Designs in Finite Geometry

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Thesis Declaration

I, Jesse Lansdown, certify that:

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Abstract

This thesis is concerned with the study of Delsarte designs in symmetric association schemes, particularly in the context of finite geometry. We prove that m-ovoids of regular near polygons satisfying certain conditions must be hemisystems, and as a consequence, that for $d \geqslant 3$ m-ovoids of $\mathsf{DH}(2d-1,q^2)$, $\mathsf{DW}(2d-1,q)$, and $\mathsf{DQ}(2d,q)$ are hemisystems. We also construct an infinite family of hemisystems of $\mathsf{Q}(2d,q)$, for q an odd prime power and $d \geqslant 2$, the first known family for $d \geqslant 4$. We generalise the AB-Lemma to constructions of m-covers other than just hemisystems. In the context of general Delsarte designs, we show that either the size of a design, or the strata in which it lies, may be constrained when certain Krein parameters vanish, and explore various consequences of this result. We also study the concept of a "witness" to the non-existence of a design, in particular by considering projection and inclusion of association schemes, and the implications this has on the existence of designs when the strata of a projected design is constrained. We furthermore introduce *strong semi-canonicity* and use it in a black-box pruned orderly algorithm for effective generation of designs and combinatorial objects. We use these techniques to find new computational results on various m-ovoids, partial ovoids, and hemisystems.

Kurzbeschreibung

Diese Dissertation behandelt Delsarte-Designs von symmetrischen Assoziationsschemata, insbesondere im Kontext der endlichen Geometrie. Wir beweisen, dass m-Ovoide in regulären Fastpolygonen, unter bestimmten Bedingungen Hemisysteme sein müssen. Als Folgerung zeigen wir, dass für $d \geqslant 3$ m-Ovoide in $\mathsf{DH}(2d-1,q^2)$, $\mathsf{DW}(2d-1,q)$, und $\mathsf{DQ}(2d,q)$, Hemisysteme sind. Darüber hinaus konstruieren wir eine unendliche Familie von Hemisystemen in $\mathsf{Q}(2d,q)$, für q ungerade und $d \geqslant 2$. Für $d \geqslant 4$ stellt dies die erste bekannte Familie dar. Wir verallgemeinern das AB-Lemma zur Konstruktion der m-Abdeckungen, über den Hemisystem Fall hinaus. Im Kontext von allgemeinen Delsarte-Designs zeigen wir, dass entweder die Größe des Designs oder die zugehörigen Eigenräume, in denen es liegt, unter der Voraussetzung, dass bestimmt Kreinparameter Null sind, gewissen Einschränkungen unterliegen, und besprechen anschließend die verschiedenen Implikationen dieses Ergebnisses. Außerdem betrachten wir Kriterien für die Nicht-Existenz eines Designs, indem wir insbesondere die Projektion und Inklusion von Assoziationsschemata und deren Bedeutung für die Existenz von Designs betrachten, wenn die Eigenräume des projizierten Designs Einschränkungen unterliegen. Des Weiteren stellen wir das Konzept der starken Halbkanonizität vor und nutzen dieses in einem ,,black-box"-eingrenzenden Orderly Algorithmus zur effektiven Erzeugung von Designs und kombinatorischen Objekten. Wir verwenden diese Methoden, um neue rechnergestützte Ergebnisse von m-Ovoiden, Teilovoiden und Hemisystemen zu erhalten.

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Outside of maths, I have been blessed with many incredible friendships. Although I could list many, I particularly want to thank Jonas and Friederike, Howard and Yveena, and Aaron for their part in my life these last few years during my PhD.

I am thankful for the love and support of my family, in particular my parents, Andrew and Susan. What I am today, I owe in large part to the foundation that they laid.

I am thankful for brothers and sisters first at Maylands Baptist, then at the Gemeinde am Michaelsberg, and currently at St Matthew's. I thank God for their faithfulness and love, and for encouraging me to keep my eyes always fixed on the cross. Above all else, I praise God, my creator and sustainer, for his wonderful grace.

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Authorship Declaration

This thesis contains work that has been published and work prepared for publication.

Details of the work: J. Bamberg, J. Lansdown, and M. Lee. On m-ovoids of regular near polygons. *Des. Codes Cryptogr.*, 86(5):997–1006, 2018
Location in thesis: Chapter 5.
Student contribution to work: Melissa was able to show computationally that there were no m-ovoids of $\mathsf{DW}(5, q)$ but that they do exist in $\mathsf{DQ}(6, q)$ for $q \in \{3, 5\}$, and observed that this implied non-existence in $\mathsf{DH}(5, q^2)$. This prompted our interest in the existence of hemisysystems for $\mathsf{DW}(5, q)$. John and I then enumerated all non-isomorphic hemisystems of $\mathsf{DQ}(6, 5)$ with automorphism groups D_{60} and D_{20}. John observed that we could use the design orthogonal vector in [48] for one count in the proof of Theorem 5.2.2, and I obtained the remaining combinatorial counts required by the proof (pages 66-69). I was responsible for writing the paper, with John and Melissa contributing to editing. Any additions since the paper are my own, notably the additional computational results.
Co-author signatures and dates: John Bamberg 15/12/2020

15/12/20

Details of the work: J. Lansdown and A. C. Niemeyer. A family of hemisystems on the parabolic quadrics. *J. Combin. Theory Ser. A*, 175:105280, 2020
Location in thesis: Chapter 6.
Student contribution to work: I am responsible for discovering that Theorem 6.3.1 held computationally for small values of q, realising the AB-Lemma applied, and geometrically determining the groups A and B. Alice suggested a key step of isolating the subspace W and helped describe A and B with respect to W. I then showed that each of the properties needed for the AB-Lemma were indeed satisfied in this construction. I was responsible for the remarks in 6.4. I was also responsible for writing the paper, with Alice contributing to editing. Any additions since the paper are my own, notably the generalisation of the AB-Lemma in Theorem 6.1.2.
Co-author signatures and dates:

Alice 15.12.2020

Details of the work: In preparation
Location in thesis: Chapter 4.
Student contribution to work: I am solely responsible for Chapter 4, in particular Lemma 4.5.1, Algorithm 3, and the various computational results. I intend to develop Chapter 4 into a sole author publication.
Co-author signatures and dates: NA

Details of the work: In preparation.
Location in thesis: Chapter 7.
Student contribution to work: I am responsible for the mathematical content of Chapter 7, notably Theorem 7.2.2 and its consequences, with the exception of Theorem 7.5.1, which was done in collaboration with John Bamberg. John also drew my attention to the existence of Theorem 7.2.1, without which I would not have arrived at Theorem 7.2.2. I intend to develop Chapter 7 into a joint publication with John Bamberg.
Co-author signatures and dates: John Bamberg 15/12/2020

Details of the work: In preparation.
Location in thesis: Chapter 8.
Student contribution to work: I am responsible for the mathematical content of Chapter 8, in particular Theorem 8.1.3, Theorem 8.2.2, Theorem 8.2.4 and Section 8.6. The exception is the application of the MacWilliams transform in Section 8.4 and to use the *MeatAxe* to find submodules in Section 8.5, which were John Bamberg's ideas. I intend to develop the ideas of Section 8.1 and Section 8.2 for publication, though significant work is still required. This may mean taking on additional authors as the ideas are developed.
Co-author signatures and dates: John Bamberg 15/12/2020

Student signature: M Lansdown
Date: 15/12/2020

I, John Bamberg certify that the student's statements regarding their contribution to each of the works listed above are correct.

Coordinating supervisor signature: John Bamberg
Date: 15/12/2020

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Chapter 1

Introduction

This thesis is largely concerned with the study of *"things which meet other things in a nice manner"*. Having determined in which *things* and which *other things* we are interested, and, moreover, what manner of meeting constitutes *nice*, we turn to the following questions: *Do such things exist? If so, how are they constructed? If not, can it be proven?*

Of course, this is a vague explanation, better served by a mathematical description. The concept of a *design* is perhaps the best suited to the task. A $t-(\nu,k,\lambda)$ *design* is a set of k-subsets on ν points such that every t-subset is contained in exactly λ different k-subsets. There are various generalisations of designs, for example we may consider their q-analogues over the field $\mathbb{F}_q$, where we consider subspaces, rather than subsets. In his PhD thesis [49], Delsarte introduced the concept of a T-*design*, which we often refer to as a *Delsarte design* in his honour. Delsarte designs provide a unifying approach to the study of various *design-like* objects from fields including coding theory, design theory, and geometry.

In Chapter 2 we cover background material on association schemes. Association schemes in a sense generalise distance regular graphs, and occur in many settings. Crucially, they allow for a decomposition of $\mathbb{C}^n$ into mutually orthogonal eigenspaces, from which we can ascribe all sorts of algebraic properties to combinatorial objects. According to Delsarte and Levenshtein [51], *'In coding theory and related subjects, an association scheme (such as the Hamming scheme) should mainly be viewed as a "structured space" in which the objects of interest (such as codes, or designs) are living'*. This philosophy permeates this thesis.

The use of symmetry also plays an important role throughout this thesis. It is present in determining the canonicity of objects in Chapter 4, in construction of objects via orbit stitching and the AB-Lemma in Chapter 6, and in the theory of witnesses in Chapter 8, with the consideration of G-modules.

Chapter 3 introduces the relevant geometric foundations. In particular, we define polar spaces, generalised quadrangles, and their duals. The various objects that we handle throughout this thesis, including m-ovoids and hemisystems, are geometric in nature. Unsurprisingly, these geometric objects are examples of Delsarte designs.

As anyone who has computationally attempted large combinatorial problems will attest, a computer is a limited tool. Typically, there is a large divide between problems which are computationally feasible, and problems which are not, with a strong correlation between their feasibility and the interest that they command. As a result, the most tantalising computational problems lie right on the bound of feasibility, meaning that efficient methods are required. Chapter 4 concerns itself with such computational methods. In particular, we introduce the concept of *strong semi-canonicity* in Lemma 4.5.1. We further explore an orderly algorithm which utilises strong semi-canonicity and a black box pruning method in Algorithm 3. We demonstrate the value of these methods by extending various bound and classification results. We also provide an overview of software developed during the course of this thesis, namely *Gurobify* [80], and *AssociationSchemes* [9].

Motivated by a problem of Vanhove [125, Theorem 6.7.8], we consider m-ovoids of regular near polygons in Chapter 5. In Theorem 5.2.2, we prove that the only m-ovoids of certain regular near polygons are hemisystems. As a consequence we obtain in Theorem 5.2.1 that m-ovoids of $\mathsf{DQ}(2d, q)$, $\mathsf{DW}(2d-1, q)$, and $\mathsf{DH}(2d-1, q^2)$, for $d \geqslant 3$, are hemisystems. We then turn to hemisystems of the parabolic quadrics in Chapter 6. We provide the first known (for $d \geqslant 4$) infinite family of hemisystems of $\mathsf{Q}(2d, q)$ for $d \geqslant 2$ and q odd. We do this by considering the orbits of suitable groups on maximals and points, and applying the AB-Lemma. We also show that the AB-Lemma can be generalised to construct other designs.

It is often very difficult to show that a design does not exist, since if it does in fact not exist, then we obviously cannot handle it concretely, and instead we must show that its existence would imply certain properties which cannot be satisfied. Sometimes this can be achieved by counting arguments, such as in Chapter 5. In Chapter 7 we introduce a method for constraining designs based on vanishing Krein parameters in Theorem 7.2.2, which often has the consequence of proving nonexistence of designs that do not consist of half of the vertex set of the association scheme. We then consider various consequences of Theorem 7.2.2 and its application to association schemes with different properties.

In Chapter 8 we develop techniques for proving nonexistence of designs by exploring the concept of *witnesses* to their nonexistence. We thereby aim to transfer the difficulty of proving nonexistence of a design to the construction of a viable witness to its nonexistence. We investigate two main techniques, both of which rely on decomposition of the strata of an association scheme. In the first method we attempt to project designs to association schemes on a smaller number of vertices, and consider the projection of the strata in which they must lie (Theorem 8.2.2). If we can show that a projected design is constrained in its strata, then by Theorem 8.2.4 we also constrain the original design, often resulting in nonexistence. This is a promising approach, since it indicates that it is often sufficient to simply constraining a projection, rather than to eliminate the possibility of feasible projections entirely. The second method considers a decomposition of antidesign strata to produce witnesses. This technique also appears promising, but requires more sophisticated algorithms than are currently available for it to scale. We illustrate both methods by respectively giving proofs that m-ovoids of $\mathsf{Q}^-(5,q)$ are hemisystems, and that ovoids of $\mathsf{W}(3,q)$ do not exist.

Finally, we note that the results of Chapter 7 and Chapter 8 are broader in scope than a purely geometric setting. They hold for general association schemes and Delsarte designs, and we hope that they will yield further, fruitful avenues of research.

Chapter 2

Algebraic Combinatorics

Many interesting geometric objects can be described algebraically as cliques or co-cliques in graphs, or as Delsarte T-designs on the vertex set of an association scheme. This chapter covers some of the fundamental theory of graphs (in particular distance-regular graphs), association schemes, and subsets of association schemes with special properties.

2.1 Graphs

A *graph* is an ordered pair $\Gamma = (V(\Gamma), E(\Gamma))$, with a non-empty set of *vertices* $V(\Gamma)$ and a set of *edges* $E(\Gamma)$, where the edges are subsets of $V(\Gamma)$ of size two. Throughout this thesis we assume that edges are undirected, there are no loops, and no multiple edges. We further assume that graphs are finite; that is, $V(\Gamma)$ and $E(\Gamma)$ are finite. The *order* of a graph is the cardinality of the vertex set, which we shall typically denote by ν.

If $\{x, y\}$ is an edge, then the vertices x and y are said to be *adjacent*, or *neighbours*. We denote adjacency between two vertices x and y by $x \sim y$. We denote non-adjacency by $\not\sim$. Since our graphs are undirected, $\sim$ is a symmetric binary relation on the vertex set. Moreover, since our graphs are loopless, $x \sim y$ indicates that x and y are distinct. The *complement* $\overline{\Gamma}$ of a graph Γ is the graph formed by replacing edges with non-edges (i.e. an unordered pair of non-adjacent vertices), and non-edges with edges. Thus the adjacency relation of $\overline{\Gamma}$ is non-adjacency in Γ.

We define the *neighbourhood* of a vertex x by

$$\Gamma(x) := \{y \in V(\Gamma) \mid y \sim x\},$$

the set of vertices which are adjacent to x. The *valency* or *degree* of a vertex x is its number of neighbours, $|\Gamma(x)|$. If every vertex has degree k, then the graph is said to be *k-regular*, or *regular* with *degree k*. A graph is *strongly regular with parameters*

$$(\nu, k, \lambda, \mu),$$

if it is regular of degree k on ν vertices, and every pair of adjacent vertices have exactly λ common neighbours, while every pair of non-adjacent vertices have exactly

μ common neighbours. The complement of a strongly regular graph is also strongly regular, and has parameters

$$(\nu, \nu - k - 1, \nu - 2 - 2k + \mu, \nu - 2k + \lambda).$$

Example 2.1.1. *Figure 2.1 depicts the Petersen graph. It is a strongly regular graph with parameters* $(10, 3, 0, 1)$*. Figure 2.2 depicts the complement of the Petersen graph. It is also a strongly regular graph, with parameters* $(10, 6, 3, 4)$*.*

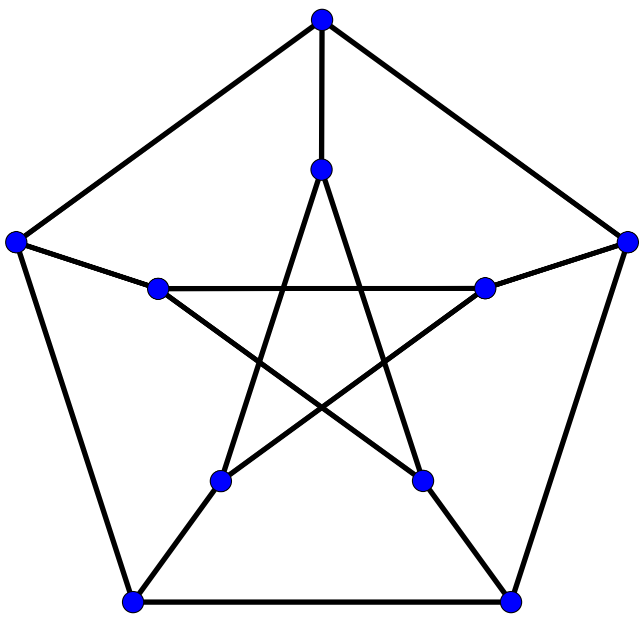


Figure 2.1: The Petersen graph.

A *complete graph* is a graph in which every pair of vertices is adjacent. A complete graph with $\nu \in \mathbb{N}$ vertices is denoted by K_ν.

A graph Γ' is a *subgraph* of Γ if $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. Γ' is an *induced subgraph* of Γ if $E(\Gamma')$ consists of all the edges of $E(\Gamma)$ which contain the elements of $V(\Gamma')$, and hence is determined by $V(\Gamma')$ and Γ.

A *clique* in a graph is a set of pairwise adjacent vertices, while a *coclique*, or *independent set*, is a set of pairwise non-adjacent vertices. A clique in a graph Γ is a coclique in the complement $\overline{\Gamma}$. Observe that a clique induces a complete subgraph, and a coclique induces an empty subgraph. A clique (respectively coclique), is called *maximal* if it is not contained in any larger clique (respectively coclique). A clique (coclique) is called *maximum* if there is no other clique (coclique) with greater cardinality. The cardinality of a maximum clique is referred to as the *clique number* of the graph, often denoted by $\omega(\Gamma)$, while the cardinality of a maximum coclique is referred to as the *coclique number*, or *independence number*, often denoted by $\alpha(\Gamma)$. If the vertex set of a graph can be divided into two disjoint cocliques, then the graph is said to be *bipartite*.

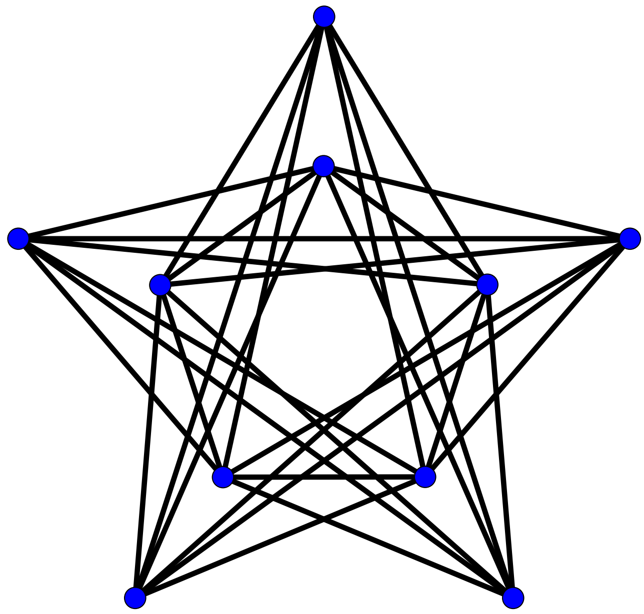


Figure 2.2: The complement of the Petersen graph.

Example 2.1.2. *Figure 2.3 highlights with red vertices a maximum coclique of the Petersen graph.*

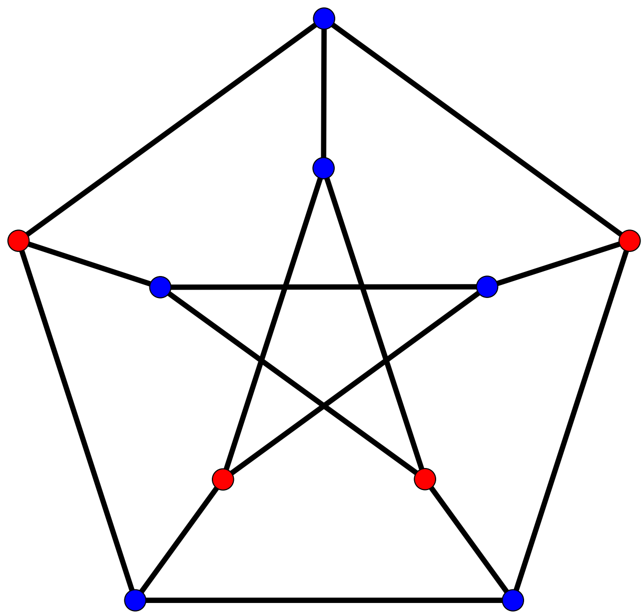


Figure 2.3: Maximum coclique in the Petersen graph.

An *isomorphism* between two graphs is a bijection between the vertex sets which preserves adjacency and non-adjacency. That is, two graphs Γ and Γ' are isomorphic if there exists a bijection $\varphi : V(\Gamma) \to V(\Gamma')$ such that $x \sim y$ if and only if $\varphi(x) \sim \varphi(y)$, for all $x, y \in V(\Gamma)$. An isomorphism from a graph to itself is an *automorphism*. It is a difficult problem in general to determine if two graphs are isomorphic. Chapter 4 explores various algorithms and computational methods which involve graph isomorphisms, and the related problem of determining canonicity.

Example 2.1.3. *Figure 2.4 and Figure 2.1 are isomorphic depictions of the Petersen graph.*

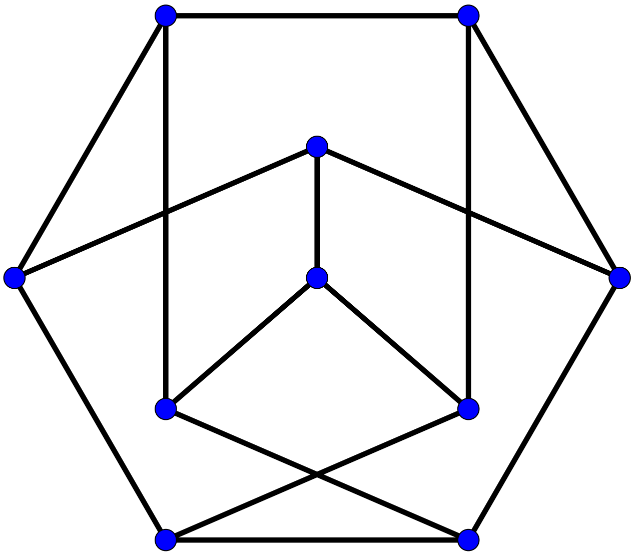


Figure 2.4: Isomorph of the Petersen graph.

The *adjacency matrix* of a graph Γ, is the $\nu \times \nu$ matrix A indexed by the elements of V, where

$$A_{xy} = \begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{if } x \not\sim y \end{cases}.$$

Since adjacency is a symmetric relation, the adjacency matrix is a symmetric matrix. Note that the adjacency matrix is given with respect to an ordering of the vertices. However, given any two orderings of the vertices, there exists a permutation matrix P such that $A' = P^\top AP$, where A is the adjacency matrix with respect to the first ordering, and A' with respect to the second. As a result, we refer to *the* adjacency matrix. In fact, two graphs are isomorphic if and only if there exists such a permutation matrix P satisfying this equation with their adjacency matrices. Moreover, the characteristic polynomials of A and A' are the same, and hence the eigenvalues and their multiplicities are the same independent of the vertex ordering. We refer to the eigenvalues of the adjacency matrix as the *eigenvalues of* Γ. The *spectrum* of a graph is the set of its eigenvalues λ_i, together with their multiplicities m_i, usually expressed

$$\{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots\}.$$

It is well known that a (connected) strongly regular graph has exactly three distinct eigenvalues,

$$\begin{aligned}\lambda_0 &= k,\\ \lambda_1 &= \frac{(\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2},\\ \lambda_2 &= \frac{(\lambda-\mu)-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2},\end{aligned}$$

with multiplicities

$$\begin{aligned}m_0 &= 1,\\ m_1 &= \frac{(\nu-1)\lambda_2+k}{\lambda_2-\lambda_1},\\ m_2 &= \frac{(\nu-1)\lambda_1+k}{\lambda_1-\lambda_2}.\end{aligned}$$

Additionally, $\lambda_1 \geqslant 0$ and $\lambda_2 \leqslant -1$.

Lemma 2.1.4 (cf. [60, Lemma 10.2.1]). *A connected, regular graph is strongly regular if and only if it has precisely three eigenvalues.*

The spectrum of a graph reveals a lot of useful information. We now present some important spectral results on the clique number $\omega(\Gamma)$ and coclique number $\alpha(\Gamma)$.

Theorem 2.1.5 (Ratio bound [67]). *Let Γ be a k-regular graph of order ν. If λ_{min} is the smallest eigenvalue of Γ, then*

$$\alpha(\Gamma) \leqslant \frac{\nu}{1-\frac{k}{\lambda_{min}}}.$$

Theorem 2.1.6 ([49]). *Let Γ be a strongly regular graph of degree k with smallest eigenvalue λ_{min}, then*

$$\omega(\Gamma) \leqslant 1-\frac{k}{\lambda_{min}}.$$

Another useful theorem for strongly regular graphs relates the clique and coclique numbers to the order of the graph.

Theorem 2.1.7 (Clique-Coclique Bound [86]). *Let Γ be a strongly regular graph with order ν and degree k, then*

$$\omega(\Gamma)\alpha(\Gamma) \leqslant \nu.$$

Note that Theorem 2.1.6 and Theorem 2.1.7 can be proved as consequences of Delsarte's linear programming bound, which we shall see in Theorem 2.4.2.

2.2 Distance-Regular Graphs

Distance-regular graphs were introduced by Biggs (cf. [18]). For greater detail, we refer to "BCN" [22], or the survey paper [122].

A *path* of *length* k in a graph is a sequence of distinct vertices $(x_0, x_1, \ldots, x_k)$ such that $x_i \sim x_{i+1}$ for $0 \leqslant i < d$. A graph is *connected* if there exists a path between any two vertices of $V(\Gamma)$, and *disconnected* otherwise. If the path between any two vertices is unique, then the graph is called a *tree*. In a connected graph, the *distance* $d(x, y)$ between two vertices x and y is defined as the length of the shortest path between x and y. The *diameter* of a connected graph is the maximum distance between vertices. We define the *i-th neighbourhood*[1] of a given vertex x by

$$\Gamma_i(x) := \{y \in V(\Gamma) \mid d(x, y) = i\}.$$

Observe that $\Gamma_1(x) = \Gamma(x)$.

A connected graph of diameter d is *distance-regular* if there exist constants $a_i, b_i, c_i \in \{0, 1, \ldots, d\}$, for $0 \leqslant i \leqslant d$, such that for any two vertices x and y with $i = d(x, y)$:

$$\begin{aligned} a_i &= |\Gamma_1(y) \cap \Gamma_i(x)|, \\ b_i &= |\Gamma_1(y) \cap \Gamma_{i+1}(x)|, \\ c_i &= |\Gamma_1(y) \cap \Gamma_{i-1}(x)|. \end{aligned}$$

The constants a_i, b_i and c_i are the called the *intersection numbers*, and the *intersection array* of a distance-regular graph is

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}.$$

Clearly distance-regular graphs are also regular of degree $k = b_0$. Moreover, $k = a_i + b_i + c_i$, for $0 \leqslant i \leqslant d$. Hence $a_i = k - b_i - c_i$, which is why the intersection array only contains the b_i and c_i intersection parameters. Clearly

$$c_0 = 0, \quad b_d = 0, \quad c_1 = 1,$$

so c_0 and b_d are also omitted from the intersection array, however c_1 is still included.

We define $k_i = |\Gamma_i(x)|$, the number of vertices at distance i from any given vertex, and have the following results:

$$\begin{aligned} k_0 &= 1, \\ k_1 &= k = b_0, \\ k_{i+1} &= \frac{k_i b_i}{c_{i+1}}. \end{aligned}$$

Since every vertex is at some distance from a given vertex, the total number of vertices can be computed in terms of k_i,

$$\nu = 1 + k_1 + \ldots + k_d.$$

[1]Typically, we define $\Gamma_{-1}(x)$ and $\Gamma_{d+1}(x)$ to be empty.

Example 2.2.1. *The Petersen graph of Figure 2.1 is distance-regular of diameter 2 with intersection array* $\{3, 2; 1, 1\}$.

A graph of diameter 2 is distance-regular if and only if it is strongly regular and connected ($\mu > 0$). In this case $\mu = c_2$ and $\lambda = k - b_1 - 1$. It has intersection array

$$\{k, k-1-\lambda; 1, \mu\}.$$

Some distance-regular graphs, such as the Petersen graph, are uniquely identifiable by their intersection array. Note that in general, however, the intersection array does not uniquely determine a distance-regular graph. For example, the 4×4 rook graph and the Shrikhande graph both have the same intersection array,

$$\{6, 3; 1, 2\},$$

yet they are non-isomorphic. We will see in Chapter 3 the dual polar spaces which give rise to non-isomorphic distance-regular graphs and yet are indistinguishable from their parameters and intersection array.

Given q, *Gaussian coefficient* for integers n and k is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1} \tag{2.1}$$

if $0 \leqslant k \leqslant n$ and 0 otherwise. We typically write $[n]_q$ instead of $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$. A graph is said to have *classical parameters* (d, b, α, β), if it has diameter d, and

$$\begin{aligned} b_i &= \Big([d]_b - [i]_b\Big)\Big(\beta - \alpha[i]_b\Big), \\ c_i &= [i]_b\Big(1 + \alpha[i-1]_b\Big), \end{aligned}$$

for $0 \leqslant i \leqslant d$. Thus (d, b, α, β) are sufficient to determine the intersection array and all other parameters.

For a distance-regular graph, Γ, with diameter d, we may define the $\nu \times \nu$ matrix

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & \textit{otherwise} \end{cases},$$

for $i \in \{0, \ldots, d\}$. The matrix A_i is called the *distance-i matrix* of Γ. Note $A_1 = A$ and $A_0 = I$. The *distance-i graph* Γ_i is the graph with vertices $V(\Gamma)$ where two vertices are x and y are adjacent if and only if $d(x, y) = i$. Thus A_i is the adjacency matrix of Γ_i.

For a distance-regular graph Γ with diameter d, we define the tridiagonal $(d+1) \times (d+1)$ *intersection matrix*,

$$L = \begin{bmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ & & & & c_d & a_d \end{bmatrix}. \tag{2.2}$$

Γ has exactly $d+1$ distinct eigenvalues, which are the eigenvalues of L. The largest eigenvalue is b_0, with multiplicity 1.

Let λ be an eigenvalue of L, and let $u = (u_0 = 1, u_1, \ldots, u_d)$, be a right eigenvector of L. We have $u_1 = \lambda/k$ and $c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \lambda u_i$. The sequence $(u_0, u_1, \ldots, u_d)$ is called the *standard sequence* of Γ with respect to λ.

Theorem 2.2.2 (Biggs' formula, cf. [122, Theorem 2.8])**.** *Let Γ be a distance-regular graph with diameter d and ν vertices. If λ an eigenvalue of Γ with standard sequence $(u_0, u_1, \ldots, u_d)$, then the multiplicity of λ is*

$$m = \frac{\nu}{\sum_{i=0}^{d} k_i u_i^2}.$$

Example 2.2.3. *Recall that for a strongly regular graph with parameters (ν, k, λ, μ) the intersection array is $\{k, k-1-\lambda; 1, \mu\}$. The intersection matrix is then*

$$L = \begin{pmatrix} 0 & k & 0 \\ 1 & \lambda & k-\lambda-1 \\ 0 & \mu & k-\mu \end{pmatrix}.$$

2.3 Association Schemes

We begin this section by remarking that the term "association scheme" is used differently throughout the literature. For example, Banai and Ito [16] use the term to mean a homogeneous coherent configuration, while Delsarte [49] uses it to mean a commutative coherent configuration, and yet other authors use it to mean a symmetric coherent configuration.

Throughout this thesis, we use "association scheme" to refer to symmetric coherent configuratons.

A coherent configuration is a more general object than an association scheme, but we do not go into further detail here, instead referring the reader to [16]. We simply note that in a coherent configuration, symmetry implies commutativity, which in turn implies homogeneity. Thus association schemes in the sense used in this thesis are a special case of coherent configurations. However, many of the results which apply for symmetric association schemes also hold in the commutative coherent configuration case. Symmetry is a natural requirement in this thesis, since the association schemes arising from geometries inherently fulfil the symmetry condition.

Association schemes originated in statistics, where they were developed in the study of the design of experiments. In addition to having connections to statistics, association schemes are also useful in the study of finite geometry, coding theory, graph theory, design theory, group theory, and representation theory.

Let Ω be a finite, non-empty set and let $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ be a set of symmetric, binary relations on Ω. Then $(\Omega, \mathcal{R})$ is a *d-class association scheme* if the following conditions are satisfied:

1. $\mathcal{R}$ forms a partition of $\Omega \times \Omega$,
2. R_0 is the identity relation,
3. There exist constants p_{ij}^k for all $i, j, k \in \{0, \ldots, d\}$ such that

$$|\{z \in \Omega : (x, z) \in R_i \text{ and } (z, y) \in R_j\}| = p_{ij}^k$$

   for all $(x, y) \in R_k$.

The cardinality of Ω is called the *order* of $(\Omega, \mathcal{R})$, the $R_i \in \mathcal{R}$ are the *relations* and the constants p_{ij}^k are called the *intersection numbers*. We say two vertices x and y are *i-th associates* if $(x, y) \in R_i$. Observe that the intersection numbers of $(\Omega, \mathcal{R})$ must be non-negative integers.

Example 2.3.1. *Strongly regular graphs yield* 2*-class association schemes.*

We define the *i-valency* $k_i = p_{ii}^0$, the number of vertices i-related to any given vertex. Clearly the sum of the i-valencies equals the number of vertices, $|\Omega|$.

Association schemes are perhaps more easily expressed in terms of matrices. The *adjacency matrix* A_i for the relation R_i is the $|\Omega| \times |\Omega|$ matrix with rows and columns indexed by Ω, such that

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \textit{otherwise} \end{cases}$$

Expressed in terms of matrices, a d-class association scheme is then a set of symmetric, non-zero, $\{0, 1\}$-matrices $\{A_0, A_1, \ldots, A_d\}$ such that

1. $J = \sum_{i=0}^d A_i$, where J is the all ones matrix,
2. $A_0 = I$,
3. There exist constants p_{ij}^k for all $i, j, k \in \{0, 1, \ldots, d\}$ such that

$$A_i A_j = \sum_{k=0}^d p_{ij}^k A_k.$$

Example 2.3.2. *The* Hamming scheme *is the association scheme with vertex set consisting of the n-tuples of elements of $\{1,\dots,k\}$, where two vertices x and y are i-th associates if they differ in exactly i positions. The* Hamming graph *is defined by R_1 in the Hamming scheme.*

Example 2.3.3. *The* Johnson scheme*, is the association scheme with vertex set consisting of the k-subsets of $\{1,\dots,n\}$. Two vertices x and y are i-th associates if $|x \cap y| = k - i$. The* Johnson graph*, $J(n,k)$, is defined by R_1 in the Johnson scheme.*

Example 2.3.4. *The* Grassmann scheme*, is the association scheme with vertex set consisting of the k-dimensional subspaces of $\mathbb{F}_q^n$. Two vertices x and y are i-th associates if $\dim(x \cap y) = k - i$. The Grassmann scheme is also called the* q-Johnson scheme*. The* Grassmann graph*, $J_q(n,k)$, is defined by R_1 in the Grassmann scheme.*

Example 2.3.5. *A group G acts* generously transitively *on a set Ω if, given any two elements ω and ω' in Ω, there exists $g \in G$ such that $\omega^g = \omega'$ and $\omega'^g = \omega$. Let G be a generously transitive permutation group acting on a set Ω. Let $\mathcal{R}$ be the orbitals of G on Ω, that is, the orbits of G on $\Omega \times \Omega$. Then $(\Omega, \mathcal{R})$ is an association scheme. Such an association scheme is called* Schurian*.*

The *automorphism group*, $\mathrm{Aut}((\Omega,\mathcal{R}))$, of an association scheme $(\Omega,\mathcal{R})$ is the group of permutations of Ω which fixes every set in $\mathcal{R}$. If $(\Omega,\mathcal{R})$ is Schurian arising from the (faithfully acting) group G, then $G \leqslant \mathrm{Aut}((\Omega,\mathcal{R}))$.

Consider the span of the adjacency matrices, $\mathcal{A}$, over $\mathbb{C}$. Then by 2), $\mathcal{A}$ contains the identity, and by 3), $\mathcal{A}$ is closed under matrix multiplication. Hence $\mathcal{A}$ is an algebra, called the *adjacency algebra*, or *Bose-Mesner algebra*, after Bose and Mesner, who were the first to construct and work with it [20]. By 1), the adjacency matrices are linearly independent and hence form a basis for $\mathcal{A}$, meaning the dimension of $\mathcal{A}$ is $(d+1)$. Moreover, because the adjacency matrices are symmetric, they also commute, since

$$A_iA_j = A_i^\top A_j^\top = (A_jA_i)^\top = \sum_{\ell=0}^{d} p_{ji}^{\ell} A_\ell^\top = \sum_{\ell=0}^{d} p_{ji}^{\ell} A_\ell = A_jA_i.$$

Thus the Bose-Mesner algebra is a commutative algebra.

The *i-th intersection matrix* B_i is the $(d+1)\times(d+1)$ matrix with entries defined by

$$(B_i)_{jk} = p_{ki}^{j}.$$

The algebra $\mathcal{B}$ spanned by the intersection matrices is called the *intersection algebra*. Furthermore, the map $A_i \mapsto B_i$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. This isomorphism is incredibly useful in computations for association schemes of large order, since the number of classes of an association scheme is generally significantly smaller than its order, and hence working with matrices in the intersection algebra is easier than working with matrices in the adjacency algebra[2]. In particular, knowing the intersection numbers of an association scheme is sufficient to calculate the matrix of eigenvalues, the matrix of dual eigenvalues, and the Krein parameters, which we shall see later in this section.

The adjacency matrices of an association scheme are symmetric, commuting, real matrices and thus are simultaneously diagonalisable. Hence $\mathbb{R}^{|\Omega|}$ decomposes into orthogonal maximal common eigenspaces of the adjacency matrices, called *strata*. Hence for a d-class association scheme,

$$\mathbb{R}^{|\Omega|} = V_0 \perp V_1 \perp \ldots \perp V_d.$$

In addition, $V_0 = \langle \mathbb{1} \rangle$, where $\mathbb{1}$ is the all ones vector. The dimension of V_i is denoted m_i. Thus $m_0 = 1$.

A matrix E in the Bose-Mesner algebra is called an *idempotent* if $E^2 = E$ and is *minimal* if it is non-zero and cannot be written as the sum of two non-zero idempotents.

Theorem 2.3.6 (cf. [16, Theorem 3.1])**.** *There is a unique basis of minimal idempotents $\{E_0, E_1, \ldots, E_d\}$ for the Bose-Mesner algebra of a d-class association scheme.*

These minimal idempotents are orthogonal, that is $E_iE_j = \delta_{ij}E_i$, where δ_{ij} denotes the Kronecker Symbol, and satisfy $\sum_{i=0}^{d} E_i = I$. Moreover, the minimal idempotents project from $\mathbb{R}^{|\Omega|}$ orthogonally onto the strata. Thus for all $j \in \{0, \ldots d\}$, $\mathrm{Im}(E_j) = V_i$ for some $i \in \{0, \ldots d\}$. Note that $\mathrm{Im}(E_i)$ is rowspace of E_i (indeed, also the column space, as E_i is symmetric). By convention, we take $E_0 = \frac{1}{|\Omega|}J$ (where J is the all ones matrix), as this is always one of the minimal idempotents, and projects onto $V_0 = \langle \mathbb{1} \rangle$.

Lemma 2.3.7 (cf. [16, Proposition 3.3])**.** *Consider a d-class association scheme $(\Omega, \mathcal{R})$ with adjacency matrices A_i and minimal idempotents E_i, for $i \in \{0, \ldots, d\}$, then*

1. $\mathrm{Trace}(A_i) = |\Omega|\delta_{i0}$,
2. $m_i = \mathrm{rank}(E_i) = \mathrm{Trace}(E_i) = \dim(V_i)$.

Since the adjacency matrices and the minimal idempotents form bases of the Bose-Mesner algebra, we may move between them. The change of basis matrix from $\{E_0, \ldots, E_d\}$ to $\{A_0, \ldots, A_d\}$, referred to as the *matrix of eigenvalues, P*, is the $(d+1) \times (d+1)$ matrix such that

$$A_i = \sum_{j=0}^{d} P_{ji} E_j.$$

[2]Indeed, this motivated statisticians in the development of association schemes: this isomorphism made working with $d \times d$ matrices much simpler than working with large covariance matrices before the advent of computers!

The name "matrix of eigenvalues" comes from the fact that P_{ji} is an eigenvalue of A_i, since

$$A_iE_j = \left(\sum_{k=0}^{d} P_{ki}E_k\right)E_j = P_{ji}E_j,$$

where the second equality follows from the fact that the idempotents are orthogonal. Indeed, since E_j projects onto the strata, if we take $v \in \mathrm{Im}(E_j)$, then we can see that $A_iv = P_{ji}v$ by multiplying v on both sides of the above equation and observing that $E_jv = v$. So v is an eigenvector for every A_i, as we would expect, since the strata are common eigenspaces of the adjacency matrices.

Similarly we have the *matrix of dual eigenvalues,* Q, is the $(d+1)\times(d+1)$ matrix such that

$$E_i = \frac{1}{|\Omega|}\sum_{j=0}^{d} Q_{ji}A_j.$$

We index the rows and columns of P and Q starting with zero. Some important properties of P and Q are:

1. $PQ = |\Omega|I = QP$,
2. $P_{i0} = Q_{i0} = 1$,
3. $P_{0j} = k_j$,
4. $Q_{0j} = m_j$,
5. $m_iP_{ij} = k_jQ_{ji}$.

In addition to the fact that the i-th column of P contains the eigenvalues of A_i, there are orthogonality relations which may help us compute P. The following lemmas follow directly from 1-5 above.

Lemma 2.3.8 (Row Orthogonality, cf. [16, Proposition 3.5])**.**

$$\sum_{v=0}^{d} \frac{1}{k_v}P_{iv}P_{jv} = \frac{|\Omega|}{m_i}\delta_{ij}.$$

Lemma 2.3.9 (Column Orthogonality, cf. [16, Proposition 3.35])**.**

$$\sum_{v=0}^{d} m_vP_{vi}P_{vj} = |\Omega|k_i\delta_{ij}.$$

In particular, the sum of the zero-th row of P is $|\Omega|$, while the sum of every other row of P is zero. It is also true that the sum of the entries in the zero-th row of Q is $|\Omega|$, and the sum of each other row of Q is zero.

We refer to the entrywise multiplication of two matrices or vectors as their *Schur*, or *Hadamard* product, denoted by $\circ$. That is, for two matrices A and B of the same size, $(A\circ B)_{ij} = A_{ij}B_{ij}$. Similarly for two vectors u and v of the same size, $(u\circ v)_i = u_iv_i$.

Note that the Bose-Mesner algebra is closed under Schur multiplication, since $A_i \circ A_j = \delta_{ij} A_i$. Hence the adjacency matrices are minimal idempotents of the Bose-Mesner algebra with respect to Schur multiplication. Thus there exist real numbers q_{ij}^h such that

$$E_i \circ E_j = \frac{1}{|\Omega|} \sum_{h=0}^{d} q_{ij}^h E_h.$$

These numbers q_{ij}^h are called the *Krein parameters*, or *dual intersection numbers*.

Theorem 2.3.10 (The Krein conditions [107])**.** *The Krein parameters of an association scheme are nonnegative real numbers.*

Although real and nonnegative, Krein parameters need not be integral or rational. The case where Krein parameters equal zero, or "vanish", is often very interesting. Vanishing Krein parameters and their consequences are discussed in greater detail later in this thesis, in Chapter 7.

We may compute the Krein parameters directly from P and Q.

Theorem 2.3.11 (cf. [22, Theorem 2.3.2])**.**

$$q_{ij}^h = \frac{1}{|\Omega| m_h} \sum_{\ell=0}^{d} k_\ell Q_{\ell i} Q_{\ell j} Q_{\ell h} = \frac{m_i m_j}{|\Omega|} \sum_{\ell=0}^{d} \frac{P_{i\ell} P_{j\ell} P_{h\ell}}{k_\ell^2}.$$

In fact, if we know one of P, Q, the set of intersection numbers, or the set of Krein parameters, we may determine each of the others.

There is a duality in the behaviour of normal matrix multiplication and Schur multiplication in the Bose-Mesner algebra. Consider:

$$\begin{array}{c|c}
A_0 = I & E_0 = \frac{1}{|\Omega|} J \\
\sum_{i=0}^{d} A_i = J & \sum_{i=0}^{d} E_i = I \\
A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k & E_i \circ E_j = \frac{1}{|\Omega|} \sum_{k=0}^{d} q_{ij}^k E_k \\
A_i \circ A_j = \delta_{ij} A_i & E_i E_j = \delta_{ij} E_i \\
A_i E_j = P_{ji} E_j & E_i \circ A_j = \frac{1}{|\Omega|} Q_{ji} A_j
\end{array}$$

We can also see where the name *matrix of dual eigenvalues* comes from, since Q_{ji} behaves like a "dual eigenvalue" for E_i under Schur multiplication.

By convention, we take $E_0 = \frac{1}{|\Omega|} J$, but there is no convention on the ordering of the other E_i. Choosing different orderings will lead to different P and Q matrices. In fact, changing the ordering of the A_i and E_i (other than A_0 and E_0) does not alter the association scheme. There are however natural ways of ordering the relations, particularly in the case where a distance-regular graph gives rise to an association scheme.

Example 2.3.12. *Let Γ be a distance-regular graph of diameter d. Define the relation R_i where $(x, y) \in R_i$ if and only if $d(x, y) = i$ in Γ, then $(V(\Gamma), \{R_0, \dots, R_d\})$ defines a d-class association scheme.*

Lemma 2.3.13 ([22, Lemma 4.1.7])**.** *The intersection numbers, p^l_{ij}, of a d-class association scheme arising from a distance-regular graph of diameter d can be calculated recursively from the intersection array by the following:*

$$p^l_{0j} = \delta_{lj}, \quad p^l_{i0} = \delta_{il}, \quad p^l_{i,d+1} = 0,$$

$$p^l_{1j} = \begin{cases} c_i & \textit{if } j = l-1, \\ a_i & \textit{if } j = l, \\ b_i & \textit{if } j = l+1, \\ 0 & \textit{otherwise} \end{cases},$$

and

$$p^l_{i+1,j} = \frac{p^{l-1}_{i,j} c_l + p^l_{i,j} a_l + p^{l+1}_{i,j} b_l - p^l_{i-1,j} b_{i-1} - p^l_{i,j} a_i}{c_{i+1}}.$$

Moreover, computing P and Q for an association scheme arising from a distance-regular graph is considerably simplified. We need only compute L as in (2.2) to compute the eigenvalues and their standard sequences. The standard sequences, appropriately scaled, then form the columns of Q.

An association scheme arising from a distance-regular graph has a natural ordering of the relations according to distance. For this reason, such association schemes are called *metric*. Formally, an association scheme $(\Omega, \mathcal{R})$ is called *metric*, or *P-polynomial*, with respect to the ordering $\{R_i\}_{i=0}^d$, if

1. $p^k_{ij} = 0$ if $i + j < k$ or $0 \leqslant k < |i - j|$,
2. $p^{i+j}_{ij} \neq 0$ for all i and j such that $i + j \leqslant d$.

It is sufficient to check the above conditions for $i = 1$, hence an association scheme is metric with respect to the ordering $\{R_i\}_{i=0}^d$, if $p^k_{1j} = 0$ implies $1 + j < k$ or $k < j - 1$, and $p^{1+j}_{1j} \neq 0$.

An association scheme is metric with respect to the ordering $\{R_i\}_{i=0}^d$, if and only if R_1 defines a distance-regular graph with $d(x, y) = i$ precisely when $(x, y) \in R_i$. Equivalently, an association scheme is metric with respect to the ordering $\{R_i\}_{i=0}^d$, if, for each $i \in \{0, \dots, d\}$, there exists a degree i polynomial v_i such that $v_i(A_1) = A_i$. This is the reason for the alternative name P-polynomial.

We say that an association scheme is *metric*, or *P-polynomial*, if it is metric with respect to some ordering of its relations (not necessarily the given ordering, $\{R_i\}_{i=0}^d$). An association scheme which does not come from a polygon can be metric with respect to at most two orderings of its relations (cf. [22, Theorem 4.2.12]), and so it can have at most two relations which define a distance-regular graph.

Example 2.3.14. *The Hamming, Johnson, and Grassmann scheme are metric with respect to their first relations. The Johnson scheme $J(2k+1, k)$ is metric with respect to the relation R_1 and R_k.*

An association scheme is metric if there is a meaningful ordering on its relations, R_i. Similarly, if an association scheme admits a meaningful ordering of its minimal idempotents, E_i, we call it *cometric*. Formally, an association scheme $(\Omega, \mathcal{R})$ is called *cometric*, or *Q-polynomial*, with respect to the ordering $\{E_i\}_{i=0}^d$, if

1. $q_{ij}^k = 0$ if $i + j < k$ or $0 \leqslant k < |i - j|$,
2. $q_{ij}^{i+j} \neq 0$ for all i and j such that $i + j \leqslant d$.

As with the metric property of an association scheme, it is sufficient to test for $i = 1$ to determine if an association scheme is cometric. Thus an association scheme is cometric if $q_{1j}^k = 0$ implies $1 + j < k$ or $k < j - 1$, and $q_{1j}^{1+j} \neq 0$. As in the metric case, an association scheme is called *cometric*, or *Q-polynomial*, if it is cometric with respect to some ordering of its minimal idempotents.

Equivalently, an association scheme is cometric if, for each $i \in \{0, \ldots, d\}$, there exists a degree i polynomial q_i such that $q_i(E_1) = E_i$, where matrix multiplication is the Schur product, that is, performed entrywise. Again, this is the reason for the alternative name Q-polynomial.

There is a duality between the algebraic properties of metric and cometric schemes, where adjacency matrices and intersection numbers in the metric case are replaced by minimal idempotents and Krein parameters in the cometric case. However, while metric association schemes corresponds with distance-regular graphs, there is no nice distance type comparison in the cometric case.

2.4 Subsets of Association Schemes

Much of the following is due to Delsarte in his PhD thesis [49].

Let $(\Omega, \mathcal{R})$ be a d-class association scheme, and let θ be a subset of Ω. We define the characteristic vector χ_θ as the $\{0, 1\}$-vector indexed by Ω, where

$$(\chi_\theta)_\omega = \begin{cases} 1 & \text{if } \omega \in \theta \\ 0 & \text{otherwise} \end{cases}$$

We define the *inner distribution* of θ by $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d)$, where $\mathbf{a}_i$ is the non-negative rational number

$$\mathbf{a}_i = \frac{|R_i \cap \theta^2|}{|\theta|}.$$

Hence the $\mathbf{a}_i$ represent the average number of vertices of θ which are i-related to a fixed vertex of θ. We can also compute $\mathbf{a}_i$ by

$$\mathbf{a}_i = \frac{\chi_\theta A_i \chi_\theta^\top}{|\theta|}.$$

We can define the inner distribution of a general vector, $v \in \mathbb{R}^{|\Omega|}\backslash\{0\}$, in a similar manner by replacing χ_θ with v and $|\theta|$ by $v \cdot \mathbb{1}$. Observe that $\mathbf{a}_0 = 1$ and $\sum_{i=0}^{d} \mathbf{a}_i = |\theta|$.

The *degree set* of θ is the subset of $\{1, \dots, d\}$ consisting of all i such that $(x, y) \in R_i$ for some distinct pair $x, y \in \theta$. Thus if χ_θ has inner distribution $\mathbf{a}$, then the degree set of χ_θ is

$$\{i \mid i \neq 0, \mathbf{a}_i \neq 0\}.$$

The *degree* of θ is the size of its degree set. The *dual degree set* of a vector v is the subset of $\{1, \dots, d\}$ consisting of all i such that $E_i v \neq 0$, and the *dual degree* is the size of the dual degree set.

Let $T \subseteq \{1, \dots, d\}$ and let $\theta \subseteq \Omega$. We say θ is an *T-clique* if the degree set of θ is a subset of T. If T is disjoint from the degree set of θ, then θ is a *T-coclique*.

Theorem 2.4.1 ([49]). *Let θ be a T-clique, and let Θ be a T-coclique. Then*

$$|\theta| \cdot |\Theta| \leqslant |\Omega|$$

with equality if and only if $|\theta \cap \Theta| = 1$.

The *MacWilliams transform* of $\mathbf{a}$ is the vector $\mathbf{a}Q$, and must have nonnegative entries. This leads to Delsarte's linear programming bound.

Theorem 2.4.2 (Delsarte's linear programming bound). *Let θ a T-clique. Then $|\theta|$ is bounded above by:*

$$\begin{aligned}
\textit{Maximise: } & \sum_{i=0}^{d} \mathbf{a_i} \\
\textit{Subject to: } & \mathbf{a_0} = 1, \\
& \mathbf{a_i} = 0 \textit{ for all } i \in \{1, 2, \dots, d\}\backslash T, \\
& \mathbf{a_i} \geqslant 0 \textit{ for all } i = 1, 2, \dots, d, \\
& (\mathbf{a}Q)_i \geqslant 0 \textit{ for all } i = 0, 1, \dots, d.
\end{aligned}$$

Analogous to T-cliques and T-cocliques are T-designs and T-antidesigns, where the dual degree set is considered instead of the degree set. Let $T \subseteq \{1, \dots, d\}$, then θ is a *T-design* if its dual degree set is disjoint from T, that is, $E_i \chi_\theta = 0$ for all $i \in T$. If the dual degree set is a subset of T, then θ is a *T-antidesign*. Thus θ is a T-antidesign if it is a T'-design, whenever $T \cap T' = \varnothing$. Moreover, θ is a T-design if and only if $(\mathbf{a}Q)_i = 0$ for all $i \in T$, where $\mathbf{a}$ is the inner distribution of θ.

Example 2.4.3. *In the Johnson scheme, θ is a $\{1,\ldots,t\}$-design if and only if it is a $t-(\nu,k,\lambda)$ design.*

We call two vectors u and v *design orthogonal* if their dual degree sets are disjoint. Subsets are design orthogonal if their characteristic vectors are design orthogonal. Observe that a T-design is design orthogonal to a T-antidesign.

Recall that the minimal idempotents E_i project onto the strata. Hence $E_i v = 0$ means that v has no component lying in V_i. That is, it lies in the orthogonal complement of V_i. Hence, if $T = \{i_1,\ldots,i_{|T|}\}$, a T-design lies in

$$\left(V_{i_1} \perp \ldots \perp V_{i_{|T|}}\right)^{\perp},$$

and a T-antidesign lies in

$$V_0 \perp V_{i_1} \perp \ldots \perp V_{i_{|T|}},$$

where $i_j \in T$.

Designs and antidesigns intersect one another in a predictable manner.

Theorem 2.4.4 ([104])**.** *If X is a T-design, and X' a T-antidesign, then*

$$|X \cap X'| = \frac{|X||X'|}{|\Omega|}.$$

More generally, for $v \in \mathbb{R}^{|\Omega|}$, then v is a *(weighted) T-design* if $E_i v = 0$ for all $i \in T$. It is weighted if it has entries other than one or zero, which would otherwise correspond to the characteristic vector of a set. We define *(weighted) T-antidesigns* for vectors similarly. Theorem 2.4.4 also holds for weighted antidesigns.

Theorem 2.4.5. *If v is a T-design, and w a T-antidesign, then*

$$v \cdot w = \frac{(v\cdot\mathbb{1})(w\cdot\mathbb{1})}{|\Omega|}.$$

Proof. Let

$$\alpha = \frac{v\cdot\mathbb{1}}{\mathbb{1}\cdot\mathbb{1}} \quad \text{and} \quad \beta = \frac{w\cdot\mathbb{1}}{\mathbb{1}\cdot\mathbb{1}}.$$

Then

$$(v-\alpha\mathbb{1})\cdot\mathbb{1} = 0 \quad \text{and} \quad (w-\beta\mathbb{1})\cdot\mathbb{1} = 0.$$

Since v and w are design-orthogonal, $(v-\alpha\mathbb{1})$ and $(w-\beta\mathbb{1})$ belong to a pair of direct sums of eigenspaces that intersect trivially and hence $(v-\alpha\mathbb{1})\cdot(w-\beta\mathbb{1}) = 0$. Thus

$$\begin{aligned} v\cdot w &= \alpha(w\cdot\mathbb{1}) + \beta(v\cdot\mathbb{1}) - \alpha\beta\mathbb{1}\cdot\mathbb{1} \\ &= \frac{(v\cdot\mathbb{1})(w\cdot\mathbb{1})}{\mathbb{1}\cdot\mathbb{1}} + \frac{(v\cdot\mathbb{1})(w\cdot\mathbb{1})}{\mathbb{1}\cdot\mathbb{1}} - \frac{(v\cdot\mathbb{1})(w\cdot\mathbb{1})}{\mathbb{1}\cdot\mathbb{1}} \\ &= \frac{(v\cdot\mathbb{1})(w\cdot\mathbb{1})}{|\Omega|}. \end{aligned}$$

□

Chapter 3

Finite Geometry

In this chapter we introduce the various geometries that we will be working with throughout the thesis. In particular, we give an overview of the classical polar spaces and generalised quadrangles, and their duals. We see that the maximal totally isotropic and totally singular subspaces of these geometries produce a metric association scheme and that interesting geometric objects such as m-ovoids, m-covers, and hemisystems are Delsarte designs of the corresponding scheme. For greater detail, we refer to [3], [46], and [53].

3.1 Point-Lines Geometries

A *point-line geometry*, $(\mathcal{P}, \mathcal{L}, \mathrm{I})$, consists of a set $\mathcal{P}$ called *points* and a set $\mathcal{L}$ called *lines*, with a binary symmetric incidence relation I between points and lines. A point-line geometry is *finite* if $\mathcal{P}$ and $\mathcal{L}$ are both finite. In this thesis, we focus only on finite geometries. We denote incidence between a point P and line ℓ by $P \,\mathrm{I}\, \ell$, and often say that P *is on* ℓ or that ℓ *passes through* P. If two points are incident with a common line, they are said to be *collinear*. We sometimes denote the line incident with two collinear points P and Q by PQ. We denote the set of all points collinear with P by $P^{\perp}$. If two lines are both incident with a common point, they are said to *meet*, or *intersect*, and are *skew* if they do not intersect.

A point-line geometry has order (s, t) if every line is incident with $s+1$ points, and every point is incident with $t+1$ lines. We say a point-line geometry has order s if its order is (s, s). A point-line geometry is *thick* if both $s \geqslant 2$ and $t \geqslant 2$. The *dual* of a point-line geometry $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is the point-line geometry $(\mathcal{L}, \mathcal{P}, \mathrm{I})$.

The *incidence graph* of a point-line geometry $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ with adjacency defined by incidence in the geometry. The *collinearity* or *point graph* of a point line geometry $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ has vertex set $\mathcal{P}$, and adjacency between two vertices is defined by collinearity in the geometry. An *isomorphism* from $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ to $(\mathcal{P}', \mathcal{L}', \mathrm{I})$ is an incidence preserving bijection from $\mathcal{P}$ to $\mathcal{P}'$ and $\mathcal{L}$ to $\mathcal{L}'$. An isomorphism from $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ to itself is an *automorphim*.

A *partial linear space* is a point line geometry such that every line is on at least two points, and any pair of distinct points are incident with at most one common line. If every pair of distinct points is collinear, then a partial linear space is in fact a *linear space.*

3.2 Projective and Polar Spaces

A *projective plane* is a linear space such that

1. every pair of distinct lines meet in a unique point,
2. there exist four points, such that no three are collinear.

In a projective plane, there are exactly $n + 1$ points on any line, and $n + 1$ lines through any point. The projective plane is said to have order n.

Example 3.2.1. *Figure 3.1 depicts* $\mathsf{PG}(2, 2)$*, the Fano plane. It is the smallest possible projective plane, and has order* 2*.*

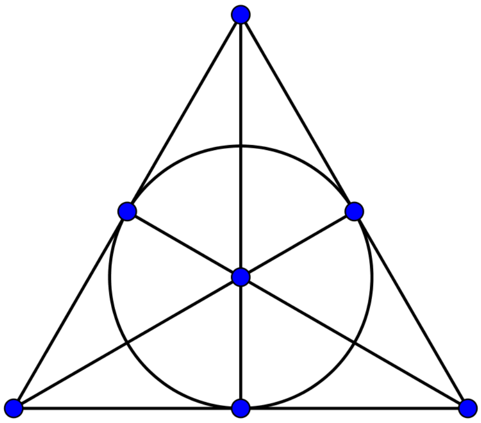


Figure 3.1: The Fano plane.

A *projective space* is a linear space such that every line is incident with at least three points, and the following axiom is satisfied.

Definition 3.2.2 (Veblen-Young axiom)**.** *If* P_1*,* P_2*,* P_3*, and* P_4 *are points such that* P_1P_2 *and* P_3P_4 *meet, then* P_1P_3 *and* P_2P_4 *also meet.*

A *subspace*, S, of a projective space is a subset of the points such that if a line meets two points of S, then every point on the line is also in S. The rank of a projective space is the largest n such that there exist subspaces $S_0, \ldots, S_n$, satisfying $\varnothing \subset S_0 \subset \ldots \subset S_n = \mathcal{P}$.

We can construct a projective space by taking the points and lines to be the 1-spaces and 2-spaces of $\mathbb{F}_q^{n+1}$, respectively, and defining incidence by inclusion of subspaces. Such a projective space is called *classical* and is denoted $\mathsf{PG}(n, q)$. A $(k+1)$-dimensional subspace of $\mathbb{F}_q^{n+1}$ forms a k-dimensional subspace of $\mathsf{PG}(n, q)$, and we say it has projective dimension k. Clearly $\mathsf{PG}(n,q)$ has rank n. Care must be taken to differentiate between vector space dimension and projective dimension. Each subspace of $\mathsf{PG}(n,q)$ of projective dimension k forms a projective space equivalent to $\mathsf{PG}(k,q)$. An $(n-1)$-dimensional projective subspace of $\mathsf{PG}(n,q)$ is called a *hyperplane*. A $(n-k)$-dimensional subspace is said to have codimension k.

Significantly, for rank at least 3, the only finite projective spaces are classical.

Theorem 3.2.3 ([126]). *A finite projective space with rank $n \geqslant 3$ is isomorphic to $\mathsf{PG}(n,q)$.*

The projective planes are precisely the rank 2 projective spaces, and as such are excluded from the previous theorem. Indeed, there exist nonclassical projective planes [129]. Classifying the projective planes is a long-standing open problem.

Recall from (2.1) that the *Gaussian coefficient* for integers n and k is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i-1}$$

if $0 \leqslant k \leqslant n$ and 0 otherwise. The value $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is the number of k-dimensional subspaces of $\mathbb{F}_q^n$, and so allowing for the change to projective dimension, also counts $(k-1)$-subspaces of $\mathsf{PG}(n-1,q)$.

A *duality* of $\mathsf{PG}(n,q)$ is a bijective, incidence preserving map which takes subspaces of dimension k to subspaces of codimension $k+1$. A *polarity* of $\mathsf{PG}(n,q)$ is an involutory duality. Given a polarity, γ, we say that a subspace W is *absolute* if $W \subseteq W^\gamma$. The absolute points and lines form special geometries of their own. They are examples of polar spaces.

The following definition of a polar space is due to Buekenhout and Shult [25], and is equivalent to definitions by Veldkamp [127] and Tits [120]. A (finite) *polar space* is a partial linear space such that:

1. For every point P and line ℓ not incident with P, either every point of ℓ is collinear with P, or there is exactly one point on ℓ which is collinear with P,
2. There is no point which is collinear with every other point,

The first axiom is sometimes referred to as the "one or all" axiom and is the defining axiom of a polar space. The second axiom ensures nondegeneracy. We can define subspaces and rank in the same manner as for projective spaces. If we were not working with finite polar spaces, then we would require an additional axiom ensuring that every strictly ascending chain of subspaces has finite length (and hence finite rank). The subspaces attaining largest rank are referred to as *maximals*. Maximals are also sometimes referred to as *generators*, though we avoid this usage in this thesis. Subspaces of projective dimension $n-2$ are referred to as *next-to-maximals*.

In order to construct almost every finite polar space, we turn now to consider vector spaces equipped with forms. Let V be an n-dimensional vector space over the field $\mathbb{F}_q$, for some prime power q. Given a field automorphism σ, a *σ-sesquilinear form* is a map $\beta : V \times V \to \mathbb{F}_q$, such that

$$\beta(\lambda u + \mu v, w) = \lambda\beta(u, w) + \mu\beta(v, w),$$

and

$$\beta(u, \lambda v + \mu w) = \lambda^{\sigma}\beta(u, v) + \mu^{\sigma} f(u, w).$$

for all $\lambda, \mu \in \mathbb{F}_q$, and $u, v, w \in V$. We will often refer to a σ-sesquilinear form simply as a *form.*

A σ-sesquilinear form β is:

- *bilinear* if $\sigma = 1$,
- *reflexive* if $\beta(v, w) = 0$ implies $\beta(w, v) = 0$ for all $v, w \in V$,
- *symmetric* if $\beta(v, w) = \beta(w, v)$ for all $v, w \in V$,
- *alternating* if $\beta(v, v) = 0$ for all $v \in V$,
- *hermitian* if $\sigma \neq \sigma^2 = 1$ and $\beta(v, w) = \beta(w, v)^{\sigma}$ for all $v, w \in V$.

Note that hermitian forms only occur for $\mathbb{F}_{q^2}$, since involutary field automorphisms only exist when the order of a field is the square of some prime power q. If the characteristic of $\mathbb{F}_q$ is 2, then an alternating form is also symmetric, since $\beta(u+v, u+v) = 0$ implies that $\beta(u, v) = -\beta(v, u) = \beta(v, u)$. If the characteristic of the field is not equal to 2, then a form is alternating if and only if $\beta(u, v) = -\beta(v, u)$. A reflexive σ-sesquilinear form β is *degenerate* if there exists a vector $v \in V\backslash\{0\}$ such that $\beta(v, w) = 0$ for all $w \in V$, and *non-degenerate* otherwise. We say that a subspace $U \leqslant V$ is *totally isotropic* if $\beta(v, w) = 0$ for all $v, w \in U$, and *non-degenerate* if the form induced on U is non-degenerate.

Clearly symmetric, hermitian, and alternating forms are reflexive. It turns out that all non-degenerate, reflexive forms are one of these three types.

Theorem 3.2.4 (Birkhoff - von Neumann (cf. [3])). *A non-degenerate, reflexive, σ-sesquilinear form β is a symmetric bilinear form, an alternating form, or a hermitian form.*

A *quadratic* form on V is a map $\kappa : V \to \mathbb{F}_q$ such that for all $\lambda \in \mathbb{F}_q$ and $v, w \in V$

$$\kappa(\lambda v) = \lambda^2\kappa(v)$$

and

$$\beta_{\kappa}(v, w) := \kappa(v + w) - \kappa(v) - \kappa(w)$$

defines a bilinear form, called the *associated bilinear form*, or the *polarisation* of κ. A quadratic form κ is *degenerate* if there exists a vector $v \in V\backslash\{0\}$ such that $\kappa(v) = 0$ and $\beta_{\kappa}(v, w) = 0$ for all $w \in V$, and *non-degenerate* otherwise.

Observe that β_κ is clearly reflexive and symmetric. Moreover, $\kappa(v) = \beta_\kappa(v,v)/2$, so the associated bilinear form uniquely determines κ when $\mathbb{F}_q$ is of odd characteristic, and there is a one-to-one correspondence between reflexive, symmetric bilinear forms and quadratic forms in odd characteristic. In even characteristic, a quadratic form cannot be recovered from its associated bilinear form. Thus the study of quadratic forms is richer than that of symmetric bilinear forms.

Let V_1 and V_2 be two vector spaces over $\mathbb{F}_q$ respectively equipped with alternating or hermitian forms β_1 and β_2, or quadratic forms κ_1 and κ_2. Let $\varphi : V_1 \to V_2$ be an invertible linear map such that $\beta_1(v,w) = \lambda\beta_2(\varphi(v), \varphi(w))$, respectively, $\kappa_1(v) = \lambda\kappa_2(\varphi(v))$, for some $0 \neq \lambda \in \mathbb{F}_q$ and all $v, w \in V_1$. We call φ an *isometry* if $\lambda = 1$, and a *similarity* otherwise.

A subspace $U \leqslant V$ is *totally singular* if $\kappa(u) = 0$ for all $u \in U$, and *anisotropic* if $\kappa(u) \neq 0$ for all non-zero vectors in U. If $u, v \in V$ are two vectors such that $\beta_\kappa(u,v) = 1$ and $\beta_\kappa(u,u) = \beta_\kappa(v,v) = 0$, then (u,v) is called a *hyperbolic pair*.

Any n-dimensional vector space, V, equipped with a quadratic form has an orthogonal decomposition

$$V = H_1 \oplus H_2 \oplus \ldots \oplus H_d \oplus X$$

where each H_i is the span of a hyperbolic pair, and X is an anisotropic subspace. Up to equivalence, there are three types of non-degenerate quadratic forms:

- *hyperbolic* when $\dim(X) = 0$,
- *parabolic* when $\dim(X) = 1$,
- *elliptic* when $\dim(X) = 2$.

Hence the parabolic case occurs when n is odd, while the hyperbolic and elliptic cases occur for n even. Hyperbolic forms are often referred to as "+" type, while elliptic forms are referred to as "−" type, providing an easy notation to distinguish between the two cases in even dimension. Since there is only one parabolic form in odd dimension it is unnecessary to indicate its type, however we may refer to it as "$\circ$" type for consistency, or to identify it when the dimension is not explicitly stated.

Theorem 3.2.5 (Witt's Theorem [130])**.** *Let V_1 and V_2 be two vector spaces equipped with non-degenerate symmetric, alternating, or hermitian forms. Any isometry between two subspaces U_1 and U_2 extends to an isometry from V_1 to V_2.*

An immediate consequence of Witt's Theorem is that any two maximal totally isotropic subspace have the same dimension. We refer to the dimension of the largest totally isotropic subspace as the *Witt index*. In the previous decomposition, the Witt index in the parabolic case is $d = \frac{1}{2}(n-1)$, the Witt index in the hyperbolic case is $d = \frac{n}{2}$, and the Witt index in the elliptic case is $d = \frac{n}{2} - 1$.

Given a non-degenerate σ-sesquilinear or quadratic form, let β be either the form itself or, for a quadratic form, its polarisation. Two vectors u and v in V are *orthogonal* if $\beta(u,v)=0$. The *orthogonal complement* or *perp* of a subspace U of V is

$$U^\perp := \{v \in V \mid \beta(u,v)=0, \forall u \in U\}.$$

If U is non-degenerate, then so too is $U^\perp$, and $U \subseteq U^\perp$ for a totally isotropic subspace U.

Moreover, $\perp$ defines a polarity of $\mathsf{PG}(n-1,q)$ by considering its elements as subspaces of an $(n+1)$-dimensional vector space where U maps to $U^\perp$. We saw previously that the absolute points and lines of a polarity define a polar space. We can define polar spaces directly by considering a vector space with a form. Given a non-degenerate alternating or hermitian form β, or quadratic form κ, the totally isotropic (respectively, totally singular) subspaces form a polar space. With respect to vector dimension, the 1-spaces form the points, 2-spaces form the lines, etc... with maximals being the spaces with dimension equal to the Witt index, and incidence defined by inclusion of subspaces. Such a polar space is called *classical* and clearly embeds in $\mathsf{PG}(n-1,q)$, which we refer to as the *ambient projective space*.

The classical polar spaces fall into three types according to their forms: *Symplectic* polar spaces arise from alternating forms, *hermitian* (or *unitary*) polar spaces arise from hermitian forms, and *quadrics* (or *orthogonal*) polar spaces arise from quadratic forms. Note that the hermitian spaces and the quadrics split into further categories depending on the dimension of the vector space from which they arise. It turns out that other than for rank 2, in which case we have generalised quadrangles, the classical polar spaces are the only examples (cf. [46, Corollary 7.137]).

Theorem 3.2.6 (Veldkamp - Tits - Buekenhout - Shult [25, 120, 127])**.** *Every finite polar space of rank $d \geqslant 3$ is classical.*

A classical polar space of rank d has *parameters* (s,t) if there are $s+1$ points incident with any line, and each $(d-1)$ space is incident with $t+1$ maximals. Moreover, we say that a polar space has *type* e if $(s,t)=(s,s^e)$. Table 3.1 below summarises some of the key information for the classical polar spaces, and introduces their notation. The first argument indicates the projective dimension of the ambient projective space, while the second term indicates the order of the finite field.

Polar space	Name	Form	(s,t)	e
$\mathsf{W}(2d-1,q)$	Symplectic	Alternating	(q,q)	1
$\mathsf{H}(2d-1,q^2)$	Hermitian	Hermitian	(q^2,q)	1/2
$\mathsf{H}(2d,q^2)$	Hermitian	Hermitian	(q^2,q^3)	3/2
$\mathsf{Q}^-(2d+1,q)$	Elliptic	Quadratic	(q,q^2)	2
$\mathsf{Q}(2d,q)$	Parabolic	Quadratic	(q,q)	1
$\mathsf{Q}^+(2d-1,q)$	Hyperbolic	Quadratic	$(q,1)$	0

Table 3.1: Summary of classical polar spaces

A finite polar space of rank d with order (s, s^e) has

$$\prod_{i=0}^{d-1}(s^{i+e}+1), \quad \text{and} \quad (s^{d-1+e}+1)[d],$$

maximals and points, respectively. The number of k-spaces is [22, Lemma 9.4.1]

$$\begin{bmatrix} d \\ k \end{bmatrix}_s \prod_{i=1}^{k}(s^{d+e-i}+1).$$

Given a finite classical polar space of rank $d \geqslant 2$, the set of all its subspaces which contain a point P forms a finite classical polar space of the same type and rank $d-1$, called the *quotient space*.

Remark 3.2.7. *We can actually quotient about larger elements of a polar space as well. If U is a subspace of V, we can define a vector space V/U on the equivalence classes of the relation $v \sim u$, where $v \sim u$ if $v-u \in U$. If β is a sesquilinear form, or κ is a quadratic form, then we can take a U to be a totally isotropic or totally singular subspace, respectively, and define the forms*

$$\beta_U(U+v_1, U+v_2) = \beta(v_1, v_2), \quad v_1, v_2 \in U^\perp,$$

and

$$\kappa_U(U+v) = \kappa(v), \quad v \in U^\perp.$$

The quotient polar space *about U is the polar space with the form β_U or κ_U, respectively, on the vector space $U^\perp/U$.*

3.3 Generalised Quadrangles and Partial Geometries

A *generalised quadrangle* is a partial linear space such that

- for every point P and line ℓ which are not incident, there exists a unique point Q incident with ℓ such that Q is collinear with P.

This axiom is often called the “generalised quadrangle” axiom. Note that it implies the “one or all” axiom, and hence generalised quadrangles are polar spaces. A generalised quadrangle has order (s, t) if there are $s+1$ points on every line, and $t+1$ lines through every point. A generalised quadrangle is a polar space of rank 2. Hence there are classical generalised quadrangles, corresponding to those in Table 3.1, however there also exist non-classical examples, for example those of Kantor [75] and Tits [53]. The classification of generalised quadrangles is still very much an open problem.

Example 3.3.1. *The smallest generalised quadrangle is* $\mathsf{W}(3,2)$ *with order* $(2,2)$*, and is depicted in Figure 3.2. It is often referred to as “the doily” because of its appearance.*

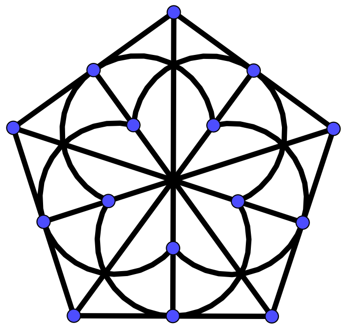


Figure 3.2: $\mathsf{W}(3,2)$.

Observe that the dual of a generalised quadrangle with order (s,t) is itself a generalised quadrangle of order (t,s). We denote the dual by prefixing a D to the notation. For example, the dual of $\mathsf{W}(3,q)$ is $\mathsf{DW}(3,q)$.

We have the following useful isomorphisms between classical generalised quadrangles.

Theorem 3.3.2 ([99, 3.2.1])**.** *$\mathsf{Q}(4,q)$ is isomorphic to $\mathsf{DW}(3,q)$. It is self-dual if and only if q is even.*

Theorem 3.3.3 ([99, 3.2.3])**.** *$\mathsf{Q}^-(5,q)$ is isomorphic to $\mathsf{DH}(3,q^2)$.*

An *m-ovoid* of a generalised quadrangle is a set of points meeting every line in exactly m points. We call a 1-ovoid an *ovoid*. An m-cover is a set of lines such that every point lies on exactly m lines in that set. A hemisystem of a generalised quadrangle is a $\frac{s+1}{2}$-ovoid. Note that an m-cover is an m-ovoid of the corresponding dual generalised quadrangle. We call a 1-cover a *spread*. Clearly the complement of an m-ovoid or m-cover, in a generalised quadrangle of order (s,t) is a $(s+1-m)$-ovoid or $(t+1-m)$-cover, respectively. A *partial ovoid* is a collection of mutually non-collinear points. Similarly, a *partial spread* is a set of pairwise skew lines. A partial ovoid or partial spread is an *ovoid* or *spread* if it partitions the line set or the point set, respectively. Moreover, note that a partial ovoid corresponds to a coclique in the collinearity graph. We refer to [42] for a survey of some key results on partial spreads and partial ovoids.

A x-tight set $\mathcal{T}$ of a generalised quadrangle of order (s,t) is a set of points such that for every point P in $\mathcal{T}$, there are $s+x$ points of $\mathcal{T}$ collinear with P (and distinct from P), and for every point P not in $\mathcal{T}$, there are x points of $\mathcal{T}$ collinear with P.

The collinearity graph of a generalised quadrangle is a strongly regular graph with parameters

$$\begin{aligned}
\nu &= (s+1)(st+1) \\
k &= s(t+1) \\
\lambda &= s-1 \\
\mu &= t+1.
\end{aligned}$$

The eigenvalues of the collinearity graph are $k = s(t+1)$, $e^+ = s-1$, and $e^- = -t-1$, so the matrix of eigenvalues is

$$P = \begin{bmatrix} 1 & s(t+1) & s^2t \\ 1 & s-1 & -s \\ 1 & -t-1 & t \end{bmatrix}.$$

The direct decomposition into eigenspaces corresponding to k, e^+ and e^- is often denoted by

$$V_0 \perp V^+ \perp V^-,$$

and we will ensure this ordering of the eigenspaces so that we may set $V_1 = V^+$ and $V_2 = V^-$ when this is more convenient.

It turns out that m-ovoids and tight sets are Delsarte T-designs [7]. If $\mathcal{O}$ is an m-ovoid, then $\chi_{\mathcal{O}} \in V_0 \perp V^-$. If $\mathcal{T}$ is a tight set, then $\chi_{\mathcal{T}} \in V_0 \perp V^+$. We can calculate the intersection of a m-ovoid and x-tight set by Theorem 2.4.4 to be equal to mx [12].

Given a generalised quadrangle $(\mathcal{P}, \mathcal{L}, \mathrm{I})$, a *generalised subquadrangle* $(\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is a generalised quadrangle such that $\mathcal{P}' \subset \mathcal{P}$, $\mathcal{L}' \subset \mathcal{L}$, and I' is I restricted to the new point and line sets.

Example 3.3.4. *The subquadrangle $\mathsf{Q}(4,q)$ of $\mathsf{Q}^-(5,q)$ is a (q^2+1)-tight set of $\mathsf{Q}^-(5,q)$ and the subquadrangle $\mathsf{H}(3,q^2)$ of $\mathsf{H}(4,q^2)$ is a (q^3+1)-tight set of $\mathsf{H}(4,q^2)$.*

A *partial geometry* $\mathsf{pg}(s,t,\alpha)$ is an incidence structure with order (s,t) and given any line ℓ and point P not incident with ℓ, there are exactly α points of ℓ which are collinear with P. Clearly when $\alpha = 1$, a partial geometry is a generalised quadrangle.

The collinearity graph of a partial geometry $\mathsf{pg}(s,t,\alpha)$ is a strongly regular graph (cf. [23]), with parameters

$$\begin{aligned}
\nu &= (s+1)(1+\frac{st}{\alpha}), \\
k &= s(t+1), \\
\lambda &= (s-1)+t(\alpha-1), \\
\mu &= \alpha(t+1).
\end{aligned}$$

The eigenvalues are thus $s(t+1)$, $s-\alpha$, and $-t-1$, and so the matrix of eigenvalues is

$$P = \begin{bmatrix} 1 & s(t+1) & \frac{st}{\alpha}(s+1-\alpha) \\ 1 & s-\alpha & \alpha-s-1 \\ 1 & -t-1 & t \end{bmatrix}.$$

3.4 Regular Near Polygons and Dual Polar Spaces

A *near polygon* is a partial linear space such that for every point P and line L, there exists a unique point Q on L nearest to P [109]. They include the *generalised polygons* of Tits [119], and the dual polar spaces of Cameron [26], which we shall return to shortly. We call a near polygon a *near* $2d$*-gon* if it has diameter d in its collinearity graph. If in addition the collinearity graph is distance-regular, we say that a near $2d$-gon is *regular*. A regular near polygon has *order* (s,t) if there are $t+1$ lines on each point and $s+1$ points on each line, and it has *parameters* $(s, t_2, t_3, \dots, t_{d-1}, t)$ such that there are $t_i + 1$ lines on a point P which are distance $i-1$ from a point Q whenever P and Q are at distance i.

Regular near 4-gons are precisely the generalised quadrangles (cf. [46, Theorem 6.1]). There are sporadic examples of regular near polygons arising, for example, from the unique Steiner system $S(5,8,24)$ [109], from the extended ternary Golay code [109], and from the Hall-Janko group [33], with parameters $(2,2,14)$, $(2,1,11)$, and $(2,0,3,4)$, respectively.

An important class of regular near polygons are the dual polar spaces. Let Π be a polar space of rank $d \geqslant 2$, and let $\mathcal{M}$ and $\mathcal{N}$ be the set of maximals and next-to-maximals of Π respectively. The point-line geometry $\mathsf{D}\Pi = (\mathcal{M}, \mathcal{N}, \mathrm{I})$, with I being inclusion of subspaces, is called a *dual polar space of rank* d. We denote the dual of a polar space by prefixing a D to its notation. Hence the classical dual polar spaces of rank d are $\mathsf{DW}(2d-1,q)$, $\mathsf{DH}(2d-1,q^2)$, $\mathsf{DH}(2d,q^2)$, $\mathsf{DQ}^+(2d-1,q)$, $\mathsf{DQ}(2d,q)$, and $\mathsf{DQ}^-(2d+1,q)$. Observe that this definition is consistent with the definition of the dual of a generalised quadrangle. Moreover, the dual polar space of rank d is a regular near $2d$-gon.

Let Π be a classical polar space or rank d and parameters (s, s^e). The collinearity graph of $\mathsf{D}\Pi$ is distance-regular with diameter d and classical parameters $(d, s, 0, s^e)$ [22, Theorem 9.4.3]. Distance is defined for two vertices $M_1, M_2 \in \mathcal{M}$ by $d(M_1, M_2) = d-1-\dim(M_1 \cap M_2)$. Since the collinearity graph is distance regular, it also produces a d-class metric association scheme. We can easily compute the eigenvalues using the following theorem.

Theorem 3.4.1 (cf. [125, Theorem 4.3.6])**.** *Given a finite classical polar space of rank d with parameters (s, s^e), the entries of the matrix of eigenvalues for the* *association scheme arising from the dual are*

$$P_{ij} = \sum_{t=max(i-j,0)}^{min(d-j,i)} (-1)^{i+t} \begin{bmatrix} d-i \\ d-j-t \end{bmatrix}_s \begin{bmatrix} i \\ t \end{bmatrix}_s s^{(t+j-i)(t+j-i+2e-1)/2+(i-t)(i-t-1)/2}.$$

Moreover the ordering of the strata defined by the matrix of eigenvalues in Theorem 3.4.1 is cometric and corresponds to the ordering given in Theorem 3.4.2.

Theorem 3.4.2 ([50] (c.f. [124])). *Let $(\Omega, \mathcal{R})$ be the association scheme corresponding to the dual of a finite classical polar space of rank d. There exists an ordering of the strata of the scheme such that for $1 \leqslant k \leqslant d$*

$$V_0 \perp V_1 \perp \ldots \perp V_k$$

is spanned by the set of all $\chi_{[X]}$, where $[X]$ is the set of maximals which contain the $(k-1)$-space X.

An m-ovoid of a regular near polygon is a set of points $\mathcal{O}$, such that every line is incident with exactly m points of $\mathcal{O}$. We call an m-ovoid a *hemisystem* if $m = \frac{s+1}{2}$. We shall look at m-ovoids of regular near polygons in Chapter 5, where we show that m-ovoids of certain regular near polygons are hemisystems. Note that m-ovoids are Delsarte designs.

Lemma 3.4.3. *An m-ovoid of a classical dual polar space is a $\{1, \ldots, d-1\}$-design.*

Proof. Let the order of the dual polar space be (s,t) and let X be a next-to-maximal, then $\big|[X]\big| = s+1$. Let $\mathcal{O}$ be an m-ovoid, then a double count reveals that $|\mathcal{O}| = \frac{m|\mathcal{M}|}{s+1}$. Now, by Theorem 3.4.2, $V_0 \perp V_1 \perp \ldots \perp V_{d-1}$ is spanned by the $\chi_{[X]}$ and so $\chi_{[X]} - \frac{s+1}{|\mathcal{M}|}j$ span $V_1 \perp \ldots \perp V_{d-1}$. Note that $\chi_{\mathcal{O}} \cdot \chi_{\{X\}} = m$ by definition of an m-ovoid, and $\chi_{\mathcal{O}} \cdot j = |\mathcal{O}| = \frac{m|\mathcal{M}|}{s+1}$. Hence $\chi_{\mathcal{O}} \cdot (\chi_{[X]} - \frac{s+1}{|\mathcal{M}|}j) = 0$. Thus $\chi_{\mathcal{O}}$ is orthogonal to $V_1 \perp \ldots \perp V_{d-1}$. □

Note that although a dual polar space is a point-line geometry, it is often helpful to remember that its elements are the maximals and next-to-maximals of the corresponding polar space. We obtain a similar result to Lemma 3.4.3 for m-covers of polar spaces. An *m-cover* of a polar space is a set of maximals $\mathcal{O}$, such that every point is contained in m elements of $\mathcal{O}$.

Lemma 3.4.4. *An m-ovoid of a classical polar space is a $\{1\}$-design.*

The proof is essentially the same as that of Lemma 3.4.3, except we take X to be a point rather than a next-to-maximal, and make the corresponding adjustments.

An m-cover of a classical polar space is a *hemisystem* if m is equal to half of the number of maximals on a point. In this case $|\mathcal{O}|$ is equal to half the total number of maximals. We shall look at hemisystems of the quadrics in detail in Chapter 6, where we construct the first known infinite family of hemisystems of $\mathsf{Q}(2d,q)$ for $d \geqslant 2$ and q odd.

3.5 Penttila-Williford Scheme

Let S be a generalised quadrangle of order (s,t) and let S' be a generalised subquadrangle of S of order (s,t'). Let P be a point of S not in S', then $P^{\perp} \cap S'$ is an ovoid of S' which we denote $\mathcal{O}_P$. We say $\mathcal{O}_P$ is *subtended* by P. If there is exactly one other point P' such that $\mathcal{O}_P$ is also subtended by P', then we say $\mathcal{O}_P$ is doubly subtended. If every ovoid of S' is doubly subtended, then we say that S' is doubly subtended in S (cf. [24]).

Let S be a generalised quadrangle of order (s,s^2), for $s > 2$ and let S' be a doubly subtended generalised quadrangle of order (s,s) contained in S, and let Ω be the set of points of $S \backslash S'$. Then a cometric, Q-bipartite association scheme $(\Omega, \{R_0, R_1, R_2, R_3, R_4\})$, is defined by the following relations on $\Omega \times \Omega$:

- R_0 is the identity relation,
- $(x,y) \in R_1$ if and only if x and y are not collinear in S and $|\mathcal{O}_x \cap \mathcal{O}_y| = 1$,
- $(x,y) \in R_2$ if and only if x and y are not collinear in S and $|\mathcal{O}_x \cap \mathcal{O}_y| = s+1$,
- $(x,y) \in R_2$ if and only if x and y are collinear in S,
- $(x,y) \in R_2$ if and only if $\mathcal{O}_x = \mathcal{O}_y$.

This association scheme is often referred to as the *Penttila-Williford scheme* after Penttila and Williford who first described it [100]. The Penttilla-Williford scheme is rare in that it is not metric and nor is it the dual of a metric association scheme. It is Q-bipartite but not Q-antipodal (see §7.4). It has the following matrix of eigenvalues:

$$P = \begin{pmatrix} 1 & (s-1)(s^2+1) & (s^2-2s)(s^2+1) & (s-1)(s^2+1) & 1 \\ 1 & s^2+1 & 0 & -(s^2+1) & -1 \\ 1 & s-1 & -2s & s-1 & 1 \\ 1 & -s+1 & 0 & s-1 & -1 \\ 1 & -(s-1)^2 & 2s(s-2) & -(s-1)^2 & 1 \end{pmatrix}.$$

There are two known families of generalised quadrangles of order (s,s^2) containing a doubly subtended quadrangle of order (s,s). They are $\mathsf{Q}^-(5,q)$ and $\mathsf{Q}(4,q)$ for all prime powers q [117] and a family of generalised quadrangles of Kantor for all odd prime powers of q (cf. [24, Lemma 3.1.5] and [76]). We shall consider the Penttila-Williford scheme again in Chapter 8.

Chapter 4

Computational Methods and Results

Computation is often the backbone of theoretical results, helping to identify trends or examples which may eventually be described from a purely theoretical standpoint. But computational results are often interesting in their own right, particularly when new examples or counterexamples are found, or when complete classifications are given.

Geometric and combinatorial problems are often amenable to computation, since they can typically be restated in terms of maximal cliques of graphs, or constraint satisfaction problems. However, such problems suffer from "combinatorial explosion", in which the complexity of a problem grows exponentially as the parameters governing the problem are increased only slightly. This renders many problems computationally infeasible, and thus to push the bounds of feasibility, efficient computational methods must be employed.

The techniques presented in this chapter are applicable to a wide range of combinatorial problems. They rely heavily on a combination of isomorph-free generation and mixed integer programming techniques. In the pursuit of efficient computational methods, we also present a number of novel developments. In particular, we introduce the notion of strong semi-canonicity in Lemma 4.5.1 and develop an orderly algorithm utilising strong semi-canonicity informed black-box pruning in Algorithm 3. We then present a number of new computational results in geometry using these methods, and conclude with a brief description of some of the software developed in the course of this thesis.

4.1 Solving Combinatorial Constraints

Many geometric objects are described in terms of their combinatorial properties. For example, consider the problem of determining the existence of an ovoid in a generalised quadrangle. Recall that an ovoid $\mathcal{O}$ is a subset of the points such that every line is incident with exactly one point in $\mathcal{O}$. This can be expressed as

$$\chi_{[\ell]} \cdot \chi_{\mathcal{O}} = 1,$$

for all lines $\ell \in \mathcal{L}$, where $[\ell]$ represents the point set of ℓ.

In fact, there exists x, a $\{0,1\}$-vector, such that

$$\chi_{\{\ell\}} \cdot x = 1,$$

for all lines $\ell \in \mathcal{L}$ if and only if there exists an ovoid $\mathcal{O}$.

Thus we can convert the geometric problem of determining the existence of an ovoid into the constraint satisfaction problem of determining the existence of a binary solution to a system of equations. This is now more easily implemented in a computer and solved. Moreover, a complete classification of ovoids can be found by determining all such binary solutions.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with some ordering on the points and lines, then the *point-line incidence matrix*, M, is the $|\mathcal{P}| \times |\mathcal{L}|$ matrix with rows indexed by the points, and columns indexed by the lines, and with entries given by

$$M_{P,\ell} = \begin{cases} 1 & \text{if } P \,\mathrm{I}\, \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the *line-point incidence matrix* is the $|\mathcal{L}|\times|\mathcal{P}|$ matrix with rows indexed by lines and columns indexed by points, with a 1 in any entry where the corresponding point and line are incident, and a zero otherwise. The point-line incidence matrix and the line-point incidence matrix are transposes of one another. We will simply use *incidence matrix* when the context is clear.

There exists an m-ovoid of $\mathcal{S}$ if and only if there exists a binary solution to $Mx = m\mathbb{1}$. Moreover, a partial ovoid corresponds to a solution to $Mx \leqslant \mathbb{1}$, where for two vectors u and v, $u \leqslant v$ if each of the entries of u is equal to or less than the corresponding entry of v. Maximising $|x|$ subject to $Mx \leqslant \mathbb{1}$ then yields a maximal partial ovoid.

In the following example, Example 4.1.1, we show how the line-point incidence matrix of the generalised quadrangle $\mathsf{W}(3,2)$ may be used to find an ovoid.

Example 4.1.1. *Consider* $\mathsf{W}(3,2)$ *with point set*

$$\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_7, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}\},$$

and line set

$$\mathcal{L} = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_7, \ell_9, \ell_{10}, \ell_{11}, \ell_{12}, \ell_{13}, \ell_{14}, \ell_{15}\},$$

where

$$\begin{array}{lll}
\ell_1 = \{P_1, P_4, P_5\}, & \ell_6 = \{P_2, P_{12}, P_{14}\}, & \ell_{11} = \{P_5, P_{11}, P_{14}\}, \\
\ell_2 = \{P_1, P_8, P_9\}, & \ell_7 = \{P_3, P_4, P_7\}, & \ell_{12} = \{P_6, P_9, P_{15}\}, \\
\ell_3 = \{P_1, P_{12}, P_{13}\}, & \ell_8 = \{P_3, P_8, P_{11}\}, & \ell_{13} = \{P_7, P_9, P_{14}\}, \\
\ell_4 = \{P_2, P_4, P_6\}, & \ell_9 = \{P_3, P_{12}, P_{15}\}, & \ell_{14} = \{P_6, P_{11}, P_{13}\}, \\
\ell_5 = \{P_2, P_8, P_{10}\}, & \ell_{10} = \{P_5, P_{10}, P_{15}\}, & \ell_{15} = \{P_7, P_{10}, P_{13}\}.
\end{array}$$

This indexing corresponds to the lexicographical ordering of subspaces, such as used in FinInG *[5]. Figure 4.1 depicts* $\mathsf{W}(3,2)$ *with the points and lines labelled according to* $\mathcal{P}$ *and* $\mathcal{L}$*. Figure 4.2 gives the corresponding incidence matrix,* M*, of* $\mathsf{W}(3,2)$*.*

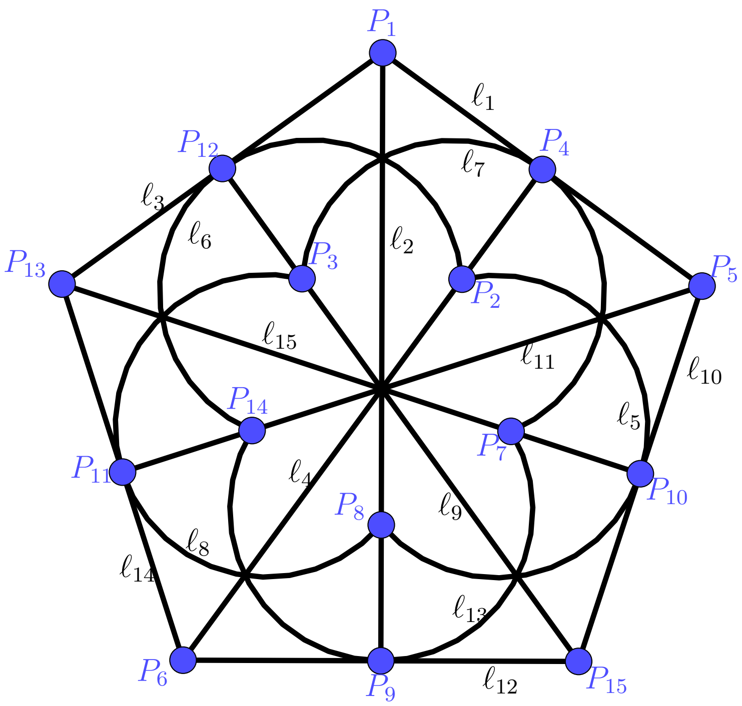


Figure 4.1: $\mathsf{W}(3,2)$ with labels.

To find an ovoid of $\mathsf{W}(3,2)$*, we require a set of points* $\mathcal{O} \subseteq \mathcal{P}$ *such that every line of* $\mathcal{L}$ *is incident with exactly one point in* $\mathcal{O}$*. Let* $\chi_{\mathcal{O}}$ *be the characteristic vector of* $\mathcal{O}$*, then* $M\chi_{\mathcal{O}} = \mathbb{1}$*. Hence solving* $Mx = \mathbb{1}$ *for a binary vector* x *yields an ovoid, while no solution would imply non-existence. Observe that we have the following solution*

$$(0,0,0,1,0,0,0,0,1,1,1,1,0,0,0)$$

and hence

$$\{P_4, P_9, P_{10}, P_{11}, P_{12}\}$$

is an ovoid of $\mathsf{W}(3,2)$*.*

$$
\begin{array}{c}
\\
\ell_1\\ \ell_2\\ \ell_3\\ \ell_4\\ \ell_5\\ \ell_6\\ \ell_7\\ \ell_8\\ \ell_9\\ \ell_{10}\\ \ell_{11}\\ \ell_{12}\\ \ell_{13}\\ \ell_{14}\\ \ell_{15}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccc}
P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} & P_{11} & P_{12} & P_{13} & P_{14} & P_{15}
\end{array}\\
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 4.2: Incidence matrix of $\mathsf{W}(3,2)$ with indices labelled.

Of course there are other constraints which may be used to compute, or exclude, geometric objects. Delsarte design conditions are particularly useful in this regard. Let θ be a T-design of an association scheme $(\Omega, \mathcal{R})$, and let v be a weighted antidesign. Then by Theorem 2.4.4,

$$v \cdot \chi_\theta = \frac{(v \cdot \mathbb{1})(\chi_\theta \cdot \mathbb{1})}{|\Omega|}.$$

Thus we have a constraint which χ_θ must satisfy in order for θ to exist. In particular, a T-design θ with $|\theta| = k$ exists if and only if there exists a binary solution, x, to the system of equations:

$$\mathbb{1} \cdot x = k,$$

and

$$E_i x = 0$$

for all $i \in T$. Moreover, linearly dependent rows of E_i can be removed to simplify solving such a system. This may save time when the system is to be repeatedly solved, as occurs in enumeration.

Note that any solution must be binary, since it must be the characteristic vector for some associated geometric object. Note too, that the constraints need not be binary. They may be weighted vectors, as is the case for weighted antidesigns.

In order to solve these systems of equations, we may formulate the problem as a *mixed integer program (MIP)* of the form

$$\begin{aligned}\text{Maximise: } & \mathbf{c}^\top \mathbf{x},\\ \text{Subject to: } & A\mathbf{x} \leqslant \mathbf{b}\end{aligned}$$

on $|\Omega|$ variables for some integer vector $x \geqslant 0$. While MIP problems may be converted to the form given above, software is often more flexible in practice with regard to the allowed input. MIPs solved in the course of this thesis used the *Gurobi Optimizer* software [65].

4.2 Tactical Decompositions and Collapsed Incidence Matrices

Sometimes it is possible to condense a problem, while retaining much of its key information, in order to make computation much simpler. A key method for doing this is to use tactical decompositions (cf. Dembowski [52]) and group orbits.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry, and let $\Omega = \{\Omega_1, \Omega_2, \dots, \Omega_m\}$ be a partition of the point set and $\Delta = \{\Delta_1, \Delta_2, \dots, \Delta_n\}$ be a partition of the line set. If for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ there exist non-negative integer constants ρ_{ij} and κ_{ij} such that:

1. every point of Ω_i is incident with ρ_{ij} lines of Δ_j,
2. every line of Δ_i is incident with κ_{ij} points of Ω_j,

then (Ω, Δ) is called a *tactical decomposition* of $\mathcal{S}$.

If M is the incidence matrix of $\mathcal{S}$, then the submatrix $[M_{k\ell}]$ for $k \in \Omega_i$ and $\ell \in \Delta_j$ has constant row sum ρ_{ij} and constant column sum κ_{ij}. If we wish, we may reorder the rows and columns of M according to the partitions Ω, and Δ, in order to obtain a block decomposition of M, such that the row and column sums of each block are constants. Hence we may talk about the tactical decomposition of the matrix, and observe that an incidence structure admits a tactical decomposition precisely when its incidence matrix does. The matrices $[\rho_{ij}]$ and $[\kappa_{ij}]$ are called the *tactical decomposition matrices* of M. In particular, if M is the line-point incidence matrix, we shall refer to $N = [\kappa_{ij}]$ as the *collapsed incidence matrix.*

Example 4.2.1. *Figure 4.3 displays the incidence matrix of* $\mathsf{W}(3, 2)$ *from Example 4.1.1, with the rows and columns permuted to highlight a tactical decomposition* (Ω, Δ).

The most obvious way to construct a tactical decomposition is to use some subgroup $H \leqslant \mathrm{Aut}(\mathcal{S})$. The orbits of H on points forms the partition Ω, and the orbits of H on lines forms the partition Δ, yielding a tactical decomposition of $\mathcal{S}$. The collapsed incidence matrix, $N^H_{\mathcal{L},\mathcal{P}}$, has entries $(N^H_{\mathcal{L},\mathcal{P}})_{ij} = |\{p \in \Omega_j : p \,\mathrm{I}\, \ell\}|$, for any $\ell \in \Delta_i$. Here $N^H_{\mathcal{L},\mathcal{P}}$ is sometimes called the *Kramer-Mesner* matrix, and we use the subscript and superscript notation to imitate the Kramer-Mesner notation, which typically has subscripts t and k, corresponding to a $t - (\nu, k, \lambda)$ design.

Let $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_m$, and $\Delta = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_n$, form a tactical decomposition of $\mathcal{S}$, with collapsed incidence matrix N. Moreover, let $S \subseteq \{1, \dots, n\}$,

$$
\begin{array}{cc}
 & \begin{array}{cccc|cccc|cccc|cc|c}
 & \Omega_1 & & & & \Omega_2 & & & & \Omega_3 & & & \Omega_4 & & \Omega_5 \\
P_1 & P_3 & P_6 & P_{14} & P_2 & P_5 & P_8 & P_{15} & P_4 & P_9 & P_{11} & P_{12} & P_7 & P_{13} & P_{10}
\end{array} \\
\begin{array}{cc}
 & \ell_1 \\
\Delta_1 & \ell_6 \\
 & \ell_8 \\
 & \ell_{12} \\
\hline
 & \ell_2 \\
\Delta_2 & \ell_4 \\
 & \ell_9 \\
 & \ell_{11} \\
\hline
 & \ell_3 \\
\Delta_3 & \ell_7 \\
 & \ell_{13} \\
 & \ell_{14} \\
\hline
\Delta_4 & \ell_5 \\
 & \ell_{10} \\
\hline
\Delta_5 & \ell_{15}
\end{array} &
\left(\begin{array}{cccc|cccc|cccc|cc|c}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right)
\end{array}
$$

Figure 4.3: Incidence matrix of $\mathsf{W}(3,2)$ highlighting a tactical decomposition.

and $\theta = \bigcup\{\Omega_i \in \Omega \colon i \in S\}$. Then clearly $N\chi_S = \lambda\mathbb{1}$ if and only if $M\chi_\theta = \lambda\mathbb{1}$. In particular, the we have the following:

Theorem 4.2.2. *There exists an m-ovoid of a point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, admitting a subgroup $H \leqslant \mathrm{Aut}(S)$ if and only if there exists a solution to*

$$N^H_{\mathcal{L},\mathcal{P}}\, x = m\mathbb{1}.$$

Theorem 4.2.2 gives rise to a sensible procedure for searching for m-ovoids, or indeed other designs. Subgroups of the full automorphism group may be considered and the MIP with coefficient matrix corresponding to the collapsed incidence matrix is solved. Thus designs may be classified or excluded up to some subgroup of their stabilisers. Clearly we can take all subgroups up to conjugacy and work our way through them. This lends itself to larger subgroups, since these will have larger orbits, and hence smaller collapsed incidence matrices. A more efficient method is to consider the cyclic subgroups generated by a representative from each conjugacy class. That way we need not consider all subgroups directly. In particular, by Cauchy's theorem, if we consider all conjugacy classes of prime order elements, then we can consider all subgroups indirectly. If no designs exist which admit a cyclic element of prime order, then any design must have a trivial automorphism group. One drawback here, is that generally speaking, the smaller the group the more difficult the computation is.

This approach is also sensible from a philosophical perspective. Most "interesting" designs have non-trivial automorphism groups. In fact, many would argue that having a non-trivial automorphism group is what makes a design interesting. In addition, large groups are typically necessary if we are to identify an infinite family. Thus it is reasonable to operate under the assumption that a design admits some automorphism. It is then a matter of discovering which automorphism is admitted. This simplifies computation drastically.

In coding theory, this approach of constructing designs with prescribed automorphism groups is often referred to as the Kramer-Mesner method [79]. This is a standard computational approach, and has been used successfully, for example, to construct the only known q-analogues of Steiner systems [21].

Of course a similar technique may be used in other settings and one need not necessarily use orbits to collapse an incidence matrix. Nor need one use the incidence matrix, or even produce a tactical decomposition. Any matrix equation of the form

$$Ax = y$$

can be collapsed by simply summing certain columns, and requiring the corresponding values of x to all be in, or all out, of the solution. For example, we can collapse minimal idempotents of an association scheme by taking orbits on the vertices and summing columns according to the orbits.

In the following example, we show how a group can be used to collapse the incidence matrix of $\mathsf{W}(3,2)$ and the corresponding MIP solved to find ovoids.

Example 4.2.3. *Consider* $\mathsf{W}(3,2)$ *as in Example 4.1.1. The orbits of the group*

$$\langle (1,3,6,14)(2,5,8,15)(4,11,9,12)(7,13)\rangle$$

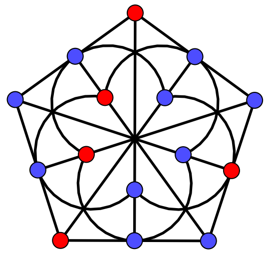
on the points and lines are:

$$\begin{aligned} A_1 &= \{\ell_1,\ell_6,\ell_8,\ell_{12}\}, & B_1 &= \{P_1,P_3,P_6,P_{14}\},\\ A_2 &= \{\ell_2,\ell_4,\ell_9,\ell_{11}\}, & B_2 &= \{P_2,P_5,P_8,P_{15}\},\\ A_3 &= \{\ell_3,\ell_7,\ell_{13},\ell_{14}\}, & B_3 &= \{P_4,P_9,P_{11},P_{12}\},\\ A_4 &= \{\ell_5,\ell_{10}\}, & B_4 &= \{P_7,P_{13}\},\\ A_5 &= \{\ell_{15}\}, & B_5 &= \{P_{10}\}. \end{aligned}$$

Indeed, these orbits correspond to the tactical decomposition given in Example 4.2.1. Consider the collapsed incidence matrix, N, given by

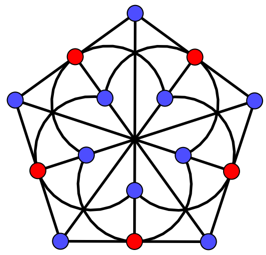
$$\begin{array}{c} \\ A_1\\ A_2\\ A_3\\ A_4\\ A_5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array}\\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 1 & 0\\ 0 & 2 & 0 & 0 & 1\\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}. \end{array}$$

Observe that there are two solutions to $Nx = \mathbb{1}$ and $(4,4,4,2,1)\cdot x = 5$, namely $(1,0,0,0,1)$, corresponding to the ovoid



,

and $(0, 0, 1, 0, 1)$*, corresponding to the ovoid*



.

This second solution is the same ovoid given in Example 4.1.1, yet the matrix equation to find this solution is considerably simpler to solve using the collapsed incidence matrix.

Remark 4.2.4. *It is worth noting here that the normaliser of the group preserves the orbits, which means that orderly generation algorithms, such as those we shall encounter later in this chapter, can be used in conjunction with the collapsed incidence matrix.*

4.3 Isomorphic and Canonical Objects

When enumerating geometric or combinatorial objects, we often want to know how many there are up to equivalence, rather than how many there are in total. For example, there are 76422528 hemisystems of $\mathsf{DQ}(6, 3)$, and yet they are all isomorphic! Thus rather than complete enumeration, we are often interested in enumerating a single representative from each isomorphism class. This is referred to as *generation*.

One method of identifying two isomorphic objects in a point-line geometry is to partition the vertex set of the collinearity graph according to those vertices belonging to each object, and then test if the two partitioned graphs are isomorphic. This converts the problem into the well studied graph isomorphism problem. There exist polynomial time algorithms for solving graph isomorphism in special classes of graphs, such as planar graphs [34, 69, 68], graphs with bounded degree [87], graphs with excluded minors [62, 101], and graphs with excluded topological subgraphs [63]. In general, however, the fastest that the graph isomorphism problem can currently be solved is in quasipolynomial time $e^{O((\log n)^b)}$ for constant b, and is due to Babai [2]. The fastest algorithm for solving graph isomorphism of graphs with maximum degree d has $e^{O((\log d)^c)}$ running time for constant c, and is due to Grohe, Neuen and Schweitzer [64].

The fastest general purpose algorithms in practice use the individualisation-refinement paradigm. The individualisation-refinement paradigm was first used by Parris and Read [98], and further developed and popularised by McKay's *nauty* software [90]. It consists of selecting (individualising) a vertex from a chosen, non-singleton cell of a partition of the vertices of a graph, and then refining the partition in an isomorphism invariant manner that results in this individualised vertex becoming a singleton in the refinement. This is repeated until a discrete partition is obtained, and then it is necessary to backtrack and individualise the other vertices in each of the chosen cells. A detailed explanation of the individualisation-refinement paradigm can be found in [92]. Neuen and Schweitzer [95] recently proved that algorithms which utilise individualisation-refinement have exponential running time in the worst case by constructing a family of difficult graphs in a manner similar to Miyazaki [94]. Despite this exponential worst case running time, the individualisation-refinement algorithms are still the best algorithms in practice, and for many graphs, particularly those arising from finite geometries, have much better running time.

Generally a specific representative from each isomorphism class of objects is chosen to be the *canonical* example for that class. Given some object, determining the canonical representative in its isomorphism class is referred to as finding the canonical labelling of that object. For convenience, we shall refer to a canonically labelled isomorph as a "canon" for its class. We can determine if two objects are isomorphic if they have the same canonical labelling. Note that this replaces isomorph testing with equality testing, which is much faster. If many objects must be tested for isomorphism it is significantly faster to find and compare their canonically labelled isomporphs than to pairwise check if they are isomorphic by other means.

Given two ordered sets $S = \{s_1, s_2, \ldots, s_n\}$ and $T = \{t_1, t_2, \ldots, t_m\}$, with respect to some total ordering, we say that S is *lexicographically less* than T, or simply $S < T$, if $s_i < t_i$ in the first instance that $s_i \neq t_i$, or if $n < m$ and $s_i = t_i$ for all $1 \leqslant i \leqslant n$. For convenience, we often abbreviate "lexicographical" simply to "lex". Lex-ordering provides a convenient method of defining canonicity, by simply taking the lexicographically least member of each isomorphism class to be the canon. Lexicographical ordering has the advantage that it is very easy for one to directly compare two sets by inspection. However it is dependent on the chosen ordering, and is not always the most efficient method for determining a canon. Often canons are chosen by some method which is internally convenient for a given algorithm. Such canons are often more computationally efficient to determine, however generally not easily understandable for humans. The definition of a canon may also vary between algorithms, so while a correct algorithm will determine a canonical labelling of a given object, it may differ from the canonical labelling given by a different algorithm. Lex-ordering also has an advantage here in that it will be consistent across algorithms.

Example 4.3.1. *Consider the ovoid found in Example 4.1.1. There exists an element*

$$g = (1,4)(2,12)(3,8)(6,13)(7,9)(10,15)$$

in the automorphism group of $\mathsf{W}(3,2)$ *acting on the points such that*

$$\{4,9,10,11,12\}^g = \{1,2,7,11,15\} < \{4,9,10,11,12\}.$$

Moreover, $\{P_1, P_2, P_7, P_{11}, P_{15}\}$ *is in fact the canonical representative of the ovoid, where the canon is the lexicographically least ordered object in the orbit under the automorphism group of* $\mathsf{W}(3,2)$*.*

Utilising automorphisms is very effective in pruning the search tree of a canonicity algorithm. This is precisely why *Traces* [92] prioritises discovering new automorphisms earlier rather than later. There are also a number of algorithms which take a permutation group directly and find the canonical image of a set upon which the group acts. Notable among these algorithms are Linton's algorithm [85] and the Ladder Game, or *Leiterspiel* in German, of Schmalz [106]. Such algorithms are well suited to problems in geometry, since the full automorphism group of the geometry is generally already known, removing the need for the algorithm to discover automorphisms.

The *Ladder Game* was developed by Schmalz [106] from the homomorphism principle (*Homomorphieprinzip*) of Laue [82]. The key development was to enable the homomorphisms to act in different directions [106], reminiscent of the game "snakes and ladders", and hence the name. Further developments have more recently been made by Koch [78]. The Ladder Game is presented in terms of double cosets, since many isomorphism problems can be represented by double cosets.

We give a simple version of the Ladder Game in Algorithm 1. Note that this is less general, as it is restricted to the action of permutation groups upon sets. Moreover, this particular implementation does not actually return the canonical image of a set. Instead, it simply checks if the set is canonical or not, with the canonical set being the lexicographically least set in the orbit of the group. Thus Algorithm 1 is suitable for an *IsMinimal* type check.

In Algorithm 1, we backtrack over all arrangements of the elements of the set s. For each candidate, we successively apply elements in the stabiliser of the first $m-1$ elements of s. This should map the first m elements of the candidate to the first m elements of s (as a set). Each of the remaining elements are also mapped to something new. Then the next transversal element is applied to bring the $(m+1)$-th element across and so on. The first m elements will be fixed, since we are using stabiliser elements. If at any point the candidate maps to something lexicographically greater than s, then the candidate is discarded and the backtrack proceeds. If however the candidate maps to something lexicographically less than s, then this implies that s is not minimal, and the algorithm terminates.

Note that orbits must be computed to determine the minimal element of each orbit, and typically the transversal group elements which map the other elements

Algorithm 1: Ladder Game (Leiterspiel)

```
   Input: Starter set s, permutation group G
   Output: True or false, depending on whether s is minimal under action of G
1  stack ← [[]]
2  while not IsEmpty(stack) do
3      c ← Pop(stack)
4      k ← |c|
5      d ← c
6      for i ∈ {1, . . . , k} do
7          g_i ← Element of G_{s_{1,...,i}} mapping c_i to the lex least element in its orbit
            under G_{s_{1,...,i}}
8          d ← d^{g_i}
9          if d_{1,...,i} < s_{1,...,i} then
10             return false
11         else if d_{1,...,i} > s_{1,...,i} then
12             break
13         else if i = k then
14             for x ∈ s\c do
15                 Append(stack, c ∪ {x})
16 return true
```

of the orbit to this minimal element are computed in the process, and hence may simply be stored. Thus the transversal elements can easily be looked up and applied directly, and Line 7 of Algorithm 1 would most likely be computed once outside of the while loop and simply accessed at this point of the algorithm. Moreover, any extension of s inherits all of the stabilisers, orbits, and transversal elements computed for s. As a result, this process lends itself to orderly generation, which is explained in Section 4.4. In addition, the lexicographical ordering used by the Ladder Game may also be exploited, as we see in Lemma 4.5.1.

4.4 Isomorph-Free Generation

The previous section dealt with determining isomorphic objects and finding canonical representatives. However this is only useful for sorting objects that have already been enumerated. It is possible to incorporate canonicity in the process of generating objects, however. When a branch of the search tree is not canonical, or fails to be constructed in the correct manner, it is abandoned and the branch is not explored any further. This tends to drastically prune the search tree, saving unnecessary computation.

McKay's *canonical construction path* method [91], also known as *canonical augmentation*, is one such way of generating nonisomorphic objects by ensuring that each canonical set will be generated exactly once in the process. The technique consists of *canonical augmentation* and *canonical deletion* steps. In the augmentation step, an object is augmented by adding a new vertex, accounting for symmetry under the automorphisms of the object. It is possible that the same object could arise from the augmentation of two different parents, however. Thus it is necessary that a child be constructed from the "correct" parent. To ensure that the correct parent is chosen, an augmented object undergoes an isomorphism invariant *canonical deletion* process which is then compared to the parent. If the canonical deletion corresponds to the parent, then it is the correct parent and the construction path is canonical. Otherwise the parent is incorrect, the construction path is not canonical, and the augmented object is abandoned at this point of the search tree. Thus the canonical construction path method does not focus on constructing canonical objects so much as ensuring that the path used in their construction is canonical, and so isomorphs are rejected based upon canonical extension rather than actually being canonical.

The canonical construction path method does not require lists of canonical objects, and only needs to test the link between an augmented object and its parent. Hence it can be tested locally, and implemented in parallel. The canonical augmentation and deletion steps may be tested using canonicity algorithms, and are typically implemented using *nauty*. They do not require a hereditary property in canonicity. This is in contrast to various orderly generation algorithms which do require a hereditary property, which is present in lexicographical ordering, for example.

Orderly generation, introduced by Faradzev [55] and Read [102], is another process of isomorph-free generation. Orderly generation has successfully been used in finite geometry, for example, by Royle [105]. It works by extending canonical objects and testing that each extension is also canonical. Any object which is not canonical is then rejected. The orderly algorithm requires that the canonical form has the hereditary property, such that if $X = \{x_1, \ldots, x_m\}$ is canonical, so too is $\{x_1, \ldots, x_{m-1}\}$. As its name implies, it requires a well defined ordering on the objects being enumerated. Typically an orderly algorithm will then use a lexicographically least canon. In this case it is only necessary to consider augmentations of an object X to $X \cup \{x\}$ for those x which are minimal in the orbits of G_X on $\Omega \backslash X$, thus using symmetry to minimise the number of augmentation steps which occur.

We present pseudocode for the orderly generation algorithm in Algorithm 2. Algorithm 2 will generate all the canonical subsets of $\{1, \ldots, n\}$ of size k under the action of some permutation group G which can be extended from a starter set s. We take the lexicographically least element in each isomorphism class to be the canon, and assume that there is some algorithm *IsMinimal* which can be applied to check whether or not a given element is canonical under the action of the permutation group G. Note that the Ladder Game algorithm presented in Algorithm 1 would be suitable for such an *IsMinimal* check.

The Ladder Game algorithm code of Algorithm 1 may easily be adapted to

Algorithm 2: The orderly algorithm

Input: Starter set s, permutation group G, number of points n, length k
Output: All canonical extensions of s of length k

Algorithm `OrderlyAlgorithm`(s, G, n, k)
    return `OrderlySubAlgorithm`$(s, G, n, k, \{\})$

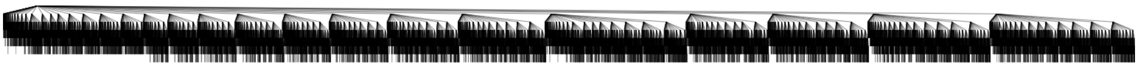
Procedure `OrderlySubAlgorithm`(s, G, n, k, S)
    if *IsMinimal(s, G)* **then**
        if $|s| = k$ **then**
            $S \leftarrow S \cup \{s\}$
        else
            $G_s \leftarrow$ setwise stabiliser of s in G
            $O \leftarrow$ orbits of G_s on $\{1, \ldots, n\}$
            for $i \in \{1, \ldots, |O|\}$ **do**
                $m \leftarrow Min(O_i)$
                $x \leftarrow s \cup \{m\}$
                $S \leftarrow$ OrderlySubAlgorithm(x, G, n, k, S)
    return S

produce an orderly algorithm. This means that the stabilisers, orbits and transversal elements need not be repeatedly calculated in the *IsMinimal* check, but rather may be stored and called upon. Since the *IsMinimal* check is the most expensive part of Algorithm 2 this may result in significant time savings. Alternatively, *IsMinimal* can be implemented in GAP [59] using the *IsMinimalImage* command of the *images* package [72], or by comparing a given set to its image under the *SmallestImageSet* command of the *GRAPE* package [112]. Note that *IsMinimalImage* is preferable to *SmallestImageSet*, since we may reject an object once we know it is not minimal, and do not need to do the extra work necessary to determine its minimal image.

Figure 4.4 illustrates the effectiveness of the orderly algorithm compared to naive generation. It depicts the search trees of both algorithms in enumerating maximal partial ovoids of $W(3, 3)$. In each case we reject any extension which is not a partial ovoid. In the naive search, we also assume that the set is ordered. If $s < s'$, then s is depicted to the right of s', and as a result, the search tree skews to the right. The contrast between the number of nodes which must be visited in naive backtracking compared to the orderly algorithm is stark. To further illustrate how few nodes need be visited in the orderly algorithm, the number of nodes at each depth of the two search trees is listed explicitly in Table 4.1.

4.5 An Effective Orderly Algorithm

Standard isomorph-free generation techniques are often much more efficient than naive generation since they so aggressively prune the search tree. However, since



(a) Naive search.

(b) Orderly algorithm.

Figure 4.4: Comparison of search trees for enumeration of maximum partial ovoids of $\mathsf{W}(3,3)$.

Depth	Naive	Orderly algorithm
1	40	1
2	540	1
3	3240	2
4	9450	3
5	13824	2
6	10080	2
7	2880	1
Total	40054	12

Table 4.1: Comparison of enumeration methods for maximum partial ovoids of $\mathsf{W}(3,3)$, showing number of nodes visited.

orderly and canonical construction path algorithms function by augmenting sets by one new element at a time, it may be that many nodes of the search tree are visited before it is discovered that a given object cannot be extended to a new canonical object with the correct size and properties. Ideally it would be discovered that this node cannot be extended and it would be pruned from the search immediately.

Search trees can be further pruned by "look ahead" techniques, where not only the node itself is considered, but also possible augmentations. For example, when searching for a coclique in a graph, one can count the vertices which are non-adjacent to the elements in the starter set. If there are insufficiently many, then the corresponding branches of the search tree need not be explored any further. We can of course use some other technique. We need some "black box" which takes the starter set corresponding to the node of the search tree and simply returns true if the search tree is to be explored further, or false otherwise. In this thesis *Gurobi* [65] is used as the black box to solve a MIP such as those described in Section 4.1 and Section 4.2, and for a given starter set S, the only additional constraint to add to the MIP is

$$\chi_S \cdot x = |S|.$$

The MIP to be solved remains the same in each case, with only the starter constraint changing at each node of the search tree. It may be that additional information can also be extracted from the starter set, and hence more complex constraints may also be added inside the black box.

Using a black box to look ahead and prune the search tree can result in a drastic reduction of the number of nodes to be visited. We may also stipulate that a given node in the search tree not only extends to a viable solution, but that it extends to a viable *canonical* solution. It is generally rather difficult to tell by inspection if an object is canonical. However, it is sometimes possible to tell with little effort when an object is not canonical, or when an augmentation would violate canonicity. This is particularly true when lexicographical orderings are used to determine canonicity. For example, any augmentation of an object cannot be canonical if the vertex being appended is lexicographically less than the maximum element of the object. We can exclude more elements by also considering orbits under the stabiliser of the object. We introduce *strong semi-canonicity* in the following lemma, Lemma 4.5.1. We name it such, as it is a strengthening of the concept of *semi-canonicity*, as described for example in [83].

Lemma 4.5.1 (Strong semi-canonicity). *Let Ω be a set and G be a permutation group acting on Ω. Let $S_k = \{s_1, \ldots, s_k\} \subseteq \Omega$, where $s_1 < s_2 < \ldots < s_k$, and for any $i \in \{1, \ldots, k\}$ define the subset $S_i = \{s_1, \ldots, s_i\}$ of S_k. Let $r \in \Omega$, and suppose that for some $j \in \{1, \ldots, k\}$, we have $r \notin S_j$ and $r < s_j$. Then $S_k \cup \{m\}$ is not minimal for any $m \in \{r^{g_j g_{j+1} \cdots g_{k-1} g_k} \mid g_i \in G_{S_i}, i = j, \ldots, k\}$.*

Proof. Let $m \in \{r^{g_j g_{j+1} \cdots g_{k-1} g_k} \mid g_i \in G_{S_i}\}$. Then there exists some $g = g_k g_{k-1} \ldots g_{j+1} g_j$, where $g_i \in G_{S_i}$, such that $m^g = r$. Now, $S_i^{g_i} = S_i$, since $g_i \in G_{S_i}$, and so $S_k^g = S_j \cup \{s_{j+1}, \ldots, s_k\}^g$. Thus $(S_k \cup \{m\})^g = S_j \cup \{s_{j+1}, \ldots, s_k\}^g \cup \{r\}$. However, $S_{j-1} \cup \{r\} < S_{j-1} \cup \{s_j\}$, since $r \notin S_j$ and $r < s_j$. Hence $(S_k \cup \{m\})^g < S_k \cup \{m\}$. □

Given some minimal set S, any augmentation of S which is not shown to be non-minimal by Lemma 4.5.1 is referred to as *strongly semi-canonical*.

Restricting to $r^{G_{S_j}} \subseteq \{r^{g_j g_{j+1} \cdots g_{k-1} g_k} \mid g_i \in G_{S_i}\}$ in Lemma 4.5.1 yields the weaker *semi-canonicity*, and hence semi-canonicity follows as a corollary, Corollary 4.5.2.

Corollary 4.5.2 (Semi-canonicity, cf. [83]). *Let $S_k = \{s_1, \ldots, s_k\}$, where $s_1 < s_2 < \ldots < s_k$, and for any $i \in \{1, \ldots, k\}$ define the subset $S_i = \{s_1, \ldots, s_i\}$ of S_k. For $r \in \Omega$, and some $j \in \{1, \ldots, k\}$, let $r \notin S_j$ and $r < s_j$. Then $S_k \cup \{m\}$ is not minimal for any $m \in r^{G_{S_j}}$.*

An augmentation of some minimal set which is not shown to be non-minimal by Corollary 4.5.2 is referred to as *semi-canonical*.

The following example, Example 4.5.3, illustrates the effect of using a black box to check if a set is extendable to a set of the right size, even if the set may be augmented canonically. This is then contrasted with using strong semi-canonicity, as described in Lemma 4.5.1, to inform the black box in excluding starter sets.

Example 4.5.3. *Consider the point set of $\mathsf{H}(4, 4)$ ordered lexicographically. Then the permutation representation of the collineation group acting on the point set is* *given by the group generated by the elements in Appendix B. Up to equivalence, there is a unique maximal partial ovoid of* $\mathsf{H}(4,4)$ *of size* 21 *with lexicographically least ordered index set given by*

$$\{1, 2, 3, 5, 12, 13, 14, 23, 24, 33, 34, 44, 46, 86, 90, 139, 140, 160, 161, 162, 163\}.$$

The partial ovoid indexed by $\{1, 2, 9, 13, 14\}$ *can be extended to a larger partial ovoid* $\{1, 2, 9, 13, 14, 23\}$*. However it cannot be extended to a partial ovoid of size* 21*, hence this starter set can be pruned by a simple look-ahead check. The partial ovoid* $\{1, 2, 3, 34\}$*, however, is extendable to a partial ovoid of size* 21*:*

$$\{1, 2, 3, 34, 42, 60, 73, 74, 79, 80, 81, 82, 83, 84, 86, 124, 139, 149, 154, 158, 162\}.$$

Thus it cannot be pruned by simply looking ahead. However the points excluded by $\{1, 2, 3, 34\}$ *according to Lemma 4.5.1 are*

$$\begin{aligned}\{&4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\\ &24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 36, 37, 39, 42, 43, 44, 45, 46\\ &47, 49, 50, 52, 53, 57, 58, 64, 65, 67, 68, 74, 75, 77, 79, 80, 81, 82\\ &83, 84, 86, 87, 88, 89, 90, 91, 92, 93, 94, 96, 98, 99, 100, 101, 103\\ &104, 106, 107, 108, 111, 112, 113, 114, 115, 128, 129, 130, 131, 132\\ &133, 134, 135, 136, 137, 138, 142, 145, 154, 157, 158, 162, 164\}.\end{aligned}$$

After excluding these points it is no longer possible to extend $\{1, 2, 3, 34\}$ *to a partial ovoid of size* 21*. Thus this starter set may in fact also be pruned.* □

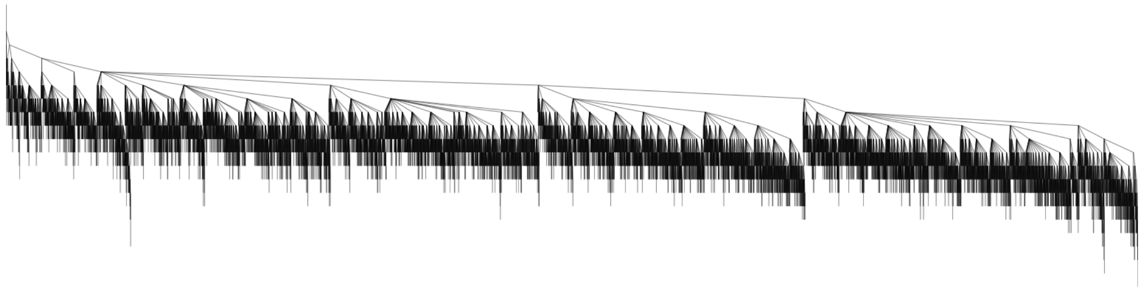
Checking canonicity is far more expensive than checking strong semi-canonicity or semi-canonicity. Thus a drastic speed gain is obtained by excluding augmentations which fail strong semi-canonicity before checking for canonicity. Moreover strong semi-canonicity is an inherited property, since any elements excluded for a given object will also be excluded for any of its augmentations. Thus a set of "bad" elements which violate Lemma 4.5.1 can also be maintained and easily updated as we move down the search tree. These "bad" elements may further be fed to the black box to be used in checking if there are viable extensions of a given starter set. These observations allow for a more effective orderly algorithm to be constructed. We give pseudocode for such an algorithm in Algorithm 3.

Algorithm 3: Orderly algorithm with black box pruning using strong semi-canonicity

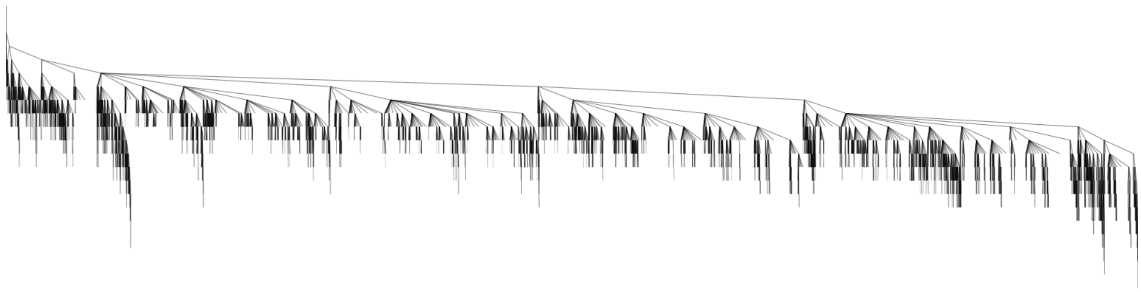
```
Input: Starter set s, permutation group G, number of points n, length k
Output: All canonical extensions of s of length k
Algorithm Alg(s, G, n, k)
    B ← {}
    for i ∈ {0, ..., |s|} do
        x ← s_{1...i} (the first i entries of s)
        G_x ← setwise stabiliser of x in G
        O ← orbits of G_x on {1, ..., n}
        for i ∈ {1, ..., |O|} do
            m ← Min(O_i)
            if m ∈ B then
                B ← B ∪ O_i
            else if m < Max(s) and m ∉ x then
                B ← B ∪ O_i
    return SubAlg(s, G, n, k, B, {})
Procedure SubAlg(s, G, n, k, B, S)
    if OtherCheck(s) and IsMinimal(s, G) then
        if |s| = k then
            S ← S ∪ {s}
        else
            B′ ← B
            C ← {}
            G_s ← setwise stabiliser of s in G
            O ← orbits of G_s on {1, ..., n}
            for i ∈ {1, ..., |O|} do
                m ← Min(O_i)
                if m ∈ B then
                    B′ ← B′ ∪ O_i
                else if |s| > 0 and m < Max(s) and m ∉ s then
                    B′ ← B′ ∪ O_i
                else
                    C ← C ∪ {s ∪ {m}}
            if BlackBox(s, B′) then
                for c ∈ C do
                    S ← SubAlg(c, G, n, k, B′, S)
    return S
```

Clearly the main part of Algorithm 3 is actually contained in the procedure *SubAlg*. The algorithm itself is simply a wrapper which calls this procedure and passes it an initially empty set of solutions, as well as computing the excluded points, B, corresponding to the starter set, s.

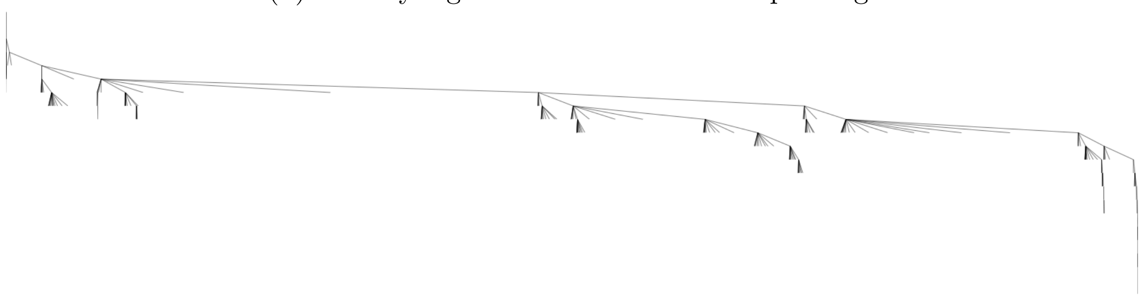
OtherCheck in Algorithm 3 allows some other property of s to be checked before continuing with the algorithm. *IsMinimal* is expensive and can sometimes be avoided by calling some less expensive *OtherCheck*. Ideally *OtherCheck* will check some inherited property, so that only the element in the augmentation step needs to be checked for violating the given property. For example, when enumerating partial ovoids, it is easy to check if an augmentation preserves non-collinearity with the set being augmented. Thus an appropriate *OtherCheck* would check that any augmentation is also a partial ovoid.



(a) Orderly algorithm.



(b) Orderly algorithm with look-ahead pruning.



(c) Orderly algorithm with look-ahead pruning informed by strong semi-canonicity.

Figure 4.5: Comparison of search trees for enumeration of maximum ovoids of $\mathsf{H}(4,4)$.

Figure 4.5 illustrates the impact of look-ahead pruning and strong semi-canonicity on the performance of an orderly algorithm. Each subfigure depicts the search tree of Algorithm 3 with different *BlackBox* implementations in generating the maximum partial ovoids of size 21 in $H(4,4)$. In each case only partial ovoids are considered, hence *OtherCheck* is set to check the validity of an augmentation as a partial ovoid. Figure 4.5(a) displays the search tree of the orderly algorithm with no additional pruning, that is, *BlackBox* simply returns true. Figure 4.5(b) displays the search tree where nodes are pruned by a black box if the corresponding starter set cannot be extended to a partial ovoid of size 21, that is, *BlackBox* is only influenced by s and not B'. Figure 4.5(c) displays the search tree where a node is pruned if it is not extendable to a strongly semi-canonical partial ovoid of size 21, that is, *BlackBox* uses both s and B'.

Table 4.2 shows the number of nodes at a given depth for each of the search trees in Figure 4.5, as well as the number of nodes which would be considered in naive backtracking, for comparison. Observe that using look-ahead pruning informed by strong semi-canonicity results in a drastic reduction in the number of nodes which must be visited in the course of the orderly generation. Note that the "orderly algorithm" column of Table 4.2 also gives the number of non-equivalent partial ovoids of $\mathsf{H}(4,4)$ with size equal to the depth of the search tree.

Depth	Naive Backtrack	Figure 4.5 a)	Figure 4.5 b)	Figure 4.5 c)
1	165	1	1	1
2	10560	1	1	1
3	352000	3	3	3
4	6906240	6	6	6
5	85295232	20	20	7
6	691616640	66	65	20
7	3788484480	237	233	37
8	14307891840	711	579	38
9	37803427200	1695	949	53
10	70532632896	3017	1177	17
11	93285084480	3923	1092	31
12	87366350720	3693	842	15
13	57854649600	2541	579	10
14	27340727040	1289	329	4
15	9552787200	523	159	4
16	2602955520	182	72	1
17	564537600	61	31	2
18	93434880	20	13	1
19	11151360	7	5	2
20	887040	2	2	1
21	42240	1	1	1
Total	405889224933	17999	6159	255

Table 4.2: Comparison of enumeration methods for maximum partial ovoids of $\mathsf{H}(4,4)$, as depicted in Figure 4.5.

There is a tension between the time saved by pruning a branch of the search tree using *BlackBox*, and the time it takes for *BlackBox* to determine if the branch should be pruned. Often it is practical to place a time limit upon *BlackBox* such that if it times out or finds a viable extension, the search tree is explored further, and a node is pruned only when *BlackBox* determines definitively that it does not extend to a canonical solution. Moreover, pruning with *BlackBox* will be more effective further down the search tree, since it can be fed a larger starter set and more excluded vertices which would violate strong semi-canonicity, drastically reducing the search space for that node. A suitable time limit that strikes a balance between pruning as many nodes as possible, while doing so as fast as possible, is essential to reduce the overall run time of Algorithm 3. Effort spent tuning the time limit parameter is often repaid by a shortening of the run time of Algorithm 3, often drastically. Typically, it is better to allow more time in *BlackBox* the larger the starter set s and excluded vertex set B' are.

Not only is a single test for canonicity expensive, but the larger the object being tested, the more expensive the test becomes, since more backtrack steps must occur. Moreover, the larger the object under consideration, the size of the setwise stabiliser tends to decrease, and correspondingly the number of its orbits increase. At a certain point, the stabiliser will be almost trivial, and thus a lot of unnecessary orbit and stabiliser computations are conducted for what essentially approaches naive backtracking. Thus in practice, it is often better to use Algorithm 3 to a certain depth, and then simply find all extensions naively. These extensions may then be canonised and duplicates removed. Choosing exactly at what depth to begin naive enumeration can vary performance greatly, and thus it is also worth taking time to tune this parameter.

Algorithm 3 is intended for enumeration, however it may easily be adapted into a search algorithm. Clearly it may be terminated upon finding a solution. But *BlackBox* may also be used to terminate the algorithm. In this instance, it is assumed that *BlackBox* is operating under a time limit, and the algorithm continues when *BlackBox* times out, backtracks when *BlackBox* returns infeasible, and terminates when *BlackBox* finds a viable extension, returning the solution that *BlackBox* finds. Used in this way, *BlackBox* is doing most of the work in the search. Algorithm 3 is essentially being used to determine starter sets for *BlackBox*, and also using symmetry information to reduce the amount of work it needs to do. When too much time elapses, the starter set is abandoned and its children are tried instead. Note that this search is being conducted under the assumptions of orderly generation, namely that every node in the search tree will be visited exactly once, or in the case of a search, at most once. Using the strong semi-canonicity information, *BlackBox* will then finish much faster. However, it is also more likely to exclude a viable solution, simply because it can't be formed canonically. In generation, extending the "right" parent is important, but this is not so when searching for existence. Thus the order of traversal of the search tree may have an impact on the speed of the search.

Observe that the search tree is very skewed. This is because lexicographically greater objects are more likely to be non-extendable, since they will exclude more vertices due to the strong semi-canonicity condition. This means that they are more likely to be pruned, and also that it is likely *BlackBox* will determine this quickly. However, it also means that they are less likely to yield a viable solution. Roughly speaking, traversing from the lexicographically greater side of the search tree is better for generation as it keeps a smaller stack in memory, while traversal from the lexicographically less side is better for search as it is more likely to yield a solution. Compare this to Figure 4.5 for an indication of how the direction of traversal of the search tree may affect the performance of Algorithm 3 in generation and in search.

In the following sections, we apply some of the computational methods described in this chapter to obtain various new results in finite geometry.

4.6 Spectral Results for m-Ovoids of $\mathsf{Q}(4,q)$

Taking the intersection of certain hyperplanes of $\mathsf{PG}(4,q)$ with $\mathsf{Q}(4,q)$ yields an elliptic quadric, which is an ovoid of $\mathsf{Q}(4,q)$. In addition, it is known that $\frac{q+1}{2}$-ovoids [12] and $\frac{q-1}{2}$-ovoids [56, 57] of $\mathsf{Q}(4,q)$ exist, for q odd. For small values of q, Bamberg, Law, and Penttila constructed m-ovoids by computer [12].

New spectral results on m-ovoids of $\mathsf{Q}(4,q)$ are presented in Table 4.3, updating the table given in [12]. The values in bold, red font indicate new results, where previously the existence of an m-ovoid was unknown. We only consider values of m for $1 \leqslant m \leqslant \frac{q+1}{2}$, since the complement of an m-ovoid is a $(q+1-m)$-ovoid.

q	m-ovoid known to exist	Unkown existence
3	1, 2	-
5	1, 2, 3	-
7	1, **2**, 3, 4	-
9	1, **2**, 3, 4, 5	-
11	1, **3**, **4**, 5, 6	2
13	1, **4**, **5**, 6, 7	2, 3
17	1, **7**, 8, 9	$2, \ldots, 6$
19	1, 9, 10	$2, \ldots, 8$

Table 4.3: Spectral results for m-ovoids of $\mathsf{Q}(4,q)$.

In addition to the results of Table 4.3, if there exists a 2-ovoid of $\mathsf{Q}(4,11)$, then any element of its stabiliser has order at most 4, and if a 3-ovoid of $\mathsf{Q}(4,13)$ exists, then any element of its stabiliser has order at most 7.

Remark 4.6.1. *A large amount of time was spent attempting to determine the existence of a 2-ovoid of $\mathsf{Q}(4,11)$, but to date it has eluded computation and remains open.*

4.7 Partial Ovoids of $\mathsf{Q}^-(5,q)$

Recall that $\mathsf{Q}^-(5,q)$ and $\mathsf{H}(3,q^2)$ are dual, and hence partial ovoids of $\mathsf{Q}^-(5,q)$ correspond to partial spreads of $\mathsf{H}(3,q^2)$. The best known upper bounds on the size of a partial ovoid of $\mathsf{Q}^-(5,q)$, for $q = p^t$ are

$$\frac{q^3+q+2}{2},$$

due to De Beule, Klein, Metsch and Storme [41], and

$$\left(\frac{2p^3+p}{3}\right)^t + 1,$$

and

$$19^t \qquad \text{when } p = 3,$$

due to Ihringer, Sin and Xiang [70]. For small values of q, the best known lower bounds on the size of an ovoid of $\mathsf{Q}^-(5,q)$ are mostly due to Cimráková and Fack [32].

In Table 4.4 we update the best known upper and lower bounds of partial ovoids of $\mathsf{Q}^-(5,q)$, for small q. New results are presented in bold, red font and the previous best bounds are listed in parentheses.

q	Lower Bound	Upper bound
3	16 [54]	16 [54]
4	25 [54]	25 [32]
5	48 [32]	**48** (66 [41])
7	98 [32]	176 [41]
8	126 [32]	217 [19, 70]
9	**148** (146 [32])	361 [70]
11	**258** (216 [32])	672 [41]
13	**330** (273 [32])	1106 [41]

Table 4.4: Bounds on the size of maximum partial ovoids of $\mathsf{Q}^-(5,q)$.

For $\mathsf{Q}^-(5,5)$, we also exhaustively enumerate all maximal partial ovoids, and discover there is one up to equivalence. This is documented in Table 4.5.

q	Size	Number	Stabiliser size
3 [54]	16	1	11520
4 [32]	25	3	300, 160, 80
5	**48**	**1**	2304

Table 4.5: Maximum partial ovoids of $\mathsf{Q}^-(5,q)$.

4.8 Partial Ovoids of $\mathsf{DH}(4, q^2)$

It is known via exhaustive computer search that there is no ovoid of $\mathsf{DH}(4, 4)$ (cf. [99, Theorem 3.1.4]). For $q > 2$, the existence of ovoids of $\mathsf{DH}(4, q^2)$ remains open.

Considering the conjugacy classes of the automorphism group of $\mathsf{DH}(4, 9)$, we have eliminated all cyclic groups except one by computer, yielding the following result.

Theorem 4.8.1. *An ovoid of $\mathsf{DH}(4, 9)$ has either a trivial stabiliser, or it admits up to conjugacy only the following element of order 3 with centraliser order 17496,*

$$\begin{pmatrix} z & 1 & 2 & 2 & z \\ z^2 & z^6 & z^7 & z^7 & 2 \\ 1 & z & z^7 & 2 & z^6 \\ 1 & z & 1 & z^7 & 2 \\ z^7 & 1 & z^6 & 1 & z^7 \end{pmatrix},$$

where z is a primitive root of $\mathbb{F}_9$.

Most of the conjugacy classes could be eliminated relatively quickly, with the exception of two elements of order 3. The element of order 3 with centraliser order 139968 took over half a year to complete! The element of order 3 with centraliser order 17496 is ongoing, and hence must be included in Theorem 4.8.1.

We were able to construct a partial ovoid of size 172, placing a lower bound on the size of a maximum partial ovoid. We therefore have the following result.

Theorem 4.8.2. *Let $\mathcal{O}$ be a partial ovoid of $\mathsf{DH}(4, 9)$ of largest size. Then $172 \leqslant |\mathcal{O}| \leqslant 244$.*

No theoretical proof of the non-existence of ovoids of $\mathsf{DH}(4, 4)$ is known. Perhaps an algebraic proof can be found, for which x-tight sets may prove useful. In Table 4.6 we list the number of unique, irreducible x-tight sets of $\mathsf{DH}(4, 4)$ for $x \in \{1, \ldots, 8\}$. By irreducible, we mean that they do not contain any tight sets of smaller size. Parentheses indicate the multiplicity of x-tight sets with given stabiliser order.

x	Number	Stabiliser Sizes
1	1	165888
2	0	-
3	1	155520
4	0	-
5	0	-
6	1	1944
7	3	72, 144 (2)
8	40	4 (6), 6 (4), 8 (3), 12 (8), 24 (3), 32, 36 (4), 48 (2), 72 (3), 96 (3), 144, 240, 288

Table 4.6: Number of irreducible x-tight sets in $\mathsf{DH}(4,4)$.

4.9 Software Developed During this Research

Two GAP [59] packages were developed in the course of this thesis. We give a brief overview here, and refer to their respective manuals for more information.

The first package, *Gurobify* [80], is an interface to the *Gurobi Optimizer* software [65]. It facilitates the solving of MIP and LP problems directly from within GAP itself. For example, to solve the MIP defined by

$$Mx = \mathbb{1},$$

where M is an $n \times m$ matrix already defined in GAP, we would run the following:

```
model := GurobiNewModel( ListWithIdenticalEntries( m, "binary" ) );;
comp := ListWithIdenticalEntries( n, "=");;
val := ListWithIdenticalEntries( n, 1 );;
GurobiAddMultipleConstraints( model, M, comp, val );;
GurobiOptimiseModel( model );;
GurobiSolution( model );
```

Moreover, the model can be modified in memory. This is particularly helpful for large models, where one only needs to add or remove a single constraint (such as a starter set if using *Gurobi* as the *black-box* in Algorithm 3). Various attributes and parameters can also be set and modified (such as a time limit).

The second package, *AssociationSchemes* [9], was developed along with John Bamberg and Akihide Hanaki, and allows one to work with homogeneous coherent configurations in GAP. These are defined by their relation matrix

$$M = \sum_{i=0}^{d} iA_i,$$

which is a weighted sum of the adjacency matrices. Various attributes of the association schemes can be computed, such as the matrix of eigenvalues and Krein parameters, and properties such as whether the scheme is metric or cometric may be checked. Minimal idempotents may also be computed in order to work with Delsarte designs. Various constructor methods are given, such as constructing a Schurian scheme from a generously transitive group. When possible, the group is used to exploit the power of GAP and to speed up computation. Hanaki's database [66] of small order homogeneous coherent configurations is also included.

Chapter 5

On m-Ovoids of Regular Near Polygons

An *m-ovoid* of a near $2d$-gon is a set of points $\mathcal{O}$ such that every line is incident with exactly m points of $\mathcal{O}$. The *trivial* m-ovoids are the empty set ($m = 0$) and the full set of points ($m = s + 1$; the number of points on a line). For dual polar spaces (that are not generalised quadrangles), the existence of 1-ovoids is mostly resolved, however, in rank 3, it is still not known whether $\mathsf{DQ}^-(7, q)$ or $\mathsf{DH}(6, q^2)$ can contain 1-ovoids. It follows from [99, 3.4.1] that there are no 1-ovoids of $\mathsf{DW}(5, q)$ for q even, and the q odd case was settled by Thomas [118, Theorem 3.2] (see [36] and [47, Appendix] for alternative proofs). De Bruyn and Vanhove reproved this result [48, Corollary 3.14] and extended it to other regular near hexagons by showing that a finite generalised hexagon of order (s, s^3) with $s \geqslant 2$ has no 1-ovoids [48, Corollary 3.19].

Another interesting case arises in the study of m-ovoids when m is exactly half of the number of points on a line. Such an m-ovoid is called a *hemisystem*. In 1965, Segre [108] showed that the only nontrivial m-ovoids of $\mathsf{DH}(3, q^2)$, for q odd, are hemisystems. Cameron, Goethals and Seidel [29] extended Segre's result to all generalised quadrangles of order (q, q^2), q odd. This was then extended further to regular near $2d$-gons of order (s, t) by Vanhove [124], also providing a generalisation of the so-called *Higman bound*: if $s > 1$ then the intersection number c_i for all $i \in \{1, \dots, d\}$ obeys the following inequality,

$$c_i \leqslant \frac{s^{2i} - 1}{s^2 - 1}.$$

Furthermore, if the bound is sharp for some c_i with $i \in \{2, \dots, d\}$ then any nontrivial m-ovoid is a hemisystem [124, Theorem 3].

The main result of this chapter is to prove that nontrivial m-ovoids of the dual polar spaces $\mathsf{DQ}(2d, q)$, $\mathsf{DW}(2d - 1, q)$ and $\mathsf{DH}(2d - 1, q^2)$ ($d \geqslant 3$) are hemisystems in Theorem 5.2.1. We also provide a more general result that holds for regular near polygons in Theorem 5.2.2.

5.1 A Problem of Vanhove

In his PhD thesis, Vanhove showed that hemisystems of $\mathsf{DH}(2d-1,q^2)$ give rise to new distance-regular graphs with classical parameters [125, Theorem 6.7.8] (also [124, Theorem 4]).

Theorem 5.1.1. *If $\mathcal{O}$ is a hemisystem of $\mathsf{DH}(2d-1,q^2)$ with q odd and $d \geqslant 2$, then the induced subgraph of $\mathcal{O}$ on the collinearity graph of $\mathsf{DH}(2d-1,q^2)$ is a distance-regular graph with classical parameters*

$$(d,b,\alpha,\beta) = \left(d, -q, -(q+1), -\left(\frac{(-q)^d+1}{2}\right)\right).$$

He also posed the natural question "Do any $(q+1)/2$-ovoids in the dual polar space on $\mathsf{H}(2d-1,q^2)$, q odd, exist if $d \geqslant 3$?" [125, Appendix B, Problem 7].

Hemisystems of the dual hermitian spaces $\mathsf{DH}(2d-1,q^2)$ produce hemisystems of the dual symplectic spaces $\mathsf{DW}(2d-1,q)$. Non-existence of hemisystems of $\mathsf{DW}(2d-1,q)$ would therefore imply non-existence of hemisystems in $\mathsf{DH}(2d-1,q^2)$. Hence the existence of hemisystems in the dual symplectic spaces is also of interest.

Lemma 5.1.2 (cf. [45, 84])**.** *A hemisystem of $\mathsf{DH}(2d-1,q^2)$ induces a hemisystem of $\mathsf{DW}(2d-1,q)$.*

Proof. $\mathsf{DW}(2d-1,q)$ embeds in $\mathsf{DH}(2d-1,q^2)$, and lines of $\mathsf{DW}(2d-1,q)$ have the same points in both. Thus the intersection of a hemisystem of $\mathsf{DH}(2d-1,q^2)$ with the points of $\mathsf{DW}(2d-1,q)$ is a hemisystem of $\mathsf{DW}(2d-1,q)$. □

Moreover, $\mathsf{DQ}(2d,q)$ and $\mathsf{DW}(2d-1,q)$ have the same combinatorial properties, since the symplectic and parabolic polar spaces are both of type $e=1$ and order $(1,1)$, and as a result their collinearity graphs have the same intersection numbers. Thus the existence of hemisystems of $\mathsf{DQ}(2d,q)$ is also worth considering.

A simple quotienting argument shows that if a hemisystem exists in the dual of a polar space, then hemisystems exist in the duals of all polar spaces of the same type and smaller rank.

Lemma 5.1.3. *Let Π be a polar space of rank $d \geqslant 3$. An m-ovoid of $\mathsf{D}\Pi$ induces an m-ovoid in the dual of the polar space of the same type and rank $d-1$.*

Proof. Let P be a point in Π, and Π/P be the quotient geometry about the point P. Now Π/P is isomorphic to the polar space of the same type as Π but with rank $d-1$. Note that $d \geqslant 3$ or the next-to-maximals of Π/P are empty. An m-ovoid of $\mathsf{D}\Pi$ is a set of maximals of Π such that exactly m of them are contained in every next-to-maximal of Π. Now, every next-to-maximal on P will therefore also meet m of the maximals of the m-ovoid which also contain P and hence the maximals meeting P form an m-ovoid of $\mathsf{D}(\Pi/P)$. □

Lemma 5.1.2 and Lemma 5.1.3 are displayed in Figure 5.1 for $\mathsf{DH}(2d-1,q^2)$, $\mathsf{DW}(2d-1,q)$, and $\mathsf{DQ}(2d,q)$ to indicate the chain of implication for the existence of hemisystems. If for a given q, hemisystems exist in any of the geometries, then they must exist in the geometries below and to the right in the diagram. Conversely, if hemisystems do not exist in geometries at any point in the diagram, then nor do they exist in the geometries above and to the left in the diagram, for a given value of q.

$$\begin{tikzcd}
{} \arrow[d, dashed] & {} \arrow[d, dashed] & {} \arrow[d, dashed] \\
\mathsf{DH}(2d-1,q^2) \arrow[r] \arrow[d] & \mathsf{DW}(2d-1,q) \arrow[d] & \mathsf{DQ}(2d,q) \arrow[d] \\
\mathsf{DH}(2d-3,q^2) \arrow[r] \arrow[d, dashed] & \mathsf{DW}(2d-3,q) \arrow[d, dashed] & \mathsf{DQ}(2d-2,q) \arrow[d, dashed] \\
\mathsf{DH}(7,q^2) \arrow[r, "embedding"] \arrow[d, "quotienting"] & \mathsf{DW}(7,q) \arrow[d] & \mathsf{DQ}(8,q) \arrow[d] \\
\mathsf{DH}(5,q^2) \arrow[r] \arrow[d] & \mathsf{DW}(5,q) \arrow[d] & \mathsf{DQ}(6,q) \arrow[d] \\
\mathsf{DH}(3,q^2) \arrow[r] & \mathsf{DW}(3,q) & DQ(4,q)
\end{tikzcd}$$

Figure 5.1: Induced hemisystems in $\mathsf{DH}(2d-1,q^2)$, $\mathsf{DW}(2d-1,q)$, and $\mathsf{DQ}(2d,q)$.

Hemisystems of $\mathsf{H}(3,q^2)$, $\mathsf{W}(3,q)$, and $\mathsf{Q}(4,q)$ are known to exist for all odd q [12, 37, 39]. Hence the bottom rung of Figure 5.1 holds. For $d>2$ the question is open, and provides motivation for this chapter.

5.2 Regular Near Polygons and m-Ovoids

It is known that m-ovoids of $\mathsf{DW}(3,q)$ exist for $m \neq \frac{q+1}{2}$ [12]. Refer also to Section 4.6 where we have considered this problem computationally. Similarly, m-ovoids of $\mathsf{DQ}(4,q)$ exist for $m \neq \frac{q+1}{2}$ [56, 57] . We prove now that this is not true for $d>2$. If m-ovoids of $\mathsf{DQ}(2d,q)$, $\mathsf{DW}(2d-1,q)$ and $\mathsf{DH}(2d-1,q^2)$ exist for $d \geqslant 3$, then they are hemisystems. In particular, this means that no m-ovoids exist when q is even.

Theorem 5.2.1. *The only nontrivial m-ovoids that exist in $\mathsf{DQ}(2d,q)$, $\mathsf{DW}(2d-1,q)$ and $\mathsf{DH}(2d-1,q^2)$, for $d \geqslant 3$, are hemisystems (i.e., $m=(q+1)/2$).*

Theorem 5.2.1 follows from a more general, but perhaps more technical result, on m-ovoids of regular near polygons. Our main theorem is:

Theorem 5.2.2. *Let $\mathcal{S}$ be a regular near $2d$-gon of order $(s, t_2, t_3, \ldots, t_{d-1}, t)$ satisfying*

$$t_i + 1 = \frac{(s^i + (-1)^i)(t_{i-1} + 1 + (-1)^i s^{i-2})}{s^{i-2} + (-1)^i}$$

for some $3 \leqslant i \leqslant d$. If a nontrivial m-ovoid of $\mathcal{S}$ exists, then it is a hemisystem.

De Bruyn and Vanhove [48, Theorem 3.2] prove that a regular near $2d$-gon (with $s, d \geqslant 2$) satisfies

$$\frac{(s^i - 1)(t_{i-1} + 1 - s^{i-2})}{s^{i-2} - 1} \leqslant t_i + 1 \leqslant \frac{(s^i + 1)(t_{i-1} + 1 + s^{i-2})}{s^{i-2} + 1} \tag{5.1}$$

for all $i \in \{3, \ldots, d\}$, and that a finite regular near $2d$-gon with $s \geqslant 2$ and $d \geqslant 3$ which attains the lower bound for $i = 3$ is isomorphic to $\mathsf{DQ}(2d, s)$, $\mathsf{DW}(2d-1, s)$ or $\mathsf{DH}(2d-1, s^2)$, where s is a prime power [48, Theorem 3.5]. Note that the hypothesis of Theorem 5.2.2 is valid when the the upper bound is met for i even, or when the lower bound is met for i odd, in the De Bruyn–Vanhove bounds (5.1). Theorem 5.2.1 follows directly from [48, Theorem 3.5] and Theorem 5.2.2.

Recall that in a finite regular near polygon, $\mathcal{S}$, with parameters $(s, t_2, \ldots, t_{d-1}, t)$, there are $t + 1$ lines on each point, $s + 1$ points on each line, and $t_i + 1$ lines on y containing a point at distance $i - 1$ from x, whenever two points x and y are at distance i. Moreover, recall that given any line ℓ and point x, there is a unique nearest point to x on ℓ. Furthermore, the collinearity graph, Γ, of $\mathcal{S}$ is distance-regular, such that for all $i \in \{0, \ldots, d\}$ we have the following correspondence between the intersection numbers of Γ and the parameters of $\mathcal{S}$,

$$a_i = (s-1)(t_i + 1), \quad b_i = s(t - t_i), \quad c_i = t_i + 1.$$

By definition, $t_0 = -1$ and $t_1 = 0$.

Given x and y at distance l, there are $p^l_{i,j}$ vertices that are at distance i from x and distance j from y. Furthermore, $p^{i-1}_{1,i} = b_{i-1}$, $p^i_{1,i} = a_i$, $p^{i+1}_{1,i} = c_{i+1}$ and $p^l_{i,j} = p^l_{j,i}$. We may calculate $p^l_{i+1,j}$ recursively using the formula in Theorem 2.3.13. We also define the i-distance *valencies* of the graph, $k_i := p^0_{i,i}$ for $i \in \{0, 1, \ldots, d\}$ (and so $k_1 = s(t + 1)$). In a regular near $2d$-gon with parameters $(s, t_2, t_3, \ldots, t_{d-1}, d)$, we have the following relations.

Lemma 5.2.3 ([22, §4.1 (7); (9); (1c)])**.**

$$\begin{aligned} k_i &= k_{i-1}\frac{b_{i-1}}{c_i} = s k_{i-1}\frac{t - t_{i-1}}{t_i + 1} \quad (1 \leqslant i \leqslant d), \\ p^l_{i,j} k_l &= p^i_{l,j} k_i \quad (0 \leqslant i, j, \ell, d), \\ p^{i+1}_{1,i} &= c_{i+1} \quad (1 \leqslant i \leqslant d). \end{aligned}$$

Lemma 5.2.3 gives the following corollary.

Corollary 5.2.4. *Let $1 \leqslant i \leqslant d$. Then*

$$p^1_{i,i-1} = \frac{k_i c_i}{k_1} = \frac{k_i(t_i+1)}{s(t+1)}.$$

Recall that a distance-regular graph yields a metric association scheme, and so the set of adjacency matrices $\{A_0, A_1, \ldots, A_d\}$ forms a basis for the Bose-Mesner algebra for Γ which also has a unique basis of minimal idempotents $\{E_0, E_1, \ldots, E_d\}$. Moreover, recall that the *dual degree set* of a vector v is the set of indices of the minimal idempotents E_i such that $vE_i \neq 0$ and $i \neq 0$, and two vectors are called *design-orthogonal* when their dual degree sets are disjoint.

The following lemma follows directly from the definition of an m-ovoid and the fact that there are $s+1$ points on every line of a finite regular near $2d$-gon $\mathcal{S}$ with parameters $(s, t_2, \ldots, t_{d-1}, t)$.

Lemma 5.2.5. *The complement of an m-ovoid of $\mathcal{S}$ is a $(s+1-m)$-ovoid.*

Lemma 5.2.6 ([124, Lemma 5])**.** *If $\mathcal{O}$ is an m-ovoid of $\mathcal{S}$, then for every $i \in \{0, 1, \ldots, d\}$ and $x \in \mathcal{O}$,*

$$|\Gamma_i(x) \cap \mathcal{O}| = k_i\left(\frac{m}{s+1} + \left(-\frac{1}{s}\right)^i\left(1 - \frac{m}{s+1}\right)\right).$$

By Lemmas 5.2.5 and 5.2.6, we have the following:

Corollary 5.2.7. *If $\mathcal{O}$ is an m-ovoid of $\mathcal{S}$, then for every $i \in \{0, 1, \ldots, d\}$ and $x \notin \mathcal{O}$,*

$$|\Gamma_i(x) \cap \mathcal{O}| = k_i \frac{m}{s+1}\left(1 - \left(\frac{-1}{s}\right)^i\right).$$

We now prove Theorem 5.2.2. Recall that we are assuming that

$$c_i = t_i + 1 = \frac{(s^i + (-1)^i)(c_{i-1} + (-1)^i s^{i-2})}{s^{i-2} + (-1)^i}$$

for some $3 \leqslant i \leqslant d$.

Proof. Let $\mathcal{O}$ be a nontrivial m-ovoid of $\mathcal{S}$. A simple double counting argument shows that $|\mathcal{O}|$ is equal to $m|\mathcal{L}|/(t+1)$ where $\mathcal{L}$ is the set of lines of $\mathcal{S}$. If we also count flags (i.e., point-line incident pairs), then $|\mathcal{P}|(t+1) = |\mathcal{L}|(s+1)$ where $\mathcal{P}$ is the set of points of $\mathcal{S}$, and hence

$$|\mathcal{O}| = \frac{mn}{s+1},$$

where $n = |\mathcal{P}|$.

Recall that $3 \leqslant i \leqslant d$. Now we fix an element $x \notin \mathcal{O}$ and count pairs (y,z) of elements of $\mathcal{O}$ such that $\mathrm{d}(x,y) = i$ and either $\mathrm{d}(y,z) = i-1$ and $\mathrm{d}(x,z) = 1$, or $\mathrm{d}(y,z) = 1$ and $\mathrm{d}(x,z) = i-1$.

Let x and y be two points at distance i. Define $v_{x,y}$ as in [48, Theorem 3.2(b)],

$$v_{x,y} := s(c_{i-1} + (-1)^i s^{i-2})(\chi_x + \chi_y) + \chi_{\Gamma_1(x)\cap\Gamma_{i-1}(y)} + \chi_{\Gamma_{i-1}(x)\cap\Gamma_1(y)}.$$

Note that

$$v_{x,y} \cdot \mathbb{1} = 2(s(c_{i-1} + (-1)^i s^{i-2}) + p^i_{1,i-1}) = 2(s(c_{i-1} + (-1)^i s^{i-2}) + c_i)$$

and furthermore that $v_{x,y}$ and $\chi_{\mathcal{O}}$ are design-orthogonal [48, Theorem 3.2] and hence by Lemma 2.4.5,

$$\mu := v_{x,y} \cdot \chi_{\mathcal{O}} = 2(s(c_{i-1} + (-1)^i s^{i-2}) + c_i)m/(s+1).$$

Let Γ be the collinearity graph of $\mathcal{S}$.

Counting first y and then z, the number of pairs is

$$\begin{aligned}
&\sum_{y\in\mathcal{O}\cap\Gamma_i(x)} (|\Gamma_1(x)\cap\Gamma_{i-1}(y)\cap\mathcal{O}| + |\Gamma_{i-1}(x)\cap\Gamma_1(y)\cap\mathcal{O}|)\\
=&\sum_{y\in\mathcal{O}\cap\Gamma_i(x)} (v_{x,y} - s(c_{i-1} + (-1)^i s^{i-2})(\chi_x + \chi_y))\cdot\chi_{\mathcal{O}}\\
=&|\mathcal{O}\cap\Gamma_i(x)|(\mu - s(c_{i-1} + (-1)^i s^{i-2}))\\
=&|\mathcal{O}\cap\Gamma_i(x)|\left(\frac{2\left(s(c_{i-1} + (-1)^i s^{i-2}) + c_i\right)m}{s+1} - s(c_{i-1} + (-1)^i s^{i-2})\right)\\
=&|\mathcal{O}\cap\Gamma_i(x)|\left(\frac{2c_i ms - (s+1-2m)\left(c_{i-1}s^2 + (-1)^i s^i\right)}{s(s+1)}\right).
\end{aligned}$$

Now, by Corollary 5.2.7 and Lemma 5.2.3,

$$|\mathcal{O}\cap\Gamma_i(x)|c_i = k_i c_i \frac{m}{s+1}\left(1 - \left(-\frac{1}{s}\right)^i\right) = sk_{i-1}(t - t_{i-1})\frac{m}{s+1}\left(1 - \left(-\frac{1}{s}\right)^i\right)$$

and hence the number of pairs (y,z) is

$$\frac{mk_{i-1}(t-t_{i-1})}{s+1}\left(1 - \left(-\frac{1}{s}\right)^i\right)\frac{2c_i ms - (s+1-2m)\left(c_{i-1}s^2 + (-1)^i s^i\right)}{c_i(s+1)}. \tag{5.2}$$

Now we consider the pairs the opposite way, namely counting z then y. The number of pairs (z,y) is equal to

$$\sum_{z\in\mathcal{O}\cap\Gamma_1(x)} |\Gamma_{i-1}(z)\cap\Gamma_i(x)\cap\mathcal{O}| + \sum_{z\in\mathcal{O}\cap\Gamma_{i-1}(x)} |\Gamma_1(z)\cap\Gamma_i(x)\cap\mathcal{O}|.$$

Suppose $\mathrm{d}(z,x) = i-1$. There are $t+1$ lines on z, and the set of points incident with these lines, other than the point z itself, form $\Gamma_1(z)$. There are $t_{i-1}+1$ lines on z incident with a unique point at distance $i-2$ from x, and the remaining points on these lines are at distance $i-1$ from x. Moreover, z is the unique nearest point to x with distance $i-1$ for the remaining $t-t_{i-1}$ lines, and hence any other point on these lines must have distance i from x. Since z is in $\mathcal{O}$, there are $m-1$ additional points of $\mathcal{O}$ on each such line. Therefore,

$$|\Gamma_1(z) \cap \Gamma_i(x) \cap \mathcal{O}| = (t - t_{i-1})(m-1).$$

Now suppose $\mathrm{d}(z,x) = 1$. We will compute $|\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}|$. Take $z \in \mathcal{O} \cap \Gamma_1(x)$ and consider a point $w \in \Gamma_{i-2}(z) \cap \Gamma_{i-1}(x)$. Note that any point y is collinear with some such point w, giving rise to the following equation:

$$\begin{aligned}\sum_{y \in \Gamma_{i-1}(z)\cap\Gamma_i(x)\cap\mathcal{O}} |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \Gamma_1(y)| &= \sum_{w \in \Gamma_{i-2}(z)\cap\Gamma_{i-1}(x)\cap\mathcal{O}} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O} \cap \Gamma_1(w)| \\ &\quad + \sum_{w \in \Gamma_{i-2}(z)\cap\Gamma_{i-1}(x)\cap\mathcal{O}^c} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O} \cap \Gamma_1(w)|\end{aligned} \tag{5.3}$$

where $\mathcal{O}^c$ is the complement of $\mathcal{O}$ within the set of points of $\mathcal{S}$.

Let ℓ be a line through w. There is a point on ℓ which is the unique closest point to x. If this point is w, then every other point must be at distance i from x. If this point is not w, then it must be distance $i-2$ from x, and every other point on ℓ is distance $i-1$ from x. There are $t_{i-1}+1$ lines on w with a unique point at distance $i-2$ from x. Hence there are $t - t_{i-1}$ lines ℓ' for which w is the unique nearest point to x and every other point on ℓ' is at distance i from x. Moreover, note that if a point y is at distance i from x, then it cannot be distance $i-2$ from z, since $\mathrm{d}(x,z) = 1$, and thus any point other than w on any line ℓ' is in $\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \Gamma_1(w)$. There are $m-1$ such points in $\mathcal{O}$ when $w \in \mathcal{O}$, otherwise there are m such points in $\mathcal{O}$.

There are t_{i-1} lines on any point y which have a unique point at distance $i-2$ from z, and hence also at distance $i-1$ from x. Recalling that $|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x)| = p^1_{i-1,i-2}$, our Equation (5.3) becomes:

$$\begin{aligned}|\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}|(t_{i-1}+1) &= |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|(t - t_{i-1})(m-1) \\ &\quad + |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}^c|(t - t_{i-1})m \\ &= |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|(t - t_{i-1})(m-1) \\ &\quad + (p^1_{i-1,i-2} - |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|)(t - t_{i-1})m \\ &= p^1_{i-1,i-2}(t - t_{i-1})m - |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|(t - t_{i-1}).\end{aligned}$$

Hence we obtain an iterative formula,

$$|\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| = p^1_{i-1,i-2}\frac{t - t_{i-1}}{t_{i-1}+1}m - \frac{t - t_{i-1}}{t_{i-1}+1}|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|,$$

which, with the help of Lemma 5.2.3 and Corollary 5.2.4, we can write as a recurrence relation

$$sf_i = m - f_{i-1}, \quad f_1 = 1$$

where $f_i := \frac{1}{p^1_{i,i-1}} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}|$ for all $i \geqslant 1$. (Note: $|\Gamma_0(z) \cap \Gamma_1(x) \cap \mathcal{O}| = 1$ and $p^1_{1,0} = 1$). Therefore, by the elementary theory of recurrence relations, we have

$$f_i = \frac{m - s\left(-\frac{1}{s}\right)^i (-m+s+1)}{s+1}$$

for all $i \geqslant 1$. Hence, by Corollary 5.2.4,

$$\begin{aligned}
|\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| &= p^1_{i,i-1}\left(\frac{m - s\left(-\frac{1}{s}\right)^i (-m+s+1)}{s+1}\right) \\
&= \frac{k_{i-1}(t-t_{i-1})}{t+1}\left(\frac{m - s\left(-\frac{1}{s}\right)^i (-m+s+1)}{s+1}\right) \\
&= \frac{k_{i-1}(t-t_{i-1})}{s^{i-1}(t+1)}\left(\frac{m}{s+1}\left(s^{i-1} + (-1)^{i-2}\right) + (-1)^{i-1}\right).
\end{aligned}$$

Now, making use of Corollary 5.2.7, we sum our two terms together:

$$\begin{aligned}
&\sum_{z \in \mathcal{O} \cap \Gamma_1(x)} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| + \sum_{z \in \mathcal{O} \cap \Gamma_{i-1}(x)} |\Gamma_1(z) \cap \Gamma_i(x) \cap \mathcal{O}| \\
=&|\mathcal{O} \cap \Gamma_1(x)||\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| + |\mathcal{O} \cap \Gamma_{i-1}(x)||\Gamma_1(z) \cap \Gamma_i(x) \cap \mathcal{O}| \\
=&s(t+1)\frac{m}{s+1}\left(1+\frac{1}{s}\right)\frac{k_{i-1}(t-t_{i-1})}{s^{i-1}(t+1)}\left(\frac{m}{s+1}\left(s^{i-1} + (-1)^{i-2}\right) + (-1)^{i-1}\right) \\
&\quad + k_{i-1}\frac{m}{s+1}\left(1 - \left(-\frac{1}{s}\right)^{i-1}\right)(t-t_{i-1})(m-1) \\
=&\frac{mk_{i-1}(t-t_{i-1})}{s+1}\left(m\left(1-\left(-\frac{1}{s}\right)^{i-1}\right) + \left(\frac{-1}{s}\right)^{i-1}(s+1) + (m-1)\left(1-\left(-\frac{1}{s}\right)^{i-1}\right)\right)
\end{aligned}$$

and therefore, the number of pairs (z, y) is

$$\frac{mk_{i-1}(t-t_{i-1})}{s+1}\left(2m - 1 + \left(\frac{-1}{s}\right)^{i-1}(s-2m+2)\right). \tag{5.4}$$

Equating the two counts, (5.2) and (5.4) yields

$$\begin{aligned}
&\left(1 - \left(-\frac{1}{s}\right)^i\right)\frac{2c_ims - (s+1-2m)\left(c_{i-1}s^2 + (-1)^i s^i\right)}{c_i(s+1)} \\
&= 2m - 1 + \left(\frac{-1}{s}\right)^{i-1}(s-2m+2).
\end{aligned}$$

Taking the difference of each side of the above equation and factoring gives

$$\frac{s^{-i}(s+1-2m)\left(c_i\left(s^i+(-1)^i(s+2)s\right)+\left((-1)^i-s^i\right)\left(c_{i-1}s^2+(-1)^is^i\right)\right)}{c_i(s+1)}=0$$

and hence

$$(s+1-2m)\left(c_i\left(s^i+(-1)^i(s+2)s\right)+\left((-1)^i-s^i\right)\left(c_{i-1}s^2+(-1)^is^i\right)\right)=0. \quad (5.5)$$

Now by assumption,

$$c_{i-1}s^2+(-1)^is^i=c_i\frac{s^i+(-1)^is^2}{s^i+(-1)^i}$$

and hence

$$\begin{aligned}
&c_i\left(s^i+(-1)^i(s+2)s\right)+\left((-1)^i-s^i\right)\left(c_{i-1}s^2+(-1)^is^i\right)\\
&=c_i\left(s^i+(-1)^i(s+2)s+\left((-1)^i-s^i\right)\frac{s^i+(-1)^is^2}{s^i+(-1)^i}\right)\\
&=c_i\frac{2(-1)^i(s+1)\left(s^i+(-1)^is\right)}{s^i+(-1)^i}.
\end{aligned}$$

Since $i>1$, we have $s^i+(-1)^is\neq 0$, and therefore, Equation (5.5) becomes $m=(s+1)/2$. □

5.3 Further Results and Computation

Theorem 5.2.1 leaves open the natural question of whether there exist hemisystems of $\mathsf{DQ}(6,q)$, $\mathsf{DW}(5,q)$ and $\mathsf{DH}(5,q^2)$. Firstly, De Bruyn and Vanhove announced in conference presentations that there are no hemisystems of $\mathsf{DW}(5,3)$, and that there is a unique example for $\mathsf{DQ}(6,3)$. For small values of (odd) q, we have found examples for $\mathsf{DQ}(6,q)$, and we have listed the known examples in Table 5.1. In particular, we could show by using the computer algebra system GAP [59], a package FinInG [4], and the mixed-integer programming software Gurobi [65] that there is a unique example up to equivalence in $\mathsf{DQ}(6,3)$. For $\mathsf{DQ}(6,5)$, there were numerous examples found admitting an element of order 5 or 9, but we were unable to enumerate them all.

Remark 5.3.1. *Our computational results showing the existence of hemisystems of* $\mathsf{DQ}(6,q)$ *for* $q\in\{3,5,7,11\}$ *answers the following question of Vanhove[125, Appendix B, Problem 6] in the affirmative for* $t=2$ *and rank* $d=3$*: "In a classical finite polar space of rank at least three, are there any non-trivial combinatorial designs of maximals with respect to* t*-spaces if* $t\geqslant 2$*?".*

For $\mathsf{DW}(5,q)$, it seems the situation is different, despite its combinatorial parameters being identical to those of $\mathsf{DQ}(6,q)$. By computer, we showed that there are no hemisystems of $\mathsf{DW}(5,q)$ for $q\in\{3,5\}$. Note that for $q\in\{3,7,11\}$ the hemisystems of $\mathsf{DQ}(6,q)$ are examples found as a subset of hemisystems of $\mathsf{Q}(6,q)$ according to the construction presented in Chapter 6. We shall discuss them further in Chapter 6. We make the following conjectures:

q	Stabiliser	Number up to equivalence
3	$2 \times A_5$	1
5	D_{60}	4
	D_{20}	16
7	$\geqslant \Omega_3(7)$	$\geqslant 1$
11	$\geqslant \Omega_3(11)$	$\geqslant 1$

Table 5.1: Some known examples of hemisystems of $\mathsf{DQ}(6, q)$, for small q.

Conjecture 5.3.2. *There are no hemisystems of* $\mathsf{DW}(5, q)$*, for all prime powers* q*.*

If true, this would also imply that there are no hemisystems of $\mathsf{DH}(5, q^2)$, for all prime powers q, answering a problem posed by Vanhove [125, Appendix B, Problem 7].

Conjecture 5.3.3. *For each odd prime power* q*, there exists a hemisystem of* $\mathsf{DQ}(6, q)$*.*

Remark 5.3.4. *Conjectures 5.3.2 and 5.3.3 were first posed in [11].*

Remark 5.3.5. *We shall return to Theorem 5.2.1 in Chapter 7, where we reprove it using techniques involving vanishing Krein parameters.*

Chapter 6

An Infinite Family of Hemisystems of the Parabolic Quadrics

In Chapter 5 we saw that an m-ovoid of $\mathsf{DQ}(2d, q)$ for q odd is a hemisystem. We now consider hemisystems of $\mathsf{Q}(2d, q)$. Firstly, a note of caution: these are not the same objects! A hemisystem of $\mathsf{Q}(2d, q)$ is a set of maximals such that exactly half of all the maximals on every point are contained in the hemisystem. Note that a hemisystem of $\mathsf{DQ}(2d, q)$ is also a hemisystem of $\mathsf{Q}(2d, q)$, but that the reverse is not always true. Hemisystems have connections to other objects in geometry, graph theory, and coding theory. In particular, they often induce new objects such as partial quadrangles, strongly regular or distance regular graphs, and association schemes [27, 123].

Recently, Cossidente and Pavese found an infinite family of hemisystems of $\mathsf{Q}(6, q)$, q odd, admitting $\mathrm{PSL}_2(q^2)$ [38]. This was previously the only known family of hemisystems of the parabolic quadrics, for $d \geqslant 3$. For $d = 2$, hemisystem constructions have been found by Feng et al. [56] as well as by Cossidente et al. [37]. In this chapter we construct an infinite family of hemisystems of $\mathsf{Q}(2d, q)$ for $d \geqslant 2$, q odd, and hence the first known construction for $d \geqslant 4$. Moreover, we provide a generalisation of the AB-Lemma which allows for non-hemisystem m-covers.

6.1 Hemisystems and the AB-Lemma

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be point-line geometry with *points* $\mathcal{P}$ and *lines* $\mathcal{L}$. We say that $\mathcal{S}$ has order (s, t) if there are $s + 1$ points incident with every line, and $t + 1$ lines incident with every point. An m-cover of S is a subset $\mathcal{O}$ of $\mathcal{L}$, such that every point is incident with exactly m lines in $\mathcal{O}$. An m-cover $\mathcal{O}$ is said to "admit" a group B if B is isomorphic to a subgroup of the stabiliser of $\mathcal{O}$ in the automorphism group of $\mathcal{S}$. We say $\mathcal{O}$ is a *hemisystem* if $m = \frac{t+1}{2}$ (thus requiring that t is odd). Moreover, a hemisystem contains half of the line set, and the complement of a hemisystem is also a hemisystem.

Remark 6.1.1. *We note that $\frac{s+1}{2}$-ovoids and $\frac{t+1}{2}$-covers are both referred to as* hemisystems. *This is unfortunate, however it is common throughout the literature. We further note that a hemisystem in one type is a hemisystem of the dual point-line geometry of the other type. Hence, there is a sense in which the distinction is simply a matter of perspective. Usually the use of* hemisystem *is clear from context, though we may sometimes try to distinguish the two uses by referring to them as* hemisystems on points *or* hemisystems on lines*, respectively. Throughout this chapter we use the term* hemisystem *to refer to a* $\frac{t+1}{2}$*-cover, in contrast to the previous chapter, where the term was used for a* $\frac{s+1}{2}$*-ovoid.*

Hemisystems were first defined by Segre on Hermitian varieties, where he demonstrated the existence of a hemisystem in $\mathsf{H}(3,3^2)$, and raised the question whether they exist in $\mathsf{H}(2d-1,q^2)$ for $d \geqslant 2, q > 3$ [108]. For a long time no new examples were found, and it was thought that Segre's example might be the only example, with Thas even conjecturing that there were no hemisystems of $\mathsf{H}(3,q^2)$, for $q > 3$ [116]. This conjecture was disproved, however, when Cossidente and Penttila constructed infinite families for $\mathsf{H}(3,q^2)$ [39] and $\mathsf{H}(5,q^2)$ [40], q odd. In his PhD thesis, Luke Bayens constructed hemisystems of $\mathsf{H}(2d-1,q^2)$, $d > 2, q$ odd [17], thus answering Segre's question. His construction introduced the so-called "AB-Lemma", which is also utilised by the construction in this chapter.

Hemisystems have also been generalised beyond Hermitian varieties. Cameron, Goethals, and Seidel extended the definition of a hemisystem to a generalised quadrangle of order (q,q^2), for q odd, and showed that the collinearity graph of such a hemisystem is strongly regular [27, 29]. Bamberg, Guidici, and Royle showed that every flock generalised quadrangle of order (s^2,s), s odd, contains a hemisystem [8], and van Dam, Martin, and Muzychuk showed that hemisystems of generalised quadrangles of order (s^2,s) give rise to 4-class cometric association schemes [123].

A common approach to the construction of geometric objects is to consider a subgroup of the automorphism group, and to stitch together its orbits on the elements of the geometry. Since elements of one type interact with an element of another type in the same manner within an orbit, far fewer elements need then be considered. This approach lends itself to large subgroups of the automorphism group, since this means there are fewer orbits, making it is easier to consider the interplay between them. By contrast, the hemisystems in the family presented in this chapter admit a small group relative to the full automorphism group of the parabolic quadric. In fact, the admitted group is dependent only on q and is constant regardless of the rank of the quadric. We construct the hemisystems by considering a parabolic plane in the ambient projective space and consider how the points and maximals of the parabolic quadric meet this plane.

The so called "AB-Lemma", which was first stated in Luke Bayens' thesis [17], allows one to prove the existence of hemisystems in an incidence structure whose automorphism group contains subgroups with certain properties, without the need to construct a tactical configuration. The following theorem generalises the AB-Lemma. We show that the same ideas may be extended to m-covers other than just the $m = \frac{t+1}{2}$ (hemisystem) case. We provide a proof which is closer to the proof of the AB-Lemma as stated in [14] than Bayens' original proof.

Theorem 6.1.2. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{M}, I)$ be a point-line geometry with order (s,t). Let A and B be two subgroups of the automorphism group of $\mathcal{S}$ such that*

1. *B is a normal subgroup of A,*
2. *A and B have the same orbits on $\mathcal{P}$,*
3. *each A-orbit on $\mathcal{M}$ splits into $k > 1$ B-orbits.*

Then for $x \in \{1, \dots, k-1\}$ there are $\binom{k}{x}^n$ $\frac{x(t+1)}{k}$-covers admitting B, where n is the number of A-orbits on lines.

Proof. Given $H \leqslant \mathrm{Aut}(S)$, $P \in \mathcal{P}$, and $\ell \in \mathcal{M}$, let

$$n(P, \ell, H) = |\{m \in \ell^H : P \,\mathrm{I}\, m\}|.$$

Note that for $h \in H$,

$$n(P, \ell, H) = n(P^h, \ell, H)$$

since $P \,\mathrm{I}\, m$ if and only if $P^h \,\mathrm{I}\, m^h$. Now, $B \lhd A$, so for any $a \in A$,

$$\begin{aligned} n(P^a, \ell^a, B) &= |\{m \in (\ell^a)^B : P^a \,\mathrm{I}\, m\}| = |\{m \in (\ell^B)^a : P^a \,\mathrm{I}\, m\}| \\ &= |\{m \in \ell^B : P \,\mathrm{I}\, m\}^a| = |\{m \in \ell^B : P \,\mathrm{I}\, m\}| \\ &= n(P, \ell, B). \end{aligned}$$

Since the orbits of B and A on $\mathcal{P}$ are the same, there exists $b \in B$ such that $P^b = P^a$. Thus

$$n(P, \ell^a, B) = n(P^b, \ell^a, B) = n(P^a, \ell^a, B) = n(P, \ell, B).$$

Hence $n(P, \ell, B) = n(P, \ell', B)$ for all $\ell' \in \ell^A$.

Since each A-orbit on $\mathcal{M}$ splits into k B-orbits, there exist $\ell_1, \dots, \ell_k \in \mathcal{M}$ such that $\ell^A = \ell_1^B \cup \dots \cup \ell_k^B$. Moreover, we can write $\ell_i = \ell^{a_i}$ for some $a_i \in A$. Thus

$$n(P, \ell, A) = \sum_{i=1}^{k} n(P, \ell^{a_i}, B) = k\, n(P, \ell, B).$$

Let $\mathcal{M} = M_1 \cup \dots \cup M_n$, where each M_i is an A-orbit. For $i \in \{1, \dots n\}$ and $j \in \{1, \dots, x\}$, let $\ell_{i,j} \in M_i$ such that $\ell_{i,j}^B \neq \ell_{i,j'}^B$ for $j' \in \{1, \dots x\} \backslash \{j\}$, and define

$$\mathcal{O} = \bigcup_{i,j} \ell_{i,j}^B.$$

Let P be any point, then for fixed j the number of lines incident with P is

$$t + 1 = \sum_{i=1}^{n} n(P, \ell_{i,j}, A) = \sum_{i=1}^{n} k\, n(P, \ell_{i,j}, B).$$

Thus

$$x(t+1) = \sum_{i=1}^{n}\sum_{j=1}^{x} n(P,\ell_{i,j},A) = \sum_{i=1}^{n}\sum_{j=1}^{x} k\, n(P,\ell_{i,j},B)$$

and so the number of lines of $\mathcal{O}$ incident with P is

$$\frac{x(t+1)}{k} = \sum_{i=1}^{n}\sum_{j=1}^{x} n(P,\ell_{i,j},B)$$

This is true for any point P, and so $\mathcal{O}$ is a $\frac{x(t+1)}{k}$-cover. Observe that there are $\binom{k}{x}$ choices of $\ell_{i,j}$ for fixed i, and hence $\binom{k}{x}^n$ different compositions of $\mathcal{O}$. □

Moreover, the proof of Theorem 6.1.2 also provides the construction. A $\frac{x(t+1)}{k}$-cover may be constructed by choosing x B-orbits from inside each of the A-orbits on lines. Note that for the conditions of Theorem 6.1.2 to apply, k must clearly divide $t+1$. Additionally, any $\frac{x(t+1)}{k}$-cover constructed in this manner partitions into $\frac{(t+1)}{k}$-covers.

Remark 6.1.3. *If $A \triangleleft B$ such that A acts semiregularly on the set of B-orbits on lines, then B is in the kernel of this action and each A-orbit on lines splits into $|A:B| = k$ B-orbits, thus implying the first two conditions of Theorem 6.1.2. If, moreover, $|A:B| = 2$, then the formulation of the AB-Lemma given in [14] follows.*

Example 6.1.4. *In $\mathsf{PG}(6,2)$ there are 127 points, 2667 lines, and 2667 projective 4-spaces. There exists a* Singer cycle *which has order 127 and its normaliser has order 889. Both are transitive on points, and semiregular on lines and 4-spaces. Hence the conditions of Theorem 6.1.2 are satisfied and so for $i \in \{1,\dots,6\}$ there exist $1-(7,2,9i)_2$ and $1-(7,5,93i)_2$ designs admitting the Singer cycle.*

When $k = 2$, Therorem 6.1.2 yields Bayens' original AB-Lemma as a direct corollary.

Lemma 6.1.5 (The AB-Lemma [17, 4.4.1])**.** *Let $\mathcal{S} = (\mathcal{P},\mathcal{M},I)$ be point-line geometry. Let A and B be two subgroups of the automorphism group of $\mathcal{S}$ such that*

1. *B is a normal subgroup of A,*
2. *A and B have the same orbits on $\mathcal{P}$,*
3. *each A-orbit on $\mathcal{M}$ splits into two B-orbits.*

Then there are 2^m hemisystems admitting B, where m is the number of A-orbits on the lines.

6.2 The Quadrics and the Orthogonal Groups

Most of the following are standard definitions and results in the subject, and can be found in, for example, [1, 77]. We also refer back to 3.2 for more on classical polar space, including the quadrics.

Let V be an n-dimensional vector space over $\mathbb{F}_q$ equipped with a nondegenerate quadratic form κ. Let q be an odd prime power throughout. Recall that we obtain a *parabolic quadric* $\mathsf{Q}(2d,q)$, a *hyperbolic quadric* $\mathsf{Q}^+(2d-1,q)$, or an *elliptic quadric* $\mathsf{Q}^-(2d+1,q)$, corresponding to the type of the form on V, by taking the totally singular subspaces of V . Here d is the Witt index and $2d$, $2d-1$, and $2d+1$ each give $n-1$, which is the projective dimension. The totally singular 1-spaces are called points, while the largest totally singular subspaces are called *maximals*. Incidence is then defined as inclusion of subspaces. The *rank* of the quadric is given by the Witt index and corresponds to the number of different types of elements in the geometry.

Recall that the order of an incidence structure is (s,t) where there are $s+1$ points on each maximal, and $t+1$ maximals on every point. Thus a hemisystem may exist only when $t+1$ is even. In the case of $\mathsf{Q}(2d,q)$,

$$s+1=\frac{q^d-1}{q-1}, \quad \text{and} \quad t+1=\prod_{i=1}^{d-1}(q^i+1).$$

Hence why we limit ourselves to the case where q is odd, since $t+1$ above is even precisely when q is odd. Recall that

$$\beta := \kappa(u+v)-\kappa(u)-\kappa(v)$$

defines a bilinear form. Now, $\kappa(v)=\beta(v,v)/2$, so for q odd we may work with the bilinear and quadratic forms interchangeably, depending on which better suits the task at hand. With respect to a basis $\{e_1,e_2,\dots,e_n\}$, the *Gram matrix* J of a bilinear form β is the $n\times n$ matrix with entries

$$J_{ij}=\beta(e_i,e_j).$$

The Gram matrix describes the bilinear form with respect to the given basis, where

$$\beta(v,w)=vJw^T.$$

The automorphism group of the quadric is the group which preserves the totally singular subspaces, $\mathrm{P\Gamma O}_n^{\epsilon}(q)$. We elaborate on groups preserving quadratic forms now.

The subgroup of $\mathrm{GL}(V)$ preserving the form is called the *orthogonal group*, denoted $\mathrm{O}(V)$. The *special orthogonal group*, $\mathrm{SO}(V)$, is the subgroup of $\mathrm{O}(V)$ consisting of the elements with determinant one, and the derived subgroup of $\mathrm{O}(V)$ is denoted by $\Omega(V)$. Each group is an index two subgroup of the previous group, that is, $|\mathrm{O}(V):\mathrm{SO}(V)|=|\mathrm{SO}(V):\Omega(V)|=2$ [77, Table 2.1.C].

Moreover, since the nondegenerate quadratic forms are unique up to a change of basis, we may write $\mathrm{O}_n^{\epsilon}(q)$, $\mathrm{SO}_n^{\epsilon}(q)$, and $\Omega_n^{\epsilon}(q)$, where $\epsilon\in\{+,\circ,-\}$ corresponds to the type of the form, n is the dimension of the vector space, and q is the order of the field. We will often still write $\mathrm{O}(V)$, $\mathrm{SO}(V)$, or $\Omega(V)$, to emphasise the vector space V. In particular, we adapt the notation to apply to a subspace of V to mean the image of the projection of the group onto the subspace. Formally, let X be O, SO, or Ω, and let W be a nondegenerate subspace of dimension m in V, where $\kappa|_W$ is the restriction of κ to W, then

$$X(W) := \{g \oplus 1_{W^\perp} \mid g \in X_m^{\epsilon'}(q) \text{ with respect to } \kappa|_W\}, \tag{6.1}$$

for some $\epsilon' \in \{+, \circ, -\}$. Note that the restriction of κ to W need not have the same type as κ itself, and hence ϵ is not necessarily equal to ϵ', for $X(V) \cong X_n^\epsilon(q)$. Moreover, $X(W)$ need not be a subgroup of X.

There also exist projective versions of each of these groups,

$$\mathrm{PX}(V) := \mathrm{X}(V)/(\mathrm{X}(V) \cap Z(\mathrm{GL}(V))),$$

for $X = \mathrm{O}, \mathrm{SO}$, or Ω. For a vector space over a field, $Z(GL(V))$ is simply all the non-zero scalar matrices, and so the projective versions of the groups are the original groups modulo the corresponding scalar matrices. As a result the projective versions of the groups act naturally on one dimensional subspaces rather than on vectors.

More on the classical groups can be found in Kleidman and Liebeck [77]. A few results which form part of more general results in Kleidman and Liebeck are collected here. Recall that q is assumed to be odd.

Lemma 6.2.1. *[77, 2.9.1] There exist the following isomorphisms:*

1. $\mathrm{PSL}_2(q) \cong \Omega_3(q)$,
2. $\mathrm{O}_2^\pm(q) \cong D_{2(q\mp1)}$,
3. $\mathrm{SO}_2^\pm(q) \cong \mathbb{Z}_{q\mp1}$,
4. $\Omega_2^\pm(q) \cong \mathbb{Z}_{(q\mp1)/2}$.

The vectors of V can be partitioned according to their value under the quadratic form κ, so for $\alpha \in \mathbb{F}$ we define

$$V_\alpha := \{v \in V \backslash \{0\} \mid \kappa(v) = \alpha\}. \tag{6.2}$$

We have the following orbit results on V_α.

Lemma 6.2.2. *[77, 2.10.5]*

1. $\mathrm{O}_n^\epsilon(q)$ *is transitive on* V_α, *for all* n, α, *and* ϵ.
2. $\Omega_3^\circ(q)$ *has two orbits on* V_0 *of size* $\frac{1}{2}(q^2-1)$ *and is transitive on* V_α *for* $\alpha \neq 0$.
3. $\Omega_2^+(q)$ *has* 4 *orbits on* V_0, *and* $\Omega_2^\pm(q)$ *has* 2 *orbits on* V_α *for* $\alpha \neq 0$.

We denote the stabiliser in $H \leqslant \mathrm{O}(V)$ of a subspace W or a vector v, by H_W or H_v, respectively. For a subgroup H fixing a subspace W, the subgroup that H induces upon W is denoted by H^W.

The following lemma describes how the orthogonal group interacts with the stabilisers of a nondegenerate subspace and its perp. It holds in more generality, but for our purposes we restrict it to $\mathrm{O}(V)$.

Lemma 6.2.3. *[77, 4.1.1] Assume that $V = U \perp W$, where U is nondegenerate, and $X = \mathrm{O}, \mathrm{SO}$, or Ω. Then:*

1. $\mathrm{O}(V)_U = \mathrm{O}(U) \times \mathrm{O}(W)$,
2. $\Omega(V)_U \geqslant \Omega(U) \times \Omega(W)$,
3. $X(U) \cap \Omega(V) = \Omega(U)$ *and* $X(W) \cap \Omega(V) = \Omega(W)$,
4. $\Omega(V)_U^U = \mathrm{O}(U)$,
5. $\Omega(V)_U^W = \mathrm{O}(W)$, *unless* $\dim(U) = 1$.

When considering the subspace spanned by a hyperbolic pair, it is easy to describe the elements of the orthogonal group explicitly. Since the special orthogonal group consists of the determinant one elements of the orthogonal group and the derived subgroup has index two in the special orthogonal group, the elements of these groups are also easily describable.

Lemma 6.2.4. *Given the quadratic form $\kappa(x_1, x_2) = x_1 x_2$ for a two dimensional vector space over $\mathbb{F}_q$,*

$$\mathrm{O}_2^+(q) = \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \gamma \\ \gamma^{-1} & 0 \end{pmatrix} : \gamma \in \mathbb{F}_q^* \right\}.$$

Moreover, $\mathrm{SO}_2^+(q)$ consists only of the diagonal elements, and $\Omega_2^+(q)$ consists of those elements with squares on the diagonal.

We summarise some of the core information relating to a vector space V equipped with a quadratic form κ in Table 6.1 below.

Type	$n = \dim(V)$	ϵ	$d =$ Witt Index	$O(V)$
Parabolic	Odd	$\circ$	$\frac{1}{2}(n-1)$	$\mathrm{O}_n^\circ(q)$
Hyperbolic	Even	$+$	$\frac{n}{2}$	$\mathrm{O}_n^+(q)$
Elliptic	Even	$-$	$\frac{n}{2} - 1$	$\mathrm{O}_n^-(q)$

Table 6.1: Vector spaces with quadratic forms

6.3 Constructing Hemisystems

In this section we construct the first known infinite family of hemisystems of $\mathsf{Q}(2d, q)$ for $d \geqslant 2$ and q.

Theorem 6.3.1. *There exist 2^n hemisystems of $\mathsf{Q}(2d, q)$ admitting $\Omega_3(q) \cong \mathrm{PSL}_2(q)$ for all odd prime powers q and all $d \geqslant 2$, where $2n$ is the number of orbits of $\Omega_3(q)$ on the maximals.*

We prove Theorem 6.3.1 by applying the AB-Lemma to two subgroups of the orthogonal group with respect to a suitable construction of the parabolic quadric. Let V be a $(2d+1)$-dimensional vector space over $\mathbb{F}_q$, for q odd, $d \geqslant 2$, with a basis

$$\mathcal{B} = \{z, e_0, f_0, x, y, e_1, f_1, e_2, f_2 \dots, e_{d-2}, f_{d-2}\}.$$

Let β be a nondegenerate bilinear form defined on V such that

$$V = \langle z \rangle \perp \langle e_0, f_0 \rangle \perp \langle x, y \rangle \perp \langle e_1, f_1 \rangle \dots \langle e_{d-2}, f_{d-2} \rangle$$

where (e_i, f_i) for $i \in \{0, \dots, d-2\}$ are hyperbolic pairs and $\langle z \rangle$ and $\langle x, y \rangle$ are anisotropic subspaces. In particular let $\beta(e_i, f_i) = 1$, $\beta(e_i, e_i) = \beta(f_i, f_i) = 0$, and $\beta(z, z) = 1$. Let κ be the associated quadratic form with β, defined by $\kappa(v) = \beta(v, v)/2$. The Gram matrix of β according to this basis is

$$J = \begin{pmatrix} 1 &&&&&&&&&& \\ &&1&&&&&&&& \\ &1&&&&&&&&& \\ &&&1&&&&&&& \\ &&&&\zeta&&&&&& \\ &&&&&&1&&&& \\ &&&&&1&&&&& \\ &&&&&&&&\ddots&& \\ &&&&&&&\ddots&&& \\ &&&&&&&&&&1 \\ &&&&&&&&&1& \end{pmatrix},$$

where

$$\zeta = \begin{cases} 1 & q \equiv 3 \pmod 4 \\ \text{primitive root of } \mathbb{F}_q & q \equiv 1 \pmod 4. \end{cases}$$

We distinguish two subspaces $W = \langle z, e_0, f_0 \rangle$ and $U = W^{\perp} = \langle x, y, e_1, f_1, \dots, e_{d-2}, f_{d-2} \rangle$. The construction makes it clear that W is a 3-dimensional parabolic subspace of V, while U is a $(2d-2)$-dimensional elliptic subspace of V.

The automorphism group of $\mathsf{Q}(2d, q)$ is $\mathrm{P\Gamma O}_{2d+1}(q)$, however it is sufficient for us to consider the matrix group $\mathrm{O}_{2d+1}(q)$ as its action on subspaces is the same as that of $\mathrm{PO}_{2d+1}(q) \leqslant \mathrm{P\Gamma O}_{2d+1}(q)$. Thus we let $G = \mathrm{O}_{2d+1}(q)$.

Let $B = \Omega(W)$, where we recall $\Omega(W)$ is defined as in (6.1). Moreover, take $\tau \in G$, where

$$\tau = \left(\begin{array}{ccc|c} -1 && & \\ & 1 & & 0 \\ && 1 & \\ \hline & 0 & & I_{2d-2} \end{array}\right), \tag{6.3}$$

and I_{2d-2} is the identity matrix, and entries not indicated are equal to zero. Note that τ fixes every basis vector except for z, which it sends to $-z$. It will be responsible for pairing and fusing B-orbits on maximals in order to apply the AB-Lemma. Lastly let

$$A = \langle B, \tau \rangle.$$

We observe that $\Omega(W) \leqslant A \leqslant O(W)$ and so every element of A (and also of B) has the form

$$\left(\begin{array}{c|c} g & 0 \\ \hline 0 & I_{2d-2} \end{array}\right)$$

for some $g \in O_3^\circ(q)$ with respect to $\kappa|_W$. Since each element of A (and therefore of B) contains this identity block, the subgroups A and B are unchanged modulo scalar matrices: $A/(A \cap Z) \cong A$.

We make the following observation,

Remark 6.3.2. *$B \cong \mathrm{PSL}_2(q)$, since $B \cong \Omega_3^\circ(q) \cong \mathrm{PSL}_2(q)$ by Lemma 6.2.1(1).*

We now give a few technical lemmas to aid in later proofs.

Lemma 6.3.3. *Let T be a 3-dimensional vector space. Let v be a non-singular vector of T, and let $T_1 = \langle v \rangle$, $T_2 = \langle v \rangle^\perp$ such that $T = T_1 \perp T_2$, then*

$$\Omega(T_2) \leqslant \Omega(T)_{T_1} = \Omega(T)_{T_2} \leqslant \mathrm{O}(T_1) \times \mathrm{O}(T_2)$$

and

$$\Omega(T)_v \cong \Omega(T_2).$$

Proof. Since v is non-singular, T_1 and T_2 are nondegenerate, thus by Lemma 6.2.3(2) and (1), $\Omega(T_1) \times \Omega(T_2) \leqslant \Omega(T)_{T_2} \leqslant \mathrm{O}(T)_{T_2} = \mathrm{O}(T_1) \times \mathrm{O}(T_2)$. Since $\dim(T_1) = 1$, $\mathrm{O}(T_1) = \{\pm 1\}$ and $\mathrm{SO}(T_1) = \Omega(T_1) = 1$. Fixing T_2 means also fixing T_1, so $\Omega(T)_{T_1} = \Omega(T)_{T_2}$, and the first part follows. If in addition, v is fixed, then $\Omega(T)_v^{T_1} = 1$, and so $\Omega(T_2) \leqslant \Omega(T)_v \leqslant \mathrm{O}(T_2)$. Now by Lemma 6.2.3(3), it follows that $\mathrm{O}(T_2) \cap \Omega(T) = \Omega(T_2)$, so the second part follows. $\square$

Note that T_2 in Lemma 6.3.3 is a nondegenerate two dimensional subspace and hence $\mathrm{O}(T_2)$ and $\Omega(T_2)$ could be of either $+$ or $-$ type, depending on T_2 itself.

Observe that W is a 3-dimensional subspace of V with the same properties as T in Lemma 6.3.3. In particular, the stabiliser in $\Omega(W)$ of a nonsingular vector v is isomorphic to $\Omega_2^\pm(q)$. Hence it is useful to know exactly how subgroups of $\mathrm{O}_2^\pm(q)$ act. Recall the definition of V_α in (6.2).

Lemma 6.3.4. *Let Δ_1 and Δ_2 be the two orbits of $\Omega_2^\epsilon(q)$ on V_α, for $\alpha \neq 0$, and let $g \in \mathrm{O}_2^\epsilon(q) \backslash \mathrm{SO}_2^\epsilon(q)$. Then $\Delta_i^g = \Delta_i$ precisely when g fixes a point of V_α, and $\Delta_i^g = \Delta_j$ otherwise, for $i \in \{1,2\}$, $j \in \{1,2\} \backslash \{i\}$.*

Proof. By Lemma 6.2.1, $\mathrm{O}_2^\pm(q)$ is dihedral of order $2(q\mp1)$, while $\mathrm{SO}_2^\pm(q)$ and $\Omega_2^\pm(q)$ are cyclic. Moreover, by Lemma 6.2.2, $\mathrm{O}_2^\pm(q)$ is transitive on V_α, while $\Omega_2^\pm(q)$ has two orbits, for $\alpha \neq 0$. Now, $|\mathrm{O}_2^\pm(q) : \mathrm{SO}_2^\pm(q)| = 2$ and $|\mathrm{SO}_2^\pm(q) : \Omega_2^\pm(q)| = 2$, and so $\mathrm{O}_2^\pm(q)$ has the following subgroup structure

$$\begin{tikzcd}[column sep=small]
 & \mathrm{O}_2^\pm(q) = \langle s, t \colon |s| = q \mp 1, \quad |t| = 2, \quad tst = s^{-1} \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathrm{SO}_2^\pm(q) = \langle s \rangle \arrow[dr, dash] & \langle s^2, t \rangle \arrow[d, dash] & \langle s^2, st \rangle \arrow[dl, dash] \\
 & \Omega_2^\pm(q) = \langle s^2 \rangle &
\end{tikzcd}$$

Note that the action of $\mathrm{O}_2^\pm(q)$ on V_α is permutation isomorphic to the action of $D_{2(q\mp1)}$ on the vertices of a regular $(q \mp 1)$-gon (which has an even number of vertices for q odd), and moreover $D_{2(q\mp1)}$ has two conjugacy classes of reflections: reflections in an axis through opposite vertices and reflections in an axis through opposite midpoints. The first type fix exactly two points and we shall call such reflections "hyperbolic", while the second fix no points and shall be called "elliptic". Considering the action of the dihedral group, it is clear that a hyperbolic reflection preserves the orbits of the cyclic subgroup $\langle s^2 \rangle$ which rotates the vertices, while an elliptic reflection interchanges the orbits. Since $\mathrm{SO}_2^\pm(q)$ is cyclic, an element $\sigma \in \mathrm{O}_2^\pm(q) \backslash \mathrm{SO}_2^\pm(q)$ must be a reflection. The existence of a fixed point on V_α determines if σ is hyperbolic or elliptic, and hence determines its action on the orbits of $\Omega_2^\pm(q)$ on V_α. □

A key observation arising from the previous lemma is that although $O_2^\pm(q)$ is permutationally isomorphic to $D_{2(q\mp1)}$, a specific reflection σ may be hyperbolic in its action upon V_α, but elliptic in its action upon $V_{\alpha'}$, for $\alpha \neq \alpha'$ and $\alpha, \alpha' \neq 0$.

Lemma 6.3.5. *B is a proper normal subgroup of A.*

Proof. Note that the derived subgroup of a group is normal, hence $B = \Omega(W) \trianglelefteq \mathrm{O}(W)$, and thus $B \trianglelefteq A$. Moreover, $\det(\tau) = -1$, so $\tau \notin B$ and $B < A$. It then follows that $B \triangleleft A$. □

We now consider the action of the two subgroups A and B on the points and maximals of $\mathsf{Q}(2d, q)$. We recall that the elements of $\mathsf{Q}(2d, q)$ are totally singular subspaces, and that A and B act on vectors of W (not just subspaces). Note that $V = W \perp U$, hence every $v \in V$ can be expressed as $v = w + u$ for some $w \in W$ and $u \in U$. Moreover, A and B fix every vector in U, so it is sufficient to consider the action of A and B on w to investigate their action on v. We recall the definition of τ in (6.3).

Lemma 6.3.6. *Let P be a point of $\mathsf{Q}(2d,q)$. Then there exists some $g \in B$ such that $P^g = P^\tau$.*

Proof. $P = \langle p \rangle$ for some $p \in V$, such that $p = w + u$ for $w \in W$ and $u \in U$. Now, $p^\tau = w^\tau + u$, since A fixes u, and hence we require $g \in B$ such that $w^g = w^\tau$.

Let $w = \gamma_1 z + \gamma_2 e_0 + \gamma_3 f_0$. If $\gamma_1 = 0$ then $w^{1_G} = w^\tau$, where $1_G \in B$ is the identity element. If $\gamma_1 \neq 0$, then without loss of generality we may take $\gamma_1 = 1$, since $P = \langle p \rangle = \langle \gamma_1^{-1} p \rangle$.

Consider now $\gamma_1 = 1$ and $\kappa|_W(w) = \alpha$. If $\alpha \neq 0$ then by Lemma 6.2.2(2) there exists some $g \in B$ such that $w^g = w^\tau$, and hence $p^g = p^\tau$ and $P^g = P^\tau$.

Consider instead $\alpha = 0$, then by Lemma 6.2.2(2), there are two orbits on W_α under B. Set $v = \gamma_2 e_0 + \gamma_3 f_0$ and $v' = \gamma_2 e_0 - \gamma_3 f_0$. Then τ fixes v, and by Lemma 6.3.3, the stabiliser of v in B is $\Omega(\langle v \rangle^\perp) = \Omega_2^\pm(q)$. Observe that τ fixes v' in $\langle v \rangle^\perp$, but not z. Note that $p = z + v$, $\kappa(p) = 0$, and $\kappa(z) = \frac{1}{2}$ imply that $\kappa(v) = \gamma_2\gamma_3 = -\frac{1}{2}$, which in turn implies that $\kappa(v') = -\gamma_2\gamma_3 = \frac{1}{2}$. Now $v', z \in \langle v \rangle^\perp$ and v' is fixed by τ, hence by Lemma 6.3.4 there exists some $g \in \Omega_2^\pm(q) \leqslant B$ such that $w^g = w^\tau$ and hence $p^g = p^\tau$ and $P^g = P^\tau$. $\square$

Corollary 6.3.7. *The orbits of A and B on the points are the same.*

Note that we are not forced to fix v in the previous proof, it simply proved convenient in demonstrating the existence of an appropriate group element g in B. However, in forthcoming proofs we will seek to show that τ is unique in its action on maximals, and it will be necessary to fix certain subspaces according to the action of τ.

Lemma 6.3.8. *Let M be a maximal totally singular subspace in V, and let M' be a maximally totally singular subspace in $\langle z \rangle^\perp$. Then M projects nontrivially onto $\langle z \rangle$ and M' projects non-trivially onto $\langle e_0, f_0 \rangle$.*

Proof. Consider $\langle z \rangle^\perp = \langle e_0, f_0 \rangle \perp \langle x, y \rangle \perp \langle e_1, f_1 \rangle \dots \langle e_{d-1}, f_{d-1} \rangle$. Since $\langle e_i, f_i \rangle$ are hyperbolic planes, and $\langle x, y \rangle$ is an anisotropic subspace, $\langle z \rangle^\perp$ is an elliptic subspace under the form $\beta|_{\langle z \rangle^\perp}$. Since $\langle z \rangle^\perp$ has dimension $2d$ it has Witt index $d-1$. However a maximal M in V has dimension d, so M cannot be entirely contained in $\langle z \rangle^\perp$. Thus there is a nonempty projection of M onto $\langle z \rangle$. Similarly, U is an elliptic subspace of dimension $2d-2$, with Witt index $d-2$. Hence M' is not completely contained in U and must have a nonempty projection onto $\langle e_0, f_0 \rangle$. $\square$

Lemma 6.3.9. *Let M be a maximal. There exists a basis $b_1, b_2, b_3, \dots, b_d$ of M such that*

$$b_1 = z + u_1, \quad b_2 = e_0 + u_2, \quad b_3 = f_0 + u_3, \quad b_i = u_i \quad \textit{for } i \in \{4, \dots, d\},$$

or

$$b_1 = z + \lambda f_0 + u_1, \quad b_2 = e_0 + \mu f_0 + u_2, \quad b_i = u_i \quad \text{for } i \in \{3, \dots, d\},$$

or

$$b_1 = z + \lambda e_0 + u_1, \quad b_2 = f_0 + u_2, \quad b_i = u_i \quad \text{for } i \in \{3, \dots, d\},$$

where in each case $u_i \in U$ *for* $i \in \{1, \dots d\}$.

Proof. Let $b_1, b_2, \dots, b_d$ be a basis of M. By Lemma 6.3.8 M projects onto $\langle z \rangle$, and so without loss of generality, we may take b_1 to project non-trivially onto $\langle z \rangle$. Since $\dim(\langle z \rangle) = 1$, there exists a basis $b_1, b_2', b_3', \dots, b_d'$ of M, where each b_i' projects trivially onto $\langle z \rangle$ for $i \in \{2, \dots, d\}$. Such a basis can be obtained by taking linear combinations of b_1 from b_i.

Now consider $M' = \langle b_2', \dots, b_d' \rangle$, a $(d-1)$-dimensional totally isotropic subspace of $\langle z \rangle^\perp$. By Lemma 6.3.8 again, M' projects non-trivially onto $\langle e_0, f_0 \rangle$. Without loss of generality, we may assume b_2' has a non-empty projection to $\langle e_0, f_0 \rangle$. Since $\dim(\langle e_0, f_0 \rangle) = 2$, we then have two cases: either there is some other basis vector, which we may take to be b_3', such that b_3' projects non-trivially onto $\langle e_0, f_0 \rangle$ and the projections of b_2' and b_3' onto $\langle e_0, f_0 \rangle$ are linearly independent, or else b_2' is the only such vector. By taking linear combinations we may then manipulate the bases into the desired form. □

Lemma 6.3.10. *Let M be a maximal. There does not exist $g \in B$ such that $M^g = M^\tau$.*

Proof. Let $\{b_1, b_2, \dots, b_d\}$ be a basis for the totally singular subspace M. Without loss of generality, we need only consider the first two bases of Lemma 6.3.9, since the argument for the third basis is identical to that of the second. Now, τ fixes each basis vector other than b_1, and for any g in B, g fixes each vector in U. Hence $M^g = M^\tau$ if and only if $\langle b_1, b_2, b_3 \rangle^g = \langle b_1^\tau, b_2, b_3 \rangle$ in the first case, or $\langle b_1, b_2 \rangle^g = \langle b_1^\tau, b_2 \rangle$ in the second case. Throughout, we recall that $\kappa(b_i) = 0$ and $\beta(b_i, b_j) = 0$, since M is totally singular. We now consider each case.

Case 1: Consider M with first three basis vectors

$$b_1 = z + u_1, \quad b_2 = e_0 + u_2, \quad b_3 = f_0 + u_3$$

for $u_1, u_2, u_3 \in U$.

Observe that $\kappa(b_1) = 0$, $\kappa(b_3) = 0$, and $\beta(b_1, b_3) = 0$ imply that $\beta(u_1, u_1) = -1$, $\kappa(u_3) = 0$, and $\beta(u_1, u_3) = 0$, respectively.

We first show that $u_2 \notin \langle u_1, u_3 \rangle$. Consider for a contradiction that $u_2 = \gamma_1 u_1 + \gamma_2 u_3$. Then $2\kappa(u_2) = \beta(\gamma_1 u_1 + \gamma_2 u_3, \gamma_1 u_1 + \gamma_2 u_3) = -\gamma_1^2$. However $\kappa(b_2) = 0$ implies that $\kappa(u_2) = 0$, and thus $\gamma_1 = 0$ and $u_2 = \gamma_2 u_3$. Now, from $\beta(b_2, b_3) = 0$ it follows that $\beta(u_2, u_3) = -1$ and therefore $\beta(\gamma_2 u_3, u_3) = 2\gamma_2\kappa(u_3) = -1$. However $\kappa(u_3) = 0$, a contradiction. So $u_2 \notin \langle u_1, u_3\rangle$.

Moreover, since $\kappa(u_1) = -\frac{1}{2}$ and $\kappa(u_3) = 0$, it follows that $u_1 \neq \gamma u_3$. Hence u_1, u_2, and u_3 are linearly independent.

Let $g \in B$ such that $b_1^g, b_2^g, b_3^g \in \langle b_1^\tau, b_2, b_3\rangle$. Then

$$b_1^g = z^g + u_1^g = z^g + u_1,$$

and

$$\begin{aligned} b_1^g &= \gamma_1 b_1^\tau + \gamma_2 b_2 + \gamma_3 b_3 \\ &= -\gamma_1 z + \gamma_1 u_1 + \gamma_2 e_0 + \gamma_2 u_2 + \gamma_3 f_0 + \gamma_3 u_3. \end{aligned}$$

However, since u_1, u_2, and u_3 are linearly independent, $\gamma_2 = \gamma_3 = 0$ and $\gamma_1 = 1$. Hence $b_1^g = -z + u_1 = b_1^\tau$. Moreover, $b_2^g = b_2$ and $b_3^g = b_3$ by the same argument. Thus

$$z^g = -z, \quad e_0^g = e_0, \quad f_0^g = f_0.$$

From this it follows that $g = \tau$. However $\tau \notin B$, so there is no such element $g \in B$ such that $M^g = M^\tau$.

Case 2: Consider M with first two basis vectors

$$b_1 = z + \lambda f_0 + u_1, \quad b_2 = e_0 + \mu f_0 + u_2.$$

Recall that $b_3 \in U$. From $\kappa(b_1) = 0$ if follows that $\kappa(u_1) = -\frac{1}{2}$, and since $\kappa(b_3) = 0$ it follows that u_1 and b_3 are linearly independent vectors in U. Consider now $u_1 = \gamma u_2$. Since u_1 and b_3 are linearly independent, so too must u_2 and b_3 be linearly independent. Thus we may take a basis $b_1', b_2, b_3, \ldots, b_d$ of M, where $b_1' = b_1 + b_3 = z + \lambda f_0 + u_1'$ and $u_1' = u_1 + b_3$. We now have u_1' is linearly independent of u_2, so without loss of generality, we may assume $u_1 \neq \gamma u_2$.

Let $g \in B$ such that $b_1^g, b_2^g \in \langle b_1^\tau, b_2\rangle$. Then

$$\begin{aligned} b_1^g &= z^g + \lambda f_0^g + u_1^g = z^g + \lambda f_0^g + u_1, \\ b_2^g &= e_0^g + \mu f_0^g + u_2^g = e_0^g + \mu f_0^g + u_2, \end{aligned}$$

and

$$\begin{aligned} b_1^g &= \gamma_1 b_1^\tau + \gamma_2 b_2 = \gamma_1(-z + \lambda f_0 + u_1) + \gamma_2(e_0 + \mu f_0 + u_2), \\ b_2^g &= \gamma_3 b_1^\tau + \gamma_4 b_2 = \gamma_3(-z + \lambda f_0 + u_1) + \gamma_4(e_0 + \mu f_0 + u_2). \end{aligned}$$

Since u_1 and u_2 are linearly independent, it follows that $\gamma_1 = \gamma_4 = 1$ and $\gamma_2 = \gamma_3 = 0$. That is, $b_2^g = b_2$ and $b_1^g = b_1^\tau$. From this it follows, that,

$$(z + \lambda f_0)^g = -z + \lambda f_0, \quad (e_0 + \mu f_0)^g = e_0 + \mu f_0.$$

Here we have three subcases: $\mu = 0$, $2\mu \in \square\backslash\{0\}$, and $2\mu \notin \square$, where $\square$ is the set of squares of $\mathbb{F}_q$.

Case 2a: Consider $\mu = 0$. Then $\kappa(b_2) = 0$ implies $\kappa(u_2) = 0$. Moreover, $\beta(u_2, b_i) = 0$ for $3 \leqslant i \leqslant d$. Observe, $b_3, \dots, b_d$ are linearly independent, since they are basis vectors, and they are all contained in U. Since U is of elliptic type, it has Witt index $d-2$, and hence u_2 is either a linear combination of $b_3, \dots, b_d$ or $u_2 = 0$. In either case, $\beta(u_1, u_2) = 0$, since $\beta(u_1, b_i) = 0$ for $3 \leqslant i \leqslant d$ and $\beta(u_1, 0) = 0$. However, $\beta(b_1, b_2) = 0$ and so $\beta(u_1, u_2) = -\lambda$, thus $\lambda = 0$.

Now, we require g such that $z^g = -z$ and $e_0^g = e_0$. It follows then that g fixes $\langle z\rangle$ and hence $g \in B_{\langle z\rangle}$, where $1 \times \Omega_2^+(q) \leqslant B_{\langle z\rangle} \leqslant \mathrm{O}_1^\circ(q) \times \mathrm{O}_2^+(q)$ by Lemma 6.3.3, since $\kappa(z) \neq 0$ and $\langle e_0, f_0\rangle$ is of hyperbolic type. Moreover, $g = (-1, h)$, where $h \in \mathrm{O}_2^+(q)$, since $z^g = -z$. However, $g \in \Omega_3^\circ(q) \leqslant \mathrm{SO}_3^\circ(q)$, and hence $\det(g) = 1$, meaning $\det(h) = -1$. By Lemma 6.2.4, h must then be an element of the form

$$\begin{pmatrix} 0 & \gamma \\ \gamma^{-1} & 0 \end{pmatrix},$$

however no such element fixes e_0. Thus there is no such $g \in B$.

In the remaining two subcases where $\mu \neq 0$, let $v = e_0 + \mu f_0$, and $\mathcal{C} = \{v, z, e_0 - \mu f_0\}$ be a basis for W. Further let $W_1 = \langle v\rangle$ and $W_2 = \langle z, e_0 - \mu f_0\rangle$. Expressed with respect to $\mathcal{C}$, $z + \lambda f_0 = z - \frac{\lambda}{2\mu}(e_0 - \mu f_0) + \frac{\lambda}{2\mu} v$. However $v^g = v$, and so $(z - \frac{\lambda}{2\mu}(e_0 - \mu f_0))^g = -z - \frac{\lambda}{2\mu}(e_0 - \mu f_0)$. Note g fixes v, hence $g \in B_v$. Moreover $\kappa(v) = \mu \neq 0$, so by Lemma 6.3.3, $B_v = \Omega(W_2) \cong \Omega_2^\epsilon(q)$.

Case 2b: Consider $2\mu \in \square\backslash\{0\}$. Then there exists $\gamma \neq 0$ such that $\gamma^2 = 2\mu$. Let $w = \gamma z + (e_0 - \mu f_0)$. Then $\kappa(w) = 0$ and hence W_2 is hyperbolic, meaning $B_v \cong \mathrm{O}_2^+(q)$. Thus we require $g = (1, h)$, where $h \in \mathrm{O}_2^+(q)$, such that $(z - \frac{\lambda}{2\mu}(e_0 - \mu f_0))^h = -z - \frac{\lambda}{2\mu}(e_0 - \mu f_0)$. By Lemma 6.2.4 h has the form $\mathrm{Diag}(\zeta^2, \zeta^{-2})$. Now, $-\frac{\lambda}{2\mu}(e_0 - \mu f_0)^h = -\frac{\lambda}{2\mu}(e_0 - \mu f_0)$ implies $h = \mathrm{Diag}(1, 1)$. However, $z^h = -z$ implies $h = \mathrm{Diag}(-1, -1)$, a contradiction. Thus there is no such $g \in B$ which replicates the action of τ on M.

Case 2c: Consider $2\mu \notin \square$. Recall that $\mathrm{O}_2^\epsilon(q)$ is dihedral and $\Omega_2^\epsilon(q)$ is cyclic, by Lemma 6.2.1. With respect to $\mathcal{C}$, τ remains unchanged, and induces the element $\tau_{W_2} = \mathrm{Diag}(-1, 1) \in \mathrm{O}(W_2) \cong \mathrm{O}_2^\epsilon(q)$ when restricted to W_2. Clearly $\det(\tau_{W_2}) = -1$ and hence $\tau_{W_2} \notin \Omega(W_2) = B_v$. Note that the determinant is unaffected by a change of basis of W_2. Moreover τ_{W_2} is not an element of $\mathrm{SO}(W_2)$ and is thus a reflection in the dihedral group $\mathrm{O}(W_2)$. Observe that τ_{W_2} fixes only elements of the form $\gamma(e_0 - \mu f_0)$. Now, $\kappa(\gamma(e_0 - \mu f_0)) = -\mu\gamma^2$, and so τ_{W_2} fixes an element of V_α for $\alpha \in -\mu\square$ and is fixed-point-free otherwise. Moreover $\lambda^2 \neq 2\mu$, hence $\kappa(z - \frac{\lambda}{2\mu}(e_0 - \mu f_0)) \neq 0$ and so $z - \frac{\lambda}{2\mu}(e_0 - \mu f_0) \notin V_0$. Thus if $\kappa(z - \frac{\lambda}{2\mu}(e_0 - \mu f_0)) =$

$\frac{1}{2} + (\frac{\lambda}{2\mu})^2(-\mu) \notin -\mu\Box$, then by Lemma 6.3.4, τ_{W_2} interchanges the orbits of $\Omega(W_2)$ on $V_{\frac{1}{2}+(\frac{\lambda}{2\mu})^2(-\mu)}$, meaning $z - \frac{\lambda}{2\mu}(e_0 - \mu f_0)$ and $-z - \frac{\lambda}{2\mu}(e_0 - \mu f_0)$ lie in different orbits of $\Omega(W_2) = B_v$.

We claim that indeed $\kappa(z - \frac{\lambda}{2\mu}(e_0 - \mu f_0)) \notin -\mu\Box$. To prove the claim, observe that $\frac{1}{2} + (\frac{\lambda}{2\mu})^2(-\mu) \notin -\mu\Box$ if and only if $\lambda^2 - 2\mu \notin \Box$. Consider for a contradiction that $\lambda^2 - 2\mu \in \Box$. Then there exists γ such that $\kappa(\gamma u_1 + u_2) = 0$, since $\kappa(\gamma u_1 + u_2) = \gamma^2\kappa(u_1) + \gamma\beta(u_1, u_2) + \kappa(u_2) = -\frac{\gamma^2}{2} - \gamma\lambda - \mu$ has discriminant $\lambda^2 - 2\mu$. Hence $\langle \gamma u_1 + u_2, u_3, \ldots, u_d\rangle$ is a totally isotropic subspace in U of dimension $d - 1$. However, as U is elliptic, its Witt index is $d - 2$, a contradiction, thus proving the claim. It follows then that there is no such $g \in B$ which acts on M the same way as τ.

After considering each case, we see that there is no element $g \in B$ such that $M^g = M^\tau$. $\Box$

As a result of Lemma 6.3.10 the image of τ on each element in an orbit of B on maximals is outside of the orbit, and since B is index 2 in A, this results in B doubling each orbit of A. This gives the immediate corollary:

Corollary 6.3.11. *Each orbit of A on maximals splits into two orbits of B on maximals.*

The proof of Theorem 6.3.1 follows directly from Lemma 6.3.5, Corollary 6.3.7, Corollary 6.3.11 and the application of the AB-Lemma, Lemma 6.1.5.

6.4 Further Remarks

Remark 6.4.1. *In [38, Remark 2.12], a sporadic example of a hemisystem of $\mathsf{Q}(6,3)$ is given which bares similarities to the construction in this chapter. The authors provide a similar decomposition of the ambient space into a conic and its perp (of elliptic type). A subgroup isomorphic to A_5 is then found by computer in the pointwise stabiliser of the conic which fixes a hemisystem. It is not known if the construction in [38] generalises to all q, or to greater rank, and so the authors describe it as sporadic. Our construction instead finds a subgroup in the pointwise stabiliser of the perp of the conic which is isomorphic to $\Omega_3(3) \cong A_4$, in the case $q = 3$, and generalises to $\Omega_3(q)$ for all odd q and rank at least 2.*

Remark 6.4.2. *As indicated in Theorem 6.3.1, the construction presented in this chapter results in 2^n hemisystems, where n is the number of A-orbits on the maximals. This number arises from the fact that there are two choices of B-orbits contained in every A-orbit, with every choice producing a hemisystem. However, while we can construct 2^n hemisystems in this manner, they are not necessarily non-isomorphic. A hemisystem $\tilde{H}$ is isomorphic to a hemisystem H if $\tilde{H}$ lies in* *the orbit of H under G. Indeed, every hemisystem is isomorphic to its complement, since τ interchanges every pair of B-orbits.*

It is interesting, then, to ask how many of the 2^n hemisystems are non-isomorphic. We can compute a very rough lower bound to show that the number is enormous. By the Orbit-Stabiliser theorem, $|G| = |H^G||G_H|$, and since $B \leqslant G_H$ it follows that $|H^G| \leqslant \frac{|G|}{|B|}$. Moreover, $n \geqslant \lfloor \frac{|\mathcal{M}|}{2|B|} \rfloor$, so the number of non-isomorphic hemisystems is bounded below by

$$\frac{2^{\lfloor \frac{|\mathcal{M}|}{2|B|} \rfloor}}{|G|/|B|}.$$

Since $2^{\lfloor \frac{|\mathcal{M}|}{2|B|} \rfloor} \gg \frac{|G|}{|B|}$ it follows that a vast number of hemisystems constructed by this method are non-isomorphic. Moreover, note that $\frac{|G|}{|B|}$ massively over counts those hemisystems which are unions of B-orbits on maximals, which is what 2^n counts.

Remark 6.4.3. *In some instances G_H, the full stabiliser of H in G, is larger than B. Finding G_H is computationally expensive, so it is easier to compute $G_H \cap N_G(B)$, the stabiliser of H in $N_G(B)$. This is moreover a reasonable concession since $N_G(B)$ preserves the orbits of B on maximals. In this case, $B \leqslant G_H \cap N_G(B) \leqslant B.K$, where $B.K$ is the group extension of some subgroup K by B. Table 6.2 presents all maximal, inequivalent K, for small values of d and q, for which there exists a hemisystem H such that $G_H \cap N_G(B) = B.K$. The contents of Table 6.2 were found by exhaustive computer search.*

d, q	K
$3, 3$	D_{10}, S_3^2
$3, 5$	$C_2^2 \times S_3$, $C_2^2 \times S_3$, $C_{13} : C_4$, $C_2 \times A_5$, $C_2 \times A_5$, $C_5^2 : (C_4 \times S_3)$
$3, 7$	S_3, D_{12}, D_{12}, D_{50}, $C_7^2 : (C_6 \times C_2)$
$3, 9$	$C_4 \times A_4$, $C_4 \times A_4$, $C_5 : (C_8 \times C_2)$, $C_5 : (C_8 \times C_2)$, $C_2^2 \times (C_5 : C_4)$, $C_2^2 \times (C_5 : C_4)$, $C_5 : ((C_8 \times C_2) : C_2)$, $C_{41} : C_8$, $C_{41} : C_8$, $C_{41} : C_8$, $D_8 \times D_{10}$, $C_2 \times A_6$, $A_6 : C_4$, $C_3^4 : (C_5 : (C_8 : C_4))$
$4, 3$	$C_4 \times C_2$, C_2^3, D_8, D_{20}, D_{20}, $C_2 \times A_4$, $\mathrm{PSL}(3,2) : C_2$, A_6, $C_3^4 : D_{20}$, $(((((C_3 \times (C_3^2 : C_3)) : C_3) : Q_8) : C_3) : C_2) : C_2$

Table 6.2: Structure description of K in Remark 6.4.3 for small values of d and q.

Remark 6.4.4. *We can give a more geometric description of the construction given in Theorem 6.3.1. Let p be a point of $PG(2d, q)$ such that $p^\perp$ is of elliptic type. Then τ is the unique involution which fixes $p^\perp$ point-wise. Let ℓ be any line of $p^\perp$ of hyperbolic type. Then B is the derived subgroup of the point-wise stabiliser of $\langle p, \ell \rangle^\perp$. It is clear that $\langle p, \ell \rangle$ as a vector subspace corresponds to W in Section 3.*

Remark 6.4.5. *The examples of hemisystems of* $\mathsf{DQ}(6,q)$ *computed in Section 5.3 for* $q \in \{3,7,11\}$ *belong to the construction corresponding to Theorem 6.3.1 (recall that a hemisystem of* $\mathsf{DQ}(6,q)$ *is also a hemisystem of* $\mathsf{Q}(6,q)$*). We note that this construction does not yield examples for* $q \in \{5,9\}$*. Thus the known examples are for* $q \equiv 3 \pmod 4$*, which is when* $\zeta = 1$ *in the Gram matrix corresponding to this construction. Thus we hope that, with additional arguments, this construction may be extended to a construction of hemisystems of* $\mathsf{DQ}(6,q)$ *for all* $q \equiv 3 \pmod 4$*, perhaps by further studying the interaction of maximals and next-to-maximals with* $\langle x,y\rangle$*. We note that next-to-maximals behave in some instances like maximals, and in other instances like points, with respect to their interaction with the subspace* W*.*

Chapter 7

Vanishing Krein Parameters and T-Designs

This chapter investigates the properties of association schemes with vanishing Krein parameters. It is well known that vanishing Krein parameters have important consequences, for example in determining the feasibility or infeasibility of parameter sets for distance-regular graphs, and in placing bounds upon the orders of generalised polygons.

Vanishing Krein parameters may also be used to say something about subsets of the vertices of an association scheme. The key result of this chapter is Theorem 7.2.2 which states that given certain vanishing Krein parameters, either the strata containing a T-design are constrained, or else the T-design consists of exactly half of the vertices of the association scheme. We then investigate various consequences and applications of Theorem 7.2.2.

Recall that in an association scheme it is sufficient to know one of P, Q, $\{p_{ij}^k\}$, or $\{q_{ij}^k\}$, since they can all be computed from one another. For example, Theorem 2.3.11 may be used to compute the Krein parameters from the matrix of eigenvalues, P, or the matrix of dual eigenvalues, Q. When computing vanishing Krein parameters, we need not concern ourselves with the constants in front of the sum, and so the following equations hold:

$$q_{ij}^h = 0 \iff \sum_{\ell=0}^{d} k_\ell Q_{\ell i} Q_{\ell j} Q_{\ell h} = \sum_{\ell=0}^{d} \frac{P_{i\ell} P_{j\ell} P_{h\ell}}{k_\ell^2} = 0,$$

in particular,

$$q_{ii}^i = 0 \iff \sum_{\ell=0}^{d} k_\ell Q_{\ell i}^3 = \sum_{\ell=0}^{d} \frac{P_{i\ell}^3}{k_\ell^2} = 0.$$

Recall also that $k_\ell = P_{0\ell}$, which makes the previous equations simpler to express in terms of the matrix of eigenvalues, P.

7.1 Triple Intersection Numbers and Feasible Parameter Sets

Cameron, Goethals, and Seidel [29] proved that strongly regular graphs with either $q_{11}^1 = 0$ or $q_{22}^2 = 0$ have strongly regular subconstituents about every vertex. Moreover, they characterised such graphs, when connected, as being a pentagon, a Smith graph (cf. [111]), or the complement of a Smith graph. Similar results have been investigated for distance-regular graphs with slightly larger diameter [61, 73].

Given $x, y, z \in V(\Gamma)$, we define the *triple intersection numbers*

$$p_{r,s,t}^{x,y,z} = |\{u : d(x,u) = r, d(y,u) = s, d(z,u) = t\}|.$$

There is an important relation between vanishing Krein parameters and the triple intersection numbers.

Theorem 7.1.1 ([35]). *Let $(\Omega, \mathcal{R})$ be a d-class association scheme. Then $q_{ij}^h = 0$ if and only if*

$$\sum_{r,s,t=0}^{d} Q_{ri} Q_{sj} Q_{th} p_{r,s,t}^{x,y,z} = 0,$$

for all $x, y, z \in \Omega$.

The triple intersection numbers are defined with respect to a given triple of vertices, and so they are more restricted than standard intersection numbers. Nevertheless, they are still very useful. For example, using Theorem 7.1.1, distance-regular graphs with the following intersection arrays have been shown to not exist:

- $\{4r^3 + 8r^2 + 6r + 1, 2r(r+1)(2r+1), 2r^2 + 2r + 1;$
  $1, 2r(r+1), (2r+1)(2r^2 + 2r + 1\}$ [35],
- $\{(2r^2 - 1)(2r + 1), 4r(r^2 - 1), 2r^2; 1, 2(r^2 - 1), r(4r^2 - 2)\}$ [74],
- $\{2r^2(2r + 1), (2r - 1)(2r^2 + r + 1), 2r^2; 1, 2r^2, r(4r^2 - 1)\} (r \geqslant 2)$ [74],
- $\{(r + 1)(r^3 - 1), r(r - 1)(r^2 + r - 1), r^2 - 1;$
  $1, r(r + 1), (r^2 - 1)(r^2 + r - 1)\} (r \geqslant 3)$ [121],
- $\{(2r + 1)(4r + 1)(4t - 1), 8r(4rt - r + 2t), (r + t)(4r + 1);$
  $1, (r + t)(4r + 1), 4r(2r + 1)(4t - 1)\} (r \geqslant 2, t \geqslant 2)$ [128],
- $\{74, 54, 15; 1, 9, 60\}$ [35],
- $\{135, 128, 16; 1, 16, 120\}$ [128],
- $\{234, 165, 12; 1, 30, 198\}$ [128],
- $\{55, 54, 50, 35, 10; 1, 5, 20, 45, 55\}$ [128]

Each of these parameter sets are otherwise feasible. Chapter 14 of "BCN" [22] lists many feasible parameter sets which are still open as to the existence of a corresponding distance-regular graph.

7.2 Schur Multiplication and Subsets of Vertices

We have already seen that the Bose-Mesner algebra is closed under both normal and Schur multiplication, and as a consequence many interesting results have a similar looking dual expression in which normal multiplication and Schur multiplication are interchanged. The two most prominent examples of this duality property are the existence of the two bases of minimal idempotents for the Bose-Mesner algebra (the adjacency matrices, A_i, for Schur multiplication, and the projection matrices, E_i, for normal multiplication), and the intersection numbers and the Krein parameters. Indeed, the Krein parameters are also referred to as dual intersection numbers, since algebraically they behave in the same manner as the intersection numbers.

This duality of association schemes is, however, largely algebraic. There is a clear combinatorial meaning for the intersection numbers; they count the number of vertices related in a certain way. Similarly, normal multiplication with the i-th adjacency matrix will yield the i-neighbours. However the combinatorial meaning of the dual concepts is less clear. Despite this, they have interesting combinatorial consequences. The celebrated Krein conditions mean that Krein parameters must be non-negative real numbers [107], for example, and we saw in Section 7.1 that Krein parameters are linked to triple intersection numbers and so can be used to determine feasibility of putative parameter sets of distance-regular graphs.

There is a bijection between the powerset of the vertices and the set of all $\{0,1\}$-vectors by taking the characteristic vector. Thus we can think of any $\{0,1\}$-vector as being the characteristic vector of some set. It is clear that Schur multiplication of two $\{0,1\}$-vectors produces another $\{0,1\}$-vector with a one in an entry if and only if the corresponding entry is one in both of the original vectors. Thus intersection of two vertex subsets is described by Schur multiplication of their characteristic vectors. That is,

$$\chi_{\theta_1\cap\theta_2} = \chi_{\theta_1} \circ \chi_{\theta_2}$$

for all $\theta_1, \theta_2 \subseteq \Omega$. In particular we make the simple yet powerful observation that for any $\theta \subseteq \Omega$,

$$\chi_\theta \circ \chi_\theta = \chi_\theta.$$

Moreover $\mathbb{1}$ is the identity under Schur multiplication, and $\mathbb{1} - \chi_\theta = \chi_{\Omega\setminus\theta}$. Hence Schur multiplication obtains a degree of combinatorial meaning when viewed as describing intersection.

The following theorem, Theorem 7.2.1, is due Cameron, Goethals and Seidel [28], however we give a short proof due to Martin [89]. It states that the Schur product of two vectors, each lying in precisely one eigenspace, projects trivially to a third eigenspace if the corresponding Krein parameter vanishes.

Theorem 7.2.1 ([28, Proposition 5.1]). *Let $u \in V_\ell$ and $v \in V_j$ and $q^i_{\ell j} = 0$, then $E_i(u \circ v) = 0$.*

Proof. Observe

$$E_i(u \circ v) = E_i \Delta_u v = E_i \Delta_u E_j v.$$

Consider the square of the Frobenius norm of $E_i \Delta_u E_j$,

$$\begin{aligned}
||E_i\Delta_u E_j||_F^2 &= \mathrm{Tr}\big((E_i\Delta_u E_j)^* E_i \Delta_u E_j\big) \\
&= \mathrm{Tr}(E_j \Delta_u E_i \Delta_u E_j) \\
&= \sum_{y\in\Omega}\sum_{z\in\Omega} u_y u_z (E_i)_{yz} \sum_{x\in\Omega} (E_j)_{xy}(E_j)_{zx} \\
&= \sum_{y\in\Omega}\sum_{z\in\Omega} u_y u_z (E_i)_{yz}(E_j)_{yz} \\
&= \frac{1}{|\Omega|}\sum_{k=0}^{d} q_{ij}^k \sum_{y\in\Omega}\sum_{z\in\Omega} u_y (E_k)_{yz} u_z \\
&= \frac{1}{|\Omega|}\sum_{k=0}^{d} q_{ij}^k u^\top E_k u.
\end{aligned}$$

Now $q_{\ell j}^i = 0$ implies that $q_{ij}^\ell = 0$. Since $u \in V_\ell$ it follows that

$$E_k u = \delta_{\ell,k} u,$$

which together with $q_{ij}^\ell = 0$ yields

$$||E_i \Delta_u E_j||_F^2 = 0,$$

and hence

$$E_i \Delta_u E_j = 0.$$

Thus $E_i(u \circ v) = 0$. □

Theorem 7.2.1 is incredibly useful, and can be used, for example, to offer an alternative proof of Theorem 2.4.4 (cf. [89, Theorem 2]).

We now reach the main theorem of this section. Theorem 7.2.2 shows how given certain vanishing Krein parameters, T-designs must either be constrained to a smaller subset of the eigenspaces in which they lie, or else consist of exactly half of the vertices.

Theorem 7.2.2. *Let $(\Omega, \mathcal{R})$ be an association scheme, and let θ be a subset of the vertices, such that*

$$\chi_\theta \in V_0 \underset{i\in S}{\perp} V_i,$$

where $S \subseteq \{1, \ldots, d\}$. If there is some $k \in S$, such that $q_{i,j}^k = 0$ for all $i, j \in S$, then either

$$\chi_\theta \in V_0 \underset{i\in S\setminus\{k\}}{\perp} V_i,$$

or $|\theta| = \frac{|\Omega|}{2}$.

Proof. Let $v = \chi_\theta$, then $v \in V_0 \perp_{i\in S} V_i$, and hence $v = h\mathbb{1} + \sum_{i\in S} v_i$, where $v_i \in V_i$ and $h = \frac{|\theta|}{|\Omega|}$. If $v_k = 0$, then $E_k v = 0$ and so

$$v \in V_0 \underset{i\in S\setminus\{k\}}{\perp} V_i.$$

Instead take $v_k \neq 0$. Now,

$$v \circ v = h^2\mathbb{1} + \sum_{i\in S} 2hv_i + \sum_{i,j\in S} v_i \circ v_j,$$

since $\mathbb{1}$ is the identity under Schur multiplication. Thus $E_k(v \circ v) = 2hv_k$, since $E_k v_i = 0$ for all $i \neq k$, and $E_k(v_i \circ v_j) = 0$ for all $i, j \in S$, by Theorem 7.2.1, since $q^k_{ij} = 0$. However, note that $v \circ v = v = h\mathbb{1} + \sum_{i\in S} v_i$, since $v = \chi_\theta$ is $\{0, 1\}$-vector. Thus $E_k(v \circ v) = v_k$. Equating these expressions yields $2hv_k = v_k$, and since v_k is non-trivial, $h = \frac{1}{2}$ and so $|\theta| = \mathbb{1} \cdot v = \frac{|\Omega|}{2}$. □

Observe that θ in Theorem 7.2.2 is a T-design, where $T = \{1, \ldots, d\}\setminus S$, in which case Theorem 7.2.2 is saying that if the conditions hold, then either $|\theta| = \frac{|\Omega|}{2}$ or θ is in fact a $T \cup \{k\}$-design. Clearly Theorem 7.2.2 can be applied repeatedly if the conditions are satisfied for multiple values of k, and hence many T-designs may be further constrained.

Remark 7.2.3. *The conditions of Theorem 7.2.2 being satisfied does not imply the existence of a T-design, θ, such that $|\theta| = \frac{|\Omega|}{2}$. Such a T-design may or may not exist. By way of example,* $\mathsf{DW}(5, 3)$ *has Krein parameter $q^3_{33} = 0$, however it has no hemisystems, while* $\mathsf{DQ}(6, 3)$ *also has Krein parameter $q^3_{33} = 0$, and does have hemisystems (cf. Section 5.3).*

Remark 7.2.4. *The converse of Theorem 7.2.2 does not hold. As a counterexample, consider $W(3, 3)$, which possesses 2-ovoids but no 1-ovoids, and yet has Krein parameters*

$$q^1_{11} = \frac{44}{3}, \quad q^2_{22} = \frac{10}{3}, \quad q^2_{12} = q^2_{21} = \frac{32}{3}, \quad q^1_{12} = q^1_{21} = \frac{25}{3}.$$

7.3 Intriguing Sets

In order to use Theorem 7.2.2 we require at a minimum that $q^k_{kk} = 0$. Moreover, the smallest such setting is for the set $S = \{k\}$, and hence we have an interesting situation for so-called intriguing sets.

A subset $\theta \in \Omega$ is called an *intriguing set of type i* if $\chi_\theta \in V_0 \perp V_i$. Hence an intriguing set of type i is a $\{1, \ldots, d\}\setminus\{i\}$-design. Intriguing sets are often known by other names in different settings. For example, *tight sets* (cf. [7, 10, 12]), *Cameron-Liebler line classes* (cf. [30, 43, 44, 93, 103]), and *Boolean degree 1 functions* (cf. [58]) are all variations of intriguing sets that are actively being researched.

We have the following special case of Theorem 7.2.2 for intriguing sets of type i.

Corollary 7.3.1. *If $q^i_{ii} = 0$, then a non-trivial intriguing set of type i contains exactly half of the vertex set.*

Example 7.3.2. *The Clebsch graph, the Schläfli graph, and the complement of the Higman-Sims graph, are all strongly regular graphs, with parameters* $(16, 10, 6, 6)$, $(27, 16, 10, 8)$, *and* $(100, 77, 60, 56)$, *respectively. In each of these graphs,* $q^1_{11} = 0$, *hence by Corollary 7.3.1 intriguing sets of type* 1 *in these graphs consist of half of the vertices.*

Corollary 7.3.1 is the minimum that can be said about T-designs in a given scheme as a result of Theorem 7.2.2, if indeed Theorem 7.2.2 is able to say anything at all. Thus we are particularly interested in the existence of Krein parameters of the form q^i_{ii} which vanish.

Table 7.1 records the number of d-class association schemes for which the corresponding number of Krein parameters of the form q^i_{ii}, for $i \in \{1, \dots, d\}$, vanish. The table considers all association schemes of order n, for $n \in \{3, \dots, 34, 38\}$. This data is computed from Hanaki's database of all small non-thin homogeneous coherent configurations of the same orders [66], where we only consider those which are symmetric, and hence association schemes by our definition. The Krein parameter computations were done in the GAP [59] package, *AssociationSchemes* [9], which was developed in the course of this thesis. Indeed, the database of small homogeneous coherent configurations is included as a library in the package.

	Number of vanishing Krein parameters, $\lvert\{i \mid q^i_{ii} = 0, 1 \leqslant i \leqslant d\}\rvert$																							
d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	81	34	2																					
3	93	51	48	7																				
4	13	2082	22	50	18																			
5	11	9	46	23	108	21																		
6	3	2	8	8	3	313	85																	
7	2		3	3	10	1	61	576																
8								26	446															
9	1					3	1		27	77														
10										5	128													
11							1				9	69												
12												1	23											
13	1								1			2	40											
14															5									
15															3	11								
16																1	3							
17																		6						
18																								
19																				6				
20																								
21																								
22																								
23																								1

Table 7.1: Number of association schemes with d-classes and $\lvert\{i \mid q^i_{ii} = 0, 1 \leqslant i \leqslant d\}\rvert$ vanishing Krein parameters, for small order.

It is interesting to note that as the number of classes, d, increases, the number of Krein parameters of the form q^i_{ii} which vanish tends to d. It would be interesting to know if this is asymptotically true for all orders, or if the small order of the schemes relative to the number of classes is skewing the result. Given Theorem 7.3.1 this would have important consequences for intriguing sets in such schemes.

7.4 Vanishing Krein Parameters in Cometric, Q-Bipartite and Q-Antipodal Schemes

The results of the previous sections constrain either the strata or the size of Delsarte T-designs based on vanishing Krein parameters. It is natural, therefore, to consider special classes of association schemes, in which many of Krein parameters vanish.

Recall that a scheme is cometric with respect to the ordering $\{E_i\}_{i=0}^d$, if

1. $q^k_{ij} = 0$ if $i + j < k$ or $0 \leqslant k < |i - j|$,
2. $q^{i+j}_{ij} \neq 0$ for all i and j.

Thus such schemes have a large proportion of vanishing Krein parameters. Indeed, Terwilliger was interested in cometric schemes for this reason [115]. Many of the classical association schemes are both metric and cometric. Schemes which are cometric but not metric are relatively rare, hence the interest in the Penttila-Williford scheme. We consider this scheme further in Section 7.7. Moreover, most association schemes arising from geometries are cometric, such as the dual polar spaces, for example.

Corollary 7.4.1. *Let $(\Omega, \mathcal{R})$ be a cometric association scheme, and let θ be a subset of the vertices, such that*

$$\chi_\theta \in V_0 \mathop{\perp}_{i\in S} V_i \perp V_k,$$

where $S \subseteq \{1, \ldots, \lceil \frac{k}{2} \rceil - 1\}$ for some k. If $q^k_{kk} = 0$ then either

$$\chi_\theta \in V_0 \mathop{\perp}_{i\in S} V_i,$$

or $|\theta| = \frac{|\Omega|}{2}$.

An association scheme is *primitive* if all of its associate graphs, Γ_i, are connected. Otherwise it is called *imprimitive*. The following conjecture, due to Bannai and Ito, if true, would imply that, for sufficiently large diameter, most of the Krein parameters of a primitive distance-regular graph vanish.

Conjecture 7.4.2 (Bannai and Ito [16, p. 312])**.** *For sufficiently large diameter, a primitive association scheme is metric if and only if it is cometric.*

Note that even in a cometric scheme there is no guarantee that $q^i_{ii} = 0$ for any i. However, we can say a bit more, particularly in the case of imprimitive cometric association schemes.

For a cometric association scheme, define $a_i^* = q_{1i}^i$, $b_i^* = q_{1,i+1}^i$, and $c_i^* = q_{1,i-1}^i$. The *Krein array*, or *dual intersection array*, is

$$\{b_0^*, b_1^*, \ldots, b_{d-1}^*; c_1^*, c_2^*, \ldots, c_d^*\}.$$

Furthermore, $a_i^* + b_i^* + c_i^* = q_{ii}^0$.

An association scheme is *bipartite* if $p_{ij}^k = 0$ when $i+j+k$ is odd, and *antipodal* if $b_j = c_{d-j}$ for all j except possibly $j = \lfloor \frac{d}{2} \rfloor$. Dually, an association scheme is *Q-bipartite* if $q_{ij}^k = 0$ when $i+j+k$ is odd, and *Q-antipodal* if $b_j^* = c_{d-j}^*$ for all j except possibly $j = \lfloor \frac{d}{2} \rfloor$. Note that $a_i^* = 0$ for all i implies that an association scheme is Q-bipartite.

Example 7.4.3. *[122, 5.5.10] For $d \geqslant 2$, a distance-regular graph is called almost Q-bipartite if $a_i^* = 0$ for $i < d$ and $a_d^* = 0$. An almost Q-bipartite distance-regular graph is either the halved $(2d+1)$-cube, the folded $(2d+1)$-cube, or the collinearity graph of $\mathsf{DH}(2d-1, q^2)$. Since $a_1^* = q_{11}^1 = 0$, it follows from Corollary 7.3.1 that an intriguing set of type 1 consists of half the vertex set.*

We can classify imprimitive metric schemes into three types. This is achieved by extending an argument of Smith on distance-transitive graphs [110].

Theorem 7.4.4 ([22], Theorem 4.2.1)**.** *An imprimitive metric association scheme other than a cycle is bipartite, or antipodal, or both.*

Suzuki [113] proved that an imprimitive cometric association scheme is Q-bipartite, or Q-antipodal, or both Q-bipartite and Q-antipodal, or has either four or six classes. Cerzo and Suzuki [31] later showed that the exceptional four class case does not exist, followed by a similar non-existence result by Tanaka and Tanaka for the exceptional six-class case [114]. Thus we have a similar classification for imprimitive cometric schemes as for imprimitive metric schemes.

Theorem 7.4.5 ([31, 113, 114])**.** *An imprimitive cometric association scheme other than a cycle is Q-bipartite, or Q-antipodal, or both.*

The Q-bipartite condition is very strong, leading to the following result for constraining Delsarte designs.

Theorem 7.4.6. *Let $(\Omega, \mathcal{R})$ be a Q-bipartite association scheme, and let θ be a nontrivial subset of the vertices, such that*

$$\chi_\theta \in V_0 \perp_{i \in S} V_i,$$

where $S \subseteq \{1, \ldots, d\}$, such that i is odd for all $i \in S$. Then $|\theta| = \frac{n}{2}$.

Proof. Since i, j, k are odd for all $i, j, k \in S$, so too is $i + j + k$. Hence $q_{ij}^k = 0$ for all $i, j, k \in S$, as $(\Omega, \mathcal{R})$ is Q-bipartite. Assume for a contradiction that $|\theta| \neq \frac{n}{2}$. If we fix some $k \in S$, then

$$\chi_\theta \in V_0 \underset{i \in S \setminus \{k\}}{\perp} V_i,$$

by Theorem 7.2.2. Repeating this argument eventually constrains χ_θ to lie in $V_0 = \langle j \rangle$, but this is a contradiction, since θ is a non-trivial subset of the vertices. Thus $|\theta| = \frac{|\Omega|}{2}$. □

Example 7.4.7. *The* Taylor graphs *are distance-regular graphs with intersection array* $\{k, \mu, 1; 1, \mu, k\}$. *They are cometric with respect to two orderings, and are Q-bipartite (cf. [22, p431]). Hence by Theorem 7.4.6, intriguing sets of type* 1 *and* 3 *consist of half of the vertex set, as do* $\{2\}$*-Designs.*

d	Number of association schemes							
	Primitive			Imprimitive				
	Cometric		Not cometric	Cometric				Not cometric
	A or B	Neither A nor B		A and B	Only A	Only B	Neither A nor B	
2		55		17	45			
3		2	10	16	16	4		151
4			3	4			1	2177
5		1	2	2				213
6		1		1				420
7			1	1			1	653
8		1		1				470
9		1		1				107
10				1			1	131
11		1		1				77
12				1			1	22
13				1			1	42
14		1		1				3
15		1		1				12
16				1			1	2
17				1				5
18								
19				1				5
20								
21								
22								
23								1

Table 7.2: Association schemes of small order according to properties.

Table 7.2 separates the association schemes from [66], considered in Table 7.1, by the properties: primitive, cometric, Q-antipodal (denoted A), and Q-bipartite (denoted B). We also replicate Table 7.1 in Table 7.3, but only for primitive association schemes. The observation made for Table 7.1 that as the rank increases most of the relevant Krein parameters vanish seems to hold for both primitive and imprimitive schemes.

	Number of vanishing Krein parameters, $|\{i \mid q_{ii}^i = 0, 1 \leqslant i \leqslant d\}|$															
d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	52	2	1													
3	8			4												
4	2				1											
5	1				1	1										
6							1									
7								1								
8									1							
9										1						
10																
11												1				
12																
13																
14															1	
15																1

Table 7.3: Vanishing Krein parameters in primitive association schemes of small order.

7.5 Partial Geometries and Generalised Quadrangles

Partial geometries are a class of point-line geometries which we encountered in Section 3.3. They have strongly regular collinearity graphs, and hence we may easily compute their Krein parameters and apply Theorem 7.2.2.

Theorem 7.5.1. *Let Γ be a thick $\mathsf{pg}(s,t,\alpha)$. Then the Krein parameter q_{ii}^i is zero if and only if $i = 2$,*

$$\alpha = \frac{s^2 - t - 1 + \sqrt{t(t+1-s^2)}}{s-1},$$

and $t \geqslant s^2$. Moreover, in the positive case, $\alpha = 1$ if and only if $t = s^2$.

Proof. Note that $1 \leqslant \alpha \leqslant s+1, t+1$.

There are $(s+1)(st+\alpha)/\alpha$ points, and the matrix of eigenvalues for the collinearity graph (which is strongly regular) is:

$$P = \begin{bmatrix} 1 & s(t+1) & \frac{st}{\alpha}(s+1-\alpha) \\ 1 & s-\alpha & \alpha - s - 1 \\ 1 & -t-1 & t \end{bmatrix}.$$

Let us calculate the q_{ii}^i:

$$q_{11}^1 = 1 + \frac{(s-\alpha)^3}{(s(t+1))^2} + \frac{(\alpha-s-1)^3}{(\frac{st}{\alpha}(s+1-\alpha))^2},$$
$$q_{22}^2 = 1 + \frac{(-t-1)^3}{(s(t+1))^2} + \frac{t^3}{(\frac{st}{\alpha}(s+1-\alpha))^2}.$$

Case $q_{11}^1 = 0$:

If $\alpha = s+1$, then
$$q_{11}^1 = 1 - \frac{1}{s^2(t+1)^2}$$
which is clearly nonzero. So assume $\alpha < s+1$. Then we can cross-multiply and take the numerator:

$$(s(t+1))^2(\tfrac{st}{\alpha}(s+1-\alpha))^2 + (s-\alpha)^3(\tfrac{st}{\alpha}(s+1-\alpha))^2 + (\alpha-s-1)^3(s(t+1))^2 = 0.$$

Since $s, t, \alpha > 0$,

$$0 = (s^2(t+1)^2 + (s-\alpha)^3)t^2 + (\alpha-s-1)(t+1)^2\alpha^2$$

and hence
$$(s^2(t+1)^2 + (s-\alpha)^3)t^2 = (s+1-\alpha)(t+1)^2\alpha^2.$$
Now $0 \leqslant (s-\alpha)^3$ and so

$$s^2(t+1)^2t^2 \leqslant (s+1-\alpha)(t+1)^2\alpha^2.$$

Hence,
$$s^2t^2 \leqslant (s+1-\alpha)\alpha^2 \leqslant (s+1-\alpha)(t+1)^2$$
since $\alpha \leqslant t+1$. So
$$s^2 \leqslant (s+1-\alpha)(\tfrac{t+1}{t})^2 < s(\tfrac{t+1}{t})^2$$
since $\alpha \geqslant 1$. Thus

$$s < (\tfrac{t+1}{t})^2 = \left(1 + \frac{1}{t}\right)^2 \leqslant \left(\frac{3}{2}\right)^2 = 2 + \frac{1}{4}.$$

Since $s \geqslant 2$, we have $s = 2$. When $t > 2$, we have $(1 + \frac{1}{t})^2 < 2$; a contradiction. Therefore, $t = 2$ and $\alpha = 1$ because $0 < \alpha < s+1$. We can now substitute in our values for s, t, α to compute $q_{11}^1 = 65/72 \neq 0$.

Case $q_{22}^2 = 0$:

If $\alpha = s+1$, then the second relation is empty, so we assume $\alpha < s+1$. As before, we cross-multiply and take the numerator:

$$(s(t+1))^2(\tfrac{st}{\alpha}(s+1-\alpha))^2 - (t+1)^3(\tfrac{st}{\alpha}(s+1-\alpha))^2 + t^3(s(t+1))^2 = 0.$$

Since $s, t, \alpha > 0$,

$$0 = s^2(s+1-\alpha)^2 - (t+1)(s+1-\alpha)^2 + t\alpha^2 = (s^2-t-1)(s+1-\alpha)^2 + t\alpha^2$$

and hence

$$\alpha = \frac{s^2-t-1 \pm \sqrt{t(t+1-s^2)}}{s-1}.$$

Now for α to be a real number, it is necessary that $t+1 \geqslant s^2$. If $t+1 = s^2$, then $\alpha = 0$ (a contradiction). Therefore, $t \geqslant s^2$. Furthermore, $t \geqslant s^2$ implies that $\sqrt{t(t+1-s^2)} \geqslant \sqrt{t} \geqslant s \geqslant 1$ and $s^2-t-1 \leqslant -1$. Now, $\alpha > 0$ and so

$$\alpha = \frac{s^2-t-1 + \sqrt{t(t+1-s^2)}}{s-1}.$$

Moreover,

$$\begin{aligned} & \alpha = 1, \\ \iff & s-1 = s^2-t-1-\sqrt{t(t+1-s^2)}, \\ \iff & t(t+1-s^2) = (s^2-t-s)^2, \\ \iff & t^2+t-s^2t = s^4-2s^2t-2s^3+t^2+2st+s^2, \\ \iff & t(s^2-2s+1) = s^2(s^2-2s+1), \\ \iff & t = s^2. \end{aligned}$$

□

By Corollary 7.3.1, Theorem 7.5.1 yields a result on hemisystems of partial geometries.

Corollary 7.5.2. *Let Γ be a $\mathsf{pg}(s, t, \frac{s^2-t-1+\sqrt{t(t+1-s^2)}}{s-1})$, where $s, t > 1$. Then any nontrivial proper m-ovoid of Γ is a hemisystem.*

As a direct consequence of this result, we reobtain a theorem by Cameron, Goethals and Seidel [29].

Corollary 7.5.3. *Let Γ be a $\mathsf{GQ}(s, s^2)$, where $s > 1$. Then any nontrivial proper m-ovoid of Γ is a hemisystem.*

For $\alpha = 2$, the only putative partial geometries with vanishing Krein parameters are $\mathsf{pg}(4, 27, 2)$, $\mathsf{pg}(5, 32, 2)$, and $\mathsf{pg}(7, 54, 2)$ [88]. Makhnev showed that there is no partial geometry $\mathsf{pg}(5, 32, 2)$ [88] while Östergård and Soicher showed that there is no partial geometry $\mathsf{pg}(4, 27, 2)$ [96], leaving the only open case for $\alpha = 2$ being $\mathsf{pg}(7, 54, 2)$.

The partial geometry $\mathsf{pg}(6, 80, 3)$ induces $\mathsf{pg}(5, 32, 2)$ in the neighbourhood of a vertex, and so there is no partial geometry $\mathsf{pg}(6, 80, 3)$ [88]. The smallest open case for a partial geometry with $\alpha = 3$ and a vanishing Krein parameter is thus $\mathsf{pg}(7, 75, 3)$. For $\alpha \geqslant 4$, the existence problem of partial geometries with $q_{22}^2 = 0$ appears to be completely open.

7.6 Dual Polar Spaces

We present here an alternative proof of Theorem 5.2.1 which utilises Theorem 7.2.2. Theorem 5.2.1 was originally given as a consequence of Theorem 5.2.2 which relied upon various combinatorial counting arguments.

Theorem 5.2.1
The only nontrivial m-ovoids that exist in $\mathsf{DQ}(2d,q)$, $\mathsf{DW}(2d-1,q)$ and $\mathsf{DH}(2d-1,q^2)$, for $d \geqslant 3$, are hemisystems (i.e., $m=(q+1)/2$).

Proof. The matrix of eigenvalues for $\mathsf{DH}(5,q^2)$, $\mathsf{DQ}(6,q)$, and $\mathsf{DW}(5,q)$, computed using Theorem 3.4.1, are

$$P_{\mathsf{DH}(5,q^2)} = \begin{pmatrix} 1 & q(q^4+q^2+1) & q^4(q^4+q^2+1) & q^9 \\ 1 & q^3+q-1 & q(q^3-q^2-1) & -q^4 \\ 1 & -q^2+q-1 & -q(q^2-q+1) & q^3 \\ 1 & -q^4-q^2-1 & q^2(q^4+q^2+1) & -q^6 \end{pmatrix},$$

and

$$P_{\mathsf{DQ}(6,q)} = P_{\mathsf{DW}(5,q)} = \begin{pmatrix} 1 & q(q^2+q+1) & q^3(q^2+q+1) & q^6 \\ 1 & q^2+q-1 & q(q^2-q-1) & -q^3 \\ 1 & -1 & -q^2 & q^2 \\ 1 & -q^2-q-1 & q(q^2+q+1) & -q^3 \end{pmatrix}.$$

Computing the Krein parameters with Theorem 2.3.11 yields $q_{11}^1 > 0$, $q_{22}^2 > 0$, and $q_{33}^3 = 0$ in each instance.

Now by Lemma 3.4.3, m-ovoids of $\mathsf{DH}(5,q^2)$, $\mathsf{DQ}(6,q)$, and $\mathsf{DW}(5,q)$ are a intriguing sets of type 3, and so by Corollary 7.3.1 the only non-trivial m-ovoids of $\mathsf{DH}(5,q^2)$, $\mathsf{DQ}(6,q)$, and $\mathsf{DW}(5,q)$ are hemisystems.

We could continue considering Krein parameters for $d > 3$ to prove the remainder of the theorem, however we instead opt to simply apply a quotienting argument, and the result follows from Lemma 5.1.3. $\square$

It is also worth noting here that nothing more can be said about designs in $\mathsf{DW}(5,q)$ than can be said about $\mathsf{DQ}(6,q)$ from a purely algebraic perspective, since the intersection numbers are the same for both. In Section 5.3, we saw computationally for small q that $\mathsf{DW}(5,q)$ has no hemisystems, while $\mathsf{DQ}(6,q)$ does have hemisystems. To disprove the existence of hemisystems in $\mathsf{DW}(5,q)$ additional combinatorial or geometric attributes need to be considered.

7.7 The Penttila-Williford Scheme

Since the Penttila-Williford scheme is Q-bipartite, we have the immediate consequence of Corollary 7.4.6.

Lemma 7.7.1. *Let θ be a non-trivial intriguing set of type 1, intriguing set of type 3, or a $\{2,4\}$-design of the Penttila-Williford association scheme. Then $|\theta| = \frac{|\Omega|}{2}$.*

Bamberg and Metsch recently described the intriguing sets of the Penttila-Williford scheme arising from $\mathsf{Q}^-(5,q)\backslash\mathsf{Q}(4,q)$, $q > 2$ [15]. They show an intriguing set of type 1 or or type 3 must consist of exactly half of the vertices of the scheme by considering a special central collineation of $\mathsf{PG}(5,q)$. Since their proof relies on a geometric property, it only holds for the Penttilla-Williford scheme arising from $\mathsf{Q}^-(5,q)\backslash\mathsf{Q}(4,q)$, however Lemma 7.7.1 uses only algebraic arguments and hence generalises the result of Bamberg and Metsch to all Penttila-Williford schemes.

The following result of Bamberg and Lee follows as a corollary of Lemma 7.7.1 and the fact that such an m-cover is an intriguing set of type 1 [13, Corollary 3.4].

Theorem 7.7.2 ([13])**.** *Given a generalised quadrangle of order (q^2,q) containing a doubly subtended quadrangle of order (q,q), a non-trivial relative m-cover is a relative hemisystem, and q is even.*

We will consider the Penttila-Williford scheme again in Section 8.3, where we apply Theorem 7.2.2. This instance differs from most of the examples considered so far, in that we only partially constrain the strata of a Delsarte design, and yet this turns out to have more far reaching consequences than one might expect.

Chapter 8

Witnesses to the Non-Existence of Delsarte Designs

We shall use the term *witness* loosely to mean some object or vector which can be shown to exist, and is itself in some way incompatible with a hypothetical design. Thus the difficulty of proving nonexistence of the design may be transferred to showing the existence of a suitable witness. Generally a few additional arguments are needed to show that the witness is in fact incompatible with a hypothetical design. For example, an antidesign may be constructed, the size of the intersection with the putative design found with Theorem 2.4.4, and then a counting argument employed to find a contradiction. Unfortunately this method often leads to tautological arguments, where counting yields no contradictions, nor further restrictions.

The properties of designs and antidesigns are really properties inherited from the strata in which they lie. Given a $T \subseteq \{1, \ldots, d\}$, for convenience we shall refer to the strata

$$\perp_{i \in \{0,\ldots,d\} \setminus T} V_i$$

as the *design strata* and

$$\perp_{i \in T} V_i$$

as the *antidesign strata*. We exclude the V_0 eigenspace from the definition of the antidesign strata, since it is common to the the design strata. Hence a design is orthogonal to every vector in the antidesign strata. If a design does not exist, then there is no $\{0, 1\}$-vector in the design strata corresponding to its characteristic vector. To show this nonexistence, we explore two directions in this chapter.

The first direction considers the design strata. If no suitable $\{0, 1\}$-vector exists in the strata, then there is something about the design strata itself which is fundamentally incompatible with such a design. Thus we attempt to find some incompatible vector from within the design strata, and which therefore acts as a witness to the nonexistence of the design.

The second direction considers the antidesign strata. We see that if a design does not exist, then its existence must be incompatible in some way with the antidesign strata. In fact, the entirety of the antidesign strata is often overkill. We ask the question: just how incompatible are the antidesign strata with such a hypothetical design? We attempt to answer this question by considering submodules of the design strata.

In each case we illustrate the potential of these techniques by providing examples.

8.1 Witnesses in the Design Strata

Submodules, and in particular, irreducible submodules, are very important in understanding Delsarte designs. A G-submodule is irreducible if it contains no smaller G-submodule, and we shall denote the G-submodule spanned by v^G as $\langle v^G \rangle$. We refer to [71] for more on submodules. The following theorem provides a useful fact about the strata of a Schurian scheme.

Theorem 8.1.1 (cf. [28, §6])**.** *Let G act generously transitively on Ω, then $R^{|\Omega|}$ decomposes into $d+1$ irreducible orthogonal G-submodules corresponding to the strata of the Schurian scheme afforded by G.*

Example 8.1.2. *The association schemes from the classical dual polar spaces are Schurian with respect to the collineation group of the polar space acting on the maximals. Hence the strata of the dual polar space schemes are irreducible.*

The antidesign strata are essentially characterised by being orthogonal to every T-design. Thus in showing non-existence of a putative T-design, we seek to show that there exists some vector in the design strata which *behaves* as if it were in the antidesign strata. That is, every putative T-design with properties of interest must be orthogonal to part of its own design strata, meaning it cannot exist. It simply "doesn't fit".

The following theorem says that if such a "design incompatible" vector exists in the design strata, then the design must be constrained to a smaller subset of strata. That is, if the antidesign space encroaches even slightly upon any of the eigenspaces of the design strata, then it must contain the entirety of that eigenspace. Clearly if the strata are completely constrained, then a non-trivial design does not exist.

Theorem 8.1.3. *Let $(\Omega, \mathcal{R})$ be a d-class association scheme and let G be the permutation representation of* $\mathrm{Aut}\big((\Omega,\mathcal{R})\big)$ *acting upon Ω. Let $S \subseteq \{1,\dots,d\}$ and*

$$\Theta = \{\theta \subseteq \Omega : \chi_\theta \in V_0 \mathop{\perp}_{i\in S} V_i \text{ and property } \delta \text{ is satisfied}\},$$

for some G-invariant property δ[1]*. If V_k is an irreducible G-submodule for some $k \in S$, and there exists $0 \neq v \in V_k$ such that $\chi_\theta \cdot v = 0$ for all $\theta \in \Theta$, then*

$$\chi_\theta \in V_0 \mathop{\perp}_{i \in S\setminus\{k\}} V_i,$$

for all $\theta \in \Theta$.

[1] That is, δ is a union of G-orbits in the action of G on the power set of Ω.

Proof. Observe that

$$V_0 \underset{i\in S}{\perp} V_i$$

is G-invariant, and since δ is G-invariant

$$\begin{aligned} & v \cdot \chi_\theta = 0 \quad \forall \theta \in \Theta \\ \iff & v^g \cdot \chi_\theta^g = 0 \quad \forall \theta \in \Theta, \quad \forall g \in G \\ \iff & v^g \cdot \chi_\theta = 0 \quad \forall \theta \in \Theta, \quad \forall g \in G. \end{aligned}$$

Now $\langle v^G \rangle = V_k$ since V_k is an irreducible G-submodule, meaning that χ_θ is orthogonal to V_k for all $\theta \in \Theta$. □

The property δ could be vacuously true, in which case Θ is the set of all $\{1, \ldots, d\} \backslash S$-designs. Or it could be more sophisticated, typically being a constraint on the size of θ, for example $|\theta| \neq \frac{|\Omega|}{2}$. Note also that δ must be G-invariant, since we want to show that v is orthogonal to every χ_θ for θ of a given type, and not simply a subset of the designs. Thus Θ amounts to a union of G-orbits.

A crucial observation is that v need not constrain all $\{1, \ldots, d\} \backslash S$-designs, but only those satisfying δ. This is intended to enable one to find targeted witnesses. In this way we can show that V_k acts as if it belongs to the antidesign space for certain designs, and not for others. This allows one to differentiate between, for example, hemisystems and all other m-ovoids in a polar space. This is intended as an improvement upon antidesigns which have the same properties for all putative T-designs.

If V_k is not irreducible, then $\langle v^G \rangle \neq V_k$ and so it is still possible that $\chi_\theta E_k \neq 0$. Thus this result is largely a consequence of the irreducibility of V_k. Many schemes arising in geometry satisfy this condition. Indeed, many interesting schemes are Schurian, and hence are afforded by a generously transitive group, which means that each of their eigenspaces are irreducible G-modules, by Theorem 8.1.1.

Remark 8.1.4. *Theorem 8.1.3 can be extended to witnesses with components in multiple strata, however care must be taken in handling the possibility of "diagonal type"*[2] *G-submodules. In Schurian schemes, the strata are multiplicity free G-submodules, and hence in this situation the "diagonal type" G-submodules are not a concern.*

Example 8.1.5. *The following example is somewhat trivial, however it illustrates the purpose of Theorem 8.1.3, and the same ideas extend to non-trivial uses. We know that the characteristic vector of an m'-cover of $\mathsf{Q}(6, q)$ lies in $V_0 \perp V_2 \perp V_3$ (cf. Lemma 3.4.4). Moreover, we know that an m-ovoid θ of $\mathsf{DQ}(6, q)$ is an m'-cover of $\mathsf{Q}(6, q)$, with the added condition that each line of $\mathsf{Q}(6, q)$ meets exactly m elements of θ', and so we consider*

$$\Theta = \{\theta \subseteq \Omega : \chi_\theta \in V_0 \perp V_2 \perp V_3 \textit{ and } \chi_\theta \cdot \chi_{[\ell]} = m \textit{ for all lines } \ell\}.$$

[2] If $\varphi : U_1 \to U_2$ is an isomorphism of G-submodules, then $\{(u, \phi(u)) : u \in U_1\}$ is a diagonal G-submodule.

Taking any line ℓ, and observing that $\chi_{[\ell]} \in V_0 \perp V_1 \perp V_2$ by Theorem 3.4.2, then $(E_2\chi_{[\ell]}) \cdot \chi_\theta = 0$ for any $\theta \in \Theta$. We discover that $\chi_\theta \in V_0 \perp V_3$ by applying Theorem 8.1.3. Of course, we knew this to begin with (cf. Lemma 3.4.3).

8.2 Finding Witnesses by Projection and Inclusion

In the previous section we saw that we could constrain the strata in which a design lies if we can show the existence of a vector in the design strata which is orthogonal to the design. The question remains of how to find such vectors. In this section we introduce a technique of projecting strata from one association scheme to another and then including a suitable vector back into the original scheme. We shall fix the following notation for the remainder of this chapter.

Let $(\Omega, \mathcal{R})$ be a d-class association scheme and let $(\Delta, \mathcal{S})$ be an e-class association scheme such that $\Delta \subset \Omega$, $d \leqslant e$ and for all $i \in \{1, \ldots, e\}$, $S_i \subset R_j$, for some $j \in \{1, \ldots, d\}$. That is, the vertices of the second association scheme are a subset of the vertices of the first, and the relations of the second scheme are formed by splitting the relations of the first scheme on their restriction to this subset of vertices.

Let $A_0, \ldots, A_d$, $E_0, \ldots, E_d$ and $V_0, \ldots, V_d$ be the adjacency matrices, minimal idempotents, and strata of $(\Omega, \mathcal{R})$, respectively, and let $B_0, \ldots, B_e$, $F_0, \ldots, F_e$, and $W_0, \ldots, W_e$ be the adjacency matrices, minimal idempotents, and strata of $(\Delta, \mathcal{S})$, respectively.

Let M be the $|\Delta| \times |\Omega|$ inclusion matrix for Δ in Ω. Let $\overline{A_i} = MA_iM^\top$ and $\overline{E_i} = ME_iM^\top$. Note that if Ω is ordered such that the first $|\Delta|$ elements are the elements of Δ, then we would have the following block decomposition of the adjacency matrices and minimal idempotents

$$A_i = \left(\begin{array}{c|c} \overline{A_i} & * \\ \hline * & * \end{array}\right), \quad E_i = \left(\begin{array}{c|c} \overline{E_i} & * \\ \hline * & * \end{array}\right). \tag{8.1}$$

Assuming such an ordering may help with visualising the behaviour of the two schemes relative to one another.

Now, since for all $i \in \{1, \ldots, e\}$, $S_i \subset R_j$, for some $j \in \{1, \ldots, d\}$, there exists some partition $\Pi = \Pi_0 \cup \ldots \cup \Pi_d$ of $\{0, \ldots, e\}$ such that

$$\overline{A_i} = \sum_{j \in \Pi_i} B_j,$$

and we can also define an onto function, $\varphi : \{0, \ldots, e\} \to \{0, \ldots, d\}$ such that $\varphi(i) = j$ if and only if $i \in \Pi_j$. Note that φ simply identifies which $\overline{A_j}$ satisfies $\overline{A_j} \circ B_i = B_i$.

Example 8.2.1. *One method of constructing such a pair of schemes, would be to take a scheme $(\Omega, \mathcal{R})$ and some subgroup $H \leqslant \mathrm{Aut}\big((\Omega, \mathcal{R})\big)$ such that H acts* *generously transitively on an H orbit of Ω, which we take to be Δ. The Schurian scheme afforded by H would then produce a suitable $(\Delta, \mathcal{S})$. Suitable fusions of the relations of $\mathcal{S}$ may yield further examples.*

Let $\pi : \mathbb{R}^{|\Omega|} \to \mathbb{R}^{|\Delta|}$ be a projection map from the larger vector space into the smaller, and let $\iota : \mathbb{R}^{|\Delta|} \to \mathbb{R}^{|\Omega|}$ be the natural inclusion map. If Ω is ordered as in (8.1), then $\pi(v) = vM^\top$ and $\iota(w) = wM$, in which case π would truncate a vector of $\mathbb{R}^{|\Omega|}$ after the first $|\Delta|$ entries, while ι would append $|\Omega| - |\Delta|$ zeros to a vector of $\mathbb{R}^{|\Delta|}$.

Theorem 8.2.2. *For $(\Delta, \mathcal{S})$ and $(\Omega, \mathcal{R})$ as defined above, then there exist λ_h such that*

$$\overline{E_j} = \sum_{h=0}^{e} \lambda_h F_h.$$

Moreover, let P be the matrix of eigenvalues of $(\Delta, \mathcal{S})$ and Q be the matrix of dual eigenvalues of $(\Omega, \mathcal{R})$, $\lambda = (\lambda_0, \ldots, \lambda_e)$ and y the vector with entries $y_i = Q_{\varphi(i),j}$, then

$$\lambda = \frac{Py}{|\Omega|}.$$

Proof. Let C be the matrix of dual eigenvalues of $(\Delta, \mathcal{S})$. Now,

$$E_j = \frac{1}{|\Omega|} \sum_{i=0}^{d} Q_{ij} A_i$$

and so

$$\overline{E_j} = \frac{1}{|\Omega|} \sum_{k=0}^{d} Q_{kj} \overline{A_k} = \frac{1}{|\Omega|} \sum_{i=0}^{e} Q_{\varphi(i)j} B_i.$$

Similarly,

$$F_h = \frac{1}{|\Delta|} \sum_{i=0}^{e} C_{ih} B_i.$$

Let

$$\overline{E_j} = \sum_{h=0}^{e} \lambda_h F_h,$$

then

$$\frac{1}{|\Omega|} \sum_{i=0}^{e} Q_{\varphi(i)j} B_i = \frac{1}{|\Delta|} \sum_{h=0}^{e} \lambda_h \left(\sum_{i=0}^{e} C_{ih} B_i \right),$$

and so we have the following equation for each B_i,

$$\frac{1}{|\Omega|} Q_{\varphi(i)j} B_i = \frac{1}{|\Delta|} \sum_{h=0}^{e} \lambda_h C_{ih} B_i.$$

To solve this we need only solve

$$\frac{|\Delta|}{|\Omega|} Q_{\varphi(i)j} = \sum_{h=0}^{e} \lambda_h C_{ih}.$$

Thus we have a system of $e+1$ linear equations corresponding to $i \in \{0, \ldots, e\}$, which can be expressed in the form $C\lambda = \frac{|\Delta|}{|\Omega|} y$, where $y_i = Q_{\varphi(i)j}$. Since C is invertible, there is always a unique solution. Indeed, $C^{-1} = \frac{1}{|\Delta|} P$ and so

$$\lambda = \frac{Py}{|\Omega|}. \qquad \square$$

Corollary 8.2.3. *Let*

$$\overline{E_j} = \sum_{i=0}^{e} \lambda_i F_i,$$

then for $S = \{i \mid \lambda_i \neq 0\}$,

$$\pi(V_j) = \perp_{i \in S} W_i.$$

Proof. Since V_j is the rowspace of E_j, it follows that $\pi(V_j)$ is the rowspace of $\overline{E_j}$. Now, if $w \in W_i$ for $i \in S$, then $wF_i = w$ and $wF_k = 0$ for $k \neq i$. Hence W_i is contained in $\pi(V_j)$. $\square$

Theorem 8.2.4. *Let $\theta \subseteq \Omega$ such that*

$$\chi_\theta \in V_0 \perp V_k.$$

Let

$$\pi(V_k) = \perp_{i \in S} W_i,$$

for $S \subseteq \{0, \ldots, e\}$. If $\pi(\chi_\theta)F_i = 0$ for some $i \in S \backslash \{0\}$, then $\iota(w)E_k \cdot \chi_\theta = 0$ for any $w \in W_i$.

Proof. Take $w \in W_i$. Since $\pi(\chi_\theta)F_i = 0$, it follows that $w \cdot \pi(\chi_\theta) = 0$, and so $\iota(w) \cdot \chi_\theta = 0$.

Now, $w \cdot j_W = 0$ implies that $\iota \cdot j_V = 0$. Hence $\iota(w)E_0 = 0$, and so $\iota(w) = \iota(w)E_k + u$, where $u \notin V_0 \perp V_k$. Since $u \notin V_0 \perp V_k$ it follows that $u \cdot \chi_\theta = 0$, and moreover, since $\iota(w) \cdot \chi_\theta = 0$ we have that $\iota(w)E_k \cdot \chi_\theta = 0$.

Lastly we show that $\iota(w)E_k$ is non-trivial. There exists $w' \in W_i$ such that $w \cdot w' \neq 0$, or else w is orthogonal to every vector in W_i and hence cannot be itself in W_i, a contradiction. Since $W_i \subseteq \pi(V_k)$, there exists some vector $\iota(w') + u \in V_k$, where $u \in \mathrm{K}ern(\pi)$, such that $\pi(\iota(w') + u) = w'$. Now $\iota(w) \cdot (\iota(w') + u) \neq 0$, since $w \cdot w' \neq 0$ and $\iota(w) \cdot u = 0$. Hence $\iota(w)E_k \neq 0$.

Thus $\iota(w)E_k$ is a non-trivial vector lying in V_k which is orthogonal to χ_θ. $\square$

Clearly nonexistence of $\pi(\chi_\theta)$ implies nonexistence of χ_θ. However Theorem 8.2.4 shows that nonexistence of $\pi(\chi_\theta)$ is excessive. Instead, it is sufficient to simply constrain $\pi(\chi_\theta)$ slightly to show nonexistence of χ_θ.

We may combine Theorem 8.1.3 and Theorem 8.2.4 in order to prove nonexistence of all θ satisfying property δ. If δ is a constraint on $|\theta|$, then it is helpful to take Δ to be an antidesign of θ, so that $|\Delta \cap \theta|$ is constant by Theorem 2.4.4.

8.3 On m-Ovoids of $\mathsf{Q}^-(5,q)$

To illustrate the theory in this chapter, we reprove a result of Segre.

Theorem 8.3.1 ([108])**.** *An m-ovoid of $\mathsf{Q}^-(5,q)$ is such that $m = \frac{q+1}{2}$.*

We provide a proof shortly. Note that reproving Theorem 8.3.1 achieves very little in and of itself. Indeed, Theorem 8.3.1 follows from Corollary 7.5.3 or from Theorem 7.2.2, since $q_{22}^2 = 0$. Instead, it provides value as an illustration for the techniques of this chapter.

Recall that the matrix of eigenvalues for the *Penttila-Williford scheme* is

$$P = \begin{pmatrix} 1 & (q-1)(q^2+1) & (q^2-2q)(q^2+1) & (q-1)(q^2+1) & 1 \\ 1 & q^2+1 & 0 & -(q^2+1) & -1 \\ 1 & q-1 & -2q & q-1 & 1 \\ 1 & -q+1 & 0 & q-1 & -1 \\ 1 & -(q-1)^2 & 2q(q-2) & -(q-1)^2 & 1 \end{pmatrix}.$$

We can use Krein parameters to restrict a non-trivial $\{2,3\}$-design to $V_0 \perp V_4$ if it does not consist of exactly half of the vertices of the scheme.

Lemma 8.3.2. *Let θ be a non-trivial $\{2,3\}$-design of the Penttila-Williford association scheme. Then either $|\theta| = \frac{|\Omega|}{2}$, or θ is a $\{1,2,3\}$-design.*

Proof. Using Theorem 2.3.11 to compute Krein parameters yields

$$q_{11}^1 = q_{14}^1 = q_{41}^4 = q_{44}^1 = 0.$$

The result then follows from Theorem 7.2.2. □

Lemma 8.3.2 cannot be improved upon by using Krein parameters, since $q_{44}^4 \neq 0$. This is an example where we are more interested in constraining the strata than in the size of the object.

We now prove Theorem 8.3.1.

Proof. Let $(\Omega, \mathcal{R})$ be the standard association scheme on the points of $\mathsf{Q}^-(5,q)$, and let $(\Delta, \mathcal{S})$ be the Penttila-Williford scheme with Δ equal to the points of $\mathsf{Q}^-(5,q)$ which are not in $\mathsf{Q}(4,q)$.

Observe that a $\frac{q+1}{2}$-ovoid, or hemisystem, of $\mathsf{Q}^-(5,q)$ must have size $\frac{|\Omega|}{2}$, and for $m \neq \frac{q+1}{2}$, the size of an m-ovoid is not equal to $\frac{|\Omega|}{2}$. Let

$$\Theta = \left\{\theta \subseteq \Omega : \chi_\theta \in V_0 \perp V_2, |\theta| \neq \frac{|\Omega|}{2}\right\}.$$

That is, Θ is the set of all m-ovoids which are not hemisystems. In this case, we have some property δ which is satisfied by θ if $|\theta| \neq \frac{|\Omega|}{2}$.

Now, $\overline{A_0} = B_0$, $\overline{A_2} = B_1 + B_2 + B_4$, and $\overline{A_1} = B_3$. Hence

$$\varphi : \{0,1,2,3,4\} \to \{0,1,2\}$$

is such that

$$\varphi(i) = \begin{cases} 0 & i = 0, \\ 1 & i = 3, \\ 2 & i = 1,2,4. \end{cases}$$

Note that $\mathsf{Q}^-(5,q)$ has $(q+1)(q^3+1)$ points, and matrix of dual eigenvalues

$$Q = \begin{pmatrix} 1 & q^3+q & q^4 \\ 1 & q-1 & -q \\ 1 & -q^2-1 & q^2 \end{pmatrix}.$$

By Theorem 8.2.2,

$$\begin{aligned}
\overline{E_0} &= \frac{q^2(q-1)}{q^3+1} F_0 \\
\overline{E_1} &= \frac{q^2+1}{q^3+1} F_0 + F_2 + F_3 + \frac{2}{q+1} F_4 \\
\overline{E_2} &= F_1 + \frac{q-1}{q+1} F_4,
\end{aligned}$$

and so by Corollary 8.2.3

$$\pi(V_0 \perp V_2) = W_0 \perp W_1 \perp W_4.$$

Thus $\pi(\chi_\theta)$ is a $0,1$-vector in $W_0 \perp W_1 \perp W_4$ for all $\theta \in \Theta$, and hence a $\{2,3\}$-design of $(\Delta, \mathcal{S})$.

The point set of $\mathsf{Q}(4,q)$ is an intriguing set (see Example 3.3.4), and hence so too is its complement, Δ. By Theorem 2.4.4,

$$|\theta \cap \Delta| = \frac{|\theta||\Delta|}{|\Omega|} \neq \frac{|\Delta|}{2},$$

since $|\theta| \neq \frac{|\Omega|}{2}$.

Now since $|\theta \cap \Delta| \neq \frac{|\Delta|}{2}$, it follows from Lemma 8.3.2 that

$$\pi(\chi_\theta) \in W_0 \perp W_4.$$

It now follows from Theorem 8.2.4 that $\iota(w)E_2 \cdot \chi_\theta = 0$ for all $\theta \in \Theta$ and any $w \in W_1$. Hence by Theorem 8.1.3 $\chi_\theta \in V_0$ for all $\theta \in \Theta$, and so there exist no nontrivial m-ovoids which are not hemisystems. □

By way of remark, we note here that Bamberg and Metsch [15] also compute the the projection of V_2, but in an ad-hoc manner. Theorem 8.2.2 presents a more general and complete technique for projecting strata. The technique involving Theorem 8.2.4 and Theorem 8.1.3 is novel.

8.4 Spreads of $\mathsf{H}(4,4)$

Famously, there is no spread of $\mathsf{H}(4,4)$ (cf. [99, Theorem 3.1.4]). This result was first found computationally by Brouwer, though as of yet no theoretical proof is known. The question of existence remains open for $\mathsf{H}(4,q^2)$ for $q > 2$.

We explore here the methods of the previous in relation to this problem. Unfortunately a proof remains beyond reach, but the methods presented here hold potential for future progress in this direction. Moreover, they are illustrative of the versatility of the approach, and demonstrate the option of using different methods for restricting the strata of a design projected from $(\Omega, \mathcal{R})$ to $(\Delta, \mathcal{S})$.

For our current purposes we consider only $\mathsf{H}(4,4)$. For larger q, a fusion of relations is required to construct $(\Delta, \mathcal{S})$. Let $(\Omega, \mathcal{R})$ be the association scheme corresponding to $\mathsf{H}(4,4)$ and let $G \cong P\Gamma U(5,4)$ be the automorphism group of $\mathsf{DH}(4,4)$ acting upon the line set, Ω, such that $(\Omega, \mathcal{R})$ is the Schurian scheme afforded by G.

Take a line ℓ of $\mathsf{H}(4,4)$, and let $\Delta = \{\ell' : (\ell', \ell) \in R_2\}$, that is, the set of lines which do not intersect ℓ. Now G_ℓ acts generously transitively on Δ. Let $(\Delta, \mathcal{S})$ be the association scheme afforded by G_ℓ acting on Δ.

The matrix of eigenvalues $(\Delta, \mathcal{S})$ and the matrix of dual eigenvalues of $(\Omega, \mathcal{R})$ are, respectively,

$$P = \begin{pmatrix} 1 & 90 & 10 & 30 & 5 & 120 \\ 1 & -6 & 10 & -2 & 5 & -8 \\ 1 & 18 & 2 & 6 & -3 & -24 \\ 1 & -6 & 2 & -2 & -3 & 8 \\ 1 & -6 & -2 & 6 & 1 & 0 \\ 1 & 6 & -2 & -6 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 120 & 176 \\ 1 & 21 & -22 \\ 1 & -15/4 & 11/4 \end{pmatrix}.$$

Now, $\overline{A_0} = B_0$, $\overline{A_1} = B_3 + B_4$, and $\overline{A_2} = B_1 + B_2 + B_5$. Hence

$$\varphi : \{0,1,2,3,4,5\} \to \{0,1,2\}$$

is such that

$$\varphi(i) = \begin{cases} 0 & i = 0, \\ 1 & i = 3,4, \\ 2 & i = 1,2,5. \end{cases}$$

By Theorem 8.2.2,

$$\begin{aligned} \overline{E_0} &= \frac{256}{297} F_0 \\ \overline{E_1} &= \frac{10}{99} F_0 + \frac{2}{3} F_1 + \frac{2}{3} F_2 + F_4 \\ \overline{E_2} &= \frac{1}{27} F_0 + \frac{1}{3} F_1 + \frac{1}{3} F_2 + F_3 + F_5 \end{aligned}$$

and so by Corollary 8.2.3

$$\pi(V_0 \perp V_2) = W_0 \perp W_1 \perp W_2 \perp W_3 \perp W_5.$$

Let θ be a spread of $\mathsf{H}(4,4)$, then it consists of 33 pairwise skew lines. Moreover, $\chi_\theta \in V_2$, and so $\pi(\chi_\theta)$ is a $\{0,1\}$-vector in $W_0 + W_1 + W_2 + W_3 + W_5$, and hence a $\{4\}$-Design of $(\Delta, \mathcal{S})$. We have two further consideration. Either $\ell \in \theta$, in which case $|\theta \cap \Delta| = 32$, or $\ell \notin \theta$, in which case $|\theta \cap \Delta| = 28$.

In the first case, we are able to constrain the strata of $(\Delta, \mathcal{S})$ in which $\pi(\chi_\theta)$ lies. Since θ is a spread of $\mathsf{H}(4,4)$, it will induce a partial spread in the subset of lines Δ. That is, χ_θ will represent a coclique with respect to $\overline{A_1}$, and hence also with respect to B_3 and B_4. The size of this coclique is 32, so we obtain the following inner distribution

$$(1, x, y, 0, 0, 31 - x - y)$$

for some $x, y \geqslant 0$. The corresponding MacWilliams transform is

$$\left(32, 16y - 16, 8y + 8x - 104, 8y - 8x + 184, -\frac{48y + 16x - 240}{3}, -\frac{48y - 16x - 240}{3}\right).$$

The MacWilliams transform must have nonnegative values, and $\pi(\chi_\theta)$ has no component in W_4, meaning the corresponding entry of the MacWilliams transform is zero. Solving the Delsarte linear program

$$\begin{aligned} 16y - 16 &\geqslant 0, \\ 8y + 8x - 104 &\geqslant 0, \\ 8y - 8x + 184 &\geqslant 0, \\ -\frac{48y + 16x - 240}{3} &= 0, \\ -\frac{48y - 16x - 240}{3} &\geqslant 0, \end{aligned}$$

yields $x = 12$, $y = 1$, and the inner distribution is

$$(1, 12, 1, 0, 0, 18)$$

and the MacWilliams transform becomes

$$(32, 0, 0, 96, 0, 128).$$

Thus for $\ell \in \theta$, $\pi(\chi_\theta) \in W_0 \perp W_3 \perp W_5$, and by Theorem 8.2.4 that $\iota(w)E_2 \cdot \chi_\theta = 0$ for any w in W_1 or W_2.

Unfortunately in the second case, $9 \leqslant x \leqslant \frac{51}{4}$ and $y = 5 - \frac{x}{3}$, so we are unable to apply Theorem 8.1.3. The problem is that $\ell \in \theta$ is not a G-invariant property. We can however, say that if $\ell \in \theta$, then χ_θ is orthogonal to all vectors in the orbit of $\iota(w)E_2$ under G_ℓ. This may be useful since we can assume that $\ell \in \theta$, but it does not appear to yield much additional information. The set of lines on a point of $\mathsf{H}(4,4)$ must clearly contain one line of θ. Now, if $\ell \in \theta$, then every point not on ℓ has a unique line which intersects ℓ, by the generalised quadrangle axiom. Thus every point not on ℓ contains exactly one line which does not intersect ℓ which is in θ. The characteristic vector of these lines projects directly to Δ and hence forms an antidesign there. These vectors span $W_0 \perp W_1 \perp W_2 \perp W_4$, and so the eigenspaces W_1 and W_2 belong to the antidesign strata and hence must be orthogonal to $\pi(\chi_\theta)$. Hence the MacWilliams transform essentially reflects this fact.

However, this also demonstrates that if we can show the existence of antidesigns in $(\Delta, \mathcal{S})$ with a non-trivial projection to W_1 or W_2, independent of whether or not $\ell \in \theta$, then we would have sufficient information to apply Theorem 8.1.3. This is perhaps not unreasonable, since $\pi(V_2)$ and $\pi(V_1)$ share W_1 and W_2. Indeed, if we can show by any other means that $\pi(\chi_\theta)$ has no component in either W_1 or W_2 when $\ell \notin \theta$, then we could apply Theorem 8.1.3.

To avoid needing case distinctions such as in this exploration of spreads of $\mathsf{H}(4,4)$, it is helpful to consider $(\Delta, \mathcal{S})$ such that Δ is an antidesign of $(\Omega, \mathcal{R})$, as was the case in the previous section. It then follows from the theorem of Roos that the intersection will be constant, so the projected design will also have constant size. Any designs will then automatically satisfy the G-invariance requirement of δ and so can be eliminated by a suitable included vector.

8.5 Rational Submodules of the Antidesign Strata as Witnesses

We now turn our attention to the antidesign strata. Recall that V_i is the rowspace of E_i. A design must be orthogonal to each vector in the antidesign strata, and hence its characteristic vector is a solution to the homogeneous system of linear equations determined by the relevant minimal idempotents. If no such design exists, then there is no binary solution (with the appropriate number of ones) to this system of linear equations. Thus the vectors of the antidesign strata collectively act as witnesses to the nonexistence of the design.

We ask the questions: just how prohibitive are the antidesign strata of designs? Is a subspace of them sufficient? To investigate these questions we consider submodules of the antidesign strata. The action of the automorphism group, G, on vertices respects the relations, and so captures much of the information of the scheme. This is particularly true when the scheme is Schurian, although it need not be for the group to provide useful information. This information is encoded algebraically as G-modules, with Theorem 8.1.1 revealing that for a Schurian scheme the irreducible submodules are the strata. Now, these submodules are irreducible under the action of G. Each decomposes into smaller submodules, however, when we consider them as H-submodules for $H \leqslant G$.

Thus we may consider H-submodules of the antidesign strata to see if they are sufficient for excluding designs. This is difficult to implement, however, since we do not have an effective method for computing submodules over $\mathbb{R}$. One method is to simply take a subgroup H and compute orbits of simple antidesigns. This approach was used by Bamberg, De Beule, and Ihringer [6], who took a sum of orbits on points to produce a 36-tight set which acted as a witness to the nonexistence of ovoids of $\mathsf{H}(5,4)$. Taking orbit sums is a crude and limited method of finding submodules, however, since it only returns the most basic examples.

Parker's *MeatAxe* algorithm [97] is capable of computing submodules over finite fields. Unfortunately it does not work over the rationals. The methods of this section therefore provide a further incentive for extending the *MeatAxe* to the rationals, or finding some other algorithm for computing rational submodules.

It is sometimes possible to compute rational submodules by computing submodules over different finite fields and extrapolating to find them over the rationals. We found that the most effective method was to simply compute many submodules and estimate them over the rationals until a check revealed that they were suitable. Checking that a submodule is in fact a submodule over the rationals is a simple task, and irreducibility can be concluded by knowing the dimension of the corresponding irreducible submodule over the finite field. This method proved effective for smaller permutation groups, but became very inefficient for larger groups. Thus we have only applied it to simple situations.

Nevertheless, we have found that there do indeed exist very small submodules which in addition to a few assumptions are sufficient to exclude designs. As a case study, we consider the existence of ovoids of $\mathsf{W}(3,3)$. This example illustrates the potential for this approach, given a feasible method of computing submodules.

We also provide a new proof of nonexistence of ovoids of $\mathsf{W}(3,q)$, q odd, inspired by our investigations. Indeed, this is perhaps a primary purpose of this method: once a suitable submodule has been found, it may then be analysed and a theoretical result extracted. Thus the submodules may be seen as an intermediary step in finding and understanding witnesses to the nonexistence of designs.

8.6 Ovoids of $\mathsf{W}(3, q)$

The following result is well known.

Theorem 8.6.1 (cf. [99, 1.8.4]). *There does not exist an ovoid of* $\mathsf{W}(3, q)$*, for q odd.*

We provide a proof of Theorem 8.6.1 for $q = 3$ below, which uses an explicit representation of a witness v which was computed using the technique described in the previous section.

Proof. Let $\mathcal{O}$ be an ovoid of $\mathsf{W}(3, 3)$. Assume that no three points of $\mathcal{O}$ span a plane, then $\mathcal{O}$ must span a line of $\mathsf{PG}(3, 3)$. This is impossible, as there are 4 points on any line of $\mathsf{PG}(3, 3)$, and $|\mathcal{O}| = 10$. Hence we have a contradiction, and there must exist three points of $\mathcal{O}$ which span a plane.

Let the point set of $\mathsf{W}(3, 3)$ be lexicographically ordered, and let G be the permutation representation of the collineation group of $\mathsf{W}(3, 3)$ acting on the point set so ordered. The relation matrix for the afforded association scheme can be found in Appendix C.

There is only one orbit on three pairwise non-collinear points which span a plane, and so without loss of generality, we may assume that $S \subset \mathcal{O}$ for any set of three such points, S. We choose the following non-collinear points,

$$S = \{P_{10}, P_{11}, P_{15}\},$$

which span a plane.

Let H be the subgroup of G isomorphic to D_{18} with the following generators:

$$\begin{aligned}
&[(1,7)(2,6)(3,12)(10,11)(14,27)(16,30)(17,37)(18,32)(19,23)(20,24)(21,35)(22,28)\\
&\qquad(25,39)(26,33)(29,38)(31,36),\\
&(1,5,7,22,31,35,21,36,28)(2,27,25,37,13,17,39,14,6)(3,18,29,38,32,12,20,9,24)\\
&\qquad(4,23,19)(8,26,33)(10,11,15)(16,30,34)].
\end{aligned}$$

There exists a 1-dimensional submodule $\langle v^H \rangle$, where

$$\begin{aligned}
v =&(31, -14, -44, -54, 31, -14, 31, 6, -44, 111, 111, -44, -14,\\
&-14, 111, -9, -14, -44, -54, -44, 31, 31, -54, -44, -14, 6,\\
&-14, 31, -44, -9, 31, -44, 6, -9, 31, 31, -14, -44, -14, 81),
\end{aligned}$$

and $v \in V_2$. For a binary solution, $\chi_{\mathcal{O}}$, the following mixed integer program,

$$\begin{aligned} \mathbb{1} \cdot \chi_{\mathcal{O}} &= 10, \\ \chi_S \cdot \chi_{\mathcal{O}} &= 3, \\ v \cdot \chi_{\mathcal{O}} &= 0, \end{aligned}$$

is infeasible! This demonstrates that there is no ovoid $\mathcal{O}$ of $\mathsf{W}(3,3)$. □

The first condition of the MIP at the end of the proof forces the size of $\chi_{\mathcal{O}}$ to be the correct size for an ovoid. The second constraint requires the points of S to be contained in the solution. Only one additional constraint is required to show that no such $\chi_{\mathcal{O}}$ and hence $\mathcal{O}$ exist.

The vector v is a weighted sum of orbits of the group H. If one desired, one could study this further to determine exactly what they are. We applied such an analysis to a different submodule and the insight this afforded led to the following simple proof of the well known result (c.f [99, 1.8.4]) that there are no ovoids of $\mathsf{W}(3,q)$, for q odd. It involves only counting arguments, however its inspiration lay in an analysis of submodules of the strata of the association scheme arising from $\mathsf{W}(3,q)$. It serves as an illustration of how the method presented in the previous section can be used to motivate theoretical results.

We require the following property, that there are at most two points of an ovoid on a hyperbolic line. This is of course implied by the fact that $\chi_\ell + \chi_{\ell^\perp}$ is 2-tight for a hyperbolic line ℓ [99, 1.8.4]. However we leave it as it is to show that weaker assumptions are required by our proof.

Property 8.6.2. *Let ℓ be a hyperbolic line in $\mathsf{PG}(3,q)$, then there are at most two points of an ovoid of $\mathsf{W}(3,q)$ on ℓ.*

We now consider the intersection of an ovoid of $\mathsf{W}(3,q)$ with hyperbolic lines in $PG(3,q)$, for q odd, and count the number of hyperbolic lines to show that such an ovoid does not exist, proving Theorem 8.6.1.

Proof. Assume for a contradiction that there exists an ovoid, $\mathcal{O}$. Let P be a point not in $\mathcal{O}$, and let $\ell_0, \ell_1, \dots, \ell_q$ be the $q+1$ lines of $\mathsf{W}(3,q)$ on P. Observe that $\ell_0, \ell_1, \dots, \ell_q$ lie in a plane, π. Since P is not in $\mathcal{O}$, there must be a distinct point of $\mathcal{O}$ on each of these lines. Let $Q_0, Q_1, \dots, Q_q$ be the points of $\mathcal{O}$ on $\ell_0, \ell_1, \dots, \ell_q$, respectively.

Consider the line Q_iQ_j of for $i, j \in \{1, \dots, q\}$, $i \neq j$. Then Q_iQ_j is a non-degenerate line of π and does not contain P. By Property 8.6.2, Q_i and Q_j are the only two points of θ on Q_iQ_j. Hence the number of such lines is equal to the number of choices of Q_i and Q_j, which is $\binom{q}{2} = \frac{q(q-1)}{2}$. Since π is a projective plane, each Q_iQ_j must meet ℓ_0.

Say R is a point of ℓ_0 other than P and Q_0. Let r be a nondegenerate line of π on R such that r contains Q_i and Q_j for $i,j \in \{1,\ldots,q\}$, and $i \neq j$. Then r contains no further points of θ, by Theorem 8.6.2. Let r' be another nondegenerate line of π on R which meets some Q_k, then $Q_k \neq Q_i$ and $Q_k \neq Q_j$, by the linear space axiom. Thus the maximum number of distinct, nondegenerate lines in π containing R and two points of θ is $\lfloor \frac{q}{2} \rfloor = \frac{q-1}{2}$, since q is odd. There are $q-1$ choices of R, and so there are at most $\frac{(q-1)(q-1)}{2}$ distinct, nondegenerate lines of π which meet ℓ_0 and contain two points of $\mathcal{O}$ other than Q_0.

However, $\frac{q(q-1)}{2} > \frac{(q-1)(q-1)}{2}$, a contradiction. □

Observe that if q is even, the count in the previous argument is tight. Indeed, we know that ovoids of $W(3,q)$ exist for q even.

Again, we reiterate that better proofs of Theorem 8.6.1 are known, but that we provide this as an example of a theoretical result which is extracted from exploration of submodules. It became clear after studying a submodule which yielded an infeasible MIP, that it described a plane and the set of nondegenerate lines on the plane.

Chapter 9

Conclusion

This thesis has been concerned with the study of designs, with an emphasis on those to be found in a geometric setting. In particular, this has meant studying m-ovoids, where in a point-line geometry $(\mathcal{P}, \mathcal{L}, \mathrm{I})$, an m-ovoid $\mathcal{O}$ is a subset of $\mathcal{P}$, such that every line is incident with precisely m points of $\mathcal{O}$. The dual concept of an m-ovoid is that of an m-cover; indeed, an m-ovoid of $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is precisely an m-cover of $(\mathcal{L}, \mathcal{P}, \mathrm{I})$, and so we work with the two objects as easily as shifting perspective. The thesis moves in a manner tending towards abstraction: we consider designs first in a concrete manner via computational techniques, then theoretically in specific settings, and ultimately we consider designs theoretically in a general setting.

In Chapter 4 we develop Algorithm 3 to effectively generate combinatorial objects. Algorithm 3 is an orderly algorithm based on the *Ladder Game* which utilises a *black-box* to prune its search tree. Crucially, the black-box is informed by symmetry information computed during the running of the orderly algorithm, via Lemma 4.5.1, allowing for drastic pruning where branches of the search tree are excluded if they are not *strongly semi-canonical.* We introduce *strong semi-canonicity* as an improvement upon semicanonicity. Implementing Algorithm 3 in *GAP*, and using *Gurobi* as a black-box, we obtained various new computational results on the spectrum of m-ovoids of $\mathsf{Q}(4, q)$, the size of partial ovoids of $\mathsf{Q}^-(5, q)$, the putative stabiliser of an ovoid of $\mathsf{DH}(4, 9)$, and partial classification of x-tight sets of $\mathsf{DH}(4, 4)$. Despite significant computational effort, the existence of 2-ovoids in $\mathsf{Q}(4, 11)$ is still open, as are many values of m for larger q. We pose the following problem:

Problem 9.0.1. *Do there exist 2-ovoids of $\mathsf{Q}(4, 11)$? For what values of m do m-ovoids exist in general in $\mathsf{Q}(4, q)$?*

Designs often have interesting properties, and sometimes produce interesting objects, such as new association schemes. Vanhove showed that a hemisystem of $\mathsf{DH}(2d-1, q^2)$ would give rise to new distance-regular graphs with classical parameters, and then posed the question of their existence. Motivated by this problem of Vanhove, for $d \geqslant 3$ we studied m-ovoids of $\mathsf{DH}(2d-1, q^2)$. Indeed, in Chapter 5 we considered m-ovoids in the more general setting of regular near polygons of order (s, t) and rank at least 3 satisfying certain parameter conditions, and discovered that for $d \geqslant 3$ the only feasible m-ovoids are $\frac{s+1}{2}$-ovoids, that is, hemisystems. Thus as a corollary we showed that for $d \geqslant 3$ the only feasible m-ovoids of $\mathsf{DQ}(2d, q)$, $\mathsf{DW}(2d-1, q)$, and $\mathsf{DH}(2d-1, q^2)$, are hemisystems. We were able to show computationally in Section 5.3 that, for small odd q, hemisystems exist in $\mathsf{DQ}(2d, q)$, while they do not in $\mathsf{DW}(2d-1, q)$ or $\mathsf{DH}(2d-1, q^2)$. We posed the following conjectures (cf. [11]):

Conjecture 5.3.2 *There are no hemisystems of* $\mathsf{DW}(5, q)$*, for all prime powers* q*.*

Conjecture 5.3.3 *For each odd prime power* q*, there exists a hemisystem of* $\mathsf{DQ}(6, q)$*.*

If the Conjecture 5.3.2 is true and there are no hemisystems of $\mathsf{DW}(5, q)$, for all prime powers q, then no hemisystems would exist for $\mathsf{DH}(2d-1, q^2)$ for all $d \geqslant 3$ and odd prime powers q, thus answering Vanhove's question in the negative.

In Chapter 6 we considered hemisystems of the parabolic quadrics. We constructed an infinite family of hemisystems of $\mathsf{Q}(2d, q)$, for q odd and $d \geqslant 2$, in Theorem 6.3.1. This is the first known family for $d \geqslant 4$. We constructed this family by considering orbit stitching via the AB-Lemma. One unusual aspect of our construction is the relatively small size of the groups used, namely $\Omega_3(q)$. Many constructions involving orbit stitching are limited to at most a dozen or so orbits on maximals, while in our case we have orders of magnitude more orbits. Since the size of the group is fixed regardless of the rank of the quadric, the number of orbits only grows drastically with the rank. We achieve our construction by a focus on the action of $\Omega_3(q)$ on the maximals of the geometry, more so than focusing on the orbits themselves.

The set of hemisystems of $\mathsf{Q}(2d, q)$ is a superset of the hemisystems of $\mathsf{DQ}(2d, q)$. In fact, we computed hemisystems of $\mathsf{DQ}(2d, q)$ for $q \in \{3, 7, 11\}$ which belonged to the construction of Theorem 6.3.1. It appears that the construction works for $q \equiv 3 \pmod 4$, when there is no primitive root of $\mathbb{F}_q$ in the basis of the anisotropic subspace isolated in our construction. Moreover, next-to-maximals fluctuate in their behaviour depending upon how they meet the parabolic plane which is fixed by our construction. Thus we pose the following question:

Problem 9.0.2. *Can the construction of Theorem 6.3.1 be extended to a construction of hemisystems of* $\mathsf{DQ}(6, q)$ *for all* $q \equiv 3 \pmod 4$*?*

Moreover in Chapter 6 we proposed a generalisation of the AB-Lemma in Lemma 6.1.2, where we showed that non-hemisystem m-covers can be constructed using the same principle as the AB-Lemma. We gave an example illustrating this generalisation by considering the action of a Singer cycle on projective subspaces, but so far have only produced non-interesting examples. We pose the following question:

Problem 9.0.3. *Can we construct interesting designs using Lemma 6.1.2?*

Attempting to resolve the geometric problems involving m-ovoids with methods from algebraic combinatorics lead to a more general consideration of Delsarte designs. Delsarte designs provide a unifying framework from which to consider many of the problems that we investigated. And if Delsarte designs are the natural objects to consider, then association schemes are the natural environment in which to consider them. We are able to draw powerful conclusions about Delsarte designs from the algebraic properties of the association scheme. We saw this in Chapter 7, particularly in Theorem 7.2.2, where we showed that when certain Krein parameters vanish, a design is either constrained in its size, or in its strata. If sufficiently constrained, this typically results in only "hemisystem-like" objects existing, that is, designs which consist of exactly half of the vertex set. This continues the trend started in Chapter 5, where m-ovoids of certain regular near polygons were shown to be hemisystems. Indeed, we reprove these results using the new method of vanishing Krein parameters. This also indicates that perhaps there is something intrinsic to a geometry about the existence of hemisystems. We showed that many known results can easily be reproved by Theorem 7.2.2, and are eager to see how it may more widely be applied:

Problem 9.0.4. *What novel results can be obtained by Theorem 7.2.2 and its corollaries?*

Theorem 7.2.2 is very much a consequence of all the relevant Krein parameters vanishing. We often find, however, that not all Krein parameters of interest do vanish, and it would be fascinating to know what more we can learn from them in this case:

Problem 9.0.5. *Can we say anything about a Delsarte design based upon the Krein parameters of an association scheme, without requiring all the relevant Krein parameters to vanish?*

Finally, in Chapter 8 we considered general methods for proving non-existence of designs, which we framed in terms of finding *witnesses* to their non-existence. The first method for constructing witnesses involved projecting the design strata of an association scheme to the strata of another association scheme on a subset of the vertices. Constraining the strata there often resulted in non-existence. This is significant in that it does not require projected designs to be entirely eliminated in order to eliminate the original design. This technique is captured in Theorem 8.1.3, Theorem 8.2.2, and Theorem 8.2.4. We successfully illustrated the technique by constraining designs by Theorem 7.2.2 and applying the MacWilliams transform.

Problem 9.0.6. *By what other methods can we constrain the strata of a design?*

We then considered a method of constructing witnesses from the antidesign strata by constructing rational submodules. This has the advantage of being a methodical process suited to computation. Such submodules may then be analysed for insight in order to obtain purely theoretical proofs. We demonstrated this technique in Section 8.6 where we give what amounts to a three line proof of the non-existence of ovoids of $\mathsf{W}(3,3)$, by providing a single vector which in combination with the size of an ovoid and a starter set of three non-collinear points in a plane, yields an infeasible MIP. This method of finding witnesses is limited in its current application by the state of the art of algorithms for working with submodules.

Problem 9.0.7. *Is there an effective computational method for computing submodules in strata of an association scheme over the rationals?*

Chapter 7 and Chapter 8 in particular introduced a number of novel methods for working with Delsarte designs. We hope that further investigation of these methods will prove fruitful.

Appendix A

GAP Code for Constructing m-Ovoids of $\mathsf{Q}(2d, q)$

The following code constructs a hemisystem of $Q(2d, q)$ in GAP [59] according to Theorem 6.3.1. It requires the FinInG [5] package.

```
LoadPackage("FinInG");
q := 3;; # must be an odd prime power
d := 3;;

n := 2*d + 1;;

if q mod 4 = 3 then
   z := 1;;
else
   z := PrimitiveRoot(GF(q));;
fi;

gram_mat := NullMat(n,n);;

gram_mat[1][1] := 1;;
gram_mat[2][3] := 1;;
gram_mat[3][2] := 1;;
gram_mat[4][4] := 1;;
gram_mat[5][5] := z;;

for i in [6 .. n] do
   if IsEvenInt(i) then
      gram_mat[i][i+1] := 1;;
   else
      gram_mat[i][i-1] := 1;;
   fi;
od;
gram_mat := gram_mat * One(GF(q));;
```

```
form := BilinearFormByMatrix(gram_mat);;
ps := PolarSpace(form);;
pg := AmbientSpace(ps);;

matrix := [[1,0,0],[0,0,1], [0,1,0]] * Z(q)^0;
form := BilinearFormByMatrix(matrix, GF(q));
pgo := SimilarityGroup( PolarSpace(form) );
gens := List(GeneratorsOfGroup(pgo), t ->
    Unpack(MatrixOfCollineation(t)));;
go := Group(gens);
B1 := DerivedSubgroup(go);;
tau1 := IdentityMat(3);;
tau1[1][1] := -1;;
tau1 := tau1 * One(GF(q));;
gensB1 := GeneratorsOfGroup(B1);;
gensA1 := Union(gensB1, [tau1]);;
A1 := Group(gensA1);;
gensB2 := List(gensB1, t -> IdentityMat(n,n));;
for i in [1 .. Size(gensB2)] do
   gensB2[i]{[1..3]}{[1..3]} := gensB1[i];;
od;;
gensB2 := gensB2*One(GF(q));;
gensA2 := List(gensA1, t -> IdentityMat(n,n));;
for i in [1 .. Size(gensA2)] do
   gensA2[i]{[1..3]}{[1..3]} := gensA1[i];;
od;;
gensA2 := gensA2*One(GF(q));;

Agens := List(gensA2, t -> CollineationOfProjectiveSpace(pg, t));;
A := Group(Agens);;
Bgens := List(gensB2, t -> CollineationOfProjectiveSpace(pg, t));;
B := Group(Bgens);;

tau := IdentityMat(n);;
tau[1][1] := -1;;
tau := tau*One(GF(q));;
tau := CollineationOfProjectiveSpace(pg, tau);;

maximals := AsSet(ElementsOfIncidenceStructure(ps, Rank(ps)));;

orbs := FiningOrbits(A, maximals);;
reps := List(orbs, t -> Representative(t));;
hemi := Concatenation(List(reps, t -> Orbit(B, t)));;

# To verify that hemi is indeed a hemisystem
Set(List(Points(ps), t -> Number(hemi, x -> x*t)));
```

Appendix B

Generators of Group

Here are the generators of the permutation representation of the collineation group of $\mathsf{H}(4,4)$ acting on the point set in its lexicographical ordering:

$$(1,14,13)(2,24,23)(4,38,35)(6,42,39)(7,20,19)(8,30,29)(10,52,53)$$
$$(11,94,96)(12,138,141)(15,26,36)(16,37,25)(17,28,40)(18,41,27)$$
$$(21,64,65)(22,106,107)(33,158,164)(34,159,165)(45,81,78)(47,85,82)$$
$$(48,61,60)(49,71,70)(50,63,62)(51,73,72)(54,139,97)(55,95,140)$$
$$(56,67,79)(57,80,66)(58,69,83)(59,84,68)(74,160,118)(75,161,119)$$
$$(76,116,162)(77,117,163)(87,123,120)(88,101,100)(89,111,110)$$
$$(91,127,124)(92,105,104)(93,115,114)(98,109,121)(99,122,108)$$
$$(102,113,125)(103,126,112)(130,143,142)(131,151,150)(132,145,144)$$
$$(133,153,152)(134,147,146)(135,155,154)(136,149,148)(137,157,156),$$

$$(1,3,50,37,61,125,156,33,116,117,41,71,79,136,35)(2,86,72,34,39)$$
$$(4,88,16,10,130)(5,51,122,49,109,22,93,121,137,120,23,128,31,129,32)$$
$$(6,89,99,65,105,102,133,98,153,67,11,131,87,110,165)$$
$$(7,38,134,25,19,74,55,84,60,97,83,148,160,140,68)$$
$$(8,123,154,159,42,135,108,30,78,13,43,9,46,63,112)$$
$$(12,85,147,103,64,104,54,69,96,151,81,146,119,17,152)$$
$$(14,90,73,66,20,124,24,44,62,161,28,106,114,141,82)$$
$$(15,144,139,58,132)(18,21,92,36,149,91,111,80,48,26,94,150,158,76,163)$$
$$(27,29,164,162,77)(40,157,45,100,75)(47,101,126,70,138)$$
$$(52,142,118,95,59)(53,143,127,155,57)(56,145,113,107,115),$$

$$(1,14)(2,24)(3,46)(4,85)(5,90)(6,127)(7,149)(8,157)(9,32)(10,53)(11,96)$$
$$(12,165)(15,84)(16,69)(17,126)(18,113)(19,148)(20,136)(21,65)(22,107)$$
$$(25,83)(26,59)(27,125)(28,103)(29,156)(30,137)(33,164)(34,141)(35,82)$$
$$(36,68)(37,58)(38,47)(39,124)(40,112)(41,102)(42,91)(43,129)(45,81)$$
$$(48,147)(49,155)(50,105)(51,115)(54,163)(55,119)(56,80)(57,67)(60,146)$$
$$(61,134)(62,104)(63,92)(66,79)(70,154)(71,135)(72,114)(73,93)(74,162)$$
$$(75,140)(76,118)(77,97)(87,123)(88,145)(89,153)(95,161)(98,122)(99,109)$$
$$(100,144)(101,132)(108,121)(110,152)(111,133)(116,160)(117,139)$$
$$(130,143)(131,151)(138,159)$$

Appendix C

Relation Matrix for the $\mathsf{W}(3,3)$ Association Scheme

[[0, 2, 2, 2, 1, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 2, 2, 2],
[2, 0, 2, 2, 1, 2, 1, 2, 2, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 2, 1, 2, 2, 2, 1, 2, 2],
[2, 2, 0, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2],
[2, 2, 2, 0, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 2, 2, 2, 1],
[1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2],
[1, 2, 2, 2, 1, 0, 2, 2, 2, 1, 2, 2, 2, 2, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2],
[2, 1, 2, 2, 1, 2, 0, 2, 2, 2, 1, 2, 2, 2, 1, 2, 1, 1, 2, 1, 2, 1, 1, 1, 2, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2],
[2, 2, 1, 2, 1, 2, 2, 0, 2, 2, 2, 1, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 1, 2, 1],
[2, 2, 2, 1, 1, 2, 2, 2, 0, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2],
[1, 2, 2, 2, 1, 1, 2, 2, 2, 0, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1],
[2, 1, 2, 2, 1, 2, 1, 2, 2, 2, 0, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 2, 1, 1, 2, 1, 2, 1, 1, 1, 2, 1, 1],
[2, 2, 1, 2, 1, 2, 2, 1, 2, 2, 2, 0, 2, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 1, 2, 1, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2],
[2, 2, 2, 1, 1, 2, 2, 2, 1, 2, 2, 2, 0, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2],
[1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2],
[1, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 0, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 1],
[2, 1, 2, 2, 2, 2, 2, 2, 2, 1, 2, 1, 1, 1, 2, 0, 2, 2, 2, 2, 2, 2, 2, 1, 2, 1, 1, 2, 1, 2, 2, 2, 2, 2, 2, 2, 1, 2, 1, 1],
[2, 2, 1, 2, 2, 2, 1, 2, 1, 1, 2, 2, 2, 1, 2, 2, 0, 2, 2, 2, 1, 2, 1, 1, 2, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 1, 1, 2, 2, 2],
[2, 2, 2, 1, 2, 1, 1, 2, 2, 2, 2, 1, 2, 1, 2, 2, 2, 0, 2, 1, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 1, 1, 2, 2, 2, 2, 1, 2],
[1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1],
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[2, 1, 2, 2, 2, 2, 2, 2, 2, 1, 2, 1, 1, 2, 1, 2, 1, 1, 1, 2, 0, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 2, 1, 1, 2, 1, 2, 2],
[2, 2, 1, 2, 2, 2, 1, 2, 1, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 0, 2, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 2, 2, 2, 2, 1, 2],
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