Fundamental solutions in modeling of vibrations radiated from tunnels with 2.5D - BEM

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Abstract
In this work, we analyze a method to model vibrations within horizontally layered orthotropic soil. After taking the Fourier transform with respect to both horizontal variables x and y, the fundamental solution of the wave propagation has a simple structure in the wavenumber (k_x, k_y)-domain. Inside a single layer, the fundamental solution can be represented by a superposition of six waves with complex wavenumbers. Their corresponding coefficients are computed by solving a linear system obtained by imposing a boundary condition at the surface, continuity conditions at the interfaces between the layers, and a radiation condition at infinity. We study stabilizing strategies for the evaluation of the fundamental solution.
Keywords: Orthotropic layered soil, Fundamental solution, BEM

1 INTRODUCTION

Vibrations in soil coming from trains moving through tunnels have significant impact on the environment. Our objective is to model such vibrations numerically in order to develop countermeasures in the future. We develop a numerical algorithm for the simulation of such vibrations in orthotropically layered soil. We use a variant of the boundary element method called 2.5D-BEM [2, 3], which is applicable under the assumption that the tunnel is straight and sufficiently long.

A crucial step in the development of the BEM is the evaluation of the fundamental solution for a linear system relating displacements and stresses. For a transversely isotropic half-space, the fundamental solution is known explicitly [4]. Our model of the soil consists of an arbitrary number of horizontal soil layers with distinct material parameters. We give a detailed description of the method for the computation of the respective fundamental solution.

2 FUNDAMENTAL SOLUTION FOR ELASTODYNAMICS

Displacements u and strains ε,

$$u = \left[u_x, u_y, u_z\right]^T, \quad \varepsilon = \left[\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}\right]^T, \tag{1}$$

are related by the symmetric linear strain tensor [1, 3.22]

$$\varepsilon = \begin{bmatrix} \partial_x & & \\ & \partial_y & \\ & & \partial_z \\ \frac{1}{2}\partial_y & \frac{1}{2}\partial_x & 0 \\ 0 & \frac{1}{2}\partial_z & \frac{1}{2}\partial_y \\ \frac{1}{2}\partial_z & 0 & \frac{1}{2}\partial_x \end{bmatrix} u. \tag{2}$$

Given the flexibility matrix F for orthotropic soil,

$$F=\begin{bmatrix}F_1 & 0\\ 0 & F_2\end{bmatrix}=\begin{bmatrix}\frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{\nu_{zx}}{E_z} & & & \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_x} & -\frac{\nu_{zx}}{E_z} & & 0 & \\ -\frac{\nu_{zx}}{E_z} & -\frac{\nu_{zx}}{E_z} & \frac{1}{E_z} & & & \\ & & & \frac{1}{2G_{xy}} & 0 & 0\\ & 0 & & 0 & \frac{1}{2G_{zx}} & 0\\ & & & 0 & 0 & \frac{1}{2G_{zx}}\end{bmatrix}, \tag{3}$$

Hooke's law [1, 3.86] relates the stresses

$$\sigma=\left[\sigma_{xx},\sigma_{yy},\sigma_{zz},\sigma_{xy},\sigma_{yz},\sigma_{zx}\right]^T \tag{4}$$

to strains by

$$\sigma=F^{-1}\varepsilon. \tag{5}$$

Cauchy's equation of motion [1, 3.56] written in the frequency domain has the following form

$$\Delta^T\sigma+\rho\omega^2u=0, \tag{6}$$

where the differential operator Δ is defined by

$$\Delta=\begin{bmatrix}\partial_x & 0 & 0 & \partial_y & 0 & \partial_z\\ 0 & \partial_y & 0 & \partial_x & \partial_z & 0\\ 0 & 0 & \partial_z & 0 & \partial_y & \partial_x\end{bmatrix}^T. \tag{7}$$

2.1 Solution of the second order ODE

In the frequency-wavenumber (k_x,k_y,z,ω) domain, the derivative operator Δ becomes

$$\widetilde{\Delta}=\begin{bmatrix}\widetilde{\Delta}_1\\ \widetilde{\Delta}_2\end{bmatrix}, \tag{8}$$

where

$$\widetilde{\Delta}_1=\begin{bmatrix}\imath k_x & & \\ & \imath k_y & \\ & & \partial_z\end{bmatrix} \tag{9}$$

and

$$\widetilde{\Delta}_2=\begin{bmatrix}\imath k_y & \imath k_x & 0\\ 0 & \partial_z & \imath k_y\\ \partial_z & 0 & \imath k_x\end{bmatrix}, \tag{10}$$

and $\imath=\sqrt{-1}$.

The linear field equations (6) can be formulated in the frequency-wavenumber domain as follows

$$\left(\widetilde{\Delta}_1^T F_1^{-1}\widetilde{\Delta}_1+\widetilde{\Delta}_2^T\begin{bmatrix}G_{xy} & & \\ & G_{zx} & \\ & & G_{zx}\end{bmatrix}\widetilde{\Delta}_2+\rho\omega^2 I_3\right)\widetilde{u}=0. \tag{11}$$

This can be written as

$$(\partial_z^2 M + \partial_z C + K)\widetilde{u} = 0, \tag{12}$$

where the matrices K, C and M are given by

$$K = \begin{bmatrix} -k_x^2 H_{11} - G_{xy}k_y^2 + \rho\omega^2 & -k_x k_y H_{12} - G_{xy}k_x k_y & 0 \\ -k_x k_y H_{12} - G_{xy}k_x k_y & -k_y^2 H_{22} - G_{xy}k_x^2 + \rho\omega^2 & 0 \\ 0 & 0 & -G_{zx}(k_x^2 + k_y^2) + \rho\omega^2 \end{bmatrix}, \tag{13}$$

$$C = \iota \begin{bmatrix} 0 & 0 & k_x(H_{13} + G_{zx}) \\ 0 & 0 & k_y(H_{23} + G_{zx}) \\ k_x(H_{13} + G_{zx}) & k_y(H_{23} + G_{zx}) & 0 \end{bmatrix}, \tag{14}$$

and

$$M = \begin{bmatrix} G_{zx} & 0 & 0 \\ 0 & G_{zx} & 0 \\ 0 & 0 & H_{33} \end{bmatrix}. \tag{15}$$

Here, $H = F_1^{-1}$, and $H_{i,j}$ only depend on the parameters E_x, E_z, ν_{xy} and ν_{zx}.
Equation (12) is a three dimensional second order ordinary differential equation (ODE). By defining $\widetilde{v} = \partial_z \widetilde{u}$, it can be converted to the following six dimensional first order ODE

$$\partial_z \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix}. \tag{16}$$

If the matrix

$$\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \tag{17}$$

has distinct eigenvalues λ_m, $m = 1, \ldots 6$, with the corresponding eigenvectors $\begin{bmatrix} \Psi_{-,m} \\ \Phi_{-,m} \end{bmatrix}$, the solution $\begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix}$ of (16) can be written as a linear combination of the form

$$\begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} = \sum_{m=1}^{6} c_m \begin{bmatrix} \Psi_{-,m} \\ \Phi_{-,m} \end{bmatrix} e^{\lambda_m z}. \tag{18}$$

It has to be noted that the eigenvalues come in pairs, i.e. if λ_m is an eigenvalue, then so is $-\lambda_m$. Multiple eigenvalues require a separate treatment.

2.2 Fundamental solution in the (k_x, k_y, z)-domain for layered soil

We are considering multiple horizontal layers of soil, with distinct material parameters. We assume layer interfaces at depths $0 = d_0 < d_1 < \ldots < d_{L-1} < d_L = \infty$, and individual flexibility matrices F^l, $l = 1, \ldots, L$ on each layer $z \in [d_{l-1}, d_l]$, displacements are given by

$$\widetilde{u}(k_x, k_y, z)^l = \sum_{m=1}^{6} c_m^l \Psi_{-,m}^l e^{\lambda_m^l z} \quad z \in [d_{l-1}, d_l], \tag{19}$$

where λ_m^l are the eigenvalues of the respective matrix (17) for $F = F^l$, and $\Psi^l_{-,m}$ are the corresponding eigenvectors. Defining the matrix D by

$$D(k_x,k_y,k_z) = \imath \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \\ \frac{1}{2}k_y & \frac{1}{2}k_x & 0 \\ 0 & \frac{1}{2}k_z & \frac{1}{2}k_y \\ \frac{1}{2}k_z & 0 & \frac{1}{2}k_x \end{bmatrix}, \tag{20}$$

We obtain from Hooke's law (5) the following relation for the respective fundamental stresses

$$\widetilde{\sigma}(k_x,k_y,z)^l = \sum_{m=1}^{6} c_m^l \left[F^{l^{-1}} D(k_x,k_y,k^l_{z,m}) \Psi^l_{-,m} \right] e^{\lambda_m^l z}, \tag{21}$$

where $k^l_{z,m} = -\imath\lambda_m^l$. We note that this representation only holds when (17) has distinct eigenvalues. For $\xi_3 \geq 0$, we compute fundamental displacements $\widetilde{u}^* = \widetilde{u}^*(k_x,k_y,z;\xi_3;i)$ and stresses $\widetilde{\sigma}^* = \widetilde{\sigma}^*(k_x,k_y,z;\xi_3;i)$ such that

$$\widetilde{\Delta}^T \widetilde{\sigma}^* + \rho\omega^2 \widetilde{u}^* = e_i \delta(z-\xi_3), \tag{22}$$

where δ denotes the Dirac delta and e_i the i-th standard unit vector. Thus $e_i\delta(z-\xi_3)$ represents a load at depth ξ_3, acting in direction e_i. We discuss how to compute the respective coefficients c_m^l in (19). For this purpose, we introduce an interface at $d_{l^*} = \xi_3$, where we assume conitinuity in the displacement and a unit jump in the stresses

$$\widetilde{u}^*(k_x,k_y,d_{l^*};\xi_3;i)^{l^*} = \widetilde{u}^*(k_x,k_y,d_{l^*};\xi_3;i)^{l^*+1}, \tag{23}$$

$$\widetilde{\sigma}^*_{-,z}(k_x,k_y,d_{l^*};\xi_3;i)^{l^*} = \widetilde{\sigma}^*_{-,z}(k_x,k_y,d_{l^*};\xi_3;i)^{l^*+1} - e_i. \tag{24}$$

Continuity in the displacement as well as the stresses across the other layer interfaces give six conditions per layer interface, i.e. for $l = 1,\ldots,L-1,\ l \neq l^*$,

$$\widetilde{u}^*(k_x,k_y,d_l;\xi_3;i)^l = \widetilde{u}^*(k_x,k_y,d_l;\xi_3;i)^{l+1}, \tag{25}$$

$$\widetilde{\sigma}^*_{-,z}(k_x,k_y,d_l;\xi_3;i)^l = \widetilde{\sigma}^*_{-,z}(k_x,k_y,d_l;\xi_3;i)^{l+1}. \tag{26}$$

Because of the compatibility conditions for the strains [1, 3.23-3.28], only the components $\sigma_{zx} = \sigma_{xz}$, σ_{yz}, and σ_{zz} are considered in (26).
From the interfaces, we therefore obtain $6(L-1)$ equations for the coefficents c_m^l, $m = 1,\ldots,6$, $l = 1,\ldots,L$. The surface condition

$$\widetilde{\sigma}^*_{-,z}(k_x,k_y,0;\xi_3;i) = 0, \tag{27}$$

contributes three conditions on the coefficients $c_1^1,\ldots,c_6^1$. The radiation condition

$$\widetilde{u}^*(k_x,k_y,z;\xi_3;i) \to 0, \qquad \text{for } z \to \infty, \tag{28}$$

eliminates three of the coefficients $c_1^L,\ldots,c_6^L$, since we only consider the exponentials with negative real parts. This leaves us with $6L-3$ conditions for the $6L-3$ unknowns c_m^l.

2.3 The coefficients c_m^l as solution of a linear system

The coefficients $c_1^1,\ldots,c_6^1$ satisfy

$$\Sigma^1(k_x,k_y,0)\begin{bmatrix}c_1^1\\ \vdots\\ c_6^1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{29}$$

where the 3×6 matrices $U(k_x,k_y,z)^l$ and $\Sigma^l(k_x,k_y,z)$ are defined by

$$U^l(k_x,k_y,z)_{-,m}=\Psi^l_{-,m}e^{\lambda_m^l z}, \tag{30}$$

$$\Sigma^l(k_x,k_y,z)_{-,m}=\left[F^{l^{-1}}D(k_x,k_y,k_{z,m}^l)\Psi^l_{-,m}\right]_{-,z}e^{\lambda_m^l z}, \tag{31}$$

for $k_{z,m}^l=-\imath\lambda_m^l$.

For $l=1,\ldots,L-1,\ l\neq l^*$, the coefficients $c_1^l,\ldots,c_6^l$ and $c_1^{l+1},\ldots,c_6^{l+1}$ satisfy the 6×12 system

$$\begin{bmatrix}U^l(k_x,k_y,d_l) & -U^{l+1}(k_x,k_y,d_l)\\ \Sigma^l(k_x,k_y,d_l) & -\Sigma^{l+1}(k_x,k_y,d_l)\end{bmatrix}\begin{bmatrix}c_1^l\\ \vdots\\ c_6^l\\ c_1^{l+1}\\ \vdots\\ c_6^{l+1}\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}. \tag{32}$$

For $l=l^*$, the coefficients $c_1^{l^*},\ldots,c_6^{l^*}$ and $c_1^{l^*+1},\ldots,c_6^{l^*+1}$ satisfy the 6×12 system

$$\begin{bmatrix}U^{l^*}(k_x,k_y,d_{l^*}) & -U^{l^*+1}(k_x,k_y,d_{l^*})\\ \Sigma^{l^*}(k_x,k_y,d_{l^*}) & -\Sigma^{l^*+1}(k_x,k_y,d_{l^*})\end{bmatrix}\begin{bmatrix}c_1^{l^*}\\ \vdots\\ c_6^{l^*}\\ c_1^{l^*+1}\\ \vdots\\ c_6^{l^*+1}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-e_i\end{bmatrix}. \tag{33}$$

Assuming the eigenvalues λ_m^L to be ordered so that the real parts are increasing, the radiation conditions implies

$$E_2\begin{bmatrix}c_1^L\\ \vdots\\ c_6^L\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{34}$$

where

$$E_2=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}. \tag{35}$$

Summarizing equations (29), (32), (33) and (34) in a single linear system for all coefficients $c^1\ldots$

c^L, we obtain, after omitting the variables k_x and k_y for the sake of readability,

$$\begin{bmatrix} \Sigma^1(0) & & & & & & & & \\ U^1(d_1) & -U^2(d_1) & & & & & & & \\ \Sigma^1(d_1) & -\Sigma^2(d_1) & & & & & & & \\ & U^2(d_2) & -U^3(d_2) & & & & & & \\ & \Sigma^2(d_2) & -\Sigma^3(d_2) & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & \ddots & \ddots & & & & \\ & & & & U^{l^*}(d_{l^*}) & -U^{l^*+1}(d_{l^*}) & & & \\ & & & & \Sigma^{l^*}(d_{l^*}) & -\Sigma^{l^*+1}(d_{l^*}) & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & U^{L-1}(d_{L-1}) & -U^L(d_{L-1}) \\ & & & & & & & \Sigma^{L-1}(d_{L-1}) & -\Sigma^L(d_{L-1}) \\ & & & & & & & & E_2 \end{bmatrix} \begin{bmatrix} c^1 \\ c^2 \\ c^3 \\ \vdots \\ c^{l*} \\ c^{l^*+1} \\ \vdots \\ c^{L-1} \\ c^L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -e_i \\ \vdots \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{36}$$

After solving (36), one obtains the fundamental displacements and the fundamental stresses by (19) and (21), respectively.

3 OUTLOOK

The derived linear system for the coefficients (36) needs to be solved. Preliminary numerical experiments indicate that the system is ill-conditioned in certain regimes. After developing stabilization techniques, we use the obtained fundamental solution in a BEM for the simulation of vibrations emanating from railway tunnels, as described in [3].

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