graph and $s \in \mathrm{DEG}^{+}(G)$. The characterization of the facets of $\mathcal{P}_G$ from Theorem 2.8 shows that we should consider inequalities of the type

$$m(G[X]) \leq s(X) \leq m(G) - m(G - X), \tag{7.1}$$

where X is a proper nonempty subset of $V(G)$. Actually, $0 \leq s \leq d_G$ is of this type and it covers all subsets of $V(G)$ consisting of a single element. Thus it seems to be useful to add the inequalities

$$1 \leq s(u) + s(v) \leq d_G(u) + d_G(v) - m_{uv},$$

for each $uv \in E(G)$ where m_{uv} denotes the number of edges between u and v. It is easy to see that these inequalities are of the type (7.1) with $X = \{u, v\}$.
Notice that for both adaptations the lattice property which holds for degree complete graphs is lost. For the first adaptation this fact is obvious because a lattice requires an unique maximal and an unique minimal element. Concerning the second adaptation we refer to Example 3.8. We have showed that $(\mathrm{DEG}^{+}(K_5), \preccurlyeq)$ is not a lattice. It is not difficult to check that the set

$$\left\{s \in \mathbb{Z}^n \mid s^l_G \preccurlyeq s \preccurlyeq s^r_G,\ 0 \leq s \leq d_G,\ 1 \leq s(u)+s(v) \leq d_G(u)+d_G(v)-1 \text{ for all } uv \in E(G)\right\}$$

equals $\mathrm{DEG}^{+}(G)$ for $G = K_5$.
The lattice property of degree complete graphs also motivates an investigation of those graphs G for which $(\mathrm{DEG}^{+}(G), \preccurlyeq)$ is a lattice. From our knowledge on degree complete graphs we should assume that this property holds for every subgraph of a graph with this condition. Based on several examples we state the following conjecture for labeled graphs.

Conjecture 7.2
Let G be a labeled graph. The poset $(\mathrm{DEG}^{+}(G), \preccurlyeq)$ is not a lattice if and only if G contains a subgraph which consists of a path P of length $l \geq 2$ and a single edge uv such that $\{u, v\} \cap V(P) = \emptyset$ and there are vertices $w_1, w_2, w_3 \in V(P)$ satisfying $w_1 < u < w_2 < v < w_3$.

Of course, it would be interesting to characterize all graphs which have a labeled version satisfying the lattice property. Such a result could be similar to our characterization of graphs with degree complete labelings. That is, it contains a structural description of the graphs in terms of forbidden subgraphs and a way to recognize such graphs.

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Ich, Sebastian Milz

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