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Fourier spectral method for acoustic simulation with domain enclosed by curved boundary

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Abstract

Curved geometries often make sound propagation complex. Such complexity may cause acoustical problems including flutter echo and sound focusing. When designing the geometry, acoustic simulation can be helpful to prevent such problems. Fourier spectral method (FSM) is a simulation method by approximation using the Fourier basis. Although FSM has many advantages such as its high convergence rate, the application range of the conventional FSM is limited to a simply-shaped domain with a specific boundary condition. In a realistic acoustics room setting, there exist a lot of room shapes beyond the scope of FSM. At the same time, the Fourier extension (FE) has been proposed for approximating a function on a complicated domain by the Fourier basis. It can be expected that the FE expands the application range of FSM. In this paper, we introduce FE into FSM for extending it to make a computational domain enclosed by curved boundaries tractable.

Keywords: Fourier spectral method, curved boundary, Fourier extension, function approximation

1 INTRODUCTION

Acoustic wave-based simulation has been widely studied for predicting or understanding acoustical phenomena. In wave-based method, a solution of the wave equation is approximated by arbitrary polynomial, which explains the wave properties well. Then, they are studied by finite difference time domain method (FDTD) [1–3] and finite element method (FEM) [4].

In the wave-based simulation for acoustics field, a complex-shaped domain is often focused because it makes sound field complex, and its prediction is difficult more than simple one. Sometimes curved boundary, associated with a complex-shaped domain, causes acoustical problems such as flutter echo or sound focusing. Some researches and simulations are conducted for preventing such problems [3, 5–7].

Fourier spectral method (FSM) is a simulation method by approximation using the Fourier basis. FSM is often used for fluid mechanics and earth science, and it has several advantages including its high convergence rate, but the application range of the conventional FSM is limited to a simply-shaped domain with a specific boundary condition because of characteristics of the Fourier basis. Since many real objects have a complex-shaped interior space, the conventional FSM cannot be applied to such space [8–12].

At the same time, the Fourier extension (FE) has been proposed for approximating a function on a complicated domain by the Fourier basis. It can approximate a nonperiodic function which is treated as a periodic function in an extended domain. Then, it can be expected that FE expands the application range of FSM [13, 14].

In this paper, we introduce FE into FSM for handling a domain enclosed by curved boundary. Since this is a first step, the Dirichlet boundary condition is considered in this paper. As a result, FSM becomes possible to handle a curved boundary defined on non-grid points. Some influence of parameter selection is evaluated and discussed by numerical simulation on a calabash-shaped domain.

2 FOURIER SPECTRAL METHOD

Spectral method is a numerical simulation method with high accuracy, because it globally approximates a function by high order polynomials. In spectral method, an unknown function $u(x)$ is approximated by N basis functions $\phi(x)$,

$$u(x) \approx u_N(x) = \sum_{k=0}^{N-1} a_k \phi_k(x). \tag{1}$$

One example of basis functions $\phi(x)$ is the Fourier basis used in the inverse discrete Fourier transform (iDFT),

$$u(x) = \sum_{k=0}^{N-1} \hat{u}(k) e^{\mathrm{i}kx\frac{2\pi}{N}}, \tag{2}$$

where $\hat{u}(k)$ is Fourier transform of $u(x)$, i means imaginary unit, and k means wave number. In this case, the differential operator can be expressed as

$$\frac{\partial^n u}{\partial x^n} = \sum_{k=0}^{N-1} \left(\frac{2\pi \mathrm{i} k}{N}\right)^n \hat{u}(k) e^{\mathrm{i}kx\frac{2\pi}{N}}. \tag{3}$$

This derivative is more accurate than the other methods as the finite difference method. In addition, it can be efficiently computed by the fast Fourier transform (FFT) algorithm. This differentiation via the Fourier bases is used to approximate the spatial derivative of a partial differential equation. However, owing to the periodicity of the Fourier bases, the standard spectral method can only be applied to simple domain and boundary condition, which is the main limitation [8–11].

2.1 Example of Fourier Spectral Method

In acoustic simulation, distribution of sound pressure can be expressed by

$$P^n = P(n\Delta t), \tag{4}$$

where Δt is discrete time steps, $n \in \mathbb{N}$ is the time index. In this paper, a grid points set constructing the P is on the Cartesian grid. Let the trapezoidal rule is considered for approximating the time integration. At first, the wave equation is denoted by

$$\frac{\partial^2 P}{\partial t^2} = c^2 \triangle P, \tag{5}$$

where c means sound speed, and $\triangle$ means the Laplace operator. When v is partial derivative of P with respect to t ($v = \partial P / \partial t$),

$$\frac{P^{n+1} - P^n}{\Delta t} = \frac{1}{2}\left(v^{n+1} + v^n\right) \tag{6}$$

is obtained by discrete integration using the trapezoidal rule. In this paper, initial condition of v (condition of v at $t = 0$) is 0. Then, the wave equation can be represented using v:

$$\frac{v^{n+1} - v^n}{\Delta t} = \frac{c^2}{2}\left(LP^{n+1} + LP^n\right), \tag{7}$$

where L is the second order spatial differential operator approximating the Laplace operator. Then, a scheme is obtained by Eq. (6) and Eq. (7),

$$\left\{1 - \left(\frac{\Delta t c}{2}\right)^2 L\right\} P^{n+1} = \left\{1 + \left(\frac{\Delta t c}{2}\right)^2 L\right\} P^n + \Delta t v^n. \tag{8}$$

Finally, Eq. (8) is calculated in FSM by,

$$\sum_{k=0}^{N-1}\left\{1-\left(\frac{\Delta tc}{2}\right)^2\hat{L}\right\}\hat{P}^{n+1}(k)e^{\mathrm{i}kx\frac{2\pi}{N}}=\sum_{k=0}^{N-1}\left[\left\{1+\left(\frac{\Delta tc}{2}\right)^2\hat{L}\right\}\hat{P}^{n}(k)+\Delta t\hat{v}^{n}(k)\right]e^{\mathrm{i}kx\frac{2\pi}{N}}, \tag{9}$$

where $\hat{L}$ is the frequency-domain representation of L [14].

3 FOURIER EXTENSION

Because of the Fourier bases's periodicity, simulations over a complex-shaped domain are difficult for the standard FSM. FE is a technique to approximate a nonperiodic function f on $[0,1]^2$ by the Fourier bases, where the periodicity of the Fourier bases is treated by extending the computational domain to $[0,T]^2$, and $T(>1)$ is an extension parameter. Therefore, FE is expected to widen the application range of the FSM.

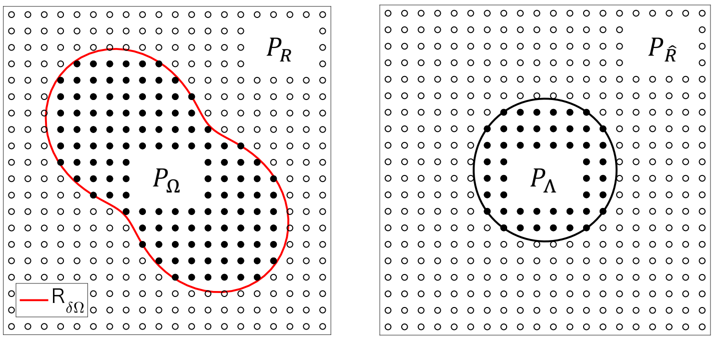
In this paper, we consider two-dimensional case. At first, we define some domains shown in Fig. 1. In this figure, P_R and $P_{\hat{R}}$ are extended spatial domain and frequency domain, respectively, P_Ω and P_Λ are set of N_Ω grid points and set of N_Λ bases, respectively, and a red line means the boundary $P_{\delta\Omega}$. Then, the main problem of FE is formulated by a least squares problem,

$$\mathbf{a}=\underset{\mathbf{c}\in\mathbb{C}^{N_\Lambda}}{\operatorname{argmin}}\sum_{\mathbf{x}\in P_\Omega}\left|f(\mathbf{x})-\sum_{\mathbf{k}\in P_\Lambda}\mathrm{c}_{\mathbf{k}}\phi_{\mathbf{k}}(\mathbf{x})\right|^2\quad\left(\phi_{\mathbf{k}}(\mathbf{x})=e^{\mathrm{i}\mathbf{k}^T\mathbf{x}\frac{2\pi}{T}}\right), \tag{10}$$

where $\mathbf{k}^T$ is the transpose of $\mathbf{k}$, $\mathbf{c}$ is the Fourier coefficients, and $\mathbf{a}$ is the solution set of $\mathbf{c}$. The solution $\mathbf{a}$ can be found by collocation, through solving the rectangular system,

$$\mathbf{A}\mathbf{a}=\mathbf{b}\quad,\mathbf{A}\in\mathbb{C}^{N_\Omega\times N_\Lambda},\quad\mathbf{b}\in\mathbb{C}^{N_\Omega}, \tag{11}$$

where $\mathbf{A}$ is the Fourier extension operator, a subblock of multi-dimensional unitary iDFT matrix. Then, $\mathbf{A}_S$ means the Fourier extension operator over spatial domain S. In addition, the boundary of the computational domain is contained inside of the extended domain. In this technique, over-sampling rate $q=N_\Omega/N_\Lambda$ is one of important parameter, $q>1$ is necessary for better accuracy [13, 14].



(a) Spatial domain (b) Frequency domain

Figure 1. Relation between the extended domain and original domain. The spatial domain Ω encompassing the grid points set P_Ω, and the frequency domain Λ encompassing the frequency points set P_Λ.

3.1 Fourier spectral method using Fourier extension

We introduce FE into FSM. According to Eq (9), this matrix system is

$$\begin{bmatrix} A_{\Omega} \\ A_{\delta\Omega_2}\left\{I-\left(\frac{\Delta tc}{2}\right)^2\hat{L}\right\}^{-1} \\ \vdots \\ A_{\delta\Omega_k}\left\{I-\left(\frac{\Delta tc}{2}\right)^2\hat{L}\right\}^{-1} \end{bmatrix} \left[\left\{I-\left(\frac{\Delta tc}{2}\right)^2\hat{L}\right\}\hat{P}_{n+1}\right] = \begin{bmatrix} A_{\Omega}\left[\left\{I+\left(\frac{\Delta tc}{2}\right)^2\hat{L}\right\}\hat{P}_n+\Delta t v_n\right] \\ wh_2 \\ \vdots \\ wh_k \end{bmatrix}, \tag{12}$$

where h means a boundary value, and w means a weight for boundary domain $\delta\Omega$. In this paper, $w = 1$ for simplicity.

4 NUMERICAL EXPERIMENT

4.1 Experimental Condition

In this paper, we calculate a sound field enclosed by the curved boundary shown in Fig. 1, which is given by

$$r(\theta) = 30\left\{\cos(2\theta + \frac{\pi}{2}) + 2.5\right\}, \quad \theta = [0, 2\pi), \tag{13}$$

and set the homogeneous Dirichlet boundary condition to it. Other experimental conditions are summarized in Table 1, and ζ_b is defined as ratio of points on boundary $N_{\delta\Omega}$ to N_{Ω},

$$\zeta_b = \frac{N_{\delta\Omega}}{N_{\Omega}}. \tag{14}$$

Table 1. Simulation condition.

N_R	2^{16}
N_{Ω}	2^{14}
N_{Λ}	$2^6, 2^8, 2^{10}$
ζ_b	1, 5, 10
Spatial discretization interval [m]	1
Time discretization interval [ms]	2
Sound speed [m/s]	340

4.2 Evaluation Error

In this paper, we evaluated three kinds of errors; Extension error, boundary error and energy error. In FE process, we solved the matrix equation using iterative method with error tolerance 1.0×10^{-10}. we defined the extension error,

$$E_{\Omega} = \frac{\|\mathbf{b} - \mathbf{A}_{\Omega}\mathbf{a}\|_2}{\|\mathbf{b}\|_2}. \tag{15}$$

where $\|\cdot\|_p$ is the ℓ_p-norm. In calculation without error, boundary value on $R_{\delta\Omega}$ is constant in every time steps. In this calculation, priority of boundary value in optimization is controlled by boundary weight. We defined the boundary error,

$$E_{\delta\Omega} = \|\mathbf{h} - \mathbf{A}_{\delta\Omega}\mathbf{a}\|_2. \tag{16}$$

This system have no theoretical decay factor, energy Q around R_Ω must be conserved. The energy around spatial domain is defined as

$$Q = \frac{(\tau + \sigma)}{Q_{max}}, \tag{17}$$

where

$$\tau = \frac{1}{2} \left\| \frac{\partial \mathbf{P}}{\partial t} \right\|_2^2, \quad \sigma = \frac{c^2}{2} \left\| \nabla \mathbf{P} \right\|_2^2, \tag{18}$$

and Q_{max} means the max value of Q over all the simulation time steps [15].

4.3 Simulation

The typical results of the wave propagation are shown in Fig. 2 with $N_\Lambda = 2^8$ and $\zeta_b = 5$. Fig. 2(a) illustrates the initial condition of this simulation, (b)-(f) show the radiation of the wave with time evolution.

At first, we performed simulation, with changing ζ_b, then, the obtained results are shown in Fig. 3. In this results, when ζ_b is 1, $E_{\delta\Omega}$ is higher than other cases, and Q is decreasing with time evolution. The cases of $\zeta_b = 5$ and $\zeta_b = 10$ suggest similar trends, but $\zeta_b = 10$ is more accurate than $\zeta_b = 5$.

In addition, the results of changing N_Λ with $\zeta_b = 5$ are in Fig. 4. According to Fig. 4(b), a trends of $E_{\delta\Omega}$ are almost same over every parameters. $N_\Lambda = 2^6$ is best condition about E_Ω, but its energy is considerably decreasing with time evolution. Then, the case of $N_\Lambda = 2^8$ has smaller value in E_Ω than the case of $N_\Lambda = 2^{10}$, but about energy Q conservation, the case of $N_\Lambda = 2^{10}$ is better than the case of $N_\Lambda = 2^8$. According to Fig. 3 and Fig. 4, ζ_b has influence of the boundary error $E_{\delta\Omega}$, and N_Ω has influence of the extension error E_Ω.

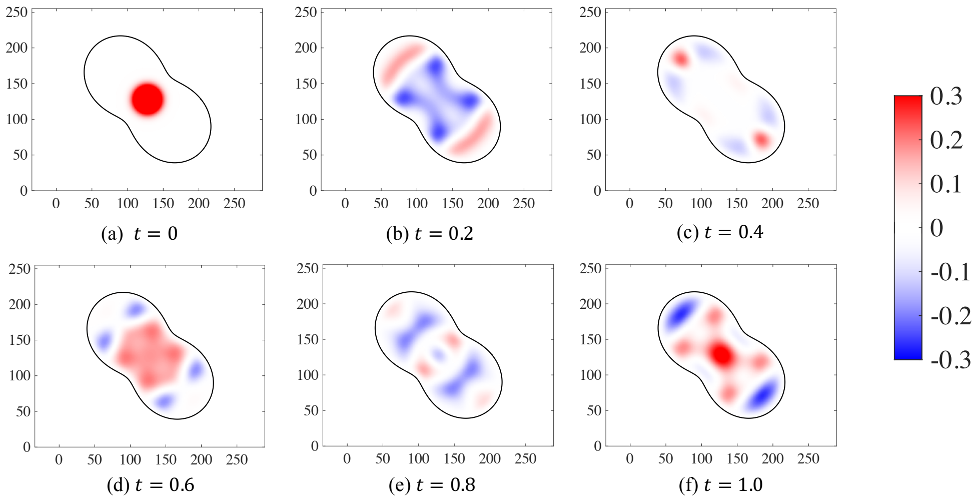
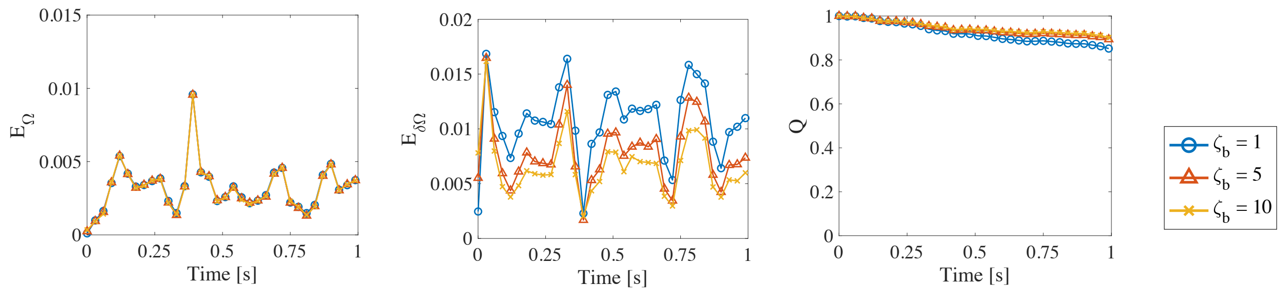


Figure 2. Snapshots of the P at t seconds, and their experimental conditions are $N_\Lambda = 2^8$ and $\zeta_b = 5$.

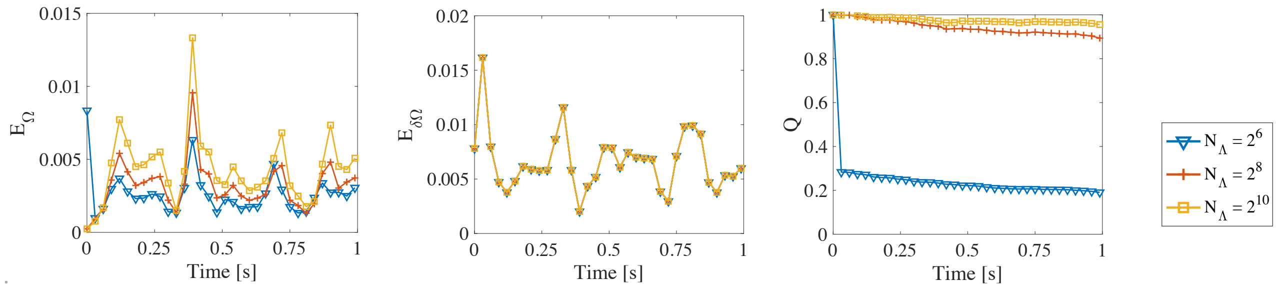
5 CONCLUSION

In this paper, we introduced FE into the FSM, and implemented simulations over the domain enclosed by the curved boundary. At first, the scheme of FSM for the wave equation is formulated by trapezoidal rule, and FE as a least squares problem is defined. In addition, the matrix system should be calculated directly was obtained by above all. Finally, we evaluated three kinds of error with changing dominant parameters for accuracy, and we confirmed the existence of parameter combinations having higher accuracy, $N_\Lambda = 2^8$, $\zeta = 5$ are one of better combination for parameters over this paper. Then, ζ_b has influence of boundary error $E_{\delta\Omega}$, and N_Ω has influence of extension error E_Ω.



(a) Extension error (b) Boundary error (c) Energy

Figure 3. Experimental results with $N_\Lambda = 2^8$ and changing ζ_b.



(a) Extension error (b) Boundary error (c) Energy

Figure 4. Experimental results with $\zeta_b = 5$ and changing N_Λ.

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