Parabolic Induction for Hecke Algebras

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7 Teile dieser Arbeit wurden zuvor veröffentlicht und zwar in:

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Contents

Introduction

One of the most natural examples of a group is the symmetric group $\mathfrak{S}_n$ consisting of bijections on the set $\{1, \dots, n\}$. Its representation theory is the study of the module categories of the group algebras $k[\mathfrak{S}_n]$, where k is a field. Despite the group's easy definition the representation theory is surprisingly complex. If the characteristic of k is zero or greater than n, the theory has been well-understood for over a century. However, if the characteristic of k is a positive prime smaller than n, the picture is much less complete.
One key question is the following: Evidently, the group algebra $k[\mathfrak{S}_{n-1}]$ embeds into $k[\mathfrak{S}_n]$ and this gives rise to induction and restriction functors between their respective module categories. What is the structure of a module obtained via induction or restriction? The answer seems to be rather complicated and to date the problem has only been partially solved. One key aspect was studied by Djoković and Malzan in 1974, when they were able to show that the induced modules are always reducible, cf. [DM74].
Perhaps the most important contribution to the investigation of these induced modules was Kleshchev's work in the 1990's, in which he was able, amongst other things, to fully describe the socle and the block decomposition of the modules obtained by inducing or restricting irreducible modules, cf. [Kle95a; Kle95b; Kle96; Kle98]. Together with Brundan he was also able to determine many of the composition factors of induced modules and their multiplicities, cf. [BK00].

The above relation between the algebras $k[\mathfrak{S}_{n-1}]$ and $k[\mathfrak{S}_n]$ and the study of the corresponding induction functor is a special case of a more general setting and more general questions:
Symmetric groups are examples of *finite Coxeter groups*. These can be realised as *real reflection groups*, that is as subgroups of general linear groups of finite-dimensional real vector spaces generated by reflections at hyperplanes. Over *complex* vector spaces such reflections have been generalised to *pseudo-reflections* and groups generated by pseudo-reflections are known as *complex reflection groups*. Both Coxeter groups and complex reflection groups contain special subgroups known as *parabolic subgroups* which are Coxeter groups or complex reflection groups, respectively, in their own right. These subgroups play a major role for example in the representation theory of the groups.
If W is a Coxeter group or a complex reflection group and k a field we can associate to W a k-algebra H that is a deformation of the group algebra $k[W]$. This is known as an *Iwahori-Hecke algebra*, if W is a Coxeter group, and a *cylcotomic Hecke algebra*, if W is a complex reflection group. The former are generalisations of certain endomorphism algebras occurring naturally in the representation theory of finite groups of Lie type, cf. [Iwa64]. The latter have been defined by Broué-Malle and, in a number of cases, by Ariki-Koike, and Ariki independently, cf. [BM93; AK94; Ari95]. We will mostly be concerned with *Ariki-Koike algebras* as defined by

Ariki-Koike, which are cylcotomic Hecke algebras of complex reflection groups isomorphic to the wreath product $C_r \wr \mathfrak{S}_n$ for a cyclic group C_r of order r and an integer n.
For this introduction we refer to H as a Hecke algebra of W.

If W' is a parabolic subgroup of W, then the corresponding Hecke algebra H' of W' embeds naturally into H and this subalgebra of H is a called a *parabolic subalgebra*.
In this thesis we are concerned with the representation theory of Hecke algebras and in particular we are interested in the *parabolic induction functor*

$$\mathrm{Ind}_{H'}^{H} : \text{mod-}H' \to \text{mod-}H,$$

arising from the embedding $H' \hookrightarrow H$, where mod-H' and mod-H are the categories of finitely generated right H'-modules and H-modules, respectively.
One key question considered in this thesis is whether the induction $\mathrm{Ind}_{H'}^{H}(M)$ of a non-zero H'-module M can be irreducible. This question is primarily motivated by the occurrence of Iwahori-Hecke algebras as endomorphism algebras of modules appearing in the representation theory of finite groups of Lie type. The parabolic induction functor $\mathrm{Ind}_{H'}^{H}$ then corresponds to Harish-Chandra induction of modules, cf. [Klu19] for the precise relationship. If H is an Iwahori-Hecke algebra or an Ariki-Koike algebra, we will show that the answer to this question is negative, just as Djoković-Malzan proved for the group algebra $k[\mathfrak{S}_n]$. Additionally, we extend this result to group algebras of an infinite family of complex reflection groups.
Once one has shown that induced modules are irreducible the next logical step is to describe their irreducible composition factors as precisely as possible. In particular we are interested in a lower bound for the number of factors, counting multiplicities.
For certain parabolic subalgebras in Ariki-Koike algebras and a number of infinite families of Iwahori-Hecke algebras this problem can be tackled via a combinatorial approach building on the categorification results by Ariki, Grojnowski, and Vazirani, cf. [Ari96; Gro99; GV01; Vaz02]. They showed that induction and restriction give rise to a module of a certain Lie algebra. Essential properties of the action on this module can be described in an entirely combinatorial manner in terms of its so-called crystal graph. This will allow us to obtain a lower bound on the number of constituents of induced modules. Later, we use Clifford theory to relate these results to induction functors of larger families of Iwahori-Hecke algebras and cyclotomic Hecke algebras.

One technique that has been essential in much of the research on Hecke algebras, particularly for computational aspects, is the concept of *generic algebras and specialisation*:
The Hecke algebra H is defined depending on a number of parameters. One can define *generic Hecke algebras* of W by choosing the parameters as algebraically independent indeterminates. The algebra H is then recovered by a change of base rings. In a suitable setting this gives rise to a *decomposition map* between the Grothendieck groups of the corresponding module categories, which once computed allows us to deduce information on H from the generic algebra via this map. This is particularly powerful since the representation theory of the generic Hecke algebra is essentially the ordinary representation theory of W, which is comparatively well-understood. We will explain the theory of specialisation, decomposition, and its connection to induction.
Finally, we will compute previously unknown decomposition maps for a number of Hecke algebras.

Before explaining the thesis' structure in detail we summarise its main results.
Theorems 3.1.1, 3.2.7, 3.2.8, and 3.3.1 are similar statements for a numbers of different structures: We prove that parabolic induction from proper parabolic subgroups or subalgebras of arbitrary Iwahori-Hecke algebras, non-exceptional complex reflection groups, and Ariki-Koike algebras will always yield reducible modules, if the underlying field is large enough.
Our second main result is Theorem 5.3.34, which concerns parabolic induction for Ariki-Koike algebras and gives a lower bound on the number of constituents of parabolically induced modules. To state it, let $\mathbf{H}_n := \mathbf{H}_{r,n}(q, Q_1, \ldots, Q_r)$ be an Ariki-Koike algebra for positive integers r, n and invertible parameters $q, Q_1, \ldots, Q_r$, see Definition 2.4.1. Then $\mathbf{H}_{n-1}$ is a parabolic subalgebra of $\mathbf{H}_n$. Theorem 5.3.34 now states that if $0 \neq M$ is an $\mathbf{H}_{n-1}$-module the induced module $\mathrm{Ind}_{\mathbf{H}_{n-1}}^{\mathbf{H}_n}(M)$ has at least $r + t$ irreducible constituents, where t is the number of q-connected components of $\{Q_1, \ldots, Q_r\}$, see Definition 5.2.1.
Finally, we compute decomposition numbers of Iwahori-Hecke algebras in positive characteristic. More precisely, we compute the decomposition numbers of Iwahori-Hecke algebras of type F_4, E_6, and E_7 in all characteristics which are bad for these types, see Theorem 9.1.17 and Appendix C. Furthermore, we compute the decomposition numbers of algebras of type H_3 and H_4 in an analogue of bad characteristic, see Theorem 9.2.1 and Appendix D. Our last main result is the computation of decomposition numbers of algebras of dihedral type in an analogue of bad characteristic, see Theorems 9.3.6 and 9.3.7.

This thesis is structured as follows:
We begin by explaining conventions and general results from representation theory to establish the language used throughout.
In Chapter 2 we give definitions and essential properties of the reflection groups discussed in this thesis, their Hecke algebras, and the corresponding parabolic substructures.
Having introduced the objects we study in this thesis we show that parabolically induced modules of large families of algebras are always reducible in Chapter 3.
In the rather short Chapter 4 we introduce combinatorial objects which are essential to the subsequent study of induction in Ariki-Koike algebras.
Chapter 5 is the very heart of this thesis. Here, we study certain parabolic induction functors in Ariki-Koike algebras using combinatorial arguments and explicit computation. We obtain a new lower bound on the number of constituents of parabolically induced modules. The chapter closes with a section on the connection to the representation theory of algebras known as cyclotomic rational Cherednik algebras.

In Chapter 6 we present the basics of specialisation, decomposition, and their relation to induction for arbitrary algebras and subalgebras.
The following Chapter 7 deals with specialisation in the case of Iwahori-Hecke algebras. In particular, the question of when an Iwahori-Hecke algebra is semisimple and how this allows us to study parabolic induction is treated. In the last section of this chapter we try to comprehensively gather known results on the decomposition maps of one-parameter Iwahori-Hecke algebras.
To date, the task of determining all decomposition maps has not yet been completed. We fill in some of the missing cases by explicit computation in Chapter 9.
Finally, Chapter 8 is devoted to the study of Clifford theory of Iwahori-Hecke algebras and

Ariki-Koike algebras. In particular, we investigate how Clifford theory can be used to relate certain decomposition and induction maps.

The thesis is supplemented by appendices:
In Appendix A we explain why seemingly different definitions appearing in the literature yield equivalent objects, allowing us to combine results from various sources for our approach in Chapter 5.
Appendix B contains a brief introduction to the theory of Specht modules of Iwahori-Hecke algebras and Ariki-Koike algebras. Furthermore, the proof of Lemma 5.3.25 can be found here.
The decomposition numbers whose computation is explained in Chapter 9 can be found in the Appendices C and D.

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Chapter 1.

Notation, Conventions, and Preliminaries

We begin by fixing certain notations and conventions, and by defining the necessary language to talk about some aspects of the representation theory of algebras. All rings and algebras in this thesis will be unitary and associative. When we refer to an object as a "ring" we will usually additionally assume that it is commutative. Even though all algebras are rings, we will not assume commutativity when specifically referring to an object as an "algebra".

Throughout this chapter let A be a ring and $\mathfrak{A}$ an A-algebra that is free of finite rank over A. We denote by mod-$\mathfrak{A}$ the category of finitely generated right modules of $\mathfrak{A}$. Its objects are all finitely generated right $\mathfrak{A}$-modules and its morphisms are $\mathfrak{A}$-module homomorphisms.
By an $\mathfrak{A}$-*module* we mean an element of mod-$\mathfrak{A}$ unless explicitly stated otherwise.

1.1. The Grothendieck Group

We follow [CR81, §16B] to introduce Grothendieck groups.

Definition 1.1.1 The *Grothendieck group* $R_0(\mathfrak{A})$ *of* $\mathfrak{A}$ is the Grothendieck group of the category mod-$\mathfrak{A}$. It is an abelian group defined in terms of generators and relations:
A generating set is given by the symbols $\{[M] \mid M \in \text{mod-}\mathfrak{A}\}$.
The relations are given by short exact sequences: If $0 \to L \to M \to N \to 0$ is a short exact sequence in mod-$\mathfrak{A}$, then $[M] - [L] - [N] = 0$. We call $[M] \in R_0(\mathfrak{A})$ the *class of M in the Grothendieck group*.

Definition 1.1.2 By $\mathrm{Irr}(\mathfrak{A})$ we denote a set of representatives of the isomorphism classes of irreducible $\mathfrak{A}$-modules.

Remark 1.1.3 If $V \cong V'$ are isomorphic $\mathfrak{A}$-modules, then $[V] = [V']$ in the Grothendieck group.

Lemma 1.1.4 ([CR81, Proposition 16.6]) *Assume* $\mathfrak{A}$ *is right artinian. Then the Grothendieck group* $R_0(\mathfrak{A})$ *is a free abelian group with a basis given by* $\{[V] \mid V \in \mathrm{Irr}(\mathfrak{A})\}$.

For two $\mathfrak{A}$-modules M and N we have $[M] = [N]$ if and only if M and N have the same irreducible composition factors, counting multiplicities. More precisely, if we denote by a_V the multiplicity of $V \in \operatorname{Irr}(\mathfrak{A})$ as a composition factor of M, then

$$[M] = \sum_{V \in \operatorname{Irr}(\mathfrak{A})} a_v \, [V].$$

We call the classes $[V]$ with $a_v \neq 0$ the composition factors of $[M]$.

Definition 1.1.5 Suppose $\mathfrak{A}$ is right artinian. Then we denote by $R_0^+(\mathfrak{A})$ the monoid $\bigoplus_{V \in \operatorname{Irr}(\mathfrak{A})} \mathbb{Z}_{\geq 0}[V]$.

Note that $\mathfrak{A}$ is right artinian if A is a field, as $\mathfrak{A}$ has finite rank over A.

Remark 1.1.6 Note that not every element of $R_0(\mathfrak{A})$ is the class of a module M, but every element can be written in the form $[M] - [M']$ for $\mathfrak{A}$-modules M and M'.
The monoid $R_0^+(\mathfrak{A})$ contains exactly the classes $[M]$, where M is an $\mathfrak{A}$-module.
If $\mathfrak{B}$ is another right artinian algebra and $\varphi : R_0^+(\mathfrak{A}) \to R_0^+(\mathfrak{B})$ is a monoid homomorphism, then φ extends to a group homomorphism $R_0(\mathfrak{A}) \to R_0(\mathfrak{B})$.

Definition 1.1.7 Suppose $\mathfrak{A}$ is right artinian and $\mathfrak{B}$ is another right artinian algebra over some ring. If $\varphi : R_0^+(\mathfrak{A}) \to R_0^+(\mathfrak{B})$ (or $\varphi : R_0(\mathfrak{A}) \to R_0(\mathfrak{B})$) is an isomorphism sending classes of irreducible modules to classes of irreducible modules we say that φ is *trivial*.

Example 1.1.8 Suppose A is a field. Then $\operatorname{Irr}(\mathfrak{A})$ can be identified with $\operatorname{Irr}(\mathfrak{A} / \operatorname{Rad}(\mathfrak{A}))$, where $\operatorname{Rad}(\mathfrak{A})$ denotes the Jacobson radical of $\mathfrak{A}$. This identification yields trivial isomorphism

$$R_0^+(\mathfrak{A}) \to R_0^+(A/\operatorname{Rad}(\mathfrak{A}))$$

and

$$R_0(\mathfrak{A}) \to R_0(A/\operatorname{Rad}(\mathfrak{A})).$$

Lemma 1.1.9 *Suppose $\mathfrak{B}$ is an algebra and $F : \text{mod-}\mathfrak{A} \to \text{mod-}\mathfrak{B}$ is an exact functor. Then*

$$R_0(\mathfrak{A}) \to R_0(\mathfrak{B}) : [M] \mapsto [F(M)]$$

is a well-defined homomorphism of abelian groups. We will often abuse notation and denote this map, too, by F.

1.2. Split Algebras

Throughout this subsection assume that A is a field.

Definition 1.2.1 Let M be an irreducible $\mathfrak{A}$-module. By Schur's lemma the $\mathfrak{A}$-endomorphism ring of M is an A-division algebra. If this is equal to A, then M is called *absolutely irreducible.* If all irreducible $\mathfrak{A}$-modules are absolutely irreducible we say that $\mathfrak{A}$ is *split*. A field extension $A \leq E$ is is called a *splitting field of* $\mathfrak{A}$ if $\mathfrak{A} \otimes_A E$ is a split E-algebra.

Often, the following definition of absolute irreducibility is given instead of the above.

Lemma 1.2.2 ([CR81, Theorem 3.43]) *An irreducible $\mathfrak{A}$-module M is absolutely irreducible if and only if $M \otimes_A E$ is an irreducible $\mathfrak{A} \otimes_A E$-module for all field extensions $A \leq E$.*

Lemma 1.2.3 (cf. [CR81, Proposition 7.13]) *There exists a field extension $A \leq E$ with $\dim_A(E) < \infty$ such that E is a splitting field of $\mathfrak{A}$.*

Proposition 1.2.4 (cf. [GP00, Exercise 7.4]) *Suppose $\mathfrak{A}$ is a split A-algebra. For M in $\mathrm{Irr}(\mathfrak{A})$ denote by χ_M the character afforded by M. Then $\{\chi_M \mid M \in \mathrm{Irr}(\mathfrak{A})\}$ is an A-linearly independent set.*

To prove Proposition 1.2.4 we follow the hints in [GP00, Exercise 7.4]. Let us first prove a lemma.

Lemma 1.2.5 *Suppose $\mathfrak{A}$ is a split A-algebra. Then for every irreducible $\mathfrak{A}$-module M there exists an idempotent e_M in $\mathfrak{A}$ such that $\dim_A(Me_M) = 1$ and $\dim_A(M'e_M = 0)$ for all irreducible $\mathfrak{A}$-modules M' that are not isomorphic to M.*

Proof If $\mathfrak{A}$ is semisimple, then the existence of e_M follows directly from Wedderburn's structure theorem. In this case $\mathfrak{A}$ is isomorphic to the direct sum of full matrix algebras with entries in A (as $\mathfrak{A}$ is split), where the direct summands correspond to the isomorphism classes of irreducible $\mathfrak{A}$-modules. For e_M we only look at the direct summand corresponding to M and there we take the matrix having 1 in the top-left position and 0 elsewhere. This element has the desired properties.
Now suppose $\mathfrak{A}$ is not semisimple. Then $\widehat{\mathfrak{A}} := \mathfrak{A} \,/\, \mathrm{Rad}\,\mathfrak{A}$ is semisimple, where $\mathrm{Rad}\,\mathfrak{A}$ is the Jacobson radical of $\mathfrak{A}$. Recall that we can identify $\mathrm{Irr}(\mathfrak{A})$ with $\mathrm{Irr}(\widehat{\mathfrak{A}})$ by the definition of the Jacobson radical, as it acts as 0 on irreducibles. Under this identification an a in $\mathfrak{A}$ acts as $a + \mathrm{Rad}(\mathfrak{A})$ on the modules in $\mathrm{Irr}(\widehat{A})$.
Now let M be in $\mathrm{Irr}(\widehat{\mathfrak{A}})$. Then by what we have just shown there exists an idempotent $\widehat{e}_M$ in $\widehat{A}$ such that $\dim_A(M\widehat{e}_M) = 1$ and $\dim_A(M'\widehat{e}_M = 0)$ for $M' \in \mathrm{Irr}(\widehat{\mathfrak{A}})$ and $M' \not\cong M$, as $\widehat{\mathfrak{A}}$ is split semisimple. By [CR81, Theorem 6.7, Theorem 6.8] we can lift $\widehat{e}_M$ to an idempotent of $\mathfrak{A}$, i.e. there exists an e_M in $\mathfrak{A}$ such that $e_M + \mathrm{Rad}(\mathfrak{A}) = \widehat{e}_M$. Since e_M acts as $\widehat{e}_M$ on M, this concludes the proof. ∎

We finish the proof of the proposition.

Proof of Proposition 1.2.4 Let M be in $\mathrm{Irr}(\mathfrak{A})$ and e_M an idempotent of $\mathfrak{A}$ as in Lemma 1.2.5. Since e_M is an idempotent and $\dim_A(Me_M) = 1$, clearly $\chi_M(e_M) = 1$, as the only eigenvalues of e_M on M are 0 and 1. Similarly, it follows that $\chi_{M'}(e_M) = 0$ if $M' \in \mathrm{Irr}(\mathfrak{A})$ is not isomorphic to M. Hence, the characters are linearly independent. ∎

1.3. Induction and Restriction

We drop the condition that A is a field, so let A be a ring. Let $\mathfrak{a} \leq \mathfrak{A}$ be a subalgebra that is also free of finite rank over A.

Definition 1.3.1

a) For an $\mathfrak{A}$-module M denote by $\mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}(M)$ the $\mathfrak{a}$-module obtained from M by restricting to the action of the subalgebra $\mathfrak{a}$. Then

$$\mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}} : \text{mod-}\,\mathfrak{A} \to \text{mod-}\,\mathfrak{a}$$

is a functor called the *restriction functor between* mod- $\mathfrak{A}$ *and* mod- $\mathfrak{a}$.

b) For an $\mathfrak{a}$-module N denote by $\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}(N)$ the $\mathfrak{A}$-module $N \otimes_{\mathfrak{a}} \mathfrak{A}$, where $\mathfrak{A}$ is a left $\mathfrak{a}$-module by multiplication in $\mathfrak{A}$. Then

$$\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}} : \text{mod-}\,\mathfrak{a} \to \text{mod-}\,\mathfrak{A}$$

is a functor called the *induction functor between* mod- $\mathfrak{a}$ *and* mod- $\mathfrak{A}$.

Induction and restriction are examples of A-linear functors.

Definition 1.3.2 A category C is called *A-linear* if for every pair X, Y of objects of C the corresponding morphism set $\mathrm{Mor}_C(X, Y)$ is an A-module and the composition of morphisms is A-bilinear.
If C and $\mathcal{D}$ are A-linear categories, then we call a functor $F : C \to \mathcal{D}$ an *A-linear functor* if the map which it induces between morphism spaces is A-linear.

Example 1.3.3 The module categories mod- $\mathfrak{A}$ and mod- $\mathfrak{a}$ are A-linear. The restriction and induction functors $\mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}$ and $\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}$ are A-linear functors.

Adjoint pairs of A-linear functors yield isomorphisms of A-modules.

Lemma 1.3.4 *Let C and $\mathcal{D}$ be A-linear categories. Furthermore, let $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to C$ be A-linear functors such that F is left adjoint to G. If X is an object of C and Z is an object of $\mathcal{D}$, then the bijection*

$$Mor_C(X, G(Z)) \to Mor_{\mathcal{D}}(F(X),\ Z)$$

afforded by the adjointness is an isomorphism of A-modules.

Proof This follows directly from the characterisation of adjointness in [Mac98, Chapter IV, Theorem 1]. ∎

By a well-known result induction and restriction constitute a pair of adjoint functors.

Lemma 1.3.5 *The induction* $\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}$ *is left adjoint to the restriction* $\mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}$.

Proof We first observe that

$$\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A}, M) \to \mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}(M);\ \psi \mapsto \psi(1)$$

defines an isomorphism of $\mathfrak{a}$-modules. The claim now follows from the well-known adjointness of the tensor product with the covariant Hom-functor, cf. [CR81, Theorem 2.19]. ∎

With Lemma 1.3.4 we immediately get the following.

Corollary 1.3.6 *Let M be in* mod-$\mathfrak{A}$ *and N in* mod-$\mathfrak{a}$. *Then* $\mathrm{Hom}_{\mathfrak{a}}(N, \mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}(M))$ *and* $\mathrm{Hom}_{\mathfrak{A}}(\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}(N),\ M)$ *are isomorphic as A-modules.*

The associativity of tensor products shows that induction and restriction are transitive.

Lemma 1.3.7 *If $\mathfrak{a}' \leq \mathfrak{a} \leq \mathfrak{A}$ is a subalgebra of $\mathfrak{a}$, then*

$$\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}'} \cong \mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}} \circ \mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{a}'} \text{ and } \mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}'} \cong \mathrm{Res}^{\mathfrak{a}}_{\mathfrak{a}'} \circ \mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}.$$

Proposition 1.3.8

a) The restriction functor $\mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}$ is exact.

b) If $\mathfrak{A}$ is flat as a left $\mathfrak{a}$-module, then $\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}$ is exact. In particular, $\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}$ is exact if $\mathfrak{A}$ is free as a left $\mathfrak{a}$-module.

Definition 1.3.9 Assume A is a field and let M be an $\mathfrak{A}$-module. Suppose $[\mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}(M)] = \sum_V \alpha_V[V]$ in the Grothendieck group of $\mathfrak{a}$, where V runs over representatives of the isomorphism classes of irreducible $\mathfrak{a}$-modules. Then we call the integers α_V the *restriction numbers of M*.
Suppose $\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}$ is exact and let N be an $\mathfrak{a}$-module. Then we call the integers β_W with $[\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}(N)] = \sum_W \beta_W[W]$ the *induction numbers of N*, where W runs over the irreducible $\mathfrak{A}$-modules.

Let us study what happens to induction and restriction under algebra automorphisms.

Definition 1.3.10 . Suppose $\varphi : \mathfrak{A} \to \mathfrak{A}$ is an automorphism of A-algebras. For an $\mathfrak{A}$-module M we denote by M^{φ} the module conjugate by φ: As an A-module this module is equal to M, but $a \in \mathfrak{A}$ acts as $\varphi(a)$ on M^{φ}.

Proposition 1.3.11 *Let $M \in$ mod-$\mathfrak{A}$ and $N \in$ mod-$\mathfrak{a}$ both be free of finite rank over A. Let $\varphi : \mathfrak{A} \to \mathfrak{A}$ an A-algebra automorphism. Assume that $\mathfrak{A}$ is free of finite rank as a left $\mathfrak{a}$-module. Suppose that $\varphi(\mathfrak{a}) = \mathfrak{a}$. Then φ restricts to an automorphism $\psi := \varphi|_{\mathfrak{a}}$ of $\mathfrak{a}$, and the following holds.*

a) $\mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}(M^{\varphi}) \cong \left(\mathrm{Res}^{\mathfrak{A}}_{\mathfrak{a}}(M)\right)^{\psi}$ *as $\mathfrak{a}$-modules.*

b) $\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}(N^{\psi}) \cong \left(\mathrm{Ind}^{\mathfrak{A}}_{\mathfrak{a}}(N)\right)^{\varphi}$ *as $\mathfrak{A}$-modules.*

Proof Part a) follows directly from the definition, so we turn to the proof of b). Throughout this proof we set $\otimes := \otimes_{\mathfrak{a}}$.
Let U be an A-basis of N and X an $\mathfrak{a}$-basis of $\mathfrak{A}$. Since $N^{\psi} = N$ as A-modules, U is also an A-basis of N^{ψ} and as φ restricts to an automorphism of $\mathfrak{a}$, $\varphi(X)$ is an $\mathfrak{a}$-basis of $\mathfrak{A}$. Therefore,

$$\{u \otimes x \mid u \in U,\ x \in X\}$$

is an A-basis of $\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(N^{\psi})$ and

$$\{u \otimes \varphi(x) \mid u \in U,\ x \in X\}$$

is an A-basis of $\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(N)$, hence also of $\left(\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(N)\right)^{\varphi}$. Thus,

$$\beta : \mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}\left(N^{\psi}\right) \to \left(\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(N)\right)^{\varphi}; u \otimes x \mapsto u \otimes \varphi(x)$$

defines an isomorphism of A-modules. It remains to show compatibility with the action of $\mathfrak{A}$. Let $a \in \mathfrak{A}$, $u \in U$, and $x \in X$. As X is an $\mathfrak{a}$-basis of $\mathfrak{A}$, there exist $c_y \in \mathfrak{a}$ for $y \in Y$ such that $xa = \sum_{y \in X} c_y y$ in $\mathfrak{A}$. Since $\psi = \varphi|_{\mathfrak{a}}$, this implies

$$\varphi(x)\varphi(a) = \sum_{y \in X} \psi(c_y)\varphi(y). \tag{1.1}$$

We denote by . the action on an unconjugate module and by $*$ that on a conjugate module. Then we have

$$\begin{aligned}
\beta\left((u \otimes x).a\right) &= \beta\left(u \otimes \sum_{y \in X} c_y y\right) \\
&= \beta\left(\sum_{y \in X} (u * c_y \otimes y)\right) \\
&= \sum_{y \in X} u.\psi(c_y) \otimes \varphi(y) \\
&= u \otimes \sum_{y \in X} \psi(c_y)\varphi(y) \\
&\overset{(1.1)}{=} u \otimes \varphi(x)\varphi(a) \\
&= (u \otimes \varphi(x)) * a.
\end{aligned}$$

Hence, β is an isomorphism of $\mathfrak{A}$-modules. ∎

We study the behaviour of induction in tensor products of algebras.

Definition 1.3.12 Let $\mathfrak{B}$ be an A-algebra, free of finite rank over A. Then $\mathfrak{A} \otimes_A \mathfrak{B}$ becomes an A-algebra via $(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2$. For $M \in$ mod-$\mathfrak{A}$ and $N \in$ mod-$\mathfrak{B}$ we define their *outer tensor product*. This is an $\mathfrak{A} \otimes_A \mathfrak{B}$-module with underlying A-module $M \otimes_A N$ and $\mathfrak{A} \otimes_A \mathfrak{B}$-module structure given by

$$(m \otimes n)(a \otimes b) := ma \otimes nb$$

for $m \in M$, $n \in N$, and $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.

The following result is a generalisation of a well-known fact about the outer tensor products of modules of group algebras.

Proposition 1.3.13 (cf. [CR81, Lemma 10.17]) *Let $\mathfrak{b} \leq \mathfrak{B}$ be A-algebras, both free of finite rank over A. Suppose that $\mathfrak{A}$ is free of finite rank over $\mathfrak{a}$ and $\mathfrak{B}$ is free of finite rank over $\mathfrak{b}$ as left modules. Then the following holds:*

a) The tensor product $\mathfrak{A} \otimes_A \mathfrak{B}$ is a free left $\mathfrak{a} \otimes_A \mathfrak{b}$-module. More precisely, if X is an $\mathfrak{a}$-basis of $\mathfrak{A}$ and Y is a $\mathfrak{b}$-basis of $\mathfrak{B}$, then $\{x \otimes_A y \mid x \in X,\ y \in X\}$ is an $\mathfrak{a} \otimes_A \mathfrak{b}$-basis of $A \otimes_A B$.

b) Let $M \in \text{mod-}\,\mathfrak{a}$ and $N \in \text{mod-}\,\mathfrak{b}$ be modules that are both free of finite rank over A. Then there is an isomorphism of $\mathfrak{a} \otimes_A \mathfrak{b}$-modules

$$\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(M) \otimes_A \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{B}}(N) \cong \mathrm{Ind}_{\mathfrak{a} \otimes_A \mathfrak{b}}^{\mathfrak{A} \otimes_A \mathfrak{B}}(M \otimes_A N).$$

We say that induction commutes with outer tensor products.

Proof The freeness of $\mathfrak{A} \otimes_R \mathfrak{B}$ as an $\mathfrak{a} \otimes_A \mathfrak{b}$-module follows from the commutativity of the tensor product with direct sums, and this also yields the statement on bases.
We now turn to the proof of b).
Let U be an A-basis of M and V an A-basis of N. Let X be an $\mathfrak{a}$-basis of $\mathfrak{A}$ and Y a $\mathfrak{b}$-basis of $\mathfrak{B}$. This yields several bases for tensor products: The set

$$\{u \otimes_{\mathfrak{a}} x \mid u \in U, x \in X\}$$

is an A-basis of $\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(M) = M \otimes_{\mathfrak{a}} \mathfrak{A}$ and

$$\{v \otimes_{\mathfrak{b}} y \mid v \in V, y \in Y\}$$

is an A-basis of $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{B}}(N) = N \otimes_{\mathfrak{b}} \mathfrak{B}$. Hence,

$$\{(u \otimes_{\mathfrak{a}} x) \otimes_A (v \otimes_{\mathfrak{b}} y) \mid u \in U, v \in V, x \in X, y \in Y\}$$

is an A-basis of $\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(M) \otimes_A \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{B}}(N)$.
Similarly, the set

$$\{u \otimes_A v \mid u \in U, v \in V\}$$

is an A-basis of $M \otimes_A N$ and

$$\{x \otimes_{\mathfrak{a} \otimes_A \mathfrak{b}} y \mid x \in X, y \in Y\}$$

is an $\mathfrak{a} \otimes_A \mathfrak{b}$-basis of $\mathfrak{A} \otimes_A B$. Hence

$$\{(u \otimes_A v) \otimes_{\mathfrak{a} \otimes_A \mathfrak{b}} (x \otimes_A y) \mid u \in U, v \in V, x \in X, y \in Y\}$$

is an A-basis of $\mathrm{Ind}_{\mathfrak{a} \otimes_A \mathfrak{b}}^{\mathfrak{A} \otimes_A \mathfrak{B}}(M \otimes_A N) = (M \otimes_A N) \otimes_{\mathfrak{a} \otimes_A \mathfrak{b}} (\mathfrak{A} \otimes_A \mathfrak{B})$. This shows that

$$\begin{aligned} \mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(M) \otimes_A \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{B}}(N) &\to \mathrm{Ind}_{\mathfrak{a} \otimes_A \mathfrak{b}}^{\mathfrak{A} \otimes_A \mathfrak{B}}(M \otimes_A N) \\ (u \otimes_{\mathfrak{a}} x) \otimes_A (v \otimes_{\mathfrak{b}} y) &\mapsto (u \otimes_A v) \otimes_{\mathfrak{a} \otimes_A \mathfrak{b}} (x \otimes_A y) \end{aligned}$$

defines an isomorphism of A-modules. An easy computation shows that it is compatible with the action of $\mathfrak{a} \otimes_A \mathfrak{b}$, which proves the claim. ∎

1.4. Symmetric Algebras

Let again A be a ring and $\mathfrak{a} \leq \mathfrak{A}$ a subalgebra of $\mathfrak{A}$ that is free of finite rank over A.

Definition 1.4.1 An A-linear map $\tau : \mathfrak{A} \to A$ with $\tau(aa') = \tau(a'a)$ for all a, a' in $\mathfrak{A}$ is called a *symmetrising trace on* $\mathfrak{A}$ if the bi-linear form

$$\mathfrak{A} \times \mathfrak{A} \to A; (a, a') \mapsto \tau(aa')$$

is non degenerate, i.e. the determinant of the corresponding Gram matrix with respect to any two A-bases of $\mathfrak{A}$ is a unit in A. If there exists a symmetrising trace on $\mathfrak{A}$, then we call $\mathfrak{A}$ a *symmetric algebra*.

Example 1.4.2 If G is a finite group, then the group algebra $A[G]$ is a symmetric algebra with a symmetrising trace defined by the linear extension of

$$g \mapsto \begin{cases} 1 \mapsto 1 \\ g \mapsto 0, & \text{if } g \neq 1 \end{cases}$$

for g in G, cf. [CR81, Proposition 9.6]. We will later see that several deformations of group algebras, too, are symmetric, with their traces being suitable generalisations of the above.

For the rest of this section we assume that τ is a symmetrising trace on $\mathfrak{A}$.

Lemma 1.4.3 *Suppose $\mathfrak{A}$ is free of finite rank n over $\mathfrak{a}$ as a left module via left multiplication in $\mathfrak{A}$. If the restriction $\tau|_{\mathfrak{a}}$ is a symmetrising trace on $\mathfrak{a}$, then $\mathfrak{A}$ is also free as a right $\mathfrak{a}$-module via right multiplication of rank n.*

Proof We go through a number of module homomorphisms using the fact that both $\mathfrak{A}$ and $\mathfrak{a}$ are symmetric. In fact, we consider these morphisms only as morphisms of right $\mathfrak{a}$-modules ignoring the fact that a number of the maps involved also respect stronger structures. Let $\{v_i \mid i \in I\}$ be an $\mathfrak{a}$-basis of $\mathfrak{A}$ as a left $\mathfrak{a}$-module via left multiplication for some finite index set I of cardinality n and let $\{u_i \mid i \in I\}$ be a basis of the free right $\mathfrak{a}$-module $\oplus_{i \in I} \mathfrak{a}_{\mathfrak{a}} = \oplus_{i \in I} u_i \, \mathfrak{a}$. Contrary to what we did before we will now have to consider left modules and we write ${}_{\mathfrak{a}}\,\mathfrak{a}$ and ${}_{\mathfrak{a}}\,\mathfrak{A}$ for $\mathfrak{a}$ and $\mathfrak{A}$ viewed as left $\mathfrak{a}$-modules via multiplication. Similarly, we write ${}_{\mathfrak{a}}\,\mathrm{Hom}$ for left $\mathfrak{a}$-module homomorphisms. Note that it does not matter whether we write ${}_{A}\,\mathrm{Hom}$ or Hom_A, as A is commutative.
There is a chain of right $\mathfrak{a}$-module isomorphisms

$$\begin{aligned} & \oplus_i \, u_i \, \mathfrak{a}_{\mathfrak{a}} && (1.2) \\ \cong\ & {}_{\mathfrak{a}}\,\mathrm{Hom}(\oplus_i \, {}_{\mathfrak{a}}\,\mathfrak{a}\, v_i,\ {}_{\mathfrak{a}}\,\mathfrak{a}) && (1.3) \\ \cong\ & {}_{\mathfrak{a}}\,\mathrm{Hom}({}_{\mathfrak{a}}\,\mathfrak{A},\ {}_{\mathfrak{a}}\,\mathfrak{a}) && (1.4) \\ \cong\ & {}_{\mathfrak{a}}\,\mathrm{Hom}({}_{\mathfrak{a}}\,\mathfrak{A},\ \mathrm{Hom}_A(\mathfrak{a}, A)) && (1.5) \\ \cong\ & \mathrm{Hom}_A(\mathfrak{a} \otimes_{\mathfrak{a}} ({}_{\mathfrak{a}}\,\mathfrak{A}),\ A) && (1.6) \\ \cong\ & \mathrm{Hom}_A({}_{\mathfrak{a}}\,\mathfrak{A},\ A) && (1.7) \end{aligned}$$

$$\cong \mathfrak{A}_{\mathfrak{a}}. \tag{1.8}$$

We explain the key steps:

((1.2) → (1.3)): The isomorphism is given by $u_i \mapsto \begin{cases} v_i \mapsto 1 \\ v_j \mapsto 0, & \text{if } i \neq j \end{cases}$. To see that this gives an isomorphism of right $\mathfrak{a}$-modules note that $\mathfrak{a}$ acts on ${}_{\mathfrak{a}}\operatorname{Hom}(\oplus_{i\mathfrak{a}}\, \mathfrak{a}\, v_i,\ {}_{\mathfrak{a}}\,\mathfrak{a})$ via right multiplication on the images.

((1.4) → (1.5)): The algebra $\mathfrak{a}$ is symmetric. By [Bro09, Section 2.B] this implies the existence of an $(\mathfrak{a}, \mathfrak{a})$-bimodule isomorphism $\mathfrak{a} \to \operatorname{Hom}_A(\mathfrak{a}, A)$.

((1.5) → (1.6)): This is the usual adjointness of the tensor product with the Hom functor, cf. [CR81, 2.19]. It is easily checked that this yields not only an isomorphism of abelian groups, but of right $\mathfrak{a}$-modules.

((1.7) → (1.8)): Since $\mathfrak{A}$, too, is symmetric, there is an isomorphism of $(\mathfrak{A}, \mathfrak{A})$-bimodules $\mathfrak{A} \to \operatorname{Hom}_A(\mathfrak{A}, A)$ which restricts to an isomorphism of right $\mathfrak{a}$-modules.

Thus, $\mathfrak{A}$ is free of rank n as a right $\mathfrak{a}$-module. ∎

Remark 1.4.4 The chain of isomorphisms in the lemma can be found similarly in [CR81, (37.1)], applied to a slightly different setup.

Proposition 1.4.5 (cf. [Sha11, Lemma 2.6]) *Suppose $\mathfrak{A}$ is free of finite rank over $\mathfrak{a}$ as a left module via left multiplication in $\mathfrak{A}$. If $\tau|_{\mathfrak{a}}$ is a symmetrising trace on $\mathfrak{a}$, then $\operatorname{Ind}_{\mathfrak{a}}^{\mathfrak{A}}$ and $\operatorname{Res}_{\mathfrak{a}}^{\mathfrak{A}}$ are bi-adjoint, i.e. they are both left and right adjoint to one another.*

Proof Clearly, the two functors $\bullet \otimes_{\mathfrak{A}} \mathfrak{A}_{\mathfrak{a}}$: mod- $\mathfrak{A}$ → mod- $\mathfrak{a}$ and $\operatorname{Res}_{\mathfrak{a}}^{\mathfrak{A}}(\bullet)$: mod- $\mathfrak{A}$ → mod- $\mathfrak{a}$ are naturally isomorphic. By the usual adjointness of the Hom-functor and the tensor product we know that $\bullet \otimes_{\mathfrak{A}} \mathfrak{A}_{\mathfrak{a}}$ is left adjoint to the *co-induction functor* $\operatorname{Hom}_{\mathfrak{a}}(\mathfrak{A},\ \bullet)$: mod- $\mathfrak{a}$ → mod- $\mathfrak{A}$. Thus, $\operatorname{Res}_{\mathfrak{a}}^{\mathfrak{A}}(\bullet)$, too, is left adjoint to co-induction, and by the uniqueness of right adjoints it suffices to show that induction and co-induction are naturally isomorphic.

We proceed in two steps. Let M be in mod- $\mathfrak{a}$.

First, note that

$$M \otimes_{\mathfrak{a}} \operatorname{Hom}_{\mathfrak{a}}(\mathfrak{A},\ \mathfrak{a}) \to \operatorname{Hom}_{\mathfrak{a}}(\mathfrak{A},\ M); m \otimes \varphi \mapsto (x \mapsto m.\varphi(x)) \tag{1.9}$$

defines an isomorphism of abelian groups as $\mathfrak{A}$ is free as a right $\mathfrak{a}$-module by Lemma 1.4.3. It is compatible with the right-action of $\mathfrak{A}$ on both sides and thus the morphism is an $\mathfrak{A}$-module isomorphism.

Now consider $\operatorname{Hom}_{\mathfrak{a}}(\mathfrak{A},\ \mathfrak{a})$. Similar to what we did in the proof of Lemma 1.4.3, there is a chain of $(\mathfrak{a},\ \mathfrak{A})$-bimodule isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{a}}(\mathfrak{A}_{\mathfrak{a}},\ \mathfrak{a}_{\mathfrak{a}}) &\cong \operatorname{Hom}_{\mathfrak{a}}(\mathfrak{A}_{\mathfrak{a}},\ \operatorname{Hom}_A(\mathfrak{a},\ A)) \\ &\cong \operatorname{Hom}_A(\mathfrak{A} \otimes_{\mathfrak{a}} \mathfrak{a},\ A) \\ &\cong \operatorname{Hom}_A(\mathfrak{A},\ A) \\ &\cong \mathfrak{A}. \end{aligned}$$

As before, the isomorphisms follow from the fact that both $\mathfrak{A}$ and $\mathfrak{a}$ are symmetric and from the adjointness of the covariant Hom-functor and the tensor product.

Thus, in (1.9) we can replace $\operatorname{Hom}_{\mathfrak{a}}(\mathfrak{A},\ \mathfrak{a})$ by $\mathfrak{A}$ to obtain an isomorphism of right $\mathfrak{A}$-modules

$$\alpha_M : M \otimes_{\mathfrak{a}} \mathfrak{A} \to \operatorname{Hom}_{\mathfrak{a}}(\mathfrak{A},\ M)$$

between the induction and the co-induction of M.
It is now easily checked that the maps α_M constitute a natural isomorphism between the functors of induction and co-induction. By the uniqueness of right adjoints, $\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}$ is right adjoint to $\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{A}}$ and by Lemma 1.3.5 it is also left adjoint. ∎

As an application we obtain the following strengthening of Corollary 1.3.6.

Corollary 1.4.6 *Let M be in* mod-$\mathfrak{A}$ *and N in* mod-$\mathfrak{a}$. *Then the bi-adjointness of induction and restriction yields isomorphisms of A-modules*

$$\mathrm{Hom}_{\mathfrak{a}}(N, \mathrm{Res}_{\mathfrak{a}}^{\mathfrak{A}}(M)) \cong \mathrm{Hom}_{\mathfrak{A}}(\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(N),\ M)$$

and

$$\mathrm{Hom}_{\mathfrak{A}}(M,\ \mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(N)) \cong \mathrm{Hom}_{\mathfrak{a}}(\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{A}}(M),\ N).$$

Proof This follows immediately from Proposition 1.4.5 and Lemma 1.3.4. ∎

Chapter 2.

Reflection Groups, Hecke Algebras, and their Parabolic Substructures

We give an introduction to Coxeter groups, (non-exceptional) complex reflection groups and their corresponding Hecke algebras. In particular, we are interested in the parabolic substructures of these groups and algebras.

2.1. Coxeter Groups

We introduce Coxeter groups and some of their properties following [GP00].

Definition 2.1.1 Let W be a group having a presentation

$$W = \langle s \in S \mid (st)^{m_{st}} \text{ for } s, t \in S \rangle$$

where $m_{st} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ for $s \neq t$ and $m_{ss} = 1$. Then W is called a *Coxeter group* and the tuple (W, S) is called a *Coxeter system*. If W is finite, we say that the Coxeter system is *finite*.
The cardinality of S is called the *rank of* W.

Remark 2.1.2 Since for all s in S we have $s^2 = 1$, we can rewrite the relation

$$(st)^{m_{st}} = 1$$

as

$$\underbrace{stst\ldots}_{m_{st}} = \underbrace{tsts\ldots}_{m_{st}}.$$

Due to their shape these relations are called *braid relations*, and

$$\left\langle s \in S \mid s^2,\ \underbrace{stst\ldots}_{m_{st}} = \underbrace{tsts\ldots}_{m_{st}} \right\rangle$$

is a presentation of W.

Lemma 2.1.3 *Let (W, S) be a Coxeter system with defining exponents m_{st} for $s, t \in S$. Then the order of st in W is exactly m_{st}.*

Remark 2.1.4 The finite Coxeter groups are exactly the finite real reflection groups.
These are defined as follows: Let V be a finite dimensional real vector space. A *reflection on V* is an element of $\mathrm{GL}(V)$ of order 2 that point-wise fixes a hyperplane. If $U \leq \mathrm{GL}(V)$ is generated by reflections, then we call U a *real reflection group*.

Example 2.1.5

a) The symmetric group $\mathfrak{S}_n$ on n letters is a Coxeter group. More precisely, $(\mathfrak{S}_n, \{(i, i+1) \mid 1 \leq i \leq n-1\})$ is a (finite) Coxeter system, where $(i, i+1)$ is the transposition interchanging i and $i+1$.

b) Dihedral groups are Coxeter groups.

The finite Coxeter groups are fully classified up to isomorphism in terms of their Coxeter graph.

Definition 2.1.6 Let (W, S) be a Coxeter system. The *Coxeter graph* of (W, S) is a graph with labeled edges and vertex set S. For s and t in S there is an edge with label m_{st} if and only if st has order m_{st} and $m_{st} \geq 3$. To simplify notation the label 3 is often ommited.
If the Coxeter graph is connected, then we say that (W, S) is an *irreducible* Coxeter system.

The Coxeter graph fully defines W up to isomorphism, as it encodes the complete presentation by Lemma 2.1.3.

Corollary 2.1.7 *Let (W, S) be a finite Coxeter system. Let $S = \coprod_i S_i$ be a disjoint union such that the S_i are exactly the vertix sets of the connected components of the Coxeter graph of (W, S). Set $W_i := \langle S_i \rangle$. Then (W_i, S_i) is itself a Coxeter system and W is isomorphic to the direct product $\times_i W_i$.*

The irreducible finite Coxeter systems are fully classified.

Proposition 2.1.8 *Let (W, S) be an irreducible finite Coxeter system. Then its Coxeter graph is one of the graphs listed in Table 2.1. Hence, every irreducible Coxeter group has a well-defined* type *listed in the first column of the table. Note that groups of type A_n, B_n and D_n have rank n, i.e. their graphs have exactly n vertices. Recall that the edge label* 3 *is omitted.*

Some subsets of the types have certain names we will use from time to time.

Definition 2.1.9

a) The groups of type A_n, B_n, D_n, E_6, E_7, E_8, F_4, and G_2 are called *Weyl groups*. These are exactly those irreducible Coxeter groups, where all exponents m_{st} are contained in the set $\{2, 3, 5, 6\}$.

b) The types A_n, B_n, and D_n are the so-called *classical types.*

c) The types E_6, E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(m)$ for $m = 5$ or $m \geq 7$ are called *exceptional*.

d) We call the groups of type H_3, H_4 and $I_2(m)$ for $m = 5$ or $m \geq 7$ *non-crystallographic.*

Remark 2.1.10

a) The group A_n is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$.

b) The group B_n is isomorphic to the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n$.

c) The group D_n is a normal subgroup of index 2 in B_n. We will study this connection in more detail in Section 8.1.

d) The groups $I_2(m)$ are dihedral groups of order $2m$.

Definition 2.1.11 Let (W, S) be a finite Coxeter system and $J \subseteq S$. Then $W_J := \langle J \rangle \leq W$ is again a Coxeter group and (W_J, J) is a Coxeter system. Moreover,

$$\langle s \in J \mid (st)^{m_{st}} \rangle ,$$

is a presentation of W_J, where the exponents m_{st} are exactly those used to define W, i.e. the order of st in W. We call W_J a *parabolic subgroup* of W.

To study the connection between parabolic subgroups and the full group we examine its length function.

Definition 2.1.12 Let (W, S) be a finite Coxeter system with $S = \{s_1, \ldots, s_n\}$ and let w in W. Since all the s_i are involutions, there exist indeces $i_1, \ldots, i_m$ such that $w = s_{i_1} \cdots s_{i_m}$. If m is minimal with this property we say that w has *length m* and we set $\ell_W(w) := m$. We will drop the subscript whenever convenient. Moreover, we call $s_{i_1} \cdots s_{i_m}$ a *reduced expression* of w.

Remark 2.1.13

a) We have $\ell(w) = \ell(w^{-1})$ for all $w \in W$.

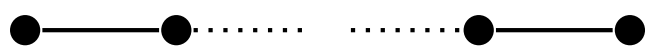
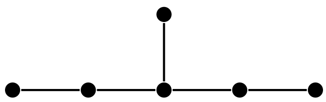
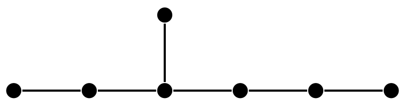
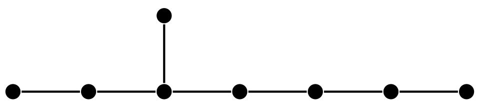
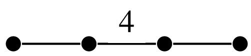
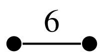
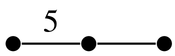
b) Prefixes and suffixes of reduced expressions are also reduced.

c) Elements of S have length 1.

d) The only element with length 0 is the identity element of W.

Lemma 2.1.14 *Let (W, S) be a finite Coxeter system and $J \subseteq S$. Then $\ell_{W_J} = \ell_W|_{W_J}$, i.e. the length function on W_J is the restriction of the length function on W to W_J.*

Table 2.1.: Coxeter graphs of irreducible finite Coxeter groups

A_n, $n \geq 1$	
B_n, $n \geq 2$	
D_n, $n \geq 4$	
E_6	
E_7	
E_8	
F_4	
G_2	
H_3	
H_4	
$I_2(m)$, $m = 5$ or $m \geq 7$	

Corollary 2.1.15 *Suppose $S = S_1 \coprod S_2$ and the elements of S_1 and S_2 commute. Then $W \cong W_1 \times W_2$ with $W_i := \langle S_i \rangle$ and for w in W we have $\ell_W(w) = \ell_{W_1}(w_1) + \ell_{W_2}(w_2)$ for $w_i \in W_i$ with $w = w_1 w_2$.*

Corollary 2.1.16 *Let s, t be elements of S and m_{st} the order of st in W. Then the product $\underbrace{sts\ldots}_{m_{st}}$ with exactly m_{st} factors is a reduced expression. We express this by saying that* braid words are reduced expressions.

We use this to define a very convenient subset of coset representatives.

Definition 2.1.17 Let (W, S) be a finite Coxeter system and $J, K \subseteq S$. Set

$$X_J := \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in S\}.$$

We call W_J the *set of distinguished right coset representatives* of W_J in W.
The set ${}_JX := X_J^{-1} = \{x^{-1} \mid x \in X_J\}$ is called the *set of distinguished left coset representatives* of W_J in W.
The *set of distinguished double coset representatives* of W_J and W_K in W is ${}_KX_J := {}_KX \cap X_J$.

We gather a number of properties of distinguished coset representatives:

Proposition 2.1.18 *Let (W, S) be a finite Coxeter system and $K, J \subseteq S$.*

a) X_J is a complete set of right coset representatives of W_J in W.

b) ${}_KX$ is a complete set of left coset representatives of W_K in W.

c) ${}_KX_J$ is a complete set of double coset representatives of (W_J, W_K) in W.

d) For every w in W there exist unique elements w' in W_J and x in X_J such that $w = w'x$ and $\ell(w) = \ell(w') + \ell(x)$.

e) For every w in W there exist unique elements w'' in W_J and y in ${}_JX$ such that $w = yw''$ and $\ell(w) = \ell(y) + \ell(w'')$.

f) For every w in W there exist unique elements v' in W_J, v'' in W_K and z in ${}_KX_J$ such that $w = v'zv''$ and $\ell(w) = \ell(v') + \ell(z) + \ell(v'')$.

g) Suppose $S = S_1 \coprod S_2$ and the elements of S_1 commute with those of S_2. Then there exist subsets $J_i \subseteq S_i$ such that $J = J_1 \coprod J_2$. If X_{1,J_1} and X_{2,J_2} are the sets of distinguished right coset representatives of $(W_1)_{J_1}$ in W_1 and of $(W_2)_{J_2}$ in W_2 respectively, then $X_J = \{xy \mid x \in X_{1,J_1},\ y \in X_{2,J_2}\}$.

2.2. Iwahori-Hecke Algebras

We follow [GP00] to define Iwahori-Hecke algebras of Coxeter groups and state some of their properties. Throughout this section let A be a commutative ring.

Definition 2.2.1 Let (W, S) be a finite Coxeter system. For s in S let u_s be an invertible element of A such that $u_s = u_t$ whenever s and t are conjugate in W for s and t in S. Then the *Iwahori-Hecke algebras of W over A with parameters* $(u_s \mid s \in S)$ is the associative unital algebra defined by the following presentation:

$$\begin{array}{rrl} & \text{Generators:} \quad T_s & \text{for } s \in S \\ \text{Relations:} & T_s^2 = u_s 1 + (u_s - 1)T_s & \text{for } s \in S \text{ (quadratic relations)} \\ & \underbrace{T_s T_t T_s \cdots}_{m_{st}} = \underbrace{T_t T_s T_t \cdots}_{m_{st}} & \text{for } s, t \in S \text{ (braid relations)} \end{array}$$

Here, $\underbrace{T_s T_t T_s \cdots}_{m_{st}}$ is a product with exactly m_{st} factors and m_{st} is the order of st in W.
We denote the Iwahori-Hecke algebra by $H_A(W, S, (u_s \mid s \in S))$.

Example 2.2.2 Let (W, S) be a finite Coxeter system. Setting $u_s := 1$ for all s in S it is easy to see that $H_A(W, S, (1 \mid s \in S))$ is equal to the group algebra $A[W]$ by identifying T_s with the generator s of W. Thus, Iwahori-Hecke algebras are often described as *deformations of the group algebras of Coxeter groups*.

To state some results on the structure of Iwahori-Hecke algebras and their parabolic substructures let us fix a finite Coxeter system (W, S), a commutative ring A, and invertible elements u_s for every s in S such that $u_s = u_t$ whenever s and t are conjugate in W. We set $\mathcal{H} := H_A(W, S, \{u_s \mid s \in S\})$.

Proposition 2.2.3

a) *Let w in W and $w = s_{i_1} \cdots s_{i_m}$ a reduced expression of w. Define $T_w := T_{s_{i_1}} \cdots T_{s_{i_m}}$. Then T_w is well-defined, i.e. independent of the chosen reduced expression of w.*

b) *The algebra $\mathcal{H}$ is free as an A-module with a basis given by $\{T_w \mid w \in W\}$. We call this basis the* standard basis of $\mathcal{H}$.

c) *Let s in S and w in W. Then*

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) > \ell(w) \\ u_s T_{sw} + (u_s - 1)T_w, & \text{if } \ell(sw) < \ell(w) \end{cases}.$$

An analogous rule holds for multiplication with T_s from the right. By induction it follows that $T_w T_v = T_{wv}$ if $\ell(wv) = \ell(w) + \ell(v)$.

Definition 2.2.4 Let $J \subseteq S$. Then $H_A(W_J, J, (u_s \mid s \in J)$ is called a *parabolic subalgebra of $\mathcal{H}$*.

Fix some $J \subseteq S$ and set $\mathcal{H}_J := H_A(W_J, J, (u_s \mid s \in J))$.

Proposition 2.2.5

a) *$\mathcal{H}_J$ embeds naturally into $\mathcal{H}$ by identifying generators of the same name.*

b) *$\mathcal{H}$ is free as an $\mathcal{H}_J$-left module with a basis $\{T_x \mid x \in X_J\}$.*

One important application of parabolic subalgebras is the following reduction to the case of Iwahori-Hecke algebras of irreducible Coxeter groups.

Proposition 2.2.6 (cf. [GP00, Exercise 8.4]) *Suppose that $S = S_1 \coprod S_2$ is a disjoint union such that the elements of S_1 commute with those of S_2. Then $W = W_1 \times W_2$ with $W_i := \langle S_i \rangle$. This yields corresponding parabolic subalgebras $\mathcal{H}_i := H_A(W_i, S_i, (u_s \mid s \in S_i)) \leq \mathcal{H}$ such that*

$$\mathcal{H} \cong \mathcal{H}_1 \otimes_A \mathcal{H}_2$$

as A-algebras, where the multiplicative structure of $\mathcal{H}_1 \otimes_A \mathcal{H}_2$ is given by $(h_1 \otimes h_2)(g_1 \otimes g_2) := h_1 g_1 \otimes h_2 g_2$ for h_i, g_i in $\mathcal{H}_i$.

Proof Set

$$\alpha : \mathcal{H}_1 \otimes_A \mathcal{H}_2 \to \mathcal{H}; T_u \otimes T_v \mapsto T_u T_v$$

for $u \in W_1$ and $v \in W_2$. Clearly, this is a well-defined A-linear map. For u in W_1 and v in W_2 we have $\ell(uv) = \ell(u) + \ell(v)$ by Corollary 2.1.15, hence $T_u T_v = T_{uv}$ by part c) of Proposition 2.2.3. As $W = W_1 \times W_2$ this shows that α is an isomorphism of A-modules, sending an A-basis of $\mathcal{H}_1 \otimes_A \mathcal{H}_2$ to an A-basis of $\mathcal{H}$ by Proposition 2.2.3 b).
Now for multiplication: Let $u, y \in W_1$ and $v, z \in W_2$. Then we have $T_u T_v T_y T_z = (T_u T_y)(T_v T_z)$, since $y \in W_1$ and $v \in W_2$ commute, and therefore $T_v T_y = T_{vy} = T_{yv} = T_y T_v$ again by Proposition 2.2.3 c). Thus,

$$\begin{aligned} \alpha(T_u \otimes T_v)\alpha(T_y \otimes T_z) &= T_u T_v T_y T_z \\ &= (T_u T_y)(T_v T_z) \\ &= \alpha(T_u T_y \otimes T_v T_z) \\ &= \alpha((T_u \otimes T_v)(T_y \otimes T_z)). \end{aligned}$$

Hence, α is an isomorphism of A-algebras. ∎

Remark 2.2.7 Combined with Proposition 2.1.8, Proposition 2.2.6 yields a classification of Hecke algebras of A.
If W is an irreducible Coxeter group of some type P (cf. Proposition 2.1.8), we say that $\mathcal{H}$, too, is of type P.

As a subalgebra $\mathcal{H}_J$ gives rise to induction and restriction functors.

Definition 2.2.8 Consider $\mathcal{H}_J$ as a subalgebra of $\mathcal{H}$ via Proposition 2.2.5. Set $\mathrm{Ind}_J^S := \mathrm{Ind}_{\mathcal{H}_J}^{\mathcal{H}}$ and $\mathrm{Res}_J^S := \mathrm{Res}_{\mathcal{H}_J}^{\mathcal{H}}$. We call these functors *parabolic induction* and *parabolic restriction* respectively. Both Res_J^S and Ind_J^S are exact, since $\mathcal{H}$ is free over $\mathcal{H}_J$, and we denote the corresponding homomorphisms of the Grothendieck groups, too, by Res_J^S and Ind_J^S.

These functors behave just like those for group algebras of Coxeter groups when it comes to the rank of free modules.

Corollary 2.2.9 *Let M be an $\mathcal{H}_J$-module that is free of finite rank r as an A-module. Then $\operatorname{Ind}_J^S(M)$ is a free A-module of rank $r \cdot |X_J| = r \cdot [W : W_J]$, where $[W : W_J]$ is the subgroup index of W_J in W. More precisely, if B is an A-basis of M, then an A-basis of $\operatorname{Ind}_J^S(M)$ is given by $\{b \otimes_A T_x \mid x \in X_J,\ b \in B\}$.*
By definition $\operatorname{Res}_J^S(M)$ is a free A-module of rank r with basis B.

Remark 2.2.10 As induction is transitive, it will often be sufficient to only study maximal parabolic subalgebras $\mathcal{H}_J$, that is parabolic subalgebras where $|S \setminus J| = 1$.

To show that the induction functors are bi-adjoint note that Iwahori-Hecke algebras are symmetric.

Definition 2.2.11 We define $\tau_{\mathcal{H}}$ as the A-linear map

$$\mathcal{H} \to A; T_w \mapsto \begin{cases} 1, & \text{if } w = 1, \\ 0, & \text{if } w \neq 1. \end{cases}$$

Proposition 2.2.12 *The homomorphism $\tau_{\mathcal{H}}$ is a symmetrising trace on $\mathcal{H}$. Hence, $\mathcal{H}$ is a symmetric algebra.*

Corollary 2.2.13 *Clearly, we have $\tau_{\mathcal{H}_J} = \tau_{\mathcal{H}}|_{\mathcal{H}_J}$, i.e. the restriction of the symmetrising trace of $\mathcal{H}$ to $\mathcal{H}_J$ is the symmetrising trace of $\mathcal{H}_J$.*

Corollary 2.2.14 *The functors Ind_J^S and Res_J^S are bi-adjoint by Proposition 1.4.5. In particular, for $M \in \text{mod-}\mathcal{H}$ and $N \in \text{mod-}\mathcal{H}_J$ there are isomorphisms of A-modules*

$$\operatorname{Hom}_{\mathcal{H}_J}\left(N, \operatorname{Res}_J^S(M)\right) \cong_A \operatorname{Hom}_{\mathcal{H}}\left(\operatorname{Ind}_J^S(N), M\right)$$

and

$$\operatorname{Hom}_{\mathcal{H}_J}\left(\operatorname{Res}_J^S(M), N\right) \cong_A \operatorname{Hom}_{\mathcal{H}}\left(M, \operatorname{Ind}_J^S(N)\right).$$

These are sometimes called Frobenius reciprocity and Nakayama relations, respectively.

An analogue of the well-known Mackey formula from group representation theory holds for Iwahori-Hecke algebras.

Proposition 2.2.15 *Let $K \subseteq S$ be a second subset of S and denote by $\mathcal{H}_K$ the corresponding parabolic subalgebra of $\mathcal{H}$. Recall that ${}_KX_J$ is the set of distinguished double coset representatives of the parabolic subgroups W_K and W_J in W. Let N be an $\mathcal{H}_J$-module. Then the following holds:*

a) For $d \in {}_KX_J$ the parabolic subalgebra $\mathcal{H}_{J^d \cap K}$ acts on $N \otimes_{\mathcal{H}_J} T_d \leq \operatorname{Ind}_J^S(N)$ by right multiplication: For $w \in W_{J^d \cap K}$ and $v \in N$ we have

$$(v \otimes T_d)T_w = vT_{dwd^{-1}} \otimes_{\mathcal{H}_J} T_d.$$

Here, $N \otimes_{\mathcal{H}_J} T_d$ is viewed as an A-submodule of $\operatorname{Ind}_J^S(N)$ via Corollary 2.2.9.

b) There is an isomorphism of $\mathcal{H}_K$-modules

$$\operatorname{Res}^S_K \circ \operatorname{Ind}^S_J(N) \cong_{\mathcal{H}_K} \bigoplus_{d \in_K X_J} \operatorname{Ind}^K_{J^d \cap K}(N \otimes_{\mathcal{H}_J} T_d)\,.$$

Finally, we rephrase Proposition 1.3.13 b) in the context of parabolic subalgebras to show that it will often suffice to only consider induction for Iwahori-Hecke algebras of irreducible Coxeter groups.

Proposition 2.2.16 *Let S_1 and S_2, W_1 and W_2, and $\mathcal{H}_1$ and $\mathcal{H}_2$ be as in Proposition 2.2.6. Then $J = K \coprod L$ for $K = J \cap S_1$ and $L = J \cap S_2$. The subalgebra H_J is isomorphic to $\mathcal{H}_K \otimes_A \mathcal{H}_L$ and we have $\mathcal{H}_K \leq \mathcal{H}_1$ and $\mathcal{H}_L \leq \mathcal{H}_2$. Let $N \in$ mod-$\mathcal{H}_K$ and $P \in$ mod-$\mathcal{H}_L$, such that both are free of finite rank as A-modules. Then there is an isomorphism of $\mathcal{H}_1 \otimes_A \mathcal{H}_2$-modules*

$$\operatorname{Ind}^{\mathcal{H}_1 \otimes_A \mathcal{H}_2}_{\mathcal{H}_K \otimes_A \mathcal{H}_L}(N \otimes_A P) \cong_{\mathcal{H}_1 \otimes_A \mathcal{H}_2} \left(\operatorname{Ind}^{\mathcal{H}_1}_{\mathcal{H}_K}\right) \otimes_A \left(\operatorname{Ind}^{\mathcal{H}_2}_{\mathcal{H}_L} P\right).$$

Remark 2.2.17 Proposition 2.2.16 is particularly helpful in the following setting: Suppose that A is an algebraically closed field or, more generally, a splitting field of all the Iwahori-Hecke algebras in the proposition. Then the irreducible modules of $\mathcal{H}_K \otimes \mathcal{H}_L$ are outer tensor products of irreducible modules of $\mathcal{H}_K$ with those of $\mathcal{H}_L$ and the proposition fully describes their induction. Since the classe of irreducible modules constitute a basis of the corresponding Grothendieck group, this fully defines the induction homomorphism on the Grothendieck group.

2.3. Complex Reflection Groups

As has already been mentioned, finite Coxeter groups are (up to isomorphism) exactly the finite real reflection groups. The definition of the latter has been generalised to the so-called complex reflection groups. An introduction to these groups can for example be found in [LT09].

Definition 2.3.1 Let V be a finite dimensional complex vector space. An element g of the general linear group $\mathrm{GL}(V)$ is called a *pseudo-reflection* if it has finite order and point-wise fixes a hyperplane in V. A subgroup $U \leq \mathrm{GL}(V)$ is called a *complex reflection group* if it is generated by pseudo-reflections. It is called *irreducible* if V is an irreducible $\mathbb{C}[U]$-module.

Example 2.3.2 All finite Coxeter groups are complex reflection groups.

Complex reflection groups were fully classified by Shephard and Todd in 1954.

Theorem 2.3.3 ([ST54]) *Every complex reflection group is the direct product of irreducible complex reflection groups, acting on the direct sum of their corresponding irreducible modules. Every irreducible complex reflection group is either isomorphic to one of the exceptional complex reflection groups denoted by G_4 up to G_{37}, or one of the groups $G(r, p, n)$ defined below.*

We first give the definition of the groups $G(r, p, n)$ in a rather abstract way that is closer to the definition of Coxeter groups and then indicate how these can be seen to be complex reflection groups.

Definition 2.3.4 Let r, p, and n be positive integers such that p divides r.

a) The group $G(r, 1, n)$ has a presentation in terms of generators and relations:

$$\begin{array}{rll} \text{Generators:} & t, s_1, \ldots, s_{n-1} & \\ \text{Relations:} & t^r = 1 & \\ & s_1^2 = \cdots = s_{n-1}^2 = 1 & \\ & ts_1ts_1 = s_1ts_1t & \\ & ts_i = s_it, & \text{for } i \geq 2 \\ & s_is_j = s_js_i, & \text{for } |i-j| \geq 2 \\ & s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, & \text{for } 1 \leq i \leq n-2 \end{array}$$

b) The group $G(r, p, n)$ is the subgroup of $G(r, 1, n)$ generated by

$$\{t^p, t^{-1}s_1t, s_1, \ldots, s_{n-1}\}$$

Proposition 2.3.5

a) The group $G(r, 1, n)$ as defined above is isomorphic to the following complex reflection group U: Let $U \leq \mathrm{GL}(\mathbb{C}^{1\times n})$ be the group generated by elements $b_0, \ldots, b_{n-1}$, where b_0 is the element multiplying the first standard basis vector with a primitive r'th root of unity and fixing every other standard basis vector, and for $1 \leq i \leq n-1$ the element b_i is uniquely defined by interchanging the i'th and $(i+1)$'st standard basis vector and fixing all the others.

b) $G(r, 1, n)$ is isomorphic to the wreath product $C_r \wr \mathfrak{S}_n$, where C_r is the cyclic group with r elements. In particular, the order of $G(r, 1, n)$ is $r^n n!$.

Lemma 2.3.6 *The group $G(r, p, n)$ is a normal subgroup of $G(r, 1, n)$ or index p. In particular, its order is $n!r^n/p$.*

Example 2.3.7

a) The group $G(1, 1, n)$ is the Coxeter group A_{n-1}, i.e. the symmetric group $\mathfrak{S}_n$.

b) The group $G(2, 1, n)$ is the Coxeter group B_n.

c) The group $G(2, 2, n)$ is the Coxeter group D_n.

d) The group $G(r, 1, 1)$ is the cyclic group of order r.

In this thesis we will only be concerned with complex reflection groups of type $G(r, p, n)$ (possibly with $p = 1$) and disregard exceptional complex reflection groups, unless they are also Coxeter groups.
From now on until the end of the section fix positive integers r, p and n, where p divides r.

Definition 2.3.8 In $G(r, 1, n)$ set $a_1 := t$ and inductively $a_{i+1} := s_i a_i s_i$ for $1 \leq i \leq n-1$. We call a_i the *i'th Jucys-Murphy element of* $G(r, 1, n)$.

Lemma 2.3.9

a) *The elements $s_1, \ldots, s_{n-1}$ generate a subgroup of $G(r, 1, n)$ that is isomorphic to the symmetric group $\mathfrak{S}_n$. Under this isomorphism s_i is the involution swapping i and $i + 1$ and fixing everything else. We identify $\mathfrak{S}_n$ with this subgroup of $G(r, 1, n)$.*

b) *As sets,*
$$G(r, 1, n) = \{a_1^{e_1} \cdots a_n^{e_n} w \mid 0 \leq e_1, \ldots, e_n < r,\ w \in \mathfrak{S}_n\},$$
and the elements of the set on the right-hand side are pairwise different.

c) *Since $s_1, \ldots, s_{n-1}$ all lie in $G(r, p, n)$ it follows that $\mathfrak{S}_n$ is also a subgroup of $G(r, p, n)$.*

d) *As sets,*
$$G(r, p, n)$$
$$= \left\{ a_1^{e_1} \cdots a_n^{e_n} w \mid 0 \leq e_1, \ldots, e_n < r,\ w \in \mathfrak{S}_n,\ \sum_{i=1}^{n} e_i \equiv 0 \pmod{p} \right\}.$$

Remark 2.3.10 Although part d) also yields a description of all elements if $p = 1$, we handle $p = 1$ separately as we can remove some superfluous conditions in this case.

When viewing $G(r, 1, n)$ as $C_r \wr \mathfrak{S}_n = (C_r)^n \rtimes \mathfrak{S}_n$, the element a_i is a generator of the i'th component of $(C_r)^n$ and s_j is the transposition $(j, j+1)$ in $\mathfrak{S}_n$. With this is mind the following contains nothing new.

Proposition 2.3.11 ([AK94, Proposition 2.1]) *Let $1 \leq j \leq n-1$. Then the following holds:*

a) *The elements $a_1, \ldots, a_n$ generate an abelian normal subgroup of $G(r, 1, n)$ of order r^n.*

b) *We have $a_i s_j = s_j a_i$ if $i \neq j$ and $i \neq j+1$.*

c) $a_i s_i = s_i a_{i+1}$.

d) $a_{i+1} s_i = s_i a_i$.

Corollary 2.3.12 *The action of the symmetric group $\mathfrak{S}_n \leq G(r,1,n)$ on $\{a_1,\dots,a_n\}$ via conjugation from the right is equivalent to the natural right action of $\mathfrak{S}_n$ on $\{1,\dots,n\}$: For $w \in \mathfrak{S}_n$ and $1 \leq i \leq n$ we have $w^{-1}a_iw = a_{(i)w}$, where $(i)w$ is the image of i under w under the natural right action.*

Definition 2.3.13 Let $W \leq \mathrm{GL}(V)$ be a complex reflection group. A *parabolic subgroup of W* is a stabiliser of a subset of V.

Remark 2.3.14 If $n = 1$, then $G(r,p,n)$ has no non-trivial parabolic subgroup, i.e. every parabolic subgroup is either trivial or the full group. Hence, from now on we will only be concerned with the case $n \geq 2$.

Definition 2.3.15 Suppose $n \geq 2$.

a) For $2 \leq j \leq n-1$ define the *j'th maximal standard parabolic subgroup of $G(r,p,n)$* as

$$G(r,p,n)_j := \left\langle t^p, t^{-1}s_1t, s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_{n-1} \right\rangle .$$

b) The 1'st maximal standard parabolic subgroup of $G(r,p,n)$ is

$$G(r,p,n)_1 := \langle t^p, s_2, \dots, s_{n-1} \rangle .$$

c) The 0'th maximal standard parabolic subgroup of $G(r,p,n)$ is

$$G(r,p,n)_0 := \langle s_1, \dots, s_{n-1} \rangle .$$

Remark 2.3.16 For $p = 1$ we have $t^p = t$ and therefore

$$G(r,1,n)_j = \left\langle t, s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_{n-1} \right\rangle$$

for $1 \leq j \leq n-1$ and

$$G(r,1,n)_0 = \langle s_1, \dots, s_{n-1} \rangle .$$

This definition of these subgroups should remind the reader of the definition of parabolic subgroups of Coxeter groups, which were also obtained by removing generators.
Often it will suffice to only work with these maximal standard parabolic subgroups, which follows from the following classification result.

Proposition 2.3.17 ([MT18, Theorem 3.6, Theorem 3.9])

a) The subgroup $G(r,p,n)_j$ is a parabolic subgroup of $G(r,p,n)$ for any $0 \leq j \leq n-1$.

b) If a subgroup $U \leq G(r,p,n)$ is parabolic, then there exists an α in $G(r,1,n)$ and $0 \leq j \leq n-1$ such that U^α is a subgroup of $G(r,p,n)_j$.

c) For $1 \leq j \leq n-1$ *we have* $G(r, p, n)_j \cong G(r, p, j) \times \mathfrak{S}_{n-j}$.

d) We have $G(r, p, n)_0 \cong \mathfrak{S}_n$.

The proper language to write down the elements of the maximal standard parabolical subgroups is in terms of Young subgroups.

Definition 2.3.18 ([GP00, 5.4.3]) A *composition of n* is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers such that $\sum_{i=1}^{\ell} \lambda_i = n$.
If λ is a composition of n we write $\mathfrak{S}_\lambda$ for the corresponding *Young subgroup of* $\mathfrak{S}_n$. This is the stabiliser of the partition

$$\left\{\{1, \ldots, \lambda_1\}, \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}, \ldots, \left\{\left(\sum_{i=1}^{\ell-1} \lambda_i\right) + 1, \ldots, n\right\}\right\}$$

in the natural action of $\mathfrak{S}_n$ on partitions of $\{1, \ldots, n\}$.
Equivalently,

$$\mathfrak{S}_\lambda = \left\langle s_i \mid 1 \leq i \leq n-1,\ i \neq \sum_{k=1}^{m} \lambda_k \text{ for all } 1 \leq m \leq \ell \right\rangle.$$

Lemma 2.3.19 ([MT18, (3.3)]) *Let* $1 \leq j \leq n-1$. *Then*

$$G(r, p, n)_j = \left\{a_1^{e_1} \cdots a_j^{e_j} w \mid w \in \mathfrak{S}_{(j,\, n-j)},\ 0 \leq e_1, \ldots, e_j < r,\ \sum_{i=1}^{j} e_i \equiv 0 \pmod{p}\right\}.$$

2.4. Ariki-Koike Algebras and Cyclotomic Hecke Algebras

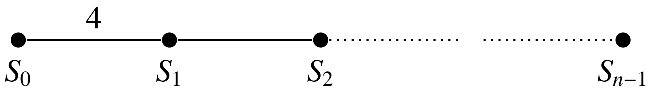
2.4.1. Ariki-Koike Algebras

As an analogoue to Iwahori-Hecke algebras as deformations of the group algebras of Coxeter groups one can define cyclotomic Hecke algebras as deformations of the group algebras of complex reflection groups. The most prominent amongst these are the so-called Ariki-Koike algebras, i.e. cyclotomic Hecke algebras of type $G(r, 1, n)$.

Definition 2.4.1 Let r and n be positive integers.
Let A be an integral domain and $q, Q_1, \ldots, Q_r$ invertible elements of A. Then the *Ariki-Koike algebra* $\mathbf{H}_{r,n}(q, Q_1, \ldots, Q_r)$ *of* $G(r, 1, n)$ *over A with parameters* $q, Q_1, \ldots, Q_r$ is defined by the following presentation:

Generators: S_i for $0 \leq i \leq n-1$
Relations: $(S_0 - Q_1)(S_0 - Q_2) \cdots (S_0 - Q_r) = 0$
$(S_i - q)(S_i + 1) = 0$ for $1 \leq i \leq n-1$

and the braid relations defined by the following Coxeter diagram:



That is

$$\begin{aligned} S_0S_1S_0S_1 &= S_1S_0S_1S_0 && \\ S_iS_{i+1}S_i &= S_{i+1}S_iS_{i+1} && \text{for } 1 \leq i \leq n-2 \\ S_iS_j &= S_jS_i && \text{for } |i-j| > 1. \end{aligned}$$

We will drop the parameters whenever they are clear, in which case we denote the algebra by $\mathbf{H}_{r,n}$.
For completeness we define $\mathbf{H}_{r,0}(q, Q_1, \ldots, Q_r)$ to be the trivial one-dimensional A-algebra.

Example 2.4.2 Just as for Iwahori-Hecke algebras we see that Ariki-Koike algebras are deformations of group algebras: If A contains a primitive r'th root of unity ζ, set $q = 1$ and $Q_i := \zeta^i$. Then $\mathbf{H}_{r,n}(q, Q_1, \ldots, Q_r)$ is isomorphic to the group algebra $A[G(r, 1, n)]$.

Some families of Iwahori-Hecke algebras are also Ariki-Koike algebras.

Proposition 2.4.3 *Let A be an integral domain.*

a) Let $q, Q_1, \ldots, Q_r$ be invertible elements of A and n and r positive integers. Let a be an invertible element of A and set $Q'_i := aQ_i$ for $1 \leq i \leq r$ Then

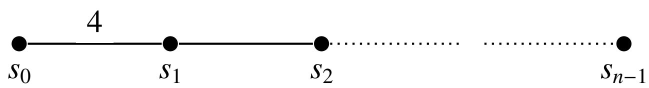
$$\mathbf{H}_{r,n}(q, Q_1, \ldots, Q_r) \cong \mathbf{H}_{r,n}(q, Q'_1, \ldots, Q'_r)$$

as A-algebras.

b) Let $q, Q_1, \ldots, Q_r$ be invertible elements of A and n and r positive integers. If π is in $\mathfrak{S}_r$, then $\mathbf{H}_{r,n}(q, Q_1, \ldots, Q_r) = \mathbf{H}_{r,n}(q, Q_{(1)\pi}, \ldots, Q_{(r)\pi})$.

c) Assume $r = 1$. Then for any choice of $q,\ Q_1 \in A^*$ the algebra $\mathbf{H}_{1,n}(q, Q_1)$ is isomorphic to the Iwahori-Hecke algebra of type A_{n-1} with parameters $u_{s_i} := q$ for all generators s_i of A_{n-1}. Indeed, $S_0 \mapsto Q_1$, $S_i \mapsto T_{s_i}$ for $1 \leq i \leq n-1$ defines an algebra isomorphism, where $s_i := (i, i+1)$ is the i'th generator of $A_{n-1} = \mathfrak{S}_n$, cf. Example 2.1.5.

d) Let $r = 2$ and q, Q_1, Q_2 be invertible elements of A. Then the Ariki-Koike algebra $\mathbf{H}_{2,n}(q,\ Q_1, Q_2)$ is isomorphic to an Iwahori-Hecke algebra of type B_n. More precisely, let the generators of the Coxeter group B_n be named as indicated by the following diagram:



Set $u_{s_0} := -Q_1Q_2^{-1}$ and $u_{s_1} := \cdots := u_{s_{n-1}} := q$. Then $S_0 \mapsto -Q_2T_{s_0}$ and $S_i \mapsto T_{s_i}$ for $1 \leq i \leq n-1$ defines an algebra isomorphism between $\mathbf{H}_{2,n}(q,\ Q_1, Q_2)$ and the Iwahori-Hecke algebra of B_n over A with parameters $(u_{s_0}, \ldots, u_{s_{n-1}})$.

Proof Everything follows from the defining relations.

a) Denote by $S_0, \dots, S_{n-1}$ the generators of $\mathbf{H}_{r,n}(q, Q_1, \dots, Q_r)$ and by $S_0', \dots, S_{n-1}$ those of $\mathbf{H}_{r,n}(q, Q_1', \dots, Q_r')$. The defining relations show immediately that $S_0 \mapsto a^{-1}S_0'$ and $S_i \mapsto S_i'$ for $1 \leq i \leq n-1$ defines an isomorphism of A-algebras.

b) Denote again by $S_0, \dots, S_{n-1}$ the generators of $\mathbf{H}_{r,n}(q, Q_1, \dots, Q_r)$. We have $(S_0 - Q_1) \cdots (S_0 - Q_r) = 0$ if and only if $(S_0 - Q_{(1)\pi}) \cdots (S_0 - Q_{(r)\pi}) = 0$, and this is the only defining relation involving the Q_i. Thus, the equality follows.

c) The defining relations of Iwahori-Hecke algebras in Definition 2.2.1 show that the given map is a well-defined homomorphism of A-algebras. We have $S_0 = Q_1$ for the generator S_0 of the Ariki-Koike algebra, thus $T_{s_i} \mapsto S_i$ for $1 \leq i \leq n-1$ defines an inverse of the given homomorphism which is therefore an isomorphism.

d) As for part c) one can show that the given map is a well-defined homomorphism of A-algebras. Since Q_2 is invertible, an inverse is given by $T_{s_i} \mapsto S_i$ for $1 \leq i \leq n-1$ and $T_{s_0} \mapsto -Q_2^{-1}S_0$. ∎

For the remainder of the section fix an integral domain A, positive integers r and n, and invertible elements q and $Q_1, \dots, Q_r$ of A.
We set $\mathbf{H} := \mathbf{H}_{r,n}(q, Q_1, \dots, Q_r)$.

As in the case of Iwahori-Hecke algebras, $\mathbf{H}$ is a free A-module. To write down a basis we define certain elements and subalgebras of $\mathbf{H}$. As for complex reflection groups we can define Jucys-Murphy elements for Ariki-Koike algebras.

Definition 2.4.4 Set $L_1 := S_0$ and inductively $L_{i+1} := q^{-1}S_iL_iS_i$ for $1 \leq i \leq n-1$. Then L_i is called the *i'th Jucys-Murphy element of* $\mathbf{H}$.

Corollary 2.4.5 *We have* $(L_1 - Q_1) \cdots (L_1 - Q_r) = 0$.
Furthermore, L_i *is invertible in* $\mathbf{H}$ *for all* $1 \leq i \leq n$.

Proof Since $L_1 = S_0$ by definition, the first equation follows immediately from the defining relations of $\mathbf{H}$ in Definition 2.4.1.
As the the Q_i are invertible, this relation already shows that L_1 is invertible: For $1 \leq j \leq r-1$ there exists elements f_j in A such that

$$L_1^r + \sum_{j=1}^{r-1} f_j L_1^j = -(Q_1 \cdots Q_r),$$

and thus

$$L_1 \left(-(Q_1 \cdots Q_r)^{-1} \left(L_1^{r-1} + \sum_{j=1}^{r-1} L_1^{j-1} \right) \right) = 1.$$

Similarly, q is invertible and we have $(S_k - q)(S_k + 1) = 0$ for $1 \leq k \leq n-1$. Therefore, each S_k is invertible with $S_k^{-1} = q^{-1}S_k + (q^{-1} - 1)$.
In total, q, the S_k, and L_1 are all invertible, and an inductive argument now shows that L_i is invertible. ∎

The Jucys-Murphy elements of $\mathbf{H}$ behave similarly to those of $G(r, 1, n)$.

Lemma 2.4.6 ([AK94, Lemma 3.3], [Mat04, Theorem 3.4]) *The subalgebra of* $\mathbf{H}$ *generated by* $L_1, \ldots, L_n$ *is abelian. The symmetric polynomials in the* L_i *are contained in the center of* $\mathbf{H}$ *and if* $\mathbf{H}$ *is semisimple then the center consists exactly of these symmetric polynomials.*

The algebra $\mathbf{H}$ contains another special subalgebra, in analogy to the symmetric group $\mathfrak{S}_n$ lying in the complex reflection group $G(r, 1, n)$.

Lemma 2.4.7 *The subalgebra of* $\mathbf{H}$ *generated by* $S_1, \ldots, S_{n-1}$ *is isomorphic via* $S_i \mapsto T_i$ *to the Iwahori-Hecke algebra of* $\mathfrak{S}_n = A_{n-1}$ *over* A *, where all the parameters* u_s *are equal to* q*. We denote this algebra by* $\mathbf{H}(\mathfrak{S}_n, q) \leq \mathbf{H}$.

In our study of Iwahori-Hecke algebras we learned that $\mathbf{H}(\mathfrak{S}_n, q)$ contains elements T_w for every $w \in \mathfrak{S}_n$ defined via reduced expressions of the elements of $\mathfrak{S}_n$. Via Lemma 2.4.7 these elements can be thought of as elements of $\mathbf{H}$: If $w = s_{i_1} \cdots s_{i_m}$ with $1 \leq i_j \leq n-1$ is a reduced expression of w in $\mathfrak{S}_n$, then we set $T_w := S_{i_1} \cdots S_{s_m}$. With this in mind we can define an A-basis of $\mathbf{H}$. This is an analogue of Proposition 2.2.3 for Iwahori-Hecke algebras and Lemma 2.3.9 for complex reflection groups.

Proposition 2.4.8 *The algebra* $\mathbf{H}$ *is free as an* A*-module. More precisely,*

$$\{L_1^{e_1} \cdots L_n^{e_n} T_w \mid w \in \mathfrak{S}_n,\ 0 \leq e_1, \ldots, e_n \leq r-1\}$$

is an A*-basis of* $\mathbf{H}$.

The interaction between the subalgebra generated by $L_1, \ldots, L_n$ and that generated by $S_1, \ldots S_{n-1}$ mirrors the behaviour of the corresponding subgroups of $G(r, 1, n)$. However, we have to account for some correction terms.

Lemma 2.4.9 ([AK94, Lemma 3.3]) *Let* $1 \leq i \leq n-1$.

a) Let k *be a natural number. Then*

$$S_i L_{i+1}^k = L_i^k S_i + (q-1) \sum_{j=1}^{k} L_i^{j-1} L_{i+1}^{k-j+1}.$$

b) Let $1 \leq j \leq n-1$*. If* $j \neq i-1$ *and* $j \neq i$*, then* $S_j L_i = L_i S_j$.

Remark 2.4.10 Note that we have to slightly adapt the formulas of [AK94], as Ariki and Koike define Jucys-Murphey elements without the factor q^{-1}. Our version improves legibility in most calculations and is also that which is used in a number of more recent articles such as [Mat04].

Once again we define parabolic substructures:

Definition 2.4.11 Every subalgebra of $\mathbf{H}$ generated by a subset of $\{S_0, \ldots, S_{n-1}\}$ is called a *parabolic subalgebra*.
Let $0 \leq j \leq n-1$. Then the subalgebra generated by $\left\{S_0, \ldots, S_{j-1}, S_{j+1}, \ldots, S_{n-1}\right\}$ is called the *j'th maximal parabolic subalgebra of* $\mathbf{H}$.

Clearly, every parabolic subalgebra of $\mathbf{H}$ is contained in at least one maximal parabolic subalgebra of $\mathbf{H}$. Note that the generating sets of maximal parabolic subalgebras are obtained by removing exactly one vertx in the Coxeter diagram in Definition 2.4.1. An analogue of Proposition 2.2.5 holds.

Proposition 2.4.12 ([KMW18, Proposition 3.4]) *Let* $\mathbf{h}$ *be a parabolic subalgebra of* $\mathbf{H}$. *Then* $\mathbf{H}$ *is a free left-module of finite rank over* $\mathbf{h}$.

For the maximal parabolic subalgebra generated by $\{S_0, \ldots, S_{n-2}\}$ this has been known for a longer time, as in this case it is easy to write down a basis using Proposition 2.4.8.

Proposition 2.4.13 (cf. [Ari02, Corollary 13.12]) *The algebra* $\mathbf{H}$ *is free as a left module over the* $(n-1)$*'st maximal parabolic subalgebra with a basis*

$$\{L_n^{e_n} T_x \mid x \in X,\ 0 \leq e_n \leq r-1\},$$

where X is the set of distinguished right coset representatives of $\mathfrak{S}_{n-1}$ *in* $\mathfrak{S}_n$.

Proposition 2.4.14 ([MM98, Theorem 3.1], [BM97, Theorem 2.8]) *The algebra* $\mathbf{H}$ *is symmetric with a symmetrising trace defined by*

$$L_1^{e_1} \cdots L_n^{e_n} T_w \mapsto \begin{cases} 1, & \text{if } e_1 = \cdots = e_n = 0,\ w = 1 \\ 0, & \text{otherwise} \end{cases}$$

for $w \in \mathfrak{S}_n$ *and* $0 \leq e_1, \ldots, e_n \leq r-1$. *The restriction to parabolic subalgebras yields a symmetrising trace on the subalgebras.*

Remark 2.4.15 The fact that the restriction to parabolic subalgebras yields a symmetrising trace there, while not explicit, follows from a close study of the proof of [MM98, Theorem 3.1].

Corollary 2.4.16 *The induction functor* $\mathrm{Ind}_{\mathbf{h}}^{\mathbf{H}}$ *is exact and bi-adjoint to* $\mathrm{Res}_{\mathbf{h}}^{\mathbf{H}}$.

Proof The algebra $\mathbf{H}$ is free of finite over $\mathbf{h}$ as a left module, $\mathbf{H}$ is symmetric and the restriction of its symmetrising traces yields a symmetrising trace on the subalgebra. More precisely: By Propositions 2.4.12 and 2.4.14, the algebras satisfy the hypotheses of Propositions 1.4.5 and 1.3.8. ∎

2.4.2. Cyclotomic Hecke Algebras of type $G(r, p, n)$

Cyclotomic Hecke algebras have been defined for all complex reflection groups by Broué and Malle, cf. [BM93]. We have already seen Ariki-Koike algebras, which are cyclotomic Hecke algebras of type $G(r, 1, n)$. Now we will focus on cyclotomic Hecke algebras of $G(r, p, n)$ where p is not necessarily 1. While these algebras can be defined in terms of generators and relations just as Ariki-Koike algebras (cf. [Ari95]) it is usually more convenient to see them as subalgebras of Ariki-Koike algebras. This is analogous to our introduction of $G(r, p, n)$ as a (normal) subgroup of $G(r, 1, n)$.

Definition 2.4.17 (cf. [Ari95, Proposition 1.6]) Suppose that r, p, and $n \geq 3$ are positive integers such that p divides r and set $d := r/p$. Let A be an integral domain containing invertible elements q and $x_1, \ldots, x_d$. Suppose that A contains elements $Q_1, \ldots, Q_r$ such that we have

$$(Y - Q_1) \cdots (Y - Q_r) = (Y^p - x_1) \cdots (Y^p - x_d)$$

in $A[Y]$. Then the *cyclotomic Hecke Algebra* $\mathbf{H}_{r,p,n}(q, x_1, \ldots, x_d)$ *of type* $G(r, p, n)$ *with parameters* $q, x_1, \ldots, x_d$ is the subalgebra of $\mathbf{H}_{r,n}(q, Q_1, \ldots, Q_r)$ generated by

$$\left\{S_0^p,\ S_0^{-1} S_1 S_0,\ S_1, \ldots, S_{n-1}\right\}.$$

Note that S_0 is invertible by Corollary 2.4.5.
As for $\mathbf{H}_{r,n}$ we will usually drop the parameters in notation.

Remark 2.4.18 Note that the condition

$$(Y - Q_1) \cdots (Y - Q_r) = (Y^p - x_1) \cdots (Y^p - x_d)$$

already implies that the Q_i are invertible: They divide the product $Q_1 \cdots Q_r$, which is equal to $x_1 \cdots x_d$ and therefore invertible, as the x_i are invertible.

For the remainder of the section fix positive integers r, p, and $n \geq 3$ such that p divides r and parameters $q, x_1, \ldots, x_d$ as well as $Q_1, \ldots, Q_r$ as in the definition. We set $\mathbf{H}' := \mathbf{H}_{r,p,n} \leq \mathbf{H} := \mathbf{H}_{r,n}$.

The cylcotomic Hecke algebra $\mathbf{H}'$ is a free A-module, and the connection between bases of $\mathbf{H}_{r,n}$ and $\mathbf{H}_{r,p,n}$ is similar to that between the elements of $G(r, 1, n)$ and those of $G(r, p, n)$. Recall that $L_k := q^{1-k} S_{k-1} \cdots S_1 S_0 S_1 \cdots S_{k-1}$ is the k'th Jucys-Murphy element of $\mathbf{H}_{r,n}$.

Proposition 2.4.19 ([Ari95, Proposition 1.6]) *The algebra* $\mathbf{H}'$ *is a free A-module with a basis given by*

$$\left\{ L_1^{e_1} \cdots L_n^{e_n} T_w \mid w \in \mathfrak{S}_n,\ 0 \leq e_1, \ldots, e_n \leq r - 1,\ \sum_{i=1}^{n} e_i \equiv 0 \pmod{p} \right\}.$$

We define parabolic subalgebras.

Definition 2.4.20 Let M be a subset of $\{S_0^p,\ S_0^{-1}S_1S_0\} \cup \{S_i \mid 1 \leq i \leq n-1\} \subseteq \mathbf{H}'$ such that M contains either both of S_0^p and $S_0^{-1}S_1S_0$ or neither of these elements. Then we call the subalgebra of $\mathbf{H}'$ generated by M a *parabolic subalgebra of* $\mathbf{H}'$.
For $1 \leq j \leq n-1$ the *j'th maximal parabolic subalgebra of* $\mathbf{H}'$ is the subalgebra of $\mathbf{H}'$ spanned by $\{S_0^p,\ S_0^{-1}S_1S_0,\ S_1, \ldots, S_{j-1},\ S_{j+1}, \ldots, S_{n-1}\}$. Similarly, the *0'th maximal parabolic subalgebra of* $\mathbf{H}'$ is the span of $\{S_i \mid 1 \leq i \leq n-1\}$.

Clearly, every parabolic subalgebra of $\mathbf{H}'$ is contained in a maximal parabolic subalgebra.

Freeness over a maximal parabolic subalgebra

To the best of the author's knowledge there is no published proof on the freeness of $\mathbf{H}'$ over parabolic subalgebras. We conclude this section by proving a freeness result for the $(n-1)$'st maximal parabolic subalgebra.

Theorem 2.4.21 *Let $n \geq 3$. Let $\mathbf{H}'_{n-1}$ be the $(n-1)$'st maximal parabolic subalgebra of $\mathbf{H}'$. Then $\mathbf{H}'$ is a free left $\mathbf{H}'_{n-1}$-module with a basis given by*

$$\mathcal{B} := \{T_x \mid x \in X\} \cup \{L_1^{r-e} L_n^e T_x \mid x \in X,\ 1 \leq e \leq r-1\},$$

where X is the set of distinguished right coset representatives of $\mathfrak{S}_{n-1}$ in $\mathfrak{S}_n$.

Corollary 2.4.22 *The induction functor $\mathrm{Ind}_{\mathbf{H}'_{n-1}}^{\mathbf{H}'}$ is exact.*

The proof of Theorem 2.4.21 is rather technical, so we split it up in several parts. While linear independence follows almost immediately, it is quite a bit harder to show that $\mathcal{B}$ actually generates $\mathbf{H}'$ as a module. By $\mathbf{H}_{n-1}$ we will denote the $(n-1)$'st maximal parabolic subalgebra of $\mathbf{H}$. It follows immediately from the definition that $\mathbf{H}'_{n-1} \leq \mathbf{H}_{n-1}$.

Lemma 2.4.23 *The set $\mathcal{B}$ is an $\mathbf{H}'_{n-1}$-linearly independent subset of $\mathbf{H}'$.*

Proof Clearly, every element of $\mathcal{B}$ is in $\mathbf{H}'$ by Proposition 2.4.19. Now suppose that for $x \in X$ and $1 \leq e \leq r-1$ there exist elements g_x and $g_{x,e}$ in $\mathbf{H}'_{n-1}$ such that

$$\left(\sum_{x \in X} g_x T_x\right) + \left(\sum_{\substack{x \in X, \\ 1 \leq e \leq r-1}} g_{x,e} L_1^{r-e} L_n^e T_x\right) = 0$$

Now L_1 is in $\mathbf{H}_{n-1}$ and so is $g_{x,e}$ for every $1 \leq e \leq r-1$ and $x \in X$. Hence, $f_{x,e} := g_{x,e} L_1^{r-e}$ is in $\mathbf{H}_{n-1}$ and the equation becomes

$$\left(\sum_{x \in X} g_x T_x\right) + \left(\sum_{\substack{x \in X \\ 1 \leq e \leq r-1}} f_{x,e} L_n^e T_x\right) = 0.$$

By Proposition 2.4.13, this implies $f_{x,e} = 0$ and $g_x = 0$ for all x and e. We know that L_1 is invertible in $\mathbf{H}$ by Corollary 2.4.5 and therefore the $g_{x,e}$, too, are 0. Thus, $\mathcal{B}$ is $\mathbf{H}'_{n-1}$-linearly independent. ∎

To show that $\mathbf{H}'$ is indeed generated by $\mathcal{B}$ as an $\mathbf{H}'_{n-1}$-module we will show that every element of the A-basis of $\mathbf{H}'$ in Proposition 2.4.19 lies in $\mathbf{H}'_{n-1}\,\mathcal{B}$.

Lemma 2.4.24 *The n'th Jucys-Murphy element L_n commutes with S_j for $1 \le j \le n-2$.*

Proof This is immediate from Lemma 2.4.9 b). ∎

Lemma 2.4.25 *Let e be an intger. Then*

$$L_1^e \in \left\langle L_1^f \mid 0 \le f \le r-1,\ f \equiv e \pmod{p} \right\rangle_A.$$

Proof Since $L_1 = S_0$ we have $(L_1 - Q_1)\cdots(L_1 - Q_r) = 0$ in $\mathbf{H}_n$, and by the conditions on the Q_i in Definition 2.4.17 this implies

$$(L_1^p - x_1)\cdots(L_1^p - x_d) = 0. \tag{2.1}$$

Thus, for $e' \ge 0$ we have $\left(L_1^p\right)^{e'} \in \left\langle \left(L_1^p\right)^{f'} \mid 0 \le f' \le (d-1)\right\rangle_A$, since larger powers of L_1^p can be reduced via the relation in (2.1). Equivalently, if $e \ge 0$ is divisible by p, then $L_1^e \in \left\langle L_1^f \mid 0 \le f \le (d-1)p,\ f \equiv 0 \pmod{p}\right\rangle_A$. Since $r = dp$, this shows the claim for all non-negative e which are divisible by p.

Now suppose $e \ge 0$ and let $k \ge 0$ and $0 \le m \le p-1$ be integers such that $e = kp + m$. We have already shown that L_1^{kp} is in $\left\langle L_1^f \mid 0 \le f \le (d-1)p,\ f \equiv 0 \pmod{p}\right\rangle_A$, and therefore we have $L_1^e = L_1^{kp}L_1^m \in \left\langle L_1^f \mid 0 \le f \le (d-1)p + m,\ f \equiv m \pmod{p}\right\rangle_A$. Since $m \equiv e \pmod{p}$ and $(d-1)p + m \le dp - 1 = r - 1$, this proves the claim for all positive e.

We turn to negative exponents: We assume $e \ge 0$ and consider L_1^{-e}.

By Corollary 2.4.5, the element L_1 is invertible in $\mathbf{H}_n$. The relation (2.1) and the fact that the x_i are invertible in A show that there exist elements g_j of A such that

$$1 = L_1\left(\sum_{j=1}^{d} g_j L_1^{pj-1}\right),$$

hence L_1^{-1} lies in $\left\langle L_1^{pk-1} \mid k \ge 1\right\rangle_A$ and therefore L_1^{-e} is in $\left\langle L_1^{pk-e} \mid k \ge e\right\rangle_A$. But $pk - e \ge 0$ for $k \ge e$, and we have already shown that L_1^{pk-e} is an element of $\left\langle L_1^f \mid 0 \le f \le r-1,\ f \equiv -e \pmod{p}\right\rangle_A$ for $pk - e \ge 0$. This shows that $L_1^{-e} \in \left\langle L_1^f \mid 0 \le f \le r-1,\ f \equiv -e \pmod{p}\right\rangle_A$. ∎

Corollary 2.4.26 *Let e be an integer and $0 \le e_2, \dots, e_n \le r-1$ such that $e + \sum_{j=2}^{n} e_j \equiv 0 \pmod{p}$. Then*

$$L_1^e L_2^{e_2} \cdots L_{n-1}^{e_{n-1}} \in \mathbf{H}'_{n-1}$$

Proof Via identifying generators of the same name it is easy to see that $\mathbf{H}'_{n-1}$ is the cyclotomic Hecke algebra of type $G(r, p, n-1)$, and hence the claim follows from Lemma 2.4.25 and Proposition 2.4.19. ∎

This allows us to move powers of L_1 past some of the generators of $\mathbf{H}'_{n-1}$:

Lemma 2.4.27 *Let $1 \le j \le n-2$ and z an integer. Then $L_1^z S_j \in \mathbf{H}'_{n-1}\, L_1^z$.*

Proof For $j \ge 2$ this is immediate from Proposition 2.4.9 b). For $j = 1$ we apply Proposition 2.4.9 a) to see that

$$L_1^z S_1 = -S_1 L_2^z + (q-1) \sum_{j=1}^{z} L_1^{j-1} L_2^{z-j+1}. \tag{2.2}$$

We have $-S_1 L_2^z = -S_1 L_2^z \left(L_1^{-z} L_1^z\right)$. By Corollary 2.4.26 the product $L_2^z L_1^{-z}$ is in $\mathbf{H}'_{n-1}$, as L_1 and L_2 commute by Lemma 2.4.6. By Definition 2.4.17 we know that S_1, too, is in $\mathbf{H}'_{n-1}$. Thus, $-S_1 L_2^z L_1^{-z}$ is an element of $\mathbf{H}'_{n-1}$, and therefore $-S_1 L_2^z = \left(-S_1 L_2^z L_1^{-z}\right) L_1^z \in \mathbf{H}'_{n-1}\, L_1^z$.
Again from the commutativity of L_1 with L_2 and Corollary 2.4.26 it follows that $L_1^{j-1} L_2^{z-j+1} = \left(L_2^{z-j+1} L_1^{j-1-z}\right) L_1^z$ is in $\mathbf{H}'_{n-1}\, L_1^z$ for $1 \le j \le z$. In total, both summands on the right-hand side of (2.2) are in $\mathbf{H}'_{n-1}\, L_1^z$ and thus also the left-hand side. ■

Corollary 2.4.28 *Let w' be in $\mathfrak{S}_{n-1}$ and z an integer. Then $L_1^z T_{w'} \in \mathbf{H}'_{n-1}\, L_1^z$.*

Proof Let $k := \ell(w')$. Then there exist integers $1 \le i_1, \ldots, i_k \le n-2$ such that $s_{i_1} \cdots s_{i_k}$ is a reduced expression of w', and therefore $T_{w'} = S_{i_1} \cdots S_{i_k}$. Hence, $L_1^z T_{w'} = L_1^z S_{i_1} \cdots S_{i_k}$, and the claim follows from Lemma 2.4.27 via induction on k. ■

This was the final ingredient to the proof of the theorem.

Proof of Theorem 2.4.21 Let $1 \le e_1, \ldots, e_n \le r-1$ with $\sum_{i=1}^n e_i \equiv 0 \pmod p$ and w in $\mathfrak{S}_n$. Then $Z := L_1^{e_1} \cdots L_n^{e_n} T_w$ is an element of the A-basis of $\mathbf{H}'$ defined in Proposition 2.4.19. By Lemma 2.4.23, the claim is proved once we show that every such Z is in $\mathbf{H}'_{n-1}\, \mathcal{B}$.
First note that there exist (unique) elements w' in $\mathfrak{S}_{n-1}$ and $x \in X$ such that $w = w'x$ and $T_w = T_{w'} T_x$. Hence, $Z = L_1^{e_1} \cdots L_n^{e_n} T_{w'} T_x$ which is equal to $L_1^{e_1} \cdots L_{n-1}^{e_{n-1}} T_{w'} L_n^{e_n} T_x$ by Lemma 2.4.24. If $e_n = 0$, then we are done, since $L_1^{e_1} \cdots L_{n-1}^{e_{n-1}} T_{w'}$ is in $\mathbf{H}'_{n-1}$ and T_x is in $\mathcal{B}$. Thus, assume $e_n \ge 1$ and set $\varepsilon := e_1 - (r - e_n)$. Then

$$Z = \left(L_1^{\varepsilon} L_2^{e_2} \cdots L_{n-1}^{e_{n-1}}\right) L_1^{r-e_n} T_{w'} L_n^{e_n} T_x,$$

using once more the fact that the L_i commute by Lemma 2.4.6. Now $L_1^{\varepsilon} L_2^{e_2} \cdots L_{n-1}^{e_{n-1}}$ is in $\mathbf{H}'_{n-1}$ by Corollary 2.4.26, since $\varepsilon \equiv e_1 + e_n \pmod p$ and $e_1 + e_n + \sum_{j=2}^{n-1} e_j \equiv 0 \pmod p$. Moreover, $L_1^{r-e_n} T_{w'}$ is in $\mathbf{H}'_{n-1}\, L_1^{r-e_n}$ by Corollary 2.4.28 and in summary this shows that Z is in $\mathbf{H}'_{n-1}\, L_1^{r-e_n} L_n^{e_n} T_x$ and therefore in $\mathbf{H}'_{n-1}\, \mathcal{B}$. ■

Remark 2.4.29 Using the explicit bases of $\mathbf{H}$ over parabolic subalgebras in [KMW18] it seems feasible to replicate the proof of Theorem 2.4.21 for the other parabolic subalgebras of $\mathbf{H}'$. However, our proof builds heavily on the fact that L_n commutes with $\mathbf{H}(\mathfrak{S}_{n-1}, q)$, and this is where things get more difficult for the other parabolic subalgebras of $\mathbf{H}'$: Here one has to be much more careful with the interaction between the different L_i's and the S_j's.

Chapter 3.

Reducibility of Induced Modules

3.1. Iwahori-Hecke Algebras

We show that parabolic induction of non-zero modules of Iwahori-Hecke algebras over splitting fields always yields reducible modules. We have already published this section's result in [Sch17]. The proof is based on analogous techniques for group algebras employed in [DM74] and stated precisely in [HHM15, Lemma 2.2]. We will later state the group algebra version explicitly in Lemma 3.2.1.

Theorem 3.1.1 *Let (W, S) be a finite Coxeter system and K a field. For s in S let u_s be an invertible element of K such that $u_s = u_t$ whenever s and t are conjugate in W. Let $\mathcal{H} := H_K(W, S, (u_s \mid s \in S))$ be the corresponding Iwahori-Hecke algebra and assume that K is a splitting field of $\mathcal{H}$. Let $J \subsetneq S$ be a proper subset of S, denote by $\mathcal{H}_J$ the parabolic subalgebra, and let M be in* mod-$\mathcal{H}_J$. *Then* $\operatorname{Ind}_J^S(M)$ *is reducible.*

Proof We will show that the endomorphism algebra of $\operatorname{Ind}_J^S(M)$ has dimension at least two from which the assumption will follow as K is a splitting field of $\mathcal{H}$. We begin with an observation on the generators. Fix an element $t \in S \setminus J$.

Claim The element t is a distinguished double coset representative of W_J and W_J in W, i.e. $t \in {}_JX_J$. Furthermore, t commutes with $J^t \cap J$.

Proof Since t is in S but not in J, we have $\ell(st) > \ell(t)$ for all s in J and therefore $t \in X_J$. As t is an involution and ${}_JX_J = X_J^{-1} \cap X_J$ by definition, it follows that t is in ${}_JX_J$.
Now let s in $J^t \cap J$ and denote by m_{st} the order of st in W. If $m_{st} = 2$, then s and t commute. Assume that m_{st} is greater than 2. Then $\underbrace{tst\cdots}_{m_{st}}$ is a reduced expression by Corollary 2.1.16 and therefore its prefix tst, too, is a reduced expression, i.e. tst has length 3 and in particular it does not lie in $J^t \cap J \subseteq S$. But tst is in $J^t \cap J$ if and only if s is, so this is contradiction to $s \in J^t \cap J$, hence m_{st} is 2 and s and t commute. ∎

Let us consider the endomorphism algebra $\operatorname{End}_{\mathcal{H}}(\operatorname{Ind}_J^S(M))$. By adjointness and the Mackey formula (cf. Corollary 2.2.14 and Proposition 2.2.15) there are isomorphisms of K-vector spaces and we have

$$\begin{aligned}
\mathrm{End}_{\mathcal{H}}(\mathrm{Ind}_J^S(M)) &= \mathrm{Hom}_{\mathcal{H}}(\mathrm{Ind}_J^S(M), \mathrm{Ind}_J^S(M)) \\
&\cong \mathrm{Hom}_{\mathcal{H}_J}(\mathrm{Res}_J^S \circ \mathrm{Ind}_J^S(M), M) \\
&\cong \bigoplus_{d \in {}_J X_J} \mathrm{Hom}_{\mathcal{H}_J}(\mathrm{Ind}^J_{J^d \cap J}(M \otimes_{\mathcal{H}_J} T_d),\ M) \\
&\cong \bigoplus_{d \in {}_J X_J} \mathrm{Hom}_{\mathcal{H}_{J^d \cap J}}(M \otimes_{\mathcal{H}_J} T_d,\ \mathrm{Res}^J_{J^d \cap J}(M)) \\
&\cong \mathrm{Hom}_{\mathcal{H}_{J^1 \cap J}}(M \otimes_{\mathcal{H}_J} T_1,\ \mathrm{Res}^J_{J^1 \cap J}(M)) \\
&\quad \oplus \mathrm{Hom}_{\mathcal{H}_{J^t \cap J}}(M \otimes_{\mathcal{H}_J} T_t,\ \mathrm{Res}^J_{J^t \cap J}(M)) \\
&\quad \oplus V
\end{aligned}$$

for some finite dimensional K-vector space V. The first two summands in the last direct sum are those for $d = 1$ and $d = t$, respectively.
Clearly, we have $\mathrm{Hom}_{\mathcal{H}_{J^1 \cap J}}(M \otimes_{H_J} T_1,\ \mathrm{Res}^J_{J^1 \cap J}(M)) = \mathrm{Hom}_{\mathcal{H}_J}(M,\ M) = \mathrm{End}_{\mathcal{H}_J}(M)$ and therefore the first summand has dimension at least 1.
Let us consider the second summand: By Proposition 2.2.15, we know that $\mathcal{H}_{J^t \cap J}$ acts on $M \otimes_{\mathcal{H}_J} T_t$ via

$$(m \otimes T_t)T_w = mT_{twt} \otimes_{\mathcal{H}_J} T_t$$

for w in $W_{J^t \cap J}$. But t commutes with every element of $J^t \cap J$ and therefore with $W_{J^t \cap J}$. Thus,

$$\mathrm{Res}^J_{J^t \cap J}(M) \to M \otimes_{\mathcal{H}_J} T_t;\ m \mapsto m \otimes_{\mathcal{H}_J} T_t$$

is an isomorphism of $\mathcal{H}_{J^t \cap J}$-modules. Hence,

$$\mathrm{Hom}_{\mathcal{H}_{J^t \cap J}}(M \otimes_{\mathcal{H}_J} T_t,\ \mathrm{Res}^J_{J^t \cap J}(M)) \cong \mathrm{End}_{\mathcal{H}_{J^t \cap J}}(\mathrm{Res}^J_{J^t \cap J}(M))$$

and therefore the second summand, too, has at least dimension 1. In total, we have shown that $\mathrm{End}_{\mathcal{H}}(\mathrm{Ind}_J^S(M))$ has dimension at least two, and as mentioned above the fact that K is a splitting field of K then implies that $\mathrm{Ind}_J^S(M)$ is reducible. ∎

Remark 3.1.2 As all Group algebras of Coxeter groups are also Iwahori-Hecke algebras, Theorem 3.1.1 shows that the usual induction from proper parabolic subgroups only yields reducible modules for group algebras of Coxeter groups over splitting fields.

3.2. Non-Exceptional Complex Reflection Groups

We prove an analogue of Theorem 3.1.1 for non-exceptional complex reflection groups by using the following lemma.

Lemma 3.2.1 ([HHM15, Lemma 2.2]) *Let G be a group and U a subgroup of G. Suppose there exists an element $z \in G \setminus U$ such that z centralises $U \cap U^z$. If K is a splitting field of G, then the induced module $\mathrm{Ind}_{K[U]}^{K[G]}(M)$ is reducible for all $K[U]$-modules $M \neq 0$.*

Let r, p, and n be positive integers such that p divides r. Moreover, let $n \geq 2$, as $G(r, p, n)$ does not have any proper parabolic subgroups for $n = 1$. Since $G(1, 1, n)$ is a Coxeter group, it suffices to consider $r \geq 2$ by Remark 3.1.2. We split up the proof into the cases $p = 1$ and $p > 1$. Hence, assume $p \geq 2$ for the rest of the section.
Set $W := G(r, 1, n)$ and for $0 \leq j \leq n-1$ denote by W_j the j'th maximal standard parabolic subgroup of W as in Definition 2.3.15.
Furthermore, set $W' := G(r, p, n)$ and W'_j its j'th maximal standard parabolic subgroup. Clearly, $W' \leq W$ and $W'_j \leq W_j$. We start with a trivial result on conjugation of Young subgroups as defined in Definition 2.3.18.

Lemma 3.2.2 *Let* $1 \leq j \leq n-1$. *Then*

$$\mathfrak{S}_{(j,n-j)} \cap \left(\mathfrak{S}_{(j,n-j)}\right)^{s_j} = \mathfrak{S}_{(j-1,1,1,n-j-1)} = \left\langle s_1, \ldots, s_{j-2}, s_{j+2}, \ldots, s_{n-1} \right\rangle.$$

Proof By definition $\mathfrak{S}_{(j,n-j)}$ is the stabiliser of the partition $\{\{1, \ldots, j\}, \{j+1, \ldots, n\}\}$ and therefore $\mathfrak{S}_{(j,n-j)}^{s_j}$ is the stabiliser of the partition

$$\{\{1, \ldots, j-1, j+1\}, \{j, j+1, \ldots, n\}\},$$

and the intersection of these stabilisers is easily seen to be $\mathfrak{S}_{(j-1,1,1,n-j-1)}$. ∎

We show that the maximal standard parabolic subgroups of W and W' satisfy the conditions of Lemma 3.2.1.

Proposition 3.2.3 *Let* $1 \leq j \leq n-1$.

a) The element s_j lies in $W \setminus W_j$ and centralises $W_j \cap W_j^{s_j}$.

b) The element t lies in $W \setminus W_0$ and centralises $W_0 \cap W_0^t$.

Proof We first prove a).
It follows from Lemmas 2.3.9 b) and 2.3.19 that s_j is not in W_j: Every element of W can be written uniquely as $a_1^{e_1} \cdots a_n^{e_n} w$ with $w \in \mathfrak{S}_n$ and $0 \leq e_1, \ldots, e_n < r$ and if this element is in W_j then w is in $\mathfrak{S}_{(j,\, n-j)}$. But clearly s_j is not in $\mathfrak{S}_{(j,\, n-j)}$, and therefore $s_j \in W \setminus W_j$.
Now suppose $g \in W_j \cap W_j^{s_j}$. Then there exist $e_1, \ldots, e_j$ with $0 \leq e_1, \ldots, e_j < r$ and w in $\mathfrak{S}_{(j,\, n-j)}$ such that $g = s_j \left(a_1^{e_1} \cdots a_j^{e_j} w\right) s_j$. By Corollary 2.3.12 this implies $g = a_1^{e_1} \cdots a_{j-1}^{e_{j-1}} a_{j+1}^{e_j} w^{s_j}$. Now g and $a_1^{e_1} \cdots a_{j-1}^{e_{j-1}}$ both lie in W_j, hence so does $a_{j+1}^{e_j} w^{s_j}$. By the description of elements of W in Lemma 2.3.9 b) and by that of elements of W_j in Lemma 2.3.19 it follows that $e_j = 0$ and $w^{s_j} \in W_j$. As w is in W_j this implies $w \in \mathfrak{S}_{(j,n-j)} \cap \left(\mathfrak{S}_{(j,n-j)}\right)^{s_j}$ which by Lemma 3.2.2 is equal to $\left\langle s_1, \ldots, s_{j-2}, s_{j+2}, \ldots, s_{n-1} \right\rangle$. By the defining relations of W we follow that s_j centralises $\mathfrak{S}_{(j,n-j)} \cap \left(\mathfrak{S}_{(j,n-j)}\right)^{s_j}$, in particular s_j commutes with w.
In total, $g = a_1^{e_1} \cdots a_{j-1}^{e_{j-1}} w$, where s_j commutes with w by the above observation and s_j commutes with $a_1, \ldots, a_{j-1}$ by Corollary 2.3.12, thus s_j centralises $W_j \cap W_j^{s_j}$.
We now turn to the proof of b).
Clearly, t is not in W_0, as $W_0 \neq W$ but $\langle W_0,\ t \rangle = W$.
Now let $g \in W_0 \cap W_0^t$. Then there exists $w \in W_0 = \mathfrak{S}_n$ such that $g = t^{-1} w t$. By Corollary

2.3.12 we have $t^{-1}wt = a_1^{-1}(w^{-1}a_1w) = a_1^{-1}a_{(1)w^{-1}}w$, since $t = a_1$. By applying Lemma 2.3.9 once more we see that therefore $t^{-1}wt$ is in W_0 if and only if $(1)w^{-1} = 1$, i.e. w fixes 1, hence $w \in \langle s_2, \dots, s_{n-1}\rangle$. By the defining relations of W it is obvious that t commutes with w and therefore also with g, so t centralises $W_0 \cap W_0^t$. ■

Corollary 3.2.4 *Let* $1 \leq j \leq n-1$.

a) The element s_j *lies in* $W' \setminus W'_j$ *and centralises* $W'_j \cap (W'_j)^{s_j}$.

b) The element $x := t^{-1}s_1t$ *lies in* $W' \setminus W'_0$ *and centralises* $W'_0 \cap (W'_0)^x$.

Proof The proof of a) follows directly from Proposition 3.2.3, since $W'_j \leq W_j$.
Let us prove b).
We have $x = t^{-1}s_1t$ in W' by definition of W'. Furthermore, $x = t^{-1}s_1t = a_1^{-1}s_1a_1 = a_1^{-1}a_2s_1$ is not in $\mathfrak{S}_n = W'_0$ by Lemma 2.3.9 d).
Now let $g := xvx$ be an element of $W'_0 \cap (W'_0)^x$, where v is an elements of W'_0 (note that x is an involution, hence $v^x = xvx$). Set $u := v^{-1}s_1$. Then

$$g = xvx = a_1^{-1}s_1a_1va_1^{-1}s_1a_1 = \underbrace{\left(a_1^{-1}a_2a_{1(u)}^{-1}a_{(2)u}\right)}_{=:z} s_1vs_1$$

by Corollary 2.3.12. But g is in W'_0 by hypothesis and therefore $z = 1$.
We now distinguish the cases $r = 2$ and $r \neq 2$, since $a_i = a_i^{-1}$ for all i if and only if $r = 2$.
First assume $r \neq 2$. Then $z = 1$ implies $(1)u = 2$ and $(2)u = 1$, or equivalently $(1)v = 1$ and $(2)v = 2$. Hence, $v \in \mathfrak{S}_{(1,\,1,\,n-2)} = \langle s_3, \dots, s_{n-1}\rangle$. Therefore, v commutes with s_1 and with t by the defining relations of $G(r,1,n)$ and thus v commutes with x. This shows $g = v$ and as g was an arbitrary element of $W'_0 \cap (W'_0)^x$ it follow that x centralises $W'_0 \cap (W'_0)^x$.

Now assume $r = 2$. Then $x = 1$ implies either

$$(1)u = 2 \quad \text{and} \quad (2)u = 1$$

or

$$(1)u = 1 \quad \text{and} \quad (2)u = 2.$$

In the first case we can proceed just as we did for $r = 2$, so assume that we are in the second case, i.e. $(1)u = 1$ and $(2)u = 2$ or, equivalently, $(2)v = 1$ and $(1)v = 2$. Then v is in $\mathfrak{S}_{(2,\,n-2)} = \langle s_1, s_3, \dots, s_{n-1}\rangle$. But then x commutes with v, since x commutes with $s_1, s_3, \dots, s_{n-1}$: For the generators $s_3, \dots, s_{n-1}$ this follows directly from the defining relations of $G(r,1,n)$, as they immediately imply that both s_1 and t commute with $s_3, \dots, s_{n-1}$. For s_1 we find

$$s_1x = s_1t^{-1}s_1t = s_1ts_1t = ts_1ts_1 = xs_1,$$

as $r = 2$ and therefore $t^{-1} = t$, and $s_1ts_1t = ts_1ts_1$ is one of the defining relations. Thus, $v = g$ commutes with x and we conclude that x centralises $W'_0 \cap (W'_0)^x$. ■

The following lemma in combination with Proposition 2.3.17 allows us to only consider maximal standard parabolic subgroups of W and W'.

Lemma 3.2.5 *Let G be a finite group and K a field. For any subgroup U of G denote by Ind_U^G the induction functor $\mathrm{Ind}_{K[U]}^{K[G]}$. Let P be a subgroup, and α a group automorphism of G, and set $Q := \alpha(P) \leq G$. For a $K[Q]$-module M denote by M^α the $K[P]$-module with underlying vector space M, on which $g \in P$ acts as $\alpha(g)$. Similarly, for a $K[G]$-module N denote by N^α the $K[G]$-module on which $h \in G$ acts via $\alpha(h)$.*
Then, for every $K[Q]$-module M there is an isomorphism of $K[G]$-modules

$$\mathrm{Ind}_P^G(M^\alpha) \cong \left(\mathrm{Ind}_Q^G(M)\right)^\alpha .$$

Proof Let $\{g_1, \ldots, g_m\} \subseteq G$ be a set of right coset representatives of P in G, i.e. $G = \coprod_{i=1}^m g_i P$. Then $\{\alpha(g_1), \ldots, \alpha(g_m)\}$ is a set of right coset representatives of $Q = \alpha(P)$ in $\alpha(G) = G$.
Now let $\{b_1, \ldots, b_\ell\}$ be a K-basis of M. Then it is well-known that

$$\{b_i \otimes_{K[P]} g_k \mid 1 \leq i \leq \ell,\ 1 \leq k \leq m\}$$

and

$$\{b_i \otimes_{K[Q]} \alpha(g_k) \mid 1 \leq i \leq \ell,\ 1 \leq k \leq m\}$$

are K-bases of $\mathrm{Ind}_P^G(M^\alpha)$ and $\left(\mathrm{Ind}_Q^G(M)\right)^\alpha$, respectively. Hence,

$$\psi : \mathrm{Ind}_P^G(M^\alpha) \to \left(\mathrm{Ind}_Q^G(M)\right)^\alpha ; b_i \otimes_{KP} g_k \mapsto b_i \otimes_{KQ} \alpha(g_k)$$

for $1 \leq i \leq \ell$ and $1 \leq k \leq m$ defines a K-vector space isomorphism. Now let h in G. For a module element v of a module V and a group element x we will write $v.x$ when we apply the action of x on V to v and we will write $v * x$ for the action of x on V^α to v, i.e. $v * x = v.\alpha(x)$. Fix $1 \leq i \leq \ell$ and $1 \leq k \leq m$ and define $1 \leq t \leq n-1$ and $\hat{h} \in P$ by $g_k h = \hat{h} g_t$. Then

$$\begin{aligned}
\psi((b_i \otimes_{K[P]} g_k)\, h) &= \psi\, (b_i \otimes_{K[P]} g_k h) \\
&= \psi\left(b_i \otimes_{K[P]} \hat{h} g_t\right) \\
&= \psi\left(b_i * \hat{h} \otimes_{K[P]} g_t\right) \\
&= \psi\left(b_i.\alpha(\hat{h}) \otimes_{K[P]} g_t\right) \\
&= b_i.\alpha(\hat{h}) \otimes_{K[Q]} \alpha(g_t) \\
&= b_i \otimes_{K[Q]} \alpha(\hat{h})\alpha(g_t) \\
&= b_i \otimes_{K[Q]} \alpha(g_k)\alpha(h) \\
&= (b_i \otimes_{K[Q]} \alpha(g_t)) * h \\
&= \psi(b_i \otimes_{K[P]} g_t) * h.
\end{aligned}$$

Hence, ψ is a $K[G]$-module isomorphism. ∎

Corollary 3.2.6 *Assume the setting of Lemma 3.2.5. Then $\mathrm{Ind}_{K[P]}^{K[G]}(N)$ is reducible for all non-zero $K[P]$-modules N if and only if $\mathrm{Ind}_{K[Q]}^{K[G]}(N')$ is reducible for all non-zero $K[Q]$-module N'.*

Theorem 3.2.7 *Let Q be a parabolic subgroup of W and K a splitting field of $K[W]$. Then $\mathrm{Ind}_{K[Q]}^{K[W]}(M)$ is reducible for every non-zero $K[Q]$-module M.*

Proof By Proposition 2.3.17 b), there exists an automorphism α of W and an integer $0 \leq j \leq n-1$ such that $\alpha^{-1}(Q)$ is a subgroup of W_j. Hence, $\alpha(W_j)$ is an overgroup of Q, and $\operatorname{Ind}_{K[Q]}^{K[W]}(M) = \operatorname{Ind}_{K[\alpha(W_j)]}^{K[W]}(\operatorname{Ind}_{K[Q]}^{K[\alpha(W_J)]}(M))$. But $\operatorname{Ind}_{K[Q]}^{K[\alpha(W_j)]}(M)$ is a non-zero $K[\alpha(W_j)]$-module and for all non-zero $K[W_j]$-modules N the induced module $\operatorname{Ind}_{K[W_j]}^{K[W]}(N)$ is reducible by Lemma 3.2.1 and Proposition 3.2.3. Now Corollary 3.2.6 proves the statement. ∎

Theorem 3.2.8 *Let Q be a parabolic subgroup of W' and K a splitting field of $K[W']$. Then $\operatorname{Ind}_{K[Q]}^{K[W']}(M)$ is reducible for every non-zero $K[Q]$-module M.*

Proof The proof works exactly as that for W, if we replace every occurrence of W with W' and apply Corollary 3.2.4 instead of Proposition 3.2.3. ∎

3.3. Ariki-Koike Algebras

We conclude this section by proving an analogue of Theorem 3.1.1 for Ariki-Koike algebras. The necessary Mackey formula was recently proved by Kuwabara, Miyachi, and Wada, cf. [KMW18].

As their notation is somewhat complicated we will not cite the full Mackey formula and all its necessary details but rather stick to what is necessary for our application.

Let $\mathbf{H} = \mathbf{H}_{n,r}(q, Q_1, \ldots, Q_r)$ be an Ariki-Koike algebra over a field K such that K is a splitting field of $\mathbf{H}$, where n and r are positive integers and $q, Q_1, \ldots, Q_r$ are invertible elements of K.

Theorem 3.3.1 *Let $\mathbf{h} \leq \mathbf{H}$ be a proper parabolic subalgebra as in Definition 2.4.11 and $0 \neq M$ an $\mathbf{h}$-module. Then $\operatorname{Ind}_{\mathbf{h}}^{\mathbf{H}}(M)$ is reducible.*

Proof For $0 \leq j \leq n-1$ denote by $\mathbf{h}_j \leq \mathbf{H}$ the j'th maximal parabolic subalgebra of $\mathbf{H}$. As induction is transitive and $\mathbf{h}$ is contained in a maximal parabolic subalgebra we can assume without loss of generality that $\mathbf{h} = \mathbf{h}_j$ for some j.

Parabolic induction and restriction between $\mathbf{h}$ and $\mathbf{H}$ are bi-adjoint by Corollary 2.4.16. Hence,

$$\begin{aligned}\operatorname{End}_{\mathbf{H}}\left(\operatorname{Ind}_{\mathbf{h}}^{\mathbf{H}}(M)\right) &= \operatorname{Hom}_{\mathbf{H}}\left(\operatorname{Ind}_{\mathbf{h}}^{\mathbf{H}}(M),\ \operatorname{Ind}_{\mathbf{h}}^{\mathbf{H}}(M)\right)\\ &\cong \operatorname{Hom}_{\mathbf{h}}\left(\operatorname{Res}_{\mathbf{h}}^{\mathbf{H}} \operatorname{Ind}_{\mathbf{h}}^{\mathbf{H}}(M),\ M\right)\end{aligned}$$

as K-vector spaces, and as for Iwahori-Hecke algebras it suffices to show that $\operatorname{Hom}_{\mathbf{h}}\left(\operatorname{Res}_{\mathbf{h}}^{\mathbf{H}} \operatorname{Ind}_{\mathbf{h}}^{\mathbf{H}}(M),\ M\right)$ is at least two-dimensional to prove the reducibility of the induced module.

Theorem 3.12 in [KMW18] states that $\operatorname{Res}_{\mathbf{h}}^{\mathbf{H}} \operatorname{Ind}_{\mathbf{h}}^{\mathbf{H}}(M)$ has a direct sum decomposition in which the direct summands are indexed by a set X of certain elements of the group $G(r, 1, n)$. This set is a generalisation of the set of distinguished double coset representatives we used for Iwahori-Hecke algebras, but the two sets do not necessarily coincide if $\mathbf{H}$ is an Iwahori-Hecke algebra. It is easy to show that 1 is always in X. If $j \geq 1$, then the generator s_j is also in X, and if $j = 0$, then the generator t of $G(r, 1, n)$ is in X. By setting $s_0 := t$ we can handle both cases simultaneously.

Thus, $N_1 \oplus N_{s_j} \leq \mathrm{Res}_{\mathbf{h}}^{\mathbf{H}} \mathrm{Ind}_{\mathbf{h}}^{\mathbf{H}}(M)$, where N_1 is the direct summand corresponding to 1 and N_{s_j} that corresponding to s_j. While N_1 is isomorphic to M, the story is slightly more complicated for N_{s_j}:
Namely, $N_{s_j} = \mathrm{Ind}_{\mathbf{g}}^{\mathbf{h}}\left(\mathrm{Res}_{\mathbf{g}}^{\mathbf{h}}(M) \otimes_{\mathbf{g}} S_j\, \mathbf{g}\right)$, where

a) S_j is the j'th standard generator of $\mathbf{H}$,

b) $\mathbf{g}$ is the subalgebra of $\mathbf{h}$ generated by $S_0, S_1, \ldots, S_{j-2}, S_{j+2}, \ldots, S_{n-1}$, if $j \neq 0$, and by $S_2, \ldots, S_{n-1}$, if $j = 0$, and

c) $S_j\, \mathbf{g}$ is a $(\mathbf{g}, \mathbf{g})$-bimodule via multiplication in $\mathbf{H}$ by [KMW18, Corollary 3.8].

Thus,

$$\mathrm{Hom}_{\mathbf{h}}\left(\mathrm{Res}_{\mathbf{h}}^{\mathbf{H}} \mathrm{Ind}_{\mathbf{h}}^{\mathbf{H}}(M),\ M\right) \geq \mathrm{Hom}_{\mathbf{h}}\left(N_1,\ M\right) \oplus \mathrm{Hom}_{\mathbf{h}}\left(N_{s_j},\ M\right).$$

As induction is left adjoint to restriction, the second summand is isomorphic to

$$\mathrm{Hom}_{\mathbf{g}}\left(\mathrm{Res}_{\mathbf{g}}^{\mathbf{h}}(M) \otimes_{\mathbf{g}} S_j\, \mathbf{g}\ ,\ \mathrm{Res}_{\mathbf{g}}^{\mathbf{h}}(M)\right)$$

as a K-vector space. By definition it is easy to see that $\mathbf{g}$ commutes with S_j and therefore $\mathrm{Res}_{\mathbf{g}}^{\mathbf{h}}(M) \otimes_{\mathbf{g}} S_j\, \mathbf{g}$ and $\mathrm{Res}_{\mathbf{g}}^{\mathbf{h}}(M)$ are isomorphic $\mathbf{g}$-modules. Hence, the dimension of $\mathrm{Hom}_{\mathbf{h}}\left(N_{s_j},\ M\right)$ is at least one, and as $\mathrm{Hom}_{\mathbf{h}}\left(N_1,\ M\right)$, too, is at least one-dimensional it follows that $\mathrm{End}_{\mathbf{H}}\left(\mathrm{Ind}_{\mathbf{h}}^{\mathbf{H}}(M)\right)$ is at least two-dimensional. Since K is a splitting field of $\mathbf{H}$ the induced module $\mathrm{Ind}_{\mathbf{h}}^{\mathbf{H}}(M)$ is reducible. ∎

Chapter 4.

Combinatorics of the Crystal Graph

In this chapter we establish the combinatorial concepts later used to study the representation theory of Ariki-Koike algebras. In particular, we study multipartitions and crystal graphs.

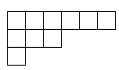
4.1. Multipartitions

Definition 4.1.1 Let n be a non-negative integer. A *partition* $\alpha = (\alpha_1, \ldots, \alpha_t)$ *of* n is a composition of n whose elements are ordered non-increasingly, i.e. $|\alpha| := \sum_{i=1}^{t} n_i = n$ and $\alpha_i \geq \alpha_{i+1}$ for all i. We write $\alpha \vdash n$ and we call α_i the *i'th part of* α.

Definition 4.1.2 Let α be a partition of n. The *Young diagram* $Y(\alpha)$ *of* $\alpha = (\alpha_1, \ldots, \alpha_t)$ is the set $\{(a, b) \mid 1 \leq a \leq t,\ 1 \leq b \leq \alpha_a\} \subseteq \mathbb{Z}^2_{>0}$. Its elements are called *boxes of the Young diagram of* α or simply *boxes of* α. For two boxes (a, b) and (a', b') we say that (a, b) *lies above* (a', b') or equivalently (a', b') *lies below* (a, b) if $a < a'$ and $b = b'$. Similarly, we say that (a, b) lies to the right of (a', b') (equivalently, (a', b') lies to the left of (a, b) if $a = a'$ and $b > b'$.

We will often not distinguish between a partition α and its Young diagram, as their is an obvious one-to-one correspondence between partitions and Young diagrams. The diagrams have a rather natural visualisation that has turned out to be very helpful to understanding the combinatorics of representations: We visualise α, or more precisely its Young diagram, by left aligned rows of boxes where the top-most row contains α_1 elements, the second one α_2 and so forth. We will identify α with this representation.

Example 4.1.3 Let $\alpha := (6, 3, 1)$. Then α is a partition of 10 and its Young diagram is visualised by the diagram below.



Definition 4.1.4 Let $r \geq 1$ be an integer. An *r-multipartition* λ *of* n is a tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ where all $\lambda^{(i)}$ are partitions and $|\lambda| := \sum_{i=1}^{r} |\lambda^{(i)}| = n$. We write $\lambda \vdash_r n$ and we call $\lambda^{(i)}$ the *i'th component of* λ.

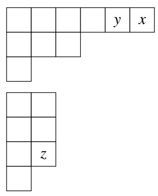
Definition 4.1.5 Let $\lambda \vdash_r n$ be an r-multipartition of n. Then the *Young diagram $Y(\lambda)$ of $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$* is defined as

$$\{(a, b, c) \mid (a, b) \text{ in the Young diagram of } \lambda^{(c)}\} \subseteq \mathbb{Z}^2_{>0} \times \{1, \dots, r\}.$$

As before the elements are called *boxes of the Young diagram of λ* or *boxes of λ*.
We visualise the Young diagram of a multipartition λ by writing the Young diagrams of its components below one another and we say that a box (a, b, c) *lies above* (a', b', c') (or equivalently that (a', b', c') *lies below* (a, b, c)) if either $c < c'$ or $c = c'$ and (a, b) lies above (a', b') in $Y(\lambda^{(c)})$ in the sense of Definition 4.1.2. Similarly, we say that (a, b, c) lies to the right of (a', b', c') if $c = c'$ and (a, b) lies to the right of (a', b') in the sense of Definition 4.1.2.

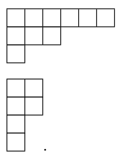
Remark 4.1.6 The definitions of boxes being above, below, to the right, or to the left of one another are simply the formal descriptions of what is read of naturally from the visualisation. With the visualisation in mind it also makes sense to talk about rows and columns of λ.

Example 4.1.7 Let $\lambda := ((6, 3, 1), (2, 2, 2, 1))$. Then λ is a 2-multipartition of 17 and its Young diagram $Y(\lambda)$ is represented by the diagram below. The box labelled by x lies above the box the labelled by z and to the right of that labelled by y.

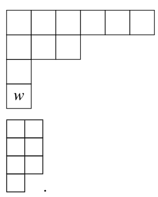


Definition 4.1.8 Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be an r-multipartition of n. A triple $(a, b, c) \in \mathbb{Z}^2_{>0} \times \{1, \dots, r\}$ is called an *addable box of λ* if (a, b, c) is not in $Y(\lambda)$ and $Y(\lambda) \cup \{(a, b, c)\}$ is the Young diagram of a partition. We write $\lambda \cup (a, b, c)$ for the partition with $Y(\lambda \cup (a, b, c)) = Y(\lambda) \cup \{(a, b, c)\}$.
Conversely, we call an element (a', b', c') of $Y(\lambda)$ a *removable box of λ*, if $Y(\lambda) \setminus \{(a', b', c')\}$ is the Young diagram of a partition. We write $\lambda \setminus (a', b', c')$ for the unique partition with $Y(\lambda \setminus (a', b', c')) = Y(\lambda) \setminus \{(a', b', c')\}$.

Example 4.1.9 Let $\lambda := ((6, 3, 1), (2, 2, 2, 1))$. Then $z := (3, 2, 2)$ as in Example 4.1.7 is a removable box of λ and the partition $\lambda \setminus (3, 2, 2)$ is



Similarly, $w := (4, 1, 1)$ is an addable box of λ and $\lambda \cup (4, 1, 1)$ has the Young diagram



4.2. The Crystal Graph

In this section we define combinatorial objects called crystal graphs and study their properties. The crystal graphs we are interested first appeared in the representation theory of the Kac-Moody algebra $\widehat{\mathfrak{sl}}_e$, i.e. the affine Lie algebra of type $A_{e-1}^{(1)}$, or rather its quantised universal enveloping algebra. Here, they appear as visualisations of *crystal bases* as defined by Kashiwara. Throughout this section we will only be interested in the combinatorial properties of crystal graphs. We refer the reader to [HK02] for details on crystal bases and crystal graphs.
A connection between crystal graphs and the representation theory of Ariki-Koike algebras has been shown to exist by what is known as *categorification results* by Ariki, Grojnowski, and Vazirani, and we will study this relation in Chapter 5.

Let $2 \leq e \in \mathbb{Z} \cup \{\infty\}$, let r be a positive integer and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{Z}^r$ an r-tuple of integers with $s_1 \leq \cdots \leq s_r$.

Definition 4.2.1 For finite e we identify $\mathbb{Z}/e\mathbb{Z}$ with the set $\{0, \ldots, e-1\}$ in the natural way and we define $\mathbb{Z}/\infty\mathbb{Z} := \mathbb{Z}$. Furthermore, for integers j and k we say that $j \equiv k \pmod{\infty}$ if and only if $j = k$.
Now let (a, b, c) be a box of an r-multipartition λ. Then its *e-residue* is defined as

$$b - a + \mathbf{s}_c \pmod{e} \in \mathbb{Z}/e\mathbb{Z},$$

A box (a, b, c) is called an *i-box* if its e-residue is equal to $i \in \mathbb{Z}/e\mathbb{Z}$. An addable i-box is called i-addable and being i-removable is defined analogously.

We begin with a first observation on i-addable and i-removable boxes in the same row.

Lemma 4.2.2 *Let $\lambda \vdash_r n$ and let $x = (a, b, c)$ be a box of λ. The box directly to its right is $y := (a, b+1, c)$, and if x is an i-box, then y is an $(i+1)$-box. In particular, if x is i-removable, then y cannot be i-addable. Hence, if a row of λ contains an i-removable box, there is no i-addable box in this row.*

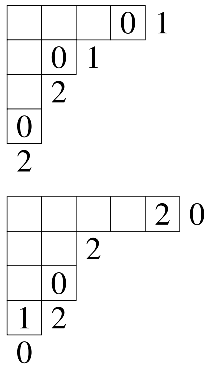
Proof We have $e \geq 2$ and thus for $k := b - a + \mathbf{s}_c$ we have $k + 1 \not\equiv k \pmod{e}$, which shows that x and y have different e-residues. ∎

The e-residue allows us to define a number of new objects necessary for the definition of the crystal graph. We follow [Ari02].

Definition 4.2.3 Let $\lambda \vdash_r n$ be an r-multipartition. Label every box of λ with its e-residue. Now fix some $i \in \mathbb{Z}/e\mathbb{Z}$. We write down a sequence of symbols $-_i$ and $+_i$ as follows: Start at the highest row of λ and move downwards. Whenever there is an i-addable box in a row, put down a symbol $+_i$, and whenever there is an i-removable box in a row put down a symbol $-_i$. Note that by Lemma 4.2.2 for each row we put down at most one symbol. In total, this will yield a sequence of $-_i$ and $+_i$ in which every symbol corresponds to an addable or removable i-box of λ. This sequence is called the *i-signature of λ*.
Now recursively remove all pairs of the shape $-_i+_i$ in the i-signature until this is no longer

possible. The resulting sequence is called the *reduced i-signature of λ*. It is independent of the order in which the pairs $-_i+_i$ were removed and hence well-defined, cf. [Ari02, Lemma 11.2]. As the reduced i-signature contains no such pairs it has the shape $(+_i)^k(-_i)^\ell$ for some non-negative integers k and ℓ. The symbols in this sequence still correspond to addable and removable boxes of λ.
The box corresponding to the left-most symbol $-_i$ is called the *i-good box of λ*.
The box corresponding to the right-most symbol $+_i$ is called the *i-co-good box of λ*.
All boxes corresponding to symbols $-_i$ in the reduced i-signature of λ are called *i-normal*.
All boxes corresponding to symbols $+_i$ in the reduced i-signature of λ are called *i-co-normal*.

Example 4.2.4 Suppose $e = 3$ and $\mathbf{s} = (0, 1)$. Let $\lambda := ((4, 2, 1, 1), (5, 2, 2, 1))$. Below we labelled the addable and removable boxes of λ with their e-residues:



Reading down from the top-right as per definition, the 0-signature of λ is $-_0 -_0 -_0 +_0 -_0 +_0$ and its reduced 0-signature is $-_0 -_0$.
The 1-signature of λ is $+_1 +_1 -_1$, and as this does not contain any subsequence $-_1+_1$ it is already reduced.
Finally, the 2-signature of λ $+_2 +_2 -_2 +_2 +_2$, therefore its reduced 2-signature is $+_2 +_2 +_2$.

Definition 4.2.5 A box of λ is called good, co-good, normal, or co-normal respectively, if it is i-good, i-co-good, i-normal, or i-co-normal for some i.

We use these new notions to define operators on multipartitions.

Definition 4.2.6 Let $i \in \mathbb{Z}/e\mathbb{Z}$. Denote by $\mathcal{P}_{r,n}$ the set of r-multipartitions of n and set $\mathcal{P}_r := \cup_{n \in \mathbb{Z}_{\geq 0}} \mathcal{P}_{r,n}$. Then we define operators

$$\widetilde{e_i} : \mathcal{P}_r \to \mathcal{P}_r \coprod \{0\}; \lambda \mapsto \begin{cases} \lambda \setminus x, & \text{if } x \text{ is the } i\text{-good box of } \lambda \\ 0, & \text{if } \lambda \text{ has no } i\text{-good box} \end{cases}$$

and

$$\widetilde{f_i} : \mathcal{P}_r \to \mathcal{P}_r \coprod \{0\}; \lambda \mapsto \begin{cases} \lambda \cup x, & \text{if } x \text{ is the } i\text{-co-good box of } \lambda \\ 0, & \text{if } \lambda \text{ has no } i\text{-good box} \end{cases}$$

Finally, for $\lambda \in \mathcal{P}_r$ we set

$$\varepsilon_i(\lambda) := \max_j \{ j \geq 0 \mid \widetilde{e_i}^j(\lambda) \neq 0 \}$$

and

$$\varphi_i(\lambda) := \max_j \{ j \geq 0 \mid \widetilde{f_i}^j(\lambda) \neq 0 \}.$$

Remark 4.2.7 The fact that any multipartition λ only has finitely many boxes shows that $\varepsilon_i(\lambda)$ is finite. It also holds that $\varphi_i(\lambda)$ is finite, but this requires more work, cf. Proposition 4.2.11.

Remark 4.2.8 Note that by 0 we denote an element that is *not* a multipartition. This is not to be confused with $\emptyset$, the multipartition of 0.

Let us study these operators.

Lemma 4.2.9 *Let $\lambda \vdash_r n$ be a multipartition and $i \in \mathbb{Z}/e\mathbb{Z}$.*
Suppose that $(+_i)^k(-_i)^\ell$ is the reduced i-signature of λ for non-negative integers k and ℓ. If $k \geq 1$, then the reduced i-signature of $\widetilde{f_i}(\lambda)$ is $(+_i)^{k-1}(-_i)^{\ell+1}$.
Similarly, if $\ell \geq 1$, then the reduced i-signature of $\widetilde{e_i}(\lambda)$ is $(+_i)^{k+1}(-_i)^{\ell-1}$.

Proof We begin with the proof for $\widetilde{f_i}(\lambda)$.
If $k \geq 1$, then λ has an i-co-normal box and in particular an i-co-good box, say x. Suppose we add this box to λ. What does this change about addable and removable boxes? Clearly, the adding of x only affects the addability and removability of boxes which lie in the same component of λ as x.

- The box x itself is still an i-box, but it is an i-removable box of $\widetilde{f_i}(\lambda) = \lambda \cup x$.
- The box directly to the right of x might become addable, depending on the length of the row above x. The box directly to the right of x has e-residue $i-1$, which is not equivalent to i, since $e \geq 2$.
- The box directly to the left of x is no longer removable if it was removable before. Its residue is $i+1 \not\equiv i \pmod e$.
- The box directly below x might become addable, depending on the the length of the row below x. The residue of this box is $i-1 \not\equiv i \pmod e$.
- The box directly above x might no longer be removable if it was removable beforehand. Its residue is $i+1 \not\equiv i \pmod e$.

These are the only differences between the addable and removable boxes of λ and of $\widetilde{f_i}(\lambda)$, respectively. But all of these boxes which are not x have e-residue not equal to i. Thus, they do not play any role in the i-signatures of λ or $\widetilde{f_i}(\lambda)$: Suppose the i-signature of λ is

$$w_1 \dots w_m +_i w_{m+1} \dots w_{m'}$$

for some non-negative integers m and m' such that $m \leq m'$, where the symbol $+_i$ corresponds to the box x and where $w_j \in \{-_i, +_i\}$ for all j. By what we have just discussed about the effects of adding x to λ, it follows that the i-signature of $\widetilde{f_i}(\lambda)$ is

$$w_1 \dots w_m -_i w_{m+1} \dots w_{m'},$$

where now $-_i$ corresponds to x, as it is a removable i-box of $\widetilde{f_i}(\lambda)$.
Now consider the procedure that computes the reduced i-signature from Definition 4.2.15. We

recursively remove pairs of the shape $-_i+_i$ until this is longer possible. As x is the i-co-good box of λ, its corresponding symbol is never removed. Suppose that two symbols w_j and $w_{j'}$ with $j < j'$ at some point in the procedure form a pair $-_i+_i$ which is then removed. Then we have either $j, j' \leq m$ or $j, j' \geq m+1$: As x is never removed, its symbol would otherwise stand between w_j and $w_{j'}$ at every step of the procedure. In other words, in the i-signature the symbols w_j and $w_{j'}$ are either both left or both right of the symbol $+_i$ corresponding to x. Therefore, the reduction process of recursively removing pairs happens independently on the sub-words $w_1 \dots w_m$ and $w_{m+1} \dots w_{m'}$. Since x is the i-co-good box of λ, the reduction of $w_1 \dots w_m$ yields $\underbrace{+_i \cdots +_i}_{k-1}$ and that of $w_{m+1} \cdots w_{m'}$ yields $\underbrace{-_i \cdots -_i}_{\ell}$ by Definition 4.2.15.

As noted above the i-signature of $\widetilde{f_i}(\lambda)$ is $w_1 \dots w_m -_i w_{m+1} \dots w_{m'}$ and by the reductions of the sub-word $w_1 \dots w_m$ and $w_{m+1} \dots w_{m'}$ we can reduce this to $\underbrace{+_i \cdots +_i}_{k-1} -_i \underbrace{-_i \cdots -_i}_{\ell}$. Since this word in the symbols $+_i$ and $-_i$ no longer contains any pairs of the form $-_i+_i$ it follows that this is the reduced i-signature of $\widetilde{f_i}(\lambda)$, thus proving the claim for $\widetilde{f_i}$.
For $\widetilde{e_i}$ everything works analogously, with the change that if x is an i-good box of λ, then x corresponds to a symbol $-_i$ in the i-signature of λ and to a symbol $+_i$ in that of $\widetilde{e_i}(\lambda)$. Now everything works just as it did for $\widetilde{f_i}$. ∎

Corollary 4.2.10 *A box x of λ is i-good if and only if x is an i-co-good box of $\lambda \setminus \{x\}$.*

Proof The boxes corresponding to the symbols $-_i$ and $+_i$ in the reduced i-signature of λ and $\lambda \setminus \{x\}$ are the same, with the exception of the box x, which corresponds to a symbol $-_i$ in the reduced i-signature of λ and to $+_i$ in the reduced i-signature of $\lambda \setminus \{x\}$. The claim now follows, as x corresponds to the left-most $-_i$ in the reduced i-signature of λ and thus to the right-most $+_i$ in the reduced i-signature of $\lambda \setminus x$ by Lemma 4.2.9. ∎

While it is clear that $\varepsilon_i(\lambda)$ is finite, the finiteness of $\varphi_i(\lambda)$ also holds.

Proposition 4.2.11 *Let $\lambda \vdash_r n$ for some positive integer n and $i \in \mathbb{Z}/e\mathbb{Z}$. By definition λ has an i-good box if and only if λ has an i-normal box. Suppose λ has $\ell \geq 1$ i-normal boxes. Then $\widetilde{e_i}(\lambda)$ has exactly $\ell - 1$ i-normal boxes. In particular, $\varepsilon_i(\lambda)$ is exactly the number of i-normal boxes of λ.*
Similarly, if λ has $k \geq 1$ i-co-normal boxes, then $\widetilde{f_i}(\lambda)$ has exactly $k-1$ i-co-normal boxes, and $\varphi_i(\lambda)$ is exactly the number of i-co-normal boxes of λ.
In particular, both $\varepsilon_i(\lambda)$ and $\varphi_i(\lambda)$ are finite.

Proof This follows directly from Lemma 4.2.9. ∎

Proposition 4.2.12 *Let $\lambda \vdash_r n$ for $n \geq 0$.*

a) *The number of addable boxes of λ is exactly by r greater than the number of removable boxes of λ.*

b) *The number of co-normal boxes of λ is exactly by r greater than the number of normal boxes of λ.*

Proof First consider only $\mu := \lambda^{(1)}$, the first component of λ. In the row exactly below every removable box there is an addable box of μ and we can pair up removable and addable boxes in this way. But there is an addable box in the very first row of μ above which there is no row and therefore no removable box, so μ has exactly one more addable boxes than removable boxes. The same holds true for the other $r-1$ components of λ and in total λ has exactly r more addable than removable boxes.

We turn to co-normal boxes.
Clearly, the number of co-normal boxes of λ is exactly the number of symbols + in the reduced signatures of λ when summing over all i, and similarly for normal boxes and the symbols −. Now suppose we are computing all reduced signatures at once. In the beginning, every box is an i-box for some i, so every addable and removable box corresponds to exactly one symbol + or − in exactly one of the i-sequences of λ. As we have just shown, the number of +'s will be exactly by r larger than the number of −'s.
Now in every step towards the reduced i-signatures we remove pairs −+, until we reach the reduced i-signatures, where the remaining symbols correspond to normal and co-normal boxes. But if in every step we only remove such pairs, then after every step the difference between the number of +'s and −'s is still equal to r. ∎

Corollary 4.2.13 *Let $\lambda \vdash_r n$ for some positive integer n. If $\widetilde{e_i}(\lambda) \neq 0$ for some $i \in \mathbb{Z}/e\mathbb{Z}$, then* $\sum_j \varphi_j(\lambda) \geq r+1$.

Proof If $\widetilde{e_i}(\lambda) \neq 0$, then λ has an i-good and therefore an i-normal box. Hence, λ has at least $r+1$ co-normal boxes, and the number of co-normal boxes of λ is exactly $\sum_j \varphi_j(\lambda)$ by Proposition 4.2.11. ∎

We finish our initial study of the operators $\widetilde{f_i}$ and $\widetilde{e_i}$ with an observation on multisets of residues.

Lemma 4.2.14 *Let $\lambda \vdash_r n$ and $i, j \in \mathbb{Z}/e\mathbb{Z}$ with $i \neq j$. If $\widetilde{f_i}(\lambda)$ and $\widetilde{f_j}(\lambda)$ are both not* 0*, then as multisets*

$$\{\{e\text{-}residue(x) \mid x \text{ a box of } \widetilde{f_i}(\lambda)\}\} \neq \{\{e\text{-}residue(x) \mid x \text{ a box of } \widetilde{f_j}(\lambda)\}\}.$$

An analogous statement holds for $\widetilde{e_i}$ and $\widetilde{e_j}$ if $\widetilde{e_i}(\lambda)$ and $\widetilde{e_j}(\lambda)$ are both not 0.

Proof Set

$$\begin{aligned}
\mathcal{R} &:= \{\!\{e\text{-residue}(x) \mid x \text{ a box of } \lambda\}\!\} \\
\mathcal{R}_i &:= \{\!\{e\text{-residue}(x) \mid x \text{ a box of } \widetilde{f_i}(\lambda)\}\!\} \\
\mathcal{R}_j &:= \{\!\{e\text{-residue}(x) \mid x \text{ a box of } \widetilde{f_j}(\lambda)\}\!\}
\end{aligned}$$

Then $\mathcal{R} \subseteq \mathcal{R}_i$ and $\mathcal{R} \subseteq \mathcal{R}_j$, as every box of λ is also a box of $\widetilde{f_i}(\lambda)$ and of $\widetilde{f_j}(\lambda)$. By Definition 4.2.6 we have

$$\mathcal{R}_i \setminus \mathcal{R} = \{\!\{y\}\!\},$$

where y is the i-co-good box of λ, and

$$\mathcal{R}_j \setminus \mathcal{R} = \{\!\{z\}\!\},$$

where z is the j-co-good box of λ. But y has e-residue i and z has e-residue j, thus $\mathcal{R}_i \neq \mathcal{R}_j$.
For $\widetilde{e_i}$ and $\widetilde{e_j}$ we can argue similarly by setting

$$\begin{aligned} \mathcal{R}'_i &:= \{\!\{e\text{-residue}(x) \mid x \text{ a box of } \widetilde{e_i}(\lambda)\}\!\} \\ \mathcal{R}'_j &:= \{\!\{e\text{-residue}(x) \mid x \text{ a box of } \widetilde{e_j}(\lambda)\}\!\} \end{aligned}$$

and then considering $\mathcal{R} \setminus \mathcal{R}'_i$ and $\mathcal{R} \setminus \mathcal{R}'_j$. ∎

We can now finally define our crystal graphs following [Ari02].

Definition 4.2.15 The *crystal graph of multicharge* **s** is the graph $\mathcal{B}_{e,\mathbf{s}}$ with vertex set $\mathcal{P}_r$ and directed, labelled edges. For λ, μ in $\mathcal{P}_r$ there is an edge $\lambda \xrightarrow{i} \mu$ with label i if and only if $\widetilde{f_i}(\lambda) = \mu$, or, equivalently, $\widetilde{e_i}(\mu) = \lambda$.
The *irreducible crystal graph of multicharge* **s** is the subgraph of $\mathcal{B}_{e,\mathbf{s}}$ defined by the connected component of the vertex $\emptyset \in \mathcal{P}_r$. We denote it by $\mathcal{B}_{e,\mathbf{s},\emptyset}$. The vertices of $\mathcal{B}_{e,\mathbf{s},\emptyset}$ are called *Kleshchev multipartitions*.

Example 4.2.16 Suppose $e = 3$ and $r = 2$. The irreducible crystal graph of multicharge $\mathbf{s} = (0, 1)$ is partially displayed in Figure 4.2.16.

Remark 4.2.17 For $e < \infty$ the graph $\mathcal{B}_{e,\mathbf{s},\emptyset}$ is in fact the so-called *crystal graph associated to an irreducible highest weight module of the quantised enveloping algebra of the affine Lie algebra* $\widehat{\mathfrak{sl}}_e$ with weight only depending on **s**, cf. [Ari02] or [HK02] for details.

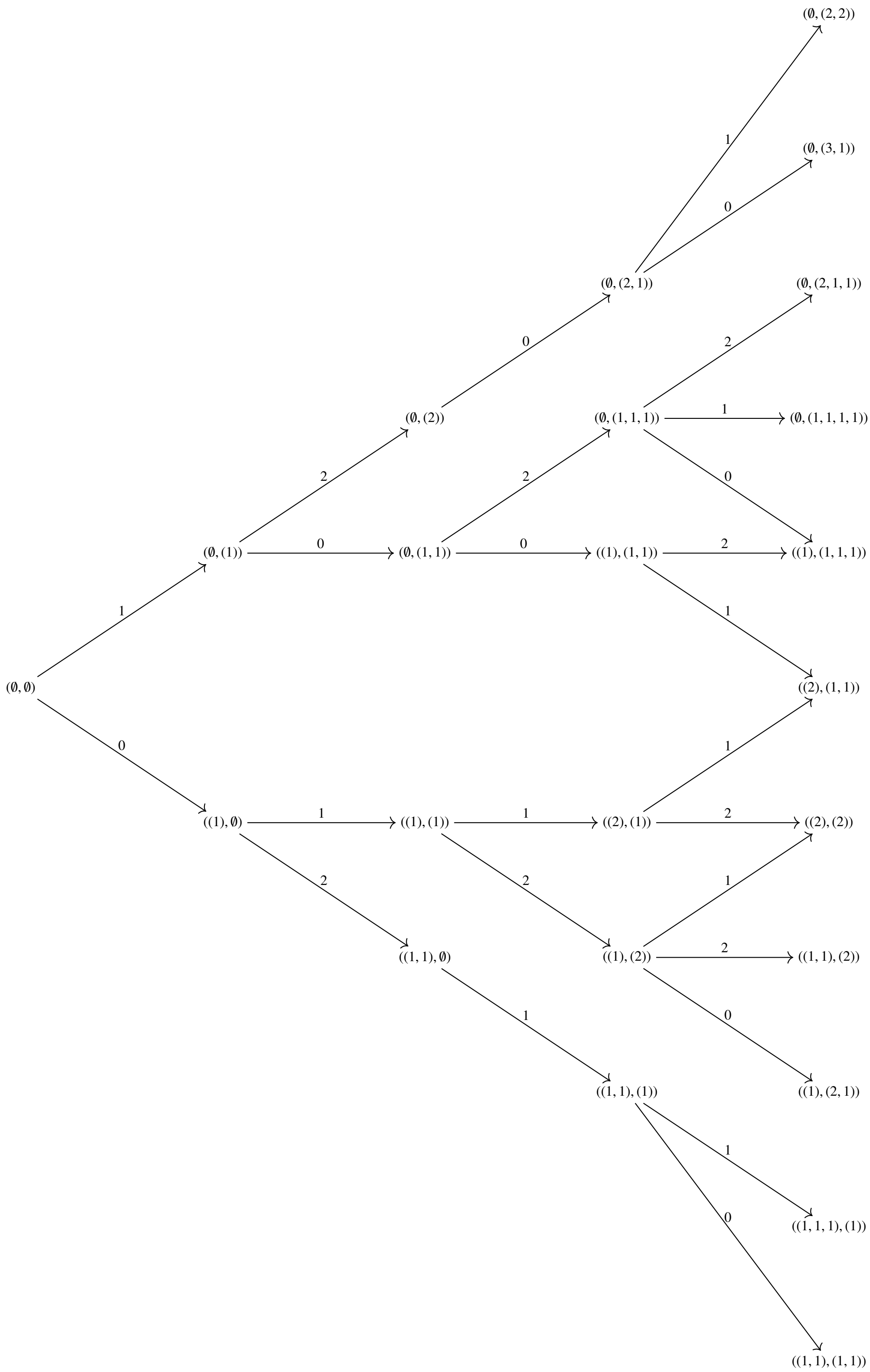
We close with some easy results following from the definition of the crystal graph.

Lemma 4.2.18 *Let λ be a Kleshchev multipartition and $\in \mathbb{Z}/e\mathbb{Z}$. Then $\widetilde{e_i}(\lambda) = 0$ for all i if and only if $\lambda = \emptyset$.*

Proof Clearly, $\emptyset$ has no good boxes, hence $\widetilde{e_i}(\emptyset) = 0$ for all i. Conversely, if λ is a Kleshchev multipartition, then it lies in the connected component of $\emptyset$. Hence, by definition there exists some path $\emptyset \xrightarrow{i_1} \mu_1 \cdots \xrightarrow{i_n} \mu_n = \lambda$ in the crystal graph for some Kleshchev multipartitions $\mu_1, \ldots, \mu_{n-1}$. Thus, $\widetilde{f_{i_n}}(\mu_{n-1}) = \lambda$, and equivalently $\widetilde{e_{i_n}}(\lambda) = \mu_{n-1} \neq 0$. ∎

Lemma 4.2.19 *Let $\lambda \in \mathcal{P}_r$ and $i \in \mathbb{Z}/e\mathbb{Z}$. By definition, $\varphi_i(\lambda)$ is the maximum length of a directed i-path in $\mathcal{B}_{e,\mathbf{s}}$ starting in the vertex λ.*

Figure 4.1.: Part of the irreducible crystal graph $\mathcal{B}_{3,(0,1),\emptyset}$



Chapter 5.

Induced Modules of Ariki-Koike Algebras

In this chapter we study certain parabolic induction functors of Ariki-Koike algebras to obtain a lower bound on the number of constituents of induced modules. We give the basics of the representation theory of Ariki-Koike algebras and then reduce the problem to what is called *q-connected parameters* using a Morita equivalence result. Subsequently, we prove our bound on the number of constituents.
We close by pointing out how these results apply to the closely related *rational Cherednik algebras*.

5.1. Standard and Simple Modules of Ariki-Koike Algebras

We outline the fundamentals of the representation theory of Ariki-Koike algebras following [Mat04].
Let r be a positive integer and K a field and *assume that K is algebraically closed*.
Let q and $Q_1, \dots, Q_r$ be invertible elements of K. For a natural number n let $\mathbf{H}_n := \mathbf{H}_{r,n}(q, Q_1, \dots, Q_r)$ be the Ariki-Koike algebra over K as in Definition 2.4.1. We study the representation theory of $\mathbf{H}_n$.
It is well-known that $\mathbf{H}_n$ is isomorphic to the n'th maximal parabolic subalgebra of $\mathbf{H}_{n+1}$ as defined in 2.4.11 and we identify the two, cf. [AK94, Corollary 3.11]. To simplify notation we set $\mathrm{Ind}_n := \mathrm{Ind}_{\mathbf{H}_n}^{\mathbf{H}_{n+1}}$ and $\mathrm{Res}_n := \mathrm{Res}_{\mathbf{H}_{n-1}}^{\mathbf{H}_n}$.

Fix a natural number n. In [DJM98], Dipper, James, and Mathas studied the representation theory of $\mathbf{H}_n$. In particular, they defined a so-called *Specht module S^λ* for every r-multipartition λ of n.

Proposition 5.1.1 ([Mat04, Definition 3.10, Theorem 3.13]) *Let λ be an r-multipartition of n. Then there exists a well-defined non-zero $\mathbf{H}_n$-module S^λ called a* Specht module. *It is finite dimensional over K and equipped with an $\mathbf{H}_n$-invariant symmetric K-bilinear form. If $\mathbf{H}_n$ is semisimple, then*

$$\{S^\mu \mid \mu \vdash_r n\}$$

is a complete set of pairwise non-isomorphic irreducible $\mathbf{H}_n$*-modules and every* S^μ *is absolutely irreducible.*

These modules contain all irreducible modules as quotients even if $\mathbf{H}_n$ is not semisimple.

Proposition 5.1.2 ([Mat04, Theorem 3.12]) *Let* $\lambda \vdash_r n$ *be an* r*-multipartition of* n *and let* $\operatorname{rad} S^\lambda$ *be the radical of the bilinear form on* S^λ*. Then*

$$D^\lambda := S^\lambda / \operatorname{rad} S^\lambda$$

is a simple module or 0*. Moreover,*

$$\{D^\lambda \mid \lambda \vdash_r n,\ D^\lambda \neq 0\}$$

is a complete set of pairwise non-isomorphic absolutely irreducible $\mathbf{H}_n$*-modules.*

Remark 5.1.3 A more detailed introduction to Specht modules and their quotients is given in Appendix B.

The relationship between D^λ and S^λ can be made more precise in terms of the so-called dominance order on the indexing partitions, a natural order on the set of r-multipartitions.

Definition 5.1.4 Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ be r-multipartitions of n. Then λ *dominates* μ if

$$\sum_{t=1}^{s-1} |\lambda^{(t)}| + \sum_{j=1}^{i} \lambda_j^{(s)} \geq \sum_{t=1}^{s-1} |\mu^{(t)}| + \sum_{j=1}^{i} \mu_j^{(s)}$$

for all $1 \leq s \leq r$ and all $i \geq 1$ (we set $\lambda_j^{(s)} = 0$ if it is not defined, i.e. we extend each $\lambda^{(s)}$ by an arbitrary number of 0's). If λ dominates μ we write $\lambda \trianglerighteq \mu$.

Lemma 5.1.5 *The dominance order is a partial order on the set of* r*-multipartitions of* n*.*

Example 5.1.6 Let $\lambda := ((2,2,1),(3,1))$, $\mu := ((7),(1,1))$ and $\nu := ((3,2,1),(3))$. Then $\nu \trianglerighteq \lambda$, but $\mu \ntrianglerighteq \nu$ and $\nu \ntrianglerighteq \mu$. In particular, this shows that $\trianglerighteq$ is not a total order in general.

Proposition 5.1.7 ([Mat04, Theorem 3.12]) *Let* λ *and* μ *be multipartitions.*

a) If $D^\lambda \neq 0$*, then the multiplicity of* D^λ *as a composition factor of* S^λ *is exactly* 1*.*

b) If $D^\lambda \neq 0 \neq D^\mu$ *and* D^μ *appears as a composition factor of* S^λ*, then* $\mu \trianglelefteq \lambda$*.*

The statement of Proposition 5.1.7 is often referred to as *unitriangularity of the decomposition matrix*. This wording will become clearer in Chapter 6. We prove a well-known fact.

Corollary 5.1.8 *The Grothendieck group* $R_0(\mathbf{H}_n)$ *is generated by*

$$\left\{\left[S^\lambda\right] \mid \lambda \vdash_r n,\ D^\lambda \neq 0\right\}$$

as an Abelian group.

Proof The classes of irreducible $\mathbf{H}_n$-modules are a $\mathbb{Z}$-basis of the Grothendieck group. By Proposition 5.1.2, every irreducible module is isomorphic to D^λ for some λ. Suppose now that $D^\lambda \neq 0$ and λ is minimal in the dominance order with respect to this property. By Proposition 5.1.2 b), we have $S^\lambda \cong D^\lambda$ and in particular $[S^\lambda] = [D^\lambda]$ in the Grothendieck group. The claim now follows via induction on the dominance order. More precisely, we can show that $[D^\mu]$ is in the subgroup of $R_0(\mathbf{H}_n)$ spanned by $\left\{[S^\lambda] \mid \lambda \trianglerighteq \mu\right\}$ via Proposition 5.1.7 and induction on the maximal length of a strictly decreasing sequence of multipartitions beginning with μ. ■

Remark 5.1.9 The above results are all due to the fundamental work of Dipper, James, and Mathas in [DJM98]. In this article they showed that Ariki-Koike algebras over fields are so-called *cellular algebras* in the sense of Graham and Lehrer. Such algebras posses special bases that are well-suited to study their representation theory. In particular, the theory of cellular algebras yields a number of so-called standard modules equipped with bilinear forms whose quotients by their radicals yield a complete set of pairwise non-isomorphic irreducibles. The theory also provides a result linking the multiplicities of simple modules in standard modules to a partial order on the standard modules.
In the case of Ariki-Koike algebras the standard modules are the Specht modules S^λ, the irreducibles the D^λ and the partial ordering is given by the dominance order on the indexing partitions.

Usually, it is much easier to study Specht modules instead of simples. For example, induction is rather well understood on Specht modules. The following is a generalisation of the well-known Young rule which describes images of induction functors in the ordinary representation theory of symmetric group.

Proposition 5.1.10 ([Mat09, Theorem A]) *Let λ be an r-multipartition of n. The induced module $\operatorname{Ind}_n(S^\lambda)$ has a filtration $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_a = \operatorname{Ind}_n(S^\lambda)$ such that for any $1 \leq j \leq a$ the quotient I_j/I_{j-1} is also a Specht module. Moreover, the Specht modules appearing as such quotients I_i/I_{i-1} are indexed exactly by the multipartitions of $n+1$ obtained by adding exactly one addable box to λ and every such multipartition appears exactly once.*

We close with an observation on the action of the Jucys-Murphy elements L_i from Definition 2.4.4 on Specht modules and irreducible modules.

Lemma 5.1.11 ([Mat04, Chapter 3]) *Let $\lambda \vdash_r n$ be a multipartition of n.*

a) All eigenvalues of L_i on the Specht modules S^λ are of the shape $q^i Q_j$ for some integers i and j with $1 \leq j \leq r$.

b) If D^λ is not 0, then all eigenvalues of L_i on D^λ are of the shape $q^i Q_j$ for some integers i and j with $1 \leq j \leq r$.

c) Let $f \in K[X_1, \ldots, X_n]^{Sym}$ be a symmetric polynomial. Then $f(L_1, \ldots, L_n)$ is in the center of $\mathbf{H}_n$ by Lemma 2.4.6 and it acts as a scalar on both S^λ and D^λ. All of its eigenvalues are symmetric polynomials evaluated at elements of the shape $q^i Q_j$ for integers i and $1 \leq j \leq r$.

We finish with a remark to retain consistency in our notation.

Remark 5.1.12 For $n = 0$, there is exactly one r-multipartition of n, namely $\emptyset \vdash_r 0$, the empty partition. The corresponding Specht module $S^{\emptyset}$ and irreducible modules $D^{\emptyset}$ are both isomorphic to the trivial one-dimensional $\mathbf{H}_0$-module.

5.2. Reducing to q-Connected Parameters

Many questions on the representation theory of Ariki-Koike algebras have only been covered for so-called q-connected parameter sets. This can be motivated by the Morita equivalence theorem by Dipper and Mathas, cf. Theorem 5.2.2 below.
We use this equivalence to reduce the problem of computing the multiplicities of irreducible modules in induced modules to Ariki-Koike algebras with q-connected parameters. This reduction has been published by the author as part of a preprint, cf. [Sch18].

Definition 5.2.1 Two elements x and y of K are called q-connected if there exists an integer k such that $x = q^k y$. We write $x \sim_q y$. Clearly, this defines an equivalence relation on K. We call a set or sequence X of elements in K q-connected if all elements of X are q-connected. Finally, if X and Y are q-connected sets (or sequences) over K we say that X and Y are q-connected if there exist elements $x \in X$ and $y \in Y$ such that x and y are q-connected.

We set $\mathbf{Q} := (Q_1, \dots, Q_r)$. As reordering of the Q_i does not change the algebra $\mathbf{H}_n$ by Proposition 2.4.3 b), we can assume without loss of generality that

$$\mathbf{Q} = \mathbf{Q}_1 \coprod \cdots \coprod \mathbf{Q}_t,$$

for q-connected sequences $\mathbf{Q}_i$ which are pairwise not q-connected, where we denote by $\coprod$ the concatenation of sequences. In particular, t is the number of $\sim_q$-equivalence classes on $\mathbf{Q}$.
For $1 \leq j \leq t$ define $r_j := |\mathbf{Q}_j|$, the length of $\mathbf{Q}_j$. Throughout this section we denote by $\otimes$ the tensor product over K.

Theorem 5.2.2 ([DM02, Theorem 1.1]) *There is a Morita equivalence*

$$\mathbf{H}_n \sim_{Morita} \mathbf{H}_n^t := \bigoplus_{\substack{0 \leq n_1, \dots, n_t \leq n, \\ \sum_i n_i = n}} {}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t},$$

where ${}^j\mathbf{H}_m := \mathbf{H}_{m,\, r_j}(q, \mathbf{Q}_j)$ *for* $0 \leq m \leq n$ *and* $1 \leq j \leq t$*. In particular, there is an exact functor*

$$F_n : \text{mod-}\mathbf{H}_n \to \text{mod-}\mathbf{H}_n^t.$$

Remark 5.2.3 As the functor F_n is exact it induces a homomorphism on the corresponding Grothendieck groups. By abuse of notation we will denote the latter, too, by F_n.
Note that the Grothendieck group of $\mathbf{H}_n^t$ is the direct sum of the Grothendieck groups of the

algebras ${}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}$. Furthermore, if M is an irreducible $\mathbf{H}^t_n$-module, then there exists exactly one composition $(n_1, \ldots, n_t)$ of n such that the subalgebra ${}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}$ acts irreducibly on M. For all other compositions $(n'_1, \ldots, n'_t) \neq (n_1, \ldots, n_t)$ of n the subalgebra ${}^1\mathbf{H}_{n'_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n'_t}$ acts as 0 on M. In this case we say that M lies in mod-${}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}$ but not in mod-${}^1\mathbf{H}_{n'_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n'_t}$.
Finally, the irreducible modules of ${}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}$ are exactly the tensor products of irreducible modules of ${}^1\mathbf{H}_{n_1}, \ldots, {}^t\mathbf{H}_{n_t}$ as all the ${}^i\mathbf{H}_{n_i}$ are split, cf. Lemma 6.1.28 below.

The functor F_n is well-understood on Specht modules and irreducibles:

Proposition 5.2.4 ([DM02, Proposition 4.11]) *Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ be an r-multipartition of n. Then define ${}^1\lambda$ to be the r_1-multipartition consisting of the first r_1 components of λ, i.e. ${}^1\lambda := (\lambda^{(1)}, \ldots, \lambda^{(r_1)})$. Let ${}^2\lambda$ be the r_2-multipartition consisting of the next r_2 components of λ, etc. In the end we have $\lambda = {}^1\lambda \coprod \cdots \coprod {}^t\lambda$.*
Then for the Specht module S^λ we have

$$F_n(S^\lambda) \cong S^{{}^1\lambda} \otimes \cdots \otimes S^{{}^t\lambda},$$

which is an $\mathbf{H}^t_n$-module by having nearly all direct summands act as zero with the exception of ${}^1\mathbf{H}_{|{}^1\lambda|} \otimes \cdots \otimes {}^t\mathbf{H}_{|{}^t\lambda|}$.
A completely analogous result holds for the module D^λ, where we just replace every S by D.

The induction Ind_n on Specht modules is well-understood by Proposition 5.1.10. We work towards an analogous result for $\mathbf{H}^t_n$. This requires the definition of a number of homomorphisms $R_0(\mathbf{H}^t_n) \to R_0(\mathbf{H}^t_{n+1})$.
For $0 \leq n_1, \ldots, n_t \leq n$ with $\sum_i n_i = n$ and $1 \leq j \leq t$ define

$${}^j\mathrm{Ind}^{(n_1,\ldots,n_t)}_{n,t} : R_0\left(\mathbf{H}^t_n\right) \to R_0\left(\mathbf{H}^t_{n+1}\right)$$

by giving its image on classes of irreducible modules. If M is an irreducible module of $\mathbf{H}^t_n$ that is not in mod-$\left({}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}\right)$, then set ${}^j\mathrm{Ind}^{(n_1,\ldots,n_t)}_{n,t}([M]) := 0$. If M is an irreducible module of $\mathbf{H}^t_n$ and in mod-$\left({}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}\right)$, then it is isomorphic to the tensor product $D_1 \otimes \cdots \otimes D_t$ for irreducible ${}^i\mathbf{H}_{n_i}$-modules D_i.
Set

$${}^j\mathrm{Ind}^{(n_1,\ldots,n_t)}_{n,t}([M]) := [D_1 \otimes \cdots \otimes D_{j-1} \otimes \left(\mathrm{Ind}^{{}^j\mathbf{H}_{n_j+1}}_{{}^j\mathbf{H}_{n_j}}\left(D_j\right)\right) \otimes D_{j+1} \otimes \cdots \otimes D_t],$$

i.e. apply parabolic induction for ${}^j\mathbf{H}_{n_j}$ in the j-th component.
By Proposition 5.1.10, this immediately yields the following.

Lemma 5.2.5 *For $1 \leq i \leq t$ with $i \neq j$ let $M_i \in$ mod-${}^i\mathbf{H}_{n_i}$. Let $\alpha \vdash_{r_j} n_j$ be a multipartition. Then*

$${}^j\mathrm{Ind}^{(n_1,\ldots,n_t)}_{n,t}([M_1 \otimes \cdots \otimes S^\alpha \otimes \cdots \otimes M_t]) = \sum_{\substack{\beta \vdash_{r_j} n_j+1, \\ |\beta \setminus \alpha|=1}} \left[M_1 \otimes \cdots \otimes S^\beta \otimes \cdots \otimes M_t\right],$$

i.e. the sum runs over all multipartitions $\beta \vdash_{r-J} n_j + 1$ obtained by adding exactly one box to α.

Now for $0 \le n_1, \ldots, n_t \le n$ with $\sum_i n_i = n$ set

$$\mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)} := \bigoplus_{j=1}^{t} {}^{j}\,\mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)}$$

and finally

$$\mathrm{Ind}_{n,t} := \bigoplus_{\substack{0\le n_1,\ldots,n_t\le n,\\ \sum_i n_i = n}} \mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)}.$$

Theorem 5.2.6 *The following diagram commutes:*

$$\begin{array}{ccc} R_0(\mathbf{H}_n) & \xrightarrow{F_n} & R_0(\mathbf{H}_n^t) \\ \big\downarrow{\scriptstyle \mathrm{Ind}_n} & & \big\downarrow{\scriptstyle \mathrm{Ind}_{n,t}} \\ R_0(\mathbf{H}_{n+1}) & \xrightarrow{F_{n+1}} & R_0(\mathbf{H}_{n+1}^t) \end{array}$$

Proof Let $\lambda \vdash_r n$ and define ${}^1\lambda, \ldots, {}^t\lambda$ as in Proposition 5.2.4. For $1 \le i \le t$ set $n_i := |{}^i\lambda|$. By definition, $\mathrm{Ind}_{n,t}([F_n(S^\lambda)]) = \mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)}([F_n(S^\lambda)])$ and by Lemma 5.2.5 and the definition of $\mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)}$ we have

$$\mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)}([F_n(S^\lambda)]) = \sum_{j=1}^{t} \sum_{\substack{\beta\vdash_{r_j} n_j+1\\ |\beta\setminus {}^j\lambda|=1}} [S^{{}^1\lambda} \otimes \cdots \otimes S^{{}^{j-1}\lambda} \otimes S^{\beta} \otimes S^{{}^{j+1}\lambda} \otimes \cdots \otimes S^{{}^t\lambda}].$$

For $1 \le j \le t$ and $\beta \vdash_{r_j} n_j + 1$ with $|\beta \setminus {}^j\lambda| = 1$ let $\mu(j,\beta) \vdash_r n + 1$ be the multipartition of $n + 1$ obtained as the concatenation ${}^1\lambda \coprod \cdots \coprod \beta \coprod \cdots \coprod {}^t\lambda$, where β is the j'th subsequence. By Proposition 5.2.4,

$$F_{n+1}([S^{\mu(j,\beta)}]) = [S^{{}^1\lambda} \otimes \cdots \otimes S^{{}^{j-1}\lambda} \otimes S^{\beta} \otimes S^{{}^{j+1}\lambda} \otimes \cdots \otimes S^{{}^t\lambda}].$$

Clearly, the $\mu(j,\beta)$ run exactly over all multipartitions of $n + 1$ which are obtained from λ by adding exactly one box and every such multipartition appears exactly once. Hence, by Proposition 5.1.10,

$$\mathrm{Ind}_n([S^\lambda]) = \sum_{j=1}^{t} \sum_{\beta\vdash_{r_j} n_j+1} [S^{\mu(j,\beta)}]$$

and we have $F_{n+1}(\mathrm{Ind}_n([S^\lambda])) = \mathrm{Ind}_n^t(F_n([S^\lambda]))$.
Since the Specht modules generate the Grothendieck group of $\mathbf{H}_n$ this already implies the commutativity of the diagram. ∎

Corollary 5.2.7 *The homomorphisms F_n and F_{n+1} obtained from the Morita equivalence preserve the number of constituents. Hence, if $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ is an r-multipartition of n such that $D^\lambda \neq 0$, then the number of irreducible constituents of the module* $\mathrm{Ind}_n(D^\lambda)$ *is equal to that of* $\mathrm{Ind}_{n,t}\left(D^{{}^1\lambda} \otimes \cdots \otimes D^{{}^t\lambda}\right)$ *by Proposition 5.2.4 and Theorem 5.2.6. By definition, this is equal to*

the number obtained by summing the number of constituents of $\operatorname{Ind}_{{}^j\mathbf{H}_{n_j}}^{{}^j\mathbf{H}_{n_j+1}}(D^{{}^j\lambda})$ *over all j, where* $n_j := |{}^j\lambda|$ *as usual.*
Since ${}^j\mathbf{H}_{n_j}$ *and* ${}^j\mathbf{H}_{n_j+1}$ *are defined for q-connected parameters we have now reduced the problem of finding the number of constituents of induced modules to the q-connected parameter case.*

Remark 5.2.8 A completely analogous result holds for restriction in place of induction. The filtration in Proposition 5.1.10 has to be replaced by that in [Mat18] for restriction of Specht modules. One has to pay attention when defining the partial restriction homomorphisms ${}^j\operatorname{Res}_{n,t}^{(n_1,\dots,n_t)}$; they are only defined for $n_j \geq 1$. Then each ${}^j\operatorname{Res}_{n,t}^{(n_1,\dots,n_t)}$ is defined on classes of irreducibles via the usual parabolic restriction in the j'th component of the tensor product.
In total, we define the restriction on the $\mathbf{H}_n^t$ side to be

$$\operatorname{Res}_{n,t} = \bigoplus_{j=1}^{t} \bigoplus_{\substack{0\leq n_1,\dots,n_t\leq n,\\ \sum_i n_i=n\\ n_j\geq 1}} {}^j\operatorname{Res}_{n,t}^{(n_1,\dots,n_t)}.$$

Then the following diagram commutes.

$$\begin{array}{ccc} R_0(\mathbf{H}_n) & \xrightarrow{F_n} & R_0(\mathbf{H}_n^t) \\ \Big\downarrow{\scriptstyle \operatorname{Res}_n} & & \Big\downarrow{\scriptstyle \operatorname{Res}_{n,t}} \\ R_0(\mathbf{H}_{n-1}) & \xrightarrow{F_{n-1}} & R_0(\mathbf{H}_{n-1}^t) \end{array}$$

The proof works completely analogous to that for induction.

5.3. On the Number of Irreducible Constituents of Induced Modules

The study of induced and restricted modules of Ariki-Koike algebras has been vastly improved by the introduction of refinements of the induction and restriction functors by Ariki in [Ari96]. These have been studied extensively by a number of authors, notably Ariki himself, Mathas, Grojnowski, and Vazirani, cf. for example [Ari06; Vaz99; Gro99; GV01; AM00]. Some of the results in this section have been published by the author as part of a preprint, cf. [Sch18].

Assume the setting and notation of Section 5.1. By Theorem 5.2.6, it is enough to only consider q-connected parameter sets, hence we assume that $\{Q_1, \dots, Q_r\}$ is q-connected.
Thus, for $1 \leq i \leq r$ there exist integers β_i such that $Q_i = q^{\beta_i} Q_1$. By Proposition 2.4.3 a), we can replace Q_i by $Q_i Q_1^{-1}$ and hence we can assume without loss of generality that all the Q_i are integer powers of q. By the same result we can replace each Q_i by $q^N Q_i$ for some fixed integer N, thus without loss of generality we can assume that all the Q_i are non-negative powers of q.

Finally, by Proposition 2.4.3 b), we can re-order the Q_i. In total, we can assume without loss of generality that there exist non-negative integers $(\gamma_1, \dots, \gamma_r)$ such that $Q_i = q^{\gamma_i}$ for $1 \le i \le r$ with $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_r$.

As we proceed we distinguish two cases depending on the parameter q. Having considered both cases we will be able to drop the condition of q-connectedness and state our result in full generality in Theorem 5.3.34.

Throughout this section denote by $e \in \mathbb{Z}_{\ge 1} \cup \{\infty\}$ the multiplicative order of q in K, that is $e := |\langle q \rangle| \le K^*$.

5.3.1. The case $e \ge 2$

Let us first assume $q \ne 1$, i.e. q is either a primitive root of unity for some $e \ge 2$ or q has infinite order in K^*. By Lemma 5.1.11, the eigenvalues of the Jucys-Murphy elements on Specht modules and irreducible modules are powers of q. Based on this observation we define a refinement of Res_n and Ind_n following [Mat04; AM00]. Recall that we set $\mathbb{Z}/\infty\mathbb{Z} := \mathbb{Z}$ in Definition 4.2.1.

Definition 5.3.1 Let M be an $\mathbf{H}_n$-module. Let $c_n := L_1 + \cdots + L_n$ and for $\alpha \in K$ define $P_{n,\alpha}(M)$ to be the generalised eigenspace of c_n on M with respect to $\alpha \in K$, that is $P_{n,\alpha}(M) = \ker(c_n - \alpha)^N \le M$ for $N \ge \dim(M)$. For $i \in \mathbb{Z}/e\mathbb{Z}$ we define *i-restriction and i-induction* functors

$$i\text{-}\mathrm{Res}_n : \text{mod-}\,\mathbf{H}_n \to \text{mod-}\,\mathbf{H}_{n-1} \quad \text{and} \quad i\text{-}\mathrm{Ind}_n : \text{mod-}\,\mathbf{H}_n \to \text{mod-}\,\mathbf{H}_{n+1}$$

as

$$i\text{-}\mathrm{Res}_n := \bigoplus_{\alpha \in K} P_{n-1,\,\alpha - q^i} \circ \mathrm{Res}_n \circ P_{n,\,\alpha} \text{ and } i\text{-}\mathrm{Ind}_n := \bigoplus_{\alpha \in K} P_{n,\,\alpha + q^i} \circ \mathrm{Ind}_n \circ P_{n-1,\,\alpha}.$$

Remark 5.3.2 This is by far not the only definition of i-Res and i-Ind found in the literature. Different authors (and sometimes the same authors in different articles) have used a number of seemingly rather different definitions of these functors. In particular, while we give Mathas's and Ariki's definition here, most of the structural results cited below are due to Grojnowski and Vazirani, whose definition is somewhat different initially. In Appendix A we show that this causes no problem as the definitions are equivalent.

It makes sense to refer to these functors as refinements of parabolic restriction and induction:

Proposition 5.3.3 ([Gro99, Lemma 8.8]) *There are natural equivalences of functors*

$$\mathrm{Res}_n \cong \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} i\text{-}\mathrm{Res}_n$$

and

$$\mathrm{Ind}_n \cong \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} i\text{-}\mathrm{Ind}_n\,.$$

This decomposition is in some way as fine as possible:

Lemma 5.3.4 ([Gro99, Corollary 9.11]) *If M is an irreducible $\mathbf{H}_n$-module, then i-$\mathrm{Res}_n(M)$ and i-$\mathrm{Ind}_n(M)$ are indecomposable for all $i \in \mathbb{Z}/e\mathbb{Z}$. In particular, $\mathrm{Res}_n(M) \cong \oplus_i\, i$-$\mathrm{Res}_n(M)$ and $\mathrm{Ind}_n(M) \cong \oplus_i\, i$-$\mathrm{Ind}_n(M)$ are decompositions into indecomposables.*

The relation of i-Ind_n and i-Res_n mirrors that of Ind_n and Res_n:

Proposition 5.3.5 *Let $i \in \mathbb{Z}/e\mathbb{Z}$, Then i-Ind is left-adjoint to i-Res.*

Proof We start with an observation on homomorphisms and eigenvalues: If N and N' are $\mathbf{H}_n$-modules and α is in K, then there is a natural isomorphism of K-vector spaces

$$\mathrm{Hom}_{\mathbf{H}_n}\left(P_{n,\alpha}(N),\ N'\right) \cong \mathrm{Hom}_{\mathbf{H}_n}\left(N, P_{n,\alpha}(N')\right) : \tag{5.1}$$

As c_n is central with all its eigenvalues lying in K, every module decomposes into a direct sum of generalised eigenspaces. An element of a direct summand corresponding to such an eigenvalue is mapped to a direct summand corresponding to that same eigenvalue under a homomorphism.

Now let M be in mod-$\mathbf{H}_{n-1}$ and N in mod-$\mathbf{H}_n$. By Corollary 2.4.16 the induction functor Ind_n is left-adjoint to Res_n and with the above isomorphism we obtain a chain of natural K-vector space isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{H}_n}\left(i\text{-}\mathrm{Ind}_n(M),\ N\right) &= \mathrm{Hom}_{\mathbf{H}_n}\left(\bigoplus_{\alpha\in K} P_{n,\alpha+q^i}\circ \mathrm{Ind}_n \circ P_{n-1,\alpha}(M),\ N\right)\\
&\cong \bigoplus_{\alpha\in K}\mathrm{Hom}_{\mathbf{H}_n}\left(P_{n,\alpha+q^i}\circ \mathrm{Ind}_n \circ P_{n-1,\alpha}(M),\ N\right)\\
&\overset{(5.1)}{\cong} \bigoplus_{\alpha\in K}\mathrm{Hom}_{\mathbf{H}_n}\left(\mathrm{Ind}_n \circ P_{n-1,\,\alpha}(M),\ P_{n,\,\alpha+q^i}(N)\right)\\
&\cong \bigoplus_{\alpha\in K}\mathrm{Hom}_{\mathbf{H}_{n-1}}\left(P_{n-1,\,\alpha}(M),\ \mathrm{Res}_n \circ P_{n,\,\alpha+q^i}(N)\right)\\
&\overset{(5.1)}{\cong} \bigoplus_{\alpha\in K}\mathrm{Hom}_{\mathbf{H}_{n-1}}\left(M,\ P_{n-1,\,\alpha}\circ \mathrm{Res}_n \circ P_{n,\,\alpha+q^i}(N)\right)\\
&\cong \mathrm{Hom}_{\mathbf{H}_{n-1}}\left(M,\ \bigoplus_{\alpha\in K} P_{n-1,\,\alpha}\circ \mathrm{Res}_n \circ P_{n,\,\alpha+q^i}(N)\right)\\
&= \mathrm{Hom}_{\mathbf{H}_{n-1}}\left(M,\ i\text{-}\mathrm{Res}_n(N)\right)
\end{aligned}$$

■

Corollary 5.3.6 *A completely analogous proof also shows that i-Ind_n is right-adjoint to i-Res_n, since Ind_n and Res_n are bi-adjoint. Hence, i-Ind_n and i-Res_n are bi-adjoint.*

Lemma 5.3.7 *The taking of generalised eigenspaces of central elements is an exact K-linear functor and so are Ind_n and Res_n. Thus, i-Res_n and i-Ind_n are K-linear exact functors for all $i \in \mathbb{Z}/e\mathbb{Z}$.*

As usual, we will denote the resulting homomorphisms between the corresponding Grothendieck groups, too, by $i\text{-Res}_n$ and $i\text{-Ind}_n$.
Just as for Ind_n the study of images of $i\text{-Ind}_n$ is rather complex, but we can study socles and heads of modules obtained via refined induction and restriction.

Definition 5.3.8 Let $\mathfrak{A}$ be a finite-dimensional K-algebra and M in mod-$\mathfrak{A}$. Then $\operatorname{soc} M$ is the submodule of M that is generated by by all its irreducible submodules, called the *socle of M*. Equivalently, $\operatorname{soc} M$ is the largest semisimple submodule of M.
Similarly, head M is the quotient of M by the intersection of all its maximal submodules, called the *head of M*. Equivalently, head M is the largest semisimple quotient of M.

Definition 5.3.9 Let $i \in \mathbb{Z}/e\mathbb{Z}$ and M in mod-$\mathbf{H}_n$. We define

$$\widetilde{E}_i(M) := \operatorname{soc} i\text{-Res}_n(M),$$

and

$$\widetilde{F}_i(M) := \operatorname{head} i\text{-Ind}_n(M).$$

The notation in terms of $\widetilde{F}_i$ and $\widetilde{F}_i$ is clearly reminiscent of the crystal operators $\widetilde{e}_i$ and $\widetilde{f}_i$ studied above. This is no accident and will become clearer later on.

By definition $\widetilde{E}_i(M)$ and $\widetilde{F}_i(M)$ are semisimple. If M is irreducible the result is much stronger.

Proposition 5.3.10 ([Gro99, Theorem 9.4]) *Let $i \in \mathbb{Z}/e\mathbb{Z}$ and M an irreducible $\mathbf{H}_n$-module.*

a) *The module $\widetilde{E}_i(M)$ is either 0 or irreducible.*

b) *The module $\widetilde{F}_i(M)$ is either 0 or irreducible.*

c) *If $N \neq 0$ is an $\mathbf{H}_{n-1}$-module, then $\widetilde{E}_i(M) = N$ if and only if $\widetilde{F}_i(N) = M$.*
In other words: If $\widetilde{E}_i(M) \neq 0$, then $\widetilde{F}_i(\widetilde{E}_i(M)) = M$. Analogously, if $\widetilde{F}_i(N) \neq 0$, then $\widetilde{E}_i(\widetilde{F}_i(N)) = N$.

The connection between $\widetilde{E}_i$ and $\widetilde{E}_j$ and between $\widetilde{F}_i$ and $\widetilde{F}_j$ for $i \neq j$ follows from the following result on direct summands of induced modules.

Lemma 5.3.11 ([Gro99, Corollary 9.10, Proposition 9.12]) *Let M be an $\mathbf{H}_n$-module. Then the socle of $\text{Res}_n(M)$ is isomorphic to its head. Similarly, the socle of $\text{Ind}_n(M)$ is isomorphic to its head. If M is irreducible, then both the socle of $\text{Res}_n(M)$ and that of $\text{Ind}_n(M)$ are multiplicity-free semisimple modules.*

Corollary 5.3.12 *Let $i, j \in \mathbb{Z}/e\mathbb{Z}$ and $i \neq j$ and M an irreducible $\mathbf{H}_n$-module.*

a) *The modules $\widetilde{E}_i(M)$ and $\widetilde{E}_j(M)$ are either both 0 or they are not isomorphic.*

b) *The modules $\widetilde{F}_i(M)$ and $\widetilde{F}_j(M)$ are either both 0 or they are not isomorphic.*

Proof We prove the claim for restriction only. The proof for induction is analogous.
By Proposition 5.3.3, and because the socle is compatible with direct sums we have

$$\begin{aligned} \operatorname{soc} \operatorname{Res}_n(M) &= \operatorname{soc} \bigoplus_{k \in \mathbb{Z}/e\mathbb{Z}} k\text{-}\operatorname{Res}_n(M) \\ &= \bigoplus_{k \in \mathbb{Z}/e\mathbb{Z}} \operatorname{soc} k\text{-}\operatorname{Res}_n(M) \\ &= \bigoplus_{k \in \mathbb{Z}/e\mathbb{Z}} \widetilde{E_k}(M). \end{aligned}$$

∎

By Lemma 5.3.11, we know that $\operatorname{soc} \operatorname{Res}_n(M)$ is multiplicity-free. If $\widetilde{E_j}(M)$ and $\widetilde{E_i}(M)$ were isomorphic, then the socle would not be multiplicity-free yielding a contradiction.

We can now state the connection between irreducible $\mathbf{H}_n$-modules and crystal graphs. Recall that irreducible modules of $\mathbf{H}_n$ are indexed by r-multipartitions and that $Q_j = q^{\gamma_j}$ for $1 \leq j \leq r$.

Theorem 5.3.13 ([Ari06, Theorem 6.1], [Vaz02, Theorem 3.5]) *Let $\lambda \vdash_r n$ be a multipartition and set $\mathbf{s} := (\gamma_1, \ldots, \gamma_r)$. Then the following holds:*

a) We have $D^\lambda \neq 0$ if and only if λ is a vertex in $\mathcal{B}_{e,\mathbf{s},\emptyset}$, i.e. if λ is a Kleshchev multipartition.

Now let $\lambda \vdash_r n$ and $\mu \vdash_r n+1$ such that $D^\lambda,\ D^\mu \neq 0$.

b) We have $\widetilde{F_i}(D^\lambda) \cong D^\mu$ if and only if $\widetilde{f_i}(\lambda) = \mu$.

c) We have $\widetilde{E_i}(D^\mu) \cong D^\lambda$ if and only if $\widetilde{e_i}(\mu) = \lambda$.

d) If $\widetilde{f_i}(\lambda) = 0$, then $\widetilde{F_i}(D^\lambda) = 0$.

e) If $\widetilde{e_i}(\lambda) = 0$, then $\widetilde{E_i}(D^\lambda) = 0$.

Remark 5.3.14 As $\widetilde{E_i}(D^\lambda)$ and $\widetilde{F_i}(D^\lambda)$ are either 0 or irreducible these operators clearly define a directed, labelled graph on the set of irreducible modules. In [Gro99] Grojnowski showed that this graph is isomorphic to $\mathcal{B}_{e,\,\mathbf{s},\,\emptyset}$. This isomorphism was then conjectured to be trivial, if one identified D^λ with λ.
The subsequent proof of this fact was then achieved first by Vazirani (for the case $e = \infty$) and then by Ariki (for the case $2 \leq e < \infty$).

Theorem 5.3.13 allows us to apply combinatorial methods to the study of i-Ind_n and i-Res_n. In particular, the number of certain constituents of reduced and restricted modules is encoded in the crystal graph.

Proposition 5.3.15 ([Gro99, Theorem 9.13, Theorem 9.15]) *Let D^λ be an irreducible $\mathbf{H}_n$-module for some $\lambda \vdash_r n$ and $i \in \mathbb{Z}/e\mathbb{Z}$.*

a) *If $\widetilde{e_i}(\lambda) \neq 0$, then the multiplicity of $\widetilde{E_i}(D^\lambda)$ as an irreducible constituent of i-$\mathrm{Res}(D^\lambda)$ is exactly $\varepsilon_i(\lambda)$.*

b) *If $\widetilde{f_i}(\lambda) \neq 0$, then the multiplicity of $\widetilde{F_i}(D^\lambda)$ as an irreducible constituent of i-$\mathrm{Ind}(D^\lambda)$ is exactly $\varphi_i(\lambda)$.*

Using a result on blocks of modules of Ariki-Koike algebras we can improve this and give the multiplicity in the whole restricted and induced modules and not only in the refined direct summands: We can replace i-Ind_n and i-Res_n by Ind_n and Res_n in Proposition 5.3.15.

Corollary 5.3.16 *Let D^λ be an irreducible $\mathbf{H}_n$-module and $i \in \mathbb{Z}/e\mathbb{Z}$.*

a) *If $\widetilde{E_i}(D^\lambda) \neq 0$, then the multiplicity of $\widetilde{E_i}(D^\lambda)$ as an irreducible constituent of $\mathrm{Res}_n(D^\lambda)$ is exactly $\varepsilon_i(\lambda)$.*

b) *If $\widetilde{F_i}(D^\lambda) \neq 0$, then the multiplicity of $\widetilde{F_i}(D^\lambda)$ as an irreducible constituent of $\mathrm{Ind}_n(D^\lambda)$ is exactly $\varphi_i(\lambda)$.*

Proof We give the proof for induction, for restriction this is proven analogously.
By Proposition 5.3.15, the multiplicity of $\widetilde{F_i}(D^\lambda)$ as a constituent of i-$\mathrm{Ind}(^\lambda_D)$ is exactly $\varphi_i(\lambda)$. Thus, as $\mathrm{Ind}(D^\lambda) = \oplus_j\ j$-$\mathrm{Ind}(D^\lambda)$ it suffices to show that $\widetilde{F_i}(D^\lambda)$ is no constituent of j-$\mathrm{Ind}_n(D^\lambda)$ for $i \neq j$. Hence, let $j \in \mathbb{Z}/e\mathbb{Z}$ such that j-$\mathrm{Ind}_n(D^\lambda)$ is not 0 and $i \neq j$. Then $\widetilde{F_i}(D^\lambda) = D^\alpha$ and $\widetilde{F_j}(D^\lambda) = D^\beta$ for some multipartitions α and β of $n + 1$.
By [Mat04, Theorem 5.5] the two modules D^α and D^β are in the same $\mathbf{H}_{n+1}$-block if and only if the multisets

$$\{\{e\text{-residue}(x) \mid x \text{ a box of } \alpha\}\}$$

and

$$\{\{e\text{-residue}(y) \mid y \text{ a box of } \beta\}\}$$

are equal. Hence, by Theorem 5.3.13, the two modules lie in different blocks by Lemma 4.2.14 and Theorem 5.3.13. Now, j-$\mathrm{Ind}_n(D^\lambda)$ is indecomposable by Lemma 5.3.4 and therefore all its constituents lie in the same block. In particular, they all lie in the same block as its head, which is isomorphic to D^β. But D^α is not in the same block as D^β and therefore D^α is not a constituent of j-$\mathrm{Ind}_n(D^\lambda)$. ∎

Now let us look at all i simultaneously.

Corollary 5.3.17 *Let D^λ be an irreducible $\mathbf{H}_n$-module for some $\lambda \vdash_r n$. Then the number of irreducible constituents of $\mathrm{Ind}_n(D^\lambda)$ is at least $\sum_{i\in\mathbb{Z}/e\mathbb{Z}} \varphi_i(\lambda)$ and the number of irreducible constituents of $\mathrm{Res}_n(D^\lambda)$ is at least $\sum_{i\in\mathbb{Z}/e\mathbb{Z}} \varepsilon_i(\lambda)$.*

Proof We only prove the first statement as the proof of the second one works completely analogous. The multiplicity of $\widetilde{F_i}(D^\lambda)$ in i-$\mathrm{Ind}_n(D^\lambda)$ is $\varphi_i(\lambda)$ and we have $\mathrm{Ind}_n(D^\lambda) \cong \oplus_{i\in\mathbb{Z}/e\mathbb{Z}}\, i$-$\mathrm{Ind}_n(D^\lambda)$. ∎

We combine this with our earlier combinatorial results to obtain this section's main theorem.

Theorem 5.3.18 *Let D^λ be an irreducible $\mathbf{H}_n$-module. Then the number of irreducible constituents of* $\mathrm{Ind}_n(D^\lambda)$ *is at least* $r+1$.

Proof This is just the combination of Corollary 4.2.13 with Corollary 5.3.17 and the fact that λ is a Kleshchev multipartition by Theorem 5.3.13. ∎

As induction is exact the following result is immediate.

Corollary 5.3.19 *Let $0 \neq M$ be an $\mathbf{H}_n$-module. Then the number of irreducible constituents of* $\mathrm{Ind}_n(M)$ *is at least $r+1$ times the number of irreducible constituents of M.*

Remark 5.3.20 By applying the results of [Gro99, Section 15] we can show that an analogue of Theorem 5.3.18 also holds for the so-called degenerate cyclotomic Hecke algebras. The proof is essentially identical to that for $\mathbf{H}_n$.

5.3.2. The case $e = 1$

Let us now assume that $q = 1$. Thus, its multiplicative order e is equal to 1.
As we assume that there exist integers γ_i such that $Q_i = q^{\gamma_i}$, we see that for $q = 1$ the Q_i are all equal to 1.

Remark 5.3.21 Note that in general $\mathbf{H}_n$ is *not* isomorphic to the so-called degenerate cyclotomic Hecke algebra as defined by Drinfel'd, cf. [Dri86].

Remark 5.3.22 For $q = 1$, the subalgebra $\mathbf{H}_n(\mathfrak{S}_n, 1) \leq \mathbf{H}_n$ from Lemma 2.4.7 is isomorphic by Example 2.2.2 to the group algebras $K[\mathfrak{S}_n]$. We identify the algebras accordingly. In particular, we will write w instead of T_w for the element of $\mathbf{H}_n(\mathfrak{S}_n, 1) \leq \mathbf{H}_n$ which corresponds to w in $\mathfrak{S}_n$ as in Lemma 2.4.7.

Proposition 5.3.23 ([Mat98, Theorem 3.7, Lemma 3.3]) *Let $D^\lambda \neq 0$ for a multipartition $\lambda = \left(\lambda^{(1)}, \dots, \lambda^{(r)}\right) \vdash_r n$.*

a) We have $\lambda^{(j)} = \emptyset$ unless $j = r$.

b) The Jucys-Murphy elements $L_1, \dots, L_n$ act trivially (i.e. as the identity) on S^λ, and hence also on D^λ.

Corollary 5.3.24 *The irreducible $\mathbf{H}_n$-modules are exactly the irreducible $K[\mathfrak{S}_n]$-modules, by letting $S_0 = L_1$ act trivially.*

Lemma 5.3.25 ([Sch18, Lemma 2.15]) *Let $\lambda = (\emptyset, \dots, \emptyset, \lambda^{(r)}) \vdash_r n$.*

a) The restriction $\mathrm{Res}^{\mathbf{H}_n}_{K[\mathfrak{S}_n]}(S^\lambda)$ is isomorphic to the Specht module $S^{\lambda^{(r)}}$ of $K[\mathfrak{S}_n]$.

b) The restriction $\mathrm{Res}^{\mathbf{H}_n}_{K[\mathfrak{S}_n]}(D^\lambda)$ is isomorphic to the $K[\mathfrak{S}_n]$-module $D^{\lambda^{(r)}}$.

Proof The proof requires a rather detailed study of Specht modules and their irreducible quotients and we postpone it to Appendix B, where this is Lemma B.2. ∎

Lemma 5.3.26 *Let* $1 \le i \le n$, $1 \le j \le n-1$, *and* $w \in \mathfrak{S}_n$.

a) We have $(L_i - 1)^r = 0$.

b) We have $S_j L_i = \begin{cases} L_{i+1}\, S_j, & \text{if } i = j, \\ L_{i-1}\, S_j, & \text{if } i-1 = j \\ L_i\, S_j, & \text{otherwise.} \end{cases}$

c) We have $wL_i = L_{(i)w^{-1}} w$.

Proof We first prove a). For $i = 1$ we have $L_1 = S_0$ by Definition 2.4.4, and $(S_0 - 1)^r = 0$ is one of the defining relations of $\mathbf{H}_n$ in Definition 2.4.1, since all the Q_i are equal to 1. As $q = 1$, we have $S_i^2 = 1$ again by Definition 2.4.1, and by Definition 2.4.4 this implies that all the L_i are conjugate in $\mathbf{H}_n$. Thus, since $(L_1 - 1)^r = 0$, we also have $(L_i - 1)^r = 0$.
We now turn to the proof of b).
The only case which does not follow directly from Lemma 2.4.9 is $i = j$, so suppose $i = j$. Since $S_i^{-1} = S_i$, we have $S_i L_i = (S_i L_i S_i)\, S_i$, and by Definition 2.4.4 we have $S_i L_i S_i = L_{i+1}$, which proves b).
Part c) follows from b) via induction: Let $s_{i_1} \cdots s_{i_k}$ be a reduced expression of w in $\mathfrak{S}_n$, where as usual s_k is the transposition $(k, k+1)$ in $\mathfrak{S}_n$. By definition we have $w = S_{i_1} \cdots S_{i_k}$ in $\mathbf{H}_n$. By b) we therefore have $wL_i = S_{i_1} \cdots S_{i_{k-1}} L_{(i)s_{i_k}} S_{i_k}$, hence a simple induction argument shows that

$$wL_i = L_{(1)s_{i_k} \cdots s_{i_1}} w.$$

This proves c), as it is well-known that $s_{i_k} \cdots s_{i_1} = w^{-1}$. ∎

Let us study induced modules.

Proposition 5.3.27 *Let* D^λ *be an irreducible* $\mathbf{H}_n$*-module for some* $\lambda = (\emptyset, \ldots, \emptyset, \lambda^{(r)}) \vdash_r n$ *and* B *a* K*-basis of* D^λ. *Furthermore, denote by* Y *the set of distinguished right coset representatives of* $\mathfrak{S}_n$ *in* $\mathfrak{S}_{n+1}$. *Finally, set* $\widehat{\operatorname{Ind}_n} := \operatorname{Ind}_{K[\mathfrak{S}_n]}^{K[\mathfrak{S}_{n+1}]}$.
For $0 \le \ell < r$ *let* M_ℓ *be the* K*-vector space spanned by*

$$\left\{ b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^j\, y \mid b \in B,\ y \in Y,\ \ell \le j < r \right\}.$$

and set $M_r := 0$. *Then the following holds.*

a) For every ℓ, M_ℓ *is an* $\mathbf{H}_{n+1}$*-module. Moreover, the above spanning set of* M_ℓ *is a* K*-basis.*

b) We have $0 = M_r \lneq M_{r-1} \lneq \cdots \lneq M_1 \lneq M_0 = \operatorname{Ind}_n(D^\lambda)$.

c) Set $N_\ell := M_\ell / M_{\ell+1}$ *for* $0 \le \ell \le r-1$. *Then* L_i *acts trivially on* N_ℓ *for any* $0 \le i \le n+1$.

d) For all ℓ, *the restriction of* N_ℓ *to* $K[\mathfrak{S}_{n+1}]$ *is isomorphic to* $\widehat{\operatorname{Ind}_n}\left(D^{\lambda^{(r)}}\right)$. *Note that* $\widehat{\operatorname{Ind}_n}\left(D^{\lambda^{(r)}}\right)$ *does* not *depend on* ℓ.

Proof Throughout the proof we denote by $\otimes$ the tensor product $\otimes_{\mathbf{H}_n}$.
We begin by showing that the generating sets given for the M_ℓ are actually K-bases.
As explained in the introduction to this chapter, $\mathbf{H}_n$ is the n'th maximal parabolic subalgebra of $\mathbf{H}_{n+1}$. By Proposition 2.4.13, this implies that $\mathbf{H}_{n+1}$ is a free left $\mathbf{H}_n$-module and a basis is given by

$$\left\{L_{n+1}^j y \mid y \in Y,\ 0 \leq j < r\right\}.$$

Thus,

$$C := \left\{b \otimes L_{n+1}^j y \mid b \in B,\ y \in Y,\ 0 \leq j < r\right\}.$$

is a K-basis of $\mathrm{Ind}_n(D^\lambda) = D^\lambda \otimes \mathbf{H}_{n+1}$. Let

$$E := \left\{b \otimes (L_{n+1} - 1)^j y \mid b \in B,\ y \in Y,\ 0 \leq j < r\right\}.$$

We show that $\langle C\rangle_K = \langle E\rangle_K$:
Since E is contained in $D^\lambda \otimes \mathbf{H}_n = \mathrm{Ind}_n(D^\lambda)$ and C is a basis of $\mathrm{Ind}_n(D^\lambda)$, we have

$$\langle E\rangle_K \leq \langle C\rangle_K, \tag{5.2}$$

and it remains to show that every element of C is in $\langle E\rangle_K$. To prove this, let $0 \leq j \leq r-1$, $b \in B$, and $y \in Y$. We proceed by induction on j.
If $j = 0$, then $b \otimes L_{n+1}^0 y = b \otimes y = b \otimes (L_{n+1} - 1)^0 y \in E$.
Now let $j \geq 1$ and assume that for all $j' < j$ we have already shown that $b \otimes L_{n+1}^{j'} y \in \langle E\rangle_K$. Then

$$b \otimes (L_{n+1} - 1)^j y = b \otimes L_{n+1}^j y + \sum_{k=0}^{j-1} (-1)^{j-k} \binom{j}{k} \left(b \otimes L_{n+1}^k y\right),$$

by simply expanding $(L_{n+1} - 1)^j$, and therefore

$$b \otimes L_{n+1}^j y = \underbrace{b \otimes (L_{n+1} - 1)^j y}_{\text{in } E \text{ by definition}} - \underbrace{\sum_{k=0}^{j-1} (-1)^{j-k} \binom{j}{k} \left(b \otimes L_{n+1}^k y\right)}_{\text{in } \langle E\rangle_K \text{ by induction assumption}}.$$

Hence, $b \otimes L_{n+1}^j y$ is in $\langle E\rangle_K$ for all j, b, and y. Thus, $\langle C\rangle_K \leq \langle E\rangle_K$, which implies $\langle C\rangle_K = \langle E\rangle_K$ by (5.2). Now C is a K-basis of $\mathrm{Ind}_n(D^\lambda)$ and by definition E contains at most as many elements as C. As $\langle E\rangle_K = \langle C\rangle_K = \mathrm{Ind}_n(D^\lambda)$, this shows that E is a K-basis of $\mathrm{Ind}_n(D^\lambda)$. Since the generating sets of the M_ℓ are subsets of E, they are K-linearly independent sets and therefore K-bases of the M_ℓ, thus proving the second statement of a). Moreover, this implies $M_{\ell+1} \subsetneq M_\ell$.

We now turn to the $\mathbf{H}_{n+1}$-action. Let $0 \leq \ell \leq r$. To show that M_ℓ is an $\mathbf{H}_{n+1}$-module it suffices to show that the vector space is closed under right multiplication with $S_0 = L_1$ and the elements of $K[\mathfrak{S}_{n+1}]$, as these together generate $\mathbf{H}_{n+1}$. To this end let $\ell \leq j < r$, $b \in B$, and $y \in Y$.
First let $w \in \mathfrak{S}_{n+1}$. Since Y is a set of coset representatives of $\mathfrak{S}_n$ in $\mathfrak{S}_{n+1}$, there exist some $z \in Y$ and $v \in \mathfrak{S}_n$ such that $yw = vz$. By Lemma 2.4.24, we have $vL_{n+1} = L_{n+1}v$, hence

$$\left(b \otimes (L_{n+1} - 1)^j y\right) w \ = b \otimes (L_{n+1} - 1)^j vz$$

$$= b \otimes v(L_{n+1} - 1)^j z$$
$$= bv \otimes (L_{n+1} - 1)^j z.$$

Since $bv \in D^\lambda = \langle B \rangle_K$, this shows that

$$\left(b \otimes (L_{n+1} - 1)^j y\right) w = bv \otimes (L_{n+1} - 1)^j z \in M_\ell. \tag{5.3}$$

We now show compatibility with multiplication with L_1. We have $yL_1 = L_{(1)y^{-1}}y$ by Lemma 5.3.26 c). Therefore,

$$\left(b \otimes (L_{n+1} - 1)^j y\right)(L_1 - 1) = b \otimes (L_{n+1} - 1)^j (L_{(1)y^{-1}} - 1)y. \tag{5.4}$$

Let us first assume $(1)y^{-1} \neq n + 1$. Then $L_{(1)y^{-1}}$ commutes with L_{n+1} by Lemma 2.4.6 and $L_{(1)y^{-1}}$ is an element of $\mathbf{H}_n$. Thus, we have

$$b \otimes (L_{n+1} - 1)^j (L_{(1)y^{-1}} - 1)y = b(L_{(1)y^{-1}} - 1) \otimes (L_{n+1} - 1)^j y.$$

By Proposition 5.3.23 b), $L_{(1)y^{-1}}$ acts trivially on D^λ, hence

$$b(L_{(1)y^{-1}} - 1) \otimes (L_{n+1} - 1)^j y = 0.$$

In total, we have $\left(b \otimes (L_{n+1} - 1)^j y\right)(L_1 - 1) = 0$, or, equivalently,

$$\left(b \otimes (L_{n+1} - 1)^j y\right) L_1 = b \otimes (L_{n+1} - 1)^j y, \tag{5.5}$$

if $(1)y^{-1} \neq n + 1$.
Now assume $(1)y^{-1} = n + 1$. Then

$$b \otimes (L_{n+1} - 1)^j (L_{(1)y^{-1}} - 1)y = b \otimes (L_{n+1} - 1)^{j+1} y,$$

hence, by (5.4), in total we have

$$\left(b \otimes (L_{n+1} - 1)^j y\right) L_1 = b \otimes (L_{n+1} - 1)^{j+1} y + b \otimes (L_{n+1} - 1)^j y, \tag{5.6}$$

if $(1)y^{-1} = n + 1$.
If $j \leq r - 2$, then $b \otimes (L_{n+1} - 1)^{j+1} y$ and $b \otimes (L_{n+1} - 1)^j y$ lie in M_ℓ by definition. If $j = r - 1$, then $b \otimes (L_{n+1} - 1)^j y$ lies in M_ℓ by definition and by Lemma 5.3.26 a) we have $(L_{n+1} - 1)^r = 0$, thus $b \otimes (L_{n+1} - 1)^{j+1} y = b \otimes (L_{n+1} - 1)^r y = 0$, and 0 is in M_ℓ.
In conclusion, (5.3), (5.5), and (5.6) show that M_ℓ is indeed an $\mathbf{H}_{n+1}$-module, thus proving a).
Since $M_{\ell+1} \subsetneq M_\ell$, this also finishes the proof of b).
We turn to the proof of c).
We have shown above that the generating sets of the M_ℓ are in fact K-bases. It follows that

$$\{b \otimes (L_{n+1} - 1)^\ell y \; + \; M_{\ell+1} \mid b \in B, \; y \in Y\} \tag{5.7}$$

is a K-basis for N_ℓ. Now (5.5) and (5.6) imply that L_1 acts trivially on the elements of this basis and therefore also on N_ℓ. By Definition 2.4.4, all the L_i are conjugate in $\mathbf{H}_{n+1}$, since $S_j^{-1} = S_j$

for all $1 \leq j \leq n$, and therefore the L_i, too, act trivially on N_ℓ. This concludes the proof of c). We finally prove d).
First recall that Y is a set of distinguished right coset representatives of $\mathfrak{S}_n$ in $\mathfrak{S}_{n+1}$ and B is a basis of D^λ and therefore also for $D^{\lambda^{(r)}}$ by Lemma 5.3.25. It follows that $\{b \otimes_{K[\mathfrak{S}_n]} y \mid b \in B,\ y \in Y\}$ is a K-basis of $\widehat{\mathrm{Ind}_n}\left(D^{\lambda^{(r)}}\right)$. Together with the K-basis of N_ℓ in (5.7) this shows that

$$\Psi : \widehat{\mathrm{Ind}_n}\left(D^{\lambda^{(r)}}\right) \to \mathrm{Res}^{\mathbf{H}_{n+1}}_{K[\mathfrak{S}_{n+1}]}(N_\ell)\,;\ b \otimes_{K[\mathfrak{S}_n]} y \mapsto b \otimes (L_{n+1} - 1)^\ell y + M_{\ell+1}$$

for $b \in B$ and $y \in Y$ defines an isomorphism of K-vector spaces.
Now let $w \in \mathfrak{S}_{n+1}$, $b \in B$, and $y \in Y$. As above there exist $v \in \mathfrak{S}_n$ and $z \in Y$ such that $yw = vz$. Thus, the action of w on $b \otimes_{K[\mathfrak{S}_n]} y$ is given by

$$(b \otimes_{K[\mathfrak{S}_n]} y)\, w = bv \otimes_{K[\mathfrak{S}_n]} z. \tag{5.8}$$

A comparison of (5.3) with (5.8) now shows that Ψ is an isomorphism of $K[\mathfrak{S}_{n+1}]$-modules. ■

Corollary 5.3.28 *Let D^λ be an irreducible $\mathbf{H}_n$-module with $\lambda = (\emptyset, \ldots, \emptyset, \lambda^{(r)}) \vdash_r n$. Suppose $t \geq 1$ is the number of irreducible constituents of the $K[\mathfrak{S}_{n+1}]$-module $\widehat{\mathrm{Ind}_n}(D^{\lambda^{(r)}})$. Then the number of irreducible constituents of* $\mathrm{Ind}_n(D^\lambda)$ *is exactly the product rt.*

Theorem 5.3.29 *Let D^λ be an irreducible $\mathbf{H}_n$-module for $\lambda = (\emptyset, \ldots, \emptyset, \lambda^{(r)}) \vdash_r n$. Then* $\mathrm{Ind}_n(D^\lambda)$ *has at least $2r$ irreducible constituents.*

Proof The induced module $\widehat{\mathrm{Ind}_n}(D^{\lambda^{(r)}})$ has at least 2 irreducible constituents by Theorem 3.1.1. The claim now follows from Corollary 5.3.28. ■

Remark 5.3.30 Note that Lemma 5.3.25 is not strictly necessary to prove Theorem 5.3.29, as we only need that $\widehat{\mathrm{Ind}_n}(\mathrm{Res}^{\mathbf{H}_n}_{K[\mathfrak{S}_n]}(D^\lambda))$ has at least 2 constituents which follows from Theorem 3.1.1 without knowing the isomorphism type of $\mathrm{Res}^{\mathbf{H}_n}_{K[\mathfrak{S}_n]}(D^\lambda)$. However, Lemma 5.3.25 makes our notation much nicer and allows us to compute the socle of induced modules in Proposition 5.3.31 below.

We conclude this subsection by describing the socle of the induced modules, as this can be obtained with barely any additional work and nicely complements the branching rules for $e \geq 2$.

Proposition 5.3.31 *Assume the setting and notation of Proposition 5.3.27. Then the socle of* $\mathrm{Ind}_n(D^\lambda)$ *is contained in M_{r-1}. More precisely, the socle of* $\mathrm{Ind}_n(D^\lambda)$ *is isomorphic to the socle of* $\widehat{\mathrm{Ind}_n}(D^{\lambda^{(r)}})$*, by letting S_0 act trivially.*

Proof The L_i act trivially on irreducible $\mathbf{H}_{n+1}$-modules by Proposition 5.3.23, and therefore also on the socle of $\mathrm{Ind}_n(D^\lambda)$, as it is semisimple. By parts a) and b) of Proposition 5.3.27, a basis of $\mathrm{Ind}_n(D^\lambda) = M_0$ is given by

$$\{b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^j y \mid b \in B,\ y \in Y,\ 0 \leq j < r\}.$$

For $b \in B$ and $y \in Y$ define $V_{b,y} := \left\langle b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^j y \mid 0 \le j < r \right\rangle_K$. Clearly, we have a vector space decomposition

$$\mathrm{Ind}_n(D^\lambda) = \bigoplus_{b \in B,\ y \in Y} V_{b,y}. \tag{5.9}$$

Let $b \in B$, $y \in Y$, $0 \le j \le r-1$, and $1 \le i \le n+1$. Completely analogously to (5.5) and (5.6) one can show that

$$\begin{aligned} &\left(b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^j y\right) L_i \\ &= \begin{cases} b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^j y, & \text{if } (i)y^{-1} \neq n+1, \\ b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^j y + b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^{j+1} y, & \text{if } (i)y^{-1} = n+1. \end{cases} \end{aligned} \tag{5.10}$$

Let $\langle L_1, \ldots, L_{n+1} \rangle$ be the subalgebra of $\mathbf{H}_{n+1}$ generated by $L_1, \ldots, L_{n+1}$. Then (5.10) shows that $V_{b,y}$ is an $\langle L_1, \ldots, L_{n+1} \rangle$-module, thus the decomposition in (5.9) is a decomposition of $\mathrm{Ind}_n(D^\lambda)$ into $\langle L_1, \ldots, L_{n+1} \rangle$-submodules.
Now set $j_y := (n+1)y$. Since $(L_{n+1} - 1)^r = 0$ by Lemma 5.3.26 a), it follows from (5.10) that $b \otimes_{\mathbf{H}_n} y = b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^0 y$ is a cyclical vector for the action of L_{j_y} on $V_{b,y}$, and that the minimal polynomial of L_{j_y} on $V_{b,y}$ is $(x-1)^r$. Since the dimension of $V_{b,y}$ is r, the minimal polynomial of L_{j_y} is equal to its characteristic polynomial. Therefore, the eigenspace of L_{j_y} on $V_{b,y}$ with respect to the eigenvalue 1 is one-dimensional, and by (5.10) it is spanned by $b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^{r-1} y$. Again by (5.10) it follows that every L_i with $i \neq j_y$ acts trivially on this eigenspace.
Since (5.9) is a decomposition into $\langle L_1, \ldots, L_{n+1} \rangle$-modules, this shows that the common eigenspace of $L_1, \ldots, L_{n+1}$ on $\mathrm{Ind}_n(D^\lambda)$ with respect to the eigenvalue 1 is exactly the K-span of

$$\{b \otimes (L_{n+1} - 1)^{r-1} y \mid b \in B,\ y \in Y\}.$$

This K-span is exactly the $\mathbf{H}_{n+1}$-module $M_{r-1} \le M_0$. Since the L_i act trivially on the socle of $\mathrm{Ind}_n(D^\lambda)$, the socle is contained in their common eigenspace with respect to the eigenvalue 1, hence we have

$$\mathrm{soc}\,\mathrm{Ind}_n(D^\lambda) = \mathrm{soc}\, M_{r-1}.$$

As the L_i act trivially on M_{r-1}, it follows that the irreducible $\mathbf{H}_{n+1}$-submodules of M_{r-1} are exactly the irreducible $K[\mathfrak{S}_{n+1}]$-submodules of M_{r-1}. Therefore, the socle of M_{r-1} is equal as a set to the socle of $\mathrm{Res}^{\mathbf{H}_{n+1}}_{K[\mathfrak{S}_{n+1}]}(M_{r-1})$. By Proposition 5.3.27 d), and because $M_r = 0$ we have $M_{r-1} \cong N_{r-1}$, and thus

$$\mathrm{soc}\,\mathrm{Res}^{\mathbf{H}_{n+1}}_{K[\mathfrak{S}_{n+1}]}(M_{r-1}) = \mathrm{soc}\,\widehat{\mathrm{Ind}}_n(D^{\lambda^{(r)}}),$$

which concludes the proof. ∎

Remark 5.3.32 The socle of $\widehat{\mathrm{Ind}}_n(D^{\lambda(r)})$ can be described either using the well-known Young rule (cf. [JK81, 2.8.2]), if $K[\mathfrak{S}_{n+1}]$ is semisimple, or Kleshchev's branching rule in the case that it is not, cf. [Kle05, Theorem 11.2.8]. Both can be encoded combinatorially in terms of Young diagrams and addable boxes.

Putting everything together yields the following.

Theorem 5.3.33 *Let D^λ be an irreducible $\mathbf{H}_n$-module. Then the socle of* $\mathrm{Ind}_n(D^\lambda)$ *is multiplicity free and can be described completely by the addition of certain boxes to λ.*

5.3.3. The general case of non-connected parameters and arbitrary $q \in K^*$

Finally, we drop the condition that the parameters $\{Q_1, \ldots, Q_r\}$ are q-connected: Let $q', Q'_1, \ldots, Q'_r$ be invertible elements in K and $n \geq 1$. Set $H_n := \mathbf{H}_{r,n}(q';\ Q'_1, \ldots, Q'_r)$ and let $\mathbf{Q}' := (Q'_1, \ldots, Q'_r)$. By Proposition 2.4.3 b) we can assume without loss of generality that $\mathbf{Q}' = \mathbf{Q}'_1 \coprod \cdots \coprod \mathbf{Q}'_t$ is a concatenation of q'-connected parameter sequences $\mathbf{Q}'_i$ which are pairwise not q-connected.

Theorem 5.3.34 *Let $0 \neq M$ be an H_n-module The induced module $\operatorname{Ind}_{H_n}^{H_{n+1}}(M)$ has at least $r + t$ constituents. In particular, it is reducible.*

Proof As $\operatorname{Ind}_{H_n}^{H_{n+1}}$ is exact it suffices to consider an irreducible module D^λ for some $\lambda \vdash_r n$.
Define ${}^1\lambda, \ldots, {}^t\lambda$ as in Proposition 5.2.4.
Recall the homomorphisms F_n and F_{n+1} arising from the Morita equivalence in Theorem 5.2.2 and let $1 \leq j \leq t$. By Theorems 5.3.18 and 5.3.29, the module ${}^j\operatorname{Ind}_{n,t}^{(n_1,\ldots,n_t)}(F_n(D^\lambda))$ has at least $|\mathbf{Q}'_j| + 1$ irreducible constituents, where $n_i := |{}^i\lambda|$ for $1 \leq i \leq t$. By Corollary 5.2.7 it follows that the number of irreducible constituents of $\operatorname{Ind}_{H_n}^{H_{n+1}}(M)$ is at least

$$\sum_{j=1}^{t} \left(|\mathbf{Q}'_j| + 1\right) = t + \sum_{j=1}^{t} |\mathbf{Q}'_j| = t + r.$$

∎

5.4. A Connection to Cherednik algebras

Throughout this section let $K = \mathbb{C}$ be the complex numbers. Fix some integer $e \geq 2$ and set $q := \exp(2\pi\sqrt{-1}/e) \in \mathbb{C}$, the "standard" primitive root of unity of order e. Now fix a positive integer r, non-negative integers $\gamma_1, \ldots, \gamma_r$, and set $Q_i := q^{\gamma_i}$. Finally, set $\mathbf{H}_n := \mathbf{H}_{n,r}(q; Q_1, \ldots, Q_r)$, the corresponding Ariki-Koike algebra. Note that parts of this section's results haven been published by the author in a preprint, cf. [Sch18].

We sketch the relation between these Ariki-Koike algebras and so-called rational Cherednik algebras following [Chl15]. This will allow us to use Theorem 5.3.18 to obtain an analogue result on elements of certain subcategories of the module categories of rational Cherednik algebras.

Rational Cherednik algebras are so-called symplectic reflection algebras associated with complex reflection groups. They were introduced by Etingof and Ginzburg, cf. [EG02].
Amongst other parameters these algebras rely on some $t \in \mathbb{C}$, but up to isomorphism it only matters whether or not this t is zero or not. In [Chl15, Chapter 5] Chlouveraki gives a definition of rational Cherednik algebras at $t = 1$ associated with groups of type $G(r, 1, n)$ depending only on the parameters e and $(\gamma_1, \ldots, \gamma_r)$. In the following we denote this algebra by $\mathbb{H}_n$. To avoid this sketch becoming overly long we will not give a precise definition but refer the reader to [Chl15, 5.2, 5.6]. Let us just say that $\mathbb{H}_n$ is a quotient of the smash product of the complex

group algebra $\mathbb{C}[G(r,1,n)]$ with the tensor algebra of the vector space $\mathbb{C}^{n\times 1} \oplus \left(\mathbb{C}^{n\times 1}\right)^*$, where $\left(\mathbb{C}^{n\times 1}\right)^*$ is the dual of $\mathbb{C}^{n\times 1}$.

The category mod-$\mathbb{H}_n$ has an important subcategory called $\mathcal{O}_n$ whose elements are those elements of mod-$\mathbb{H}_n$ on which a certain subalgebra of $\mathbb{H}_n$ acts locally nilpotent. We will now study the relation between $\mathcal{O}_n$ and mod-$\mathbf{H}_n$. Let us begin by first outlining some generalities.

Lemma 5.4.1 ([Chl15, Remark 5.2]) *For every irreducible complex $G(r,1,n)$-module ξ there exists a* standard *or* Verma module *of $\mathbb{H}_n$ denoted by $\Delta(\xi)$. This module is an element of $\mathcal{O}_n$. The head of $\Delta(\xi)$ is a simple $\mathbb{H}_n$-module denoted by $L(\xi)$ and it, too, lies in $\mathcal{O}_n$. Even stronger, the set*

$$\{L(\xi) \mid \xi \in \mathrm{Irr}(\mathbb{C}[G(r,1,n)]\}$$

is a complete set of pairwise non-isomorphic irreducible elements of $\mathcal{O}_n$.

Proposition 5.4.2 ([Chl15, 5.3]) *There is an exact functor*

$$\mathrm{KZ}_n : \mathcal{O}_n \to \text{mod-}\mathbf{H}_n$$

known as the Knizhnik-Zamaldochikov functor.
Set $\mathcal{B} := \{\xi \mid \xi \in \mathrm{Irr}(\mathbb{C}[G(r,1,n)]),\ \mathrm{KZ}_n(L(\xi)) \neq 0)\}$. Then

$$\{\mathrm{KZ}_n(L(\xi)) \mid \xi \in \mathcal{B}\}$$

is a complete set of pairwise non-isomorphic irreducible $\mathbf{H}_n$-modules. In particular, the KZ *functor yields a bijection $\mathcal{B} \leftrightarrow \mathrm{Irr}(\mathbf{H}_n)$.*

Since KZ_n is exact it defines a morphism of Grothendieck groups which we also denote by KZ_n.

Remark 5.4.3 While we only defined the Grothendieck group for the categories of all finitely generated modules of an algebra the same construction can also be carried out for other categories of modules such as $\mathcal{O}_n$, cf. [CR81, §16 B] for details.

Corollary 5.4.4 *Let $M \in \mathcal{O}_n$. Then the number of constituents of* $\mathrm{KZ}_n(M)$ *is a lower bound for the number of constituents of M. More precisely, if*

$$[M] = \sum_{\xi\in\mathcal{B}} a_\xi\,[L(\xi)] + \sum_{\xi\notin\mathcal{B}} a_\xi\,[L(\xi)]$$

is the decomposition of $[M]$ in the Grothendieck group of $\mathcal{O}_n$, then $\mathrm{KZ}_n(M)$ has exactly $\sum_{\xi\in\mathcal{B}} a_\xi$ constituents.

As always we want to study induced modules. Induction and restriction functors on the categories $\mathcal{O}_n$ have been defined by Bezrukavnikov and Etingof, cf. [BE09]. Subsequently, Shan has shown that just as for Ariki-Koike algebras there exist refinements of both induction and restriction, cf. [Sha11]. We follow her outline on induction and restriction for rational Cherednik algebras.

Lemma 5.4.5 ([Sha11, 1.6]) *There exist exact induction and restriction functors*

$$F(n) : \mathcal{O}_n \to \mathcal{O}_{n+1} \quad \text{and} \quad E(n) : \mathcal{O}_n \to \mathcal{O}_{n-1}.$$

Remark 5.4.6 Note that what we denote by $F(n)$ is denoted by $F(n+1)$ in [Sha11]. We make this change to be in line with our earlier definitions Ind_n : mod-$\mathbf{H}_n \to$ mod-$\mathbf{H}_{n+1}$ and Res_n : mod-$\mathbf{H}_n \to$ mod-$\mathbf{H}_{n-1}$.

We will not go into detail about these functors but rather focus on the study of the number of constituents of (in particular) induced modules. The induction and restriction functors are compatible with the KZ_n functor:

Lemma 5.4.7 ([Sha11, Theorem 2.1, Corollary 2.3]) *There are natural equivalences of functors*

$$\mathrm{KZ}_{n+1} \circ F(n) \cong \mathrm{Ind}_n \circ \mathrm{KZ}_n$$

and

$$\mathrm{KZ}_{n-1} \circ E(n) \cong \mathrm{Res}_n \circ \mathrm{KZ}_n .$$

We say that the KZ functor intertwines with induction and restriction.

This already yields the aforementioned analogue of Theorem 5.3.18.

Theorem 5.4.8 *Let $M \in \mathcal{O}_n$ such that $\mathrm{KZ}_n(M) \neq 0$. Then the number of constituents of $F(n)(M)$ is at least that of $\mathrm{Ind}_n(\mathrm{KZ}_n(M))$. In particular, $F(n)(M)$ has at least $r+1$ irreducible constituents, counting multiplicities.*

Proof By Theorem 5.3.18, the induced module $\mathrm{Ind}_n(\mathrm{KZ}_n(M))$ has at least $r+1$ irreducible constituents. As induction and KZ intertwine the same is true for $\mathrm{KZ}_{n+1}(F(n)(M))$. The claim now follows with Corollary 5.4.4. ∎

Remark 5.4.9 Suppose $L(\xi) \in \mathcal{O}_n$ is killed by KZ_n. Then $F(n)(L(\xi))$ still has at least r irreducible constituents, cf. [Sch18, Theorem 1.18]. It follows that for any $0 \neq M \in \mathcal{O}_n$, the induced modules $F(n)(M)$ has at least r constituents.

We can make this slightly more precise and also fully classify all $\xi \in \mathfrak{B}$ for which $F(n)(L(\xi))$ has exactly two irreducible constituents. To this end we consider Shan's refined functors. Their behaviour mirrors that of the functor i-Ind and i-Res for Ariki-Koike algebras.

Proposition 5.4.10 ([Sha11, Definition 4.2, Proposition 4.4]) *For $0 \leq i \leq e-1$ there exist exact functors*

$$F_i(n) : \mathcal{O}_n \to \mathcal{O}_{n+1}$$

and

$$E_i(n) : \mathcal{O}_n \to \mathcal{O}_{n-1}.$$

They are defined in terms of generalised eigenspaces of central elements and are refinements of $F(n)$ and $E(n)$ in the following way:
For $M \in \mathcal{O}_n$,

$$F(n)(M) \cong \oplus_{i=0}^{e-1} F_i(n)(M)$$

and

$$E(n)(M) \cong \oplus_{i=0}^{e-1} E_i(n)(M).$$

Hence, as for Ariki-Koike algebras it suffices to count the constituents of the $F_i(n)(M)$ to determine the number of constituents of $F(n)(M)$. The refined functors, too, intertwine with the KZ functors.

Lemma 5.4.11 ([Sha11, Proposition 4.3]) *For $0 \leq i \leq e-1$ there are natural equivalences of functors*

$$\mathrm{KZ}_{n+1} \circ F_i(n) \cong i\text{-}\mathrm{Ind}_n \circ \mathrm{KZ}_n$$

and

$$\mathrm{KZ}_{n-1} \circ E_i(n) \cong i\text{-}\mathrm{Res}_n \circ \mathrm{KZ}_n .$$

Just as for the full induction functor one can prove the following.

Corollary 5.4.12 *Let $M \in \mathcal{O}_n$ such that $\mathrm{KZ}_n(M) \neq 0$. Then the number of irreducible constituents of $F_i(n)(M)$ is at least that of $i\text{-}\mathrm{Ind}_n(M)$. In particular, if M is irreducible, then the number of irreducible constituents of $F_i(n)(M)$ is at least $\varphi_i(\mathrm{KZ}_n(M))$, cf. Proposition 5.3.15.*

The structure of $F_i(n)(M)$ is rather restrictive and well-understood:

Proposition 5.4.13 ([Sha11, Theorem 6.2], [CR08, Proposition 5.20]) *Let $L(\xi) \in \mathcal{O}_n$ be irreducible. Then the following holds:*

a) The socle of $E_i(n)(L(\xi))$ is isomorphic to its head and this is either 0 or an irreducible module, denoted by $\widetilde{E}_i(n)(L(\xi))$.

b) The socle of $F_i(n)(L(\xi))$ is isomorphic to its head and this is either 0 an irreducible module, denoted by $\widetilde{F}_i(n)(L(\xi))$.

c) If $E_i(n)(L(\xi)) \neq 0$, then $\widetilde{F}_i(n-1)(\widetilde{E}_i(n)(L(\xi))) \cong L(\xi)$.

d) If $F_i(n)(L(\xi)) \neq 0$, then $\widetilde{E}_i(n+1)(\widetilde{F}_i(n)(L(\xi))) \cong L(\xi)$.

In particular, both $F_i(n)(L(\xi))$ and $E_i(n)(L(\xi))$ are indecomposable.

The classification all $\xi \in \mathcal{B}$ such that $F(n)(L(\xi))$ has exactly two constituents now can be deduced from the following result on self-extension of irreducible modules in $\mathcal{O}_n$.

Proposition 5.4.14 ([BEG03, Proposition 1.12]) *Let $\xi \in \mathrm{Irr}(\mathbb{C}[G(\ell, 1, n)])$. Then*

$$Ext^1_{\mathcal{O}}(L(\xi), L(\xi)) = 0,$$

i.e. $L(\xi)$ only has trivial self-extensions in $\mathcal{O}_n$: If $M \in \mathcal{O}_n$ decomposes as $[M] = 2[L(\xi)]$ in the Grothendieck group, then M is isomorphic to the direct sum $L(\xi) \oplus L(\xi)$.

Corollary 5.4.15 *Let $L(\xi) \in \mathcal{O}_n$ be irreducible and let $M \in \mathcal{O}_{n+1}$ be an irreducible constituent of $F_i(n)(L(\xi))$ for some i. If the multiplicity of M as a constituent of $F_i(n)(L(\xi))$ is at least 2, then the number of constituents of $F_i(n)(L(\xi))$ is at least 3.*

Proof The statement is clear if the multiplicity of M is 3 in $F_i(n)(L(\xi))$, so the only case left to consider is $[F_i(n)(L(\xi))] = 2[M]$ in the Grothendieck group, as $F_i(n)(L(\xi))$ has more than 2 constituents otherwise. But then $F_i(n)(L(\xi)) \cong M \oplus M$ by Proposition 5.4.14. This is a contradiction to the indecomposability of $F_i(n)(L(\xi))$. ∎

Lemma 5.4.16 (cf. [GM13, 4.15]) *Let $\xi \in \mathcal{B}$ and $0 \leq i \leq e-1$.*

a) If $F_i(n)(L(\xi)) \neq 0$, then

$$\mathrm{KZ}_{n+1}(\widetilde{F}_i(n)(L(\xi))) \neq 0$$

and

$$\mathrm{KZ}_{n+1}(F_i(n)(L(\xi))) \neq 0.$$

b) If $E_i(n)(L(\xi)) \neq 0$, then

$$\mathrm{KZ}_n(\widetilde{E}_i(n)(L(\xi))) \neq 0$$

and

$$\mathrm{KZ}_n(E_i(n)(L(\xi))) \neq 0.$$

Proof We give the proof of a), as b) follos analogously.
By Proposition 5.4.13 d), we have $\widetilde{E}_i(n+1)(\widetilde{F}_i(n)(L(\xi))) = L(\xi)$. Thus,

$$\mathrm{KZ}_n(\widetilde{E}_i(n+1)(\widetilde{F}_i(n)(L(\xi)))) \neq 0 \tag{5.11}$$

because $\xi \in \mathcal{B}$.
Assume $\mathrm{KZ}_{n+1}(\widetilde{F}_i(n)(L(\xi))) = 0$. Then $i\text{-Res}_{n+1}(\mathrm{KZ}_{n+1}(\widetilde{F}_i(n)(L(\xi)))) = 0$ and, by Lemma 5.4.11, also $\mathrm{KZ}_n(E_i(n+1)(\widetilde{F}_i(n)(L(\xi)))) = 0$. But $\widetilde{E}_i(n+1)(\widetilde{F}_i(n)(L(\xi))) = \mathrm{soc}\, E_i(n+1)(\widetilde{F}_i(n)(L(\xi)))$ is a submodule of $E_i(n+1)(\widetilde{F}_i(n)(L(\xi)))$ and KZ_n is an exact functor. Thus, this is a contradiction to (5.11), hence $\mathrm{KZ}_{n+1}(\widetilde{F}_i(n)(L(\xi))) \neq 0$.
As $\widetilde{F}_i(n)(L(\xi))$ is a submodule of $F_i(n)(L(\xi))$ and KZ_{n+1} is exact, this implies $\mathrm{KZ}_{n+1}(F_i(n)(L(\xi))) \neq 0$. ∎

Theorem 5.4.17 *Let $\xi \in \mathcal{B}$ and define $\lambda \vdash_r n$ by $\mathrm{KZ}_n(L(\xi)) \cong D^\lambda$. Then $F(n)(L(\xi))$ has exactly two irreducible constituents if and only if the following conditions are satisfied:*

i) There are exactly two integers $i \neq j$ with $0 \leq i, j \leq e-1$ such that $\varphi_i(\lambda) \neq 0$ and $\varphi_j(\lambda) \neq 0$, and it is $\varphi_k(\lambda) = 0$ for all $0 \leq k \leq e-1$ with $k \neq i, j$.

ii) We have $\varphi_i(\lambda) = \varphi_i(\lambda) = 1$.

If this is the case, then $F(n)(L(\xi))$ is semisimple and r is necessarily equal to 1.

Proof From Theorem 5.4.8 it follows that $F(n)(L(\xi))$ can only have two constituents if $r = 1$.
First suppose $F(n)(L(\xi))$ has exactly two constituents.
We have $F(n)(L(\xi)) \cong \oplus_k F_k(n)(L(\xi))$ by Proposition 5.3.3. If $F_k(n)(L(\xi)) \neq 0$ for some k, then the head and socle of $F_k(n)(L(\xi))$ are isomorphic and irreducible by Proposition 5.4.13 b). Thus, if $F_k(L(\xi))$ were not irreducible, then by Corollary 5.4.15 it would have at least three irreducible

constituents. But $F(n)(L(\xi))$ has only two constituents, so this is a contradiction and $F_k(n)(L(\xi))$ is irreducible if it is not zero. It follows that there are exactly two distinct indeces i and j such that $F_i(n)(L(\xi))$ and $F_j(n)(L(\xi))$ are non-zero and both are irreducible. Therefore, $F(n)(L(\xi))$ is semisimple and

$$F(n)(L(\xi)) \cong F_i(n)(L(\xi)) \oplus F_j(n)(L(\xi)). \tag{5.12}$$

Since both summands on the right-hand side are irreducible, we have $F_i(n)(L(\xi)) = \widetilde{F}_i(n)(L(\xi))$ and $F_j(n)(L(\xi)) = \widetilde{F}_j(n)(L(\xi))$. Thus, it follows from Lemma 5.4.16 a) that $\mathrm{KZ}_{n+1}(F_i(n)(L(\xi)))$ and $\mathrm{KZ}_{n+1}(F_j(n)(L(\xi)))$ are irreducible $\mathbf{H}_{n+1}$-modules. By Lemma 5.4.11, they are isomorphic to $i\text{-}\mathrm{Ind}_n(D^\lambda)$ and $j\text{-}\mathrm{Ind}_n(D^\lambda)$, respectively. Hence, by Lemma 5.4.7, the application of KZ_{n+1} on both sides of (5.12) yields

$$\mathrm{Ind}_n(D^\lambda) \cong i\text{-}\mathrm{Ind}_n(D^\lambda) \oplus j\text{-}\mathrm{Ind}_n(D^\lambda).$$

By Propositions 5.3.3 and 5.3.15, and Corollary 5.3.12, together with the irreducibility of $i\text{-}\mathrm{Ind}_n(D^\lambda)$ and $j\text{-}\mathrm{Ind}_n(D^\lambda)$ this implies $\varphi_i(\lambda) = \varphi_j(\lambda) = 1$ and $\varphi_k(\lambda) = 0$ for all $k \neq i, j$.

To prove the converse, assume that i) and ii) hold.
Again by Propositions 5.3.3 and 5.3.15, and Corollary 5.3.12, this implies that

$$\mathrm{Ind}_n(D^\lambda) \cong i\text{-}\mathrm{Ind}_n(D^\lambda) \oplus j\text{-}\mathrm{Ind}_n(D^\lambda), \tag{5.13}$$

and both direct summands are irreducible.
On the other hand, we have $F(n)(L(\xi)) \cong \oplus_k F_k(n)(L(\xi))$. Therefore, by Lemma 5.4.7

$$\mathrm{Ind}_n(D^\lambda) \cong \oplus_k \mathrm{KZ}_{n+1}(F_k(n)(L(\xi))).$$

If $F_k(n)(L(\xi)) \neq 0$ for some k, then $\mathrm{KZ}_{n+1}(F_k(n)(L(\xi))) \neq 0$ by Lemma 5.4.16 a). Thus, the exactness of KZ_{n+1}, Lemma 5.4.11, and the decomposition in (5.13) imply that $F_k(n)(L(\xi)) = 0$ for all $k \neq i, j$, and therefore

$$F(n)(L(\xi)) \cong F_i(n)(L(\xi)) \oplus F_j(n)(L(\xi)).$$

It remains to show that the two direct summands are irreducible.
To prove this assume the opposite, i.e. without loss of generality assume $F_i(n)(L(\xi))$ is reducible. The socle and the head of $F_i(n)(L(\xi))$ are both isomorphic to the irreducible module $\widetilde{F}_i(n)(L(\xi))$, thus if $F_i(n)(L(\xi))$ is reducible, then multiplicity of $\widetilde{F}_i(n)(L(\xi))$ in $F_i(n)(L(\xi))$ is at least two. By Lemma 5.4.16, this would imply that $\mathrm{KZ}_{n+1}(F_i(n)(L(\xi))) \cong i\text{-}\mathrm{Ind}_n(D^\lambda)$ has at least two constituents, which is a contradiction to the irreducibility of $i\text{-}\mathrm{Ind}(D^\lambda)$. ∎

We can rephrase the condition via the combinatorial description of the crystal graph.

Corollary 5.4.18 *Let $\xi \in \mathcal{B}$ and define $\lambda \vdash_r n$ by $\mathrm{KZ}_n(L(\xi)) \cong D^\lambda$. Then $F(n)(L(\xi))$ has exactly two irreducible constituents if and only if the vertex λ in the crystal graph has exactly two outgoing edges labelled by i and j respectively, the vertex μ with $\lambda \xrightarrow{i} \mu$ has no outgoing edge with label i and the vertex ν with $\lambda \xrightarrow{j} \nu$ has no outgoing edge with label j.*
Equivalently, $r = 1$ and λ has exactly two co-normal boxes both of which are co-good, in particular labelled differently.

Chapter 6.

Decomposition and Induction

We study what happens if we change the underlying ring of an algebra and how this yields a relation between different Grothendieck groups. Subsequently, we study how this affects induction and restriction functors.

6.1. Specialisation and Decomposition

This section closely follows [GP00, Section 7.4].
Let A be an integral domain contained in a field K and $\mathfrak{A}$ an A-algebra that is free of finite rank over A.

Definition 6.1.1 Let B be an integral domain and $\theta : A \to B$ a ring homomorphism. Then the *specialisation of* $\mathfrak{A}$ *via* θ is defined as

$$B\mathfrak{A} := \mathfrak{A} \otimes_A B,$$

where B is a left A-module via θ. The specialisation is a B-algebra via $(h \otimes b)(h' \otimes b') := hh' \otimes bb'$ for b, b' in B and h, h' in $\mathfrak{A}$.

Now let M be an $\mathfrak{A}$-module that is free of finite rank over A. Then the *specialisation of M via θ* is

$$BM := M \otimes_A B.$$

This is a $B\mathfrak{A}$-module and free of finite rank over B, since the tensor product commutes with direct sums.

Example 6.1.2 The embedding $A \hookrightarrow K$ is a ring homomorphism and therefore $K\mathfrak{A}$ is a K-algebra which is finite dimensional over K. Note that $\mathfrak{A}$ embeds naturally into $K\mathfrak{A}$ because $\mathfrak{A}$ is free over A.

The specialisation of modules and algebras will allow us to relate Grothendieck groups of different algebras to one another. An well-known first example are specialisations arising from ring isomorphisms.

Proposition 6.1.3 *Suppose $\theta : A \to B$ is a ring isomorphism. Then there is a trivial isomorphism*

$$d : R_0(\mathfrak{A}) \to R_0(B\mathfrak{A}); [M] \mapsto [BM].$$

Proof Since θ is an isomorphism, the ring B is free and therefore flat as an A-module. Thus, mod-$\mathfrak{A} \to$ mod-$B\mathfrak{A}$; $M \mapsto BM$ is an exact functor and d is well-defined. As $\theta^{-1} : B \to A$ is also a ring isomorphism, it follows that $f : R_0(B\mathfrak{A}) \to R_0(A(B\mathfrak{A})); [N] \mapsto [AN]$ is also a well-defined homomorphism of Grothendieck groups. Now let $\psi : A(B\mathfrak{A}) \to \mathfrak{A}; (h \otimes_A 1) \otimes_B 1 \mapsto h$ for $h \in \mathfrak{A}$. It is easily checked that ψ is an A-algebra isomorphism. Similarly, one can show that for every $\mathfrak{A}$-module M we have $A(BM) \cong M$ as $\mathfrak{A}$-modules and therefore $f(d([M])) = [M]$ and d is an isomorphism.
It remains to show that d is trivial, i.e. that it sends classes of irreducible modules to classes of irreducibles. Let $M \in$ mod-$\mathfrak{A}$. Suppose M has some proper submodule M'. Then $0 \to N \to M$ is exact, and therefore $0 \to BM' \to BM$ is exact. Since we have $[0] \neq [M'] \neq [M]$ and d is an isomorphism, we also have $[0] \neq [BM'] \neq [BM]$, so BM' is a proper submodule of BM. Therefore, BM is irreducible only if M is irreducible. Applying the same argument to f in place of d shows that BM is irreducible if $M \cong A(BM)$ is irreducible, and therefore d is a trivial isomorphism. ∎

We now consider specialisations via field extensions. To do this we associate $K\mathfrak{A}$-modules to characteristic polynomials. Recall from Definition 1.1.5 that $R_0^+(K\mathfrak{A})$ is the submonoid of $R_0(K\mathfrak{A})$ generated by the classes of irreducible modules.

Definition 6.1.4 Let X be an indeterminate over K and $\mathrm{Maps}(\mathfrak{A}, K[X])$ the set of maps from $\mathfrak{A}$ to $K[X]$. By pointwise multiplication of maps this becomes a K-algebra. Define the map

$$\mathfrak{p}_K : R_0^+(K\mathfrak{A}) \to \mathrm{Maps}(\mathfrak{A}, K[X]),$$

which sends the class $[M]$ of a $K\mathfrak{A}$-module M to the map that sends an h in $\mathfrak{A}$ to its characteristic polynomial on M. More precisely, if ρ_M is a representation of $K\mathfrak{A}$ afforded by M we define $\mathfrak{p}_K([M])(h)$ as the characteristic polynomial of $\rho_M(h)$. Clearly, this is both independent of the choice of ρ_M and of the choice of the representative M of the class $[M]$, since the characteristic polynomial is invariant under base changes and only depends on the irreducible constituents and their multiplicities.

Remark 6.1.5 Addition in the Grothendieck translates to multiplication of the corresponding characteristic polynomials, thus $\mathfrak{p}_K$ is a homomorphism of monoids, where $\mathrm{Maps}(\mathfrak{A}, K[X])$ is a monoid via pointwise multiplication.

With this map we can take a look at decomposition maps arising from field extensions.

Lemma 6.1.6 ([GP00, Lemma 7.3.4]) *Let $K \leq E$ be a field extension and t_K^E :* $\mathrm{Maps}\,\mathfrak{A}, K[X] \hookrightarrow \mathrm{Maps}(\mathfrak{A}, E[X])$ *the natural embedding. Then there exists a well-defined canonical map*

$$d_K^E : R_0^+(K\mathfrak{A}) \to R_0^+(E\mathfrak{A});\ [M] \mapsto [M \otimes_K E]$$

for a $K\mathfrak{A}$-module M. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} R_0^+(K\mathfrak{A}) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(\mathfrak{A},\ K[X]) \\ \big\downarrow d_K^E & & \big\downarrow t_K^E \\ R_0^+(E\mathfrak{A}) & \xrightarrow{\mathfrak{p}_E} & \mathrm{Maps}(\mathfrak{A},\ E[X]) \end{array}$$

If $K\mathfrak{A}$ is split, i.e. K is a splitting field of $K\mathfrak{A}$, then d_K^E is a trivial isomorphism.

To work with fields that are not necessarily field extensions of K we need the language of valuation rings.

Definition 6.1.7 A subring $\mathcal{O} \subseteq K$ is called a *valuation ring* if for every x in K either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. A valuation ring is always local and we denote its maximal ideal by $J(\mathcal{O})$.

Proposition 6.1.8 ([GP00, 7.3.5, 7.3.7])

a) If $I \subseteq A$ is a prime ideal, then there exists a valuation ring $\mathcal{O}$ in K such that $A \subseteq \mathcal{O}$ and $J(\mathcal{O}) \cap A = I$.

b) Every finitely generated torsion-free module over a valuation ring $\mathcal{O}$ in K is free over $\mathcal{O}$.

c) The intersection of all valuation rings in K that contain A is equal to the integral closure of A in K. In particular, valuation rings in K are integrally closed in K.

d) Let M be a $K\mathfrak{A}$-module and $A \subseteq \mathcal{O} \subseteq K$ a valuation ring. Then there exists an $\mathcal{O}\mathfrak{A}$-module M' which is free and finitely generated over $\mathcal{O}$, such that M is isomorphic to $KM' = M' \otimes_{\mathcal{O}} K$. In particular, if $\rho : K\mathfrak{A} \to K^{n\times n}$ is a representation of $K\mathfrak{A}$, then ρ is equivalent to a representation ρ' of $K\mathfrak{A}$ with $\rho'(h) \in \mathcal{O}^{n\times n} \subseteq K^{n\times n}$ for all h in $\mathcal{O}\mathfrak{A}$. We say that M can be realised over $\mathcal{O}$.

Definition 6.1.9 Let $A \subseteq \mathcal{O} \subseteq K$ be a valuation ring. An $\mathcal{O}\mathfrak{A}$-module M' which is free and finitely generated over $\mathcal{O}$ is called an $\mathcal{O}\mathfrak{A}$-*lattice*.

Corollary 6.1.10 ([GP00, Proposition 7.3.8]) *Let M be a $K\mathfrak{A}$-module and denote by A^* the integral closure of A in K. Then $\mathfrak{p}_K([M])(h)$ is in $A^*[X] \subseteq K[X]$ for all h in $\mathfrak{A}$.*

Proof Let ρ be a representation afforded by M and $A \subseteq \mathcal{O} \subseteq K$ a valuation ring. Let h in $\mathfrak{A}$. By Proposition 6.1.8 d), the characteristic polynomial of h is in $\mathcal{O}[X]$. Since this holds for all valuation rings in K that contain A, the claim now follows from the fact that their intersection is A^*. ∎

Building on this we can re-define the map $\mathfrak{p}_K$ by restricting its co-domain, i.e.

$$\mathfrak{p}_K : R_0^+(K\mathfrak{A}) \to \mathrm{Maps}(\mathfrak{A}, A^*[X]).$$

We are now prepared to relate $K\mathfrak{A}$ to algebras over other fields. For the remainder of the section fix the following setting:

Setting 6.1.11

- A is integrally closed in K.
- k is a field and $\theta : A \to k$ is a ring homomorphism
- $A \subseteq \mathcal{O} \subseteq K$ is a valuation ring such that $J(\mathcal{O}) \cap A = \ker(\theta)$.
- $\pi : \mathcal{O} \to \overline{\mathcal{O}} := \mathcal{O}/J(\mathcal{O})$ is the natural epimorphism to the residue class field of $\mathcal{O}$.
- The specialised algebra $k'\mathfrak{A}$ is split, where k' is the field of fractions of $\theta(A)$.

Remark 6.1.12 As $\ker(\theta) = J(\mathcal{O}) \cap A = \ker(\pi) \cap A$ it follows that $\pi(A)$ is isomorphic to $A/\ker(\theta)$ which in turn is isomorphic to $\theta(A)$. Since k' is the field of fractions of $\theta(A)$ it follows that k' can be viewed as a subfield of $\overline{\mathcal{O}}$. This is best visualised by the following commuting diagram:

$$\begin{array}{ccccc} A & \hookrightarrow & \mathcal{O} & \hookrightarrow & K \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\pi} & & \\ k' & \hookrightarrow & \overline{\mathcal{O}} & & \\ \downarrow & & & & \\ k & & & & \end{array}$$

Lemma 6.1.13 (cf. [GP00, 7.4.1]) *Under the above setting the map $d_{k'}^{\overline{\mathcal{O}}}$ from Lemma 6.1.6 is a trivial isomorphism. Similarly, the map $d_{k'}^{k}$ is a trivial isomorphism. Hence, we can identify $R_0^+(k\mathfrak{A})$ with $R_0^+(\overline{\mathcal{O}}\mathfrak{A})$.*

Remark 6.1.14 Our setting differs slightly from what is considered in Chapter 7 of [GP00], where it is always assumed that $k' = k$, but with the trivial isomorphism $d_{k'}^{k}$ in mind everything holds in our version.

In this setting we can relate $K\mathfrak{A}$-modules and $\overline{\mathcal{O}}\mathfrak{A}$-modules via $\mathcal{O}\mathfrak{A}$-modules.

Definition 6.1.15 Let M be a $K\mathfrak{A}$-module. By Proposition 6.1.8 d), there exists an $\mathcal{O}\mathfrak{A}$-lattice M' such that $M' \otimes_{\mathcal{O}} K$ is isomorphic to M. The field $\overline{\mathcal{O}}$, too, is an $\mathcal{O}$-module via π and so we can define the *modular reduction* of M as

$$M' \otimes_{\mathcal{O}} \overline{\mathcal{O}}.$$

Due to the associativity of the tensor product it is clear that this is an $\overline{\mathcal{O}}\mathfrak{A}$-module: $\overline{\mathcal{O}}\mathfrak{A} = \mathfrak{A} \otimes_A \overline{\mathcal{O}} \cong \mathfrak{A} \otimes_A \mathcal{O} \otimes_{\mathcal{O}} \overline{\mathcal{O}} \cong \mathcal{O}\mathfrak{A} \otimes_{\mathcal{O}} \overline{\mathcal{O}}$. As before we will write KM' and $\overline{\mathcal{O}}M'$ for the modules $M' \otimes_{\mathcal{O}} K$ and $M' \otimes_{\mathcal{O}} \overline{\mathcal{O}}$ respectively.

Remark 6.1.16 Note that if $\hat{M}$ and M' are $\mathcal{O}\mathfrak{A}$-modules such that $M \cong KM' \cong K\hat{M}$, then M' is not necessarily isomorphic to $\hat{M}$.

Next we consider representations and characters of modular reductions of modules.

Remark 6.1.17 (cf. [GP00, Remark 7.4.4]) Let M' be an $\mathcal{O}\mathfrak{A}$-lattice and $\rho : \mathcal{O}\mathfrak{A} \to \mathcal{O}^{n\times n}$ a representation afforded by M'. Then the K-linear map

$$K\mathfrak{A} \to K^{n\times n}; h \otimes 1 \mapsto \rho(h)$$

and the $\overline{\mathcal{O}}$-linear map

$$\overline{\mathcal{O}}\mathfrak{A} \to \overline{\mathcal{O}}^{n\times n}; h \otimes 1 \mapsto \pi(\rho(h))$$

for h in $\mathfrak{A}$ define representations of KM' and $\overline{\mathcal{O}}M'$, respectively, where $\pi(\rho(h))$ is the matrix obtained by applying π to all entries of $\rho(h)$.
Let χ be the character of M'. Then $K\mathfrak{A} \to K; h \otimes 1 \mapsto \chi(h)$ defines the character of KM' and $\overline{\mathcal{O}}H \to \overline{\mathcal{O}}; h \otimes 1 \mapsto \pi(\chi(h))$ defines that of $\overline{\mathcal{O}}M'$. Note that by Corollary 6.1.10 the characteristic polynomial of $h \in \mathfrak{A}$ on M' only has entries in A, as A is integrally closed in K, hence the character of $\overline{\mathcal{O}}M'$ can also be written as $\mathcal{O}\mathfrak{A} \to \mathcal{O}; h \otimes 1 \mapsto \theta(\chi(h))$.

Theorem 6.1.18 ([GP00, Theorem 7.4.3]) *We define $d_\theta : R_0^+(K\mathfrak{A}) \to R_0^+(k\mathfrak{A})$ via modular reduction: For a $K\mathfrak{A}$-module M let M' be an $\mathcal{O}\mathfrak{A}$-module such that $M' \otimes_{\mathcal{O}} K \cong M$ and set $d_\theta([M]) = d_\theta([KM']) := [\overline{\mathcal{O}}M']$, where we consider $[\overline{\mathcal{O}}M']$ an element of $R_0^+(k\mathfrak{A})$ via the identification in Lemma 6.1.13. The following holds.*

a) The map d_θ is a well-defined monoid homomorphism. It does not depend on the choice of M'.

b) The following diagram commutes:

$$\begin{array}{ccc} R_0^+(K\mathfrak{A}) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(\mathfrak{A},\ A[X]) \\ \downarrow{\scriptstyle d_\theta} & & \downarrow{\scriptstyle t_\theta} \\ R_0^+(k\mathfrak{A}) & \xrightarrow{\mathfrak{p}_k} & \mathrm{Maps}(\mathfrak{A},\ k[X]) \end{array}$$

where $t_\theta : \mathrm{Maps}(\mathfrak{A},\ A[X]) \to \mathrm{Maps}(\mathfrak{A},\ k[X])$ is the map induced by θ, i.e. applying θ to all coefficients of the polynomials. Note that A is integrally closed in K and therefore the co-domain of $\mathfrak{p}_K$ is $\mathrm{Maps}(\mathfrak{A},\ A[X])$.

c) The map d_θ is the unique map between $R_0^+(K\mathfrak{A})$ and $R_0^+(k\mathfrak{A})$ with the property that the above diagram commutes. In particular, d_θ does not depend on the choice of $\mathcal{O}$, but only on θ.

Remark 6.1.19 Clearly, the decomposition map d_θ extends to a homomorphism of full Grothendieck groups, i.e.

$$d_\theta : R_0(K\mathfrak{A}) \to R_0(k\mathfrak{A}),$$

cf. Remark 1.1.6.

Definition 6.1.20 Assume the setting of Theorem 6.1.18. By Lemma 1.1.4, a basis of $R_0(K\mathfrak{A})$ is given by $\{[V] \mid V \in \mathrm{Irr}(K\mathfrak{A})\}$, and $\{[V'] \mid V' \in \mathrm{Irr}(k\mathfrak{A})\}$ is a basis of $R_0(k\mathfrak{A})$. We call the matrix of d_θ with respect to these two bases the *decomposition matrix of d_θ*.

If $k\mathfrak{A}$ is split semi-simple then the representation theory of $k\mathfrak{A}$ is essentially that of $K\mathfrak{A}$. This is known as *Tits' Deformation Theorem*.

Theorem 6.1.21 ([GP00, Theorem 7.4.6]) *If $k\mathfrak{A}$ is split semisimple and $K\mathfrak{A}$ is split, then $K\mathfrak{A}$, too, is semisimple and the decomposition map is a trivial isomorphism.*

This can be slightly generalised. For this we need a useful technical statement on flat modules.

Definition 6.1.22 Let B be a commutative ring and

$$\mathcal{E} : 0 \to L \xrightarrow{\varphi} M \to N \to 0$$

an exact sequence of (not necessarily finitely generated) right B-modules. Then we say that $\mathcal{E}$ is *pure* if $\mathcal{E} \otimes_B C'$ is an exact sequence of abelian groups for any left B-module C'. In this case we say that $\varphi(L)$ is a *pure submodule* of M.

Proposition 6.1.23 ([Lam99, Theorem 4.85]) *Let B be a commutative ring. A (not necessarily finitely generated) right B-module C is flat if and only if any short exact sequence*

$$0 \to L \to M \to C \to 0$$

of (not necessarily finitely generated) right B-modules is pure.

Theorem 6.1.24 (cf. [Gec98, Theorem 2.4]) *Assume that both $k\mathfrak{A}$ and $K\mathfrak{A}$ are split and that the Jacobson radical of $K\mathfrak{A}$ has the same dimension as that of $k\mathfrak{A}$. Then d_θ is a trivial isomorphism.*

Proof Let $J := \mathrm{Rad}(K\mathfrak{A}) \cap \mathcal{O}\mathfrak{A}$, where $\mathrm{Rad}(K\mathfrak{A})$ is the Jacobson radical of $K\mathfrak{A}$, and set $\overline{\mathfrak{A}} := \mathcal{O}\mathfrak{A}/J$. As $\mathcal{O}$ is a valuation ring in K, clearly K is the field of fractions of $\mathcal{O}$ and thus $\mathrm{Rad}(K\mathfrak{A})$ contains a K-basis that is entirely contained in $\mathcal{O}\mathfrak{A} \subseteq K\mathfrak{A}$. Hence $\mathrm{Rad}(K\mathfrak{A}) = KJ$ and $K\mathfrak{A}/KJ$ is semisimple.

Claim: As an $\mathcal{O}$-module, $\overline{\mathfrak{A}}$ is free.

Proof Since $\mathcal{O}$ is a valuation ring it suffices to show that $\overline{\mathfrak{A}}$ is torsion-free, since $\mathcal{O}\mathfrak{A}$ and therefore $\overline{\mathfrak{A}}$ is finitely generated as an $\mathcal{O}$-module. Let $0 \neq x + J$ be an element of $\overline{\mathfrak{A}}$ for some x in $\mathcal{O}\mathfrak{A} \setminus J$. In particular, x acts non-trivially on some irreducible $K\mathfrak{A}$-module M, as $x \notin \mathrm{Rad}(K\mathfrak{A})$. Clearly, zx, too, acts non-trivially on M for every non-zero element z of $\mathcal{O}$ and thus zx is in $\mathcal{O}\mathfrak{A} \setminus J$ which shows $zx + J \neq 0$. Therefore, $\overline{\mathfrak{A}}$ is torsion free. ∎

In particular, $\overline{\mathfrak{A}}$ is a flat $\mathcal{O}$-module and by Proposition 6.1.23 this implies that every short exact sequence

$$\mathcal{E} : 0 \to M \to N \to \overline{\mathfrak{A}} \to 0$$

is pure. In particular this holds for the short exact sequence

$$0 \to J \to \mathcal{O}\mathfrak{A} \to \overline{\mathfrak{A}} \to 0$$

and by tensoring over $\mathcal{O}$ with K and $\overline{\mathcal{O}}$ respectively this shows that $K\overline{\mathfrak{A}} \cong K\mathfrak{A}\,/KJ$ and $\overline{\mathcal{O}}\,\overline{\mathfrak{A}} \cong \overline{\mathcal{O}}\mathfrak{A}\,/\overline{\mathcal{O}}J$.

Now J is nilpotent and thus $\overline{\mathcal{O}}J$ is contained in $\mathrm{Rad}(\overline{\mathcal{O}}\mathfrak{A})$. Both $\overline{\mathcal{O}}\mathfrak{A}$ and $k\mathfrak{A}$ are split and therefore the dimension of $\mathrm{Rad}(\overline{\mathcal{O}}\mathfrak{A})$ is equal to that of $\mathrm{Rad}(k\mathfrak{A})$ (cf. [CR81, Theorem 7.9]) which by hypothesis is equal to that of $\mathrm{Rad}(K\mathfrak{A})$. But $\mathrm{Rad}(K\mathfrak{A}) = KJ$, and thus the dimension of $\mathrm{Rad}(\overline{\mathcal{O}}\mathfrak{A})$ is equal to the rank of J as an $\mathcal{O}$-module. In total, this shows $\mathrm{Rad}(\overline{\mathcal{O}}\mathfrak{A}) = \overline{\mathcal{O}}J$ and therefore $\overline{\mathcal{O}}\,\overline{\mathfrak{A}}$ is semi-simple.
Since $K\mathfrak{A}$ and $\overline{\mathcal{O}}\mathfrak{A}$ are split, the algebras $K\overline{\mathfrak{A}} \cong K\mathfrak{A}\,/\,\mathrm{Rad}(K\mathfrak{A})$ and $\overline{\mathcal{O}}\,\overline{\mathfrak{A}} \cong \overline{\mathcal{O}}\mathfrak{A}\,/\,\mathrm{Rad}(\overline{\mathcal{O}}\mathfrak{A})$, too, are split, and by Tits' Deformation Theorem 6.1.21 the decomposition map

$$R_0^+(K\overline{\mathfrak{A}}) \to R_0^+(\overline{\mathcal{O}}\,\overline{\mathfrak{A}})$$

is trivial.

As Jacobson radicals act trivially on simple modules there exist trivial isomorphisms $R_0^+(K\mathfrak{A}) \cong R_0^+(K\overline{\mathfrak{A}})$ and $R_0^+(k\mathfrak{A}) \cong R_0^+(\overline{\mathcal{O}}\mathfrak{A}) \cong R_0^+(\overline{\mathcal{O}}\,\overline{\mathfrak{A}})$ and the composition of these isomorphisms ultimately yields the triviality of d_θ. ∎

Under certain conditions a factorisation of ring homomorphisms yields a factorisation of corresponding decomposition maps.

Proposition 6.1.25 (cf. [Gec98, Proposition 2.6]) *Suppose $\iota : A \to B$ is a surjective ring homomorphism to an integral domain B that is integrally closed in L, its field of fractions. Suppose further that θ factors through B via ι and some $\kappa : B \to k$, that is*

$$\begin{array}{ccc} A & \xrightarrow{\quad\theta\quad} & k \\ & {\scriptstyle\iota}\searrow \quad \nearrow{\scriptstyle\kappa} & \\ & B & \end{array}$$

commutes. If the specialised algebra $L\mathfrak{A}$ is split, then the decomposition maps in the following diagram are all well-defined and the diagram commutes:

$$\begin{array}{ccc} R_0(K\mathfrak{A}) & \xrightarrow{\quad d_\theta\quad} & R_0(k\mathfrak{A}) \\ & {\scriptstyle d_\iota}\searrow \quad \nearrow{\scriptstyle d_\kappa} & \\ & R_0(L\mathfrak{A}) & \end{array}$$

The decomposition map d_κ is sometimes called an adjustment map*, and its decomposition matrix an* adjustment matrix.

Proof By Theorem 6.1.18, the decomposition maps d_κ and d_ι are uniquely defined by the commutativity of the following diagrams:

$$\begin{array}{ccc} R_0^+(K\mathfrak{A}) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(\mathfrak{A},\ A[X]) \\ \downarrow{\scriptstyle d_\iota} & & \downarrow{\scriptstyle t_\iota} \\ R_0^+(L\mathfrak{A}) & \xrightarrow{\mathfrak{p}_L} & \mathrm{Maps}(\mathfrak{A},\ L[X]) \end{array} \qquad \begin{array}{ccc} R_0^+(L\mathfrak{A}) & \xrightarrow{\mathfrak{p}'_L} & \mathrm{Maps}(B\mathfrak{A},\ B[X]) \\ \downarrow{\scriptstyle d_\kappa} & & \downarrow{\scriptstyle t'_\kappa} \\ R_0^+(k\mathfrak{A}) & \xrightarrow{\mathfrak{p}'_k} & \mathrm{Maps}(B\mathfrak{A},\ k[X]) \end{array}$$

Note that in the second diagram the role of $\mathfrak{A}$ has been taken by $B\mathfrak{A}$. The maps $\mathfrak{p}'_k$ and $\mathfrak{p}'_L$ are defined completely analogous to the maps $\mathfrak{p}_k$ and $\mathfrak{p}_L$, and t'_κ is induced by applying κ to the coefficients of polynomials.
The first key observation is that as B is integrally closed in L, we can replace $\mathrm{Maps}(\mathfrak{A}, L[X])$ by $\mathrm{Maps}(\mathfrak{A}, B[X])$, cf. Corollary 6.1.10.
Secondly, the map $\alpha : \mathfrak{A} \to B\mathfrak{A}; a \mapsto a \otimes_A 1$ induces two morphisms $\beta_1 : \mathrm{Maps}(B\mathfrak{A},\ B[X]) \to \mathrm{Maps}(\mathfrak{A}, B[X]);\ f \mapsto f \circ \alpha$ and $\beta_2 : \mathrm{Maps}(B\mathfrak{A},\ k[X]) \to \mathrm{Maps}(\mathfrak{A}, k[X]);\ f \mapsto f \circ \alpha$. If we denote by $t : \mathrm{Maps}(\mathfrak{A}, B[X]) \to \mathrm{Maps}(\mathfrak{A}, k[X])$ the map induced by applying κ to the coefficients of polynomials we obtain the following commuting diagram:

$$\begin{array}{ccc} \mathrm{Maps}(B\mathfrak{A}, B[X]) & \xrightarrow{\beta_1} & \mathrm{Maps}(\mathfrak{A}, B[X]) \\ \downarrow{\scriptstyle t'_\kappa} & & \downarrow{\scriptstyle t} \\ \mathrm{Maps}(B\mathfrak{A}, k[X]) & \xrightarrow{\beta_2} & \mathrm{Maps}(\mathfrak{A},\ k[X]) \end{array}$$

Furthermore, $\beta_1 \circ \mathfrak{p}'_L = \mathfrak{p}_L$ as well as $\beta_2 \circ \mathfrak{p}'_k = \mathfrak{p}_k$ and finally $t \circ t_\iota = t_\theta$, where t_θ is the map in the commuting diagram defining d_θ.
This allows us to glue together the above two commuting diagram to obtain a new commuting diagram:

$$\begin{array}{ccc} R_0^+(K\mathfrak{A}) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(\mathfrak{A},\ A[X]) \\ \downarrow{\scriptstyle d_\kappa \circ d_\iota} & & \downarrow{\scriptstyle t_\theta} \\ R_0^+(k\mathfrak{A}) & \xrightarrow{\mathfrak{p}_L} & \mathrm{Maps}(\mathfrak{A},\ k[X]) \end{array}$$

The claim now follows from the uniqueness of d_θ in Proposition 6.1.18 c). ∎

Remark 6.1.26 The proofs of both Theorem 6.1.24 and Proposition 6.1.25 are sketched in [Gec98] only for the case that O is a discrete valuation ring. Our proofs work in the more general setting of valuation rings.

Finally, we study decomposition maps of tensor products of algebras. Let $\mathfrak{B}$ be an A-algebra that is free of finite rank and suppose that the algebras $K\mathfrak{A}$, $k'\mathfrak{A}$, $K\mathfrak{B}$ and $k'\mathfrak{B}$ are split.
We first need some technical results.

Lemma 6.1.27

a) There is an isomorphism of B-algebras

$$(B\,\mathfrak{A}) \otimes_B (B\,\mathfrak{B}) \cong B(\mathfrak{A} \otimes_A \mathfrak{B})$$

given by multiplication.

b) Let M be in mod-$\mathfrak{A}$ *and N in* mod-$\mathfrak{B}$. *Then there is an isomorphism of* $B(\mathfrak{A} \otimes_A \mathfrak{B})$*-modules*

$$B(M \otimes_A N) \cong (BM) \otimes_B (BN).$$

Proof As B-modules the isomorphisms follow immediately from the associativity of the tensor product and its commutativity when taken over commutative rings. The compatibility with multiplication then follows directly from the definition. ∎

Lemma 6.1.28 ([CR81, Lemma 10.37, Theorem 10.38]) *Let E be in* $\{K, \overline{O}, k', k\}$. *Then the specialisation* $E(\mathfrak{A} \otimes_A \mathfrak{B})$ *is split and a complete set of pairwise non-isomorphic irreducible* $E(\mathfrak{A} \otimes_A \mathfrak{B})$*-modules is given by*

$$\{M \otimes_E N \mid M \in \mathrm{Irr}(E\,\mathfrak{A}),\ N \in \mathrm{Irr}(E\,\mathfrak{B})\}.$$

It follows that

$$R_0(E\,\mathfrak{A}) \otimes_{\mathbb{Z}} R_0(E\,\mathfrak{B}) \to R_0(E(\mathfrak{A} \otimes_A \mathfrak{B}));\ [M] \otimes_{\mathbb{Z}} [N] \mapsto [M \otimes_E N]$$

for $M \in \mathrm{Irr}(E\,\mathfrak{A})$ *and* $N \in \mathrm{Irr}(E\,\mathfrak{B})$ *defines an isomorphism of abelian groups.*

Theorem 6.1.29 *By Theorem 6.1.18, there exist well-defined decomposition maps* $d_\theta^{\mathfrak{A}} : R_0(K\,\mathfrak{A}) \to R_0(k\,\mathfrak{A})$ *and* $d_\theta^{\mathfrak{B}} : R_0(K\,\mathfrak{B}) \to R_0(k\,\mathfrak{B})$. *As* $K(\mathfrak{A} \otimes_A \mathfrak{B})$ *and* $k'(\mathfrak{A} \otimes_A \mathfrak{B})$ *are split by Lemma 6.1.28 there is also a decomposition map* $d_\theta^{\mathfrak{A} \otimes \mathfrak{B}} : R_0(K(\mathfrak{A} \otimes \mathfrak{B})) \to R_0(k(\mathfrak{A} \otimes \mathfrak{B}))$. *Then*

$$d_\theta^{\mathfrak{A} \otimes \mathfrak{B}} = d_\theta^{\mathfrak{A}} \otimes_{\mathbb{Z}} d_\theta^{\mathfrak{B}},$$

where we identify $R_0(K(\mathfrak{A} \otimes_A \mathfrak{B}))$ *with* $R_0(K\,\mathfrak{A}) \otimes_{\mathbb{Z}} R_0(K\,\mathfrak{B})$ *and* $R_0(k(\mathfrak{A} \otimes_A \mathfrak{B}))$ *with* $R_0(k\,\mathfrak{A}) \otimes_{\mathbb{Z}} R_0(k\,\mathfrak{B})$ *via the isomorphisms in Lemma 6.1.28.*

Proof Clearly, it suffices to show this on simple modules.
Let $M \in \mathrm{Irr}(K\,\mathfrak{A})$ and $N \in \mathrm{Irr}(K\,\mathfrak{B})$ realised by $O\,\mathfrak{A}$- and $O\,\mathfrak{B}$-lattices M' and N', respectively. Then

$$\begin{aligned} d_\theta^{\mathfrak{A} \otimes \mathfrak{B}}([M \otimes_K N]) &= d_\theta^{\mathfrak{A} \otimes \mathfrak{B}}([K(M' \otimes_O N')]) \\ &= [\overline{O}(M' \otimes_O N')] \\ &= [(\overline{O}M') \otimes_{\overline{O}} (\overline{O}N')] \\ &= d_\theta^{\mathfrak{A}} \otimes d_\theta^{\mathfrak{B}}([KM' \otimes_K KN']), \end{aligned}$$

by repeatedly applying Lemma 6.1.27. ∎

6.2. Specialisation, Decomposition, and Induction

We keep the initial setting of Section 6.1. Recall in particular that $\mathfrak{A}$ is an A-algebra which is free of finite rank over A.
Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{A}$ that is also free of finite rank over A and suppose $\mathfrak{A}$ is free of finite rank as a left $\mathfrak{a}$- module via multiplication in $\mathfrak{A}$. Assume furthermore that $\mathfrak{a}$ is a pure A-submodule of $\mathfrak{A}$, see Definition 6.1.22.

Remark 6.2.1 By Proposition 6.1.23, our conditions hold for example if $\mathfrak{A}\,/\,\mathfrak{a}$ is a flat A-module, which in turn holds for example if $\mathfrak{a}$ is a direct summand of $\mathfrak{A}$ as an A-module: If $\mathfrak{a}$ is a direct summand of $\mathfrak{A}$, then so is $\mathfrak{A}\,/\,\mathfrak{a}$, hence it is projective and thus flat.

We assume $\mathfrak{a}$ to be pure in $\mathfrak{A}$ to make sure that induction still makes sense after specialisation.

Remark 6.2.2 If $\theta : A \to B$ is a ring homomorphism, then

$$0 \to B\,\mathfrak{a} \to B\,\mathfrak{A} \to B\,\mathfrak{A}\,/B\,\mathfrak{a} \to 0$$

is exact by Proposition 6.1.23. In particular, $B\,\mathfrak{a}$ is a B-subalgebra of $B\,\mathfrak{A}$.

Remark 6.2.3 Note that we cannot arbitrarily weaken the conditions: Clearly, $\mathbb{Z}$ is a $\mathbb{Z}$-subalgebra of $\mathbb{Q}$ and there is a ring homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Then it is well-known that $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z} \cong \{0\}$ as abelian groups, but $\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$, so clearly the latter cannot be a subalgebra of the former.

Example 6.2.4 If H is an Iwahori-Hecke algebra over A and H' is a parabolic subalgebra of H, then both H and H' are free of finite rank over A and H/H' is a flat A-module, as H has an A-basis containing an A-basis of H', cf. Proposition 2.2.3 b).

Lemma 6.2.5 *Let $\theta : A \to B$ be a ring homomorphism. Then for every $\mathfrak{a}$-module M which is free of finite rank over A there is an isomorphism of $B\,\mathfrak{A}$-modules*

$$\mathrm{Ind}_{B\,\mathfrak{a}}^{B\,\mathfrak{A}}(BM) \cong B(\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(M)).$$

We say that induction commutes with specialisation.

Proof Let M be an $\mathfrak{a}$-module, free of finite rank over A with an A-basis U, and let Y be an $\mathfrak{a}$-basis of $\mathfrak{A}$. Then $\{u \otimes_A 1 \mid u \in U\}$ is a B-basis of BM and $\{y \otimes_A 1 \mid y \in Y\}$ is an $B\,\mathfrak{a}$-basis of $B\,\mathfrak{A}$ as a left $B\,\mathfrak{a}$-module. In total, the set $\{(u \otimes_A 1) \otimes_{B\,\mathfrak{a}} (y \otimes_A 1) \mid y \in Y,\ u \in U\}$ is a B-basis of $(M\otimes_A B)\otimes_{B\,\mathfrak{a}} B\,\mathfrak{A}$. Similarly, a B-basis of $(M\otimes_{\mathfrak{a}}\mathfrak{A})\otimes_A B$ is given by $\{(u\otimes_{\mathfrak{a}} y)\otimes_A 1 \mid u \in U,\ y \in Y\}$. Hence, there is a well-defined isomorphism of B-modules

$$\Psi : \mathrm{Ind}_{B\,\mathfrak{a}}^{B\,\mathfrak{A}}(BM) = (M \otimes_A B) \otimes_{B\,\mathfrak{a}} B\,\mathfrak{A} \to B(\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{A}}(M)) = (M \otimes_{\mathfrak{a}} \mathfrak{A}) \otimes_A B$$

given by

$$\Psi((u \otimes_A 1) \otimes_{B\,\mathfrak{a}} (y \otimes_A 1)) \;:=\; (u \otimes_{\mathfrak{a}} y) \otimes_A 1$$

for $u \in U$ and $y \in Y$.
We check that this map is compatible with multiplication by $B\,\mathfrak{A}$.

Let h be in $\mathfrak{A}$ and $(u \otimes_A 1) \otimes_{B\,\mathfrak{a}} (y \otimes_A 1)$ a basis element of $\mathrm{Ind}_{B\,\mathfrak{a}}^{B\,\mathfrak{A}}(BM)$ as described above, with $u \in U$ and $y \in Y$. Since Y is an $\mathfrak{a}$-basis of $\mathfrak{A}$ there exist (unique) elements $g_z \in \mathfrak{a}$ for $z \in Y$ such that $yh = \sum_{z\in Y} g_z z$. Furthermore, since M is free as an A-module, there exist elements $a_{z,v} \in A$ for every $z \in Y$ and $v \in U$ such that $u.g_z = \sum_{v\in U} a_{z,v} v$. Now let $b \in B$. Then

$$
\begin{aligned}
&\Psi\left((u \otimes_A 1) \otimes_{B\,\mathfrak{a}} (y \otimes_A 1).(h \otimes_A b)\right)\\
&= \Psi\left((u \otimes_A 1) \otimes_{B\,\mathfrak{a}} (yh \otimes_A b)\right)\\
&= \Psi\left((u \otimes_A 1) \otimes_{B\,\mathfrak{a}} (\sum_{z\in Y} g_z z \otimes_A b)\right)\\
&= \Psi\left(\sum_{z\in Y} (ug_z \otimes_A 1) \otimes_{B\,\mathfrak{a}} (z \otimes_A b)\right)\\
&= \Psi\left(\sum_{z\in Y} \left(\sum_{v\in U} a_{z,v} v \otimes_A 1\right) \otimes_{B\,\mathfrak{a}} (z \otimes_A b)\right)\\
&= \Psi\left(\sum_{z\in Y} \sum_{v\in U} \theta(a_{z,v}) b \left((v \otimes_A 1) \otimes_{B\,\mathfrak{a}} (z \otimes_A 1)\right)\right)\\
&= \sum_{z\in Y} \sum_{v\in U} \theta(a_{z,v}) b\, \Psi\left(((v \otimes_A 1) \otimes_{B\,\mathfrak{a}} (z \otimes_A 1))\right)\\
&= \sum_{z\in Y} \sum_{v\in U} \theta(a_{z,v}) b\, (v \otimes_{\mathfrak{a}} z) \otimes_A 1\\
&= \sum_{z\in Y} (\sum_{v\in U} a_{z,v} v \otimes_{\mathfrak{a}} z) \otimes_A b\\
&= \sum_{z\in Y} (ug_z \otimes_{\mathfrak{a}} z) \otimes_A b\\
&= (u \otimes_{\mathfrak{a}} \sum_{z\in Y} g_z z) \otimes_A b\\
&= (u \otimes_{\mathfrak{a}} yh) \otimes_A b\\
&= ((u \otimes_{\mathfrak{a}} y) \otimes_A 1).(h \otimes_A b)\\
&= \Psi\left((u \otimes_A 1) \otimes_{B\,\mathfrak{a}} (y \otimes_A 1)\right).(h \otimes_A b).
\end{aligned}
$$

■

Once again we assume Setting 6.1.11. Additionally assume that $k'\,\mathfrak{a}$ is split.

Theorem 6.2.6 *Denote by d'_θ the decomposition map $R_0(K\,\mathfrak{a}) \to R_0(k\,\mathfrak{a})$ and by d_θ the decomposition map $R_0(K\,\mathfrak{A}) \to R_0(k\,\mathfrak{A})$. Furthermore, let $\mathrm{Ind}_K := \mathrm{Ind}_{K\,\mathfrak{a}}^{K\,\mathfrak{A}}$ and $\mathrm{Ind}_k := \mathrm{Ind}_{k\,\mathfrak{a}}^{k\,\mathfrak{A}}$ denote the maps on the Grothendieck groups defined by the exact induction functors. Then the diagram*

$$
\begin{array}{ccc}
R_0(K\,\mathfrak{a}) & \xrightarrow{d'_\theta} & R_0(k\,\mathfrak{a})\\
\Big\downarrow{\scriptstyle \mathrm{Ind}_K} & & \Big\downarrow{\scriptstyle \mathrm{Ind}_k}\\
R_0(K\,\mathfrak{A}) & \xrightarrow{d_\theta} & R_0(k\,\mathfrak{A})
\end{array}
$$

commutes. We say that induction and decomposition commute.

Proof For any specialisation $A \to B$ set $\mathrm{Ind}_B := \mathrm{Ind}_{B\,\mathfrak{a}}^{B\,\mathfrak{A}}$ throughout this proof, noting that this functor is well-defined by Remark 6.2.2.
Everything now follows from the commutativity of specialisation and induction. Note first, that for any field extension $k' \leq E$ we identified $R_0(k'\,\mathfrak{a})$ with $R_0(E\,\mathfrak{a})$ and $R_0(k'\,\mathfrak{A})$ with $R_0(E\,\mathfrak{A})$ via Lemma 6.1.6. Under this identifications the induction map $\mathrm{Ind}_{k'}$ is mapped to Ind_E because specialisation and induction commute. Hence, the identifications $R_0(k\,\mathfrak{a}) \cong R_0(\overline{O}\,\mathfrak{a})$ and $\mathbb{R}_0(k\,\mathfrak{A}) \to R_0(\overline{O}\,\mathfrak{A})$ identify Ind_k with $\mathrm{Ind}_{\overline{O}}$.
Now let M be a $K\,\mathfrak{a}$-lattice and M' an $O\,\mathfrak{a}$-module such that $M' \otimes_O K \cong M$. Then

$$\begin{aligned}
&\mathrm{Ind}_k \circ d'_\theta([M]) \\
&= \mathrm{Ind}_k\left(\left[M' \otimes_O \overline{O}\right]\right) \\
&= \left[\mathrm{Ind}_{\overline{O}}\left(M' \otimes_O \overline{O}\right)\right] \\
&= \left[(\mathrm{Ind}_O(M')) \otimes_O \overline{O}\right] \\
&= d_\theta\left([(\mathrm{Ind}_O(M')) \otimes_O K]\right) \\
&= d_\theta\left([\mathrm{Ind}_K\left(M' \otimes_O K\right)]\right) \\
&= d_\theta \circ \mathrm{Ind}_K\left([M]\right).
\end{aligned}$$

∎

Corollary 6.2.7 *Assume the setting of Theorem 6.2.6 and suppose that d'_θ is surjective. If c'_θ is a right inverse homomorphism of d'_θ, then*

$$\mathrm{Ind}_k = d_\theta \circ \mathrm{Ind}_K \circ c'_\theta,$$

i.e. the induction map Ind_k *can be computed from the decomposition maps d'_θ, d_θ, and the induction map* Ind_K.

Proof Let m be in $R_0(k\,\mathfrak{a})$. Then by the commutativity of the diagram in Theorem 6.2.6 we have

$$\begin{aligned}
d_\theta \circ \mathrm{Ind}_K \circ c'_\theta(m) &= (d_\theta \circ \mathrm{Ind}_K)\,(c'_\theta(m)) \\
&= \mathrm{Ind}_k\,(\,d'_\theta(c'_\theta(m))) \\
&= \mathrm{Ind}_k(m).
\end{aligned}$$

∎

We conclude by briefly stating the analogous results for restriction in place of induction.

Lemma 6.2.8 *Let $A \to B$ be a ring homomorphism. Then there is an isomorphism of $B\,\mathfrak{a}$-modules*

$$\mathrm{Res}_{B\,\mathfrak{a}}^{B\,\mathfrak{A}}(BM) \cong B(\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{A}}(M))$$

for every $\mathfrak{A}$-module M that is free of finite rank over A. We say that restriction commutes with specialisation.

Proof This follows immediately from the definition of the action of $B\,\mathfrak{A}$ on BM. ∎

The proof of the following result is completely analogous to those of Theorem 6.2.6 and Corollary 6.2.7.

Theorem 6.2.9 *Assume the setting of Theorem 6.2.6 and denote by* $\operatorname{Res}_K := \operatorname{Res}^{K\mathfrak{A}}_{K\mathfrak{a}}$ *and* $\operatorname{Res}_k := \operatorname{Res}^{k\mathfrak{A}}_{k\mathfrak{a}}$ *the maps on the Grothendieck groups defined by the exact restriction functors. Then the following diagram commutes:*

$$\begin{array}{ccc} R_0(K\mathfrak{A}) & \xrightarrow{d_\theta} & R_0(k\mathfrak{A}) \\ \downarrow{\scriptstyle \operatorname{Res}_K} & & \downarrow{\scriptstyle \operatorname{Res}_k} \\ R_0(K\mathfrak{a}) & \xrightarrow{d'_\theta} & R_0(k\mathfrak{a}) \end{array}$$

Furthermore, if d_θ *is surjective and* c_θ *is a right inverse homomorphism of* d_θ*, then*

$$\operatorname{Res}_k = d'_\theta \circ \operatorname{Res}_K \circ c_\theta.$$

Chapter 7.

Decomposition of Iwahori-Hecke algebras

We apply the specialisation results to Iwahori-Hecke algebras to study them over different rings. This gives rise to splitness and semisimplicity criteria. Finally, we try to summarise the current scientific status of the computation of certain decomposition maps.

7.1. Specialisation and Decomposition of Generic Iwahori-Hecke algebras

The key observation is that specialisation of Iwahori-Hecke algebras yields Iwahori-Hecke algebras.

Lemma 7.1.1 (cf. [GP00, 8.1.2]) *Let (W, S) be a Coxeter system and u_s invertible elements of a ring A for s in S such that $u_s = u_t$ whenever s and t are conjugate in W. Denote by $H := H_A(W, S, (u_s \mid s \in S))$ the corresponding Iwahori-Hecke algebra. If $\theta : A \to B$ is a ring homomorphism, then there is a B-algebra isomorphism*

$$BH = H \otimes_A B \cong H_B(W, S, (\theta(u_s) \mid s \in S)).$$

If $\{T_w \mid w \in W\}$ is the standard basis of H and $\{T'_w \mid w \in W\}$ that of $H_B(W, S, (\theta(u_s) \mid s \in S))$ (cf. Proposition 2.2.3 b)), then this isomorphisms maps $T_w \otimes_A 1$ to T'_w.

This is an instance of the following more general result.

Proposition 7.1.2 *Let A be a ring and $\theta : A \to B$ a ring homomorphism. Denote by $A\langle X_1, \ldots, X_n\rangle$ and $B\langle Y_1, \ldots, Y_n\rangle$ the free algebras with n (non-commuting) generators over A and B, respectively. For f in $A\langle X_1, \ldots, X_n\rangle$ denote by $\theta(f)$ the element of $B\langle Y_1, \ldots, Y_n\rangle$ obtained by applying θ to the coefficients and replacing every X_i with Y_i. If $\mathfrak{A}$ is an A-algebra with a presentation $\langle X_1, \ldots, X_n \mid f_1, \ldots, f_r\rangle$ for some elements $f_1, \ldots, f_r$ of $A\langle X_1, \ldots, X_n\rangle$ and $\mathfrak{A}$ is flat as an A-module, then the specialisation $\mathfrak{A} \otimes_A B$ is isomorphic as a B-algebra to $\langle Y_1, \ldots, Y_n \mid \theta(f_1), \ldots, \theta(f_r)\rangle$.*

Proof Clearly, the B-algebras $A\langle X_1,\dots,X_n\rangle \otimes_A B$ and $B\langle Y_1,\dots,Y_n\rangle$ are isomorphic and the isomorphism maps $(\langle f_1,\dots,f_r\rangle)\otimes_A B$ to the ideal $\langle\theta(f_1),\dots,\theta(f_r)\rangle \trianglelefteq B\langle Y_1,\dots,Y_n\rangle$.
As $\mathfrak{A}$ is a flat A-module, by Proposition 6.1.23 the short exact sequence sequence

$$0 \to \langle f_1,\dots,f_r\rangle \to A\langle X_1,\dots,X_n\rangle \to \mathfrak{A} \to 0$$

remains exact after tensoring with B over A, as it is a pure sequence. ∎

Remark 7.1.3 By Proposition 7.1.2, an analogue of Lemma 7.1.1 also holds for Ariki-Koike algebras and, more general, cyclotomic Hecke algebras of type $G(r,p,n)$ as these, too, are free over the ring over which they are defined.

Example 7.1.4 Let (W,S) be a finite Coxeter system and A a ring containing units u_s for every $s \in S$ with $u_s = u_t$ whenever s and t are conjugate in W. Set $\mathbf{A} := \mathbb{Z}\left[X_s^{\pm 1} \mid s \in S\right]$ with indeterminants X_s satisfying $X_s = X_t$ if and only if s is conjugate to t in W. Clearly, there is a ring homomorphism $\mathbf{A} \to A$; $X_s \mapsto u_s$. Hence, the Iwahori-Hecke algebra $H_A(W,S,(u_s \mid s \in S))$ is a specialisation of $H_{\mathbf{A}}(W,S,(X_s \mid s \in S))$.

We see that rings of Laurent polynomials are a natural starting point for specialisation and therefore also for decomposition, due to their universal properties. We call Iwahori-Hecke algebras whose parameters are indeterminates *generic algebras*.

Definition 7.1.5 Let (W,S) be a finite Coxeter system and A a ring. A set $\{V_s \mid s \in S\}$ of indeterminates over A is called (W,S)-*compatible* if $V_s = V_t$ whenever s and t are conjugate in W.

Remark 7.1.6 Suppose (W,S) is an irreducible finite Coxeter system. If W is not of type B_n, F_4 or $I_2(m)$ for even m it is well-known that all the generators in S are conjugate. In this case a set $\{V_s \mid s \in S\}$ of indeterminates is (W,S)-compatible if and only if they are all equal. If W has type B_n, F_4 or $I_2(m)$ for even m, then the conjugacy relation partitions S into two subsets, thus a subset of (W,S)-compatible variables has cardinality at most 2 in this case.

When are generic algebras over fields of characteristic zero split? In general, the necessary ingredients are a splitting field of the Coxeter group and square roots of the parameters.

Proposition 7.1.7 ([GP00, Theorem 9.3.5]) *Let (W,S) be a finite Coxeter system. Let A be an integral domain containing $\mathbb{Z}$ such that its field of fractions is a splitting field of W. Let $\mathbf{A} := A\left[V_s^{\pm 1} \mid s \in S\right]$ for (W,S)-compatible indeterminates V_s and denote by $\mathbb{K}$ the field of fractions of $\mathbf{A}$. Then the generic Iwahori-Hecke algebra $H_{\mathbb{K}}(W,S,(U_s := V_s^2 \mid s \in S))$ is split.*

Corollary 7.1.8 (cf. [GP00, Theorem 8.1.7]) *Assume the setting of Proposition 7.1.7 and suppose that A is integrally closed in its field of fractions. Then the generic Iwahori-Hecke algebra $H_{\mathbb{K}}(W,S,(U_s := V_s^2 \mid s \in S))$ is split semisimple and there is a trivial isomorphism*

$$R_0(H_{\mathbb{K}}(W,S,(U_s \mid s \in S))) \to R_0(\mathbb{C}[W]).$$

This isomorphism induces a labelling of the irreducible $H_{\mathbb{K}}(W,S,(U_s \mid s \in S))$-modules by the irreducible ordinary characters of W.

Proof Let $H := H_{\mathbf{A}}(W, S, (U_s \mid s \in S))$. Then $H_{\mathbb{K}}(W, S, (U_s \mid s \in S))$ is isomorphic to $\mathbb{K}\,H$ by Lemma 7.1.1. Now denote by $\mathbb{K}_0$ the field of fractions of A. Then there is a well-defined ring homomorphism $\theta : \mathbf{A} \to \mathbb{K}_0; V_s \mapsto 1$ and the specialised algebra $\mathbb{K}_0\,H$ is isomorphic to the group algebra $\mathbb{K}_0[W]$ by Example 2.2.2. By hypothesis $\mathbb{K}_0$ is a splitting field of W and as $\mathbb{K}_0$ has characteristic zero the algebra $\mathbb{K}_0\,H$ is split semisimple. Thus, there is a well-defined decomposition map $d_\theta : R_0(\mathbb{K}\,H) \to R_0(\mathbb{K}_0[W])$ and by Tits' Deformation Theorem 6.1.21 this is a trivial isomorphism and $\mathbb{K}\,H$ is semisimple.
Now as $\mathbb{K}_0\,W$ is split semi-simple there also exists a trivial isomorphism $R_0(\mathbb{K}_0[W]) \to R_0(\mathbb{C}[W])$ by Lemma 6.1.6, yielding the labelling of irreducible $\mathbb{K}\,H$-modules. ∎

Proposition 7.1.7 requires characteristic zero splitting fields of Coxeter groups as one key ingredient. These can be rather small. Since a splitting field of a direct product of groups is given by any field containing splitting fields for all factors we only consider irreducible Coxeter groups.

Proposition 7.1.9 ([GP00, Theorem 6.3.8]) *If W is a Weyl group, then $\mathbb{Q}$ is a splitting field of W. If W is of type H_3 or H_4, then a splitting field of W is given by $\mathbb{Q}(\cos(2\pi/5))$. If W has type $I_2(m)$, then $\mathbb{Q}(\cos(2\pi/m))$ is a splitting field of W.*
All of the above splitting fields are minimal in the sense that they are contained in every characteristic zero splitting field of their respective groups.

Because it will make our lives easier in the long run we will usually consider slightly larger splitting fields.

Corollary 7.1.10 *For an integer $n \geq 2$ let ζ_n be a primitive n'th root of unity over $\mathbb{Q}$. Then $\mathbb{Q}(\zeta_5)$ is a splitting field of both H_3 and H_4 and $\mathbb{Q}(\zeta_m)$ is a splitting field of $I_2(m)$.*
In particular, for every Coxeter group W a splitting field is given by $\mathbb{Q}(\zeta_{m_{st}} \mid s, t \in S,\ m_{st} \neq 2, 3, 4, 6)$, where as usual m_{st} denotes the order of the product st in W. Clearly, this field is equal to $\mathbb{Q}(\zeta_m)$ where m is the least common multiple of all exponents $m_{s,t}$ that are not $2, 3, 4,$ or 6.

The next corollary, too, follows from Tits' deformation theorem and Corollary 7.1.8. While this seems rather obvious the proof is quite technical as we have to make sure that we are actually in a situation to apply those results.

Corollary 7.1.11 *Let (W, S) be a Coxeter group. Let k be a field containing invertible elements u_s such that $u_s = u_t$ whenever s and t are conjugate in W. If $H' := H_k(W, S, (u_s \mid s \in S))$ is split semisimple, then there is a trivial isomorphism*

$$R_0(H') \to R_0(\mathbb{C}[W]).$$

Proof This is just a matter of carefully checking the existence of a suitable ring homomorphism:
Let F be the prime field of k. We adjoin several elements to F.
First, for every u_s choose a square root v_s, that is a solution of the equation $X^2 - u_s = 0$, such that

$v_s = v_t$ if $u_s = u_t$. Then let ζ_m be a primitive root of unity, where m is the least common multiple of the exponents m_{st} with $m_{st} \neq 2, 3, 4, 6$, and choose α such that $\mathbb{Z}[\zeta_m] \to F[\alpha] : \zeta_m \mapsto \alpha$ defines a ring homomorphism. This can for example be achieved by choosing α as a root of the image of the m'th cyclotomic polynomial Φ_m in $F[X]$. Finally, there exist non-zero elements x_i with i in some index set I such that k is contained in $F(x_i \mid i \in I)$. Set $E := F(\alpha, v_s, x_i \mid i \in I,\ s \in S) \subseteq \widetilde{k}$ and $\mathbf{A} := \mathbb{Z}\left[\zeta_m,\ V_s^{\pm 1},\ X_i^{\pm 1} \mid i \in I,\ s \in S\right]$ for indeterminates V_s and X_i, where $V_s = V_t$ if and only if s and t are conjugate in W. By the choice of the various elements there exists a ring homomorphism $\theta : \mathbf{A} \to E : \zeta_m \mapsto \alpha,\ V_s \mapsto v_s,\ X_i \mapsto x_i$ and E is the field of fractions of $\theta(\mathbf{A})$.

Let $H := H_{\mathbf{A}}(W, S, (U_s := V_s \mid s \in S))$ be the corresponding generic Iwahori-Hecke algebra. Then the specialisation EH is the Iwahori-Hecke algebra of W with parameters u_s over E and therefore it is the specialisation EH' of H' via the embedding $k \hookrightarrow E$. Since H' is split semisimple, it is separable and hence EH, too, is split semisimple. If we denote by $\mathbb{K}$ the field of fractions of $\mathbf{A}$ it follows from Tits' Deformation Theorem 6.1.21 that the decomposition map

$$d_\theta : R_0(\mathbb{K}H) \to R_0(EH)$$

is a trivial isomorphism. Note that $\mathbb{K}H$ is split by Corollary 7.1.10 and Proposition 7.1.7.
Finally, by Lemma 6.1.6, we can identify $R_0(kH')$ with $R_0(EH)$ and by Corollary 7.1.8 we can also identify $R_0(\mathbb{K}H)$ with $R_0(\mathbb{C}[W])$ via a trivial isomorphism. ■

In general it is hard to say whether a given Iwahori-Hecke algebra is split. One powerful criterion can be stated in terms of L_0-good rings.

Definition 7.1.12 (cf. [GJ11, Remark 1.3.5, Table 3.1]) Let (W, S) be a finite Coxeter system. An integral domain A containing $\mathbb{Z}$ as a subring is called *L_0-good for W* if it is L_0-good for all irreducible factors of W. The conditions on A to be L_0-good if W is irreducible are given in the table below, where we denote by $U(A)$ the invertible elements in A.

Type of W	Conditions on A
A_n or B_n	-
D_n or G_2	$2 \in U(A)$
F_4, E_6, or E_7	$2, 3 \in U(A)$
E_8	$2, 3, 5 \in U(A)$
H_3	$2, 5 \in U(A)$ and $2\cos(2\pi/5) \in A$
H_4	$2, 3, 5 \in U(A)$ and $2\cos(2\pi/5) \in A$
$I_2(m)$	$m \in U(A)$ and $2\cos(2\pi/m) \in A$

Theorem 7.1.13 ([GJ11, 3.1.12, Lemma 3.1.13, Theorem 3.1.14]) *Let A be an L_0-good ring for W and assume that A is Noetherian and integrally closed in its field of fractions. Let $\mathbf{A} := A[V_s^{\pm 1} \mid s \in S,]$ for (W, S)-compatible indeterminants V_s and denote by $\mathbb{K}$ its field of fractions. Let $\mathcal{H} := H_{\mathbf{A}}(W, S, (U_s := V_s^2 \mid s \in S)$. Then the generic algebra $\mathbb{K}\mathcal{H} = \mathcal{H} \otimes_{\mathbf{A}} \mathbb{K}$ is split semisimple.*
Suppose additionally that $\theta : \mathbf{A} \to k$ is a ring homomorphism to a field k. Then the specialised algebra $k\mathcal{H}$ is split and the decomposition map $d_\theta : R_0(\mathbb{K}\mathcal{H}) \to R_0(k\mathcal{H})$ is surjective.

Remark 7.1.14 Note that, by Proposition 7.1.7 and Corollary 7.1.8, we already knew that $\mathbb{K}\mathcal{H}$ was split semisimple.

The theorem makes it clear that L_0-good rings which are Noetherian and integrally closed are desirable.

Definition 7.1.15 For an integer $m \geq 2$ denote by ζ_m a primitive root of unity over $\mathbb{Q}$. For elements $a_1, \ldots, a_r$ of $\widetilde{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, denote by $\mathbb{Z}[a_1, \ldots, a_r]$ the subring of $\widetilde{\mathbb{Q}}$ containing $\mathbb{Z}$ and $a_1, \ldots, a_r$. Then for an irreducible Coxeter group W define the ring $\mathbb{Z}_W$ depending on the type of W as follows:

Type of W	$\mathbb{Z}_W$
A_n or B_n	$\mathbb{Z}$
D_n or G_2	$\mathbb{Z}[1/2]$
F_4, E_6, or E_7	$\mathbb{Z}[1/2,\ 1/3]$
E_8	$\mathbb{Z}[1/2,\ 1/3,\ 1/5]$
H_3	$\mathbb{Z}[\zeta_5,\ 1/2,\ 1/5]$
H_4	$\mathbb{Z}[\zeta_5, 1/2,\ 1/3,\ 1/5]$
$I_2(m)$	$\mathbb{Z}[\zeta_m, 1/m]$

For a reducible Coxeter group W denote by $\mathbb{Z}_W$ the smallest ring extension of $\mathbb{Z}$ containing $\mathbb{Z}_{W_i}$ for all irreducible factors W_i of W.

Lemma 7.1.16 *Let (W, S) be a finite Coxeter system. Then $\mathbb{Z}_W$ is L_0-good for W, Noetherian, and algebraically closed in its field of fractions.*

Let us consider these result in the context of induction. In particular the surjectivity of the decomposition map is a nice additional bit of information in Theorem 7.1.13, since this can be used to compute induction maps by Corollary 6.2.7.

Corollary 7.1.17 *If an integral A is L_0-good for W, then it is also L_0-good for all its parabolic subgroups W_J for $J \subseteq S$. Hence, in the setting of Theorem 7.1.13 everything still holds for all parabolic subalgebras $\mathcal{H}_J$ of $\mathcal{H}$ and their specialisations. In particular, the decomposition map $d'_\theta : R_0(\mathbb{K}\mathcal{H}_J) \to R_0(k\mathcal{H}_J)$ has a right inverse c'_θ and if we let Ind_k be the induction map $R_0(\mathcal{H}_J) \to R_0(k\mathcal{H})$ and $\mathrm{Ind}_\mathbb{K}$ the induction map $R_0(\mathbb{K}\mathcal{H}_J) \to R_0(\mathbb{K}\mathcal{H})$, then $\mathrm{Ind}_k = d_\theta \circ \mathrm{Ind}_\mathbb{K} \circ c'_\theta$.*

Proof We only have to consider irreducible Coxeter groups to show that an L_0-good ring is L_0-good for its parabolic subgroups and this follows immediately from the definition. Everything else follows from Theorem 7.1.13 and Corollary 6.2.7. ∎

By Corollary 7.1.8 we have the following.

Corollary 7.1.18 *The map $\mathrm{Ind}_\mathbb{K}$ from Corollary 7.1.17 can be computed by only using methods from the ordinary representation theory of $\mathbb{C}[W]$ and $\mathbb{C}[W_J]$.*

Proof In the setting of Theorem 7.1.13 there are trivial decomposition maps $d_1 : R_0(\mathbb{K}\mathcal{H}) \to R_0(\mathbb{C}[W])$ and $d_1' : R_0(\mathbb{K}\mathcal{H}_J) \to R_0(\mathbb{C}[W_J])$. Denote by $\mathrm{Ind}_{\mathbb{C}}$ the induction map for the Grothendieck groups of the group algebras $\mathbb{C}[W]$ and $\mathbb{C}[W_J]$. The,n by Theorem 6.2.6, there is a commuting diagram

$$\begin{array}{ccc} R_0(\mathbb{K}\mathcal{H}_J) & \xrightarrow{d_1'} & R_0(\mathbb{C}[W_J]) \\ \downarrow{\scriptstyle \mathrm{Ind}_{\mathbb{K}}} & & \downarrow{\scriptstyle \mathrm{Ind}_{\mathbb{C}}} \\ R_0(\mathbb{K}\mathcal{H}) & \xrightarrow{d_1} & R_0(\mathbb{C}[W]) \end{array}$$

and by identifying $R_0(\mathbb{K}\mathcal{H})$ with $R_0(\mathbb{C}[W])$ and $R_0(\mathbb{K}\mathcal{H})$ with $R_0(\mathbb{C}[W_J])$ via the trivial isomorphisms the claim follows. ∎

Remark 7.1.19 Suppose the setting of Theorem 7.1.13 and the notation of the proof of Corollary 7.1.18. Then the following diagram commutes:

$$\begin{array}{ccccc} R_0(\mathbb{C}[W_J]) & \xrightarrow{(d_1')^{-1}} & R_0(\mathbb{K}\mathcal{H}_J) & \xrightarrow{d_\theta'} & R_0(k\mathcal{H}_J) \\ \downarrow{\scriptstyle \mathrm{Ind}_{\mathbb{C}}} & & \downarrow{\scriptstyle \mathrm{Ind}_{\mathbb{K}}} & & \downarrow{\scriptstyle \mathrm{Ind}_k} \\ R_0(\mathbb{C}[W]) & \xrightarrow{d_1^{-1}} & R_0(\mathbb{K}\mathcal{H}) & \xrightarrow{d_\theta} & R_0(k\mathcal{H}) \end{array}$$

Note that $d_\theta' \circ (d_1')^{-1}$ is surjective, indicating how to compute Ind_k without explicitly computing $\mathrm{Ind}_{\mathbb{K}}$.

Example: Parabolic induction of Iwahori-Hecke algebras of dihedral type

As an example for the applications of the commuting diagram in Remark 7.1.19 we study Iwahori-Hecke algebras of type $I_2(m)$. Throughout this subsection fix an integer $m \geq 3$ and set $W := I_2(m)$.

A presentation of W is given by $\left\langle s, t \mid s^2,\ t^2, (st)^m \right\rangle$ and $(W,\ \{s,t\})$ is a Coxeter system. Clearly, the only maximal parabolic subgroups of W are $\langle s \rangle$ and $\langle t \rangle$, respectively, and by symmetry it does not matter which one we consider. Thus, let $J := \{s\}$. Clearly, W_J is a cyclic group of order 2.

Let $\mathbf{A} := \mathbb{Z}_W[V_s^{\pm 1}, V_t^{\pm 1}]$ for indeterminates V_s and V_t. If m is odd, then assume $V_s = V_t$, otherwise assume that V_s and V_t are different indeterminates. Set $\mathbb{K}$ to be the field of fractions of $\mathbf{A}$. Suppose that $\theta : \mathbf{A} \to k$ is a ring homomorphism to a field k. Note that $1/m$ is in $\mathbf{A}$ and thus the characteristic of k does not divide m. Denote by $\mathcal{H} := H_{\mathbf{A}}(W, \{s,t\},\ (U_s := V_s^2, U_t := V_t^2))$ the corresponding generic Iwahori-Hecke algebra and, as usual, by $\mathcal{H}_J$ its parabolic subalgebra.

We want to exploit the commutative diagram in Remark 7.1.19 to compute Ind_k. To this end, we begin by computing $\mathrm{Ind}_{\mathbb{C}}$. Let us first give the irreducible representations of $\mathbb{C}[W]$.

Lemma 7.1.20

a) *Suppose m is even.*

   i) *A set of representatives of the conjugacy classes of W is given by* $\{1, s, t, (st)^k \mid 1 \leq k \leq m/2\}$.

   ii) *The group algebra* $\mathbb{C}[W]$ *has four one-dimensional,* $(m-2)/2$ *two-dimensional, and no other irreducible representations. The representations are uniquely defined by their characters. We denote the one-dimensional characters by* $\chi_{-1,-1}, \chi_{-1,1}, \chi_{1,-1}$, *and* $\chi_{1,1}$, *and the two dimensional characters by* χ_j *for* $1 \leq j \leq (m-2)/2$. *For allowed values of* a, b, j, *and* k *the values on conjugacy class representatives are as follows:*

$$\chi_{a,b}(1) = 1$$
$$\chi_{a,b}(s) = a$$
$$\chi_{a,b}(t) = b$$
$$\chi_{a,b}((st)^k) = (ab)^k$$
$$\chi_j(1) = 2$$
$$\chi_j(s) = \chi_j(t) = 0$$
$$\chi_j((st)^k) = \zeta_m^{jk} + \zeta_m^{-jk}.$$

b) *Suppose m is odd.*

   i) *A set of representatives of the conjugacy classes of W is given by* $\{1, s, (st)^k \mid 1 \leq k \leq (m-1)/2\}$.

   ii) *The group algebra* $C[W]$ *has two one-dimensional,* $(m-1)/2$ *two-dimensional, and no other irreducible representations. They are uniquely defined by their characters. We denote the one-dimensional characters by* $\chi_{1,1}$ *and* $\chi_{-1,-1}$ *and the two-dimensional characters by* χ_j *for* $1 \leq j \leq (m-1)/2$. *For allowed values of* a, j, *and* k *their values on conjugacy class representatives are as follows:*

$$\chi_{a,a}(1) = 1$$
$$\chi_{a,a}(s) = a$$
$$\chi_{a,a}(t) = a$$
$$\chi_{a,a}((st)^k) = a^{2k}$$
$$\chi_j(1) = 2$$
$$\chi_j(s) = \chi_j(t) = 0$$
$$\chi_j((st)^k) = \zeta_m^{jk} + \zeta_m^{-jk}.$$

Clearly, the conjugacy classes of W_J are $\{1\}$ and $\{s\}$. The only irreducible representations of $\mathbb{C}[W_J]$ are $\varphi : s \mapsto 1$ and $\psi : s \mapsto -1$. Let us compute their inductions. For an irreducible character χ of $\mathbb{C}[W]$ (or $\mathbb{C}[W_J]$) we will write $[\chi]$ for the class in the Grothendieck group of a $\mathbb{C}[W]$-module (or $\mathbb{C}[W_J]$-module) with character χ.

Proposition 7.1.21

a) Suppose m is even. Then

$$\mathrm{Ind}_{\mathbb{C}}([\varphi]) = [\chi_{1,-1}] + [\chi_{1,1}] + \sum_{j=1}^{(m-2)/2} [\chi_j]$$

and

$$\mathrm{Ind}_{\mathbb{C}}([\psi]) = [\chi_{-1,1}] + [\chi_{-1,-1}] + \sum_{j=1}^{(m-2)/2} [\chi_j].$$

b) Suppose m is odd. Then

$$\mathrm{Ind}_{\mathbb{C}}([\varphi]) = [\chi_{1,1}] + \sum_{j=1}^{(m-1)/2} [\chi_j]$$

and

$$\mathrm{Ind}_{\mathbb{C}}([\psi]) = [\chi_{-1,-1}] + \sum_{j=1}^{(m-1)/2} [\chi_j].$$

Proof Denote by (,) the usual scalar product of characters.
We handle both cases "m even" and "m odd" simultaneously. Let a, b and j such that $\chi_{a,b}$ and χ_j are irreducible characters of $\mathbb{C}[W]$.
By Frobenius reciprocity

$$\begin{aligned}(\chi_{a,b},\ \mathrm{Ind}_{\mathbb{C}}(\varphi)) &= (\mathrm{Res}_{\mathbb{C}}(\chi_{a,b}),\ \varphi)\\ &= \frac{1}{2}\left(\chi_{a,b}(1) + \chi_{a,b}(s)\right)\\ &= \frac{1}{2}(1+a)\\ &= \begin{cases}0, & a=-1\\ 1, & a=1\end{cases}\end{aligned}$$

and

$$\begin{aligned}(\chi_j, \mathrm{Ind}_{\mathbb{C}}(\varphi)) &= (\mathrm{Res}_{\mathbb{C}}(\chi_j), \varphi)\\ &= \frac{1}{2}\left(\chi_j(1) + \chi_j(s)\right)\\ &= \frac{1}{2}(2+0) \ = \ 1.\end{aligned}$$

For the induction of ψ this works completely analogous. ■

To compute Ind_k we still need decomposition maps. The following is immediate from the quadratic relation satisfied by T_s, the fact that $d'_1 : R_0(\mathbb{K}\mathcal{H}_J) \to R_0(\mathbb{C}[W_J])$ is a trivial isomorphism, and the surjectivity of d'_θ.

Lemma 7.1.22 *The algebra $\mathbb{K}\mathcal{H}_J$ has exactly two irreducible representations, namely $T_s \mapsto -1$ and $T_s \mapsto U_s$. Thus, $k\mathcal{H}_J$ has exactly two irreducible representations if $-1 \neq \theta(U_s)$ in k, and exactly one irreducible representation if $-1 = \theta(U_s)$ in k. These representations are M_{-1}, given by $T_s \mapsto -1$, and M_{U_s}, given by $T_s \mapsto \theta(U_s)$.*

Corollary 7.1.23 *The morphism*

$$c : R_0(k\mathcal{H}_J) \to \mathbb{C}[W_J]; [M_{-1}] \mapsto [\psi],\ [M_{U_s}] \mapsto [\varphi]$$

is a right-inverse of $d'_\theta \circ (d'_1)^{-1}$.

The only missing ingredient is d_θ, or rather $f_\theta := d_\theta \circ d_1^{-1}$:

Lemma 7.1.24 ([GJ11, Section 7.2]) *For an irreducible character of $\mathbb{C}[W]$ set $[\widetilde{\chi}] := f_\theta([\chi])$.*

a) If $\theta(U_s) = -1$ and $\theta(U_t) = -1$, then $[\widetilde{\chi_{-1,-1}}] = [\widetilde{\chi_{-1,1}}] = [\widetilde{\chi_{1,-1}}] = [\widetilde{\chi_{1,1}}]$

b) If $\theta(U_s) = -1$ and $\theta(U_t) \neq -1$, then $[\widetilde{\chi_{-1,1}}] = [\widetilde{\chi_{1,1}}] \neq [\widetilde{\chi_{-1,-1}}] = [\widetilde{\chi_{1,-1}}]$.

c) If $\theta(U_s) \neq -1$ and $\theta(U_t) = -1$, then $[\widetilde{\chi_{1,-1}}] = [\widetilde{\chi_{1,1}}] \neq [\widetilde{\chi_{-1,-1}}] = [\widetilde{\chi_{-1,1}}]$

d) If $\theta(U_s) \neq -1$ and $\theta(U_t) \neq -1$, then $[\widetilde{\chi_{1,-1}}]$ $[\widetilde{\chi_{1,1}}]$, $[\widetilde{\chi_{-1,-1}}]$, and $[\widetilde{\chi_{-1,1}}]$ are pairwise different.

e) Let j be such that χ_j is an irreducible character of $\mathbb{C}[W]$. Then $[\widetilde{\chi_j}]$ is irreducible, unless $\theta(U_sU_t) = \theta(\zeta_m)^{\pm j}$ or $\theta(U_sU_t^{-1}) = -\theta(\zeta_m)^{\pm j}$. In the first case $[\widetilde{\chi_j}] = [\widetilde{\chi_{1,1}}] + [\widetilde{\chi_{-1,-1}}]$. The second case can only occur if m is even, and if it occurs, then $[\widetilde{\chi_j}] = [\widetilde{\chi_{1,-1}}] + [\widetilde{\chi_{-1,1}}]$.

Furthermore, if $j \neq j'$, then $[\widetilde{\chi_j}] \neq [\widetilde{\chi_{j'}}]$.
This gives a full description of f_θ.

Thus, we can now compute Ind_k, since $\mathrm{Ind}_k = f \circ \mathrm{Ind}_{\mathbb{C}} \circ c$ for c as in Corollary 7.1.23 by Remark 7.1.19.

Theorem 7.1.25

a) Suppose m is odd and let $\{1, \dots, (m-1)/2\} = I_1 \coprod I_2$ a disjoint union such that I_2 contains exactly all j with $\theta(U_sU_t) = \theta(\zeta_m)^{\pm j}$. Then

$$\mathrm{Ind}_k\,[M_{-1}] = [\widetilde{\chi_{-1,-1}}] + |I_2|\left([\widetilde{\chi_{1,1}}] + [\widetilde{\chi_{-1,-1}}]\right) + \sum_{j\in I_1}[\widetilde{\chi_j}]$$

and

$$\mathrm{Ind}_k\,[M_{U_s}] = [\widetilde{\chi_{1,1}}] + |I_2|\left([\widetilde{\chi_{1,1}}] + [\widetilde{\chi_{-1,-1}}]\right) + \sum_{j\in I_1}[\widetilde{\chi_j}],$$

where all summands on the right-hand sides are classes of irreducible modules.

b) *Suppose m is even and let $\{1,\dots,(m-2)/2\} = I_1 \coprod I_2 \coprod I_3$ a disjoint union such that I_2 contains exactly all j with $\theta(U_sU_t) = \theta(\zeta_m)^{\pm j}$ and I_3 those with $\theta(U_sU_t^{-1}) = -\theta(\zeta_m)^{\pm j}$. Then*

$$\begin{aligned}\operatorname{Ind}_k([M_{-1}]) &= [\widetilde{\chi_{-1,-1}}] + [\widetilde{\chi_{-1,1}}] + |I_2|\,([\widetilde{\chi_{1,1}}] + [\widetilde{\chi_{-1,-1}}]) \\ &\quad + |I_3|\,([\widetilde{\chi_{-1,1}}] + [\widetilde{\chi_{1,-1}}]) + \sum_{j\in I_1}[\widetilde{\chi_j}]\end{aligned}$$

and

$$\begin{aligned}\operatorname{Ind}_k([M_{U_s}]) &= [\widetilde{\chi_{1,1}}] + [\widetilde{\chi_{1,-1}}] + |I_2|\,([\widetilde{\chi_{1,1}}] + [\widetilde{\chi_{-1,-1}}]) \\ &\quad + |I_3|\,([\widetilde{\chi_{-1,1}}] + [\widetilde{\chi_{1,-1}}]) + \sum_{j\in I_1}[\widetilde{\chi_j}],\end{aligned}$$

where all summands on the right-hand sides are classes of irreducible modules.

7.2. Semisimplicity of Specialised Iwahori-Hecke algebras and Bad Primes

As seen in Corollary 7.1.11 the specialisation of generic Iwahori-Hecke algebras to semisimple algebras only yields trivial decomposition maps. Thus, the interesting cases are specialisations to Iwahori-Hecke algebras that are not semisimple. In this section we give a powerful criterion to discern these cases.
Let (W, S) be a finite Coxeter system and $\mathbb{Z} \leq A$ an integrally closed integral domain such that the field of fractions of A is a splitting field of W. Let $\mathbf{A} := A\left[V_s^{\pm 1} \mid s \in S\right]$ for (W, S)-compatible indeterminates V_s. Denote by $\mathbb{K}$ its field of fractions. Set $\mathcal{H} := H_{\mathbf{A}}(W, S, (U_s := V_s^2 \mid s \in S))$. Then by Corollary 7.1.8 the generic Iwahori-Hecke algebra $\mathbb{K}\mathcal{H}$ is split semi-simple.
Suppose now that $\theta : \mathbf{A} \to k$ is a ring homomorphism to a field k such that $k'\mathcal{H}$ is split, where k' is the field of fractions of $\theta(\mathbf{A})$. In the following we will develop a semisimplicity criterion for $k\mathcal{H}$ that derives from $\mathcal{H}$ being a symmetric algebra.

Definition 7.2.1 (cf. [GP00, Definition 7.2.1, Theorem 7.2.6]) Let $\operatorname{Irr}(\mathbb{K}\mathcal{H})$ be the set of irreducible modules of $\mathbb{K}\mathcal{H}$ up to isomorphism and for $M \in \operatorname{Irr}(\mathbb{K}\mathcal{H})$ denote by χ_M its character. By Proposition 2.2.12, there is a symmetrising trace τ on $\mathbb{K}\mathcal{H}$. Since $\mathbb{K}\mathcal{H}$ is split semisimple there exist unique elements $c'_M \in \mathbb{K}$ such that

$$\tau = \sum_{M\in\operatorname{Irr}(\mathbb{K}\mathcal{H})} c'_M \chi_M.$$

For $M \in \operatorname{Irr}(\mathbb{K}\mathcal{H})$ we have $c'_M \neq 0$ and we call $c_M := (c'_M)^{-1}$ the *Schur element of M*.

Every Iwahori-Hecke algebras has a certain one-dimensional representation.

Lemma 7.2.2 ([GP00, Remark 8.1.3]) *For any Iwahori-Hecke algebra over some ring with parameters u_s the map* ind : $T_s \mapsto u_s$ *defines a one-dimensional representation of the Iwahori-Hecke algebra called the* index representation.

Proof It is trivial to check that the braid relations and the quadratic relations of the Iwahori-Hecke algebra are satisfied by the images $\mathrm{ind}(T_s)$. ∎

Remark 7.2.3 (cf. [GP00, 8.1.8]) The index representation of $\mathbb{K}\mathcal{H}$ is one-dimensional and hence irreducible. Its Schur element is called the *Poincaré polynomial of W* and denoted by P_W. We have $P_W = c_{\mathrm{Ind}} = \sum_{w \in W} \mathrm{ind}(T_w)$. In particular, $P_w \in \mathbf{A}$.

This shows, that we can apply θ to the Poincaré polynomial of W. But we can actually do this for all irreducible representations of $\mathbb{K}\mathcal{H}$.

Lemma 7.2.4 (cf. [GP00, Theorem 9.3.5]) *Let $M \in \mathrm{Irr}(\mathbb{K}\mathcal{H})$. Then c_M is in $\mathbf{A}$. Even stronger: Let $w_0 \in W$ be the unique longest word in W. Then $\mathrm{ind}(T_{w_0})c_M$ lies in $A\,[V_s \mid s \in S]$, i.e. all the V_s have non-negative exponent.*

This allows us to state the aforementioned semisimplicity criterion:

Proposition 7.2.5 ([GP00, Theorem 7.4.7]) *The algebra $k\mathcal{H}$ is semisimple if and only if $\theta(c_M) \neq 0$ for all irreducible $\mathbb{K}\mathcal{H}$-modules M.*

As the Schur elements are known whenever W is irreducible (cf. [GP00, Theorems 8.3.4, 10.5.2, 10.5.3, Proposition 10.5.6, Appendix E]), this proposition yields a straightforward test for semisimplicity of $\mathbb{K}\mathcal{H}$.

The most common setting is the so-called equal parameter case, in which all parameters of the Iwahori-Hecke algebra are equal. Hence, until the end of the section assume that $V_s = V_t$ for all s and t in S. Set $V := V_s$ and $U := U_s$.

Proposition 7.2.6 ([GP00, Corollary 9.3.6]) *Suppose W is an irreducible Weyl group. Let $M \in \mathrm{Irr}(\mathbb{K}\mathcal{H})$. Then there exists a positive integer f_M and a monic polynomial r_M in $\mathbb{Z}[U]$ such that*

$$\frac{1}{f_M} c_M r_M = P_W.$$

Since the Poincaré polynomial itself is a Schur element, the following is immediate from Proposition 7.2.5.

Corollary 7.2.7 *Suppose W is an irreducible Weyl group. If the characteristic of k does not divide f_M for any $M \in \mathrm{Irr}(\mathbb{K}\mathcal{H})$, then the specialised algebra $k\mathcal{H}$ is semisimple if and only if* $\theta(P_W) \neq 0$.

Hence, the characteristics dividing any of the positive integers f_M are of particular importance.

Definition 7.2.8 Suppose W is an irreducible Weyl group. The prime numbers that divide f_M for some $M \in \mathrm{Irr}(\mathbb{K}\mathcal{H})$ are called *bad primes for W*. All other prime numbers are called *good primes for W*. To get rid of cumbersome exceptions, we say that zero, too, is good for W.

Lemma 7.2.9 *The bad primes for irreducible Weyl groups are given in the table below.*

W	*Bad primes*
A_n	–
B_n, D_n	2
G_2, F_4, E_6, E_7	2, 3
E_8	2, 3, 5

Remark 7.2.10 Note that the characteristic of k is necessarily a good prime for W if the ring A is L_0-good for W.

Proposition 7.2.11 ([GP00, Theorem 10.5.1, Appendix E]) *Suppose W is an irreducible Weyl group. Denote by Φ_m the m'th cyclotomic polynomial over $\mathbb{Q}$ in the indeterminate U. Then the Poincaré polynomial P_W is as follows:*

W	P_W
A_n	$\prod_{i=1}^{n+1}(1 + U + \cdots + U^{i-1})$
B_n	$P_{A_{n-1}} \prod_{i=1}^{n}(1 + U^i)$
D_n	$P_{A_{n-1}} \prod_{i=1}^{n-1}(1 + U^i)$
G_2	$\Phi_2^2\Phi_3\Phi_6$
F_4	$\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$
E_6	$\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$
E_7	$\Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
E_8	$\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$

Taking into account that

$$(1 + U + \cdots + U^{i-1}) = \prod_{\substack{d \mid i \\ d>1}} \Phi_d$$

and

$$1 + U^i = \prod_{\substack{d \mid 2i \\ d \nmid i}} \Phi_d,$$

it follows that the Poincaré polynomial has a factorisation into cylcotomic polynomials. In particular, if the characteristic of k is a good prime for W, then $k\mathcal{H}$ is semisimple unless $\theta(U)$ is a root of unity.

Example 7.2.12 Suppose $W = A_{n-1}$. By Proposition 7.2.11 and Lemma 7.2.9, the algebra $k\mathcal{H}$ is semisimple unless $\theta(U)$ is an i'th root of unity for some $i \leq n$. In particular, for fixed characteristic of k there is only a finite number of specialisations θ such that $k\mathcal{H}$ is not semisimple. Thus, for fixed characteristic there is only a finite number of specialisations for which d_θ is not trivial.
For example, if k has characteristic 0, then $k\mathcal{H}$ is semisimple if and only if $\theta(U)$ is not a root of Φ_i for some $i \leq n$.

7.3. Known Decomposition Numbers of Iwahori-Hecke algebras

We attempt to summarise the current status of computation of decomposition numbers of one-parameter Iwahori-Hecke algebras. We assume the setting of Section 7.2 and assume additionally that W is irreducible. By Theorem 6.1.29 and Proposition 2.2.6, this is can be done without loss of generality. Moreover, we only consider the case of one-parameter Iwahori-Hecke algebras, that is we assume $V := V_s = V_t$ for all s and t in S. Set $U := V^2$.

One-parameter Algebras of Weyl Groups in Characteristic zero

Let W be a Weyl group and suppose k has characteristic zero.
If $\theta(P_W(U)) \neq 0$, then by Corollary 7.2.7 and Theorem 6.1.21 the decomposition map d_θ is a trivial isomorphism. Hence, assume $\theta(P_W(U)) = 0$.

a) Type A_n: The decomposition map d_θ can be computed via the *LLT algorithm*. This algorithm computes numbers which Lascoux, Leclerc, and Thibon conjectured to be the decomposition numbers of Ariki-Koike algebras in characteristic 0, in particular for Iwahori-Hecke algebras of type A, cf. [LLT96]. This conjecture was proved by Ariki, cf. [Ari96].

b) Type B_n: The decomposition map d_θ can be computed via the LLT-algorithm or Jacon's generalisation thereof, cf. [Jac05]: Either $\mathbb{K}\mathcal{H}$ is already an Ariki-Koike algebra with q-connected parameters, in which case one can use Jacon's algorithm, or the decomposition map is the block diagonal matrix of decomposition maps of Iwahori-Hecke algebras of type A_n, cf. Theorem 5.2.2 and Proposition 5.2.4.

c) Type D_n: If n is odd and $\prod_{i=1}^{n-1}\left(1 + \theta(U)^i\right) \neq 0$, then the problem of computing d_θ has been reduced to the computation of decompositions numbers for type A by Pallikaros, cf. [Pal94].

d) Type G_2: The decomposition numbers are easily computed directly as all irreducible representations of $\mathbb{K}\mathcal{H}$ have degree at most 2, cf. for example [GJ11, Section 7.2].

e) Type F_4: The decomposition numbers have been computed by Geck and Lux, cf. [GL91].

f) Type E_6: The decomposition numbers have been computed by Geck, cf. [Gec93].

g) Type E_7: The decomposition numbers have been computed by Geck and Müller, cf. [Gec95; Mül95].

h) Type E_8: The decomposition numbers have been computed by Geck and Müller, cf. [Mül95; GM09].

The decomposition numbers of the exceptional Weyl groups F_4, E_6, E_7, and E_8 are printed in [GJ11, Chapter 7]. Here Geck and Jacon omit so-called blocks of defect 1, but this does not pose a problem as decomposition numbers of such blocks can be easily obtained: By [GJ11, Theorem 3.3.13] they can be read of the blocks' structure as printed in [GP00, Appendix F]. The blocks of defect 0 are also omitted. For these, the corresponding diagonal block of the decomposition matrix is of the form (1).
Lastly, note that for type E_8 there is a non-trivial block missing from the decomposition numbers in [GJ11]. It can however be found in Müller's dissertation thesis, cf. [Mül95], but one has to be careful as another labelling of irreducible characters of $\mathbb{K}\mathcal{H}$ is employed by Müller than by Geck and Jacon.

One-Parameter Algebras of Weyl Groups in Good Positive Characteristic

Assume as above that $V := V_s = V_t$ for all s and t in S and set $U := U_s$ and W is an irreducible Weyl group. Now suppose that the characteristic p of k is a good prime for W.
As W is a Weyl group we can assume without loss of generality that $\mathbb{K} = \mathbb{Q}(V)$ since this is a splitting field of $\mathbb{K}\mathcal{H}$.
We follow [GM09] to study this setting via the factorisation of decomposition maps.
By Tits' Deformation Theorem 6.1.21, Theorem 7.1.13, and Proposition 7.2.11 the only non-trivial decomposition maps appear in the case that $\theta(U)$ is a root of unity, so we assume that we are in this case.
Set $e \geq 2$ to be the minimal integer such that $1 + \theta(U) + \cdots + \theta(U)^{e-1} = 0$. For an integer $m \geq 1$ denote by Φ_m the m'the cyclotomic polynomial over $\mathbb{Q}$. Then $\Phi_e(\theta(U)) = \Phi_e(\theta(V)^2) = 0$ by definition of e. Now

$$\Phi_e(\theta(V)^2) = \begin{cases} \Phi_{2e}(\theta(V)) = \Phi_{2e}(-\theta(V)), & \text{if } e \text{ is even} \\ \Phi_e(\theta(V))\Phi_e(-\theta(V)), & \text{if } e \text{ is odd} \end{cases}$$

and for odd e we have $\Phi_e(-\theta(V)) = 0$ if and only if $\Phi_{2e}(\theta(V)) = 0$. Thus, after possibly replacing V by $-V$ in the definition of $\mathbf{A}$ and $\mathbb{K}$ we can assume $\Phi_{2e}(\theta(V)) = 0$.
Now let ζ be a primitive $2e$'th root of unity over $\mathbb{Q}$ and set $L := \mathbb{Z}[\zeta]$ and $\mathbb{L}$ its field of fractions. Then θ factors as $\theta = \theta_2 \circ \theta_1$ with

$$\theta_1 : \mathbf{A} \to L;\ V \mapsto \zeta$$
$$\theta_2 : L \to k;\ \zeta \mapsto \theta(V).$$

By Proposition 6.1.25, this also gives a factorisation of the corresponding decomposition maps:

$$\begin{array}{ccc} R_0(\mathbb{K}\mathcal{H}) & \xrightarrow{\quad d_\theta \quad} & R_0(k\mathcal{H}) \\ & {\scriptstyle d_{\theta_1}}\searrow \qquad \nearrow{\scriptstyle d_{\theta_2}} & \\ & R_0(\mathbb{L}\mathcal{H}) & \end{array}$$

Proposition 7.3.1 ([Gec92, Proposition 5.5]) *There exists an integer $N > 0$ depending only on e and W such that d_{θ_2} is trivial if the characteristic p of k does not divide N.*

a) Type A_n: It has been a long-standing conjecture by James that d_{θ_2} is trivial if $ep > n+1$, cf. [Jam90]. However, this has since been disproved by Williamson, cf [Wil17].
A number of rules partially determining d_θ are given in [Mat99, Section 6.4].

b) Type B_n: If $\prod_{i=-(n-1)}^{n-1}(U + U^i) \neq 0$, then decomposition map d_θ can be obtained from decomposition maps of Iwahori-Hecke algebras of type A, cf. [DJ92, Theorem 5.8].

c) Type D_n: If n is odd and $2\prod_{i=1}^{n-1}\left(1 + \theta(U)^i\right) \neq 0$, the decomposition map d_θ can be expressed in terms of the decomposition numbers of algebras of type A, but these equalities are only proven to be correct modulo p, i.e. they are equalities in the group $\mathbb{F}_p \otimes_{\mathbb{Z}} (R_0(\mathbb{L}\mathcal{H}))$, cf. [Hu09].

Now for exceptional Weyl groups.

Theorem 7.3.2 ([GM09, Theorem 3.10]) *Let W be an exceptional Weyl group, i.e an element of $\{G_2, F_4, E_6, E_7, E_8\}$ and denote by p the characteristic of k. Then d_{θ_2} is trivial unless*

- *$p = 5$, $e = 2$, and $W = E_7$, or*
- *$p = 7$, $e = 2$, and $W = E_7$, or*
- *$p = 7$, $e = 2$, and $W = E_8$.*

One-Parameter Algebras of Non-Crystallographic Groups

Let us consider non-crystallographic Coxeter groups: Assume as before that we are in the one-parameter case, i.e. $V := V_s = V_t$ for all s and t in S and set $U := U_s$.
Let $\mathbf{A} = \mathbb{Z}_W[V^{\pm 1}]$ with $\mathbb{Z}_W$ as in 7.1.15.

For non-crystallographic groups the concept of good and bad primes is more complicated than for Weyl groups. However, characteristic zero is still manageable, thus suppose k has characteristic zero. We obtain the following analogue of Proposition 7.2.11.

a) Type H_3: The algebra $k\mathcal{H}$ is semisimple unless $\theta(U)$ is a primitive root of unity of order $2, 3, 5, 6$, or 10.

b) Type H_4: The algebra $k\mathcal{H}$ is semisimple unless $\theta(U)$ is a primitive root of unity of order $2, 3, 4, 5, 6, 10, 12, 15, 20$, or 30.

c) Type $I_2(m)$: The algebra $k\mathcal{H}$ is semisimple unless $\theta(U)$ is a primitive root of unity j where $j \geq 2$ divides m.

In all cases Müller has computed the corresponding decomposition maps, cf. [Mül97]. For type $I_2(m)$ the decomposition maps have also been computed in the case that k has a characteristic not dividing m, cf. [GJ11, Section 7.2]. We consider specialisations to fields whose characteristic divides m in Section 9.3.
To the best of the author's knowledge the decomposition numbers for Iwahori-Hecke algebras of type H_3 and H_4 over fields of positive characteristic have not been published so far. We partially fill this gap in Section 9.2.

Chapter 8.

Induction of Hecke Algebras and Clifford Theory

In this chapter we investigate the relation between Iwahori-Hecke algebras and Ariki-Koike algebras and subalgebras which give rise to a Clifford system. We outline how this yields a connection between different induction and decomposition maps.

8.1. Clifford Theory of Iwahori-Hecke Algebras of Type D

It is well-known that groups of type D_n are normal subgroups of index 2 of groups of type B_n. Similarly, Iwahori-Hecke algebras of type D_n embed into certain Iwahori-Hecke algebras of type B_n. This gives rise to a connection between their respective representation theories and in particular the study of their decomposition maps, which this section is devoted to.

We begin by considering the underlying groups: The Coxeter graph of $W_n := B_n$ is given below.

The subgroup generated by $u := ts_1t$ and $s_1, \dots, s_{n-1}$ is a Coxeter group of type D_n and the generators correspond to the standard generators of D_n. This subgroup has index 2 in W_n and is therefore normal. We identify D_n with this normal subgroup of W_n and denote it by W_n'.

Now let $\mathbf{A} := \mathbb{Z}[V^{\pm 1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in one variable and define $\mathcal{H}_n$ as the Iwahori-Hecke algebra of W_n over $\mathbf{A}$ with parameters $(1, U, U, \dots, U)$, where $U := V^2$, and standard generators $T_t, T_1 := T_{s_1}, \dots, T_{n-1} := T_{s_{n-1}}$.
By [GJ11, Proposition 2.4.5] the subalgebra generated by $T_0 := T_tT_1T_t$ and $T_1, \dots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra of type D_n over $\mathbf{A}$ with parameter U, and under this isomorphism the generators $T_0, \dots, T_{n-1}$ are mapped to the standard generators of the latter Iwahori-Hecke algebra. We we denote this subalgebra by $\mathcal{H}_n'$.

Let $\mathbb{K}$ be the field of fractions of $\mathbf{A}$. By Corollary 7.1.8 both $\mathbb{K}\mathcal{H}_n$ and $\mathbb{K}\mathcal{H}'_n$ are split semisimple.

Proposition 8.1.1 (cf. [GJ11, Section 2.4], [Hu02, Section 4]) *The* $\mathbf{A}$*-submodules* $\{\mathcal{H}'_n, T_t\mathcal{H}'_n\}$ *form a* $\mathbb{Z}/2\mathbb{Z}$*-graded Clifford system of* $\mathcal{H}_n$ *in the sense of [CR81, Definition 11.12].*
In particular, $\mathcal{H}'_n$ *is a direct summand of* $\mathcal{H}_n$ *as an* $\mathbf{A}$*-module and by Remark 6.2.2 every ring homomorphism* $\mathbf{A} \to B$ *yields an embedding* $B\mathcal{H}'_n \leq B\mathcal{H}_n$*. Moreover,* $\{B\mathcal{H}'_n,\ T_tB\mathcal{H}'_n\}$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded Clifford system of* $B\mathcal{H}_n$*.*

Remark 8.1.2 Let $\mathbf{A} \to B$ be a ring homomorphism and denote the corresponding exact restriction functor by $R_B := \mathrm{Res}^{B\mathcal{H}_n}_{B\mathcal{H}'_n}$. We will abuse notation and also use this to denote the restriction functor $\mathrm{Res}^{B\mathcal{H}_{n+1}}_{B\mathcal{H}'_{n+1}}$, but it will always be clear from context which functor we are talking about. Note that this is *not* a *parabolic* restriction functor.
We will study this restriction's compatibility with a certain parabolic induction. To this end denote by Ind_B the parabolic induction functor $\mathrm{Ind}^{B\mathcal{H}_{n+1}}_{B\mathcal{H}_n}$ and by Ind'_B the parabolic induction functor $\mathrm{Ind}^{B\mathcal{H}'_{n+1}}_{B\mathcal{H}'_n}$.
By Theorems 6.2.9 and 6.2.6, the homomorphisms on the Grothendieck groups arising from the above functors commute with the decomposition maps, if B is a field and all involved algebras are split over the field of fractions of $\theta(B)$.

We will show that under certain conditions the restriction from algebras of type W to type W' commutes with the parabolic induction Ind respectively Ind$'$. This is done step by step, going through group algebras, generic algebras, and finally Iwahori-Hecke algebras over almost arbitrary fields.
We start with ordinary representation theory.

Lemma 8.1.3 *Consider the specialisation* $\mathbf{A} \to \mathbb{C}; V \mapsto 1$*. Then the specialised algebras are the group algebras of their respective Coxeter group and the following diagram commutes:*

$$\begin{array}{ccc} R_0(\mathbb{C}[W_n]) & \xrightarrow{R_{\mathbb{C}}} & R_0(\mathbb{C}[W'_n]) \\ \downarrow{\scriptstyle \mathrm{Ind}_{\mathbb{C}}} & & \downarrow{\scriptstyle \mathrm{Ind}'_{\mathbb{C}}} \\ R_0(\mathbb{C}[W_{n+1}]) & \xrightarrow{R_{\mathbb{C}}} & R_0(\mathbb{C}[W'_{n+1}]) \end{array}$$

Proof The subgroup W'_{n+1} is normal in W_{n+1}, and thus $W'_{n+1}W_n = W_nW'_{n+1} \leq W_{n+1}$. Furthermore, the generator t lies in W_n, and $s_1, \ldots, s_n$ lie in W'_{n+1}. Hence, $W'_{n+1}W_n = W_{n+1}$. Moreover, $W'_{n+1} \cap W_n = W'_n$, since $W'_n \subseteq W'_{n+1} \cap W_n$ and $|W'_{n+1} \cap W_n| = \left(|W'_{n+1}||W_n|\right)/|W_{n+1}| = |W'_n|$.
If M is a $\mathbb{C}[W_n]$-module, then the usual Mackey formula for group algebras (cf. [CR81, Theorem 10.13]) now shows that

$$\mathrm{Res}^{\mathbb{C}[W_{n+1}]}_{\mathbb{C}[W'_{n+1}]}\left(\mathrm{Ind}^{\mathbb{C}[W_{n+1}]}_{\mathbb{C}[W_n]}(M)\right) \cong \mathrm{Ind}^{\mathbb{C}[W'_{n+1}]}_{\mathbb{C}[W'_n]}\left(\mathrm{Res}^{\mathbb{C}[W_n]}_{\mathbb{C}[W'_n]}(M)\right),$$

which proves the claim. ■

Proposition 8.1.4 *The following diagram commutes:*

$$\begin{array}{ccc} R_0(\mathbb{K}\mathcal{H}_n) & \xrightarrow{R_{\mathbb{K}}} & R_0(\mathbb{K}\mathcal{H}'_n) \\ \Big\downarrow{\scriptstyle \mathrm{Ind}_{\mathbb{K}}} & & \Big\downarrow{\scriptstyle \mathrm{Ind}'_{\mathbb{K}}} \\ R_0(\mathbb{K}\mathcal{H}_{n+1}) & \xrightarrow{R_{\mathbb{K}}} & R_0(\mathbb{K}\mathcal{H}'_{n+1}) \end{array}$$

Proof By Corollary 7.1.8 the decomposition maps $d_n : R_0(\mathbb{K}\mathcal{H}_n) \to R_0(\mathbb{C}[W_n])$ and $d_{n+1} : R_0(\mathbb{K}\mathcal{H}_{n+1}) \to R_0(\mathbb{C}[W_{n+1}])$ are trivial isomorphisms, and so are $d'_n : R_0(\mathbb{K}\mathcal{H}'_n) \to R_0(\mathbb{C}[W'_n])$ and $d'_{n+1} : R_0(\mathbb{K}\mathcal{H}'_{n+1}) \to R_0(\mathbb{C}[W'_{n+1}])$. Decomposition commutes with induction by Theorem 6.2.6 and with restriction by Theorem 6.2.9. Thus, by Lemma 8.1.3, we have

$$\begin{aligned} & d'_{n+1} \circ R_{\mathbb{K}} \circ \mathrm{Ind}_{\mathbb{K}} \\ \overset{6.2.9}{=} \ & R_{\mathbb{C}} \circ d_{n+1} \circ \mathrm{Ind}_{\mathbb{K}} \\ \overset{6.2.6}{=} \ & R_{\mathbb{C}} \circ \mathrm{Ind}_{\mathbb{C}} \circ d_n \\ \overset{8.1.3}{=} \ & \mathrm{Ind}'_{\mathbb{C}} \circ R_{\mathbb{C}} \circ d_n \\ \overset{6.2.9}{=} \ & \mathrm{Ind}'_{\mathbb{C}} \circ d'_n \circ R_{\mathbb{K}} \\ \overset{8.1.3}{=} \ & d'_{n+1} \circ \mathrm{Ind}'_{\mathbb{K}} \circ R_{\mathbb{K}}. \end{aligned}$$

The claim now follows because d'_{n+1} is an isomorphism. ∎

This now enables us to finally prove the general version:

Theorem 8.1.5 *Let $\theta : \mathbf{A} \to k$ be a ring homomorphism to a field k such that k', the field of fractions of $\theta(\mathbf{A})$, is a splitting field of $k'\mathcal{H}_n$, $k'\mathcal{H}_{n+1}$, $k'\mathcal{H}'_n$, and $k'\mathcal{H}'_{n+1}$. Suppose the decomposition map $d_n : R_0(\mathbb{K}\mathcal{H}_n) \to R_0(k\mathcal{H}_n)$ is surjective. Then the following diagram commutes:*

$$\begin{array}{ccc} R_0(k\mathcal{H}_n) & \xrightarrow{R_k} & R_0(k\mathcal{H}'_n) \\ \Big\downarrow{\scriptstyle \mathrm{Ind}_k} & & \Big\downarrow{\scriptstyle \mathrm{Ind}'_k} \\ R_0(k\mathcal{H}_{n+1}) & \xrightarrow{R_k} & R_0(k\mathcal{H}'_{n+1}) \end{array}$$

Proof Denote by c a right inverse of d_n. There are a number of well-defined decomposition maps, namely $d_{n+1} : R_0(\mathbb{K}\mathcal{H}_{n+1}) \to R_0(k\mathcal{H}_{n+1})$, $d'_n : R_0(k\mathcal{H}'_n) \to R_0(k\mathcal{H}'_n)$ and $d'_{n+1} : R_0(\mathbb{K}\mathcal{H}'_{n+1}) \to R_0(k\mathcal{H}'_{n+1})$, all of which we need in the proof.
By Corollary 6.2.7 we have

$$\mathrm{Ind}_k = d_{n+1} \circ \mathrm{Ind}_{\mathbb{K}} \circ c \tag{8.1}$$

and similarly, by Theorem 6.2.9

$$R_k = d'_n \circ R_{\mathbb{K}} \circ c. \tag{8.2}$$

The statement now follows from the commuting diagrams in Theorem 6.2.6, Theorem 6.2.9, and Proposition 8.1.4, as we have

$$\mathrm{Ind}'_k \circ R_k \overset{(8.2)}{=} \mathrm{Ind}'_k \circ d'_n \circ R_{\mathbb{K}} \circ c$$

$$\begin{aligned} &\overset{6.2.6}{=} d'_{n+1} \circ \operatorname{Ind}'_{\mathbb{K}} \circ R_{\mathbb{K}} \circ c \\ &\overset{8.1.4}{=} d'_{n+1} \circ R_{\mathbb{K}} \circ \operatorname{Ind}_{\mathbb{K}} \circ c \\ &\overset{6.2.9}{=} R_k \circ d_{n+1} \circ \operatorname{Ind}_{\mathbb{K}} \circ c \\ &\overset{(8.1)}{=} R_k \circ \operatorname{Ind}_k . \end{aligned}$$

■

Remark 8.1.6 Note that, by Theorem 7.1.13, the hypotheses of Theorem 8.1.5 are automatically satisfied if the characteristic of k is not 2.

It follows that we can exploit our results on Iwahori-Hecke algebras of type B and maps R_k to study the induction for Iwahori-Hecke algebras of type D. In the following let $\theta : \mathbf{A} \to k$ be a ring homomorphism to a field k of a characteristic other than 2. By Theorem 7.1.13, the algebras $k\mathcal{H}_n$ and $k\mathcal{H}'_n$ are split for all n.
To relate the parabolic induction functors Ind and Ind′ we exploit the fact that we have a Clifford system.

Definition 8.1.7 Define a k-algebra automorphism $\tau_n : k\mathcal{H}_n \to k\mathcal{H}_n$ by $\tau_n(T_1) := T_0$, $\tau_n(T_i) := T_i$ for all $i > 1$, and $\tau_n(T_t) = T_t$.

Remark 8.1.8 Note that the automorphism τ_n is given by conjugation with T_t: The parameter of $\mathcal{H}_n$ corresponding to the generator T_t is 1, and therefore we have $T_t^2 = 1 + (1-1)T_t = 1$, thus $T_t^{-1} = T_t$. By the braid relations satisfied by the generators of $\mathcal{H}_n$ we have $T_tT_iT_t = T_i$ for all $i > 1$ and by definition $T_tT_1T_t = T_0$.

Remark 8.1.9 On $k\mathcal{H}'_n$, the automorphism τ is induced by the unique non-trivial graph automorphism of the Coxeter graph of the group W'_n. In particular, $k\mathcal{H}_n$ is an *Iwahori-Hecke algebra of an extended Weyl group, cf. [Gec00].*

Remark 8.1.10 We have $\tau_n(k\mathcal{H}'_n) = k\mathcal{H}'_n$, hence the restriction of τ_n to $k\mathcal{H}'_n$ is an algebra automorphism. Moreover, $\tau_{n+1}|_{k\mathcal{H}_n} = \tau_n$, so we will drop the subscript in the following.

Recall that for a $k\mathcal{H}'_n$-module N denote by N^τ the module conjugate by the automorphism τ of $k\mathcal{H}'_n$, cf. Definition 1.3.10. By Proposition 1.3.11, we see τ commutes with Ind'_k.

Proposition 8.1.11 *Let N be an $k\mathcal{H}'_n$-module. Then* $\operatorname{Ind}'_k(N)^\tau \cong \operatorname{Ind}'_k(N^\tau)$.

This will become important since τ also controls R_k.

Lemma 8.1.12 ([Hu02, Lemma 4.3, Corollary 4.4])

a) *Every irreducible $k\mathcal{H}'_n$-module N appears as a constituent of $R_k(M)$ for some irreducible $k\mathcal{H}_n$-module M.*

Now let N be an irreducible $k\mathcal{H}'_n$-module and M an irreducible $k\mathcal{H}_n$-module such that N is a constituent of $R_k(M)$.

b) If $N \ncong N^\tau$, then $R_k(M) \cong N \oplus N^\tau$.

c) If $N \cong N^\tau$, then $R_k(M) \cong N$.

The case $N \ncong N^\tau$ does not occur for all n.

Lemma 8.1.13 ([GP00, 5.6.1], [Hu09, Lemma 1.4, Lemma 3.3]) *Let N be an irreducible $k\mathcal{H}'_n$-module. If $N \ncong N^\tau$, then n is even.*

This allows us to once again obtain bounds on the number of constituents of induced modules.

Theorem 8.1.14 *Let N be an irreducible $k\mathcal{H}'_n$-module and M an irreducible $k\mathcal{H}_n$-module such that N is a constituent of $R_k(M)$.*

a) If n is odd, then $\mathrm{Ind}'_k(N)$ has at least as many constituents as $\mathrm{Ind}_k(M)$.

b) If n is even and $N \cong N^\tau$, then $\mathrm{Ind}'_k(N)$ has exactly as many constituents as $\mathrm{Ind}_k(M)$.

c) If n is even and $N \ncong N^\tau$, then the number of irreducible constituents of $\mathrm{Ind}_k(M)$ is exactly twice the number of irreducible constituents of $\mathrm{Ind}'_k(N)$. In particular, the number of constituents of $\mathrm{Ind}_k(M)$ is even.

Proof a) If n is odd, then $N \cong N^\tau$ by Lemma 8.1.13, and therefore $R_k(M) \cong N$. By Theorem 8.1.5, we have $[\mathrm{Ind}'_k(N)] = [R_k(\mathrm{Ind}_k(M))]$ in the Grothendieck group and the claim follows.

b) Here, $R_k(M) \cong N$ just as above. But $n+1$ is odd and by Lemmas 8.1.13 and 8.1.12 c), the number of irreducible constituents of $R_k(\mathrm{Ind}_k(M))$ is exactly that of $\mathrm{Ind}_k(M)$.

c) By Lemma 8.1.12 b), Proposition 8.1.11, and Theorem 8.1.5, we have

$$\begin{aligned} R_k(\mathrm{Ind}_k([M])) &\overset{8.1.5}{=} \mathrm{Ind}'_k(R_k([M])) \\ &\overset{8.1.12}{=} \mathrm{Ind}'_k([N] + [N^\tau]) \\ &\overset{8.1.11}{=} [\mathrm{Ind}'_k(N)] + [\mathrm{Ind}'_k(N)^\tau]. \end{aligned}$$

Clearly, $\mathrm{Ind}'_k(N)$ and $\mathrm{Ind}'_k(N)^\tau$ have the same number of constituents. As $n+1$ is odd, by Lemma 8.1.13, the number of constituents of $R_k(\mathrm{Ind}_k([M]))$ is equal to that of $\mathrm{Ind}_k([M])$. ∎

Corollary 8.1.15 *Let M be an irreducible $k\mathcal{H}_n$-module. If $R_k(M)$ is decomposable, then $\mathrm{Ind}_k(M)$ has an even number of constituents. As this number of constituents is at least 3 by Theorem 5.3.34, it follows that $\mathrm{Ind}_k(M)$ has at least 4 constituents.*

Proof If $R_k(M)$ is decomposable, then by Lemma 8.1.12 b) and Lemma 8.1.13, it follows that n is even, and $R_k(M) \cong N \oplus N^\tau$ for an irreducible $k\mathcal{H}'_n$-module N. Since $n+1$ is odd, it follows from Lemma 8.1.12 c) and Lemma 8.1.13 that $R_k(M')$ is irreducible for every irreducible constituent

M' of $\mathrm{Ind}_k(M)$. Thus, the number of constituents of $\mathrm{Ind}_k(M)$ is equal to that of $R_k(\mathrm{Ind}_k(M))$. By Proposition 8.1.11, and the commuting diagram in Theorem 8.1.5, we have

$$\begin{aligned} R_k(\mathrm{Ind}_k([M])) &= \mathrm{Ind}'_k(R_k([M])) \\ &= \mathrm{Ind}'_k([N] + [N^\tau]) \\ &= [\mathrm{Ind}'_k(N)] + [(\mathrm{Ind}'_k(N))^\tau], \end{aligned}$$

which clearly has an even number of constituents. ■

Remark 8.1.16 The Iwahori-Hecke algebra $k\mathcal{H}_n$ of type B is isomorphic to the Ariki-Koike algebra $\mathbf{H}_{2,n}(\theta(U), -1,\ 1)$ over k. Thus, we can apply Theorem 5.3.34, and for an $k\mathcal{H}_n$-module $M \neq 0$ the induced module $\mathrm{Ind}_k(M)$ has at least 3 constituents, if -1 is a power of $\theta(U)$, and at least 4 constituents, if this is not the case.

Remark 8.1.17 While the parabolic induction of $k\mathcal{H}_n$-modules always yields modules with at least 3 constituents, this is *not* true for $k\mathcal{H}'_n$: Suppose $L = \mathbb{Q}$, $n = 6$ and $U = 1$. The Specht module S^λ of $k\mathcal{H}_n$ indexed by the multipartition $\lambda := ((1,1,1),(1,1,1))$ yields a module with 4 constituents after parabolic induction, but $R_k(S^\lambda)$ has two irreducible constituents, usually denoted by S^λ_- and S^λ_+, whose respective parabolic inductions only have 2 constituents each.

8.2. Clifford Theory of Cyclotomic Hecke Algebras

Some results of the last section can be carried over to cyclotomic Hecke algebras. The crucial step is finding the correct setting. In particular, we need to consider generic Ariki-Koike algebras.

Let r and $p \neq 1$ be positive integers where p divides r and set $d := r/p$. Let η be a primitive p'th root of unity of order p in $\mathbb{C}$ and set $\mathbf{B} := \mathbb{Z}\left[\eta, q^{\pm 1}, Y_1^{\pm 1}, \ldots, Y_d^{\pm 1}\right]$ for independent indeterminates $q,\ Y_1, \ldots, Y_d$. Finally, for $1 \leq i \leq r$ set

$$Q_i := \eta^{i-1} Y_{\lfloor \frac{i-1}{p} \rfloor + 1}.$$

This choice is such that for $0 \leq j \leq d-1$ we have

$$(Z - Q_{jp+1}) \cdots (Z - Q_{jp+p}) = (Z^p - Y_{j+1}^p),$$

an equality of polynomials in the indeterminate Z.

Set $X_j := Y_j^p$. By Definition 2.4.17 the cyclotomic Hecke algebra $\mathbf{H}'_n := \mathbf{H}_{r,p,n}(q; X_1, \ldots, X_d)$ over $\mathbf{B}$ is a subalgebra of the Ariki-Koike algebra $\mathbf{H}_n := \mathbf{H}_{r,n}(q; Q_1, \ldots, Q_r)$ over $\mathbf{B}$.

Remark 8.2.1 By Propositions 2.4.19 and 2.4.8, it is clear that $\mathbf{H}'_n$ is a direct summand of $\mathbf{H}_n$ as a $\mathbf{B}$-module. In particular, it is a pure submodule and therefore for any ring homomorphism $\mathbf{B} \to B$ the embedding $\mathbf{H}'_n \leq \mathbf{H}_n$ yields an embedding $B\mathbf{H}'_n \leq B\mathbf{H}_n$, cf. Remark 6.2.2.

Denote by R_B the exact restriction functor $\mathrm{Res}^{B\mathbf{H}_n}_{B\mathbf{H}'_n}$ and by Ind_B and Ind'_B the induction functors $\mathrm{Ind}^{B\mathbf{H}_{n+1}}_{B\mathbf{H}_n}$ and $\mathrm{Ind}^{B\mathbf{H}'_{n+1}}_{B\mathbf{H}'_n}$, respectively.

Proposition 8.2.2 ([Gen04, Section 4.1]) *For any ring homomorphism* $\mathbf{B} \to B$*, the submodules* $\left\{B\,\mathbf{H}'_n, S_0 B\,\mathbf{H}'_n, \ldots, S_0^{p-1} B\,\mathbf{H}'_n\right\}$*, constitute a* $\mathbb{Z}\,/\,p\,\mathbb{Z}$*-graded Clifford system of* $B\,\mathbf{H}_n$*.*

Let $\mathbb{L}$ be the field of fractions of $\mathbf{B}$. Denote by $\overline{\mathrm{W}}_n$ and $\overline{\mathrm{W}}'_n$ the complex reflection groups corresponding to $\mathbf{H}_n$ and $\mathbf{H}'_n$, respectively, i.e. $\overline{\mathrm{W}}_n$ has type $G(r,1,n)$ and $\overline{\mathrm{W}}'_n$ type $G(r,p,n)$.

Proposition 8.2.3 (cf. [Ari94, Main Theorem], [Ari95, Theorem 2.6], [LT09, Section 1.7])

a) The algebra $\mathbb{L}\,\mathbf{H}_n$ *is split.*

b) The algebra $\mathbb{L}\,\mathbf{H}'_n$ *is split.*

Let ζ *be a primitive* r*-th root of unity in* $\mathbb{C}$ *such that* $\zeta^d = \eta$ *and define a ring homomorphism* $\theta : \mathbf{B} \to \mathbb{C}$ *by* $\theta(\eta) := \eta$ *and* $\theta(Y_j) := \zeta^j$*, and finally* $\theta(q) := 1$*.*

c) The specialisations $\mathbb{C}\,\mathbf{H}_n$ *and* $\mathbb{C}\,\mathbf{H}'_n$ *are isomorphic to the complex group algebras* $\mathbb{C}[\overline{\mathrm{W}}_n]$ *and* $\mathbb{C}[\overline{\mathrm{W}}'_n]$*, respectively.*

d) The field of fractions of $\theta(\mathbf{B})$ *is a splitting field of the complex group algebras of both* $\overline{\mathrm{W}}_n$ *and* $\overline{\mathrm{W}}'_n$*.*

e) The decomposition maps $R_0(\mathbb{L}\,\mathbf{H}_n) \to R_0(\mathbb{C}\,\mathbf{H}_n)$ *and* $R_0(\mathbb{L}\,\mathbf{H}'_n) \to R_0(\mathbb{C}\,\mathbf{H}'_n)$ *are well-defined trivial isomorphisms and both* $\mathbb{L}\,\mathbf{H}_n$ *and* $\mathbb{L}\,\mathbf{H}'_n$ *are semisimple.*

Proof The statements on splitting fields of $\mathbb{L}\,\mathbf{H}_n$ and $\mathbb{L}\,\mathbf{H}'_n$ can be found in [Ari94] and [Ari95],respectively. Since η is a primitive p'th root of unity, the Q_i are pairwise different and we have

$$\{Q_1, \ldots, Q_r\} = \{\eta^i Y_j \mid 0 \le i < p,\ 1 \le j \le d\}.$$

As $\eta = \zeta^d$, this implies

$$\{\theta(Q_1), \ldots, \theta(Q_r)\} = \{\zeta^{di+j} \mid 0 \le i < p,\ 1 \le j \le d\}.$$

Since $r = dp$, this set is equal to $\{\zeta^m \mid 0 \le m < r\}$. The specialised Ariki-Koike algebra $\mathbb{C}\,\mathbf{H}_n$ is the Ariki-Koike algebra over $\mathbb{C}$ with parameters $q, \theta(Q_1), \ldots, \theta(Q_r)$ and by Example 2.4.2 and Proposition 2.4.3 b) it is therefore isomorphic to the group algebra $\mathbb{C}[\overline{\mathrm{W}}_n]$. A comparison of the generators of $\mathbb{C}\,\mathbf{H}'_n$ in Definition 2.4.17 as a subalgebra of $\mathbb{C}\,\mathbf{H}_n$ with those of $\overline{\mathrm{W}}'_n$ as a subgroup of $\overline{\mathrm{W}}_n$ in Definition 2.3.4 then shows $\mathbb{C}\,\mathbf{H}'_n \cong \mathbb{C}[\overline{\mathrm{W}}'_n]$. By [LT09, Section 1.7], the field of fractions of $\theta(\mathbf{B})$ is a splitting field of both $\overline{\mathrm{W}}_n$ and $\overline{\mathrm{W}}'_n$.
Finally, the decomposition maps are well-defined as all involved algebras are split, and by Tits' Deformation Theorem 6.1.21 it follows that the decomposition maps are trivial isomorphisms and that $\mathbb{L}\,\mathbf{H}_n$ and $\mathbb{L}\,\mathbf{H}'_n$ are semisimple. ∎

With this in mind we can proceed just as we did for Iwahori-Hecke algebras. The essential step is proving an analogue of Lemma 8.1.3.

Lemma 8.2.4 *Consider the specialisations $\mathbb{C}\,\mathbf{H}_n$ and $\mathbb{C}\,\mathbf{H}'_n$ from Proposition 8.2.3, which are isomorphic to the group algebras $\mathbb{C}[\overline{\mathrm{W}}_n]$ and $\mathbb{C}[\overline{\mathrm{W}}'_n]$. The following diagram commutes:*

$$\begin{array}{ccc} R_0(\mathbb{C}[\overline{\mathrm{W}}_n)] & \xrightarrow{R_\mathbb{C}} & R_0(\mathbb{C}[\overline{\mathrm{W}}'_n]) \\ \downarrow \mathrm{Ind}_\mathbb{C} & & \downarrow \mathrm{Ind}'_\mathbb{C} \\ R_0(\mathbb{C}[\overline{\mathrm{W}}_{n+1}]) & \xrightarrow{R_\mathbb{C}} & R_0(\mathbb{C}[\overline{\mathrm{W}}'_{n+1}]) \end{array}$$

Proof We adapt the proof of Lemma 8.1.3 to our new situation.
The subgroup $\overline{\mathrm{W}}_{n+1}$ is normal in $\overline{\mathrm{W}}_{n+1}$ and therefore normalised by $\overline{\mathrm{W}}_n$, thus $\overline{\mathrm{W}}_n\,\overline{\mathrm{W}}'_{n+1} = \overline{\mathrm{W}}'_{n+1}\,\overline{\mathrm{W}}_n \le \overline{\mathrm{W}}_{n+1}$. Since $\overline{\mathrm{W}}_n$ contains the generator t and $\overline{\mathrm{W}}'_{n+1}$ contains the generators $s_1,\dots,s_1,\dots,s_n$ of $\overline{\mathrm{W}}_{n+1}$, it follows that $\overline{\mathrm{W}}'_{n+1}\,\overline{\mathrm{W}}_n = \overline{\mathrm{W}}_{n+1}$. By Lemma 2.3.9 b) and d), $\overline{\mathrm{W}}_n \cap \overline{\mathrm{W}}'_{n+1} = \overline{\mathrm{W}}'_n$.
Now let M be a $\mathbb{C}[\overline{\mathrm{W}}_n]$-module. Just as in the proof of Lemma 8.1.3 we apply Mackey's formula and see that

$$\mathrm{Res}^{\mathbb{C}[\overline{\mathrm{W}}_{n+1}]}_{\mathbb{C}[\overline{\mathrm{W}}'_{n+1}]}\left(\mathrm{Ind}^{\mathbb{C}[\overline{\mathrm{W}}_{n+1}]}_{\mathbb{C}[\overline{\mathrm{W}}_n]}(M)\right) \cong \mathrm{Ind}^{\mathbb{C}[\overline{\mathrm{W}}'_{n+1}]}_{\mathbb{C}[\overline{\mathrm{W}}'_n]}\left(\mathrm{Res}^{\mathbb{C}[\overline{\mathrm{W}}_n]}_{\mathbb{C}[\overline{\mathrm{W}}'_n]}(M)\right).$$

■

The following two results are proved just as their analogues, Proposition 8.1.4 and Theorem 8.1.5.

Proposition 8.2.5 *The following diagram commutes:*

$$\begin{array}{ccc} R_0(\mathbb{L}\,\mathbf{H}_n) & \xrightarrow{R_\mathbb{L}} & R_0(\mathbb{L}\,\mathbf{H}'_n) \\ \downarrow \mathrm{Ind}_\mathbb{L} & & \downarrow \mathrm{Ind}'_\mathbb{L} \\ R_0(\mathbb{L}\,\mathbf{H}_{n+1}) & \xrightarrow{R_\mathbb{L}} & R_0(\mathbb{L}\,\mathbf{H}'_{n+1}) \end{array}$$

Theorem 8.2.6 *Suppose $\iota : \mathbf{B} \to k$ is a ring homomorphism to a field k such that k', the field of fractions of $\iota(\mathbf{B})$ is a splitting field of $k'\,\mathbf{H}_n$, $k'\,\mathbf{H}_{n+1}$, $k'\,\mathbf{H}'_n$, and $k'\,\mathbf{H}'_{n+1}$. If the decomposition map $R_0(\mathbb{L}\,\mathbf{H}_n) \to R_0(\mathbb{L}\,\mathbf{H}_n)$ is surjective, then the following diagram commutes:*

$$\begin{array}{ccc} R_0(k\,\mathbf{H}_n) & \xrightarrow{R_k} & R_0(k\,\mathbf{H}'_n) \\ \downarrow \mathrm{Ind}_k & & \downarrow \mathrm{Ind}'_k \\ R_0(k\,\mathbf{H}_{n+1}) & \xrightarrow{R_k} & R_0(k\,\mathbf{H}'_{n+1}) \end{array}$$

Remark 8.2.7 Clearly, the theorem still holds if we replace $\mathbf{B}$ by any integrally closed integral domain $\widehat{\mathbf{B}}$ containing $\mathbf{B}$ and $\mathbb{L}$ by the field of fractions of $\widehat{\mathbf{B}}$, since $\mathbb{L}$ is already a splitting field of all involved algebras.

Let $\widehat{\mathbf{B}}$ be an integrally closed integral domain containing $\mathbf{B}$ and fix some ring homomorphism $\iota : \widehat{\mathbf{B}} \to k$ satisfying the hypotheses of Theorem 8.2.6 and assume that k is algebraically closed and has characteristic zero.

Proposition 8.2.8 ([GJ06, Lemma 2.2]) *Let η be a primitive p'th root of unity in k.*

a) The map $f_n : k\,\mathbf{H}_n \to k\,\mathbf{H}_n : S_0^j h \mapsto \eta^j S_0^j h$ for all h in $k\,\mathbf{H}'_n$ and $0 \leq j \leq p-1$ is an algebra automorphism of order p. Note that every element of $k\,\mathbf{H}_n$ can be written uniquely as $S_0^j h$ due to the $\mathbb{Z}/p\mathbb{Z}$-grading.

b) The map $g_n : k\,\mathbf{H}'_n \to k\,\mathbf{H}'_n : h \mapsto S_0^{-1} h S_0$ is an algebra automorphism of order p.

If V is an irreducible $k\,\mathbf{H}_n$-module, and U is a an irreducible constituent of $R_k(V)$, then $R_k(V)$ is the direct sum of $\langle g_n \rangle$-conjugates of U. More precisely: Let $\{U_1, \ldots, U_s\}$ be a set of pairwise non-isomorphic representatives of the $k\,\mathbf{H}'_n$-modules $\left\{U^{g_n^i} \mid 0 \leq i \leq p\right\}$. Then $R_k(V)$ is isomorphic to $U_1 \oplus \cdots \oplus U_s$.

Note that the map g_n assumes the role played by τ_n in the case of Iwahori-Hecke algebras. Furthermore, g_{n+1} restricts to g_n on $\mathbf{H}'_n$. Thus, the following analogue of Proposition 8.1.11 follows immediately from Proposition 1.3.11.

Corollary 8.2.9 *Let U be a $k\,\mathbf{H}'_n$-module. Then*

$$\mathrm{Ind}'_k\left(U^{g_n}\right) \cong \left(\mathrm{Ind}'_k(U)\right)^{g_{n+1}}.$$

Remark 8.2.10 By the remarks after [Gen04, Proposition 2.4.1] we can drop the hypothesis that k has characteristic zero and replace it with the condition that the characteristic of k does not divide p. The proof of [GJ06, Lemma 2.2] carries over to this setting if we keep the requirement that k is algebraically closed.

Chapter 9.

Computation of Decomposition Numbers in Bad Characteristic

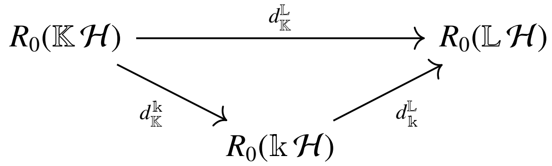
9.1. Exceptional Weyl Groups

Let (W, S) be a finite Coxeter system and suppose that W is an exceptional Weyl group, i.e. W is in $\{G_2,\ F_4,\ E_6,\ E_7,\ E_8\}$.
Let $\mathbb{K}$ be the function field $\mathbb{Q}(V)$ and $\mathbf{K} := \mathbb{Z}[V^{\pm 1}]$ the ring of Laurent polynomials over $\mathbb{Z}$ in the indeterminate V. As in earlier chapters define the generic one-parameter Iwahori-Hecke algebra $\mathcal{H} := H_{\mathbf{K}}(W, S, (U_s := U := V^2 \mid s \in S))$. By Theorem 7.1.13, the specialisation $\mathbb{K}\mathcal{H}$ is split semisimple.
Suppose $0 \neq \varepsilon$ is an element of a field extension of $\mathbb{F}_p$ for some prime p and set $\mathbb{L} := \mathbb{F}_p(\varepsilon)$. If the specialisation $\mathbb{L}\mathcal{H}$ via $\theta_{\mathbf{K},\mathbb{L}} : V \mapsto \varepsilon$ is split, then there exists a well-defined corresponding decomposition map $d_{\mathbb{K}}^{\mathbb{L}} : R_0(\mathbb{K}\mathcal{H}) \to R_0(\mathbb{L}\mathcal{H})$. This section's goal is to compute this decomposition map for all choices of ε. If p is a good prime for W, then by Theorem 7.1.13, the algebra $\mathbb{L}\mathcal{H}$ is split and $d_{\mathbb{K}}^{\mathbb{L}}$ is a surjective homomorphism. Therefore, we focus on the cases in which p is a bad prime for W.
The main tool in the investigation of $d_{\mathbb{K}}^{\mathbb{L}}$ is what we call the $\mathbb{F}_p$*-generic algebra*. Let y be an indeterminate over $\mathbb{F}_p$ and set $\mathbf{k} := \mathbb{F}_p[y^{\pm 1}]$ and $\Bbbk := \mathbb{F}_p(y)$ its field of fractions. We call $\Bbbk\mathcal{H}$ the $\mathbb{F}_p$*-generic algebra*. It can be considered an approximation of $\mathbb{L}\mathcal{H}$, because $\theta_{\mathbf{K},\mathbb{L}}$ factors through $\mathbf{k}$. In total, the situation looks as follows:

$$\begin{array}{ccccc} \mathbb{K} & \hookleftarrow & \mathbf{K} & & V \\ & & \downarrow \theta_{\mathbf{K},\mathbf{k}} & & \downarrow \\ \Bbbk & \hookleftarrow & \mathbf{k} & \theta_{\mathbf{K},\mathbb{L}} & y \\ & & \downarrow \theta_{\mathbf{k},\mathbb{L}} & & \downarrow \\ & & \mathbb{L} & & \varepsilon \end{array}$$

Remark 9.1.1 The factorisation of $\theta_{\mathbf{K},\mathbb{L}}$ should remind the reader of the factorisation $\theta = \theta_2 \circ \theta_1$ we introduced in Section 7.3, just before Proposition 7.3.1. However, there is one key difference: In both cases we want to go from a generic parameter over a field of characteristic zero to an arbitrary parameter over a field of positive characteristic. In Section 7.3 we first change the parameter to a root of unity while remaining in characteristic zero. Then we change the characteristic in the second step. For the factorisation of $\theta_{\mathbf{K},\mathbb{L}}$ we employ in this chapter we begin by changing the characteristic of the field while keeping the parameter generic, and then go over to an arbitrary parameter in a second step

Proposition 9.1.2 (cf. [Gyo96, Theorem C]) *Suppose $W \neq E_8$.*
The specialised algebra $\Bbbk\mathcal{H}$ is split. Thus, there exists a well-defined decomposition map $d_{\mathbb{K}}^{\Bbbk} : R_0(\mathbb{K}\mathcal{H}) \to R_0(\Bbbk\mathcal{H})$.
If $\mathbb{L}\mathcal{H}$, too, is split, then the decomposition map $d_{\Bbbk}^{\mathbb{L}} : R_0(\Bbbk\mathcal{H}) \to R_0(\mathbb{L}\mathcal{H})$ is also well-defined and the diagram



commutes.

Proof If p is a good prime for W, this follows from Lemma 7.1.16 and Theorem 7.1.13, and also holds for $W = E_8$. For bad characteristic and $W \neq E_8$ this is [Gyo96, Theorem C].
The commutativity of the diagram is Proposition 6.1.25. ∎

Definition 9.1.3 If $\Bbbk\mathcal{H}$ is split we call $d_{\mathbb{K}}^{\Bbbk}$ the *$\mathbb{F}_p$-generic decomposition map of $\mathcal{H}$.*

We want to study the decomposition map $d_{\mathbb{K}}^{\mathbb{L}}$ for all possible values of ε. Initially, this is an infinite task and therefore hard to tackle computationally. However, we will show that in almost all cases the decomposition map is fully determined by the decomposition map $d_{\mathbb{K}}^{\Bbbk}$. The key argument to compute the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ is given by the following adaptation of a result by Geck, see Proposition 7.3.1.

Proposition 9.1.4 (cf. [Gec92, Proposition 5.5]) *Suppose $\Bbbk\mathcal{H}$ is split. Then there exists a polynomial $g \in \mathbb{F}_p[y]$ with the following property: If $g(\varepsilon) \neq 0$, then $\mathbb{L}\mathcal{H}$ is split and the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ is a trivial isomorphism.*

The proof of Proposition 9.1.4 requires some preliminary steps.

We will have to consider concrete representations, so let us study how they behave under specialisation, similarly to what we did in Remark 6.1.17.

Remark 9.1.5 All representations of $\Bbbk\mathcal{H}$ can be realised over $\mathbb{F}_p[y] \subseteq \mathbf{k}$ and thus over $\mathbf{k}$: The polynomial ring $\mathbb{F}_p[y] \subseteq \mathbf{k}$ is a principal ideal domain and $\Bbbk$ its field of fractions. Every finitely generated torsion-free module over $\mathbb{F}_p[y]$ is free and we can proceed as in [GP00, 7.3.7].
If N is an $\Bbbk\mathcal{H}$-module and $\rho_N : \Bbbk\mathcal{H} \to \Bbbk^{n\times n}$ is a representation of N such that $\rho_N(h)$ is

in $\mathbf{k}^{n\times n}$ for all h in $\mathbf{k}\mathcal{H}$, then a representation of the specialised module $\mathbb{L}N$ is given by $\rho_{\mathbb{L}N} : \mathbb{L}\mathcal{H} \to \mathbb{L}^{n\times n}$; $h \otimes_{\mathbf{k}} 1 \mapsto \theta_{\mathbf{k},\mathbb{L}}(\rho_N(h))$ for $h \in \mathbf{k}\mathcal{H}$, where $\theta_{\mathbf{k},\mathbb{L}}(\rho_N(h))$ is the matrix obtained by applying $\theta_{\mathbf{k},\mathbb{L}}$ to all entries of $\rho_N(h)$.
Similarly, every $\mathbb{K}\mathcal{H}$-module can be realised over $\mathbf{K}$, and if ρ_M is a representation of an $\mathbb{K}\mathcal{H}$-module M realised over $\mathbf{K}$a representation of $\Bbbk M$ (respectively $\mathbb{L}M$) can be obtained by applying $\theta_{\mathbf{K},\Bbbk}$ (respectively $\theta_{\mathbf{K},\mathbb{L}}$) to the representing matrices.

Before stating Lemma 9.1.8, which gives a testable criterion whether $\Bbbk\mathcal{H}$ or $\mathbb{L}\mathcal{H}$ are split, we need two small technical results on splitting fields and absolutely irreducible modules.

Lemma 9.1.6 *Let $\mathfrak{A}$ be a finite-dimensional algebra over a field D and $D \subseteq E$ a field extension such that $E\mathfrak{A}$ is split. If every irreducible $E\mathfrak{A}$-module is realisable over D, then $\mathfrak{A}$ is already split over D.*

Proof Let M be an irreducible $\mathfrak{A}$-module. Then EM is irreducible: Suppose N is an irreducible submodule of EM. Then N is realisable over D, thus there exists an irreducible $\mathfrak{A}$-module N' such that $EN' \cong N$. By [CR62, 29.5], we have $E(\mathrm{Hom}_{\mathfrak{A}}(N', M)) \cong \mathrm{Hom}_{E\mathfrak{A}}(EN', EM)$ as E-vector spaces. Since EN' embeds into EM, the right-hand side is at least one-dimensional and therefore so is the left-hand side. Thus, there exists a non-zero homomorphism in $\mathrm{Hom}_{\mathfrak{A}}(N', M)$ and as both N' and M are irreducible it is an isomorphism. Then EM is isomorphic to $EN' \cong N$ which was irreducible by hypothesis, thus EM is irreducible.
Since EM is irreducible and $E\mathfrak{A}$ is split, we have $\dim_E(\mathrm{End}_{E\mathfrak{A}}(EM)) = 1$. As $E \otimes_D \mathrm{End}_{\mathfrak{A}}(M) \cong \mathrm{End}_{E\mathfrak{A}}(EM)$ as E-vector spaces, it follows that $\mathrm{End}_{\mathfrak{A}}(M)$, too, is one-dimensional, and therefore M is a split $\mathfrak{A}$-module. Because M was arbitrary, this shows that $\mathfrak{A}$ is split over D. ■

Lemma 9.1.7 *Let $\mathfrak{A}$ be finite-dimensional algebra over a field D and $D \subseteq E$ a field extension. Let M be an $\mathfrak{A}$-module. If every irreducible constituent of M is absolutely irreducible, then every irreducible constituent of EM can be realised over D.*

Proof By Lemma 6.1.6, there exists a homomorphism $f : R_0(\mathfrak{A}) \to R_0(E\mathfrak{A})$ given by $f([N]) := [N \otimes_D E]$ for $N \in$ mod-$\mathfrak{A}$. Suppose $[M] = \sum_{U\in\mathrm{Irr}(\mathfrak{A})} a_U[U]$ is the decomposition of $[M]$ into classes of irreducibles, where the a_U are non-negative integers. If $a_U > 0$ for some U, then U is a constituent of M and by hypothesis every such U is absolutely irreducible. Therefore, EU is also irreducible. Thus, $f([M]) = \sum_{U\in\mathrm{Irr}(\mathfrak{A})} a_U[EU]$ is a decomposition of $f([M]) = [EM]$ into irreducibles. Hence, every irreducible constituent of EM is isomorphic to EU for some irreducible $\mathfrak{A}$-module U and thus realisable over D. ■

Lemma 9.1.8

a) *Let $\mathcal{M}$ be a set of $\mathcal{H}$-modules such that $\mathbb{K}\mathcal{M} := \{\mathbb{K}M \mid M \in \mathcal{M}\}$ is a complete set of pairwise non-isomorphic irreducible $\mathbb{K}\mathcal{H}$-modules. Suppose that for every M in $\mathcal{M}$ all irreducible constituents of $\Bbbk M$ are absolutely irreducible. Then $\Bbbk\mathcal{H}$ is split and every irreducible $\Bbbk\mathcal{H}$-module appears as a constituent of $\Bbbk M$ for some M in $\mathcal{M}$.*

b) *Suppose $\Bbbk\mathcal{H}$ is split. Let $\mathcal{N}$ be a set of $\mathbf{k}\mathcal{H}$-modules such that $\Bbbk\mathcal{N}$ is a complete set of irreducible $\Bbbk\mathcal{H}$-modules. Suppose that for every N in $\mathcal{N}$ all irreducible constituents of $\mathbb{L}N$ are absolutely irreducible. Then $\mathbb{L}\mathcal{H}$ is split and every irreducible $\mathbb{L}\mathcal{H}$-module appears as a constituent of $\mathbb{L}N$ for some N in $\mathcal{N}$.*

Proof We only prove a), as b) is proved analogously.
By Lemma 1.2.3, there exists a finite field extension $E = \mathbb{F}_p(y, \alpha_1, \ldots, \alpha_s)$ of $\Bbbk$ with $\alpha_i \neq 0$ such that $E\mathcal{H}$ is split. Define $\widehat{\mathbf{K}} := \mathbb{Z}[V^{\pm 1}, X_1^{\pm 1}, \ldots, X_s^{\pm 1}]$ and denote by $\widehat{\mathbb{K}}$ its field of fractions. Clearly, $\widehat{\mathbb{K}}$ is a field extension of $\mathbb{K}$ and thus $\widehat{\mathbb{K}}\mathcal{H}$ is split.
The ring homomorphism $\nu : \widehat{\mathbf{K}} \to E; V \mapsto y,\ X_i \mapsto \alpha_i$ extends $\theta_{\mathbf{K},\mathbf{k}}$ and the field of fractions of $\nu(\widehat{\mathbf{K}})$ is E. Thus, there exists a well-defined decomposition map $d : R_0(\widehat{\mathbb{K}}\mathcal{H}) \to R_0(E\mathcal{H})$.
By the definition of decomposition maps in Theorem 6.1.18 we have $d([\widehat{\mathbb{K}}M]) = [EM]$ for every M in $\mathcal{M}$.
Now denote by V the regular $\widehat{\mathbb{K}}\mathcal{H}$-module. As $\widehat{\mathbb{K}}\mathcal{M}$ is a complete set of irreducible $\widehat{\mathbb{K}}\mathcal{H}$-modules there exist positive integers a_M for M in $\mathcal{M}$ such that

$$[V] = \sum_{M \in \mathcal{M}} a_M[\widehat{\mathbb{K}}M].$$

On the other hand, $V \cong \mathcal{H} \otimes_{\mathbf{K}} \widehat{\mathbb{K}}$, hence $d([V]) = [\mathcal{H} \otimes_{\mathbf{K}} E]$, and $\mathcal{H} \otimes_{\mathbf{K}} E$ is the regular $E\mathcal{H}$-module. As d is a homomorphism, we have

$$[E\mathcal{H}] = d([V]) = \sum_{M \in \mathcal{M}} a_M \cdot d([\widehat{\mathbb{K}}M]) = \sum_{M \in \mathcal{M}} a_M \cdot [EM].$$

Every irreducible $E\mathcal{H}$-module is a constituent of EV and thus every irreducible $E\mathcal{H}$-module is a constituent of EM for some M in $\mathcal{M}$. By hypothesis, every irreducible constituent of $\Bbbk M$ is absolutely irreducible, and, by Lemma 9.1.7, this shows that every irreducible constituent of $EM \cong E(\Bbbk M)$ is realisable over $\Bbbk$. Therefore, every irreducible $E\mathcal{H}$-module is realisable over $\Bbbk$ and $\Bbbk\mathcal{H}$ is split by Lemma 9.1.6.
Finally, let U be an irreducible $\Bbbk\mathcal{H}$-module. As $\Bbbk\mathcal{H}$ is split, this implies that EU is an irreducible $E\mathcal{H}$-module. We have shown above that then EU is a constituent of EM for some $M \in \mathcal{M}$. Choose such an M. By Lemma 6.1.6, there is a trivial isomorphism of Grothendieck groups $f : R_0(\Bbbk\mathcal{H}) \to R_0(E\mathcal{H}); N \mapsto [EN]$, as we have already shown that $\Bbbk\mathcal{H}$ is split. Since this is a trivial isomorphism, it follows that U is a constituent of $\Bbbk M$: Let $[\Bbbk M] = \sum_{T \in \mathrm{Irr}(\Bbbk\mathcal{H})} b_T[T]$ be the decomposition of $[\Bbbk M]$ in the Grothendieck group of $\Bbbk\mathcal{H}$, where the b_T are non-negative integers. Since f is a trivial isomorphism, it follows that

$$[EM] = \sum_{T \in \mathrm{Irr}(\Bbbk\mathcal{H})} b_T[ET] \tag{9.1}$$

is the decomposition of $[EM]$ in the Grothendieck group of $E\mathcal{H}$, and the ET form a complete set of pairwise non-isomorphic irreducible $E\mathcal{H}$-modules. As EU is an irreducible constituent of EM, the decomposition in (9.1) shows that there exists some $T' \in \mathrm{Irr}(\Bbbk\mathcal{H})$, such that $b_{T'} > 0$ and $EU \cong ET'$. But this implies that $f([U]) = f([T'])$, and since f is an isomorphism and both T' and U are irreducible, this shows that $T' \cong U$. Since $b_{T'} > 0$, we see that T' and therefore U is an irreducible constituent of $\Bbbk M$. ∎

The key argument to check a module for irreducibility is *Norton's Irreducibility Criterion*:

Proposition 9.1.9 ([LP10, Theorem 1.3.3]) *Let F be a field and $\mathfrak{A}$ a finite dimensional F-algebra. Let $\rho : \mathfrak{A} \to F^{n\times n}$ be a representation of $\mathfrak{A}$. Suppose b is an element of $\mathfrak{A}$ such that $\rho(b)$ does not have full rank. Then ρ is irreducible if and only if* *a)* $F^{1\times n} = v\,\rho(\mathfrak{A})$ *for all* $0 \neq v$ *in* $\ker(\rho(b))$ *and*

b) $F^{1\times n} = x\,\rho(\mathfrak{A})^{\mathrm{Tr}}$ *for some* x *in* $\ker(\rho(b)^{\mathrm{Tr}})$*, where* A^{Tr} *is the transposed matrix of* A *for* $A \in F^{n\times n}$.

Corollary 9.1.10 *Assume the setting of Proposition 9.1.9 and suppose that* $\rho(b)$ *has co-rank* 1 *for some* $b \in \mathfrak{A}$*. Then* ρ *is* absolutely irreducible *if and only if*

a′) $F^{1\times n} = v_0\,\rho(\mathfrak{A})$ *for some* $0 \neq v_0$ *in* $\ker(\rho(b))$ *and*

b′) $F^{1\times n} = x\,\rho(\mathfrak{A})^{\mathrm{Tr}}$ *for some* x *in* $\ker(\rho(b)^{\mathrm{Tr}})$.

Proof If ρ is an absolutely irreducible representation, then *a′*) and *b′*) hold by Proposition 9.1.9. Conversely, assume that *a′*) and *b′*) hold. Since $\rho(b)$ has co-rank 1, the kernel $\ker(\rho(b))$ is one-dimensional, hence *a′*) holds not only for v_0 but for all $0 \neq v \in \ker(\rho(b))$ and ρ is irreducible by Proposition 9.1.9, so it remains to show absolute irreducibility.
Suppose $F \subseteq E$ is a field extension and denote by ρ_E the extension of ρ to $E\,\mathfrak{A}$ via $\rho_E : E\,\mathfrak{A} \to E^{n\times n}$; $a \otimes_F 1 \mapsto \rho(a) \otimes_F 1$. The rank of $\rho_E(b \otimes_F 1)$ is $n-1$ and thus the kernel $\ker(\rho_E(b \otimes_F 1))$ has dimension 1 and is spanned by $v_0 \otimes 1$. Since *a′*) holds, $E^{1\times n} = (v_0 \otimes_F 1)\rho_E(E\,\mathfrak{A})$, and since $v_0 \otimes_F 1$ spans $\ker(\rho_E(b \otimes_F 1))$ it follows that $E^{1\times n} = v\,\rho_E(E\,\mathfrak{A})$ holds for all $0 \neq v$ in $\ker(\rho_E(b \otimes_F 1))$.
Similarly, if x is in $\ker(\rho(b)^{\mathrm{Tr}})$ such that $F^{1\times n} = x\,\rho(\mathfrak{A})^{\mathrm{Tr}}$, then $E^{1\times n} = (x \otimes_F 1)\,\rho_E(E\,\mathfrak{A})^{\mathrm{Tr}}$. By Proposition 9.1.9, this implies irreducibility of ρ_E and therefore ρ is irreducible and remains irreducible under field extensions, thus it is absolutely irreducible. ∎

When we are given two irreducible representations of a split algebra we can use the fact that their characters are linearly independent to check whether they are equivalent, cf. Proposition 1.2.4.
However, bases of the algebras considered here can become quite large which might make it hard to actually compute characters. For the Iwahori-Hecke algebras we are studying the situation is made much less cumbersome by the following result.

Lemma 9.1.11 ([GP00, Corollary 8.2.6]) *Let* H *be an Iwahori-Hecke algebra of* W *over some commutative ring* A *with standard basis* $\{T_w \mid w \in W\}$ *and let* $\rho : H \to \mathfrak{A}^{n\times n}$ *be a representation of* H*. Denote by* $\chi : H \to A$; $h \mapsto Trace(\rho(h))$ *the corresponding character. Now let* $\mathcal{C}$ *be the conjugacy classes of* W *and for every class* C *in* $\mathcal{C}$ *let* w_C *be a shortest representative with respect to the usual length function on* W*. Then* χ *is uniquely determined by the images on the* w_C*, and if* w'_C *is another shortest element in* C*, then* $\chi(w_C) = \chi(w'_C)$.

Thus, we can define a character table.

Definition 9.1.12 (cf. [GP00, Definition 8.2.9]) Suppose H is an Iwahori-Hecke algebra of W over a field F. Then the *character table of* H is a matrix with values in F with rows indexed by the elements of $\mathrm{Irr}(H)$ and columns indexed by the conjugacy classes of W. The entry indexed by the conjugacy class C and the irreducible H-module M is the value of the character afforded by M on a shortest representative of C. We denote the character table by $\mathfrak{X}(H)$.

Note that the character table is only defined up to re-ordering of rows and columns.

We are fully prepared to prove Proposition 9.1.4.

Proof of Proposition 9.1.4 Let $\rho : \Bbbk\mathcal{H} \to \Bbbk^{n\times n}$ be an irreducible representation and suppose that ρ is realised over $\mathbf{k}$.
By [CR81, Theorem 3.32] there exists a linear combination $\sum_{w\in W} a_w T_w$ with $a_w \in \Bbbk$ such that $\rho(\sum_{w\in W} a_w T_w)$ has co-rank 1. As $\Bbbk$ is the field of fractions of $\mathbb{F}_p[y]$ we can choose the elements a_w in $\mathbb{F}_p[y]$.
Now let v and v' in $\mathbb{F}_p[y]^{1\times n} \subseteq \Bbbk^{1\times n}$ be vectors spanning the one-dimensional nullspaces of $\rho(\sum_{w\in W} a_w T_w)$ and of $\rho(\sum_{w\in W} a_w T_w)^{\mathrm{Tr}}$, respectively. By Norton's Irreducibility Criterion, cf. Proposition 9.1.9, the irreducibility of ρ implies that by repeatedly multiplying v with the $\rho(T_s)$ and v' with the $\rho(T_s)^{\mathrm{Tr}}$ for s in S we obtain bases $(v_1 := v, v_2, \dots, v_n)$ and $(v'_1 := v', v'_2, \dots, v'_n)$ of $\Bbbk^{1\times n}$.
Let g_ρ be the product of the following Laurent polynomials in y:

- The determinant of an $(n-1)\times(n-1)$-submatrix of $\rho(\sum_{w\in W} a_w T_w)$ having full rank, that is rank $n-1$,
- the determinant of the matrix having the vectors v_i as its rows, and
- the determinant of the matrix having the vectors v'_i as its rows.

Since ρ is realised over $\mathbf{k}$ the $\rho(T_s)$ all lie in $\mathbf{k}^{n\times n}$ and thus g_ρ is indeed a Laurent polynomial in y.
Now assume $g_\rho(\varepsilon) \neq 0$ and denote by $\rho_{\mathbb{L}}$ the corresponding representation of $\mathbb{L}\mathcal{H}$, that is $\rho_{\mathbb{L}} : \mathbb{L}\mathcal{H} \to \mathbb{L}^{n\times n}; h \otimes_{\mathbf{k}} 1 \mapsto \theta_{\mathbf{k},\,\mathbb{L}}(\rho(h))$.
Then $\rho_{\mathbb{L}}(\sum_{w\in W} a_w T_w \otimes_{\mathbf{k}} 1)$ has co-rank 1. Furthermore, $(v \otimes_{\mathbf{k}} 1)\rho_{\mathbb{L}}(T_w \otimes_{\mathbf{k}} 1) = (vT_w) \otimes_{\mathbf{k}} 1$, that is the multiplication with $\rho(T_w)$ essentially commutes with specialisation. Since $\mathbf{g}_\rho(\varepsilon)$ is not zero, $(v_1 \otimes_{\mathbf{k}} 1, \dots, v_n \otimes_{\mathbf{k}} 1)$ is linearly independent and $\mathbb{L}^{1\times n} = (v \otimes_{\mathbf{k}} 1)\rho_{\mathbb{L}}(\mathbb{L}\mathcal{H})$. Clearly, $0 \neq (v_1 \otimes_{\mathbf{k}} 1)$ is in the kernel of $\rho_{\mathbb{L}}(\sum_{w\in W} a_w T_w \otimes_{\mathbf{k}} 1)$.
The argument works completely analogous for $v'_1 \otimes_{\mathbf{k}} 1$ and by Corollary 9.1.10 the specialisation $\rho_{\mathbb{L}}$ is absolutely irreducible.

Now consider the character table $\mathfrak{X}(\Bbbk\mathcal{H})$. The irreducible characters of $\Bbbk\mathcal{H}$ are linearly independent by Proposition 1.2.4 and therefore the rank of $\mathfrak{X}(\Bbbk\mathcal{H})$ is exactly equal to the number of rows. Furthermore, all entries of $\mathfrak{X}(\Bbbk\mathcal{H})$ lie in $\mathbf{k}$ as all irreducible representations of $\Bbbk\mathcal{H}$ can be realised over $\mathbf{k}$. Hence, there exists a suitable subset of the columns of $\mathfrak{X}(\Bbbk\mathcal{H})$ such that the corresponding submatrix is quadratic and has non-zero determinant $g_{\det} \in \mathbf{k}$.

We set

$$g' := g_{\det} \prod_{\rho} g_\rho,$$

where ρ runs over representations of the elements of $\mathrm{Irr}(\Bbbk\mathcal{H})$ and every such ρ is realised over $\mathbf{k}$. Choose some large enough $N \in \mathbb{Z}_{\geq 0}$ such that $g := g'y^N$ is in $\mathbb{F}_p[y] \subseteq \mathbf{k}$.
Assume $g(\varepsilon) \neq 0$. Then $\rho_{\mathbb{L}}$ is absolutely irreducible for every irreducible representation ρ by

the above observation and by Lemma 9.1.8 the algebra $\mathbb{L}\mathcal{H}$ is split. Moreover, as $g_{\det}(\varepsilon)$ is not zero, it follows that the characters of $\rho_{\mathbb{L}}$ and $\rho'_{\mathbb{L}}$ are equal if and only if ρ and ρ' are equivalent representations of $\mathbb{L}\mathcal{H}$.
Expressed in modules this means that $d_{\Bbbk}^{\mathbb{L}}([M])$ is the class of an irreducible $\mathbb{L}\mathcal{H}$-module for every irreducible $\Bbbk\mathcal{H}$-module M, that is $d_{\Bbbk}^{\mathbb{L}}$ sends classes of irreducible modules to classes of irreducible modules. Moreover, if M and M' are non-isomorphic $\Bbbk\mathcal{H}$-modules, then $d_{\Bbbk}^{\mathbb{L}}([M]) \neq d_{\Bbbk}^{\mathbb{L}}([M'])$, thus $d_{\Bbbk}^{\mathbb{L}}$ is injective, and by Lemma 9.1.8 it is also surjective. ■

Remark 9.1.13 If a polynomial g as in Proposition 9.1.4 is explicitly known, the computation of the decomposition maps $d_{\mathbb{K}}^{\mathbb{L}}$ for all choices of ε is a finite problem: For all ε with $g(\varepsilon) \neq 0$ the decomposition map $d_{\mathbb{K}}^{\mathbb{L}}$ is equal to $d_{\mathbb{K}}^{\Bbbk}$ up to trivial isomorphism and since g has finite degree there is only a finite number of roots. For all of these roots the computation of $d_{\mathbb{K}}^{\mathbb{L}}$ can be achieved via a standard application of Parker's MEATAXE, cf. [Par84], as $\mathbb{L}$ is a finite field in this case.

Using the above results we were able to compute the decomposition numbers for all values of ε for all bad primes p for all exceptional Weyl groups with the exception of $W = E_8$. We describe our approach in the following. The computations were carried out using a combination of the computer algebra systems GAP3, GAP4, and MAGMA, cf. [Sch+97; GAP18; BCP97]. Particularly, the GAP3 package CHEVIE and its development version by Michel have played a crucial role, cf. [Gec+96; Mic15], as well as the MEATAXE algorithm in GAP4, cf. [Par84]. Our approach is similar to what Geck and Lux did in [GL91] to compute decomposition numbers of Iwahori-Hecke algebras of type F_4.
Recall that p is a bad prime for W and W is either F_4, E_6, or E_7.

Computational Steps 9.1.14

a) Take a set $\mathcal{M}$ of $\mathcal{H}$-representations such that $\mathbb{K}\mathcal{M}$ is a complete set of pairwise non-isomorphic irreducible $\mathbb{K}\mathcal{H}$-modules. Such a set is available in the development version of CHEVIE.

b) Specialise the elements of $\mathcal{M}$ via $\theta_{\mathbb{K},\mathbf{k}}$ to a set $\Bbbk\mathcal{M}$ of $\Bbbk\mathcal{H}$-modules with representations $\rho_{\Bbbk M}$ realised over $\mathbf{k}$. From the decomposition maps in [Gyo96] we know how the elements of $\Bbbk\mathcal{M}$ decompose into irreducibles. (Alternatively, we could have specialised further via $y \mapsto \omega$ for some ω of large multiplicative order in a finite field extension of $\mathbb{F}_p$. Using the standard MEATAXE we can decompose the elements of $\mathbb{F}_p[\omega]\mathcal{M}$ into irreducibles and by Proposition 9.1.4, this also yields the decomposition of $\Bbbk$ if the order of ω was chosen large enough. However, we do not know how to choose such an order a priori.) Luckily, the decomposition numbers are small and for almost all irreducible $\Bbbk\mathcal{H}$-modules N there exists an element M of $\mathcal{M}$ such that $\Bbbk M \cong N$. For those irreducible $\Bbbk\mathcal{H}$-modules N where no such M exists there still exists an M such that N is a submodule of $\Bbbk M$. In the representations used in CHEVIE these submodules have generators that are either standard basis vectors or sums thereof, thus it is easy to compute the corresponding representation of N via a spinning algorithm (cf. for example [LP10, Section 1.3]) and it is easy to ensure that these representation are already realised over $\mathbf{k}$.
In total, we obtain a set of $\mathbf{k}\mathcal{H}$-modules $\mathcal{N}$ such that $\Bbbk\mathcal{N}$ is a complete set of pairwise non-isomorphic irreducible $\Bbbk\mathcal{H}$-modules.

c) For every N in $\Bbbk\mathcal{N}$ find an element b_N in $\Bbbk\mathcal{H}$ such that $\rho_N(b)$ has co-rank 1. Such an element exists by [CR81, Theorem 3.32], as $\Bbbk\mathcal{H}$ is split by Proposition 9.1.2, but $\Bbbk\mathcal{H}$ has infinitely many elements so it is not clear how this should be done. Rather surprisingly, short sums of short words in the generators T_s of $\Bbbk\mathcal{H}$ suffice. A similar phenomenon has been observed in related contexts, cf. [GL91; Par98].
For every N, determine vectors v and v' in $\mathbb{F}_p[y]^{1\times\dim(N)}$ spanning the nullspace of $\rho_N(b_N)$ and $\rho_N(b_N)^{\mathrm{Tr}}$, respectively, and by repeatedly multiplying with the $\rho_N(T_w)$ and the $\rho_N(T_w)^{\mathrm{Tr}}$, respectively, obtain bases $V_N := (v_1, \ldots, v_{\dim(N)})$ and $V'_N := (v'_1, \ldots, v'_{\dim(N)})$ as in the proof of Proposition 9.1.4.

d) The character table $\mathcal{X}(\mathbb{K}\mathcal{H})$ is available in CHEVIE and via the decomposition map $d_{\mathbb{K}}^{\Bbbk}$ in [Gyo96] this yields the character table $\mathcal{X}(\Bbbk\mathcal{H})$. We use this to compute the polynomial $g_{\det}$ from the proof of Proposition 9.1.4 and subsequently the polynomial g using the data V_N, V'_N and $\rho_N(b_N)$.

e) Factorise the polynomial g. For every irreducible factor g_i of g let ε be a root of g_i. Specialise the elements of $\mathcal{N}$ to $\mathbb{F}_p[\varepsilon_i]$-modules via $y \mapsto \varepsilon_i$ and compute the decomposition of the resulting $\mathbb{F}_p[\varepsilon_i]\mathcal{H}$-modules into irreducibles using the MEATAXE in GAP and the linear independence of characters to differentiate between non-isomorphic modules. The MEATAXE is also able to verify that all appearing irreducible constituents are absolutely irreducible and by Lemma 9.1.8 this implies that $\mathbb{F}_p[\varepsilon_i]\mathcal{H}$ is split.
Note that we only have to consider one root of each g_i, as specialisation at another root $\varepsilon_i{}'$ of g_i is related to specialisation at ε_i via a field automorphism which yields an isomorphism of Grothendieck groups by Proposition 6.1.3.

Some remarks on the computation are in order.

Remark 9.1.15 Computation in $\mathbf{k}$ is difficult and time consuming and in particular the computation of V_N and V'_N is very time-consuming. Hence, it is often more efficient to specialise to some finite field $\mathbb{F}_p[\omega]$ via $y \mapsto \omega$ and compute over $\mathbb{F}_p[\omega]$ instead, and then use the results there to draw conclusions for $\mathbf{k}$.
Another computational problem is finding the elements b_N. As has already been mentioned it is not clear how these elements can be found systematically. To check whether a given candidate has the correct rank we can use specialisation just as for the computation of the V_N and V'_N. However, this is still not good enough. It might happen that we found a suitable element b_N such that the determinant of V_N or V'_N has huge degree, or similarly the determinant of an $(n-1)\times(n-1)$ submatrix of $\rho_N(b_N)$ might have large degree in y. This in turn would cause g to have large degree. In general this leads to more irreducible factors of g of large degree and thus to more cases in step e) which need to be handled by the MEATAXE over possibly large fields. As this can take very long we have to choose the elements b_N carefully to avoid too large irreducible factors of g.

Remark 9.1.16 We can use the elements b_N to re-check Gyoja's decomposition result:
Corollary 9.1.10 and the elements b_N allow us to check that $\mathcal{N}$ is indeed a set of absolutely irreducible $\Bbbk\mathcal{H}$-modules. Via their characters we check that they are pairwise non-isomorphic. To compute the decomposition map we use the following observation:

Suppose that for every M in $\mathcal{M}$ there exist elements $a_{M,N}$ in $\mathbb{F}_p$ such that

$$\chi_{\Bbbk M} = \sum_{N \in \mathcal{N}} a_{M,N}\, \chi_N.$$

Then denote by $0 \leq a'_{M,N} \leq p-1$ the integer such that $a'_{M,N} \equiv a_{M,N} \pmod p$. If $\dim(\Bbbk M) = \sum_{N\in\mathcal{N}} a'_{M,N} \dim(N)$, then $[\Bbbk M] = \sum_{N\in\mathcal{N}} a'_{M,N}\,[N]$ in the Grothendieck group, and in particular all irreducible constituents of $\Bbbk M$ lie in $\mathcal{N}$.
This observation is strong enough to compute the decomposition of $\Bbbk M$ with one single exception in the case $W = F_4$ and $p = 2$. Here we found a submodule such that both the decomposition of the submodule and the quotient were already known using the above observation, which yields the decomposition of the whole module.
In total, we were able to recover Gyoja's result and also find one error in his work.
In the case $W = E_7$ and $p = 2$ the decomposition numbers for the block containing the irreducible $\mathbb{K}\mathcal{H}$-modules called 512_a and $512'_a$ in CHEVIE are given incorrectly in [Gyo96]. In fact, this block contains only these two irreducible representations and their specialisations to $\Bbbk\mathcal{H}$-modules are isomorphic, irreducible $\Bbbk\mathcal{H}$-modules.

Combining our computational results yields the following.

Theorem 9.1.17 *Suppose W is either F_4, E_6, or E_7 and p is a bad prime for W. Then $\mathbb{L}\mathcal{H}$ is split for any choice of ε and the decomposition map $d^{\mathbb{L}}_{\Bbbk}$ is trivial unless ε is a root of the polynomial $\widehat{g}$ listed in the table below. We print $d^{\Bbbk}_{\mathbb{K}}$ and all non-trivial $d^{\mathbb{L}}_{\Bbbk}$ in Appendix C.*

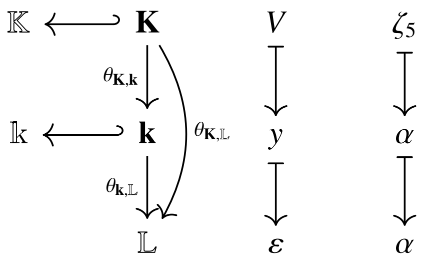
W	p	$\widehat{g}$
F_4	2	y^3+1
	3	$y^{16}-1$
E_6	2	$y^{13}+y^{12}+y^{11}+y^{10}+y^9+y^4+y^3+y^2+y+1$
	3	$y^{24}+y^{22}+y^{20}+y^{18}+y^{16}-y^8-y^6-y^4-y^2-1$
E_7	2	$y^{19}+y^{17}+y^{15}+y^{14}+y^{13}+y^{11}+y^{10}+y^9+y^8+y^6+y^5+y^4+y^2+1$
	3	$y^{56}-y^{52}+y^{44}+y^{40}-y^{32}+y^{24}-y^{16}-y^{12}+y^4-1$

Remark 9.1.18 The case $W = G_2$ will be handled in Section 9.3.

9.2. Iwahori-Hecke Algebras of Type H_3 and H_4

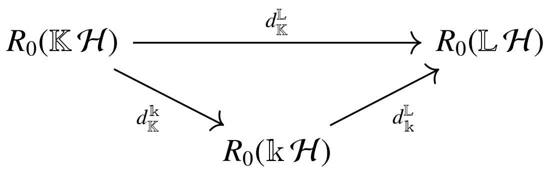
We can handle the non-crystallographic Coxeter group H_3 and H_4 similarly to exceptional Weyl groups. In this section let (W, S) be a Coxeter system where W is either H_3 and H_4. To handle the corresponding Iwahori-Hecke algebras we have to slightly adjust the underlying rings.
We set $\mathbf{K} := \mathbb{Z}[V^{\pm 1}, \zeta_5]$, where ζ_5 is a primitive fifth root of unity, and denote by $\mathbb{K}$ its field of fractions. If p is a prime and α is a root of $y^4+y^3+y^2+y+1$ over $\mathbb{F}_p$, and ε is an invertible element in a field extension of $\mathbb{F}_p$, then there is a well-defined ring homomorphism

$\theta_{\mathbf{K},\mathbb{L}} : \mathbf{K} \to \mathbb{L} := \mathbb{F}_p(\varepsilon, \alpha); V \mapsto \varepsilon,\ \zeta_5 \mapsto \alpha.$
As for Weyl groups this morphism factors through another ring, namely $\mathbf{k} := \mathbb{F}_p[y^{\pm 1},\ \alpha]$ as visualised in the following updated diagram, where $\Bbbk$ is the field of fractions of $\mathbf{k}$:



We define $\mathcal{H} := H_{\mathbf{K}}(W, S, (U := V^2 \mid s \in S))$ as before. If $p \notin \{2, 5\}$ (respectively $p \notin \{2, 3, 5\}$) and $W = H_3$ (respectively $W = H_4$), then, by Theorem 7.1.13, the algebras $\mathbb{K}\mathcal{H}$, $\Bbbk\mathcal{H}$, and $\mathbb{L}\mathcal{H}$ are all split and $\mathbb{K}\mathcal{H}$ is semisimple. Furthermore, the decomposition maps $d_{\mathbb{K}}^{\mathbb{L}} : R_0(\mathbb{K}\mathcal{H}) \to R_0(\mathbb{L}\mathcal{H})$ and $d_{\mathbb{K}}^{\Bbbk} : R_0(\mathbb{K}\mathcal{H}) \to R_0(\Bbbk\mathcal{H})$ are surjective. If $p \in \{2, 5\}$ respectively $p \in \{2, 3, 5\}$ Theorem 7.1.13 is not applicable and thus we investigate specialisations and decomposition numbers in these cases in the following.
First note that analogues of Proposition 9.1.4 and Lemma 9.1.8 hold for our new versions of $\Bbbk$ and $\mathbb{L}$. From CHEVIE we obtain a complete set of irreducible $\mathbb{K}\mathcal{H}$-modules, realised as representations over $\mathbf{K}$ and using the version of Norton's Criterion in Corollary 9.1.10, Lemma 9.1.8, and the observation on characters in Remark 9.1.16 we can compute the decomposition map $d_{\mathbb{K}}^{\Bbbk}$. This is aided in no small way by the surprising fact that for every irreducible $\Bbbk\mathcal{H}$-module N there exists an irreducible $\mathbb{K}\mathcal{H}$-module M such that $\Bbbk M \cong N$. This fact also allows us to immediately obtain a complete set of irreducible $\Bbbk\mathcal{H}$-modules, realised over $\mathbf{k}$, by simply specialising the corresponding irreducible representations of $\mathbb{K}\mathcal{H}$ realised over $\mathbf{K}$.
Then we proceed as in the Computational Steps 9.1.14 to obtain the following.

Theorem 9.2.1 *Let $p \in \{2, 5\}$ if $W = H_3$ and $p \in \{2, 3, 5\}$ if $W = H_4$. Then $\Bbbk\mathcal{H}$ is split and $\mathbb{L}\mathcal{H}$, too, is split, for any choice of ε. Thus, there are well-defined decomposition maps $d_{\mathbb{K}}^{\Bbbk}$, $d_{\mathbb{K}}^{\mathbb{L}}$, and $d_{\Bbbk}^{\mathbb{L}}$ and the following diagram commutes:*



The decomposition map $d_{\Bbbk}^{\mathbb{L}}$ is trivial unless ε is a root of the polynomial $\hat{g}$ given in the table below.

W	p	$\hat{g}$
H_3	2	$y^7 + y^6 + y^5 + y^2 + y + 1$
	3	$y^{12} - 1$
H_4	2	$y^{15} + 1$
	3	$y^{40} - 1$
	5	$y^{24} - 1$

Furthermore, $d^{\Bbbk}_{\mathbb{K}}$, $d^{\mathbb{L}}_{\Bbbk}$, and $d^{\mathbb{L}}_{\mathbb{K}}$ are all surjective. We print $d^{\Bbbk}_{\mathbb{K}}$ and all non-trivial $d^{\mathbb{L}}_{\Bbbk}$ in Appendix D.

Remark 9.2.2 To the best of our knowledge the decomposition maps $d^{\mathbb{L}}_{\mathbb{K}}$ have not yet been computed in the cases where the characteristic p of $\mathbb{L}$ is a good prime for W.

9.3. Iwahori-Hecke Algebras of Dihedral Groups

Let us finally study decomposition maps of Iwahori-Hecke algebras of groups of type $I_2(m)$ for $m \geq 3$, that is dihedral groups of order $2m$.
Evidently, our explicit computational methods are useless when it comes to handling an infinite family of groups. But the representations of Iwahori-Hecke algebras of $I_2(m)$ are not overly complicated and for varying m representations of the algebras are constructed uniformly.

Throughout this section let $m \geq 3$ be an integer and W the dihedral group $I_2(m)$ with presentation $\left\langle s, t \mid s^2,\ t^2,\ (st)^m \right\rangle$. Clearly, $(W, \{s, t\})$ is a Coxeter system.
Let ζ be a primitive $2m$'th root of unity over $\mathbb{Q}$ and in analogy to the preceding sections set $\mathbf{K} := \mathbb{Z}[V^{\pm 1},\ \zeta]$ and denote by $\mathbb{K}$ its field of fractions.
Let p be a prime and let m' be co-prime to p such that $m = p^k m'$ for some non-negative integer k.
Let α be an element of order m' in a field extension of $\mathbb{F}_p$ and let ε be an invertible element of a field extension of $\mathbb{F}_p$. Set $\mathbb{L} := \mathbb{F}_p[\varepsilon, \alpha]$.
Then $\theta_{\mathbf{K},\mathbb{L}} : \mathbb{K} \to \mathbb{L}; V \mapsto \varepsilon,\ \zeta \mapsto \alpha$ defines a ring homomorphism. As before, it factors through $\Bbbk := \mathbb{F}_p[y^{\pm 1},\ \alpha]$:

$$\begin{array}{ccccccc}
\mathbb{K} & \hookleftarrow & \mathbf{K} & & V & & \zeta \\
 & & \downarrow{\scriptstyle \theta_{\mathbf{K},\mathbf{k}}} & & \downarrow & & \downarrow \\
\Bbbk & \hookleftarrow & \mathbf{k} & {\scriptstyle \theta_{\mathbf{K},\mathbb{L}}} & y & & \alpha \\
 & & \downarrow{\scriptstyle \theta_{\mathbf{k},\mathbb{L}}} & & \downarrow & & \downarrow \\
 & & \mathbb{L} & & \varepsilon & & \alpha
\end{array}$$

Now set $\mathcal{H} := H_{\mathbf{K}}(W, \{s, t\},\ (U_s := U := V^2, U_t := U))$.
By Lemma 7.1.16 and Theorem 7.1.13, the algebra $\mathbb{K}\mathcal{H}$ is split semisimple and if p does not

divide m, then $\Bbbk\mathcal{H}$ and $\mathbb{L}\mathcal{H}$ are both split. In this case, the corresponding decomposition maps $d_{\mathbb{K}}^{\Bbbk}$ and $d_{\Bbbk}^{\mathbb{L}}$ are computed in [GJ11, Section 7.2]. Using similar techniques we do the same in the case that p divides m.

We follow [GP00, Section 8.3] for the representation theory of $\mathbb{K}\mathcal{H}$. Recall that we have a one-to-one correspondence between the irreducible representations of $\mathbb{K}\mathcal{H}$ and those of $\mathbb{C}[W]$, whose characters we gave in Lemma 7.1.20.

Proposition 9.3.1 *Let $1 \leq j \leq (m-2)/2$ if m is even and $1 \leq j \leq (m-1)/2$ if m is odd. Then*

$$\rho_j : T_s \mapsto \begin{pmatrix} -1 & V(\zeta^j + \zeta^{-1}) \\ 0 & V^2 \end{pmatrix},\ T_t \mapsto \begin{pmatrix} V^2 & 0 \\ V(\zeta^j + \zeta^{-j}) & -1 \end{pmatrix}$$

defines a representation of $\mathbb{K}\mathcal{H}$.
Let a and b be in $\{-1, V^2\}$, and if m is odd let $a = b$. Then

$$\varepsilon_{a,b} : T_s \mapsto a,\ T_t \mapsto b$$

defines a representation of $\mathbb{K}\mathcal{H}$.
If m is even, then $\{\varepsilon_{-1,V^2},\ \varepsilon_{-1,-1},\ \varepsilon_{V^2,V^2},\ \varepsilon_{V^2,-1},\ \rho_j \mid 1 \leq j \leq (m-2)/2\}$ is a complete set of pairwise non-isomorphic $\mathbb{K}\mathcal{H}$-representations.
If m is odd, then $\{\varepsilon_{-1,-1},\ \varepsilon_{V^2,V^2},\ \rho_j \mid 1 \leq j \leq (m-1)/2\}$ is a complete set of pairwise non-isomorphic irreducible representations of $\mathbb{K}\mathcal{H}$.
Note that all the above representations are realised over **K**.

Recall from Lemma 9.1.11 that the character of a representation of an Iwahori-Hecke algebra is determined by its images of standard basis elements corresponding to shortest representatives of the conjugacy classes in W.

Lemma 9.3.2 *If m is even, then*

$$\{1, s, t, (st)^k \mid 1 \leq k \leq m/2\}$$

is a complete set of shortest representatives of conjugacy classes in W.
If m is odd, then

$$\{1, s, (st)^k \mid 1 \leq k \leq (m-1)/2\}$$

is such a set.

Lemma 9.3.3 *For a representation φ denote by χ_φ its character. For all allowed values of a, b, j, and k we have*

$$\begin{aligned}
\chi_{\varepsilon_{a,b}}(T_t) &= a, \\
\chi_{\varepsilon_{a,b}}(T_s) &= b, \\
\chi_{\varepsilon_{a,b}}(T_{(st)^k}) &= (ab)^k, \\
\chi_{\rho_j}(T_s) = \chi_{\rho_j}(T_t) &= V^2 - 1, \\
\chi_{\rho_j}(T_{(st)^k}) &= V^{2k}(\zeta^{2jk} + \zeta^{-2jk}).
\end{aligned}$$

Remark 9.3.4 Note that our notation for irreducible characters of $\mathbb{K}\mathcal{H}$ is reminiscent of that which we used for $\mathbb{C}[W]$ in Lemma 7.1.20 to indicate the relationship between the two sets. Under the specialisation $\mathfrak{A} \to \mathbb{C}; V \mapsto 1$, the character we denote here by $\chi_{\varepsilon_{a,b}}$ is mapped to $\chi_{a,b}$ from Lemma 7.1.20, and χ_{ρ_j} is mapped to χ_j.

Denote by M_j the $\mathcal{H}$-module afforded by ρ_j and by $M_{a,b}$ the module afforded by the representation $\varepsilon_{a,b}$ for all allowed values of j, a, and b and let $\mathcal{M}$ be the set containing all M_j and $M_{a,b}$. By Proposition 9.3.1, the set $\mathbb{K}\mathcal{M}$ is a complete set of pairwise non-isomorphic irreducible $\mathbb{K}\mathcal{H}$-modules.

Lemma 9.3.5 *Let $\mathbb{A} \in \{\mathbb{k}, \mathbb{L}\}$ and let $\theta := \theta_{\mathbf{K},\mathbb{L}}$ if $\mathbb{A} = \mathbb{L}$ and $\theta := \theta_{\mathbf{K},\mathbf{k}}$ if $\mathbb{A} = \mathbb{k}$.*
The specialisation $\mathbb{A}M_{a,b}$ is absolutely irreducible for all allowed values of a and b. If $\mathbb{A} \subseteq \mathbb{A}'$ is a field extension then every one-dimensional representation of $\mathbb{A}'\mathcal{H}$ is isomorphic to some $\mathbb{A}'M_{a,b}$. In particular, this holds for $\mathbb{A}' = \mathbb{A}$.
If $\theta(V^2) \neq -1$, then $\mathbb{k}M_{a,b}$ and $\mathbb{k}M_{a',b'}$ for allowed values a, b, a', b' are non-isomorphic unless $a = a'$ and $b = b'$.
Finally, if M in $\mathcal{M}$ is two-dimensional, then $\mathbb{A}M$ is absolutely irreducible unless its character is the sum of two characters of one-dimensional $\mathbb{A}\mathcal{H}$-modules.

Proof Since $\mathbb{A}M_{a,b}$ is one-dimensional it is clearly absolutely irreducible.
It follows from the quadratic relations satisfied by T_s and T_t that every one-dimensional irreducible representation is isomorphic to some $\mathbb{A}M_{a,b}$ and field extensions yield no new one-dimensional representations.
The non-isomorphism of $\mathbb{A}M_{a,b}$ and $\mathbb{A}M_{a',b'}$ follows from the fact that they afford different characters by Lemma 9.3.3.
Finally, let M in $\mathcal{M}$ be two-dimensional. If $\mathbb{A}M$ is not absolutely irreducible, then there exists a field extension $\mathbb{A} \subseteq \mathbb{A}'$ such that $\mathbb{A}'M$ has a one-dimensional submodule and a corresponding one-dimensional quotient. But all one-dimensional representations of $\mathbb{A}'\mathcal{H}$ can be realised over $\mathbb{A}$ and therefore the character of $\mathbb{A}M$ is the sum of two characters of one-dimensional $\mathbb{A}\mathcal{H}$-modules. ∎

Let us first study the $\mathbb{F}_p$-generic decomposition.

Theorem 9.3.6

a) *If m is odd, then $\mathbb{k}M$ is absolutely irreducible for all M in $\mathcal{M}$. In particular, $\mathbb{k}\mathcal{H}$ is split by Lemma 9.1.8.*
   For $1 \leq j, j' \leq (m-1/2)$ with $j \neq j'$ we have $\mathbb{k}M_j \cong \mathbb{k}M_{j'}$ if and only if m' divides $2(j-j')$ or m' divides $2(j+j')$. In combination with Lemma 9.3.5 this fully defines $d^{\mathbb{k}}_{\mathbb{K}}$.

b) *If m is even, then $\mathbb{k}M$ is absolutely irreducible for all M in $\mathcal{M}$ unless $M = M_j$ for some j such that $\alpha^{2j} = -1$, in which case $\mathbb{k}M_j$ is isomorphic to the direct sum $\mathbb{k}(M_{-1,V^2} \oplus M_{V^2,-1})$. By Lemma 9.1.8, the algebra $\mathbb{k}\mathcal{H}$ is split.*
   For $1 \leq j, j' \leq (m-1/2)$ with $j \neq j'$ we have $\mathbb{k}M_j \cong \mathbb{k}M_{j'}$ if and only if m' divides $2(j-j')$ or m' divides $2(j+j')$. In combination with Lemma 9.3.5 this fully defines $d^{\mathbb{k}}_{\mathbb{K}}$.

Proof By Lemma 9.3.5 we only have to consider two-dimensional modules in $\mathcal{M}$.
Denote by χ_j the character of $\Bbbk M_j$ and by $\chi_{a,b}$ that of $\Bbbk M_{a,b}$.
From the values of the characters on T_s and T_{st} given in Lemma 9.3.3 it follows that $\chi_j \neq \chi_{a,b} + \chi_{a',b'}$ for all allowed values of a, b, a', b', unless m is even and $\chi_j = \chi_{-1,V^2} + \chi_{V^2,-1}$. In particular, by Lemma 9.3.5, the module $\Bbbk M_j$ is absolutely irreducible unless m is even and $\chi_j = \chi_{-1,V^2} + \chi_{V^2,-1}$.
By the conjugacy class representatives in Lemma 9.3.2 we have

$$\begin{aligned} && \chi_j &= \chi_{-1,V^2} + \chi_{V^2,-1} & \\ \Leftrightarrow && 2(-y^2)^k &= y^{2k}(\alpha^{2jk} + \alpha^{-2jk}) && \text{for all } 1 \leq k \leq \frac{m}{2} \\ \Leftrightarrow && \alpha^{2jk} + \alpha^{-2jk} &= 2(-1)^k && \text{for all } 1 \leq k \leq \frac{m}{2}. \end{aligned}$$

By plugging $k = 1$ into the last line we see that $\chi_j = \chi_{-1,V^2} + \chi_{V^2,-1}$ if and only if $\alpha^{2j} = -1$. In this case the specialisations of the defining matrices $\rho_j(T_s)$ and $\rho_j(T_t)$ are diagonal matrices. Thus, $\Bbbk M_j$ is a direct sum of two one-dimensional $\Bbbk\mathcal{H}$-modules and as $\chi_j = \chi_{-1,V^2} + \chi_{V^2,-1}$ we have $\Bbbk M_j \cong \Bbbk(M_{-1,V^2} \oplus M_{V^2,-1})$ if $\alpha^{2j} = -1$.

Now suppose that $\Bbbk M_j$ and $\Bbbk M_{j'}$ are isomorphic. By the linear independence of characters of irreducible modules of split algebras this is the case if and only if $\chi_j = \chi_{j'}$. Via the character values provided in Lemma 9.3.3 we can show that this is the case if and only if m' divides $2k(j - j')$ or $2k(j + j')$ for all allowed values of k and this in turn is the case if and only if m' divides $2(j - j')$ or $2(j + j')$. ∎

Similarly, we can compute the decomposition map $d^{\mathbb{L}}_{\mathbb{K}}$. Let $\xi := \varepsilon^2$.

Theorem 9.3.7

a) *Suppose m is odd and $\xi \neq -1$.*
For $1 \leq j \leq (m-1)/2$ the module $\mathbb{L} M_j$ is absolutely irreducible unless $\alpha^{2j} \in \{\xi, \xi^{-1}\}$. In the latter case $\mathbb{L} M_j$ is reducible but not semisimple. It has a proper submodule isomorphic to $\mathbb{L} M_{-1,-1}$ and the corresponding quotient is isomorphic to $\mathbb{L} M_{V^2,V^2}$.

b) *Suppose m is odd and $\xi = -1$.*
For $1 \leq j \leq (m-1)/2$ the module $\mathbb{L} M_j$ is absolutely irreducible unless $\alpha^{2j} = -1$, in which case $\mathbb{L} M_j$ is isomorphic to the direct sum of two copies of $\mathbb{L} M_{-1,-1}$.

c) *Suppose m is even and $\xi \neq -1$*
For $1 \leq j \leq (m-2)/2$ the module $\mathbb{L} M_j$ is absolutely irreducible unless $\alpha^{2j} = -1$ or $\alpha^{2j} \in \{\xi, \xi^{-1}\}$.
If $\alpha^{2j} = -1$, then $\mathbb{L} M_j$ is isomorphic to the direct sum $\mathbb{L}(M_{-1,V^2} \oplus M_{V^2,-1})$.
If $\alpha^{2j} \in \{\xi, \xi^{-1}\}$, then $\mathbb{L} M_j$ is reducible but not semisimple. It has a proper submodule isomorphic to $\mathbb{L} M_{-1,-1}$ and the corresponding quotient is isomorphic to $\mathbb{L} M_{V^2,V^2}$.

d) *Suppose m is even and $\xi = -1$.*
For $1 \leq j \leq (m-2)/2$ the module $\mathbb{L} M_j$ is absolutely irreducible unless $\alpha^{2j} = -1$, in which case $\mathbb{L} M_j$ is isomorphic to the direct sum $\mathbb{L}(M_{-1,V^2} \oplus M_{V^2,-1})$.

In all cases the algebra $\mathbb{L}\mathcal{H}$ is split by Lemmas 9.1.8 and 9.3.5.
For allowed values of j and j' we have $\mathbb{L}\, M_j \cong \mathbb{L}\, M_{j'}$ if and only if m' divides $2(j-j')$ or $2(j+j')$ and in combination with Lemma 9.3.5 this fully defines the decomposition map $d^{\mathbb{L}}_{\mathbb{K}}$

Proof We proceed just as in the proof of Theorem 9.3.6. In particular, the case $\alpha^{2j} = -1$ is handled completely analogously.
Denote by χ_j the character of $\mathbb{L}\, M_j$ and by $\chi_{a,b}$ that of $\mathbb{L}\, M_{a,b}$ for all allowed values of a, b, j.
First assume $\xi \neq -1$. From the character values in Lemma 9.3.3 it follows that χ_j is not the sum of two characters of one-dimensional representations unless $\chi_j = \chi_{-1,-1} + \chi_{V^2,V^2}$, which is the case if and only if $\alpha^{2j} \in \{\xi, \xi^{-1}\}$.
Assume $\alpha^{2j} \in \{\xi, \xi^{-1}\}$ and denote by $\theta(\rho_j)$ the representation of $\mathbb{L}\, M_j$ obtained by applying $\theta_{\mathbf{K},\mathbb{L}}$ to all entries of the representing matrices of ρ_j.
Then the eigenspace to the eigenvalue -1 of both $\theta(\rho_j)(T_s)$ and $\theta(\rho_j)(T_t)$ is spanned by the vector

$$\left(1,\ \frac{-\xi - 1}{\varepsilon(\alpha^j + \alpha^{-j})}\right) \in \mathbb{L}^{1\times 2}$$

and thus $\mathbb{L}\, M_j$ has a submodule isomorphic to $\mathbb{L}\, M_{-1,-1}$. From the character of $\mathbb{L}\, M_j$ we have already deduced that the corresponding quotient is isomorphic to $\mathbb{L}\, M_{V^2,V^2}$. But the eigenspaces to the eigenvalue ξ of $\theta(\rho_j)(T_s)$ and $\theta(\rho_j)(T_t)$ are spanned by the first and the second standard basis vector, respectively, and therefore $\mathbb{L}\, M_j$ does not have a submodule isomorphic to $\mathbb{L}\, M_{V^2,V^2}$.

Now assume $\xi = -1$.
As all one-dimensional representations are isomorphic in this case a computation analogous to that in the proof of Theorem 9.3.6 shows that LM_j is absolutely irreducible unless $\alpha^{2j} = -1$ and that in this case $\mathbb{L}\, M_j$ is isomorphic to the given direct sum.

The conditions on isomorphisms $\mathbb{L}\, M_j \cong \mathbb{L}\, M_{j'}$ follow just as in the proof of Theorem 9.3.6. ∎

We give the "adjustment map" $d^{\mathbb{L}}_{\Bbbk}$.

Remark 9.3.8 By Theorems 9.3.6, 9.3.7 and 7.1.13, we can apply Proposition 6.1.25 to see that we have our usual factorisation of decomposition maps, i.e. $d^{\mathbb{L}}_{\Bbbk} \circ d^{\Bbbk}_{\mathbb{K}} = d^{\mathbb{L}}_{\mathbb{K}}$. By Theorem 9.3.6, every irreducible $\Bbbk\mathcal{H}$-module is of the form $\Bbbk\, M$ for some M in $\mathcal{M}$. Thus, we have essentially already computed $d^{\mathbb{L}}_{\Bbbk}$: For an irreducible $\Bbbk\mathcal{H}$-module N, use Theorem 9.3.6 to choose some M in $\mathcal{M}$ such that $\Bbbk\, M \cong N$. Then we have $d^{\mathbb{L}}_{\Bbbk}([N]) = d^{\mathbb{L}}_{\mathbb{K}}(\mathbb{K}\, M)$, and we can read of the right-hand side from Theorem 9.3.7. In total, this yields the following.

a) If $\xi \neq -1$, then the one-dimensional modules remain pairwise non-isomorphic under specialisation. If $\xi = -1$, then their specialisations are all isomorphic to the unique one-dimensional $\mathbb{L}\mathcal{H}$-module.

b) Suppose m is odd.
If $\alpha^{2j} \notin \{\xi, \xi^{-1}\}$, then both $\Bbbk\, M_j$ and $\mathbb{L}\, M_j$ are irreducible.
If $\alpha^{2j} \in \{\xi, \xi^{-1}\}$, then $\Bbbk\, M_j$ is irreducible and we have $d^{\mathbb{L}}_{\Bbbk}([\Bbbk\, M_j]) = [\mathbb{L}\, M_{-1,-1}] + [\mathbb{L}\, M_{V^2,V^2}]$.

c) Suppose m is even.
If $\alpha^{2j} \notin \{\xi, \xi^{-1}\}$, then $\Bbbk\, M_j$ and $\mathbb{L}\, M_j$ are both irreducible.

If $\alpha^{2j} \in \{\xi, \xi^{-1}\}$ and $\xi \neq -1$, then $\Bbbk\, M_j$ is irreducible and we have $d_{\Bbbk}^{\mathbb{L}}([\Bbbk\, M_j]) = [\mathbb{L}\, M_{-1,-1}] + [\mathbb{L}\, M_{V^2,V^2}]$.

We close with a remark on splitting fields in the case that $W = G_2$.

Remark 9.3.9 If $W = G_2$, then the representations ρ_j and $\varepsilon_{a,b}$ are realised over $\mathbb{Q}[V]$ for all allowed values of a, b, and j:
The sixth cyclotomic polynomial is $\Phi_6 = X^2 - X + 1$ and as ξ is a root of Φ_6, this implies $\xi + \xi^{-1} = 1$.
Thus, all the irreducible representations of $\Bbbk\,\mathcal{H}$ and $\mathbb{L}\,\mathcal{H}$ are realised over $\mathbb{F}_p(y)$ and $\mathbb{F}_p(\varepsilon)$, respectively, cf. Theorems 9.3.6 and 9.3.7. Hence, $\mathbb{F}_p(y)\,\mathcal{H}$ and $\mathbb{F}_p(\varepsilon)\,\mathcal{H}$ are split for any choice of ε, just as for the other Weyl groups F_4, E_6, and E_7, cf. Theorem 9.1.17.

Appendix

Appendix A.

On Different Definitions of i-restriction and i-induction

We study the relation between different definitions of i-Res and i-Ind. To this end, let K be an algebraically closed field, q an invertible element of K, r and n positive integers, and $Q_1, \ldots, Q_r$ powers of q. As usual, we denote by $\mathbf{H}_n$ the Ariki-Koike algebra $\mathbf{H}_{n,r}(q, Q_1, \ldots, Q_r)$.
The common denominator of all definitions of refined induction and restriction functors is that they combine the usual parabolic restriction and induction with the operation of taking generalised eigenspaces of certain elements constructed from Jucys-Murphy elements. Some examples:

- In [AM00; Ari96] these elements are sums of Jucys-Murphy elements.
- In [Ari02] these elements are symmetric polynomials of Jucys-Murphy elements.
- In [Gro99; GV01] only one specific Jucys-Murphy element is considered.
- In [Ari06; Fay08] all symmetric polynomials in the Jucys-Murphy elements are considered at once.

All of these articles and books claim that these are definitions of i-Res. In [Vaz99] Vazirani indicates why the definitions in [AM00] (i.e. the one we, too, gave in Definition 5.3.1) and that in [Gro99; GV01] are equivalent. We expand this proof below. The equivalence to the definition in [Ari02] can be shown analogously. We close with a remark on the equivalence of different definitions of i-Ind.

Definition A.1 ([GV01; Gro99]) Denote by $\widehat{P}_{n,\,\alpha}$ the operator that takes the generalised eigenspace of the Jucys-Murphy element L_n with respect to the eigenvalue $\alpha \in K$ on an $\mathbf{H}_n$-module. Then define a refined restriction functor R_α as

$$R_\alpha(M) := \operatorname{Res}^n_{n-1}\left(\widehat{P}_{n,\,\alpha}(M)\right)$$

for an $\mathbf{H}_n$-module M.

This is the definition of refined restriction found in [Gro99; GV01; Vaz99].

Remark A.2 Definition A.1 yields a well-defined functor R_α : mod- $\mathbf{H}_n \to$ mod- $\mathbf{H}_{n-1}$: The n'th Jucys-Murphy element L_n commutes with $\mathbf{H}_{n-1}$ by Lemma 2.4.9. Hence, any generalised eigenspace of L_n is indeed an $\mathbf{H}_{n-1}$-module. By Lemma 5.1.11, and because the Q_s are powers of q it is obvious that $R_\alpha(M)$ is 0 unless α is a power of q.

We show the equivalence to our definition.

Proposition A.3 *Let $i \in \mathbb{Z}/e\mathbb{Z}$, where $e \in \mathbb{Z} \cup \{\infty\}$ is the multiplicative order of q. Then*

$$R_{q^i} = i\text{-Res}_n .$$

Proof Let M be an $\mathbf{H}_n$-module. Since both R_{q^i} and i-Res$_n$ are compatible with direct sums we can assume without loss of generality that M is an indecomposable $\mathbf{H}_n$-module. Denote by d the K-dimension of M.
Recall that for $\alpha \in K$ we defined $P_{n,\alpha}$ to be the operator that takes the generalised eigenspace of the element $c_n = L_1 + \cdots + L_n$ with respect to the eigenvalue α, cf. Definition 5.3.1. Since c_n is a central element in $\mathbf{H}_n$ and M is indecomposable it follows from Lemma 5.1.11 that there is exactly one β in K such that $M = P_{n,\beta}(M)$ and $P_{n,\alpha}(M) = 0$ for all $\beta \neq \alpha$ in K.
Thus, by Definition 5.3.1 we have

$$i\text{-Res}(M) = P_{n-1,\, \beta-q^i} \circ \text{Res}_n \circ P_{n,\, \beta}(M) = P_{n-1,\beta-q^i}(\text{Res}_n(M)).$$

Hence, it suffices to show that the generalised eigenspace of L_n with respect to q^i is equal to the generalised eigenspace of c_{n-1} with respect to $\beta - q^i$.
As M is equal to $P_{n,\beta}(M)$ it is by definition also equal to the kernel (more precisely the torsion) of $(c_n - \beta)^d$. Hence, M is equal to the kernel of $(A+B)^d$ for $A := c_{n-1} - (\beta - q^i)$ and $B := L_n - q^i$, since $c_n = c_{n-1} + L_n$. As the Jucys-Murphy elements commute, Lemma A.4 below shows that the kernel of A^d is equal to the kernel of B^d which is exactly the equality of the generalised eigenspaces. ∎

Lemma A.4 *Let V be a finite dimensional K-vector space and $A, B \in End(V)$ commuting endomorphisms. Let $d := \dim(V)$. If $(A + B)^d = 0$, then*

$$\ker(A^d) = \ker(B^d).$$

Proof Consider A and B as matrices by fixing a K-basis of V. Since they commute we can assume without loss of generality that both A and B are in Jordan normal form. In particular, by going over to the algebraic closure of K we can further assume that they are of upper triangular form where the diagonal entries are exactly the eigenvalues of A and B respectively, and the multiplicity of an eigenvalue on the diagonal is exactly the dimension of the generalised eigenspace to that eigenvalue.
As $(A + B)^d = 0$, the i'th diagonal entry of A is the negative of the i'th diagonal entry of B for $1 \leq i \leq d$. The kernel of A^d is spanned by those basis vectors corresponding to the diagonal entries that are 0, and since $0 = -0$, it follows that they also span the kernel of B^d. ∎

We have shown that different definitions of the refined restriction functors are equal. Now for induction.

Corollary A.5 *The definition of i-induction employed by Grojnowski and Vazirani is rather technical. However, in [Gro99, Proposition 8.4] it is shown that it is left-adjoint to their i-restriction functor* R_{q^i} *which we have just shown to be isomorphic to* $i\text{-}\mathrm{Res}_n$*. By the uniqueness of left-adjoints up to isomorphisms it follows that the definition of i-induction used by Grojnowski and Vazirani yields a functor which is isomorphic to* $i\text{-}\mathrm{Ind}_n$ *as we defined it in Definition 5.3.1.*

Appendix B.

Specht Modules and the Proof of Lemma 5.3.25

In this appendix we expand on the theory of Specht modules. In particular, we proof (a stronger version of) Lemma 5.3.25. To this end, we briefly introduce the theory of Specht modules, first for Iwahori-Hecke algebras of type A and then for Ariki-Koike algebras, before bringing everything together.

B.1. Specht and Irreducible Modules of Iwahori-Hecke Algebras of Type A_{n-1}

We closely follow [Mat99, Chapter 3]. Let $n \geq 1$ be an integer, R an integral domain, and q an invertible element of R. Let $\mathcal{H}$ be the Iwahori-Hecke algebra of type A_{n-1} over R with parameter q. The theory of Specht modules is based on the theory of *tableaux*:
Let $\alpha \vdash n$. A *tableau of shape* α is a bijection $\mathfrak{t} : [\alpha] \to \{1, \ldots, n\}$, where $[\alpha]$ is the Young diagram of α. This can be visualised by labelling every box x of $[\alpha]$ with $\mathfrak{t}(x)$. In this sense, we can speak of *entries, rows, and columns of* $\mathfrak{t}$.
We say that $\mathfrak{t}$ is is *standard*, if the entries of $\mathfrak{t}$ in every row increase from left to right and in every column from top to bottom. By $\mathrm{Std}(\alpha)$ we denote the set of all standard tableaux of shape α. and we denote by $\mathfrak{t}^\alpha$ the tableau of shape α that is obtained by increasingly labelling the boxes of $[\alpha]$ with $\{1, \ldots, n\}$ from left to right, from top to bottom.

Example B.1 Let $\alpha := 4, 2, 2, 1$. Then

4	2	3	5
6	1		
9	7		
8			

is a tableau of shape α, but not standard. The tableau t^α is given by

1	2	3	4
5	6		
7	8		
9			

and this tableau is standard. ■

Clearly, the symmetric group $\mathfrak{S}_n$ acts naturally on the set of all tableaux of shape α. For any tableau $\mathfrak{t}$ denote by $d(\mathfrak{t})$ the unique element of $\mathfrak{S}_n$ satisfying $\mathfrak{t} = \mathfrak{t}^\alpha d(\mathfrak{t})$.
Let $* : \mathcal{H} \to \mathcal{H}$ be the R-linear anti-automorphism of $\mathcal{H}$ determined by $T_{s_i}^* = T_{s_i}$ for the standard generators $T_{s_1}, \ldots, T_{s_{n-1}}$ of $\mathcal{H}$. For two standard tableau $\mathfrak{s}$ and $\mathfrak{t}$ of shape α set $m'_{\mathfrak{s}\mathfrak{t}} := T^*_{d(\mathfrak{s})} m'_\alpha T_{d(\mathfrak{t})}$, where $m'_\alpha := \sum_{w \in \mathfrak{S}_\alpha} T_w$ (recall that $\mathfrak{S}_\alpha$ is the Young subgroup of $\mathfrak{S}_n$ corresponding to α).
Then

$$\{m'_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\beta) \text{ for some partition } \beta \text{ of } n\}$$

is an R-basis of $\mathcal{H}$.
Now let $\mathcal{H}^\alpha$ be the R-span of

$$\{m'_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\beta) \text{ for some partition } \beta \text{ of } n \text{ with } \beta \rhd \alpha\},$$

where $\rhd$ is the dominance order of partitions.
Then $\mathcal{H}^\alpha$ is a two-sided ideal of $\mathcal{H}$ and the *Specht module* S^α is defined as

$$S^\alpha := (m'_\alpha + \mathcal{H}^\alpha)\mathcal{H} \leq \mathcal{H} / \mathcal{H}^\alpha .$$

The set

$$\{m'_{\mathfrak{t}^\alpha \mathfrak{t}} + \mathcal{H}^\alpha \mid \mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\alpha)\}$$

is an R-basis of S^α.

Now for D^α:
There is a R-bilinear map $\langle\ ,\ \rangle_\alpha$ on S^α, which is uniquely defined by the equations

$$\langle m'_{\mathfrak{t}^\alpha \mathfrak{s}} + \mathcal{H}^\alpha,\, m'_{\mathfrak{t}^\alpha \mathfrak{t}} + \mathcal{H}^\alpha \rangle_\alpha\, m_\alpha \equiv m'_{\mathfrak{t}^\alpha \mathfrak{s}} m'_{\mathfrak{t}\mathfrak{t}^\alpha} \pmod{\mathcal{H}^\alpha}$$

for all standard tableaux $\mathfrak{s}$ and $\mathfrak{t}$ of shape α. Denote by $\mathrm{rad}(S^\alpha)$ its radical and set

$$D^\alpha := S^\alpha / \mathrm{rad}(S^\alpha).$$

If R is a field, then

$$\{D^\beta \mid \beta \vdash n,\ D^\beta \neq 0\}$$

is a complete set of pairwise non-isomorphic irreducible $\mathcal{H}$-modules.

B.2. Specht modules and irreducible modules of Ariki-Koike algebras

We carry out what is essentially the same construction for Ariki-Koike algebras, closely following [Mat04, Chapters 2&3].
Let $n, r \geq 1$ be integers and R an integral domain. Let q and $Q_1, \ldots, Q_r$ be invertible elements of R and set $\mathbf{H} := \mathbf{H}_{r,n}(q, Q_1, \ldots, Q_r)$, the corresponding Ariki-Koike algebra.

Let $\lambda \vdash_r n$. A *tableau of shape* λ is a bijection $\mathfrak{t} : [\lambda] \to \{1, \dots, n\}$, where $[\lambda]$ is the Young diagram of λ. As for the case $r = 1$ discussed above this can be visualised by labelling the boxes of $[\lambda]$ with their corresponding image under $\mathfrak{t}$ and it makes sense to speak of *entries, components, columns, and rows of* $\mathfrak{t}$.
We call $\mathfrak{t}$ *standard* if in every component of $\mathfrak{t}$ the entries in every row increase from left to right and in every column from top to bottom. The set of all standard tableaux of shape λ is denoted by $\mathrm{Std}(\lambda)$.
We denote by $\mathfrak{t}^\lambda$ the standard tableau of shape λ that is obtained by labelling the boxes of $[\lambda]$ increasingly by $\{1, \dots, n\}$, going left to right, top to bottom in each component, starting with the first component.
The symmetric group $\mathfrak{S}_n$ acts naturally on the set of tableaux of shape λ and for a tableau $\mathfrak{s}$ we denote by $d(\mathfrak{s})$ the unique element of $\mathfrak{S}_n$ such that $\mathfrak{s} = \mathfrak{t}^\lambda d(\mathfrak{s})$.
Denote by $\mathfrak{S}_\lambda$ the *Young subgroup of* $\mathfrak{S}_n$ *corresponding to* λ. It is defined completely analogously to the case $r = 1$: It is the stabiliser of the partition of n given by the rows of $\mathfrak{t}^\lambda$.
Set

$$x_\lambda := \sum_{w \in \mathfrak{S}_\lambda} T_w \quad \text{and} \quad u_\lambda := \prod_{s=2}^{r} \prod_{k=1}^{|\lambda^{(1)}|+\cdots+|\lambda^{(s-1)}|} (L_k - Q_s)\,,$$

where L_k is the k'th Jucys-Murphy element of $\mathbf{H}$. Furthermore, set $m_\lambda := x_\lambda u_\lambda$.
Let $* : \mathbf{H} \to \mathbf{H}$ be the R-linear anti-automorphism defined by $S_i^* := S_i$ for $0 \le i \le n-1$. For $\mathfrak{s}$ and $\mathfrak{t}$ in $\mathrm{Std}(\lambda)$ set $m_{\mathfrak{st}} := T^*_{d(\mathfrak{s}} m_\lambda T_{d(\mathfrak{t})}$.
Then

$$\{m_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\mu) \text{ for some } \mu \vdash_r n\}$$

is an R-basis of $\mathbf{H}$.
Now let $\mathbf{H}^\lambda$ be the R-span of all $m_{\mathfrak{st}}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are standard tableau of some shape μ with $\mu \rhd \lambda$, i.e. μ strictly dominates λ. Then $\mathbf{H}^\lambda$ is a two-sided ideal of $\mathbf{H}$ and the *Specht module* S^λ is defined as

$$S^\lambda := m_\lambda + \mathbf{H}^\lambda \le \mathbf{H} / \mathbf{H}^\lambda\,.$$

The set

$$\{m_{\mathfrak{t}^\lambda\mathfrak{t}} + \mathcal{H}^\lambda \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$$

is an R-basis of S^λ.

We discuss the module D^λ.
There is an R-bilinear form $\langle\ ,\ \rangle_\lambda$ on S^λ uniquely defined by

$$\langle m_{\mathfrak{t}^\lambda\mathfrak{s}} + \mathbf{H}^\lambda,\, m_{\mathfrak{t}^\lambda\mathfrak{t}} + \mathbf{H}^\lambda\rangle_\lambda\, m_\lambda \equiv m_{\mathfrak{t}^\lambda\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^\lambda} \pmod{\mathbf{H}^\lambda}.$$

Denote by $\mathrm{rad}(S^\lambda)$ its radical. Then D^λ is defined as $D^\lambda := S^\lambda / \mathrm{rad}(S^\lambda)$. If R is a field, then

$$\{D^\mu \mid \mu \vdash_r n,\ D^\mu \neq 0\}$$

is a complete set of pairwise non-isomorphic irreducible $\mathbf{H}$-modules.

B.3. Proof of Lemma 5.3.25

Instead of proving Lemma 5.3.25 we prove the following slightly stronger result, in which we do not require q to be 1.

Lemma B.2 Let $n, r \geq 1$ be integers.
Let K be a field and $q, Q_1, \ldots, Q_r$ invertible elements of K. Set $\mathbf{H} := \mathbf{H}_{r,n}(q, Q_1, \ldots, Q_r)$ as the corresponding Ariki-Koike algebra and $\mathcal{H} := \mathbf{H}(\mathfrak{S}_n, q) \leq \mathbf{H}$ to be its subalgebra generated by $S_1, \ldots, S_{n-1}$. Let $\lambda \vdash_r n$ be a multipartition with $\lambda = (\emptyset, \ldots, \emptyset, \lambda^{(r)})$. Then the following holds.

a) The restriction $\mathrm{Res}^{\mathbf{H}}_{\mathcal{H}}\left(S^{\lambda}\right)$ is isomorphic to the Specht module $S^{(\lambda^{(r)})}$ of $\mathcal{H}$.

b) The restriction $\mathrm{Res}^{\mathbf{H}}_{\mathcal{H}}\left(D^{\lambda}\right)$ is isomorphic to the $\mathcal{H}$-module $D^{(\lambda^{(r)})}$. ∎

Proof Set $\alpha := \lambda^{(r)}$. We can identify the set of tableaux of shape α with the set of tableaux of shape λ, as we can trivially identify $[\lambda]$ with $[\alpha]$. This identification maps standard tableaux to standard tableaux. In particular, it identifies $\mathfrak{t}^{\alpha}$ with $\mathfrak{t}^{\lambda}$.
From the structure of λ it follows that $u_\lambda = 1$ and therefore $m_\lambda = x_\lambda$. Once more from the structure of λ we see that $\mathfrak{S}_\lambda = \mathfrak{S}_\alpha$ and therefore $m_\lambda = m'_\alpha$.
If $\mathfrak{s}$ is a tableaux of shape λ, then by the above identification $\mathfrak{s}$ can be seen as a tableaux of shape α and the two notions of $d(\mathfrak{s})$ coincide. Therefore, it is $m_{\mathfrak{st}} = m'_{\mathfrak{st}}$ for standard tableaux $\mathfrak{s}$ and $\mathfrak{t}$ of shape λ.
By the bases given above for $\mathbf{H}$, $\mathbf{H}^{\lambda}$, $\mathcal{H}$, and $\mathcal{H}^{\alpha}$ it is easy to see that $\mathbf{H}^{\lambda} \cap \mathcal{H} = \mathcal{H}^{\alpha}$.
The identification of $\mathrm{Std}(\lambda)$ with $\mathrm{Std}(\alpha)$ and the respective bases for S^{λ} and S^{α} above imply that

$$\Psi : S^{\lambda} \to S^{\alpha};\ m_{\mathfrak{t}^{\lambda},\mathfrak{s}} + \mathbf{H}^{\lambda} \mapsto m_{\mathfrak{t}^{\lambda},\ \mathfrak{s}} + \mathcal{H}^{\alpha}$$

is a K-vector space isomorphism.
We check that Ψ is compatible with the $\mathcal{H}$-action on both sides.
Let $\mathfrak{s}$ be a standard tableau of shape λ and h in $\mathcal{H}$. Then we have

$$m_{\mathfrak{t}^{\lambda}\mathfrak{s}}\, h + \mathbf{H}^{\lambda} = \sum_{\mathfrak{u} \in \mathrm{Std}(\lambda)} a_{\mathfrak{u}} m_{\mathfrak{t}^{\lambda}\mathfrak{u}} + \mathbf{H}^{\lambda}$$

for some $a_{\mathfrak{u}}$ in K, and therefore $z := (m_{\mathfrak{t}^{\lambda}\mathfrak{s}}\, h) - \sum_{\mathfrak{u}\in\mathrm{Std}(\lambda)} a_{\mathfrak{u}} m_{\mathfrak{t}^{\lambda}\mathfrak{u}}$ is in $\mathbf{H}^{\lambda}$. But the $m_{\mathfrak{t}^{\lambda}\mathfrak{u}}$ all lie in $\mathcal{H}$ and so does $m_{\mathfrak{t}^{\lambda}\mathfrak{s}}\, h$. Thus, the element z is in $\mathbf{H}^{\lambda} \cap \mathcal{H} = \mathcal{H}^{\alpha}$ and therefore

$$m_{\mathfrak{t}^{\lambda}\mathfrak{s}}\, h + \mathcal{H}^{\alpha} = \sum_{\mathfrak{u} \in \mathrm{Std}(\lambda)} a_{\mathfrak{u}} m_{\mathfrak{t}^{\lambda}\mathfrak{u}} + \mathcal{H}^{\alpha}$$

and Ψ is an isomorphism of $\mathcal{H}$-modules.

Now for the modules D^{λ}.
The bilinear form on S^{λ} is defined by

$$\langle m_{\mathfrak{t}^{\lambda}\mathfrak{s}} + \mathbf{H}^{\lambda},\ m_{\mathfrak{t}^{\lambda}\mathfrak{t}} + \mathbf{H}^{\lambda}\rangle_{\lambda}\, m_{\lambda} \equiv m_{\mathfrak{t}^{\lambda}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^{\lambda}} \pmod{\mathbf{H}^{\lambda}},$$

whereas that on S^α is defined by

$$\langle m_{\mathfrak{t}^\lambda\mathfrak{s}} + \mathcal{H}^\alpha,\, m_{\mathfrak{t}^\lambda\mathfrak{t}} + \mathcal{H}^\alpha\rangle_\alpha\, m_\lambda \equiv m_{\mathfrak{t}^\lambda\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^\lambda} \pmod{\mathcal{H}^\alpha}.$$

Now m_λ, $m_{\mathfrak{t}^\lambda\mathfrak{s}}$, and $m_{\mathfrak{t}\mathfrak{t}^\lambda}$ all lie in $\mathcal{H}$. Hence, for $k \in K$ we see that $km_\lambda - m_{\mathfrak{t}^\lambda\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^\lambda}$ is in $\mathbf{H}^\lambda$ if and only if it is in $\mathbf{H}^\lambda \cap \mathcal{H} = \mathcal{H}^\alpha$. This shows that

$$\langle m_{\mathfrak{t}^\lambda\mathfrak{s}} + \mathbf{H}^\lambda,\, m_{\mathfrak{t}^\lambda\mathfrak{t}} + \mathbf{H}^\lambda\rangle_\lambda = \langle m_{\mathfrak{t}^\lambda\mathfrak{s}} + \mathcal{H}^\alpha,\, m_{\mathfrak{t}^\lambda\mathfrak{t}} + \mathcal{H}^\alpha\rangle_\alpha,$$

and thus Ψ is compatible with the bilinear forms and therefore also with their radicals. ■

Appendix C.

Decomposition Numbers of Iwahori-Hecke Algebras of Type F_4, E_6, and E_7 in bad Characteristic

We give the results of the computation described in Section 9.1, i.e. we print the decomposition matrices of the $\mathbb{F}_p$-generic decomposition maps $d_{\mathbb{K}}^{\Bbbk}$ and of the adjustment maps $d_{\Bbbk}^{\mathbb{L}}$ for Iwahori-Hecke algebras of type F_4, E_6, and F_7 for bad primes p. Assume the setting and notation of Section 9.1.

While the decomposition maps $d_{\mathbb{K}}^{\Bbbk}$ have already been published by Gyoja (with one mistake, cf. Remark 9.1.16), it takes some effort to extract the decomposition matrices from [Gyo96]: To use the results as Gyoja published them one has to first associate the irreducible $\mathbb{K}\mathcal{H}$-modules to corresponding elements of the so-called *associated finite groups* as given in [Lus84, §4]. This is rather cumbersome to do, so for convenience we print the decomposition matrices below.

Let us explain our notation. As decomposition matrices can be arranged to have block diagonal shape we only give these diagonal blocks, and we omit all trivial blocks, i.e. 1×1-blocks of the shape (1). The names by which we refer to the irreducible $\mathbb{K}\mathcal{H}$-modules are those used in CHEVIE. We will replace every 0 in the matrices by the symbol $\cdot$ to improve legibility.

Example C.1 As an example on how to read the following results consider the matrices given in Table C.2. As explained above, these results given there imply that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is a diagonal block of the decomposition matrix $d_{\mathbb{K}}^{\Bbbk}$ whose rows correspond to the irreducible $\mathbb{K}\mathcal{H}$-modules $\varphi'_{6,6}$, $\varphi''_{6,6}$, and $\varphi_{12,4}$. Furthermore, all other diagonal blocks have shape (1). Therefore, there exist two different irreducible $\Bbbk\mathcal{H}$-modules, say $\widehat{\varphi'_{6,6}}$ and $\widehat{\varphi''_{6,6}}$, such that $d_{\mathbb{K}}^{\Bbbk}([\varphi'_{6,6}]) = [\widehat{\varphi'_{6,6}}]$, $d_{\mathbb{K}}^{\Bbbk}([\varphi''_{6,6}]) = [\widehat{\varphi''_{6,6}}]$, and $d_{\mathbb{K}}^{\Bbbk}([\varphi_{12,4}]) = [\widehat{\varphi'_{6,6}}] + [\widehat{\varphi''_{6,6}}]$. For all irreducible $\mathbb{K}\mathcal{H}$-modules φ with $\varphi \notin \{\varphi'_{6,6}, \varphi''_{6,6}, \varphi_{12,4}\}$ there exists a unique irreducible $\Bbbk\mathcal{H}$-module $\widehat{\varphi}$ with $d_{\mathbb{K}}^{\Bbbk}[\varphi] = [\widehat{\varphi}]$ and $\widehat{\varphi} \notin \{\widehat{\varphi'_{6,6}}, \widehat{\varphi''_{6,6}}\}$. Furthermore, if ψ is an irreducible $\mathbb{K}\mathcal{H}$-module which is not isomorphic to φ and with $\psi \notin \{\varphi'_{6,6}, \varphi''_{6,6}, \varphi_{12,4}\}$, then $\widehat{\psi} \not\cong \widehat{\varphi}$.

We also print the non-trivial diagonal blocks of the decomposition maps $d_{\Bbbk}^{\mathbb{L}}$ whenever these maps are not trivial isomorphisms. By Theorem 9.1.17, we know that $d_{\Bbbk}^{\mathbb{L}}$ is trivial, unless $\varepsilon = \theta_{\mathbf{K},\mathbb{L}}(V)$ is a root of a polynomial g specified in said theorem. By the remarks in part e) of the Computational Steps 9.1.14, we only have to consider one root for every irreducible factor of g. Therefore, we give the diagonal blocks of the decomposition matrix of $d_{\Bbbk}^{\mathbf{L}}$ for every such irreducible factor of g.

For this, we have to give names to the irreducible modules of $\Bbbk\mathcal{H}$, which will index the rows of these blocks. We do this by the following convention: As can be seen from the $\mathbb{F}_p$-generic decomposition numbers, for nearly every irreducible $\Bbbk\mathcal{H}$-module $\widehat{\varphi}$ we can choose some irreducible $\mathbb{K}\mathcal{H}$-module φ such that $d_{\mathbb{K}}^{\Bbbk}([\varphi]) = [\widehat{\varphi}]$. This works in all but three cases. Hence, we have to introduce the following additional convention:

- For $W \cong E_6$ and $p = 3$ we denote by $\widehat{\varphi_{90,8}}$ the unique irreducible module that is a constituent of the specialisation of $\varphi_{90,8}$ and not isomorphic to $\widehat{\varphi_{20,10}}$.

- For $W \cong E_7$ and $p = 3$ we denote by $\widehat{\varphi_{280,9}}$ the unique irreducible module that is a constituent of the specialisation of $\varphi_{280,9}$ and not isomorphic to $\widehat{\varphi_{35,13}}$.

- For $W \cong E_7$ and $p = 3$ we denote by $\widehat{\varphi_{280,18}}$ the unique irreducible module that is a constituent of the specialisation of $\varphi_{280,18}$ and not isomorphic to $\widehat{\varphi_{35,22}}$.

Finally, note that some of the matrices were too large to print in the usual fashion, as they have too many rows. We split these matrices horizontally and indicate by double vertical lines where to glue them back together again.

C.1. $\mathbb{F}_p$-Generic Decomposition Matrices

Table C.1.: Non-trivial blocks of the $\mathbb{F}_2$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = F_4$

$\varphi''_{1,12}$	1	·	·	·	·
$\varphi'_{1,12}$	·	1	·	·	·
$\varphi_{4,8}$	·	·	1	·	·
$\varphi''_{9,6}$	1	·	1	1	·
$\varphi'_{9,6}$	·	1	1	·	1
$\varphi''_{6,6}$	1	1	1	·	·
$\varphi_{12,4}$	·	·	1	1	1
$\varphi''_{4,7}$	·	·	·	1	·
$\varphi'_{4,7}$	·	·	·	·	1
$\varphi_{16,5}$	·	·	2	1	1

$\varphi''_{2,4}$	1	·
$\varphi'_{2,4}$	·	1
$\varphi_{4,1}$	1	1

$\varphi'_{2,16}$	1	·
$\varphi''_{2,16}$	·	1
$\varphi_{4,13}$	1	1

Table C.2.: Non-trivial blocks of the $\mathbb{F}_3$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = F_4$

$\varphi'_{6,6}$	1	·
$\varphi''_{6,6}$	·	1
$\varphi_{12,4}$	1	1

Table C.3.: Non-trivial blocks of the $\mathbb{F}_2$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = E_6$

$\varphi_{20,10}$	1	·
$\varphi_{60,8}$	·	1
$\varphi_{80,7}$	1	1

$\varphi_{15,5}$	1	·
$\varphi_{15,4}$	·	1
$\varphi_{30,3}$	1	1

$\varphi_{15,17}$	1	·
$\varphi_{15,16}$	·	1
$\varphi_{30,15}$	1	1

Table C.4.: Non-trivial blocks of the $\mathbb{F}_3$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = E_6$

$\varphi_{10,9}$	1	·	·
$\varphi_{20,10}$	·	1	·
$\varphi_{80,7}$	1	·	1
$\varphi_{90,8}$	·	1	1

Table C.5.: Non-trivial blocks of the $\mathbb{F}_2$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = E_7$

$\varphi_{15,28}$	1	·
$\varphi_{105,26}$	·	1
$\varphi_{120,25}$	1	1

$\varphi_{15,7}$	1	·
$\varphi_{105,5}$	·	1
$\varphi_{120,4}$	1	1

$\varphi_{21,6}$	1	·
$\varphi_{35,4}$	·	1
$\varphi_{56,3}$	1	1

$\varphi_{21,33}$	1	·
$\varphi_{35,31}$	·	1
$\varphi_{56,30}$	1	1

$\varphi_{35,22}$	1	·
$\varphi_{280,17}$	·	1
$\varphi_{315,16}$	1	1

$\varphi_{35,13}$	1	·
$\varphi_{280,8}$	·	1
$\varphi_{315,7}$	1	1

$\varphi_{84,12}$	1	·
$\varphi_{336,11}$	·	1
$\varphi_{420,10}$	1	1

$\varphi_{84,15}$	1	·
$\varphi_{336,14}$	·	1
$\varphi_{420,13}$	1	1

$\varphi_{189,10}$	1	·
$\varphi_{216,9}$	·	1
$\varphi_{405,8}$	1	1

$\varphi_{189,17}$	1	·
$\varphi_{216,16}$	·	1
$\varphi_{405,15}$	1	1

$\varphi_{512,12}$	1
$\varphi_{512,11}$	1

Table C.6.: Non-trivial blocks of the $\mathbb{F}_3$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = E_7$

$\varphi_{35,22}$	1	·	·
$\varphi_{70,18}$	·	1	·
$\varphi_{280,18}$	1	·	1
$\varphi_{315,16}$	·	1	1

$\varphi_{35,13}$	1	·	·
$\varphi_{70,9}$	·	1	·
$\varphi_{280,9}$	1	·	1
$\varphi_{315,7}$	·	1	1

C.2. Adjustment Matrices

Table C.7.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=F_4, p=2, and ε a root of $g := y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi''_{1,12}}$	1	·	·	·
$\widehat{\varphi'_{1,12}}$	1	·	·	·
$\widehat{\varphi_{1,24}}$	1	·	·	·
$\widehat{\varphi''_{2,4}}$	·	1	·	·
$\widehat{\varphi'_{2,16}}$	·	1	·	·
$\widehat{\varphi'_{2,4}}$	·	·	1	·
$\widehat{\varphi''_{2,16}}$	·	·	1	·
$\widehat{\varphi_{4,8}}$	·	·	·	1
$\widehat{\varphi_{9,2}}$	1	1	1	1
$\widehat{\varphi''_{4,7}}$	·	1	1	·
$\widehat{\varphi'_{4,7}}$	·	1	1	·
$\widehat{\varphi_{9,10}}$	1	1	1	1
$\widehat{\varphi'_{6,6}}$	2	·	·	1
$\widehat{\varphi''_{8,3}}$	2	1	·	1
$\widehat{\varphi'_{8,9}}$	2	1	·	1
$\widehat{\varphi'_{8,3}}$	2	·	1	1
$\widehat{\varphi''_{8,9}}$	2	·	1	1

Table C.8.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=F_4, p=2, and ε a root of $g := y^2 + y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi''_{1,12}}$	·	1	·	·	·	·
$\widehat{\varphi'_{1,12}}$	·	·	1	·	·	·
$\widehat{\varphi_{1,24}}$	·	·	·	1	·	·
$\widehat{\varphi''_{2,4}}$	1	1	·	·	·	·
$\widehat{\varphi'_{2,16}}$	·	·	1	1	·	·
$\widehat{\varphi'_{2,4}}$	1	·	1	·	·	·
$\widehat{\varphi''_{2,16}}$	·	1	·	1	·	·
$\widehat{\varphi_{4,8}}$	1	1	1	1	·	·
$\widehat{\varphi''_{4,7}}$	·	·	·	·	1	·
$\widehat{\varphi'_{4,7}}$	·	·	·	·	·	1
$\widehat{\varphi''_{8,3}}$	2	1	1	·	1	·
$\widehat{\varphi'_{8,9}}$	·	1	1	2	·	1
$\widehat{\varphi'_{8,3}}$	2	1	1	·	·	1
$\widehat{\varphi''_{8,9}}$	·	1	1	2	1	·

Table C.9.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=F_4, p=3, and ε a root of $g := y^2 + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi''_{1,12}}$	1	·	·	·
$\widehat{\varphi'_{1,12}}$	1	·	·	·
$\widehat{\varphi_{1,24}}$	1	·	·	·
$\widehat{\varphi''_{2,4}}$	·	1	·	·
$\widehat{\varphi'_{2,16}}$	·	1	·	·
$\widehat{\varphi'_{2,4}}$	·	·	1	·
$\widehat{\varphi''_{2,16}}$	·	·	1	·
$\widehat{\varphi_{9,2}}$	·	1	1	1
$\widehat{\varphi''_{9,6}}$	·	1	1	1
$\widehat{\varphi'_{9,6}}$	·	1	1	1
$\widehat{\varphi_{9,10}}$	·	1	1	1
$\widehat{\varphi'_{6,6}}$	1	·	·	1
$\widehat{\varphi''_{6,6}}$	1	·	·	1
$\widehat{\varphi''_{8,3}}$	1	1	·	1
$\widehat{\varphi'_{8,9}}$	1	1	·	1
$\widehat{\varphi'_{8,3}}$	1	·	1	1
$\widehat{\varphi''_{8,9}}$	1	·	1	1

$\widehat{\varphi_{4,1}}$	1
$\widehat{\varphi''_{4,7}}$	1
$\widehat{\varphi'_{4,7}}$	1
$\widehat{\varphi_{4,13}}$	1

Table C.10.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=F_4, p=3, and ε a root of $g \in \{y + 2,\ y + 1\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi''_{1,12}}$	·	1	·	·
$\widehat{\varphi'_{1,12}}$	·	·	1	·
$\widehat{\varphi_{1,24}}$	·	·	·	1
$\widehat{\varphi''_{2,4}}$	1	1	·	·
$\widehat{\varphi'_{2,16}}$	·	·	1	1
$\widehat{\varphi'_{2,4}}$	1	·	1	·
$\widehat{\varphi''_{2,16}}$	·	1	·	1
$\widehat{\varphi_{4,8}}$	1	1	1	1

$\widehat{\varphi_{4,1}}$	1	·	·	·
$\widehat{\varphi''_{4,7}}$	·	1	·	·
$\widehat{\varphi'_{4,7}}$	·	·	1	·
$\widehat{\varphi_{4,13}}$	·	·	·	1
$\widehat{\varphi''_{8,3}}$	1	1	·	·
$\widehat{\varphi'_{8,9}}$	·	·	1	1
$\widehat{\varphi'_{8,3}}$	1	·	1	·
$\widehat{\varphi''_{8,9}}$	·	1	·	1
$\widehat{\varphi_{16,5}}$	1	1	1	1

Table C.11.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=F_4, p=3, and ε a root of $g \in \{y^2 + 2y + 2,\ y^2 + y + 2\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,24}}$	·	1	·	·	·
$\widehat{\varphi_{4,8}}$	·	·	1	·	·
$\widehat{\varphi_{9,2}}$	2	·	1	1	·
$\widehat{\varphi_{9,10}}$	·	2	1	·	1
$\widehat{\varphi'_{6,6}}$	1	1	1	·	·
$\widehat{\varphi''_{6,6}}$	·	·	·	1	1
$\widehat{\varphi_{4,1}}$	1	·	·	1	·
$\widehat{\varphi_{4,13}}$	·	1	·	·	1

$\widehat{\varphi''_{2,4}}$	1	·
$\widehat{\varphi''_{2,16}}$	·	1
$\widehat{\varphi''_{4,7}}$	1	1

$\widehat{\varphi'_{2,16}}$	1	·
$\widehat{\varphi'_{2,4}}$	·	1
$\widehat{\varphi'_{4,7}}$	1	1

Table C.12.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=F_4, p=3, and ε a root of $g \in \{y^4 + 2y^2 + 2,\ y^4 + y^2 + 2\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi_{1,24}}$	·	1	·	·
$\widehat{\varphi_{9,2}}$	1	·	1	·
$\widehat{\varphi_{9,10}}$	·	1	·	1
$\widehat{\varphi_{16,5}}$	·	·	1	1

Table C.13.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_6, p=2, and ε a root of $g := y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,36}}$	1	·	·	·	·
$\widehat{\varphi_{10,9}}$	2	1	·	·	·
$\widehat{\varphi_{6,1}}$	·	·	1	·	·
$\widehat{\varphi_{6,25}}$	·	·	1	·	·
$\widehat{\varphi_{20,10}}$	·	1	2	·	·
$\widehat{\varphi_{15,5}}$	1	·	·	1	·
$\widehat{\varphi_{15,17}}$	1	·	·	1	·
$\widehat{\varphi_{15,4}}$	1	1	1	·	·
$\widehat{\varphi_{15,16}}$	1	1	1	·	·
$\widehat{\varphi_{20,2}}$	·	·	1	1	·
$\widehat{\varphi_{20,20}}$	·	·	1	1	·
$\widehat{\varphi_{24,6}}$	2	1	·	1	·
$\widehat{\varphi_{24,12}}$	2	1	·	1	·
$\widehat{\varphi_{60,8}}$	·	1	2	·	1
$\widehat{\varphi_{90,8}}$	2	1	2	2	1
$\widehat{\varphi_{60,5}}$	·	·	1	1	1
$\widehat{\varphi_{60,11}}$	·	·	1	1	1
$\widehat{\varphi_{81,6}}$	1	1	3	1	1
$\widehat{\varphi_{81,10}}$	1	1	3	1	1

$\widehat{\varphi_{64,4}}$	1
$\widehat{\varphi_{64,13}}$	1

Table C.14.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_6, p=2, and ε a root of $g := y^2 + y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·
$\widehat{\varphi_{1,36}}$	·	1	·	·	·	·	·	·
$\widehat{\varphi_{10,9}}$	·	·	1	1	·	·	·	·
$\widehat{\varphi_{6,1}}$	1	·	·	1	·	·	·	·
$\widehat{\varphi_{6,25}}$	·	1	1	·	·	·	·	·
$\widehat{\varphi_{20,10}}$	·	·	·	·	1	1	·	·
$\widehat{\varphi_{15,5}}$	·	·	·	1	·	1	·	·
$\widehat{\varphi_{15,17}}$	·	·	1	·	1	·	·	·
$\widehat{\varphi_{15,4}}$	1	1	·	·	·	·	1	·
$\widehat{\varphi_{15,16}}$	1	1	·	·	·	·	·	1
$\widehat{\varphi_{20,2}}$	2	·	·	1	·	·	1	·
$\widehat{\varphi_{20,20}}$	·	2	1	·	·	·	·	1
$\widehat{\varphi_{24,6}}$	1	·	·	·	1	·	1	·
$\widehat{\varphi_{24,12}}$	·	1	·	·	·	1	·	1
$\widehat{\varphi_{60,8}}$	2	2	1	1	1	1	1	1
$\widehat{\varphi_{60,5}}$	2	2	1	1	1	1	2	·
$\widehat{\varphi_{60,11}}$	2	2	1	1	1	1	·	2
$\widehat{\varphi_{64,4}}$	2	1	·	1	1	2	2	·
$\widehat{\varphi_{64,13}}$	1	2	1	·	2	1	·	2

Table C.15.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_6, p=2, and ε a root of $g := y^4 + y^3 + y^2 + y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi_{6,25}}$	·	1	·	·
$\widehat{\varphi_{24,6}}$	1	·	1	·
$\widehat{\varphi_{64,13}}$	·	1	·	1
$\widehat{\varphi_{81,10}}$	·	·	1	1

$\widehat{\varphi_{1,36}}$	1	·	·	·
$\widehat{\varphi_{6,1}}$	·	1	·	·
$\widehat{\varphi_{24,12}}$	1	·	1	·
$\widehat{\varphi_{64,4}}$	·	1	·	1
$\widehat{\varphi_{81,6}}$	·	·	1	1

Table C.16.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_6, p=2, and ε a root of $g := y^6 + y^3 + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{1,36}}$	·	1	·	·	·	·
$\widehat{\varphi_{20,2}}$	1	·	1	·	·	·
$\widehat{\varphi_{20,20}}$	·	1	·	1	·	·
$\widehat{\varphi_{90,8}}$	·	·	·	·	1	1
$\widehat{\varphi_{64,4}}$	·	·	1	·	·	1
$\widehat{\varphi_{64,13}}$	·	·	·	1	1	·

Table C.17.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_6, p=3, and ε a root of $g := y^2 + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{1,36}}$	1	·	·	·	·	·
$\widehat{\varphi_{10,9}}$	·	1	·	·	·	·
$\widehat{\varphi_{6,1}}$	·	·	1	·	·	·
$\widehat{\varphi_{6,25}}$	·	·	1	·	·	·
$\widehat{\varphi_{20,10}}$	·	·	1	1	·	·
$\widehat{\varphi_{15,5}}$	2	·	·	·	1	·
$\widehat{\varphi_{15,17}}$	2	·	·	·	1	·
$\widehat{\varphi_{15,4}}$	1	·	·	1	·	·
$\widehat{\varphi_{15,16}}$	1	·	·	1	·	·
$\widehat{\varphi_{20,2}}$	1	·	1	·	1	·
$\widehat{\varphi_{20,20}}$	1	·	1	·	1	·
$\widehat{\varphi_{24,6}}$	1	1	·	·	1	·
$\widehat{\varphi_{24,12}}$	1	1	·	·	1	·
$\widehat{\varphi_{30,3}}$	1	1	1	·	1	·
$\widehat{\varphi_{30,15}}$	1	1	1	·	1	·
$\widehat{\varphi_{60,8}}$	·	·	·	2	·	1
$\widehat{\varphi_{90,8}}$	2	1	·	·	2	1
$\widehat{\varphi_{60,5}}$	1	·	·	1	1	1
$\widehat{\varphi_{60,11}}$	1	·	·	1	1	1
$\widehat{\varphi_{81,6}}$	2	·	1	2	1	1
$\widehat{\varphi_{81,10}}$	2	·	1	2	1	1

$\widehat{\varphi_{64,4}}$	1
$\widehat{\varphi_{64,13}}$	1

Table C.18.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_6, p=3, and ε a root of $g \in \{y+1,\ y+2\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{1,36}}$	·	1	·	·	·	·	·	·	·	·
$\widehat{\varphi_{10,9}}$	·	·	1	1	·	·	·	·	·	·
$\widehat{\varphi_{6,1}}$	1	·	·	1	·	·	·	·	·	·
$\widehat{\varphi_{6,25}}$	·	1	1	·	·	·	·	·	·	·
$\widehat{\varphi_{20,10}}$	·	·	·	·	1	1	·	·	·	·
$\widehat{\varphi_{15,5}}$	·	·	·	1	·	1	·	·	·	·
$\widehat{\varphi_{15,17}}$	·	·	1	·	1	·	·	·	·	·
$\widehat{\varphi_{15,4}}$	·	1	·	·	·	·	1	·	·	·
$\widehat{\varphi_{15,16}}$	1	·	·	·	·	·	·	1	·	·
$\widehat{\varphi_{20,2}}$	1	·	·	1	·	·	1	·	·	·
$\widehat{\varphi_{20,20}}$	·	1	1	·	·	·	·	1	·	·
$\widehat{\varphi_{24,6}}$	·	·	·	·	1	·	1	·	·	·
$\widehat{\varphi_{24,12}}$	·	·	·	·	·	1	·	1	·	·
$\widehat{\varphi_{30,3}}$	·	·	·	1	·	·	·	·	1	·
$\widehat{\varphi_{30,15}}$	·	·	1	·	·	·	·	·	·	1
$\widehat{\varphi_{60,8}}$	1	1	1	1	1	1	1	1	·	·
$\widehat{\varphi_{90,8}}$	·	·	·	·	1	1	·	·	1	1
$\widehat{\varphi_{60,5}}$	·	1	1	1	1	·	1	·	1	·
$\widehat{\varphi_{60,11}}$	1	·	1	1	·	1	·	1	·	1
$\widehat{\varphi_{64,4}}$	·	·	·	1	1	1	1	·	1	·
$\widehat{\varphi_{64,13}}$	·	·	1	·	1	1	·	1	·	1

Table C.19.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_6, p=3, and ε a root of $g \in \{y^2+y+2,\ y^2+2y+2\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·
$\widehat{\varphi_{1,36}}$	·	1	·	·	·	·	·	·
$\widehat{\varphi_{10,9}}$	1	1	1	·	·	·	·	·
$\widehat{\varphi_{6,1}}$	1	·	·	1	·	·	·	·
$\widehat{\varphi_{6,25}}$	·	1	·	·	1	·	·	·
$\widehat{\varphi_{20,10}}$	·	·	·	·	·	1	1	·
$\widehat{\varphi_{15,5}}$	·	·	·	1	·	·	1	·
$\widehat{\varphi_{15,17}}$	·	·	·	·	1	1	·	·
$\widehat{\varphi_{15,4}}$	2	·	1	1	·	·	·	·
$\widehat{\varphi_{15,16}}$	·	2	1	·	1	·	·	·
$\widehat{\varphi_{90,8}}$	·	·	·	1	1	·	·	1
$\widehat{\varphi_{81,6}}$	1	·	·	2	·	·	1	1
$\widehat{\varphi_{81,10}}$	·	1	·	·	2	1	·	1

$\widehat{\varphi_{20,2}}$	1	·	·
$\widehat{\varphi_{20,20}}$	·	1	·
$\widehat{\varphi_{60,5}}$	1	·	1
$\widehat{\varphi_{60,11}}$	·	1	1

Table C.20.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_6, p=3, and ε a root of $g \in \{y^4 + y^3 + y^2 + y + 1,\ y^4 + 2y^3 + y^2 + 2y + 1\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi_{6,25}}$	·	1	·	·
$\widehat{\varphi_{24,6}}$	1	·	1	·
$\widehat{\varphi_{64,13}}$	·	1	·	1
$\widehat{\varphi_{81,10}}$	·	·	1	1

$\widehat{\varphi_{1,36}}$	1	·	·	·
$\widehat{\varphi_{6,1}}$	·	1	·	·
$\widehat{\varphi_{24,12}}$	1	·	1	·
$\widehat{\varphi_{64,4}}$	·	1	·	1
$\widehat{\varphi_{81,6}}$	·	·	1	1

Table C.21.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_6, p=3, and ε a root of $g \in \{y^4 + 2y^2 + 2,\ y^4 + y^2 + 2\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,36}}$	·	1	·	·	·
$\widehat{\varphi_{30,3}}$	1	·	1	·	·
$\widehat{\varphi_{30,15}}$	·	1	·	1	·
$\widehat{\varphi_{81,6}}$	·	·	1	·	1
$\widehat{\varphi_{81,10}}$	·	·	·	1	1

Table C.22.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_7, p=2, and ε a root of $g := y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{1,63}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{7,46}}$	1	1	·	·	·	·	·
$\widehat{\varphi_{7,1}}$	1	1	·	·	·	·	·
$\widehat{\varphi_{15,28}}$	1	1	1	·	·	·	·
$\widehat{\varphi_{15,7}}$	1	1	1	·	·	·	·
$\widehat{\varphi_{21,6}}$	1	1	·	1	·	·	·
$\widehat{\varphi_{21,33}}$	1	1	·	1	·	·	·
$\widehat{\varphi_{21,36}}$	1	1	·	1	·	·	·
$\widehat{\varphi_{21,3}}$	1	1	·	1	·	·	·
$\widehat{\varphi_{27,2}}$	1	2	·	1	·	·	·
$\widehat{\varphi_{27,37}}$	1	2	·	1	·	·	·
$\widehat{\varphi_{35,22}}$	1	2	1	1	·	·	·
$\widehat{\varphi_{35,13}}$	1	2	1	1	·	·	·
$\widehat{\varphi_{35,4}}$	1	2	1	1	·	·	·
$\widehat{\varphi_{35,31}}$	1	2	1	1	·	·	·
$\widehat{\varphi_{70,18}}$	2	1	·	1	1	·	·
$\widehat{\varphi_{70,9}}$	2	1	·	1	1	·	·
$\widehat{\varphi_{84,12}}$	2	2	1	1	1	·	·
$\widehat{\varphi_{84,15}}$	2	2	1	1	1	·	·
$\widehat{\varphi_{105,26}}$	1	2	·	2	·	1	·
$\widehat{\varphi_{105,5}}$	1	2	·	2	·	1	·
$\widehat{\varphi_{105,6}}$	3	3	1	2	1	·	·
$\widehat{\varphi_{105,21}}$	3	3	1	2	1	·	·
$\widehat{\varphi_{105,12}}$	3	3	1	2	1	·	·
$\widehat{\varphi_{105,15}}$	3	3	1	2	1	·	·
$\widehat{\varphi_{168,6}}$	2	2	·	3	1	1	·
$\widehat{\varphi_{168,21}}$	2	2	·	3	1	1	·
$\widehat{\varphi_{189,10}}$	3	4	1	3	1	1	·
$\widehat{\varphi_{189,17}}$	3	4	1	3	1	1	·
$\widehat{\varphi_{189,22}}$	3	4	1	3	1	1	·
$\widehat{\varphi_{189,5}}$	3	4	1	3	1	1	·
$\widehat{\varphi_{189,20}}$	3	4	1	3	1	1	·
$\widehat{\varphi_{189,7}}$	3	4	1	3	1	1	·
$\widehat{\varphi_{210,6}}$	4	5	1	4	1	1	·
$\widehat{\varphi_{210,21}}$	4	5	1	4	1	1	·
$\widehat{\varphi_{210,10}}$	2	3	2	1	1	·	1
$\widehat{\varphi_{210,13}}$	2	3	2	1	1	·	1
$\widehat{\varphi_{216,16}}$	2	3	1	2	1	·	1
$\widehat{\varphi_{216,9}}$	2	3	1	2	1	·	1
$\widehat{\varphi_{280,18}}$	6	6	1	5	2	1	·
$\widehat{\varphi_{280,9}}$	6	6	1	5	2	1	·
$\widehat{\varphi_{280,8}}$	2	3	1	2	1	1	1
$\widehat{\varphi_{280,17}}$	2	3	1	2	1	1	1
$\widehat{\varphi_{336,14}}$	4	6	2	4	1	1	1
$\widehat{\varphi_{336,11}}$	4	6	2	4	1	1	1
$\widehat{\varphi_{378,14}}$	4	5	2	4	2	1	1
$\widehat{\varphi_{378,9}}$	4	5	2	4	2	1	1

Table C.23.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_7, p=2, and ε a root of $g := y^2 + y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{1,63}}$	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{7,46}}$	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{7,1}}$	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{15,28}}$	1	1	·	·	1	·	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{15,7}}$	1	1	·	·	·	1	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{21,6}}$	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
$\widehat{\varphi_{21,33}}$	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,36}}$	·	1	1	·	1	·	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{21,3}}$	1	·	·	1	·	1	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{35,22}}$	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
$\widehat{\varphi_{35,13}}$	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
$\widehat{\varphi_{35,4}}$	1	·	·	1	·	·	·	·	·	·	1	·	·	·	·
$\widehat{\varphi_{35,31}}$	·	1	1	·	·	·	·	·	·	·	·	1	·	·	·
$\widehat{\varphi_{70,18}}$	·	1	·	·	·	·	1	1	·	·	·	1	·	·	·
$\widehat{\varphi_{70,9}}$	1	·	·	·	·	·	1	1	·	·	1	·	·	·	·
$\widehat{\varphi_{84,12}}$	1	1	·	1	1	·	·	·	1	·	1	·	·	·	·
$\widehat{\varphi_{84,15}}$	1	1	1	·	·	1	·	·	·	1	·	1	·	·	·
$\widehat{\varphi_{105,26}}$	·	2	1	·	1	·	·	1	1	·	·	1	·	·	·
$\widehat{\varphi_{105,5}}$	2	·	·	1	·	1	1	·	·	1	1	·	·	·	·
$\widehat{\varphi_{105,6}}$	2	1	1	1	·	2	·	·	·	1	1	·	·	·	·
$\widehat{\varphi_{105,21}}$	1	2	1	1	2	·	·	·	1	·	·	1	·	·	·
$\widehat{\varphi_{105,12}}$	·	1	·	·	1	·	·	·	·	·	·	·	1	·	·
$\widehat{\varphi_{105,15}}$	1	·	·	·	·	1	·	·	·	·	·	·	·	1	·
$\widehat{\varphi_{168,6}}$	3	1	·	2	·	2	·	·	1	1	2	·	·	·	·
$\widehat{\varphi_{168,21}}$	1	3	2	·	2	·	·	·	1	1	·	2	·	·	·
$\widehat{\varphi_{210,6}}$	2	1	·	1	·	2	1	·	·	1	1	·	1	·	·
$\widehat{\varphi_{210,21}}$	1	2	1	·	2	·	·	1	1	·	·	1	·	1	·
$\widehat{\varphi_{210,10}}$	2	2	1	1	·	2	1	1	1	1	1	1	·	·	·
$\widehat{\varphi_{210,13}}$	2	2	1	1	2	·	1	1	1	1	1	1	·	·	·
$\widehat{\varphi_{280,18}}$	1	2	·	·	1	·	·	1	1	·	·	1	·	1	1
$\widehat{\varphi_{280,9}}$	2	1	·	·	·	1	1	·	·	1	1	·	1	·	1
$\widehat{\varphi_{280,8}}$	3	2	1	2	1	2	·	·	1	1	2	·	1	·	·
$\widehat{\varphi_{280,17}}$	2	3	2	1	2	1	·	·	1	1	·	2	·	1	·
$\widehat{\varphi_{336,14}}$	1	2	·	·	2	·	·	·	1	·	·	·	1	1	1
$\widehat{\varphi_{336,11}}$	2	1	·	·	·	2	·	·	·	1	·	·	1	1	1
$\widehat{\varphi_{512,12}}$	4	4	1	1	2	2	1	1	1	1	1	1	1	1	1

$\widehat{\varphi_{27,2}}$	1	·	·	·
$\widehat{\varphi_{189,17}}$	·	1	·	·
$\widehat{\varphi_{189,5}}$	1	·	1	·
$\widehat{\varphi_{189,20}}$	·	·	·	1
$\widehat{\varphi_{216,16}}$	1	·	·	1
$\widehat{\varphi_{378,9}}$	1	1	1	·

$\widehat{\varphi_{27,37}}$	1	·	·	·
$\widehat{\varphi_{189,10}}$	·	1	·	·
$\widehat{\varphi_{189,22}}$	1	·	1	·
$\widehat{\varphi_{189,7}}$	·	·	·	1
$\widehat{\varphi_{216,9}}$	1	·	·	1
$\widehat{\varphi_{378,14}}$	1	1	1	·

Table C.24.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_7, p=2, and ε a root of $g := y^4 + y^3 + y^2 + y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,33}}$	·	1	·	·	·	·	·	·
$\widehat{\varphi_{21,3}}$	1	·	1	·	·	·	·	·
$\widehat{\varphi_{35,31}}$	·	·	·	1	·	·	·	·
$\widehat{\varphi_{84,12}}$	1	·	·	·	1	·	·	·
$\widehat{\varphi_{189,17}}$	·	1	·	·	·	1	·	·
$\widehat{\varphi_{189,22}}$	·	1	·	1	·	·	1	·
$\widehat{\varphi_{189,7}}$	1	·	1	·	·	·	·	1
$\widehat{\varphi_{216,16}}$	·	·	·	·	1	·	1	·
$\widehat{\varphi_{336,11}}$	·	·	·	·	·	1	·	1

$\widehat{\varphi_{1,63}}$	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,6}}$	·	1	·	·	·	·	·	·
$\widehat{\varphi_{21,36}}$	1	·	1	·	·	·	·	·
$\widehat{\varphi_{35,4}}$	·	·	·	1	·	·	·	·
$\widehat{\varphi_{84,15}}$	1	·	·	·	1	·	·	·
$\widehat{\varphi_{189,10}}$	·	1	·	·	·	1	·	·
$\widehat{\varphi_{189,5}}$	·	1	·	1	·	·	1	·
$\widehat{\varphi_{189,20}}$	1	·	1	·	·	·	·	1
$\widehat{\varphi_{216,9}}$	·	·	·	·	1	·	1	·
$\widehat{\varphi_{336,14}}$	·	·	·	·	·	1	·	1

$\widehat{\varphi_{7,46}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{7,1}}$	·	1	·	·	·	·	·
$\widehat{\varphi_{27,2}}$	·	1	1	·	·	·	·
$\widehat{\varphi_{27,37}}$	1	·	·	1	·	·	·
$\widehat{\varphi_{168,6}}$	·	1	1	·	1	·	·
$\widehat{\varphi_{168,21}}$	1	·	·	1	·	1	·
$\widehat{\varphi_{378,14}}$	1	·	·	·	·	1	1
$\widehat{\varphi_{378,9}}$	·	1	·	·	1	·	1
$\widehat{\varphi_{512,12}}$	·	·	·	·	1	1	1

Table C.25.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_7, p=2, and ε a root of $g \in \{y^3 + y + 1,\ y^3 + y^2 + 1\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{1,63}}$	·	1	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{15,28}}$	·	·	1	·	·	·	·	·	·	·	·
$\widehat{\varphi_{15,7}}$	·	·	·	1	·	·	·	·	·	·	·
$\widehat{\varphi_{27,2}}$	1	·	·	·	1	·	·	·	·	·	·
$\widehat{\varphi_{27,37}}$	·	1	·	·	·	1	·	·	·	·	·
$\widehat{\varphi_{105,26}}$	·	·	·	·	·	1	1	·	·	·	·
$\widehat{\varphi_{105,5}}$	·	·	·	·	1	·	·	1	·	·	·
$\widehat{\varphi_{189,10}}$	·	·	·	·	·	·	·	1	1	·	·
$\widehat{\varphi_{189,17}}$	·	·	·	·	·	·	1	·	1	·	·
$\widehat{\varphi_{216,16}}$	·	·	1	·	·	·	·	·	·	1	·
$\widehat{\varphi_{216,9}}$	·	·	·	1	·	·	·	·	·	·	1
$\widehat{\varphi_{512,12}}$	·	·	·	·	·	·	·	·	1	1	1

Table C.26.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_7, p=2, and ε a root of $g := y^6 + y^3 + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{1,63}}$	·	1	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{7,46}}$	·	1	1	·	·	·	·	·	·	·	·
$\widehat{\varphi_{7,1}}$	1	·	·	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,6}}$	·	·	·	1	1	·	·	·	·	·	·
$\widehat{\varphi_{21,33}}$	·	·	1	·	·	1	·	·	·	·	·
$\widehat{\varphi_{35,22}}$	·	·	·	·	·	1	1	·	·	·	·
$\widehat{\varphi_{35,13}}$	·	·	·	·	1	·	1	·	·	·	·
$\widehat{\varphi_{35,4}}$	1	·	·	·	·	·	·	1	·	·	·
$\widehat{\varphi_{35,31}}$	·	1	·	·	·	·	·	·	1	·	·
$\widehat{\varphi_{280,8}}$	·	·	·	·	·	·	·	1	·	1	·
$\widehat{\varphi_{280,17}}$	·	·	·	·	·	·	·	·	1	·	1
$\widehat{\varphi_{512,12}}$	·	·	·	·	·	·	1	·	·	1	1

Table C.27.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_7, p=3, and ε a root of $g := y^2 + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{1,63}}$	1	·	·	·	·	·
$\widehat{\varphi_{7,46}}$	1	1	·	·	·	·
$\widehat{\varphi_{7,1}}$	1	1	·	·	·	·
$\widehat{\varphi_{15,28}}$	1	·	1	·	·	·
$\widehat{\varphi_{15,7}}$	1	·	1	·	·	·
$\widehat{\varphi_{21,6}}$	2	1	·	1	·	·
$\widehat{\varphi_{21,33}}$	2	1	·	1	·	·
$\widehat{\varphi_{21,36}}$	2	1	·	1	·	·
$\widehat{\varphi_{21,3}}$	2	1	·	1	·	·
$\widehat{\varphi_{27,2}}$	2	2	·	1	·	·
$\widehat{\varphi_{27,37}}$	2	2	·	1	·	·
$\widehat{\varphi_{35,22}}$	2	1	1	1	·	·
$\widehat{\varphi_{35,13}}$	2	1	1	1	·	·
$\widehat{\varphi_{35,4}}$	2	1	1	1	·	·
$\widehat{\varphi_{35,31}}$	2	1	1	1	·	·
$\widehat{\varphi_{70,18}}$	1	·	1	1	1	·
$\widehat{\varphi_{70,9}}$	1	·	1	1	1	·
$\widehat{\varphi_{84,12}}$	1	·	2	1	1	·
$\widehat{\varphi_{84,15}}$	1	·	2	1	1	·
$\widehat{\varphi_{105,26}}$	3	2	·	1	·	1
$\widehat{\varphi_{105,5}}$	3	2	·	1	·	1
$\widehat{\varphi_{105,6}}$	3	1	2	2	1	·
$\widehat{\varphi_{105,21}}$	3	1	2	2	1	·
$\widehat{\varphi_{105,12}}$	3	1	2	2	1	·
$\widehat{\varphi_{105,15}}$	3	1	2	2	1	·

$\widehat{\varphi_{168,6}}$	3	1	1	2	1	1
$\widehat{\varphi_{168,21}}$	3	1	1	2	1	1
$\widehat{\varphi_{189,10}}$	4	2	2	2	1	1
$\widehat{\varphi_{189,17}}$	4	2	2	2	1	1
$\widehat{\varphi_{189,22}}$	4	2	2	2	1	1
$\widehat{\varphi_{189,5}}$	4	2	2	2	1	1
$\widehat{\varphi_{189,20}}$	4	2	2	2	1	1
$\widehat{\varphi_{189,7}}$	4	2	2	2	1	1
$\widehat{\varphi_{210,6}}$	6	3	2	3	1	1
$\widehat{\varphi_{210,21}}$	6	3	2	3	1	1
$\widehat{\varphi_{210,10}}$	3	·	3	3	3	·
$\widehat{\varphi_{210,13}}$	3	·	3	3	3	·
$\widehat{\varphi_{280,18}}$	5	2	2	3	2	1
$\widehat{\varphi_{280,9}}$	5	2	2	3	2	1
$\widehat{\varphi_{378,14}}$	6	1	4	5	4	1
$\widehat{\varphi_{378,9}}$	6	1	4	5	4	1
$\widehat{\varphi_{405,8}}$	8	3	4	6	4	1
$\widehat{\varphi_{405,15}}$	8	3	4	6	4	1
$\widehat{\varphi_{420,10}}$	9	3	5	6	4	1
$\widehat{\varphi_{420,13}}$	9	3	5	6	4	1

$\widehat{\varphi_{56,30}}$	1	·	·
$\widehat{\varphi_{56,3}}$	1	·	·
$\widehat{\varphi_{120,4}}$	1	1	·
$\widehat{\varphi_{120,25}}$	1	1	·
$\widehat{\varphi_{216,16}}$	·	·	1
$\widehat{\varphi_{216,9}}$	·	·	1
$\widehat{\varphi_{280,8}}$	·	1	1
$\widehat{\varphi_{280,17}}$	·	1	1
$\widehat{\varphi_{336,14}}$	1	1	1
$\widehat{\varphi_{336,11}}$	1	1	1

Table C.28.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_7, p=3, and ε a root of $g := y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{7,46}}$	·	1	·	·	·	·	·	·	·	·
$\widehat{\varphi_{15,28}}$	1	·	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,6}}$	·	·	·	1	·	·	·	·	·	·
$\widehat{\varphi_{21,36}}$	·	1	1	·	·	·	·	·	·	·
$\widehat{\varphi_{35,22}}$	·	·	·	·	1	·	·	·	·	·
$\widehat{\varphi_{35,4}}$	1	·	·	·	·	1	·	·	·	·
$\widehat{\varphi_{56,30}}$	·	1	·	·	·	·	1	·	·	·
$\widehat{\varphi_{70,18}}$	·	·	·	1	·	·	1	·	·	·
$\widehat{\varphi_{84,12}}$	1	·	1	·	1	1	·	·	·	·
$\widehat{\varphi_{105,26}}$	·	1	1	·	1	·	1	·	·	·
$\widehat{\varphi_{105,6}}$	·	1	·	·	·	·	·	1	·	·
$\widehat{\varphi_{105,12}}$	·	·	1	·	·	·	·	·	1	·
$\widehat{\varphi_{120,4}}$	1	·	·	1	·	·	·	1	·	·
$\widehat{\varphi_{168,6}}$	1	·	·	·	1	1	·	1	·	·
$\widehat{\varphi_{210,6}}$	·	·	·	1	·	·	·	1	1	·
$\widehat{\varphi_{210,10}}$	·	1	·	1	1	·	1	1	·	·
$\widehat{\varphi_{280,18}}$	·	·	·	·	·	·	1	·	·	1
$\widehat{\varphi_{280,8}}$	1	1	1	·	1	1	·	1	1	·
$\widehat{\varphi_{336,14}}$	·	·	1	·	1	·	·	·	1	1
$\widehat{\varphi_{420,10}}$	·	·	·	·	1	·	·	1	1	1
$\widehat{\varphi_{512,11}}$	1	1	1	1	1	·	1	1	1	1

$\widehat{\varphi_{1,63}}$	1	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{7,1}}$	·	1	·	·	·	·	·	·	·	·
$\widehat{\varphi_{15,7}}$	1	·	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,33}}$	·	·	·	1	·	·	·	·	·	·
$\widehat{\varphi_{21,3}}$	·	1	1	·	·	·	·	·	·	·
$\widehat{\varphi_{35,13}}$	·	·	·	·	1	·	·	·	·	·
$\widehat{\varphi_{35,31}}$	1	·	·	·	·	1	·	·	·	·
$\widehat{\varphi_{56,3}}$	·	1	·	·	·	·	1	·	·	·
$\widehat{\varphi_{70,9}}$	·	·	·	1	·	·	1	·	·	·
$\widehat{\varphi_{84,15}}$	1	·	1	·	1	1	·	·	·	·
$\widehat{\varphi_{105,5}}$	·	1	1	·	1	·	1	·	·	·
$\widehat{\varphi_{105,21}}$	·	1	·	·	·	·	·	1	·	·
$\widehat{\varphi_{105,15}}$	·	·	1	·	·	·	·	·	1	·
$\widehat{\varphi_{120,25}}$	1	·	·	1	·	·	·	1	·	·
$\widehat{\varphi_{168,21}}$	1	·	·	·	1	1	·	1	·	·
$\widehat{\varphi_{210,21}}$	·	·	·	1	·	·	·	1	1	·
$\widehat{\varphi_{210,13}}$	·	1	·	1	1	·	1	1	·	·
$\widehat{\varphi_{280,9}}$	·	·	·	·	·	·	1	·	·	1
$\widehat{\varphi_{280,17}}$	1	1	1	·	1	1	·	1	1	·
$\widehat{\varphi_{336,11}}$	·	·	1	·	1	·	·	·	1	1
$\widehat{\varphi_{420,13}}$	·	·	·	·	1	·	·	1	1	1
$\widehat{\varphi_{512,12}}$	1	1	1	1	1	·	1	1	1	1

$\widehat{\varphi_{27,2}}$	1	·
$\widehat{\varphi_{189,20}}$	·	1
$\widehat{\varphi_{216,16}}$	1	1

$\widehat{\varphi_{27,37}}$	1	·
$\widehat{\varphi_{189,7}}$	·	1
$\widehat{\varphi_{216,9}}$	1	1

$\widehat{\varphi_{189,10}}$	1	·
$\widehat{\varphi_{189,22}}$	·	1
$\widehat{\varphi_{378,14}}$	1	1

$\widehat{\varphi_{189,17}}$	1	·
$\widehat{\varphi_{189,5}}$	·	1
$\widehat{\varphi_{378,9}}$	1	1

Table C.29.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_7, p=3, and ε a root of $g := y + 2$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{7,46}}$	·	1	·	·	·	·	·	·	·	·
$\widehat{\varphi_{15,28}}$	1	·	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,6}}$	·	·	·	1	·	·	·	·	·	·
$\widehat{\varphi_{21,36}}$	·	1	1	·	·	·	·	·	·	·
$\widehat{\varphi_{35,22}}$	·	·	·	·	1	·	·	·	·	·
$\widehat{\varphi_{35,4}}$	1	·	·	·	·	1	·	·	·	·
$\widehat{\varphi_{56,30}}$	·	1	·	·	·	·	1	·	·	·
$\widehat{\varphi_{70,18}}$	·	·	·	1	·	·	1	·	·	·
$\widehat{\varphi_{84,12}}$	1	·	1	·	1	1	·	·	·	·
$\widehat{\varphi_{105,26}}$	·	1	1	·	1	·	1	·	·	·
$\widehat{\varphi_{105,6}}$	·	1	·	·	·	·	·	1	·	·
$\widehat{\varphi_{105,12}}$	·	·	1	·	·	·	·	·	1	·
$\widehat{\varphi_{120,4}}$	1	·	·	1	·	·	·	1	·	·
$\widehat{\varphi_{168,6}}$	1	·	·	·	1	1	·	1	·	·
$\widehat{\varphi_{210,6}}$	·	·	·	1	·	·	·	1	1	·
$\widehat{\varphi_{210,10}}$	·	1	·	1	1	·	1	1	·	·
$\widehat{\varphi_{280,18}}$	·	·	·	·	·	·	1	·	·	1
$\widehat{\varphi_{280,8}}$	1	1	1	·	1	1	·	1	1	·
$\widehat{\varphi_{336,14}}$	·	·	1	·	1	·	·	·	1	1
$\widehat{\varphi_{420,10}}$	·	·	·	·	1	·	·	1	1	1
$\widehat{\varphi_{512,12}}$	1	1	1	1	1	·	1	1	1	1

$\widehat{\varphi_{1,63}}$	1	·	·	·	·	·	·	·	·	·
$\widehat{\varphi_{7,1}}$	·	1	·	·	·	·	·	·	·	·
$\widehat{\varphi_{15,7}}$	1	·	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,33}}$	·	·	·	1	·	·	·	·	·	·
$\widehat{\varphi_{21,3}}$	·	1	1	·	·	·	·	·	·	·
$\widehat{\varphi_{35,13}}$	·	·	·	·	1	·	·	·	·	·
$\widehat{\varphi_{35,31}}$	1	·	·	·	·	1	·	·	·	·
$\widehat{\varphi_{56,3}}$	·	1	·	·	·	·	1	·	·	·
$\widehat{\varphi_{70,9}}$	·	·	·	1	·	·	1	·	·	·
$\widehat{\varphi_{84,15}}$	1	·	1	·	1	1	·	·	·	·
$\widehat{\varphi_{105,5}}$	·	1	1	·	1	·	1	·	·	·
$\widehat{\varphi_{105,21}}$	·	1	·	·	·	·	·	1	·	·
$\widehat{\varphi_{105,15}}$	·	·	1	·	·	·	·	·	1	·
$\widehat{\varphi_{120,25}}$	1	·	·	1	·	·	·	1	·	·
$\widehat{\varphi_{168,21}}$	1	·	·	·	1	1	·	1	·	·
$\widehat{\varphi_{210,21}}$	·	·	·	1	·	·	·	1	1	·
$\widehat{\varphi_{210,13}}$	·	1	·	1	1	·	1	1	·	·
$\widehat{\varphi_{280,9}}$	·	·	·	·	·	·	1	·	·	1
$\widehat{\varphi_{280,17}}$	1	1	1	·	1	1	·	1	1	·
$\widehat{\varphi_{336,11}}$	·	·	1	·	1	·	·	·	1	1
$\widehat{\varphi_{420,13}}$	·	·	·	·	1	·	·	1	1	1
$\widehat{\varphi_{512,11}}$	1	1	1	1	1	·	1	1	1	1

$\widehat{\varphi_{27,2}}$	1	·
$\widehat{\varphi_{189,20}}$	·	1
$\widehat{\varphi_{216,16}}$	1	1

$\widehat{\varphi_{27,37}}$	1	·
$\widehat{\varphi_{189,7}}$	·	1
$\widehat{\varphi_{216,9}}$	1	1

$\widehat{\varphi_{189,10}}$	1	·
$\widehat{\varphi_{189,22}}$	·	1
$\widehat{\varphi_{378,14}}$	1	1

$\widehat{\varphi_{189,17}}$	1	·
$\widehat{\varphi_{189,5}}$	·	1
$\widehat{\varphi_{378,9}}$	1	1

Table C.30.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=E_7, p=3, and ε a root of $g \in \{y^2 + y + 2,\ y^2 + 2y + 2\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,36}}$	·	1	·	·	·	·	·	·
$\widehat{\varphi_{35,22}}$	·	·	1	·	·	·	·	·
$\widehat{\varphi_{56,3}}$	1	·	·	1	·	·	·	·
$\widehat{\varphi_{70,18}}$	1	1	·	·	1	·	·	·
$\widehat{\varphi_{105,6}}$	2	·	·	1	1	·	·	·
$\widehat{\varphi_{120,25}}$	·	1	·	·	·	1	·	·
$\widehat{\varphi_{189,10}}$	·	·	1	·	·	·	1	·
$\widehat{\varphi_{189,22}}$	·	2	·	·	1	1	·	·
$\widehat{\varphi_{210,6}}$	1	·	·	1	·	·	1	·
$\widehat{\varphi_{280,18}}$	·	·	·	·	·	1	·	1
$\widehat{\varphi_{336,11}}$	1	·	1	·	·	·	1	1
$\widehat{\varphi_{405,8}}$	2	·	·	1	1	·	1	1

$\widehat{\varphi_{1,63}}$	1	·	·	·	·	·	·	·
$\widehat{\varphi_{21,3}}$	·	1	·	·	·	·	·	·
$\widehat{\varphi_{35,13}}$	·	·	1	·	·	·	·	·
$\widehat{\varphi_{56,30}}$	1	·	·	1	·	·	·	·
$\widehat{\varphi_{70,9}}$	1	1	·	·	1	·	·	·
$\widehat{\varphi_{105,21}}$	2	·	·	1	1	·	·	·
$\widehat{\varphi_{120,4}}$	·	1	·	·	·	1	·	·
$\widehat{\varphi_{189,17}}$	·	·	1	·	·	·	1	·
$\widehat{\varphi_{189,5}}$	·	2	·	·	1	1	·	·
$\widehat{\varphi_{210,21}}$	1	·	·	1	·	·	1	·
$\widehat{\varphi_{280,9}}$	·	·	·	·	·	1	·	1
$\widehat{\varphi_{336,14}}$	1	·	1	·	·	·	1	1
$\widehat{\varphi_{405,15}}$	2	·	·	1	1	·	1	1

$\widehat{\varphi_{7,46}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{15,28}}$	1	1	·	·	·	·	·
$\widehat{\varphi_{21,6}}$	·	·	1	·	·	·	·
$\widehat{\varphi_{27,2}}$	·	·	·	1	·	·	·
$\widehat{\varphi_{35,4}}$	·	1	·	1	·	·	·
$\widehat{\varphi_{105,26}}$	·	·	·	·	1	·	·
$\widehat{\varphi_{105,12}}$	·	·	1	·	·	1	·
$\widehat{\varphi_{189,20}}$	·	·	·	·	1	1	·
$\widehat{\varphi_{210,10}}$	1	1	·	1	·	·	1
$\widehat{\varphi_{216,9}}$	·	·	1	1	·	·	1
$\widehat{\varphi_{280,17}}$	1	·	·	·	1	·	1
$\widehat{\varphi_{378,14}}$	·	·	1	·	1	1	1

$\widehat{\varphi_{7,1}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{15,7}}$	1	1	·	·	·	·	·
$\widehat{\varphi_{21,33}}$	·	·	1	·	·	·	·
$\widehat{\varphi_{27,37}}$	·	·	·	1	·	·	·
$\widehat{\varphi_{35,31}}$	·	1	·	1	·	·	·
$\widehat{\varphi_{105,5}}$	·	·	·	·	1	·	·
$\widehat{\varphi_{105,15}}$	·	·	1	·	·	1	·
$\widehat{\varphi_{189,7}}$	·	·	·	·	1	1	·
$\widehat{\varphi_{210,13}}$	1	1	·	1	·	·	1
$\widehat{\varphi_{216,16}}$	·	·	1	1	·	·	1
$\widehat{\varphi_{280,8}}$	1	·	·	·	1	·	1
$\widehat{\varphi_{378,9}}$	·	·	1	·	1	1	1

Table C.31.: Non-trivial blocks of the decomposition map $d^{\mathbb{L}}_{\mathbb{k}}$ for W=E_7, p=3, and ε a root of $g := y^4 + 2y^3 + y^2 + 2y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi_{56,30}}$	·	1	·	·
$\widehat{\varphi_{84,12}}$	1	·	1	·
$\widehat{\varphi_{189,22}}$	·	1	·	1
$\widehat{\varphi_{216,16}}$	·	·	1	1

$\widehat{\varphi_{1,63}}$	1	·	·	·
$\widehat{\varphi_{56,3}}$	·	1	·	·
$\widehat{\varphi_{84,15}}$	1	·	1	·
$\widehat{\varphi_{189,5}}$	·	1	·	1
$\widehat{\varphi_{216,9}}$	·	·	1	1

$\widehat{\varphi_{7,46}}$	1	·	·	·
$\widehat{\varphi_{27,2}}$	·	1	·	·
$\widehat{\varphi_{168,6}}$	·	1	1	·
$\widehat{\varphi_{378,14}}$	1	·	·	1
$\widehat{\varphi_{512,11}}$	·	·	1	1

$\widehat{\varphi_{7,1}}$	1	·	·	·
$\widehat{\varphi_{27,37}}$	·	1	·	·
$\widehat{\varphi_{168,21}}$	·	1	1	·
$\widehat{\varphi_{378,9}}$	1	·	·	1
$\widehat{\varphi_{512,12}}$	·	·	1	1

$\widehat{\varphi_{21,6}}$	1	·	·	·
$\widehat{\varphi_{21,36}}$	·	1	·	·
$\widehat{\varphi_{189,10}}$	1	·	1	·
$\widehat{\varphi_{189,20}}$	·	1	·	1
$\widehat{\varphi_{336,14}}$	·	·	1	1

$\widehat{\varphi_{21,33}}$	1	·	·	·
$\widehat{\varphi_{21,3}}$	·	1	·	·
$\widehat{\varphi_{189,17}}$	1	·	1	·
$\widehat{\varphi_{189,7}}$	·	1	·	1
$\widehat{\varphi_{336,11}}$	·	·	1	1

Table C.32.: Non-trivial blocks of the decomposition map $d^{\mathbb{L}}_{\mathbb{k}}$ for W=E_7, p=3, and ε a root of $g := y^4 + y^3 + y^2 + y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi_{56,30}}$	·	1	·	·
$\widehat{\varphi_{84,12}}$	1	·	1	·
$\widehat{\varphi_{189,22}}$	·	1	·	1
$\widehat{\varphi_{216,16}}$	·	·	1	1

$\widehat{\varphi_{1,63}}$	1	·	·	·
$\widehat{\varphi_{56,3}}$	·	1	·	·
$\widehat{\varphi_{84,15}}$	1	·	1	·
$\widehat{\varphi_{189,5}}$	·	1	·	1
$\widehat{\varphi_{216,9}}$	·	·	1	1

$\widehat{\varphi_{7,46}}$	1	·	·	·
$\widehat{\varphi_{27,2}}$	·	1	·	·
$\widehat{\varphi_{168,6}}$	·	1	1	·
$\widehat{\varphi_{378,14}}$	1	·	·	1
$\widehat{\varphi_{512,12}}$	·	·	1	1

$\widehat{\varphi_{7,1}}$	1	·	·	·
$\widehat{\varphi_{27,37}}$	·	1	·	·
$\widehat{\varphi_{168,21}}$	·	1	1	·
$\widehat{\varphi_{378,9}}$	1	·	·	1
$\widehat{\varphi_{512,11}}$	·	·	1	1

$\widehat{\varphi_{21,6}}$	1	·	·	·
$\widehat{\varphi_{21,36}}$	·	1	·	·
$\widehat{\varphi_{189,10}}$	1	·	1	·
$\widehat{\varphi_{189,20}}$	·	1	·	1
$\widehat{\varphi_{336,14}}$	·	·	1	1

$\widehat{\varphi_{21,33}}$	1	·	·	·
$\widehat{\varphi_{21,3}}$	·	1	·	·
$\widehat{\varphi_{189,17}}$	1	·	1	·
$\widehat{\varphi_{189,7}}$	·	1	·	1
$\widehat{\varphi_{336,11}}$	·	·	1	1

Table C.33.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_7, p=3, and ε a root of $g \in \{y^4 + 2y^2 + 2,\ y^4 + y^2 + 2\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{21,36}}$	·	1	·	·	·
$\widehat{\varphi_{105,6}}$	1	·	1	·	·
$\widehat{\varphi_{189,22}}$	·	1	·	1	·
$\widehat{\varphi_{216,9}}$	·	·	1	·	1
$\widehat{\varphi_{280,17}}$	·	·	·	1	1

$\widehat{\varphi_{1,63}}$	1	·	·	·	·
$\widehat{\varphi_{21,3}}$	·	1	·	·	·
$\widehat{\varphi_{105,21}}$	1	·	1	·	·
$\widehat{\varphi_{189,5}}$	·	1	·	1	·
$\widehat{\varphi_{216,16}}$	·	·	1	·	1
$\widehat{\varphi_{280,8}}$	·	·	·	1	1

$\widehat{\varphi_{7,46}}$	1	·	·	·	·
$\widehat{\varphi_{27,2}}$	·	1	·	·	·
$\widehat{\varphi_{56,3}}$	·	1	1	·	·
$\widehat{\varphi_{105,12}}$	·	·	1	1	·
$\widehat{\varphi_{120,25}}$	1	·	·	·	1
$\widehat{\varphi_{189,20}}$	·	·	·	1	1

$\widehat{\varphi_{7,1}}$	1	·	·	·	·
$\widehat{\varphi_{27,37}}$	·	1	·	·	·
$\widehat{\varphi_{56,30}}$	·	1	1	·	·
$\widehat{\varphi_{105,15}}$	·	·	1	1	·
$\widehat{\varphi_{120,4}}$	1	·	·	·	1
$\widehat{\varphi_{189,7}}$	·	·	·	1	1

Table C.34.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_7, p=3, and ε a root of $g \in \{y^4 + 2y^3 + y + 1,\ y^4 + y^3 + 2y + 1\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{21,3}}$	1	1	·	·	·	·
$\widehat{\varphi_{35,31}}$	·	·	1	·	·	·
$\widehat{\varphi_{189,22}}$	·	·	1	1	·	·
$\widehat{\varphi_{189,7}}$	·	1	·	·	1	·
$\widehat{\varphi_{405,15}}$	·	·	·	1	·	1
$\widehat{\varphi_{420,10}}$	·	·	·	·	1	1

$\widehat{\varphi_{1,63}}$	1	·	·	·	·	·
$\widehat{\varphi_{21,36}}$	1	1	·	·	·	·
$\widehat{\varphi_{35,4}}$	·	·	1	·	·	·
$\widehat{\varphi_{189,5}}$	·	·	1	1	·	·
$\widehat{\varphi_{189,20}}$	·	1	·	·	1	·
$\widehat{\varphi_{405,8}}$	·	·	·	1	·	1
$\widehat{\varphi_{420,13}}$	·	·	·	·	1	1

$\widehat{\varphi_{7,46}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{7,1}}$	·	1	·	·	·	·	·
$\widehat{\varphi_{27,2}}$	·	1	1	·	·	·	·
$\widehat{\varphi_{27,37}}$	1	·	·	1	·	·	·
$\widehat{\varphi_{168,6}}$	·	·	1	·	1	·	·
$\widehat{\varphi_{168,21}}$	·	·	·	1	·	1	·
$\widehat{\varphi_{378,14}}$	·	·	·	·	·	1	1
$\widehat{\varphi_{378,9}}$	·	·	·	·	1	·	1

Table C.35.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_7, p=3, and ε a root of $g := y^6 + 2y^5 + y^4 + 2y^3 + y^2 + 2y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{15,28}}$	·	1	·	·	·	·
$\widehat{\varphi_{27,2}}$	1	·	1	·	·	·
$\widehat{\varphi_{120,4}}$	·	·	1	1	·	·
$\widehat{\varphi_{216,16}}$	·	1	·	·	1	·
$\widehat{\varphi_{405,8}}$	·	·	·	1	·	1
$\widehat{\varphi_{512,11}}$	·	·	·	·	1	1

$\widehat{\varphi_{1,63}}$	1	·	·	·	·	·
$\widehat{\varphi_{15,7}}$	·	1	·	·	·	·
$\widehat{\varphi_{27,37}}$	1	·	1	·	·	·
$\widehat{\varphi_{120,25}}$	·	·	1	1	·	·
$\widehat{\varphi_{216,9}}$	·	1	·	·	1	·
$\widehat{\varphi_{405,15}}$	·	·	·	1	·	1
$\widehat{\varphi_{512,12}}$	·	·	·	·	1	1

Table C.36.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_7, p=3, and ε a root of $g := y^6 + y^5 + y^4 + y^3 + y^2 + y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{15,28}}$	·	1	·	·	·	·
$\widehat{\varphi_{27,2}}$	1	·	1	·	·	·
$\widehat{\varphi_{120,4}}$	·	·	1	1	·	·
$\widehat{\varphi_{216,16}}$	·	1	·	·	1	·
$\widehat{\varphi_{405,8}}$	·	·	·	1	·	1
$\widehat{\varphi_{512,12}}$	·	·	·	·	1	1

$\widehat{\varphi_{1,63}}$	1	·	·	·	·	·
$\widehat{\varphi_{15,7}}$	·	1	·	·	·	·
$\widehat{\varphi_{27,37}}$	1	·	1	·	·	·
$\widehat{\varphi_{120,25}}$	·	·	1	1	·	·
$\widehat{\varphi_{216,9}}$	·	1	·	·	1	·
$\widehat{\varphi_{405,15}}$	·	·	·	1	·	1
$\widehat{\varphi_{512,11}}$	·	·	·	·	1	1

Table C.37.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=E_7, p=3, and ε a root of $g \in \{y^6 + y^5 + y^3 + y + 1,\ y^6 + 2y^5 + 2y^3 + 2y + 1\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{1,63}}$	·	1	·	·	·	·	·
$\widehat{\varphi_{27,2}}$	1	·	1	·	·	·	·
$\widehat{\varphi_{27,37}}$	·	1	·	1	·	·	·
$\widehat{\varphi_{105,26}}$	·	·	·	1	1	·	·
$\widehat{\varphi_{105,5}}$	·	·	1	·	·	1	·
$\widehat{\varphi_{189,10}}$	·	·	·	·	·	1	1
$\widehat{\varphi_{189,17}}$	·	·	·	·	1	·	1

Appendix D.

Decomposition Numbers of Iwahori-Hecke Algebras of Type H_3 and H_4 in bad Characteristic

We give the results of our computation described in Section 9.2, i.e. we give the decomposition matrices of the $\mathbb{F}_p$-generic decomposition maps $d_{\mathbb{K}}^{\mathbb{k}}$ and of the adjustment maps $d_{\mathbb{k}}^{\mathbb{L}}$ for Iwahori-Hecke algebras of type H_3 and H_5 for the primes discussed in that section. Our notation is essentially that which we already used for the analogous results for Iwahori-Hecke algebras of Weyl groups in Appendix C. For every irreducible $\mathbb{k}\mathcal{H}$-module $\widehat{\varphi}$ we can choose an irreducible $\mathbb{K}\mathcal{H}$-module φ such that $d_{\mathbb{K}}^{\mathbb{k}}([\varphi]) = [\widehat{\varphi}]$ and we use this scheme to lable the irreducible $\mathbb{k}\mathcal{H}$-modules.

We have to explain an additional convention which was not necessary in the case of Iwahori-Hecke algebras of Weyl groups. Recall from Section 9.2 that the definition of $\theta_{\mathbf{K},\mathbb{k}}$ and $\theta_{\mathbf{K},\mathbb{L}}$ relied on some α in a finite field extension of $\mathbb{F}_p$ such that α is a root of $Y^4 + Y^3 + Y^2 + Y + 1 \in \mathbb{F}_p[Y]$. For our computation we chose such an α and we give our choices below. However, it should be noted that it does not actually matter which root of $Y^4 + Y^3 + Y^2 + Y + 1$ we choose: The fields over which we have computed the decomposition numbers have characteristic 2, 3, or 5. Over $\mathbb{F}_5$, the poolynomial $Y^4 + Y^3 + Y^2 + Y + 1 = (Y^5 - 1)/(Y - 1)$ has exactly one root, namely 1. Over $\mathbb{F}_2$ and $\mathbb{F}_3$, the polynomial $Y^4 + Y^3 + Y^2 + Y + 1$ is irreducible. Therefore, if α and α' are two roots of the polynomial, then there exists a field automorphism of $\mathbb{F}_p[\alpha] = \mathbb{F}_p[\alpha']$ which sends α to α'. This extends to a field automorphism of $\mathbb{k} = \mathbb{F}_p[\alpha](y)$. By Proposition 6.1.3, it follows that the decomposition maps for the specialisations $\mathbf{K} \to \mathbb{k}; V \mapsto y,\ \zeta_5 \mapsto \alpha$ and $\mathbf{K} \to \mathbb{k}; V \mapsto y,\ \zeta_5 \mapsto \alpha'$ differ only by a trivial isomorphism of Grothendieck groups. Thus, it suffices to consider only one root α of $Y^4 + Y^3 + Y^2 + Y + 1$ for each prime p to compute the $\mathbb{F}_p$-generic decomposition maps.

To write down our choices for α, let ω be a root of the irreducible Conway polynomial of degree 4 over $\mathbb{F}_p$ for $p \in \{2, 3\}$. Then, ω is a generator of the multiplicative group of $\mathbb{F}_p$. For $p = 2$, we set $\alpha := \omega^3$, and for $p = 3$ we set $\alpha := \omega^{48}$.

D.1. $\mathbb{F}_p$-Generic Decomposition Matrices

Table D.1.: Non-trivial blocks of the $\mathbb{F}_2$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = H_3$

$\varphi_{4,3}$	1
$\varphi_{4,4}$	1

Table D.2.: Non-trivial blocks of the $\mathbb{F}_5$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = H_3$

$\varphi_{3,6}$	1
$\varphi_{3,8}$	1

$\varphi_{3,1}$	1
$\varphi_{3,3}$	1

Table D.3.: Non-trivial blocks of the $\mathbb{F}_2$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = H_4$

$\varphi_{6,12}$	1	·
$\varphi_{24,11}$	·	1
$\varphi''_{30,10}$	1	1

$\varphi_{6,20}$	1	·
$\varphi_{24,7}$	·	1
$\varphi'_{30,10}$	1	1

$\varphi_{8,12}$	1	·	·
$\varphi_{16,11}$	·	1	·
$\varphi_{16,13}$	·	·	1
$\varphi_{24,12}$	1	·	1
$\varphi_{24,6}$	1	1	·
$\varphi_{40,8}$	1	1	1
$\varphi_{48,9}$	2	1	1

$\varphi_{8,13}$	1	·
$\varphi_{10,12}$	·	1
$\varphi_{18,10}$	1	1

$\varphi_{16,3}$	1
$\varphi_{16,6}$	1

$\varphi_{16,21}$	1
$\varphi_{16,18}$	1

Table D.4.: Non-trivial blocks of the $\mathbb{F}_3$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = H_4$

$\varphi_{6,12}$	1	·
$\varphi_{24,12}$	·	1
$\varphi'_{30,10}$	1	1

$\varphi_{6,20}$	1	·
$\varphi_{24,6}$	·	1
$\varphi''_{30,10}$	1	1

$\varphi_{24,11}$	1	·
$\varphi_{24,7}$	·	1
$\varphi_{48,9}$	1	1

Table D.5.: Non-trivial blocks of the $\mathbb{F}_5$-generic decomposition map $d_{\mathbb{K}}^{\Bbbk}$ for $W = H_4$

$\varphi_{4,1}$	1
$\varphi_{4,7}$	1

$\varphi_{4,31}$	1
$\varphi_{4,37}$	1

$\varphi_{6,12}$	1	·
$\varphi_{6,20}$	1	·
$\varphi_{18,10}$	·	1
$\varphi_{24,12}$	1	1
$\varphi_{24,6}$	1	1

$\varphi_{8,13}$	1	·
$\varphi_{16,11}$	·	1
$\varphi_{16,13}$	·	1
$\varphi_{24,11}$	1	1
$\varphi_{24,7}$	1	1

$\varphi_{9,2}$	1
$\varphi_{9,6}$	1

$\varphi_{9,22}$	1
$\varphi_{9,26}$	1

$\varphi_{10,12}$	1	·
$\varphi'_{30,10}$	·	1
$\varphi''_{30,10}$	·	1
$\varphi_{40,8}$	1	1

D.2. Adjustment Matrices

Table D.6.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_3, p=2, and ε a root of $g := y + 1$

$\widehat{\varphi_{1,15}}$	1	·	·
$\widehat{\varphi_{1,0}}$	1	·	·
$\widehat{\varphi_{5,5}}$	1	1	1
$\widehat{\varphi_{5,2}}$	1	1	1
$\widehat{\varphi_{3,6}}$	1	·	1
$\widehat{\varphi_{3,8}}$	1	1	·
$\widehat{\varphi_{3,1}}$	1	·	1
$\widehat{\varphi_{3,3}}$	1	1	·

Table D.7.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_3, p=2, and ε a root of $g \in \{y + \omega^6,\ y + \omega^9\}$

$\widehat{\varphi_{1,15}}$	1	·	·
$\widehat{\varphi_{1,0}}$	·	1	·
$\widehat{\varphi_{3,8}}$	1	·	1
$\widehat{\varphi_{3,3}}$	·	1	1
$\widehat{\varphi_{4,3}}$	1	1	1

Table D.8.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_3, p=2, and ε a root of $g \in \{y + \omega^{12},\ y + \omega^3\}$

$\widehat{\varphi_{1,15}}$	1	·	·
$\widehat{\varphi_{1,0}}$	·	1	·
$\widehat{\varphi_{3,6}}$	1	·	1
$\widehat{\varphi_{3,1}}$	·	1	1
$\widehat{\varphi_{4,3}}$	1	1	1

Table D.9.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_3, p=2, and ε a root of $g \in \{y + \omega^5,\ y + \omega^{10}\}$

$\widehat{\varphi_{1,15}}$	1	·	·
$\widehat{\varphi_{1,0}}$	·	1	·
$\widehat{\varphi_{5,5}}$	1	·	1
$\widehat{\varphi_{5,2}}$	·	1	1
$\widehat{\varphi_{4,3}}$	·	·	1

Table D.10.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_3, p=5, and ε a root of $g \in \{y+2,\ y+3\}$

$\widehat{\varphi_{1,15}}$	1	·
$\widehat{\varphi_{1,0}}$	1	·
$\widehat{\varphi_{5,5}}$	1	2
$\widehat{\varphi_{5,2}}$	1	2
$\widehat{\varphi_{3,6}}$	1	1
$\widehat{\varphi_{3,1}}$	1	1

Table D.11.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_3, p=5, and ε a root of $g := y+1$

$\widehat{\varphi_{1,15}}$	1	·
$\widehat{\varphi_{3,1}}$	·	1
$\widehat{\varphi_{4,4}}$	1	1

$\widehat{\varphi_{1,0}}$	1	·
$\widehat{\varphi_{3,6}}$	·	1
$\widehat{\varphi_{4,3}}$	1	1

Table D.12.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_3, p=5, and ε a root of $g := y+4$

$\widehat{\varphi_{1,15}}$	1	·
$\widehat{\varphi_{3,1}}$	·	1
$\widehat{\varphi_{4,3}}$	1	1

$\widehat{\varphi_{1,0}}$	1	·
$\widehat{\varphi_{3,6}}$	·	1
$\widehat{\varphi_{4,4}}$	1	1

Table D.13.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_3, p=5, and ε a root of $g := y$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{5,5}}$	1	1	1	2	·
$\widehat{\varphi_{5,2}}$	3	·	·	1	1
$\widehat{\varphi_{3,6}}$	·	1	1	1	·
$\widehat{\varphi_{3,1}}$	2	·	·	·	1
$\widehat{\varphi_{4,3}}$	1	·	1	1	1
$\widehat{\varphi_{4,4}}$	1	·	1	1	1

Table D.14.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_3, p=5, and ε a root of $g := y^2 + 4y + 1$

$\widehat{\varphi_{1,15}}$	1	·
$\widehat{\varphi_{5,5}}$	1	1
$\widehat{\varphi_{4,4}}$	·	1

$\widehat{\varphi_{1,0}}$	1	·
$\widehat{\varphi_{5,2}}$	1	1
$\widehat{\varphi_{4,3}}$	·	1

Table D.15.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_3, p=5, and ε a root of $g := y^2 + y + 1$

$\widehat{\varphi_{1,15}}$	1	·
$\widehat{\varphi_{5,5}}$	1	1
$\widehat{\varphi_{4,3}}$	·	1

$\widehat{\varphi_{1,0}}$	1	·
$\widehat{\varphi_{5,2}}$	1	1
$\widehat{\varphi_{4,4}}$	·	1

Table D.16.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_3, p=5, and ε a root of $g \in \{y^2 + 3y + 4,\ y^2 + 2y + 4\}$

$\widehat{\varphi_{1,15}}$	1	·	·
$\widehat{\varphi_{1,0}}$	·	1	·
$\widehat{\varphi_{5,5}}$	1	·	1
$\widehat{\varphi_{5,2}}$	·	1	1

Table D.17.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=2, and ε a root of $g := y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	1	·	·	·	·	·
$\widehat{\varphi_{4,1}}$	·	1	·	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	·	·	·
$\widehat{\varphi_{4,7}}$	·	·	1	·	·	·
$\widehat{\varphi_{4,37}}$	·	·	1	·	·	·
$\widehat{\varphi_{6,12}}$	2	·	·	1	·	·
$\widehat{\varphi_{6,20}}$	2	·	·	·	1	·
$\widehat{\varphi_{8,13}}$	·	·	·	·	·	1
$\widehat{\varphi_{9,2}}$	1	·	1	1	·	·
$\widehat{\varphi_{9,22}}$	1	·	1	1	·	·
$\widehat{\varphi_{9,6}}$	1	1	·	·	1	·
$\widehat{\varphi_{9,26}}$	1	1	·	·	1	·
$\widehat{\varphi_{10,12}}$	2	·	·	1	1	·
$\widehat{\varphi_{24,11}}$	·	2	·	·	2	1
$\widehat{\varphi_{24,7}}$	·	·	2	2	·	1
$\widehat{\varphi_{25,4}}$	1	1	1	1	1	1
$\widehat{\varphi_{25,16}}$	1	1	1	1	1	1
$\widehat{\varphi_{36,5}}$	4	1	1	2	2	1
$\widehat{\varphi_{36,15}}$	4	1	1	2	2	1

$\widehat{\varphi_{16,3}}$	1
$\widehat{\varphi_{16,21}}$	1

Table D.18.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=2, and ε a root of $g \in \{y + \omega^6,\ y + \omega^9\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·	·	·	·
$\widehat{\varphi_{4,1}}$	·	·	1	·	·	·	·	·
$\widehat{\varphi_{4,31}}$	·	·	·	1	·	·	·	·
$\widehat{\varphi_{6,20}}$	·	·	·	·	1	·	·	·
$\widehat{\varphi_{8,12}}$	1	1	·	·	1	·	·	·
$\widehat{\varphi_{9,2}}$	1	·	·	·	·	1	·	·
$\widehat{\varphi_{9,22}}$	·	1	·	·	·	·	1	·
$\widehat{\varphi_{16,11}}$	·	·	·	·	·	1	1	·
$\widehat{\varphi_{16,13}}$	·	·	·	·	·	·	·	1
$\widehat{\varphi_{16,3}}$	2	·	·	·	1	1	·	·
$\widehat{\varphi_{16,21}}$	·	2	·	·	1	·	1	·
$\widehat{\varphi_{24,7}}$	·	·	1	1	·	·	·	1
$\widehat{\varphi_{36,5}}$	2	·	1	·	1	1	·	1
$\widehat{\varphi_{36,15}}$	·	2	·	1	1	·	1	1

$\widehat{\varphi_{4,7}}$	1	·	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	·
$\widehat{\varphi_{8,13}}$	1	1	·	·
$\widehat{\varphi_{9,6}}$	1	·	1	·
$\widehat{\varphi_{9,26}}$	·	1	·	1
$\widehat{\varphi_{10,12}}$	·	·	1	1

Table D.19.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=2, and ε a root of $g \in \{y + \omega^{12},\ y + \omega^{3}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·	·	·	·
$\widehat{\varphi_{4,7}}$	·	·	1	·	·	·	·	·
$\widehat{\varphi_{4,37}}$	·	·	·	1	·	·	·	·
$\widehat{\varphi_{6,12}}$	·	·	·	·	1	·	·	·
$\widehat{\varphi_{8,12}}$	1	1	·	·	1	·	·	·
$\widehat{\varphi_{9,6}}$	1	·	·	·	·	1	·	·
$\widehat{\varphi_{9,26}}$	·	1	·	·	·	·	1	·
$\widehat{\varphi_{16,11}}$	·	·	·	·	·	·	·	1
$\widehat{\varphi_{16,13}}$	·	·	·	·	·	1	1	·
$\widehat{\varphi_{16,3}}$	2	·	·	·	1	1	·	·
$\widehat{\varphi_{16,21}}$	·	2	·	·	1	·	1	·
$\widehat{\varphi_{24,11}}$	·	·	1	1	·	·	·	1
$\widehat{\varphi_{36,5}}$	2	·	1	·	1	1	·	1
$\widehat{\varphi_{36,15}}$	·	2	·	1	1	·	1	1

$\widehat{\varphi_{4,1}}$	1	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	·
$\widehat{\varphi_{8,13}}$	1	1	·	·
$\widehat{\varphi_{9,2}}$	1	·	1	·
$\widehat{\varphi_{9,22}}$	·	1	·	1
$\widehat{\varphi_{10,12}}$	·	·	1	1

Table D.20.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=2, and ε a root of $g \in \{y + \omega^{5},\ y + \omega^{10}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·	·	·	·
$\widehat{\varphi_{4,1}}$	·	·	1	·	·	·	·	·
$\widehat{\varphi_{4,31}}$	·	·	·	1	·	·	·	·
$\widehat{\varphi_{4,7}}$	·	·	·	·	1	·	·	·
$\widehat{\varphi_{4,37}}$	·	·	·	·	·	1	·	·
$\widehat{\varphi_{8,12}}$	·	·	·	·	·	·	1	·
$\widehat{\varphi_{8,13}}$	·	·	·	·	·	·	·	1
$\widehat{\varphi_{10,12}}$	1	1	·	·	·	·	1	·
$\widehat{\varphi_{16,11}}$	·	·	1	1	·	·	·	1
$\widehat{\varphi_{16,13}}$	·	·	·	·	1	1	·	1
$\widehat{\varphi_{16,3}}$	·	·	1	·	1	·	·	1
$\widehat{\varphi_{16,21}}$	·	·	·	1	·	1	·	1
$\widehat{\varphi_{25,4}}$	1	·	1	·	1	·	1	1
$\widehat{\varphi_{25,16}}$	·	1	·	1	·	1	1	1

Table D.21.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=2, and ε a root of $g \in \{y+\omega^7,\ y+\omega^2,\ y+\omega^{13},\ y+\omega^8\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·	·
$\widehat{\varphi_{4,7}}$	1	·	1	·	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	1	·	·
$\widehat{\varphi_{6,20}}$	·	·	1	1	·	·
$\widehat{\varphi_{16,3}}$	1	·	1	·	1	·
$\widehat{\varphi_{16,21}}$	·	1	·	1	·	1
$\widehat{\varphi_{24,7}}$	·	·	·	·	1	1

Table D.22.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=2, and ε a root of $g \in \{y+\omega^{11},\ y+\omega^1,\ y+\omega^{14},\ y+\omega^4\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·	·
$\widehat{\varphi_{4,1}}$	1	·	1	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	1	·	·
$\widehat{\varphi_{6,12}}$	·	·	1	1	·	·
$\widehat{\varphi_{16,3}}$	1	·	1	·	1	·
$\widehat{\varphi_{16,21}}$	·	1	·	1	·	1
$\widehat{\varphi_{24,11}}$	·	·	·	·	1	1

Table D.23.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y + \omega^{20},\ y + \omega^{60}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	1	·	·	·	·	·
$\widehat{\varphi_{4,1}}$	·	1	·	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	·	·	·
$\widehat{\varphi_{4,7}}$	·	·	1	·	·	·
$\widehat{\varphi_{4,37}}$	·	·	1	·	·	·
$\widehat{\varphi_{6,12}}$	1	·	·	1	·	·
$\widehat{\varphi_{6,20}}$	1	·	·	·	1	·
$\widehat{\varphi_{9,2}}$	·	·	1	1	·	·
$\widehat{\varphi_{9,22}}$	·	·	1	1	·	·
$\widehat{\varphi_{9,6}}$	·	1	·	·	1	·
$\widehat{\varphi_{9,26}}$	·	1	·	·	1	·
$\widehat{\varphi_{10,12}}$	·	·	·	1	1	·
$\widehat{\varphi_{18,10}}$	2	·	·	·	·	1
$\widehat{\varphi_{24,12}}$	·	·	2	·	·	1
$\widehat{\varphi_{24,6}}$	·	2	·	·	·	1
$\widehat{\varphi_{25,4}}$	1	1	1	·	·	1
$\widehat{\varphi_{25,16}}$	1	1	1	·	·	1
$\widehat{\varphi_{36,5}}$	2	1	1	1	1	1
$\widehat{\varphi_{36,15}}$	2	1	1	1	1	1

$\widehat{\varphi_{16,3}}$	1
$\widehat{\varphi_{16,18}}$	1

$\widehat{\varphi_{16,21}}$	1
$\widehat{\varphi_{16,6}}$	1

Table D.24.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y + \omega^{56},\ y + \omega^{24}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{6,20}}$	·	·	1	·	·
$\widehat{\varphi_{8,12}}$	1	1	1	·	·
$\widehat{\varphi_{9,2}}$	·	·	·	1	·
$\widehat{\varphi_{9,22}}$	·	·	·	·	1
$\widehat{\varphi_{16,6}}$	1	·	1	1	·
$\widehat{\varphi_{16,18}}$	·	1	1	·	1
$\widehat{\varphi_{24,6}}$	·	·	1	1	1

$\widehat{\varphi_{4,1}}$	1	·	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	·	·
$\widehat{\varphi_{16,13}}$	·	·	1	·	·
$\widehat{\varphi_{16,3}}$	1	·	·	1	·
$\widehat{\varphi_{16,21}}$	·	1	·	·	1
$\widehat{\varphi_{24,11}}$	·	·	·	1	1
$\widehat{\varphi_{24,7}}$	1	1	1	·	·
$\widehat{\varphi_{36,5}}$	2	·	1	1	·
$\widehat{\varphi_{36,15}}$	·	2	1	·	1

$\widehat{\varphi_{4,7}}$	1	·
$\widehat{\varphi_{4,37}}$	·	1
$\widehat{\varphi_{8,13}}$	1	1

$\widehat{\varphi_{9,6}}$	1	·
$\widehat{\varphi_{9,26}}$	·	1
$\widehat{\varphi_{18,10}}$	1	1

Table D.25.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y+\omega^{16},\ y+\omega^{64}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{6,20}}$	·	·	1	·	·
$\widehat{\varphi_{8,12}}$	1	1	1	·	·
$\widehat{\varphi_{9,2}}$	·	·	·	1	·
$\widehat{\varphi_{9,22}}$	·	·	·	·	1
$\widehat{\varphi_{16,3}}$	1	·	1	1	·
$\widehat{\varphi_{16,21}}$	·	1	1	·	1
$\widehat{\varphi_{24,6}}$	·	·	1	1	1

$\widehat{\varphi_{4,1}}$	1	·	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	·	·
$\widehat{\varphi_{16,13}}$	·	·	1	·	·
$\widehat{\varphi_{16,6}}$	1	·	·	1	·
$\widehat{\varphi_{16,18}}$	·	1	·	·	1
$\widehat{\varphi_{24,11}}$	·	·	·	1	1
$\widehat{\varphi_{24,7}}$	1	1	1	·	·
$\widehat{\varphi_{36,5}}$	2	·	1	1	·
$\widehat{\varphi_{36,15}}$	·	2	1	·	1

$\widehat{\varphi_{4,7}}$	1	·
$\widehat{\varphi_{4,37}}$	·	1
$\widehat{\varphi_{8,13}}$	1	1

$\widehat{\varphi_{9,6}}$	1	·
$\widehat{\varphi_{9,26}}$	·	1
$\widehat{\varphi_{18,10}}$	1	1

Table D.26.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y+\omega^{8},\ y+\omega^{72}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{6,12}}$	·	·	1	·	·
$\widehat{\varphi_{8,12}}$	1	1	1	·	·
$\widehat{\varphi_{9,6}}$	·	·	·	1	·
$\widehat{\varphi_{9,26}}$	·	·	·	·	1
$\widehat{\varphi_{16,6}}$	1	·	1	1	·
$\widehat{\varphi_{16,18}}$	·	1	1	·	1
$\widehat{\varphi_{24,12}}$	·	·	1	1	1

$\widehat{\varphi_{4,1}}$	1	·
$\widehat{\varphi_{4,31}}$	·	1
$\widehat{\varphi_{8,13}}$	1	1

$\widehat{\varphi_{4,7}}$	1	·	·	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	·	·
$\widehat{\varphi_{16,11}}$	·	·	1	·	·
$\widehat{\varphi_{16,3}}$	1	·	·	1	·
$\widehat{\varphi_{16,21}}$	·	1	·	·	1
$\widehat{\varphi_{24,11}}$	1	1	1	·	·
$\widehat{\varphi_{24,7}}$	·	·	·	1	1
$\widehat{\varphi_{36,5}}$	2	·	1	1	·
$\widehat{\varphi_{36,15}}$	·	2	1	·	1

$\widehat{\varphi_{9,2}}$	1	·
$\widehat{\varphi_{9,22}}$	·	1
$\widehat{\varphi_{18,10}}$	1	1

Table D.27.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y + \omega^{32},\ y + \omega^{48}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{6,12}}$	·	·	1	·	·
$\widehat{\varphi_{8,12}}$	1	1	1	·	·
$\widehat{\varphi_{9,6}}$	·	·	·	1	·
$\widehat{\varphi_{9,26}}$	·	·	·	·	1
$\widehat{\varphi_{16,3}}$	1	·	1	1	·
$\widehat{\varphi_{16,21}}$	·	1	1	·	1
$\widehat{\varphi_{24,12}}$	·	·	1	1	1

$\widehat{\varphi_{4,1}}$	1	·
$\widehat{\varphi_{4,31}}$	·	1
$\widehat{\varphi_{8,13}}$	1	1

$\widehat{\varphi_{4,7}}$	1	·	·	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	·	·
$\widehat{\varphi_{16,11}}$	·	·	1	·	·
$\widehat{\varphi_{16,6}}$	1	·	·	1	·
$\widehat{\varphi_{16,18}}$	·	1	·	·	1
$\widehat{\varphi_{24,11}}$	1	1	1	·	·
$\widehat{\varphi_{24,7}}$	·	·	·	1	1
$\widehat{\varphi_{36,5}}$	2	·	1	1	·
$\widehat{\varphi_{36,15}}$	·	2	1	·	1

$\widehat{\varphi_{9,2}}$	1	·
$\widehat{\varphi_{9,22}}$	·	1
$\widehat{\varphi_{18,10}}$	1	1

Table D.28.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g := y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{8,12}}$	·	·	1	·	·
$\widehat{\varphi_{10,12}}$	1	1	1	·	·
$\widehat{\varphi_{16,3}}$	·	·	·	1	·
$\widehat{\varphi_{16,21}}$	·	·	·	·	1
$\widehat{\varphi_{25,4}}$	1	·	1	1	·
$\widehat{\varphi_{25,16}}$	·	1	1	·	1
$\widehat{\varphi_{40,8}}$	·	·	1	1	1

$\widehat{\varphi_{4,1}}$	1	·	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	·	·
$\widehat{\varphi_{4,7}}$	·	·	1	·	·
$\widehat{\varphi_{4,37}}$	·	·	·	1	·
$\widehat{\varphi_{8,13}}$	·	·	·	·	1
$\widehat{\varphi_{16,11}}$	1	1	·	·	1
$\widehat{\varphi_{16,13}}$	·	·	1	1	1
$\widehat{\varphi_{16,6}}$	1	·	1	·	1
$\widehat{\varphi_{16,18}}$	·	1	·	1	1

Table D.29.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g := y + 2$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{8,12}}$	·	·	1	·	·
$\widehat{\varphi_{10,12}}$	1	1	1	·	·
$\widehat{\varphi_{16,6}}$	·	·	·	1	·
$\widehat{\varphi_{16,18}}$	·	·	·	·	1
$\widehat{\varphi_{25,4}}$	1	·	1	1	·
$\widehat{\varphi_{25,16}}$	·	1	1	·	1
$\widehat{\varphi_{40,8}}$	·	·	1	1	1

$\widehat{\varphi_{4,1}}$	1	·	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	·	·
$\widehat{\varphi_{4,7}}$	·	·	1	·	·
$\widehat{\varphi_{4,37}}$	·	·	·	1	·
$\widehat{\varphi_{8,13}}$	·	·	·	·	1
$\widehat{\varphi_{16,11}}$	1	1	·	·	1
$\widehat{\varphi_{16,13}}$	·	·	1	1	1
$\widehat{\varphi_{16,3}}$	1	·	1	·	1
$\widehat{\varphi_{16,21}}$	·	1	·	1	1

Table D.30.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y+\omega^{12},\ y+\omega^{28},\ y+\omega^{68},\ y+\omega^{52}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·	·	·
$\widehat{\varphi_{4,7}}$	1	·	1	·	·	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	1	·	·	·
$\widehat{\varphi_{6,20}}$	·	·	1	1	·	·	·
$\widehat{\varphi_{9,6}}$	·	·	·	·	1	·	·
$\widehat{\varphi_{9,26}}$	·	·	·	·	·	1	·
$\widehat{\varphi_{24,6}}$	1	1	·	·	·	·	1
$\widehat{\varphi_{36,5}}$	2	·	1	·	1	·	1
$\widehat{\varphi_{36,15}}$	·	2	·	1	·	1	1
$\widehat{\varphi_{40,8}}$	·	·	·	·	1	1	1

$\widehat{\varphi_{4,1}}$	1	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	·
$\widehat{\varphi_{9,2}}$	1	·	1	·
$\widehat{\varphi_{9,22}}$	·	1	·	1
$\widehat{\varphi_{10,12}}$	·	·	1	1

Table D.31.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y+\omega^{76},\ y+\omega^{36},\ y+\omega^{44},\ y+\omega^{4}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·	·	·
$\widehat{\varphi_{4,1}}$	1	·	1	·	·	·	·
$\widehat{\varphi_{4,31}}$	·	1	·	1	·	·	·
$\widehat{\varphi_{6,12}}$	·	·	1	1	·	·	·
$\widehat{\varphi_{9,2}}$	·	·	·	·	1	·	·
$\widehat{\varphi_{9,22}}$	·	·	·	·	·	1	·
$\widehat{\varphi_{24,12}}$	1	1	·	·	·	·	1
$\widehat{\varphi_{36,5}}$	2	·	1	·	1	·	1
$\widehat{\varphi_{36,15}}$	·	2	·	1	·	1	1
$\widehat{\varphi_{40,8}}$	·	·	·	·	1	1	1

$\widehat{\varphi_{4,7}}$	1	·	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	·
$\widehat{\varphi_{9,6}}$	1	·	1	·
$\widehat{\varphi_{9,26}}$	·	1	·	1
$\widehat{\varphi_{10,12}}$	·	·	1	1

Table D.32.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y+\omega^{30},\ y+\omega^{70},\ y+\omega^{50},\ y+\omega^{10}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·	·	·
$\widehat{\varphi_{8,13}}$	·	·	1	·	·	·	·
$\widehat{\varphi_{9,2}}$	1	·	·	1	·	·	·
$\widehat{\varphi_{9,22}}$	·	1	·	·	1	·	·
$\widehat{\varphi_{9,6}}$	1	·	·	·	·	1	·
$\widehat{\varphi_{9,26}}$	·	1	·	·	·	·	1
$\widehat{\varphi_{24,11}}$	·	·	1	·	·	1	1
$\widehat{\varphi_{24,7}}$	·	·	1	1	1	·	·
$\widehat{\varphi_{25,4}}$	1	·	1	1	·	1	·
$\widehat{\varphi_{25,16}}$	·	1	1	·	1	·	1

Table D.33.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y+\omega^{38},\ y+\omega^{62},\ y+\omega^{22},\ y+\omega^{42},\ y+\omega^{78},\ y+\omega^{58},\ y+\omega^{2},\ y+\omega^{18}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·
$\widehat{\varphi_{9,6}}$	1	·	1	·
$\widehat{\varphi_{9,26}}$	·	1	·	1
$\widehat{\varphi_{16,13}}$	·	·	1	1

Table D.34.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=3, and ε a root of $g \in \{y+\omega^{46},\ y+\omega^{66},\ y+\omega^{54},\ y+\omega^{74},\ y+\omega^{6},\ y+\omega^{14},\ y+\omega^{34},\ y+\omega^{26}\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·
$\widehat{\varphi_{9,2}}$	1	·	1	·
$\widehat{\varphi_{9,22}}$	·	1	·	1
$\widehat{\varphi_{16,11}}$	·	·	1	1

Table D.35.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=5, and ε a root of $g \in \{y+2,\ y+3\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi_{1,60}}$	1	·	·	·
$\widehat{\varphi_{4,7}}$	·	1	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	·
$\widehat{\varphi_{6,12}}$	1	·	1	·
$\widehat{\varphi_{9,2}}$	·	1	1	·
$\widehat{\varphi_{9,22}}$	·	1	1	·
$\widehat{\varphi_{10,12}}$	·	·	2	·
$\widehat{\varphi_{18,10}}$	1	·	·	1
$\widehat{\varphi_{25,4}}$	·	2	·	1
$\widehat{\varphi_{25,16}}$	·	2	·	1
$\widehat{\varphi''_{30,10}}$	·	2	1	1
$\widehat{\varphi_{36,5}}$	1	2	2	1
$\widehat{\varphi_{36,15}}$	1	2	2	1

$\widehat{\varphi_{16,3}}$	1
$\widehat{\varphi_{16,18}}$	1

$\widehat{\varphi_{16,21}}$	1
$\widehat{\varphi_{16,6}}$	1

Table D.36.: Non-trivial blocks of the decomposition map $d_{\Bbbk}^{\mathbb{L}}$ for W=H_4, p=5, and ε a root of $g := y+4$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{6,12}}$	·	·	1	·	·
$\widehat{\varphi_{8,12}}$	1	1	1	·	·
$\widehat{\varphi_{9,2}}$	·	·	·	1	·
$\widehat{\varphi_{9,22}}$	·	·	·	·	1
$\widehat{\varphi_{16,6}}$	1	·	1	1	·
$\widehat{\varphi_{16,18}}$	·	1	1	·	1
$\widehat{\varphi_{18,10}}$	·	·	·	1	1

$\widehat{\varphi_{4,7}}$	1	·	·	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	·	·
$\widehat{\varphi_{8,13}}$	1	1	·	·	·
$\widehat{\varphi_{16,13}}$	·	·	1	·	·
$\widehat{\varphi_{16,3}}$	·	·	·	1	·
$\widehat{\varphi_{16,21}}$	·	·	·	·	1
$\widehat{\varphi_{36,5}}$	1	·	1	1	·
$\widehat{\varphi_{36,15}}$	·	1	1	·	1
$\widehat{\varphi_{48,9}}$	·	·	1	1	1

Table D.37.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=5, and ε a root of $g := y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{6,12}}$	·	·	1	·	·
$\widehat{\varphi_{8,12}}$	1	1	1	·	·
$\widehat{\varphi_{9,2}}$	·	·	·	1	·
$\widehat{\varphi_{9,22}}$	·	·	·	·	1
$\widehat{\varphi_{16,3}}$	1	·	1	1	·
$\widehat{\varphi_{16,21}}$	·	1	1	·	1
$\widehat{\varphi_{18,10}}$	·	·	·	1	1

$\widehat{\varphi_{4,7}}$	1	·	·	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	·	·
$\widehat{\varphi_{8,13}}$	1	1	·	·	·
$\widehat{\varphi_{16,13}}$	·	·	1	·	·
$\widehat{\varphi_{16,6}}$	·	·	·	1	·
$\widehat{\varphi_{16,18}}$	·	·	·	·	1
$\widehat{\varphi_{36,5}}$	1	·	1	1	·
$\widehat{\varphi_{36,15}}$	·	1	1	·	1
$\widehat{\varphi_{48,9}}$	·	·	1	1	1

Table D.38.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=5, and ε a root of $g := y^2 + y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{8,12}}$	·	·	1	·	·
$\widehat{\varphi_{10,12}}$	1	1	1	·	·
$\widehat{\varphi_{16,6}}$	1	·	·	1	·
$\widehat{\varphi_{16,18}}$	·	1	·	·	1
$\widehat{\varphi_{25,4}}$	2	·	1	1	·
$\widehat{\varphi_{25,16}}$	·	2	1	·	1
$\widehat{\varphi''_{30,10}}$	·	·	·	1	1

$\widehat{\varphi_{4,7}}$	1	·	·
$\widehat{\varphi_{4,37}}$	·	1	·
$\widehat{\varphi_{8,13}}$	·	·	1
$\widehat{\varphi_{16,13}}$	1	1	1
$\widehat{\varphi_{16,3}}$	2	·	1
$\widehat{\varphi_{16,21}}$	·	2	1

Table D.39.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=5, and ε a root of $g := y^2 + 4y + 1$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{8,12}}$	·	·	1	·	·
$\widehat{\varphi_{10,12}}$	1	1	1	·	·
$\widehat{\varphi_{16,3}}$	1	·	·	1	·
$\widehat{\varphi_{16,21}}$	·	1	·	·	1
$\widehat{\varphi_{25,4}}$	2	·	1	1	·
$\widehat{\varphi_{25,16}}$	·	2	1	·	1
$\widehat{\varphi''_{30,10}}$	·	·	·	1	1

$\widehat{\varphi_{4,7}}$	1	·	·
$\widehat{\varphi_{4,37}}$	·	1	·
$\widehat{\varphi_{8,13}}$	·	·	1
$\widehat{\varphi_{16,13}}$	1	1	1
$\widehat{\varphi_{16,6}}$	2	·	1
$\widehat{\varphi_{16,18}}$	·	2	1

Table D.40.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=5, and ε a root of $g \in \{y^2+3,\ y^2+2\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{8,13}}$	·	·	1	·	·
$\widehat{\varphi_{9,2}}$	1	·	·	1	·
$\widehat{\varphi_{9,22}}$	·	1	·	·	1
$\widehat{\varphi_{16,13}}$	·	·	·	1	1
$\widehat{\varphi_{25,4}}$	1	·	1	2	·
$\widehat{\varphi_{25,16}}$	·	1	1	·	2

Table D.41.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=5, and ε a root of $g \in \{y^2+3y+4,\ y^2+2y+4\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·	·
$\widehat{\varphi_{4,7}}$	1	·	1	·	·
$\widehat{\varphi_{4,37}}$	·	1	·	1	·
$\widehat{\varphi_{6,12}}$	·	·	1	1	·
$\widehat{\varphi_{18,10}}$	1	1	·	·	1
$\widehat{\varphi_{25,4}}$	3	·	2	·	1
$\widehat{\varphi_{25,16}}$	·	3	·	2	1

Table D.42.: Non-trivial blocks of the decomposition map $d_{\mathbb{k}}^{\mathbb{L}}$ for W=H_4, p=5, and ε a root of $g \in \{y^2+4y+2,\ y^2+2y+3,\ y^2+3y+3,\ y^2+y+2\}$

$\widehat{\varphi_{1,0}}$	1	·	·	·
$\widehat{\varphi_{1,60}}$	·	1	·	·
$\widehat{\varphi_{25,4}}$	1	·	1	·
$\widehat{\varphi_{25,16}}$	·	1	·	1
$\widehat{\varphi_{48,9}}$	·	·	1	1

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