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Structural topology optimization for repeated eigenvalues with the adjoint sensitivity analysis

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Abstract

The sensitivity analysis of the eigenvectors corresponding to multiple eigenvalues is a challenging problem. The main difficulty is that for given multiple eigenvalues, the eigenvector derivatives can only be computed for a specific eigenvector basis, the so-called *adjacent* eigenvector basis. These adjacent eigenvectors depend on individual variables, which makes the eigenvector derivative calculation very elaborate and expensive from a computational perspective for problems. We present a new method avoiding passing through adjacent eigenvectors in the calculation of the partial derivatives of any prescribed eigenvector basis. Some examples, including one in the topology optimization, are provided to validate the present approach. The present paper is based on our research [1]

Keywords: eigenvector sensitivity repeated eigenvalue topology optimization

1 INTRODUCTION

Eigenproblems, i.e., problems regarding eigenvalues and/or eigenvectors are widely present today in differing research fields [1, 2, 3]. Whenever they constitute or form a part of the physics of optimization problems with numerous design variables, then a computationally efficient sensitivity analysis must be performed to enhance the performance. Topology optimization problems [4], in which a prescribed amount of a material is distributed in a design space to optimize an objective function while fulfilling some constraints, need to be particularly highlighted. Their discretized forms are finite-dimensional mathematical programming problems in which the number of design variables (called densities) is large because it typically coincides with the number of finite elements of the mesh. In such cases, the cost associated with the computation of the derivatives may be critical sometimes. In general, computing eigenvalue derivatives with respect to either a single or vector design variable is inexpensive, whereas computing eigenvector sensitivities is an issue that is relatively much more complex and expensive. In the case of single eigenvalues, the eigenvectors are differentiable, and there are well established methods for computing the eigenvector derivatives. Comparatively, the case of eigenvectors corresponding to multiple eigenvalues is more difficult because in this scenario, there exist infinitely many eigenvectors basis (orthonormal with respect to the mass matrix) associated with the multiple eigenvalues. This adversely affects the differentiability, and even the continuity, of the eigenvectors with respect to the design variables. In this case, when a single design variable is perturbed, the multiple eigenvalues split into several single eigenvalues. Moreover, it can be shown that the eigenvector basis corresponding to the multiple eigenvalues (before perturbing the design variable) *closer* to the eigenvectors of the single eigenvalues (after perturbing the design variable) is differentiable. This particular basis is called the *adjacent basis*, and in general, and it is a very important feature, which depends on each design variable. Any other eigenvector corresponding to the multiple eigenvalues can be expressed as a linear combination of the eigenvectors of the adjacent basis, for which the partial derivatives may be computed. Moreover, the partial derivatives of the target eigenvector can be obtained as a linear combination of those of the adjacent eigenvectors. In practical cases, e.g., in optimization, we are interested in computing the sensitivities of a functional depending of certain eigenvectors, which probably *follows* a prescribed reference mode shape, and with the existing methods in the literature, this can be achieved using the adjacent eigenvectors basis. From the implementation perspective, when we have to compute the partial derivatives with respect to numerous design variables, obtaining these results becomes computationally expensive and elaborate because as highlighted above, the adjacent eigenvectors depend on each design variable. In this paper, we propose a method that overcomes this difficulty.

2 AN EFFICIENT GENERAL METHOD FOR SENSITIVITY ANALYSIS OF EIGENVECTORS

Let λ be an eigenvalue of multiplicity m and Φ a matrix whose columns $\phi_1, \phi_2, \cdots, \phi_m$ are a basis of M-orthonormalized eigenvectors ($\Phi^{\mathrm{T}} M \Phi = I_m$) associated with λ. In such a case, the eigenproblem may be expressed as follows:

$$K\Phi = M\Phi\Lambda, \quad \Phi \in \mathbb{R}^{n\times m}, \quad \Lambda = \lambda I_m \in \mathbb{R}^{m\times m} \tag{1}$$

We start by computing the derivatives of the adjacent eigenvectors, Z, verifying

$$KZ = MZ\Lambda, \quad Z^{\mathrm{T}} M Z = I_m \tag{2}$$

For this purpose, let us consider the following objective function:

$$\tilde{G} = Z, \tag{3}$$

which indeed is differentiable, and is next modified to conveniently consider the augmented function,

$$\tilde{G} = Z + P^{\mathrm{T}}(KZ - MZ\Lambda) - \frac{1}{2}\Xi^{\mathrm{T}}(Z^{\mathrm{T}}MZ - I_m), \tag{4}$$

with the help of the adjoint method, where P and Ξ are the vector Lagrange multipliers. Differentiating and rearranging the terms, we arrive at

$$\tilde{G}' = P^{\mathrm{T}}(K' - \lambda M')Z - \frac{1}{2}\Xi^{\mathrm{T}} Z^{\mathrm{T}} M' Z, \tag{5}$$

being the pair (P, Ξ), the unique solution of the so-called adjoint system,

$$\begin{bmatrix} K - \lambda M & -MZ \\ -Z^{\mathrm{T}}M & 0 \end{bmatrix} \begin{bmatrix} P \\ \Xi \end{bmatrix} = \begin{bmatrix} -I_n \\ 0_{m\times n} \end{bmatrix} \tag{6}$$

which is not singular, as already proved using different techniques in [3].

Two-element cantilever beam with multiple eigenvalues

A two-element cantilever beam with multiple eigenvalues is considered, as shown in Fig. 1. This is a reference example in the computation of sensitivities of eigenvectors with repeated eigenvalues [2, 3]. The design variable is the z-axis area moment of the second element with the material properties shown in Fig. 1.

Without loss of generality, both the global stiffness and mass matrices with clamped boundary conditions are given by

$$K = \frac{E}{L}\begin{bmatrix} \frac{24I_z}{L^2} & 0 & 0 & 0 & -\frac{12I_z}{L^2} & 0 & 0 & \frac{6I_z}{L} \\ & \frac{24I_y}{L^2} & 0 & 0 & 0 & -\frac{12I_y}{L^2} & -\frac{6I_y}{L} & 0 \\ & & 8I_y & 0 & 0 & \frac{6I_y}{L} & 2I_y & 0 \\ & & & 8I_z & -\frac{6I_z}{L} & 0 & 0 & 2I_z \\ & & & & \frac{12I_z}{L^2} & 0 & 0 & -\frac{6I_z}{L} \\ & & sym & & & \frac{12I_y}{L^2} & \frac{6I_y}{L} & 0 \\ & & & & & & 4I_y & 0 \\ & & & & & & & 4I_z \end{bmatrix} \tag{7}$$

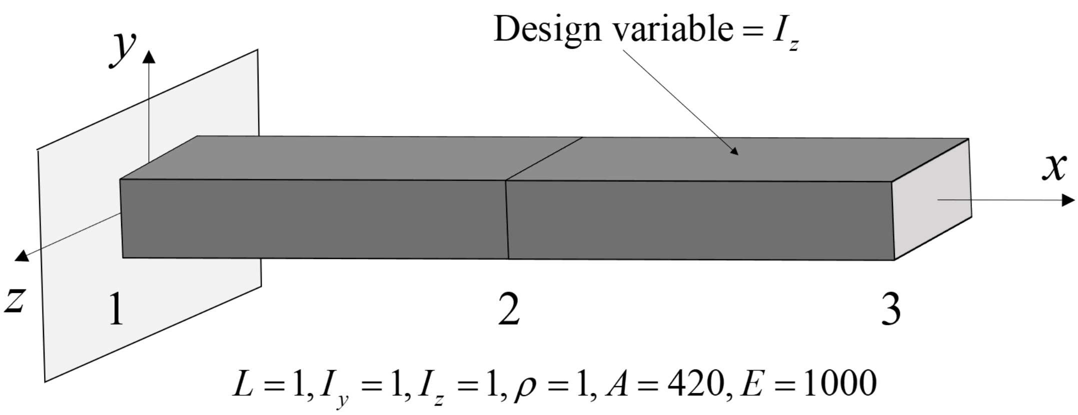

Figure 1. Two-element cantilever beam with multiple eigenvalues.

$$M = \frac{\rho AL}{420}\begin{bmatrix} 312 & 0 & 0 & 0 & 54 & 0 & 0 & -13L \\ & 312 & 0 & 0 & 0 & 54 & 13L & 0 \\ & & 8L^2 & 0 & 0 & -13L & -3L^2 & 0 \\ & & & 8L^2 & 13L & 0 & 0 & -3L^2 \\ & & & & 156 & 0 & 0 & -22L \\ & & sym & & & 156 & 22L & 0 \\ & & & & & & 4L^2 & 0 \\ & & & & & & & 4L^2 \end{bmatrix} \tag{8}$$

The displacement vector for the second and third elements is

$$q = [y_2, z_2, \theta_{y2}, \theta_{z2}, y_3, z_3, \theta_{y3}, \theta_{z3}]^T \tag{9}$$

Taking the z-axis area moment of the second element as the design variable, $b = I_z$, and using the material properties shown in Fig.1, the differentiation of the above matrices is as follows:

$$\frac{dK}{db} = \frac{E}{L}\begin{bmatrix} \frac{12}{L^2} & 0 & 0 & \frac{6}{L} & -\frac{12}{L^2} & 0 & 0 & \frac{6}{L} \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 4 & -\frac{6}{L} & 0 & 0 & 2 \\ & & & & \frac{12}{L^2} & 0 & 0 & -\frac{6}{L} \\ & & sym & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 4 \end{bmatrix}, \quad \frac{dM}{db} = 0 \tag{10}$$

In our computation environment, the eigenvectors and eigenvalues obtained are the following

$$\tilde{\Phi} = [\tilde{\phi_1}\ \tilde{\phi_2}] = \begin{bmatrix} -0.0026 & 0.0233 \\ 0.0233 & 0.0026 \\ -0.0399 & -0.0044 \\ -0.0044 & 0.0399 \\ -0.0075 & 0.0687 \\ 0.0687 & 0.0075 \\ -0.0473 & -0.0052 \\ -0.0052 & 0.0473 \end{bmatrix} \tag{11}$$

and $\lambda_1 = \lambda_2 = 1.8414$. Notice that the eigenvectors in (11) are different from the adjacent vectors reported [2, 3], but the rotation of 83.7 degrees after switching the columns of the eigenvectors converts them into the adjacent vectors found in [2, 3], i.e.,

$$\mathscr{Z} = \tilde{\Phi} \times \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{12}$$

where θ= 83.7 °, with $\mathscr{Z}$ the adjacent basis,

$$\mathscr{Z} = \begin{bmatrix} 0 & -0.0234 \\ -0.0234 & 0 \\ 0.0401 & 0 \\ 0 & -0.0401 \\ 0 & -0.0690 \\ -0.0690 & 0 \\ 0.0475 & 0 \\ 0 & -0.0475 \end{bmatrix} \tag{13}$$

The adjacent eigenvector sensitivities are

$$\mathscr{Z}' = \left[\frac{\partial \mathscr{Z}_1}{\partial b} \; \frac{\partial \mathscr{Z}_2}{\partial b} \right] = \begin{bmatrix} 0 & -0.0012 \\ 0 & 0 \\ 0 & 0 \\ 0 & -0.0019 \\ 0 & 0.0020 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.0051 \end{bmatrix} \tag{14}$$

and the sensitivities of the eigenvalues are given by

$$*' = \begin{bmatrix} 0 & 0 \\ 0 & 0.0918 \end{bmatrix} \tag{15}$$

Note that the off-diagonal values of Λ' are zero. The above sensitivity analysis implies that a change in the design variable changes one of the eigenvalues associated with Z. To show the accuracy of the computed sensitivity of the adjacent vectors with respect to the design variable, b, it is common to compute the ratio, R, of the values predicted by the design sensitivity analysis and those by the central finite difference method (approximated sensitivity values). In this work, we use the following formula:

$$R = \frac{(\mathscr{Z}(b+\Delta b) - \mathscr{Z}(b-\Delta b))/(2\Delta b)}{d\mathscr{Z}/db} \times 100(\%), \;\; \Delta b = 10^{-4} \tag{16}$$

It is considered that adjacent vectors should be employed to compute the sensitivity of the eigenvectors. However, as we stated in the previous section, this is actually not correct. To prove this, which is one of the major contributions of the present study, let us consider the following reference vectors which are the original eigenvectors, i.e., $\Phi^{ref} = \tilde{\Phi}$.

To determine the rotational coefficients aligning the computed eigenvectors in (11) to the target vectors, the optimization problem is solved. To compute the sensitivity analysis, i.e., $d\tilde{\phi_1}/db$, $d\tilde{\phi_2}/db$, the adjoint sensitivity analysis is performed, and the values are summarized in Table 2. For the central difference approach, it is necessary to compute the eigenvectors $\tilde{\Phi}(b+\Delta b)$ and $\tilde{\Phi}(b-\Delta b)$. To perform this, again two optimization problems have to be solved for $\tilde{\Phi}(b+\Delta b)$ and $\tilde{\Phi}(b-\Delta b)$ with the perturbation $\Delta b = 10^{-4}$.

Table 1. Design sensitivity coefficients of the two-beam problem and their comparison with the central finite difference method

$d\mathscr{Z}_2/db$	R (%)
-0.0012	100.00
-0.0000	100.00
0.0000	100.00
-0.0020	100.00
0.0021	100.00
-0.0000	100.00
0.0000	100.00
0.0051	100.00

Note that the double type digits are used for accurate representation:

$$\Phi(b+\Delta b) = \begin{bmatrix} -0.002557245554096 & 0.023312001772246 \\ 0.023311882479460 & 0.002557232468092 \\ -0.039928315387980 & -0.004379997393876 \\ -0.004380019189142 & 0.039928514074914 \\ -0.007531949884064 & 0.068661700775857 \\ 0.068661904719745 & 0.007531972255999 \\ -0.047257847757893 & -0.005184021614949 \\ -0.005183965914722 & 0.047257339991335 \end{bmatrix} \quad (17)$$

$$\Phi(b-\Delta b) = \begin{bmatrix} -0.002557219522960 & 0.023311763147863 \\ 0.023311882463730 & 0.002557232611496 \\ -0.039928315361036 & -0.004379997639496 \\ -0.004379975840005 & 0.039928116635597 \\ -0.007531995054570 & 0.068662108656116 \\ 0.068661904673411 & 0.007531972678375 \\ -0.047257847726001 & -0.005184021905656 \\ -0.005184077616588 & 0.047258355590119 \end{bmatrix} \quad (18)$$

Table 2 compares the adjoint sensitivity values with the central difference method using the ratio of the sensitivity values. As illustrated, the ratios are almost 100%, showing that the sensitivities are accurately computed. Interestingly, the derivatives of the eigenvalues, Λ'_{aux}, become as follows

$$\Lambda'_{aux} = \begin{bmatrix} 0.0011 & -0.0100 \\ -0.0100 & 0.0908 \end{bmatrix} \quad (19)$$

and it has the non-zero off-diagonal terms because the eigenvectors are not the adjacent ones. However, there exists a matrix, which we call here Γ_{aux} , being a solution of a small eigenvalue problem, 2×2, which relates Λ' (which is diagonal) with Λ'_{aux} in the following manner

$$\Lambda'_{aux}\Gamma_{aux} = \Gamma_{aux}\Lambda' \quad (20)$$

and, indeed, $Z' = \tilde{\Phi}'\Gamma_{aux}$ with

$$\Gamma_{aux} = \begin{bmatrix} -0.9940 & 0.1090 \\ -0.1090 & -0.9940 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \theta = 83.7^\circ \quad (21)$$

Table 2. Design sensitivity analysis of the two-beam problem with an arbitrary reference vector

$d\tilde{\phi}_1/db$	$d\tilde{\phi}_2/db$	$\Delta\tilde{\phi}_1/\Delta b$	$\Delta\tilde{\phi}_2/\Delta b$	R_1 (%)	R_2 (%)
-0.0001	0.0012	-0.0001	0.0012	100.0000	100.0066
-0.0000	0.0000	0.0000	-0.0000	100.0000	100.0000
0.0000	-0.0000	-0.0000	0.0000	100.0000	100.0000
-0.0002	0.0020	-0.0002	0.0020	100.0000	100.0068
0.0002	-0.0020	0.0002	-0.0020	100.9435	100.0000
-0.0000	0.0000	0.0000	-0.0000	100.0000	100.0000
0.0000	-0.0000	-0.0000	0.0000	100.0000	100.0000
0.0006	-0.0051	0.0006	-0.0051	100.2605	100.0000

Currently, the results of our present approach match with those of Dailey's method, but our method neither requires the second-order derivative terms of both the stiffness and mass matrices nor needs computing the adjacent basis.

3 CONCLUSIONS

In this paper, a highly efficient method for computing the derivatives of general functionals, which could eventually be vectorial magnitudes depending on eigenvectors, is developed. Our method includes the cases of both eigenvectors with and without repeated eigenvalues, but the emphasis is on the remarkably more difficult case of multiple eigenvalues. Our method overcomes the main difficulties of the sensitivity analysis of eigenvectors with repeated eigenvalues, providing a simple, direct, and highly efficient procedure from a computational perspective.

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