A physically and geometrically nonlinear formulation for isogeometric analysis of solids in boundary representation

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This contribution concerns a physically and geometrically nonlinear formulation for isogeometric analysis of solids in boundary representation. The parametrization of the proposed approach is inspired by the scaled boundary finite element method [3]. The domain is partitioned into sections \( \Omega \), which are partitioned into sections \( \Omega_{\gamma} \) an additive split of the strains and the associative flow rule read

Here \( \mu \) denotes the shear modulus and \( \Lambda \) the Lamé parameter. The second Piola-Kirchhoff stress is derived as

\[
\sigma = \mathcal{E} + \mathcal{E}^p,
\]

where \( \gamma \) denotes the consistency parameter and \( \mathcal{E}^p \) the plastic strain rate. The parametrization of the domain follows the idea of the scaled boundary finite element method [3]. The domain is partitioned into sections \( \Omega_{\Omega_0} \) in relation to a central point, the scaling center. Each section is parametrized by the boundary coordinate \( \eta \) and a radial scaling parameter \( \xi \), which runs from the scaling center with \( \xi = 0 \) to the boundary with \( \xi = 1 \). This approach requires the domain to be star-shaped. In case of complex geometries, sub-structuring can be applied [1,3]. The position vector on the boundary \( x \) from the scaling center with \( \xi = 1 \), which runs

\[
P = \int_{\Omega_0} \delta F^T : P \delta A - \int_{\Omega_0} \delta u^T : b_0 \delta A - \int_{\partial \Omega_0} \delta u^T : t_0 \delta S = 0 \quad \text{in} \quad \Omega_0.
\]

The Dirichlet and Neumann boundary condition is defined as \( u = \bar{u} \) on \( \partial_\Omega \Omega_0 \) and \( P \) on \( \partial_\Omega \Omega_0 \) respectively, where \( N \) represents the normal outward vector in the reference configuration. Nonlinear kinematics are considered where the deformation gradient and Green-Lagrange strain tensor are given as

\[
F = \text{Grad} x = I + \text{Grad} u, \quad E = \frac{1}{2}(F^T F - I).
\]

It is remarked that \( x \) is the position vector in the current configuration and \( u \) is the displacement vector. We consider a St. Venant-Kirchhoff material model with the strain energy function defined as

\[
W(E) = \frac{\Lambda}{2} (tr E)^2 + \mu (tr E^2).
\]

Here \( \mu \) denotes the shear modulus and \( \Lambda \) the Lamé parameter. The second Piola-Kirchhoff stress is derived as \( S = 2 \frac{\partial W}{\partial E} \). In order to account for physical nonlinearity we consider a J2 plasticity model with the von Mises yield condition \( f(S) \). The additive split of the strains and the associative flow rule read

\[
E = E^e + E^p, \quad \dot{E}^p = \gamma \frac{\partial f}{\partial S},
\]
domain \( x \) is given by \( x_s = N_b(\eta)X_s \) on \( \partial\Omega_{0,s} \) and \( x = x_0 + \xi(x_s(\eta) - x_0) \) in \( \Omega_{0,s} \) respectively, where \( x_0 \) is the position vector of the scaling center, \( N_b \) is the NURBS shape functions matrix and \( X_s \) contains the coordinates of the boundary control points. The transformation of the geometry in scaled boundary coordinates results in a multiplicative structure of the Jacobian matrix, see [1] for a more detailed description. For the approximation of the displacement solution we employ NURBS in both parametric directions according to the isogeometric paradigm [4]. Therefore it holds for the displacements and virtual displacements \( u(\eta, \xi) = N_b(\eta)N_s(\xi)U_j \) and \( \delta u(\eta, \xi) = N_b(\eta)\xi U_j \) respectively. Here, \( U_j \) denotes the displacement vector of the control points per radial scaling line in the interior of the domain. Employing the approximation with NURBS to Eq.(1), the discretized weak form reads

\[
\sum_{s=1}^{\text{nsec}} \delta U_j^T \left[ \int_{\Omega_{0,s}} \xi D^h T \hat{F} T S \ J \xi \xi d\eta U_j - \int_{\Omega_{0,s}} \xi N^T b J \xi \xi d\eta - \int_{\partial\Omega_{0,s}} N^T b^2 \hat{F} S l \xi U_j \right] = 0. \tag{5}
\]

Here, \( N = N_b(\eta)N_s(\xi) \) are the shape functions per radial scaling line, \( \hat{F} \) contains the components of the deformation gradient and \( b_2 \) the derivatives of \( x \) in respect to \( \xi \). Furthermore, \( D^h \) is the discretized differential operator, \( J \) the determinant of the Jacobian matrix \( J(\eta) \) and \( l \) the scaling length in analogy to [2]. The linearization of the weak form can be achieved by applying the iterative Newton-Raphson scheme. We have gathered so far all necessary equations to perform analysis of solid surfaces including physical and geometrical nonlinearities.

## 2 Numerical Example

We apply the proposed formulation for the analysis of a wrench with material and geometrical nonlinearities. The geometry, loading and boundary conditions are shown in Fig. 1. The depicted dimensions of the open-jaw and handle are \( R_1 = 13 \) mm, \( R_2 = 16 \) mm, \( R_3 = 47 \) mm and \( R_4 = 14.5 \) mm respectively. The thickness is assumed as \( t = 1 \) mm. The material properties are the Young’s modulus \( E = 2.1 \times 10^5 \) MPa and Poisson’s ratio \( \nu = 0.3 \). Plane strain condition is applied. The system is modeled by cubic B-Splines and sub-structured as depicted in Fig. 1. The proposed formulation is compared to multi-patch isogeometric analysis (IGA). First we perform elastic analysis including geometrical nonlinearities. Additionally, we also consider elasto-plastic behavior with a yield stress of \( \sigma_y = 1000 \) MPa. Fig. 1 depicts the vertical displacement at the top of the handle over the length of the handle. In both cases, we observe a very good agreement with IGA which indicates the capability of the proposed formulation to analyze complex geometries with physical and geometrical nonlinearities.

![Fig. 1: Left: Problem definition (top), parametrization (bottom). Right: Vertical displacements [mm] over the length of the handle.](image)

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**References**
