Novel variants and applications of the single-item lot-sizing problem

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Aachen den, April 9, 2019

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Abstract

In this thesis, we study novel variants of the single-item lot-sizing problem, in the context of energy supply, with a focus on heat and power co-production in combination with heat storage units. These variants result from the inclusion of storage bounds, storage deterioration and/or production bounds in the problem formulation. The corresponding models are primarily regarded in the form of mixed integer linear programs.

First, the problems are analyzed with respect to computational complexity. We particularly consider not only linear objective functions, but also study the problems for non-negative and non-decreasing concave ones. Additionally, we study cost functions with the Wagner-Whitin property, that is the absence of speculative motive. Here, all considered problems except those that combine stationary production bounds and stationary deterioration rates are identified as either polynomially solvable or $\mathcal{NP}$-complete, when including known results from the literature.

Moreover, the structure of the underlying convex polytope is studied in the case of only storage deterioration. We modify known ideal and extended formulations for the standard single-item lot-sizing problem to include the deterioration.

Lastly, we explore optimization approaches for a real life problem based on a combined heat and power plant in combination with a heat storage tank. Here, the focus lies on ways to handle uncertainty in the heat demand data, based on a given forecast, primarily in a robust manner. Furthermore, a measure for the realized quality of solutions, i.e., their application to the actual realization of the data, is designed. Finally, we compare static and adaptive approaches in an extensive computational study, also incorporating stochastic objective functions and a regularization method, to prevent unstable behavior.

Zusammenfassung

In dieser Arbeit studieren wir neuartige Varianten des Single-Item Lot-Sizing Problems im Kontext von Energieerzeugung, mit einem Fokus auf Koproduktion von Hitze und Strom, in Verbindung mit Wärmespeichern. Diese Varianten resultieren aus der Einbindung von Speichergrenzen, Speicherverlusten und/oder Produktionsgrenzen in die Problemformulierung. Die korrespondierenden Modelle werden hierbei zumeist in der Form ganzzahliger linearer Programme aufgestellt.

Zuerst untersuchen wir die Probleme im Hinblick auf ihre theoretische Komplexität. Wir betrachten insbesondere nicht nur lineare Zielfunktionen, sondern auch nichtnegative und nicht-fallende konkave Funktionen. Zudem studieren wir Kostenfunktionen mit der Wagner-Whitin Eigenschaft, welche spekulative Motive ausschließt. Hierbei werden alle Probleme, außer diejenigen, welche die Kombination von stationären Produktionsgrenzen und stationären Verfallsraten kombinieren, entweder als polynomiell lösbar oder $\mathcal{NP}$-vollständig identifiziert, wobei Ergebnisse aus der Literatur mit einbezogen werden.

Weiterhin untersuchen wir die Struktur des zugrundeliegenden Polytops für die Variante, welche nur Speicherverluste einbindet. Wir modifizieren eine bekannte vollständige Formulierung sowie eine erweiterte Formulierung für das normale Single-Item Lot-Sizing Problem, sodass die Verluste abgedeckt werden.

Zuletzt behandeln wir Optimierungsansätze for ein Realbeispiel basierend auf einem Blockheizkraftwerk in Verbindung mit einem Wärmespeicher. Hierbei liegt der Fokus auf dem Umgang mit Unsicherheiten in den Wärmedaten, basierend auf einer gegebenen Vorhersage, primär in einer robusten Art und Weise. Weiterhin wird ein Maß für die realisierte Qualität von Lösungen, beziehungsweise deren Anwendung auf die tatsächliche Realisierung der Daten, entwickelt. Schlussendlich vergleichen wir statische und adaptive Ansätze in einer ausführlichen Rechenstudie, wobei auch stochastische Zielfunktionen und Regularisierungsmethoden eingebunden werden, um instabiles Verhalten zu unterbinden.

Contents

Introduction

This thesis revolves around the *single-item lot-sizing problem*, which is a central element of most production or ordering, and demand based problems with means of storing goods. In the basic single-item version, only a single product is considered. Given a demand sequence for this product over a finite and discrete time horizon, the problem calls for a feasible production plan which minimizes accumulated storage and production costs. Here, solutions are primarily characterized by the chosen amount of product to produce/order in each individual period. Literature on this topic is manifold and often concerns adaptations, respectively extensions, of the basic case. See, for instance, the surveys [20, 30, 19] and the book [39].

A great deal of variations of the original problem are considered in theory, as well as in practical applications. In our work, we focus on the inclusion of storage deterioration, storage bounds and production bounds. The motivation for these problem variants are energy supply systems in which the internal demand can be covered by a small scale on-site energy plant in combination with a limited storage device. We particularly consider systems with certain kinds of heat and power co-generation units, heat storage tanks and power market access. Here, the heat energy can be produced in advance and stored, but a deterioration of the held value over time is immanent. In practice, this is the case for most types of energy storage devices like batteries, spinning wheels, heat tanks, or even pumped storage hydroelectric plants which suffer from evaporation. This application opens a new research direction in the context of single-item lot-sizing in which both production and storage are bounded and proportional storage losses in form of deterioration occur and are modeled accordingly.

The main focus of this work lies on developing methods to optimize these problems under heat demand uncertainty which is identified as the dominating factor for rendering solutions based only on forecasts infeasible in practice. To tackle this issue, we mainly employ techniques from the field of robust optimization. Still, we also incorporate other concepts, for example from stochastic optimization. Outside these application-driven approaches, we study the structure of the resulting models, especially the computational complexity, but also the polyhedral structure in the case of only storage deterioration.

While we study variants of the single-item lot-sizing problem, we explicitly exclude backlogging, i.e., demand can be postponed to later time periods, and lost sales, i.e., demand can be neglected, each at a given cost. These variants are popular extensions of the classical single-item lot-sizing problem, but can not simply be applied to the intended applications in the energy sector, due to technical restrictions, environmental regulations and the immediate nature of the demands.

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Contributions

In the following, we list a selection of the most significant results achieved in the context of this thesis. As is common in the field of applied mathematical optimization, some of the presented results, i.e., parts of Chapter 4, are already published, see Coniglio et al. [23, 24].

- The single-item lot-sizing problem with stationary production bounds and time-dependent deterioration rates is shown to be $\mathcal{NP}$-complete.
- Polynomial time algorithms for the single-item lot-sizing problem with time-dependent storage bounds and time-dependent deterioration rates, and the single-item lot-sizing problem with time-dependent storage bounds and stationary production bounds are presented.
- An ideal formulation for the single-item lot-sizing problem with storage deterioration is presented under the assumption of zero final storage value.
- As the above ideal formulation is exponential in size, an extended formulation of polynomial size is considered.
- A static robust single-item lot-sizing model is formulated and shown to be polynomially reducible to a nominal single-item lot-sizing problem.
- An adaptive robust model is developed based on affine rules, which are further adjusted with regularization terms to improve stability when employing realized robustness, as explained later on.
- A robust-stochastic hybrid approach with robust constraints and stochastic objective function is introduced.
- The above approaches are compared in an extensive computational study, by the means of realized robustness, in which robust solutions are methodically applied to the actual realization of the uncertain data.

Outline

This thesis is composed of four chapters.

In the first chapter – *Basic concepts: single-item lot-sizing and data uncertainty* – we introduce the general concepts used in this work. These are the single-item lot-sizing problem with its considered variants, and methods to tackle data uncertainty via robust solutions in mathematical programming. In particular, we study single-item lot-sizing variants with production bounds, storage bounds and storage deterioration. For these, we consider concave costs functions, which are more general than the standard linear ones. We also regard the Wagner-Whitin cost structure, that is the absence of speculative motive, which we generalize to non-linear cost functions. For all variants, we present the standard formulations as well as useful reformulations in the form of mathematical programs. With respect to data uncertainty, we summarize methods to cope with it. Here, we specifically focus on robust optimization under linearity assumptions.

In the second chapter – *Complexity of single-item lot-sizing variants* – we consider the computational complexity of the considered single-item lot-sizing problems under the additional assumption of non-negative and non-decreasing costs. I.e., we aim to categorize the variants as polynomially solvable or $\mathcal{NP}$-complete. Although we do not fully succeed, we solve three critical cases, not yet covered in the literature, for all considered cost functions. We identify the single-item lot-sizing problem with stationary production bounds and time depended deterioration rates as $\mathcal{NP}$-complete. Also, we show that the variants with time-dependent storage bounds and either time-dependent deterioration rates or stationary production bounds are polynomially solvable.

In the third chapter – *Polyhedral study for single-item lot-sizing with storage deterioration* – we perform a polyhedral study for the single-item lot-sizing problem variant with only storage deterioration, under the assumption of zero final storage bounds. An ideal formulation for the standard model via so-called (ℓ, S)-inequalities is extended to include deterioration. As this formulation exhibits an exponential number of constraints, we also adapt an extended formulation which adds only a polynomial number of constraints, but also a polynomial number of auxiliary variables, to achieve a polynomially-sized linear programming formulation of the problem.

In the fourth chapter – *Robustness approaches* – we explore the real life problem that motivated the contents of this thesis. We reformulate it to a single-item lot-sizing problem with storage and production bounds, storage deterioration and linear costs. Subsequently to modeling the nominal case, we formulate multiple approaches to tackle uncertainty, primarily in a robust manner. Here, we employ static robust, adaptive robust and hybrid robust-stochastic concepts. We compare them in a computational study, applying the actual realization of the uncertain data. This is done by employing the concept of realized robustness. That is, solutions for the robust problem are adjusted for scenarios outside the considered uncertainty set, allowing for predefined violations, which are then measured. For the static robust approach, we present a polynomial reduction to a nominal single-item lot-sizing problem, which is viable for many relevant forms of uncertainty sets. As an adaptive robust approach, we employ affine rules, for which we introduce regularization terms, which prove to greatly stabilize the obtained solutions with respect to realized robustness. Furthermore, a robust-stochastic hybrid model, in which the constraints are handled in a robust manner, while the objective costs are stated as an expected value, is developed.

Finally, we complete this thesis with a *Conclusion*, which discusses the achieved contributions of this work, open questions and potential topics of future research.

1 Basic concepts: single-item lot-sizing and data uncertainty

In this section, we introduce the problem-specific and non-standard concepts employed in this work. For the employed standard mathematical programming techniques, we refer to the textbooks [18], [45] and [52].

1.1 General notation

As we are working on a period based problem, we introduce some abbreviating notation motivated by sequences of consecutive periods. For $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$ with $m \leq n$ we let

$$[m, n] := \{m, m+1, \dots, n\}$$

and

$$[n] := [1, n] = \{1, 2, \dots, n\}.$$

We are aware that this notation conflicts with the one of closed continuous intervals, which is why we restrain from using abbreviated notations for continuous intervals.

Also, the mappings

$$a \in A^{[m,n]}$$

are denoted as vectors $(a_m, \dots, a_n)$ with $a_i \in A$ for all $i \in [m, n]$.

When considering a function $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and $n \in \mathbb{Z}_{>0}$, we call f *affinely linear* if for some $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$

$$f(x) = \sum_{i=1}^{n} c_i x_i + d$$

holds for all $x \in A$, and *strictly linear* if additionally $d = 0$ holds.

Furthermore, in any formula, with “const” we denote a fixed computable constant term in the given context.

1.2 Single-item lot-sizing variants

We are studying variants of the single-item lot-sizing problem over a time horizon of $T \in \mathbb{Z}_{>0}$ periods. In each period $t \in [T]$ a given demand $d_t \in \mathbb{R}_{\geq 0}$ has to be met by a combination of that periods production or ordering value $x_t \in \mathbb{R}_{\geq 0}$ and available stored product or inventory value $u_{t-1} \in \mathbb{R}_{\geq 0}$ at the beginning of the period. For the first period the initial storage value u_0 is a given constant, while for all subsequent periods it is equal to the storage value at the end of the previous period. At the end of each period, all excessive product amount $x_t + u_{t-1} - d_t$ is stored as u_t and is transported to the next period. Costs are accounted separately as combined production and setup costs $p_t : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for the production value x_t and holding costs $h_t : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for the storage value u_t in each period, and then accumulated.

Thus, we formulate the standard single-item lot-sizing problem as the following mathematical program:

$$\min \quad \sum_{t=1}^{T} p_t(x_t) + \sum_{t=1}^{T} h_t(u_t) \tag{1.1}$$

$$\text{s.t.} \quad x_t + u_{t-1} - u_t = d_t \qquad \forall\, t \in [T] \tag{1.2}$$

$$x_t, u_t \geq 0 \qquad \forall\, t \in [T] \tag{1.3}$$

Here, constraints (1.2) enforce the product balance. That is, the sum of the product value x_t produced in the current period $t \in [T]$ and the storage value u_{t-1} carried over from the former period is distributed to satisfy the current demand d_t and to contribute to the new storage value u_t.

Note that all feasible solutions are fully characterized by the production sequence x via

$$u_k = u_0 + \sum_{t=1}^{k} (x_t - d_t) \qquad \forall\, k \in [T]. \tag{1.4}$$

This implies that the influence of each production period on any following storage value is linear and, thus, independent of all other production values.

In the case of including storage bounds $0 \leq s_t \leq S_t \leq \infty$ for $t \in [T]$, we add the constraints

$$s_t \leq u_t \leq S_t \qquad \forall\, t \in [T], \tag{1.5}$$

which limit the storage values, to the basic problem formulation (1.1)-(1.3).

When including production bounds or capacities $0 \leq L_t \leq U_t \leq \infty$ for $t \in [T]$, we postulate that if non-zero production takes place in period t, then its value has to be between the lower production bound L_t and the upper production bound U_t. Alternatively, a production value of zero is also feasible. Hence, we add the constraints

$$L_t y_t \leq x_t \leq U_t y_t \qquad \forall\, t \in [T] \tag{1.6}$$

$$y_t \in \{0, 1\} \qquad \forall\, t \in [T], \tag{1.7}$$

in which y_t can be interpreted as a commitment indicator, to the basic problem formulation (1.1)-(1.3). For all $t \in [T]$ this forces $x_t = 0$ for $y_t = 0$, and $L_t \leq x_t \leq U_t$ for $y_t = 1$.

Feasible solutions are still fully characterized under the assumption that $x_t = 0$ is equivalent to $y_t = 0$ for all $t \in [T]$. While this is not true for $L_t = 0$, this case is unproblematic since y_t does not directly impose any costs. Hence, without loss of generality, we can only consider solutions with minimal viable y, which corresponds to the assumed equivalence. We remark that this property only holds due to our convention of defining the production costs p_t to implicitly cover the popular concept of "setup costs", for which we refer to Definition (1.17).

Furthermore, we consider proportional storage deterioration or losses per period. Therefore, we let $0 \leq \alpha_t \leq 1$ denote the conservation rate in period $t \in [T]$, that is the percentage of product that is not lost when stored from period t to period $t+1$. In this case, we substitute the balance constraints (1.2) with

$$x_t + \alpha_t u_{t-1} - u_t = d_t \qquad \forall\, t \in [T]. \tag{1.8}$$

Employing the notation

$$\alpha_{m,n} := \prod_{k=m+1}^{n} \alpha_k \tag{1.9}$$

for $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$, so in particular $\alpha_{m,m} = 1$ and

$$\alpha_{\ell,m}\alpha_{m,n} = \alpha_{\ell,n}$$

for $\ell \in \mathbb{Z}_{\geq 0}$ with $\ell \leq m \leq n$, the production sequence x characterizes the storage values u via

$$u_k = \alpha_{0,k} u_0 + \sum_{t=1}^{k} \alpha_{t,k} \left(x_t - d_t\right) \qquad \forall\, k \in [T] \tag{1.10}$$

in any feasible solution. The linear influence of each production period on any following storage value is maintained in this variation.

We now present a comprehensive formulation that includes all variations, so production bounds, storage bounds and storage deterioration, in the form of the following mathematical program (the generalized single-item lot-sizing problem):

$$
\begin{aligned}
(\mathrm{P}_1)\ \ \min\quad & \sum_{t=1}^{T} p_t(x_t) + \sum_{t=1}^{T} h_t(u_t) && && (1.11)\\
\text{s.t.}\quad & x_t + \alpha_t u_{t-1} - u_t = d_t && \forall\, t \in [T] && (1.12)\\
& s_t \leq u_t \leq S_t && \forall\, t \in [T] && (1.13)\\
& L_t y_t \leq x_t \leq U_t y_t && \forall\, t \in [T] && (1.14)\\
& x_t, u_t \geq 0 && \forall\, t \in [T] && (1.15)\\
& y_t \in \{0,1\} && \forall\, t \in [T] && (1.16)
\end{aligned}
$$

We obtain any desired variant by setting the characterizing parameters of the undesired variations to their trivial values, that is $\alpha = 1$ for total conservation, $s = 0$ and $S = \infty$ for unrestricted storage and $L = 0$ and $U = \infty$ for unrestricted production. Note that for $U_t = \infty$, the former can be substituted with a sufficiently big constant $M \in \mathbb{R}$ in all models. In the remainder of the chapter, we generally consider formulation (P_1) for all variants.

1.2.1 Cost structures

Among the considered cost structures in this work, the classic and most studied one is the affinely linear one (LIN). It consists of periodic production cost functions

$$p_t(x_t) := \begin{cases} 0 & \text{if } x_t = 0 \\ c^{\mathrm{v}}_t x_t + c^{\mathrm{f}}_t & \text{otherwise} \end{cases} \qquad \forall\, t \in [T] \tag{1.17}$$

with fixed production or setup costs $c^{\mathrm{f}} \in \mathbb{R}^T_{\geq 0}$ and variable production costs $c^{\mathrm{v}} \in \mathbb{R}^T$, and periodic storage cost functions

$$h_t(u_t) := c^{\mathrm{i}}_t u_t + \text{const} \qquad \forall\, t \in [T]$$

with variable production costs $c^{\mathrm{i}} \in \mathbb{R}^T$. The standard mixed integer linear programming reformulation for model (P_1) reads

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T} \left(c^{\mathrm{v}}_t x_t + c^{\mathrm{f}}_t y_t + c^{\mathrm{i}}_t u_t \right) && && (1.18)\\
\text{s.t.} \quad & x_t + \alpha_t u_{t-1} - u_t = d_t && \forall\, t \in [T] && (1.19)\\
& s_t \leq u_t \leq S_t && \forall\, t \in [T] && (1.20)\\
& L_t y_t \leq x_t \leq U_t y_t && \forall\, t \in [T] && (1.21)\\
& x_t, u_t \geq 0 && \forall\, t \in [T] && (1.22)\\
& y_t \in \{0,1\} && \forall\, t \in [T]. && (1.23)
\end{aligned}$$

Here, the commitment variable $y \in \{0,1\}^T$ is used to linearize the cost function (1.17).

We also consider the more general concave cost functions (CONC), in which for $t \in [T]$ both p_t and h_t are concave for $0 \leq x_t \leq U_t$ and $s_t \leq u_t \leq S_t$. Mark that, due to the non-negativity of c^{f}_t, all affinely linear cost functions can be extended to concave cost functions on $\mathbb{R}^T_{\geq 0}$. Thus, (LIN) is a subclass of (CONC).

Additionally, we consider the concept of the Wagner-Whitin property for costs (WW), see Wagner and Whitin [51], that is the absence of speculative motive. I.e., for fixed commitment $y \in \{0,1\}^T$, as induced by constraint (1.6), whenever it is feasible to substitute a positive amount of production value in a period by a corresponding (for $\alpha = 1$ identical) amount of production in a later period, the overall costs do not increase by doing so. Intuitively, this implies that when commitment is fixed, we prefer to produce as late as possible.

Since feasible solutions are fully characterized by the production values $x \in \mathbb{R}^T_{\geq 0}$ via equalities (1.10), it holds that if some production value $P \in \mathbb{R}_{>0}$ in period j can be feasibly replaced with an additional production value $P' \in \mathbb{R}$ in period k with $1 \leq j < k \leq T$, then the latter has to suffice $P' = \alpha_{j,k}P$. We formally define the Wagner-Whitin property for costs as follows.

Theorem 1. *An instance of the single-item lot-sizing problem holds the Wagner-Whitin property for costs, when for all feasible solutions* $(x, y, u) \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T \times \mathbb{R}^T_{\geq 0}$ *the following holds:*
For all choices of periods $j, k \in [T]$ *with* $1 \leq j < k \leq T$ *and a production value* $P \in \mathbb{R}_{\geq 0}$*, let* $(x', y, u') \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T \times \mathbb{R}^T_{\geq 0}$ *be the solution characterized by*

$$x'_i = \begin{cases} x_i - P & \text{if } i = j \\ x_i + \alpha_{j,k}P & \text{if } i = k \\ x_i & \text{otherwise.} \end{cases}$$

If (x', y, u') *is feasible, then the objective value of* (x', y, u') *is not larger than the objective value of* (x, y, u)*. That is,*

$$p_j(x_j - P) + \sum_{i=j}^{k-1} h_i(u_i - \alpha_{ji}P) + p_k(x_k + \alpha_{jk}P) \leq p_j(x_j) + \sum_{i=j}^{k-1} h_i(u_i) + p_k(x_k)$$

holds.

In the former, we exploit that for all considered solutions above, $u'_i = u_i$ holds for all $i \in [1, j-1] \cup [k, T]$. Thus, the difference in the total costs is given by the sum of differences in production costs in periods j and k and the differences in storage costs in periods $[j, k]$.

We stress that this cost structure does not imply our formerly stated concave cost structure, as it, for example, allows for discontinuities in the storage costs. Hence (WW) is not a subclass of (CONC).

Lastly, our results also cover the linear Wagner-Whitin (WWL) case, which is the intersection of (LIN) and (WW). This class matches the original definition by Wagner and Whitin [51]. Here, given an affinely linear cost structure, the condition for it to hold the Wagner-Whitin property simplifies to

$$c^{\mathrm{v}}_t + c^{\mathrm{i}}_t \geq \alpha_t c^{\mathrm{v}}_{t+1} \qquad \forall\, t \in [T]$$

with $c^{\mathrm{v}}_0 := 0$ and $c^{\mathrm{v}}_{T+1} := 0$.

1.2.2 Reduced single-item lot-sizing formulations

In this section, we discuss methods to reformulate (P_1), such that the resulting models facilitate a structural analysis of the problem.

First, we consider a linear reduction which allows us to assume zero lower storage bounds in all results concerning theoretical complexity. This kind of reduction is well known for the standard single-item lot-sizing problem with stationary lower storage bounds and general upper storage bounds, and can easily be expanded to cover production bounds and storage deterioration. Van Den Heuvel and Wagelmans [49] expanded the approach to also cover general storage and production bounds. The combination of non-stationary lower storage bounds and storage deterioration rates prevents us from directly adapting their result, because they rely on a monotonicity property in a reformulation of the problem, which is not preserved when adding storage deterioration. Still, we achieve a similar result.

Lemma 1. *Any instance of* (P_1) *can be linearly reduced to an instance of* (P_1) *with zero lower storage bounds, zero initial storage value and (adjusted) non-negative demands, by modifying the upper storage bounds, the storage variables and the storage costs.*

Proof. The core of the reduction concept is to shift the storage bound constraints by the lower bound in each period. Here, the difference in consecutive lower bounds is compensated by adjusting the demands. To do so without rendering the adjusted demands negative, we tighten the lower storage bounds prior. This is done by iteratively increasing them as much as possible without compromising the set of feasible solutions. The aim of this is to limit the difference in consecutive lower storage bounds. Lastly, the initial storage value is distributed to the first available demand values.

Given an instance of (P_1), we iteratively determine

$$\begin{aligned} &\hat{s}_0 := u_0 \\ &\hat{s}_t := \max\{s_t, \alpha \hat{s}_{t-1} - d_t\} && \forall\, t \in [T] \end{aligned}$$

in linear time. Then, we can substitute the lower storage bounds s in the original problem with $\hat{s}$ without influencing the set of feasible solutions. This is due to the fact that the storage balance constraints

$$x_t + \alpha_t u_{t-1} = u_t + d_t$$

imply

$$\alpha_t s_{t-1} - d_t \leq u_t,$$

as $s_t \leq u_t$ and $x_t \geq 0$ hold for all $t \in [T]$. Hence,

$$\hat{s}_t \leq u_t$$

iteratively dominates

$$s_t \leq u_t$$

for $t = 1, \ldots, T$, by the definition of $\hat{s}_t$.

We also iteratively determine

$$\begin{aligned} &h'_t(x) := h_t(x + \hat{s}_t) && \forall\, t \in [T] \\ &S'_t := S_t - \hat{s}_t && \forall\, t \in [T] \\ &d'_t := d_t + \hat{s}_t - \alpha_t \hat{s}_{t-1} && \forall\, t \in [T] \end{aligned}$$

in linear time. By shifting the storage variables via $u'_t = u_t - \hat{s}_t$ for all $t \in [T]$, an equivalent single-item lot-sizing problem is given by zero lower storage bounds $s' = 0$, demands d', upper storage bounds S', storage variables u', storage costs h' and all other values adopted from the original formulation. This can be easily verified by substituting the values. Mark that the demands d' are non-negative by definition, as

$$\hat{s}_t \geq \alpha \hat{s}_{t-1} - d_t$$

holds for all $t \in T$.

Lastly, we define

$$\begin{aligned}
&H_0 := u_0 && \\
&d''_t := \max\{d'_t - \alpha_t H_{t-1}, 0\} && \forall\, t \in [T] \\
&H_t := \alpha_t H_{t-1} - (d'_t - d''_t) && \forall\, t \in [T] \\
&S''_t := S'_t - H_t && \forall\, t \in [T] \\
&h''_t(x) := h'_t(x + H_t) && \forall\, t \in [T].
\end{aligned}$$

Here, shifting the storage via $u''_t = u'_t - H_t$ for all $t \in [T]$ yields the proposed formulation with zero initial storage value $u''_0 = 0$, demands d'', upper storage bounds S'', storage costs h'' and all other values adopted from the previous formulation. Again, this is easily verified by substituting the values. □

We illustrate the above reduction in the following example.

Example 1. *Starting with values*

$$u_0 = 4,\ \ \alpha = (1,1,1),\ \ d = (2,2,2),\ \ s = (5,1,2)\ \textit{and}\ S = (6,6,4),$$

the first reduction stage results in

$$\hat{s} = (5,3,2),$$

the second reduction stage results in

$$S' = (1,3,2)\ \textit{and}\ d' = (7,0,1)$$

and the third stage results in

$$S'' = (1,3,2)\ \textit{and}\ d'' = (3,0,1),$$

as depicted in Figure 1.1.

Beside the upper storage bounds, which turn from stationary to time-dependent ones, the storage costs are modified, as stated in the reduction. This is essential, since the cost structure is often fundamental to efficient algorithms.

Lemma 2. *The above reductions preserve all cost structures considered in this work.*

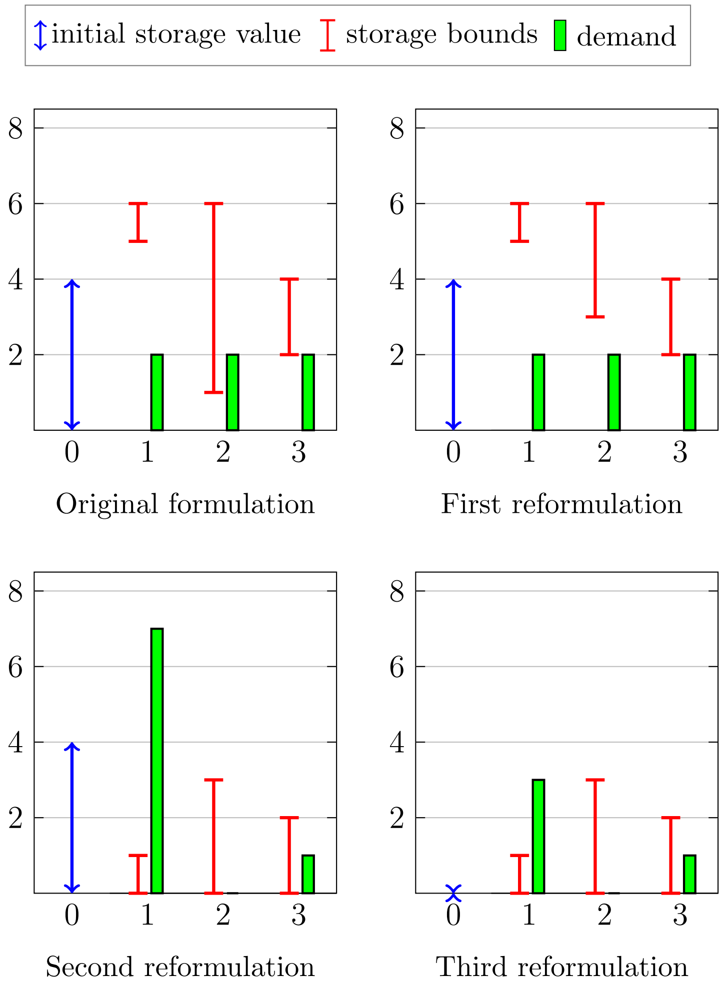


Figure 1.1: A three period example

Proof. The concave and Wagner-Whitin structures are invariant under increasing the actual storage value. For the affinely linear case, only constant additional costs occur and affine linearity is maintained. □

While the presented reduction can be used to eliminate lower storage bounds, which is necessary in Section 2.3.2, it barely simplifies most presented algorithms. Thus, we still include lower storage bounds unless stated otherwise, to retain a broader applicability.

In the following, we formulate a well known equivalent mathematical program to (P_1), in which all redundant "bookkeeping" variables are eliminated. As the storage value sequence $u \in \mathbb{R}^T_{\geq 0}$ is uniquely induced by the production-value sequence $x_t \in \mathbb{R}^T_{\geq 0}$ in any feasible solution to (P_1), as described in (1.10), we derive the more compact formulation

$$(\mathrm{P}_2)\ \min \quad \sum_{t=1}^{T} p_t(x_t) + \sum_{t=1}^{T} h_t \left(\alpha_{0,t} + \sum_{i=1}^{t} \alpha_{i,t} \left(x_i - d_i \right) \right)$$

$$\begin{aligned} \text{s.t.} \quad & s_t \leq \alpha_{0,t} + \sum_{i=1}^{t} \alpha_{i,t} \left(x_i - d_i \right) \leq S_t && \forall\, t \in [T] \\ & L_t y_t \leq x_t \leq U_t y_t && \forall\, t \in [T] \\ & x_t \geq 0 && \forall\, t \in [T] \\ & y_t \in \{0,1\} && \forall\, t \in [T] \end{aligned}$$

by eliminating the storage variables u_t. Consequently, the feasible solutions of (P_1) and (P_2) are equivalent. That is, any feasible solution to (P_2) implies a corresponding feasible solution with identical value to (P_1) by employing (1.10), and any feasible solution to (P_1) implies a corresponding feasible solution with identical value to (P_2) by discarding the storage values u.

The cost structures (LIN), (CONC), (WW) and (WWL) are maintained by

$$f_t(x_t) := p_t(x_t) + h_t \left(\alpha_{0,t} + \sum_{i=1}^{t} \alpha_{i,t} \left(x_i - d_i \right) \right) \tag{1.24}$$

in the canonical sense. Furthermore, the objective function in (P_1) can directly be substituted with (1.24). This is useful in cases, in which we want to maintain the storage variable for accessibility, but still prefer simplifying the objective in order to be dependent only on the production values.

1.3 Tackling uncertainty in mathematical programming

Uncertain data is inherent in most practical optimization problems. It occurs due to measurement errors, imperfect information, or external decisions and factors. There are different approaches to tackle this issue, the arguably simplest being to optimize against a single data forecast (determined by the practitioner) and then heuristically adjusting it to the realized values. More elaborate approaches are discussed in the following section.

1.3.1 A word on stochastic optimization

Historically, classical approaches to handle uncertainty in single-item lot-sizing are stochastic in nature, dating back as early as 1960, see [43]. Here, the idea is to first assign a probability distribution to the uncertain data. Then, a solution of minimum expected costs which is feasible with sufficiently high probability is determined.

In this context, Liyanage and Shanthikumar [36] pointed out an issue of the solutions yielded by this method, when the latter are implemented with the demand that realizes in practice. That is, even if the distribution is estimated within high precision, they can be substantially more costly than they were predicted to be (or infeasible, depending on how the problem is formulated). Moreover, and regardless of the accuracy of the estimation, such stochastic techniques are in many cases intrinsically doomed to suffer from the curse of dimensionality, see [13]. The reason for this is that they usually require a computational time, which is at least linear with respect to the size of the probability space (assumed to be finite) to which the realized demand belongs. Typically, such space itself is exponential in size with respect to the size of the nominal problem instance.

For an alternative way of modeling stochastic demands, via so-called service-level constraints, see [17, 46, 47]. Also, see [21] for a stochastic programming approach in which feasibility is enforced in expectation.

1.3.2 Robust optimization

A different and from a computational perspective often more affordable option is resorting to a *robust optimization* approach. The idea is to look for a single solution which is feasible for any realization belonging to a given *uncertainty set*, and which among all such feasible solutions is of minimum costs in the *worst-case*. So, given a deterministic mathematical programming problem

$$\min_{x \in X} \{c(x)\},$$

with $X \subseteq \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$, and introducing an uncertainty set U which represents scenarios $X_u \subseteq \mathbb{R}^n$ and $c_u : \mathbb{R}^n \to \mathbb{R}$ for all $u \in U$, the robust problem reads

$$\min_{x \in \bigcap_{u \in U} X_u} \left\{ \sup_{u \in U} \{c_u(x)\} \right\}.$$

It is common to assume that there is a deterministic problem motivating the robust problem and that there exists a nominal scenario $\bar{u} \in U$ with $(X, c) = (X_{\bar{u}}, c_{\bar{u}})$. Clearly, the complexity of the robust problem is no less than that of the deterministic problem, and it heavily depends on the set of the (possibly infinitely many) scenarios. Conversely, quite simple uncertainty sets can render well-tractable deterministic problems rather hard to solve in their robust version.

Example 2. *The shortest-path problem can be solved in quadratic time in the nominal case. It is rendered $\mathcal{NP}$-hard when considering the robust version, even for only two scenarios (i.e. $|U| = 2$), see Yu and Yang [53].*

1.3.3 Linear data uncertainty in mixed integer linear programs

Consider a standard mixed integer linear programming formulation

$$\begin{aligned} \min \quad & c'^T x' \\ \text{s.t.} \quad & A'x' \leq b' \\ & x' \geq 0 \\ & x_i \in \mathbb{Z} && \forall\, i \in [\ell]. \end{aligned}$$

with $c' \in \mathbb{R}^{n'}$, $A' \in \mathbb{R}^{m' \times n'}$, $b' \in \mathbb{R}^{m'}$ and $\ell \in [n']$. In this context, uncertainties are commonly modeled in a more specific form based on an uncertainty set $D \subseteq \mathbb{R}^k$. This is done via

$$c'(d) = (c'_i(d))_{i\in[n']}, \qquad A'(d) = (a'_{j,i}(d))_{j\in[m'],i\in[n']}, \qquad b'(d) = (b'_j(d))_{j\in[m']}$$

for $d \in D$ with $c'_i : D \to \mathbb{R}$, $a'_{j,i} : D \to \mathbb{R}$ and $b'_j : D \to \mathbb{R}$ being affinely linear functions for all $j \in [m']$ and $i \in [n']$. Thus, we can formulate the robust counterpart as the following incremental semi-infinite mathematical program:

$$\begin{array}{lll} \min & \sup\limits_{d\in D}\{c'(d)^T x'\} & \\ \text{s.t.} & A'(d)x' \leq b'(d) & \forall\, d \in D \\ & x' \geq 0 & \\ & x'_i \in \mathbb{Z} & \forall\, i \in [\ell] \end{array}$$

This can be reformulated as

$$\begin{array}{lll} \min & \eta & \\ \text{s.t.} & A'(d)x' - b'(d)y \leq 0 & \forall\, d \in D \\ & \eta - c'(d)^T x \geq 0 & \forall\, d \in D \\ & y = 1 & \\ & x', y \geq 0 & \\ & \eta \text{ free} & \\ & x'_i \in \mathbb{Z} & \forall\, i \in [\ell], \end{array}$$

that is

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & A(d)x \leq b & \forall\, d \in D \\ & x \geq 0 & \\ & x_i \in \mathbb{Z} & \forall\, i \in [\ell] \end{array}$$

for suitable $c \in \mathbb{R}^n$, $A : D \to \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ with

$$A(d) = (a_{j,i}(d))_{j\in[m],i\in[n]}$$

again consisting of affinely linear functions $a_{j,i} : D \to \mathbb{R}$ for all $j \in [m]$ and $i \in [n]$. Thus, all influence of the uncertainty is transferred to the constraint matrix. This will be the base for all further considerations.

In the following, we consider the special case of uncertainty sets D, in which the convex hull of the closure

$$\tilde{D} := \operatorname{conv}(\overline{D})$$

is a convex polytope. This particularly includes finite and bounded polyhedral uncertainty sets.

Let $D' \subseteq \tilde{D}$ denote the finite set of extreme points of $\tilde{D}$. Then for fixed $j \in [m]$ the constraints imposed by the j-th row of A

$$\sum_{i \in [n]} a_{j,i}(d) x_i \leq b_j \qquad \forall\, d \in D$$

are equivalent to

$$\sum_{i \in [n]} a_{j,i}(d) x_i \leq b_j \qquad \forall\, d \in \tilde{D},$$

which are equivalent to

$$\sum_{i \in [n]} a_{j,i}(d) x_i \leq b_j \qquad \forall\, d \in D'.$$

The latter induce an incremental mixed integer linear programming formulation of the robust problem by individually imposing each of the m' rows only for the finitely many scenarios in D'.

Furthermore, the above constraints are equivalent to the single but non-linear constraint

$$\max_{d \in \tilde{D}} \left\{ \sum_{i \in [n]} a_{j,i}(d) x_i \right\} \leq b_j, \tag{1.25}$$

which we aim to reformulate to a system of linear expressions to obtain an alternative mixed integer linear programming formulation. As $A(d)_j^T x$ is an affinely linear function in d for any fixed x, the former can be reformulated via

$$\sum_{i \in [n]} a_{j,i}(d) x_i = f(x)^T d + h(x) \tag{1.26}$$

with

$$f(x) = (f_i(x))_{i \in [n]}$$

and $f_i : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ being affinely linear functions. Given an (extended) representation

$$\tilde{D} = \left\{ d \in \mathbb{R}^k_{\geq 0} \;\middle|\; \exists\, q \in \mathbb{R}^r_{\geq 0} :\; Gd + Hq \leq \ell \right\}$$

with $r \in \mathbb{Z}_{\geq 0}$, $G \in \mathbb{R}^{p \times k}$, $H \in \mathbb{R}^{p \times r}$ and $\ell \in \mathbb{R}^p$, we reformulate the maximum in constraint (1.25) as the linear program

$$\begin{aligned} \max \quad & f(x)^T d + h(x) \\ \text{s.t.} \quad & Gd + Hq \leq \ell \\ & d, q \geq 0. \end{aligned}$$

By fixing x, we can employ linear programming duality to reformulate it as

$$\begin{aligned} \min \quad & \ell^T \lambda + h(x) \\ \text{s.t.} \quad & G^T \lambda \geq f(x) \\ & H^T \lambda \geq 0 \\ & \lambda \geq 0. \end{aligned}$$

Thus, constraint (1.25) can be substituted by

$$\begin{aligned} & \ell^T \lambda + h(x) \leq b_j \\ & G^T \lambda - f(x) \geq 0 \\ & H^T \lambda \geq 0 \\ & \lambda \geq 0, \end{aligned}$$

i.e., a finite number of new continuous variables and linear constraints.

We stress that after the substitution x is regarded as a variable again. This is viable as the above reformulation is well posed for each individual fixed x. Hence, given an extended formulation of $\tilde{D}$, we obtain an alternative mixed integer linear programming reformulation with respect to the aforementioned incremental one.

In the following, we consider the popular concept of *budgeted* or Γ-robustness for modeling uncertainty, introduced by Bertsimas and Sim [12]. Here, each uncertain parameter $d_i \in \mathbb{R}$ is assumed deviate by at most $\hat{d}_i \in \mathbb{R}_{\geq 0}$ from its nominal value $\bar{d}_i \in \mathbb{R}$, that is

$$\bar{d} - \hat{d} \leq d \leq \bar{d} + \hat{d}.$$

As it is deemed overly-pessimistic to expect each uncertain parameter to deviate maximally from its nominal value, the idea is to only consider scenarios in which the sum of absolute relative deviations

$$\sum_{i=1}^{k} \left| \frac{d_i - \bar{d}_i}{\hat{d}_i} \right|$$

is limited by a chosen value $\Gamma \in \mathbb{R}_{\geq 0}$. This particularly includes the scenarios in which up to Γ values deviate by the maximum amount while the other values coincide with the nominal value.

We fit this concept in this section's notation, while imposing $\Gamma \in \mathbb{Z}_{\geq 0}$ for notational convenience.

Example 3. *For $\Gamma \in \mathbb{Z}_{\geq 0}$, nominal values $\bar{d} \in \mathbb{R}^k$ and maximal deviations $\hat{d} \in \mathbb{R}^k_{\geq 0}$, the corresponding uncertainty set is defined as the bounded polytope*

$$\tilde{D} = D := \left\{ d \in \mathbb{R}^k \;\middle|\; \begin{array}{l} \sum_{i=1}^{k} \left| \frac{d_i - \bar{d}_i}{\hat{d}_i} \right| \leq \Gamma \\ \bar{d}_i - \hat{d}_i \leq d_i \leq \bar{d}_i + \hat{d}_i \quad \forall\, i \in [k] \end{array} \right\}.$$

As the extreme points of D are given by

$$D' = \left\{ d \in \mathbb{R}^k \;\middle|\; \begin{array}{l} \sum_{i=1}^{k} \left| \frac{d-\bar{d}}{\hat{d}} \right| = \Gamma \\ d_i \in \{\bar{d}_i - \hat{d}_i, \bar{d}_i, \bar{d}_i + \hat{d}_i\} \quad \forall\, i \in [k] \end{array} \right\},$$

the critical scenarios are those in which exactly Γ values deviate by the maximum amount in one direction.

An extended linear programming formulation is given by

$$\tilde{D} = \left\{ d \in \mathbb{R}^k \;\middle|\; \exists\, (y^+, y^-) \in \mathbb{R}^k_{\geq 0} \times \mathbb{R}^k_{\geq 0} : \begin{array}{ll} d_i = \bar{d}_i + \left(y_i^+ - y_i^-\right) \hat{d}_i & \forall\, i \in [k] \\ y_i^+ + y_i^- \leq 1 & \forall\, i \in [k] \\ \sum_{i=1}^{k} \left(y_i^+ + y_i^-\right) \leq \Gamma & \end{array} \right\}.$$

Consequently, constraint (1.25) *can be substituted with*

$$\begin{array}{ll} \sum_{i=1}^{k} \bar{d}_i \kappa_i + \sum_{i=1}^{k} \chi_i + \Gamma\tau + h(x) \leq b_j & \\ \kappa_i = f(x)_i & \forall\, i \in [k] \\ -\hat{d}_i \kappa_i + \chi_i + \tau \geq 0 & \forall\, i \in [k] \\ \hat{d}_i \kappa_i + \chi_i + \tau \geq 0 & \forall\, i \in [k] \\ \kappa_i \text{ free} & \forall\, i \in [k] \\ \chi_i, \tau \geq 0 & \forall\, i \in [k], \end{array}$$

with $f_i : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ being the affinely linear functions derived from identity (1.26)*, as discussed above.*

We remark that throughout this example we retain the expandable variable d in favor of notational consistence. This directly translates to the introduction of the likewise expandable variable κ in the dualization. Eliminating either of these variables during the reformulation results in the arguably more accessible system

$$\begin{array}{ll} \sum_{i=1}^{k} \bar{d}_i f(x)_i + \sum_{i=1}^{k} \chi_i + \Gamma\tau + h(x) \leq b_j & \\ \chi_i + \tau \geq \hat{d}_i f(x)_i & \forall\, i \in [k] \\ \chi_i + \tau \geq -\hat{d}_i f(x)_i & \forall\, i \in [k] \\ \chi_i, \tau \geq 0 & \forall\, i \in [k]. \end{array}$$

Two seminal papers applying a robust optimization approach to single-item lot-sizing are [9, 13]. They address the uncapacitated version of single-item lot-sizing with backlogging, static storage and production costs, with demands subject to Γ-robustness [11, 12].

Among other results, the authors of [9, 13] show that the Γ-robust counterpart of their considered variant of single-item lot-sizing (including backlogging) can be solved as an instance of the deterministic problem with modified demands. This still applies if production bounds are in place. Notice that for this result to hold, bounds on the storage can not be enforced. Unfortunately, such bounds are necessary in many applications, for example in the energy sector. Backlogging is not viable in this context as the demand of energy must be satisfied when issued.

1.3.4 Adaptive robust optimization

The concept of robust solutions can be extended to adaptive solutions, which are not given by a single solution vector, but by a function f from a given class

$$F \subseteq \mathbb{R}[U] := \{f : U \to \mathbb{R}^n \mid f(u) \in X_u \;\forall\, u \in U\}\,.$$

Thus the function f provides a feasible solution dependent on the realization. For mathematical programming problems, the corresponding adaptive robust counterpart can be stated as

$$\min_{f\in F}\left\{\sup_{u\in U}\{c_u(f(u))\}\right\}.$$

In the case of $F = \mathbb{R}[U]$ a canonical optimal solution is given by

$$f(u) \in \operatorname*{argmin}_{x\in X_u}\{c_u(x)\}$$

for all $u \in U$. Hence, the robust problem can be decomposed to separately solving the deterministic problems for all $u \in U$. Thus, this general approach is redundant for problems in which there are no restrictions on the times between the realization of the data and the implementation of the solution, as we could simply wait for the realization and then solve the induced nominal problem.

In practice, the class of considered functions F is often restricted to abide causality or other application induced limitations, or to avoid issues like tractability or even computability. We remark that adaptive robust optimization approaches tend to be very problem specific.

Example 4. *In the case of recoverable robustness, the form of f is restricted to a base solution, which is the solution for the nominal case, and a predefined recourse action dependent on the realization (inside the uncertainty set).*

For instance, consider a design problem under uncertainty, where the designer has to decide on a selection of component to fulfill some structural requirements. After the installation, the uncertainty unveils in that up to Γ of components can fail. The designer then can react by repairing up to $R < \Gamma$ of the failed components.

Hence the solution function f consists of two parts. The first part is a design feasible decision of the nominal problem, which is independent of the realization of the uncertainty. The second part depends on the set Z of at most Γ failing machines and returns a subset $Z' \subseteq Z$ of at most R components that are to be repaired.

Mark that, for a feasible robust solution, the first part has to be chosen such that it allows for a valid repair action (that restores the structural requirements) for each realization of the uncertainty.

For a more detailed definition and more explicit examples see [35].

In particular, one can consider the following form of recoverable robustness:

Example 5. *In two-stage robust optimization the variables are partitioned in two sets. The first-stage (or here-and-now) variables are required to take a value independently of the realization of the uncertain parameters. The second-stage (or wait-and-see) variables are allowed to vary as a function of the specific realization that takes or has taken place, depending on whether the problem has a dynamical aspect or not. In linear programming, variables whose value is uniquely given as a linear function of the other variable values in each feasible solution via equality constraints, are predestinated as second stage variables.*

The way we employ adaptive robust optimization is via affine rules, see Section 4.4.1.

2 Complexity of single-item lot-sizing variants

The aim of this chapter is to classify arbitrary variants of the single-item lot-sizing problem, considering combinations of storage bounds, production bounds and storage deterioration, with different degrees of flexibility in the variant-characterizing parameters, as polynomially solvable or $\mathcal{NP}$-complete.

For convenience, we work with model (P_1). Although all approaches can be adjusted to the equivalent formulation (P_2), the existence of the, generally redundant, storage variables proves useful for a convenient block structure as defined later on. This block structure then serves as the basis for constructing a dominant set of solutions over which we can efficiently optimize in certain cases.

In addition to the general definitions in Chapter 1, we make the assumption of non-negative and non-decreasing costs in this chapter. Otherwise the set of feasible solution might be unbounded and only solutions with redundant production values, i.e., production values that can be lowered without compromising the feasibility of the induced solution, might be optimal. While this is a non-trivial assumption, it is often met in practical applications, as an additional production value seldom results in lower production or storage costs.

We consider any combination of trivial, stationary or time-dependent bounds and deterioration rates as described in the following. The problem structure is denoted in the form COST-STOR-CAP-DET, with:

- COST describes the cost structure as defined in Section 1.2.1 as concave CONC, affinely linear LIN, Wagner-Whitin WW or Wagner-Whitin and affinely linear WWL. Recall that costs are always assumed to be non-negative and non-decreasing in this chapter.

- STOR describes the storage bounds as either redundant $(0, \infty)$, stationary (s, S) or time dependent (s_t, S_t).

- CAP describes the production bounds as either redundant $(0, \infty)$, stationary (L, U) or time dependent (L_t, U_t).

- DET describes the storage conservation rates as either redundant 1, stationary α or time dependent α_t.

For example, the standard single-item lot-sizing problem with concave costs is donoted as CONC-$(0, \infty)$-$(0, \infty)$-1.

2.1 Results and contributions

Table 2.1 shows an overview of currently known results for the single-item lot-sizing problem with various settings of storage bounds, production bounds and storage deterioration rates. We fit existing results into our notation and integrate our own contributions as [*·], with · referencing the corresponding subsection in which the result is derived. Original results are underlined, while the others are simple implications derived from the original result in the same box. The $\mathcal{O}(T^2)$ run-time of the standard single-item lot-sizing problem with concave costs is considered folklore and $\mathcal{NP}$-complete problems are denoted as $\mathcal{NPC}$. Although we do not explicitly study affinely linear costs ourselves, we do include them to better cover the existing literature.

Table 2.1: Complexity results

(s_t, S_t)	(L_t, U_t)	α_t	CONC	LIN	WWL	WW
$(0,\infty)$	$(0,\infty)$	1	$\underline{\mathcal{O}(T^2)}$	$\underline{\mathcal{O}(T\log(T))}$ [25]	$\underline{\mathcal{O}(T)}$ [25]	$\mathcal{O}(T^2)$
(s_t, S_t)	$(0,\infty)$	1	$\underline{\mathcal{O}(T^3)}$ [37]	$\underline{\mathcal{O}(T^2)}$ [3]	$\mathcal{O}(T^2)$	$\mathcal{O}(T^2)$
$(0,\infty)$	$(0,\infty)$	α_t	$\mathcal{O}(T^3)$	$\underline{\mathcal{O}(T^3)}$ [29]	$\mathcal{O}(T^2)$	$\mathcal{O}(T^2)$
$(0,\infty)$	$(0,\infty)$	α	$\mathcal{O}(T^3)$	$\mathcal{O}(T^3)$	$\mathcal{O}(T^2)$	$\mathcal{O}(T^2)$
(s_t, S_t)	$(0,\infty)$	α_t	$\underline{\mathcal{O}(T^3)}$ [*2.3.1]	$\mathcal{O}(T^3)$	$\mathcal{O}(T^2)$	$\underline{\mathcal{O}(T^2)}$ [*2.3.2]
(s_t, S_t)	$(0,\infty)$	α	$\mathcal{O}(T^3)$	$\mathcal{O}(T^3)$	$\mathcal{O}(T^2)$	$\mathcal{O}(T^2)$
(s, S)	$(0,\infty)$	α_t	$\mathcal{O}(T^3)$	$\mathcal{O}(T^3)$	$\mathcal{O}(T^2)$	$\mathcal{O}(T^2)$
(s, S)	$(0,\infty)$	α	$\mathcal{O}(T^3)$	$\mathcal{O}(T^3)$	$\mathcal{O}(T^2)$	$\mathcal{O}(T^2)$
·	(L_t, U_t)	·	$\mathcal{NPC}$	$\mathcal{NPC}$	$\underline{\mathcal{NPC}}$ [26]	$\mathcal{NPC}$
$(0,\infty)$	(L, U)	1	$\underline{\mathcal{O}(T^6)}$ [27, 28]	$\mathcal{O}(T^6)$	$\mathcal{O}(T^6)$	$\mathcal{O}(T^6)$
(s_t, S_t)	(L, U)	1	$\underline{\mathcal{O}(T^6)}$ [*2.3.3]	$\mathcal{O}(T^6)$	$\mathcal{O}(T^6)$	$\underline{\mathcal{O}(T^6)}$ [*2.3.4]
(s, S)	(L, U)	1	$\mathcal{O}(T^6)$	$\mathcal{O}(T^6)$	$\mathcal{O}(T^6)$	$\mathcal{O}(T^6)$
·	(L, U)	α_t	$\mathcal{NPC}$	$\mathcal{NPC}$	$\underline{\mathcal{NPC}}$ [*2.2.2]	$\mathcal{NPC}$
·	(L, U)	α	open	open	open	open

[*·] denotes new results as shown in Section · of this work.

Since time-dependent production bounds are already shown to render the problem $\mathcal{NP}$-complete, we only consider stationary ones in our study. This is a plausible assumption in many applications.

We observe that for trivial production bounds all considered problems are polynomially solvable. For stationary production bounds, we found that the polynomial solvability primarily depends on the nature of the conservation rates in interaction with the former. Here, for time dependent deterioration rates the problem is rendered $\mathcal{NP}$-complete.

Conversely, when considering no storage deterioration, all resulting problems are polynomially solvable (albeit with different complexities). Only in the case of stationary conservation rates and stationary production bounds we did not manage to achieve any complexity results. Hence, this problem is left for further research.

There is a number of works which we could not include, as they do not quite fit our scheme, for examples see Bitran and Yanasse [15]. This particularly affects models that cover non-trivial upper but only zero lower production bounds like van Hoesel and Wagelmans et al. [50]. A similar issue applies to models with strictly linear cost structures instead of the more commonly considered affinely linear ones. For instance, see Sedeno-Noda et al. [44], who consider strictly linear production and holding costs, and Toczylowsky [48] who assumes zero constant costs for storage levels other than the lower bound.

Other works, especially Hsu [29], can be considered unintentionally underrepresented since they employ more general concepts that can not be properly integrated in our classification.

2.2 $\mathcal{NP}$-complete variants

Florian et al. [26] showed that the single-item lot-sizing problem with affinely linear production cost functions, zero holding costs and time-dependent upper production bounds is $\mathcal{NP}$-complete. This is done by polynomially reducing the subset-sum problem. The resulting single-item lot-sizing problem has non-zero demand only in period T, and time-dependent production bounds.

The aim of this brief section is to tweak the aforementioned result to cover stationary upper production bounds in combination with general storage deterioration rates, and, in particular, affinely linear Wagner-Whitin cost functions. Recently and independently, a similar result was shown by Sargut and Işık [42] with a slightly more general form of storage deterioration and affinely linear costs. Still, our result is stronger since it implies the result from the literature and is more general as it shows a more restricted problem to be $\mathcal{NP}$-hard.

2.2.1 Stationary production bounds and time-dependent deterioration rates with concave costs

We induce the situation described above with an auxiliary (stationary) upper production bound by choosing the deterioration rates accordingly and adjusting the production costs.

Theorem 2. *The single-item lot-sizing problem with stationary production bounds, time-dependent storage deterioration rates and linear costs is $\mathcal{NP}$-complete.*

Proof. We polynomially reduce the following $\mathcal{NP}$-complete problem, see Karp [31], to the decision version of single-item lot-sizing with stationary production bounds and time-dependent conservation rates.

SUBSET-SUM: Given positive integers $T \in \mathbb{Z}_{>0}$, $B \in \mathbb{Z}_{>0}$ and $b_t \in \mathbb{Z}_{>0}$ for $t \in [T]$, does there exist a subset $S \subseteq [T]$ such that $\sum_{t \in S} b_t = B$ holds?

Without loss of generality, let the entries of b be sorted in ascending order, that is $b_t \leq b_{t+1}$ holds for $t \in [T-1]$ with $b_0 := 1$. The considered instance of single-item lot-sizing with stationary production bounds and storage deterioration reads:

$$
\begin{aligned}
& U = b_T, \quad L = 0, \quad u_0 = 0, \quad d_T = B && \\
& d_t = 0 && \forall\, t \in [T-1] \\
& \alpha_t = \frac{b_{t-1}}{b_t} && \forall\, t \in [T] \\
& p_t(x_t) = \begin{cases} \frac{b_t}{b_T}\left(1 + \frac{b_{T-1}}{b_T} x_t\right) & \text{if } x_t > 0 \\ 0 & \text{if } x_t = 0 \end{cases} && \forall\, t \in [T] \\
& h_t(u_t) = 0 && \forall\, t \in [T]
\end{aligned}
$$

SINGLE-ITEM LOT-SIZING: Is there a solution with objective value at most B to the given problem?

Here, the only non-zero demand occurs in period T, and production costs are strictly positive and increasing. Thus, in any optimal solution all production is implemented to meet demand d_T. Recall that, by equalities (1.10), all production periods and their influence on period T can be analyzed separately. A production of x_t in period t results in an available amount of

$$z_t = \alpha_{t,T} x_t = \frac{b_t}{b_T} x_t$$

in period T. Hence, we have potential values $0 \leq z_t \leq b_t$, and the production costs p_t depending on z_T (instead of x_t) are

$$p_t(z_t) = \begin{cases} \frac{b_t}{b_T}\left(1 + \frac{b_{T-1}}{b_T}\frac{b_T}{b_t} z_t\right) = \frac{b_t}{b_T} + \frac{b_{T-1}}{b_T} z_t & \text{if } 0 < z_t \leq b_t \\ 0 & \text{if } z_t = 0. \end{cases}$$

Thus,

$$p_t(z_t) \begin{cases} = z_t & \text{if } z_t \in \{0, b_t\} \\ > z_t & \text{if } 0 < z_t < b_t \end{cases}$$

holds. Thereby, each solution in which a product amount of at least B is available in period T induces a cost of at least B. Here, the value B can be achieved if and only if there exists a subset $S \subseteq [T]$ such that $\sum_{t \in S} b_t = B$ holds, by setting

$$
\begin{aligned}
& x_t = b_T && (\Leftrightarrow \quad z_t = b_t) && \text{if } t \in S \\
& x_t = 0 && (\Leftrightarrow \quad z_t = 0) && \text{if } t \notin S.
\end{aligned}
$$

Lastly, SINGLE-ITEM LOT-SIZING belongs to $\mathcal{NP}$ as it is a subproblem of mixed integer linear programming. $\square$

This proof can be adjusted, such that the demand and storage bound values exhibit popular additional properties in the context of single-item lot-sizing, as shown below.

Remark. *Theorem 2 can be adjusted to exhibit either strictly positive demands or non-trivial lower and trivial upper production bounds.*

Proof. For strictly positive demands we introduce an auxiliary initial period $t_0 = 0$. This period t_0 gets assigned the same maximum production capacity $U = b_T$ at zero (or sufficiently small) costs. Then periods $[0, T-1]$ are given strictly positive demands, that are exactly met by the costless production U in period t_0.

For trivial upper production bounds, we define the lower production bounds to be b_T and adjust the cost function to only be efficient (with $p_t(z_t) = z_t$) for the lower production bound. □

2.2.2 Stationary production bounds and time-dependent deterioration rates with Wagner-Whitin costs

Since the constructed cost structure already holds the Wagner-Whitin property, the result can also be applied in this case.

Corollary 1. *The single-item lot-sizing problem with stationary production bounds and time-dependent storage deterioration is $\mathcal{NP}$-complete in the Wagner-Whitin case.*

2.3 Polynomial time solvable variants

In this section, we show that the single-item lot-sizing variant with time-dependent storage bounds and time-dependent storage deterioration rates, and the variant with stationary production bounds and time-dependent storage bounds can be solved in polynomial time, by extending results from the literature. We exploit and combine algorithms by Love [37] and Hellion et al. [27, 28], using notation of Atamtürk and Küçükyavuz [3]. In both algorithms, the concept is to characterize a dominant set of solutions which have some kind of block structure (see below) with at most one *exceptional* production period per block. Therefore, we formally define the concept of dominant solution subsets in the considered context of mathematical programming problems.

Definition 1. *Let a class of mathematical programming problems*

$$\min_{x \in X} \{f(x)\} \quad \text{for} \quad f \in F$$

with a set of feasible solutions X and a class of objective functions $F \subseteq \{f \mid f : X \to \mathbb{R}\}$ be given. We call a subset $S \subseteq X$ dominant if for every $f \in F$, whenever there exists an optimal solution in X, there also exists an optimal solution in S. That is,

$$\min_{x \in X} \{f(x)\} = \min_{x \in S} \{f(x)\} \qquad \forall\ f \in F$$

holds.

Thus, it is sufficient to only consider a dominant subset S of solutions for solving a considered problem instance to optimality (where $f \in F$ determines the instance). We recall the following elementary property:

Lemma 3. *Using the notation of the above definition, let $X \subset \mathbb{R}^n$ be closed with* $\operatorname{conv}(X)$ *being a bounded polytope and $F := \{f \mid f : \operatorname{conv}(X) \to \mathbb{R},\ f \text{ is concave}\}$. Then the set of extreme points $C \subseteq X$ of* $\operatorname{conv}(X)$ *is dominant.*

Proof. Each point in $x \in X$ can be written as a convex combination

$$x = \sum_{c \in C} \lambda_c c,$$

i.e., $\lambda_c \in \mathbb{R}$ and $0 \leq \lambda_c \leq 1$ for all $c \in C$, and $\sum_{c \in C} \lambda_c = 1$.
Let $f \in F$ be given. Assuming $f(c) > f(x)$ for all $c \in C$, then

$$f(x) = f\left(\sum_{c \in C} \lambda_c c\right) \geq \sum_{c \in C} \lambda_c f(c) > \sum_{c \in C} \lambda_c f(x) = f(x)$$

follows, which is a contradiction. □

To identify such dominant sets of solutions, we consider the following block structure to base a characterization of relevant production sequences on.

Definition 2. *Let $(x, u, y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \times \{0, 1\}^T$ be a feasible solution to* (P_1). *Setting $s_0 = S_0 = u_0$, a consecutive sequence of time periods $[j, \ell]$ is called a block $[j, \ell]^\beta_\gamma$ with $\beta \in \{s_{j-1}, S_{j-1}\}$ and $\gamma \in \{s_\ell, S_\ell\}$ if $u_{j-1} = \beta$, $u_\ell = \gamma$ and $s_t < u_t < S_t$ hold for all $t \in [j, \ell - 1]$.*

With this, we are able to characterize solutions with suitable block structure as extreme points of the polyhedral convex hull of feasible solutions. Thus, having determined such a dominant set of solutions, we can reduce our problem to shortest-path problem on an acyclic directed graph by exploiting said block structure. Here, we use the following well known result, see [1, Theorem 4.3].

Lemma 4. *Let $G = (V, A)$ be an acyclic directed graph with a cost function $w : A \to \mathbb{R} \cup \{\infty\}$. Then, any shortest path in G can be computed in $\mathcal{O}(|A|)$.*

Thus, we obtain a polynomial time algorithm for solving the considered problem, by restricting the feasible solutions to a dominant subset.

2.3.1 Time-dependent storage bounds and time-dependent deterioration rates with concave costs

First, we demonstrate the concepts described above, by presenting a polynomial time algorithm for the single-item lot-sizing problem with time-dependent storage bounds, time-dependent deterioration rates and concave costs. Without loss of generality, we make the assumption

$$s_t \geq \alpha_t s_{t-1} - d_t \qquad \forall\, t \in [2, T], \tag{2.1}$$

since no feasible solution can exhibit a storage value less than $\alpha_t s_{t-1} - d_t$ at the end of period t. Thus, we can update the lower storage bounds s, such that (2.1) is satisfied, as done in the first step of the reduction in Lemma 1, in linear time. With this, we deduce the following.

Observation 1. *Since all costs are increasing and non negative, there exists an optimal solution with minimal final storage value for any instance. Thus, without loss of generality, we add the constraint $u_T = s_T$ to our formulation. Consequently, any feasible solution is rendered a sequence of blocks.*

While similar adjustments can be performed for the upper storage bounds S_t, this is not needed for the algorithm. Now, we can introduce a dominant set of solutions exhibiting the following block structure.

Lemma 5. *The set of solutions of CONC-(s_t, S_t)-$(0, \infty)$-α_t with minimum final storage value and at most one non-zero production period per block is dominant.*

Proof. Consider a feasible solution $(x, u, y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \times \{0, 1\}^T$ with a block $[j, \ell]^\beta_\gamma$ that contains at least two distinct non-zero production periods $n, m \in [j, \ell]$ with $m < n$. Let

$$\delta = \min_{t \in [m,n]} \{S_t - u_t, u_t - s_t, x_m, x_n\} > 0,$$

then

$$x'_t = \begin{cases} x_t + \delta & \text{for } t = m \\ x_t - \alpha_{m,n}\delta & \text{for } t = n \\ x_t & \text{otherwise} \end{cases}$$

and

$$x''_t = \begin{cases} x_t - \delta & \text{for } t = m \\ x_t + \alpha_{m,n}\delta & \text{for } t = n \\ x_t & \text{otherwise} \end{cases}$$

induce feasible solutions $(x', u', y), (x'', u'', y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \times \{0, 1\}$ which satisfy

$$(x, u, y) = \frac{1}{2}(x', u', y) + \frac{1}{2}(x'', u'', y).$$

Thus, (x, u, y) is not an extreme point of the convex hull of feasible solutions. As the considered objective function can be embedded in a concave function on the convex hull of feasible solutions, Lemma 3 states that the set solutions with at most one non-zero production period per block is dominant. $\square$

Over this dominant set, we are able to optimize efficiently.

Theorem 3. *The single-item lot-sizing problem with time-dependent storage bounds and time-dependent deterioration rates can be solved in* $\mathcal{O}(T^3)$.

Proof. Let

$$d_{p,q} := \sum_{t=p}^{q} \frac{1}{\alpha_{p,t}} d_t$$

denote the amount of available (production or storage) value needed in period $p \in [T]$ to meet the demands from periods $[p,q]$ with $q \in [p,T]$.

Consider a block $[j,\ell]_\gamma^\beta$ in an (optimal) solution $(x,u,y) \in \mathbb{R}_{\geq 0}^T \times \mathbb{R}_{\geq 0}^T \times \{0,1\}^T$ as described in Lemma 5. In this, we analyze the potential position of the non-zero production period $k \in [j,\ell]$. The production in period k has to suffice

$$x_k = d_{k,\ell} - \alpha_{j,k}(\alpha_j \beta - d_{j,k-1}) + \frac{1}{\alpha_{k,\ell}}\gamma,$$

while $x_i = 0$ has to hold for all $i \in [j,\ell] \backslash \{k\}$.

Furthermore, consider the induced subsolution $(x',u',y') \in \mathbb{R}_{\geq 0}^{[j,\ell]} \times \mathbb{R}_{\geq 0}^{[j,\ell]} \times \{0,1\}^{[j,\ell]}$ for block $[j,\ell]$. We can check this subsolution for feasibility, by examining whether the induced storage values u' fulfill the given bounds. If the induced subsolution is feasible we compute its combined production and holding costs

$$\sum_{t=j}^{k-1} h_t \left(\alpha_{j,t} \left(\alpha_j \beta - d_{j,t}\right)\right) + p_k(x_k) + \sum_{t=k}^{\ell} h_t \left(\alpha_{k,t} \left(x_k + \alpha_{j,k}(\alpha_j \beta - d_{j,k-1}) - d_{k,t}\right)\right).$$

Then, we define $\varphi_{j,k,\ell}^{\beta,\gamma}$ to be the above costs if the induced solution is feasible, and to be infinity otherwise. Thus,

$$c_{j,\ell}^{\beta,\gamma} := \min_{k \in [j,\ell]} \left\{ \varphi_{j,k,\ell}^{\beta,\gamma} \right\}$$

represents the costs for an optimal production subsequence $x' \in \mathbb{R}^{[j,\ell]}$ on block $[j,\ell]_\gamma^\beta$. We exploit this to find an optimal solution as a sequence of optimal blocks. Note that all $\varphi_{j,k,\ell}^{\beta,\gamma}$ can be iteratively computed in $\mathcal{O}(T^3)$, and each of the $\mathcal{O}(T^2)$ values $c_{j,\ell}^{\beta,\gamma}$ can be determined in $\mathcal{O}(T)$.

Now, we reduce the problem to a shortest-path problem on an acyclic directed graph $G = (V,A)$ with

$$V = \left\{v_1^{u_0}\right\} \cup \bigcup_{t=1}^{T-1} \left\{v_{t+1}^{s_t}, v_{t+1}^{S_t}\right\} \cup \left\{v_{T+1}^{s_T}\right\}$$

$$A = \left\{ (v_j^\beta, v_\ell^\gamma) \in V \times V \;\middle|\; j < \ell \right\},$$

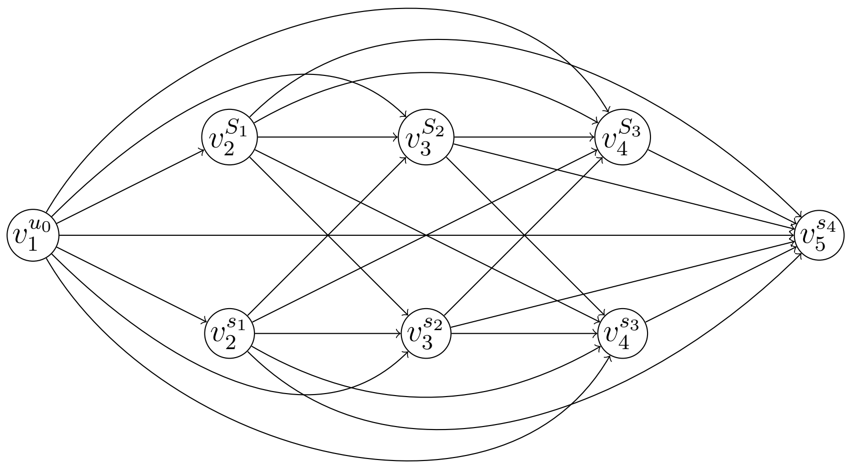


Figure 2.1: Example for $T = 4$

see Figure 2.1, and the weight function

$$w : A \to \mathbb{R}_{\geq 0} \cup \{\infty\},\ (v_j^{\beta}, v_{\ell+1}^{\gamma}) \mapsto c_{j,\ell}^{\beta,\gamma}.$$

Here, we ask for the shortest path from $v_1^{u_0}$ to $v_{T+1}^{s_T}$. The Graph $G = (V, A)$ has $2T$ vertices and

$$(2T - 1) + \sum_{i=2}^{T} 2(2(T - i) + 1) = \mathcal{O}(T^2)$$

arcs. As shown, the total preprocessing runs in $\mathcal{O}(T^3)$. According to Lemma 4, the shortest-path problem can be solved in $\mathcal{O}(|A|) = \mathcal{O}(T^2)$, i.e., the overall run-time is $\mathcal{O}(T^3)$. An optimal solution to the shortest-path problem induces an optimal solution to the given single-item lot-sizing instance as follows. The visited vertices in the optimal shortest-path solution yield the sequence of blocks in the single-item lot-sizing solution. Furthermore, the used arcs imply the optimal production periods by recalling the arguments minimizing the corresponding values $c_{j,\ell}^{\beta,\gamma}$. □

2.3.2 Time-dependent storage bounds and time-dependent deterioration rates with Wagner-Whitin costs

In the following, we consider the single-item lot-sizing problem with time-dependent storage bounds and time-dependent deterioration rates with Wagner-Whitin costs instead of concave ones. Due to the Wagner-Whitin cost structure, we can modify the former algorithm to achieve a run-time of $\mathcal{O}(T^2)$.

First, we eliminate lower storage bounds and negative demands in linear time as shown in Lemma 1, while maintaining the Wagner-Whitin property. Thus, we obtain an equivalent problem on which we can efficiently optimize.

Thus, without loss of generality, we assume that we are considering the single-item lot-sizing problem with time-dependent upper storage bounds and time-dependent deterioration rates, but with zero lower storage bounds, zero initial storage value and the Wagner-Whitin cost structure.

At this point, we can exploit the following *zero-inventory* property, introduced by Wagner and Whitin [51], to characterize a dominant set of solutions.

Definition 3. *A solution to the single-item lot-sizing problem holds the zero-inventory property if for all periods in which production takes place, the storage is empty.*

Now, the Wagner-Whitin cost structure in the given context obviously yields:

Lemma 6. *The set of solutions of WW-*(s_t, S_t)*-*$(0, \infty)$*-*α_t *with the zero-inventory property and zero final storage value is dominant.*

Proof. Considering a feasible solution, we construct a feasible solution with at most equal value and the zero-inventory property by iteratively postponing as much production as possible without affecting the commitment. □

Hence, we obtain a characterization for a dominant set of solutions similar to the former block structure, but with the following additional properties: Production always takes place at the beginning of the block, and blocks always begin and end with the (zero) lower storage bound as storage value. Thus, we can employ the same shortest-path formulation as before to solve the problem.

Corollary 2. *The single-item lot-sizing problem with time-dependent storage bounds and time-dependent deterioration rates can be solved in* $\mathcal{O}(T^2)$ *in the Wagner-Whitin case.*

Proof. We use the same shortest-path formulation as before, but discard the vertices for the upper storage bounds. Furthermore, we can compute the weights in linear time. □

2.3.3 Time-dependent storage bounds and stationary production bounds with concave costs

When considering the single-item lot-sizing problem with time-dependent storage bounds and stationary production bounds with concave costs, we exploit a slightly adjusted characterization for a dominant set of solutions. Note that the situation $u_T \neq s_T$ can arise in all optimal solutions, due to strictly-positive lower production bounds.

Observation 2. *All solutions to CONC-*(s_t, S_t)*-*(L, U)*-1 are sequences of blocks possibly up to a turning point* $h \in [T]$*, from whereon* $s_t < u_t < S_t$ *holds for all* $t \in [h, T]$*.*

To utilize the block structure we need additional terminology.

Definition 4. *A period $t \in [T]$ is called a fractional production period if $L < x_t < U$ holds.*

This property yields the form of the exceptional production period stated before.

Lemma 7. *The set of solutions of CONC-(s_t, S_t)-(L, U)-1 with at most one fractional production period per block and the property that if there is a turning point $h \in [T]$, then $x_t \in \{0, L\}$ holds for all $t \in [h, T]$, is dominant.*

Proof. Consider a feasible solution $(x, u, y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \times \{0,1\}^T$ with a block $[j, \ell]^\beta_\gamma$ that contains at least two distinct fractional production periods $m, n \in [j, \ell]$. Let

$$\delta = \min_{t \in [m,n], q \in \{m,n\}} \{S_t - u_t, u_t - s_t, U - x_q, x_q - L\} > 0,$$

then

$$x'_t = \begin{cases} x_t + \delta & \text{for } t = m \\ x_t - \delta & \text{for } t = n \\ x_t & \text{otherwise} \end{cases}$$

and

$$x''_t = \begin{cases} x_t - \delta & \text{for } t = m \\ x_t + \delta & \text{for } t = n \\ x_t & \text{otherwise} \end{cases}$$

induce feasible solutions $(x', u', y), (x'', u'', y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \times \{0,1\}^T$ which satisfy

$$(x, u, y) = \frac{1}{2}(x', u', y) + \frac{1}{2}(x'', u'', y).$$

Thus, (x, u, y) is not an extreme point of the convex hull of feasible solutions. As the considered objective function can be embedded in a concave function on the convex hull of feasible solutions, Lemma 3 states that the set of solutions with at most one fractional production period per block is dominant.

Now, assume that there was an instance in which no optimal solution as proposed in the statement exists. Consider the (non empty) set of optimal solutions with at most one fractional production period per block which also possesses a turning point. Let one such solution s with maximal $\hat{t} \in [T]$, with $\hat{t}$ denoting the first period after the turning point of s such that $x_{\hat{t}} \notin \{0, L\}$ holds, be given.

Conversely, consider the solution s' induced by lowering $x_{\hat{t}}$ as much as possible without creating an infeasibility (updating the storage values for $[\hat{t}, T]$). Then, s' exhibits at most equal costs as s, due to the cost functions being non negative and non decreasing. Hence, s' is also optimal, and its potential turning point possibly occurs later than that of s, due to the updated storage values and thus updated block structure. Overall, s' either holds the structure proposed in the statement, or the first period $\hat{t}' \in [T]$ of s', which is after its turning point and exhibits $x_{\hat{t}'} \notin \{0, L\}$, is strictly bigger than $\hat{t}$. In both cases we observe a contradiction. Hence, the statement follows. □

Consider an optimal solution as stated in Lemma 7 and one of its blocks $[j,\ell]^{\beta}_{\gamma}$. Let

$$d = d^{\beta;\gamma}_{j,\ell} = \sum_{t=j}^{\ell} d_t - \beta + \gamma$$

be this block's accumulated demand. This demand has to be met by the production inside the block, while there is at most one fractional production period. Thus the production subsequence on said block induces a triplet $(\pi, \rho, \varepsilon) \in [0,T] \times [0,T] \times (\{0\} \cup (L,U))$ which satisfies

$$\pi U + \rho L + \varepsilon = d. \tag{2.2}$$

Triplets with this property we call *valid triplets*. In the following, we deduce a limit to their number.

Lemma 8. *Per block $[j,\ell]^{\beta}_{\gamma}$, the number of valid triplets (π, ρ, ε) is in $\mathcal{O}(T)$.*

Proof. Given a value for $\rho' \in [0,T]$, if there exists a valid triplet $(\pi', \rho', \varepsilon')$, the values of π' and ε' are uniquely determined as

$$\pi' := \left\lfloor \frac{d - \rho' L}{U} \right\rfloor$$

and

$$\varepsilon' := d - \pi' U - \rho' L.$$

Furthermore, the triplet $(\pi', \rho', \varepsilon')$ is valid if and only if $\rho' L \leq d$ holds, and $\varepsilon' = 0$ or $L < \varepsilon' < U$ holds. □

Thus, we have further restricted the types of potentially optimal blocks we have to consider.

Lemma 9. *For every potential block, a production subsequence of minimal costs can be computed in $\mathcal{O}(T^4)$.*

Proof. Given a block $[j,\ell]^{\beta}_{\gamma}$ and a valid triplet (π, ρ, ε), we consider the minimum costs $f((\pi', \rho', \varepsilon'), k) \in \mathbb{R}$ induced by a feasible production subsequence associated to the subtriplet $(\pi', \rho', \varepsilon')$ on $[j,k]$ with $\pi' \in [0,\pi], \rho' \in [0,\rho]$ and $\varepsilon' \in \{0, \varepsilon\}$. If there exists no such feasible production subsequence, we define $f((\pi', \rho', \varepsilon'), k) \in \mathbb{R}$ to be infinity. Let

$$u'_k(\pi', \rho', \varepsilon') := \pi' U + \rho' L + \varepsilon' + \beta - \sum_{t=j}^{k} d_t$$

be the induced storage value in period k for $(\pi', \rho', \varepsilon')$. For $f((\pi', \rho', \varepsilon'), k) < \infty$ to hold, a subtriplet has to satisfy

$$s_k \leq u'_k(\pi', \rho', \varepsilon') \leq S_k,$$

as well as

$$\pi' + \rho' + 1_{\mathbb{R}_{>0}}(\varepsilon') \leq k - j + 1$$

with $1_{\mathbb{R}_{>0}} : \mathbb{R} \to \{0, 1\}$ denoting the indicator function that indicates whether a value is strictly positive.

If both conditions hold, we compute the optimal production subsequence $f((\pi, \rho, \varepsilon), l)$ for the considered block recursively via

$$f((\pi', \rho', \varepsilon'), k) = \min \left\{ \begin{array}{l} f((\pi', \rho', \varepsilon'), k-1) + h_{k-1}\left(u'_{k-1}(\pi', \rho', \varepsilon')\right), \\ f((\pi'-1, \rho', \varepsilon'), k-1) + p_k(U) + h_{k-1}\left(u'_{k-1}(\pi'-1, \rho', \varepsilon')\right), \\ f((\pi', \rho'-1, \varepsilon'), k-1) + p_k(L) + h_{k-1}\left(u'_{k-1}(\pi', \rho'-1, \varepsilon')\right), \\ f((\pi', \rho', 0), k-1) + p_k(\varepsilon') + h_{k-1}\left(u'_{k-1}(\pi', \rho', 0)\right) \end{array} \right\},$$

with $f((0, 0, 0), j-1) := 0$.

There are at most $T+1$ potential values for π' and ρ' respectively, at most T potential values for k and at most two potential values for ε'. Thus, the optimal cost corresponding to a given valid triplet (π, ρ, ε) can be computed in $\mathcal{O}(T^3)$. With $\mathcal{O}(T)$ valid triplets, the total computing time for an overall cost optimal production subsequence is $\mathcal{O}(T^4)$. □

This allows for the following result:

Theorem 4. *The single-item lot-sizing problem with stationary production bounds and time-dependent storage bounds with concave costs can be solved in $\mathcal{O}(T^6)$.*

Proof. This proof is similar to the one of Theorem 3, but we have to consider a potential breaking point h, after which the block structure does not apply anymore. Hence, for each potential block $[j, \ell]_\gamma^\beta$ let $c_{j,\ell}^{\beta,\gamma}$ be the cost induced by a production subsequence of minimum cost as described in Lemma 9 (infinity if such a sequence does not exist). To cover the possibility of a breaking point h, we observe that, as described in Lemma 7, there exists an optimal production sequence that exhibits exactly

$$\left\lceil \frac{\sum_{t=h}^{T} d_t - u_{h-1} + s_T}{L} \right\rceil$$

non-zero production periods (with production value L) after h with $u_{h-1} \in \{s_{h-1}, S_{h-1}\}$ being given by the preceding block structure. Thus, we can compute an optimal production subsequence which preserves the breaking point property of h in $\mathcal{O}(T^2)$ with he recursion from Lemma 9. We let $c_{h,T+1}^{u_{h-1},*}$ denote the combined production and holding costs of this optimal production subsequence and we let $c_{T+1,T+1}^{s_T,*} := 0$. Then, we reduce the problem to a shortest-path problem on the acyclic directed graph $G = (V, A)$ with

$$\begin{aligned} V &= \{v_1^{u_0}\} \cup \bigcup_{t=1}^{T-1} \{v_{t+1}^{s_t}, v_{t+1}^{S_t}\} \cup \{v_{T+1}^{s_T}\} \cup \{v_{T+2}^{*}\} \\ A &= \left\{ (v_j^\beta, v_\ell^\gamma) \in V \times V \;\middle|\; j < \ell \right\}, \end{aligned}$$

see Figure 2.2, and the weight function

$$w : A \to \mathbb{R}_{\geq 0} \cup \{\infty\},\ (v_j^\beta, v_{\ell+1}^\gamma) \mapsto c_{j,\ell}^{\beta,\gamma}.$$

Here, we ask for the shortest path from $v_1^{u_0}$ to v_{T+2}^*. The Graph G has $2T+1$ vertices and

$$2T + \sum_{i=2}^{T} 2(2(T-i+1)) = \mathcal{O}(T^2)$$

arcs. Note that the arcs to v_{T+2}^* cover the potentially optimal ending sequence after a

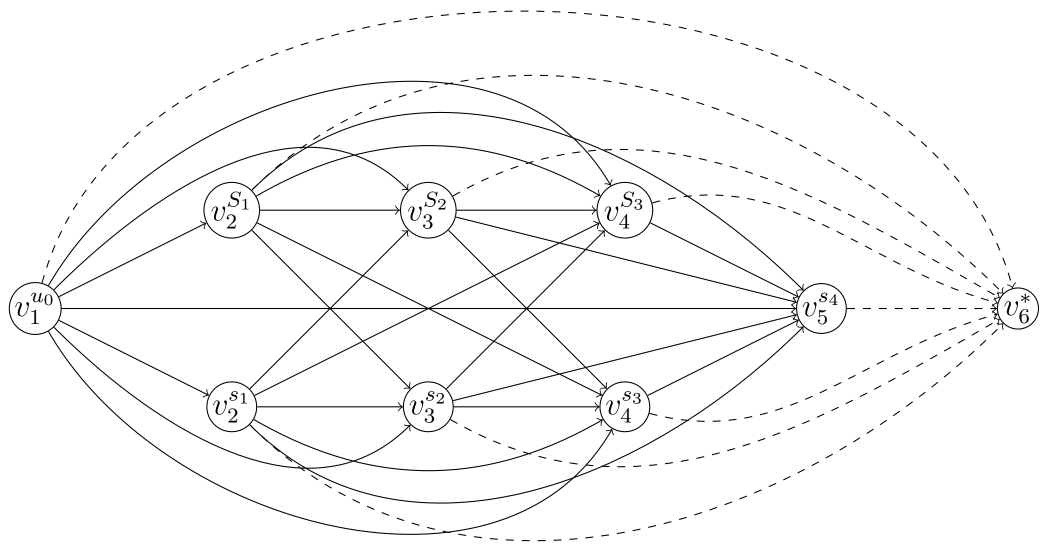


Figure 2.2: Example for $T = 4$

breaking point h from whereon the production plan does not exhibit the block structure anymore. Since each of the $\mathcal{O}(T^2)$ arcs need a preprocessing of $\mathcal{O}(T^4)$, the total preprocessing runs in $\mathcal{O}(T^6)$. According to Lemma 4, the shortest-path problem can be solved in $\mathcal{O}(|A|) = \mathcal{O}(T^2)$. Hence, the overall run-time is $\mathcal{O}(T^6)$. □

2.3.4 Time-dependent storage bounds and stationary production bounds with Wagner-Whitin costs

For time-dependent storage bounds and stationary production bounds the Wagner-Whitin cost structure once more yields a tighter characterization for a dominant set of solutions than the concave one. But in this case it does not affect the run-time.

Lemma 10. *A set of dominant solutions to WW-(s_t, S_t)-(L, U)-1 is given by the feasible solutions that exhibit*

- *at most one fractional production period per block,*
- *only production values in $\{0, L\}$ before, and only production values in $\{0, U\}$ after the potential fractional period per block,*
- *the property that, if there is a turning point $h \in [T]$, then $x_t \in \{0, L\}$ holds for all $t \in [h, T]$.*

Proof. First, we show that the set of solutions which satisfy the first two properties is dominant. For a given feasible solution we denote the number of blocks as B and the number of blocks that do not satisfy the first two properties as $\hat{B}$. Now consider an optimal solution with maximal $B - \hat{B}$.

Assuming the critical case $\hat{B} > 0$, we construct a feasible solution with at most equal costs by choosing the first two periods which contradict the considered properties, and postponing as much production as possible from the first to the second. Thus, we render a previously non-viable block viable or create a new viable block. In both cases we increase $B - \hat{B}$, which is a contradiction.

Furthermore, given an optimal solution, which satisfies the first two properties, we construct an optimal solution with at most equal costs, which also satisfies the third property. That is, we iteratively reduce the production after the turning point as much as possible without affecting the commitment. □

Clearly, the above dominant set of solutions is a subset of that described in Lemma 7. Thus, we can still use the exact same algorithm we formerly used and obtain an analogous result.

Corollary 3. *The single-item lot-sizing problem with stationary production bounds and time-dependent storage bounds can be solved in $\mathcal{O}(T^6)$ in the Wagner-Whitin case.*

2.4 Concluding remarks

In this chapter, we studied the computational complexity of single-item lot-sizing variants, covering different degrees of flexibility in the variant-characterizing parameters when including production bounds, storage bounds or conservation rates. Here, some fairly general classes of cost functions were considered.

Since time-dependent production bounds were already known to be $\mathcal{NP}$-complete, we restricted our study to stationary ones. In all cases in which we extended polynomial time algorithms to more general problems, we maintained the complexity of the original one. While the single-item lot-sizing problem with stationary production bounds turned out to be polynomially solvable, even when adding time-dependent storage bounds, we did not manage to achieve any complexity results by restricting the deterioration rates to be stationary in combination with stationary production bounds.

When considering the unsolved problem with static deterioration rates and production bounds, we struggle to achieve a result, as motivated in the following.

On the one hand, for a $\mathcal{NP}$-hardness proof we would need to reduce a $\mathcal{NP}$-hard problem to the decision version of the single-item lot-sizing variant. Here, we aim to induce a discrete integer subproblem, which is severely hindered by the deterioration rate α and its powers.

On the other hand, when trying to prove polynomial solvability, we would like to extend a polynomial time algorithm to one of the above cases. Here, considering the most elaborate case, of static production bounds and arbitrary storage bounds, see Subsection 2.3.3, we cannot properly limit the size of the dominant set of solutions when adding non-trivial deterioration rates.

Firstly, the concept of valid triplets is no longer viable, as the total demand for a block is no longer directly distributed to the production periods, since storage losses also have to be covered. In particular, considering the concept of valid triplets, ε is not uniquely given by the rest of the triplet.

Also, we cannot properly limit the number of production periods inside a block, as for very long blocks any production in an early period may have infinitesimal influence on late periods, due to the accumulated losses.

In principle, an analogous dominant set of solutions to those in Subsection 2.3.3 can be established, but we did not manage to limit it to polynomial size, or to explore it in polynomial time.

While we assumed all costs to be non negative and non decreasing, all polynomial results can easily be extended to cover the cases without these assumptions in the linear case when excluding unbounded cases. This holds, as due to the non-negative fixed costs, if any superfluous production value in a single period is efficient (which requires negative variable costs), it is also efficient to increase the production value in this period as much as possible. Also, this property allows us to identify unbounded instances. Hence, the formerly explored dominant sets of solutions remain so, and thus the given algorithms are still applicable.

3 Polyhedral study for single-item lot-sizing with storage deterioration

In this chapter, we study the polyhedra associated with the single-item lot-sizing problem with storage deterioration under the assumption of strictly positive demands. This is based on the companion papers [5] and [6] by Barany et al.. Here, the class of (ℓ, S) inequalities is identified as facet defining for a formulation of the single-item lot-sizing polytope. Incorporating these inequalities, an ideal formulation of the uncapacitated single-item lot-sizing problem is derived. Furthermore, a description via an extended formulation is showcased. We extend these results to include storage deterioration rates.

In order to achieve this, we use standard mixed integer linear programming concepts, again see [45] and [52]. Here, we particularly rely on primal-dual techniques in which the optimality of a solution to a linear program is verified by constructing a solution of equal value to the dual linear program.

3.1 Problem formulation

The classic linear programming formulation for the single-item lot-sizing problem with storage deterioration reads

$$\begin{aligned}
\text{(A)}\quad \min \quad & \sum_{t=1}^{T} (\tilde{c}_t x_t + p_t u_t + f_t y_t) && && (3.1)\\
\text{s.t.} \quad & x_t + \alpha_t u_{t-1} = d_t + u_t && \forall\, t \in [T] && (3.2)\\
& x_t \leq M y_t && \forall\, t \in [T] && (3.3)\\
& u_T = 0 && && (3.4)\\
& x_t, u_t \geq 0 && \forall\, t \in [T] && (3.5)\\
& y_t \in \{0,1\} && \forall\, t \in [T] && (3.6)
\end{aligned}$$

with a time horizon $T \in \mathbb{Z}_{>0}$, variable production costs $\tilde{c} \in \mathbb{R}^T$, variable storage costs $p \in \mathbb{R}^T$, commitment costs $f \in \mathbb{R}^T$ conservation rates $0 < \alpha_t \leq 1$ for $t \in [T]$, strictly positive demands $d \in \mathbb{R}^T_{>0}$, zero initial storage value $u_0 := 0$, a sufficiently big constant $M \in \mathbb{R}$ and final storage constraint (3.4) which we discuss later on. Again, for each period $t \in [T]$ the variable x_t denotes the production value, the binary variable y_t indicates if production takes place and the variable u_t denotes the storage value at the end of the period.

We base our study on the equivalent (up to a constant cost term) model

$$
\begin{aligned}
\text{(B)}\ \min\quad & \sum_{t=1}^{T} (c_t x_t + f_t y_t) && && (3.7)\\
\text{s.t.}\quad & \sum_{i=1}^{t} \alpha_{i,t} x_i \geq d_{1,t} && \forall\, t \in [T-1] && (3.8)\\
& x_t \leq M y_t && \forall\, t \in [T] && (3.9)\\
& \sum_{t=1}^{T} \alpha_{t,T} x_t = d_{1,T} && && (3.10)\\
& x_t \geq 0 && \forall\, t \in [T] && (3.11)\\
& y_t \in \{0,1\} && \forall\, t \in [T] && (3.12)
\end{aligned}
$$

with

$$
d_{j,k} := \sum_{i=j}^{k} \alpha_{i,k} d_i \qquad \begin{array}{l} \forall\, k \in [T] \\ \forall\, j \in [k] \end{array} \qquad (3.13)
$$

and

$$
c_t := \tilde{c}_t + \sum_{k=t}^{T} \alpha_{t,k} p_k \qquad \forall\, t \in [T].
$$

This we derive, analogously to model (P_2) in Section 1.2.2, via

$$
u_t = \sum_{i=1}^{t} \alpha_{i,t} (x_i - d_i) \qquad \forall\, t \in [T].
$$

We stress that, while notation (3.13) differs from the very similar one in Theorem 3, it severely simplifies the following notations and proofs. Employing it, we observe that for $k \in [T]$ and $j \in [k]$ the term

$$
\frac{1}{\alpha_{j,k}} d_{j,k} = \sum_{i=j}^{k} \frac{1}{\alpha_{j,i}} d_i
$$

denotes the minimum amount of product needed in period i to cover the demands in periods j to k, including the induced losses. As we only consider strictly positive demands, this implies

$$
\frac{1}{\alpha_{j,k}} d_{j,k} < \frac{1}{\alpha_{j,m}} d_{j,m}
$$

for $k \in [T]$ and $j, m \in [k]$ with $j < m$.

For $j, k, m \in [T]$ with $j \leq k < m$, we further establish the recursion

$$\alpha_{k,m} d_{j,k} + d_{k+1,m} = d_{j,m},$$

which we will extensively employ later on.

Mark that, for equivalent solutions to problems (B) and (A), objective (3.7) exhibits a constant additional value

$$\sum_{t=1}^{T} p_t \sum_{i=1}^{t} \alpha_{i,t} d_i \tag{3.14}$$

when compared to objective (3.1). Thus, we can interpret objective (3.7) to be conceived by assuming that all product value that is distributed to satisfy some demand stays in the storage but is considered unavailable to meet further demand from this point forward. Hence, it implies storage costs in all following periods but does not influence the remainder of the solution. We stress that term (3.14) is indeed constant, as it is independent of x and y.

Remark. *In model* (A) *the final storage value is forced to be zero with constraint* (3.4)*, which is a non-trivial additional assumption on the single-item lot-sizing definition as used before. This, in combination with the positivity assumption on* d*, allows us to determine the dimension of the considered polytope, see Lemma 12.*

Still, from an optimization perspective this is not a relevant restriction as in the case that no optimal solution with zero final storage value to the problem without constraint (3.4) *exists, the problem is unbounded due to the linearity of the objective function. This is the case if and only if there exists a negative component in* c*, which can easily be checked.*

The aim of this chapter is to determine formulations, respectively representations, of the convex hull of the set of feasible solutions

$$X' := \left\{ (x,y) \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T \;\middle|\; \begin{array}{ll} \sum\limits_{i=1}^{t} \alpha_{i,t} x_i \geq d_{1,t} & \forall\, t \in [T-1] \\ \sum\limits_{t=1}^{T} \alpha_{t,T} x_t = d_{1,T} & \\ x_t \leq M y_t & \forall\, t \in [T] \end{array} \right\}$$

to model (B). It is obvious that X' is bounded, and thus $\operatorname{conv}(X')$ is a convex polytope.

3.2 Valid inequalities and facets

In the following, we generalize the class of (ℓ, S)-inequalities introduced by Barany et al. [6]. For $\alpha = 1$, they state

$$\sum_{i \in S} x_i + \sum_{i \in [\ell] \setminus S} d_{i,\ell} y_i \geq d_{1,\ell}$$

for $\ell \in [T]$ and $S \subseteq [\ell]$. Here, the value of $d_{j,k}$, which in our definition (3.13) depends on the conservation rate α, simplifies to $\sum_{i=j}^{k} d_i$.

These inequalities serve as the based for the ideal formulation.

Theorem 5. *The (generalized) (ℓ, S)-inequalities*

$$\sum_{i \in S} \alpha_{i,\ell} x_i + \sum_{i \in [\ell] \setminus S} d_{i,\ell} y_i \geq d_{1,\ell} \tag{3.15}$$

with $\ell \in [T]$ and $S \subseteq [\ell]$ are valid for X'.

Proof. Let a feasible solution $(x, y) \in X'$ to model (B) be given. Then, we consider two cases.

If $y_i = 0$ holds for all $i \in [\ell] \setminus S$, and thus $x_i = 0$ holds for all $i \in [\ell] \setminus S$, then

$$\sum_{i \in S} \alpha_{i,\ell} x_i + \sum_{i \in [\ell] \setminus S} d_{i,\ell} y_i = \sum_{i=1}^{\ell} \alpha_{i,\ell} x_i \geq d_{1,\ell}$$

is implied.

Otherwise, we consider $k := \min \{i \in [\ell] \setminus S \mid y_i = 1\}$. Hence, $x_i = 0$ holds for all $i \in [k-1] \setminus S$. This yields

$$\sum_{i \in S} \alpha_{i,\ell} x_i + \sum_{i \in [\ell] \setminus S} d_{i,\ell} y_i \geq \alpha_{k,\ell} \sum_{i=1}^{k-1} \alpha_{i,k} x_i + d_{k,\ell} \geq \alpha_{k,\ell} d_{1,k-1} + d_{k,\ell} = d_{1,\ell}.$$

□

This class of inequalities contains some of the constraints stated in model (B).

Remark 1. *The (generalized) (ℓ, S)-inequalities include constraints* (3.8) *via $\ell := t$ and $S := [t]$ for $t \in [T-1]$.*

As the number of (generalized) (ℓ, S)-inequalities is exponential in the instance size, it is intuitive to consider whether there is a way to polynomially separate them.

Lemma 11. *Let a solution $(x, y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0}$ to the linear programming relaxation of model* (B) *be given. It suffices to only consider the T pairs (ℓ, S_ℓ) with $\ell \in [T]$ and*

$$S_\ell := \{i \in [\ell] \mid \alpha_{i,\ell} x_i < d_{i,\ell} y_i\}$$

to separate the (generalized) (ℓ, S)-inequalities. Thus, the separation process can be implemented with quadratic run-time.

Proof. For any given $\ell \in [T]$ we consider which choices of $S \subseteq [\ell]$ are critical for the feasibility of the (ℓ, S)-inequality (3.15).

Separately, for each $i \in [\ell]$ with $i \in S$ or $i \notin S$ it is determined whether the addend $\alpha_{i,\ell} x_i$ or the addend $d_{i,\ell} y_i$ is contributed to the left-hand side. Conversely, the right-hand side is unaffected by S. Thus, the given choice S_ℓ maximizes the left-hand side, and the statement is implied. □

Now, we explore the affine dimension of the convex hull of X' to base all further proofs on.

Lemma 12. *The affine dimension of* $\operatorname{conv}(X')$ *is*

$$\dim(\operatorname{conv}(X')) = \dim(X') = 2T - 2.$$

Proof. As we assume $d_t > 0$ for all $t \in [T]$, all valid solutions satisfy $x_1 > 0$, which implies $y_1 = 1$. Furthermore, given the constraint

$$\sum_{t=1}^{T} \alpha_{t,T} x_t = d_{1,T}$$

in the formulation of X', we conclude

$$\dim(X') \leq 2T - 2$$

since model (B) contains $2T$ variables.

To show $\dim(X') \geq 2T - 2$, we present $2T - 1$ affinely independent points in X'. These are given by the following two classes. For $j \in [2, T]$ let w_j be in the form $(x, y) \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T$ with

$$\begin{aligned}
x_1 &= \frac{1}{\alpha_{1,T}} d_{1,T}, & y_1 &= 1 & & \\
x_j &= 0, & y_j &= 1 & & \\
x_t &= 0, & y_t &= 0 & & \text{for } t \in [T] \backslash \{1, j\},
\end{aligned}$$

and for $j \in [T]$ let v_j be in the form $(x, y) \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T$ with

$$\begin{aligned}
x_t &= d_t, & y_t &= 1 & & \text{for } t \in [j-1] \\
x_j &= \frac{1}{\alpha_{j,T}} d_{j,T}, & y_j &= 1 & & \\
x_t &= 0, & y_t &= 0 & & \text{for } t \in [j+1, T].
\end{aligned}$$

These vectors clearly belong to X'. Rearranging all considered vectors $u = (x, y)$ to $u' = (x_1, y_1, y_2, \ldots, y_T, x_2, x_3, \ldots, x_T)^T$, we conclude that they are linearly independent and thus affinely independent as the matrix

$$(w'_2, w'_3, \ldots, w'_T, v'_1, v'_2 \ldots, v'_T)$$

yields a column echelon form. $\square$

While we do assume strictly positive demands, as they are necessary for some concepts later on, one can consider the influence of zero-demand periods on the dimension of $\operatorname{conv}(X')$.

Observation 3. *Consider the case of allowing for zero-demand periods. Let $\rho \in \{0,1\}$ indicate if the first period has zero demand and τ denote the number of consecutive zero-demand periods at the end of the time horizon $[T]$. Then*

$$\dim(X') = 2T + \rho - \tau - 2$$

holds.

Proof. Consider the case $\rho = 0$. For $t \in [T-\tau+1, T]$ we have $x_t = 0$ due to the zero final storage assumption. Hence, including $y_1 = 1$, we observe $\tau+1$ affinely independent valid equalities for X' and conclude

$$\dim(\operatorname{conv}(X')) = \dim(X') \leq 2T - \tau - 2.$$

Now, we construct $2T-\tau-2$ affinely independent points in X' via the following auxiliary structure. For each zero-demand period in $t \in [T-\tau]$ we adjust the single-item lot-sizing problem induced by the parameters of periods $[T-\tau]$ by distributing a sufficiently small strictly positive amount of demand from period $T-\tau$ to period t, while discounting for storage losses. That is, we decrease $d_{T-\tau}$ by $\varepsilon_t > 0$, and increase d_t to $\frac{1}{\alpha_{t,T-\tau}}\varepsilon_t$, while $d_{T-\tau}$ stays positive. Each solution for this auxiliary problem with strictly positive demands can be extended to a solution of the original problem by letting $x_t = y_t = 0$ for all $t \in [T-\tau+1, t]$. Hence, the application of Lemma 12 to the auxiliary problem on $[T-\tau]$ yields $2(T-\tau)-1$ affinely independent valid solutions to the original problem. Another τ affinely independent valid solutions are received by considering such a solution and separately letting $y_t = 1$ for each $t \in [T-\tau+1, t]$.

In the case of $\rho = 1$, the above arguments only change by losing the valid equality $y_1 = 1$, but gaining an affinely independent point via constructing a solution with $x_1 = y_1 = 0$. □

Still, from now on we assume strictly positive demands again. With the dimensionality result from Lemma 12, we identify most of the (generalized) (ℓ, S)-inequalities as facet defining, so in a way necessary for an ideal formulation.

Theorem 6. *For $\ell \in [T-1]$, $1 \in S$ and $S \neq [\ell]$, the (generalized) (ℓ, S)-inequalities define facets of* $\operatorname{conv}(X')$.

Proof. Given an (generalized) (ℓ, S)-inequality (3.15) under the given assumptions, we construct $2T-2$ affinely independent points in X' which satisfy the considered (generalized) (ℓ, S)-inequality with equality.

Therefore, let $k \in [\ell]$ be the smallest index with $k \notin S$, so in particular $k > 1$. Now, consider the subproblem H_1 on periods $[k-1]$ but with adjusted demands

$$d'_t := \begin{cases} d_t & \text{for } t \in [1, k-2] \\ \frac{1}{\alpha_{k-1,\ell}} d_{k-1,\ell} & \text{for } t = k-1, \end{cases} \tag{3.16}$$

that is

$$
\begin{aligned}
(\mathrm{H}_1)\ \min\ & \sum_{t=1}^{k-1}(c_t x_t + f_t y_t) && && (3.17)\\
\text{s.t.}\ & \sum_{i=1}^{t} \alpha_{i,t} x_i \geq d'_{1,t} && \forall\, t \in [k-2] && (3.18)\\
& \sum_{i=1}^{k-1} \alpha_{i,k-1} x_i = d'_{1,k-1} && && (3.19)\\
& x_t \leq M y_t && \forall\, t \in [k-1] && (3.20)\\
& x_t \geq 0 && \forall\, t \in [k-1] && (3.21)\\
& y_t \in \{0,1\} && \forall\, t \in [k-1] && (3.22)
\end{aligned}
$$

and the subproblem H_2 on periods $[\ell+1, T]$, that is

$$
\begin{aligned}
(\mathrm{H}_2)\ \min\ & \sum_{t=\ell+1}^{T} (c_t x_t + f_t y_t) && \\
\text{s.t.}\ & \sum_{i=\ell+1}^{t} \alpha_{i,t} x_i \geq d_{\ell+1,t} && \forall\, t \in [\ell+1, T]\\
& \sum_{i=\ell+1}^{k-1} \alpha_{i,k-1} x_i = d_{\ell+1,k-1} && \\
& x_t \leq M y_t && \forall\, t \in [\ell+1, T]\\
& x_t \geq 0 && \forall\, t \in [\ell+1, T]\\
& y_t \in \{0,1\} && \forall\, t \in [\ell+1, T].
\end{aligned}
$$

Consider a feasible solution $(x^1, y^1) \in \mathbb{R}_{\geq 0}^{k-1} \times \{0,1\}^{k-1}$ to (H_1) and a feasible solution $(x^2, y^2) \in \mathbb{R}_{\geq 0}^{[\ell+1,T]} \times \{0,1\}^{[\ell+1,T]}$ to (H_2). Then, $(x', y') \in \mathbb{R}_{\geq 0}^{T} \times \{0,1\}^T$ with

$$
(x'_t, y'_t) = \begin{cases} (x^1_t, y^1_t) & \text{for } t \in [1, k-1] \\ (0,0) & \text{for } t \in [k, \ell] \\ (x^2_t, y^2_t) & \text{for } t \in [\ell+1, T] \end{cases} \tag{3.23}
$$

is a feasible solution to the original problem, as $d'_{k-1} = \frac{1}{\alpha_{k-1,\ell}} d_{k-1,\ell}$ constitutes the minimum amount of product needed in period $k-1$ to cover all original demands from periods $k-1$ to ℓ. Mark that any solution of this kind satisfies $x'_{\ell+1} > 0$ and thus $y'_{\ell+1} = 1$, as $d_{\ell+1} > 0$ holds. As Lemma 12 establishes $2(k-1)-1$ affinely independent feasible solutions to H_1 and $2(T-\ell)-1$ affinely independent feasible solutions to H_2, this construction yields $2T-2(\ell-k)-5$ affinely independent feasible solutions to the original problem. We denote these solutions (in form of (3.23)) as u_j with $j \in [2T-2(\ell-k)-5]$.

Furthermore, we iteratively construct another two affinely independent feasible solutions v_j, w_j in the form $(x', y') \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T$ for each $j \in [k, \ell]$. For these, we always let

$$(x'_t, y'_t) = \begin{cases} (0,0) & \text{for } t \in [\ell] \backslash \{1, j\} \\ (d_t, 1) & \text{for } t \in [\ell + 2, T]. \end{cases} \tag{3.24}$$

The first solution v_j is completed by

$$(x'_t, y'_t) = \begin{cases} (\frac{1}{\alpha_{1,j-1}} d_{1,j-1}, 1) & \text{for } t = 1 \\ (\frac{1}{\alpha_{j,\ell}} d_{j,\ell}, 1) & \text{for } t = j \\ (d_{\ell+1}, 1) & \text{for } t = \ell + 1. \end{cases}$$

For the second solution w_j, we consider two cases. If $j \in S$ holds, we let

$$(x'_t, y'_t) = \begin{cases} (\frac{1}{\alpha_{1,\ell}} d_{1,\ell}, 1) & \text{for } t = 1 \\ (0, 1) & \text{for } t = j \\ (d_{\ell+1}, 1) & \text{for } t = \ell + 1, \end{cases}$$

and if $j \notin S$ holds, we let

$$(x'_t, y'_t) = \begin{cases} (\frac{1}{\alpha_{1,j-1}} d_{1,j-1}, 1) & \text{for } t = 1 \\ (\frac{1}{\alpha_{j,\ell+1}} d_{j,\ell+1}, 1) & \text{for } t = j \\ (0, 0) & \text{for } t = \ell + 1. \end{cases}$$

To obtain one final affinely independent feasible solution w', we modify the former one for $j = k$ $(\notin S)$ by setting $y'_{\ell+1} = 1$, that is

$$(x'_t, y'_t) = \begin{cases} (\frac{1}{\alpha_{1,j-1}} d_{1,j-1}, 1) & \text{for } t = 1 \\ (\frac{1}{\alpha_{j,\ell+1}} d_{j,\ell+1}, 1) & \text{for } t = j \\ (0, 1) & \text{for } t = \ell + 1. \end{cases}$$

Thus, we have constructed an additional $2(\ell - k + 1) + 1$ solutions. As the feasibility of all solutions is easily verified, we only prove that they satisfy the considered (generalized) (ℓ, S)-inequality with equality and their affine independence.

For u_j with $j \in [2T - 2(\ell - k) - 5]$, the considered (generalized) (ℓ, S)-inequality is satisfied with equality, as, due to k being the smallest index outside S,

$$\begin{aligned}
\sum_{i\in S}\alpha_{i,\ell}x_i+\sum_{i\in[\ell]\setminus S}d_{i,\ell}y_i &= \sum_{i=1}^{k-1}\alpha_{i,\ell}x_i+\sum_{\substack{i\in[k,\ell]:\\ i\in S}}\alpha_{i,\ell}x_i+\sum_{\substack{i\in[k,\ell]:\\ i\notin S}}d_{i,\ell}y_i\\
&\overset{(3.16)}{=} \alpha_{k-1,\ell}\sum_{i=1}^{k-1}\alpha_{i,k-1}x_i\\
&\overset{(3.19)}{=} d'_{1,k-1}\\
&= \alpha_{k-1,\ell}\sum_{i=1}^{k-1}\alpha_{i,k-1}d'_i\\
&= \sum_{i=1}^{k-1}\alpha_{i,\ell}d'_i\\
&= \sum_{i=1}^{k-2}\alpha_{i,\ell}d'_i+\alpha_{k-1,\ell}d'_{k-1}\\
&\overset{(3.16)}{=} \sum_{i=1}^{k-2}\alpha_{i,\ell}d_i+d_{k-1,\ell}\\
&= \alpha_{k-2,\ell}d_{1,k-2}+d_{k-1,\ell}\\
&= d_{1,\ell}
\end{aligned}$$

holds.

For all solutions v_j, w_j and w' for $j\in[k,\ell]$ ($j=k$ for w') we have

$$\sum_{i\in S}\alpha_{i,\ell}x_i+\sum_{i\in[\ell]\setminus S}d_{i,\ell}y_i=\alpha_{1,\ell}x_1+\begin{cases}\alpha_{j,\ell}x_j & \text{if } j\in S\\ d_{j,\ell}y_j & \text{if } j\notin S\end{cases}$$

due to construction (3.24) and the assumption $1\in S$. Here, for w_j with $j\in S$ we deduce

$$\alpha_{1,\ell}x_1+\begin{cases}\alpha_{j,\ell}x_j & \text{if } j\in S\\ d_{j,\ell}y_j & \text{if } j\notin S\end{cases}=\alpha_{1,\ell}\frac{1}{\alpha_{1,\ell}}d_{1,\ell}=d_{1,\ell}.$$

All other solutions v_j for $j\in[k,\ell]$, w' and w_j with $j\notin S$ exhibit

$$\alpha_{1,\ell}x_1+\begin{cases}\alpha_{j,\ell}x_j & \text{if } j\in S\\ d_{j,\ell}y_j & \text{if } j\notin S\end{cases}=\alpha_{1,\ell}\frac{1}{\alpha_{1,j-1}}d_{1,j-1}+d_{j,\ell}=\alpha_{j-1,\ell}d_{1,j-1}+d_{j,\ell}=d_{1,\ell}.$$

Thus, the considered (generalized) (ℓ,S)-inequality is satisfied with equality by all given solutions.

For the affine independence, we consider an affine combination

$$\sum_{j=1}^{2T-2(\ell-k)-5}\lambda_j u_j+\sum_{j=k}^{\ell}\rho_j v_j+\sum_{j=k}^{\ell}\tau_j w_j+\pi w'=0 \tag{3.25}$$

with weights $(\lambda, \rho, \tau, \pi) \in \mathbb{R}^{2T-2(\ell-k)-5} \times \mathbb{R}^{\ell} \times \mathbb{R}^{\ell} \times \mathbb{R}$ which satisfy

$$\sum_{j=1}^{2T-2(\ell-k)-5} \lambda_j + \sum_{j=k}^{\ell} \rho_j + \sum_{j=k}^{\ell} \tau_j + \pi = 0. \tag{3.26}$$

For the affine independence we aim to show that the only feasible weights are all zero. In the following, we denote addends u_j, v_j, w_j or w' with weights that are not yet proven to be zero as *remaining addends*.

For $j \in [k+1, \ell]$, we conclude

$$\rho_j + \tau_j = 0,$$

as v_j and w_j are the only addends with $y'_j \neq 0$, namely $y'_j = 1$. Since further, all addends but v_j and w_j satisfy $x'_j = 0$ and v_j and w_j exhibit different values for x'_j,

$$\rho_j = \tau_j = 0 \qquad \forall\ j \in [k+1, \ell]$$

is implied.

Hence, w_k is rendered the only remaining addend with and $y'_{\ell+1} = 0$. Since all other remaining addend exhibit $y'_{\ell+1} = 1$, we deduce

$$\sum_{j=1}^{2T-2(\ell-k)-5} \lambda_j + \rho_k + \pi = 0$$

from the component corresponding to $y'_{\ell+1}$ in (3.25), and further

$$\sum_{j=1}^{2T-2(\ell-k)-5} \lambda_j + \rho_k + \tau_k + \pi = 0$$

from (3.25). Thus,

$$\tau_k = 0$$

is implied. Further, for $j = k$ $(\notin S)$ we conclude

$$\rho_k + \pi = 0,$$

as v_k and w' are only remaining addends with $y'_k \neq 0$, namely $y'_k = 1$. Since further, all remaining addends but v_k and w' satisfy $x'_k = 0$ and v_k and w' exhibit different values for x'_k,

$$\rho_k = \pi = 0$$

is implied. Lastly, all remaining addends v_j are affinely independent among each other, such that

$$\lambda_j = 0 \qquad \forall\ j \in [2T - 2(\ell - k) - 5]$$

is implied. $\square$

Furthermore, we discuss two more classes of facets, which are already implicitly contained in model (B). The first is given by the relaxation of the binary constraints of the commitment variables y.

Theorem 7. *For $t' \in [2, T]$, the inequalities*

$$y_{t'} \leq 1$$

define facets of $\operatorname{conv}(X')$*.*

Proof. For a given $t' \in [2, T]$, consider the auxiliary problem in which the demand of period t' is merged into the demand of period $t' - 1$ and period t' is dismissed. That is

$$\begin{aligned}
\text{(H)}\ \min\quad & \sum_{t=1}^{T-1} (c'_t x_t + f'_t y_t) & \\
\text{s.t.}\quad & \sum_{i=1}^{t} \alpha'_{i,t} x_i \geq d'_{1,t} & \forall\, t \in [T-2] \\
& \sum_{i=1}^{k-1} \alpha'_{i,k-1} x_i = d'_{1,T-1} & \\
& x_t \leq M y_t & \forall\, t \in [T-1] \\
& x_t \geq 0 & \forall\, t \in [T-1] \\
& y_t \in \{0, 1\} & \forall\, t \in [T-1]
\end{aligned}$$

with

$$(c'_t, f'_t, \alpha'_t, d'_t) = \begin{cases} (c_t, f_t, \alpha_t, d_t) & \text{for } t \in [t'-2] \\ (c_t, f_t, \alpha_t, d_t + \frac{1}{\alpha'_t} d_{t'}) & \text{for } t = t'-1 \\ (c_{t+1}, f_{t+1}, \alpha_{t'}\alpha_{t+1}, d_{t+1}) & \text{for } t = t' \\ (c_{t+1}, f_{t+1}, \alpha_{t+1}, d_{t+1}) & \text{for } t \in [t'+1, T-1]. \end{cases}$$

Lemma 12 establishes $2(T-1)-1$ affinely independent feasible solutions $(x', y') \in \mathbb{R}^{T-1} \times \mathbb{Z}^{T-1}$ to (H). These induce $2(T-1)-1$ solutions $(x, y) \in X'$ in the following form:

$$(x_t, y_t) = \begin{cases} (x'_t, y'_t) & \text{for } t \in [t'-1] \\ (0, 1) & \text{for } t = t' \\ (x'_{t-1}, y'_{t-1}) & \text{for } t \in [t'+1, T] \end{cases}$$

Another affinely independent solution solution is given by $(x, y) = (d, 1)$. All these $2T - 2$ solutions satisfy the considered valid inequality with equality. $\square$

The last set of facets that we consider is obtained by determining the smallest valid value for M for each constraint in (3.9).

Theorem 8. *For $t' \in [2, T]$ the inequalities*

$$x_{t'} \leq \frac{1}{\alpha_{t',T}} d_{t',T} y_{t'}$$

define facets of $\operatorname{conv}(X')$.

Proof. First, we show that $\frac{1}{\alpha_{t',T}} d_{t',T}$ is a valid choice for M. Intuitively, with $\frac{1}{\alpha_{t',T}} d_{t',T}$, the upper limit we impose on $x_{t'}$ amounts to the minimum production value in period t' needed to fulfill all following demands. This is valid as the final storage value is forced to be zero. Formally, this directly follows from combining constraint (3.10) with constraint (3.8) for $t = t' - 1$ as

$$\begin{aligned}
\alpha_{t',T} x_{t'} &\leq \sum_{t=t'}^{T} \alpha_{t,T} x_t \\
&= \sum_{t=1}^{T} \alpha_{t,T} x_t - \sum_{i=1}^{t'-1} \alpha_{i,T} x_i \\
&= \sum_{t=1}^{T} \alpha_{t,T} x_t - \alpha_{t'-1,T} \sum_{i=1}^{t'-1} \alpha_{i,t'-1} x_i \\
&\leq d_{1,T} - \alpha_{t'-1,T} d_{1,t'-1} \\
&= d_{t',T}
\end{aligned}$$

holds.

For the affine independence, again consider the $2(T-1)-1$ affinely independent feasible solutions $(x', y') \in \mathbb{R}^{T-1} \times \mathbb{Z}^{T-1}$ to (H) from the proof of Theorem 7. These yield $2(T-1)-1$ affinely independent solutions $(x, y) \in X'$ in the following form:

$$(x_t, y_t) = \begin{cases} (x'_t, y'_t) & \text{for } t \in [t'-1] \\ (0, 0) & \text{for } t = t' \\ (x'_{t-1}, y'_{t-1}) & \text{for } t \in [t'+1, T] \end{cases}$$

Another affinely independent solution is given by

$$(x_t, y_t) = \begin{cases} (d_t, 1) & \text{for } t \in [t'-1] \\ (\frac{1}{\alpha_{t,T}} d_{t,T}, 1) & \text{for } t = t' \\ (0, 0) & \text{for } t \in [t'+1, T]. \end{cases}$$

All these $2T - 2$ solutions satisfy the considered valid inequality with equality. □

While the above inequalities define facets, they do not need to be included explicitly in an ideal formulation, as the considered polyhedron is not full dimensional.

Remark 2. *Although the inequalities in Theorem 8 are facet defining, they are not explicitly stated in our ideal formulation, see Theorem 9. Still, they are implicitly induced by substituting constraint* (3.8)*, so*

$$d_{1,T} = \sum_{t=1}^{T} \alpha_{t,T} x_t,$$

in the (generalized) (ℓ, S)-inequality with $\ell = T$ and $S = [T] \backslash \{t'\}$.

3.3 Auxiliary structure

In order to derive optimal dual solutions in the primal-dual proofs, an auxiliary structure inspired by dynamic programming is introduced. We exploit the following "Interval Division Property" for the single-item lot-sizing problem with storage deterioration, as shown by Hsu [29] for a more general model. Mark that this is equivalent to the zero-inventory property in Lemma 6.

Lemma 13. *There exists an optimal solution $(x^*, y^*) \in X'$ which exhibits*

$$\begin{aligned} x^*_{t_i} &= \sum_{k=t_i}^{t_{i+1}} \frac{1}{\alpha_{t_i,k}} d_k = \frac{1}{\alpha_{t_i,t_{i+1}}} d_{t_i,t_{i+1}} && \forall\ i \in [k] \\ x^*_t &= 0 && \forall\ t \notin \{t_1, \dots, t_k\} \end{aligned}$$

for some increasing sequence of periods $1 = t_1 \leq t_2 \leq \ldots \leq t_k \leq t_{k+1} = T$.

To establish the auxiliary structure, we first eliminate all negative constant costs f_t for $t \in [T]$. Here, we exploit that the corresponding commitment variables y_t are set to 1 in any optimal solution independently of the production value x_t. Therefore, we incorporate the notation

$$z^+ := \max\{z, 0\}, \qquad z^- := \min\{z, 0\}$$

for $z \in \mathbb{R}$ and define

$$\begin{aligned} G(\ell) := \min \quad & \sum_{t=1}^{\ell} \left(c_t x_t + f_t^+ y_t \right) \\ \text{s.t.} \quad & \sum_{i=1}^{t} \alpha_{i,t} x_i \geq d_{1,t} && \forall\ t \in [\ell - 1] \\ & \sum_{t=1}^{n} \alpha_{t,\ell} x_t = d_{1,\ell} \\ & x_t \leq M y_t && \forall\ t \in [\ell] \\ & x_t \geq 0 && \forall\ t \in [\ell] \\ & y_t \in \{0, 1\} && \forall\ t \in [\ell], \end{aligned}$$

for $\ell \in [T]$, and in particular

$$G(0) := 0.$$

Thus, $G(T) + \sum_{t=1}^{T} f_t^-$ is the optimal value of model (B). We derive the following structural properties.

Lemma 14. *For $(f, c) \in \mathbb{R}^T \times \mathbb{R}^T_{\geq 0}$ we have*

(i) $G(\ell) \leq G(\ell + 1)$ for $\ell \in [T-1]$,

(ii) $G(k) \leq G(t-1) + f_t^+ + c_t \frac{1}{\alpha_{t,k}} d_{t,k}$ for $k \in [T]$ and $t \in [k]$,

(ii) $\alpha_{t,\ell} \frac{G(\ell)-G(\ell-1)}{d_\ell} \geq \alpha_{t,\ell+1} \frac{G(\ell+1)-G(\ell)}{d_{\ell+1}}$ for $\ell \in [T-1]$ and $t \in [\ell]$.

Proof. Let $\mathcal{G}(\ell)$ denote the problem corresponding to $G(\ell)$ for $1 \leq \ell \leq T$.

(i) Any optimal solution to $\mathcal{G}(\ell + 1)$ yields a less costly feasible solution to $\mathcal{G}(\ell)$ by suitably decreasing the production values.

(ii) Given an optimal solution to $\mathcal{G}(t-1)$, it can be extended to a feasible solution to $\mathcal{G}(k)$ by producing a value of $\frac{1}{a_{t,k}} d_{t,k}$ in period t and nothing thereafter. This increases the costs by $f_t^+ + c_t \frac{1}{a_{t,k}} d_{t,k}$.

(iii) Consider an optimal solution s^* to $\mathcal{G}(\ell)$ in which all demand for period ℓ is produced as $\frac{1}{\alpha_{t',\ell}} d_\ell$ in a single period $t' \leq \ell$. This exists according to Lemma 13. We extend s^* to a feasible solution of $\mathcal{G}(\ell + 1)$ by also producing demand $d_{\ell+1}$ in period t' (resulting in an additional value of $\frac{1}{\alpha_{t',\ell+1}} d_{\ell+1}$) and nothing in period $\ell + 1$. Hence

$$G(\ell) + c_{t'} \frac{1}{\alpha_{t',\ell+1}} d_{\ell+1} \geq G(\ell + 1)$$

holds. Now, we aim to show that

$$G(\ell) - c_{t'} \frac{1}{\alpha_{t',\ell}} d_\ell \geq G(\ell - 1)$$

also holds since it implies

$$\alpha_{t',\ell} \frac{G(\ell) - G(\ell - 1)}{d_\ell} \geq c_{t'} \geq \alpha_{t',\ell+1} \frac{G(\ell + 1) - G(\ell)}{d_{\ell+1}},$$

which yields the statement (by multiplying positive constants to the inequality).

If $t' < \ell$ holds, then s^* exhibits no production in period ℓ, and restricting s^* to periods $[\ell]$, and reducing the production value in period t' by $\frac{1}{\alpha_{t',\ell}} d_\ell$ yields a feasible solution to $\mathcal{G}(\ell - 1)$. Thus,

$$G(\ell) - c_{t'} \frac{1}{\alpha_{t',\ell}} d_\ell \geq G(\ell - 1)$$

holds.

If $t' = \ell$ holds, the production value of s^* in period ℓ is d_ℓ, and restricting s^* to periods $[\ell]$ yields a feasible solution to $\mathcal{G}(\ell - 1)$. Hence,

$$G(\ell-1) \leq G(\ell) - (f_\ell^+ + c_\ell \frac{1}{\alpha_{\ell,\ell}} d_\ell) = G(\ell) - f_{t'}^+ - c_{t'} \frac{1}{\alpha_{t',\ell}} d_\ell \leq G(\ell) - c_{t'} \frac{1}{\alpha_{t',\ell}} d_\ell$$

holds.

□

3.4 Ideal formulation

Given the previous structural results, we are able to determine a formulation for the convex hull of X'.

Theorem 9. *The convex hull conv*(X') *of the feasible solutions of model* (B) *is given by*

$$X := \left\{ (x,y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \;\middle|\; \begin{array}{ll} \sum_{t=1}^{T} \alpha_{t,T} x_t = d_{1,T} & \\ \sum_{i \in S} \alpha_{i,\ell} x_i + \sum_{i \in [\ell] \setminus S} d_{i,\ell} y_i \geq d_{1,\ell} & \begin{array}{l} \forall\, \ell \in [T], \\ \forall\, S \subseteq [\ell] \end{array} \\ y_t \leq 1 & \forall\, t \in [T] \end{array} \right\}.$$

Proof. Since the inequalities describing X which are not explicitly stated in the description of X' are valid for X', it is obvious that $\text{conv}(X') \subseteq X$ holds.

To show $X \subseteq \text{conv}(X')$, consider that for any $(f, c') \in \mathbb{R}^T \times \mathbb{R}^T$ replacing c'_t with $c_t = c'_t + \alpha_{t,T} M$ in model (B) only results in constant additional costs as

$$c_t x_t = \sum_{t=1}^{T} (c'_t + \alpha_{t,T} M) x_t = \sum_{t=1}^{T} c'_t x_t + M \sum_{t=1}^{T} \alpha_{t,T} x_t = \sum_{t=1}^{T} c'_t x_t + M d_{1,T}$$

holds. Thus, assuming $X \not\subseteq \text{conv}(X')$, there would exist cost factors $(f, c) \in \mathbb{R}^T \times \mathbb{R}^T_{\geq 0}$ with non-negative values for c such that

$$\text{(P)} \;\; \min \quad \sum_{t=1}^{T} (c_t x_t + f_t y_t) \tag{3.27}$$

$$\text{s.t.} \quad \sum_{t=1}^{T} \alpha_{t,T} x_t = d_{1,T} \tag{3.28}$$

$$\sum_{i \in S} \alpha_{i,\ell} x_i + \sum_{i \in [\ell] \setminus S} d_{i,\ell} y_i \geq d_{1,\ell} \qquad \forall\, \ell \in [T], \; \forall\, S \subseteq [\ell] \tag{3.29}$$

$$y_t \leq 1 \qquad \forall\, t \in [T] \tag{3.30}$$

$$x_t, y_t \geq 0 \qquad \forall\, t \in [T], \tag{3.31}$$

has a better solution value than model (B). This we aim to contradict. We remark, that we explicitly demand c to be non negative, to, for example, be able to apply Lemma 14 in the course of the proof.

Let an optimal solution to model (B) with cost factors $(f, c) \in \mathbb{R}^T \times \mathbb{R}^T_{\geq 0}$ and value $G(T) + \sum_{t=1}^{T} f_t^-$ be given. We show that this solution is not only feasible but also optimal to (P). To this end, we construct a feasible solution with identical value to the dual linear program

$$\text{(D)} \quad \max \quad -\sum_{t=1}^{T} z_t + d_{1,T} u_0 + \sum_{\ell=1}^{T} \sum_{S \subseteq [\ell]} d_{1,\ell} u_{\ell,S} \tag{3.32}$$

$$\text{s.t.} \quad \alpha_{t,T} u_0 + \sum_{\ell=t}^{T} \alpha_{t,\ell} \sum_{\substack{S \subseteq [\ell]: \\ t \in S}} u_{\ell,S} \leq c_t \qquad \forall\, t \in [T] \tag{3.33}$$

$$- z_t + \sum_{\ell=t}^{T} d_{t,\ell} \sum_{\substack{S \subseteq [\ell]: \\ t \notin S}} u_{\ell,S} \leq f_t \qquad \forall\, t \in [T] \tag{3.34}$$

$$z_t, u_{\ell,S} \geq 0 \qquad \forall\, t \in [T],\ \forall\, \ell \in [T],\ \forall\, S \subseteq [\ell] \tag{3.35}$$

$$u_0 \text{ free} \tag{3.36}$$

of (P). Here, the dual variable u_0 corresponds to primal constraint (3.28), the dual variables $u_{\ell,S}$ correspond to primal constraints (3.29) and the dual variables z_t correspond to primal constraints (3.30).

In order to find such a solution to (D), we define

$$v_\ell := \frac{G(\ell) - G(\ell - 1)}{d_\ell} \qquad \forall\, \ell \in [T] \tag{3.37}$$

$$v_{T+1} := 0 \tag{3.38}$$

$$\alpha_{T+1} := 0 \tag{3.39}$$

$$v_{t,\ell} := (\alpha_{t,\ell} v_\ell - c_t)^+ - (\alpha_{t,\ell+1} v_{\ell+1} - c_t)^+ \qquad \forall\, \ell \in [T-1],\ \forall\, t \in [\ell] \tag{3.40}$$

$$w_{t,\ell} := \alpha_{t,\ell} v_\ell - \alpha_{t,\ell+1} v_{\ell+1} - v_{t,\ell} \qquad \forall\, \ell \in [T-1],\ \forall\, t \in [\ell]. \tag{3.41}$$

Note that, due to the non-negativity of c and with Lemma 14, we identify all these values as non negative since:

- $G(\ell) - G(\ell - 1) \geq 0$ holds for $\ell \in [T]$.
- For $t \in [\ell]$ we have $\alpha_{t,\ell} v_\ell \geq \alpha_{t,\ell+1} v_{\ell+1}$.

- For $\beta, \gamma, \delta \in \mathbb{R}_{\geq 0}$ with $\beta \geq \gamma$

$$(\beta - \delta)^+ - (\gamma - \delta)^+ = \begin{cases} \beta - \gamma & \text{if } \delta \leq \gamma \\ \beta - \delta & \text{if } \gamma < \delta < \beta \\ 0 & \text{if } \beta \leq \delta, \end{cases}$$

  is satisfied, and thus,

$$0 \leq (\beta - \delta)^+ - (\gamma - \delta)^+ \leq \beta - \gamma.$$

  is implied.

Now, let

$$u_0 = 0, \qquad z_t = -f_t^- \tag{3.42}$$

for $t \in [T]$, and u such that

$$\alpha_{t,\ell} \sum_{\substack{S \subseteq [\ell]: \\ t \notin S}} u_{\ell,S} = v_{t,\ell} \qquad \begin{array}{l} \forall\ \ell \in [T], \\ \forall\ t \in [\ell] \end{array} \tag{3.43}$$

$$\alpha_{t,\ell} \sum_{\substack{S \subseteq [\ell]: \\ t \in S}} u_{\ell,S} = w_{t,\ell} \qquad \begin{array}{l} \forall\ \ell \in [T], \\ \forall\ t \in [\ell] \end{array} \tag{3.44}$$

holds, which is proven to exists at the end of the proof.

Given such a solution we now prove it to be feasible and optimal. With respect to the objective (3.32), the choices (3.42)-(3.44) imply a solution value of

$$\begin{aligned}
& -\sum_{t=1}^{T} z_t + d_{1,T} u_0 + \sum_{\ell=1}^{T} \sum_{S \subseteq [\ell]} d_{1,\ell} u_{\ell,S} \\
= & \sum_{t=1}^{T} f_t^- + \sum_{\ell=1}^{T} \sum_{S \subseteq [\ell]} d_{1,\ell} u_{\ell,S} \\
= & \sum_{t=1}^{T} f_t^- + \sum_{\ell=1}^{T} d_{1,\ell} \left(\sum_{\substack{S \subseteq [\ell]: \\ 1 \notin S}} u_{\ell,S} + \sum_{\substack{S \subseteq [\ell]: \\ 1 \in S}} u_{\ell,S} \right) \\
= & \sum_{t=1}^{T} f_t^- + \sum_{\ell=1}^{T} d_{1,\ell} \left(\frac{v_{1,\ell}}{\alpha_{1,\ell}} + \frac{w_{1,\ell}}{\alpha_{1,\ell}} \right) \\
\overset{(3.41)}{=} & \sum_{t=1}^{T} f_t^- + \sum_{\ell=1}^{T} d_{1,\ell} \left(v_\ell - \alpha_{\ell+1} v_{\ell+1} \right)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(3.38)}{=} \sum_{t=1}^{T} f_t^- + \sum_{\ell=1}^{T} d_{1,\ell} v_\ell - \sum_{\ell=1}^{T-1} \alpha_{\ell+1} d_{1,\ell} v_{\ell+1} \\
&= \sum_{t=1}^{T} f_t^- + \sum_{\ell=1}^{T} d_{1,\ell} v_\ell - \sum_{\ell=2}^{T} \alpha_{\ell} d_{1,\ell-1} v_{\ell} \\
&= \sum_{t=1}^{T} f_t^- + d_{1,1} v_1 + \sum_{\ell=2}^{T} \left(d_{1,\ell} - \alpha_\ell d_{1,\ell-1}\right) v_\ell \\
&= \sum_{t=1}^{T} f_t^- + d_1 v_1 + \sum_{\ell=2}^{T} \left(\sum_{k=1}^{\ell} \alpha_{k,\ell} d_k - \alpha_\ell \sum_{k=1}^{\ell-1} \alpha_{k,\ell-1} d_k \right) v_\ell \\
&= \sum_{t=1}^{T} f_t^- + d_1 v_1 + \sum_{\ell=2}^{T} \alpha_{\ell,\ell} d_\ell v_\ell \\
&= \sum_{t=1}^{T} f_t^- + \sum_{\ell=1}^{T} d_\ell v_\ell \\
&\overset{(3.37)}{=} \sum_{t=1}^{T} f_t^- + \sum_{\ell=1}^{T} \left(G(\ell) - G(\ell-1)\right) \\
&= \sum_{t=1}^{T} f_t^- + G(T),
\end{aligned}
$$

which matches the optimal solution value of the primal problem (P).

To account for feasibility, we show that constraints (3.33) and (3.34) are satisfied. Regarding constraints (3.33), choices (3.42),(3.44) and the non-negativity of c_T imply

$$
\begin{aligned}
&\alpha_{t,T} u_0 + \sum_{\ell=t}^{T} \alpha_{t,\ell} \sum_{\substack{S \subseteq [\ell]: \\ t \in S}} u_{\ell,S} \\
&= \sum_{\ell=t}^{T} w_{t,\ell} \\
&\overset{(3.41)}{=} \sum_{\ell=t}^{T} \left(\alpha_{t,\ell} v_\ell - \alpha_{t,\ell+1} v_{\ell+1} - v_{t,\ell}\right) \\
&\overset{(3.40)}{=} \sum_{\ell=t}^{T} \left(\alpha_{t,\ell} v_\ell - \alpha_{t,\ell+1} v_{\ell+1} - (\alpha_{t,\ell} v_l - c_t)^+ + (\alpha_{t,\ell+1} v_{\ell+1} - c_t)^+\right) \\
&= \sum_{\ell=t}^{T} \left(\left(\alpha_{t,\ell} v_\ell - (\alpha_{t,\ell} v_\ell - c_t)^+\right) - \left(\alpha_{t,\ell+1} v_{\ell+1} - (\alpha_{t,\ell+1} v_{\ell+1} - c_t)^+\right)\right) \\
&= \left(\alpha_{t,t} v_t - (\alpha_{t,t} v_t - c_t)^+\right) - \left(\alpha_{t,T+1} v_{T+1} - (\alpha_{t,T+1} v_{T+1} - c_T)^+\right) \\
&\overset{(3.38)}{=} \alpha_{t,t} v_t - (\alpha_{t,t} v_t - c_t)^+
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha_{t,t} v_t - (\alpha_{t,t} v_t - c_t) \\
&= c_t
\end{aligned}
$$

for $t \in [T]$. With respect to constraints (3.34), choice (3.43) and the non-negativity of c_T imply

$$
\begin{aligned}
& -z_t + \sum_{\ell=t}^{T} d_{t,\ell} \sum_{\substack{S \subseteq [\ell]: \\ t \notin S}} u_{\ell,S} \\
= \; & -z_t + \sum_{\ell=t}^{T} d_{t,\ell} \frac{v_{t,\ell}}{\alpha_{t,\ell}} \\
\overset{(3.40)}{=} \; & -z_t + \sum_{\ell=t}^{T} \frac{1}{\alpha_{t,\ell}} d_{t,\ell} \left((\alpha_{t,\ell} v_\ell - c_t)^+ - (\alpha_{t,\ell+1} v_{\ell+1} - c_t)^+ \right) \\
\overset{(3.38)}{=} \; & -z_t + \sum_{\ell=t}^{T} \frac{1}{\alpha_{t,\ell}} d_{t,\ell} (\alpha_{t,\ell} v_\ell - c_t)^+ - \sum_{\ell=t}^{T-1} \frac{1}{\alpha_{t,\ell}} d_{t,\ell} (\alpha_{t,\ell+1} v_{\ell+1} - c_t)^+ \\
= \; & -z_t + d_t (v_t - c_t)^+ + \sum_{\ell=t+1}^{T} \frac{1}{\alpha_{t,\ell}} d_{t,\ell} (\alpha_{t,\ell} v_\ell - c_t)^+ - \sum_{\ell=t+1}^{T} \frac{1}{\alpha_{t,\ell-1}} d_{t,\ell-1} (\alpha_{t,\ell} v_\ell - c_t)^+ \\
= \; & -z_t + d_t (v_t - c_t)^+ + \sum_{\ell=t+1}^{T} \left(\frac{1}{\alpha_{t,\ell}} \sum_{k=t}^{\ell} a_{k,\ell} d_k - \frac{1}{\alpha_{t,\ell-1}} \sum_{k=t}^{\ell-1} a_{k,\ell-1} d_k \right) (\alpha_{t,\ell} v_\ell - c_t)^+ \\
= \; & -z_t + d_t (v_t - c_t)^+ + \sum_{\ell=t+1}^{T} \left(\sum_{k=t}^{\ell} \frac{1}{\alpha_{t,k}} d_k - \sum_{k=t}^{\ell-1} \frac{1}{\alpha_{t,k}} d_k \right) (\alpha_{t,\ell} v_\ell - c_t)^+ \\
= \; & -z_t + d_t (v_t - c_t)^+ + \sum_{\ell=t+1}^{T} \frac{1}{\alpha_{t,\ell}} d_\ell (\alpha_{t,\ell} v_\ell - c_t)^+ \\
= \; & -z_t + \sum_{\ell=t}^{T} \frac{1}{\alpha_{t,\ell}} d_\ell (\alpha_{t,\ell} v_\ell - c_t)^+
\end{aligned}
$$

for $t \in [T]$. Thus, it suffices to show that

$$\sum_{\ell=t}^{T} \frac{1}{\alpha_{t,\ell}} d_\ell (\alpha_{t,\ell} v_\ell - c_t)^+ \leq f_t + z_t = f_t^+$$

holds. Again exploiting the non-negativity of c, according to Lemma 14, the term $(\alpha_{t,\ell} v_\ell - c_t)^+ \geq 0$ is non increasing in ℓ and hence for all $t \geq \ell$ the equality $(\alpha_{t,\ell} v_\ell - c_t)^+ = 0$ implies $(\alpha_{t,t} v_t - c_t)^+ = 0$. If $(\alpha_{t,t} v_t - c_t)^+ = 0$ holds, we have

$$\sum_{\ell=t}^{T} \frac{1}{\alpha_{t,\ell}} d_\ell (\alpha_{t,\ell} v_\ell - c_t)^+ = 0 \leq f_t^+.$$

Otherwise, we let $k \in [t, T]$ be the largest index which satisfies $(\alpha_{t,k} v_k - c_t)^+ > 0$. This choice, again with Lemma 14, yields

$$\begin{aligned}
&\sum_{\ell=t}^{T} \frac{1}{\alpha_{t,\ell}} d_\ell (\alpha_{t,\ell} v_\ell - c_t)^+ \\
= &\sum_{\ell=t}^{k} \frac{1}{\alpha_{t,\ell}} d_\ell (\alpha_{t,\ell} v_\ell - c_t)^+ \\
\overset{(3.37)}{=} &\sum_{\ell=t}^{k} \frac{1}{\alpha_{t,\ell}} d_\ell \left(\alpha_{t,\ell} \frac{G(\ell) - G(\ell-1)}{d_\ell} - c_t \right) \\
= &\sum_{\ell=t}^{k} (G(\ell) - G(\ell-1)) - \sum_{\ell=t}^{k} \frac{1}{\alpha_{t,\ell}} d_\ell c_t \\
= &\; G(k) - G(t-1) - c_t \frac{1}{\alpha_{t,k}} d_{t,k} \\
\le &\; f_t^+.
\end{aligned}$$

Thus, the feasibility and optimality of the solution is verified.

Finally, we construct values for u such that (3.43) and (3.44) is satisfied. As the sums are independent for different $\ell \in [T]$, we provide this in the form of Algorithm 1, which iteratively computes suitable components of u for fixed ℓ. Here, we start with all-zero values of u and suitably increase components until (3.43) and (3.44) are met, while retaining the satisfaction of (3.33) and (3.34).

In the algorithm, we denote the current choice of components as $u'_{\ell,S}$ and define an auxiliary value $\bar{v}_\ell$ with

$$\alpha_{1,\ell} \bar{v}_\ell \ge \alpha_{1,\ell+1} v_{\ell+1} \ge 0. \tag{3.45}$$

This proves useful in the verification of the algorithm's correctness. Furthermore, we introduce the functions

$$\begin{aligned}
\tilde{v}_{t,\ell}(\bar{v}_\ell) &:= (\alpha_{1,\ell} \bar{v}_\ell - \alpha_{1,t} c_t)^+ - (\alpha_{1,\ell+1} v_{\ell+1} - \alpha_{1,t} c_t)^+ && \forall\, t \in [\ell] \\
\tilde{w}_{t,\ell}(\bar{v}_\ell) &:= \alpha_{1,\ell} \bar{v}_\ell - \alpha_{1,\ell+1} v_{\ell+1} - \tilde{v}_{t,\ell}(\bar{v}_\ell) && \forall\, t \in [\ell]
\end{aligned}$$

for all such values $\bar{v}_\ell$.

Then, for fixed ℓ, Algorithm 1 provides values $u_{\ell,S}$ for $S \subseteq [\ell]$, that hold the required properties (3.43) and (3.44), as shown in the following. By construction, the functions $\tilde{v}_{t,\ell}$ and $\tilde{w}_{t,\ell}$ have values which satisfy

$$0 \le \tilde{v}_{t,\ell}(\bar{v}_\ell) \le \alpha_{1,\ell} \bar{v}_\ell - \alpha_{1,\ell+1} v_{\ell+1}$$

and thus

$$0 \le \tilde{w}_{t,\ell}(\bar{v}_\ell) \le \alpha_{1,\ell} \bar{v}_\ell - \alpha_{1,\ell+1} v_{\ell+1}$$

Data: $\bar{v}_\ell = v_\ell$ and $u_{\ell,S} = 0$ for all $S \subseteq [\ell]$
Result: $u_{\ell,S}$ for $S \subseteq [\ell]$, satisfying (3.43) and (3.44)
while $\forall\ t \in [\ell] : \tilde{w}_{t,\ell}(\bar{v}_\ell) > 0$ **do**

$$\Delta := \min \left\{ \min_{t\in[\ell]} \{\tilde{v}_{t,\ell}(\bar{v}_\ell) \mid \tilde{v}_{t,\ell}(\bar{v}_\ell) > 0\}, \min_{t\in[\ell]} \{\tilde{w}_{t,\ell}(\bar{v}_\ell)\} \right\}$$

if $\exists\ t \in [\ell] : \Delta = \tilde{v}_{t,\ell}(\bar{v}_\ell)$ **then**

$$R := \{t \in [\ell] \mid \tilde{v}_{t,\ell}(\bar{v}_\ell) > 0\}$$

$$u_{\ell,[\ell]\setminus R} := u_{\ell,[\ell]\setminus R} + \tfrac{\Delta}{\alpha_{1,\ell}}$$

$$\bar{v}_\ell := \bar{v}_\ell - \tfrac{\Delta}{\alpha_{1,\ell}}$$

else if $\forall\ t \in [\ell] : \tilde{v}_{t,\ell}(\bar{v}_\ell) > 0$ **then**

$$\Omega := \min_{t\in[\ell]} \{\tilde{v}_{t,\ell}(\bar{v}_\ell)\}$$

$$u_{\ell,\emptyset} := u_{\ell,\emptyset} + \tfrac{\Omega}{\alpha_{1,\ell}}$$

$$\bar{v}_\ell := \bar{v}_\ell - \tfrac{\Omega}{\alpha_{1,\ell}}$$

else

$$u_{\ell,[\ell]} := u_{\ell,[\ell]} + \tfrac{\Delta}{\alpha_{1,\ell}}$$

$$\bar{v}_\ell := \bar{v}_\ell - \tfrac{\Delta}{\alpha_{1,\ell}}$$

end

Algorithm 1: Constructing suitable components for u with fixed ℓ

for all $t \in [\ell]$. We aim to show, that for our choices of $u_{\ell,S}$ for $S \subseteq [\ell]$ they denote the current discounted deficits

$$\tilde{v}_{t,\ell}(\bar{v}_\ell) = \alpha_{1,t} \left(v_{t,\ell} - \sum_{\substack{S\subseteq[\ell]:\\ t\notin S}} \alpha_{t,\ell} u_{\ell,S} \right) = \alpha_{1,t} v_{t,\ell} - \sum_{\substack{S\subseteq[\ell]:\\ t\notin S}} \alpha_{1,\ell} u_{\ell,S} \qquad \forall\ t \in [\ell] \tag{3.46}$$

$$\tilde{w}_{t,\ell}(\bar{v}_\ell) = \alpha_{1,t} \left(w_{t,\ell} - \sum_{\substack{S\subseteq[\ell]:\\ t\in S}} \alpha_{t,\ell} u_{\ell,S} \right) = \alpha_{1,t} w_{t,\ell} - \sum_{\substack{S\subseteq[\ell]:\\ t\in S}} \alpha_{1,\ell} u_{\ell,S} \qquad \forall\ t \in [\ell] \tag{3.47}$$

of (3.43) and (3.44), as functions of $\bar{v}_\ell$ in each step of Algorithm 1.

For the sake of notation, let x' denote the value of variable x at the beginning of the current step and x'' denote its value at the end of the step. As conditions (3.45), (3.46) and (3.47) obviously are initially satisfied, it remains to show:

(i) *Conditions* (3.45)*,* (3.46) *and* (3.47) *are preserved with the choices of* $\bar{v}''_\ell$ *and* $u''_{\ell,S}$ *for* $S \subseteq [\ell]$ *in each step of Algorithm 1.*

We assume $\tilde{w}_{t,\ell}(\bar{v}'_\ell) > 0$ for all $t \in [\ell]$ as otherwise the statement is trivial.

If there exists an index $t \in [\ell]$ with $\Delta = \tilde{v}_{t,\ell}(\bar{v}'_\ell)$, then for $t \in [\ell]$ we verify

$$\Delta \leq \alpha_{t,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1},$$

which implies that the $\alpha_{t,\ell}\bar{v}''_\ell$ satisfy condition (3.45). For $t \in R$

$$\Delta \leq \tilde{v}_{t,\ell}(\bar{v}'_\ell) \leq \alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1}$$

holds and for $t \notin R$

$$\Delta \leq \tilde{w}_{t,\ell}(\bar{v}'_\ell) \leq \alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1}$$

holds. Furthermore, this choice of $\alpha_{t,\ell}\bar{v}''_\ell$ with $t \in [\ell]$ implies

$$\begin{aligned}
\tilde{v}_{t,\ell}(\bar{v}''_\ell) &= \tilde{v}_{t,\ell}(\bar{v}'_\ell) && \forall\, t \notin R\\
\tilde{v}_{t,\ell}(\bar{v}''_\ell) &= \tilde{v}_{t,\ell}(\bar{v}'_\ell) - \Delta && \forall\, t \in R\\
\tilde{w}_{t,\ell}(\bar{v}''_\ell) &= \tilde{w}_{t,\ell}(\bar{v}'_\ell) - \Delta && \forall\, t \notin R\\
\tilde{w}_{t,\ell}(\bar{v}''_\ell) &= \tilde{w}_{t,\ell}(\bar{v}'_\ell) && \forall\, t \in R,
\end{aligned}$$

which implies conditions (3.46) and (3.47) since

$$\begin{aligned}
u''_{\ell,R} &= u'_{\ell,R} + \frac{\Delta}{\alpha_{1,\ell}}\\
u''_{\ell,S} &= u'_{\ell,S} && \forall\, S \subseteq [\ell] : S \neq R
\end{aligned}$$

hold. In particular, this yields $\tilde{v}_{t',\ell}(\bar{v}'_\ell) > 0$ and $\tilde{v}_{t',\ell}(\bar{v}''_\ell) = 0$.

If there exists no index $t \in [\ell]$ such that $\Delta = \tilde{v}_{t,\ell}(\bar{v}'_\ell)$ and $\tilde{v}_{t,\ell}(\bar{v}'_\ell) > 0$ hold for all $t \in [\ell]$, analogously to the former case, the values $\alpha_{t,\ell}\bar{v}''_\ell$ with $t \in [\ell]$ satisfy condition (3.45) as for $t \in [\ell]$

$$\Omega \leq \tilde{v}_{t,\ell}(\bar{v}'_\ell) \leq \alpha_{t,\ell}\bar{v}'_\ell - \alpha_{t,\ell+1}v_{\ell+1}$$

holds and

$$\begin{aligned}
\tilde{v}_{t,\ell}(\bar{v}''_\ell) &= \tilde{v}_{t,\ell}(\bar{v}'_\ell) - \Omega && \forall\, t \in [\ell]\\
\tilde{w}_{t,\ell}(\bar{v}''_\ell) &= \tilde{w}_{t,\ell}(\bar{v}'_\ell) && \forall\, t \in [\ell],
\end{aligned}$$

i.e., conditions (3.46) and (3.47) are also satisfied. Again, in particular, $\tilde{v}_{t',\ell}(\bar{v}'_\ell) > 0$ and $\tilde{v}_{t',\ell}(\bar{v}''_\ell) = 0$ hold for some $t' \in [\ell]$ with $\tilde{v}_{t',\ell}(\bar{v}'_\ell) = \Omega$.

If none of the above cases realizes, there exists no index $t \in [\ell]$ with $\Delta = \tilde{v}_{t,\ell}(\bar{v}'_\ell)$ and there exists an index $k' \in [\ell]$ with $\tilde{v}_{k',\ell}(\bar{v}'_\ell) = 0$. Then, letting $t' \in [\ell]$ with $\tilde{w}_{t',\ell}(\bar{v}'_\ell) = \Delta$ yields

$$\Delta = \tilde{w}_{t',\ell}(\bar{v}'_\ell) = \alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1} - \tilde{v}_{t',\ell}(\bar{v}'_\ell) \geq \alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1} = \tilde{w}_{k',\ell}(\bar{v}'_\ell) \geq \Delta,$$

which implies $\alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1} = \Delta$. Thus, the validity of condition (3.45) follows and we have

$$0 = \alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1} - \Delta \geq \alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1} - \tilde{w}_{t,\ell}(\bar{v}'_\ell) = \tilde{v}_{t,\ell}(\bar{v}'_\ell) \geq 0 \quad \forall\, t \in [\ell],$$

so $\tilde{v}_{t,\ell}(\bar{v}'_\ell) = 0$ and $\tilde{w}_{t,\ell}(\bar{v}'_\ell) = \Delta$ hold for all $t \in [\ell]$. Hence

$$\begin{aligned} \tilde{v}_{t,\ell}(\bar{v}''_\ell) &= \tilde{v}_{t,\ell}(\bar{v}'_\ell) = 0 && \forall\, t \in [\ell] \\ \tilde{w}_{t,\ell}(\bar{v}''_\ell) &= \tilde{w}_{t,\ell}(\bar{v}'_\ell) - \Delta = 0 && \forall\, t \in [\ell] \end{aligned}$$

are implied, i.e., conditions (3.46) and (3.47) are satisfied.

(ii) *Algorithm 1 terminates with $\tilde{v}_{t,\ell}(\bar{v}'_\ell) = \tilde{w}_{t,\ell}(\bar{v}'_\ell) = 0$ for all $t \in [\ell]$.*

By construction, the number of zero values among $\tilde{v}_{t,\ell}(\bar{v}'_\ell), \tilde{w}_{t,\ell}(\bar{v}'_\ell)$ with $t \in [\ell]$ strictly increases in each non-trivial step, as shown above. Furthermore, in the trivial step, in which $\tilde{w}_{t,\ell}(\bar{v}'_\ell) = 0$ holds for an index $t' \in [\ell]$,

$$0 = \tilde{w}_{t',\ell}(\bar{v}'_\ell) = \alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1} - \tilde{v}_{t',\ell}(\bar{v}'_\ell) \geq \alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1} \geq 0$$

is satisfied. Hence, $\alpha_{1,\ell}\bar{v}'_\ell - \alpha_{1,\ell+1}v_{\ell+1} = 0$ holds, and

$$\tilde{v}_{t,\ell}(\bar{v}'_\ell) = \tilde{w}_{t,\ell}(\bar{v}'_\ell) = 0 \qquad \forall\, t \in [\ell]$$

is implied.

This conclude the proof. □

Mark that the resulting ideal formulation of the problem consists of an exponential number of constraints with respect to the input-size of the corresponding single-item lot-sizing problem. Employing the above, we obtain a linear programming formulation of (A).

Corollary 4. *The mixed integer linear program* (A) *can be optimally solved by optimizing* *the linear program*

$$
\begin{aligned}
\text{(A')}\ \ \min\quad & \sum_{t=1}^{T} (\tilde{c}_t x_t + p_t u_t + f_t y_t) & \\
\text{s.t.}\quad & \sum_{t=1}^{T} \alpha_{t,T} x_t = d_{1,T} & \\
& \sum_{i\in S} \alpha_{i,\ell} x_i + \sum_{i\in[\ell]\setminus S} d_{i,\ell} y_i \geq d_{1,\ell} & \begin{array}{l}\forall\ \ell \in [T],\\ \forall\ S \subseteq [\ell]\end{array} \\
& y_t \leq 1 & \forall\ t \in [T] \\
& u_t = \sum_{i=1}^{t} \alpha_{i,t}(x_i - d_i) & \forall\ t \in [T] \\
& x_t, y_t, u_t \geq 0 & \forall\ t \in [T].
\end{aligned}
$$

Proof. Here, we implicitly assume an optimal solution in form of an extreme point for (A'). This is always the case when using the simplex method, and can also be incorporated into other algorithms like the ellipsoid method, see [16]. □

Despite the exponential number of constrains in the ideal formulation, the results of this chapter yield an alternative prove for the polynomial complexity of the single-item lot-sizing problem with storage deterioration.

Corollary 5. *The given single-item lot-sizing problem with storage deterioration can be solved in polynomial time.*

Proof. As the constraints of the ideal formulation are separable in polynomial time, see Lemma 11, the "equivalence of optimization and separation" [38, Theorem 3.3] yields the statement. □

Note that the assumptions in this chapter are slightly more restrictive than those in Chapter 2. In particular, they only cover linear objectives.

3.5 An extended formulation

To circumvent the issue of exponentially many additional constraints in the ideal formulation, one can consider an alternative representation of X' induced by the following extended formulation:

$$
\begin{aligned}
\text{(C)}\ \ \min\quad & \sum_{t=1}^{T} (c_t x_t + f_t y_t) && && (3.48)\\
\text{s.t.}\quad & \sum_{k=1}^{t} q_{k,t} = d_t && \forall\ t \in [T] && (3.49)\\
& x_t - \sum_{m=t}^{T} \frac{1}{\alpha_{t,m}} q_{t,m} = 0 && \forall\ t \in [T] && (3.50)\\
& x_t \leq M y_t && \forall\ t \in [T] && (3.51)\\
& x_t, q_{k,t} \geq 0 && \forall\ t \in [T],\ \forall\ k \in [T] && (3.52)\\
& y_t \in \{0,1\} && \forall\ t \in [T] && (3.53)
\end{aligned}
$$

Here, $q_{k,t}$ denotes the amount of demand in period t that is contributed by period k. To provide this demand portion in period t, we have to implement a production value of $\frac{1}{\alpha_{k,t}} q_{k,t}$ in period k. Accordingly, constraints (3.49) enforce the demand balance and constraints (3.50) enforce the production balance in each period t.

While $q_{k,t}$ is only employed for $k \leq t$, we do not impose this in our model for notational convenience. We observe that the set of feasible solutions to model (B) can be represented as the projection of the feasible solutions to (C) on the x and y variables.

Lemma 15. *With*

$$
Y' := \left\{ (x,y,q) \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T \times \mathbb{R}^{T^2}_{\geq 0} \ \middle|\ \begin{array}{ll} \sum\limits_{k=1}^{t} q_{k,t} = d_t & \forall\ t \in [T] \\ x_t - \sum\limits_{m=t}^{T} \frac{1}{\alpha_{t,m}} q_{t,m} = 0 & \forall\ t \in [T] \\ x_t \leq M y_t & \forall\ t \in [T] \end{array} \right\},
$$

we have

$$
X' = \left\{ (x,y) \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T \ \middle|\ \exists\ q \in \mathbb{R}^{T^2}_{\geq 0} :\ (x,y,q) \in Y' \right\}.
$$

Proof. To show $(x,y) \in X'$, we construct $(x,y,q) \in Y'$ via the “first in - first out” policy

$$
q_{k,t} := \min\left\{ d_t, \max\left\{ 0, \sum_{i=1}^{k} \alpha_{i,t} x_i - \sum_{i=1}^{t-1} \alpha_{i,t} d_i \right\} \right\} \qquad \forall\ t \in [T],\ \forall\ k \in [t].
$$

Furthermore, for each $(x,y,q) \in Y'$ we have $(x,y) \in X'$. □

Thus, model (B), and consequently formulation (A), can be solved by optimizing formulation (C). To efficiently optimize the relevant instances of (C), we need a tighter formulation of Y' than its linear relaxation. Here, we exploit the following class of constraints.

Lemma 16. *The inequalities*

$$d_t y_k - q_{k,t} \geq 0 \qquad \begin{array}{l}\forall\ t \in [T],\\ \forall\ k \in [t]\end{array}$$

are valid for Y'.

We remark that we do not necessarily aim to determine an ideal formulation for the feasible solutions of (C) as we are only interested in solving instances induced by a single-item lot-sizing instance, so a quite restricted class of linear objectives. Thus, we are content with a formulation whose projection on the x and y variables is equal to X.

Theorem 10. *With*

$$Y := \left\{ (x,y,q) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \times \mathbb{R}^{T^2}_{\geq 0} \;\middle|\; \begin{array}{ll} \sum\limits_{k=1}^{t} q_{k,t} = d_t & \forall\ t \in [T] \\ x_t - \sum\limits_{m=t}^{T} \frac{1}{\alpha_{t,m}} q_{t,m} = 0 & \forall\ t \in [T] \\ d_t y_k - q_{k,t} \geq 0 & \begin{array}{l}\forall\ t \in [T],\\ \forall\ k \in [t]\end{array} \\ y_i \leq 1 & \forall\ t \in [T] \end{array} \right\},$$

we have

$$X = \operatorname{conv}(X') = \left\{ (x,y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \;\middle|\; \exists\ q \in \mathbb{R}^{T^2}_{\geq 0} :\ (x,y,q) \in Y \right\}.$$

Proof. Let

$$\tilde{Y} := \left\{ (x,y) \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T \;\middle|\; \exists\ q \in \mathbb{R}^{T^2}_{\geq 0} :\ (x,y,q) \in Y \right\}.$$

By applying Lemmata 15 and 16, we determine

$$\begin{aligned}
\operatorname{conv}(X') &= \operatorname{conv}\left(\left\{ (x,y) \in \mathbb{R}^T_{\geq 0} \times \{0,1\}^T \;\middle|\; \exists\ q \in \mathbb{R}^{T^2}_{\geq 0} :\ (x,y,q) \in Y' \right\} \right) \\
&\subseteq \operatorname{conv}\left(\left\{ (x,y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \;\middle|\; \exists\ q \in \mathbb{R}^{T^2}_{\geq 0} :\ (x,y,q) \in Y \right\} \right) \\
&= \left\{ (x,y) \in \mathbb{R}^T_{\geq 0} \times \mathbb{R}^T_{\geq 0} \;\middle|\; \exists\ q \in \mathbb{R}^{T^2}_{\geq 0} :\ (x,y,q) \in Y \right\} \\
&= \tilde{Y}
\end{aligned}$$

Now, assume

$$\operatorname{conv}(X') \subsetneq \tilde{Y}.$$

As both sets are bounded polytopes, there exists an extreme point $(x^\circ, y^\circ) \in \tilde{Y}$ which is not contained in $\operatorname{conv}(X')$. Since

$$\begin{aligned}
\sum_{t=1}^{T} \alpha_{t,T} x_t &= \sum_{t=1}^{T} \alpha_{t,T} \sum_{m=t}^{T} \frac{1}{\alpha_{t,m}} q_{t,m} \\
&= \sum_{t=1}^{T} \sum_{m=t}^{T} \alpha_{m,T} q_{t,m} \\
&= \sum_{t=1}^{T} \sum_{k=1}^{t} \alpha_{t,T} q_{k,t} \\
&= \sum_{t=1}^{T} \alpha_{t,T} \sum_{k=1}^{t} q_{k,t} \\
&= \sum_{t=1}^{T} \alpha_{t,T} d_t \\
&= d_{1,T}
\end{aligned}$$

is satisfied for all $(x, y, q) \in Y$, similar to the proof of Theorem 9, there exists an objective function $(f, c) \in \mathbb{R}^n \times \mathbb{R}^n_{\geq 0}$ such that (x°, y°) is the unique optimal solution to

$$\text{(E)} \quad \min \quad \sum_{t=1}^{T} (c_t x_t + f_t y_t) \tag{3.54}$$

$$\text{s.t.} \quad \sum_{k=1}^{t} q_{k,t} = d_t \qquad \forall\, t \in [T] \tag{3.55}$$

$$x_t - \sum_{m=t}^{T} \frac{1}{\alpha_{t,m}} q_{t,m} = 0 \qquad \forall\, t \in [T] \tag{3.56}$$

$$d_t y_k - q_{k,t} \geq 0 \qquad \forall\, t \in [T],\ \forall\, k \in [T] \tag{3.57}$$

$$y_i \leq 1 \qquad \forall\, t \in [T] \tag{3.58}$$

$$x_t, y_t, q_{k,t} \geq 0 \qquad \forall\, t \in [T],\ \forall\, k \in [T] \tag{3.59}$$

with optimal solution value z°. Let $(x^*, y^*) \in X'$ be an optimal solution to model (B) for the given choice (f, c), that is to

$$\min_{(x,y) \in X'} \left\{ \sum_{t=1}^{T} (c_t x_t + f_t y_t) \right\}.$$

As $\operatorname{conv}(X') \subseteq \tilde{Y}$ holds, (x^*, y^*) is also feasible to (E) and by choice of (f, c) has solution value

$$z^* = G(T) + \sum_{t=1}^{T} f_t^- > z^\circ.$$

We aim to show that (x^*, y^*) is also optimal to (E), which is a contradiction. The dual linear program to (E) is given by

$$
\begin{aligned}
\text{(F)}\ \min\quad & \sum_{t=1}^{T} (d_t v_t - z_t) && && (3.60)\\
\text{s.t.}\quad & \sum_{i=t}^{T} d_k w_{t,i} - z_t \le f_t && \forall\, t \in [T] && (3.61)\\
& u_t \le c_t && \forall\, t \in [T] && (3.62)\\
& v_t - \frac{1}{\alpha_{k,t}} u_k - w_{k,t} \le 0 && \forall\, t \in [T],\ \forall\, k \in [t] && (3.63)\\
& w_{k,t}, z_t \ge 0 && \forall\, t \in [T],\ \forall\, k \in [t] && (3.64)\\
& v_t, u_t \text{ free} && \forall\, j \in [T],\ \forall\, t \in [T]. && (3.65)
\end{aligned}
$$

Here, the dual variables v_t correspond to primal constraints (3.55), the dual variables u_t correspond to primal constraints (3.56), the dual variables $w_{k,t}$ correspond to primal constraints (3.57) and the dual variables z_t correspond to primal constraints (3.58).

Letting

$$v_t = \frac{G(t) - G(t-1)}{d_t}, \qquad u_t = c_t, \qquad w_{k,t} = (v_t - c_k)^+, \qquad z_t = -f_t^-$$

for all $t \in [T]$ yields an optimal solution with value $z^* = G(T) + \sum_{t=1}^{T} f_t^-$ to (F) as shown with similar but simpler arguments as the ones in the proof for Theorem 9. Note that for the proof of feasibility of constraint (3.63) we exploit $\frac{1}{\alpha_{k,t}} > 1$ and the non-negativity of c, in that

$$v_t - \frac{1}{\alpha_{k,t}} u_k = v_t - \frac{1}{\alpha_{k,t}} c_k < v_t - c_k \le (v_t - c_k)^+ = w_{k,t}$$

is implied for all $t \in [T]$ and $k \in [t]$. □

Hence, we can solve the problem by an extended linear programming formulation.

Corollary 6. *The mixed integer linear program* (B) *can be solved by optimizing the linear program*

$$\min \left\{ \sum_{t=1}^{T} (c_t x_t + f_t y_t) \;\middle|\; (x, y, q) \in Y \right\}.$$

Unlike the ideal formulation in the original space, this extended formulation only adds a polynomial number of constraints but also a polynomial number of new variables. With respect to the original formulation, we conclude the following:

Corollary 7. *The mixed integer linear program* (A) *can be optimally solved by optimizing the linear program*

$$
\begin{aligned}
\text{(A")} \quad \min \quad & \sum_{t=1}^{T} (\tilde{c}_t x_t + p_t u_t + f_t y_t) && \\
\text{s.t.} \quad & \sum_{k=1}^{t} q_{k,t} = d_t && \forall\ t \in [T] \\
& x_t - \sum_{m=t}^{T} \frac{1}{\alpha_{t,m}} q_{t,m} = 0 && \forall\ t \in [T] \\
& d_t y_k - q_{k,t} \geq 0 && \forall\ t \in [T],\ \forall\ k \in [T] \\
& y_i \leq 1 && \forall\ t \in [T] \\
& u_t = \sum_{i=1}^{t} \alpha_{i,t}(x_i - d_i) && \forall\ t \in [T] \\
& x_t, y_t, u_t, q_{k,t} \geq 0 && \forall\ t \in [T],\ \forall\ k \in [T],
\end{aligned}
$$

which exhibits polynomial size with respect to (A)*.*

Thus, this formulation induces an explicitly polynomial time algorithm for the presented problem.

3.6 Concluding remarks

In this chapter, we studied the polytope of the single-item lot-sizing problem with storage deterioration under the additional assumption of zero final storage value and strictly positive demands. While we do work on a reduced formulation, all results directly translate to the equivalent incremental formulation. Mark that both assumptions made by us are standard ones when considering single-item lot-sizing polytopes as they are essential to determining their dimension.

The reason for not including storage or production bounds is that the presented results are based on the knowledge of the considered polytope's dimension, respectively the dynamic programming motivated auxiliary structure presented in Section 3.3. Both these concepts are compromised by the addition of bounds, especially lower ones, in combination with zero final storage value, as the number of viable commitment decisions can be severely limited and additional demands in the form of additional periods can result in lower optimal costs in the considered reformulation. This, in turn, prevents us from constructing an equivalent dual solution in the presented stacking manner.

These issues are also reflected in the results from the literature as, to the best of our knowledge, no ideal formulation for a single-item lot-sizing polytope with any kind of non-trivial bounds is known. Mark that this statement only refers to the general formulation as considered in this work, but not necessarily to (non-equivalent) formulations that are only viable under additional assumptions. For example, consider Wagner-Within costs or the inclusion of additional concepts like backlogging or lost sales. To clarify this point, we summarize some relevant results circumventing or illustrating the above issues.

Pochet and Wolsey [41] study the single-item lot-sizing problem with linear Wagner-Within costs. Here, an extended formulation that does not represent all feasible solutions in the standard formulation but (only) a dominant subset, is considered. For this, they present ideal formulations for a number of variants, that is the standard problem and the variants with either start up costs or backlogging or constant upper production bounds, by exploiting the total unimodularity of the corresponding constraint matrices. This is extended to the case of time-dependent upper storage bounds by Atamtürk and Küçükyavuz [2]. In their work, they also provide valid inequalities which cut off all fractional points of the LP relaxation of the standard formulation with upper storage bounds and/or start up costs. Additionally, they provide conditions for these inequalities to define facets.

Küçükyavuz and Pochet [33] consider the standard single-item lot-sizing problem with backlogging, extending a class of infinitely many valid inequalities by Pochet and Wolsey [40], and prove that they induce an ideal formulation.

Furthermore, there is a number of results, besides the work of Atamtürk and Küçükyavuz [2] mentioned above, that do not provide an ideal formulation but classes of valid inequalities, respectively facets. Leung et al. [34] determine valid inequalities for the single-item lot-sizing problem with constant upper production bounds and provide conditions for these to define facets. All of these works exhibit quite convoluted conditions on the presented valid inequalities to be facet defining. These conditions primarily ensure certain dimensionalities of the polytope and of the faces corresponding to the considered inequalities. From our perspective, this reinforces the assessment that the lack of knowledge of the corresponding polytope's dimensionality is a critical factor in the structural analysis.

At this point, we have to mention the work of Atamtürk and Muñoz [4], as it allegedly defies our assessment on the issue of determining the dimension of the polytope of the single-item lot-sizing problem with upper production bounds. It claims to present an equivalent (non-extended) reduced full dimension reformulation of fixed solution space. We stress that this formulation is not equivalent but a relaxation. With respect to the product balance, it only enforces upper but no lower limits on the production values. In particular, it allows for the zero production solution to be feasible. Additionally, it clearly exhibits a strictly bigger dimension than the original polytope. Still, all presented "bottleneck cover" inequalities for the relaxation are naturally also valid for the original formulation, but the facet properties do not simply translate.

4 Robustness approaches

In this chapter, we study the practical optimization problem that motivated the single-item lot-sizing variants considered in this work, namely a heat and power co-production system with a heat storage unit. We model this system hourly over the time span of one day. The presented results are already partially published in [23] and [24].

Since the deterministic modeling and solving of this kind of problem is well explored, we focus on tackling uncertain data, namely heat demand uncertainty. Here, we introduce static and adaptive robust models, as well as robust-stochastic hybrid models. These models are based on uncertainty sets constructed from a given forecast for the current day and historical forecast and realization data. The resulting solutions are tested against the actual data realization of the considered day. To do so, we establish strategies to implement the solutions also for realizations outside the considered uncertainty set. As this is often hindered by infeasibilities, we resort to relaxing certain constraints and introducing a measure of the resulting violations. These violations, in turn, indicate the practical quality of the found solution. This also allows us to compare different approaches in a computational study incorporating real-life data provided by our business partner.

4.1 Combined heat and power plants as an application

We consider the real-life optimization problem of controlling a combined heat and power production (CHP) plant. CHP plants are production units in which the heat that is generated when cooling down the plant is extracted and, at least partially, utilized for heating purposes. These plants can be coupled with a storage tank in which the excess heat can be temporarily stored subject to proportional losses due to dissipation effects. From a production planning perspective, two products and two demands are given, one of heat and one of power.

In this work, we consider a period based model, as suggested in Figure 4.1, in which a CHP plant and a storage tank are integrated in a system with internal power and heat demands.

We assume that this system is connected to a general power network in which arbitrary amounts of power can be bought or sold subject to external market prizes. With respect to power, we assume the system to be self sufficient. The fuel required for the CHP plant is assumed to be available at a fixed prize, and heat and power are to be produced at a fixed ratio. Note that the storage value is the only component linking the periods.

Formally, we define the problem as a mixed integer linear program (L_1) over $T \in \mathbb{Z}_{>0}$ periods. Here, we introduce the following variables and parameters for each period

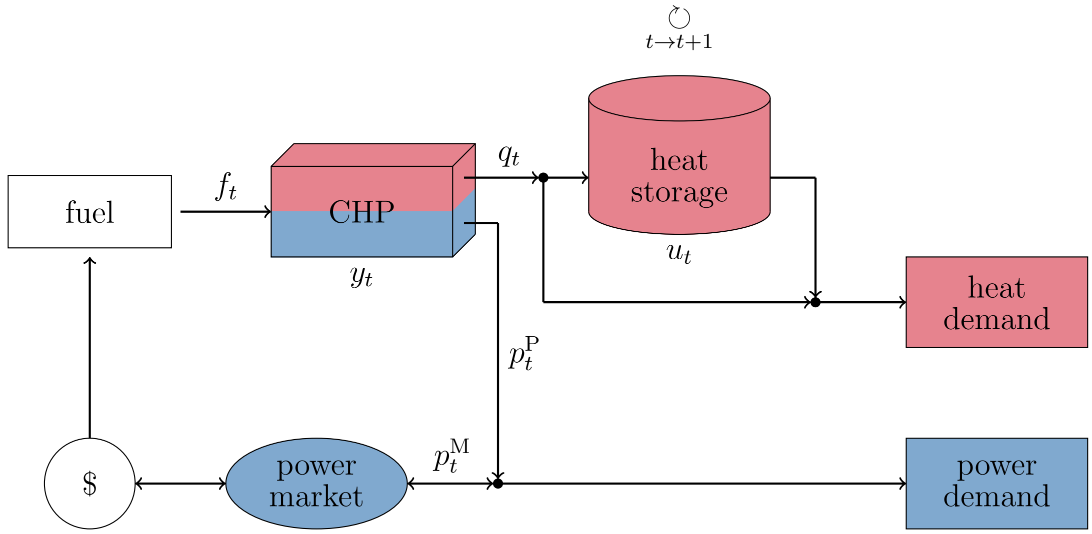


Figure 4.1: CHP system for a single period $t \in [T]$

$t \in [T]$. The values $q_t, u_t, d_t \in \mathbb{R}_{\geq 0}$ represent the amount of heat which is, respectively, produced, stored, and required as a demand. Regarding power, we introduce a second production variable p_t^{P} indicating the amount of power that is generated, a market variable p_t^{M} representing the amount of power that is bought/sold and a demand vector d_t^{P} at each point in time. Furthermore, we let c_t^{M} denote the market prices for both buying and selling a unit of power and c^{F} denote the fuel prices. Fuel consumption, given by the variable f_t, is modeled as the linear function $sq_t + hy_t$ with h being a constant fuel demand corresponding to the activation of the CHP unit. We assume that heat and power are produced at a fixed proportion $\rho \in \mathbb{R}_{>0}$. Finally, α_t again denotes the conservation rate from period t to period $t+1$. Thus, we obtain the following model:

$$(\mathrm{L}_1) \quad \min \quad \sum_{t=1}^{T} \left(c_t^{\mathrm{M}} p_t^{\mathrm{M}} + c_t^{\mathrm{F}} f_t + c_t^{\mathrm{I}} u_t \right) \tag{4.1}$$

$$\text{s.t.} \quad \alpha_t u_{t-1} + q_t = u_t + d_t \qquad \forall\, t \in [T] \tag{4.2}$$

$$p_t^{\mathrm{P}} + p_t^{\mathrm{M}} = d_t^{\mathrm{P}} \qquad \forall\, t \in [T] \tag{4.3}$$

$$\underline{U}_t \leq u_t \leq \overline{U}_t \qquad \forall\, t \in [T] \tag{4.4}$$

$$\underline{Q}_t y_t \leq q_t \leq \overline{Q}_t y_t \qquad \forall\, t \in [T] \tag{4.5}$$

$$f_t = sq_t + hy_t \qquad \forall\, t \in [T] \tag{4.6}$$

$$p_t^{\mathrm{P}} = \rho q_t \qquad \forall\, t \in [T] \tag{4.7}$$

$$q_t, u_t, f_t, p_t^{\mathrm{P}}, q_t \geq 0 \qquad \forall\, t \in [T] \tag{4.8}$$

$$p_t^{\mathrm{M}} \text{ free} \qquad \forall\, t \in [T] \tag{4.9}$$

$$y_t \in \{0, 1\} \qquad \forall\, t \in [T] \tag{4.10}$$

This model can be reduced to the single-item lot-sizing problem

$$
\begin{aligned}
(\mathrm{L}_2)\ \min \quad & \sum_{t=1}^{T} \Bigg(c_t^{\mathrm{M}} \left(\rho q_t - d_t^{\mathrm{P}}\right) + c_t^{\mathrm{F}} \left(s q_t + h y_t\right) && (4.11)\\
& \qquad + c_t^{\mathrm{I}} \left(\alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} \left(q_i - d_i\right) \right) \Bigg) \\
\text{s.t.} \quad & \underline{U}_t \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} \left(q_i - d_i\right) \leq \overline{U}_t \qquad \forall\, t \in [T] && (4.12)\\
& \underline{Q}_t y_t \leq q_t \leq \overline{Q}_t y_t \qquad \forall\, t \in [T] && (4.13)\\
& q_t, u_t \geq 0 \qquad \forall\, t \in [T] && (4.14)\\
& y_t \in \{0,1\} \qquad \forall\, t \in [T] && (4.15)
\end{aligned}
$$

and further to

$$
\begin{aligned}
(\mathrm{L}_3)\ \min \quad & \sum_{t=1}^{T} \left(c_t^{\mathrm{v}} q_t + c_t^{\mathrm{f}} y_t + c_t^{\mathrm{d}} d_t + c_t^{\mathrm{c}} \right) && (4.16)\\
\text{s.t.} \quad & \underline{U}_t \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} \left(q_i - d_i\right) \leq \overline{U}_t \qquad \forall\, t \in [T] && (4.17)\\
& \underline{Q}_t y_t \leq q_t \leq \overline{Q}_t y_t \qquad \forall\, t \in [T] && (4.18)\\
& q_t \geq 0 \qquad \forall\, t \in [T] && (4.19)\\
& y_t \in \{0,1\} \qquad \forall\, t \in [T] && (4.20)
\end{aligned}
$$

via

$$
\begin{aligned}
p_t^{\mathrm{P}} &= \rho q_t \qquad \forall\, t \in [T] && (4.21)\\
p_t^{\mathrm{M}} &= \rho q_t - d_t^{\mathrm{P}} \qquad \forall\, t \in [T] && (4.22)\\
f_t &= s q_t + h y_t \qquad \forall\, t \in [T] && (4.23)\\
u_t &= \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} \left(q_i - d_i\right) \qquad \forall\, t \in [T] && (4.24)\\
& && (4.25)
\end{aligned}
$$

and

$$
\begin{aligned}
c_t^{\mathrm{v}} &:= c_t^{\mathrm{M}} \rho + c_t^{\mathrm{F}} s + \sum_{j=t}^{T} c_j^{\mathrm{I}} \alpha_{t,j} \qquad \forall\, t \in [T] && (4.26)\\
c_t^{\mathrm{f}} &:= c_t^{\mathrm{F}} s \qquad \forall\, t \in [T] && (4.27)\\
c_t^{\mathrm{d}} &:= -\sum_{j=t}^{T} c_j^{\mathrm{I}} \alpha_{t,j} \qquad \forall\, t \in [T] && (4.28)\\
c_t^{\mathrm{c}} &:= c_t^{\mathrm{I}} \alpha_{0,t} u_0 - c_t^{\mathrm{M}} d_t^{\mathrm{P}} \qquad \forall\, t \in [T]. && (4.29)
\end{aligned}
$$

The new parameters $c_t^{\mathrm{v}}, c_t^{\mathrm{f}}, c_t^{\mathrm{d}}$ and c_t^{c} are purposely defined independent of the demand d, as the latter is considered an uncertain parameter later on.

In the remainder of this work, model (L_3) serves as the basis for all robust approaches, as the eliminated variables $p_t^{\mathrm{P}}, p_t^{\mathrm{M}}, f_t$ and u_t in the models (L_1) and (L_2), from a practical perspective, are expendable bookkeeping variables. Aside from rendering the problem more convoluted, these variables hinder a straight-forward adaptation of the robustness techniques as described in Section 1.3. As they are both uniquely induced by the remaining variable values and dependent on the realization of the uncertain values in any given feasible solution, they lack the necessary slack to simultaneously guaranty feasibility for different scenarios.

Due to our application, we consider the heat demand as the single source of uncertainty in the studied problem. We remark that we lay our focus on the feasibility of solutions rather than the worst-case objective guarantee. The reason for this is that in the considered application, feasibility in the production and storage bound constraints cannot be violated physically due to technical constraints.

4.2 Static production plans

Let an uncertainty set $D \subset \mathbb{R}^T$ be given. Applying the standard robustness concept, see Sections 1.3.2 and 1.3.3, to formulation (L_3), we duplicate constraint (4.17) for each $d \in D$. We derive the following semi-infinite mixed integer mathematical programming formulation:

$$
\begin{aligned}
(\mathrm{L}_{\mathrm{stat}})\quad \min\quad & \sup_{d \in D} \left\{ \sum_{t=1}^{T} \left(c_t^{\mathrm{v}} q_t + c_t^{\mathrm{f}} y_t + c_t^{\mathrm{d}} d_t + c_t^{\mathrm{c}} \right) \right\} \\
\text{s.t.}\quad & \underline{U}_t \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} \left(q_i - d_i \right) \leq \overline{U}_t && \forall\, t \in [T],\ \forall\, d \in D && (4.30) \\
& \underline{Q}_t y_t \leq q_t \leq \overline{Q}_t y_t && \forall\, t \in [T] && (4.31) \\
& q_t \geq 0 && \forall\, t \in [T] && (4.32) \\
& y_t \in \{0,1\} && \forall\, t \in [T] && (4.33)
\end{aligned}
$$

This model can easily be reformulated as a mixed integer linear program both for finite uncertainty sets and Γ-uncertainty sets as discussed in Section 1.3.4. For the explicit reformulations we refer to the subsequent models ($\mathrm{L}'_{\mathrm{aff}}$) in the finite case and ($\mathrm{L}^{\Gamma}_{\mathrm{aff}}$) in the Γ case, as the considered models are special cases of those, as shown later on.

While this application is very straight forward, one has to consider that this is due to the former reformulation. For formulation (L_1) the duplication of constraints would result in substituting constraints (4.2) with

$$
\alpha_t u_{t-1} + q_t = u_t + d_t \qquad \forall\, t \in [T],\ \forall\, d \in D,
$$

which clearly would render the problem infeasible in all non-trivial cases. This is not the case in formulations (L_2) and (L_3), as the storage variables are eliminated and all substitutions are well posed. In particular the substitution of (4.12) = (4.17) with

$$\underline{U}_t \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} (q_i - d_i) \leq \overline{U}_t \qquad \forall\, t \in [T],\ \forall\, d \in D$$

in (L_1) respectively (L_2) is equivalent to introducing a separate storage variable vector $u(d) \in \mathbb{R}^T_{\geq 0}$ for every realization $d \in D$, i.e., substituting (4.2) with

$$\alpha_t u(d)_{t-1} + q_t = u(d)_t + d_t \qquad \forall\, t \in [T],\ \forall\, d \in D$$

and (4.4) = (4.12) with

$$\underline{U}_t \leq u(d)_t \leq \overline{U}_t \qquad \forall\, t \in [T],\ \forall\, d \in D.$$

Here, the unique valid value of $u(d)_t$ in every feasible solution is given by

$$u(d)_t = \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} (q_i - d_i) \qquad \forall\, t \in [T],\ \forall\, d \in D.$$

Implicitly, the prior reformulation to model (L_3) qualifies as a trivial adaptive robust approach, in particular a two-stage approach, as the bookkeeping variables are affinely dependent on the realization. Still, we consider this approach as static, as we assume to start with model (L_3).

4.2.1 Reduction to a deterministic problem

The static robust formulation (L_{stat}) is clearly equivalent to

$$
\begin{aligned}
(L'_{\text{stat}})\ \min\quad & \sum_{t=1}^{T} \left(c^{\text{v}}_t q_t + c^{\text{f}}_t y_t + c^{\text{c}}_t \right) + \sup_{d \in D} \left\{ \sum_{t=1}^{T} c^{\text{d}}_t d_t \right\} && && (4.34)\\
\text{s.t.}\quad & \underline{U}_t + \sup_{d \in D} \left\{ \sum_{i=1}^{t} \alpha_{i,t} d_i \right\} \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} q_i && \forall\, t \in [T] && (4.35)\\
& \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} q_i \leq \overline{U}_t + \inf_{d \in D} \left\{ \sum_{i=1}^{t} \alpha_{i,t} d_i \right\} && \forall\, t \in [T] && (4.36)\\
& \underline{Q} y_t \leq q_t \leq \overline{Q} y_t && \forall\, t \in [T] && (4.37)\\
& q_t \geq 0 && \forall\, t \in [T] && (4.38)\\
& y_t \in \{0, 1\} && \forall\, t \in [T]. && (4.39)
\end{aligned}
$$

Thus, it constitutes a deterministic single-item lot-sizing problem with zero demands and adjusted lower and upper storage bounds

$$\underline{U}'_t := \underline{U}_t + \sup_{d \in D} \left\{ \sum_{i=1}^{t} \alpha_{i,t} d_i \right\}, \qquad \overline{U}'_t = \overline{U}_t + \inf_{d \in D} \left\{ \sum_{i=1}^{t} \alpha_{i,t} d_i \right\}, \tag{4.40}$$

since

$$\sup_{d \in D} \left\{ \sum_{t=1}^{T} c^{\mathrm{d}}_t d_t \right\}$$

is a constant term.

As all-zero demands intuitively defy the nature of the single-item lot-sizing problem (for instance non-zero demands have to be assumed for the results in Chapter 3 to hold) we also consider another equivalent deterministic problem in the following. Here, all parameters but the upper storage bounds $\overline{U}''$ and demands d'', which are given by

$$d''_t := \sup_{d \in D} \left\{ d_t - \sum_{i=1}^{t-1} \alpha_{i,t} \left(d''_i - d_i \right) \right\} \qquad \forall\, t \in [T] \tag{4.41}$$

$$\Delta_t := \sup_{d \in D} \left\{ \sum_{i=1}^{t} \alpha_{i,t} \left(d''_i - d_i \right) \right\} \qquad \forall\, t \in [T] \tag{4.42}$$

$$\overline{U}''_t := \overline{U}_t - \Delta_t \qquad \forall\, t \in [T], \tag{4.43}$$

are inherited from the robust problem. The idea of this reformulation is similar to that of Lemma 1. That is, the newly introduced demand d''_t specifies a lower bound on the product amount, which has to be available at time $t \in [T]$ to ensure that every potential demand $d \in D$ can be met. The value Δ_t reduces the storage upper bounds of the transformed problem to prevent that, if a large production is realized but a large deficit in demand occurs, the storage upper bound $\overline{U}_t$ is not exceeded. Formally, we have:

Theorem 11. *If d''_t and Δ_t are computable and finite for each $t \in [T]$, then each instance* $(\mathrm{L}_{\mathrm{stat}})$ *can be reduced to an equivalent instance of* (L_3) *with non-negative demands $d = d''$ and upper storage bounds $\overline{U} = \overline{U}''$, as defined in equations* (4.41)-(4.43).

Proof. In the following we consider $t \in [T]$. First, we show that d'' is non negative if d is non negative. As

$$d_{t-1} - \left(\sum_{i=1}^{t-2} \alpha_{i,t-1} \left(d''_i - d_i \right) \right)$$

is an affinely linear term in d, there exists an element $\check{d}$ in the closure $\bar{D}$ of D such that

$$\begin{aligned} d''_{t-1} &= \sup_{d\in D}\left\{ d_{t-1} - \left(\sum_{i=1}^{t-2} \alpha_{i,t-1} \left(d''_i - d_i\right)\right)\right\} \\ &= \max_{d\in \bar{D}}\left\{ d_{t-1} - \left(\sum_{i=1}^{t-2} \alpha_{i,t-1} \left(d''_i - d_i\right)\right)\right\} \\ &= \check{d}_{t-1} - \left(\sum_{i=1}^{t-2} \alpha_{i,t-1} \left(d''_i - \check{d}_i\right)\right) \end{aligned}$$

holds, and we deduce

$$\begin{aligned} d''_t =& \sup_{d\in D}\left\{ d_t - \sum_{i=1}^{t-1} \alpha_{i,t} \left(d''_i - d_i\right)\right\} \\ =& \max_{d\in \bar{D}}\left\{ d_t - \alpha_t \left(\sum_{i=1}^{t-2} \alpha_{i,t-1} \left(d''_i - d_i\right) + d''_{t-1} - d_{t-1}\right)\right\} \\ =& \max_{d\in \bar{D}}\left\{ d_t - \alpha_t \left(d''_{t-1} - \left(d_{t-1} - \sum_{i=1}^{t-2} \alpha_{i,t-1} \left(d''_i - d_i\right)\right)\right)\right\} \\ \geq& \check{d}_t - \alpha_t \left(d''_{t-1} - \left(\check{d}_{t-1} - \sum_{i=1}^{t-2} \alpha_{i,t-1} \left(d''_i - \check{d}_i\right)\right)\right) \\ =& \check{d}_t - \alpha_t \left(d''_{t-1} - d''_{t-1}\right) \\ =& \check{d}_t \\ \geq& 0. \end{aligned} \tag{*}$$

With this, we show

$$\begin{aligned} & \inf_{d\in D}\left\{ \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t}(q_i - d_i)\right\} \\ =& \inf_{d\in D}\left\{ \alpha_{0,t} u_0 + \sum_{i=1}^{t-1} \alpha_{i,t}(q_i - d_i) + (q_t - d_t)\right\} \\ =& \inf_{d\in D}\left\{ \alpha_{0,t} u_0 + \sum_{i=1}^{t-1} \alpha_{i,t}q_i - \sum_{i=1}^{t-1} \alpha_{i,t}d_i + q_t - d_t + \sum_{i=1}^{t-1} \alpha_{i,t}d''_i - \sum_{i=1}^{t-1} \alpha_{i,t}d''_i\right\} \\ =& \inf_{d\in D}\left\{ \alpha_{0,t} u_0 + \sum_{i=1}^{t-1} \alpha_{i,t}q_i - \sum_{i=1}^{t-1} \alpha_{i,t}d''_i + q_t - d_t + \sum_{i=1}^{t-1} \alpha_{i,t}d''_i - \sum_{i=1}^{t-1} \alpha_{i,t}d_i\right\} \\ =& \inf_{d\in D}\left\{ \alpha_{0,t} u_0 + \sum_{i=1}^{t-1} \alpha_{i,t}(q_i - d''_i) + q_t - d_t + \sum_{i=1}^{t-1} \alpha_{i,t}(d''_i - d_i)\right\} \end{aligned}$$

$$
\begin{aligned}
&=\alpha_{0,t}u_0 + \sum_{i=1}^{t-1} \alpha_{i,t}(q_i - d_i'') + q_t + \inf_{d \in D} \left\{ -d_t + \sum_{i=1}^{t-1} \alpha_{i,t}(d_i'' - d_i) \right\} \\
&=\alpha_{0,t}u_0 + \sum_{i=1}^{t-1} \alpha_{i,t}(q_i - d_i'') + q_t - \sup_{d \in D} \left\{ d_t - \sum_{i=1}^{t-1} \alpha_{i,t}(d_i'' - d_i) \right\} \\
&=\alpha_{0,t}u_0 + \sum_{i=1}^{t-1} \alpha_{i,t}(q_i - d_i'') + (q_t - d_t'') \\
&=\alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}(q_i - d_i'').
\end{aligned}
$$

Similarly, we derive

$$
\begin{aligned}
&\sup_{d \in D} \left\{ \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}(q_i - d_i) \right\} \\
=&\sup_{d \in D} \left\{ \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}q_i - \sum_{i=1}^{t} \alpha_{i,t}d_i \right\} \\
=&\sup_{d \in D} \left\{ \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}q_i - \sum_{i=1}^{t} \alpha_{i,t}d_i + \sum_{i=1}^{t} \alpha_{i,t}d_i'' - \sum_{i=1}^{t} \alpha_{i,t}d_i'' \right\} \\
=&\sup_{d \in D} \left\{ \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}q_i - \sum_{i=1}^{t} \alpha_{i,t}d_i'' + \sum_{i=1}^{t} \alpha_{i,t}d_i'' - \sum_{i=1}^{t} \alpha_{i,t}d_i \right\} \\
=&\sup_{d \in D} \left\{ \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}(q_i - d_i'') + \sum_{i=1}^{t} \alpha_{i,t}(d_i'' - d_i) \right\} \\
=&\alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}(q_i - d_i'') + \sup_{d \in D} \left\{ \sum_{i=1}^{t} \alpha_{i,t}(d_i'' - d_i) \right\} \\
=&\alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}(q_i - d_i'') + \Delta_t.
\end{aligned}
$$

Furthermore, the considered problems have identical feasible solutions, as

$$
\begin{aligned}
& \underline{U}_t \le \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}\left(q_i - d_i\right) && \forall\ d \in D \\
\Leftrightarrow\quad & \underline{U}_t \le \inf_{d \in D} \left\{ \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}(q_i - d_i) \right\} \\
\Leftrightarrow\quad & \underline{U}_t \le \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}(q_i - d_i'')
\end{aligned}
$$

and

$$
\begin{aligned}
& \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} \left(q_i - d_i \right) \leq \overline{U}_t && \forall\, d \in D \\
\Leftrightarrow \quad & \sup_{d \in D} \left\{ \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} (q_i - d_i) \right\} \leq \overline{U}_t \\
\Leftrightarrow \quad & \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} (q_i - d''_i) + \Delta_t \leq \overline{U}_t \\
\Leftrightarrow \quad & \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} (q_i - d''_i) \leq \overline{U}''_t
\end{aligned}
$$

hold. Lastly, the objective functions are equivalent up to the constant term

$$
\sup_{d \in D} \left\{ \sum_{t=1}^{T} c^{\mathrm{d}}_t d_t \right\} - \sum_{t=1}^{T} c^{\mathrm{d}}_t d''_t .
$$

□

This more elaborate reduction yields a more applicable and natural reformulation than the one motivated in (4.40).

Remark. *If all demand scenarios contained in the closure of the uncertainty set are strictly positive, then the corresponding deterministic problem's demand values are also strictly positive.*

Proof. In the proof of Theorem 11 we can substitute the $\geq$ in (*) by $>$. □

From a computational complexity perspective, Theorem 11 allows us to establish a direct relationship between the robust problem and the corresponding deterministic problem.

Corollary 8. *Let an uncertainty set D over which linear functions can be optimized in polynomial time be given. Then, the robust problem is in $\mathcal{P}$ (respectively $\mathcal{NP}$-hard) if and only if the corresponding deterministic problem is in $\mathcal{P}$ (respectively $\mathcal{NP}$-hard).*

Proof. On the one hand, the nominal problem can be stated as a trivial robust one. On the other hand, under the given assumption, the reduction in Theorem 11 is a polynomial one. □

Note that the result still holds when we introduce additional constraints on z and q or assumptions on the given parameters (except for d and $\overline{U}$). It is also valid for $\overline{U} = \infty$ and for not necessarily non-negative demands d.

The assumptions on the uncertainty set in Corollary 8 subsume many widely employed robustness models, including finite uncertainty sets, polyhedral uncertainty sets (such as the Γ-robustness one) and ellipsoidal uncertainty sets as employed in [7].

4.3 An alternative static approach

In this section, we present an alternative static but not strictly robust approach, motivated by the following observation.

When tackling uncertainty with models based on uncertainty sets, we often implicitly assume that the given set is representative of a wider range of potential realizations. Given this property, we hope that the solution from the optimization can be adjusted to be applicable also for realizations outside the uncertainty set. For our implementation of this concept in this work, we refer to Section 4.6.

Depending on the source of the uncertainty set, especially regarding finite ones, some of the covered scenarios may not properly represent the nature of the uncertainty. Conversely, exhibiting this irregularity, these particular scenarios may significantly influence on the form of the solution, due to the worst-case objective. Thus, we may achieve better results by excluding them. Hence, the idea of our alternative approach is to consider finite uncertainty sets, but to allow the model to neglect a given number of scenarios.

For finite uncertainty sets D, we employ the following mixed integer linear program, in which $B \in [0, |D| - 1]$ denotes the permitted number of exceptions and $M \in \mathbb{R}$ denotes a sufficiently big number:

$$
\begin{aligned}
(\mathrm{L}_{\mathrm{alt}})\ \min\quad & \sum_{t=1}^{T} \left(c_t^{\mathrm{v}} q_t + c_t^{\mathrm{f}} y_t + c_t^{\mathrm{c}} \right) + \eta && && (4.44)\\
\text{s.t.}\quad & \eta \geq \sum_{t=1}^{T} c_t^{\mathrm{d}} d_t - M z_d && \forall\, d \in D && (4.45)\\
& \underline{U}_t + \sum_{i=1}^{t} \alpha_{i,t} d_i - M z_d \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} q_i && \forall\, t \in [T],\ \forall\, d \in D && (4.46)\\
& \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} q_i \leq \overline{U}_t + \sum_{i=1}^{t} \alpha_{i,t} d_i + M z_d && \forall\, t \in [T],\ \forall\, d \in D && (4.47)\\
& \underline{Q} y_t \leq q_t \leq \overline{Q} y_t && \forall\, t \in [T] && (4.48)\\
& \sum_{d \in D} z_d \leq B && && (4.49)\\
& q_t \geq 0 && \forall\, t \in [T] && (4.50)\\
& \eta \text{ free} && && (4.51)\\
& y_t, z_d \in \{0, 1\} && \forall\, t \in [T],\ \forall\, d \in D && (4.52)
\end{aligned}
$$

Here, the binary indicator variable z_d denotes weather scenario d is disregarded, which is the case for $z_d = 1$. We remark that constraint (4.49) can be replaced by the stricter version

$$\sum_{d \in D} z_d = B$$

as there always exists an optimal solution that fully exploits its exception potential.

A note on complexity

To the best of our knowledge, the general complexity of this problem is yet unknown. Note that the problem generally decomposes into solving the standard static robust problem for all viable subsets of covered scenarios. Thus, in the case of a polynomial number of exception choices, the problem is rendered polynomially solvable (respectively, $\mathcal{NP}$-hard) if and only if the corresponding standard static robust problem is polynomially solvable (respectively, $\mathcal{NP}$-hard). This is, for example, the case for a fixed number of exceptions. As we assumed the uncertainty sets to be finite, we can further apply Corollary 8. We conclude that, for a polynomial number of exception choices, the complexity of the considered problem qualitatively coincides with that of the corresponding nominal problem.

Hence, to explore the general complexity, one has to consider instance classes with superpolynomial many exception choices. This is, for example, the case for the instances considered in our computational study, see Section 4.8. There, we consider finite uncertainty sets of increasing size and allow a fixed percentage of them to be neglected. Here, we lack structural knowledge to tackle the problem's complexity.

4.4 Adaptive solutions

While static robustness approaches are the standard ones, as they tend to be well applicable and tractable (provided suitable uncertainty sets), they are quite restrictive. Thus, we also incorporate the concept of adaptive solutions, in which the solution of the problem depends on the realization of the data, see Section 1.3.4. Similar to the static approach, we work on formulation (L_3). Thus, the incremental infinite mathematical program reads

$$
\begin{aligned}
(\mathrm{L}_{\mathrm{adapt}})\ \min \quad & \sup_{d \in D} \left\{ \sum_{t=1}^{T} \left(c^{\mathrm{v}}_t q(d)_t + c^{\mathrm{f}}_t y(d)_t + c^{\mathrm{d}}_t d_t + c^{\mathrm{c}}_t \right) \right\} && \\
\text{s.t.} \quad & \underline{U}_t \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} \left(q(d)_i - d_i \right) \leq \overline{U}_t && \forall\, t \in [T],\ \forall\, d \in D && (4.53) \\
& \underline{Q}_t y_t \leq q(d)_t \leq \overline{Q}_t y_t && \forall\, t \in [T],\ \forall\, d \in D && (4.54) \\
& q(d)_t \geq 0 && \forall\, t \in [T],\ \forall\, d \in D && (4.55) \\
& y(d)_t \in \{0,1\} && \forall\, t \in [T],\ \forall\, d \in D. && (4.56)
\end{aligned}
$$

This is easily transformed to an infinite mixed integer linear program via the epigraph formulation

$$
\begin{aligned}
(\mathrm{L}'_{\mathrm{adapt}})\quad \min\quad & \eta && && (4.57)\\
\text{s.t.}\quad & \eta \geq \sum_{t=1}^{T} \left(c^{\mathrm{v}}_t q(d)_t + c^{\mathrm{f}}_t y(d)_t + c^{\mathrm{d}}_t d_t + c^{\mathrm{c}}_t\right) && \forall\, d \in D && (4.58)\\
& \underline{U}_t \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t}\left(q(d)_i - d_i\right) \leq \overline{U}_t && \forall\, t \in [T],\ \forall\, d \in D && (4.59)\\
& \underline{Q}_t y_t \leq q(d)_t \leq \overline{Q}_t y_t && \forall\, t \in [T],\ \forall\, d \in D && (4.60)\\
& q(d)_t \geq 0 && \forall\, t \in [T],\ \forall\, d \in D && (4.61)\\
& \eta \text{ free} && && (4.62)\\
& y(d)_t \in \{0,1\} && \forall\, t \in [T],\ \forall\, d \in D. && (4.63)
\end{aligned}
$$

Clearly, this problem can be decomposed into solving the deterministic problem for every realization inside the uncertainty set. Also, for finite uncertainty sets it obviously is finite.

With this model, we encounter a number of issues. On the theoretical side, we have no way to solve this problem in a tractable way in the general setting, as we are possibly given an infinite number of variables and constraints, depending on the uncertainty set. On the practical side, we need to find applicable solutions. Thus, we have to decide what strategy to employ in case of a realization outside the considered uncertainty set, as we do in Section 4.6. Additionally we are studying a multi-period problem and are facing the restriction of causality. I.e., in each period we only learn the heat demand value as it realizes, but not for any future periods. Thus, the robust solution $q(d)_t$ must not depend on values d_k with $k \in [t+1, T]$, which occurs in $(\mathrm{L}_{\mathrm{adapt}})$ as demonstrated in the following.

Proposition 1. *Optimal solutions to* $(\mathrm{L}_{\mathrm{adapt}})$ *can be acausal, with* $q_t(d)$ *depending on* $d_{t'}$ *with* $t' \in [t+1, T]$ *for some* $t \in [T]$.

Proof. Consider the single-item lot-sizing problem instance given by the parameters

$$
\begin{aligned}
&T = 3, && c^{\mathrm{v}} = (3,2,1), && c^{\mathrm{f}} = c^{\mathrm{d}} = c^{\mathrm{c}} = (0,0,0),\\
&\alpha = (1,1,1), && \underline{U} = \underline{Q} = (0,0,0), && \overline{U} = \overline{Q} = (2,2,2),\\
&D = \{d^1, d^2\}, && d^1 = (1,3,1), && d^2 = (1,1,3),\\
&u_0 = 0.
\end{aligned}
$$

Since the set-up costs and lower production bounds are all zero, without loss of generality we assume $y_t = 1$ for all $t \in [T]$ in all optimal solutions. Thus, all optimal solutions are fully characterized by their production sequence q. Optimizing $(\mathrm{L}_{\mathrm{adapt}})$ as two independent instances of $(\mathrm{L}_{\mathrm{stat}})$ yields:

- For $d^1 = (1,3,1)$, the unique optimal production vector is given by $q^1 = (2,2,1)$.
- For $d^2 = (1,1,3)$, the unique optimal production vector is given by $q^2 = (1,2,2)$.

Note that q_1^1 and q_1^2 differ, even though d_1^1 and d_1^2 coincide. To be able to decide whether to employ q_1 as $q_1^1 = 2$ or as $q_1^2 = 1$ at time $t = 1$, we need to know the value of d_2 (to be able to determine whether d^1 or d^2 realizes). Thus, the (unique) optimal solution to the problem is acausal. □

We remark that, theoretically, causality could be enforced via the (possibly infinitely many) constraints

$$q(d^1)_t = q(d^2)_t \qquad \forall\ t \in [T],\ d^1, d^2 \in D:\ d_i^1 = d_i^2\ \forall\ i \in [t]$$

which proves severely unpractical in most applications.

4.4.1 Affine rules

To account for the above issues, we resort to *affine rules*, a technique which has been successfully adopted in a number of optimization and control problems, see [8, 21, 10]. While still considering $q(d)$ to be dependent of d, the idea is to require q_t for each $t \in [T]$ to be an affinely linear function of the uncertain demand vector d, that is

$$q(d)_t := \sum_{j=1}^{T} \xi_{j,t} d_j + \Xi_t \tag{4.64}$$

with $\xi_{j,t}, \Xi_t \in \mathbb{R}$ for all $j \in [T]$. Furthermore, we consider the commitment variable y to be static. Thus, the solution's realization is fully characterized by the production sequence. Given a nominal realization $\bar{d}$, the affine rules can equivalently be formulated as

$$\begin{aligned} q(d)_t &= \sum_{j=1}^{T} \xi_{j,t} \left((d_j - \bar{d}_j) + \bar{d}_j \right) + \Xi_t \\ &= \sum_{j=1}^{T} \xi_{j,t} \left(d_j - \bar{d}_j \right) + \underbrace{\sum_{\substack{j \in [T]: \\ j \le t}} \bar{d}_j + \Xi_t}_{:= \Xi'_j} . \end{aligned}$$

Here, we observe that the solution is composed of a fixed solution value Ξ' for the nominal problem $\bar{d}$ and a linear term in the deviation $d - \bar{d}$ for any given scenario d. Hence this, again, constitutes a two stage robustness and thus a recoverable robustness approach, see Subsection 1.3.4, Example 4 and Example 5.

As, in this form, the affine rules are clearly acausal, they can not be applied to our specific problem in practice. Still, they are relevant for problems in which all uncertainty reveals just before the point of implementation of the solution. Here, all time-intensive optimization has to be finished beforehand. Hence, in this work we decided to include and run them for our given data, in favor of theoretical insights outside the explicit application. To enforce causality, we restrict $\xi_{j,t} = 0$ for $t \in [T]$ and $j \in [t+1, T]$, and thus the causal affine rules read

$$q(d)_t := \sum_{j=1}^{t} \xi_{j,t} d_j + \Xi_t. \tag{4.65}$$

Quite importantly, affine rules allow to compute the value for the production variables $q(d)$ as a function of the realization $d \in \mathbb{R}^T$ (even for $d \notin D$) very quickly, namely in linear time with respect to T. Note though that if $d \notin D$ holds, the approach does not guarantee that the obtained solution is feasible. This approach generalizes that of ($\mathrm{L}_{\mathrm{stat}}$) with a first-stage production variable q, as the latter is obtained when restricting $\xi_{j,t} = 0$ for all $j, t \in [T]$. Overall, we obtain the following, potentially infinite, mathematical program:

$$
\begin{aligned}
(\mathrm{L}_{\mathrm{aff}})\ \min \quad & \sup_{d \in D} \left\{ \sum_{t=1}^{T} \left(c_t^{\mathrm{v}} q(d)_t + c_t^{\mathrm{f}} y_t + c_t^{\mathrm{d}} d_t + c_t^{\mathrm{c}} \right) \right\} && && (4.66)\\
\text{s.t.} \quad & \underline{U}_t \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} \left(q(d)_i - d_i \right) \leq \overline{U}_t && \forall\, t \in [T],\ \forall\, d \in D && (4.67)\\
& \underline{Q}_t y_t \leq q(d)_t \leq \overline{Q}_t y_t && \forall\, t \in [T],\ \forall\, d \in D && (4.68)\\
& q(d)_t = \sum_{j=1}^{T} \xi_{j,t} d_j + \Xi_t && \forall\, t \in [T],\ \forall\, d \in D && (4.69)\\
& q(d)_t \geq 0 && \forall\, t \in [T],\ \forall\, d \in D && (4.70)\\
& \xi_{j,t} = 0 && \forall\, t \in [T],\ \forall\, j \in [t+1, T] && (4.71)\\
& \xi_{j,t}, \Xi_t \text{ free} && \forall\, j \in [T],\ \forall\, t \in [T] && (4.72)\\
& y_t \in \{0, 1\} && \forall\, t \in [T] && (4.73)
\end{aligned}
$$

In this model, the coefficients ξ_{it} and Ξ_t, together with y_t, are considered as decision variables. Hence, this model can be considered a standard static robustness approach for an auxiliary formulation. Thus, similarly to the static model, this one can be reformulated to the (semi-infinite) mixed integer linear program

$$
\begin{aligned}
(\mathrm{L}'_{\mathrm{aff}})\ \min \quad & \eta && && (4.74)\\
\text{s.t.} \quad & \eta \geq \sum_{t=1}^{T} \left(c_t^{\mathrm{v}} \left(\sum_{j=1}^{T} \xi_{j,t} d_j + \Xi_t \right) + c_t^{\mathrm{f}} y_t + c_t^{\mathrm{d}} d_t + c_t^{\mathrm{c}} \right) && \forall\, d \in D && (4.75)
\end{aligned}
$$

$$\underline{U}_t \leq \alpha_{0,t} u_0 + \sum_{i=1}^{t} \alpha_{i,t} \left(\sum_{j=1}^{T} \xi_{j,i} d_j + \Xi_i - d_i \right) \leq \overline{U}_t \qquad \forall\, t \in [T],\ \forall\, d \in D \tag{4.76}$$

$$\underline{Q}_t y_t \leq \sum_{j=1}^{T} \xi_{j,t} d_j + \Xi_t \leq \overline{Q}_t y_t \qquad \forall\, t \in [T],\ \forall\, d \in D \tag{4.77}$$

$$\xi_{j,t} = 0 \qquad \forall\, t \in [T],\ \forall\, j \in [t+1, T] \tag{4.78}$$

$$\xi_{j,t}, \Xi_t, \eta \text{ free} \qquad \forall\, t \in [T],\ \forall\, j \in [T] \tag{4.79}$$

$$y_t \in \{0, 1\} \qquad \forall\, t \in [T]. \tag{4.80}$$

Remark. *In the former models causality is enforced by additional constraints* (4.71) *and* (4.78) *instead of dropping the acausal parameters, for notational convenience. As discussed, the static model is obtained by forcing all affine coefficients ξ to zero. Here, the terms Ξ adopt the role of the former production variables q.*

To enforce further properties on the model or to achieve a smaller but restricted model, one could impose additional constraints on the affine factors ξ.

Example 6. *By adding $\xi_{t,t} = 0$ for all $t \in T$ to our causal formulation, we obtain a stricter version of causality, in which the current values are not used. A smaller model can be attained by eliminating affine factors ξ. This can be done directly by forcing $\xi_{j,t} = 0$ for certain $j, t \in [T]$. Also, imposing $\xi_{j_1,t} = \xi_{j_2,t} = \ldots = \xi_{j_k,t}$, i.e., accumulating the effect of periods $j_1, j_2, \ldots, j_k$ on period t, practically eliminates $k-1$ affine factors.*

Mark that we cannot hope for a result similar to that of Theorem 11, as the influence of the realization d is by definition no longer independent from the production values. In general, the computational complexity of this problem is yet unexplored.

For completeness, we fully report the our mixed integer linear programming formulation ($\mathrm{L}^{\Gamma}_{\mathrm{aff}}$) for the Γ-uncertainty model with general affine rules. This is achieved by the methods described in Subsection 1.3.3, see Example 3. In it, we employ the slight abuse of notation $\alpha_{k,t} := 0$ for $k > t$.

$$
\begin{aligned}
(\mathrm{L}^{\Gamma}_{\mathrm{aff}})\ \min\quad & \eta \\
\text{s.t.}\quad & \eta \geq \sum_{t=1}^{T} \left(c^{\mathrm{d}}_t + \sum_{k=1}^{T} c^{\mathrm{v}}_k \xi_{t,k} \right) \bar{d}_t + \sum_{t=1}^{T} \left(c^{\mathrm{v}}_t \Xi_t + c^{\mathrm{f}}_t y_t + c^{\mathrm{c}}_t \right) \\
& \qquad + \sum_{t=1}^{T} \pi_t + \Gamma \pi_0 \\
& \pi_0 + \pi_t \geq \left(c^{\mathrm{d}}_t + \sum_{k=1}^{T} c^{\mathrm{v}}_k \xi_{t,k} \right) \hat{d}_t && \forall\, t \in [T] \\
& \pi_0 + \pi_t \geq -\left(c^{\mathrm{d}}_t + \sum_{k=1}^{T} c^{\mathrm{v}}_k \xi_{t,k} \right) \hat{d}_t && \forall\, t \in [T]
\end{aligned}
$$

$$
\begin{aligned}
&\underline{U}_t \leq \sum_{i=1}^{T} \left(\sum_{k=1}^{t} \alpha_{k,t}\xi_{i,k} - \alpha_{i,t} \right) \bar{d}_i + \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}\Xi_i \\
&\qquad - \sum_{i=1}^{T} \rho_{t,i} - \Gamma\rho_{t,0} && \forall\ t \in [T] \\
&\rho_{t,0} + \rho_{t,i} \geq \left(\sum_{k=1}^{t} \alpha_{k,t}\xi_{i,k} - \alpha_{i,t} \right) \hat{d}_i && \forall\ t \in [T], \forall\ i \in [T] \\
&\rho_{t,0} + \rho_{t,i} \geq - \left(\sum_{k=1}^{t} \alpha_{k,t}\xi_{i,k} - \alpha_{i,t} \right) \hat{d}_i && \forall\ t \in [T], \forall\ i \in [T] \\
&\overline{U}_t \geq \sum_{i=1}^{T} \left(\sum_{k=1}^{t} \alpha_{k,t}\xi_{i,k} - \alpha_{i,t} \right) \bar{d}_i + \alpha_{0,t}u_0 + \sum_{i=1}^{t} \alpha_{i,t}\Xi_i \\
&\qquad + \sum_{i=1}^{T} \psi_{t,i} + \Gamma\psi_{t,0} && \forall\ t \in [T] \\
&\psi_{t,0} + \psi_{t,i} \geq \left(\sum_{k=1}^{t} \alpha_{k,t}\xi_{i,k} - \alpha_{i,t} \right) \hat{d}_i && \forall\ t \in [T], \forall\ i \in [T] \\
&\psi_{t,0} + \psi_{t,i} \geq - \left(\sum_{k=1}^{t} \alpha_{k,t}\xi_{i,k} - \alpha_{i,t} \right) \hat{d}_i && \forall\ t \in [T], \forall\ i \in [T] \\
&\underline{Q}_t y_t \leq \sum_{j=1}^{T} \xi_{j,t}\bar{d}_j + \Xi_t - \Gamma\tau_{t,0} - \sum_{i=1}^{t} \tau_{t,i} && \forall\ t \in [T] \\
&\tau_{t,0} + \tau_{i,t} \geq \xi_{i,t}\hat{d}_i && \forall\ t \in [T], \forall\ j \in [T] \\
&\tau_{t,0} + \tau_{i,t} \geq -\xi_{i,t}\hat{d}_i && \forall\ t \in [T], \forall\ j \in [T] \\
&\overline{Q}_t y_t \geq \sum_{j=1}^{T} \xi_{j,t}d_j + \Xi_t + \Gamma\sigma_{t,0} + \sum_{i=1}^{t} \sigma_{t,i} && \forall\ t \in [T] \\
&\sigma_{t,0} + \sigma_{t,i} \geq \xi_{i,t}\hat{d}_i && \forall\ t \in [T], \forall\ i \in [T] \\
&\sigma_{t,0} + \sigma_{t,i} \geq -\xi_{i,t}\hat{d}_i && \forall\ t \in [T], \forall\ i \in [T] \\
&\pi_t, \rho_{t,i}, \psi_{t,i}, \tau_{i,t}, \sigma_{t,i} \geq 0 && \forall\ t \in [T], \forall\ i \in [T] \\
&\xi_{j,t}, \Xi_t, \eta \text{ free} && \forall\ t \in [0,T], \forall\ j \in [0,T] \\
&y_t \in \{0,1\} && \forall\ t \in [T]
\end{aligned}
$$

Regularization

As observed in, among others, [22], the adoption of affine rules induces an affine regression subproblem to the robust counterpart of a deterministic problem. We now illustrate how this observation translates into our problem, and propose a regularization approach to obtain affine rules which, empirically, yield more stable results when implemented with demands outside of the uncertainty set.

For convenience, we consider general (anticipative) affine rules and finite uncertainty sets D in the following. Let $(q^*, z^*) \in \left\{q \;\middle|\; q : D \to \mathbb{R}^T_{\geq 0}\right\} \times \{0,1\}^T$ be an optimal solution to ($\mathrm{L}_{\mathrm{adapt}}$). If ($\mathrm{L}_{\mathrm{aff}}$) admits a solution with coefficients $(\xi^*, \Xi^*, z^*) \in \mathbb{R}^{T\times T} \times \mathbb{R}^T \times \{0,1\}^T$ satisfying

$$\sum_{j=1}^{T} \xi^*_{j,t} d_j + \Xi^*_t = q^*_t(d) \qquad \begin{array}{l}\forall\, t \in [T],\\ \forall\, d \in D,\end{array} \tag{4.81}$$

then the affine rules practically pose no restriction to the problem and (ξ^*, Ξ^*, z^*) constitutes an optimal solution to ($\mathrm{L}_{\mathrm{aff}}$) with identical solution value to that of (q^*, z^*) to ($\mathrm{L}_{\mathrm{adapt}}$).

Now, we interpret (4.81) as a system of linear equations with variables (ξ, Ξ). The space of valid solutions to (4.81) is either empty or an affine subspace of $\mathbb{R}^{T\times T} \times \mathbb{R}^T$. Hence, if there exists a valid solution to (4.81) and it is not unique, there exist infinitely many valid solutions and the magnitude of their parameters is unbounded. This unboundedness is problematic from a practical perspective as arbitrarily small deviations from one of the covered scenarios $d \in D$ can result in arbitrarily large changes in the corresponding production plan given by the affine rules.

In a number of machine learning works, see [14], it is observed that, both in theory and in practice, a better generalization is obtained by introducing a *regularization* term which limits the magnitude of the coefficients of the function that is being learned. For these reasons, we consider a modified approach with limited coefficients

$$-1 \leq \xi_{j,t} \leq 1 \qquad \forall\, j, t \in [T]. \tag{4.82}$$

This corresponds to introducing the ∞-norm constraint $||\xi_t||_\infty \leq 1$ on the vector of coefficients ξ_t for all $t \in [T]$. The bound of 1 is motivated by the intuition that any change in demand should not affect the production plan by more than its own value. While this idea is not enforced globally but individually for the production value of each period, empirically it proves to be efficient, as shown in our computational study, see Section 4.8.

Here, we also observe that the above issue is largely more prevalent for small uncertainty sets, i.e. those with $|D| \leq T$. In the following we motivate the causes of this phenomenon.

The system (4.81) is (uniquely) solvable if and only if the systems

$$\sum_{j=1}^{T} \xi_{j,t'} d_j + \Xi_{t'} = q^*_{t'}(d) \qquad \forall\, d \in D \tag{4.83}$$

are (uniquely) solvable for each $t' \in [T]$. For fixed $t' \in [T]$, let

$$v(d) := \begin{pmatrix} d \\ 1 \end{pmatrix} \in \mathbb{R}^{T+1}, \qquad w(d) := \begin{pmatrix} d \\ 1 \\ q^*_{t'}(d) \end{pmatrix} \in \mathbb{R}^{T+2}, \qquad u(d) := \begin{pmatrix} d \\ q^*_{t'}(d)) \end{pmatrix} \in \mathbb{R}^{T+1}$$

for all $d \in D$ and

$$V := \bigcup_{d \in D} \{v(d)\}, \qquad W := \bigcup_{d \in D} \{w(d)\}, \qquad U := \bigcup_{d \in D} \{u(d)\}.$$

Then system (4.83) is solvable if and only if

$$\dim_{\text{lin}}(\text{span}_{\text{lin}}(V)) = \dim_{\text{lin}}(\text{span}_{\text{lin}}(W))$$

holds, which is the case if and only if

$$\dim_{\text{aff}}(\text{span}_{\text{aff}}(D)) = \dim_{\text{aff}}(\text{span}_{\text{aff}}(U))$$

is satisfied.

Furthermore, we rely on the statement that for a given set of random vectors $R \subseteq \mathbb{R}^n$, it is "likely" that the affine dimension of $\text{span}(R)$ is $\min\{|R| - 1, n\}$. In other words, it is likely that given a random choice of vectors from $\mathbb{R}^n$, as many as possible of them are affinely independent. Even though the vectors d and $u(d)$ are not random, we are not aware of any property that influences their affine dependence. Thus, the former argument suggests that, likely, the affine dimension of $\text{span}(D)$ is $\min\{|D| - 1, T\}$ and the affine dimension of $\text{span}(U)$ is $\min\{|U| - 1, T + 1\}$.

Hence, for $|D| \leq T$ it is likely that

$$\dim_{\text{aff}}(\text{span}_{\text{aff}}(D)) = |D| - 1$$

holds. As adding another component to each vector can not cause affine dependence,

$$\dim_{\text{aff}}(\text{span}_{\text{aff}}(U)) = |U| = |D| - 1$$

is implied. Thus system (4.83) is solvable for each $t' \in [T]$. Furthermore, the system can not be uniquely solvable, as there are more variables than equalities. Hence, the space of valid solutions is infinite.

For $|D| > T$ we likely have

$$\dim_{\text{aff}}(\text{span}_{\text{aff}}(U)) = T + 1.$$

As by the elimination of one component at most one dimension is lost, which is guaranteed in this case due to $\dim_{\text{aff}}(\text{span}_{\text{aff}}(D)) \leq T$,

$$\dim_{\text{aff}}(\text{span}_{\text{aff}}(D)) = T$$

is implied. Thus system (4.83) is not solvable for some $t' \in [T]$.

In conclusion, system (4.81) is likely to be solvable for $|D| \leq T$, with its solution spaces unboundedness being implied, and likely to be unsolvable for $|D| > T$.

In the case of (4.83) not being solvable, we are considering a "proper" affine regression subproblem, in which no comparable instabilities are observed.

4.5 A robust-stochastic hybrid model

A purely robust approach, tends not to be competitively viable in a free market, in which the decision maker is facing a number of non risk-averse adversaries. While we explicitly demand solutions which are feasible in every case, minimizing the objective function in expectation, as opposed to in the worst-case, is likely to offer solutions which, by being less risk-averse, can be more profitable in practice.

As customary in many results in robust optimization, for example those in [12] on Γ-robustness, we assume that $d \in D$ is a symmetrically distributed random variable about $\bar{d} := \mathbb{E}[d]$. With this, the following holds:

Theorem 12. *Let $D \subseteq \mathbb{R}^n$ be a set that is centrally symmetric about $\bar{d} \in D$. Further, let $d \in D$ be a random variable with density function $p : D \to [0,1]$ which is symmetric about $\bar{d}$ and $g : D \to \mathbb{R}$ an affine function. Then, $\mathbb{E}[g(d)] = g(\bar{d})$ holds.*

Proof. Let $D' := D - \bar{d}$, then D' is centrally symmetric about 0 and $D' = -D'$ holds. Furthermore, let $p'(x) := p(x + \bar{d})$ for $x' \in D'$. Hence, p' is symmetric about 0. For any strictly linear function $f : D \to \mathbb{R}$, we have

$$\begin{aligned}
\int_{x' \in D'} p'(x')f(x')dx' &= -\int_{x' \in D'} -p'(x')f(x')dx' \\
&= -\int_{x' \in D'} p'(x')f(-x')dx' && \text{(by the linearity of } f\text{)} \\
&= -\int_{(-x') \in (-D')} p'(-x')f(x')dx' && \text{(subs. } -x' \text{ for } x'\text{)} \\
&= -\int_{(-x') \in (-D')} p'(x')f(x')dx' && \text{(by the symm. of } p'\text{)} \\
&= -\int_{x' \in D'} p'(x')f(x')dx' && \text{(due to } -D' = D'\text{)}.
\end{aligned}$$

Thus, we conclude $\int_{D'} p(x')g(x')dx' = 0$. Since $g(x) - g(0)$ is a linear function, we deduce

$$\begin{aligned}
\mathbb{E}[g(d)] &= \int_{x \in D} p(x)g(x)dx \\
&= \int_{x \in D} p(x)\big(g(x) - g(0)\big)dx + \int_D p(x)g(0)dx \\
&= \int_{x' \in D'} p\big(\bar{d} + x'\big)\big(g(\bar{d} + x') - g(0)\big)dx' + g(0) \\
&= \int_{x' \in D'} p'(x')\big(g(\bar{d} + x') - g(0)\big)dx' + g(0) \\
&= \underbrace{\int_{x' \in D'} p'(x')\big(g(x') - g(0)\big)dx'}_{=0} + \underbrace{\int_{x' \in D'} p'(x')\big(g(\bar{d}) - g(0)\big)dx'}_{=g(\bar{d})-g(0)} + g(0) \\
&= g(\bar{d}),
\end{aligned}$$

which concludes the proof. □

So for uncertainty sets as described above, we obtain the robust-stochastic hybrid version of ($\mathrm{L_{aff}}$) by substituting the objective (4.66) with

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{T}\left(c_t^{\mathrm{v}}p(d)_t + c_t^{\mathrm{f}}y_t + c_t^{\mathrm{d}}d_t + c_t^{\mathrm{c}}\right)\right]\\
&= \mathbb{E}\left[\sum_{t=1}^{T}\left(c_t^{\mathrm{v}}\left(\sum_{j=1}^{T}\xi_{j,t}d_j + \Xi_t\right) + c_t^{\mathrm{f}}y_t + c_t^{\mathrm{d}}d_t + c_t^{\mathrm{c}}\right)\right]\\
&= \sum_{t=1}^{T}\left(c_t^{\mathrm{v}}\left(\sum_{j=1}^{T}\xi_{j,t}\bar{d}_j + \Xi_t\right) + c_t^{\mathrm{f}}y_t + c_t^{\mathrm{d}}\bar{d}_t + c_t^{\mathrm{c}}\right).
\end{aligned}$$

This we apply in the case of Γ robustness.

Furthermore, we adopt a similar approach in the case of finite uncertainty sets which, by construction, are not symmetric in our application, see Section 4.7. Here, we assume a uniform distribution with first moment $\hat{d} := \sum_{d\in D}\frac{1}{|D|}d$. Then, we obtain the robust-stochastic hybrid version of ($\mathrm{L_{aff}}$) by substituting the objective (4.66) with

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{T}\left(c_t^{\mathrm{v}}p(d)_t + c_t^{\mathrm{f}}y_t + c_t^{\mathrm{d}}d_t + c_t^{\mathrm{c}}\right)\right]\\
&= \mathbb{E}\left[\sum_{t=1}^{T}\left(c_t^{\mathrm{v}}\left(\sum_{j=1}^{T}\xi_{j,t}d_j + \Xi_t\right) + c_t^{\mathrm{f}}y_t + c_t^{\mathrm{d}}d_t + c_t^{\mathrm{c}}\right)\right]\\
&= \sum_{d\in D}\frac{1}{|D|}\sum_{t=1}^{T}\left(c_t^{\mathrm{v}}\left(\sum_{j=1}^{T}\xi_{j,t}d_j + \Xi_t\right) + c_t^{\mathrm{f}}y_t + c_t^{\mathrm{d}}d_t + c_t^{\mathrm{c}}\right)\\
&= \sum_{t=1}^{T}\left(c_t^{\mathrm{v}}\left(\sum_{j=1}^{T}\xi_{j,t}\hat{d}_j + \Xi_t\right) + c_t^{\mathrm{f}}y_t + c_t^{\mathrm{d}}\hat{d}_t + c_t^{\mathrm{c}}\right).
\end{aligned}$$

We remark that the hybrid approach is different from applying affine rules in a stochastic optimization setting as we enforce all constraints, except for the one representing the original objective function, in the worst-case.

Also, the robust-stochastic hybrid approach only differs from the purely robust one when adaptive solutions are employed as for static q the stochastic objective function reads

$$\mathbb{E}\left[\sum_{t=1}^{T}\left(c_t^{\mathrm{v}}p_t + c_t^{\mathrm{f}}y_t + c_t^{\mathrm{d}}d_t + c_t^{\mathrm{c}}\right)\right] = \sum_{t=1}^{T}\left(c_t^{\mathrm{v}}p_t + c_t^{\mathrm{f}}y_t + c_t^{\mathrm{c}}\right) + \mathbb{E}\left[\sum_{t=1}^{T}c_t^{\mathrm{d}}d_t\right].$$

Thus, the robust-stochastic objective function and the purely robust objective function only differ by the constant term

$$\mathbb{E}\left[\sum_{t=1}^{T}c_t^{\mathrm{d}}d_t\right] - \sup_{d\in D}\left\{\sum_{t=1}^{T}c_t^{\mathrm{d}}d_t\right\}.$$

This additive term can be dropped without affecting the optimality of the solutions that are found.

4.6 Realized robustness

Given a source of uncertainty, robust optimization aims to determine a solution that guarantees feasibility for any realization of the uncertain parameters. In practice, this property heavily relies on forecasts and suitable uncertainty sets and is often unrealistic. Furthermore, we face the concept of the "price of robustness" introduced by Bertsimas and Sim [12], especially in the case of static solutions. On the one hand, extending a given uncertainty set increases the chance of any realization being included and thus being guaranteed feasibility of the solution. On the other hand, the extension naturally increases the (worst-case) robust objective value, and, empirically, also tends to increase the objective value for the majority of individual scenarios. Hence, one may be willing to compromise the guarantee of feasibility in favor of better objective values. Also, large uncertainty sets may not allow for feasible robust solutions at all. Consequently, in this case we have to settle for smaller/different uncertainty sets, risking infeasibility in the implementation.

In this work, for a given day we consider uncertainty sets based on given forecasts and historical data, see Section 4.7, and evaluate them with respect to the actual realization of that day. Due to the limitations of our given data, we face infeasible solutions in most cases when implementing the actual realizations. Thus, we need a more elaborate approach to evaluate the quality of our solutions.

To this end, we adopt the concept of *realized robustness*, see [32]. Here, we simulate a real-world situation in which a solution found via robust optimization has to be implemented for a realization outside the considered uncertainty set. That is, the solution is heuristically adjusted if the realized demand $\tilde{d}$ does not belong to the uncertainty set D, and infeasibilities are measured. As this procedure is highly problem specific, we base it on the original formulation (L_1). Given a robust solution (ξ, Ξ, y) to a variant of $(\mathrm{L}_{\mathrm{aff}})$ and the actual data realization $\tilde{d}$ (possibly outside D), we construct a solution $(\tilde{q}, \tilde{u}, \tilde{f}, \tilde{p}^{\mathrm{P}}, \tilde{q}, \tilde{p}^{\mathrm{M}}, y)$ to (L_1), which satisfies all constraints but the balance constraints (4.2). First, we transfer any violation occurring in the lower and upper bound constraints, imposed on u and q, to the balance constraints. We achieve this by defining the auxiliary production and storage values

$$\tilde{q}_t := \min \left\{ \max \left\{ \sum_{j \in [T]} \xi_{j,t} \tilde{d}_j + \Xi_t, \underline{Q}_t y_t \right\}, \overline{Q}_t y_t \right\} \tag{4.84}$$

$$\tilde{u}_t := \min \left\{ \max \left\{ \alpha_t \tilde{u}_{t-1} + \tilde{q}_t - \tilde{d}_t, \underline{U}_t \right\}, \overline{U}_t \right\} \tag{4.85}$$

for all $t \in [T]$, with $\tilde{u}_0 := u_0$. Thus, the lower and upper bound constraints on storage and production are always satisfied. The remaining values $\tilde{f}, \tilde{p}^{\mathrm{P}}, \tilde{q}, \tilde{p}^{\mathrm{M}}$ are induced by (4.21) - (4.24). We then define the violation of the balance constraints for the newly introduced $\tilde{q}$ and $\tilde{u}$ as

$$v_t := \alpha_t \tilde{u}_{t-1} + \tilde{q}_t - \tilde{d}_t - \tilde{u}_t.$$

In the case of a static approach with solution (q, z), we simply let

$$\tilde{q}_t := q_t$$

for all $t \in [T]$ while adopting the same definition (4.85) for $\tilde{u}$.

We remark that if $v \leq 0$ and $\tilde{d} + v \geq 0$ hold, the solution $(\tilde{q}, \tilde{u}, \tilde{f}, \tilde{p}^{\mathrm{P}}, \tilde{q}, \tilde{p}^{\mathrm{M}}, z)$ corresponds to what a practitioner would arguably do when an infeasibility is detected. Production and storage values are adjusted to stay within their prescribed bounds (which physically cannot be violated) while the infeasibility is transferred, if possible, to the customer, by satisfying only the reduced demand $\tilde{d} + v$ as opposed to the original $\tilde{d}$.

For each instance of the problem (i.e., for each realization $\tilde{d}$), we evaluate the quality of a robust solution in terms of how it performs when implemented with the given realization $\tilde{d}$. Here, we consider a measure of accumulated *realized violation*

$$\tilde{v} := \sum_{t=1}^{T} |v_t|$$

as well as the *realized cost*

$$\tilde{c} := \sum_{t=1}^{T} \left(c_t^{\mathrm{M}} \tilde{p}_t^{\mathrm{M}} + c_t^{\mathrm{F}} \tilde{f}_t + c_t^{\mathrm{I}} \tilde{u}_t \right).$$

Note that the realized cost can be quite different from the objective function value achieved by the (optimal) robust solution. Indeed, the latter accounts for either the worst-case or the expected cost over D (depending on whether a worst-case or a stochastic objective function is employed), as opposed to the former, which amounts to the cost that realizes for a given $\tilde{d}$. Furthermore, the adoption of adaptive approaches, although advantageous in terms of the extra flexibility offered, introduces an element of so-called *nervousness* to the solutions. For different realizations of $\tilde{d}$, the realized production values $\tilde{q}_t$ can be quite distinct. Here, the nervousness of $\tilde{d}$ is given as the difference in production values for $\tilde{d}$ and the production values Ξ for the nominal demand $\bar{d}$. As mentioned in [47], large nervousness values can have a negative impact on managerial decisions in the context of single-item lot-sizing problems. Thus, we also report a cumulative measure of nervousness, defined as

$$n(\tilde{d}) = \sum_{t=1}^{T} |\Xi_t - \tilde{q}_t|.$$

Naturally, this is only considered for adaptive solutions.

4.7 Data and considered uncertainty models

We consider a real-life data set spanning a period of two years (with some missing months) with hourly time steps, provided by our industrial partner ProCom GmbH. The first half of the data set (338 days) is used as a *training set*, the second half (232 days) as a *testing set*. The heat demand is taken from historical data of a portion of Frankfurt (of around 50,000 households). For all periods $t \in [T]$, the storage bounds are set to $\underline{U}_t = 0$ MWh and $\overline{U}_t = 120$ MWh, while we set the production bounds to $\underline{Q}_t = 37.5$ MWh and $\overline{Q}_t = 125$ MWh. We also set $u_0 = 36$ MWh. The market prices for the power market are taken from EPEX SPOT (the European Power Exchange).

We consider two cases, finite uncertainty sets and Γ uncertainty sets. Both sets are built based on the training set as well as on a heat demand forecast which also provided by our industrial partner. This forecast is generated in a two-stage fashion, with an auto-regressive and a neural network component, relying on temperature and calendar events as main influence factors.

The finite uncertainty sets D are constructed as follows. From the heat demand forecast $\bar{d}$ for the current day, we single out the (up to 70) days from the set of historical time series (the training set), in which the heat demand forecast in 1-norm is closest to $\bar{d}$. For each of these days i, we compute the (historical) forecast error as the difference $\delta^i := d^i - \bar{d}^i$ of the heat demand realization d^i and its forecast $\bar{d}^i$ for that day. Then, we create a heat demand scenario $\bar{d} + \delta^i$ by adding such error δ^i to the forecast $\bar{d}$ of the current day (for which the problem is being solved). Thus, we obtain new scenarios, which we add to the initial uncertainty set consisting only of the forecast $\bar{d}$ for the day ahead. The aim of this construction is to learn and cover systematic, but possibly unavoidable forecast errors.

In the alternative static approach we consider the same above finite uncertainty set, but allow to neglect up to 10% of the added scenarios, i.e., we let

$$B := \left\lfloor \frac{|D| - 1}{10} \right\rfloor .$$

For the Γ-robustness approach, we combined different hours into groups and considered quantiles of their forecasts errors, to compensate for our limited scope of data. The corresponding values were determined as the most successful ones in a narrow pre-study for static solution approaches. We partition the hours of the training set days into the three groups *morning* $= [1, 6]$, *midday* $= [7, 20]$ and *evening* $= [21, 24]$. Similarly, the demand forecasts are partitioned into the groups *low*, *medium*, and *high*. This way, we can associate each pair (t, d_t) with a category in $\{\text{morning}, \text{midday}, \text{evening}\} \times \{\text{low}, \text{medium}, \text{high}\}$. Here, we consider all hours in a given category over all the training days and determine the deviation (i.e. the forecast error) between their historical demand and the forecast $\bar{d}_t$. Then, we take their 95%-quantiles with respect to positive and negative deviations, and let the deviation $\hat{d}_t$ at time t be equal to their maximum (in absolute).

4.8 Computational study

The experiments in this work are run on an Intel i7-3770 3.40 GHz machine with 32 GB RAM using CPLEX 12.6 and AMPL as modeling language. All the instances are solved to optimality within default precision.

For adaptive robust solutions, we consider two cases, the causal one in which the realization of $\tilde{d}$ unfolds one step at a time, an the anticipative one, in which we assume that $\tilde{d}$ reveals completely at time $t = 1$. When affine rules are in place, all the experiments with either $|D| = 1$ or $\Gamma = 0$ are obtained by letting $\xi_{j,t} = 0$ for all $j, t \in [T]$, i.e., solving the deterministic problem for the forecast (which is the only member of the uncertainty set).

We present our results in the form of the sum over all the instances in our data set. I.e., we consider the sum of realized violation, the sum of realized nervousness and the relative realized cost, for each value of $|D| - 1$ or Γ. For the relative realized cost, we state the sum of realized costs in the given setting, divided by the sum of realized cost for the nominal model (for the forecast). Thus, the relative realized costs denote the overall change in realized costs when switching from the nominal model to the given robust model. The charts that we report show the violation-versus-cost and nervousness-versus-cost curves, thus allowing for, visually, assessing the trade-off between these quantities, that is achieved with the different approaches. The curves are obtained by joining each pair of points obtained with two consecutive values of $|D| - 1$ or Γ. Here, the nominal case, coinciding with $|D| = 1$ or $\Gamma = 0$, is the (singular) point in which all curves meet and exhibits relative realized cost of 1. We consider Γ in steps of 1 until the given approach exhibits at least one unsolvable instance day, and $|D| - 1$ in steps of 10 up to 70 additional scenarios (in addition to the forecast).

We report the computational results in Tables 4.1 to 4.5. For reference, we state the the realized costs of 2647 for the nominal model with perfect information, that is the forecast exactly matches the realization. In contrast, the realized costs are 2675 for the nominal model with the given forecast. Thus, under the unrealistic assumption of perfect information we save circa 1% in costs, without suffering any violations or nervousness at all, when compared to the standard nominal model.

Partially, these results are already published in [24], but some remodeling was performed, which naturally can and did influence the realized values. This is due to the potential ambiguity of optimal solution, especially when considering affine rules, and limited default precision of the implementation. While some values were slightly affected quantitatively, in our analysis, their qualitative properties remained the same.

4.8.1 Static robust solutions

For the standard static approaches, we test both the model yielded by Theorem 11 and the incremental formulation ($\mathrm{L}_{\mathrm{stat}}$). Here, we obtain identical solutions, but at different computation times. Note that we do not go beyond $\Gamma = 6$, as the problem becomes infeasible with larger values, so no feasible solution to the corresponding robust problem exists, at least for some instances. This is the case, as the storage volume does not suffice to cover the difference in demand for solutions with very high and solutions with very low demand. When considering the alternative approach, we employ the (incremental) formulation (L_{alt}).

An illustration in terms of realized costs and violations is provided in Figure 4.2, while complete results are deferred to Table 4.1. Furthermore, we showcase a comparison of the individual computation times for each day, considering the incremental formulation and the one employing the reduction from Theorem 11 for $|D| - 1 = 70$ and $\Gamma = 6$ in Figure 4.3. Nervousness is not reported as, due to q being first stage, it is equal to zero for all instances.

	$\lvert D\rvert - 1$							
	0	10	20	30	40	50	60	70
viol.	3802	1562	1226	1097	1017	999	969	965
costs	2675	2900	2953	2996	3017	3037	3054	3063
time - incremental	15	36	47	59	75	88	105	117
time - reduction	15	15	15	15	15	15	15	15

(a) Results for the standard static approach with finite uncertainty sets.

	Γ						
	0	1	2	3	4	5	6
viol.	3802	2635	1982	1507	1202	972	882
costs	2675	2741	2807	2877	2951	3026	3105
time - incremental	15	41	50	55	58	59	58
time - reduction	15	15	15	15	15	15	14

(b) Results for the standard static approach with Γ uncertainty sets.

	$\lvert D\rvert - 1$							
	0	10	20	30	40	50	60	70
viol.	3802	1874	1580	1333	1299	1210	1230	1182
costs	2675	2840	2870	2901	2902	2904	2908	2914
time - incremental	15	92	238	644	1549	3449	7937	18345

(c) Results for the alternative static approach with finite uncertainty sets.

Table 4.1: Static solution results.

Violations comparison for static robust solutions

We observe that, with the expansion of the uncertainty sets, the violations mostly decrease, while the realized costs increase. This showcases that the intuition of the "price of robustness" tends to carry over to realized robustness. Hence, lower rates of violation can be achieved when employing bigger uncertainty sets, but come at the trade-off of higher costs.

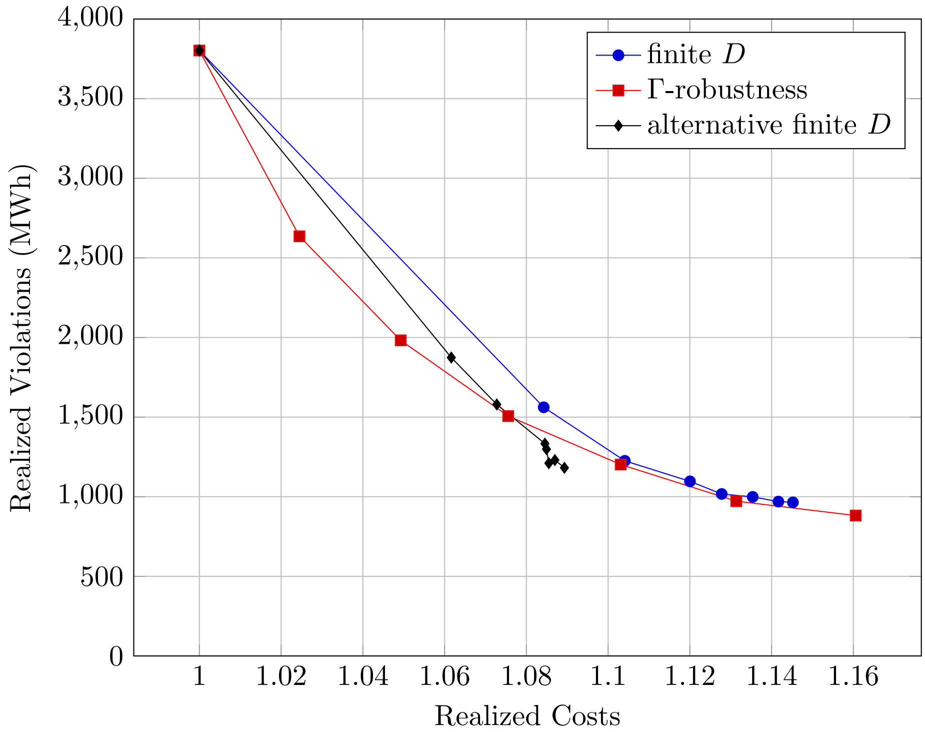


Figure 4.2: Violation-versus-cost curves for static robust solutions.

For the standard approaches, we observe that the adoption of Γ-robustness provides slightly better results, inducing a curve dominating the one for finite scenarios. That is, it provides smaller violations at lower costs, especially for small values of $|D|$ and Γ. Interestingly, as the uncertainty set sizes increase, the two approaches become almost equivalent. Both seem to exhibit a limit to solution quality in terms of violations. This could be caused by a combination of limitations to the static robust approach and external (possibly unavoidable) forecast errors. As we see in the remainder of this section, both of those factors are affirmed, as adaptive approaches yield superior results, but are still limited in quality. Furthermore, all static approaches, by definition, do not suffer from nervousness.

The alternative approach exhibits results in magnitudes comparable to the standard approaches. For the violations it manages to achieve, it does so at severely lower costs. We remark that the limit of achievable violation values cannot be conclusively evaluated due to the restricted scope of our data, which prevents us from properly constructing more scenarios. Thus, we may underestimate the potential of this approach. Still, from the given data, we observe that the achievable violation values are clearly larger than those of the standard approaches. We substantiate this assessment by the fact that the alternative approach for $|D| - 1 = 70$, in which at least 64 (selected) scenarios are covered, exhibits larger violations when compared to the standard approach for $|D| - 1 = 30$, in which only (but all) 31 scenarios are covered. Conversely, covering the above 64 scenarios in the alternative approach yields lower costs and lower violations than the standard approach for $|D| - 1 = 20$.

Computation time comparison for static robust solutions

For the standard static approaches, we observe significant speed-ups when employing the reduction, for both considered types of uncertainty set. Contrary to the incremental model, the computation times for the reduced model are almost independent of the actual uncertainty set. With expanding uncertainty sets, the computation times for incremental finite uncertainty set models monotonically increase nearly linearly. The computation times for incremental Γ uncertainty set models also tend to increase, but much more moderately, and seem to converge. Thus, the speed-ups increase with bigger uncertainty set sizes, and between the Γ and finite uncertainty sets, the speed-up is more substantial for the finite ones.

To elaborate, in Figure 4.3 we consider the individual computation times for all instances for the maximum considered uncertainty set sizes, for which we observe the biggest improvements. Here, we sort the instances in increasing order with respect to their computation times for the incremental model.

We observe that the reduction models need less computation time than the incremental model in each single instance.

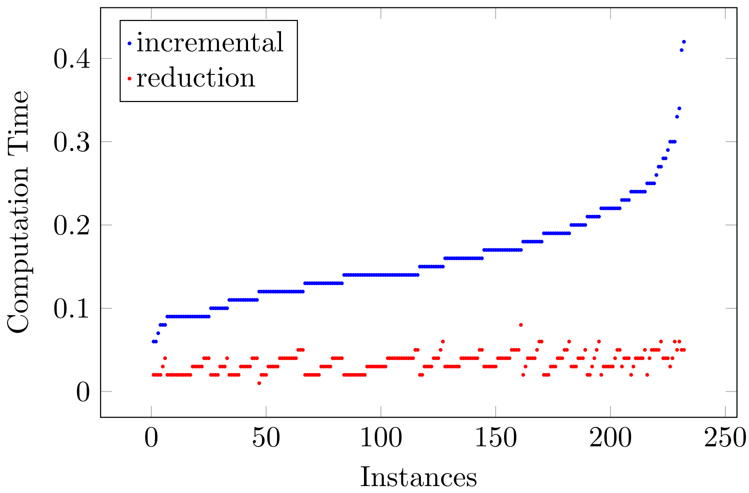
The alternative approach exhibits severely bigger computation times, by factors up to 150 when compared to the approaches for the same finite uncertainty sets, even when comparing it to the incremental implementation of the standard approach. In general, its computation times show exponential behavior, roughly doubling each step of 10 additional scenarios.

Summary for static robust solutions

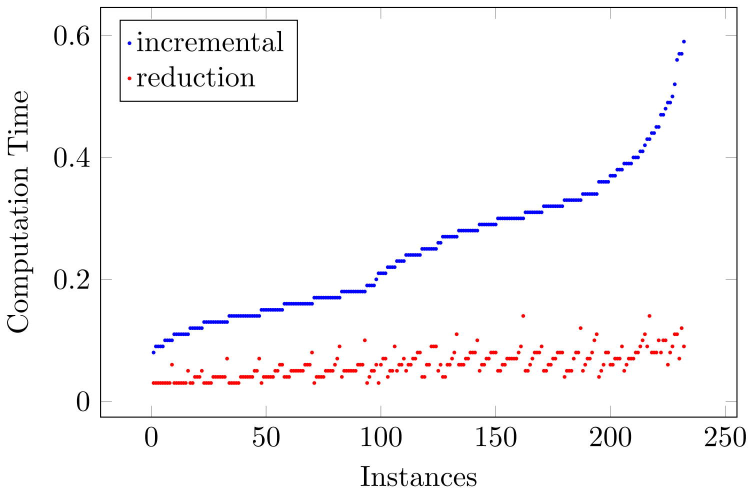
Given our application, we focus on the feasibility aspect of robust optimization rather than on the objective guarantee. This is particularly the case in the context of realized robustness where the objective guarantee is not considered at all. The feasibility (or lack thereof) on the other hand is reflected in the violations. Overall, we prefer solutions which exhibit lower violations, even if the reduction in violations comes at considerably higher costs. Hence, we favor the standard static approach with Γ uncertainty sets.

4.8.2 Adaptive solutions via causal affine rules

In the following, we compare the causal robust and causal hybrid robust-stochastic approaches both among themselves, and against the above static approaches, which are intrinsically causal. Thus, this and the previous subsection cover all approaches which are actually viable for our application, while later on we also consider anticipative approaches in favor of theoretical insight. We study the problem with finite uncertainty sets and Γ uncertainty sets, considering both the cases of a worst-case objective function and a stochastic one. Furthermore, we test all approaches both with regularization, i.e.,



(a) Computation times for finite uncertainty set models for $|D| - 1 = 70$.

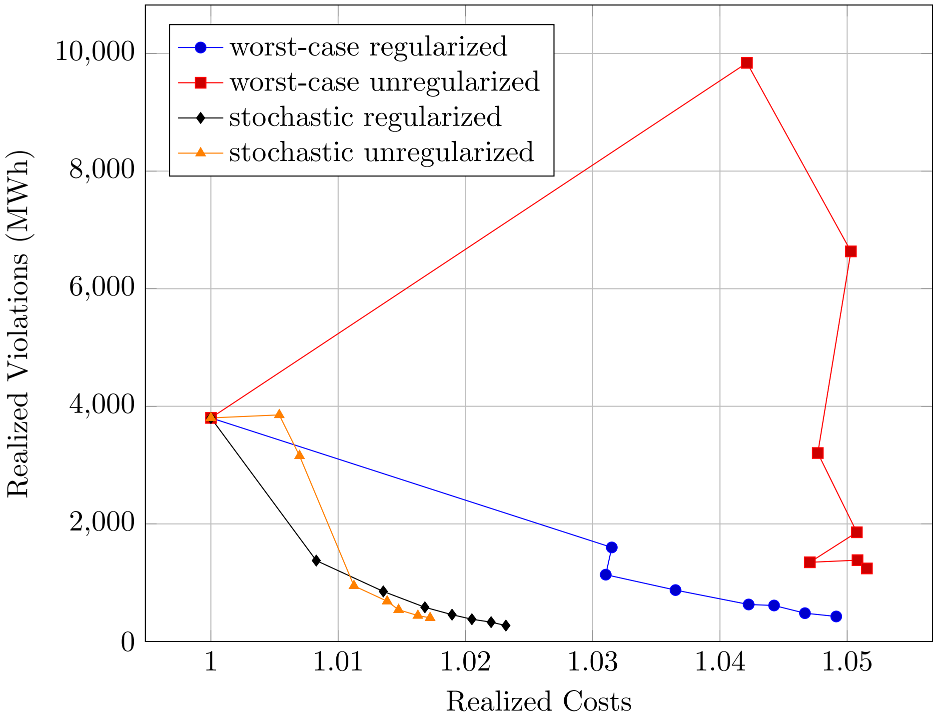


(b) Computation times for Γ uncertainty set models for $\Gamma = 6$.

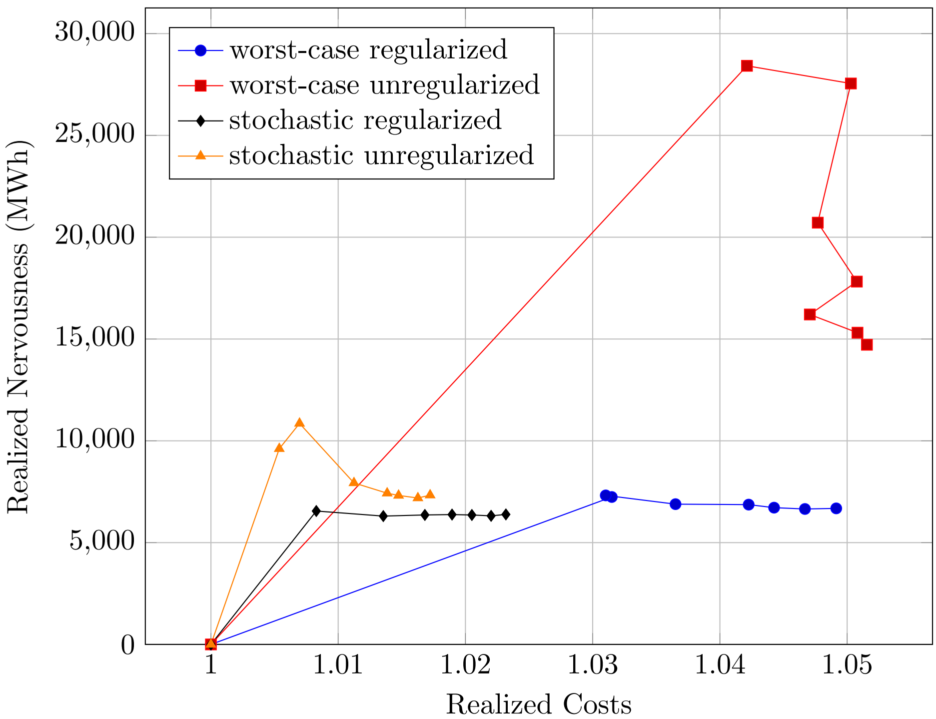
Figure 4.3: Computation time comparison between the reduction and the incremental model for the maximum considered uncertainty set sizes.

coefficients $\xi_{t,j}$ limited to $-1 \leq \xi_{t,j} \leq 1$ for all $j, t \in [T]$, and without regularization, i.e., with unlimited coefficients.

The results for finite uncertainty sets are reported in Table 4.2 and illustrated in Figure 4.4. Analogous experiments for Γ-uncertainty sets are reported in Table 4.3 and illustrated in Figure 4.5. We remark that with the added flexibility, due to the adaptivity, the Γ model remains solvable up to $\Gamma = 12$ in contrast to the value of 6 for the static models.



(a) Violation-versus-cost curves for causal affine rules with finite uncertainty sets.



(b) Nervousness-versus-cost curves for causal affine rules with finite uncertainty sets.

Figure 4.4: Violation-versus-cost and nervousness-versus-cost curves for causal affine rules with finite uncertainty sets.

Table 4.2: Results for causal affine rules with finite uncertainty sets.

			$\lvert D\rvert - 1$							
			0	10	20	30	40	50	60	70
worst-case obj.	reg.	viol.	3802	1600	1136	874	629	612	481	425
		nerv.	0	7241	7319	6889	6866	6717	6652	6682
		costs	2675	2759	2758	2773	2788	2793	2800	2806
		time	15	1541	7108	12026	19424	27493	34566	41562
	unreg.	viol.	3802	9841	6635	3205	1857	1347	1383	1242
		nerv.	0	28417	27549	20714	17819	16205	15309	14718
		costs	2675	2788	2810	2803	2811	2801	2811	2813
		time	15	862	4580	11247	19564	27369	34417	39647
stochastic obj.	reg.	viol.	3802	1376	851	582	456	378	329	271
		nerv.	0	6552	6303	6359	6378	6357	6315	6385
		costs	2675	2697	2711	2720	2726	2730	2734	2737
		time	15	1445	5885	12117	16543	23101	29357	35916
	unreg.	viol.	3802	3854	3157	946	684	540	441	402
		nerv.	0	9612	10854	7933	7427	7317	7192	7332
		costs	2675	2689	2694	2705	2712	2714	2719	2721
		time	15	767	2842	6975	9971	13792	17358	21161

Violations comparison for causal affine rules with finite uncertainty sets

First, we notice that in the case of worst-case objectives without regularization the solutions are highly unstable. That is, realized violations reach extremely high values and costs do not process (even nearly) monotonically with expanding uncertainty sets for small sizes. This phenomenon is motivated in Subsection 4.4.1, and is largely predominant for $|D| \leq T = 24$. While the solutions are still somewhat unstable beyond this point, they are not nearly as volatile as before.

The solutions that are obtained when imposing $-1 \leq \xi_{tj} \leq 1$ for all $j, t \in [T]$ almost completely circumvent this issue and overall clearly dominate the unregularized solutions for worst-case objectives.

Furthermore, the adoption of a stochastic objective quite effectively mitigates the stability issue and produces far superior solutions, both when regularized or unregularized. In terms of achieved violation value, the regularized stochastic objectives perform slightly better, but at noticeably higher costs. Still, the results for the regularized worst-case approach are comparable for large values of $|D|$. Here, the regularization approach is not strictly necessary from a stability perspective, but yields slightly better results for uncertainty set sizes beyond the threshold of $|D| = T = 24$.

In summary, the stochastic objective approaches are clearly superior to the worst-case objective approaches, and the unregularized worst-case one is dominated on all accounts.

Nervousness comparison for causal affine rules with finite uncertainty sets

As for nervousness, we mostly observe behavior similar to that for violations. That is, the nervousness values decrease with expanding uncertainty sets and stagnate even quicker than the violation values. Hence, interestingly, there is no trade-off between violations and nervousness. Thus, we can simultaneously achieve better violation values and lower nervousness values with larger uncertainty sets, but at the price of higher costs.

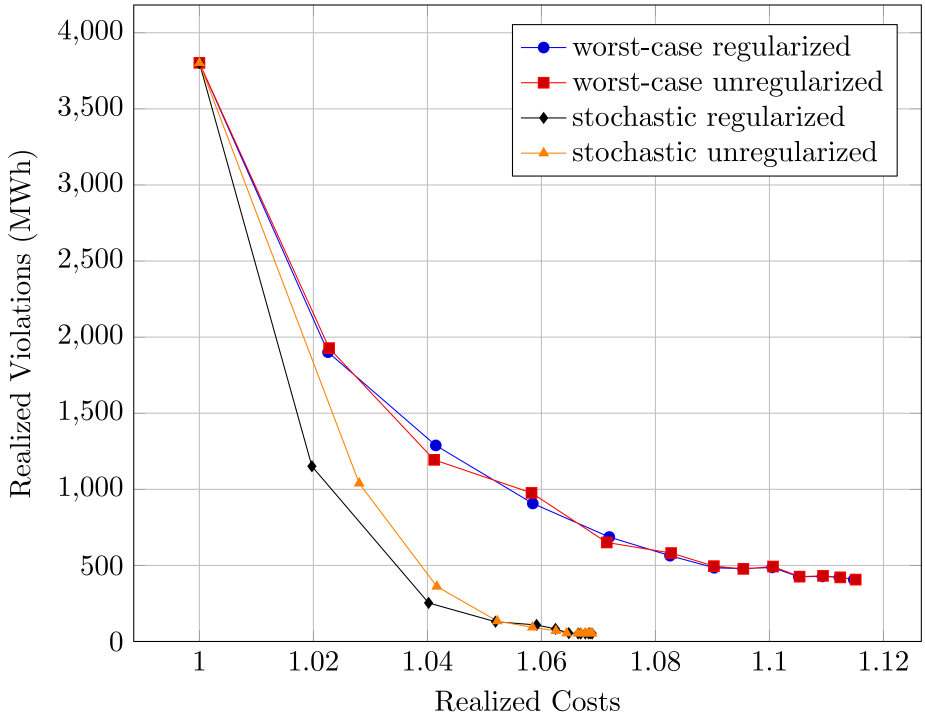
Again, the stochastic objectives are superior, at least for $|D| > T = 24$, while one could argue that the regularized worst-case objective is also viable, as it yields low nervousness values, albeit at very high costs. Overall, the regularized stochastic approach dominates all other approaches, and the unregularized worst-case one yields the worst results.

Summary for causal affine rules with finite uncertainty sets

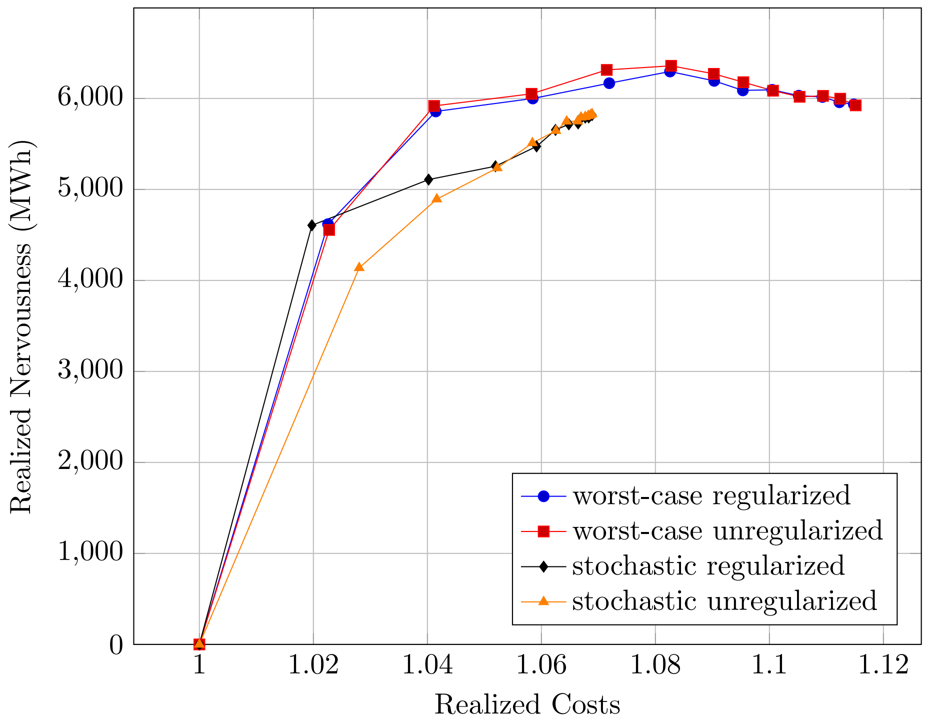
Since we primarily focus on low violation values, but still prefer low costs and nervousness, we favor the regularized stochastic approach. The only genuine competitor would be the unregularized stochastic approach, as it achieves most intermediate violation values at slightly lower costs. Still, it exhibits higher nervousness values and higher minimum violation values.

Table 4.3: Results for causal affine rules with Γ uncertainty sets.

			Γ 0	1	2	3	4	5	6	7	8	9	10	11	12
worst-case obj.	reg.	viol.	3802	1901	1289	907	686	564	484	478	487	425	428	422	407
		nerv.	0	4619	5857	6000	6168	6296	6193	6089	6095	6030	6016	5958	5935
		costs	2675	2735	2786	2831	2867	2896	2917	2930	2944	2956	2967	2975	2982
		time	15	745	1487	2392	3145	3520	4106	3918	3658	3568	3592	3405	3227
	unreg.	viol.	3802	1928	1194	977	651	581	495	477	493	426	431	421	406
		nerv.	0	4555	5918	6051	6314	6359	6271	6179	6085	6018	6028	5997	5924
		costs	2675	2736	2785	2831	2866	2896	2916	2930	2944	2957	2968	2976	2983
		time	15	534	1098	1792	2365	2848	3291	3627	3676	3677	3721	3724	3587
stochastic obj.	reg.	viol.	3802	1152	253	130	108	83	54	53	53	53	53	54	53
		nerv.	0	4606	5108	5255	5473	5655	5716	5726	5781	5790	5792	5813	5818
		costs	2675	2728	2783	2814	2833	2842	2848	2853	2854	2856	2858	2859	2859
		time	15	404	602	990	1203	1231	1335	1281	1188	1139	1107	1044	1030
	unreg.	viol.	3802	1040	362	134	93	70	54	53	53	53	53	53	54
		nerv.	0	4138	4893	5236	5511	5645	5749	5750	5792	5798	5809	5819	5829
		costs	2675	2750	2786	2815	2831	2842	2847	2853	2854	2856	2858	2859	2859
		time	15	296	499	713	910	1048	1189	1235	1297	1289	1328	1281	1301



(a) Violation-versus-cost curves for causal affine rules with Γ uncertainty sets.



(b) Nervousness-versus-cost curves for causal affine rules with Γ uncertainty sets.

Figure 4.5: Violation-versus-cost and nervousness-versus-cost curves for causal affine rules with Γ uncertainty sets.

Violations comparison for causal affine rules with Γ uncertainty sets

Here, in contrast to the finite uncertainty sets, all approaches are quite stable, even with unrestricted coefficients. This is likely due to the fact that Γ uncertainty sets are continuous and of full dimension, so the phenomenon described in Subsection 4.4.1 cannot occur. Interestingly, the regularization does not hinder the quality of the solutions. Rather, it initially slightly improves the results for both approaches, especially for small values of Γ. For increasing values of Γ or $|D|$, the effect of the regularization becomes unnoticeable. Between the objective function options, the stochastic ones once more dominate the worst-case ones.

Accordingly, the restricted stochastic approach performs best, and the restricted worst-case approach performs worst.

Nervousness comparison for causal affine rules with Γ uncertainty sets

As for nervousness, we observe different behavior than in the case of the finite uncertainty sets, as the values increase initially for the worst-case objectives and permanently for the stochastic objective. For the worst-case objectives, the nervousness values begin to slightly improve beyond $\Gamma = 5$. Conversely to the violations, the regularization produces slightly worse values for the stochastic objective with small Γ. Nevertheless, for all approaches the nervousness values seem to converge to a similar value. Once more, the effect of the regularization becomes unnoticeable for bigger values of Γ.

Summary for causal affine rules with Γ uncertainty sets

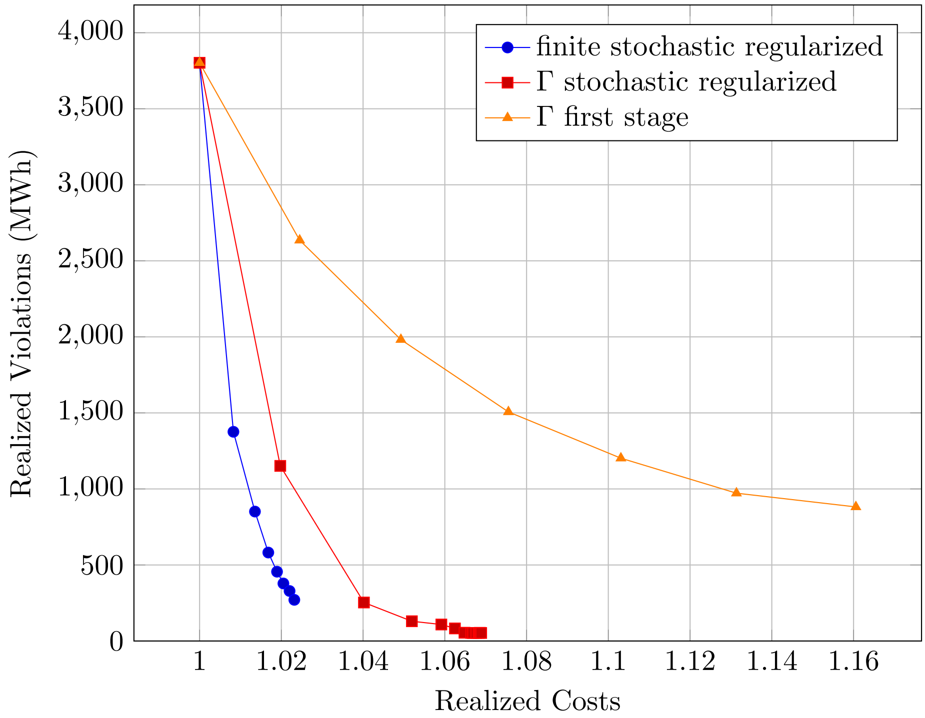
Again, we favor the regularized stochastic approach, as it clearly dominates the worst-case approaches and beats the unregularized stochastic approach for low Γ, with respect to violation values. This is a close call, as the regularized and unregularized stochastic approaches behave very similar for most Γ values. The only relevant trade-off is that of lower violation values for the regularized one, against lower nervousness values for the unregularized one, for small Γ.

4.8.3 Comparison of causal approaches

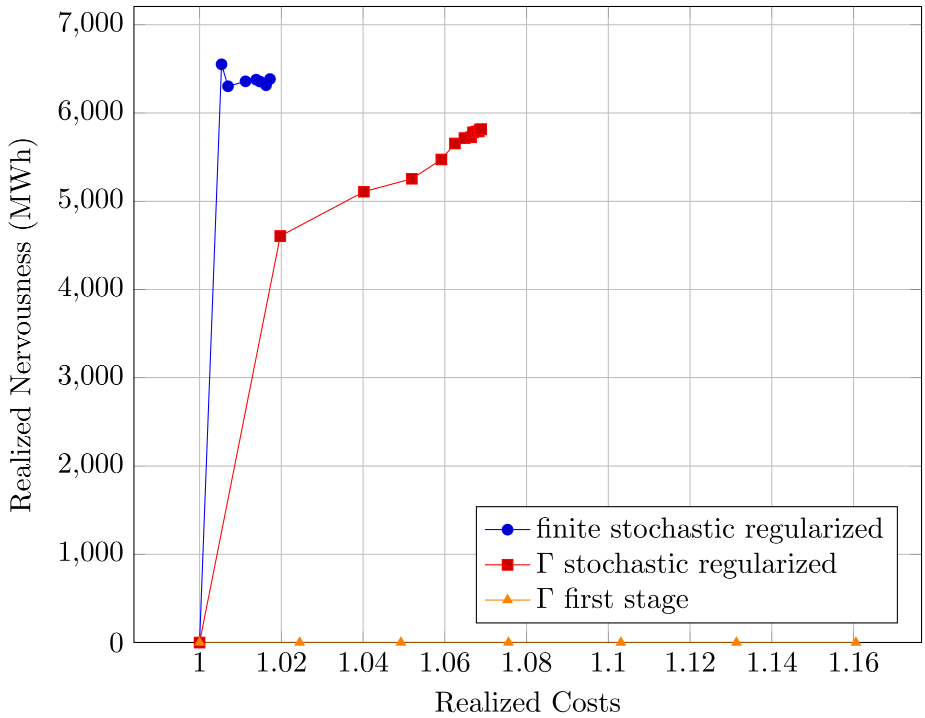
In the following, we compare the arguably best approaches considered above, as illustrated in Figure 4.6. Here, we choose the Γ uncertainty sets for the static solutions, as they were slightly superior. For the causal affine rules, we choose the regularized stochastic approach both for the finite and the Γ uncertainty sets.

On the violation side, we observe that the static approach is clearly outperformed by the affine rules. While the causal affine rules for finite uncertainty set produce reasonably good violation values at small costs, the absolute achieved values when employing Γ uncertainty sets is much lower, albeit at much higher costs.

On the nervousness side, the static approach is naturally optimal due to the general absence of it. Regarding the causal affine rule approaches, the Γ uncertainty sets yield considerably lower nervousness values.



(a) Comparison of causal violation-versus-cost curves



(b) Comparison of causal nervousness-versus-cost curves

Figure 4.6: Comparison of violation-versus-cost and nervousness-versus-cost curves for causal approaches.

Consequently, we clearly favor the causal affine rule with Γ uncertainty sets as their high costs, in our opinion, are compensated by the far superior violation and nervousness values.

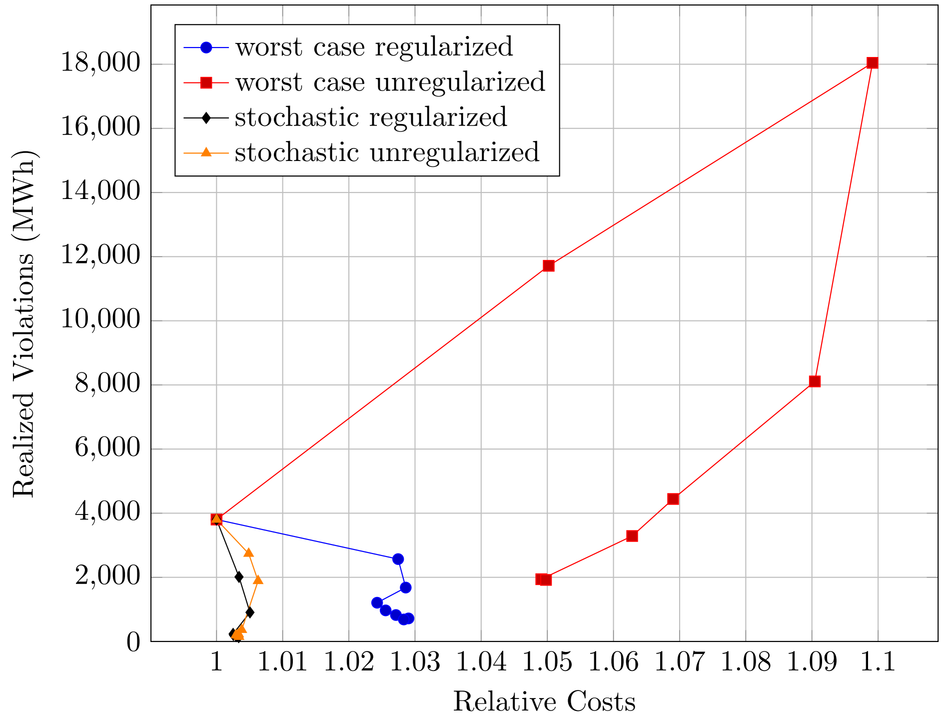
4.8.4 Adaptive solutions via anticipative affine rules

In this subsection, we consider how the results change when we allow the solutions to be anticipative, that is the realization of all data is available from the first period. The resulting solutions are clearly not viable for our specific problem. Still, as argued before, this is a relevant model for problems in which all data realizes at once, but there is no more time available to solve the nominal problem for the realized values before a solution has to be implemented. This also applies if only parts of the solution, like bids or orders in a market environment, have to be implemented on time. Naturally, this model yields the potential for substantial improvements.

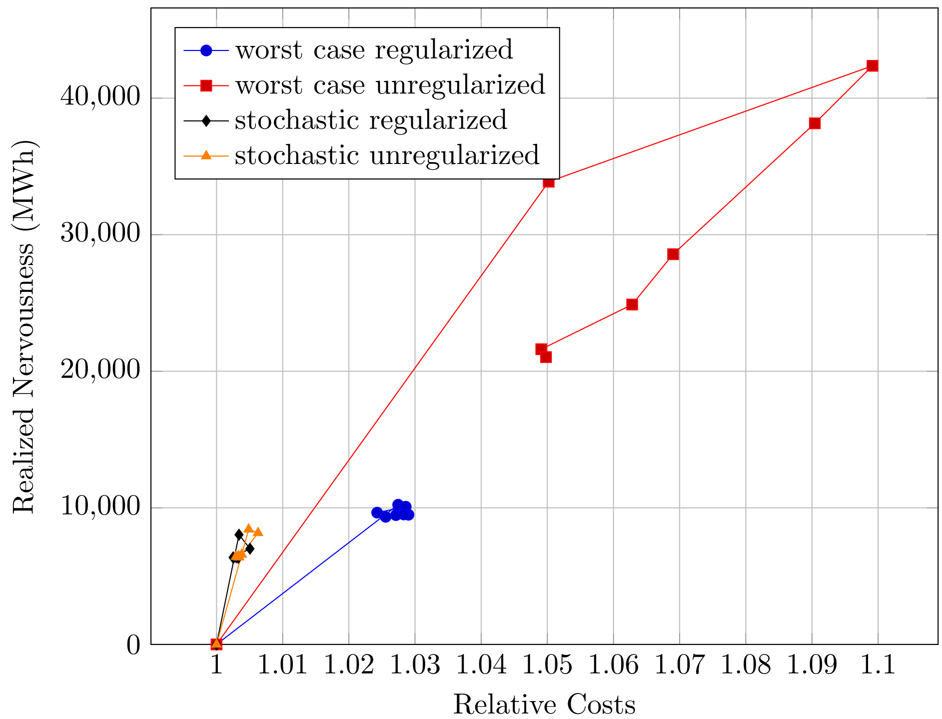
The results for finite uncertainty sets are reported in Table 4.4 and illustrated in Figure 4.7. Analogously, the results for Γ uncertainty sets are reported in Table 4.5 and illustrated in Figure 4.8.

Table 4.4: Solutions for anticipative affine rules with finite uncertainty sets.

			$\lvert D\rvert - 1$							
			0	10	20	30	40	50	60	70
worst-case obj.	reg.	viol.	3802	2569	1681	1208	969	824	719	686
		nerv.	0	10229	10084	9650	9342	9467	9495	9516
		costs	2675	2748	2752	2740	2743	2748	2753	2751
		time	15	2081	9166	24303	39842	55781	71732	95580
	unreg.	viol.	3802	11715	18043	8109	4443	3289	1942	1920
		nerv.	0	33881	42366	38156	28571	24887	21616	21039
		costs	2675	2809	2940	2917	2860	2843	2806	2808
		time	15	1256	2153	11539	28265	43610	57904	74760
stochastic obj.	reg.	viol.	3802	2013	906	237	202	166	157	136
		nerv.	0	8035	7010	6374	6336	6377	6350	6378
		costs	2675	2684	2689	2682	2682	2683	2684	2684
		time	15	2277	12509	22498	31936	40760	51866	66478
	unreg.	viol.	3802	2740	1889	365	229	202	188	147
		nerv.	0	8426	8168	6591	6474	6425	6415	6379
		costs	2675	2688	2692	2685	2684	2683	2684	2684
		time	15	1145	2829	7762	13699	21263	29390	37846



(a) Violation-versus-cost curves for anticipative affine rules with finite uncertainty sets.



(b) Nervousness-versus-cost curves for anticipative affine rules with finite uncertainty sets.

Figure 4.7: Violation-versus-cost and nervousness-versus-cost curves for causal affine rules with finite uncertainty sets.

Violations comparison for anticipative affine rules with finite uncertainty sets

Considering the violation-versus-cost curves for anticipative affine rules with finite uncertainty sets, we observe stability issues similar to the ones in the causal case. The initial unstable behavior of the unregularized worst-case approach is even more severe than in the causal setting. Again, the regularization renders the results much more stable and generally improves the results for the worst-case objective. Noticeably, all approaches but the regularized worst-case one arguably defy the concept of the "price of robustness". That is, both violation and cost values keep improving with expanding uncertainty sets. For the stochastic approaches this is the case initially, but not conclusively observable beyond $\Gamma = 3$ as the values converge rapidly. For the unregularized worst case approach this is the case only beyond the initial unstable phase, after which the values also converge, but much slower. For the regularized worst-case approach the violation values also monotonically improve, but the costs increase from $|D| - 1 = 30$ onward.

The stochastic objective approaches are clearly superior to both worst-case objective approaches and yield no unstable behavior. Between them, the regularized stochastic approach performs marginally better for small uncertainty sets and achieves slightly lower minimal violation values at almost identical costs.

Overall, the regularized stochastic approach yields the best results, and the unregularized worst-case is clearly the worst approach.

Nervousness comparison for anticipative affine rules with finite uncertainty sets

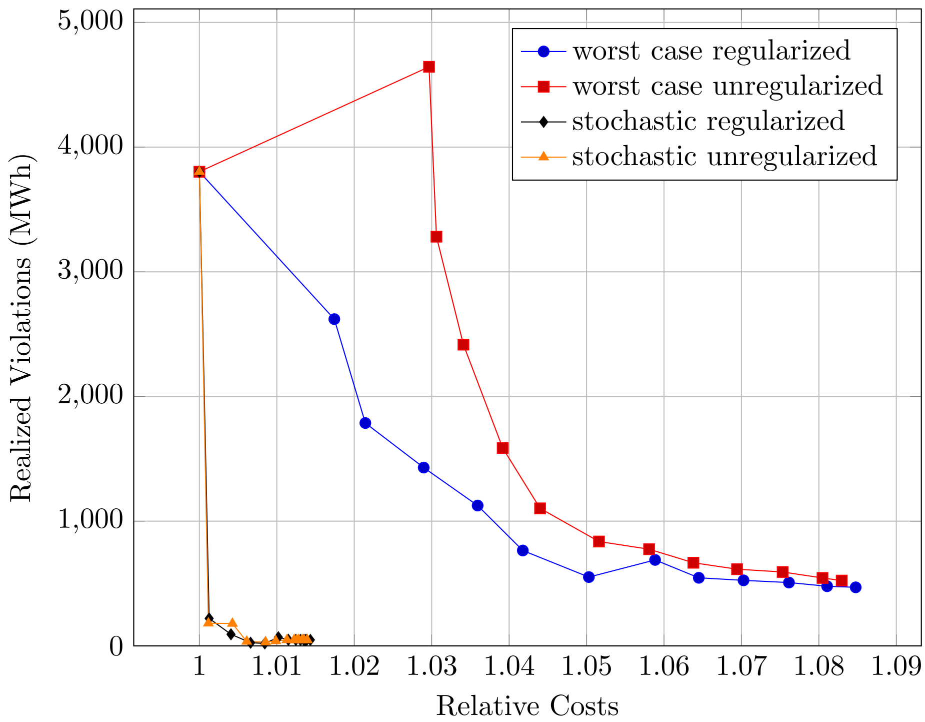
With respect to nervousness, again we observe very similar behavior to the violation results. The unregularized worst-case approach displays the same extreme values and behavior for nervousness as for the violations. Once more, the stochastic approaches are more stable and yield better results than the worst-case ones and the regularized stochastic approach is slightly superior. Similar to the violation values, the nervousness values rapidly converge for all stable approaches.

Summary for anticipative affine rules with finite uncertainty sets

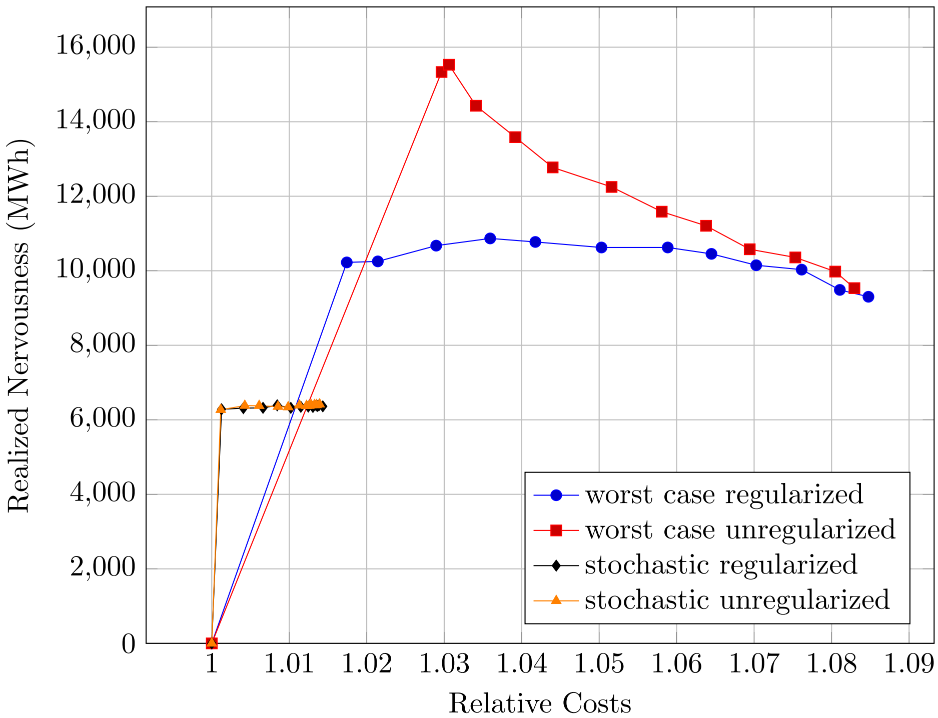
Once more, we favor the regularized stochastic approach. In this setting it is superior on all accounts, while the unregularized stochastic approach once again comes very close and generally performs very similar.

Violations comparison for anticipative affine rules with Γ uncertainty sets

We observe relatively stable and monotone behavior with respect to the violation values, as with increasing size of $|D|$ they mostly improve, while the cost values increase. Still, the regularization considerably improves the worst-case objective approaches for small $|D|$, while both worst-case objective approaches converge toward an identical violation value. The stochastic objective approaches clearly perform better than the worst-case objective approaches. Among each other they perform practically identical. That is,



(a) Violation-versus-cost curves for anticipative affine rules with Γ uncertainty sets.



(b) Nervousness-versus-cost curves for anticipative affine rules with Γ uncertainty sets.

Figure 4.8: Violation-versus-cost and nervousness-versus-cost curves for causal affine rules with Γ uncertainty sets.

Table 4.5: Results for anticipative rules with Γ uncertainty sets.

			Γ												
			0	1	2	3	4	5	6	7	8	9	10	11	12
worst-case obj.	reg.	viol.	3802	2620	1787	1430	1125	765	552	689	546	525	508	478	469
		nerv.	0	10224	10252	10674	10868	10773	10625	10625	10454	10148	10030	9486	9302
		costs	2675	2722	2732	2753	2771	2787	2810	2832	2848	2863	2879	2892	2902
		time	15	8599	15394	16475	14190	12641	12893	13083	13493	15134	14869	15722	16920
	unreg.	viol.	3802	4644	3282	2416	1587	1103	837	775	667	615	592	545	523
		nerv.	0	15335	15530	14429	13586	12774	12250	11586	11207	10577	10358	9978	9534
		costs	2675	2754	2757	2766	2780	2793	2813	2830	2846	2861	2877	2890	2897
		time	15	9144	14116	13993	11517	10671	10852	12092	14059	14956	16067	16907	17309
stochastic obj.	reg.	viol.	3802	220	93	25	20	69	49	48	48	48	48	48	48
		nerv.	0	6283	6313	6321	6389	6320	6348	6356	6345	6388	6372	6396	6362
		costs	2675	2678	2686	2693	2698	2702	2706	2708	2710	2711	2712	2712	2713
		time	15	1108	1480	1615	1614	1960	2235	2210	2237	2286	2408	2454	2501
	unreg.	viol.	3802	181	179	35	30	40	48	48	48	48	48	48	48
		nerv.	0	6270	6381	6377	6353	6341	6390	6375	6379	6399	6398	6399	6417
		costs	2675	2678	2686	2691	2698	2701	2705	2708	2709	2709	2710	2711	2712
		time	15	927	1156	1457	1597	1747	1917	2158	2170	2246	2331	2318	2270

with expanding Γ uncertainty sets, they very quickly converge with respect to violation values, while the costs keep on increasing. Thus, bigger uncertainty sets yield strictly worse results with respect to violations and costs (and nervousness as shown later on). The worst-case objective approaches display similar, but more moderate behavior.

Nervousness comparison for anticipative affine rules with Γ uncertainty sets

Considering nervousness, as expected, the unregularized worst-case objective approach starts off with the largest values, which steadily decrease and converge to the values of the regularized worst-case objective approach. The regularized worst-case objective approach itself dominates the unregularized one. It slightly worsens in the beginning, but improves beyond $\Gamma = 4$.

Again, both stochastic objective approaches exhibit almost identical results. They immediately converge and clearly dominate the worst-case ones.

Summary for anticipative affine rules with Γ uncertainty sets

As the stochastic approaches are superior to the worst-case ones and exhibit identical results we are indifferent about them. Again, we remark that, with respect to realized robustness, they yield the best results for small values of Γ. In the following, we consider the regularized stochastic approaches as our preferred one, as thus the regularized stochastic approach is chosen for each setting.

4.8.5 Comparison of all approaches

In this subsection, we compare the arguably best approaches for both finite and Γ uncertainty sets for the causal and the anticipative affine rules, as illustrated in Figure 4.9. As argued before, we always identify the regularized stochastic objective approach as the best one, as we prioritize low violation values. We do not consider static solutions, as they are already strongly dominated among the causal approaches. Still, their advantage of not suffering from nervousness remains.

First we consider the violation values. As expected, the anticipative approaches have a clear advantage over the causal approaches and exhibit much lower violation values at much lower costs. Among the anticipative approaches, the Γ uncertainty sets produce slightly lower violation values, albeit at noticeably higher costs. Interestingly, the causal Γ uncertainty set approach yields minimum violation values at least comparable to the anticipative ones. The causal finite uncertainty set approach exhibits the highest violation values by far, but at severely lower cost compared to the Γ uncertainty set approaches, as discussed before.

Considering nervousness, the only approach that stands out is the causal Γ one, as for the considered uncertainty set sizes, it yields the lowest values. Still, it appears to converge to a similar value as achieved by the other approaches. The other approaches mostly remain the same level. Only the anticipative finite approach starts off with quite high values for very small uncertainty set sizes. While similar behavior could be argued for the causal finite approach, the observable volatility is too weak to be considered conclusive.

In summary, we observe clearly superior violation and cost values for the anticipative approaches. Remarkably, the causal Γ approach results come quite close with respect to violations and even excel with respect to nervousness, but at much higher costs.

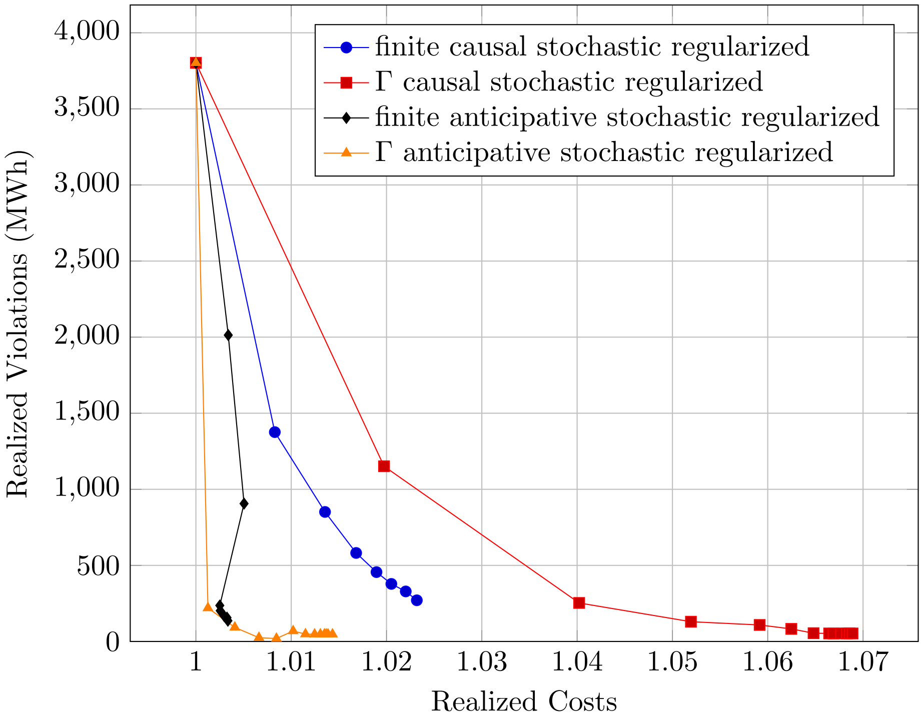
Overall, the anticipative Γ uncertainty set approach with stochastic objective yields the best results at very low costs. Once more, we stress that, with respect to realized robustness, it performs best when a small Γ value, i.e. $\Gamma = 3$, is employed.

4.8.6 A note on computation times

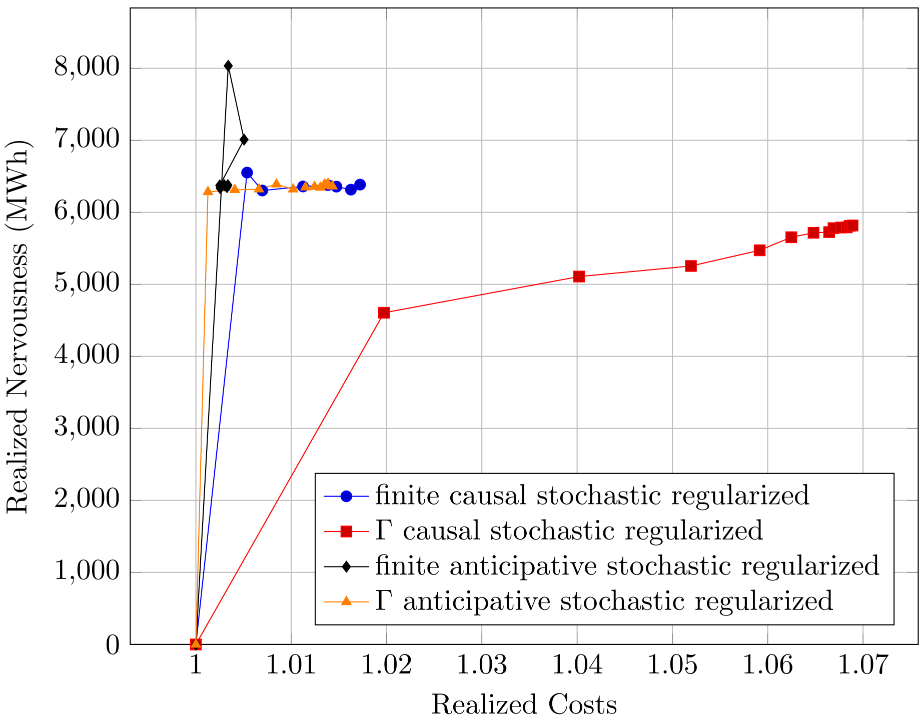
Although the focus of our computational study does not lie on computation times, but on the quality of the solutions in terms of realized robustness, we briefly analyze the behavior of the former.

In the following, we do not further consider the standard static approaches, as the results were already studied in the context of the reduction, which renders them practically constant. Furthermore, the computation times of the standard static approaches are negligibly small compared to the adaptive approaches. The alternative static approach will be covered last.

Considering all adaptive approaches, we observe that, with increasing uncertainty set size, the progression of the corresponding computation times is much more distinctive for finite uncertainty sets. Here, the values monotonically increase with the uncertainty set size in all cases. For Γ uncertainty sets, a bigger Γ value, which induces a lager



(a) Comparison of causal violation-versus-cost curves



(b) Comparison of causal nervousness-versus-cost curves

Figure 4.9: Comparison of violation-versus-cost and nervousness-versus-cost curves for general approaches.

uncertainty set, does not necessarily imply higher computation times. This behavior meets our intuition, as for lager finite uncertainty sets the corresponding model ($\mathrm{L}'_{\mathrm{aff}}$) substantially grows in size, whereas the size of ($\mathrm{L}^{\Gamma}_{\mathrm{aff}}$) does not depend on Γ (beside $\Gamma = 0$, which induces the nominal problem for the forecast).

Between the two kinds of considered uncertainty sets (albeit the values of $|D|$ and Γ not being directly comparable) we observe that the Γ uncertainty set models are solved much quicker than the finite uncertainty set models for big values of $|D|$ and Γ. Again, this is most likely due to the fact that, with a finite uncertainty set D, we replicate each constraint that contains the heat demand d exactly $|D|$ times, whereas with Γ-robustness we employ the compact reformulation ($\mathrm{L}^{\Gamma}_{\mathrm{aff}}$).

When comparing worst-case objectives to stochastic objectives, the stochastic ones consistently yield optimal solutions substantially faster, especially for Γ uncertainty sets.

In general, the computation times with anticipative rules are larger than those with causal ones. We affiliate this behavior with the larger models that are solved in the anticipative case. That is, with anticipative rules we introduce almost twice as many coefficients $\xi_{j,t}$ (i.e., variables), as compared to the causal case.

With respect to regularization we observe a strong impact for finite uncertainty set, severely increasing the computation times. For the Γ uncertainty sets the impact is much weaker and overall inconclusive, with a slight trend to decrease the computation times.

Finally, the alternative static approach is faster than all adaptive approaches based on the same considered finite uncertainty sets. Remarkably, in the long run the increase in computation time is much more rapid for the alternative static approach. I.e., the times at least double for each step (10 scenarios) in uncertainty set size, with nearly constant rates. While for all adaptive approaches with finite uncertainty sets the initial increase in computation time from $|D| = 11$ to $|D| = 21$ is much more severe than that of the alternative static approach, the rates quickly decrease well below that of the latter. Hence, we expect the alternative approach to be much more time intensive when employing bigger uncertainty sets than our data allowed for. The lower computation times for the alternative static approach can be attributed to the original increase in complexity for the adaptive approaches when introducing the affine rules. Consequently, when comparing the alternative static approach with the adaptive approaches with Γ uncertainty sets, the latter initially are more time intensive but are quickly and clearly overtaken by the former.

4.9 Concluding remarks

In this chapter, we studied a variant of the single-item lot-sizing problem with storage losses, subject to demand uncertainty, stemming form a combined heat and power application. Here, we employed concepts of robust optimization and modified them with tools from the fields of machine learning and stochastic optimization. I.e., we considered two types of uncertainty sets, finite ones and Γ ones, which we constructed based on a given forecast and historical forecast and realization data. The developed mixed integer linear programming models for the application are based on static solutions and adaptive solutions via causal affine rules. Furthermore, adaptive solutions via anticipative affine rules are considered for theoretical insights, albeit not being viable for the given setting in practice.

To evaluate the quality of the achieved solutions we define violation, cost and nervousness measures with respect to the actual realization in the form of realized robustness. Using this, the models are tested in a computational study based on real life historical data.

Comparing the basic robust models to the model of just optimizing with respect to the given forecast, we observe severe improvements for sufficiently large uncertainty sets. Between the static and the adaptive solutions, the adaptive ones clearly outperform the static ones.

For small finite uncertainty sets the solutions display extremely unstable behavior. To tackle this issue, we propose a regularization approach by limiting the affine factors ξ. This regularization proves very effective with respect to stability, and generally has a positive influence on the results.

Additionally, we explore the concept of stochastic objectives, motivated by the limitations of (with respect to costs) purely risk averse strategies in an economic environment. Incorporating these clearly improves all models with respect to all values of the realized robustness. We remark, that our practice oriented approach of measuring the sum of realized cost, which is equivalent to measuring the average realized costs, inherently favors approaches employing expected-value objectives. This is not the case for the sum of violations or nervousness, as all approaches but the alternative static one demand strict robustness in the constraints. Still, the expected-value objectives consistently strictly dominate with respect to minimum achieved violations and weakly dominate with respect to minimum achieved nervousness.

Altogether, the models in which both regularization and stochastic objectives are employed, consistently perform best from our perspective. Regarding the two studied uncertainty sets, the Γ ones are clearly superior. Consequently the regularized Γ uncertainty set approach with stochastic objectives is identified as the best causal one. We remark that the regularized finite uncertainty set approach with stochastic objectives, albeit being dominated, performs sufficiently good to be considered in practice, on accounts of being more intuitive and thus more practitioner friendly.

When allowing the solutions to be acausal, in the form of anticipative affine rules, as expected, we observe substantial improvements. Once more, the models with regularization and stochastic objectives consistently perform best.

In conclusion, we proposed static and robust mixed integer programming models based both on user-friendly finite uncertainty sets and better performing Γ uncertainty sets. All models were clearly improved by introducing regularization constraints and substituting the worst-case objectives by stochastic ones.

Conclusion

While we did already formulate concluding remarks in each individual chapter, here we aim to give a more comprehensive final conclusion in a broader context. In particular, we want to elaborate how the theoretical models and results originated from the practice-driven application and what fields of interest remain unexplored.

This thesis was motivated by a problem of optimally controlling combined heat and power plants with a storage unit, under heat demand uncertainty. We formulated the corresponding nominal problem in the form of a mixed integer linear program. Said program was reduced to an equivalent single-item lot-sizing problem with storage deterioration, production bounds and storage bounds. Based on this model, we formulated static robust versions to tackle heat demand uncertainty. Furthermore, we introduced an alternative static approach and adaptive approaches via affine rules.

On the theoretical side, we explored the complexity and structure of the above generalization of the single-item lot-sizing problem. We showed that, for the standard static solution approach, the corresponding robust single-item lot-sizing problem can be reduced to an auxiliary nominal single-item lot-sizing problem under reasonable assumptions. Hence, the intuitive approach of first studying the nominal problem allows for a direct transfer of results to a (static) robust case. Thus, we first conducted a structural and complexity analysis for the nominal problem.

Regarding the complexity, we covered different degrees of flexibility in the parameters characterizing, each, the storage deterioration, production bounds and storage bounds. I.e., we aimed to characterize all resulting problem variants as either polynomially solvable or $\mathcal{NP}$-hard. Here, we succeeded to identify the problem with static production bounds and time dependent conservation rates as $\mathcal{NP}$-hard. Also, we characterized the problems with time-dependent storage bounds and time-dependent deterioration rates, or time-dependent storage bounds and stationary production bounds, as polynomially solvable. Due to the aforementioned reduction, these results transfer to the static robust problem variant defined in this work.

With respect to the structural analysis of the considered problems, we first characterized dominant sets of solutions in the above complexity reductions. Also, the presented reduction from a static robust single-item lot-sizing instance to a nominal single-item lot-sizing instance allows for structural insights. Similarly, we formulated a polynomial reduction to eliminate non trivial lower production bounds while preserving all relevant properties. Furthermore, we exceeded to extend an ideal formulation of the single-item lot-sizing problem to include storage deterioration. Analogously, we extended a polynomial sized extended formulation in the same context.

On the practical side, we defined a method to evaluate the quality of the obtained solutions with respect to the actual realization. This we did in the form of realized robustness. Thus, we were also able to test and compare different robustness approaches. Here, we observed severely unstable behavior for finite uncertainty set for the adaptive approaches. We affiliated this issue to regression problems with an insufficient scope of data, see Section 4.4.1. To tackle this phenomenon, we introduced a regularization approach, which proves quite successful without compromising the solution quality in the long run. As an alternative, we established a robust-stochastic hybrid approach, in which the standard worst-case objective is replaced by a stochastic one. This stochastic objective can be formulated as a linear term under some standard assumptions. It succeeded both in preventing unstable behavior and in improving the general solution quality with respect to realized robustness.

Outlook

Although we present a broad range of results, there are still open questions, especially concerning the structure of the considered problems. As discussed in Section 2.4, the single-item lot-sizing variants with the combination of stationary deterioration rates and stationary production bounds could not yet be characterized to be polynomially solvable or $\mathcal{NP}$-hard. Furthermore, we still struggle to tackle the complexity of the presented adaptive robust approaches. This is due to the lack of structural inside regarding optimal affine rules, because we are not aware of exploitable dominant sets of solutions. Regarding the alternative static approach, we face a similar issue, see Section 4.3.

With respect to polyhedral results, we still lack approaches for single-item lot-sizing problems with bounded storage or bounded production. As discussed in Section 3.6, we are of the opinion that a promising first step to tackle this problem, is to determine a formula for the dimension of the single-item lot-sizing polyhedron in the bounded cases.

Furthermore, several model extensions could be studied, especially since all model instances considered in this work were solvable to optimality in reasonable time.

Notably, we would be interested in exploring the concept of reserve energy. That is, a power market mechanism where, from a consumer perspective, any digression in ones power balance can instantly be resolved, but at a high price. From the supplier perspective, power production sides can offer capacities by which they increase or decrease their production instantly when demanded. In our models we see potential to include either the consumer or supplier role.

On the consumer side, the inclusion of reserve energy provides another adaptive component in our models, which could greatly increase the model-flexibility, even if reserve energy is quite costly. Here, when canonically introducing a reserve energy variable for each realization, we face the issue of it contributing to the otherwise linear objective function in absolute for each period. This, we did not yet manage to reformulate to a properly solvable model.

On the supplier side, the combination of multiple of the presented systems into a virtual power plant could enable us to effectively accumulate capacities to offer to the power market. Here, the main issue is to account for the actual reserve energy demand as a new non-trivial source of uncertainty. This model, including a combination of multiple interlinked systems, could prove much harder to optimize.

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