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Seshadri stratifications and standard monomial theory

Rocco Chirivì[1] · Xin Fang[2] · Peter Littelmann[3]

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Abstract
We introduce the notion of a Seshadri stratification on an embedded projective variety. Such a structure enables us to construct a Newton-Okounkov simplicial complex and a flat degeneration of the projective variety into a union of toric varieties. We show that the Seshadri stratification provides a geometric setup for a standard monomial theory. In this framework, Lakshmibai-Seshadri paths for Schubert varieties get a geometric interpretation as successive vanishing orders of regular functions.

1 Introduction

We fix throughout the article an algebraically closed field $\mathbb{K}$.

The aim of the article is to develop a theory parallel to that of Newton-Okounkov bodies, built on a web rather than a flag of subvarieties. The other ingredient making our approach different from that in Newton-Okounkov theory is a finite collection of functions with a prescribed set-theoretical vanishing behavior, leading to the notion of a Seshadri stratification. Compared to the Newton-Okounkov theory: instead of a valuation we have a quasi-valuation, but with values in the positive orthant; instead of a single monoid we obtain a fan of monoids, but the monoids in this fan are always finitely generated; instead of a body in an Euclidean space we get a simplicial complex with a rational structure.

As an example of this comparison, in the case of flag varieties, in the same way in which the string cones and their associated monoids [5, 53] show up in the application

✉ X. Fang
xinfang.math@gmail.com

R. Chirivì
rocco.chirivi@unisalento.it

P. Littelmann
peter.littelmann@math.uni-koeln.de

[1] Dipartimento di Matematica e Fisica “Ennio De Giorgi”, Università del Salento, Lecce, Italy

[2] Lehrstuhl für Algebra und Darstellungstheorie, RWTH Aachen University, Pontdriesch 10-16, 52062 Aachen, Germany

[3] Department Mathematik/Informatik, Universität zu Köln, 50931, Cologne, Germany

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of Newton-Okounkov theory [36], our setup leads to a polyhedral geometric version of the Lakshmibai-Seshadri path model [51].

Before going into the details, we review a few aspects of the various versions of standard monomial theory, which have been part of the motivating background for this article.

1.1 Theories of standard monomials

One of the motivations for theories of standard monomials is the computation of the Hilbert function of a graded finitely generated algebra R over a field $\mathbb{K}$. As their name suggests, there are two choices to make: what are the generators of the algebra R, and which monomials in the generators are chosen to be "standard".

1.1.1 Algebraic setting

To the best of our knowledge, the first work in this direction is by Macaulay [55]. He considered the case $R = \mathbb{K}[x_1, \ldots, x_n]/I$ where I is a homogeneous ideal. The important idea of Macaulay is to mix the structure of an order into the algebraic structure, transforming R to a "simpler" algebra sharing the same Hilbert function as R.

The theory of Gröbner basis, introduced by Buchberger [11], associates a unique reduced Gröbner basis $\mathrm{GB}(I, >)$ to the ideal I and a fixed monomial order $>$. Monomials in $x_1, \ldots, x_n$, which are not contained in the initial ideal $\mathrm{in}_>(I)$, are chosen to be *standard*. The standard monomials form a basis of $\mathbb{K}[x_1, \ldots, x_n]/\mathrm{in}_>(I)$, which share the same Hilbert polynomial as R. Determining the Hilbert function of R is thus reduced to a purely combinatorial problem of counting standard monomials.

The reduced Gröbner basis contains further information: each element in $\mathrm{GB}(I, >)$ has the form

a non-standard monomial $+$ a linear combination of standard monomials.

Not only does it tell which monomials are standard, but also how to rewrite a non-standard one as a linear combination of standard monomials. Elements in a reduced Gröbner basis are called straightening laws.

1.1.2 Algebro-geometric setting

Hodge [33] studied this problem when R is the homogeneous coordinate ring of a Grassmann variety or a Schubert subvariety in the Plücker embedding. The monomials are those in Plücker coordinates; a monomial is standard if the associated Young tableau is standard. He proved that the standard monomials form a basis of R, which allows him to deduce the postulation formula describing the Hilbert function. The Plücker relations have been used to write the non-standard monomials as linear combinations of standard ones.

The idea of Hodge is extracted in the work of De Concini, Eisenbud and Procesi [23] (see also [26]), where they coined the name "Hodge algebra" (a.k.a. algebra with straightening laws). Such an algebra R is defined with the following choices: a generating set of R indexed by a partially ordered set (poset), a linear basis consisting of *standard* monomials, i.e. those supported on a maximal chain in the poset, and a rule how to write non-standard monomials into a linear combination of standard ones. Verifying several geometric properties (Gorenstein property, Cohen-Macaulayness, etc) of the algebra R can be reduced to combinatorial problems. Later De Concini and Lakshmibai [20] generalized Hodge algebras to doset algebras.

1.1.3 Geometric setting

Seshadri [65] generalizes the work of Hodge from Grassmann varieties to a partial flag variety G/P with G a reductive algebraic group and P a minuscule maximal parabolic subgroup in G. In this work he took a geometric approach to the standard monomials in order to avoid applying the explicit straightening relations, and set up the paradigm of deducing geometric properties, such as vanishing of higher cohomology, normality and singular locus of Schubert varieties in G/P from the existence of a *standard monomial theory*. Motivated by the work of De Concini and Procesi [21], in collaboration with Lakshmibai and Musili [44, 45, 47, 48], Seshadri succeeded in generalizing the results in [65] to Schubert varieties in G/Q, where Q is a parabolic subgroup of classical type in G, by introducing the notion of admissible pairs. This case corresponds to the doset algebras above.

Going beyond classical type, the definition of admissible pairs becomes involved. Lakshmibai (see [46], or Appendix C of [68] for a reprint) made a conjecture on a possible index system of a basis of $\mathrm{H}^0(X(\tau), \mathcal{L}_\lambda)$ where $\tau \in W$ is an element in the Weyl group W of G, $X(\tau) \subseteq G/Q$ is the corresponding Schubert variety and $\mathcal{L}_\lambda$ is an ample line bundle on G/Q associated to a dominant weight λ. Such an index system consists of a chain of elements in the Bruhat graph of W/W_Q below τ, with W_Q the Weyl group of Q, together with a sequence of rational numbers. It was meanwhile asked to associate an explicit global section to each element in the index system.

1.2 Path models and standard monomial bases

The conjecture of Lakshmibai on the indexing systems is established by the third author in [51, 52] as a special path model consisting of Lakshmibai-Seshadri (LS)-paths of shape λ. The LS-paths are piece-wise linear paths starting from the origin in the dual space of a fixed Cartan subalgebra in the Lie algebra of G, and their endpoints coincide with the weights appearing in $V(\lambda)$, the Weyl module of G of highest weight λ, counted with multiplicities. This gives a type-free positive character formula of $V(\lambda)$, i.e. without cancellations like in the Weyl character formula.

Later in [54], the third author solved the question about the construction of global sections. For each LS-paths π of shape λ, using the Frobenius map of Lusztig in quantum groups at roots of unity, he constructed a path vector $p_\pi \in \mathrm{H}^0(G/Q, \mathcal{L}_\lambda)$ such that when π runs over all LS-paths of shape λ, the path vectors p_π form a basis of the space of global sections. Such constructions are compatible with Schubert varieties (in the sense of *loc.cit*).

The first author [13] introduced LS-algebras, which further generalized the above-mentioned work on Hodge algebras and doset algebras, to establish an algebro-geometric setting of the constructions in [51, 52, 54]. An LS-algebra can be degenerated to a much simpler LS-algebra (called discrete LS-algebra). Together with results in [54], this gives a proof of normality and Koszul property of the Schubert varieties by transforming these properties to combinatorics of the discrete LS-algebra (see also [14]).

The LS-paths were defined in a combinatorial way, and their geometric interpretation was missing. We quote the following observation/question by Seshadri from [66]:

> *"The character formula via paths or standard diagrams is a formula which involves only the cellular decomposition and its topological properties. It leads one to suspect that there could be a «cellular Riemann-Roch» which could also explain the character formula."*

One of the goals of this article is to build up a framework to provide a geometric interpretation of LS-paths, and at the same time, generalizing them from Schubert varieties to projective varieties with a Seshadri stratification (see below). An algebraic approach has been already taken in [16] by establishing a connection between LS-algebras and valuation theory, generalizing results in [13].

1.3 Newton-Okounkov theory

Newton-Okounkov bodies first appeared in the work of Okounkov [59]. His construction has been systemized by Kaveh-Khovanskii [37] and Lazarsfeld-Mustaţă [50] into the theory of Newton-Okounkov bodies. These discrete geometric objects received great attention in the past ten years.

To be more precise, the inputs of this machinery are an embedded projective variety Y, a flag $Y_\bullet := (Y = Y_r \supseteq Y_{r-1} \supseteq \cdots \supseteq Y_0 = \{\mathrm{pt}\})$ of normal subvarieties (we assume the normality only for simplicity), a collection of rational functions $u_r, \ldots, u_1 \in \mathbb{K}(Y)$ such that the restriction of u_k to Y_k is a uniformizer in $\mathcal{O}_{Y_k, Y_{k-1}}$, and a total order on $\mathbb{Z}^r$. For a non-zero rational function f in $\mathbb{K}(Y)$, one first looks at the vanishing order a_r of f at Y_{r-1}, then considers the function $f_{r-1} := f u_r^{-a_r}|_{Y_{r-1}}$ to eliminate the zero or pole, and repeats this procedure for f_{r-1} and the flag starting from Y_{r-1}. The outcome is a point $\nu_{Y_\bullet}(f) := (a_r, \ldots, a_1) \in \mathbb{Z}^r$; taking into account the total order on $\mathbb{Z}^r$, one obtains a valuation $\nu_{Y_\bullet} : \mathbb{K}(Y) \setminus \{0\} \to \mathbb{Z}^r$.

Extending the valuation to the homogeneous coordinate ring $\mathbb{K}[Y]$ of Y by sending a homogeneous function $f \in \mathbb{K}[Y]$ of degree m to $(m, \nu_{Y_\bullet}(f))$ yields a valuation $\tilde{\nu}_{Y_\bullet} : \mathbb{K}[Y] \setminus \{0\} \to \mathbb{Z} \times \mathbb{Z}^r$. The image of $\tilde{\nu}_{Y_\bullet}$ is a monoid. One of the most important questions in Newton-Okounkov theory is to determine when this monoid is finitely generated. If it happens to be so, Anderson [3] obtains a toric degeneration[1] of Y to the toric variety associated to this monoid.

Motivated by seeking for an interpretation of the LS-paths in the above setup as vanishing order of functions, the second and the third author in [27] studied the case of Grassmann varieties. Instead of a flag of subvarieties, a web of subvarieties consisting of Schubert varieties is fixed. They constructed in *loc.cit.* a quasi-valuation by choosing the minimum of all possible vanishing orders at each step. The graded algebra associated to the filtration arising from this quasi-valuation coincides with

[1] In this article, toric varieties are irreducible but not necessarily normal.

the discrete Hodge algebra in [23] (a.k.a. the Stanley-Reisner algebra of the poset arising from the web). It was asked in [27, 28] how to generalize this construction to Schubert varieties in a partial flag variety.

1.4 Seshadri stratification, Newton-Okounkov bodies and standard monomial theory

In this article we introduce the notion of a Seshadri stratification. In such a framework we construct a Newton-Okounkov simplicial complex and the associated semi-toric degeneration: this enables us to prove a formula on the degree of X and to give a new geometric setup for standard monomial theory.

1.4.1 Semi-toric degeneration from Seshadri stratification

The geometric setting in the entire article is encoded in the concept of a Seshadri stratification (Definition 2.1) of an embedded projective variety $X \subseteq \mathbb{P}(V)$,[2] where V is a finite dimensional vector space. Such a stratification consists of a collection of subvarieties X_p in X which are smooth in codimension one, together with homogeneous functions f_p on V, both indexed by a finite set A, i.e. $p \in A$. The set A is naturally endowed with a poset structure from the inclusion of subvarieties, such that covering relation $q < p$[3] means that X_q is a divisor in X_p. The poset A is assumed to have a maximal element $p_{\max}$ with $X_{p_{\max}} = X$. These subvarieties and the functions are compatible in the following sense:

– the vanishing set of the restriction of f_p to X_p is the union of all divisors in X_p which are of form X_q;
– f_p vanishes on X_r for $p \not\leq r$.

Typical examples of this setting are Schubert varieties in a flag variety and extremal weight functions (Sect. 16.6). More examples, varying from quadrics and elliptic curves to Grassmann varieties and group compactifications, will be discussed in Sect. 16.

The requirements above seem to be restrictive. It is natural to ask for the existence and the uniqueness of Seshadri stratifications on an embedded projective variety.

Concerning the existence: the definition of a Seshadri stratification demands the variety to be smooth in codimension one, and this is in fact sufficient:

Proposition 1 (Proposition 2.11) *Every embedded projective variety $X \subseteq \mathbb{P}(V)$, smooth in codimension one, admits a Seshadri stratification.*

The Seshadri stratification is far away from being unique: examples will be discussed in the article (Example 2.7, Remark 16.4).

[2] Such a setup is more suitable to Standard Monomial Theory, line bundles or linear systems will be discussed in a future work.

[3] A relation $q < p$ in A is called a *covering relation*, if there is no $r \in A$ satisfying $q < r < p$.

One of the purposes of this article is to prove the following theorem, constructing semi-toric[4] degenerations of X from a Seshadri stratification on X.

Theorem 1 (Theorem 12.2) *Let $X \subseteq \mathbb{P}(V)$ be a projective variety and X_p, f_p, $p \in A$ defines a Seshadri stratification on X. There exists a flat degeneration of X into a reduced union of projective toric varieties X_0. Moreover, X_0 is equidimensional, and its irreducible components are in bijection with maximal chains in A.*

Combining with Proposition 1 gives the following

Corollary 1 (Corollary 12.3) *Every embedded projective variety, which is smooth in codimension one, admits a flat degeneration into a reduced union of projective toric varieties, the number of irreducible components coincides with its degree.*

An important problem in the study of toric degenerations is to construct degenerations of projective varieties into projective toric varieties. The above theorem does not go precisely in this direction: our aim is rather to seek for degenerations of a projective variety which are compatible with a prescribed collection of subvarieties. Generally speaking, such a degeneration can not be toric, as being pointed out by Olivier Mathieu already for Schubert varieties (see the introduction of [12]). The above theorem provides an answer to this problem if the projective variety admits a Seshadri stratification. In other words, such a degeneration exists if there are regular functions with prescribed set-theoretic vanishing locus. This condition is in the same vein as the Riemann-Roch theorem: the geometry gets controlled by the existence of certain functions.

The proof of the above theorem occupies a large part of the article. The general idea is similar to the one in [3], as soon as an analogue of a Newton-Okounkov polytope (not just a body) can be associated to the Seshadri stratification. In fact, we will construct a Newton-Okounkov simplicial complex from a Seshadri stratification, and the semi-toric variety X_0 is determined by this simplicial complex together with a lattice in each simplex.

This article is influenced by the idea of Allen Knutson [41] to use Rees valuations. Later in the work of Alexeev and Knutson [1], they suggested to apply this idea to recover the degenerations in [13] for Schubert varieties.

1.4.2 Newton-Okounkov simplicial complex

In a Seshadri stratification, all maximal chains in A have the same length, which is $\dim X$. We start with a naïve idea. Fix a maximal chain $\mathfrak{C} : p_r > p_{r-1} > \cdots > p_1 > p_0$ in A, we aim to produce a convex body as in [37, 50] from the flag of subvarieties

$$X = X_{p_r} \supseteq X_{p_{r-1}} \supseteq \cdots \supseteq X_{p_1} \supseteq X_{p_0}$$

and the functions $f_{p_r}, \ldots, f_{p_0}$.

[4]Different to the terminology in symplectic geometry, a semi-toric variety is a variety whose irreducible components are toric varieties [12].

We switch to the affine picture: let $\hat{X}_{p_k}$ be the affine cone over X_{p_k}. The problem is that the restriction of f_{p_k} to $\hat{X}_{p_k}$ is not necessarily a uniformizer in the local ring $\mathcal{O}_{\hat{X}_{p_{k-1}},\hat{X}_{p_k}}$. As in the first step of the construction of a valuation associated to the flag, for a rational function $g \in \mathbb{K}(\hat{X})$, there is no reason why there exists $m \in \mathbb{Z}$ such that the function gf_p^m, when restricted to $\hat{X}_{p_{r-1}}$, yields a well-defined and non-zero rational function. One could think about choosing a uniformizer in $\mathcal{O}_{\hat{X}_{p_{k-1}},\hat{X}_{p_k}}$, however, we will not concentrate on only one maximal chain $\mathfrak{C}$ but take into account all of them, there is no control of this uniformizer on the flag associated to other maximal chains in A.

We adjust the construction of the Newton-Okounkov body by keeping track of the vanishing multiplicities of the functions f_p along a divisor. We consider the Hasse graph of the poset A and colour an edge arising from the covering relation $q < p$ by the vanishing order of f_p on X_q (see Sect. 2.3). These colours are called bonds.

We fix N to be the l.c.m. of all bonds appearing in the coloured Hasse graph.

For a non-zero rational function $g_r := g \in \mathbb{K}(\hat{X})$ with vanishing order a_r along the divisor $\hat{X}_{p_{r-1}}$ in $\hat{X} = \hat{X}_{p_r}$, we define a rational function

$$h := \frac{g_r^N}{f_{p_r}^{N\frac{a_r}{b_r}}} \in \mathbb{K}(\hat{X}_{p_r}),$$

where b_r is the vanishing order of f_{p_r} along $\hat{X}_{p_{r-1}}$. The restriction of h to $\hat{X}_{p_{r-1}}$, denoted by g_{r-1}, gives rise to a well-defined non-zero rational function in $\mathbb{K}(\hat{X}_{p_{r-1}})$ (Lemma 4.1). This procedure can be henceforth iterated, yielding a sequence of rational functions $g_{\mathfrak{C}} := (g_r, g_{r-1}, \dots, g_0)$ with $g_k \in \mathbb{K}(\hat{X}_{p_k}) \setminus \{0\}$. The vanishing order of g_k (resp. f_{p_k}) on $\hat{X}_{p_{k-1}}$ will be denoted by a_k (resp. b_k).

Similar to the Newton-Okounkov theory, the vanishing orders will be collected to define a valuation. In view of the N-th powers appearing in the sequence of rational functions, we define a map $\mathcal{V}_{\mathfrak{C}} : \mathbb{K}[\hat{X}] \setminus \{0\} \to \mathbb{Q}^{\mathfrak{C}}$ in the following way:

$$g \mapsto \frac{a_r}{b_r} e_{p_r} + \frac{1}{N}\frac{a_{r-1}}{b_{r-1}} e_{p_{r-1}} + \cdots + \frac{1}{N^r}\frac{a_0}{b_0} e_{p_0},$$

where e_{p_k} is the coordinate function in $\mathbb{Q}^{\mathfrak{C}}$ corresponding to $p_k \in \mathfrak{C}$. Such a map is indeed a valuation (Proposition 6.10) having at most one-dimensional leaves (Theorem 6.16). As in the situation of Sect. 1.3, we do not know whether the image of $\mathcal{V}_{\mathfrak{C}}$ is a finitely generated monoid. In general, the finite generation property is not expected in general as the flag of subvarieties reveals rather the local geometry.

In order to pass from local to global, we define a quasi-valuation (Definition 3.1) $\mathcal{V} : \mathbb{K}[\hat{X}] \setminus \{0\} \to \mathbb{Q}^A$ by taking the minimum over all maximal chains in A. For this we choose a total order $>^t$ on A refining the partial order (Equation (17)), extend lexicographically to $\mathbb{Q}^A$, and define

$$\mathcal{V}(g) := \min\{\mathcal{V}_{\mathfrak{C}}(g) \mid \mathfrak{C} \text{ is a maximal chain in } A\},$$

where $\mathbb{Q}^{\mathfrak{C}}$ is naturally embedded into $\mathbb{Q}^A$.

The first nice property of this quasi-valuation is its positivity: the image of $\mathcal{V}$ is contained in $\mathbb{Q}^A_{\geq 0}$ (Proposition 8.6). Such a property is guaranteed by the Valuation

Theorem of Rees (Theorem 3.8), whose spirit is already incorporated as part of the Seshadri stratification, as well as the valuation $\mathcal{V}_{\mathfrak{C}}$. This positivity encodes in fact the regularity: for a non-zero regular function $g \in \mathbb{K}[\hat{X}]$ and a maximal chain $\mathfrak{C}$ on which the minimum of $\mathcal{V}(g)$ is attained, the function g_k in the sequence of rational functions $g_{\mathfrak{C}}$ is regular in the normalization of $\hat{X}_{p_k}$ for all $k = 0, 1, \ldots, r$.

For such a function $g \in \mathbb{K}[\hat{X}]$, there could be many maximal chains on which the minimum $\mathcal{V}(g)$ is attained. We will prove (Proposition 8.7) that these maximal chains are precisely those containing the support of $\mathcal{V}(g)$, defined as the set of elements in A on which $\mathcal{V}(g)$ takes non-zero (hence positive) value. As a consequence of this characterization via supports, we are able to decompose the image Γ of the quasi-valuation $\mathcal{V}$ into a finite union of (finitely generated) monoids $\Gamma_{\mathfrak{C}}$ (Corollary 9.1) where $\mathfrak{C}$ runs over all maximal chains in A and $\Gamma_{\mathfrak{C}}$ consists of elements in Γ supported on $\mathfrak{C}$.

The set Γ encodes rather the global aspects of X: first, for a given regular function, it tells how to smooth out its zeros using the functions f_p and keeping the regularity simultaneously; secondly, the quasi-valuation $\mathcal{V}$ has at most one-dimensional leaves (Lemma 10.2), hence $\mathbb{K}[\hat{X}]$ and Γ have the same "size"; moreover, the monoids $\Gamma_{\mathfrak{C}}$ are finitely generated (Lemma 9.6).

The finite generation of $\Gamma_{\mathfrak{C}}$ allows us to investigate the geometry of Γ. We will define a fan algebra $\mathbb{K}[\Gamma]$ by gluing different $\Gamma_{\mathfrak{C}}$ in a Stanley-Reisner way (Definition 9.3). The affine variety $\mathrm{Spec}(\mathbb{K}[\Gamma])$ associated to the fan algebra is an irredundant union of affine toric varieties $\mathrm{Spec}(\mathbb{K}[\Gamma_{\mathfrak{C}}])$ where $\mathfrak{C}$ runs over all maximal chains in A, each of dimension $\dim \hat{X}$ (Proposition 9.8).

In order to prove Theorem 1, we need to construct a flat family over $\mathbb{A}^1$ with special fibre $\mathrm{Proj}(\mathbb{K}[\Gamma])$. The quasi-valuation $\mathcal{V}$ induces an algebra filtration on $R := \mathbb{K}[\hat{X}]$. The associated graded algebra $\mathrm{gr}_{\mathcal{V}} R$ is finitely generated and reduced (Corollary 10.6). Different to the toric case as in [3], some work is needed in proving that the fan algebra $\mathbb{K}[\Gamma]$ and the associated graded algebra $\mathrm{gr}_{\mathcal{V}} R$ are indeed isomorphic as algebras (Theorem 11.1). Once this isomorphism is established, the machinery of the Rees algebra associated to a filtration can be applied to construct the flat family in Theorem 1.

As an application to Theorem 1, we show that if the poset A is Cohen-Macaulay over $\mathbb{K}$ and the monoids $\Gamma_{\mathfrak{C}}$ are saturated, then the embedded projective variety is projectively normal (Theorem 14.1).

Out of the set Γ we define the associated Newton-Okounkov simplicial complex $\Delta_{\mathcal{V}}$ (Definition 13.1), where each monoid $\Gamma_{\mathfrak{C}}$ for a maximal chain $\mathfrak{C}$ in A contributes a simplex. Different to the Newton-Okounkov theory [37, 50], where the collection of rational functions are uniformizers, to each simplex we associate a natural lattice $\mathcal{L}^{\mathfrak{C}}$ from the quasi-valuation. The degree $\deg(X)$ of the embedded projective variety $X \hookrightarrow \mathbb{P}(V)$ can be computed as a sum of volumes of simplexes:

Theorem 2 (Theorem 13.6) *For each maximal chain $\mathfrak{C}$, we provide an r-dimensional simplex with rational vertices $D_{\mathfrak{C}}$ such that*

$$\deg(X) = r! \sum \mathrm{vol}(D_{\mathfrak{C}}),$$

where the sum runs over all maximal chains $\mathfrak{C}$ in A.

For Schubert varieties, such a formula was obtained by Knutson [40] in the symplectic geometric setting, and the first author [13] in the algebro-geometric setting.

When the monoids $\Gamma_{\mathfrak{C}}$ are saturated, the Hilbert function can be calculated in the same way as Ehrhart functions of simplexes (Corollary 13.8).

1.4.3 The case of a totally ordered poset

In order to help the reader compare our approach with the usual Netwon-Okounkov context, we want to shortly discuss the case of a Seshadri stratification with a totally ordered poset $A = \{p_r > p_{r-1} > \cdots > p_1 > p_0\}$. The quasi-valuation $\mathcal{V}$ coincides with the valuation $\mathcal{V}_A$ for the unique maximal chain A.

Further, the zero locus of the extremal function f_k in $\hat{X}_{p_k}$ is the divisor $\hat{X}_{p_{k-1}}$. It is then clear that $f_k|_{\hat{X}_{p_k}}$ is the b_k–th power of a uniformizer u_k in $\mathcal{O}_{\hat{X}_{p_k},\hat{X}_{p_{k-1}}}$. In particular the first r components of $\mathcal{V}(g) \in \mathbb{Q}^{r+1}$, for homogeneous $g \in \mathbb{K}[\hat{X}] \setminus \{0\}$, are just renormalizations of the components of the usual valuation $\nu_{X_\bullet}(g) \in \mathbb{Z}^r$ associated to the flag of subvarieties $X_\bullet = (X = X_{p_r} \supseteq X_{p_{r-1}} \supseteq \cdots \supseteq X_{p_0})$ and uniformizer u_k, $k = 1, \ldots, r$, in the Newton-Okounkov theory. More precisely: if $\nu_{X_\bullet}(g) = (n_r, \ldots, n_1)$ then

$$\mathcal{V}(g) = \left(\frac{n_r}{b_r}, \frac{n_{r-1}}{b_{r-1}}, \ldots, \frac{n_0}{b_0}\right),$$

where n_0 is such that $\sum_{j=0}^{r} \deg f_j \frac{n_j}{b_j} = \deg g$.

Finally, the Newton-Okounkov simplicial complex of $\mathcal{V}$ is just a simplex in this case.

If such a setup arises from a generic hyperplane stratification (see Sect. 2.4), the finite generation result in this article is closely related to [4, Proposition 14]. Indeed, the assumptions in that proposition allow to construct a Seshadri stratification with a linear poset, which is exactly the proof in *loc.cit.*

1.4.4 Standard monomial theory

Seshadri stratifications provide geometric setups for standard monomial theories on $R = \mathbb{K}[\hat{X}]$.

In view of the calculation of Hilbert functions (Proposition 13.8), in order to have a standard monomial theory, the monoids $\Gamma_{\mathfrak{C}}$ should be assumed to be saturated. Under this assumption, each element $\underline{a} \in \Gamma_{\mathfrak{C}}$ can be uniquely decomposed into a sum of indecomposable elements (Definition 15.2) in $\Gamma_{\mathfrak{C}}$ (Proposition 15.4).

From the one-dimensional leaf property of the quasi-valuation $\mathcal{V}$, for each indecomposable element $\underline{a} \in \Gamma_{\mathfrak{C}}$ we choose a regular function $x_{\underline{a}} \in R$ with $\mathcal{V}(x_{\underline{a}}) = \underline{a}$. The condition of being standard will be defined on monomials in these regular functions: a monomial $x_{\underline{a}_1} \cdots x_{\underline{a}_k}$ with $\underline{a}_1, \ldots, \underline{a}_k \in \Gamma_{\mathfrak{C}}$ is standard if for $i = 1, \ldots, k-1$, $\min \operatorname{supp} \underline{a}_i \geq \max \operatorname{supp} \underline{a}_{i+1}$. This defines a standard monomial theory on R (Proposition 15.6).

One of the tasks in Seshadri's paradigm is to construct standard monomial bases compatible with all strata X_p. We call a standard monomial $x_{\underline{a}_1} \cdots x_{\underline{a}_k}$ standard on

X_p if the maximal element in supp $\underline{a}_1, \ldots,$ supp $\underline{a}_k$ is $\leq p$. This compatibility requires extra conditions on the independence to the choice of the total order $>^t$ in the definition of $\mathcal{V}$. To eliminate this dependency, we introduce the *balanced* conditions on Seshadri stratifications (Definition 15.7); this extra structure allows us to show:

Theorem 3 (Theorem 15.12) *In the above situation, the following hold:*

i) *All standard monomials on X which are standard on X_p form a basis of $\mathbb{K}[\hat{X}_p]$.*
ii) *Standard monomials on X which are not standard on X_p are precisely those vanishing on X_p. They form a linear basis of the defining ideal of X_p in X.*
iii) *For any $p, q \in A$, the scheme-theoretic intersection $X_p \cap X_q$ is a reduced union of strata contained in both.*

1.4.5 L-S paths as vanishing orders

We explain to what extent the framework in this article answers Seshadri's question in Sect. 1.2.

Let G be a simple simply connected algebraic group, B a Borel subgroup, T a maximal torus and W the Weyl group of G, viewed as a poset with the Bruhat order. The assumption on G is made only to simplify the statements: the results will be proved in [15] for Schubert varieties in a symmetrizable Kac-Moody group. The Schubert varieties $X(\sigma)$, together with the extremal weight functions p_σ for $\sigma \in W$, form a Seshadri stratification of the flag variety $X := G/B$, embedded in $\mathbb{P}(V(\lambda))$ for a regular dominant weight λ.

To describe the image of the associated quasi-valuation $\mathcal{V}$, we introduce for each maximal chain $\mathfrak{C}$ an explicit lattice $L_{\mathfrak{C},\lambda}$ (Equation (26)) and a fan of (saturated) monoids L_λ^+, which is in an easy bijection with the Lakshmibai-Seshadri paths (LS-paths) of shape λ (Lemma 16.10).

Theorem 4 (Theorem 16.14, Sect. 16.6.5, Proposition 16.15)

i) *The image of $\mathcal{V}$ coincides with L_λ^+.*
ii) *The degree of $X(\sigma)$ is a sum of products of bonds.*
iii) *The Seshadri stratification is balanced, hence all statements in Sect. 1.4.4 hold for Schubert varieties too.*
iv) *The Schubert varieties $X(\sigma) \hookrightarrow \mathbb{P}(V(\lambda))$ are projectively normal.*

The difficult part is i), and the key point to the proof is Theorem 16.12: for any element π in L_λ^+ (looked as an LS-path) we seek for a regular function p_π with $\mathcal{V}(p_\pi) = \pi$. A candidate for p_π has already been constructed by the third author in [54], but to prove the desired vanishing properties requires results and techniques from representation theory of algebraic groups and quantum groups at roots of unity. The complete proof is given in a separate article [15].

As a consequence of the theorem, the LS-paths get interpreted as vanishing orders of regular functions. This fits perfectly into Seshadri's expectation of "cellular Riemann-Roch".

1.5 Outline of the article

In Sect. 2 we introduce the Seshadri stratifications of an embedded projective variety and the associated Hasse graph coloured by bonds. A few quick examples are discussed therein as running examples for the article. In Sect. 3 we collect a few standard facts about valuations and quasi-valuations, and, in particular, we recall the homogenized quasi-valuation arising from ideal filtration, and the Rees valuation theorem investigating their structures. In Sect. 4 and 5 we prepare the procedure used in Sect. 6 to define a valuation associated to a maximal chain in the poset A. Further studies regarding these valuations are carried out in Sect. 7. In Sect. 8 we introduce the main point of this article: a quasi-valuation, defined as the minimum over the collection of valuations introduced in Sect. 6.

We introduce in Sect. 9 the notion of fan monoids and fan algebras to describe the associated graded algebra. In Sect. 10 we discuss some nice properties of this quasi-valuation. Section 11 is devoted to proving that the associated graded algebra and the fan algebra are isomorphic as algebras, which is the crucial step in the construction of the semi-toric degeneration in Sect. 12.

In Sect. 13 we associate to the Seshadri stratification a Newton-Okounkov simplicial complex to investigate the discrete geometry behind the semi-toric variety. This complex, unlike the usual Newton-Okounkov setting, is not necessarily a convex body. We compensate this difference by endowing the complex with a rational or integral structure. As an application, we prove a criterion on the projective normality in Sect. 14. Under certain hypothesis on the Seshadri stratification, we define a standard monomial theory for the homogeneous coordinate ring in Sect. 15.

Examples such as Schubert varieties in partial flag varieties, compactification of torus and $\mathrm{PSL}_2(\mathbb{C})$, quadrics and elliptic curves are discussed in Sect. 16.

The structure of the article is rather linear, the notations used throughout the article are gathered in a list of notations after Sect. 16.

1.6 Recent development

The Seshadri stratification of a Schubert variety consisting of its Schubert subvarieties is studied in [15], where results announced in Sect. 16.6 are proved with the help of quantum groups at roots of unity. A different approach without using quantum groups is given in [18]. The algebraic counterpart of this article is studied in [16] in the framework of valuations on LS-algebras, the connection to the current article is made clear in [17]. More results on normal Seshadri stratification, especially its connection to Gröbner theory, are topics of *loc.cit*.

2 Seshadri stratifications

2.1 Conventions

Let V be a finite dimensional $\mathbb{K}$-vector space. For a homogeneous polynomial function $f \in \mathrm{Sym}(V^*)$, we denote its vanishing set $\mathcal{H}_f := \{[v] \in \mathbb{P}(V) \mid f(v) = 0\}$. For a partially ordered set (poset) $(A, \leq)$ and $p \in A$, we denote $A_p := \{q \in A \mid q \leq p\}$: $(A_p, \leq)$ is a poset. A relation $q < p$ in A is called a covering relation, if there is no $r \in A$ such that $q < r < p$.

In this article, varieties are assumed to be irreducible. Toric varieties are not necessarily normal. When it is necessary to emphasize on the normality, we use the term "normal toric variety".

2.2 Definition and examples

Let $X \subseteq \mathbb{P}(V)$ be an embedded projective variety with graded homogeneous coordinate ring $R = \mathbb{K}[X]$. The degree k component of R will be denoted by $R(k)$: $R = \bigoplus_{k\geq 0} R(k)$.

Let $\{X_p \mid p \in A\}$ be a collection of projective subvarieties X_p in X indexed by a finite set A. The set A is naturally endowed with a partial order $\leq$ by: for $p, q \in A$, $p \leq q$ if and only if $X_p \subseteq X_q$. We assume that there exists a unique maximal element $p_{\max} \in A$ with $X_{p_{\max}} = X$.

For each $p \in A$, we fix a homogeneous function f_p on V of degree larger or equal to 1.

Definition 2.1 The collection of subvarieties X_p and homogeneous functions f_p for $p \in A$ is called a *Seshadri stratification*, if the following conditions are fulfilled:

(S1) the projective varieties X_p, $p \in A$, are all smooth in codimension one; for each covering relation $q < p$ in A, $X_q \subseteq X_p$ is a codimension one subvariety;
(S2) for any $p \in A$ and any $q \not\leq p$, f_q vanishes on X_p;
(S3) for $p \in A$, it holds: set theoretically

$$\mathcal{H}_{f_p} \cap X_p = \bigcup_{q \text{ covered by } p} X_q.$$

The subvarieties X_p will be called *strata*, and the functions f_p are called *extremal functions*.

The following lemma will be used throughout the article often without mention.

Lemma 2.2

i) *The function f_p does not identically vanish on X_p.*
ii) *All maximal chains in A have the same length, which coincides with* $\dim X$. *In particular, the poset A is graded.*
iii) *The intersection of two strata is a union of strata; in particular for each $p, q \in A$ we have*

$$X_p \cap X_q = \bigcup_{t \leq p,q} X_t.$$

Proof If X_p is just a point, then (S3) implies the intersection $\mathcal{H}_{f_p} \cap X_p$ is empty, which implies i) in this case. If X_p is not just a point, then the intersection $\mathcal{H}_{f_p} \cap X_p$

is not empty, and (S1) and (S3) enforce the intersection to be a union of divisors. In particular: $X_p \nsubseteq \mathcal{H}_{f_p}$, which implies i), and there must exist elements q in A covered by p, which implies ii).

From the definition of the partial order on A it follows that $\bigcup_{t\leq p,q} X_t \subseteq X_p \cap X_q$. We prove by induction on the length of a maximal chain joining p with a minimal element in A that the intersection $X_p \cap X_q$ is a union of strata. Such a length is well-defined by the part ii).

When $p = p_0$ is a minimal element in A, it follows that either $p_0 \leq q$ and $X_{p_0} \cap X_q = X_{p_0}$, or $p_0 \nleq q$ and $X_{p_0} \cap X_q = \emptyset$; in both cases the claim is proved.

For an arbitrary $p \in A$, if $p \leq q$ then $X_p \cap X_q = X_p$, so we can assume that $p \nleq q$; hence $f_p|_{X_q} = 0$ by (S2). In particular $f_p|_{X_p\cap X_q} = 0$. But, for $x \in X_p$, (S3) implies that $f_p(x) = 0$ if and only if $x \in \bigcup_{p'\leq p} X_{p'}$, which gives the inclusion $X_p \cap X_q \subseteq \bigcup_{p'<p}(X_{p'} \cap X_q)$. Since the reverse inclusion clearly holds, we have proved

$$X_p \cap X_q = \bigcup_{p'<p} (X'_p \cap X_q).$$

By induction, each intersection $X_{p'} \cap X_q$ is a union of strata, hence $X_p \cap X_q$ is a union of strata. □

Thanks to the part ii) of the lemma we can define the length of an element in A.

Definition 2.3 Let $p \in A$. The *length* $\ell(p)$ of p is the length of a (hence any) maximal chain joining p with a minimal element in A.

It is clear that $\ell(p) = \dim X_p$.

Remark 2.4 For a fixed $p \in A$, by Lemma 2.2 (ii), the poset A_p has a unique maximal element, and all maximal chains have the same length. The collection of varieties X_q, $q \in A_p$, and the extremal functions f_q, $q \in A_p$ satisfy the conditions (S1)-(S3), and hence defines a Seshadri stratification for $X_p \hookrightarrow \mathbb{P}(V)$.

Before going further we look at some examples of Seshadri stratifications. Further examples will be given in Sect. 16.

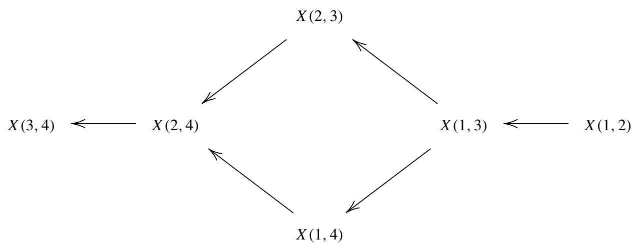
Example 2.5 Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{K}^4$. The wedge products $e_i \wedge e_j$, $1 \leq i < j \leq 4$, form a basis of $\bigwedge^2 \mathbb{K}^4$. Denote the indexing set of the basis by $I_{2,4}$, it consists of pairs of positive numbers (i, j), strictly increasing, and smaller or equal to 4. The corresponding elements $\{x_{i,j} \mid (i, j) \in I_{2,4}\}$ of the dual basis are called Plücker coordinates.

The set $I_{2,4}$ is endowed with a partial order: $(i, j) \leq (k, \ell)$ if and only if $i \leq k$ and $j \leq \ell$. Let $X := \mathrm{Gr}_2\mathbb{K}^4 \subseteq \mathbb{P}(\bigwedge^2 \mathbb{K}^4)$ be the Grassmann variety of 2-planes in $\mathbb{K}^4$, emdedded into $\mathbb{P}(\bigwedge^2 \mathbb{K}^4)$ via the Plücker embedding. Set theoretically, the Schubert varieties $X(i,j) \subseteq \mathrm{Gr}_2\mathbb{K}^4$ for $(i,j) \in I_{2,4}$ are defined by

$$X(i,j) := \{[v] \in \mathrm{Gr}_2\mathbb{K}^4 \mid x_{k,\ell}([v]) = 0 \text{ for all } (k,\ell) \in I_{2,4} \text{ such that } (k,\ell) \nleq (i,j)\}.$$

The collection of subvarieties $X(i,j)$, $(i,j) \in I_{2,4}$, together with the functions $f_{(i,j)} := x_{i,j}$, $(i,j) \in I_{2,4}$, define a Seshadri stratification on $\mathrm{Gr}_2\mathbb{K}^4$.

Below the Hasse diagram showing the inclusion relations between the Schubert varieties, here $X(i,j) \to X(k,\ell)$ means $X(i,j)$ is contained in $X(k,\ell)$ of codimension one. It depicts meanwhile the Hasse diagram of the partial order on $I_{2,4}$.



A Seshadri stratification of $\mathrm{Gr}_2\mathbb{C}^4$ is not necessarily given by Schubert varieties, see Remark 16.4.

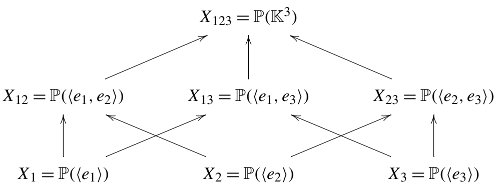
Example 2.6 More generally, consider the Grassmann variety $X := \mathrm{Gr}_d\mathbb{K}^n \subseteq \mathbb{P}(\bigwedge^d \mathbb{K}^n)$ of d-dimensional subspaces in $\mathbb{K}^n$, embedded into $\mathbb{P}(\bigwedge^d \mathbb{K}^n)$ via the Plücker embedding. The Schubert varieties in $\mathrm{Gr}_d\mathbb{K}^n$ are indexed by the set $I_{d,n} := \{\underline{i} = (i_1, \ldots, i_d) \mid 1 \le i_1 < \cdots < i_d \le n\}$ of strictly increasing sequences of length d. For $\underline{i} \in I_{d,n}$ let $X(\underline{i})$ denote the corresponding Schubert variety in $\mathrm{Gr}_d\mathbb{K}^n$. The partial order on $I_{d,n}$ induced by the inclusion of subvarieties coincides with the usual partial order on $I_{d,n}$: $\underline{i} \le \underline{j}$ if and only if $i_1 \le j_1, \ldots, i_d \le j_d$. For $\underline{i} \in I_{d,n}$ let $f_{\underline{i}} := x_{\underline{i}}$ be the Plücker coordinate. The collection of Schubert varieties $X(\underline{i})$, $\underline{i} \in I_{d,n}$ and the Plücker coordinates $x_{\underline{i}}$, $\underline{i} \in I_{d,n}$ define a Seshadri stratification on X.

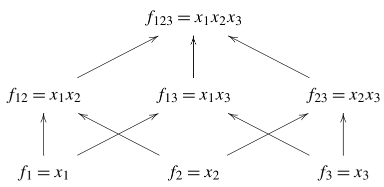
Example 2.7 Even on simple varieties such as projective spaces, there may exist *several* non-trivial Seshadri stratifications. One has been given in Example 2.6 as the special case $d = 1$. We define now a different stratification.

We consider the projective space $X := \mathbb{P}(\mathbb{K}^3)$. By fixing the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{K}^3$, the homogeneous coordinate ring $\mathbb{K}[X]$ can be identified with $\mathbb{K}[x_1, x_2, x_3]$.

As indexing set we take the power set $A := \mathfrak{P}(\{1,2,3\}) \setminus \emptyset$ by omitting the empty set. As the collection of subvarieties we set for a subset $p \in A$: $X_p := \mathbb{P}(\langle e_j \mid j \in p\rangle_{\mathbb{K}})$. The poset structure on A induced by the inclusion of subvarieties coincides with that arising from inclusion of sets. For $p \in A$, let $f_p = \prod_{j \in p} x_j$ be the extremal function.

We leave it to the reader to verify that the collection of subvarieties X_p, together with the monomials f_p, $p \in A$, define a Seshadri stratification on X. Below the inclusion diagram of the varieties, and, in the same scheme, the functions f_p corresponding to the subvariety X_p.



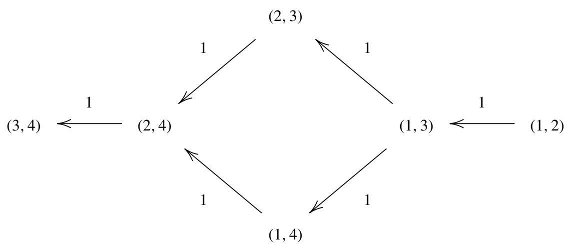


2.3 A Hasse diagram with bonds

For a given Seshadri stratification of a projective variety X consisting of subvarieties X_p and extremal functions f_p for $p \in A$, we associate to it in this section an edge-coloured directed graph.

Let $\mathcal{G}_A$ be the Hasse diagram of the poset A. The edges in $\mathcal{G}_A$ are covering relations in A. If p covers q, then the affine cone $\hat{X}_q$ of X_q is a prime divisor in the affine cone $\hat{X}_p$ of X_p. We denote by $b_{p,q} \geq 1$ the vanishing multiplicity of f_p at the prime divisor $\hat{X}_q$ (see Sect. 3.1 for the definition of the vanishing multiplicity), it is called the *bond* between p and q. The Hasse diagram with bonds is the diagram with edges coloured with the corresponding bonds: $q \xrightarrow{b_{p,q}} p$.

Example 2.8 For $X = \mathrm{Gr}_2\mathbb{K}^4 \subseteq \mathbb{P}(\bigwedge^2 \mathbb{K}^4)$ as in Example 2.5, the corresponding Hasse diagram with bonds is:



More generally, for the Grassmann variety $\mathrm{Gr}_d\mathbb{K}^n$ in Example 2.6, all the bonds are 1.

As we will see in the following example, the bonds in a Seshadri stratification are not necessarily one.

Example 2.9 To avoid technical details in small characteristics, we assume in this example that the characteristic of $\mathbb{K}$ is zero or a large prime number.

Let SL_3 be the group of 3×3 matrices over $\mathbb{K}$ having determinant 1. Its Lie algebra $\mathfrak{sl}_3$ consists of traceless 3×3 matrices over $\mathbb{K}$. Let $B \subseteq \mathrm{SL}_3$ be the subgroup consisting of upper triangular matrices.

The group SL_3 acts linearly on its Lie algebra $\mathfrak{sl}_3$ via the adjoint representation: for $g \in \mathrm{SL}_3$ and $M \in \mathfrak{sl}_3$, $g \cdot M := gMg^{-1}$. By choosing M to be a root vector for the highest root, this action induces an embedding $\mathrm{SL}_3/B \hookrightarrow \mathbb{P}(\mathfrak{sl}_3)$.

Let S_3 be the Weyl group of SL_3: it is the symmetric group acting on three letters. By abuse of notation we identify $\sigma \in S_3$ with an appropriately chosen representative $\sigma \in SL_3$; it is, up to the sign of the entries, the corresponding permutation matrix.

In the Bruhat decomposition $SL_3 = \bigsqcup_{\sigma \in S_3} B\sigma B$ of SL_3, the class of the closure of each cell in SL_3/B

$$X(\sigma) := \overline{B\sigma B}/B \subseteq SL_3/B$$

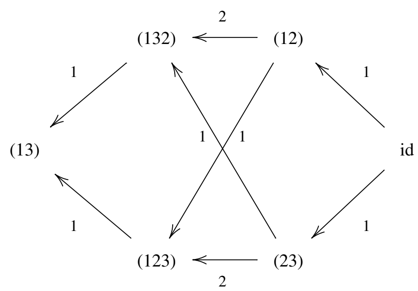
is the Schubert variety associated to $\sigma \in S_3$. We fix a basis of $\mathfrak{sl}_3$ as follows:

$$v_{(13)} := E_{3,1},\ v_{(132)} := E_{3,2},\ v_{(123)} := E_{2,1},\ v_{(23)} := E_{1,2},\ v_{(12)} := E_{2,3},\ v_{\mathrm{id}} = E_{1,3},$$

$$h_1 := E_{1,1} - E_{2,2},\ h_2 := E_{2,2} - E_{3,3},$$

where $E_{i,j}$ stands for the matrix whose only non-zero entry is a 1 at the i-th row and j-th column and the indexes of v are permutations. For $\sigma \in S_3$, let f_σ be the dual basis of v_σ. Then the Schubert varieties $X(\sigma)$ and the extremal functions f_σ for $\sigma \in S_3$ define a Seshadri stratification on the embedded projective variety SL_3/B.

The bonds can be determined using the Pieri-Chevalley formula [9]. The Hasse diagram with bonds for this Seshadri stratification is depicted below:



For the example of a Seshadri stratification of a Schubert variety in a flag variety, see Sect. 16.6.

Remark 2.10 Later in the article, we will mainly consider the affine cone $\hat{X}$ of X, so it is helpful to extend the stratification one step further.

For a minimal element $p_0 \in A$, the affine cone $\hat{X}_{p_0} \simeq \mathbb{A}^1$ is an affine line. Let $\hat{X}_{p_{-1}}$ denote the origin of V: it is contained in the affine cone of X_p for any minimal element $p \in A$. The bond $b_{p_0,p_{-1}}$ is defined to be the vanishing multiplicity of f_{p_0} at $\hat{X}_{p_{-1}}$, which coincides with the degree of f_{p_0}.

The set $\hat{A} := A \cup \{p_{-1}\}$ admits a poset structure by requiring p_{-1} to be the unique minimal element. This poset structure is compatible with the containment relations between the affine cones $\hat{X}_p$, $p \in \hat{A}$. Similarly we have the Hasse diagram $\mathcal{G}_{\hat{A}}$, the bonds on it are described as above.

2.4 Generic hyperplane stratifications

According to the following proposition, the requirements in a Seshadri stratification are not as restrictive as it looks.

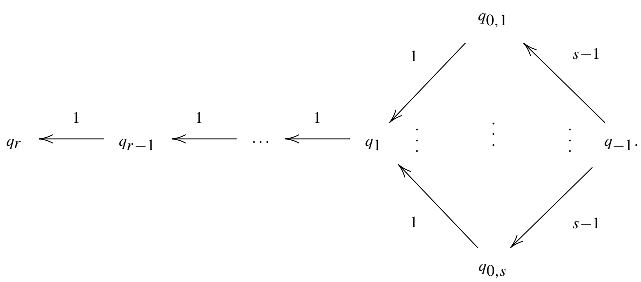
Proposition 2.11 *Every embedded projective variety $X \subseteq \mathbb{P}(V)$, smooth in codimension one, admits a Seshadri stratification.*

Proof Let r be the dimension of X. For $r \geq 2$, applying the Bertini theorem [35, Théorème 6.3] to the smooth locus of $X_r := X$, there (generically) exists $f_r \in V^*$ such that $X_{r-1} := X \cap \mathcal{H}_{f_r}$ is an irreducible variety which is again smooth in codimension one and has the same degree as X. Repeating this construction gives irreducible varieties $X_{r-2}, \ldots, X_1$ and $f_{r-1}, \ldots, f_2 \in V^*$ such that $X_k = X_{k+1} \cap \mathcal{H}_{f_{k+1}}$ is smooth in codimension one and the degree of X_k is the same as the degree of X for $k = 1, \ldots, r-2$. Now X_1 is a smooth curve. A generic hyperplane $\mathcal{H}_{f_1}$ for $f_1 \in V^*$ intersects X_1 at finitely many points $X_{0,1} = \{\varpi_1\}, \ldots, X_{0,s} := \{\varpi_s\}$. It suffices to choose homogeneous functions $f_{0,k}$, $1 \leq k \leq s$ on V satisfying: $f_{0,k}$ vanishes on ϖ_ℓ for $\ell \neq k$, but $f_{0,k}$ is non-zero at ϖ_k. □

Since the hyperplanes are chosen generically (i.e. in an open set), the geometric definition of the degree of an embedded variety implies $\deg X = s$, the number of points we get in the last intersection.

A Seshadri stratification arising from generic hyperplanes in this way will be termed a *generic hyperplane stratification*. Let $A = \{q_r, \ldots, q_1, q_{0,1}, \ldots, q_{0,s}\}$ be the indexing poset with $X_{q_k} := X_k$ and $X_{q_{0,\ell}} := \{\varpi_\ell\}$.

Example 2.12 The functions $f_{0,k}$ in the above proposition can be chosen in the following precise way. Let $h_i \in V^*$ be such that $h_i(\varpi_j) \neq 0$ for $1 \leq j \leq s$ with $j \neq i$ and $h_i(\varpi_i) = 0$. We define $g_k = \prod_{i \neq k} h_i$: it is homogeneous of degree $s-1$. These functions satisfy the requirement on $f_{0,k}$ in the above proof. With this choice, the extended Hasse diagram $\mathcal{G}_{\hat{A}}$ with bonds is depicted below:



Remark 2.13 In [30] Hibi proved that every finitely generated positively graded ring admits a Hodge algebra structure. The construction in Proposition 2.11 resembles a geometric version of the algebraic approach in *loc.cit.* under the smooth in codimension one assumption. In our setup, this extra condition allows us to extract a convex

geometric skeleton of X (see Corollary 12.3 and Sect. 13.5), and to deduce geometric properties of X (see Corollary 16.2).

In [39] a similar Bertini-type argument is applied to construct flat degenerations of embedded projective varieties into complexity-one T-varieties.

3 Generalities on valuations

From now on until Sect. 15, we fix a Seshadri stratification of $X \subseteq \mathbb{P}(V)$ with subvarieties X_p and extremal functions f_p for $p \in A$.

3.1 Definition and example

We recall the definition and some basic properties of valuations and quasi-valuations.

Definition 3.1 Let $\mathcal{R}$ be a $\mathbb{K}$-algebra. A *quasi-valuation* on $\mathcal{R}$ with values in a totally ordered abelian group G is a map $\nu : \mathcal{R} \setminus \{0\} \to \mathrm{G}$ satisfying the following conditions:

(a) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in \mathcal{R} \setminus \{0\}$ with $x + y \neq 0$;
(b) $\nu(\lambda x) = \nu(x)$ for all $x \in \mathcal{R} \setminus \{0\}$ and $\lambda \in \mathbb{K}^*$;
(c) $\nu(xy) \geq \nu(x) + \nu(y)$ for all $x, y \in \mathcal{R} \setminus \{0\}$ with $xy \neq 0$.

The map ν is called a *valuation* if the inequality in (c) can be replaced by an equality:

(c′) $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in \mathcal{R} \setminus \{0\}$ with $xy \neq 0$.

Remark 3.2 Quasi-valuations on $\mathcal{R}$ can be thought of as synonyms of algebra filtrations on $\mathcal{R}$ (see Sect. 2.4 in [38]).

The following properties are well-known for valuations (see [56, Lemma 2.1.1]). Since the proof uses only axioms (a) and (b) of a valuation, it holds also for quasi-valuations.

Lemma 3.3 *Let* $\nu : \mathcal{R} \setminus \{0\} \to \mathrm{G}$ *be a quasi-valuation and* $x, y \in \mathcal{R} \setminus \{0\}$.

i) *If* $\nu(x) \neq \nu(y)$, *then* $\nu(x + y) = \min\{\nu(x), \nu(y)\}$.
ii) *If* $x + y \neq 0$ *and* $\nu(x + y) > \nu(x)$, *then* $\nu(x) = \nu(y)$.

Lemma 3.4 ([27, Proposition 4.1]) *Let* $\nu_1, \ldots, \nu_k : \mathcal{R} \setminus \{0\} \to \mathrm{G}$ *be a family of quasi-valuations. The map* $\nu : \mathcal{R} \setminus \{0\} \to \mathrm{G}$ *defined by*

$$g \mapsto \min\{\nu_j(g) \mid j = 1, \ldots, k\}$$

is a quasi-valuation on $\mathcal{R}$.

In algebraic geometry, valuations usually arise from vanishing orders of rational functions.

We come back to the setup in Sect. 2.2. Let $R_p := \mathbb{K}[X_p]$ be the homogeneous coordinate ring of X_p with respect to the embedding $X_p \subseteq X \subseteq \mathbb{P}(V)$. In the following we consider R_p often as the coordinate ring of the affine cone $\hat{X}_p \subseteq V$ over X_p.

If p covers q in $\hat{A}$, then $\hat{X}_q \subseteq \hat{X}_p$ is a prime divisor in $\hat{X}_p$. The local ring $\mathcal{O}_{\hat{X}_p,\hat{X}_q}$ is a discrete valuation ring because $\hat{X}_p$ is smooth in codimension one by (S1). Let $\nu_{p,q}$ be the associated valuation. We refer to the value $\nu_{p,q}(f)$ for $f \in R_p \setminus \{0\}$ as the *vanishing multiplicity* of f in the divisor $\hat{X}_q$:

$$\nu_{p,q} : R_p \setminus \{0\} \to \mathbb{Z}. \tag{1}$$

The valuation $\nu_{p,q}$ can be naturally extended to a valuation on $\mathbb{K}(\hat{X}_p) = \operatorname{Quot} R_p$, the quotient field of R_p, by the rule:

$$\nu_{p,q}\left(\frac{g}{h}\right) := \nu_{p,q}(g) - \nu_{p,q}(h) \text{ for } g, h \in R_p \setminus \{0\}.$$

Remark 3.5 As a continuation of Remark 2.10, for a minimal element $p_0 \in A$, R_{p_0} is a polynomial ring. We define $\nu_{p_0,p_{-1}}$ to be the vanishing multiplicity of a polynomial in R_{p_0} in $\{0\}$.

3.2 Valuation under normalization

The following Lemma compares the prime divisors in the normalization $\tilde{X}_p$ of $\hat{X}_p$ with those in $\hat{X}_p$. Note that X_p is smooth in codimension one.

Lemma 3.6 *The normalization map $\omega : \tilde{X}_p \to \hat{X}_p$ induces a bijection $\omega_D : \tilde{Z} \mapsto \omega(\tilde{Z})$ between the set of prime divisors $\tilde{Z} \subseteq \tilde{X}_p$ and the set of prime divisors $Z \subseteq \hat{X}_p$. In addition, for all non-zero $f \in \mathbb{K}(\hat{X}_p) = \mathbb{K}(\tilde{X}_p)$ and all prime divisors $\tilde{Z} \subseteq \tilde{X}_p$ holds: $\nu_{\tilde{Z}}(f) = \nu_{\omega(\tilde{Z})}(f)$.*

Proof Let $\hat{U} \subseteq \hat{X}_p$ be the open and dense subset of smooth points and denote by $\tilde{U} \subseteq \tilde{X}_p$ the open and dense subset obtained as preimage $\omega^{-1}(\hat{U})$.

By axiom (S1), $\hat{X}_p \setminus \hat{U}$ is of codimension greater or equal to 2. Since the normalization map is finite, the same holds for $\tilde{X}_p \setminus \tilde{U}$. It follows that any prime divisor as well as the induced valuation in $\hat{X}_p$ respectively $\tilde{X}_p$ is completely determined by its intersection with $\hat{U}$ respectively $\tilde{U}$. The claim follows now since the normalization map $\omega : \tilde{X}_p \to \hat{X}_p$ induces an isomorphism $\omega|_{\tilde{U}} : \tilde{U} \to \hat{U}$. □

3.3 Rees quasi-valuations

For an element $p \in A$ let $I_p = (f_p|_{\hat{X}_p})$ be the principal ideal in R_p generated by the restriction of the extremal function f_p (see Definition 2.1) to $\hat{X}_p$. By abuse of notation we write in the following often just f_p instead of $f_p|_{\hat{X}_p}$. We define a map $\nu_{I_p} : R_p \setminus \{0\} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by:

$$\text{for any } g \in R_p \setminus \{0\},\ \nu_{I_p}(g) := \max\left\{m \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \,\middle|\, g \in I_p^m\right\}. \tag{2}$$

In our situation the value ∞ is never attained.

Lemma 3.7 ([63, Sect. 2.2]) *ν_{I_p} defines a quasi-valuation $\nu_{I_p} : R_p \setminus \{0\} \to \mathbb{Z}$.*

Such a quasi-valuation is called *homogeneous*, if for any $g \in R_p \setminus \{0\}$ and $m \in \mathbb{N}$, $\nu_{I_p}(g^m) = m\nu_{I_p}(g)$. The quasi-valuation ν_{I_p} is not necessarily homogeneous. Samuel [64] introduced a limit procedure to homogenize such a quasi-valuation.

We denote by $\bar{\nu}_{I_p}$ the corresponding *homogenized quasi-valuation* ([63, Lemma 2.11]): by definition, for $g \in R_p \setminus \{0\}$, we set

$$\bar{\nu}_{I_p}(g) := \lim_{k\to\infty} \frac{\nu_{I_p}(g^k)}{k} \in \mathbb{Q}_{\geq 0} \cup \{\infty\}. \tag{3}$$

The limit exists and is a non-negative rational number (see [58, 62]). The homogenized quasi-valuation has an interpretation as the minimum function over some rescaled discrete valuations:

Theorem 3.8 ([62, Valuation Theorem]) *There exists a finite set of discrete, integer-valued valuations $\eta_1, \ldots, \eta_k$ on R_p and integers $e_1, \ldots, e_k$ such that for all $g \in R_p \setminus \{0\}$:*

$$\bar{\nu}_{I_p}(g) := \min \left\{ \frac{\eta_j(g)}{e_j} \;\middle|\; j = 1, \ldots, k \right\}.$$

In our setting, the valuations η_j get an interpretation as vanishing multiplicities at the prime divisors occurring in $X_p \cap \mathcal{H}_{f_p}$. Let $q_1, \ldots, q_k \in A$ be such that

$$X_p \cap \mathcal{H}_{f_p} = \bigcup_{i=1,\ldots,k} X_{q_i}.$$

The affine cones $\hat{X}_{q_i}$ are prime divisors in $\hat{X}_p$ inducing valuations $\nu_{p,q_i} : R_p \setminus \{0\} \to \mathbb{Z}$ (see (1)). Let b_{p,q_i}, $i = 1, \ldots, k$, be the corresponding bonds, which are the vanishing multiplicities of f_p at the prime divisors $\hat{X}_{q_i}$ (see Sect. 2.3). The Valuation Theorem gets the following interpretation:

Proposition 3.9 *Let $g \in R_p \setminus \{0\}$. We have*

$$\bar{\nu}_{I_p}(g) = \min \left\{ \frac{\nu_{p,q_i}(g)}{b_{p,q_i}} \;\middle|\; i = 1, \ldots, k \right\}. \tag{4}$$

Proof Let $\tilde{R}_p$ be the normalization of R_p (in Quot R_p), $\tilde{I}_p$ be the principal ideal in $\tilde{R}_p$ generated by f_p and set $\tilde{Y}_p := \operatorname{Spec} \tilde{R}_p$, the normalization of $\hat{X}_p$. We can construct the quasi-valuation $\nu_{\tilde{I}_p} : \tilde{R}_p \setminus \{0\} \to \mathbb{Z}$ and its homogenized version $\bar{\nu}_{\tilde{I}_p}$ in the same way as in (2) and (3). The constructions are related by Lemma 2.2 in [62]: one has for $g \in R_p \setminus \{0\}$, $\bar{\nu}_{\tilde{I}_p}(g) = \bar{\nu}_{I_p}(g)$.

Let Σ be the set of all valuations $\nu_Z : \operatorname{Quot} \tilde{R}_p \setminus \{0\} \to \mathbb{Z}$ induced by prime divisors $Z \subseteq \tilde{Y}_p$. Since $\tilde{R}_p$ is an integrally closed noetherian ring, this set of valuations makes $\tilde{R}_p$ into a finite discrete principal order (a.k.a. Krull domain, see [8] Chapter VII,

§1.3), so $\tilde{R}_p$ satisfies the assumptions for Lemma 2.3 in [62]. The proof of the lemma identifies the valuations in Theorem 3.8 as the valuations ν_Z such that $\nu_Z(f_p|_{\hat{X}_p}) > 0$. Keeping in mind the bijection in Lemma 3.6 and (S3) in Definition 2.1, we see that in such a case $\omega(Z) = \hat{X}_{q_i}$ is an irreducible component of the vanishing set of $f_p|_{\hat{X}_p}$, and the scaling factor is $b_{p,q_i}^{-1} = (\nu_{p,q_i}(f_p))^{-1}$ (see the proof of Lemma 2.3 in [62]). □

Remark 3.10 The appropriate reformulation of the proposition above holds for elements in the normalization too. The proof above together with Lemma 3.6 shows for $g \in \tilde{R}_p \setminus \{0\}$:

$$\bar{\nu}_{\tilde{I}_p}(g) = \min\left\{ \frac{\nu_{p,q_i}(g)}{b_{p,q_i}} \,\middle|\, i = 1, \ldots, k \right\}. \tag{5}$$

4 Codimension one

In this section we modify the procedure in [37, 50] in codimension one to overcome the difficulty mentioned in the introduction. In the next two sections we will link the codimension one constructions to obtain a higher rank valuation.

Let $p, q \in A$ be such that p covers q and $g \in \mathbb{K}(\hat{X}_p) \setminus \{0\}$ be a rational function. Let $b_{p,q} = \nu_{p,q}(f_p)$ be the bond between p and q (see Sect. 2.3). Let N be the least common multiple of all bonds in $\mathcal{G}_A$, so the number $N\frac{\nu_{p,q}(g)}{b_{p,q}} \in \mathbb{Z}$. We set

$$h := \frac{g^N}{f_p^{N\frac{\nu_{p,q}(g)}{b_{p,q}}}} \in \mathbb{K}(\hat{X}_p). \tag{6}$$

Lemma 4.1 *The restriction $h|_{\hat{X}_q}$ is a well-defined, non-zero rational function on $\hat{X}_q$.*

Proof Note that

$$\nu_{p,q}(h) = \nu_{p,q}\left(\frac{g^N}{f_p^{N\frac{\nu_{p,q}(g)}{b_{p,q}}}}\right) = N\nu_{p,q}(g) - N\frac{\nu_{p,q}(g)}{b_{p,q}}b_{p,q} = 0,$$

so this rational function is an element of the local ring $\mathcal{O}_{\hat{X}_p,\hat{X}_q}$ of the prime divisor $\hat{X}_q \subseteq \hat{X}_p$. But it is not in its maximal ideal $\mathfrak{m}_{\hat{X}_q,\hat{X}_p}$ and hence its restriction gives a non-zero element in the residue field $\mathcal{O}_{\hat{X}_q,\hat{X}_p}/\mathfrak{m}_{\hat{X}_q,\hat{X}_p}$, which is the field $\mathbb{K}(\hat{X}_q)$ of rational functions on $\hat{X}_q$. □

Remark 4.2 Instead of taking the function h as above one could take $\tilde{h} := g^{b_{p,q}}/f_p^{\nu_{p,q}(g)}$. Lemma 4.1 holds for $\tilde{h}$ with the same proof. In fact, $h = \tilde{h}^{\frac{N}{b_{p,q}}}$. Since it is later more convenient to work uniformly with the N-th power instead of the $b_{p,q}$-th power, we will stick to this construction. For the valuation which will be defined in

Sect. 6.2 the choice makes no difference, one just has to rescale the values appropriately, see Remark 6.5.

What can we say about h if the starting function $g \in R_p$ is a regular function?

For the inductive procedure we will use later it is necessary to take a slightly more general point of view. Consider again the setting in Proposition 3.9 respectively Remark 3.10 and let $q_1, \ldots, q_k \in A$ be such that $X_p \cap \mathcal{H}_{f_p} = \bigcup_{i=1,\ldots,k} X_{q_i}$. Let $g \in \mathbb{K}(\hat{X}_p) \setminus \{0\}$ be a rational function which is integral over $\mathbb{K}[\hat{X}_p]$. By Lemma 3.6, this property is equivalent to $\nu_Z(g) \geq 0$ for all prime divisors $Z \subseteq \hat{X}_p$.

Proposition 4.3 *Let $g \in \mathbb{K}(\hat{X}_p) \setminus \{0\}$ be a rational function which is integral over $\mathbb{K}[\hat{X}_p]$. We assume that the enumeration of the divisors $\hat{X}_{q_1}, \ldots, \hat{X}_{q_k}$ is such that*

$$\forall i = 1, \ldots k, \ \bar{\nu}_{\tilde{I}_p}(g) = \frac{\nu_{p,q_1}(g)}{b_{p,q_1}} \leq \frac{\nu_{p,q_i}(g)}{b_{p,q_i}}. \tag{7}$$

Set $h = g^N f_p^{-N\frac{\nu_{p,q_1}(g)}{b_{p,q_1}}}$ (as in (6)). Then h is integral over $\mathbb{K}[\hat{X}_p]$, and $h|_{\hat{X}_{q_1}} \in \mathbb{K}(\hat{X}_{q_1})$ is integral over $\mathbb{K}[\hat{X}_{q_1}]$.

Proof Given a prime divisor $Z \subseteq \hat{X}_p$, we have

$$\nu_Z(h) = N\left(\nu_Z(g) - \nu_Z(f_p)\frac{\nu_{p,q_1}(h)}{b_{p,q_1}}\right).$$

By assumption we have $\nu_Z(g) \geq 0$ for all prime divisors $Z \subseteq \hat{X}_p$ and $\nu_Z(f_p) = 0$ for $Z \neq \hat{X}_{q_i}$, $j = 1, \ldots, k$. It follows: if $Z \neq \hat{X}_{q_j}$, then $\nu_Z(h) \geq 0$.

For the prime divisors $\hat{X}_{q_j}$, $j = 1, \ldots, k$, and the associated valuations ν_{p,q_j} we obtain:

$$\begin{aligned} \nu_{p,q_j}(h) = \nu_{p,q_j}\left(\frac{g^N}{f_p^{N(\frac{\nu_{p,q_1}(g)}{b_{p,q_1}})}}\right) &= N(\nu_{p,q_j}(g) - \tfrac{\nu_{p,q_1}(g)}{b_{p,q_1}}\nu_{p,q_j}(f_p)) \\ &= N\nu_{p,q_j}(f_p)\left(\frac{\nu_{p,q_j}(g)}{b_{p,q_j}} - \frac{\nu_{p,q_1}(g)}{b_{p,q_1}}\right) \geq 0 \end{aligned}$$

by the choice of q_1 (see (7)). Hence $\nu_Z(h) \geq 0$ for all prime divisors $Z \subseteq \hat{X}_p$, so h is integral in $\mathbb{K}(\hat{X}_p)$ over $\mathbb{K}[\hat{X}_p]$. By Lemma 4.1, $h|_{\hat{X}_{q_1}}$ is a well-defined, non-zero rational function, and hence $h|_{\hat{X}_{q_1}} \in \mathbb{K}(\hat{X}_{q_1})$ is also integral over $K[\hat{X}_{q_1}]$. □

5 Maximal chains and sequences of rational functions

We fix a maximal chain in A joining $p_{\max}$ with a minimal element p_0:

$$\mathfrak{C} : p_{\max} = p_r > \cdots > p_2 > p_1 > p_0. \tag{8}$$

In particular, r is the length of every maximal chain. To avoid double indexes as much as possible, we use abbreviations. Since we have fixed a maximal chain, an element $p \in \mathfrak{C}$ is either the minimal element, or there is a unique element in $\mathfrak{C}$, say q, covered by p. It makes sense to omit the second index and write ν_p and b_p instead of $\nu_{p,q}$ and $b_{p,q}$. Moreover, when $p = p_i$ we will simplify the notation further by just writing

$$\nu_i \text{ (resp. } b_i,\ f_i,\ X_i) \text{ instead of } \nu_{p_i,p_{i-1}} \text{ (resp. } b_{p_i,p_{i-1}},\ f_{p_i},\ X_{p_i}). \tag{9}$$

We use the procedure in (6) to attach to a non-zero function $g \in R$ inductively a sequence $(g_r, \dots, g_0)$ of non-zero rational functions $g_j \in \mathbb{K}(\hat{X}_j)$, $j = 0, 1, \dots, r$. For each $j = 0, 1, \dots, r$, the function g_j will depend on g and $f_{j+1}, \dots, f_r$. Starting with a regular function $g \in R \setminus \{0\}$, the following inductive procedure is well defined by Lemma 4.1: we set

$$g_r := g \quad \text{and} \quad D_r = \frac{\nu_r(g_r)}{b_r},$$

and then inductively for $j = r-1, \dots, 1, 0$:

$$g_j = \left.\frac{g_{j+1}^N}{f_{j+1}^{ND_{j+1}}}\right|_{\hat{X}_j} \quad \text{and} \quad D_j = \frac{\nu_j(g_j)}{b_j}. \tag{10}$$

Remark 5.1 We can provide a *nearly* closed formula for g_i: for $r \geq j \geq i+1$ let D_j be as defined above in (10), then

$$g_i = g^{N^{r-i}} f_r^{-N^{r-i}D_r} f_{r-1}^{-N^{r-i-1}D_{r-1}} \cdots f_{i+1}^{-ND_{i+1}}|_{\hat{X}_i}. \tag{11}$$

It is only nearly closed because the valuations $\nu_j(g_j)$, $j > i$, show up in the formula for the numbers D_j. But for our purpose this will be good enough.

We give this sequence of functions a name:

Definition 5.2 The tuple $g_{\mathfrak{C}} := (g_r, \dots, g_1, g_0)$ associated to $g \in R \setminus \{0\}$ is called the *sequence of rational functions* associated to g along $\mathfrak{C}$.

Before going further to define a higher rank valuation from this sequence of rational functions, we look at some concrete examples.

Example 5.3 The constant function a, $a \in \mathbb{K}^*$, vanishes nowhere, so $\nu_j(a|_{\hat{X}_j}) = 0$ for all $j = 1, \dots, r$, and the sequence associated to the constant function is $a_{\mathfrak{C}} = (a, a^N, \dots, a^{N^r})$. Note that this holds independent of the choice of the maximal chain $\mathfrak{C}$.

Example 5.4 We consider the case when g is the extremal function f_i for some $0 \leq i \leq r$. By Lemma 2.2, f_i does not vanish identically on $\hat{X}_j$ for $j \geq i$, one determines inductively:

$$g_r = f_i, D_r = 0, \text{ and } g_{r-1} = f_i^N, D_{r-1} = 0, \text{ and } \dots, \text{ and}$$
$$g_{i+1} = f_i^{N^{r-i-1}}, D_{i+1} = 0.$$

Next consider the function $g_i = f_i^{N^{r-i}}$. This function vanishes on the divisor $\hat{X}_{i-1}$ and we have $D_i = \frac{\nu_i(f_i^{N^{r-i}})}{b_i} = N^{r-i}$ by the definition of b_i. It follows

$$g_{i-1} = \frac{g_i^N}{(f_i)^{ND_i}} = \frac{f_i^{N^{r-i+1}}}{f_i^{N^{r-i+1}}} = 1 \text{ and } D_{i-1} = 0.$$

The procedure implies now $g_j = 1$ for all $j < i$. Summarizing the above computation gives

$$(f_i)_{\mathfrak{C}} = (f_i, f_i^N, f_i^{N^2}, \dots, f_i^{N^{r-i}}, 1, \dots, 1).$$

This holds for all choices of a maximal chain $\mathfrak{C}$ as long as $p_i \in \mathfrak{C}$. If $p \notin \mathfrak{C}$, then $(f_p)_{\mathfrak{C}}$ looks rather different, as the following example shows.

Example 5.5 Let $X = \mathrm{Gr}_2\mathbb{K}^4$ with the Seshadri stratification defined in Example 2.5 and 2.8. The bonds are all equal to 1 and hence $N = 1$. We fix as maximal chain

$$\mathfrak{C} : (3, 4) > (2, 4) > (2, 3) > (1, 3) > (1, 2).$$

For the Plücker coordinates we get as sequences of rational functions:

$$\begin{aligned}
(x_{3,4})_{\mathfrak{C}} &= (x_{3,4}, 1, 1, 1, 1), & (x_{2,4})_{\mathfrak{C}} &= (x_{2,4}, x_{2,4}, 1, 1, 1),\\
(x_{2,3})_{\mathfrak{C}} &= (x_{2,3}, x_{2,3}, x_{2,3}, 1, 1), & (x_{1,4})_{\mathfrak{C}} &= (x_{1,4}, x_{1,4}, \tfrac{x_{1,4}}{x_{2,4}}, \tfrac{x_{2,3}x_{1,4}}{x_{2,4}}, 1),\\
(x_{1,3})_{\mathfrak{C}} &= (x_{1,3}, x_{1,3}, x_{1,3}, x_{1,3}, 1), & (x_{1,2})_{\mathfrak{C}} &= (x_{1,2}, x_{1,2}, x_{1,2}, x_{1,2}, x_{1,2}).
\end{aligned}$$

The Plücker coordinate $x_{1,4}$ is the only extremal function whose index is not in $\mathfrak{C}$. By Example 5.4, the sequence $(x_{1,4})_{\mathfrak{C}} = (g_4, \dots, g_0)$ is the only one which needs an explanation.

Recall that for the sequence $(g_4, \dots, g_0)$ of rational functions associated to $x_{1,4}$ along $\mathfrak{C}$, we denote $D_j = \nu_j(g_j)/b_j$ for $j = 0, 1, \dots, 4$. In this example all bonds are equal to 1, hence $D_j = \nu_j(g_j)$.

The restriction of $x_{1,4}$ to $\hat{X}(2, 4)$ does not vanish identically and hence $D_4 = 0$; this gives $g_4 = g_3 = x_{1,4}$.

In order to compute the vanishing order, we introduce coordinates on an open subset of $\hat{X}(2, 4)$. For this we choose a maximal torus and a Borel subgroup for $\mathrm{SL}_4(\mathbb{K})$ as in Example 2.9. We use the standard enumeration of the simple roots, i.e. $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$ and $\alpha_3 = \epsilon_3 - \epsilon_4$.

For a root α, let $U_\alpha \subseteq \mathrm{SL}_4(\mathbb{K})$ be the corresponding root subgroup. The root subgroup associated to the a (negative) simple root is given by

$$U_{-\alpha_i}(t) = \mathbb{1} + tE_{i+1,i}, \quad i = 1, 2, 3.$$

We consider in $\hat{X}(2,4)$ the open subset $U_{-\alpha_3}(d)U_{-\alpha_1}(c)U_{-\alpha_2}(b)(ae_1 \wedge e_2)$, it is equal to

$$\{ae_1 \wedge e_2 + abe_1 \wedge e_3 + abce_2 \wedge e_3 + abde_1 \wedge e_4 + abcde_2 \wedge e_4 \mid a, b, c, d \in \mathbb{K}\}. \tag{12}$$

This open subset is compatible with the other Schubert varieties in the maximal chain $\mathfrak{C}$ contained in $\hat{X}(2,4)$, i.e. we get open subsets in the affine cones over these Schubert varieties by setting some of the coordinates equal to 0. For instance, by setting $d = 0$ we obtain an open subset in the affine cone $\hat{X}(2,3)$.

The restricted Plücker coordinate $g_3 = x_{1,4}|_{\hat{X}(2,4)}$ vanishes on the divisor $\hat{X}(2,3)$ with multiplicity 1, so $D_3 = 1$. To get g_2 in the sequence $(x_{1,4})_{\mathfrak{C}}$ we have to divide g_3 by $x_{2,4}|_{\hat{X}(2,4)}$ and get $g_2 = \frac{x_{1,4}}{x_{2,4}}|_{\hat{X}(2,3)}$. The rational function $\frac{x_{1,4}}{x_{2,4}}$ takes on the open set in (12) the value $\frac{1}{c}$, so $\frac{x_{1,4}}{x_{2,4}}|_{\hat{X}(2,3)}$ has a pole of order 1 at the divisor $\hat{X}(1,3) \subseteq \hat{X}(2,3)$, which implies $D_2 = -1$. Hence, for the term g_1 in the sequence $(x_{1,4})_{\mathfrak{C}}$, one has to multiply g_2 by $x_{2,3}$ and obtains $g_1 = \frac{x_{1,4}x_{2,3}}{x_{2,4}}|_{\hat{X}(1,3)}$. This rational function takes on the open set in (12) the same values as the Plücker coordinate $x_{1,3}$, and hence vanishes with multiplicity 1 on the divisor $\hat{X}(1,2) \subseteq \hat{X}(1,3)$, so we have $D_1 = 1$. We have hence to divide g_1 by $x_{1,3}$ to get g_0, which is the constant function 1.

The calculations may be simplified by using the Plücker relation $x_{1,2}x_{3,4} + x_{2,3}x_{1,4} - x_{1,3}x_{2,4} = 0$.

Even if one starts with a regular function $g \in R \setminus \{0\}$, for a fixed maximal chain $\mathfrak{C}$ there is no reason why a function g_j occurring in the sequence $g_{\mathfrak{C}} = (g_r, \ldots, g_0)$ should be a regular function on the respective subvariety $\hat{X}_{p_j}$, see Example 5.5 with $g = x_{1,4}$. Nevertheless, Proposition 4.3 shows that for a fixed function, poles can be avoided if one chooses the maximal chain carefully:

Corollary 5.6 *For every regular function $g \in R \setminus \{0\}$ there exists a maximal chain $\mathfrak{C}$ so that the associated tuple $g_{\mathfrak{C}} = (g_r, \ldots, g_0)$ consists of rational functions $g_j \in \mathbb{K}(\hat{X}_{p_j})$ such that $\nu_j(g_j) \geq 0$ for all $j = 1, \ldots, r$.*

6 Valuations from maximal chains

Throughout this section, we will work with a fixed maximal chain in A:

$$\mathfrak{C} : p_{\max} = p_r > p_{r-1} > \cdots > p_0.$$

As in the previous section, to avoid the usage of double indexes as much as possible, we keep the conventions in (9). Moreover, if not mentioned otherwise, we will write e_i instead of e_{p_i}.

Let $\mathbb{Q}^{\mathfrak{C}}$ be the $\mathbb{Q}$-vector space with basis $\{e_j \mid j = 0, \ldots, r\}$. We will write $v = (a_r, \ldots, a_0)$ for the vector $v = \sum_{j=0}^{r} a_j e_j \in \mathbb{Q}^{\mathfrak{C}}$.

Definition 6.1 We endow $\mathbb{Q}^{\mathfrak{C}}$ with the *lexicographic order*, i.e.

$(a_r, \dots, a_0) \geq (b_r, \dots, b_0)$ if and only if $a_r > b_r$, or $a_r = b_r$ and $a_{r-1} > b_{r-1}$, or etc.

This total order is compatible with the addition of vectors: for any $u, v, w \in \mathbb{Q}^{\mathfrak{C}}$, if $u \geq v$, then $u + w \geq v + w$ holds.

6.1 Linking up valuations

We start with linking together the rank one valuations arising from the covering relations in the poset A.

Definition 6.2 For $g \in R \setminus \{0\}$ let $g_{\mathfrak{C}} = (g_r, \dots, g_0)$ be the associated sequence of rational functions (see Definition 5.2). We attach to the maximal chain $\mathfrak{C}$ the map

$$\mathcal{V}_{\mathfrak{C}} : R \setminus \{0\} \to \mathbb{Q}^{\mathfrak{C}}, \quad g \mapsto \sum_{j=0}^{r} \left(\frac{\nu_j(g_j)}{N^{r-j} b_j} \right) e_j.$$

Remark 6.3 In terms of the numbers D_j, $j = 0, \dots, r$, defined in (10), we have:

$$\mathcal{V}_{\mathfrak{C}}(g) = \sum_{j=0}^{r} \frac{D_j}{N^{r-j}} e_j.$$

We will prove in Proposition 6.10 that $\mathcal{V}_{\mathfrak{C}}$ is a valuation. Before that we introduce a lattice containing the image of $\mathcal{V}_{\mathfrak{C}}$ and discuss some examples.

Remark 6.4 The construction is similar to the one in Newton-Okounkov theory which associates a valuation to a flag of subvarieties (see Introduction). The scaling factor $\frac{1}{N^{r-j}}$ in our construction shows up due to the fact that g occurs with the power N^{r-j} in g_j, see Remark 5.1. The scaling factor $\frac{1}{b_j}$ occurs due to the fact that we divide by the function f_j and not, as usual, by a (local) equation defining the prime divisor scheme theoretically.

Remark 6.5 For the construction of the functions $g_r, g_{r-1}, \dots$ we have made use of the procedure described in (6). One could as well define these functions by using the algorithm described in Remark 4.2: $\tilde{g}_r = g$ and, inductively, $\tilde{g}_{i-1} = \tilde{g}_i^{b_i} / f_i^{\nu_i(\tilde{g}_i)}|_{\hat{X}_{i-1}}$ for $i = 1, \dots, r$. As map one would take instead:

$$\widetilde{\mathcal{V}}_{\mathfrak{C}}(g) = \sum_{j=0}^{r} \left(\frac{\nu_j(\tilde{g}_j)}{\prod_{k=j}^{r} b_k} \right) e_j.$$

One sees easily from the definition that these two maps coincide: for all $g \in R \setminus \{0\}$ one has $\widetilde{\mathcal{V}}_{\mathfrak{C}}(g) = \mathcal{V}_{\mathfrak{C}}(g)$.

An immediate consequence of Remark 6.5 is:

Lemma 6.6 *The map $\mathcal{V}_{\mathfrak{C}}$ takes values in the lattice*:

$$L^{\mathfrak{C}} = \{\ell = (\ell_r, \dots, \ell_0) \in \mathbb{Q}^{\mathfrak{C}} \mid b_r \cdots b_j \ell_j \in \mathbb{Z},\ 0 \le j \le r\}. \tag{13}$$

Remark 6.7 If all bonds in the extended Hasse diagram $\mathcal{G}_{\hat{A}}$ are equal to 1, then $L^{\mathfrak{C}} \cong \mathbb{Z}^{r+1}$.

Example 6.8 Let p_i be an element in the maximal chain $\mathfrak{C}$. The renormalization is chosen so that $\mathcal{V}_{\mathfrak{C}}(f_i) = e_i$.

Indeed, by Example 5.4 we know that $(f_i)_{\mathfrak{C}} = (f_i, f_i^N, f_i^{N^2}, \dots, f_i^{N^{r-i}}, 1, \dots, 1)$. Let $\mathcal{V}_{\mathfrak{C}}(f_i) = (a_r, \dots, a_0)$. Since $f_i|_{\hat{X}_j} \not\equiv 0$ for $j \ge i$, it follows that $a_j = 0$ for $j > i$. Since 1 is a nowhere vanishing function, one has $a_j = 0$ for $j < i$. It remains to determine a_i. Now

$$\nu_i(f_i^{N^{r-i}}) = N^{r-i}\nu_i(f_i) = N^{r-i} b_i,$$

so the renormalization implies $a_i = 1$, and hence $\mathcal{V}_{\mathfrak{C}}(f_i) = e_i$. Note that this holds no matter which maximal chain one chooses as long as p_i shows up in the chain. The situation becomes different if $p \notin \mathfrak{C}$, as the following example shows.

Example 6.9 Let $X = \mathrm{Gr}_2\mathbb{K}^4$ be as in the Examples 2.5, 2.8 and 5.5. We fix the maximal chain $\mathfrak{C} : (3,4) > (2,4) > (2,3) > (1,3) > (1,2)$ and $N = 1$ as in Example 2.8. The calculations in Example 5.5 and Example 6.8 imply that the values of $\mathcal{V}_{\mathfrak{C}}$ on the Plücker coordinates are given by:

$$\begin{aligned} &\mathcal{V}_{\mathfrak{C}}(x_{3,4}) = (1,0,0,0,0), && \mathcal{V}_{\mathfrak{C}}(x_{2,4}) = (0,1,0,0,0), && \mathcal{V}_{\mathfrak{C}}(x_{2,3}) = (0,0,1,0,0),\\ &\mathcal{V}_{\mathfrak{C}}(x_{1,4}) = (0,1,-1,1,0), && \mathcal{V}_{\mathfrak{C}}(x_{1,3}) = (0,0,0,1,0), && \mathcal{V}_{\mathfrak{C}}(x_{1,2}) = (0,0,0,0,1). \end{aligned}$$

6.2 $\mathcal{V}_{\mathfrak{C}}$ is a valuation

We extend the map $\mathcal{V}_{\mathfrak{C}}$ to $\mathbb{K}(\hat{X}) = \mathrm{Quot} R$ by:

$$\mathcal{V}_{\mathfrak{C}}(f/g) := \mathcal{V}_{\mathfrak{C}}(f) - \mathcal{V}_{\mathfrak{C}}(g), \text{ for } f, g \in R \setminus \{0\}.$$

The goal of this subsection is to show that the map is well defined and

Proposition 6.10 *$\mathcal{V}_{\mathfrak{C}}$ is an $L^{\mathfrak{C}}$-valued valuation on $\mathbb{K}(\hat{X})$.*

Definition 6.11 We denote by $\mathbb{V}_{\mathfrak{C}}(X)$ the *valuation monoid* associated to X by $\mathcal{V}_{\mathfrak{C}}$, i.e. $\mathbb{V}_{\mathfrak{C}}(X) = \{\mathcal{V}_{\mathfrak{C}}(g) \mid g \in R \setminus \{0\}\} \subseteq L^{\mathfrak{C}}$.

Proof It suffices to consider elements in $R \setminus \{0\}$. Given $g, h \in R \setminus \{0\}$, one verifies by induction (using Remark 5.1) that if $g_{\mathfrak{C}} = (g_r, \dots, g_0)$ and $h_{\mathfrak{C}} = (h_r, \dots, h_0)$, then $(gh)_{\mathfrak{C}} = (g_r h_r, \dots, g_0 h_0)$. Since the ν_j, $j = 0, \dots, r$, are valuations, property (c′) in Definition 3.1 follows.

If one replaces $g \neq 0$ by a non-zero scalar multiple λg, then the components g_j in the tuple $g_{\mathfrak{C}}$ are replaced by some non-zero scalar multiples (see Remark 5.1). Since the ν_j, $j = 0, \dots, r$, are valuations, we see that property (b) in Definition 3.1 holds.

It remains to verify (a) in Definition 3.1. Let $-1 \leq j \leq r$ be minimal such that $\nu_i((g+h)_i) = \nu_i(g_i) = \nu_i(h_i)$ for all $i > j$. If $j = -1$, then obviously property (a) holds.

Suppose $j \geq 0$ and set $F_j = \prod_{\ell=j+1}^{r} f_\ell^{-N^{\ell-j} D_\ell}$, where $F_r = 1$ because we have an empty product. Remark 5.1 implies that $(g+h)_j$ is a linear combination of functions of the form $g^t h^{N^{r-j}-t} F_j$, $t = 0, \ldots, N^{r-j}$. A small calculation shows: for all $t = 0, \ldots, N^{r-j}$, one has

$$\begin{aligned} \nu_j((g+h)_j) &\geq \min\{\nu_j(g^t h^{N^{r-j}-t} F_j) \mid t = 0, \ldots, N^{r-j}\} \\ &= \min\{\tfrac{t}{N^{r-j}} \nu_j(g_j) + (1 - \tfrac{t}{N^{r-j}}) \nu_j(h_j) \mid t = 0, \ldots, N^{r-j}\}. \end{aligned} \tag{14}$$

and hence: $\nu_j((g+h)_j) \geq \min\{\nu_j(g_j), \nu_j(h_j)\}$. If the inequality is strict, then property (a) holds.

If the inequality is not strict, then (by the assumption on j) we have $\nu_j(g_j) \neq \nu_j(h_j)$. Without loss of generality assume $\nu_j(g_j) < \nu_j(h_j)$. But then:

$$(g+h)_{j-1} = \left.\frac{(g+h)_j^N}{f_j^{ND_j}}\right|_{\hat{X}_{j-1}} = \left.\frac{g_j^N}{f_j^{ND_j}}\right|_{\hat{X}_{j-1}} = g_{j-1}$$

because the restrictions of all the other terms in the expansion of $(g+h)_{j-1}$ vanish. But this implies $(g+h)_\ell = g_\ell$ for $0 \leq \ell < j$ and hence $\mathcal{V}_{\mathfrak{C}}(g+h) \geq \min\{\mathcal{V}_{\mathfrak{C}}(g), \mathcal{V}_{\mathfrak{C}}(h)\}$. □

6.3 The lattice generated by the image of $\mathcal{V}_{\mathfrak{C}}$

The lattice $L^{\mathfrak{C}}$ introduced in Lemma 6.6 should be considered as a first approximation of the lattice generated by the image of $\mathcal{V}_{\mathfrak{C}}$. The valuation monoid $\mathbb{V}_{\mathfrak{C}}(X) = \{\mathcal{V}_{\mathfrak{C}}(g) \mid g \in R \setminus \{0\}\}$ may be contained in a proper sublattice of $L^{\mathfrak{C}}$.

In this section we propose an approach to determine the sublattice $L^{\mathfrak{C}}_{\mathcal{V}} \subseteq L^{\mathfrak{C}}$ generated by $\mathbb{V}_{\mathfrak{C}}(X)$. This strategy highlights the strong connection between the point of view in this article and the usual procedure in the theory of Newton-Okounkov bodies. The difference between these approaches will only become evident in Sect. 8.

We fix a maximal chain $\mathfrak{C} : p_r > p_{r-1} > \cdots > p_0$.

Lemma 6.12 *There exist rational functions $F_r, \ldots, F_0 \in \mathbb{K}(\hat{X}) \setminus \{0\}$ such that*

$$\mathcal{V}_{\mathfrak{C}}(F_j) = (\underbrace{0, \ldots, 0}_{r-j}, 1/b_j, *, \ldots, *),$$

where the $*$ are certain numbers in $\mathbb{Q}$.

Proof Suppose $r \geq j \geq 0$. By assumption, the variety $\hat{X}_j$ is smooth in codimension 1, so the local ring $\mathcal{O}_{\hat{X}_j, \hat{X}_{j-1}}$ is a discrete valuation ring. Let η_j be a uniformizer in the maximal ideal. It is a rational function on $\hat{X}_j$ with the property $\nu_j(\eta_j) = 1$.

As a rational function on $\hat{X}_j$, η_j can be represented as the restriction to $\hat{X}_j$ of a rational function on V: there exist $g, h \in \mathbb{K}[V]$ such that both do not identically vanish on $\hat{X}_j$, and $\eta_j = \frac{g}{h}|_{\hat{X}_j}$. In particular, $F_j := \frac{g}{h}|_{\hat{X}} \in \mathbb{K}(\hat{X})$ is a well defined rational function, which has neither poles nor vanishes on non-empty open subsets in $\hat{X} = \hat{X}_r, \hat{X}_{r-1}, \ldots, \hat{X}_j$. It follows that the first $r-j$ entries in $\mathcal{V}_{\mathfrak{C}}(F_j)$ are equal to zero. One gets for the associated sequence of rational functions along $\mathfrak{C}$:

$$(F_j)_{\mathfrak{C}} = (F_j, F_j^N|_{\hat{X}_{r-1}}, \ldots, F_j^{N^{r-j}}|_{\hat{X}_j}, \ldots),$$

and hence the $(r-j+1)$-th entry is $\nu_j(F_j^{N^{r-j}}|_{\hat{X}_j})/(N^{r-j}b_j) = \frac{\nu_j(\eta_j)}{b_j} = \frac{1}{b_j}$. □

Let $L^{\mathfrak{C}}_{\mathcal{V}} \subseteq L^{\mathfrak{C}}$ be the sublattice generated by the valuation monoid $\mathbb{V}_{\mathfrak{C}}(X)$.

Proposition 6.13 *Let $F_r, \ldots, F_0 \in \mathbb{K}(\hat{X}) \setminus \{0\}$ be rational functions such that*

$$\mathcal{V}_{\mathfrak{C}}(F_j) = (\underbrace{0, \ldots, 0}_{r-j}, 1/b_j, *, \ldots, *),$$

where the $*$ are certain numbers in $\mathbb{Q}$. Then $L^{\mathfrak{C}}_{\mathcal{V}} = \langle \mathcal{V}_{\mathfrak{C}}(F_r), \ldots, \mathcal{V}_{\mathfrak{C}}(F_0) \rangle_{\mathbb{Z}}$.

Proof Let $\bar{L}^{\mathfrak{C}}_{\mathcal{V}} = \langle \mathcal{V}_{\mathfrak{C}}(F_r), \ldots, \mathcal{V}_{\mathfrak{C}}(F_0) \rangle_{\mathbb{Z}} \subseteq L^{\mathfrak{C}}$ be the lattice generated by the valuations of the rational functions $F_r, \ldots, F_0$. It is obvious that $\bar{L}^{\mathfrak{C}}_{\mathcal{V}} \subseteq L^{\mathfrak{C}}_{\mathcal{V}}$.

To prove the reverse inclusion, it suffices to show $\mathcal{V}_{\mathfrak{C}}(g) \in \bar{L}^{\mathfrak{C}}_{\mathcal{V}}$ for $g \in \mathbb{K}(\hat{X}) \setminus \{0\}$. Fix a rational function g. Note that without loss of generality we can modify g by multiplying it by a power of F_j for some $j = 0, \ldots, r$. Indeed, since $\mathcal{V}_{\mathfrak{C}}$ is a valuation, on has $\mathcal{V}_{\mathfrak{C}}(g) \in \bar{L}^{\mathfrak{C}}_{\mathcal{V}}$ if and only if $\mathcal{V}_{\mathfrak{C}}(gF_j^a) \in \bar{L}^{\mathfrak{C}}_{\mathcal{V}}$ for some $a \in \mathbb{Z}$.

We proceed by induction on the number of entries equal to zero at the beginning of $\mathcal{V}_{\mathfrak{C}}(g)$. Suppose the first entry is non-zero: it is equal to $\frac{\nu_r(g)}{b_r}$. After replacing g by $gF_r^{-\nu_r(g)}$, we can assume that $g \in \mathbb{K}(\hat{X}) \setminus \{0\}$ is such that the first entry in $\mathcal{V}_{\mathfrak{C}}(g)$ is equal to zero.

Suppose g is a non-zero rational function such that the first $r-j$ entries in $\mathcal{V}_{\mathfrak{C}}(g)$ are equal to zero. If $j = -1$, then we are done. If $j \geq 0$, then $(g)_{\mathfrak{C}} = (g, g^N, \ldots, g^{N^{r-j}}, \ldots)$, and the $(r-j+1)$-st entry in $\mathcal{V}_{\mathfrak{C}}(g)$ is $\nu_j(g^{N^{r-j}})/(N^{r-j}b_j)$, which is equal to $\frac{\nu_j(g)}{b_j}$. So by multiplying g with an appropriate power of F_j, we get a new rational function having one more entry equal to zero in its valuation. □

The rational functions $F_r, \ldots, F_0 \in \mathbb{K}(\hat{X})$ used above are far from being unique. But all possible choices have one common feature. Let $A_{\mathfrak{C}}$ be the rational $(r+1) \times (r+1)$ matrix having as columns the valuations $\mathcal{V}_{\mathfrak{C}}(F_r), \ldots, \mathcal{V}_{\mathfrak{C}}(F_0)$, this is a lower triangular matrix. Let $B_{\mathfrak{C}}$ be its inverse. It is of the following form:

$$B_{\mathfrak{C}} = \begin{pmatrix} b_r & & & \\ * & b_{r-1} & & \\ \vdots & \ddots & \ddots & \\ * & \cdots & * & b_0 \end{pmatrix},$$

where the $*$ are certain numbers in $\mathbb{Q}$.

Proposition 6.14 *Let $v \in \mathbb{Q}^{\mathfrak{C}}$. Then $v \in L^{\mathfrak{C}}_{\mathcal{V}}$ if and only if $B_{\mathfrak{C}} \cdot v \in \mathbb{Z}^{r+1}$. Moreover, the entries of $B_{\mathfrak{C}}$ are integers.*

Proof The first part is just a reformulation of the previous proposition. For the integrality property note that $e_j = \nu_{\mathfrak{C}}(f_j)$ is an element of $L^{\mathfrak{C}}_{\mathcal{V}}$, hence the j-th column of $B_{\mathfrak{C}}$ must have integral entries. □

6.4 $\mathbb{K}^*$-Action versus valuation

The varieties $\hat{X}$ and $\hat{X}_p$ are all endowed with a $\mathbb{K}^*$-action, and the algebra R as well as the algebras R_p are correspondingly graded. For $g \in R_p$ and $\lambda \in \mathbb{K}^*$ denote by g^λ the function $g^\lambda(y) := g(\lambda y)$ for $y \in \hat{X}_p$. This $\mathbb{K}^*$-action on R_p naturally extends to $\mathbb{K}(\hat{X}_p)$.

Lemma 6.15

i) *For any $g \in R \setminus \{0\}$ and $\lambda \in \mathbb{K}^* : \mathcal{V}_{\mathfrak{C}}(g^\lambda) = \mathcal{V}_{\mathfrak{C}}(g)$.*
ii) *Let $h = h_1 + \cdots + h_t \in R = \bigoplus_{i \geq 0} R(i)$ be a decomposition of $h \neq 0$ into its homogeneous parts. Then*

$$\mathcal{V}_{\mathfrak{C}}(h) = \min\{\mathcal{V}_{\mathfrak{C}}(h_j) \mid 1 \leq j \leq t \text{ such that } h_j \neq 0\}.$$

Proof The $\mathbb{K}^*$-action on $\hat{X}_j$ stabilizes the divisor $\hat{X}_{j-1}$. The associated algebra automorphisms of R_{p_j} stabilize hence the vanishing ideal of $\hat{X}_{j-1}$ as well as the local ring $\mathcal{O}_{\hat{X}_j, \hat{X}_{j-1}} \subseteq \mathbb{K}(\hat{X}_j)$ and its maximal ideal. So for all $g \in R_{p_j}$: $\nu_j(g^\lambda) = \nu_j(g)$.

For $g \in R \setminus \{0\}$ let $g_{\mathfrak{C}} = (g_r, \ldots, g_0)$ be the associated sequence of rational functions. We can use the $\mathbb{K}^*$-action to construct two new tuples: the λ-twisted tuple ${}^\lambda g_{\mathfrak{C}} = (g_r^\lambda, \ldots, g_0^\lambda)$ obtained by twisting component-wise each of the rational functions occurring in $g_{\mathfrak{C}}$. And we can consider $g_{\mathfrak{C}}^\lambda = (g_r', \ldots, g_0')$, the tuple associated to the function g^λ.

The functions $\{f_p \mid p \in A\}$ are homogeneous, so by Remark 5.1 the j-th component g_j^λ in the λ-twisted sequence ${}^\lambda g_{\mathfrak{C}}$ and the j-th component g_j' in $g_{\mathfrak{C}}^\lambda$ differ only by a scalar multiple. It follows: $\mathcal{V}_{\mathfrak{C}}(g^\lambda) = \mathcal{V}_{\mathfrak{C}}(g)$.

Let $h = h_1 + \cdots + h_t$ be a decomposition of $h \neq 0$ into its homogeneous parts. We know: $\mathcal{V}_{\mathfrak{C}}(h) \geq \min\{\mathcal{V}_{\mathfrak{C}}(h_j) \mid 1 \leq j \leq t$ such that $h_j \neq 0\}$ by the minimum property of a valuation (see Definition 3.1 a)). To prove the equality, note that one can find pairwise distinct, non-zero scalars $\lambda_1, \ldots, \lambda_t \in \mathbb{K}^*$ such that the linear span of the following functions coincide:

$$\langle h_1, \ldots, h_t \rangle_{\mathbb{K}} = \langle h^{\lambda_1}, \ldots, h^{\lambda_t} \rangle_{\mathbb{K}}.$$

So one can express the homogeneous function h_i as a linear combination of the functions $h^{\lambda_1}, \ldots, h^{\lambda_t}$, and hence by part i) of the lemma:

$$\mathcal{V}_{\mathfrak{C}}(h_i) \geq \min\{\mathcal{V}_{\mathfrak{C}}(h^{\lambda_j}) \mid j = 1, \ldots, t\} = \mathcal{V}_{\mathfrak{C}}(h),$$

and hence $\mathcal{V}_{\mathfrak{C}}(h) = \min\{\mathcal{V}_{\mathfrak{C}}(h_j) \mid j = 1, \ldots, t\}$. □

6.5 Leaves

Let $L^{\mathfrak{C},\dagger} \subseteq L^{\mathfrak{C}}$ be the submonoid of tuples such that the first non-zero entry is positive. Let $g \in R \setminus \{0\}$ be a regular function on $\hat{X}$. The first non-zero entry of $\mathcal{V}_{\mathfrak{C}}(g)$ is a rescaled vanishing multiplicity, so $\mathcal{V}_{\mathfrak{C}} : R \setminus \{0\} \to L^{\mathfrak{C}}$ is a valuation with values in $L^{\mathfrak{C},\dagger}$. The elements in this submonoid satisfy an additional compatibility property: for any $\underline{a}, \underline{b}, \underline{c} \in L^{\mathfrak{C},\dagger}$ we have:

$$\text{if } \underline{a} \geq \underline{b}, \text{ then } \underline{a} + \underline{c} \geq \underline{b} + \underline{c} \geq \underline{b}. \tag{15}$$

This property ensures that the subspaces

$$R^{\mathfrak{C}}_{\geq \underline{a}} := \{g \in R \setminus \{0\} \mid \mathcal{V}_{\mathfrak{C}}(g) \geq \underline{a}\} \cup \{0\},$$

respectively

$$R^{\mathfrak{C}}_{> \underline{a}} := \{g \in R \setminus \{0\} \mid \mathcal{V}_{\mathfrak{C}}(g) > \underline{a}\} \cup \{0\}$$

for $\underline{a} \in L^{\mathfrak{C},\dagger}$ are ideals. It follows that the associated graded vector space

$$\mathrm{gr}_{\mathfrak{C}} R = \bigoplus_{\underline{a} \in L^{\mathfrak{C},\dagger}} R^{\mathfrak{C}}_{\geq \underline{a}} / R^{\mathfrak{C}}_{> \underline{a}}$$

is a $\mathbb{K}$-algebra. The subquotient $R^{\mathfrak{C}}_{\geq \underline{a}} / R^{\mathfrak{C}}_{> \underline{a}}$ for $\underline{a} \in L^{\mathfrak{C},\dagger}$ is called a *leaf of the valuation* $\mathcal{V}_{\mathfrak{C}}$.

According to Example 6.8, $\mathcal{V}_{\mathfrak{C}}$ is a full rank valuation, i.e. the lattice generated by $\mathbb{V}_{\mathfrak{C}}(X)$ has rank $\dim X + 1$. As a consequence of the Abhyankar's inequality one obtains:

Theorem 6.16 ([38]) *The valuation $\mathcal{V}_{\mathfrak{C}}$ has at most one-dimensional leaves, i.e. for any $\underline{a} \in L^{\mathfrak{C},\dagger}$, $\dim_{\mathbb{K}} R^{\mathfrak{C}}_{\geq \underline{a}} / R^{\mathfrak{C}}_{> \underline{a}} \leq 1$.*

Corollary 6.17 *Let $g, h \in R \setminus \{0\}$. Assume that for a maximal chain $\mathfrak{C} \subseteq A$, $\mathcal{V}_{\mathfrak{C}}(g) = \mathcal{V}_{\mathfrak{C}}(h)$ holds. Then there exists $\lambda \in \mathbb{K}^*$ and $h' \in R$ such that $g = \lambda h + h'$. If $h' \neq 0$, then $\mathcal{V}_{\mathfrak{C}}(h') > \mathcal{V}_{\mathfrak{C}}(h) = \mathcal{V}_{\mathfrak{C}}(g)$ holds.*

Proof This is just a reformulation of Theorem 6.16. □

Lemma 6.18 *Let $\mathfrak{C} \subseteq A$ be a maximal chain and let $g_1, g_2, g_3, g_4 \in R \setminus \{0\}$. If $\mathcal{V}_{\mathfrak{C}}(g_1) + \mathcal{V}_{\mathfrak{C}}(g_2) = \mathcal{V}_{\mathfrak{C}}(g_3) + \mathcal{V}_{\mathfrak{C}}(g_4)$, then there exist $\lambda \in \mathbb{K}^*$ and $h' \in R$ such that $g_1 g_2 = \lambda g_3 g_4 + h'$. If $h' \neq 0$, then $\mathcal{V}_{\mathfrak{C}}(h') > \mathcal{V}_{\mathfrak{C}}(g_1) + \mathcal{V}_{\mathfrak{C}}(g_2)$ holds. If the functions g_1, g_2, g_3, g_4 are homogeneous, so is h'.*

Proof For the first statement it suffices to apply Corollary 6.17 to $g = g_1 g_2$ and $h = g_3 g_4$. Assume that g_1, g_2, g_3, g_4 are homogeneous, and let $h' = h_1 + h_2 + \cdots + h_t$, $h_k \neq 0$ for $k = 1, \ldots, t$, be a decomposition of h' into its homogeneous components. If $g_1 g_2$ and $g_3 g_4$ are not of the same degree, then the equality $g_1 g_2 = \lambda g_3 g_4 + h'$ is only possible if $\lambda g_3 g_4 = -h_j$ for some $1 \le j \le t$. By Lemma 6.15, this is impossible because

$$\mathcal{V}_{\mathfrak{C}}(h_j) \ge \min\{\mathcal{V}_{\mathfrak{C}}(h_\ell) \mid \ell = 1, \dots, t\} = \mathcal{V}_{\mathfrak{C}}(h') > \mathcal{V}_{\mathfrak{C}}(g_1 g_2) = \mathcal{V}_{\mathfrak{C}}(g_3 g_4).$$

Therefore $g_1 g_2$ and $g_3 g_4$ are necessarily of the same degree, and so is h'. □

Lemma 6.18 suggests that the structure of $\mathrm{gr}_{\mathfrak{C}} R$ is very similar to that of the semigroup algebra $\mathbb{K}[\mathbb{V}_{\mathfrak{C}}(X)]$. Indeed, it follows from Definition 3.1 (c′) that $\mathrm{gr}_{\mathfrak{C}} R$ has no zero divisors. Applying [10, Remark 4.13] (see also [38, Proposition 2.4]) gives:

Corollary 6.19 *As* $\mathbb{K}$*-algebra*, $\mathrm{gr}_{\mathfrak{C}} R$ *is isomorphic to* $\mathbb{K}[\mathbb{V}_{\mathfrak{C}}(X)]$.

7 Localization and finite generation

In this section we introduce the core of the valuation $\mathcal{V}_{\mathfrak{C}}$ associated to a maximal chain $\mathfrak{C}$: it is a finitely generated submonoid of the valuation monoid $\mathbb{V}_{\mathfrak{C}}(X)$. Further results on the relation between the core and standard monomials will be given in Sect. 9.

Definition 7.1 The *core* $P_{\mathfrak{C}}(X)$ of the valuation monoid $\mathbb{V}_{\mathfrak{C}}(X)$ is defined as its intersection with the positive orthant:

$$P_{\mathfrak{C}}(X) := \mathbb{V}_{\mathfrak{C}}(X) \cap \mathbb{Q}^{\mathfrak{C}}_{\ge 0}.$$

As the intersection of two monoids, $P_{\mathfrak{C}}(X)$ is a monoid. By Example 6.8 we know $\mathcal{V}_{\mathfrak{C}}(f_i) = e_i$ and hence $\mathbb{N}^{\mathfrak{C}} \subseteq P_{\mathfrak{C}}(X)$. So the monoid has an additional structure, it is endowed with a natural $\mathbb{N}^{\mathfrak{C}}$-action:

$$\mathbb{N}^{\mathfrak{C}} \times P_{\mathfrak{C}}(X) \to P_{\mathfrak{C}}(X), \quad (\underline{n}, \underline{k}) \mapsto \underline{n} \circ \underline{k} := \underline{n} + \underline{k}. \tag{16}$$

Lemma 7.2 *The core* $P_{\mathfrak{C}}(X)$ *is a finitely generated* $\mathbb{N}^{\mathfrak{C}}$*-module.*

Proof For the proof we use Dickson's Lemma ([25], Lemma A) which states (the formulation has been adapted to our situation): every monomial ideal in the polynomial ring $\mathbb{K}[y_p | p \in \mathfrak{C}]$ is finitely generated. By the standard bijection $\underline{n} = (n_p)_{p \in \mathfrak{C}} \mapsto \underline{y}^{\underline{n}} = \prod_{p \in \mathfrak{C}} y_p^{n_p}$ between $\mathbb{N}^{\mathfrak{C}}$ and the monomials in $\mathbb{K}[y_p | p \in \mathfrak{C}]$ we get a bijection between monomial ideals in $\mathbb{K}[y_p | p \in \mathfrak{C}]$ and $\mathbb{N}^{\mathfrak{C}}$-submodules in $\mathbb{N}^{\mathfrak{C}}$, where the latter is acting on itself by addition. And Dickson's Lemma can be reformulated as: every $\mathbb{N}^{\mathfrak{C}}$-submodule $M \subseteq \mathbb{N}^{\mathfrak{C}}$ is finitely generated as $\mathbb{N}^{\mathfrak{C}}$-module.

Since $e_r, \dots, e_1, e_0 \in L^{\mathfrak{C}}$, adding elements of $\mathbb{Z}^{\mathfrak{C}}$: $\underline{n} \circ \underline{\ell} := \underline{n} + \underline{\ell}$ defines an action of $\mathbb{Z}^{\mathfrak{C}}$ on the lattice $L^{\mathfrak{C}}$. We get an induced action by $\mathbb{N}^{\mathfrak{C}}$ on the intersection $L^{\mathfrak{C}} \cap \mathbb{Q}^{\mathfrak{C}}_{\ge 0}$, which by (16) stabilizes the submonoid $P_{\mathfrak{C}}(X)$.

We decompose $L^{\mathfrak{C}}$ into a finite number of $\mathbb{Z}^{\mathfrak{C}}$-submodules (with respect to the action "∘" defined above). Since $L^{\mathfrak{C}}$ is a lattice, the intersection

$$Q := L^{\mathfrak{C}} \cap \{\underline{a} = (a_r, \dots, a_0) \in \mathbb{Q}^{\mathfrak{C}}_{\ge 0} \mid 0 \le a_j < 1\}$$

is a finite set. By construction we have $L^{\mathfrak{C}} = \bigoplus_{\underline{a}\in Q} \mathbb{Z}^{\mathfrak{C}} \circ \underline{a}$ and correspondingly $L^{\mathfrak{C}} \cap \mathbb{Q}^{\mathfrak{C}}_{\geq 0} = \bigoplus_{\underline{a}\in Q} \mathbb{N}^{\mathfrak{C}} \circ \underline{a}$. It follows:

$$P_{\mathfrak{C}}(X) = \bigoplus_{\underline{a}\in Q} \left(\mathbb{N}^{\mathfrak{C}} \circ \underline{a} \cap P_{\mathfrak{C}}(X)\right) \subseteq L^{\mathfrak{C}} \cap \mathbb{Q}^{\mathfrak{C}}_{\geq 0}.$$

Since Q is a finite set, to show that $P_{\mathfrak{C}}(X)$ is a finitely generated $\mathbb{N}^{\mathfrak{C}}$-module it suffices to show that the intersection $(\mathbb{N}^{\mathfrak{C}} \circ \underline{a}) \cap P_{\mathfrak{C}}(X)$ is a finitely generated $\mathbb{N}^{\mathfrak{C}}$-module for all $\underline{a} \in Q$. As $\mathbb{N}^{\mathfrak{C}}$-modules, $\mathbb{N}^{\mathfrak{C}} \circ \underline{a}$ and $\mathbb{N}^{\mathfrak{C}}$ are isomorphic, and $(\mathbb{N}^{\mathfrak{C}} \circ \underline{a}) \cap P_{\mathfrak{C}}(X)$ is an intersection of submodules and hence a submodule. So one can apply (the reformulated version of) Dickson's Lemma and hence $(\mathbb{N}^{\mathfrak{C}} \circ \underline{a}) \cap P_{\mathfrak{C}}(X)$ is finitely generated as $\mathbb{N}^{\mathfrak{C}}$-module, which in turn implies $P_{\mathfrak{C}}(X)$ is a finitely generated $\mathbb{N}^{\mathfrak{C}}$-module. □

We introduce an open affine subset of $\hat{X}$ to pinch the valuation monoid $\mathbb{V}_{\mathfrak{C}}(X)$. Let $U_{\mathfrak{C}}$ be the open affine subset of $\hat{X}$ defined by

$$U_{\mathfrak{C}} = \{x \in \hat{X} \mid \prod_{p\in\mathfrak{C}} f_p(x) \neq 0\}.$$

Its coordinate ring will be denoted by $\mathbb{K}[U_{\mathfrak{C}}]$: it is the localization of $\mathbb{K}[\hat{X}]$ at $\prod_{p\in\mathfrak{C}} f_p$.

The valuation $\mathcal{V}_{\mathfrak{C}} : \mathbb{K}(\hat{X}) \setminus \{0\} \to L^{\mathfrak{C}}$ induces by restriction a valuation $\mathcal{V}_{\mathfrak{C}} : \mathbb{K}[U_{\mathfrak{C}}] \setminus \{0\} \to L^{\mathfrak{C}}$ which has one-dimensional leaves. Let $\mathbb{V}_{\mathfrak{C}}(U_{\mathfrak{C}}) := \{\mathcal{V}_{\mathfrak{C}}(g) \mid g \in \mathbb{K}[U_{\mathfrak{C}}] \setminus \{0\}\} \subseteq L^{\mathfrak{C}}$ denote the associated valuation monoid. These monoids are contained in each other: $P_{\mathfrak{C}}(X) \subseteq \mathbb{V}_{\mathfrak{C}}(X) \subseteq \mathbb{V}_{\mathfrak{C}}(U_{\mathfrak{C}})$. The core $P_{\mathfrak{C}}(X)$ can be thought of as a condensed version of the valuation monoid $\mathbb{V}(U_{\mathfrak{C}})$:

Lemma 7.3 *We have*

$$\mathbb{V}_{\mathfrak{C}}(U_{\mathfrak{C}}) = P_{\mathfrak{C}}(X) + \mathbb{Z}^{\mathfrak{C}} = \{\underline{a} + \underline{m} \mid \underline{a} \in P_{\mathfrak{C}}(X),\ \underline{m} \in \mathbb{Z}^{\mathfrak{C}}\}.$$

In particular, as $\mathbb{Z}^{\mathfrak{C}}$-module, $\mathbb{V}(U_{\mathfrak{C}})$ is finitely generated.

Proof For $g \in R \setminus \{0\}$, there exists an $m \in \mathbb{N}$ such that

$$\mathcal{V}_{\mathfrak{C}}(g(\prod_{p\in\mathfrak{C}} f_p)^m) = \mathcal{V}_{\mathfrak{C}}(g) + \sum_{p\in\mathfrak{C}} m e_p \in P_{\mathfrak{C}}(X).$$

Hence every element in $\mathbb{K}[U_{\mathfrak{C}}] \setminus \{0\}$ can be written as a quotient $\frac{h}{(\prod_{p\in\mathfrak{C}} f_p)^n}$, $h \in R \setminus \{0\}$, in such a way that $\mathcal{V}_{\mathfrak{C}}(h) \in P_{\mathfrak{C}}(X)$, which proves the claim. □

Corollary 7.4 *Let $g \in R \setminus \{0\}$. There exist $m \in \mathbb{N}$, $\lambda \in \mathbb{K}^*$ and an element $g' \in R$ with $\mathcal{V}_{\mathfrak{C}}(g') > m\mathcal{V}_{\mathfrak{C}}(g)$ as long as $g' \neq 0$, such that in $\mathbb{K}[U_{\mathfrak{C}}]$ we have*

$$g^m = \lambda f_{p_r}^{ma_r} \cdots f_{p_0}^{ma_0} + g'.$$

If g is homogeneous, then so are $f_{p_r}^{ma_r}\cdots f_{p_0}^{ma_0}$ and g', and they are of the same degree.

Proof Fix $m\in\mathbb{Z}_{>0}$ such that $m\mathcal{V}_{\mathfrak{C}}(g)=(ma_r,\dots,ma_0)\in\mathbb{Z}^{\mathfrak{C}}$. It follows by Example 6.8: $\mathcal{V}_{\mathfrak{C}}(g^m)=m\underline{a}=\mathcal{V}_{\mathfrak{C}}(f_{p_r}^{ma_r}\cdots f_{p_0}^{ma_0})$. The remaining part of the proof is the same as Lemma 6.18, where we only used the fact that the leaves are one dimensional. □

As a consequence we get the following formula recovering the degree of a homogeneous element from its valuation (one easily verifies it in Example 6.9):

Corollary 7.5 *If $g\in R\setminus\{0\}$ is homogeneous and $\mathcal{V}_{\mathfrak{C}}(g)=\underline{a}$, then*

$$\deg g=a_r\deg f_{p_r}+\cdots+a_1\deg f_{p_1}+a_0\deg f_{p_0}.$$

Proof By Corollary 7.4, we can find an $m\in\mathbb{N}$ such that $g^m=\lambda f_{p_r}^{ma_r}\cdots f_{p_0}^{ma_0}+g'$ for some $\lambda\in\mathbb{K}^*$, and all are of the same degree. It follows that the degree of g is $\deg g=\frac{1}{m}\deg(f_{p_r}^{ma_r}\cdots f_{p_0}^{ma_0})$, which proves the claimed formula. □

The valuation $\mathcal{V}_{\mathfrak{C}}$ induces a filtration on $\mathbb{K}[U_{\mathfrak{C}}]$, let $\mathrm{gr}_{\mathfrak{C}}\mathbb{K}[U_{\mathfrak{C}}]$ be the associated graded algebra. The same arguments as for Corollary 6.19 imply: the latter is isomorphic to the semigroup algebra $\mathbb{K}[\mathbb{V}_{\mathfrak{C}}(U_{\mathfrak{C}})]$, and hence by Lemma 7.2 and 7.3, this algebra is finitely generated and integral.

Corollary 7.6 $\mathrm{Spec}(\mathrm{gr}_{\mathfrak{C}}\mathbb{K}[U_{\mathfrak{C}}])\simeq\mathrm{Spec}(\mathbb{K}[\mathbb{V}_{\mathfrak{C}}(U_{\mathfrak{C}})])$ *is a toric variety.*

We finish this section by taking a different point of view: looking at $U_{\mathfrak{C}}$ as being fibered over a torus.

Corollary 7.7 *The map*

$$\phi_{\mathfrak{C}}:U_{\mathfrak{C}}\to(\mathbb{K}^*)^{r+1},\quad u\mapsto(f_{p_r}(u),\dots,f_{p_0}(u))$$

is a finite morphism.

Proof According to Example 6.8, the valuations $\mathcal{V}_{\mathfrak{C}}(f_{p_r}),\dots,\mathcal{V}_{\mathfrak{C}}(f_{p_0})$ are linearly independent, hence the functions $f_{p_r},\dots,f_{p_0}$ are algebraically independent. The induced map between the coordinate rings $\phi_{\mathfrak{C}}^*:\mathbb{K}[x_r^{\pm1},\dots,x_0^{\pm1}]\to\mathbb{K}[U_{\mathfrak{C}}]$ sending x_j to f_{p_j} is therefore injective and, by Lemma 7.3 and Corollary 7.4, it makes $\mathbb{K}[U_{\mathfrak{C}}]$ into a finite $\mathbb{K}[x_r^{\pm1},\dots,x_0^{\pm1}]$-module. □

8 Globalization: a non-negative quasi-valuation

To better understand the role of the finitely generated submonoid $P_{\mathfrak{C}}(X)$ of $\mathbb{V}_{\mathfrak{C}}(X)$, we consider in the following all the valuations $\mathcal{V}_{\mathfrak{C}}$ at once, where $\mathfrak{C}$ runs over all maximal chains in A. To do so, let $\mathbb{Q}^A$ be the $\mathbb{Q}$-vector space spanned by the basis elements $\{e_q \mid q \in A\}$. If $\mathfrak{C}$ is a maximal chain in A, we identify $\mathbb{Q}^{\mathfrak{C}}$ with the subspace of $\mathbb{Q}^A$ spanned by the basis elements $\{e_p \mid p \in \mathfrak{C}\}$. To be able to compare for a given $g \in R \setminus \{0\}$ the various valuations $\mathcal{V}_{\mathfrak{C}}(g) \in \mathbb{Q}^{\mathfrak{C}} \subseteq \mathbb{Q}^A$, we need an order on $\mathbb{Q}^A$.

To define a total order on $\mathbb{Q}^A$, fix a total order "$>^t$" on A refining the given partial order, and such that $\ell(p) > \ell(q)$ (see Definiton 2.3) implies $p >^t q$.[5] Such a total order exists since A is a graded poset. Let

$$q_M >^t q_{M-1} >^t \cdots >^t q_0 \tag{17}$$

be an enumeration of the elements of A depicting the total order.

Writing $v = (a_M, \ldots, a_0)$ for the vector $v = \sum_{j=0}^{M} a_j e_{q_j} \in \mathbb{Q}^A$, we endow $\mathbb{Q}^A$ with the lexicographic order as total order. This total order is compatible with the addition of vectors.

We will denote by $\mathcal{C}$ the set of all maximal chains in A.

8.1 A non-negative quasi-valuation

By Lemma 3.4, taking the minimum value over a finite number of valuations defines a quasi-valuation. Recall that we think of $\mathbb{Q}^{\mathfrak{C}}$ as the subspace of $\mathbb{Q}^A$ spanned by the e_p, $p \in \mathfrak{C}$. So it makes sense to write $\mathcal{V}_{\mathfrak{C}}(g) \in \mathbb{Q}^A$ for a regular function $g \in R \setminus \{0\}$. Note that the total order on $\mathbb{Q}^{\mathfrak{C}}$ defined in Definition 6.1 coincides with the total order on $\mathbb{Q}^{\mathfrak{C}}$ induced by that defined on $\mathbb{Q}^A$.

Definition 8.1

(1) We define the quasi-valuation $\mathcal{V} : R \setminus \{0\} \to \mathbb{Q}^A$ by

$$\mathcal{V}(g) := \min\{\mathcal{V}_{\mathfrak{C}}(g) \mid \mathfrak{C} \in \mathcal{C}\}.$$

(2) For $g \in R \setminus \{0\}$ with $\mathcal{V}(g) = (a_p)_{p \in A}$, the *support* of g is defined by

$$\operatorname{supp} \mathcal{V}(g) := \{p \in A \mid a_p \neq 0\}.$$

Remark 8.2 Let $g \in R \setminus \{0\}$. Unless $\operatorname{supp} \mathcal{V}(g)$ is a maximal chain, there might be several maximal chains $\mathfrak{C}$ such that $\mathcal{V}(g) = \mathcal{V}_{\mathfrak{C}}(g)$.

As an example let us consider an extremal function.

Lemma 8.3 *For any $q \in A$, $\mathcal{V}(f_q) = e_q$.*

Proof By Example 6.8 we know $\mathcal{V}_{\mathfrak{C}}(f_q) = e_q$ as long as $q \in \mathfrak{C} = (p_r, \ldots, p_0)$. If $q \notin \mathfrak{C}$, then let $p_k \in \mathfrak{C}$ be the unique element such that $\ell(p_k) = \ell(q)$. Since p_k and q are not comparable with respect to the partial order on A, f_q vanishes on X_{p_k}. But f_q is not the zero function, so there exist elements $p_j > p_{j-1} \geq p_k$ in $\mathfrak{C}$ such

[5] This second condition on the length is in fact not necessary. Results in the article hold with this condition removed. For details, see [18, Sect. 2.6].

that p_j covers p_{j-1}, $f_q|_{X_{p_j}} \not\equiv 0$, but f_q vanishes on $X_{p_{j-1}}$. It follows (compare with Example 5.4):

$$(f_q)_{\mathfrak{C}} = (f_q, f_q^N, \dots, f_q^{N^{r-j}}, \dots) \Longrightarrow \mathcal{V}_{\mathfrak{C}}(f_q) = \frac{\nu_{p_j,p_{j-1}}(f_q^{N^{r-j}})}{N^{r-j} b_{p_j,p_{j-1}}} e_{p_j} + \sum_{i<j} a_i e_{p_i}$$

for some rational numbers $a_i \in \mathbb{Q}$, $0 \le i \le j-1$. Since $\nu_{p_j,p_{j-1}}(f_q) > 0$ and $\ell(p_j) > \ell(q)$, this implies $\mathcal{V}_{\mathfrak{C}}(f_q) > e_q$, and hence $\mathcal{V}(f_q) = e_q$. □

Remark 8.4 In general, the value $\mathcal{V}(g)$ for $g \in R \setminus \{0\}$ depends on the choice of the total order $>^t$ in the construction of $\mathcal{V}$. According to the lemma, for $q \in A$, $\mathcal{V}(f_q)$ is independent of the choice.

Example 8.5 Consider the generic hyperplane stratification introduced in Sect. 2.4. We study the quasi-valuation $\mathcal{V}$ on some particular functions.

First we compute $\mathcal{V}(g_k)$ for the function g_k, where $1 \le k \le s$, defined in Example 2.12. Let $\mathfrak{C} = (p_r, \dots, p_0)$ be a maximal chain in A.

(1) If $p_0 \neq q_{0,k}$, the function g_k vanishes in X_{p_0}. This implies that the support $\operatorname{supp} \mathcal{V}_{\mathfrak{C}}(g_k)$ contains an element p_k for some $k \ge 1$.
(2) If $p_0 = q_{0,k}$, then the non-vanishing of g_k in $X_{q_{0,k}}$ implies that $\operatorname{supp} \mathcal{V}_{\mathfrak{C}}(g) = \{p_0\}$.

It follows for the quasi-valuation: $\operatorname{supp} \mathcal{V}(g_k) = \{p_0\}$, and $\mathcal{V}_{\mathfrak{C}}(g_k) = \mathcal{V}(g_k)$ if and only if $q_{0,k} \in \mathfrak{C}$. In this case we have $g_{\mathfrak{C}} = (g, g^N, \dots, g^{N^r})$. Recall that the valuation ν_0 in the last step is just the degree of a corresponding homogeneous function (see Remark 2.10). We obtain:

$$\mathcal{V}(g_k) = \mathcal{V}_{\mathfrak{C}}(g_k) = \frac{N^r(s-1)}{N^r b_{p_0,p_{-1}}} e_0 = \frac{s-1}{\deg f_{0,k}} e_0.$$

Then we consider a linear function $h \in V^*$ which does not vanish in any of the points $\varpi_1, \dots, \varpi_s$, the same arguments as above imply: if $q_{0,k} \in \mathfrak{C}$, then

$$\mathcal{V}(gh) = \mathcal{V}_{\mathfrak{C}}(gh) = \frac{s}{\deg f_{0,k}} e_0.$$

The following non-negativity property of the quasi-valuation will be crucial in determining relations in R in the spirit of Corollary 7.4 (see Corollary 10.4).

Proposition 8.6 *For all $g \in R \setminus \{0\}$, $\mathcal{V}(g) \in \mathbb{Q}^A_{\ge 0}$.*

Proof For $\mathcal{V}(g) = \mathfrak{m} = (m_p)_{p \in A}$ let $\mathfrak{C} = (p_r, \dots, p_0)$ be a maximal chain such that $\mathcal{V}_{\mathfrak{C}}(g) = \mathfrak{m}$. Then $m_p = 0$ for all $p \notin \mathfrak{C}$. We denote by $\mathfrak{m}_{\mathfrak{C}}$ the vector $(m_{p_r}, \dots, m_{p_0}) \in \mathbb{Q}^{r+1}$. To prove the proposition, it remains to show that $\mathfrak{m}_{\mathfrak{C}} \in \mathbb{Q}^{r+1}_{\ge 0}$.

Let $g_{\mathfrak{C}} = (g_r, \dots, g_0)$ be the sequence of rational functions associated to g and $\mathfrak{C}$. Since g is regular, we know $\nu_{p_r,p_{r-1}}(g) \ge 0$, and hence $m_{p_r} \ge 0$. We proceed by decreasing induction.

Assume that for some $1 \le j \le r$, we know that $g_j \in \mathbb{K}(\hat{X}_{p_j})$ is integral over $\mathbb{K}[\hat{X}_{p_j}]$ and hence $m_{p_j} \ge 0$. If $q \in A$ is covered by p_j, then we can find a maximal chain of the form $\mathfrak{C}' = (p_r, \ldots, p_j, q, \ldots)$. By the minimality assumption we know $\mathcal{V}_{\mathfrak{C}}(g) \le \mathcal{V}_{\mathfrak{C}'}(g)$. In $\mathfrak{C}$ and $\mathfrak{C}'$, the entries with indexes $r, r-1, \ldots, j$ coincide, which implies by the minimality assumption:

$$\frac{\nu_{p_j, p_{j-1}}(g_j)}{b_{p_j, p_{j-1}}} \le \frac{\nu_{p_j, q}(g_j)}{b_{p_j, q}}.$$

By Proposition 4.3, the rational function $g_{j-1} \in \mathbb{K}(\hat{X}_{p_{j-1}})$ is hence integral over $\mathbb{K}[\hat{X}_{p_{j-1}}]$ and thus $m_{p_{j-1}} \ge 0$, which finishes the proof by induction. □

8.2 A characterization via support

Given a function $g \in R \setminus \{0\}$, let $\mathcal{C}(g) \subseteq \mathcal{C}$ be the set of maximal chains along which the quasi-valuation attains its minimum:

$$\mathcal{C}(g) := \{\mathfrak{C} \in \mathcal{C} \mid \mathcal{V}_{\mathfrak{C}}(g) = \mathcal{V}(g)\}.$$

Proposition 8.7 *The set $\mathcal{C}(g)$ consists of maximal chains in A containing* $\mathrm{supp}\mathcal{V}(g)$, *i.e. $\mathfrak{C} \in \mathcal{C}(g)$ if and only if* $\mathrm{supp}\, \mathcal{V}(g) \subseteq \mathfrak{C}$.

Proof If $\mathfrak{C} \in \mathcal{C}(g)$, then $\mathrm{supp}\, \mathcal{V}(g) = \mathrm{supp}\, \mathcal{V}_{\mathfrak{C}}(g) \subseteq \mathfrak{C}$.

To prove the opposite inclusion, let $\mathfrak{C}' = (p'_r, \ldots, p'_0)$ be a maximal chain in A satisfying $\mathrm{supp}\mathcal{V}(g) \subseteq \mathfrak{C}'$, we show that $\mathcal{V}(g) = \mathcal{V}_{\mathfrak{C}'}(g)$, hence $\mathfrak{C}' \in \mathcal{C}(g)$. We proceed by induction on the length of the maximal chain in a poset. If all maximal chains in A have length 0, then A has only one element and there is nothing to prove.

In the inductive procedure, the functions showing up are not necessarily regular but those in $\mathbb{K}(\hat{X})$ which are integral over $\mathbb{K}[\hat{X}]$, so we start with such a function g and keep in mind the setup in Sect. 3.2.

Let $\mathfrak{C} = (p_r, \ldots, p_0)$ be a maximal chain such that $\mathfrak{C} \in \mathcal{C}(g)$, i.e. $\mathcal{V}(g) = \mathcal{V}_{\mathfrak{C}}(g)$. We will denote $g_{\mathfrak{C}} = (g_r, \ldots, g_0)$, $g_{\mathfrak{C}'} = (g'_r, \ldots, g'_0)$, $\mathcal{V}(g) = (a_r, \ldots, a_0)$ and $\mathcal{V}_{\mathfrak{C}'}(g) = (a'_r, \ldots, a'_0)$.

Both maximal chains $\mathfrak{C}$ and $\mathfrak{C}'$ start with $p_{\max} = p_r = p'_r$. We consider the following cases:

(1) Assume that $p_{\max} \notin \mathrm{supp}\, \mathcal{V}_{\mathfrak{C}}(g)$, then let $0 \le j < r$ be maximal such that $p_j \in \mathrm{supp}\, \mathcal{V}_{\mathfrak{C}}(g)$. It follows that $a_r = \cdots = a_{j+1} = 0$ and the sequence of function is $g_{\mathfrak{C}} = (g, g^N, \ldots, g^{N^{r-j}}, \ldots)$. In particular, $g|_{\hat{X}_{p_j}}$ is a rational function, integral over $\mathbb{K}[\hat{X}_{p_j}]$, which does not vanish on an open and dense subset of $\hat{X}_{p_j}$. Since $\mathrm{supp}\mathcal{V}(g) \subseteq \mathfrak{C}'$, $p_j = p'_j$ and hence $\hat{X}_{p_j} = \hat{X}_{p'_j}$. This implies that g does not vanish on an open and dense subset of the subvarieties $\hat{X}_{p'_k}$ for $k \ge j$. It follows that $a'_r = \cdots = a'_{j+1} = 0$ and $g_{\mathfrak{C}'} = (g, g^N, \ldots, g^{N^{r-j}}, \ldots)$. Replacing X by X_{p_j}, g by $g^{N^{r-j}}$, and considering the subposet A_{p_j} whose maximal chains have smaller length, we can proceed by induction.

(2) Now we assume that $p_{\max} \in \operatorname{supp} \mathcal{V}_{\mathfrak{C}}(g)$. There are two cases to consider:
   (a) $p_{r-1} \in \operatorname{supp} \mathcal{V}_{\mathfrak{C}}(g)$: In this case, $p_{\max}$ and p_{r-1} are contained in both $\mathfrak{C}$ and $\mathfrak{C}'$. Recall that the function g_{r-1} satisfies the desired assumption by Proposition 4.3 because $\mathfrak{C} \in \mathcal{C}(g)$. Replacing X by $X_{p_{r-1}}$ and g by g_{r-1}, we can proceed by induction.
   (b) $p_{r-1} \notin \operatorname{supp} \mathcal{V}_{\mathfrak{C}}(g)$: Let $0 \le j < r-1$ be maximal such that $p_j \in \operatorname{supp} \mathcal{V}_{\mathfrak{C}}(g)$. From this assumption we know $a_{r-1} = \cdots = a_{j+1} = 0$. We look at the functions g_{r-1} and g'_{r-1}:

$$g_{r-1} = g^N / f_{p_r}^{N a_r}, \quad g'_{r-1} = g^N / f_{p_r}^{N a'_r},$$

then the sequence of functions is of the form

$$g_{\mathfrak{C}} = (g, g_{r-1}, g_{r-1}^N, \ldots, g_{r-1}^{N^{r-j-1}}, \ldots).$$

In particular, $g_{r-1}^{N^{r-j-1}}$, and hence $g_{r-1}|_{\hat{X}_{p_j}}$ is a rational function which is integral over $\mathbb{K}[\hat{X}_{p_j}]$, and it does not vanish on an open and dense subset of $\hat{X}_{p_j}$.

From $\mathfrak{C} \in \mathcal{C}(g)$ one knows $a_r \le a'_r$. If the inequality were strict, then g_{r-1} would vanish on the open and dense subset of $\hat{X}_{p'_{r-1}}$ where g_{r-1} is defined. Notice that $\hat{X}_{p'_{r-1}} \supseteq \hat{X}_{p'_j} = \hat{X}_{p_j}$, and g_{r-1} does not vanish identically on the latter set, we get a contradiction.

It follows that $a_r = a'_r$ and hence $g_{r-1} = g'_{r-1}$. Since $g_{r-1} = g'_{r-1}$ does not vanish on an open and dense subset of $\hat{X}_{p'_j} = \hat{X}_{p_j}$, the inclusions

$$\hat{X}_{p'_j} \subseteq \hat{X}_{p'_{j+1}} \subseteq \cdots \subseteq \hat{X}_{p'_{r-2}}$$

imply $a'_{r-1} = \cdots = a'_{j+1} = 0$, and hence $g_j = g_{r-1}^{N^{r-j-1}} = g'_j$. Now replacing X by X_{p_j}, g by $g_{r-1}^{N^{r-j-1}}$ and consider the subposet A_{p_j}, we can proceed again by induction. □

8.3 Some consequences

We give some consequences of Proposition 8.7 which will be used later. As an immediate corollary of Proposition 8.7, we have:

Corollary 8.8 *If $g, h \in R \setminus \{0\}$, then $\mathcal{C}(g) \cap \mathcal{C}(h) \subseteq \mathcal{C}(gh)$.*

Proof If $\mathfrak{C} \in \mathcal{C}(g) \cap \mathcal{C}(h)$, then $\mathcal{V}_{\mathfrak{C}}(gh) = \mathcal{V}_{\mathfrak{C}}(g) + \mathcal{V}_{\mathfrak{C}}(h) = \mathcal{V}(g) + \mathcal{V}(h)$. This implies $\mathcal{V}(gh) \ge \mathcal{V}(g) + \mathcal{V}(h) = \mathcal{V}_{\mathfrak{C}}(gh)$ and hence $\mathcal{V}(gh) = \mathcal{V}_{\mathfrak{C}}(gh)$, i.e. $\mathfrak{C} \in \mathcal{C}(gh)$. □

Since $\mathcal{V}$ is a quasi-valuation, the inequality $\mathcal{V}(gh) \ge \mathcal{V}(g) + \mathcal{V}(h)$ holds. We can be more precise:

Proposition 8.9 *For* $g, h \in R \setminus \{0\}$, $\mathcal{V}(gh) = \mathcal{V}(g) + \mathcal{V}(h)$ *if and only if* $\mathcal{C}(g) \cap \mathcal{C}(h) \neq \emptyset$.

Proof For a maximal chain $\mathfrak{C}$ one has:

$$\begin{array}{ccc} \mathcal{V}_{\mathfrak{C}}(gh) & \geq & \mathcal{V}(gh) \\ \parallel & & \text{IV} \\ \mathcal{V}_{\mathfrak{C}}(g) + \mathcal{V}_{\mathfrak{C}}(h) & \geq & \mathcal{V}(g) + \mathcal{V}(h) \end{array} \tag{18}$$

If there exists a maximal chain $\mathfrak{C} \in \mathcal{C}(g) \cap \mathcal{C}(h)$, then by Corollary 8.8,

$$\mathcal{V}(g) + \mathcal{V}(h) = \mathcal{V}_{\mathfrak{C}}(g) + \mathcal{V}_{\mathfrak{C}}(h) = \mathcal{V}_{\mathfrak{C}}(gh) = \mathcal{V}(gh).$$

On the other hand, if $\mathcal{V}(gh) = \mathcal{V}(g) + \mathcal{V}(h)$ and $\mathfrak{C}$ is a maximal chain such that $\mathcal{V}_{\mathfrak{C}}(gh) = \mathcal{V}(gh)$, then (18) implies $\mathcal{V}_{\mathfrak{C}}(g) + \mathcal{V}_{\mathfrak{C}}(h) = \mathcal{V}(g) + \mathcal{V}(h)$. But this is only possible when $\mathcal{V}_{\mathfrak{C}}(g) = \mathcal{V}(g)$ and $\mathcal{V}_{\mathfrak{C}}(h) = \mathcal{V}(h)$, and hence $\mathfrak{C} \in \mathcal{C}(g) \cap \mathcal{C}(h)$. □

As an immediate consequence of Proposition 8.9 we get:

Corollary 8.10 *Let* $p_1, \ldots, p_s \in A$. *For* $a_1, \ldots, a_s \in \mathbb{N}$, $\mathcal{V}(f_{p_1}^{a_1} \cdots f_{p_s}^{a_s}) = \sum_{i=1}^{s} a_i e_{p_i}$ *if and only if there exists a maximal chain* $\mathfrak{C}$ *containing* $p_1, \ldots, p_s$.

9 Fan monoids associated to quasi-valuations

As suggested by Proposition 8.9, the image of the quasi-valuation $\mathcal{V}$ in $\mathbb{Q}^A$ is no longer a monoid. Nevertheless, by Corollary 8.10, it is not too far away from being a monoid. In the next sections we will study the algebraic and geometric structures of this image.

9.1 The fan algebra

We start with fixing notations. Let

$$\Gamma := \{\mathcal{V}(g) \mid g \in R \setminus \{0\}\} \subseteq \mathbb{Q}^A$$

denote the image of the quasi-valuation in $\mathbb{Q}^A$: as mentioned, Γ is in general not a monoid. Let $\mathcal{K}$ be the set of all chains in A. To every (not necessarily maximal) chain $C \in \mathcal{K}$ we associate the cone K_C in $\mathbb{R}^A$ defined as

$$K_C = \sum_{p \in C} \mathbb{R}_{\geq 0} e_p.$$

The collection $\{K_C \mid C \text{ is a chain in } A\}$, together with the origin $\{0\}$, defines a fan $\mathcal{F}_A$ in $\mathbb{R}^A$. By Lemma 2.2, the fan $\mathcal{F}_A$ is pure (i.e. all its maximal cones share the same dimension) of dimension $\dim X + 1$. Its maximal cones are the cones $K_{\mathfrak{C}}$ associated to the maximal chains $\mathfrak{C} \in \mathcal{C}$.

For $\underline{a} \in \Gamma$ recall that $\operatorname{supp} \underline{a} := \{p \in A \mid a_p \neq 0\}$. From the definition of $\mathcal{V}$, $\operatorname{supp} \underline{a} \subseteq \mathfrak{C}$ for some maximal chain $\mathfrak{C}$ and hence by Proposition 8.6, $\underline{a} \in K_{\mathfrak{C}}$. Conversely, for a maximal chain $\mathfrak{C}$ let $\Gamma_{\mathfrak{C}} \subseteq \Gamma$ be the subset

$$\Gamma_{\mathfrak{C}} := \{\underline{a} \in \Gamma \mid \operatorname{supp} \underline{a} \subseteq \mathfrak{C}\} = K_{\mathfrak{C}} \cap \Gamma.$$

Proposition 8.7, together with Proposition 8.9, implies: if $g, h \in R \setminus \{0\}$ are such that $\mathcal{V}(g), \mathcal{V}(h) \in \Gamma_{\mathfrak{C}}$, then

$$\mathfrak{C} \in \mathcal{C}(g) \cap \mathcal{C}(h) \Longrightarrow \mathcal{V}(g) + \mathcal{V}(h) = \mathcal{V}(gh) \in \Gamma_{\mathfrak{C}}.$$

In other words:

Corollary 9.1 *The quasi-valuation image* Γ *is a finite union of monoids*:

$$\Gamma = \bigcup_{\mathfrak{C} \in \mathcal{C}} \Gamma_{\mathfrak{C}}.$$

Definition 9.2 We call Γ the *fan of monoids* associated to the quasi-valuation $\mathcal{V}$.

We associate to the fan of monoids Γ the *fan algebra* given in terms of generators and relations. Since the various monoids $\Gamma_{\mathfrak{C}}$ are submonoids of possibly different lattices, to avoid confusion with the group algebra of one lattice, we prefer to write the elements of Γ as lower indexes instead of using them as exponents (upper indexes).

Definition 9.3 The fan algebra $\mathbb{K}[\Gamma]$ associated to the fan of monoids Γ is defined as

$$\mathbb{K}[\Gamma] := \mathbb{K}[x_{\underline{a}} \mid \underline{a} \in \Gamma]/I(\Gamma),$$

where $I(\Gamma)$ is the ideal generated by the following elements:

$$\begin{cases} x_{\underline{a}} \cdot x_{\underline{b}} - x_{\underline{a}+\underline{b}}, & \text{if there exists a chain } C \subseteq A \text{ such that } \underline{a}, \underline{b} \in K_C; \\ x_{\underline{a}} \cdot x_{\underline{b}}, & \text{if there exists no such a chain.} \end{cases}$$

To simplify the notation, we will write $x_{\underline{a}}$ also for its class in $\mathbb{K}[\Gamma]$. For a maximal chain $\mathfrak{C}$ denote by $\mathbb{K}[\Gamma_{\mathfrak{C}}]$ the subalgebra:

$$\mathbb{K}[\Gamma_{\mathfrak{C}}] := \bigoplus_{\underline{a} \in \Gamma_{\mathfrak{C}}} \mathbb{K} x_{\underline{a}} \subseteq \mathbb{K}[\Gamma],$$

then $\mathbb{K}[\Gamma_{\mathfrak{C}}]$ is naturally isomorphic to the usual semigroup algebra associated to the monoid $\Gamma_{\mathfrak{C}}$.

We endow the algebra $\mathbb{K}[\Gamma]$ with a grading inspired by Corollary 7.5: for $\underline{a} \in \mathbb{Q}^A$, the degree of $x_{\underline{a}}$ is defined by

$$\deg x_{\underline{a}} = \sum_{p \in A} a_p \deg f_p.$$

9.2 Weakly positivity versus standardness

For a fixed maximal chain $\mathfrak{C}$, we start with an algebraic relation between the monoid $\Gamma_{\mathfrak{C}}$ and the core $P_{\mathfrak{C}}$.

Definition 9.4 A regular function $g \in R \setminus \{0\}$ is called

(1) *weakly positive* along $\mathfrak{C}$ if $\mathcal{V}_{\mathfrak{C}}(g) \in P_{\mathfrak{C}}$, i.e. $\mathcal{V}_{\mathfrak{C}}(g) \in \mathbb{Q}^{\mathfrak{C}}_{\geq 0}$.
(2) *standard* along $\mathfrak{C}$ if $\mathcal{V}_{\mathfrak{C}}(g) \in \Gamma_{\mathfrak{C}}$, i.e. $\mathcal{V}(g) = \mathcal{V}_{\mathfrak{C}}(g)$.

By Proposition 8.6, if $g \neq 0$ is standard along $\mathfrak{C}$, then g is also positive along $\mathfrak{C}$. So we have a natural inclusion of submonoids of the cone $K_{\mathfrak{C}}$:

$$\Gamma_{\mathfrak{C}} \subseteq P_{\mathfrak{C}} \subseteq K_{\mathfrak{C}}.$$

Remark 9.5 A regular function, which is weakly positive along $\mathfrak{C}$, is not necessarily standard along $\mathfrak{C}$ (see Example 16.7). One should think of the *weak positivity property* as a local property, whereas the *standardness property* involves all maximal chains, and it is in this sense a global property.

One might ask up to what extent the two submonoids differ.

Lemma 9.6

i) *The monoid* $\Gamma_{\mathfrak{C}}$ *is finitely generated.*
ii) *The ring extension* $\mathbb{K}[\Gamma_{\mathfrak{C}}] \subseteq \mathbb{K}[P_{\mathfrak{C}}]$ *is finite and integral.*

Proof By Corollary 8.10, one has $\mathbb{N}^{\mathfrak{C}} \subseteq \Gamma_{\mathfrak{C}} \subseteq P_{\mathfrak{C}}$, and hence $\Gamma_{\mathfrak{C}}$ is a natural $\mathbb{N}^{\mathfrak{C}}$-submodule of $P_{\mathfrak{C}}$. Let Q be as in the proof of Lemma 7.2. We have:

$$\Gamma_{\mathfrak{C}} = \bigcup_{\underline{a} \in Q} \left(\mathbb{N}^{\mathfrak{C}} \circ \underline{a} \cap \Gamma_{\mathfrak{C}}\right) \subseteq \mathbb{Q}^{\mathfrak{C}}_{\geq 0}.$$

The same arguments as in the proof of Lemma 7.2 show that $\mathbb{N}^{\mathfrak{C}} \circ \underline{a} \cap \Gamma_{\mathfrak{C}}$ is finitely generated as $\mathbb{N}^{\mathfrak{C}}$-module for all $\underline{a}$ in Q.

For each $\underline{a} \in Q$ fix a finite generating system $\mathbb{B}_{\underline{a}}$ for $\mathbb{N}^{\mathfrak{C}} \circ \underline{a} \cap \Gamma_{\mathfrak{C}}$ as $\mathbb{N}^{\mathfrak{C}}$-module. Then the union of the $\mathbb{B}_{\underline{a}}$, $\underline{a} \in Q$, together with $\{e_0, \ldots, e_r\}$, is a finite generating system for the monoid $\Gamma_{\mathfrak{C}}$.

Since $P_{\mathfrak{C}}$ is a finitely generated module over $\mathbb{N}^{\mathfrak{C}}$, it is hence a finitely generated module over $\Gamma_{\mathfrak{C}}$, and thus $\mathbb{K}[P_{\mathfrak{C}}]$ is a finite $\mathbb{K}[\Gamma_{\mathfrak{C}}]$-module. So one can find a finite number of elements $\underline{a}^1, \ldots, \underline{a}^s$ in $P_{\mathfrak{C}}$ such that $\mathbb{K}[P_{\mathfrak{C}}] = \mathbb{K}[\Gamma_{\mathfrak{C}}][x_{\underline{a}^1}, \ldots, x_{\underline{a}^s}]$. To prove *ii)*, it remains to show that these generators are integral over $\mathbb{K}[\Gamma_{\mathfrak{C}}]$.

So suppose $\underline{a} \in P_{\mathfrak{C}}$, and let $g \neq 0$ be a regular function such that $\mathcal{V}_{\mathfrak{C}}(g) = \underline{a} = \sum_{i=0}^{r} a_i e_{p_i} \in \mathbb{Q}^{\mathfrak{C}}_{\geq 0}$. By Corollary 7.4, one can find an $m \geq 1$ such that

$$\mathcal{V}_{\mathfrak{C}}(g^m) = \mathcal{V}_{\mathfrak{C}}(f_{p_r}^{ma_r} \cdots f_{p_0}^{ma_0}) = m\underline{a} \in \mathbb{N}^{\mathfrak{C}},$$

and hence $m\underline{a} \in \Gamma_{\mathfrak{C}}$. It follows that $x_{\underline{a}} \in \mathbb{K}[P_{\mathfrak{C}}]$ satisfies the equation $p(x^{\underline{a}}) = 0$ for the monic polynomial $p(y) = y^m - x_{m\underline{a}} \in \mathbb{K}[\Gamma_{\mathfrak{C}}][y]$, which finishes the proof. □

Remark 9.7 The proof of Lemma 9.6 does not imply that g^m is standard along $\mathfrak{C}$. But one knows from Corollary 8.10 that the function $f_{p_r}^{ma_r} \cdots f_{p_0}^{ma_0}$ is standard along $\mathfrak{C}$.

Since $\mathbb{K}[\Gamma_{\mathfrak{C}}]$ is the algebra associated to a finitely generated submonoid of the real cone $K_{\mathfrak{C}}$, it is a finitely generated integral domain, and the associated variety $\mathrm{Spec}(\mathbb{K}[\Gamma_{\mathfrak{C}}])$ is an affine toric variety. Corollary 9.1 implies hence that $\mathbb{K}[\Gamma]$ is a reduced, finitely generated algebra, so $\mathrm{Spec}(\mathbb{K}[\Gamma])$ is an affine variety. The geometry of the fan of monoids Γ is summarized in the following proposition:

Proposition 9.8 *The affine variety* $\mathrm{Spec}(\mathbb{K}[\Gamma])$ *is scheme-theoretically the irredundant union of the toric varieties* $\mathrm{Spec}(\mathbb{K}[\Gamma_{\mathfrak{C}}])$ *with* $\mathfrak{C}$ *running over the set of maximal chains in* A*. Each of these toric varieties is irreducible and of dimension* $\dim X + 1$.

Proof Let $I_{\mathfrak{C}} = \mathrm{Ann}(x_{\mathfrak{C}})$ be the annihilator of the element $x_{\mathfrak{C}} = \prod_{p\in\mathfrak{C}} x_{e_p} \in \mathbb{K}[\Gamma]$. The multiplication rules in Definition 9.3 imply that $\mathbb{K}[\Gamma_{\mathfrak{C}}] \simeq \mathbb{K}[\Gamma]/I_{\mathfrak{C}}$. Since $\mathbb{K}[\Gamma_{\mathfrak{C}}]$ has no zero divisors, it follows that $I_{\mathfrak{C}}$ is a prime ideal.

For each maximal chain $\mathfrak{C}$ the canonical map $\mathbb{K}[\Gamma] \to \mathbb{K}[\Gamma_{\mathfrak{C}}]$ induces an embedding of a toric variety of dimension $\dim X + 1$:

$$\mathrm{Spec}(\mathbb{K}[\Gamma_{\mathfrak{C}}]) \hookrightarrow \mathrm{Spec}(\mathbb{K}[\Gamma]).$$

We show that the intersection of the prime ideals $I_{\mathfrak{C}}$ is equal to zero: $\bigcap_{\mathfrak{C}} I_{\mathfrak{C}} = (0)$. Indeed, let $h \in \bigcap_{\mathfrak{C}} I_{\mathfrak{C}}$ be a linear combination of monomials. For a monomial $\prod_{p\in A} x_{n_p e_p}$ let the support be the set $\{p \in A \mid n_p > 0\}$. The support of a monomial is, by the definition of the fan algebra, always contained in a maximal chain. Since h has no constant term, if h is non-zero, there would be at least one non-zero monomial in the linear combination. Let $\mathfrak{C}$ be a maximal chain containing the support of one of these monomials. In the product $x_{\mathfrak{C}} h$, all monomials are supported in $\mathfrak{C}$ and they stay linearly independent. It follows $x_{\mathfrak{C}} h \neq 0$, and thus $h \notin \bigcap_{\mathfrak{C}} I_{\mathfrak{C}}$, in contradiction to the assumption.

It remains to show the minimality of the intersection. Given a maximal chain $\mathfrak{C}$ in A, we have $x_{\mathfrak{C}} \in I_{\mathfrak{C}'}$ for any $\mathfrak{C}' \neq \mathfrak{C}$, while $x_{\mathfrak{C}} \notin I_{\mathfrak{C}}$, which finishes the proof. □

We close the section with some comments on Hilbert quasi-polynomials and a remark on connections to structures similar to the fan algebra. Let R_1 be an $\mathbb{N}$-graded integral $\mathbb{K}$-algebra and let $\mathrm{Quot}(R_1)$ be its quotient field. Denote by $Q(R_1) \subseteq \mathrm{Quot}(R_1)$ the subalgebra generated by the elements $\frac{h_1}{h_2}$, $h_1, h_2 \in R_1$ homogeneous, it is a $\mathbb{Z}$-graded algebra.

Lemma 9.9 *Suppose that* $R_1 \subseteq R_2 \subseteq Q(R_1)$ *with* R_2 *a* $\mathbb{Z}$*-graded* $\mathbb{K}$*-algebra, which is finitely generated as an* R_1*–module. If the Hilbert quasi-polynomial of* R_2 *has constant leading coefficient, then the leading term of the Hilbert quasi-polynomial of* R_1 *is the same as that of* R_2.

Proof Let $h_1, \ldots, h_q \in R_2$ be the generators of R_2 as R_1-module. Without loss of generality we may assume that they are homogeneous and of the form $h_i = \frac{h_{i,1}}{h_{i,2}}$,

$h_{i,1}, h_{i,2} \in R_1$ homogeneous. Let h be the product of the denominators. The equality $R_2 = \sum_{i=1}^{q} R_1 h_i$ implies $hR_2 \subseteq R_1$. The Hilbert quasi-polynomial of the shifted algebra hR_2 has the same leading term as the one of R_2. Now the inclusions of graded algebras $hR_2 \subseteq R_1 \subseteq R_2$ show that the Hilbert quasi-polynomials of all three have the same leading term. □

Lemma 9.10 *Let $\mathcal{L}^{\mathfrak{C}} \subseteq \mathbb{Q}^{\mathfrak{C}}$ be the lattice generated by $\Gamma_{\mathfrak{C}}$ and let $\tilde{\Gamma}_{\mathfrak{C}} = \mathcal{L}^{\mathfrak{C}} \cap K_{\mathfrak{C}}$ be the saturation of the monoid. We endow it with the same grading as $\Gamma_{\mathfrak{C}}$. The Hilbert quasi-polynomials associated to $\Gamma_{\mathfrak{C}}$ and $\tilde{\Gamma}_{\mathfrak{C}}$ have the same leading term.*

Proof The quasi-polynomial of $\mathbb{K}[\tilde{\Gamma}_{\mathfrak{C}}]$ is by construction an Ehrhart quasi-polynomial with a constant leading coeffcient. Since $\mathbb{K}[\tilde{\Gamma}_{\mathfrak{C}}]$ is the normalization of $\mathbb{K}[\Gamma_{\mathfrak{C}}]$, the inclusions $\mathbb{K}[\Gamma_{\mathfrak{C}}] \subseteq \mathbb{K}[\tilde{\Gamma}_{\mathfrak{C}}] \subseteq \mathbb{K}[\mathcal{L}^{\mathfrak{C}}]$ fulfill the conditions of Lemma 9.9. □

Remark 9.11 A structure similar to the fan algebra appears in [29, Definition 1.1 (3)] as part of their definition of a toric degeneration of Calabi-Yau varieties. We will see later (Theorem 11.1) that the quasi-valuation $\mathcal{V}$ induces a filtration on the homogeneous coordinate ring R of X, such that the associated graded ring $\mathrm{gr}_{\mathcal{V}} R$ is isomorphic to the fan algebra defined above. This leads to a flat degeneration (Theorem 12.2) of X into X_0, a reduced union of equidimensional projective toric varieties. If the Seshadri stratification is normal (see Definition 13.7) and the partially ordered set A in the Seshadri stratification is shellable, then X_0 is Cohen-Macaulay (Theorem 14.1), and the flat degeneration fulfills the conditions in [29, Definition 1.1(3)].

Also the *toric bouquets* associated to a quasifan and a lattice in [2] lead to a structure similar to the fan algebra. They start with a lattice M and a quasifan (see [2] for definitions and details) and associate to this pair the *fan ring*. This ring is defined in a similar way as in Definition 9.3, see [2, Definition 7.1]. The differences between the two approaches are: (1). we do not make the assumption that the semigroups are saturated; (2). we do not need, and hence do not assume to have a lattice $M \subseteq \mathbb{Q}^A$ such that for all maximal chains we have $\mathcal{L}^{\mathfrak{C}} = M \cap \mathbb{Q}^{\mathfrak{C}}$. Recall that $\mathcal{L}^{\mathfrak{C}}$ is the lattice generated by the semigroup $\Gamma_{\mathfrak{C}}$ in $\mathbb{Q}^{\mathfrak{C}}$. One can find a lattice $M \subseteq \mathbb{Q}^A$ such that the $\mathcal{L}^{\mathfrak{C}}$ are of finite index in the intersection $M \cap \mathbb{Q}^{\mathfrak{C}}$, but it is an open question to find conditions on the Seshadri stratification ensuring the equality $\mathcal{L}^{\mathfrak{C}} = M \cap \mathbb{Q}^{\mathfrak{C}}$ for all maximal chains.

10 Leaves, the associated graded ring and the fan algebra

The quasi-valuation $\mathcal{V} : R \setminus \{0\} \to \mathbb{Q}^A$ induces a filtration on the homogeneous coordinate ring $R = \mathbb{K}[\hat{X}]$. In this section we further investigate the associated graded algebra $\mathrm{gr}_{\mathcal{V}} R$: we will prove properties parallel to those for valuations in Sects. 6 and 7 and those for fan algebras in Sect. 9.

Definition 10.1 For $\underline{a} \in \Gamma \subseteq \mathbb{Q}^A_{\geq 0}$ we set

$$R_{\geq \underline{a}} := \{g \in R \setminus \{0\} \mid \mathcal{V}(g) \geq \underline{a}\} \cup \{0\} \text{ and } R_{> \underline{a}} := \{g \in R \setminus \{0\} \mid \mathcal{V}(g) > \underline{a}\} \cup \{0\}.$$

Since the quasi-valuation has only non-negative entries, these subspaces are ideals. Denote the associated graded algebra by:

$$\mathrm{gr}_{\mathcal{V}} R = \bigoplus_{\underline{a} \in \Gamma} R_{\geq \underline{a}} / R_{> \underline{a}}.$$

Each $R_{\geq \underline{a}} / R_{> \underline{a}}$ for $a \in \Gamma$ will be called a *leaf* of the quasi-valuation.

We start with the one-dimensional leaves property of this quasi-valuation.

Lemma 10.2 *The leaves $R_{\geq \underline{a}} / R_{> \underline{a}}$, $\underline{a} \in \Gamma$, are one dimensional.*

Proof Let f, g be non-zero regular functions on $\hat{X}$ such that $\mathcal{V}(g) = \mathcal{V}(f) = \underline{a}$. Let $\mathfrak{C} \in \mathcal{C}(g)$ and $\mathfrak{C}' \in \mathcal{C}(f)$ be such that

$$\mathcal{V}(g) = \mathcal{V}_{\mathfrak{C}}(g) = \underline{a} = \mathcal{V}_{\mathfrak{C}'}(f) = \mathcal{V}(f).$$

These equalities imply $\operatorname{supp} \mathcal{V}_{\mathfrak{C}}(g) = \operatorname{supp} \mathcal{V}_{\mathfrak{C}'}(f)$ and hence $\operatorname{supp} \mathcal{V}_{\mathfrak{C}'}(f) \subseteq \mathfrak{C}$. By Proposition 8.7, this implies $\mathfrak{C} \in \mathcal{C}(f)$ and hence we can assume $\mathfrak{C} = \mathfrak{C}'$. By Corollary 6.17 there exist a scalar $\lambda \in \mathbb{K}^*$ and $h \in R$ such that $g = \lambda f + h$ with $\mathcal{V}_{\mathfrak{C}}(h) > \underline{a}$ when $h \neq 0$.

It remains to show that $\mathcal{V}(h) > \underline{a}$ if $h \neq 0$. Now $\mathcal{V}$ is a quasi-valuation and hence $\mathcal{V}(h) \geq \min\{\mathcal{V}(g), \mathcal{V}(f)\} = \underline{a}$. If we have equality, then there exists a maximal chain $\mathfrak{C}_2$ such that $\mathcal{V}_{\mathfrak{C}_2}(h) = \mathcal{V}(h) = \underline{a} = \mathcal{V}_{\mathfrak{C}}(g)$. A similar argument as above implies $\operatorname{supp} \mathcal{V}_{\mathfrak{C}}(g) = \operatorname{supp} \mathcal{V}_{\mathfrak{C}_2}(h)$ and hence $\operatorname{supp} \mathcal{V}_{\mathfrak{C}_2}(h) \subseteq \mathfrak{C}$. As a consequence one has $\mathfrak{C} \in \mathcal{C}(h)$ and hence $\mathcal{V}(h) = \mathcal{V}_{\mathfrak{C}}(h) = \underline{a}$, which is a contradiction. □

As a consequence we can prove a version of Corollary 7.4 for the quasi-valuation. The proof relies on the following lemma, which is a generalization of Lemma 6.15 to the quasi-valuation $\mathcal{V}$:

Lemma 10.3

i) *For any $g \in R \backslash \{0\}$ and $\lambda \in \mathbb{K}^*$, $\mathcal{V}(g^\lambda) = \mathcal{V}(g)$.*
ii) *Let $g = g_1 + \cdots + g_t \in R = \bigoplus_{i \geq 0} R(i)$ be a decomposition of $g \neq 0$ into its homogeneous parts. Then*

$$\mathcal{V}(g) = \min\{\mathcal{V}(g_j) \mid j = 1, \ldots, t\}.$$

Proof Let $\mathfrak{C} \in \mathcal{C}(g)$, then $\mathcal{V}(g) = \mathcal{V}_{\mathfrak{C}}(g) = \mathcal{V}_{\mathfrak{C}}(g^\lambda) \geq \mathcal{V}(g^\lambda)$. And if $\mathfrak{C} \in \mathcal{C}(g^\lambda)$, then we get vice versa: $\mathcal{V}(g^\lambda) = \mathcal{V}_{\mathfrak{C}}(g^\lambda) = \mathcal{V}_{\mathfrak{C}}(g) \geq \mathcal{V}(g)$, which proves part i).

To prove ii), recall that $\mathcal{V}(g) \geq \min\{\mathcal{V}(g_j) \mid j = 1, \ldots, r\}$ by property (a) in Definition 3.1. But we can also find pairwise distinct, non-zero scalars $\lambda_1, \ldots, \lambda_t$ such that the linear spans of the following functions coincide: $\langle g_1, \ldots, g_t \rangle_{\mathbb{K}} = \langle g^{\lambda_1}, \ldots, g^{\lambda_t} \rangle_{\mathbb{K}}$, and hence $\mathcal{V}(g_j) \geq \min\{\mathcal{V}(g^{\lambda_j}) \mid j = 1, \ldots, r\} = \mathcal{V}(g)$ by part i) of the lemma. □

Corollary 10.4 *Let $g \in R \setminus \{0\}$ and suppose $\mathcal{V}(g) = \sum_{p \in A} a_p e_p$. If m is such that $ma_p \in \mathbb{N}$ for all $p \in A$, then there exist $\lambda \in \mathbb{K}^*$ and $g' \in R$ such that*

$$g^m = \lambda \prod_{p \in A} f_p^{ma_p} + g'$$

with $\mathcal{V}(g') > \mathcal{V}(g^m)$ when $g' \neq 0$. If g is homogeneous and $g' \neq 0$, then $f_{p_r}^{ma_r} \cdots f_{p_0}^{ma_0}$ and g' are homogeneous of the same degree as g.

Before investigating $\mathrm{gr}_{\mathcal{V}} R$, we fix a maximal chain $\mathfrak{C}$ and look at the subspace $\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$ consisting of leaves supported in $\mathfrak{C}$, i.e.

$$\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R = \bigoplus_{\underline{a} \in \Gamma_{\mathfrak{C}}} R_{\geq \underline{a}} / R_{> \underline{a}} \subseteq \mathrm{gr}_{\mathcal{V}} R.$$

This is actually a subalgebra: for $\underline{a}, \underline{b} \in \Gamma_{\mathfrak{C}}$ let $g, h \in R$ be representatives of $\bar{g} \in R_{\geq \underline{a}} / R_{> \underline{a}} \setminus \{0\}$ and $\bar{h} \in R_{\geq \underline{b}} / R_{> \underline{b}} \setminus \{0\}$. Since $\mathcal{V}(g), \mathcal{V}(h) \in \Gamma_{\mathfrak{C}}$, we know by Proposition 8.7: $\mathcal{V}(g) = \mathcal{V}_{\mathfrak{C}}(g)$, $\mathcal{V}(h) = \mathcal{V}_{\mathfrak{C}}(h)$, and hence by Lemma 8.9, $\mathcal{V}(gh) = \mathcal{V}_{\mathfrak{C}}(gh)$. This implies $\mathcal{V}(gh) = \mathcal{V}(g) + \mathcal{V}(h) = \underline{a} + \underline{b}$, and therefore

$$(R_{\geq \underline{a}} / R_{> \underline{a}}) \cdot (R_{\geq \underline{b}} / R_{> \underline{b}}) \subseteq R_{\geq \underline{a}+\underline{b}} / R_{> \underline{a}+\underline{b}} \subseteq \mathrm{gr}_{\mathcal{V},\mathfrak{C}} R.$$

It follows that $\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$ is a subalgebra of $\mathrm{gr}_{\mathcal{V}} R$ graded by the monoid $\Gamma_{\mathfrak{C}}$, without zero divisors and every graded component is one dimensional. This allows us to apply again [10, Remark 4.13]:

Lemma 10.5 *There exists an isomorphism of algebras $\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R \simeq \mathbb{K}[\Gamma_{\mathfrak{C}}]$.*

By Corollary 9.6 one has the following immediate consequence:

Corollary 10.6

i) *The $\mathbb{K}$-algebra $\mathrm{gr}_{\mathcal{V}} R$ is finitely generated and reduced.*
ii) *For any maximal chain $\mathfrak{C}$, the $\mathbb{K}$-algebra $\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$ is a finitely generated integral domain.*

It follows that $\mathrm{Spec}(\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R)$ is an irreducible affine variety, Lemma 10.5 implies that it is a toric variety.

Similarly, $\mathrm{Spec}(\mathrm{gr}_{\mathcal{V}} R)$ is an affine variety. We give a decomposition of $\mathrm{Spec}(\mathrm{gr}_{\mathcal{V}} R)$ into irreducible components as in Proposition 9.8. The following proposition can be deduced as a corollary of Theorem 11.1. We sketch a direct proof which is similar to Proposition 9.8.

For a maximal chain $\mathfrak{C} = (p_r, \ldots, p_0)$ we set $x_{\mathfrak{C}} = f_{p_0} \cdots f_{p_r} \in R$ and $I_{\mathfrak{C}} := \mathrm{Ann}(\overline{x}_{\mathfrak{C}}) \subseteq \mathrm{gr}_{\mathcal{V}} R$.

Proposition 10.7 *Given a maximal chain $\mathfrak{C}$, the quotient $\mathrm{gr}_{\mathcal{V}} R / I_{\mathfrak{C}}$ is isomorphic to $\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$. In particular, $I_{\mathfrak{C}}$ is a homogeneous prime ideal of $\mathrm{gr}_{\mathcal{V}} R$. Moreover $\bigcap_{\mathfrak{C}} I_{\mathfrak{C}} = (0)$ is the minimal prime decomposition of the zero ideal in $\mathrm{gr}_{\mathcal{V}} R$.*

Proof First note that $\mathcal{V}(x_{\mathfrak{C}}) = e_{p_r} + e_{p_{r-1}} + \cdots + e_{p_0}$, hence $\operatorname{supp}\mathcal{V}(x_{\mathfrak{C}}) = \mathfrak{C}$ and $\mathcal{C}(x_{\mathfrak{C}}) = \{\mathfrak{C}\}$ by Proposition 8.7. Now, for $g \in R$ we have $\overline{g} \cdot \overline{x_{\mathfrak{C}}} = 0$ in $\mathrm{gr}_{\mathcal{V}} R$ if and only if $\mathcal{V}(g \cdot x_{\mathfrak{C}}) > \mathcal{V}(g) + \mathcal{V}(x_{\mathfrak{C}})$ and, by Proposition 8.9, this is equivalently to $\operatorname{supp}\mathcal{V}(g) \not\subseteq \mathfrak{C}$, i.e. $\mathcal{V}(g) \notin \Gamma_{\mathfrak{C}}$.

This characterization of the elements of $I_{\mathfrak{C}} = \mathrm{Ann}(x_{\mathfrak{C}})$ proves that

$$I_{\mathfrak{C}} = \bigoplus_{\underline{a} \in \Gamma \setminus \Gamma_{\mathfrak{C}}} R_{\geq \underline{a}} / R_{> \underline{a}}.$$

It follows: $\mathrm{gr}_{\mathcal{V}} R / I_{\mathfrak{C}} \simeq \mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$ and, since $\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$ is an integral domain, $I_{\mathfrak{C}}$ is a prime ideal. It also follows that $\bigcap_{\mathfrak{C}} I_{\mathfrak{C}} = (0)$.

We want to show that $I_{\mathfrak{C}}$ is a minimal prime. So, suppose that I is an ideal properly contained in $I_{\mathfrak{C}}$. The quotient $\mathrm{gr}_{\mathcal{V}} R / I$ contains an element $\overline{g}$ with $\mathcal{V}(g) \notin \Gamma_{\mathfrak{C}}$, in particular we have $\overline{g} \cdot \overline{x_{\mathfrak{C}}} = 0$ in $\mathrm{gr}_{\mathcal{V}} R / I$ again by Proposition 8.9. Hence this quotient is not a domain and I is not prime.

Finally, note that, for a maximal chain $\mathfrak{D}$, $\overline{x_{\mathfrak{D}}}$ is a non-zero element in the intersection $\bigcap_{\mathfrak{C} \neq \mathfrak{D}} I_{\mathfrak{C}}$. This shows that the intersection $\bigcap_{\mathfrak{C}} I_{\mathfrak{C}}$ is non-redundant. □

Corollary 10.8 *The variety* $\mathrm{Spec}(\mathrm{gr}_{\mathcal{V}} R)$ *is scheme-theoretically the irredundant union of the irreducible varieties* $\mathrm{Spec}(\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R)$ *with* $\mathfrak{C}$ *running over the set of maximal chains of* A; *each of these varieties is irreducible and of dimension* $\dim X + 1$.

Proof This follows by Lemma 10.5 and Proposition 10.7. □

11 The fan algebra and the degenerate algebra

The goal of this section is to give a description of the associated graded algebra $\mathrm{gr}_{\mathcal{V}} R$ (Theorem 11.1). It is a wide generalization of the conjecture on special LS-algebra stated in [13, Remark 1].

We fix throughout this section a vector space basis $\mathbb{B}$ of the degenerate algebra $\mathrm{gr}_{\mathcal{V}} R$:

$$\mathbb{B} = \{\bar{g}_{\underline{a}} \mid \underline{a} \in \Gamma, 0 \neq \bar{g}_{\underline{a}} \in R_{\geq \underline{a}} / R_{> \underline{a}}\}.$$

By Proposition 8.9 we know for $\underline{a}, \underline{b} \in \Gamma$: $\bar{g}_{\underline{a}} \cdot \bar{g}_{\underline{b}} \neq 0$ in $\mathrm{gr}_{\mathcal{V}} R$ if and only if one can find a chain C in A (not necessarily maximal) such that $\operatorname{supp}\underline{a}, \operatorname{supp}\underline{b} \subseteq C$. If this holds, then one finds some non-zero $c_{\underline{a},\underline{b}} \in \mathbb{K}^*$ such that $\bar{g}_{\underline{a}} \cdot \bar{g}_{\underline{b}} = c_{\underline{a},\underline{b}} \bar{g}_{\underline{a}+\underline{b}}$. This list of non-zero coefficients $c_{\underline{a},\underline{b}}$ provides a complete description of $\mathrm{gr}_{\mathcal{V}} R$ in terms of generators and relations.

Theorem 11.1 *The degenerate algebra* $\mathrm{gr}_{\mathcal{V}} R$ *is isomorphic to the fan algebra* $\mathbb{K}[\Gamma]$.

Proof First note that if we fix constants $c_{\underline{a}} \in \mathbb{K}^*$ for all $\underline{a} \in \Gamma$, the linear map

$$\mathbb{K}[\Gamma] \to \mathrm{gr}_{\mathcal{V}} R, \quad x_{\underline{a}} \mapsto c_{\underline{a}} \bar{g}_{\underline{a}}$$

is a vector space isomorphism between the fan algebra $\mathbb{K}[\Gamma]$ and the degenerate algebra $\mathrm{gr}_{\mathcal{V}} R$.

Suppose we have already had for all $\underline{a} \in \Gamma$ non-zero elements $c_{\underline{a}} \in \mathbb{K}^*$ with the following property: whenever $\mathrm{supp}\,\underline{a}, \mathrm{supp}\,\underline{b} \subseteq C$ for some chain C in A, then

$$c_{\underline{a}} \cdot c_{\underline{b}} = c_{\underline{a}+\underline{b}} \cdot c_{\underline{a},\underline{b}}. \tag{19}$$

We rescale the basis $\mathbb{B}$ and get a new basis: $\mathbb{B}' = \{\bar{h}_{\underline{a}} = \frac{1}{c_{\underline{a}}} \bar{g}_{\underline{a}} \mid \underline{a} \in \Gamma\}$. The corresponding rescaled vector space isomorphism is defined on the new basis as follows:

$$\chi : \mathbb{K}[\Gamma] \to \mathrm{gr}_{\mathcal{V}} R, \quad x_{\underline{a}} \mapsto \bar{h}_{\underline{a}} \quad \text{for all } \underline{a} \in \Gamma.$$

This is in fact an algebra isomorphism: if $\underline{a}, \underline{b} \in \Gamma$ are such that there is no chain in A containing both $\mathrm{supp}\,\underline{a}$ and $\mathrm{supp}\,\underline{b}$, then $\bar{h}_{\underline{a}} \cdot \bar{h}_{\underline{b}} = 0$, and otherwise we get

$$\bar{h}_{\underline{a}} \cdot \bar{h}_{\underline{b}} = \frac{1}{c_{\underline{a}} c_{\underline{b}}} \bar{g}_{\underline{a}} \bar{g}_{\underline{b}} = \frac{c_{\underline{a},\underline{b}}}{c_{\underline{a}} c_{\underline{b}}} \bar{g}_{\underline{a}+\underline{b}} = \bar{h}_{\underline{a}+\underline{b}}.$$

It remains to prove the existence of the $c_{\underline{a}} \in \mathbb{K}^*$ for $\underline{a} \in \Gamma$, satisfying the condition in (19). This will be done in the next subsections. □

11.1 Existence of the rescaling coefficients

To prove the existence of the rescaling coefficients $c_{\underline{a}}$, $\underline{a} \in \Gamma$, we construct an affine variety Z having the following properties:

(1) There is a natural interpretation of the $\bar{g}_{\underline{a}}$, $\underline{a} \in \Gamma$, as functions on Z.
(2) The relations $\bar{g}_{\underline{a}} \bar{g}_{\underline{b}} = c_{\underline{a},\underline{b}} \bar{g}_{\underline{a}+\underline{b}}$ for $\underline{a}, \underline{b} \in \Gamma$ hold also in $\mathbb{K}[Z]$ whenever there exists a chain C in A such that $\mathrm{supp}\,\underline{a}, \mathrm{supp}\,\underline{b} \subseteq C$.
(3) There exists a point $z \in Z$ such that $\bar{g}_{\underline{a}}(z) \neq 0$ for all $\underline{a} \in \Gamma$.

Proof of the existence of the rescaling coefficients Suppose Z is an affine variety having the above properties. Let $z \in Z$ be a point as in 11.1.3. By 11.1.1 it is possible to define $c_{\underline{a}} = \bar{g}_{\underline{a}}(z)$ for all $\underline{a} \in \Gamma$, and these are all elements in $\mathbb{K}^*$. Then property 11.1.2 implies: if there exists a chain C in A such that $\mathrm{supp}\,\underline{a}, \mathrm{supp}\,\underline{b} \subseteq C$, then

$$c_{\underline{a}} \cdot c_{\underline{b}} = g_{\underline{a}}(z) \cdot g_{\underline{b}}(z) = c_{\underline{a},\underline{b}} g_{\underline{a}+\underline{b}}(z) = c_{\underline{a},\underline{b}} \cdot c_{\underline{a}+\underline{b}}$$

So this collection of non-zero rescaling coefficients has the desired property (19). □

11.2 The varieties $Z_{\mathcal{M}}$

It remains to construct a variety Z with the properties 11.1.1 – 11.1.3. This will be done using an inductive procedure. Let $C \subseteq A$ be a chain, not necessarily maximal. We associate to C the submonoid $\Gamma_C = \{\underline{a} \in \Gamma \mid \mathrm{supp}(\underline{a}) \subseteq C\}$. Denote by $M_C = \langle \Gamma_C \rangle_{\mathbb{Z}} \subseteq \mathbb{Q}^A$ the lattice generated by Γ_C in $\mathbb{Q}^A$, and let N_C be the dual lattice of M_C. Set

$$Q_C = \bigoplus_{\underline{a} \in \Gamma_C} R_{\geq \underline{a}} / R_{> \underline{a}} \subseteq \mathrm{gr}_{\mathcal{V}} R.$$

In Sect. 10 we have discussed the case where $C = \mathfrak{C}$ is a maximal chain. The same arguments can be applied to show that Q_C is a finitely generated integral domain. Denote $Y_C = \operatorname{Spec} Q_C$ the associated affine variety. Again, as in Sect. 10, Y_C is a toric variety for the torus $T_C := T_{N_C}$, i.e. the torus associated to the lattices N_C. The $\mathbb{K}$-algebra Q_C is positively graded, so the affine variety Y_C has a unique vertex which we denote by 0.

The set of all chains in A is partially ordered with respect to the inclusion relation.

We fix an inclusion of chains $C_1 \subseteq C_2$. It induces an inclusion of monoids $i_{C_1,C_2} : \Gamma_{C_1} \hookrightarrow \Gamma_{C_2}$, which in turn induces a morphism of algebras: $i_{C_1,C_2} : Q_{C_1} \hookrightarrow Q_{C_2}$. By [19], Proposition 1.3.14, we get an induced toric morphism between the associated varieties $\psi_{C_1,C_2} : Y_{C_2} \to Y_{C_1}$ and a group homomorphism $\phi_{C_1,C_2} : T_{C_2} \to T_{C_1}$. All these maps are compatible: for a sequence of inclusions of chains $C_1 \hookrightarrow C_2 \hookrightarrow C_3$, one has $i_{C_2,C_3} \circ i_{C_1,C_2} = i_{C_1,C_3}$ and $\psi_{C_1,C_2} \circ \psi_{C_2,C_3} = \psi_{C_1,C_3}$ and so on.

Moreover, we also have a closed immersion $\Psi_{C_1,C_2} : Y_{C_1} \hookrightarrow Y_{C_2}$ of varieties: the subspace

$$I_{C_1,C_2} = \bigoplus_{\underline{a} \in \Gamma_{C_2} \setminus \Gamma_{C_1}} R_{\geq \underline{a}} / R_{> \underline{a}} \subseteq Q_{C_2}$$

is an ideal and we have a natural isomorphism of algebras $Q_{C_2}/I_{C_1,C_2} \to Q_{C_1}$. This isomorphism induces the desired closed immersion $\Psi_{C_1,C_2} : Y_{C_1} = \operatorname{Spec} Q_{C_1} \hookrightarrow Y_{C_2} = \operatorname{Spec} Q_{C_2}$. The composition of algebra morphisms

$$Q_{C_1} \hookrightarrow Q_{C_2} \to Q_{C_1} \simeq Q_{C_2}/I_{C_1,C_2}$$

is the identity map on Q_{C_1}. The composition of morphisms of affine varieties:

$$Y_{C_1} \xhookrightarrow{\Psi_{C_1,C_2}} Y_{C_2} \xrightarrow{\psi_{C_1,C_2}} Y_{C_1}$$

is hence the identity map on Y_{C_1}.

Remark 11.2 We can extend the above definitions to the case when $C = \emptyset$ is the empty set. In this case $\Gamma_\emptyset = \{0\} \in \Gamma$, $M_\emptyset = \{0\} \in \mathbb{Q}^A$, $Q_\emptyset = \mathbb{K}$ and $Y_\emptyset$ is the vertex 0 in all toric varieties Y_C. Such definitions are compatible with the inclusion $\emptyset \subseteq C$ for any chain C; this allows us to define $i_{\emptyset,C}$, $\psi_{\emptyset,C}$, $\Psi_{\emptyset,C}$, etc.

Definition 11.3 A subset $\mathcal{M}$ of the set of all chains in A is called *saturated* if for any chain $C \in \mathcal{M}$ and a subset $C' \subseteq C$, we have $C' \in \mathcal{M}$. The set of all saturated subsets in A will be denoted by $\mathcal{K}^s$.

Notice that the empty set is contained in any saturated subset. The set $\mathcal{K}^s$ is partially ordered with respect to inclusion.

We now associate to $\mathcal{M} \in \mathcal{K}^s$ an affine variety $Z_\mathcal{M}$ as follows:

Definition 11.4 For $\mathcal{M} \in \mathcal{K}^s$, we define $Z_{\mathcal{M}}$ as the following closed subset of the product of varieties $\prod_{C \in \mathcal{M}} Y_C$:

$$Z_{\mathcal{M}} := \left\{ (y_C)_C \in \prod_{C \in \mathcal{M}} Y_C \mid \forall\, C_1 \subseteq C_2 \in \mathcal{M},\ \psi_{C_1,C_2}(y_{C_2}) = y_{C_1} \right\}.$$

We endow $Z_{\mathcal{M}}$ with the induced reduced structure as an affine variety.

For $C \in \mathcal{M}$, the canonical projection $p_C : \prod_{C' \in \mathcal{M}} Y_{C'} \to Y_C$ restricts to a morphism $p_C : Z_{\mathcal{M}} \to Y_C$.

Lemma 11.5 *The morphism $p_C : Z_{\mathcal{M}} \to Y_C$ is surjective.*

Proof To prove the lemma, we associate to a chain $C \in \mathcal{M}$ a subvariety $Z_C \subseteq Z_{\mathcal{M}}$ having the property: $p_C|_{Z_C} : Z_C \to Y_C$ is an isomorphism. As a first step we define Z_C as a subvariety of $\prod_{C' \in \mathcal{M}} Y_{C'}$ and set:

$$Z_C = \left\{ (y_{C'})_{C'} \in \prod_{C' \in \mathcal{M}} Y_{C'} \;\middle|\; \begin{array}{l} y_C \in Y_C \\ y_{C'} = \Psi_{C \cap C', C'} \circ \psi_{C \cap C', C}(y_C) \quad \forall C' \in \mathcal{M} \end{array} \right\}.$$

The subset Z is closed, we endow Z_C with the induced reduced structure.

We show $Z_C \subseteq Z_{\mathcal{M}}$. Given a point $(y_{C'})_{C'} \in Z_C$ and two elements $C_1 \subseteq C_2$ in $\mathcal{M}$, we have to verify that $\psi_{C_1,C_2}(y_{C_2}) = y_{C_1}$. Indeed, set $C_i' = C \cap C_i$ for $i = 1, 2$. The inclusion $C_1 \subseteq C_2$ induces an inclusion $C_1' \subseteq C_2'$. The two inclusions induce two algebra homomorphisms: $i_{C_1,C_2} : Q_{C_1} \hookrightarrow Q_{C_2}$ followed by the quotient $q_{C_2,C_2'} : Q_{C_2} \to Q_{C_2'} \simeq Q_{C_2}/I_{C_2',C_2}$, and the quotient map: $q_{C_1,C_1'} : Q_{C_1} \to Q_{C_1'} \simeq Q_{C_1}/I_{C_1',C_1}$ followed by the monomorphism $i_{C_1',C_2'} : Q_{C_1'} \hookrightarrow Q_{C_2'}$. By construction, these two algebra homomorphisms are the same, which in terms of morphisms of varieties implies:

$$\psi_{C_1,C_2} \circ \Psi_{C_2',C_2} = \Psi_{C_1',C_1} \circ \psi_{C_1',C_2'}.$$

And hence we conclude:

$$\begin{aligned} \psi_{C_1,C_2}(y_{C_2}) = \psi_{C_1,C_2} \circ (\Psi_{C_2',C_2} \circ \psi_{C_2',C}(y_C)) &= \Psi_{C_1',C_1} \circ \psi_{C_1',C_2'} \circ \psi_{C_2',C}(y_C) \\ &= \Psi_{C_1',C_1} \circ \psi_{C_1',C}(y_C) \\ &= y_{C_1}, \end{aligned}$$

which shows: $Z_C \subseteq Z_{\mathcal{M}}$.

The restriction of p_C to Z_C induces a bijection between Z_C and Y_C. The above description of Z_C can be used to define an inverse map. It follows that Z_C is isomorphic to Y_C. ☐

The varieties $Z_{\mathcal{M}}$ satisfy the usual universal property of fibre products in the category of affine varieties:

Lemma 11.6 *Let V be an affine variety. Given morphisms $\Phi_C : V \to Y_C$ for all $C \in \mathcal{M}$ such that for all $C_1 \subseteq C_2$ in $\mathcal{M}$, $\Phi_{C_1} = \psi_{C_1,C_2} \circ \Phi_{C_2}$ holds. Then there exist a unique morphism $\eta : V \to Z_{\mathcal{M}}$ such that $p_C \circ \eta = \Phi_C$.*

Proof The image of the obvious map $\prod \Phi_C : V \to \prod_{C\in\mathcal{M}} Y_C$ is contained in $Z_{\mathcal{M}}$ and this map has the desired property. □

When writing down a saturated subset, we usually omit the empty set.

Example 11.7 If $\mathcal{M} = \{C\}$ is just a chain of length zero, then $Z_{\mathcal{M}} = Y_C$ is a one-dimensional (not necessarily normal) toric variety. If $\mathcal{M} = \{C_1, \ldots, C_s\}$ is a collection of chains, all of length zero, then the fibre product becomes a direct product and $Z_{\mathcal{M}} = Y_{C_1} \times \cdots \times Y_{C_s}$ is an s-dimensional toric variety.

Lemma 11.8 *If $\mathcal{M}$ has a unique maximal element C, then $Z_{\mathcal{M}} \simeq Y_C$.*

Proof By Lemma 11.6, the morphisms $\psi_{C',C}$ for $C' \subseteq C$ induce a morphism $\phi : Y_C \to \prod_{C'\in\mathcal{M}} Y_{C'}$ with image in $Z_{\mathcal{M}}$. This image is in fact the entire $Z_{\mathcal{M}}$ by construction. The compositions $p_C \circ \phi$ and $\phi \circ p_C$, where $p_C : Z_{\mathcal{M}} \to Y_C$ is the surjective map in Lemma 11.5, are identity maps. □

Let $\mathcal{M}', \mathcal{M} \in \mathcal{K}^s$ be such that $\mathcal{M}' \subseteq \mathcal{M}$. The projection $\prod_{C''\in\mathcal{M}} Y_{C''} \to \prod_{C''\in\mathcal{M}'} Y_{C''}$ induces a morphism between the fibered products:

Lemma 11.9 *Any inclusion $\mathcal{M}' \subseteq \mathcal{M}$ induces a morphism $\psi_{\mathcal{M}',\mathcal{M}} : Z_{\mathcal{M}} \to Z_{\mathcal{M}'}$. These morphisms are compatible with the inclusion relations, i.e. for $\mathcal{M}'' \subseteq \mathcal{M}' \subseteq \mathcal{M}$ one has $\psi_{\mathcal{M}'',\mathcal{M}} = \psi_{\mathcal{M}'',\mathcal{M}'} \circ \psi_{\mathcal{M}',\mathcal{M}}$.*

These morphisms can be used to describe an inductive procedure to construct $Z_{\mathcal{M}}$. Let $C \in \mathcal{M}$ be a chain which is maximal in $\mathcal{M}$ with respect to the inclusion relation. Let $\mathcal{M}_C = \{C' \in \mathcal{M} \mid C' \subseteq C\}$, $\mathcal{M}' = \mathcal{M} \setminus \{C\}$ and $\mathcal{M}'_C = \mathcal{M}_C \setminus \{C\}$. These sets are all saturated. Recall that the fibre products are considered in the category of varieties.

Lemma 11.10 *There exists an isomorphism of varieties $Z_{\mathcal{M}} \simeq Y_C \times_{Z_{\mathcal{M}'_C}} Z_{\mathcal{M}'}$.*

Proof If C is the only chain in $\mathcal{M}$ which is maximal with respect to the inclusion relation, then $\mathcal{M} = \mathcal{M}_C$ and $\mathcal{M}' = \mathcal{M}'_C$, and hence $Y_C \times_{Z_{\mathcal{M}'}} Z_{\mathcal{M}'} \simeq Y_C \simeq Z_{\mathcal{M}}$ by Lemma 11.8. So without loss of generality we may assume in the following that C is not the only chain in $\mathcal{M}$ which is maximal with respect to the inclusion relation.

By the universal property of the fibre product, the commutative square on the right hand side, induced by the commutative square on the left hand side (Lemma 11.9), gives rise to a morphism $Z_{\mathcal{M}} \to Y_C \times_{Z_{\mathcal{M}'_C}} Z_{\mathcal{M}'}$.

$$
\begin{array}{ccc}
\mathcal{M}'_C & \longrightarrow & \mathcal{M}' \\
\downarrow & & \downarrow \\
\mathcal{M}_C & \longrightarrow & \mathcal{M}
\end{array}
\qquad\qquad
\begin{array}{ccc}
Z_{\mathcal{M}} & \longrightarrow & Z_{\mathcal{M}_C} \cong Y_C \\
\downarrow & & \downarrow \\
Z_{\mathcal{M}'} & \longrightarrow & Z_{\mathcal{M}'_C}
\end{array}
$$

By construction, the fibre product is a subvariety of

$$Y_C \times \prod_{C' \in \mathcal{M}'_C} Y_{C'} \times \prod_{C' \in \mathcal{M}'_C} Y_{C'} \times \prod_{C'' \in \mathcal{M}' \setminus \mathcal{M}'_C} Y_{C''},$$

where, in addition to the maps defining $Z_{\mathcal{M}'} \subseteq \prod_{C' \in \mathcal{M}'_C} Y_{C'} \times \prod_{C'' \in \mathcal{M}' \setminus \mathcal{M}'_C} Y_{C''}$ (here we take the second copy of $\prod_{C' \in \mathcal{M}'_C} Y_{C'}$) and $Z_{\mathcal{M}'_C} \subseteq \prod_{C' \in \mathcal{M}'_C} Y_{C'}$ (here we take the first copy of $\prod_{C' \in \mathcal{M}'_C} Y_{C'}$) and so on, we have the identity map between the two copies of the product $\prod_{C' \in \mathcal{M}'_C} Y_{C'}$. So we may omit one copy of the product $\prod_{C' \in \mathcal{M}'_C} Y_{C'}$, what is left is the variety $Z_{\mathcal{M}}$. □

For $\mathcal{M} \in \mathcal{K}^s$ let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the subset of all chains of length zero.

Proposition 11.11 *The morphism* $\psi_{\mathcal{M}_0, \mathcal{M}} : Z_{\mathcal{M}} \to Z_{\mathcal{M}_0}$ *is finite.*

Before going to the proof, we discuss some consequences of the proposition.

Corollary 11.12 *We have*: $\dim Z_{\mathcal{M}} = \#\mathcal{M}_0$.

If we have saturated sets such that $\mathcal{M}_0 \subseteq \mathcal{M}' \subseteq \mathcal{M}$, then, by Proposition 11.11, the morphisms $\psi_{\mathcal{M}_0, \mathcal{M}} : Z_{\mathcal{M}} \to Z_{\mathcal{M}_0}$ and $\psi_{\mathcal{M}_0, \mathcal{M}'} : Z_{\mathcal{M}'} \to Z_{\mathcal{M}_0}$ are finite. Since $\psi_{\mathcal{M}_0, \mathcal{M}}$ is the composition of $\psi_{\mathcal{M}', \mathcal{M}}$ and $\psi_{\mathcal{M}_0, \mathcal{M}'}$, this implies:

Corollary 11.13 *Let* $\mathcal{M}' \subseteq \mathcal{M}$ *be such that* $\mathcal{M}_0 \subseteq \mathcal{M}'$. *Then* $\psi_{\mathcal{M}_0, \mathcal{M}} : Z_{\mathcal{M}} \to Z_{\mathcal{M}'}$ *is a finite morphism.*

More generally, for $\mathcal{M}' \subseteq \mathcal{M}$ we set $\mathcal{M}'' = \mathcal{M}' \cup \mathcal{M}_0$. The morphism $\psi_{\mathcal{M}', \mathcal{M}}$ is the composition of the finite morphism $\psi_{\mathcal{M}'', \mathcal{M}}$ and the projection $\psi_{\mathcal{M}', \mathcal{M}''}$, and hence:

Corollary 11.14 *If* $\mathcal{M}' \subseteq \mathcal{M}$, *then* $\psi_{\mathcal{M}', \mathcal{M}} : Z_{\mathcal{M}} \to Z_{\mathcal{M}'}$ *is a surjective morphism.*

Proof of Proposition 11.11 The proof is by induction on the number of elements in $\mathcal{M}$. If $\#\mathcal{M} = 1$, then $\mathcal{M} = \mathcal{M}_0$, and there is nothing to prove. Suppose now $\#\mathcal{M} > 1$ and the claim holds for all saturated sets $\mathcal{M}'$ having strictly fewer elements than $\mathcal{M}$. Let $C \in \mathcal{M}$ be a chain which is maximal in $\mathcal{M}$ with respect to the inclusion relation. The set $\mathcal{M}' = \mathcal{M} \setminus \{C\}$ is a saturated set. There are three possible cases:

(a). C is a chain of length zero. In this case the fibre product in Lemma 11.10 becomes a cartesian product, i.e. $Z_{\mathcal{M}} \simeq Y_C \times Z_{\mathcal{M}'}$ and $Z_{\mathcal{M}_0} \simeq Y_C \times Z_{\mathcal{M}'_0}$. In this case the morphism $Z_{\mathcal{M}} \to Z_{\mathcal{M}_0}$ is finite by induction.

(b). C is the only chain in $\mathcal{M}$ which is maximal with respect to the inclusion relation (hence of length ≥ 1). In this case we know $\mathcal{M}_0 = \mathcal{M}'_0$ and by Lemma 11.8, $Z_{\mathcal{M}} \simeq Y_C$ is an irreducible toric variety.

Chains of length zero in $\mathcal{M}$ are just elements of A. For $p \in A$, we denote by C_p the chain in $\mathcal{M}$ consisting of p, i.e. $C_p = \{p\}$.

For every chain $C_p \in \mathcal{M}_0$, the algebra $Q_{C_p} = \bigoplus_{\underline{a}\in\Gamma_{C_p}} R_{\geq\underline{a}}/R_{>\underline{a}} \simeq \mathbb{K}[Y_{C_p}]$ is a subalgebra of $\mathbb{K}[Y_C]$. As a matter of fact, from the construction, the composition of the morphisms $Y_C = Z_{\mathcal{M}} \to Z_{\mathcal{M}_0}$ and $Z_{\mathcal{M}_0} \to Y_{C_p}$ identifies $\mathbb{K}[Y_{C_p}]$ with this subalgebra. It follows that the morphism $\psi^*_{\mathcal{M}_0,\mathcal{M}}$ identifies $\mathbb{K}[Z_{\mathcal{M}_0}] = \bigotimes_{C_p\in\mathcal{M}_0} \mathbb{K}[Y_{C_p}]$ with the subalgebra of $\mathbb{K}[Y_C]$ generated by the Q_{C_p}, $C_p \in \mathcal{M}_0$. In particular, the morphism $\psi^*_{\mathcal{M}_0,\mathcal{M}}$ is injective and $\psi_{\mathcal{M}_0,\mathcal{M}}$ is hence a dominant morphism.

By adding some elements to a (finite) generating system of $\mathbb{K}[Y_C]$ if necessary, we may without loss of generality assume it to have the form

$$\{\bar{f}_p \mid C_p \in \mathcal{M}_0\} \cup \{\bar{g}_{\underline{a}^1}, \dots, \bar{g}_{\underline{a}^t}\}.$$

By Corollary 10.4, each of the generators $\bar{g}_{\underline{a}^j}$ is integral over the polynomial subring generated by the $\{\bar{f}_p \mid C_p \in \mathcal{M}_0\}$. It follows that $\mathbb{K}[Z_{\mathcal{M}}] = \mathbb{K}[Y_C]$ is a finite module over the subalgebra generated by the $\bar{f}_p$, $C_p \in \mathcal{M}_0$, and hence $\mathbb{K}[Z_{\mathcal{M}}]$ is a finite module over $\mathbb{K}[Z_{\mathcal{M}_0}]$.

(c). C is a chain of length ≥ 1 and C is not the only chain in $\mathcal{M}$ which is maximal with respect to the inclusion relation. We still have $\mathcal{M}_0 = \mathcal{M}'_0$. Set $\mathcal{M}_C = \{C' \in \mathcal{M} \mid C' \subseteq C\}$ and $\mathcal{M}'_C = \mathcal{M}_C \setminus \{C\}$. These sets are saturated, and $Z_{\mathcal{M}} \simeq Y_C \times_{Z_{\mathcal{M}'_C}} Z_{\mathcal{M}'}$ by Lemma 11.10. In this case one can use base change arguments. By induction, $Y_C = Z_{\mathcal{M}_C}$ is finite over $Z_{\mathcal{M}_{C,0}}$ and so is $Z_{\mathcal{M}'_C}$ over $Z_{\mathcal{M}'_{C,0}} = Z_{\mathcal{M}_{C,0}}$. Since $\psi_{\mathcal{M}_{C,0},\mathcal{M}_C} = \psi_{\mathcal{M}_{C,0},\mathcal{M}'_C} \circ \psi_{\mathcal{M}'_C,\mathcal{M}_C}$, $Z_{\mathcal{M}_C}$ is finite over $Z_{\mathcal{M}'_C}$. By base change, this implies $Z_{\mathcal{M}} \simeq Y_C \times_{Z_{\mathcal{M}'_C}} Z_{\mathcal{M}'}$ is finite over $Z_{\mathcal{M}'}$. By induction, $Z_{\mathcal{M}'}$ is finite over $Z_{\mathcal{M}_0} = Z_{\mathcal{M}'_0}$. Since $\psi_{\mathcal{M}_0,\mathcal{M}}$ is the composition of $\psi_{\mathcal{M}',\mathcal{M}}$ and $\psi_{\mathcal{M}_0,\mathcal{M}'}$, $Z_{\mathcal{M}}$ is finite over $Z_{\mathcal{M}_0}$. □

11.3 The variety $Z_{\mathcal{K}^s}$

The candidate for the variety Z mentioned in Sect. 11.1 is the variety $Z_{\mathcal{K}^s}$.

Proposition 11.15 *The affine variety $Z_{\mathcal{K}^s}$ satisfies the three properties* 11.1.1–11.1.3.

Proof Given $\underline{a} \in \Gamma$, let C, C_1, C_2 be chains such that $C = \operatorname{supp}\underline{a}$ and $C \subseteq C_1, C_2$. Denote by $\mathcal{M}_C$ (resp. $\mathcal{M}_{C_1}$, $\mathcal{M}_{C_2}$) the smallest saturated sets in $\mathcal{K}^s$ containing C (resp. C_1, C_2). By Lemma 11.8, we have $Z_{\mathcal{M}_C} \simeq Y_C$ and $Z_{\mathcal{M}_{C_i}} \simeq Y_{C_i}$ for $i = 1, 2$. The morphisms $\psi_{\mathcal{M}_C,\mathcal{M}_{C_i}}$ are just the same as the morphisms ψ_{C,C_i}, $i = 1, 2$, and hence $\psi^*_{\mathcal{M}_C,\mathcal{M}_{C_i}}$ is just the inclusion $i_{C,C_i} : Q_C \hookrightarrow Q_{C_i}$, $i = 1, 2$.

By construction, $\bar{g}_{\underline{a}} \in \mathbb{K}[Y_C] = \mathbb{K}[Z_{\mathcal{M}_C}] = Q_C$, and it is also an element in $\mathbb{K}[Y_{C_1}]$ and $\mathbb{K}[Y_{C_2}]$. By the compatibility of the dominant morphisms:

$$\psi_{M_C,\mathcal{K}^s} = \psi_{M_C,\mathcal{M}_{C_1}} \circ \psi_{\mathcal{M}_{C_1},\mathcal{K}^s} = \psi_{M_C,\mathcal{M}_{C_2}} \circ \psi_{\mathcal{M}_{C_2},\mathcal{K}^s}$$

we see that $\psi^*_{\mathcal{M}_C,\mathcal{K}^s}(\bar{g}_{\underline{a}}) = \psi^*_{\mathcal{M}_{C_1},\mathcal{K}^s} \circ \psi^*_{\mathcal{M}_C,\mathcal{M}_{C_1}}(\bar{g}_{\underline{a}}) = \psi^*_{\mathcal{M}_{C_2},\mathcal{K}^s} \circ \psi^*_{\mathcal{M}_C,\mathcal{M}_{C_2}}(\bar{g}_{\underline{a}})$.

(1). The variety $Z_{\mathcal{K}^s}$ satisfies the property 11.1.1: given $\underline{a} \in \Gamma$, we can view $\bar{g}_{\underline{a}}$ as a function on $Z_{\mathcal{K}^s}$: just take any chain C in A such that $\operatorname{supp}\underline{a} \subseteq C$, then $\psi^*_{\mathcal{M}_C,\mathcal{K}^s}(\bar{g}_{\underline{a}}) \in \mathbb{K}[Z_{\mathcal{K}^s}]$ is well defined and independent of the choice of C.

(2). The variety $Z_{\mathcal{K}^s}$ satisfies the property 11.1.2: if there exists a chain C in A such that $\operatorname{supp}\underline{a}$, $\operatorname{supp}\underline{b} \subseteq C$, then we have

$$\psi^*_{\mathcal{M}_C,\mathcal{K}^s}(\bar{g}_{\underline{a}}) \cdot \psi^*_{\mathcal{M}_C,\mathcal{K}^s}(\bar{g}_{\underline{b}}) = \psi^*_{\mathcal{M}_C,\mathcal{K}^s}(\bar{g}_{\underline{a}} \cdot \bar{g}_{\underline{b}}) = c_{\underline{a},\underline{b}}\psi^*_{\mathcal{M}_C,\mathcal{K}^s}(\bar{g}_{\underline{a}+\underline{b}}).$$

So by abuse of notation we write for $\underline{a} \in \Gamma$ just $\bar{g}_{\underline{a}} \in \mathbb{K}[Z_{\mathcal{K}^s}]$ for the function $\psi^*_{\mathcal{M}_C,\mathcal{K}^s}(\bar{g}_{\underline{a}})$, where C is a chain in A containing $\operatorname{supp}\underline{a}$.

(3). The variety $Z_{\mathcal{K}^s}$ satisfies the property 11.1.3: By Proposition 11.11, the map $\psi_{\mathcal{K}^s_0,\mathcal{K}^s} : Z_{\mathcal{K}^s} \to Z_{\mathcal{K}^s_0}$ is finite. We adopt the notation C_p from the proof of Proposition 11.11.

For an element $q \in A$ consider the projection $p_{C_q} : Z_{\mathcal{K}^s_0} = \prod_{C_p \in \mathcal{K}^s_0} Y_{C_p} \to Y_{C_q}$. The coordinate ring Y_{C_q} contains the class of the extremal function $\bar{f}_q$, by abuse of notation we also write $\bar{f}_q \in \mathbb{K}[Z_{\mathcal{K}^s_0}]$ for its image via $p^*_{C_q}$. Due to the tensor product structure of $\mathbb{K}[Z_{\mathcal{K}^s_0}]$ (compare to Example 11.7), the classes of the extremal functions $\bar{f}_p$, $p \in A$, generate in $\mathbb{K}[Z_{\mathcal{K}^s_0}]$ a polynomial algebra of dimension $\#A$.

By the finiteness of $\psi_{\mathcal{K}^s_0,\mathcal{K}^s}$, the classes of the extremal functions $\bar{f}_p \in \mathbb{K}[Z_{\mathcal{K}^s}]$ (or rather their images via $\psi^*_{\mathcal{K}^s_0,\mathcal{K}^s}$) generate a polynomial algebra in $\mathbb{K}[Z_{\mathcal{K}^s}]$, which by Corollary 11.12 is of the same dimension as $\dim Z_{\mathcal{K}^s}$. It follows: there exists an open and dense subset $U \subseteq Z_{\mathcal{K}^s}$ such that $\bar{f}_p(z) \neq 0$ for all $z \in U$.

Let $\underline{a}$ be an element in Γ. By property 11.1.2, the multiplication relations also hold in $\mathbb{K}[Z_{\mathcal{K}^s}]$. An appropriate power of $\bar{g}_{\underline{a}}$ (viewed as a function on Z) is hence in $\mathbb{K}[Z]$ a non-zero scalar multiple of a product the $\bar{f}_p$'s with $p \in \operatorname{supp}\underline{a}$. It follows: $\bar{g}_{\underline{a}}(z) \neq 0$ for all $z \in U$ and all $\underline{a} \in \Gamma$. □

The proof that $Z_{\mathcal{K}^s}$ has the desired properties finishes thus the proof of Theorem 11.1.

12 Flat degenerations

In this section we construct a flat degeneration from X to the reduced projective scheme $\operatorname{Proj}(\operatorname{gr}_{\mathcal{V}} R)$ using a Rees algebra construction. We start by extending the valuation image by a total degree.

For $\underline{a} \in \Gamma$, there exists a homogeneous function $g \in R$ (Lemma 10.3) such that $\mathcal{V}(g) = \underline{a}$ with degree given by the formula in Corollary 7.5. This motivates the following definition of an extra $\mathbb{N}$-grading on Γ. Let $\mathcal{J} \subseteq \mathbb{N} \times \Gamma$ be the subset

$$\mathcal{J} = \left\{ (m, \underline{a}) \in \mathbb{N} \times \Gamma \;\middle|\; \underline{a} \in \Gamma, m = \sum_{p \in \operatorname{supp}\underline{a}} a_p \deg f_p \right\}.$$

Denote by $\succ$ the lexicographic order on the set $\mathbb{N} \times \Gamma$. For a pair $(m, \underline{a}) \in \mathbb{N} \times \Gamma$ let $\mathcal{I}_{\succeq(m,\underline{a})}$ be the following homogeneous ideal in R:

$$\mathcal{I}_{\succeq(m,\underline{a})} = \left\langle g \in R \mid g \text{ homogeneous and } \begin{cases} \text{either } \deg g = m \text{ and } \mathcal{V}(g) \geq \underline{a} \\ \text{or } \deg g > m \end{cases} \right\rangle.$$

It follows $\mathcal{I}_{\succeq(m,\underline{a})}\mathcal{I}_{\succeq(k,\underline{b})} \subseteq \mathcal{I}_{\succeq(m+k,\underline{a}+\underline{b})}$, so the ideals define a filtration on the ring R.

We define $\mathcal{I}_{\succ(m,\underline{a})}$ similarly. By Lemma 10.2, the quotient

$$\mathcal{I}_{\succeq(m,\underline{a})}/\mathcal{I}_{\succ(m,\underline{a})} = \begin{cases} \{0\}, & \text{if } (m,\underline{a}) \notin \mathcal{J}; \\ R_{\geq\underline{a}}/R_{>\underline{a}}, & \text{if } (m,\underline{a}) \in \mathcal{J}. \end{cases} \tag{20}$$

In particular, if $(m,\underline{a}) \in \mathcal{J}$, the quotient space $\mathcal{I}_{\succeq(m,\underline{a})}/\mathcal{I}_{\succ(m,\underline{a})}$ is one-dimensional.

Let $\pi : \mathcal{J} \to \mathbb{N}$ be an enumeration of the countable many elements respecting the total order, i.e. $\pi((m,\underline{a})) < \pi((k,\underline{b}))$ if and only if $(m,\underline{a}) \prec (k,\underline{b})$ with respect to the lexicographic order. We write sometimes just $\mathcal{I}_j$ if $\pi((m,\underline{a})) = j$ instead of $\mathcal{I}_{\succeq(m,\underline{a})}$. In this way we get a decreasing filtration:

$$R = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots.$$

Let

$$\mathcal{A} = \cdots \oplus Rt^2 \oplus Rt \oplus R \oplus \mathcal{I}_1 t^{-1} \oplus \mathcal{I}_2 t^{-2} \oplus \cdots \subseteq R[t,t^{-1}]$$

be the associated Rees algebra. The natural inclusion $\mathbb{K}[t] \hookrightarrow \mathcal{A}$ induces a morphism $\phi : \mathrm{Spec}(\mathcal{A}) \to \mathbb{A}^1$.

The ring $R = \mathbb{K}[\hat{X}]$ is clearly graded since $\hat{X}$ is the cone over the projective variety X. We extend this grading to $R[t,t^{-1}]$ by declaring that the degree of t is 0. Being generated by homogeneous elements, the ideals $\mathcal{I}_j$ are homogeneous, hence the subalgebra $\mathcal{A} \subseteq R[t,t^{-1}]$ is graded. So we have natural $\mathbb{G}_m$-actions on $\mathrm{Spec}(\mathcal{A})$ and on $\mathbb{A}^1$; note that the latter is trivial since $\mathbb{K}[t]$ is in degree 0.

Theorem 12.1 *The morphism ϕ is flat and $\mathbb{G}_m$–equivariant. The general fibre for $t \neq 0$ is isomorphic to $\hat{X}$, the special fibre for $t = 0$ is isomorphic to* $\mathrm{Spec}(\mathrm{gr}_{\mathcal{V}} R)$.

Proof The ring $\mathcal{A}$ is a torsion-free $\mathbb{K}[t]$ module by the inclusion $\mathcal{A} \subseteq R[t,t^{-1}]$, and hence it is a flat module. For $b \neq 0$ it is easily seen that $\mathcal{A}/(t-b) \simeq R$, and for $t = 0$ we get

$$\mathcal{A}/(t) \simeq R/\mathcal{I}_1 \oplus \mathcal{I}_1/\mathcal{I}_2 \oplus \mathcal{I}_2/\mathcal{I}_3 \oplus \cdots \simeq \bigoplus_{(\sum_{p\in A} a_p \deg f_p, \underline{a}) \in \mathcal{J}} R_{\geq\underline{a}}/R_{>\underline{a}} \simeq \mathrm{gr}_{\mathcal{V}} R.$$

The morphism ϕ is $\mathbb{G}_m$–equivariant simply because its fibers are stable by the $\mathbb{G}_m$–action on $\mathrm{Spec}(\mathcal{A})$ since $\mathbb{K}[t]$ is in degree 0. □

An immediate consequence of the $\mathbb{G}_m$–equivariance of ϕ is the existence of an induced morphism $\psi : \mathrm{Proj}(\mathcal{A}) \to \mathbb{A}^1$; in particular we get the following result.

Theorem 12.2 *The morphism ψ is flat. The general fibre for $t \neq 0$ is isomorphic to X, and the special fibre for $t = 0$ is isomorphic to* $\mathrm{Proj}(\mathrm{gr}_{\mathcal{V}} R)$.

Combining Theorem 12.2 with the existence of a generic hyperplane stratification in Proposition 2.11, we have:

Corollary 12.3 *Every embedded projective variety $X \subseteq \mathbb{P}(V)$, smooth in codimension one, admits a flat degeneration into X_0, a reduced union of projective toric varieties. Moreover, X_0 is equidimensional, the number of irreducible components in X_0 coincides with the degree of X.*

The varieties appearing as irreducible components of X_0 are in general not linear because the valuations $\mathcal{V}_{\mathfrak{C}}$ may take values in different lattices. See next section, especially Sect. 13.5 for a convex geometric aspect of this corollary.

As mentioned in Remark 2.13, the above corollary resembles a geometric counterpart of the result by Hibi [31] that finitely generated positively graded rings admit Hodge algebra structures.

13 The Newton-Okounkov simplicial complex

In the theory of Newton-Okounkov bodies [37, 50, 59], one associates a convex body to a valuation on a positively graded algebra by taking the convex closure of the degree-normalized valuation images. The (normalized) volume of this convex body computes the leading coefficient of the Hilbert polynomial of the graded algebra.

In our setup, the image $\Gamma := \{\mathcal{V}(g) \mid g \in R \setminus \{0\}\} \subseteq \mathbb{Q}^A$ of the quasi-valuation $\mathcal{V}$ is not necessarily a monoid. We need variations of the approaches in *loc.cit.* to define an analogue of the Newton-Okounkov body for a quasi-valuation, so that its volume computes the degree of the embedded projective variety X in $\mathbb{P}(V)$.

From now on we will work in real vector spaces with the usual Euclidean topology. The spirit of the construction is much the same as in [24, Sect. 3].

13.1 The order complex

Recall that in Sect. 9.1, to every chain C in A we have associated a cone K_C in $\mathbb{R}^A$ defined by

$$K_C := \sum_{p \in C} \mathbb{R}_{\geq 0} e_p.$$

The collection of these cones, together with the origin $\{0\}$, is the fan $\mathcal{F}_A$ associated to the poset A. Its maximal cones are the cones $K_{\mathfrak{C}}$ associated to the maximal chains in A. Each of them comes endowed with a submonoid, the monoid $\Gamma_{\mathfrak{C}} \subseteq K_{\mathfrak{C}}$ (see Sect. 9.1).

The order complex $\Delta(A)$ associated to the poset $(A, \leq)$ is the simplicial complex having A as vertices and all chains $C \subseteq A$ as faces, i.e. $\Delta(A) = \{C \subseteq A \mid C \text{ is a chain}\}$. A geometric realization of $\Delta(A)$ can be constructed by intersecting the cones K_C with appropriate hyperplanes: for a chain $C \subseteq A$ denote by $\Delta_C \subseteq \mathbb{R}^A$ the simplex:

$$\Delta_C := \text{convex hull} \left\{ \frac{1}{\deg f_p} e_p \mid p \in C \right\}. \tag{21}$$

The union of the simplexes

$$|\Delta(A)| = \bigcup_{C\subseteq A \text{ chain}} \Delta_C \subseteq \mathbb{R}^A$$

is the desired *geometric realization* of $\Delta(A)$. The maximal simplexes are those $\Delta_{\mathfrak{C}}$ arising from maximal chains $\mathfrak{C}$ in A.

13.2 The Newton-Okounkov simplicial complex

Let $\mathfrak{C} = \{p_r > \cdots > p_0\}$ be a maximal chain. To avoid an inundation with indexes, we abbreviate throughout this section the natural basis e_{p_j} of $\mathbb{R}^{\mathfrak{C}}$ by e_j, for $j = 0, \ldots, r$. Similarly, the functions f_{p_j} (resp. the bonds $b_{p_j, p_{j-1}}$) will be simplified to f_j (resp. b_j) for $j = 0, \ldots, r$.

Let $\deg : \mathbb{R}^A \to \mathbb{R}$ be the degree function defined by: for $\underline{a} \in \mathbb{R}^A$, set

$$\deg \underline{a} := \sum_{p\in A} a_p \deg f_p.$$

If $g \in R \setminus \{0\}$ is homogeneous, then Corollary 7.5 implies that $\deg \mathcal{V}(g)$ is the degree of g.

Recall that $\mathcal{C}$ is the set of all maximal chains in A.

Definition 13.1 The *Newton-Okounkov simplicial complex* $\Delta_{\mathcal{V}}$ associated to the quasi-valuation $\mathcal{V}$ is defined as

$$\Delta_{\mathcal{V}} := \overline{\left\{ \frac{\underline{a}}{\deg \underline{a}} \mid \underline{a} \in \Gamma \setminus \{0\} \right\}} \subseteq \mathbb{R}^A.$$

Remark 13.2

(1) It is straightforward to show that

$$\Delta_{\mathcal{V}} = \bigcup_{\mathfrak{C}\in\mathcal{C}} \overline{\bigcup_{m\geq 1} \left\{ \frac{1}{m}\underline{a} \mid \underline{a} \in \Gamma_{\mathfrak{C}}, \deg \underline{a} = m \right\}}.$$

(2) In the definition of the Newton-Okounkov bodies in [37, 50, 59] as closed convex hulls of points, the convex hull operation is not necessary (see for example [60]).

Recall that a simplicial complex Δ of dimension r is called *homogeneous* when for any face F of Δ, there exists an r-simplex containing F as a face.

Proposition 13.3 *We have $\Delta_{\mathcal{V}} = |\Delta(A)|$. In particular, $\Delta_{\mathcal{V}}$ is a homogeneous simplicial complex of dimension r.*

Proof We show $\Delta_{\mathcal{V}} \subseteq |\Delta(A)|$. Let $g \in R \setminus \{0\}$ be a homogeneous element with $\mathcal{V}(g) = \underline{a} \in \Gamma_{\mathfrak{C}}$. By Corollary 10.4, there exists $m \in \mathbb{N}$ such that for any $p \in \mathfrak{C}$,

$ma_p \in \mathbb{N}$ and $\mathcal{V}(g^m) = \mathcal{V}(\prod_{p\in\mathfrak{C}} f_p^{ma_p})$. Noticing that

$$\frac{\mathcal{V}(g)}{\deg g} = \frac{\mathcal{V}(g^m)}{m \deg g},$$

it suffices to assume that $g = \prod_{p\in C} f_p^{n_p}$ for $n_p \in \mathbb{N}$ and a chain $C \subseteq A$. Let $m = \deg g = \sum_{p\in C} n_p \deg f_p$ be the degree of g. If $\mathfrak{C}$ is a maximal chain containing C, then

$$\frac{\mathcal{V}(g)}{\deg g} = \frac{\mathcal{V}_{\mathfrak{C}}(g)}{m} = \sum_{p\in C} \frac{n_p \deg f_p}{m}\left(\frac{1}{\deg f_p} e_p\right) \in \Delta_C \subseteq |\Delta(A)|.$$

Conversely, we show $|\Delta(A)| \cap \mathbb{Q}^A \subseteq \Delta_{\mathcal{V}}$. For a chain $C \subseteq A$, a point in $\Delta_C \cap \mathbb{Q}^A \subseteq |\Delta(A)| \cap \mathbb{Q}^A$ is a convex linear combination

$$\sum_{p\in C} a_p \frac{1}{\deg f_p} e_p, \text{ with } a_p \in \mathbb{Q}_{\geq 0} \text{ and } \sum_{p\in C} a_p = 1.$$

Choose a non-zero $m \in \mathbb{N}$ to be such that for all $p \in C$: $m\frac{a_p}{\deg f_p} \in \mathbb{N}$. Let $g = \prod_{p\in C} f_p^{m\frac{a_p}{\deg f_p}}$. It is now easy to verify that

$$\frac{\mathcal{V}(g)}{\deg g} = \frac{\mathcal{V}(g)}{m} = \sum_{p\in C} a_p \frac{1}{\deg f_p} e_p.$$

This terminates the proof since $\Delta_{\mathcal{V}}$ is closed. □

13.3 Rational and integral structures on simplexes

Let $H_R(t)$ be the Hilbert polynomial for the homogeneous cooordinate ring $R = \mathbb{K}[\hat{X}]$. Our aim is to translate the calculation of its leading coefficient into a problem of calculating the leading coefficient of certain Ehrhart quasi-polynomials. Let $\Gamma_n = \{\underline{a} \in \Gamma \mid \deg \underline{a} = n\}$ be the elements in Γ of degree n. Since the leaves of the quasi-valuation are one dimensional, one has $H_R(n) = \#\Gamma_n = \#\Delta_{\mathcal{V}}(n)$ for n large, where $\Delta_{\mathcal{V}}(n)$ is defined as the intersection $n\Delta_{\mathcal{V}} \cap \Gamma$.

We have already pointed out that the lattice $L^{\mathfrak{C}}$ defined in (13) is in general too large compared to the monoid $\Gamma_{\mathfrak{C}}$. In the following let $\mathcal{L}^{\mathfrak{C}} \subseteq L^{\mathfrak{C}}$ be the sublattice generated by $\Gamma_{\mathfrak{C}}$. The affine span of the simplex $\Delta_{\mathfrak{C}}$ is the affine subspace $U_1 = \frac{1}{\deg f_0} e_0 + U_0$ of $\mathbb{R}^{\mathfrak{C}}$, where U_0 is the linear subspace

$$U_0 := \operatorname{span}_{\mathbb{R}}\left\{\frac{1}{\deg f_j} e_j - \frac{1}{\deg f_0} e_0 \mid j = 1, \ldots, r\right\}.$$

This linear subspace $U_0 \subseteq \mathbb{R}^{\mathfrak{C}}$ can also be characterized as the kernel of the degree function $\deg : \mathbb{R}^{\mathfrak{C}} \to \mathbb{R}$, $\underline{a} \mapsto \sum_{p\in\mathfrak{C}} a_p \deg f_p$. Hence for $m \in \mathbb{Z}$, the affine subspace U_m of $\mathbb{R}^{\mathfrak{C}}$ of elements of degree m is given by:

$$U_m = \frac{m}{\deg f_0} e_0 + U_0 = \{\underline{a} \in \mathbb{R}^{\mathfrak{C}} \mid \deg \underline{a} = m\}.$$

13.3.1 The projection

Let $\mathfrak{C} = (p_r, \ldots, p_0)$ be a maximal chain in A. The degree function $\deg : \mathbb{R}^{\mathfrak{C}} \to \mathbb{R}$ takes integral values on $\Gamma_{\mathfrak{C}}$, and hence the function deg takes also integral values on $\mathcal{L}^{\mathfrak{C}}$. Set $\mathcal{L}^{\mathfrak{C}}_0 = U_0 \cap \mathcal{L}^{\mathfrak{C}}$. For later purpose it is important to know that $U_m \cap \mathcal{L}^{\mathfrak{C}} \neq \emptyset$ for all $m \in \mathbb{Z}$, this is a consequence of the following lemma:

Lemma 13.4 *We have*: $\frac{1}{\deg f_0} e_0 \in U_1 \cap \mathcal{L}^{\mathfrak{C}}$.

Proof We fix an enumeration of the length 0 elements in A, say: $q_1 = p_0, q_2, \ldots, q_s$, and identify q_j with the unique point in X_{q_j}. For $i = 2, \ldots, s$ let $h_i \in V^*$ be a linear function such that $h_i(q_j) \neq 0$ for $j = 1, \ldots, s$, $j \neq i$, and $h_i(q_i) = 0$. The function

$$g = \prod_{j=2,\ldots,s} h_j \tag{22}$$

vanishes in $q_2, \ldots, q_s$, but not in $p_0 = q_1$.

Let $\mathfrak{C}' = (p'_r, \ldots, p'_0)$ be another maximal chain and $h \in V^*$ a linear function which does not vanish in any of the points $q_1, \ldots, q_s$. The same calculation as in Example 8.5 shows that

$$\mathcal{V}(g) = \frac{s-1}{\deg f_{p_0}} e_0 \quad \text{and} \quad \mathcal{V}(gh) = \frac{s}{\deg f_{p_0}} e_0.$$

These elements are both in $\Gamma_{\mathfrak{C}}$, hence $\frac{1}{\deg f_{p_0}} e_0$ is an element in the lattice $\mathcal{L}^{\mathfrak{C}}$ generated by $\Gamma_{\mathfrak{C}}$. □

Set $\ell_1 = \frac{1}{\deg f_{p_0}} e_0$. The projection $\mathrm{pr}_{\ell_1} : \mathbb{R}^{\mathfrak{C}} \to \mathbb{R}^{\mathfrak{C}\setminus\{p_0\}}$ ($\simeq \mathbb{R}^{\mathfrak{C}}/\mathbb{R}\ell_1$) induces an isomorphism of vector spaces:

$$\mathrm{pr}_{\ell_1}|_{U_0} : U_0 \xrightarrow{\sim} \mathbb{R}^{\mathfrak{C}\setminus\{p_0\}}.$$

We identify for convenience in the following U_0 sometimes with $\mathbb{R}^{\mathfrak{C}\setminus\{p_0\}}$, and we identify $\mathcal{L}^{\mathfrak{C}}_0 \subseteq U_0$ with its image in $\mathbb{R}^{\mathfrak{C}\setminus\{p_0\}}$. Note that ℓ_1 is primitive, so the image of $\mathcal{L}^{\mathfrak{C}}_0$ is isomorphic to $\mathcal{L}^{\mathfrak{C}}/\mathbb{Z}\ell_1$.

For all $m \geq 1$, the restriction $\mathrm{pr}_{\ell_1}|_{U_m} : U_m \to U_0$ induces a bijection between the affine subspace U_m and the vector space U_0. An element $\ell_m \in \mathcal{L}^{\mathfrak{C}} \cap U_m$ can always be written as $\ell_m = m\ell_1 + \ell_0$ for some $\ell_0 \in \mathcal{L}^{\mathfrak{C}}_0$. The bijection $\mathrm{pr}_{\ell_1}|_{U_m}$ induces hence in this case a bijection between $\mathcal{L}^{\mathfrak{C}} \cap U_m$ and the lattice $\mathcal{L}^{\mathfrak{C}}_0 \subseteq U_0$.

The map pr_{ℓ_1} is $\mathbb{R}$-linear, so the restriction $\mathrm{pr}_{\ell_1}|_{U_1}$ preserves the notion of the affine convex linear hull of elements in U_1. Set $\Delta^0_{\mathfrak{C}} := \mathrm{pr}_{\ell_1}|_{U_1}(\Delta_{\mathfrak{C}}) \subseteq U_0$, then

$$\begin{aligned}\Delta^0_{\mathfrak{C}} &= \text{convex hull}\left\{\frac{1}{\deg f_p}\mathrm{pr}_{\ell_1}(e_p) \mid p \in \mathfrak{C}\right\}\\ &= \text{convex hull}\left\{0, \frac{1}{\deg f_p} e_{p_j} \mid j = 1, \ldots, r\right\}.\end{aligned}$$

The same arguments hold for all $m \geq 1$:

$$\mathrm{pr}_{\ell_1}|_{U_m}(m\Delta_{\mathfrak{C}}) = m\Delta^0_{\mathfrak{C}}.$$

From this construction we have:

Lemma 13.5 *For all* $n \geq 1$, $\#\left(n\Delta_{\mathfrak{C}} \cap \mathcal{L}^{\mathfrak{C}}\right) = \#\left(n\Delta^0_{\mathfrak{C}} \cap \mathcal{L}^{\mathfrak{C}}_0\right)$.

13.3.2 Rational and integral structure

By fixing an ordered basis of $\mathcal{L}^{\mathfrak{C}}_0$, one gets an isomorphism $\Psi : U_0 \to \mathbb{R}^r$ which identifies the lattice $\mathcal{L}^{\mathfrak{C}}_0 \subseteq U_0$ with the lattice $\mathbb{Z}^r \subseteq \mathbb{R}^r$. The simplex $\Delta^0_{\mathfrak{C}} \subseteq U_0$ can be identified in this way with the simplex $D_{\mathfrak{C}} \subseteq \mathbb{R}^r$, defined as:

$$D_{\mathfrak{C}} = \text{convex hull } \left\{0, \frac{1}{\deg f_p}\Psi(e_p) \mid p \in \mathfrak{C} \setminus \{p_0\}\right\} \subseteq \mathbb{R}^r.$$

The simplex has rational vertices. Now Lemma 13.5 implies for all $n \in \mathbb{N}$:

$$\#\left(n\Delta_{\mathfrak{C}} \cap \mathcal{L}^{\mathfrak{C}}\right) = \#\left(nD_{\mathfrak{C}} \cap \mathbb{Z}^r\right). \tag{23}$$

The correspondence in (23) generalizes [24, Definition 3.1]; we call it a *rational structure* on $\Delta_{\mathfrak{C}}$. In case the vertices of $D_{\mathfrak{C}}$ are integral points, it will be called an *integral structure* on $\Delta_{\mathfrak{C}}$.

The map Ψ and the simplex $D_{\mathfrak{C}}$ depend on the choice of the ordered basis of $\mathcal{L}^{\mathfrak{C}}_0$. But different choices of ordered bases lead to unimodular equivalent simplexes.

13.4 The degree formula

Let $\Delta_{\mathcal{V}}$ be the Newton-Okounkov simplicial complex associated to X and the quasi-valuation $\mathcal{V}$. For each maximal chain $\mathfrak{C}$ let $\Delta_{\mathfrak{C}} \subseteq \Delta_{\mathcal{V}}$ be the associated simplex given by the simplicial decomposition in Proposition 13.3. We fix for all maximal chains $\mathfrak{C}$ in A a rational structure on $\Delta_{\mathfrak{C}}$ (see Sect. 13.3.2), denote by $D_{\mathfrak{C}} \subseteq \mathbb{R}^r$ the corresponding simplex with rational vertices. Let $\mathrm{vol}(D_{\mathfrak{C}})$ be the Euclidean volume of the simplex. This is also the normalized volume of $\Delta^0_{\mathfrak{C}}$ with respect to the lattice $\mathcal{L}^{\mathfrak{C}}_0$.

Theorem 13.6 *The degree of the embedded variety* $X \hookrightarrow \mathbb{P}(V)$ *is equal to*

$$\deg X = r! \sum_{\mathfrak{C} \in \mathcal{C}} vol(D_{\mathfrak{C}}).$$

Proof Let $H_R(t)$ be the Hilbert polynomial of the homogeneous coordinate ring $R = \mathbb{K}[\hat{X}]$. The degree of X can be computed as $r! \cdot c_r$, where c_r is the leading coefficient of $H_R(t)$. By Theorem 11.1, the Hilbert polynomial $H_R(t)$ and the Hilbert polynomial $H_\Gamma(t)$ associated to the fan algebra $\mathbb{K}[\Gamma]$ coincide.

Let $H_{\mathfrak{C}}(t)$ be the Hilbert quasi-polynomial of the algebra $\mathbb{K}[\Gamma_{\mathfrak{C}}]$. The leading coefficient of $H_\Gamma(t)$ is the sum of the leading coefficients of the $H_{\mathfrak{C}}(t)$, where the sum runs over all maximal chains $\mathfrak{C}$ in A. By Lemma 9.10, the leading coefficient of $H_{\mathfrak{C}}(t)$ is equal to the leading coefficient of the Hilbert quasi-polynomial $\tilde{H}_{\mathfrak{C}}(t)$ of the algebra $\mathbb{K}[\tilde{\Gamma}_{\mathfrak{C}}]$.

In the Sects. 13.2, 13.3.1 and 13.3.2 we construct a simplex $D_{\mathfrak{C}} \subseteq \mathbb{Q}^r$ such that $\#(\tilde{\Gamma}_{\mathfrak{C}})_m = \#\left(m\Delta_{\mathfrak{C}} \cap \mathcal{L}^{\mathfrak{C}}\right)$ is equal to $\#(mD_{\mathfrak{C}} \cap \mathbb{Z}^r)$, which implies that the leading coefficient of $\tilde{H}_{\mathfrak{C}}(t)$ is the same as the leading coefficient of the Ehrhart quasi-polynomial $\mathrm{Ehr}_{D_{\mathfrak{C}}}(t)$ associated $D_{\mathfrak{C}}$. In this case the leading coefficient is the Euclidean volume of $D_{\mathfrak{C}}$ divided by the co-volume of the lattice $\mathbb{Z}^r$, which finishes the proof of the theorem. □

The estimates above can be made more precise if the monoid $\Gamma_{\mathfrak{C}}$ is saturated, that is to say, $\mathcal{L}^{\mathfrak{C}} \cap K_{\mathfrak{C}} = \Gamma_{\mathfrak{C}}$, for all maximal chains $\mathfrak{C}$. Algebraically it is equivalent to say that the algebra $\mathbb{K}[\Gamma_{\mathfrak{C}}]$ is normal. Geometrically this condition is equivalent to the normality of all the irreducible components of the degenerate variety $\mathrm{Spec}(\mathrm{gr}_{\mathcal{V}} R)$. We give a name to such a situation:

Definition 13.7 A Seshadri stratification is called *normal* if $\Gamma_{\mathfrak{C}}$ is saturated for every maximal chain $\mathfrak{C}$.

Further results on normal Seshadri stratifications can be found in [17]. If this condition is fulfilled, then the number of points $\#(n\Delta_{\mathcal{V}} \cap \Gamma)$ can be determined as an alternating sum of the set of lattice points in all possible intersections of the simplexes.

Proposition 13.8 *If the Seshadri stratification is normal, then the Hilbert function $H_R(t)$ of the graded ring $R = \mathbb{K}[X]$ is an alternating sum of Ehrhart quasi-polynomials. More precisely, let $\mathfrak{C}_1, \ldots, \mathfrak{C}_t$ be an enumeration of the maximal chains in A, then*

$$H_R(t) = \sum_{1 \le i_1 < \cdots < i_\ell \le r} (-1)^{\ell-1} \mathrm{Ehr}_{\Delta_{\mathfrak{C}_{i_1}} \cap \cdots \cap \Delta_{\mathfrak{C}_{i_\ell}}}(t).$$

13.5 The degree formula and the generic hyperplane stratification

We fix a generic hyperplane stratification as in the proof of Proposition 2.11 and choose the functions $f_{0,k}$ as in Example 2.12.

To see the connection with the degree formula above, note that by Lemma 6.6, for every maximal chain $\mathfrak{C} = (q_r, \ldots, q_1, q_{0,j})$ the valuation $\mathcal{V}_{\mathfrak{C}}$ takes values in the lattice:

$$L^{\mathfrak{C}} = \{(a_r, \ldots, a_0) \in \mathbb{Q}^{\mathfrak{C}} \mid (s-1)a_0 \in \mathbb{Z},\ a_j \in \mathbb{Z},\ 1 \le j \le r\}.$$

Since $\mathcal{L}^{\mathfrak{C}} \subseteq L^{\mathfrak{C}}$, Example 6.8 and Lemma 13.4 imply that one has indeed equality $\mathcal{L}^{\mathfrak{C}} = L^{\mathfrak{C}}$.

It follows that the projection $\mathrm{pr}_{\ell_1} : \mathbb{R}^{\mathfrak{C}} \to \mathbb{R}^{\mathfrak{C}\setminus\{q_{0,j}\}}$ (Sect. 13.3.1) induces an identification of $\mathcal{L}_0^{\mathfrak{C}} \simeq \mathcal{L}^{\mathfrak{C}}/\mathbb{Z}\ell_1$ with $\mathbb{Z}^r$, and $\Delta_{\mathfrak{C}}^0 := \mathrm{pr}_{\ell_1}|_{U_1}(\Delta_{\mathfrak{C}})$ is just the standard simplex with vertices $0, e_{q_1}, \ldots, e_{q_r}$. Therefore in this case we can take Ψ as the identity map, and $D_{\mathfrak{C}} = \Delta_{\mathfrak{C}}^0$ is a standard simplex of dimension r in $\mathbb{R}^{\mathfrak{C}\setminus\{q_{0,j}\}}$.

As a summary: For every maximal chain in A we get as simplex a standard simplex, its Euclidean volume is $\frac{1}{r!}$. There are s maximal chains in A. The degree formula in Theorem 13.6 reproduces hence the predicted number.

14 Projective normality

In this section, we give a criterion on the projective normality of the projective variety $X \subseteq \mathbb{P}(V)$ using the normality of the toric varieties $\mathrm{Spec}(\mathbb{K}[\Gamma_{\mathfrak{C}}])$ and the topology of the poset A.

Let $\mathrm{SR}(A)$ be the *Stanley-Reisner algebra* of the poset A. By definition

$$\mathrm{SR}(A) := \mathbb{K}[t_p \mid p \in A]/(t_p t_q \mid p \text{ and } q \text{ are incomparable}).$$

The Stanley-Reisner algebra has a linear basis consisting of $t_{\underline{a}} := t_{p_r}^{a_r} \cdots t_{p_0}^{a_0}$ where $\mathfrak{C} = (p_r, \ldots, p_1, p_0)$ runs over all maximal chains in A and $\underline{a} = (a_r, \ldots, a_0) \in \mathbb{N}^{\mathfrak{C}}$.

The poset A is called *Cohen-Macaulay* over $\mathbb{K}$, if the algebra $\mathrm{SR}(A)$ is Cohen-Macaulay. As an example, a shellable poset A is Cohen-Macaulay over any field $\mathbb{K}$ [6].

The goal of this section is to prove the following result:

Theorem 14.1 *If the Seshadri stratification is normal and the poset A is Cohen-Macaulay over $\mathbb{K}$, then*

i) *the ring R is normal, hence $X \subseteq \mathbb{P}(V)$ is projectively normal*;
ii) *the special fibre X_0 is Cohen-Macaulay.*

Proof The proof follows the ideas in [13]. We divide it into several steps.

Step 1. We construct an embedding of $\mathbb{K}$-algebras $\phi : \mathbb{K}[\Gamma] \to \mathrm{SR}(A)$. This endows $\mathrm{SR}(A)$ with a $\mathbb{K}[\Gamma]$-module structure.

Let M be the product of all bonds in $\mathcal{G}_A$. For a fixed maximal chain $\mathfrak{C}$, by Lemma 6.6, the monoid $\Gamma_{\mathfrak{C}}$ is contained in the lattice $L^{\mathfrak{C}}$, hence $M\Gamma_{\mathfrak{C}} \subseteq \mathbb{N}^{\mathfrak{C}}$. The embedding of monoids $\Gamma_{\mathfrak{C}} \hookrightarrow \mathbb{N}^{\mathfrak{C}}$, $\underline{a} \mapsto M\underline{a}$ induces injective $\mathbb{K}$-algebra morphism $\phi_{\mathfrak{C}} : \mathbb{K}[\Gamma_{\mathfrak{C}}] \hookrightarrow \mathbb{K}[\mathbb{N}^{\mathfrak{C}}]$. In view of Corollary 9.1, we define a morphism of $\mathbb{K}$-algebra $\phi : \mathbb{K}[x_{\underline{a}} \mid \underline{a} \in \Gamma] \to \mathrm{SR}(A)$ in such a way that for a maximal chain $\mathfrak{C}$ and $\underline{a} \in \Gamma_{\mathfrak{C}}$, $\phi(x_{\underline{a}}) := \phi_{\mathfrak{C}}(x_{\underline{a}}) = t_{M\underline{a}}$. This map is clearly independent of the choice of the maximal chain so it is well-defined. Since the $\phi_{\mathfrak{C}}$ are injective, ϕ passes through $I(\Gamma)$, yields an injective $\mathbb{K}$-algebra morphism from $\mathbb{K}[\Gamma]$ to $\mathrm{SR}(A)$, which is also denoted by ϕ.

Step 2. We define a $\mathbb{K}[\Gamma]$-module morphism $\psi : \mathrm{SR}(A) \to \mathbb{K}[\Gamma]$.

We start from considering a single maximal chain $\mathfrak{C}$. Let $\psi_{\mathfrak{C}} : \mathbb{K}[\mathbb{N}^{\mathfrak{C}}] \to \mathbb{K}[\Gamma_{\mathfrak{C}}]$ be the $\mathbb{K}$-linear map sending $t_{\underline{a}} \in \mathbb{K}[\mathbb{N}^{\mathfrak{C}}]$ to $x_{\underline{a}/M}$ if $\underline{a} \in M\Gamma_{\mathfrak{C}}$ or to 0 otherwise. We show that $\psi_{\mathfrak{C}}$ is a morphism of $\mathbb{K}[\Gamma_{\mathfrak{C}}]$-modules, where the $\mathbb{K}[\Gamma_{\mathfrak{C}}]$-module structure on $\mathbb{K}[\mathbb{N}^{\mathfrak{C}}]$ is defined by: for $\underline{a} \in \Gamma_{\mathfrak{C}}$ and $\underline{b} \in \mathbb{N}^{\mathfrak{C}}$, $x_{\underline{a}} \cdot t_{\underline{b}} := t_{M\underline{a}+\underline{b}}$. It suffices to show that for $\underline{a} \in \Gamma_{\mathfrak{C}}$ and $\underline{b} \in \mathbb{N}^{\mathfrak{C}}$, $\psi_{\mathfrak{C}}(t_{M\underline{a}} t_{\underline{b}}) = x_{\underline{a}} \psi_{\mathfrak{C}}(t_{\underline{b}})$. If $\underline{b} \in M\Gamma_{\mathfrak{C}}$ then there is nothing to show. Assume that $\underline{b} \notin M\Gamma_{\mathfrak{C}}$, then $\psi_{\mathfrak{C}}(t_{\underline{b}}) = 0$. We need to show that $M\underline{a}+\underline{b} \notin M\Gamma_{\mathfrak{C}}$.

Assume the contrary, since $M\Gamma_{\mathfrak{C}}$ is saturated, $\underline{b} = (M\underline{a} + \underline{b}) - M\underline{a} \in M\Gamma_{\mathfrak{C}}$ gives a contradiction.

For the general case, take a monomial $t_{\underline{b}}$ in $\mathrm{SR}(A)$ supported on a maximal chain $\mathfrak{C}$ and define $\psi(t_{\underline{b}}) := \psi_{\mathfrak{C}}(t_{\underline{b}})$. It is independent of the choice of the maximal chain $\mathfrak{C}$, hence the map ψ is well-defined.

To show that ψ is a morphism of $\mathbb{K}[\Gamma]$-modules, take $\underline{a} \in \Gamma$, there exists a maximal chain $\mathfrak{C}'$ such that $\mathrm{supp}(\underline{a}) \subseteq \mathfrak{C}'$. Take a monomial $t_{\underline{b}}$ in $\mathrm{SR}(A)$ supported on a maximal chain $\mathfrak{C}$. If the support of $t_{\underline{b}}$ is not contained in $\mathfrak{C}'$, $\phi(x_{\underline{a}}) \cdot t_{\underline{b}} = 0$. Otherwise we assume that they both supported in $\mathfrak{C}$, and $\psi(\phi(x_{\underline{a}}) \cdot t_{\underline{b}}) = \psi_{\mathfrak{C}}(t_{M\underline{a}+\underline{b}}) = \phi(x_{\underline{a}}) \cdot \psi(t_{\underline{b}})$.

One easily verifies that as linear maps $\psi \circ \phi = \mathrm{id}_{\mathbb{K}[\Gamma]}$.

Step 3. The $\mathbb{K}$-algebra $\mathbb{K}[\Gamma]$ is Cohen-Macaulay over $\mathbb{K}$.

We start with the Stanley-Reisner algebra, which is Cohen-Macaulay by assumption. Consider the following elements in $\mathrm{SR}(A)$: for $i = 0, 1, \ldots, r$ with $r = \dim X$,

$$\ell_i := \sum_{p \in A,\ \ell(p) = i} t_p.$$

Since $\mathrm{SR}(A)$ is Cohen-Macaulay, $\ell_0, \ell_1, \ldots, \ell_r$ form a regular sequence in $\mathrm{SR}(A)$.

We prove the Cohen-Macaulay-ness of $\mathbb{K}[\Gamma]$ by constructing a regular sequence in $\mathbb{K}[\Gamma]$ of length $r+1$. Then the depth of $\mathbb{K}[\Gamma]$ is greater or equal than $r+1$, while the other inequality always holds.

Since any two elements in A having the same length are incomparable: we consider the elements

$$\ell_i^M = \sum_{p \in A,\ \ell(p) = i} t_p^M, \quad i = 0, 1, \ldots, r.$$

These elements form a regular sequence in $\mathrm{SR}(A)$, and they are contained in the image of ϕ.

We choose the unique element $u_i \in \mathbb{K}[\Gamma]$ such that $\phi(u_i) = \ell_i^M$, and show that the image of u_i is not a zero divisor in $\mathbb{K}[\Gamma]/(u_0, \ldots, u_{i-1})$. Assume the contrary, there exist $h_0, \ldots, h_{i-1} \in \mathbb{K}[\Gamma]$ such that $u_i h = h_0 u_0 + \cdots + h_{i-1} u_{i-1}$. Applying ϕ gives $\ell_i^M \phi(h) = \phi(h_0)\ell_0^M + \cdots + \phi(h_{i-1})\ell_{i-1}^M$. If $\phi(h) = s_0 \ell_0^M + \cdots + s_{i-1}\ell_{i-1}^M$ for some $s_0, \ldots, s_{i-1} \in \mathrm{SR}(A)$, applying $\mathbb{K}[\Gamma]$-module morphism ψ gives $h = \psi(s_0)u_0 + \cdots + \psi(s_{i-1})u_{i-1}$, contradicting to the assumption that $h \notin (u_0, \ldots, u_{i-1})$. Therefore $\phi(h) \notin (\ell_0^M, \ldots, \ell_{i-1}^M)$, contradicts to the fact that $\ell_0^M, \ell_1^M, \ldots, \ell_r^M$ is a regular sequence.

Step 4. The ring R is normal, hence $X \subseteq \mathbb{P}(V)$ is projectively normal.

By Theorem 11.1, $\mathrm{gr}_{\mathcal{V}} R$ is isomorphic to the fan algebra $\mathbb{K}[\Gamma]$, it is therefore Cohen-Macaulay. Since being Cohen-Macaulay is an open property, Theorem 12.1 implies that R is Cohen-Macaulay. According to the axiom (S1) of a Seshadri stratification, the ring R is smooth in codimension one. By Serre's criterion, R is normal. □

15 Standard monomial theory

In this section we first review the results from the previous sections in terms of a weak form of a standard monomial theory for the ring $R = \mathbb{K}[\hat{X}]$. Imposing additional conditions on the Seshadri stratification gives stronger versions of this theory. We will discuss two of these enhancements: the normality and the balancing of the stratification. In case these requirements are fulfilled, the ring R admits a structure closely related to the LS-algebras [13].

15.1 A basis associated to the leaves

For $\underline{a} \in \Gamma$ we choose a regular function $x_{\underline{a}} \in R$ with quasi-valuation $\mathcal{V}(x_{\underline{a}}) = \underline{a}$. By Lemma 10.2, $R = \bigoplus_{\underline{a}\in\Gamma} \mathbb{K} x_{\underline{a}}$ as a $\mathbb{K}$–vector space. In particular, for each pair $\underline{a}, \underline{b} \in \Gamma$, there exists a relation, called *straigthening relation*,

$$x_{\underline{a}} \cdot x_{\underline{b}} = \sum_{\underline{a}+\underline{b}\leq^t \underline{c}} u_{\underline{c}}^{\underline{a},\underline{b}} x_{\underline{c}} \tag{24}$$

expressing the product $x_{\underline{a}} \cdot x_{\underline{b}}$ in terms of the $\mathbb{K}$–basis elements. In what follows, each time we refer to the coefficients $u_{\underline{c}}^{\underline{a},\underline{b}}$ of a straightening relation, we will simply write $u_{\underline{c}}$ by omitting the dependence on the leaves $\underline{a}$ and $\underline{b}$ if this does not create any ambiguity.

The restriction on the indexes of the possibly non-zero terms in the expression above comes from the fact that $\mathcal{V}$ is a quasi-valuation (Lemma 3.3). This implies that $u_{\underline{c}} \neq 0$ only for those $\underline{c} \in \Gamma$ such that $\underline{a} + \underline{b} \leq^t \underline{c}$. We recall that $\underline{a} + \underline{b} \in \Gamma$ if and only if $\operatorname{supp} \underline{a} \cup \operatorname{supp} \underline{b}$ is a chain in A (Corollary 9.1). In such a case, the term $x_{\underline{a}+\underline{b}}$ does appear in the straightening relation for $x_{\underline{a}} \cdot x_{\underline{b}}$, i.e. $u_{\underline{a}+\underline{b}} \neq 0$. This term is clearly the leading term in the above straightening relation.

Remark 15.1 In analogy to the theory of LS-algebras developed in [13] and [14], one can call the data consisting of the generators $x_{\underline{a}}$ with $\underline{a} \in \Gamma$ and the straightening relations an *algebra with leaf basis* over Γ for the ring R. Results in previous sections can be proved in the general algebraic context of an algebra with leaf basis; this will appear in a forthcoming article.

15.2 Normality and standard monomials

Imposing the normality to the Seshadri stratification allows us to make a natural choice to the monomials, and to define the condition of being standard in a standard monomial theory. Let $\mathfrak{C}$ be a maximal chain in A.

Definition 15.2 An element $\underline{a} \in \Gamma_{\mathfrak{C}}$ is called *decomposable* if it is 0, or if there exist $\underline{a}_1, \underline{a}_2 \in \Gamma_{\mathfrak{C}} \setminus \{0\}$ with $\min \operatorname{supp} \underline{a}_1 \geq \max \operatorname{supp} \underline{a}_2$ such that $\underline{a} = \underline{a}_1 + \underline{a}_2$. We say that $\underline{a}$ is *indecomposable* if it is not decomposable.

Proposition 15.3 *Each* $\underline{a} \in \Gamma_{\mathfrak{C}}$ *has a decomposition* $\underline{a} = \underline{a}_1 + \underline{a}_2 + \cdots + \underline{a}_n$ *with* $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n \in \Gamma_{\mathfrak{C}}$ *indecomposable such that* $\min \operatorname{supp} \underline{a}_j \geq \max \operatorname{supp} \underline{a}_{j+1}$ *for each* $j = 1, 2, \ldots, n-1$.

Proof If $\underline{a}$ is 0 or indecomposable, then there is nothing to prove. So we suppose that $\underline{a}$ is decomposable and proceed by induction on the degree of $\underline{a}$.

Let $\underline{a} = \underline{a}_1 + \underline{a}_2$ with $\underline{a}_1, \underline{a}_2 \neq 0$ and $\min \operatorname{supp} \underline{a}_1 \geq \max \operatorname{supp} \underline{a}_2$. By induction, both $\underline{a}_1$ and $\underline{a}_2$ have decompositions into indecomposable elements. Putting them together gives a decomposition of $\underline{a}$. □

The decomposition as in the above proposition may not be unique. The uniqueness holds in an important special case. Recall that $\Gamma_{\mathfrak{C}}$ is *saturated* if $\mathcal{L}^{\mathfrak{C}} \cap K_{\mathfrak{C}} = \Gamma_{\mathfrak{C}}$ (see the paragraphs before Proposition 13.8).

Proposition 15.4 *If $\Gamma_{\mathfrak{C}}$ is saturated, the decomposition of any element in $\Gamma_{\mathfrak{C}}$ is unique.*

Proof Let $\underline{a} = \underline{b}_1 + \cdots + \underline{b}_n = \underline{c}_1 + \cdots + \underline{c}_m$ be two different decompositions of $\underline{a}$ into indecomposable elements. Let k be minimal such that $\underline{b}_k \neq \underline{c}_k$. We can assume that either $\min \operatorname{supp} \underline{b}_k > \min \operatorname{supp} \underline{c}_k$ or that $p = \min \operatorname{supp} \underline{b}_k = \min \operatorname{supp} \underline{c}_k$ but the entries of $\underline{b}_k$ in p is strictly less than that of $\underline{c}_k$. In both cases $\underline{d} = \underline{c}_k - \underline{b}_k \neq 0$ is an element of $\mathcal{L}^{\mathfrak{C}} \cap K_{\mathfrak{C}} = \Gamma_{\mathfrak{C}}$ and hence $\underline{c}_k = \underline{b}_k + \underline{d}$ with $\min \operatorname{supp} \underline{b}_k \geq \max \operatorname{supp} \underline{d}$. This is impossible since $\underline{c}_k$ is indecomposable. □

Assume that the Seshadri stratification is normal (Definition 13.7). Let $\mathbb{G} \subseteq \Gamma$ be the set of indecomposable elements in Γ. By Proposition 15.3, $\mathbb{G}$ is a generating set of Γ. For each $\underline{a} \in \mathbb{G}$, we fix a regular function $x_{\underline{a}} \in R$ satisfying $\mathcal{V}(x_{\underline{a}}) = \underline{a}$ and denote $\mathbb{G}_R := \{x_{\underline{a}} \mid \underline{a} \in \mathbb{G}\}$.

Definition 15.5 A monomial $x_{\underline{a}_1} \cdots x_{\underline{a}_n}$ with $\underline{a}_1, \ldots, \underline{a}_n \in \mathbb{G}$ is called *standard* if for each j we have $\min \operatorname{supp} \underline{a}_j \geq \max \operatorname{supp} \underline{a}_{j+1}$.

When writing down a standard monomial $x_{\underline{a}_1} \cdots x_{\underline{a}_n}$, it is understood that for each j, $\min \operatorname{supp} \underline{a}_j \geq \max \operatorname{supp} \underline{a}_{j+1}$ holds.

By Proposition 15.4, any element $\underline{a} \in \Gamma$ has a unique decomposition $\underline{a} = \underline{a}_1 + \cdots + \underline{a}_n$ into indecomposable elements. By Proposition 15.3, there exists a maximal chain $\mathfrak{C}$ such that $\operatorname{supp} \underline{a}_j \subseteq \mathfrak{C}$ for all $j = 1, \ldots, n$. By Proposition 8.9, this implies that the quasi-valuation is additive on standard monomials. Summarizing we have:

Proposition 15.6

i) *The set $\mathbb{G}_R$ is a generating set for R.*
ii) *The set of standard monomials in $\mathbb{G}_R$ is a vector space basis for R.*
iii) *If $\underline{a} = \underline{a}_1 + \underline{a}_2 + \cdots + \underline{a}_n$ is the decomposition of $\underline{a} \in \Gamma$ into indecomposables, then the standard monomial $x_{\underline{a}} := x_{\underline{a}_1} \cdots x_{\underline{a}_n}$ is such that $\mathcal{V}(x_{\underline{a}}) = \underline{a}$.*
iv) *If a monomial $x_{\underline{a}_1} \cdots x_{\underline{a}_n}$ is not standard, then there exists a straightening relation expressing it as a linear combination of standard monomials*

$$x_{\underline{a}_1} \cdots x_{\underline{a}_n} = \sum_h u_h x_{\underline{a}_{h,1}} \cdots x_{\underline{a}_{h,n_h}},$$

where, as in (24), $u_h \neq 0$ only if $\underline{a}_1 + \cdots + \underline{a}_n \leq^t \underline{a}_{h,1} + \cdots + \underline{a}_{h,n_h}$.

v) *If in* iv) *there exists a chain* $\mathfrak{C}$ *such that* $\operatorname{supp} \underline{a}_i \subseteq \mathfrak{C}$ *for all* $i = 1, \ldots, n$, *and* $\underline{a}'_1 + \cdots + \underline{a}'_m$ *is the decomposition of* $\underline{a}_1 + \cdots + \underline{a}_n \in \Gamma$ *then the standard monomial* $x_{\underline{a}'_1} \cdots x_{\underline{a}'_m}$ *appears in the right side of the straightening relation in* iv) *with a non-zero coefficient.*

We must note here that it is *not* true in general that, given $p \in A$, the element $e_p \in \Gamma$ is indecomponsable. But there exists a positive integer m_p such that $\frac{1}{m_p} e_p$ is the unique indecomponsable element with support $\{p\}$. Indeed, assume that $p \in \mathfrak{C}$ for a maximal chain $\mathfrak{C}$, then $e_p \in \Gamma_{\mathfrak{C}}$ and there exist indecomponsable elements $u_1 e_p, \ldots, u_n e_p$, with $u_1, \ldots, u_n \in \mathbb{Q}_{>0}$, such that

$$e_p = \sum_{i=1}^{n} u_i e_p$$

is the decomposition of e_p. Now let $\frac{a}{m} e_p$, with $a, m \in \mathbb{Q}_{>0}$, be an indecomponsable element. Then we have

$$\sum_{i=1}^{n} a u_i e_p = a e_p = m \cdot \frac{a}{m} e_p$$

and, since the decomposition of $a e_p$ is unique, we have $a u_i = \frac{a}{m}$ for each i. So $u_i = \frac{1}{m}$, $a = 1$ and $\frac{1}{m} e_p$ is the unique indecomponsable element with support $\{p\}$.

In the following, whenever the stratification is normal, we fix a function $h_p \in R$ such that $\mathcal{V}(h_p) = \frac{1}{m_p} e_p$ and we set $x_{e_p/m_p} = h_p \in \mathbb{G}_R$. The function h_p does not vanish identically on X_p since $\mathcal{V}_{\mathfrak{C}}(h_p) = \frac{1}{m_p} e_p$, where $\mathfrak{C}$ is any maximal chain in A containing p. In some special situations, as will be shown in the proof of Theorem 15.12 below, the functions h_p, $p \in A$, have vanishing properties similar to those of the functions f_p.

15.3 Balanced stratification

In the process of associating a quasi-valuation to a Seshadri stratification, the only choice we made is a total order $\leq^t$ on A refining the given partial order (see Sect. 8 for all possible choices for such a refinement). To emphasize this dependence, we write $\mathcal{V}_{\leq^t}$ for the quasi-valuation and $\Gamma_{\leq^t}$ for the fan of monoids in this subsection.

Definition 15.7 A Seshadri stratification of the variety X is called *balanced*[6] if the following two properties hold:

(1) the set $\Gamma_{\leq^t}$ of leaves for the quasi-valuation $\mathcal{V}_{\leq^t}$ is independent of the choice of the total order $\leq^t$: this allows us to simply write Γ instead of $\Gamma_{\leq^t}$;
(2) for each $\underline{a} \in \Gamma$ there exists a regular function $x_{\underline{a}} \in R$ such that $\mathcal{V}_{\leq^t}(x_{\underline{a}}) = \underline{a}$ for each possible total order $\leq^t$.[7]

[6] The notion of a balanced Seshadri stratification got generalized in [18, Sect. 2.9], where the length-preserving condition on the refinements of the partial order is removed.

[7] This second condition is a consequence of the first one, see [18, Sect. 2.9] for a proof.

Remark 15.8 If the Seshadri stratification is normal and balanced, the second condition above can be weakened: it suffices to require this condition to hold for $\underline{a} \in \mathbb{G}$, the indecomposable elements in Γ.

For a balanced stratification the order requirement in the straightening relations (24) is much stronger. It carries more the spirit of the classical Plücker relations.

Definition 15.9 Let $\underline{a}, \underline{b} \in \mathbb{Q}^A$. We write $\underline{a} \trianglelefteq \underline{b}$ if $\underline{a} \leq^t \underline{b}$ for each total order $\leq^t$ on A extending the given partial order $\leq$ and such that $p <^t q$ if $\ell(p) < \ell(q)$.

Note that the partial order $\trianglelefteq$ is not necessarily a total order on $\mathbb{Q}^A$. Indeed, if p and q are not comparable in A and $\ell(p) = \ell(q)$ then $e_p \ntrianglelefteq e_q$ and $e_q \ntrianglelefteq e_p$.

Proposition 15.10 *If the Seshadri stratification is balanced, then in the straightening relation* (24) *with the choice of* $x_{\underline{a}}$ *as in Definition* 15.8, *we have*: $u_{\underline{c}} \neq 0$ *only if* $\underline{a} + \underline{b} \trianglelefteq \underline{c}$.

15.4 Compatibility with the strata

We have seen in Remark 2.4 that each stratum X_p in the Seshadri stratification of X is naturally endowed with a Seshadri stratification. Let $\mathcal{V}$ be the quasi-valuation on $R = \mathbb{K}[\hat{X}]$ and let $\geq^t$ be the total order on A chosen in the construction of $\mathcal{V}$. We denote by $\geq^t$ the induced total order in A_p, and let $\mathcal{V}_p$ be the associated quasi-valuation on $R_p = \mathbb{K}[\hat{X}_p]$. So we get an associated fan of monoids etc. It is natural to ask under which conditions these objects are compatible with the corresponding ones for X. The best result is obtained in case the Seshadri stratification of X is balanced and normal. Indeed, in this case one gets automatically a standard monomial theory for each subvariety X_p.

We assume for the rest of this subsection: the Seshadri stratification of X is balanced and normal.

Let $\mathbb{G} \subseteq \Gamma$ be the subset of indecomposable elements and $\mathbb{G}_R := \{x_{\underline{a}} \mid \underline{a} \in \mathbb{G}\} \subseteq R$ be a set of regular functions chosen as in Definition 15.7 (2) (see Remark 15.8).

Definition 15.11 Let $p \in A$. A standard monomial $x_{\underline{a}_1} \cdots x_{\underline{a}_n}$ on X is called *standard on* X_p if $\max \operatorname{supp} \underline{a}_1 \leq p$.

By fixing an element $p \in A$ one gets various natural objects associated to A_p: $\mathbb{G}_p := \{\underline{a} \in \mathbb{G} \mid \operatorname{supp} \underline{a} \subseteq A_p\}$, $\mathbb{G}_{R_p} := \{x_{\underline{a}}|_{X_p} \mid \underline{a} \in \mathbb{G}_p\}$, and $\Gamma_p := \{\underline{a} \in \Gamma \mid \operatorname{supp} \underline{a} \subseteq A_p\}$. We consider the vector space $\mathbb{Q}^{A_p}$ as a subspace of $\mathbb{Q}^A$. Since every maximal chain in A_p is a chain in A, by abuse of notation we write $\mathcal{V}_p(f) \in \mathbb{Q}^A$ for a non-zero function $f \in \mathbb{K}[\hat{X}_p]$.

Theorem 15.12 *If the Seshadri stratification of* X *is balanced and normal*, *then the following holds*:

i) *For all* $p \in A$, *the induced Seshadri stratification on* X_p *is balanced and normal.*

ii) *The fan of monoids associated to $\mathcal{V}_p$ is equal to Γ_p, $\mathbb{G}_p$ is its generating set of indecomposables and $\mathbb{G}_{R_p}$ is a generating set for $R_p = \mathbb{K}[\hat{X}_p]$.*
iii) *If $x_{\underline{a}}$ is a standard monomial, standard on X_p, then $\mathcal{V}_p(x_{\underline{a}}|_{X_p}) = \mathcal{V}(x_{\underline{a}}) = \underline{a}$.*
iv) *The restrictions of the standard monomials $x_{\underline{a}}|_{X_p}$, standard on X_p, form a basis of $\mathbb{K}[\hat{X}_p]$.*
v) *A standard monomial $x_{\underline{a}}$ on X vanishes on the subvariety X_p if and only if $x_{\underline{a}}$ is not standard on X_p.*
vi) *The vanishing ideal $\mathcal{I}(X_p) \subseteq R = \mathbb{K}[\hat{X}]$ is generated by the elements in $\mathbb{G}_R \setminus \mathbb{G}_{R_p}$, and the ideal has as vector space basis the set of all standard monomials on X which are not standard on X_p.*
vii) *For all pairs of elements $p, q \in A$, the scheme theoretic intersection $X_p \cap X_q$ is reduced. It is the union of those subvarieties X_r such that $r \leq p$ and $r \leq q$, endowed with the induced reduced structure.*

Proof We start with the most important property v). Let $x_{\underline{a}} := x_{\underline{a}_1} \cdots x_{\underline{a}_n}$ be a standard monomial and $q := \max \operatorname{supp} \underline{a}_1$.

The monomial $x_{\underline{a}}$ is standard on X_p if and only if $q \leq p$. In this case, by Proposition 15.6 iii), we have $\mathcal{V}(x_{\underline{a}}) = \underline{a}$. Let $\mathfrak{C}$ be a maximal chain containing $\operatorname{supp} \underline{a}$ and p (the existence is guaranteed by $q \leq p$). By Proposition 8.7, $\mathcal{V}(x_{\underline{a}}) = \mathcal{V}_{\mathfrak{C}}(x_{\underline{a}})$ and hence $x_{\underline{a}}$ does not vanish on X_p.

For the other implication, assuming $q \not\leq p$, we show that $x_{\underline{a}}$ vanishes on X_p. The proof is by descending induction on the length of q. There are two cases:

(a). q and p are comparable, i.e. $q > p$. Notice that the base step $q = p_{\max}$ of the induction is included in this case. Let M be a positive integer such that $M\underline{a} \in \mathbb{N}^A$.

(a.1). Assume that $\operatorname{supp} \underline{a} = \{q\}$. Then $\underline{a} = n \cdot \frac{1}{m_q} e_q$ by the discussion after Proposition 15.6; so we can choose $M = m_q$. The monomial $x_{\underline{a}}^M = x_{M\underline{a}}$ is standard and it is equal to h_q^{nM}. Note that f_q^n and $x_{M\underline{a}}$ have the same value ne_q for each quasi-valuation $\mathcal{V}_{\leq^t}$; hence we have

$$f_q^n = c \cdot h_q^{nM} + g,$$

where $c \in \mathbb{K}^*$ and g is a linear combination of standard monomials $x_{\underline{b}}$ of the same degree of f_q^n with $ne_q \trianglelefteq \underline{b}$, $ne_q \neq \underline{b}$. So, if we set $q' = \max \operatorname{supp} \underline{b}$ we have $\ell(q') > \ell(q)$. Hence $q' \not\leq p$ and, by induction, g vanishes on X_p. Since f_q vanishes on X_p as well, it follows that h_q, and hence $x_{\underline{a}}$ vanish on X_p too.

(a.2). We assume that $\operatorname{supp} \underline{a} \neq \{q\}$. Since the Seshadri stratification is normal, there exist $0 \neq \underline{a}' \in \Gamma$ and $n \geq 1$ such that $M\underline{a} = ne_q + \underline{a}'$ and $\max \operatorname{supp} \underline{a}' < q$. In this case $x_{M\underline{a}} = h_q^{nm_q} x_{\underline{a}'}$. Notice that $x_{\underline{a}}^M$ is *not* a standard monomial: in its straightening relation we choose a standard monomial $x_{\underline{b}}$ with non-zero coefficient. By Proposition 15.10, $M\underline{a} \trianglelefteq \underline{b}$. We denote $q' := \max \operatorname{supp} \underline{b}$, then either $q = q'$ or $\ell(q') > \ell(q)$.

In the first case, we can write $\underline{b} = me_q + \underline{b}'$ with a rational number $m \geq n$. Since $\underline{b} - ne_q \in \Gamma$, $x_{\underline{b}} = h_q^{nm_q} x_{\underline{b} - ne_q}$. Since we have already proved in (a.1) that h_q vanishes on X_p, so does $x_{\underline{b}}$. In the second case, $x_{\underline{b}}$ vanishes on X_p by induction.

(b). q and p are not comparable. In this case $x_{\underline{a}}h_p$ is not standard: in its straightening relation we take a standard monomial $x_{\underline{b}}$ with non-zero coeffcient. By Proposition 15.10, $\underline{a}+e_p/m_p \trianglelefteq \underline{b}$. We denote $q' := \max \operatorname{supp} \underline{b}$, then $\ell(q') > \ell(q)$ and $\ell(q') > \ell(p)$. The last inequality implies $q' \not\leq p$, and the induction can be applied: $x_{\underline{b}}$ vanishes on X_p. This shows that $x_{\underline{a}}h_p$ vanishes on X_p. Since h_p does not vanish on X_p and X_p is irreducible, $x_{\underline{a}}$ vanishes on X_p.

This completes the proof of v).

We prove iii). Let $x_{\underline{a}}$ be a standard monomial, standard on X_p. By v) we know that the restriction $x_{\underline{a}}|_{X_p}$ does not identically vanish, so it makes sense to consider the quasi-valuation $\mathcal{V}_p(x_{\underline{a}}|_{X_p})$. Let $\mathfrak{C}' \subseteq A_p$ be a maximal chain in A_p and let $\mathfrak{C}$ be an extension of the chain to a maximal chain in A. Keeping in mind the identification of $\mathbb{Q}^{A_p}$ as a subspace of $\mathbb{Q}^A$, the renormalizing coefficients of the valuation in Definition 6.2 are chosen such that $\mathcal{V}_{p,\mathfrak{C}'}(x_{\underline{a}}|_{X_p}) = \mathcal{V}_{\mathfrak{C}}(x_{\underline{a}})$. Conversely, since $\operatorname{supp} \underline{a} \subseteq A_p$, one can always find a maximal chain $\mathfrak{C}'$ in A_p containing the support and extend this chain to a maximal chain $\mathfrak{C}$ in A. Since $\mathfrak{C}$ contains $\operatorname{supp} \underline{a}$ we get: $\mathcal{V}(x_{\underline{a}}) = \mathcal{V}_{\mathfrak{C}}(x_{\underline{a}})$, and the latter is by the above equal to $\mathcal{V}_{p,\mathfrak{C}'}(x_{\underline{a}}|_{X_p})$. It follows: $\mathcal{V}_p(x_{\underline{a}}|_{X_p}) = \mathcal{V}(x_{\underline{a}})$.

Parts i), ii), iv) follow from part iii). Part *vi)* is an immediate consequence of iv) and v).

It remains to prove vii). Let $Y \subseteq X$ be the union of the subvarieties X_r such that $r \leq p$ and $r \leq q$, endowed with the induced reduced structure. We say a standard monomial $x_{\underline{a}}$ on X is standard on Y if it is standard on at least one of its irreducible components. It follows that the restriction $x_{\underline{a}}|_Y$ of a standard monomial vanishes identically on Y if and only if it is not standard on Y. So the restrictions $x_{\underline{a}}|_Y$ of the standard monomials, standard on Y, span the homogeneous coordinate ring $\mathbb{K}[Y]$ as a vector space. Indeed, it is a basis: given a linear dependence relation between standard monomials $x_{\underline{a}}|_Y$, standard on Y, fix a $t \in A$ such that at least one summand with a non-zero coefficient is standard on X_t. All summands which are not standard on X_t vanish on X_t, so after restricting the linear dependence relation to X_t one gets a non-trivial linear dependence relation between standard monomials, standard on X_t, which is not possible by *iv)*. It follows that the homogeneous coordinate ring of Y has as vector space basis the standard monomials, standard on Y, and the vanishing ideal $\mathcal{I}(Y) \subseteq R$ has as vector space basis the standard monomials which are not standard on Y. Using the decomposition of an element $\underline{a} \in \Gamma$ into indecomposables, we see that $\mathcal{I}(Y)$ is generated by those $x_{\underline{a}} \in \mathbb{G}_R$ such that $x_{\underline{a}}$ is not standard on Y. But this implies $x_{\underline{a}}$ is either not standard on X_p and hence $x_{\underline{a}} \in \mathcal{I}(X_p)$, or $x_{\underline{a}}$ is not standard on X_q and hence $x_{\underline{a}} \in \mathcal{I}(X_q)$. It follows by *vi)*: $\mathcal{I}(X_q) + \mathcal{I}(X_p) = \mathcal{I}(Y)$, which finishes the proof. □

15.5 Algorithmic aspects of standard monomials

We restate some of the above results in the language of Khovanskii basis and discuss an implementation of the subduction algorithm (see for example [38]) to write a mononial in $R = \mathbb{K}[\hat{X}]$ into a linear combination of standard monomials. We keep notations as in previous subsections.

Definition 15.13

(1) A subset $\mathbb{B} \subseteq R$ is called a *Khovanskii basis for the quasi-valuation* $\mathcal{V}_{\leq^t}$, if the image of $\mathbb{B}$ in $\mathrm{gr}_{\mathcal{V}_{\leq^t}} R$ generates the algebra $\mathrm{gr}_{\mathcal{V}_{\leq^t}} R$.
(2) A subset $\mathbb{B} \subseteq R$ is called a *Khovanskii basis for the Seshadri stratification*, if it is a Khovanskii basis for all possible $\mathcal{V}_{\leq^t}$, where $\leq^t$ is a linear extensions of $\leq$ satisfying: if $\ell(p) < \ell(q)$ then $p <^t q$.

Combining Lemma 9.6 and Theorem 11.1 gives

Corollary 15.14

i) *For any total order* $\leq^t$, *there exists a finite Khovanskii basis* $\mathbb{B}_{\leq^t}$ *for the quasi-valuation* $\mathcal{V}_{\leq^t}$.
ii) *If the Seshadri stratification is balanced, there exists a finite Khovanskii basis for the Seshadri stratification.*
iii) *If the Seshadri stratification is normal and balanced, the set* $\mathbb{G}_R$ *is a Khovanskii basis for the Seshadri stratification.*

Assume hereafter that the Seshadri stratification is normal. For $g \in R$, denote by $\overline{g}$ its image in $\mathrm{gr}_{\mathcal{V}} R$.

Algorithm 15.15 (Subduction algorithm)
Input: A non-zero homogeneous element $f \in R$.
Output: $f = \sum c_{\underline{a}_1,\ldots,\underline{a}_n} x_{\underline{a}_1} \cdots x_{\underline{a}_n}$ where $c_{\underline{a}_1,\ldots,\underline{a}_n} \in \mathbb{K}^*$ and $x_{\underline{a}_1} \cdots x_{\underline{a}_n}$ is a standard monomial.
Algorithm:

(1). Compute $\underline{a} := \mathcal{V}(f)$ and choose a maximal chain $\mathfrak{C}$ such that $\underline{a} \in \Gamma_{\mathfrak{C}}$.
(2). Decompose $\underline{a}$ into a sum of indecomposable elements $\underline{a} = \underline{a}_1 + \cdots + \underline{a}_n$ such that $\min \mathrm{supp}\, \underline{a}_i \geq \max \mathrm{supp}\, \underline{a}_{i+1}$.
(3). Compute $\overline{f}$ and $\overline{x}_{\underline{a}_1} \cdots \overline{x}_{\underline{a}_n}$ in $\mathrm{gr}_{\mathcal{V}} R$ to find $\lambda \in \mathbb{K}^*$ such that $\overline{f} = \lambda \overline{x}_{\underline{a}_1} \cdots \overline{x}_{\underline{a}_n}$.
(4). Print $\lambda x_{\underline{a}_1} \cdots x_{\underline{a}_n}$ and set $f_1 := f - \lambda x_{\underline{a}_1} \cdots x_{\underline{a}_n}$. If $f_1 \neq 0$ then return to Step (1) with f replaced by f_1.
(5). Done.

Proposition 15.16 *For any valid input, the algorithm terminates and prints out the output as in the description.*

Proof We first prove the termination of the algorithm. Assume the contrary. In the first step $\mathcal{V}(f) < \mathcal{V}(f_1)$ holds by construction. Iterating this argument gives an infinite long sequence

$$\mathcal{V}(f) < \mathcal{V}(f_1) < \mathcal{V}(f_2) < \cdots,$$

hence $f, f_1, f_2, \ldots$ are linearly independent. In the first step f and $x_{\underline{a}_1} \cdots x_{\underline{a}_n}$ have the same degree, so does f_1. Repeating this argument shows that $f, f_1, f_2, \ldots$ are in the same homogeneous component of R, contradicting to the linear independency.

Once the termination is established, the correctness holds by Lemma 10.2 and Proposition 15.6. □

The subduction algorithm allows us to lift relations in $\mathrm{gr}_{\mathcal{V}}R$ to R.

Let S denote the polynomial ring in variables $y_{\underline{a}}$ for $\underline{a} \in \mathbb{G}$. There is a surjective algebra morphism $\widetilde{\varphi} : S \to R$ sending the variable $y_{\underline{a}}$ to the regular function $x_{\underline{a}} \in R$. Let I denote the kernel of $\widetilde{\varphi}$.

We apply the subduction algorithm to lift relations from $\mathrm{gr}_{\mathcal{V}}R$ to R. Let $\varphi : \mathbb{K}[t_{\underline{a}} \mid \underline{a} \in \mathbb{G}] \to \mathrm{gr}_{\mathcal{V}}R$ be the algebra morphism defined by $t_{\underline{a}} \mapsto \overline{x}_{\underline{a}}$ and $I_{\mathcal{V}}$ be its kernel. By Proposition 15.6, φ is surjective. For $r(t_{\underline{a}}) \in I_{\mathcal{V}}$ we set $g := r(y_{\underline{a}}) \in S$. The setup is summarized in the following diagram:

$$\begin{array}{ccc} S := \mathbb{K}[y_{\underline{a}}] & \xrightarrow{\widetilde{\varphi}} & R \\ \downarrow & & \downarrow \\ \mathbb{K}[t_{\underline{a}}] & \xrightarrow{\varphi} & \mathrm{gr}_{\mathcal{V}}R. \end{array}$$

We apply the subduction algorithm to $\widetilde{\varphi}(g)$ and denote the output by h. Being a linear combination of standard monomials, h can be looked as an element in S. We set $\widetilde{r} := g - h \in \ker\widetilde{\varphi} \subseteq S$.

From the subduction algorithm, all these relations $\widetilde{r}$ for $r \in I_{\mathcal{V}}$ are sufficient to rewrite a product of elements in the generating set $\mathbb{G}_R$ as a linear combination of standard monomials. This proves

Corollary 15.17 *The ideal I is generated by $\{\widetilde{r} \mid r \in I_{\mathcal{V}}\}$.*

As a matter of fact, lifting a particular generating set of $I_{\mathcal{V}}$ gives a Gröbner basis of I. Details on this lifting, applications to the Koszul and Gorenstein properties, as well as the relations to Lakshmibai-Seshadri algebras, can be found in [17].

16 Examples

16.1 Seshadri stratifications of Hodge type

We consider a special case of a Seshadri stratification where all bonds appearing in the extended Hasse diagram $\mathcal{G}_{\hat{A}}$ are 1; such a Seshadri stratification will be called of *Hodge type*. Their properties are summarized in the following proposition.

Proposition 16.1 *The following statements hold for a Seshadri stratification of Hodge type.*

i) *For any maximal chain $\mathfrak{C}$, the monoid $\Gamma_{\mathfrak{C}}$ coincides with $\mathbb{N}^{\mathfrak{C}}$, hence the Seshadri stratification is normal.*
ii) *The degenerate algebra $\mathrm{gr}_{\mathcal{V}}R$ is isomorphic to the Stanley-Reisner algebra* $\mathrm{SR}(A)$.
iii) *There exists a flat degeneration of X into a union of weighted projective spaces, one for each maximal chain in $\mathcal{C}$.*

iv) *If the poset A is Cohen-Macaulay over $\mathbb{K}$, $X \subseteq \mathbb{P}(V)$ is projectively normal.*
v) *The degree of the embedded variety $X \subseteq \mathbb{P}(V)$ is*

$$\sum_{\mathfrak{C} \in C} \frac{1}{\prod_{q \in \mathfrak{C}} \deg f_q}.$$

vi) *The Hilbert polynomial of R is given by the formula in Proposition 13.8.*
vii) *The Seshadri stratification is balanced.*
viii) *If $\deg(f_p) = 1$ for all $p \in A$, then the subvarieties X_p are defined in X by linear equations.*
ix) *For any $p, q \in A$, $X_p \cap X_q$ is a reduced union of the subvarieties contained in both of them.*

Proof By Remark 6.7, for any maximal chain $\mathfrak{C}$ in A, the lattice $L^{\mathfrak{C}} \cong \mathbb{Z}^{\mathfrak{C}}$. Together with Example 6.8, it implies $\Gamma_{\mathfrak{C}} = \mathbb{N}^{\mathfrak{C}}$. Therefore $\mathbb{K}[\Gamma_{\mathfrak{C}}]$ are polynomial algebras, and the associated (projective) toric varieties are weighted projective spaces. The first six statements follow from Proposition 9.8, Theorem 11.1, Theorem 13.6, Theorem 14.1 and Proposition 13.8.

It remains to show vii), then viii) and ix) follow from Theorem 15.12. By i), all monoids $\Gamma_{\mathfrak{C}} = \mathbb{N}^{\mathfrak{C}}$ do not depend on the possible choices of the total order $\leq^t$. The set $\mathbb{G}$ of indecomposable elements is given by $\{e_p \mid p \in A\}$. In view of Remark 15.8, we choose f_p as the regular function, its image under the quasi-valuation $\mathcal{V}_{\leq^t}$ is e_p, which is independent of the possible choices of $\leq^t$. □

According to Remark 2.4, for any $p \in A$, the induced Seshadri stratification on X_p is of Hodge type, so results in the above proposition hold for X_p.

We apply it to low degree projective varieties.

Corollary 16.2 *If $X \subseteq \mathbb{P}(V)$ is a projective variety of degree 2 which is smooth in codimension one, then X is projectively normal.*

Proof Under the degree 2 assumption, the generic hyperplane stratification for X in Proposition 2.11 with choices of $f_{0,k}$ as in Example 2.12 is of Hodge type. The statement follows from Proposition 16.1 iv). □

Example 16.3 We consider the Seshadri stratification of the Grassmann variety $\mathrm{Gr}_d\mathbb{K}^n$ with the Plücker embedding in Example 2.6 where all the bonds are 1 and the extremal functions are of degree 1. In this case the poset A is a distributive lattice, hence for any $p \in A$, the subposet A_p is shellable and hence Cohen-Macaulay over any field $\mathbb{K}$ ([6]). Applying Proposition 16.1, we obtain:

i) a degeneration of Schubert varieties $X(\underline{i}) \subseteq \mathrm{Gr}_d\mathbb{K}^n$ for $\underline{i} \in I_{d,n}$ into a union of projective spaces using quasi-valuations, recovering the main results in [27];
ii) the projective normality of the Schubert varieties $X(\underline{i})$ for $\underline{i} \in I_{d,n}$ in the Plücker embedding;
iii) the degree of the embedded Schubert varieties $X(\underline{i})$ as the cardinality of $C_{\underline{i}}$ for $\underline{i} \in I_{d,n}$;

iv) the Schubert varieties are defined by linear equations in the Grassmann variety;
v) the intersection of two Schubert varieties $X(\underline{i}) \cap X(\underline{j})$ is a reduced union of Schubert varieties.

The projective normality of the Schubert varieties in Grassmann varieties are proved by Hochster [32], Laksov [49], and Musili [57] (see also the work of Igusa [34] for the Grassmann varieties themselves). Our approach is a geometrization of the one in the framework of Hodge algebra by De Concini, Eisenbud and Procesi in [23].

Remark 16.4 On Grassmann varieties there exist many Seshadri stratifications. For example, all positroid varieties [42] in X, together with some well-chosen extremal functions, form a Seshadri stratification on $\mathrm{Gr}_2\mathbb{C}^4$. Details will be given in a forthcoming work.

Example 16.5 For any finite lattice $\mathcal{L}$ with meet operation $\wedge$ and join operation $\vee$, Hibi [31] introduced a graded $\mathbb{K}$-algebra

$$\mathcal{R}_{\mathbb{K}}(\mathcal{L}) := \mathbb{K}[x_\ell \mid \ell \in \mathcal{L}]/(x_\ell x_{\ell'} - x_{\ell\wedge\ell'} x_{\ell\vee\ell'} \mid \ell, \ell' \in \mathcal{L} \text{ non comparable}),$$

which is an integral domain if and only if $\mathcal{L}$ is a distributive lattice. In this case, one obtains a projective toric variety $Y_{\mathcal{L}} \subseteq \mathbb{P}(\mathbb{K}^{|\mathcal{L}|})$, called Hibi toric variety. For $p \in \mathcal{L}$, $\mathcal{L}_p := \{\ell \in \mathcal{L} \mid \ell \leq p\}$ is again a distributive lattice. We leave to the reader to verify that the collection of projective subvarieties $Y_{\mathcal{L}_p}$ and $f_p := x_p$ for $p \in \mathcal{L}$ defines a Seshadri stratification of Hodge type on $Y_{\mathcal{L}}$. Proposition 16.1 recovers the well-known degeneration of the Hibi toric variety into the variety associated to the Stanley-Reisner algebra $\mathrm{SR}(\mathcal{L})$ and its projective normality.

16.2 Compactification of a maximal torus

The action of the torus T of $\mathrm{PSL}_3(\mathbb{C})$ on $\mathfrak{sl}_3(\mathbb{C})$ defines an embedding of T in $\mathbb{P}(\mathrm{End}(\mathfrak{sl}_3(\mathbb{C})))$. Since this is a diagonal action and the weights of $\mathfrak{sl}_3(\mathbb{C})$ are the union Φ_0 of 0 and the root system Φ, we get an embedding

$$T \ni t \longmapsto [t^\gamma \mid \gamma \in \Phi_0] \in \mathbb{P}^6(\mathbb{C}).$$

Let $X \subseteq \mathbb{P}^6(\mathbb{C})$ be the torus compactification given by the closure of the image of this embedding. We want to show that the T–orbit closures in X are the strata for a Seshadri stratification.

If we denote by $[x_\gamma \mid \gamma \in \Phi_0]$ the homogeneous coordinates in $\mathbb{P}^6(\mathbb{C})$ then the equations defining X in $\mathbb{P}^6(\mathbb{C})$ are

$$x_\gamma x_\delta = x_\epsilon x_\eta, \quad \gamma, \delta, \epsilon, \eta \in \Phi_0 \text{ such that } \gamma + \delta = \epsilon + \eta.$$

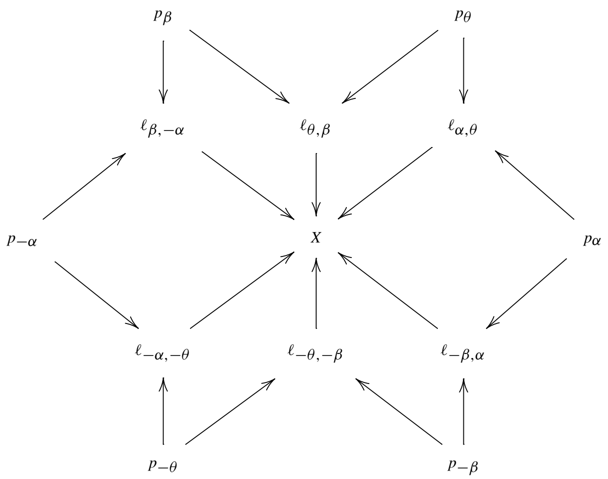
It is well known that the T-orbit closures in X are in bijection with the faces of the polyhedral decomposition of $\mathbb{R}^2$ given by the Weyl chambers. Further, it is easy to check that:

(i) we have six 0–dimensional (closed) orbits $\{p_\gamma\}$, $\gamma \in \Phi$, where $p_\gamma \in X$ has coordinates $x_\gamma(p_\gamma) = 1$ and $x_\delta(p_\gamma) = 0$ for $\delta \neq \gamma$,

(ii) the six 1–dimensional orbit closures are the projective lines $\ell_{\gamma,\delta}$ passing through p_γ, p_δ, with $\gamma, \delta \in \Phi$, $\gamma \neq \delta$ and $\gamma + \delta \notin \Phi_0$.
(iii) X is the unique 2–dimensional orbit closure.

Clearly these varieties are all smooth in codimension one.

Let us denote by α, β, θ the three positive roots of Φ such that $\alpha + \beta = \theta$. The inclusion relations among these orbits, i.e. the same inclusion relations among the faces of the polyhedral decomposition of $\mathbb{R}^2$, are as follows:



If we choose as extremal functions

(i) x_γ for the orbit p_γ,
(ii) $x_\gamma x_\delta$ for the orbit closure $\ell_{\gamma,\delta}$,
(iii) x_0 for X

then we get a Seshadri stratification with all bonds equal to 1.

Remark 16.6 A normal projective toric variety admits a Seshadri stratification where the subvarieties are orbit closures arising from the torus action. We will return to this example in a separate work.

16.3 A compactification of $\mathrm{PSL}_2(\mathbb{C})$

The De Concini–Procesi compactification X of $G = \mathrm{SL}_2(\mathbb{C})$ is the projective space of all 2×2 matrices (see [22]). The group $G \times G$ acts on X by left and right multiplication

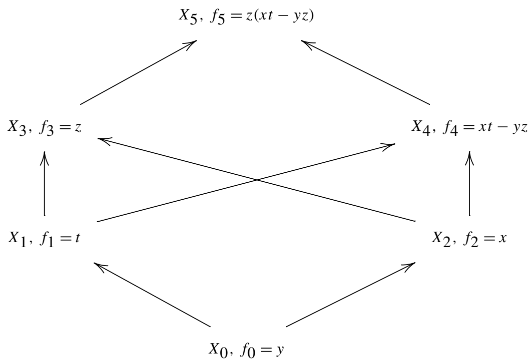
$$(g, h) \cdot [A] = [gAh^{-1}].$$

If B is the Borel subgroup of upper triangular matrices of G, then the $B \times B$–orbit closures in X are: $X_0 = \left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$, a unique point, $X_1 = \left\{\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}\right\}$ and $X_2 =$

$\left\{\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}\right\}$, two projective lines, $X_3 = \left\{\begin{bmatrix} * & * \\ * & * \end{bmatrix} \text{ of rank one}\right\}$, a smooth quadric, $X_4 = \left\{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}\right\}$, a projective plane, and $X_5 = X \simeq \mathbb{P}^3(\mathbb{C})$.

Let $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ be the coordinates on the space of matrices and define the following functions: $f_0 = y$, $f_1 = t$, $f_2 = x$, $f_3 = z$, $f_4 = xt - yz$ and $f_5 = z(xt - yz)$.

The inclusion relations for the subvarieties and the associated functions are



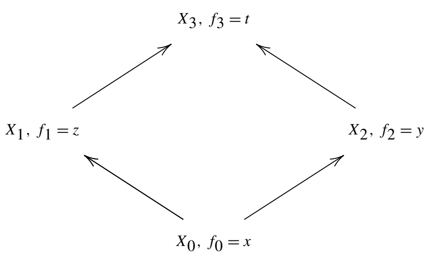
It is easy to check that the orbit closures $X_0, \dots, X_5$ with the corresponding extremal functions $f_0, \dots, f_5$ give a Seshadri stratification of X.

16.4 A family of quadrics

We see a Seshadri stratification for certain quadrics. Let $[x : y : z : t]$ be the homogeneous coordinates on $\mathbb{P}^3(\mathbb{C})$, let $h = ax + by + cz$ be a linear polynomial where $(a, b) \neq (0, 0)$ and $(a, c) \neq (0, 0)$ and consider the quadric X defined by $yz - th = 0$ in $\mathbb{P}^3(\mathbb{C})$.

The quadric is smooth if $a \neq 0$ and has $[1 : 0 : 0 : 0]$ as unique singular point if $a = 0$. So it is smooth in codimension one.

We define the strata as follows: $X_3 := X$, $X_2 := \{[x : y : z : t] \in X \mid z = t = 0\}$, $X_1 := \{[x : y : z : t] \in X \mid y = t = 0\}$ and $X_0 := \{[1 : 0 : 0 : 0]\}$. It is clear that X_2 is the projective line in $\mathbb{P}^3$ defined by $z = t = 0$ and X_1 is the projective line in $\mathbb{P}^3$ defined by $y = t = 0$. We take as extremal functions: $f_3 := t$, $f_2 := y$, $f_1 := z$ and $f_0 := x$. One easily verifies that these data give a Seshadri stratification for X.



Example 16.7 We consider the special case $a = 0$, $b = c = 1$: the quadric X is defined by the homogeneous equation $yz - t(y + z)$. The poset $A = \{p_3, p_2, p_1, p_0\}$ with $X_{p_i} := X_i$ for $i = 0, 1, 2, 3$. Let $\mathfrak{C} = (p_3, p_1, p_0)$ be the maximal chain to the left of the above diagram.

We claim that the valuation $\mathcal{V}_{\mathfrak{C}}(y) = (1, 0, 0)$. Indeed, since all bonds are 1, we set $N = 1$. The function y vanishes on X_1 with order 1; and the function $\frac{y}{t} = 1 + \frac{y}{z}$, once restricted to X_1, is the constant function 1.

The function y is positive along the chain $\mathfrak{C}$, but by Lemma 8.3, it is not standard along the chain $\mathfrak{C}$. In fact, we have the following relation in $\mathbb{K}[U_{\mathfrak{C}}]$:

$$y = t + \frac{ty}{z}, \quad \text{with } \mathcal{V}_{\mathfrak{C}}\left(\frac{ty}{z}\right) = (2, -1, 0) > (1, 0, 0),$$

which explains the equality $\mathcal{V}_{\mathfrak{C}}(y) = \mathcal{V}_{\mathfrak{C}}(t)$.

16.5 An elliptic curve

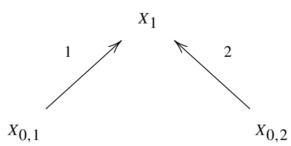
Let X be the elliptic curve defined by the homogeneous equation $y^2z - x^3 + xz^2$ in $\mathbb{P}^2$. Let $R := \mathbb{K}[x, y, z]/(y^2z - x^3 + xz^2)$ be its homogeneous coordinate ring. We abuse the notation and use the same variables x, y, z for their classes in R.

We present two Seshadri stratifications on X.

(1) We consider the Seshadri stratification on X defined as follows. The subvarieties are $X_1 := X$ and $X_0 := \{[0 : 1 : 0]\}$. As extremal functions we choose $f_1 = z$ and $f_0 = y$. The Hasse graph is a chain $X_1 \leftarrow X_0$. The vanishing order of z at X_0 is 3, which is the bond between X_1 and X_0. The quasi-valuation $\mathcal{V}$ is the valuation associated to this unique chain; it is determined by $\mathcal{V}(x) = (1/3, 2/3)$, $\mathcal{V}(y) = (0, 1)$ and $\mathcal{V}(z) = (1, 0)$. Having different valuations, the monomials $x^p y^q z^r$, where $p = 0, 1, 2$, $q, r \in \mathbb{N}$ are linearly independent, hence form a basis of R. It follows that the valuation monoid is generated by $\mathcal{V}(x)$, $\mathcal{V}(y)$ and $\mathcal{V}(z)$. Theorem 12.2 produces a toric degeneration of X to the toric variety $\mathrm{Proj}(\mathbb{K}[x, y, z]/(y^2z - x^3))$. The Newton-Okounkov complex is in this case the segment connecting $(1, 0)$ and $(0, 1)$.

    This toric degeneration is obtained in [39] using a filtration arising from symbolic powers.

(2) We give another Seshadri stratification on X. The subvarieties are $X_1 := X$, $X_{0,1} := \{[0 : 1 : 0]\}$ and $X_{0,2} := \{[0 : 0 : 1]\}$. The extremal functions are $f_1 = x$, $f_{0,1} := y$ and $f_{0,2} := z$. The Hasse graph with bonds is depicted below:



We leave to the reader to verify that this is indeed a Seshadri stratification, and the degeneration predicted by Theorem 12.2 is the union of two toric varieties, one of the two is a $\mathbb{P}^1$, and the normalization of the other one is a twisted cubic. The

Newton-Okounkov complex is a union of two segments at a point. The degree formula reads: $3 = 1 + 2$.

16.6 Flag varieties and Schubert varieties

An important motivation for introducing Seshadri stratifications was the aim to understand standard monomial theory on Schubert varieties and the associated combinatorics (as developed by Lakshmibai, Musili, Seshadri and others [43–48, 54, 65–68]) in the framework of Newton-Okounkov theory.

We present in this subsection just an announcement, the detailed proofs are published in a separate article [15]. A different approach without using quantum groups at roots of unity is carried out in [18]. We stick in the following for simplicity to the case of flag varieties G/B for G a simple simply connected algebraic group, though the results hold, with an appropriate reformulation, also for Schubert varieties (more generally, unions thereof) contained in a partial flag variety in the symmetrizable Kac-Moody case (see *loc.cit*).

We fix a Borel subgroup B of G, a maximal torus $T \subseteq B$ and a regular dominant weight λ. The associated line bundle $\mathcal{L}_\lambda$ on $X := G/B$ is ample, and we have a corresponding embedding $G/B \hookrightarrow \mathbb{P}(V(\lambda))$, where $V(\lambda)$ is the Weyl module of highest weight λ. Let W be the Weyl group of G, endowed with the partial order given by its structure as Coxeter group.

16.6.1 Seshadri stratification of G/B

The Bruhat decomposition $G = \bigcup_{w \in W} BwB$ of G implies: G/B has a decomposition into cells $C(w) := BwB/B$, called Schubert cells. The closure of a cell is called a Schubert variety $X(w) := \overline{C(w)}$. These varieties have an induced cellular decomposition: $X(w) = \bigcup_{u \leq w} C(u)$. Schubert varieties are known to be smooth in codimension one, see for example [68, Corollary 4.4.5].

Let $\Lambda := \Lambda(T)$ be the character group of T and let $\Lambda_\mathbb{R}$ be the Euclidean vector space $\Lambda \otimes_\mathbb{Z} \mathbb{R}$, endowed with the Killing form as a W-invariant scalar product. Let $P_\lambda \subseteq \Lambda_\mathbb{R}$ be the polytope obtained as the convex hull of all characters with a non-zero weight space in $V(\lambda)$. The vertices of this polytope are the characters $\{\sigma(\lambda) \mid \sigma \in W\}$.

Fix a highest weight vector $v_\lambda \in V(\lambda)$ and for $\sigma \in W$ set $v_\sigma = n_\sigma(v_\lambda)$, where $n_\sigma \in N_G(T)$ is a representative of σ in the normalizer in G of T. Then v_σ is a T-eigenvector for the character $\sigma(\lambda)$. The cell $C(\sigma) \subseteq G/B \subseteq \mathbb{P}(V(\lambda))$ can be identified with the B-orbit $B.[v_\sigma]$.

Fix a lowest weight vector $\ell_{-\lambda} \in V(\lambda)^*$. For $\sigma \in W$ let $f_\sigma = n_\sigma(\ell_{-\lambda}) \in V(\lambda)^*$, then $f_\sigma(v_\tau) \neq 0$ if and only if $\tau = \sigma$. Using the cellular decomposition, one sees that $\mathcal{H}_{f_\sigma} \cap X(\sigma)$ is the union of all codimension one Schubert varieties in $X(\sigma)$. And the property of the weights $\sigma(\lambda)$ to be vertices in P_λ can be used to show that $f_\sigma|_{C(\tau)} \not\equiv 0$ implies $[v_\sigma] \in X(\tau)$. It follows $X(\sigma) = \overline{B.[v_\sigma]} \subseteq X(\tau)$, and hence $\tau \geq \sigma$. So for any $\sigma \in W$, f_σ vanishes on $X(\tau)$ if $\sigma \not\leq \tau$.

Therefore the collection of subvarieties $X(\tau)$ and functions f_τ, $\tau \in W$, satisfies the three axioms for a Seshadri stratification of $X = G/B$.

It is clearly evident that to establish a standard monomial theory in such a general setting, one needs good candidates: for the monoid $\Gamma_\mathfrak{C}$ we present a candidate $L^+_{\mathfrak{C},\lambda}$, which is a reincarnation of the path model given by the Lakshmibai-Seshadri paths (*LS-paths* for short) of shape λ. And for each element $\pi \in L^+_{\mathfrak{C},\lambda}$ of degree one, we present a linear function $p_\pi \in V(\lambda)^*$, a candidate for a representative of a leaf of the quasi-valuation $\mathcal{V}$. The points to prove are: $L^+_{\mathfrak{C},\lambda} = \Gamma_{\mathfrak{C}}$, $\mathcal{V}(p_\pi) = \pi$, and the standard monomials (as in Sect. 15) form a basis of the homogeneous coordinate ring $\mathbb{K}[\hat{X}]$.

16.6.2 LS-paths

We recall first the notion of an LS-path π of shape λ (see [46, 51]). Given a maximal chain $\sigma = \sigma_t > \cdots > \sigma_0 = \tau$ joining two elements $\sigma, \tau \in W$, $\sigma > \tau$, there exist positive roots $\beta_1, \ldots, \beta_r$ such that $s_{\beta_i}\sigma_{i-1} = \sigma_i$ and $\ell(\sigma_i) = \ell(\sigma_{i-1}) + 1$, $i = 1, \ldots, t$. Given a rational number a, the chain is called an (a, λ)-*chain* joining σ and τ if in addition $a\langle\sigma_i(\lambda), \beta_i^\vee\rangle \in \mathbb{Z}$ for all $i = 1, \ldots, t$. It has been shown in [24] that if one maximal chain between σ and τ has the property of being an (a, λ)-chain, then all maximal chains joining σ and τ are (a, λ)-chains.

Definition 16.8 An *LS-path* $\pi = (\underline{\sigma} : \sigma_p > \sigma_{p-1} > \cdots > \sigma_1; \underline{a} : 0 < a_p < \cdots < a_1 = m)$ of shape λ and degree $m \geq 1$ is a pair of sequences of linearly ordered elements in W and rational numbers such that for all $i = 2, \ldots, p$ there exists an (a_i, λ)-chain joining σ_{i-1} and σ_i. Let $\mathrm{LS}(\lambda)$ denote the set of LS path of shape λ and arbitrary degree $m \geq 0$.

An LS-path $\pi = (\sigma_p > \sigma_{p-1} > \cdots > \sigma_1; \underline{a}) \in \mathrm{LS}(\lambda)$ is said to be *supported in* $\mathfrak{C}$, where $\mathfrak{C}$ is a maximal chain in W, if $\{\sigma_1, \ldots, \sigma_p\} \subseteq \mathfrak{C}$.

The LS-path π is said to be of shape λ *on a Schubert variety* $X(\tau)$ if in addition $\tau \geq \sigma_p$. We call σ_p the *initial direction* of the path, and σ_1 the *final direction* of π.

For convenience we add the empty path () as an LS-path of shape λ and degree 0. The weight of an LS-path is defined as $\pi(1) := \sum_{j=1}^p (a_j - a_{j+1})\sigma_j(\lambda)$, where $a_{p+1} := 0$. The weight of the empty path is set to be 0. The weight $\pi(1)$ is a weight appearing in the Weyl module $V(m\lambda)$, and the character of this representation is given by the following sum, running over all LS-paths of shape λ and degree m [51, 52]:

$$\mathrm{Char}\, V(m\lambda) = \sum_\pi e^{\pi(1)}. \tag{25}$$

Remark 16.9 The character formula will play a role later because it gives a dimension bound for the graded part $\mathbb{K}[G/B]_m$ of the homogeneous coordinate ring. We have a natural map from the homogeneous coordinate ring $\mathbb{K}[G/B]$ to the ring of sections $\bigoplus_{m\geq 0} \mathrm{H}^0(G/B, \mathcal{L}_\lambda^{\otimes m})$. The graded part of degree m: $\mathrm{H}^0(G/B, \mathcal{L}_\lambda^{\otimes m})$ is, as representation, the dual $V(m\lambda)^*$ of the Weyl module $V(m\lambda)$. So the dimension of $V(m\lambda)$ is an upper bound for the dimension of $\mathbb{K}[G/B]_m$.

16.6.3 LS-paths as fan of monoids

We translate the definition of LS-paths into the language of lattices and Hasse diagrams with bonds: the indexing set for the Schubert varieties in X is the Weyl group

W, endowed with the Bruhat order as partial order. The bonds are given by the Pieri-Chevalley formula: if σ covers τ and $\tau = s_\beta\sigma$ for a positive root β, then the bond is $b_{\sigma,\tau} = \langle\tau(\lambda), \beta^\vee\rangle$. Fix a maximal chain $\mathfrak{C}:\ w_0 = \sigma_r > \cdots > \sigma_0 = \mathrm{id}$, where w_0 is the maximal element in W. As before, for such a fixed maximal chain, we simplify the notation by writing b_i instead of $b_{\sigma_i,\sigma_{i-1}}$ for the bonds. Note that all extremal functions have degree 1 and hence $b_0 = 1$. Let $L_{\mathfrak{C},\lambda} \subseteq \mathbb{Q}^{\mathfrak{C}}$ be the lattice

$$L_{\mathfrak{C},\lambda} = \left\{ u = \begin{pmatrix} u_r \\ \vdots \\ u_0 \end{pmatrix} \in \mathbb{Q}^{\mathfrak{C}} \;\middle|\; \begin{array}{r} b_r u_r \in \mathbb{Z} \\ b_{r-1}(u_r + u_{r-1}) \in \mathbb{Z} \\ \cdots \\ b_1(u_r + u_{r-1} + \cdots + u_1) \in \mathbb{Z} \\ u_0 + u_1 + \cdots + u_r \in \mathbb{Z} \end{array} \right\}. \tag{26}$$

We call the sum $u_0 + u_1 + \cdots + u_r$ the degree of u.

As before, we view $\mathbb{Q}^{\mathfrak{C}}$ as a subspace of $\mathbb{Q}^W$, and we consider the union of the lattices $L_\lambda := \bigcup_{\mathfrak{C}} L_{\mathfrak{C},\lambda} \subseteq \bigcup_{\mathfrak{C}} \mathbb{Q}^{\mathfrak{C}}$ as a subset of $\mathbb{Q}^W$. The set $L_\lambda^+ = L_\lambda \cap \mathbb{Q}^W_{\geq 0}$ is our candidate for the fan of monoids Γ. Indeed, the results in [24] mentioned above show that the set L_λ^+ is a fan of monoids.

It remains to give an explicit bijection between the set of LS-paths $\mathrm{LS}(\lambda)$ and the fan of monoids L_λ^+. The proof of the following lemma will be given in [15], see [13, Proposition 1] for a proof in a slightly different language. In the following formula we set $a_{p+1} = 0$.

Lemma 16.10 *The map ν defined below*:

$$\begin{array}{llll} \nu: & \mathrm{LS}(\lambda) & \longrightarrow & L_\lambda^+ = L_\lambda \cap \mathbb{Q}^W_{\geq 0}; \\ & \pi = (\sigma_p, \ldots, \sigma_1; 0, a_p, \ldots, a_1 = m) & \mapsto & \nu(\pi) := \sum_{j=1}^p (a_j - a_{j+1}) e_{\sigma_j}, \end{array}$$

induces a bijection between the set of LS-paths of shape λ and degree $m \geq 0$, and the elements in L_λ^+ of degree m.

It is understood that the empty path is mapped to 0.

Consider again the lattice $L_{\mathfrak{C},\lambda}$ and the intersection $L^+_{\mathfrak{C},\lambda} := L_{\mathfrak{C},\lambda} \cap \mathbb{Q}^{\mathfrak{C}}_{\geq 0}$. By the above Lemma, we can identify this monoid with the set of all LS-paths of shape λ and supported in $\mathfrak{C}$. This monoid is normal, and hence by Proposition 15.4, every path in $L^+_{\mathfrak{C},\lambda}$ can be decomposed in a unique way into a sum of indecomposable paths. One can show (see [15] or [13, Proposition 3]) that the only indecomposable paths are those of degree 1.

Following Sect. 15.2, one can view a path π of degree $m > 1$ as a tuple $\pi = (\pi_1, \ldots, \pi_m)$ of m LS-paths of shape λ and degree 1, satisfying the additional condition: for all $i = 1, \ldots, m-1$, the final direction of π_i is larger or equal to the initial direction of π_{i+1}.

Example 16.11 Let $G = \mathrm{SL}_3(\mathbb{K})$ and $\lambda = \omega_1 + \omega_2$.

The pair $\pi = (s_1s_2s_1, s_2s_1, s_1; 0, 1, \frac{3}{2}, 2, 3)$ is an LS-paths of shape λ and degree 3. Then $\pi = (\pi_1, \pi_2, \pi_3)$, where $\pi_1 = (s_1s_2s_1; 0, 1)$, $\pi_2 = (s_2s_1, s_1; 0, \frac{1}{2}, 1)$ and $\pi_3 = (s_1; 0, 1)$ is the decomposition of π into LS-paths of shape λ and degree 1.

16.6.4 The candidates for the leaves

It was shown in [54] that one can associate to every LS-path of shape λ and degree 1 a linear function $p_\pi \in V(\lambda)^*$, called a path vector. Roughly speaking, one fixes for $\pi = (\sigma_p, \ldots, \sigma_1; 0, a_p, \ldots, a_1 = 1)$ a natural number n such that $na_j \in \mathbb{N}$ for all $j = 1, \ldots, p$. The function p_π is then intuitively defined as $\sqrt[n]{f_{\sigma_p}^{na_p} \cdots f_{\sigma_1}^{na_1}}$, i.e. an n-th root of this product of extremal functions. Indeed, we need a representation-theoretic trick: Lusztig's quantum Frobenius map at an appropriate root of unity makes it possible to define analogues of the n-th roots of these kind of functions.

By construction, these linear functions have the following property: p_π vanishes identically on a Schubert variety $X(\tau) \subseteq X$ if and only if:

$$\pi = (\sigma_p, \sigma_{p-1}, \ldots, \sigma_1; 0, a_p, \ldots, a_1 = 1) \text{ is such that } \sigma_p \not\leq \tau. \tag{27}$$

Let $\mathcal{V} : \mathbb{K}[\hat{X}] \to \mathbb{Q}^W$ be the quasi-valuation provided by the Seshadri stratification described above. The proof of the following theorem can be found in [15]:

Theorem 16.12 *For all LS-paths π of shape λ, degree 1 and support in $\mathfrak{C}$ one has*

$$\mathcal{V}(p_\pi) = \mathcal{V}_{\mathfrak{C}}(p_\pi) = \nu(\pi).$$

The value $\mathcal{V}(p_\pi)$ is independent of the choice of the total order in the construction of $\mathcal{V}$.

One has as immediate consequences of Theorem 16.12:

Corollary 16.13

i) *The fan of monoids L_λ^+ of LS-paths of shape λ and degree $m \geq 1$ is contained in the fan of monoids Γ.*
ii) *The set $\mathbb{B} = \{p_\pi \mid \pi$ LS-paths of shape λ and degree $1\}$ is a basis for $V(\lambda)^*$.*

Proof The first claim follows from the fact that the monoids in the fan of monoids L_λ^+ are generated by the degree 1 elements. The elements in $\mathbb{B}$ are linearly independent because vectors with different quasi-valuations are linearly independent [37], and the character formula in (25) implies that $\mathbb{B}$ is a basis for $V(\lambda)^*$. □

Since $L_\lambda^+ \subseteq \Gamma$, we know by Theorem 16.12 that a path vector $p_\pi \in V(\lambda)^*$ is a representative for the leaf associated to $\nu(\pi)$. Since these elements are all of degree 1, they are indecomposable. So we can talk about standard monomials in the sense of Definition 15.5. Reformulated into the language of LS-paths this gives: a monomial of degree m of path vectors: $p_{\pi_1} \cdots p_{\pi_m}$ is called a *standard monomial*, if the tuple $\underline{\pi} = (\pi_1, \ldots, \pi_m)$ is an LS-path of shape λ and degree m.

Theorem 16.14

i) *The above Seshadri stratification of G/B is normal and balanced.*

ii) *The standard monomials in the path vectors form a basis of the homogeneous coordinate ring of the embedded flag variety* $G/B \hookrightarrow \mathbb{P}(V(\lambda))$.
iii) *This basis is compatible with the quasi-valuation* $\mathcal{V}$.
iv) *We have* $\Gamma = L_\lambda^+$.

Proof Since the paths occurring in a standard monomial $p_\pi = p_{\pi_1} \cdots p_{\pi_m}$ have support in a common maximal chain, we get by Proposition 8.9 and Theorem 16.12:

$$\mathcal{V}(p_\pi) = \sum_{i=1}^{m} \nu(\pi_j),$$

and the image is independent of the choice of the total order in the definition of $\mathcal{V}$.

The uniqueness of the decomposition of a LS-path of degree m into paths of degree 1 implies the linear independence of the set of standard monomials of degree m. Indeed, vectors with different quasi-valuations are linearly independent [37]. The character formula in (25) implies that $\dim \mathbb{K}[X]_m \geq \dim V(m\lambda)^*$, which by Remark 16.9 implies that the standard monomials of degree m in the path vectors form a basis of $\mathbb{K}[X]_m$. By construction, the basis is compatible with the leaves of the quasi-valuation, which implies the claim in the theorem. □

As an immediate consequence, Proposition 15.10 provides straightening relations expressing a non-standard monomial as a linear combination of standard monomials.

16.6.5 Schubert varieties

By Remark 2.4, each stratum $X(\tau)$ in the Seshadri stratification of $X = G/B$ is naturally endowed with a Seshadri stratification. Since the Seshadri stratification is normal and balanced, by Theorem 15.12, the induced Seshadri stratification of a Schubert variety is normal and balanced. In particular, we get a standard monomial theory for each Schubert variety which is compatible in the sense of Sect. 15.4 with the standard monomial theory on G/B.

As consequences we recover the following known results: Schubert varieties are projectively normal (Theorem 14.1 and [7]; see also [61] for a different proof using Frobenius splitting); the degenerate variety is a union of normal toric varieties; the scheme theoretic intersection of Schubert varieties is reduced (Theorem 15.12). In this case, further results like (1) vanishing theorems for higher cohomology; (2) the identification of the Newton-Okounkov simplicial complex to the polytopes equipped with an integral structure in [24]; (3) the connection between the flat degeneration in [13] and that in Sect. 12; will be discussed in [15].

We conclude this section with a degree formula, which is an application of Theorem 13.6, but now reformulated into the setting of Schubert varieties. Denote by $V(\lambda)_\tau \subseteq V(\lambda)$ the subspace generated by the affine cone $\hat{X}(\tau)$ over the Schubert variety $X(\tau) \subseteq G/B \subseteq \mathbb{P}(V(\lambda))$. This subspace is called the Demazure submodule of $V(\lambda)$ associated to τ. The following formula can also be found in [13], and in [40] in a symplectic context:

Proposition 16.15 *The degree of the embedded Schubert variety $X(\tau) \subseteq \mathbb{P}(V(\lambda)_\tau)$ is equal to the sum $\sum_{\mathfrak{C}} \prod_{j=1}^{s} b_{j,\mathfrak{C}}$ running over all maximal chains $\mathfrak{C}$ in A_τ, and $\prod_{j=1}^{s} b_{j,\mathfrak{C}}$ is the product of over all bonds along a maximal chain $\mathfrak{C}$.*

List of notations

A	poset indexing strata in Seshadri stratification
$\hat{A}$	extended poset
A_p	subposet of A
$b_{p,q}$	bonds between p and q
$\mathfrak{C}$	maximal chain in A
$\mathcal{C}$	set of all maximal chains in A
$\mathcal{C}(g)$	maximal chains hitting minimum
$\Delta_{\mathcal{V}}$	Newton-Okounkov complex associated to Seshadri stratification
f_p	extremal functions
$\mathcal{G}_A$	Hasse graph with bonds
Γ	the fan of monoid associated to $\mathcal{V}$
$\Gamma_{\mathfrak{C}}$	monoid in Γ
$g_{\mathfrak{C}}$	sequence of rational function associated to g along $\mathfrak{C}$
$>^t$	fixed total order on A
$\mathrm{gr}_{\mathfrak{C}} R$	associated graded algebra of $\mathcal{V}_{\mathfrak{C}}$
$\mathrm{Gr}_d \mathbb{K}^n$	Grassmann variety
$\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$	subalgebra of $\mathrm{gr}_{\mathcal{V}} R$
$\mathrm{gr}_{\mathcal{V}} R$	associated graded algebra of quasi-valuation $\mathcal{V}$
$I_{\mathfrak{C}}$	annihilating ideal associated to $\mathfrak{C}$
$\mathcal{K}$	set of all chains in A
K_C	cone associated to chain C in A
$\mathbb{K}[\Gamma]$	the fan algebra associated to Γ
$\mathcal{K}^s$	set of all saturated chains in A
ℓ	length function on A
$L^{\mathfrak{C}}$	a lattice
$L^{\mathfrak{C},\dagger}$	submonoid of lattice $L^{\mathfrak{C}}$
$\mathcal{L}^{\mathfrak{C}}$	lattice generated by $\Gamma_{\mathfrak{C}}$
N	l.c.m of all bonds in $\mathcal{G}_A$
$P_{\mathfrak{C}}(X)$	core of valuation monoid $\mathcal{V}_{\mathfrak{C}}(X)$
R_p	homogeneous coordinate ring of X_p
$\mathrm{supp}\, \mathcal{V}(g)$	support of function g
$U_{\mathfrak{C}}$	an open affine subset of $\hat{X}$
$\mathcal{V}_{\mathfrak{C}}$	normalized valuation associated to a maximal chain
$\widetilde{\mathcal{V}}_{\mathfrak{C}}$	a variation of the valuation $\mathcal{V}_{\mathfrak{C}}$
$\mathbb{V}_{\mathfrak{C}}(U_{\mathfrak{C}})$	valuation monoid on $U_{\mathfrak{C}}$
$\mathbb{V}_{\mathfrak{C}}$	valuation monoid of $\mathcal{V}_{\mathfrak{C}}$
$\mathcal{V}(g)$	minimized quasi-valuation of function g
$\nu_{p,q}$	valuation from vanishing multiplicity
X_p	strata in Seshadri stratification
$\hat{X}_p$	affine cone over X_p

$\tilde{X}_p$	normalization of X_p
$Z_{\mathcal{M}}$	affine variety associated to saturated set $\mathcal{M}$

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